MATH7501: Mathematics for Data Science I

Unit 5: Sequences, Limits and Series

Slides by Timothy Buttsworth (2021)

Motivation: The Need for Sequences

What is the area of a circle of radius 1?

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$3.1415 \underbrace{\ldots\ldots}$

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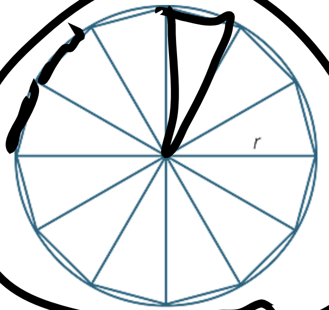

http://www.amsi.org.au/teacher_modules/the_circle.html

We can approximate this number, perhaps using geometry, or other sophisticated methods.

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As technology improves, we will get arbitrarily good approximations of π, but we will never know what it is exactly!

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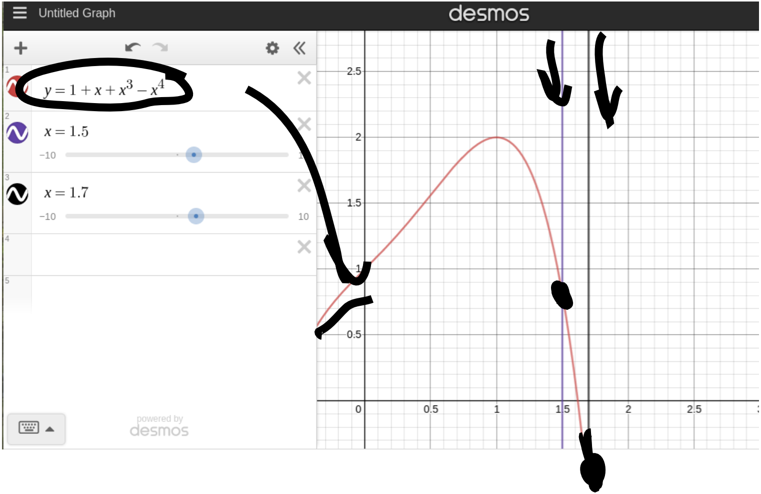
Consider the function $f(x) = 1 + x + x^3 - x^4$. Find a positive x so that $f(x) = 0$.



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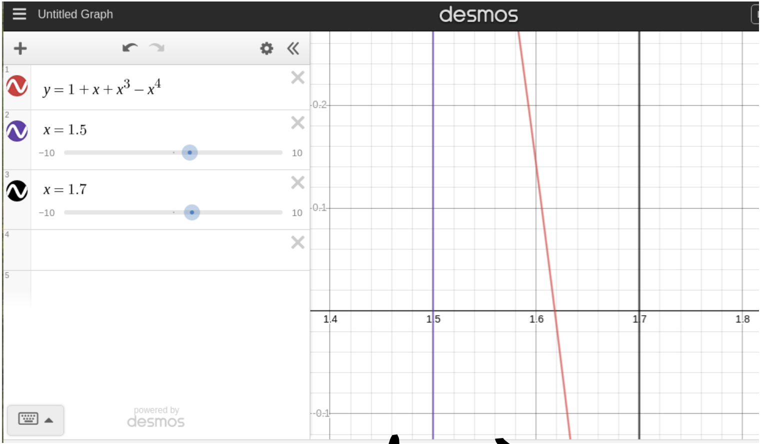

zoomed in.

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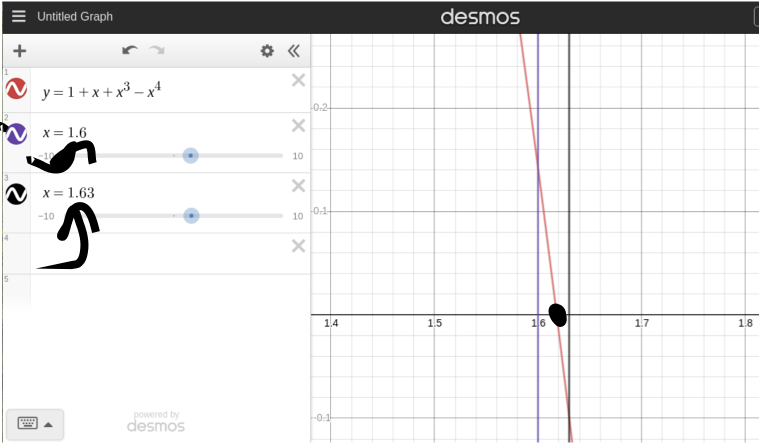
1.5
1.7



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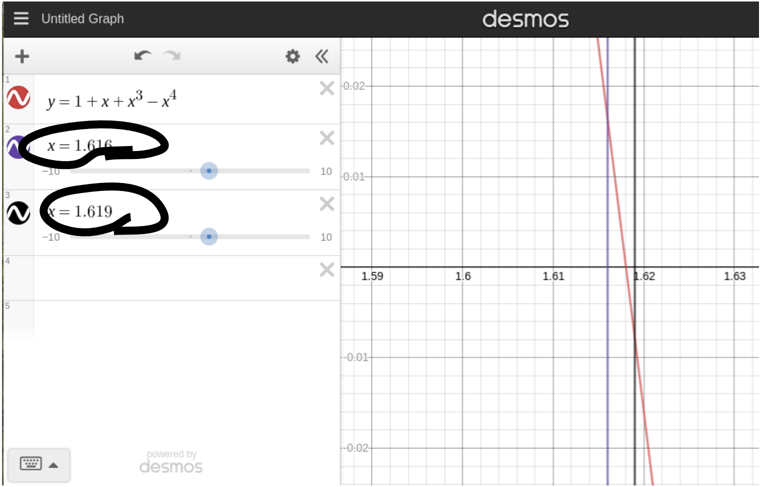

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Some things to note about these approximations:

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- The numbers $1.7, 1.63, 1.619$ are all higher than the true solution, but are also getting close.
- We could continue in this manner, with accuracy increasing.
- The true solution is $\frac{1+\sqrt{5}}{2}$, but $\sqrt{5}$ is like π, in the sense that we cannot describe it exactly.

exact solution.

5.1 Sequences of Real Numbers

If we have a large collection of real numbers (possibly infinite), and their order is important, they can be stored in a sequence.

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A *sequence* is an ordered set of real numbers. Suppose the ordered numbers are $a_1, a_2, a_3, \cdots, a_n$ (n could be ∞).

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We do not insist that the real numbers are subscripted with 1 at the start. For example, we could have a sequence $a_0, a_1, a_2, \cdots$; this time, a_i is the $i+1$th term of the sequence.

a_0 is the 1st

Example (Digits of π)

Let $\{a_k\}_{k=0}^{\infty}$ be the sequence so that a_k is π correct to the first k decimal places. So the sequence is

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415, \cdots$$

The first term of the sequence is 3, the second term of the sequence is 3.1 and so on.

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$$\boxed{3},\ 3.1,\ 3.14,\ 3.141,\ 3.1415, \cdots$$

The first term of the sequence is 3, the second term of the sequence is 3.1 and so on. Although there are infinitely-many terms in this sequence, we can only write down the ones for $k \leq$ ~~31.4 trillion~~

50 trillion

Example (Ancestors)

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$$\boxed{a_k = 2^k} \text{ for each } k. \quad \in \mathbb{N}$$

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In this case, we see that

$$\boxed{a_k = 2^k} \text{ for each } k.$$

This is an example of a *general formula for a sequence*. These arise any time we can write an arbitrary term a_k explicitly in terms of k.

Question

Suppose we have a sequence $\{a_k\}_{k=1}^{\infty}$ described explicitly by the following general formula:

$$\boxed{a_k = (-1)^{k+1}\frac{k}{k+3}.}$$

Compute the first four terms of this sequence.

$\Leftrightarrow$ Compute a_1, a_2, a_3, a_4.

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Solution

- The first term is $a_1 = (-1)^{1+1}\frac{1}{1+3} = \frac{1}{4}$.

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- The fourth term is $a_4 = (-1)^{4+1}\frac{4}{4+3} = -\frac{4}{7}$.

Question

Find a general formula for a sequence $\{a_k\}_{k=1}^{\infty}$ whose first five terms are $-1, 4, -27, 256, -3125.$

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Solution

The alternating minus signs means the sequence can look like

$$a_k = (-1)^k \boxed{b_k,}$$

for some sequence of positive numbers $\{b_k\}_{k=1}^{\infty}$ with $b_1 = 1, b_2 = 4, b_3 = 27, b_4 = 256, b_5 = 3125.$

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1^1 2^2 3^3 4^4 5^5

The b_k sequence can be $b_k = k^k$, so

$$\boxed{a_k = (-1)^k k^k}$$

is a possibility.

Sigma Sum Notation

Suppose we have a sequence

$\{a_k\}_{k=1}^{n}$ $\quad a_1, \; a_2, \; a_3, \cdots$

and we wanted to add up all of the terms between a_m and a_n (inclusive).

Sigma Sum Notation

Suppose we have a sequence

$$a_1,\ a_2,\ a_3, \cdots$$

and we wanted to add up all of the terms between a_m and a_n (inclusive). If $m < n$, we denote this with

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Σ = 'sum'

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$$\sum_{k=m}^{n} a_k \leftarrow \sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

The letter i is called a *dummy variable* because it can be replaced with any other letter, and the expression would still mean the same thing. On the other hand, the choice of n and m affect this expression as they tell us where to start and stop the adding.

5.1 Sequences of Real Numbers

$$\sum_{n=1}^{100} \frac{1}{n^2}$$

In Mathematica, use the following:

```
In[2]:= Sum[1/n^2, {n, 1, 100}]

Out[2]= 1589508694133037873112297928517553859702383498543709859889432834803818131090⋱
          369901/
        972186144434381030589657976672623144161975583995746241782720354705517986⋱
          165248000
```

Use `1.` to get the decimal representation:

```
In[3]:= Sum[1./n^2, {n, 1, 100}]

Out[3]= 1.63498
```

Question (Convert Sigma Sum to Expanded Notation)

Write out the summation $\sum_{i=1}^{n}(2i-1)$ in expanded form by writing out the first five terms and the last term.

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Solution

$2 \cdot 1 - 1$

$$\sum_{i=1}^{\boxed{n}}(2i-1) = 1 + 3 + 5 + 7 + 9 + \cdots + (2n-1).$$

$2 \cdot 2 - 1$

Question (Convert Expanded to Sigma Sum Notation)

Express the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \boxed{\cdots} + \frac{1}{n \cdot (n+1)}$$

in Sigma Sum notation.

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Terms appear to be of the form $\frac{1}{i(i+1)}$ for i running from 1 to n,

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$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}=\sum_{i=1}^{n}\frac{1}{i(i+1)}.$$

Question (Sum of Squares)

It is well-known that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Use this expression to find a simplified expression for $\boxed{\sum_{i=1}^{n+1} i^2}$ in terms of n.

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Solution

Since $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ holds for each n, we can replace n with $n+1$ to find

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

Product Notation

For a given sequence $\{a_k\}$, the expression $\boxed{\prod_{k=m}^{n} a_k}$ means Product

$$\underline{a_m \times a_{m+1} \times \cdots \times a_n},$$

i.e., $\prod_{k=m}^{n} a_k$ is the product of the terms in the sequence between a_m and a_n (with $n > m$).

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Definition (Factorials)

For a given positive integer n, the number $\boxed{n!}$ (pronounced 'n factorial') is the product of all positive integers up to and including n, i.e.,

$$\prod_{k=1}^{n} k = \boxed{n!} = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

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By convention $\boxed{0! = 1.}$

Question

Simplify the following expressions as much as possible:

$$\prod_{j=2}^{5} 2^j, \qquad \frac{10!}{8!2!}, \qquad \frac{(n+2)!}{n!}.$$

5.1 Sequences of Real Numbers

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Solution

- $\prod_{j=2}^{5} 2^j = 2^2 \times 2^3 \times 2^4 \times 2^5 = 2^{14} = \boxed{16384}$

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- $\prod_{j=2}^{5} 2^j = 2^2 \times 2^3 \times 2^4 \times 2^5 = 2^{14} = 16384$
- $\frac{10!}{8!2!} = \frac{10\times 9\times \cancel{8}\times \cancel{7}\times\cdots\times \cancel{2}\times \cancel{1}}{(\cancel{8}\times \cancel{7}\times\cdots\times \cancel{2}\times \cancel{1})\times(2\times 1)} = \frac{10\times 9}{2\times 1} = 45$

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Solution

- $\prod_{j=2}^{5} 2^j = 2^2 \times 2^3 \times 2^4 \times 2^5 = 2^{14} = 16384$
- $\frac{10!}{8!2!} = \frac{10\times9\times8\times7\times\cdots\times2\times1}{(8\times7\times\cdots\times2\times1)\times(2\times1)} = \frac{10\times9}{2\times1} = 45$
- $\frac{(n+2)!}{n!} = \frac{(n+2)\times(n+1)\times\cancel{n}\times\cdots\times\cancel{2}\times\cancel{1}}{\cancel{n}\times\cancel{(n-1)}\times\cdots\times\cancel{2}\times\cancel{1}} = (n+2)(n+1)$

Definition (Recursion Relation)

A sequence $\{a_k\}$ so that each term can be written explicitly as a function of the preceding terms is known as a *recursive sequence*, and the explicit formula itself is known as a *recursion relation*.

Example

Consider the recursive sequence $\{a_k\}_{k=1}^{\infty}$ so that $a_1 = 2$, and for each $k \geq 1$,

$$a_{k+1} = \frac{1}{3 - a_k} \text{ (recursive formula).}$$

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$$a_3 = \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2}$$

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$$a_{k+1} = \frac{1}{3 - a_k} \text{ (recursive formula).}$$

The first four terms of this sequence are 2, 1, $\frac{1}{2}$, $\frac{2}{5}$.

$$a_2 = a_{1+1} = \frac{1}{3-a_1} = \frac{1}{3-2} = 1$$

Example (The Fibonacci Sequence)

The Fibonacci sequence is a sequence of integers $\{F_k\}_{k=1}^{\infty}$ so that

$$F_1 = 1$$
$$F_2 = 1$$
$$F_k = F_{k-1} + F_{k-2}, \textit{ for } k \geq 3.$$

The first 8 terms are $1, 1, 2, 3, 5, 8, 13, 21$.

$F_3 = F_1 + F_2 = 2$

$F_4 = F_3 + F_2 = 3$

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The first 8 terms are $1, 1, 2, 3, 5, 8, 13, 21$. It turns out there is a general formula for the Fibonacci sequence:

$$* \quad F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^k.$$

$\frac{1+\sqrt{5}}{2}$ = Golden Ratio

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$$\boxed{F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k.}$$

This can be proven using strong mathematical induction.

Some Remarks About Sequences

So far we have seen three ways of describing sequences:

1. Writing out the first few terms in the sequence, and 'hoping' that the general pattern is obvious. For example, consider the sequence whose first few terms are $\underbrace{3, 5, 7,} \cdots$. What is the next term?

$3, 5, 7, 9, 11, \ldots$ the odd integers

$3, 5, 7, 11, 13, 17 \ldots$

prime numbers starting at 3.

5.1 Sequences of Real Numbers

Some Remarks About Sequences

So far we have seen three ways of describing sequences:

1. Writing out the first few terms in the sequence, and 'hoping' that the general pattern is obvious. For example, consider the sequence whose first few terms are $3, 5, 7, \cdots$. What is the next term?
2. Giving a general formula which explains how each term is calculated. For example, consider the sequence $\{a_n\}_{n=0}^{\infty}$ so that $a_n = \frac{(-1)^n}{n+2}$.

5.1 Sequences of Real Numbers

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3. Writing out the first few terms, and then giving a formula which explains how any terms can be constructed from the preceding terms. For example, a sequence $\{c_i\}_{i=0}^{\infty}$ with $c_0 = 1$, $c_1 = 2$ and $c_s = c_{s-1} + 3c_{s-2} + 1$ for all $s \geq 2$.

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It is sometimes possible to move between descriptions for a given sequence.

Question

Let $\{c_k\}_{k=0}^{\infty}$ be a sequence so that

$$c_k = (k-1)c_{k-1} + kc_{k-2} + k, \text{ for } k \geq 2$$
$$c_0 = 1, \qquad c_1 = 2.$$

Calculate c_2, c_3, c_4.

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Calculate c_2, c_3, c_4.

Solution

- $c_2 = (2-1)c_1 + 2c_0 + 2 = 2 + 2 + 2 = 6.$

Question

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$$c_0 = 1, \qquad c_1 = 2.$$

Calculate c_2, c_3, c_4.

Solution

- $c_2 = (2-1)c_1 + 2c_0 + 2 = 2 + 2 + 2 = 6.$
- $c_3 = (3-1)c_2 + 3c_1 + 3 = 12 + 6 + 3 = 21.$

Question

Let $\{c_k\}_{k=0}^{\infty}$ be a sequence so that

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$$c_0 = 1, \qquad c_1 = 2.$$

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Solution

- $c_2 = (2-1)c_1 + 2c_0 + 2 = 2 + 2 + 2 = 6.$
- $c_3 = (3-1)c_2 + 3c_1 + 3 = 12 + 6 + 3 = 21.$
- $c_4 = (4-1)c_3 + 4c_2 + 4 = 63 + 24 + 4 = 91.$

Question

Let $\{b_i\}_{i=0}^{\infty}$ be a sequence so that

$$b_i = 5b_{i-1} - 6b_{i-2}, \text{ for } i \geq 2,$$

with b_0, b_1 unspecified. Find expressions for b_{i+1}, b_{i+2}.

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Solution

The given formula holds for all $i \geq 2$, so

$$b_{i+1} = 5b_{(i+1)-1} - 6b_{(i+1)-2} = 5b_i - 6b_{i-1}.$$

5.1 Sequences of Real Numbers

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Let $\{b_i\}_{i=0}^{\infty}$ be a sequence so that

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with b_0, b_1 unspecified. Find expressions for b_{i+1}, b_{i+2}.

Solution

The given formula holds for all $i \geq 2$, so

$$b_{i+1} = 5b_{(i+1)-1} - 6b_{(i+1)-2} = 5b_i - 6b_{i-1}.$$

Similarly,

$$b_{i+2} = 5b_{(i+2)-1} - 6b_{(i+2)-2} = 5b_{i+1} - 6b_i.$$

Question

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n = 5 \cdot 2^n$. Show that this can be defined recursively with

$$\boxed{\begin{aligned} a_n &= 2a_{n-1}, \text{ for } n \geq 1, \\ a_0 &= 5. \end{aligned}}$$

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$$\boxed{a_0 = 5.}$$

Solution

Clearly $a_0 = 5$ and $a_1 = 10$.

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LH

$$a_n = 2a_{n-1}, \text{ for } n \geq 1,$$

RHS

$$a_0 = 5.$$

Solution

Clearly $a_0 = 5$ and $a_1 = 10$. Also by definition of the sequence,

$$a_n = 5 \cdot 2^n = 2 \cdot (5 \cdot 2^{n-1}) = 2a_{n-1}.$$

LHS

$= a_{n-1}$

Question

The Catalan numbers can be defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all integers $n \geq 1$. Find C_i for $i = 1, 2, 3$, and then show that $\boxed{C_k = \frac{4k-2}{k+1} C_{k-1}}$ for all $k \geq 2$.

"$2n$ choose n"

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}$$

5.1 Sequences of Real Numbers

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Solution

$C_1 = \frac{2!}{2 \cdot 1! \cdot 1!} = 1,$

$$C_1 = \frac{1}{1+1}\binom{2}{1} = \frac{1}{2} \times \frac{2!}{1!(2-1)}$$

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Solution

$C_1 = \frac{2!}{2 \cdot 1! \cdot 1!} = 1$, $C_2 = \frac{4!}{2! \cdot 2! \cdot 3} = 2$,

$$\frac{1}{2+1}\binom{4}{2} = \frac{1}{3} \times \frac{4!}{2!(4-2)!}$$

Question

The Catalan numbers can be defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all integers $n \geq 1$. Find C_i for $i = 1, 2, 3$, and then show that $C_k = \frac{4k-2}{k+1}C_{k-1}$ for all $k \geq 2$.

LHS RHS

Solution

$C_1 = \frac{2!}{2 \cdot 1! \cdot 1!} = 1,\ C_2 = \frac{4!}{2! \cdot 2! \cdot 3} = 2,\ C_3 = \frac{6!}{3! \cdot 3! \cdot 4} = 5.$

Question

The Catalan numbers can be defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all integers $n \geq 1$. Find C_i for $i = 1, 2, 3$, and then show that $C_k = \boxed{\frac{4k-2}{k+1}} C_{k-1}$ for all $k \geq 2$.

Solution

$C_1 = \frac{2!}{2 \cdot 1! \cdot 1!} = 1$, $C_2 = \frac{4!}{2! \cdot 2! \cdot 3} = 2$, $C_3 = \frac{6!}{3! \cdot 3! \cdot 4} = 5$. In general

LHS

$$
\begin{aligned}
C_n &= \frac{2n!}{n!n!(n+1)} \\
&= \frac{2n(2n-1)}{(n+1)n} \cdot \frac{(2(n-1))!}{n(n-1)!(n-1)!} \\
&= \boxed{\frac{4n-2}{n+1}} C_{n-1}.
\end{aligned}
$$

$C_{k-1} = \frac{1}{k} \frac{(2(k-1))!}{(k-1)! \cdot (k-1)!}$

C_{n-1}

5.2 Limits of Sequences

Definition (Limits)

Consider a sequence $\{a_n\}_{n=0}^{\infty}$. We say that the sequence *converges to the number l* if the terms a_n can be made arbitrarily close to l by having n arbitrarily large.

5.2 Limits of Sequences

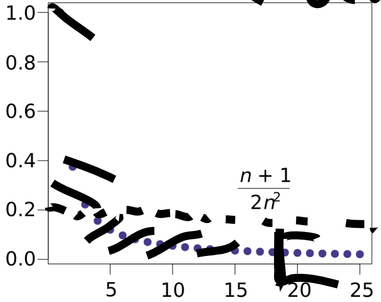
Definition (Limits)

Consider a sequence $\{a_n\}_{n=0}^{\infty}$. We say that the sequence *converges to the number l* if the terms a_n can be made arbitrarily close to l by having n arbitrarily large. More formally, the sequence converges to l if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that $\boxed{|a_n - l| < \epsilon}$ whenever $n \geq N$.

(level of accuracy)

$\epsilon = 0.2$

$\frac{n+1}{2n^2}$ converges to 0.



https://en.wikipedia.org/wiki/Limit_of_a_sequence

5.2 Limits of Sequences

Conventions (Limits)

If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number l, then l is said to be the *limit* of the sequence, written $l = \lim_{n\to\infty} a_n$.

"l is the limit of a_n as n tends/goes to infinity"

5.2 Limits of Sequences

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If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number l, then l is said to be the *limit* of the sequence, written $l = \lim_{n\to\infty} a_n$. In this case, the sequence is said to be *convergent*.

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Conventions (Limits)

If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number l, then l is said to be the *limit* of the sequence, written $l = \lim_{n\to\infty} a_n$. In this case, the sequence is said to be *convergent*. A sequence which does not have a limit is said to be *divergent*.

Question

Determine convergence/divergence of the following sequences. Find the limits (if they exist).

- $a_n = \frac{1}{n}$; $n \geq 1$
- b_n defined recursively with $b_0 = -1$, $b_n = \frac{b_{n-1}}{2}$ for $n \geq 1$.

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- The terms $\frac{1}{n}$ become arbitrarily small by having n large, so $\lim_{n\to\infty} a_n = 0$. The sequence is therefore convergent.

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Solution

- The terms $\frac{1}{n}$ become arbitrarily small by having n large, so $\lim_{n\to\infty} a_n = 0$. The sequence is therefore convergent.
- Converting to direct form, we find $b_n = -\frac{1}{2^n}$. The terms $\frac{-1}{2^n}$ become arbitrarily small by having n large, so $\lim_{n\to\infty} b_n = 0$. The sequence is convergent.

Question

Determine convergence/divergence of the following sequences. Find the limits (if they exist).

- $c_n = (-1)^n$; $n \geq 1$
- $d_n = r^n$ for some fixed real number r. $n \geq 1$

5.2 Limits of Sequences

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Determine convergence/divergence of the following sequences. Find the limits (if they exist).

- $c_n = (-1)^n$;
- $d_n = r^n$ for some fixed real number r.

Solution

- Since $(-1)^n$ is 1 for even n and -1 for odd n, the sequence $\{c_n\}$ oscillates repeatedly between -1 and 1. While the terms get close to both -1 and 1, the sequence does not **stay** close to either one of them, so the sequence is divergent.

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$1^n = 1$

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- We have sub cases: for $\{d_n\}$:
  - If $|r| < 1$ the terms shrink and $\lim_{n\to\infty} d_n = 0$.
  - If $r = 1$, $d_n = 1$ for all n, so $\lim_{n\to\infty} d_n = 1$.

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  - If $r = 1$, $d_n = 1$ for all n, so $\lim_{n\to\infty} d_n = 1$.
  - If $r = -1$, then $d_n = (-1)^n = c_n$, so this sequence is divergent.

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  - If $r = -1$, then $d_n = (-1)^n = c_n$, so this sequence is divergent.
  - If $|r| > 1$, then the size of the terms grows, becoming arbitrarily large, so the sequence is divergent.

Theorem (Properties of Sequences)

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$. Then for any constant $c \in \mathbb{R}$, the following are true:

- $\lim_{n\to\infty}(a_n \pm b_n) = l \pm m$;
- $\lim_{n\to\infty} ca_n = cl$;
- $\lim_{n\to\infty}(a_n b_n) = lm$;
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{l}{m}$, provided $\boxed{m \neq 0}$.

"limit of sum is the sum of the limits"

5.2 Limits of Sequences

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Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$. Then for any constant $c \in \mathbb{R}$, the following are true:

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Remark (Dividing by Sequences Covnerging to 0)

If $m = 0$, then really anything could happen in the last point. For example, we could take the convergent sequences $b_n = \frac{1}{n^2}$ and $a_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = n$ so $\{\frac{a_n}{b_n}\}$ is divergent.

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On the other hand, if $b_n = \frac{1}{n^2}$ and $a_n = \frac{1}{n^2}$, then $\frac{a_n}{b_n} = 1$ for all n, so the sequence converges to 1.

Theorem (Squeeze Theorem)

Suppose that $\{a_n\}, \{c_n\}$ are sequences, both convergent to l. If $\{b_n\}$ is a sequence so that there is an $n_0 \in \mathbb{N}$ so that $a_n \leq b_n \leq c_n$ whenever $n \geq n_0$, then $\lim_{n\to\infty} b_n = l$.

Theorem (Squeeze Theorem)

Suppose that $\{a_n\}, \{c_n\}$ are sequences, both convergent to l. If $\{b_n\}$ is a sequence so that there is an $n_0 \in \mathbb{N}$ so that $a_n \leq b_n \leq c_n$ whenever $n \geq n_0$, then $\lim_{n\to\infty} b_n = l$.

not same

Example

Let $a_n = \frac{\sin(n)}{n}$. Then $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$.

$$-1 \leq \sin(n) \leq 1$$

5.2 Limits of Sequences

Theorem (Squeeze Theorem)

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Example

Let $a_n = \frac{\sin(n)}{n}$. Then $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$. Both outer sequences converge to 0, so $\lim_{n\to\infty} a_n = 0$.

5.2 Limits of Sequences

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Let $a_n = \frac{\sin(n)}{n}$. Then $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$. Both outer sequences converge to 0, so $\lim_{n\to\infty} a_n = 0$.

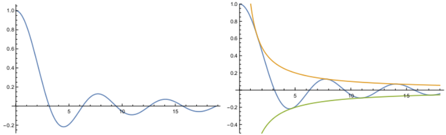

Figure 37: A plot of $f(x) = \frac{\sin(x)}{x}$ and $\pm\frac{1}{x}$ together with $f(x)$.

5.2 Limits of Sequences

Examples (Well-Known Limits)

- For a constant c, $\lim_{n\to\infty} c^n = \begin{cases} 0 \text{ if } |c| < 1, \\ 1 \text{ if } c = 1. \end{cases}$ The sequence is divergent for other c.

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$c^0 = 1$

5.2 Limits of Sequences

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- $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

more power ful

$n^n \to \infty$ loses

$n \to \infty$

$n^{\frac{1}{n}} \to 1$ wins

$n \to \infty$

5.2 Limits of Sequences

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- $\lim_{n\to\infty} \frac{1}{n!} = 0$.

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- $\lim_{n\to\infty} \frac{1}{n!} = 0$.
- For any constant c, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

5.2 Limits of Sequences

Examples (Well-Known Limits)

- For a constant c, $\lim_{n\to\infty} c^n = \begin{cases} 0 \text{ if } |c| < 1, \\ 1 \text{ if } c = 1. \end{cases}$ The sequence is divergent for other c.
- For constant $c > 0$, $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$.
- $\lim_{n\to\infty} \frac{1}{n^r} = 0$ for any $r > 0$.
- $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
- $\lim_{n\to\infty} \frac{1}{n!} = 0$.
- For any constant c, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.
- $\boxed{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.}$

$r^n \to \infty$

$\left(1+\frac{1}{n}\right)^{n \to \infty} \to 1$ as $n \to \infty$

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- For any constant c, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.
- For any real a, $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$.

5.3 Series

Definition (Infinite Sum)

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence. The *series* of $\{a_i\}$ is the new sequence $\{s_n\}_{n=1}^{\infty}$ so that $s_n = \sum_{i=1}^{n} a_i$, i.e., the first few terms of $\{s_n\}$ are

$$a_1, \ a_1 + a_2, \ a_1 + a_2 + a_3, \ a_1 + a_2 + a_3 + a_4, \cdots .$$

The term s_n is called the nth *partial sum*.

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The limit of this sequence is called the *infinite sum of* $\{a_n\}$, and is denoted $\sum_{i=1}^{\infty} a_i$.

→ ((the infinit sum of a_i))

((the sum of a_i from 1 to infinity))

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sequence $\neq$ series, (not the same concept)

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Note that the original sequence does not necessarily have to start at $i = 1$.

$$\{a_i\}_{i=3}^{\infty} \to \sum_{i=3}^{\infty} a_i$$

$\text{Sum}\left[\frac{1}{n^2}, \{n, 1, 100\}\right] \approx 1.6$

Mathematica can sometimes compute infinite sums:

In[1]:= Sum[1/n^2, {n, 1, Infinity}] $= \sum_{n=1}^{\infty} \frac{1}{n^2}$

Out[1]= $\frac{\pi^2}{6}$

$S_n = 1,\ 1+\frac{1}{4},\ 1+\frac{1}{4}+\frac{1}{9},\ 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} \ldots$

$\lim_{n\to\infty} S_n = \frac{\pi^2}{6}.$

5.3 Series

Applications in Infinite Sums

- The number π can be expressed as the infinite sum $3 + 0.1 + 0.04 + 0.001 + 0.0005 + \cdots$. More generally, solutions to equations can often be approximated by adding smaller and smaller numbers; the infinite sum should be the true solution.

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Applications in Infinite Sums

- The number π can be expressed as the infinite sum $3 + 0.1 + 0.04 + 0.001 + 0.0005 + \cdots$. More generally, solutions to equations can often be approximated by adding smaller and smaller numbers; the infinite sum should be the true solution.
- Exponential and trigonometric functions can be expressed as infinite sums of polynomials:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Question

Does $\sum_{n=0}^{\infty}(-1)^n$ converge or diverge?

$$\{a_n\}_{n=0}^{\infty}, \quad a_n=(-1)^n$$

$$\sum_{n=0}^{\infty}(-1)^n$$

$$\lim_{n\to\infty} S_n \, ?$$

Question

Does $\sum_{n=0}^{\infty}(-1)^n$ converge or diverge?

Solution

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + (-1) = 0 \\ s_2 &= 1 + (-1) + 1 = 1 \\ &\vdots \end{aligned}$$

and this pattern repeats.

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$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + (-1) = 0 \\ s_2 &= 1 + (-1) + 1 = 1 \\ &\vdots \end{aligned}$$

and this pattern repeats. In general, $s_{2m} = 1$ and $s_{2m+1} = 0$, so the limit of $\{s_n\}$ does not exist, and the infinite sum diverges.

Question (An example of a telescoping series)

Does $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ converge or diverge?

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First, note that $\{\frac{1}{(n+1)(n+2)}\}_{n=0}^{\infty}$ is the same as $\{\frac{1}{n(n+1)}\}_{n=1}^{\infty}$. So it suffices to examine convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

$\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3} \ldots$

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$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

We compute the partial sums:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1}$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right).$$

(telescoping sum

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&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).
\end{aligned}
$$

We notice some cancellation, so $s_n = 1 - \frac{1}{n+1}$, and $\lim_{n\to\infty} s_n = 1$; the series converges.

Theorem (p-test for series)

For any $p \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent whenever $p > 1$, and is divergent is $p \leq 1$.

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If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

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If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

CAUTION: this Theorem can only ever be used to conclude that a series diverges. It can NEVER be used to conclude that a series converges.

If $\lim_{n\to\infty} a_n \neq 0$ (or if it DNE), then $\sum$ is divergent.

Example (Harmonic Series)

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To see why this series diverges, we can write the sum as

$$
\begin{aligned}
\sum_{n=1}^{\infty} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .
\end{aligned}
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$\left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots \frac{1}{32}\right)$

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Theorem (Geometric Series)

If a is a real number and $|r| < 1$, then $\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.}$ The series is divergent if $|r| \geq 1$.

$a_n = a r^n$

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$$\$ \quad 100 + 0.95(100 + 0.95(\cdots)) = \$\sum_{i=0}^{n} 100 \cdot 0.95^i.$$

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As $n \to \infty$, this number increases to $\frac{100}{1-0.95} = \frac{10000}{5} = 2000.$

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Problem (The Set Cover Problem)

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Example

Let $U = \{1, 2, 3, 4, 5\}$, and consider a collection of subsets

$n = 5$

$$S = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

1 2 3 4

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{1,2,3,4,5}

Then U can be covered with only to elements of S, namely $\{1, 2, 3\}$ and $\{4, 5\}$.

How do we find these sets generally?

The Greedy Algorithm

At each stage, include the subset in S that contains the largest number of elements of U that have not already been covered.

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Then the first set to be chosen is $\{1, 2, 3\}$, and the second set is $\{4, 5\}$, so we are done!

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then the greedy algorithm will select all three sets.

1 $\{2, 3, 4, 5\}$

2 3 $\{1, 2, 3\}$ $\{4, 5, 6\}$

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In general, the greedy algorithm will produce an answer that could be as large as $H(n)$ times the true answer, where $\boxed{H(n) = \sum_{i=1}^{n} \frac{1}{i}}$ is the nth harmonic number.

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In general, the greedy algorithm will produce an answer that could be as large as $H(n)$ times the true answer, where $H(n) = \sum_{i=1}^{n} \frac{1}{i}$ is the nth harmonic number. Recall that the corresponding infinite sum diverges, so there is no limit to just how poor the answer could be!