MATH7501: Mathematics for Data Science I

Unit 6: Real functions, Limits and Continuity

Slides by Timothy Buttsworth (2021)

Motivation: The Need for Continuity

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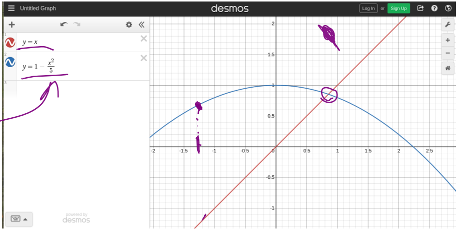
For example, how could we find where the following two curves intersect?

Choose x_0

Define

$x_{n+1} = 1 - \frac{x_n^2}{5}$



Could our methods still work if the curves had 'holes' in them?

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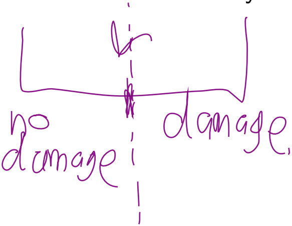
A discontinuity means 'near enough is not good enough'.



Falling off the roof in *The Nice Guys (2016)*

Intuition on Real Function Limits

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The limit of a function $f(x)$ at a point a describes its behaviour for values of x close to a, but not a itself. We can therefore think of the limit as a 'best guess' of what the function should be at a, without actually knowing $f(a)$. For example, the two functions below have the same limit as x approaches 5, even though $f(5)$ is different for both.

Limit as x tends to 5 is 10 for both,

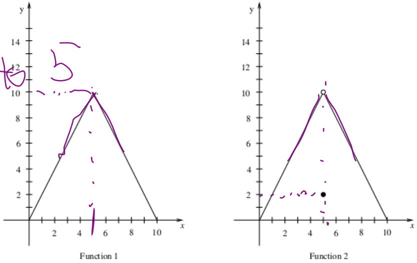
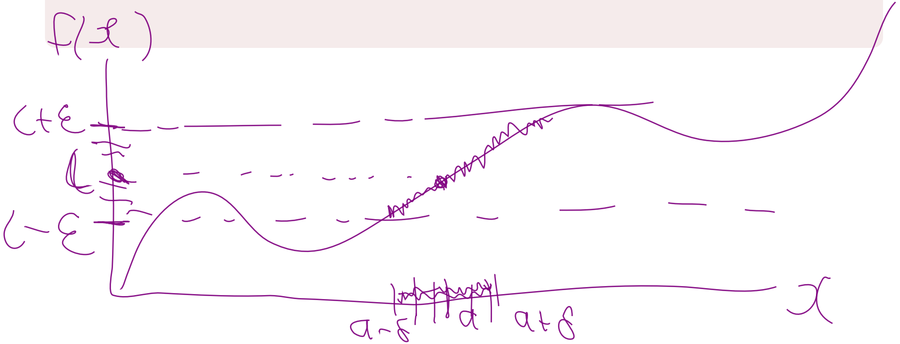


Figure 39: Two functions with limit equal to 10 as $x \to 5$.

6.1 Limits of Real Functions

Definition (Real Function Limits)

Suppose the function f is defined on some open interval containing a, except at possibly a itself. We say that the *limit of f as x approaches a* is the number l if for all $\epsilon > 0$, there is a $\delta > 0$ so that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.



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$$\lim_{x \to a} f(x) = l.$$

"the limit of $f(x)$ as x approaches a is l."

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ϵ

Remark (Limits)

A more casual way of stating the above definition is to say that $\lim_{x \to a} f(x) = l$ if $f(x)$ becomes arbitrarily close to l by having $x \neq a$ sufficiently close to a.

δ.

Example

The functions graphed earlier are

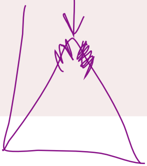
$f(5) = 10$

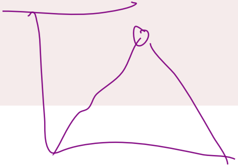
$$f(x) = \begin{cases} 2x \text{ for } 0 \leq x \leq 5, \\ -2x + 20 \text{ for } 5 < x \leq 10, \end{cases}$$

and

don't care.

$$g(x) = \begin{cases} 2x \text{ for } 0 \leq x < 5, \\ 2 \text{ for } x = 2, \\ -2x + 20 \text{ for } 5 < x \leq 10. \end{cases}$$





Example

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$$f(x) = \begin{cases} 2x \text{ for } 0 \leq x \leq 5, \\ -2x + 20 \text{ for } 5 < x \leq 10, \end{cases}$$

$f(2) = 4$

$f(5) = 10$

and

$$g(x) = \begin{cases} 2x \text{ for } 0 \leq x < 5, \\ 2 \text{ for } x = 2, \\ -2x + 20 \text{ for } 5 < x \leq 10. \end{cases}$$

This example illustrates the general principle that $\lim_{x \to a} f(x)$ does not depend on $f(a)$ at all (in fact, $f(a)$ need not even be defined).

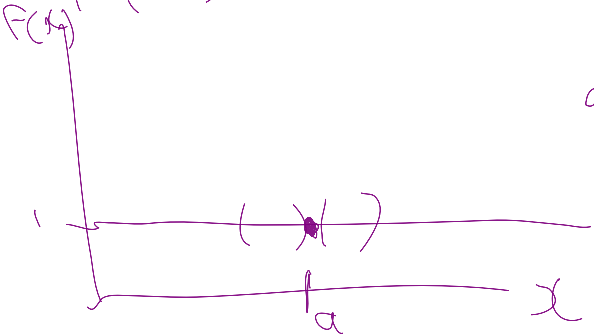
6.1 Limits of Real Functions

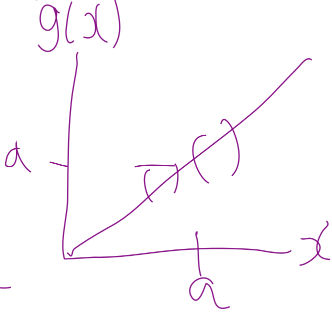
Example (The simplest limits)

For any $a \in \mathbb{R}$, $\lim_{x \to a} 1 = 1$ and $\lim_{x \to a} x = a$.

$g(x) = x$

$f(x) = 1$





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Theorem (Properties of Limits)

Suppose c is constant, $l = \lim_{x \to a} f(x)$ and $m = \lim_{x \to a} g(x)$. Then:

- $\lim_{x \to a}(f(x) \pm g(x)) = l \pm m$;

"limit of sums is the sum of the limits".

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- $\lim_{x \to a}(f(x) \pm g(x)) = l \pm m$;
- $\lim_{x \to a} cf(x) = cl$;
- $\lim_{x \to a} f(x) \cdot g(x) = l \cdot m$; and

"limit of products is the product of limits".

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- $\lim_{x\to a} cf(x) = cl$;
- $\lim_{x\to a} f(x) \cdot g(x) = l \cdot m$; and
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ provided $m \neq 0$.

Sometimes, Mathematica can compute limits:

```
In[1]:= Limit[Sin[x] / x, x -> 0]

Out[1]= 1
```

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

$$\frac{\sin(0)}{0}$$

DNE.

6.1 Limits of Real Functions

Sometimes, Mathematica can compute limits:

$$\lim_{x \to 1} (ax^2+4x+7) = 11+a$$

In[1]:= **Limit[Sin[x] / x, x -> 0]**

Out[1]= 1

In[1]:= **Limit[a x^2 + 4 x + 7, x → 1]**

Out[1]= 11 + a

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty.$$

In[2]:= $\text{Limit}\left[\dfrac{1}{(x-2)^2}, x \to 2\right]$

Out[2]= ∞

Example

Consider the function $f(x) = x^2 + 1$. Then $\lim_{x\to 1} f(x) = (1^2 + 1) = 2$.

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$$\lim_{x\to 1}(x^2+1) = \lim_{x\to 1}(x^2) + \lim_{x\to 1}(1)$$

if both limits exist.

$$= \lim_{x\to 1}(x)\cdot\lim_{x\to 1}(x) + 1$$

if both limits exist.

$$= 1\cdot 1 + 1$$

$$= 2.$$

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Example

Consider the function $f(x) = \frac{x-1}{x^2-1}$. In evaluating $\lim_{x \to 1} f(x)$, we cannot use the ratio rule for limits because the denominator is going to 0.

(technically not defined at $x=1, x=-1$.

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Consider the function $f(x) = x^2 + 1$. Then $\lim_{x \to 1} f(x) = (1^2 + 1) = 2$. This can be seen using $\lim_{x \to 1} x = 1$ and the addition, product rules for limits.

Example

Consider the function $f(x) = \frac{x-1}{x^2-1}$. In evaluating $\lim_{x \to 1} f(x)$, we cannot use the ratio rule for limits because the denominator is going to 0. Instead, note that

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1} \text{ for } x \neq 1,$$

so $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$.

6.1 Limits of Real Functions

lower

upper

Theorem (Squeeze Theorem)

Suppose $\lim_{x\to a} g(x) = l = \lim_{x\to a} h(x)$ and $g(x) \leq f(x) \leq h(x)$ for x close to a, but not equal to a. Then $\lim_{x\to a} f(x) = l$.

want to find limit.

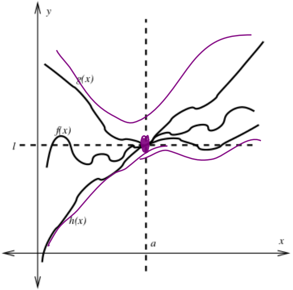

Figure 40: The squeeze principle

Question (Application of the Squeeze Theorem)

Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

$f(x) = x^2 \sin\left(\frac{1}{x}\right)$

$f(0)$ not defined

Question (Application of the Squeeze Theorem)

Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution

Since the sin function is always between -1 and 1, and x^2 is non-negative, we find that

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \text{ for all } x \neq 0.$$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

Question (Application of the Squeeze Theorem)

Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution

Since the sin function is always between -1 and 1, and x^2 is non-negative, we find that

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \text{ for all } x \neq 0.$$

Since both $-x^2$ and x^2 converge to 0 as x goes to 0, the Squeeze Theorem implies that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Example (Some other important limits)

We have:

- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$;

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Example (Some other important limits)

We have:

- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$;
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$;
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

none of these functions defined at $x=0$.

Definition (One-sided limits)

Suppose $f(x)$ is defined for values of x which are close to and lower than a. Then $\lim_{x\to a^-} f(x) = l$ means that $f(x)$ gets arbitrarily close to l for values of x that are both close to a AND less than a.

("the limit of $f(x)$ as x approaches a from below")

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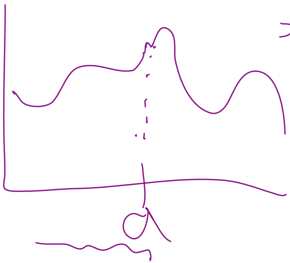
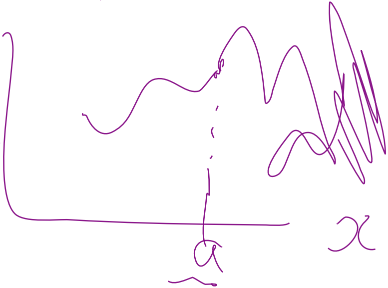
"from above"

lim $x \to a$ are the same.



x



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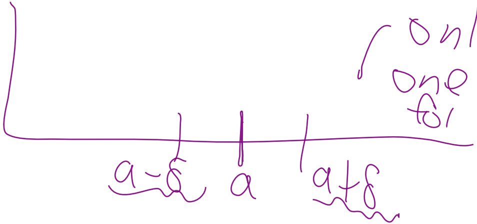
Example

Consider the piece-wise defined function $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -2 \text{ if } x < 0. \end{cases}$



only one side for one-sided limits.

6.2 Additional real function limit concepts

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Example

Consider the piece-wise defined function $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -2 \text{ if } x < 0. \end{cases}$

Then $\lim_{x \to 0^+} = 1$, $\lim_{x \to 0^-} = -2$ but $\lim_{x \to 0}$ does not exist.

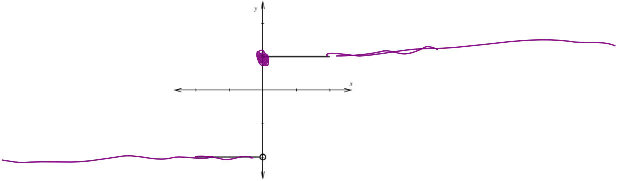


Figure 41: The limit of this function as $x \to 0$ does not exist.

6.2 Additional real function limit concepts

Mathematica can also deal with one-sides limits:

$\frac{|x|}{x}$

Limit from above:

```
In[1]:= Limit[RealAbs[x]/x, x → 0, Direction → "FromAbove"]
Out[1]= 1
```

$\lim_{x \to 0^+} \frac{|x|}{x} = 1$

Limit from below:

```
In[2]:= Limit[RealAbs[x]/x, x → 0, Direction → "FromBelow"]
Out[2]= -1
```

$\lim_{x \to 0} \frac{|x|}{x} = -1$

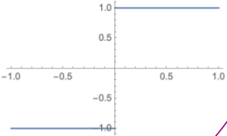
The two-sided limit does not exist:

```
In[3]:= Limit[RealAbs[x]/x, x → 0]
Out[3]= Indeterminate
```

$\lim_{x \to 0} \frac{|x|}{x} = \text{DNE}.$

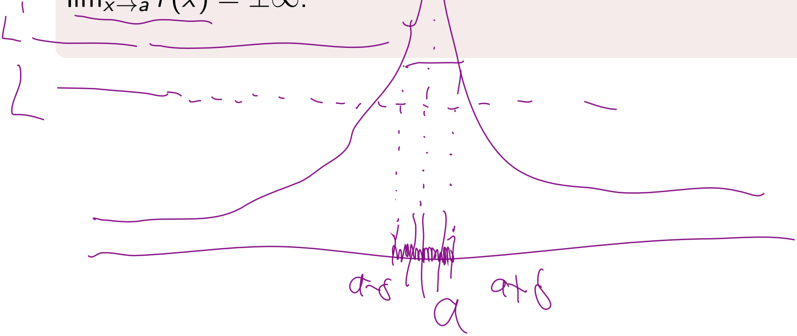
```
In[4]:= Plot[RealAbs[x]/x, {x, -1, 1}]
```

Out[4]=



Definition (Limits to $\pm\infty$)

Suppose f is defined in a neighbourhood of a (possibly excluding a itself). Then we say that $f(x)$ approaches ∞ $(-\infty)$ as x approaches a if for all $L > 0$, there exists a $\delta > 0$ so that $f(x) > L$ $(f(x) < -L)$ whenever $0 < |x - a| < \delta$. We write this as $\lim_{x \to a} f(x) = \pm\infty$.



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Suppose f is defined in a neighbourhood of a (possibly excluding a itself). Then we say that $f(x)$ approaches ∞ ($-\infty$) as x approaches a if for all $L > 0$, there exists a $\delta > 0$ so that $f(x) > L$ ($f(x) < -L$) whenever $0 < |x - a| < \delta$. We write this as $\lim_{x\to a} f(x) = \pm\infty$. The one-sided limits to $\pm\infty$ are defined analogously.

$$\lim_{x \to a^+} f(x) = +\infty$$

6.2 Additional real function limit concepts

Definition (Limits to $\pm\infty$)

Suppose f is defined in a neighbourhood of a (possibly excluding a itself). Then we say that $f(x)$ approaches ∞ ($-\infty$) as x approaches a if for all $L > 0$, there exists a $\delta > 0$ so that $f(x) > L$ ($f(x) < -L$) whenever $0 < |x - a| < \delta$. We write this as $\lim_{x\to a} f(x) = \pm\infty$. The one-sided limits to $\pm\infty$ are defined analogously.

A more intuitive description (Limits to $\pm\infty$)

$\lim_{x\to a} f(x) = \infty$ means that f becomes arbitrarily large as x gets arbitrarily close to a.

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A more intuitive description (Limits to $\pm\infty$)

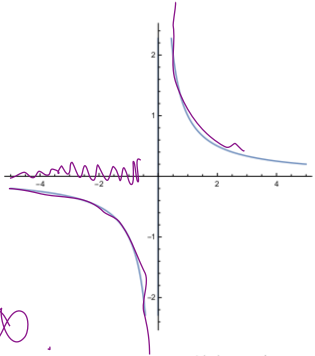
$\lim_{x\to a} f(x) = \infty$ means that f becomes arbitrarily large as x gets arbitrarily close to a. Similarly $\lim_{x\to a} f(x) = -\infty$ means that $-f(x)$ gets arbitrarily large as x gets arbitrarily close to a.

6.2 Additional real function limit concepts

Example

Consider the function $f(x) = \frac{1}{x}$. Then $\lim_{x \to 0} f(x)$ does not exist. However, we can see $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = +\infty$.

f(0)
undefined



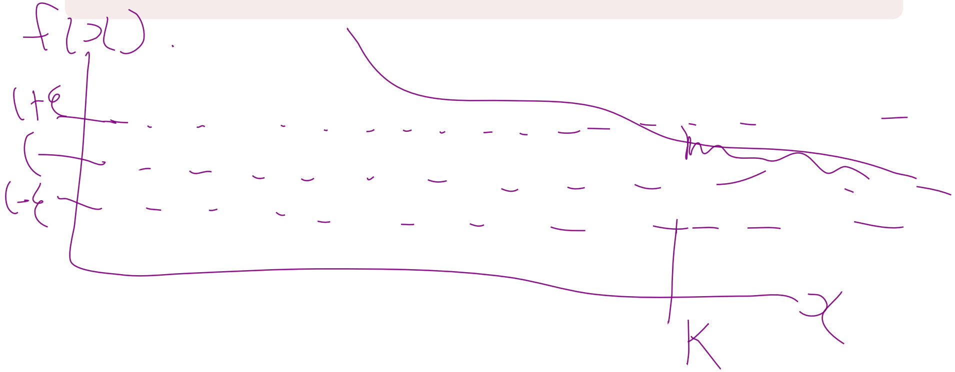
$-\infty$

Figure 42: Graph of $f(x) = 1/x$.

6.2 Additional real function limit concepts

Definition (Limits as x approaches $\pm\infty$)

Consider a function $f(x)$ defined for all sufficiently large values of x. Then $\lim_{x\to\infty} f(x) = l$ means that for all $\epsilon > 0$, there exists a $K \in \mathbb{R}$ so that $|f(x) - l| < \epsilon$ whenever $x > K$.



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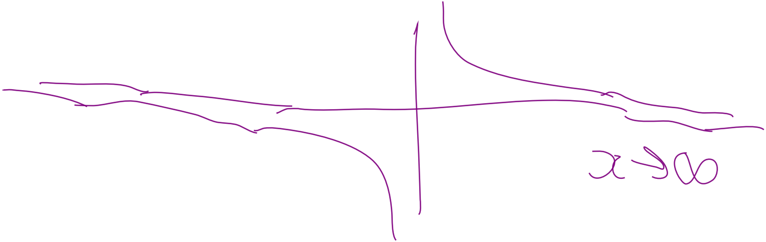
Example

Take $f(x) = \frac{1}{x}$ as before. Then $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$.



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Example

Take $f(x) = \frac{1}{x}$ as before. Then $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$.

Remark (More infinities)

We can combine previous definitions to make sense of the expressions $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

Example (Ratio of Polynomials with Identical Degree)

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x^2 + x$. Then we cannot use the quotient rule for limits to find $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ because both top and bottom limits are ∞.

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$$\frac{(2x^2+3)/x^2}{(3x^2+x)/x^2} = $$

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x^2}}{\frac{g(x)}{x^2}} = \frac{2 + \frac{3}{x^2}}{3 + \frac{1}{x}}.$$

Example (Ratio of Polynomials with Identical Degree)

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x^2 + x$. Then we cannot use the quotient rule for limits to find $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ because both top and bottom limits are ∞. However,

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x^2}}{\frac{g(x)}{x^2}} = \frac{2 + \frac{3}{x^2}}{3 + \frac{1}{x}}.$$

We can now use the quotient limit law to find $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{2}{3}$.

Example (Polynomial Divided by Polynomial of Lower Degree)

Let $f(x) = x^2 + 5$ and $g(x) = x + 1$. Then $\frac{f(x)}{g(x)} = \frac{x+\frac{5}{x}}{1+\frac{1}{x}}$.

$$\lim_{x \to \infty} \frac{\left(x^2+5\right)/x}{\left(x+1\right)/x}$$

Example (Polynomial Divided by Polynomial of Lower Degree)

Let $f(x) = x^2 + 5$ and $g(x) = x + 1$. Then $\frac{f(x)}{g(x)} = \frac{x + \frac{5}{x}}{1 + \frac{1}{x}}$. Therefore $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$.

$\lim_{x \to \infty} 1 + \frac{1}{x} = 1$

$\lim_{x \to \infty} x + \frac{5}{x} = +\infty$.

6.2 Additional real function limit concepts

Example (Polynomial Divided by Polynomial of Lower Degree)

Let $f(x) = x^2 + 5$ and $g(x) = x + 1$. Then $\frac{f(x)}{g(x)} = \frac{x+\frac{5}{x}}{1+\frac{1}{x}}$. Therefore $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

Example (Polynomial Divided by Polynomial of Higher Degree)

Let $f(x) = \underbrace{x + 1}_{x^2}$ and $g(x) = \underbrace{x^2 + 1}_{x^2}$. Then $\frac{f(x)}{g(x)} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}$.

Example (Polynomial Divided by Polynomial of Lower Degree)

Let $f(x) = x^2 + 5$ and $g(x) = x + 1$. Then $\frac{f(x)}{g(x)} = \frac{x+\frac{5}{x}}{1+\frac{1}{x}}$. Therefore $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

Example (Polynomial Divided by Polynomial of Higher Degree)

Let $f(x) = x + 1$ and $g(x) = x^2 + 1$. Then $\frac{f(x)}{g(x)} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}$.

Therefore $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(quotient rule for limits).

The Problem of Different Approaches

Consider $f(x, y) = \frac{x^2}{x^2+y^2}$, defined for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Does f have a limit as (x, y) approaches $(0, 0)$?

$\boxed{\text{Definition}}$ $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) \to L$

for all $\varepsilon > 0$, there exist a $\delta > 0$

so that $|f(x,y) - L| < \varepsilon$,

provided $|(x,y) - (a,b)| < \delta$.

$= \sqrt{(x-a)^2 + (y-b)^2}$

The Problem of Different Approaches

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- If $y = 0$, then $f(x,y) = 1$, so f approaches 1 as (x,y) approaches the origin along the x axis.

$$f(x,0) = \frac{x^2}{x^2+0^2} = 1 \quad (\text{if } x \neq 0)$$

6.3 Multivariate Limits

The Problem of Different Approaches

Consider $f(x,y) = \frac{x^2}{x^2+y^2}$, defined for $(x,y) \in \mathbb{R}^2 \setminus \{0,0\}$. Does f have a limit as (x,y) approaches $(0,0)$?

- If $y = 0$, then $f(x,y) = 1$, so f approaches 1 as (x,y) approaches the origin along the x axis.
- If $x = 0$, then $f(x,y) = 0$, so f approaches 0 as (x,y) approaches the origin along the y-axis.

$$f(0,y) = \frac{0^2}{0^2+y^2} = 0$$

(if $y \neq 0$).

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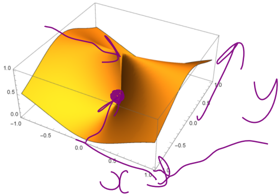
- If $y = 0$, then $f(x, y) = 1$, so f approaches 1 as (x, y) approaches the origin along the x axis.
- If $x = 0$, then $f(x, y) = 0$, so f approaches 0 as (x, y) approaches the origin along the y-axis.

These values are different, so f does not have a limit as (x, y) approaches $(0, 0)$. In general, for a multi-varied limit to exist, it must approach the same value, irrespective of the path taken.



Ambiguities Associated with Domains

Sometimes our domain of interest might be smaller than the given domain, in which case there is a weaker definition of limit that we might be interested in.

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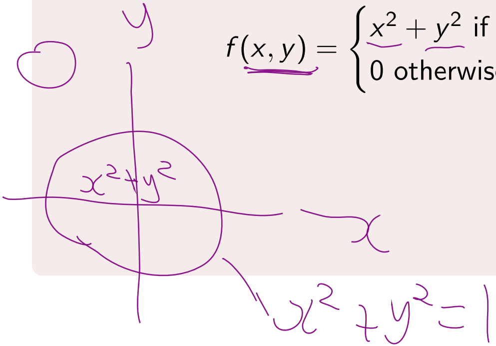
$$f(x, y) = \begin{cases} x^2 + y^2 \text{ if } x^2 + y^2 < 1 \\ 0 \text{ otherwise.} \end{cases}$$



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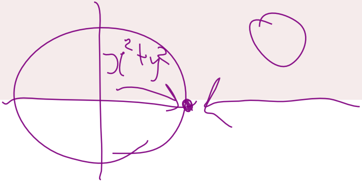
Then $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist,



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Sometimes our domain of interest might be smaller than the given domain, in which case there is a weaker definition of limit that we might be interested in. For example, consider

$$f(x,y) = \begin{cases} x^2 + y^2 \text{ if } x^2 + y^2 < 1 \\ 0 \text{ otherwise.} \end{cases}$$

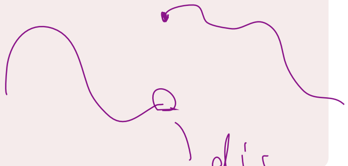
Then $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist, but $\lim_{(x,y)\to(1,0)} f(x,y) = 1$ if we are only interested in the domain $D = \{(x,y)|x^2 + y^2 < 1\}$ (this could perhaps be denoted $\lim_{(x,y)\to_D(1,0)} f(x,y)$).

6.4 Continuity

Definition (Continuity of Functions at a Point)

A function f is said to be *continuous at the point a* if:

- f is defined at a, and in a neighbourhood of a;
- $\lim_{x \to a} f(x)$ exists and
- $\lim_{x \to a} f(x) = f(a)$.



discontinuity.

This time, it matters what f(a) is (didn't matter for limits).

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- $\lim_{x \to a} f(x) = f(a)$.

If f is not continuous at a, we say that f *is discontinuous at* a, or f *has a discontinuity at* a.

$f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

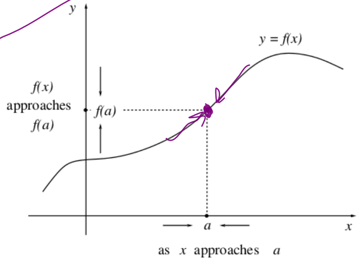

Figure 43: Graphical representation of continuity at $x = a$.

Example

Consider the function

$$f(x) = \begin{cases} x \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

Example

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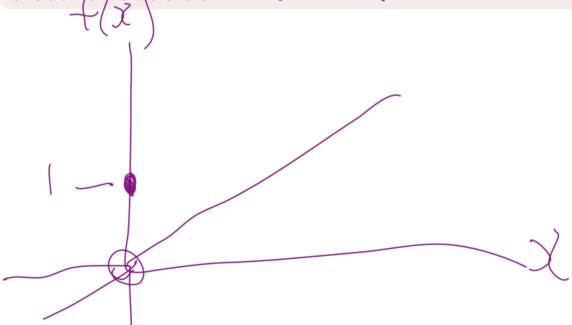
$$f(x) = \begin{cases} x \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$, ~~whole~~ while $f(0) = 1$. Therefore, f is discontinuous at $x = 0$.



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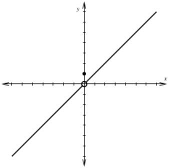

Figure 44: An example of a discontinuous function with discontinuity at $x = 0$.

Example

Consider the function $f(x) = \frac{1}{x^2}$. Then $f(0)$ is not defined, so f is discontinuous at 0.

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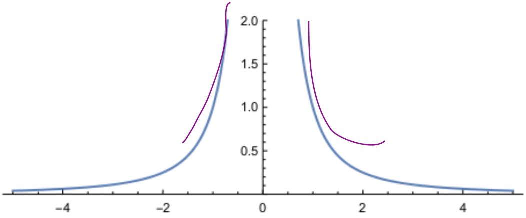

Figure 45: The function $f(x) = 1/x^2$ has a discontinuity at $x = 0$.

$\lim_{x \to 0} \frac{1}{x^2} = +\infty$. $f(0) = +\infty$

not allowed.

Example

Consider the function

$$f(x) = \begin{cases} x + 1 \text{ if } x \geq 0, \\ x^2 \text{ if } x < 0. \end{cases}$$

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$$f(x) = \begin{cases} x + 1 \text{ if } x \geq 0, \\ x^2 \text{ if } x < 0. \end{cases}$$

Then $\lim_{x \to 0} f(x)$ does not exist, so the function is not continuous at $x = 0$.

$$\lim_{x \to 0^-} f(x) = 0$$

$$\lim_{x \to 0^+} f(x) = 1 \longleftarrow = f(0).$$

Example

Consider the function

$$f(x) = \begin{cases} x + 1 \text{ if } x \geq 0, \\ x^2 \text{ if } x < 0. \end{cases}$$

Then $\lim_{x \to 0} f(x)$ does not exist, so the function is not continuous at $x = 0$. Even though the limit does not exist, we have $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = 0$.

6.4 Continuity

Definition (Continuity on an Interval)

A function f defined on an open interval (a, b) is said to be *continuous on the open interval* (a, b) if f is continuous at any $c \in (a, b)$.

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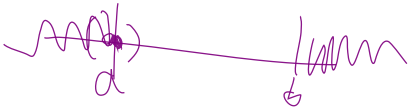
A function f defined on a closed interval $[a, b]$ is said to be *continuous on the closed interval* $[a, b]$ if it is continuous on (a, b) and

$$\lim_{x \to a^+} f(x) = f(a), \qquad \lim_{x \to b^-} f(x) = f(b).$$



Examples

- The function $f(x) = x$ is continuous on $\mathbb{R}$. Using this and the multiplication/addition laws for limits, we find that *any* polynomial is continuous on $\mathbb{R}$.

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- e^x, $|x|$, $\sin(x)$, $\cos(x)$ and $\arctan(x)$ are all continuous on $\mathbb{R}$.

$|x|$ is continuous.

Exercise

6.4 Continuity

Examples

- The function $f(x) = x$ is continuous on $\mathbb{R}$. Using this and the multiplication/addition laws for limits, we find that *any* polynomial is continuous on $\mathbb{R}$.
- e^x, $|x|$, $\sin(x)$, $\cos(x)$ and $\arctan(x)$ are all continuous on $\mathbb{R}$.
- $f(x) = \ln(x)$ is continuous on $(0, \infty)$.

$\ln(x) =$ inverse of e^x.

Theorem (Exchanging Limits)

Suppose that f is continuous at b and $\lim_{x\to a} g(x) = b$. Then

$$\lim_{x\to a} f(g(x)) = f(b).$$

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Theorem (Exchanging Limits)

Suppose that f is continuous at b and $\lim_{x \to a} g(x) = b$. Then

$$\lim_{x \to a} f(g(x)) = f(b).$$

Remark

This theorem essentially tells us that we can pull limits in and out of functions at will, provided they are continuous at the point in question.

6.4 Continuity

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Remark

This theorem essentially tells us that we can pull limits in and out of functions at will, provided they are continuous at the point in question.

Corollary (Composition of Continuous Functions)

If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

6.4 Continuity

Definition (Multi-Variate Continuity)

Suppose we have a function $f : D \to \mathbb{R}$ on a multi-variate open domain $D \subseteq \mathbb{R}^2$. The function is said to be *continuous at* $(a, b) \in D$ if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

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Example

Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ with $\cancel{f(x)} = \frac{x^2-y^2}{x^2+y^2}$.
This function is continuous on its domain. $f(x, y)$

Definition (Multi-Variate Continuity)

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Example

Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ with $f(x) = \frac{x^2-y^2}{x^2+y^2}$. This function is continuous on its domain.

Example

Take the last function, and extend its domain of definition to include $(0, 0)$, with $f(0, 0) = 0$.

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Example

Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ with $f(x) = \frac{x^2-y^2}{x^2+y^2}$. This function is continuous on its domain.

Example

Take the last function, and extend its domain of definition to include $(0, 0)$, with $f(0, 0) = 0$. Then $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, so f is not continuous at $(0, 0)$, though it is continuous everywhere else.

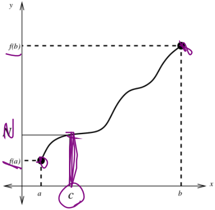
$x = 0$ path

$y = 0$ path

6.5 The Intermediate Value Theorem

Theorem (IVT)

Suppose that f is continuous on the closed interval $[a, b]$, and $f(a) \neq f(b)$. Then for any N between $f(a)$ and $f(b)$, there is a $c \in (a, b)$ so that $f(c) = N$.



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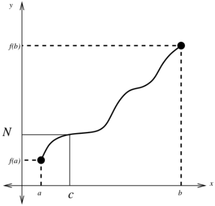
$f(x) = 0$



$N = 0$

Remark

Suppose we are trying to find roots of a function f, so $N = 0$. Then the IVT is applicable on $[a, b]$ if and only if $f(a)f(b) < 0$, because this implies that $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$.

!!

Example

Suppose that f is a $\boxed{\text{continuous}}$ function with

$$f(-2) = 3,\ f(-1) = -1,\ f(0) = -4,\ f(1) = 1,\ f(2) = 5.$$

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Then the IVT guarantees the existence of a root of f on the intervals $[-2, -1]$, $[0, 1]$ and $[0, 2]$.

Example

Suppose that f is a continuous function with

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Then the IVT guarantees the existence of a root of f on the intervals $[-2, -1]$, $[0, 1]$ and $[0, 2]$.

The IVT cannot be used on the intervals $[-1, 0]$ and $[1, \cancel{3}]$.

The Bisection Method

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function so that $f(a)f(b) < 0$, so that the IVT implies that there is a $c \in (a, b)$ so that $f(c) = 0$.

6.5 The Intermediate Value Theorem

The Bisection Method

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- Choose $d = \boxed{\frac{a+b}{2}}$. If $f(d) = 0$ we are done.

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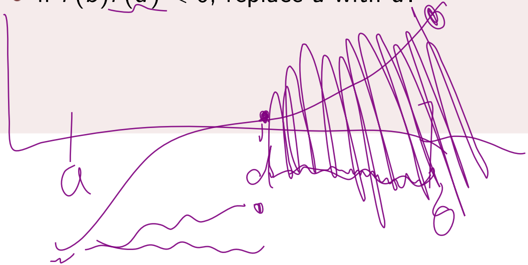
- Choose $d = \frac{a+b}{2}$. If $f(d) = 0$ we are done.
- Otherwise, $f(d) \neq 0$:
    - if $f(a)f(d) < 0$, replace b with d,
    - if $f(b)f(d) < 0$, replace a with d.



6.5 The Intermediate Value Theorem

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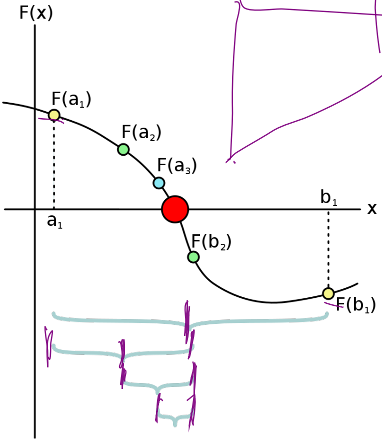
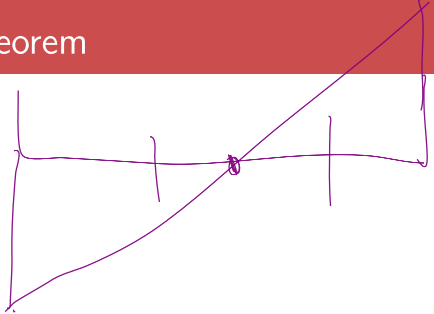
Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function so that $f(a)f(b) < 0$, so that the IVT implies that there is a $c \in (a, b)$ so that $f(c) = 0$. To approximate c, we use the following algorithm:

- Choose $d = \frac{a+b}{2}$. If $f(d) = 0$ we are done.
- Otherwise, $f(d) \neq 0$:
  - if $f(a)f(d) < 0$, replace b with d,
  - if $f(b)f(d) < 0$, replace a with d.
- There must be a zero of f on the new interval $[a, b]$. If you are happy with the accuracy on this new interval, then we are done. Otherwise, repeat the process.

6.5 The Intermediate Value Theorem



https://en.wikipedia.org/wiki/Bisection_method



6.6 Application: CDF and ECDF

Definition (Cumulative Distribution Function)

Let X be a continuous random variable. The CDF of X is the function $F : \mathbb{R} \to [0, 1]$ so that

$$F(x) = \mathbb{P}(X \leq x).$$

Eg if X is the height of a random person in cm.

$F(150) =$ (probability that the height of the selected person is ≤ 150cm).

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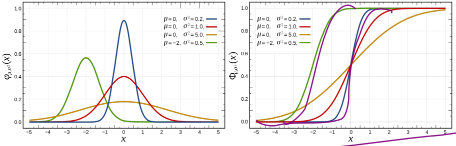
Definition (Cumulative Distribution Function)

Let X be a continuous random variable. The CDF of X is the function $F : \mathbb{R} \to [0, 1]$ so that

$$F(x) = \mathbb{P}(X \leq x).$$

Example

A normally-distributed random variable X with mean μ and standard deviation σ has cdf $F(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right)$.



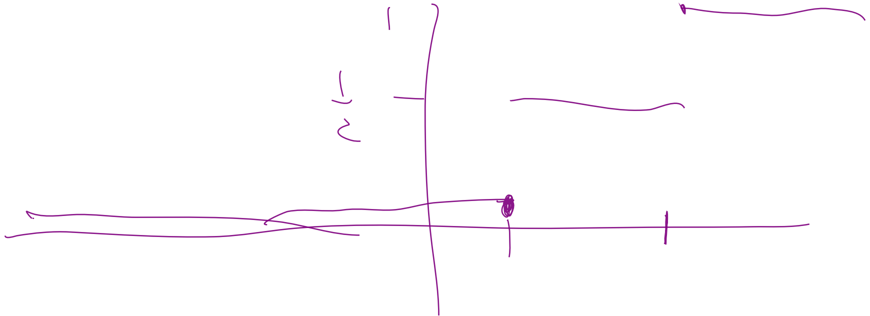
pdfs (left) and cdfs (right) of the normal distribution; taken from https://en.wikipedia.org/wiki/Normal_distribution

Definition (Empirical Cumulative Distribution Function)

Let X be a continuous random variable, and take samples $X_1, \cdots, X_n$. The ECDF is the function

$$\boxed{F(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{x \geq X_i\}},}$$

where $\mathbb{1}_{\{x \geq X_i\}}$ is 1 if $x \geq X_i$, and is zero otherwise.



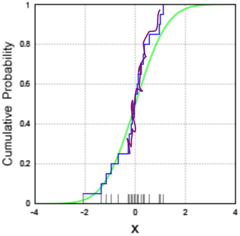
6.6 Application: CDF and ECDF

Definition (Empirical Cumulative Distribution Function)

Let X be a continuous random variable, and take samples $X_1, \cdots, X_n$. The ECDF is the function

$$F(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x \geq X_i\}},$$

where $\mathbb{1}_{\{x \geq X_i\}}$ is 1 if $x \geq X_i$, and is zero otherwise.



The ECDF is a discontinuous approximation of the true CDF; graph taken from https://en.wikipedia.org/wiki/Empirical_distribution_function