MATH7501 Practical 4, Semester 1-2021

Topic: Matrices, Sets Counting and Cardinality, Logic

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Pre-Tutorial Activity

- Students must have familiarised themselves with units 1 to 3 contents (Matrices, Sets, Logic) of the reading materials for MATH7501

Resources

- Chapters 1 to 3 of reading material

Q1. Properties of Sets

Consider the universal set, $U = \{x \in \mathbb{Z}^+ : x \leq 10\}$ and the sets $A = \{x \in U : x > 7\}$ and $B = \{1, 2, 3\}$

(a) Write out all the elements of U explicitly

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(b) Write out all of the elements of A explicitly

$A = \{8, 9, 10\}$

(c) What is $A \cap B$? (set of common elements from A and B)

$A \cap B = \{\ \}$ or $\emptyset$

(d) What is $A \cup B$? (set of all elements from A and B)

$A \cup B = \{1, 2, 3, 8, 9, 10\}$

(e) what is $A^c \cap B$?

$A^c = \{1, 2, 3, 4, 5, 6, 7\}$

$A^c \cap B = \{1, 2, 3\} = B$

(f) What is $A \times B$? (Cartesian Product, which is a set of ordered pairs from A and B)

$A \times B = \{(8, 1), (8, 2), (8, 3), (9, 1), (9, 2), (9, 3), (10, 1), (10, 2), (10, 3)\}$

(g) Write the elements of $\mathcal{P}(B)$? (Power set of B, which contains all subsets of B)

$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

(h) What is $|\mathcal{P}(B)|$? (number of elements (or cardinality) in the power set of B)

$2^{|B|} = 2^3 = 8$

Q2. Properties of Sets

Consider the set $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Determine the following:

(a) $A \bigcup B$

$A \bigcup B = \{0, 1, 2, 3, 4\}$

(b) $A \bigcap B$

$A \bigcap B = \{1, 2, 3\}$

(c) $A \times B$

$A \times B = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (1,2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4),\}$

(d) $2^{|A|}$

$2^4 = 16$

(e) $\{C \subset A : |C| = 3\} \setminus \{C \in 2^A : |C| = 3\}$

$\{\{0, 1 ,2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\} \setminus \{\{0, 1 ,2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\} = \{ \}$

(f) $|\{R \subset A \times B : \text{where } R \text{ is a function}\}|$

This counts how many binary relations on A and B are functions. For each value of A, there are 4 values in B. Thus the number of functions is 4^4

Q3. Logic

Consider the following logical expression $(A \vee B) \wedge \neg(A \wedge B)$

Here, ($\vee$ = OR, $\wedge$ = AND $\neg$ = NOT)

(a) Write the truth table for the above expression

A	B	$A \vee B$	$A \wedge B$	$\neg(A \wedge B)$	$(A \vee B) \wedge \neg(A \wedge B)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

(b) Write an expression using only ANDs, ORs and NOTs that is logically equivalent to the above expression

Lots of possible answers! It’s equivalent to an XOR operation (last column has the order F, T, T, F).
Say
$\neg\neg ((A \vee B) \wedge \neg (A \wedge B))$, or
$(A \wedge \neg B) \vee (\neg A \wedge B)$.
Verify with a truth table to show equivalence, e.g

A	B	$A \wedge \neg B$	$\neg A \wedge B$	$(A \wedge \neg B) \vee (\neg A \wedge B)$
T	T	F	F	F
T	F	T	F	T
F	T	F	T	T
F	F	F	F	F

Q4. Proof

Prove $\sum_{i=1}^{n} (2\,i - 1) = n^2$

Method 1 : Use mathematical induction

Step 1 : Show that the stament is true for n = 1

$$\sum_{i=1}^{1} (2\,i - 1) = 1 = 1^2. \text{ Therefore true}$$

Step 2 : Assume the statement is true for $n = k$. That is

$$\sum_{i=1}^{k} (2\,i - 1) = k^2$$

Step 3 : Show the statement is true for $n = k + 1$

$$\sum_{i=1}^{k+1} (2\,i - 1) = \sum_{i=1}^{k} (2\,i - 1) + (2\,(k + 1) - 1)$$
$$= k^2 + 2\,(k + 1) - 1$$
$$= k^2 + 2\,k + 2 - 1$$
$$= k^2 + 2\,k + 1$$
$$= (k + 1)^2$$

Therefore by the principle of mathematical induction the statement is true $(n \geq 1)$

Method 2 : Expanding the sum

$$\sum_{i=1}^{n} (2\,i - 1) = \sum_{i=1}^{n} 2\,i - \sum_{i=1}^{n} 1$$

$$= 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$= 2 \times \frac{1}{2} n (n+1) - n, \quad \left(\text{From Assignment 1, Q2 (b)}, \ \sum_{i=1}^{n} i = \frac{1}{2} n (n+1)\right)$$

$$= n^2 + n - n$$

$$= n^2$$

Hence the result follows.

Q5. Proof

Prove $\sum_{k=1}^{n} k^2 = \frac{n (n+1) (2 n+1)}{6}$

Step 1 : Show that the stament is true for $n = 1$

$$\text{LHS} = \sum_{i=1}^{1} k^2 = 1^2$$

$$\text{RHS} = \frac{1}{6} \{1 (1+1) (2 \times 1+1)\} = \frac{1}{6} \times 2 \times 3 = 1 = 1^2 = \text{LHS}$$

Step 2 : Assume the statement is true for $n = k$. That is

$$\sum_{k=1}^{n=k} k^2 = \frac{k (k+1) (2 k+1)}{6}$$

Step 3 : Show the statement is true for $n = k+1$

$$\text{LHS} = \sum_{i=1}^{n=k+1} k^2 = \sum_{i=1}^{k} k^2 + (k+1)^2$$

$$= \frac{1}{6} k (k+1) (2 k+1) + (k+1) (k+1)$$

$$= \frac{1}{6} (k+1) \{k (2 k+1) + 6 (k+1)\}$$

$$= \frac{1}{6} (k+1) \left\{2 k^2 + 7 k + 6\right\}$$

$$= \frac{1}{6} (k+1) \{(k+2) (2 k+3)\}$$

$$\text{RHS} = \frac{1}{6} (k+1) (k+1+1) (2 (k+1)+1) = \frac{1}{6} (k+1) (k+2) (2 k+3) = \text{LHS}$$

Therefore by the principle of mathematical induction the statement is true $(n \geq 1)$

Q6. Counting

Three identical dice are rolled, each with 6 sides labelled 1, 2, 3, 4, 5, 6. How many possible outcomes are there?

Number of possible outcomes (with no repetitions) = $\binom{n+k-1}{k}$, with n=6, k=3.

Q7. Matrices and Linear Algebra

Let α and β be two real numbers and consider the matrices,

$A = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \beta & -\alpha\beta \\ -\beta & \beta \end{pmatrix}$

(i) Set $\alpha = -1$ and determine x and y in the system of equations

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ x+y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we have to find x and y such that the following system of equations are satisfied.

$x - y = 1$ ---------- Eq1

$x + y = 1$ --------- Eq2

Eq1 + Eq2 gives $2x = 2$ and so $x = 1$. Substituting $x = 1$ into Eq2 gives $y = 0$.

(b) Determine the product A A

$$AA = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha & 2\alpha \\ 2 & 1+\alpha \end{pmatrix}$$

(c) Determine the product A B

$$AB = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta & -\alpha\beta \\ -\beta & \beta \end{pmatrix} = \beta\begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ -1 & 1 \end{pmatrix} = \beta\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$$

(d) Given a value of α, with α not equal to 1, for what value of β does $B = A^{-1}$?

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} 1 & -\alpha \\ -1 & 1 \end{pmatrix} = \frac{1}{1-\alpha}\begin{pmatrix} 1 & -\alpha \\ -1 & 1 \end{pmatrix}$$

So $B = A^{-1}$ implies

$$\beta\begin{pmatrix} 1 & -\alpha \\ -1 & 1 \end{pmatrix} = \frac{1}{1-\alpha}\begin{pmatrix} 1 & -\alpha \\ -1 & 1 \end{pmatrix}$$

Therefore $\beta = \dfrac{1}{1-\alpha}$, provided α is not 1.

(e) Set $\alpha = \dfrac{1}{2}$ and $\beta = 2$. Determine $A^9 B^8$

If $\alpha = \dfrac{1}{2}$ and $\beta = 2$,

then from part (c) $B = A^{-1} = 2\begin{pmatrix} 1 & \frac{-1}{2} \\ -1 & 1 \end{pmatrix}$. You can check this using the fact that:

$$AB = 2\begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{-1}{2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $A^9 B^8 = A^9 (A^{-1})^8 = A^9 A^{-8} = A = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$

Q8. Matrices and Linear Algebra

Given that $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(a) Define the sequence of vectors $V_1, V_2, \ldots$via $V_{n+1} = AV_n$. That is $V_2 = AV_1$, $V_3 = AV_2$, etc.

$\Rightarrow \Longrightarrow$

For $n = 1$, $V_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $n = 2$, $V_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = V_2 + V_1$

For $n = 3$, $V_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = V_3 + V_2$

For $n = 4$, $V_5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix} = V_4 + V_3$

Thus, the sequence has the following pattern:

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ V_{n+1} = V_n + V_{n-1},\ \text{for } n \geqslant 3.$$

(b) Consider now the sequence of Fibonacci numbers $x_0 = 1$,
$x_1 = 1$ and $x_{n=1} = x_n + x_{n-1}$ for $n = 1, 2, \ldots.$

Determine a matrix B such that $x_n = BV_n$

Note that as x_n is a single number, in order to satisfy the equation $x_n = BV_n$,
B must be a single row matrix. Let $B = [a, b]$. Then

For $n = 1$, $x_1 = BV_1 \Longrightarrow 1 = [\,a \;\; b\,]\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow a + b = 1$ ---------- Eq1

For n = 2, x_2 = $BV_2 \Longrightarrow 2 = [\,a\ \ b\,] \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Longrightarrow 2a + b = 2$ ---------- Eq2

For n = 3, x_3 = $BV_3 \Longrightarrow 3 = [\,a\ \ b\,] \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Longrightarrow 3a + 2b = 3$ ---------- Eq3

Solving the system of equations using Eq1 and Eq2,

gives a = 1 and b = 0. These values satisfy Eq3. Thus, B = [1, 0]

Q9. Matrices and Trigonometric Identities

For any angle θ in $[0, 2\pi]$, consider the rotation matrix:

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Compute the followings.

You may use the following trigonometric identities for the following tasks:

(T1) $\mathbf{\cos^2\theta + \sin^2\theta = 1}$

(T2) $\mathbf{\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2}$

(T3) $\mathbf{\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2}$

(a) The determinant of A_θ

$$\det(A_\theta) = \cos(\theta)\cos(\theta) + \sin(\theta)\sin(\theta)$$
$$= \cos^2(\theta) + \sin^2(\theta)$$
$$= 1, \text{ using (T1)}$$

(b) The product $A_{\theta_1} A_{\theta_2}$, for θ_1, θ_2 in $[0, 2\pi]$

$$A_{\theta_1} A_{\theta_2} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{pmatrix}$$
$$= \begin{pmatrix} \cos((\theta_1 + \theta_2)) & -\sin((\theta_1 + \theta_2)) \\ \sin((\theta_1 + \theta_2)) & \cos((\theta_1 + \theta_2)) \end{pmatrix}, \text{ using (T1) and (T2)}$$

(c) Given some A_θ, find A_η, with η in $[0, 2\pi]$ such that $A_\eta = A_\Theta{}^{-1}$

$$A_\eta = (A_\theta)^{-1} = \frac{1}{\det(A_\theta)} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$= 1 \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$