MATH7501: Mathematics for Data Science I

Unit 7: Derivatives, Optimisation and basic ODEs

Slides by Timothy Buttsworth (2021)

Motivation: The Need for Derivatives

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Fundamentally, these are important because everything in life is changing, and we need rigorous ways of describing and quantifying these changes.

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Why does the speedometer give us this information instead of, for example, our average speed over the last minute?

Possible answer: the instantaneous car speed at the precise moment of a crash is much more significant than the average speed over the last minute.

7.1 Derivatives

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$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h},$$

where $b = a + h$.

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$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h},$$

where $b = a + h$. This slope should be a good approximation to the actual tangent line by making b close to a (equivalently, having h close to 0).

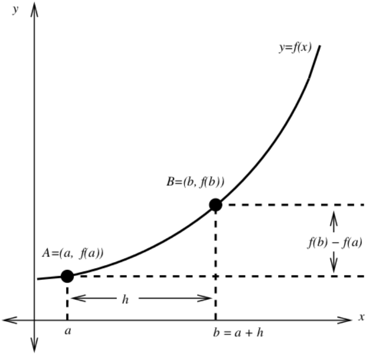

Figure 47: We want to determine the tangent line at the point A.

7.1 Derivatives

Definition (Derivative at a Point)

The *derivative of a function f at a point a* is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

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If f is differentiable at every point on an open interval (a, b), then f is said to be *differentiable on* (a, b).

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Definition (Differentiability)

If f is differentiable at every point on an open interval (a, b), then f is said to be *differentiable on* (a, b). In this case, we quite often write $\frac{df}{dx}$ to denote the function which assigns to each x the derivative at that point.

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Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = x^2 + x$.

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Solution

For each x, we have

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^2+(x+h)-(x^2+x)}{h}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-x^2-x}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2+h}{h}\\
&= 2x+1.
\end{aligned}$$

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\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} \\
&= 2x + 1.
\end{aligned}$$

Therefore, f is differentiable on $\mathbb{R}$ and $f'(x) = 2x + 1$.

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For each x, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}.
\end{aligned}$$

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For each x, we have

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Recall that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$, so $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = e^x$.

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$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}.$$

Recall that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, so $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = e^x$.
Therefore, f is differentiable on $\mathbb{R}$ and $f'(x) = e^x$.

7.2 Working with Derivatives

Mathematica can compute derivatives:

```
In[7]:= D[x^2 + x^3, x]

Out[7]= 2 x + 3 x^2

In[8]:= f[x_] = x^2 + x^3
        D[f[x], x]

Out[8]= x^2 + x^3

Out[9]= 2 x + 3 x^2

In[10]:= f'[x]

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```

Some Useful Derivatives

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Higher Derivatives

If f' is the derivative of f, then f'' is the derivative of f'. Repeating this gives the nth order derivative of f, denoted $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$.

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Higher Derivatives

If f' is the derivative of f, then f'' is the derivative of f'. Repeating this gives the nth order derivative of f, denoted $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$. For example, $\frac{d^2 \sin(x)}{dx^2} = -\sin(x)$. If $f(t)$ is the displacmement of an object after time t, then $f'(t)$ is the object's velocity, and $f''(t)$ is its acceleration.

Mathematica can compute higher derivatives:

```
In[13]:= D[x^3, {x, 2}]

Out[13]= 6 x

In[14]:= D[x^3, {x, n}]

Out[14]= x^(3-n) FactorialPower[3, n]

In[15]:= D[x^k, {x, n}]

Out[15]= x^(k-n) FactorialPower[k, n]

In[16]:= f[x_] = x^k
         D[f[x], {x, n}]

Out[16]= x^k

Out[17]= x^(k-n) FactorialPower[k, n]
```

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- $\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$;
- $\frac{d(f/g)}{dx} = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}$.

Question (Derivative of the tangent function)

Consider the function $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$. Calculate $f'(x)$, whenever it exists.

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Solution

If $x = (n + \frac{1}{2})\pi$ for some integer n, then $\cos(x) = 0$ and the function is not defined.

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Consider the function $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$. Calculate $f'(x)$, whenever it exists.

Solution

If $x = (n + \frac{1}{2})\pi$ for some integer n, then $\cos(x) = 0$ and the function is not defined. Otherwise, we can use the quotient rule:

$$\begin{aligned}
(\tan(x))' &= \frac{(\sin(x))' \cos(x) - (\cos(x))' \sin(x)}{\cos(x)^2} \\
&= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\
&= \frac{1}{\cos(x)^2} \\
&= \sec(x)^2.
\end{aligned}$$

Theorem (The Chain Rule)

Suppose g and h are differentiable functions. Then the function $f = g \circ h$ (i.e., $f(x) = g(h(x))$) is differentiable, and

$$f'(x) = g'(h(x))h'(x).$$

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In Leibniz notation, we write $u = h(x)$ and $y = g(u)$ so that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Question

Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

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Solution

Let $u = \sin(x)$ so that $y = \sqrt{u}$.

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Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

Solution

Let $u = \sin(x)$ so that $y = \sqrt{u}$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{u}} \frac{du}{dx} \\ &= \frac{\cos(x)}{2\sqrt{\sin(x)}}, \end{aligned}$$

as long as $\sin(x) > 0$.

Inverse Function Derivatives

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$$\begin{aligned} 1 &= \frac{df(y)}{dx} \\ &= \frac{df}{dy}\frac{dy}{dx}. \end{aligned}$$

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This implies that $\frac{dy}{dx} = \frac{1}{\frac{df}{dy}}$. In Newton's notation this is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

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Question (Derivative of Log)

Let $g(x) = ln(x)$, with domain $(0, \infty)$ and range $\mathbb{R}$. Find $g'(x)$.

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Solution

Let $f(x) = e^x$, so that $g(x) = f^{-1}(x)$.

Question (Derivative of Log)

Let $g(x) = ln(x)$, with domain $(0, \infty)$ and range $\mathbb{R}$. Find $g'(x)$.

Solution

Let $f(x) = e^x$, so that $g(x) = f^{-1}(x)$. Then

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{e^{\ln(x)}} \\ &= \frac{1}{x}, \end{aligned}$$

whenever $x > 0$.

Question (Derivative of arcsin)

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Solution

Let $f(x) = \sin(x)$ so that $g(x) = f^{-1}(x)$. Then

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin(x))} \\ &= \frac{1}{\sqrt{1 - \sin(\arcsin(x))^2}} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

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Similar techniques show that

$$(\text{arcos})'(x) = -\frac{1}{\sqrt{1 - x^2}}, \ (\arctan)'(x) = \frac{1}{1 + x^2}.$$

Theorem (L'Hôpital's Rule)

Suppose that f and g are defined and differentiable, and $g'(x) \neq 0$ near a (except possibly at a itself).

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Suppose that f and g are defined and differentiable, and $g'(x) \neq 0$ near a (except possibly at a itself). Suppose that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

or

$$\lim_{x \to a} f(x) = \pm\infty, \ \lim_{x \to a} g(x) = \pm\infty.$$

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$$\lim_{x \to a} f(x) = \pm\infty, \; \lim_{x \to a} g(x) = \pm\infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the right hand side exists.

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Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

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Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x\to 1}(x-1) = 0$, so the quotient rule for limits is not applicable.

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Question

Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x\to 1}(x-1) = 0$, so the quotient rule for limits is not applicable. However, l'Hôpital's rule implies that

$$lim_{x\to 1}\frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1},$$

provided the right hand limit exists.

Question

Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x\to 1}(x-1) = 0$, so the quotient rule for limits is not applicable. However, l'Hôpital's rule implies that

$$lim_{x\to 1}\frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1},$$

provided the right hand limit exists. But this limit does exist, and is equal to 1.

7.2 Working with Derivatives

Question

Evaluate $\lim_{x \to 0^+} x \ln(x)$.

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Solution

Write $x\ln(x) = \frac{\ln(x)}{1/x}$.

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Solution

Write $x\ln(x) = \frac{\ln(x)}{1/x}$. Then by L'Hôpital's rule,

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{1/x}{-1/x^2},$$

provided this right-hand-side limit exists.

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Write $x \ln(x) = \frac{\ln(x)}{1/x}$. Then by L'Hôpital's rule,

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{1/x}{-1/x^2},$$

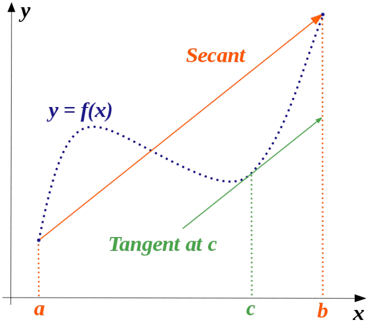
provided this right-hand-side limit exists. This limit does exist, and is equal to 0.

7.2 Working with Derivatives

Theorem (The Mean Value Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). Then there exists a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



https://en.wikipedia.org/wiki/Mean_value_theorem

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

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Solution

- Step One: Show there are at least two solutions.

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Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.

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Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
  - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.

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- Step Two: Show there are no more than two solutions.

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- Step Two: Show there are no more than two solutions.
  - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in (-1, \infty)$.

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  - If a and b were two distinct roots for f in $(-\infty, -1]$, the MVT would imply the existence of $c \in (-\infty, -1)$ with $f'(c) = 0$, which is a contradiction. Thus f can have no more than one root in $(-\infty, -1]$.

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- Step Two: Show there are no more than two solutions.
  - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in (-1, \infty)$.
  - If a and b were two distinct roots for f in $(-\infty, -1]$, the MVT would imply the existence of $c \in (-\infty, -1)$ with $f'(c) = 0$, which is a contradiction. Thus f can have no more than one root in $(-\infty, -1]$.
  - Similarly, there is at most one root of f on $[-1, \infty)$, so f can have no more than two real roots.

7.3 Smoothness

Theorem (Differentiability implies Continuity)

If f is differentiable at a, then f is continuous at a.

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To see this, note that differentiability implies that the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. But then

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = f'(a) \cdot 0 = 0,$$

so f is continuous at a.

7.3 Smoothness

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Solution

Note that

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

It is clear that f is continuous at $x = 0$.

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Solution

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It is clear that f is continuous at $x = 0$. To see that f is not differentiable at $x = 0$, note that

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1, \qquad \text{while } \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

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These one-sided limits do not agree, so the overall limit, i.e., the derivative, does not exist.

We can also find multi-variable functions that do not have tangent planes everywhere:

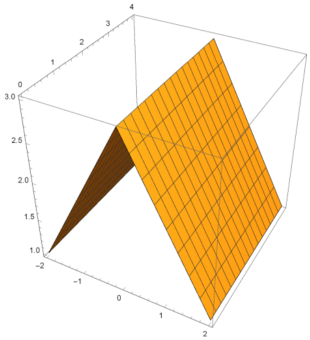

Figure 49: The function $z = f(x) = 3 - |x|$. $f_x(0)$ is undefined.

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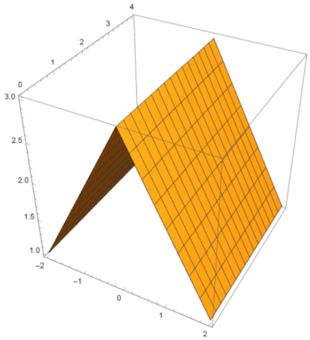
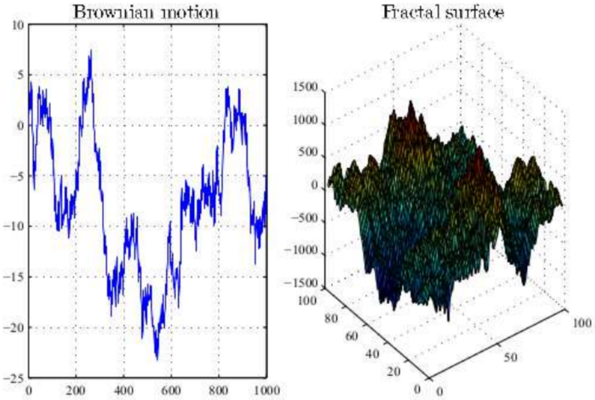


Figure 49: The function $z = f(x) = 3 - |x|$. $f_x(0)$ is undefined.

In general, we say that a surface $z = f(x, y)$ is smooth if f, f_x, f_y all exist and are continuous.

7.3 Smoothness

Many fractals (like Brownian motion) are continuous, but not smooth.



Definition (Increasing)

A function defined on a closed interval I is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.

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A function defined on a closed interval I is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.

Definition (Decreasing)

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7.4 Optimisation

Theorem (Derivatives of Increasing/Decreasing Functions)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b).

1. If $f'(x) > 0$ on (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ on (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

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Proof

Choose any $x_1 < x_2$ in $[a, b]$.

7.4 Optimisation

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Proof

Choose any $x_1 < x_2$ in $[a, b]$. Then by applying the MVT, we can find $c \in (x_1, x_2) \subseteq (a, b)$ so that

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$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If $f' > 0$, then $f(x_2) - f(x_1) > 0$, so item 1 is true.

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Theorem (Derivatives of Increasing/Decreasing Functions)

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$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If $f' > 0$, then $f(x_2) - f(x_1) > 0$, so item 1 is true. The other claims follow similarly.

Question

Find the intervals on which $f(x) = x^3 + x$ is increasing or decreasing.

7.4 Optimisation

Question

Find the intervals on which $f(x) = x^3 + x$ is increasing or decreasing.

Solution

We compute $f'(x) = x^2 + 1 > 0$. Therefore, f is increasing on $\mathbb{R}$.

Definition (Local Extrema)

A function f is said to have a *local maximum* (*local minimum*) at a point a if there is an open interval I containing a so that $f(a) \geq f(x)$ ($f(a) \leq f(x)$) for all $x \in I \cap dom(f)$.

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A function f is said to have a *global maximum* (*global minimum*) at a point a $f(a) \geq f(x)$ $(f(a) \leq f(x))$ for all $x \in dom(f)$.

All global maximums (minimums) are local maximums (minimums).

7.4 Optimisation

Definition (Critical Point)

A function f is said to have a *critical point* at $a \in dom(f)$ if $f'(a)$ vanishes, or does not exist.

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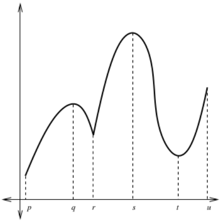

Figure 50: What do the points $x = p, q, r, s, t, u$ represent?

7.4 Optimisation

Theorem (Local Extrema are Critical Points)

If $a \in dom(f)$ is a local mininum or maximum of f, then a is a critical point of f.

Theorem (Local Extrema are Critical Points)

If $a \in dom(f)$ is a local mininum or maximum of f, then a is a critical point of f.

Proof

If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a.

7.4 Optimisation

Theorem (Local Extrema are Critical Points)

If $a \in dom(f)$ is a local mininum or maximum of f, then a is a critical point of f.

Proof

If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a. If $f'(a)$ does not exist, then a is a CP.

7.4 Optimisation

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Proof

If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a. If $f'(a)$ does not exist, then a is a CP. If $f'(a)$ does exist, then

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

7.4 Optimisation

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$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

and

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so $f'(a) = 0$ as required.

7.4 Optimisation

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If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a. If $f'(a)$ does not exist, then a is a CP. If $f'(a)$ does exist, then

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

and

$$f'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0,$$

so $f'(a) = 0$ as required. The case that a is a local minimum is almost identical.

Theorem (First Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b). Choose a point $c \in (a, b)$.

1. If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then f has a local maximum at c.
2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.

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2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.

Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c), so $f(x) \leq f(c)$ for all $a < x < c$.

7.4 Optimisation

Theorem (First Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b). Choose a point $c \in (a, b)$.

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2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.

Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c), so $f(x) \leq f(c)$ for all $a < x < c$. Similarly, f is decreasing on (c, b), so $f(x) \leq f(c)$ for all $x \in (c, b)$, as required.

Theorem (Second Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f.
2. If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f.

Theorem (Second Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

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2. If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f.

Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''.

7.4 Optimisation

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1. If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f.
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Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''. Then $f'(x)$ is a decreasing function for x close to c.

7.4 Optimisation

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Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''. Then $f'(x)$ is a decreasing function for x close to c. Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$.

7.4 Optimisation

Theorem (Second Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f.
2. If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f.

Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''. Then $f'(x)$ is a decreasing function for x close to c. Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$. The first derivative test implies that c is a local maximum of f.

7.4 Optimisation

Question

Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

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Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

Solution

- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.

7.4 Optimisation

Question

Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

Solution

- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.
- Therefore the critical points are $x = -2$ and $x = 1$.

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- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.
- Therefore the critical points are $x = -2$ and $x = 1$.
- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.

7.4 Optimisation

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Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

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- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.
- Therefore the critical points are $x = -2$ and $x = 1$.
- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.
- We have $f''(1) > 0$, so the $x = 1$ critical point is a local minimum. We have $f(1) = -3$.

Question

Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

Solution

- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.
- Therefore the critical points are $x = -2$ and $x = 1$.
- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.
- We have $f''(1) > 0$, so the $x = 1$ critical point is a local minimum. We have $f(1) = -3$.
- All local extrema are critical points, so we have found all local extrema.

7.4 Optimisation

Theorem (Extreme Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then f achieves a global maximum and a global minimum. Each of these values is achieved at an interior critical point, or at an end point.

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This still works if a is $-\infty$, or if b is ∞.

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$$f(0) = 1,\ f(2) = -3,\ f(\frac{1}{2}) = \frac{1}{8},\ f(4) = 17,$$

so $(4, 17)$ is the global maximum, and $(2, -3)$ is the global minimum.

7.5 Basic Ordinary Differential Equations

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The *order* of a differential equation is the order of the highest derivative that appears in the equation.

Example (Population Modelling)

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If $k > 0$, the population is growing; if $k < 0$, the population is shrinking; if $k = 0$, the population is constant.

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Solution

For any A, $y'(x) = xAe^{\frac{x^2}{2}}$, and $xy = xAe^{\frac{x^2}{2}}$ as required.

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Solutions of differential equations

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Solve the ODE $y'(x) = x^2$.

Solution

By integrating, we find that $y(x) = \frac{x^3}{3} + C$ for some arbitrary real constant C.

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Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

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Integrating once more and using the initial position gives

$$x(t) = ut, \qquad y(t) = vt - \frac{gt^2}{2}.$$

7.5 Basic Ordinary Differential Equations

Numerical v Analytic Solutions of ODEs

To solve an ODE analytically means to explicitly find the solution in terms well-understood continuous functions (like polynomials, trigonometric functions, exponentials, as well as sums, products, quotients and inverses of these functions).

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This is not always possible, so sometimes, it is useful to solve an ODE numerically, which means to use an algorithm to generate a function which is *almost* solution.

7.6 Application: Maximising Profits

Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a rectangular paddock for grazing. What is the maximal area of the paddock?

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- Let x, y be the side lengths of the rectangle in meters.
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- We find

$$A = x(200 - x);$$

  we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.

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  we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.
- The answer is $A = 10000$ in m^2 (achieved by the square of side length 100m).

7.6 Application: Maximising Profits

Bonus Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?

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Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?

The best shape is the circle, but why?