MATH7501: Mathematics for Data Science I

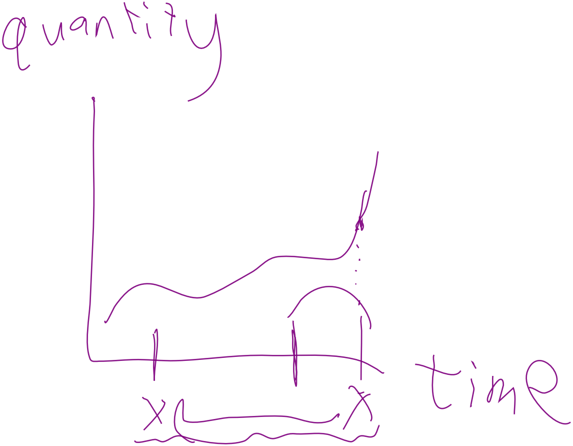
Unit 7: Derivatives, Optimisation and basic ODEs

(Differential
Calculus)

Slides by Timothy Buttsworth (2021)

Motivation: The Need for Derivatives

Derivatives are "instantaneous rates of change."



Motivation: The Need for Derivatives

Derivatives are "instantaneous rates of change."

Fundamentally, these are important because everything in life is changing, and we need rigorous ways of describing and quantifying these changes.

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There is often a practical need for derivatives. For example, when we are driving, our speedometer gives us the instantaneous rate of change of our location.

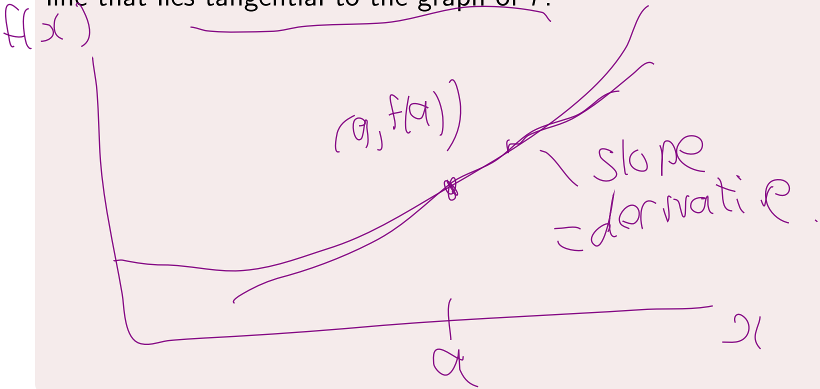
Why does the speedometer give us this information instead of, for example, our average speed over the last minute?

Possible answer: the instantaneous car speed at the precise moment of a crash is much more significant than the average speed over the last minute.

7.1 Derivatives

The idea of 'instantaneous rate of change'

The rate of change of a function f at a point a is the slope of a line that lies tangential to the graph of f.



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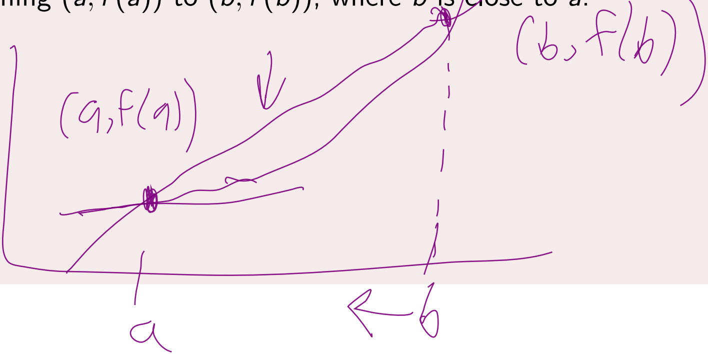
Such a tangent line can be approximated with a straight line joining $(a, f(a))$ to $(b, f(b))$, where b is close to a.



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rise

run.

$$\frac{f(b) - f(a)}{b - a} = \frac{\overset{f(b)}{f(a+h)} - f(a)}{\underset{b-a=}{h}},$$

$(b, f(b))$

where $b = a + h$.

$(a, f(a))$

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Such a tangent line can be approximated with a straight line joining $(a, f(a))$ to $(b, f(b))$, where b is close to a. Using the 'rise over run' formula, the slope of this line is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h},$$

where $b = a + h$. This slope should be a good approximation to the actual tangent line by making b close to a (equivalently, having h close to 0).

7.1 Derivatives

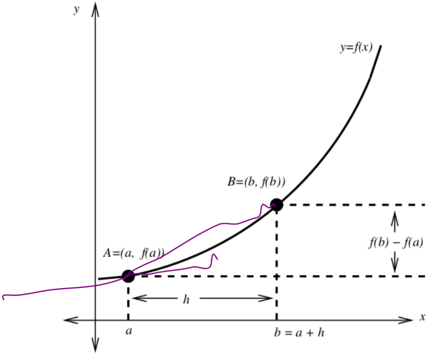


Figure 47: We want to determine the tangent line at the point A.

7.1 Derivatives

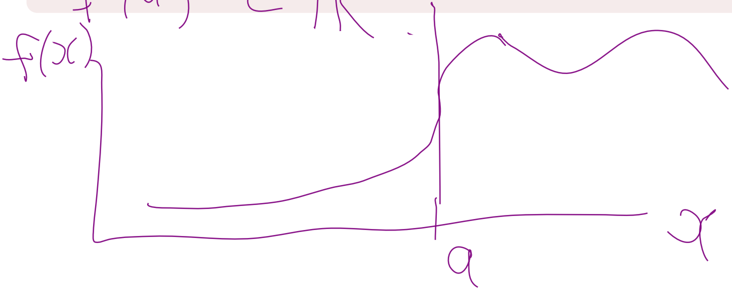
Definition (Derivative at a Point)

The *derivative of a function f at a point a* is defined to be

$$f'(a) = \underbrace{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}, \quad \leftarrow \text{slope}$$

provided this limit exists.

$f'(a) \in \mathbb{R}$



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Newtonian notation".

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Definition (Differentiability)

If f is differentiable at every point on an open interval (a, b), then f is said to be *differentiable on* (a, b). In this case, we quite often write $\frac{df}{dx}$ to denote the function which assigns to each x the derivative at that point.

Leibniz notation. $\frac{\Delta f \to}{\Delta x \to}$ $\frac{df}{dx}$.

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = x^2 + x$.

domain of $f = \mathbb{R}$

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = x^2 + x$.

$f(x+h) = (x+h) + (x+h)^2$

Solution

For each x, we have

$$
\begin{aligned}
\boxed{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}} &= \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} \\
&= 2x + 1.
\end{aligned}
$$

DNE for $h=0$.

$\frac{2xh + h^2 + h}{h}$

$= 2x + h + 1, h \neq 0.$

7.1 Derivatives

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = x^2 + x$.

Solution

For each x, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} \\
&= 2x + 1.
\end{aligned}$$

Therefore, f is differentiable on $\mathbb{R}$ and $f'(x) = 2x + 1$.

7.1 Derivatives

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Solution

For each x, we have $f(x+h) = e^{x+h}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}.$$

index rules

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = e^x$.

Solution

For each x, we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}.$$

Recall that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$, so $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = e^x$.

$f(x) = e^x$ is the unique function so that $f'(x) = f(x)$
$f(0) = 1$.

7.1 Derivatives

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$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}.$$

Recall that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, so $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = e^x$.
Therefore, f is differentiable on $\mathbb{R}$ and $f'(x) = e^x$.

7.2 Working with Derivatives

Mathematica can compute derivatives:

$$\frac{d(x^2+x^3)}{dx}$$

```
In[7]:= D[x^2 + x^3, x]
Out[7]= 2 x + 3 x^2

In[8]:= f[x_] = x^2 + x^3
        D[f[x], x]
Out[8]= x^2 + x^3

Out[9]= 2 x + 3 x^2

In[10]:= f'[x]
Out[10]= 2 x + 3 x^2
```

$f'[x]$

$= 2x + 3x^2$

7.2 Working with Derivatives

Some Useful Derivatives

- The derivative of a constant is 0 (because it does not change);

$f(x) = C \leftarrow$ constant

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) = \lim_{h \to 0} \frac{C - C}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0.$$

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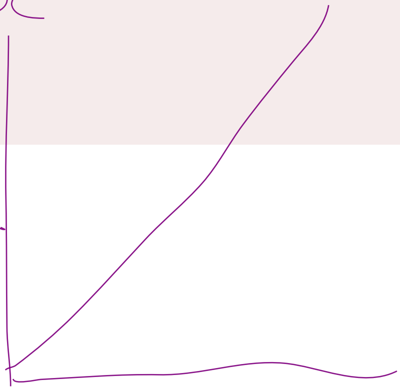
- The derivative of a constant is 0 (because it does not change);
- $\frac{d(x)}{dx} = 1$, i.e., the function $f(x) = x$ has a slope of 1 everywhere;

$$f(x) = x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1.$$



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- $\frac{d(x)}{dx} = 1$, i.e., the function $f(x) = x$ has a slope of 1 everywhere;
- $\boxed{\frac{d(x^{\alpha})}{dx} = \alpha x^{\alpha-1}}$ whenever these expressions are well-defined (be careful with expressions like $(-1)^{\frac{1}{2}}$); i

$$(x^{\alpha})' \quad \alpha x^{\alpha-1}$$

not dealing with this.

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- $\frac{d(\sin(x))}{dx} = \cos(x)$;

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Higher Derivatives

If f' is the derivative of f, then f'' is the derivative of f'. Repeating this gives the nth order derivative of f, denoted $f^{(n)}(x)$ or $\boxed{\frac{d^n f}{dx^n}}$.

(Newtonian

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Higher Derivatives

If f' is the derivative of f, then f'' is the derivative of f'. Repeating this gives the nth order derivative of f, denoted $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$. For example, $\frac{d^2 \sin(x)}{dx^2} = -\sin(x)$. If $f(t)$ is the displacmement of an object after time t, then $f'(t)$ is the object's velocity, and $f''(t)$ is its acceleration.

Mathematica can compute higher derivatives:

$f(x) = x^3$

$f'(x) = 3x^{3-1} = 3x^2$

```
In[13]:= D[x^3, {x, 2}]
```

Out[13]= $6\,x$

$f''(x) = 6x,$

```
In[14]:= D[x^3, {x, n}]
```

Out[14]= x^{3-n} FactorialPower[3, n]

$3 \times 2 \times 1 \times 0 \times \ldots$ n times

```
In[15]:= D[x^k, {x, n}]
```

Out[15]= x^{k-n} FactorialPower[k, n]

```
In[16]:= f[x_] = x^k
         D[f[x], {x, n}]
```

Out[16]= x^k

Out[17]= x^{k-n} FactorialPower[k, n]

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If f and g are differentiable functions and c is a constant, then:

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$\Rightarrow$ Derivative is a linear operator.

((derivative of a sum is the sum of the derivatives)).

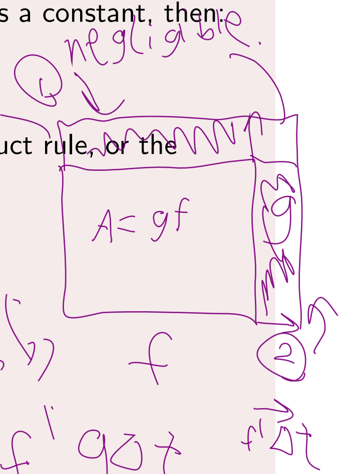
Theorem (Rules for Differentiation)

If f and g are differentiable functions and c is a constant, then:

- $(cf)' = cf'$;
- $(f \pm g)' = f' \pm g'$; $\Delta t\, g'$
- $(fg)' = f'g + fg'$ (this is called the product rule, or the Leibniz rule);

NOT TRUE.

"derivative of a product is the product of derivatives"



$① + ② \approx \Delta t\, f g' + f' g \Delta t$

7.2 Working with Derivatives

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Newtonian

- $(cf)' = cf'$;
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- $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$, if $g \neq 0$ (this is called the quotient rule). $\to 1$

Not defined at $g = 0$.

$g^{-1}(x)$ $g(x)$

inverse function $\neq$ $= \frac{1}{g(x)}$.

$\frac{f(x)}{g(x)} = f(x) g(x)^{-1}$.

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If f and g are differentiable functions and c is a constant, then:

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- $\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$;
- $\frac{d(f/g)}{dx} = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}$.

$$\left(\frac{f}{g}\right)' = (f g^{-1})'$$
$$= f' g^{-1} + (g^{-1})' f$$

Need chain rule.

Question (Derivative of the tangent function)

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Solution

If $x = (n + \frac{1}{2})\pi$ for some integer n, then $\cos(x) = 0$ and the function is not defined.

Question (Derivative of the tangent function)

Consider the function $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$. Calculate $f'(x)$, whenever it exists.

Solution

If $x = (n + \frac{1}{2})\pi$ for some integer n, then $\cos(x) = 0$ and the function is not defined. Otherwise, we can use the quotient rule:

$$\begin{aligned}(\tan(x))' &= \frac{(\sin(x))'\cos(x) - (\cos(x))'\sin(x)}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\ &= \frac{1}{\cos(x)^2} \\ &= \sec(x)^2.\end{aligned}$$

Theorem (The Chain Rule)

Suppose g and h are differentiable functions. Then the function $f = g \circ h$ (i.e., $f(x) = g(h(x))$) is differentiable, and

$$f'(x) = g'(h(x))h'(x).$$

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no ambiguities

$$\boxed{f'(x) = g'(h(x))h'(x).}$$

In Leibniz notation, we write $u = h(x)$ and $y = g(u)$ so that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

$y = f(x)$
$= g(h(x))$
$= g(u).$

(very easy to make mistakes.

Question

Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

Leibniz notation.

Let $f(x) = \sqrt{\sin(x)}$.

Calculate $f'(x)$.

7.2 Working with Derivatives

Question

Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

Solution

Let $u = \sin(x)$ so that $y = \sqrt{u}$.

7.2 Working with Derivatives

$$\left(u^{\frac{1}{2}}\right)' = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$$

Question

Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

Solution

Let $u = \sin(x)$ so that $y = \sqrt{u}$. Then

$$\frac{dy}{du}\,\frac{du}{dx} = \frac{dy}{dx} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$
$$= \frac{\cos(x)}{2\sqrt{\sin(x)}},$$

as long as $\sin(x) > 0$.

7.2 Working with Derivatives

Inverse Function Derivatives

Suppose f is a function, and $y = f^{-1}(x)$ is the inverse function.

$$(\neq \frac{1}{f(x)}$$

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(y)) = y$$

Inverse Function Derivatives

Suppose f is a function, and $y = f^{-1}(x)$ is the inverse function. Then, for all x on an appropriate domain,

$$\boxed{x = f(f^{-1}(x)).}$$

Differentiate using the chainrule.

Inverse Function Derivatives

Suppose f is a function, and $y = f^{-1}(x)$ is the inverse function. Then, for all x on an appropriate domain,

$$x = f(f^{-1}(x)).$$

Then by differentiating, we obtain

$$\begin{aligned} 1 &= \frac{df(y)}{dx} \\ &= \frac{df}{dy}\frac{dy}{dx}. \end{aligned}$$

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This implies that $\frac{dy}{dx} = \frac{1}{\frac{df}{dy}}$.

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Then by differentiating, we obtain

$$\begin{aligned} 1 &= \frac{df(y)}{dx} \\ &= \frac{df}{dy}\frac{dy}{dx}. \end{aligned}$$

This implies that $\frac{dy}{dx} = \frac{1}{\frac{df}{dy}}$. In Newton's notation this is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

7.2 Working with Derivatives

Question (Derivative of Log)

Let $g(x) = ln(x)$, with domain $(0, \infty)$ and range $\mathbb{R}$. Find $g'(x)$.

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Solution

Let $f(x) = e^x$, so that $g(x) = f^{-1}(x)$.

"natural log is the inverse of the exponential function".

Question (Derivative of Log)

Let $g(x) = ln(x)$, with domain $(0, \infty)$ and range $\mathbb{R}$. Find $g'(x)$.

Solution

Let $f(x) = e^x$, so that $g(x) = f^{-1}(x)$. Then

$(f^{-1})'(x)$

$= f'(f^{-1}(x))$

$f^{-1} = g$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$= \frac{1}{e^{\ln(x)}}$$

$e^{g(x)}$

$$= \frac{1}{x},$$

whenever $x > 0$.

$g'(x)$ DNE if $x < 0$.

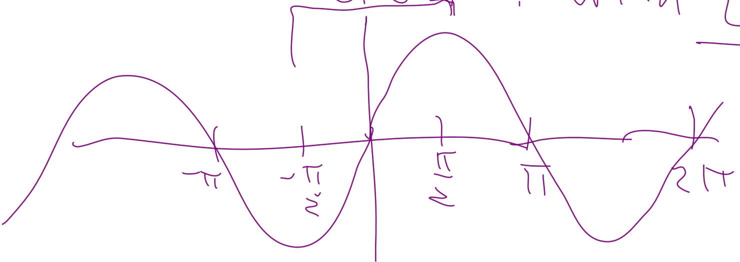
7.2 Working with Derivatives

Question (Derivative of arcsin)

Let $g(x) = \arcsin(x)$, with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find $g'(x)$.

"arcsin is the inverse of sin".

sin is injective on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, with $[-1, 1]$



7.2 Working with Derivatives

Question (Derivative of arcsin)

Let $g(x) = \arcsin(x)$, with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find $g'(x)$.

Solution

Let $f(x) = \sin(x)$ so that $g(x) = f^{-1}(x)$.

$(x \in [-\frac{\pi}{2}, \frac{\pi}{2}])$ $\quad x \in [-1, 1]$.

Question (Derivative of arcsin)

Let $g(x) = \arcsin(x)$, with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find $g'(x)$.

Solution

Let $f(x) = \sin(x)$ so that $g(x) = f^{-1}(x)$. Then

$f'(x) = \cos(x)$

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin(x))} \\ &= \frac{1}{\sqrt{1-\sin(\arcsin(x))^2}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

$g(x) = \arcsin(x).$

$\cos = \sqrt{\cos^2} = \sqrt{1-\sin^2}$

$\cos(\arcsin(x))$

$\in [-\frac{\pi}{2}, \frac{\pi}{2}].$

Question (Derivative of arcsin)

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Solution

Let $f(x) = \sin(x)$ so that $g(x) = f^{-1}(x)$. Then

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin(x))} \\ &= \frac{1}{\sqrt{1 - \sin(\arcsin(x))^2}} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

Similar techniques show that

$$(\text{arcos})'(x) = -\frac{1}{\sqrt{1-x^2}}, \quad (\arctan)'(x) = \frac{1}{1+x^2}.$$

Theorem (L'Hôpital's Rule)

Suppose that f and g are defined and differentiable, and $g'(x) \neq 0$ near a (except possibly at a itself).

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Theorem (L'Hôpital's Rule)

Suppose that f and g are defined and differentiable, and $g'(x) \neq 0$ near a (except possibly at a itself). Suppose that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

or

$$\lim_{x \to a} f(x) = \pm\infty, \ \lim_{x \to a} g(x) = \pm\infty.$$

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or

$$\lim_{x \to a} f(x) = \pm\infty, \ \lim_{x \to a} g(x) = \pm\infty.$$

"Lim of quotients is lim of quotients of the derivatives!"

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the right hand side exists.

7.2 Working with Derivatives

Question

Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Question

Evaluate $\lim_{x \to 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x \to 1}(x-1) = 0$, so the quotient rule for limits is not applicable.

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Question

Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x\to 1}(x-1) = 0$, so the quotient rule for limits is not applicable. However, l'Hôpital's rule implies that

$$lim_{x\to 1}\frac{\ln(x)}{x-1} = \lim_{x\to 1}\frac{1/x}{1},$$

$(\ln(x))' = \frac{1}{x}$

$(x-1)' = 1$

provided the right hand limit exists.

Question

Evaluate $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x\to 1} \ln(x) = \lim_{x\to 1}(x-1) = 0$, so the quotient rule for limits is not applicable. However, l'Hôpital's rule implies that

$$lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1}, \quad = 1.$$

provided the right hand limit exists. But this limit does exist, and is equal to 1.

Question

Evaluate $\lim_{x\to 0^+} x\ln(x)$.

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} = 1$$

L'Hop:

$$\frac{1+\cos(x)}{1} = 1+\cos(x),$$

limit DNE.

$$= \lim_{x\to\infty} \frac{x}{x} + \lim_{x\to\infty} \frac{\sin(x)}{x}$$

1 $\quad$ 0 squeeze theorem.

7.2 Working with Derivatives

Question

Evaluate $\lim_{x \to 0^+} x \ln(x)$.

not obviously a quotient.

Solution

Write $x \ln(x) = \frac{\ln(x)}{1/x}$.

$$\frac{1}{\left(\frac{1}{x}\right)} = x$$

7.2 Working with Derivatives

Question

Evaluate $\lim_{x \to 0^+} x \ln(x)$.

Solution

Write $x \ln(x) = \frac{\ln(x)}{1/x}$. Then by L'Hôpital's rule,

$$\ln(x)' = \frac{1}{x}$$

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{1/x}{-1/x^2},$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

provided this right-hand-side limit exists.

Question

Evaluate $\lim_{x\to 0^+} x\ln(x)$.

Solution

Write $x\ln(x) = \frac{\ln(x)}{1/x}$. Then by L'Hôpital's rule,

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}, \quad = 0$$

provided this right-hand-side limit exists. This limit does exist, and is equal to 0.

$$\frac{1/x}{-1/x^2} = \frac{-x^2}{x}$$

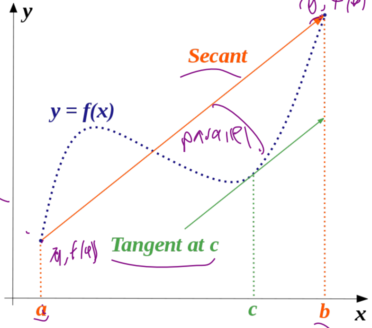
Theorem (The Mean Value Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b). Then there exists a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$\frac{\text{rise}}{\text{run}}$

MVT
"the average rate of change is acheived as a derivative somewhere".



"average rate of change on (a,b)".

https://en.wikipedia.org/wiki/Mean_value_theorem

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

7.2 Working with Derivatives

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions. and distinct.

Solution

- Step One: Show there are at least two solutions. distinct
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
  - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
  - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.
- Step Two: Show there are no more than two solutions.

7.2 Working with Derivatives

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
  - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.
- Step Two: Show there are no more than two solutions.
  - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in (1, \infty)$.

Assume at least 3 solutions.

Then at least 2 sol^{n}s

on $(-\infty, -1)$ or $(-1, \infty)$.

-1

7.2 Working with Derivatives

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
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- Step Two: Show there are no more than two solutions.
  - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in (1, \infty)$.
  - If a and b were two distinct roots for f in $(-\infty, -1]$, the MVT would imply the existence of $c \in (-\infty, -1)$ with $f'(c) = 0$, which is a contradiction. Thus f can have no more than one root in $(-\infty, -1]$.

$$\frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

7.2 Working with Derivatives

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
  - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
  - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.
- Step Two: Show there are no more than two solutions.
  - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in \cancel{(1, \infty)}$. $(-1, \infty)$
  - If a and b were two distinct roots for f in $(-\infty, -1]$, the MVT would imply the existence of $c \in (-\infty, -1)$ with $f'(c) = 0$, which is a contradiction. Thus f can have no more than one root in $(-\infty, -1]$.
  - Similarly, there is at most one root of f on $[-1, \infty)$, so f can have no more than two real roots.

7.3 Smoothness

Theorem (Differentiability implies Continuity)

If f is differentiable at a, then f is continuous at a.

Differentiablity

$\Rightarrow$ Continuity.

Theorem (Differentiability implies Continuity)

If f is differentiable at a, then f is continuous at a.

To see this, note that differentiability implies that the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

Theorem (Differentiability implies Continuity)

If f is differentiable at a, then f is continuous at a.

To see this, note that differentiability implies that the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

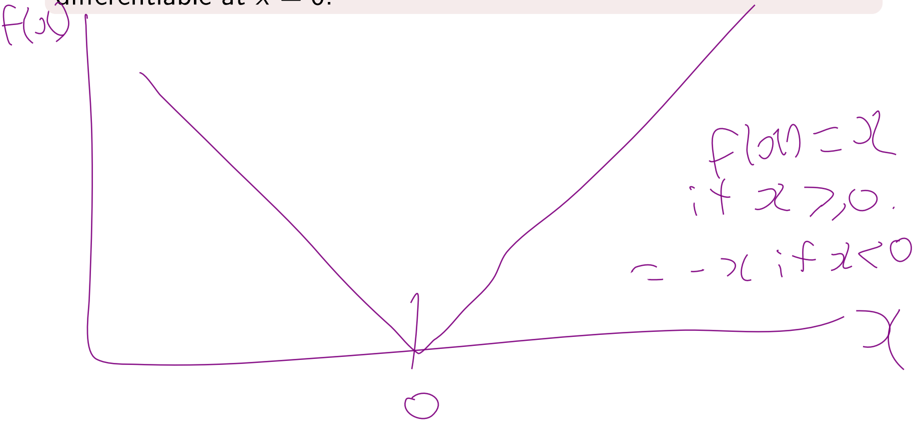
exists. But then

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) = f'(a) \cdot 0 = 0,$$

so f is continuous at a.

Question

Show that the function $f(x) = |x|$ is continuous, but not differentiable at $x = 0$.



$f(x) = x$ if $x \geq 0$.
$= -x$ if $x < 0$.

Question

Show that the function $f(x) = |x|$ is continuous, but not differentiable at $x = 0$.

Solution

Note that

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

It is clear that f is continuous at $x = 0$.

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0 = f(0).$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} (+x) = 0$$

7.3 Smoothness

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Show that the function $f(x) = |x|$ is continuous, but not differentiable at $x = 0$.

Solution

Note that

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

It is clear that f is continuous at $x = 0$. To see that f is not differentiable at $x = 0$, note that

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1, \quad \text{while} \quad \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

7.3 Smoothness

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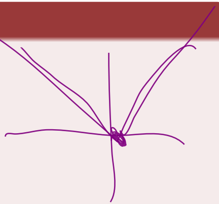

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These one-sided limits do not agree, so the overall limit, i.e., the derivative, does not exist.

7.3 Smoothness

We can also find multi-variable functions that do not have tangent planes everywhere:

$z = f(x,y)$
$= 3 - |x|$

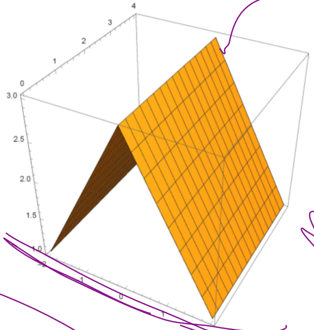
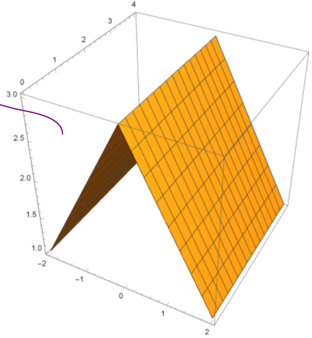


Figure 49: The function $z = f(x) = 3 - |x|$. $f_x(0)$ is undefined.

7.3 Smoothness

We can also find multi-variable functions that do not have tangent planes everywhere:

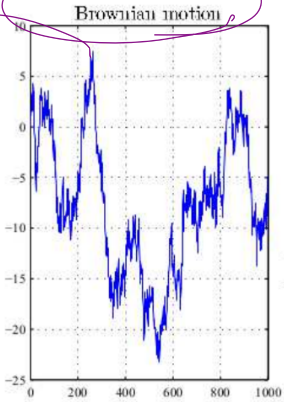


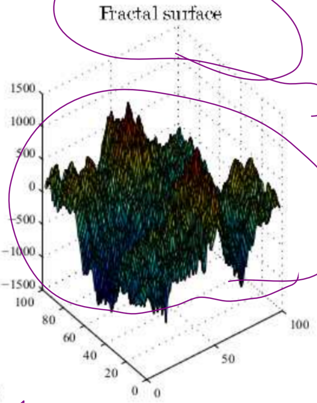
have defined set.

Figure 49: The function $z = f(x) = 3 - |x|$. $f_x(0)$ is undefined.

In general, we say that a surface $z = f(x, y)$ is smooth if f, f_x, f_y all exist and are continuous.

Many fractals (like Brownian motion) are continuous, but not smooth.



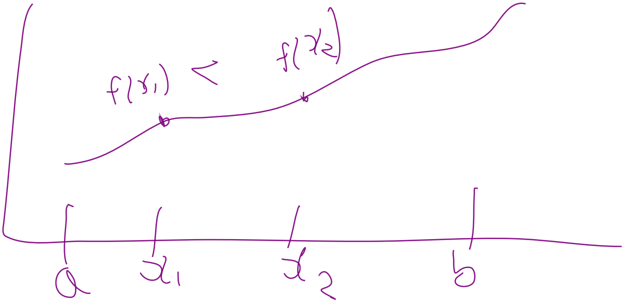


7.4 Optimisation

$[a, b]$

Definition (Increasing)

A function defined on a closed interval I is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.



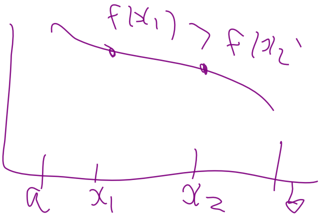
7.4 Optimisation

Definition (Increasing)

A function defined on a closed interval I is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.

Definition (Decreasing)

A function defined on an closed interval I is said to be *decreasing* on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.



Theorem (Derivatives of Increasing/Decreasing Functions)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b).

1. If $f'(x) > 0$ on (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ on (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

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3. If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

Proof

Choose any $x_1 < x_2$ in $[a, b]$.

To show that f is increasing, we need to show that $f(x_1) < f(x_2)$.

MVT on $[x_1, x_2]$

Theorem (Derivatives of Increasing/Decreasing Functions)

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Proof

Choose any $x_1 < x_2$ in $[a, b]$. Then by applying the MVT, we can find $c \in (x_1, x_2) \subseteq (a, b)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

7.4 Optimisation

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Choose any $x_1 < x_2$ in $[a, b]$. Then by applying the MVT, we can find $c \in (x_1, x_2) \subseteq (a, b)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \qquad > 0$$

If $f' > 0$, then $f(x_2) - f(x_1) > 0$, so item 1 is true.

7.4 Optimisation

Theorem (Derivatives of Increasing/Decreasing Functions)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b).

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3. If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

Proof

Choose any $x_1 < x_2$ in $[a, b]$. Then by applying the MVT, we can find $c \in (x_1, x_2) \subseteq (a, b)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \quad = 0$$

If $f' > 0$, then $f(x_2) - f(x_1) > 0$, so item 1 is true. The other claims follow similarly. $\Longrightarrow f(x_2) = f(x_1).$

Question

Find the intervals on which $f(x) = x^3 + x$ is increasing or decreasing.

Question

Find the intervals on which $f(x) = x^3 + x$ is increasing or decreasing.

Solution

We compute $f'(x) = x^2 + 1 > 0$. Therefore, f is increasing on $\mathbb{R}$.

$\mathbb{R} = (-\infty, \infty)$

is closed 'clopen'.
open.

Definition (Local Extrema)

A function f is said to have a *local maximum* (*local minimum*) at a point a if there is an open interval I containing a so that $f(a) \geq f(x)$ ($f(a) \leq f(x)$) for all $x \in I \cap dom(f)$.

(Extrema = maximums and minimums).

7.4 Optimisation

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7.4 Optimisation

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A function f is said to have a *local maximum* (*local minimum*) at a point a if there is an open interval I containing a so that $f(a) \geq f(x)$ $(f(a) \leq f(x))$ for all $x \in I \cap dom(f)$.

Definition (Global Extrema)

A function f is said to have a *global maximum* (*global minimum*) at a point a $f(a) \geq f(x)$ $(f(a) \leq f(x))$ for all $x \in dom(f)$.

All global maximums (minimums) are local maximums (minimums).

Definition (Critical Point)

A function f is said to have a *critical point* at $a \in dom(f)$ if $f'(a)$ vanishes, or does not exist.

$f'(a) = 0$

or $f'(a)$ does not exist.

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Definition (Critical Point)

A function f is said to have a *critical point* at $a \in dom(f)$ if $f'(a)$ vanishes, or does not exist.

p, is CP because $f'(p)$ DNE

" " $f(p)$

q, s, t, slope is 0

r, $f'(r)$ DNE

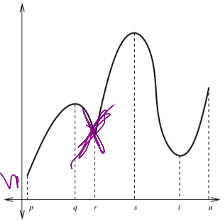

Figure 50: What do the points $x = p, q, r, s, t, u$ represent?

They are all CPs.

Theorem (Local Extrema are Critical Points)

If $a \in dom(f)$ is a local mininum or maximum of f, then a is a critical point of f.

"All local and global extrema are critical points"

BUT

"not all CPs are extrema",

7.4 Optimisation

Theorem (Local Extrema are Critical Points)

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Proof

If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a.

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If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a. If $f'(a)$ does not exist, then a is a CP. If $f'(a)$ does exist, then

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

If $x > a$ then $x - a > 0$

and $f(x) - f(a) \leq 0$.

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so $f'(a) = 0$ as required.

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so $f'(a) = 0$ as required. The case that a is a local minimum is almost identical.

Theorem (First Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b). Choose a point $c \in (a, b)$.

1. If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then f has a local maximum at c.
2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c.

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Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c), so $f(x) \leq f(c)$ for all $a < x < c$.

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Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c), so $f(x) \leq f(c)$ for all $a < x < c$. Similarly, f is decreasing on (c, b), so $f(x) \leq f(c)$ for all $x \in (c, b)$, as required.

Theorem (Second Derivative Test)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f.
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If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''.

7.4 Optimisation

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If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''. Then $f'(x)$ is a decreasing function for x close to c. Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$.

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If $f''(c) < 0$, then $f''(x) < 0$ for x close to c, by continuity of f''. Then $f'(x)$ is a decreasing function for x close to c. Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$. The first derivative test implies that c is a local maximum of f.

Question

Find all local maximums and minimums of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

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- Therefore the critical points are $x = -2$ and $x = 1$.
- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.
- We have $f''(1) > 0$, so the $x = 1$ critical point is a local minimum. We have $f(1) = -3$.
- All local extrema are critical points, so we have found all local extrema.

7.4 Optimisation

Theorem (Extreme Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then f achieves a global maximum and a global minimum. Each of these values is achieved at an interior critical point, or at an end point.

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This still works if a is $-\infty$, or if b is ∞.

Question

Find the global maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.

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The derivative is

$$f'(x) = 3x^2 - 6x,$$

so the only critical points are $x = 0, 2$.

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so the only critical points are $x = 0, 2$. Therefore, the global maximums and minimums have to occur at $0, 2$, $-\frac{1}{2}$ or 4 (being the end points of the interval). We compute:

$$f(0) = 1,\ f(2) = -3,\ f(\frac{1}{2}) = \frac{1}{8},\ f(4) = 17,$$

so $(4, 17)$ is the global maximum, and $(2, -3)$ is the global minimum.

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7.5 Basic Ordinary Differential Equations

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The *order* of a differential equation is the order of the highest derivative that appears in the equation.

Example (Population Modelling)

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If $k > 0$, the population is growing; if $k < 0$, the population is shrinking; if $k = 0$, the population is constant.

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Solution

For any A, $y'(x) = xAe^{\frac{x^2}{2}}$, and $xy = xAe^{\frac{x^2}{2}}$ as required.

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Question

Solve the ODE $y'(x) = x^2$.

Solution

By integrating, we find that $y(x) = \frac{x^3}{3} + C$ for some arbitrary real constant C.

Question

Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

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The ODEs arise by specifying acceleration:

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g \approx -9.81.$$

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Integrating once more and using the initial position gives

$$x(t) = ut, \qquad y(t) = vt - \frac{gt^2}{2}.$$

Numerical v Analytic Solutions of ODEs

To solve an ODE analytically means to explicitly find the solution in terms well-understood continuous functions (like polynomials, trigonometric functions, exponentials, as well as sums, products, quotients and inverses of these functions).

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To solve an ODE analytically means to explicitly find the solution in terms well-understood continuous functions (like polynomials, trigonometric functions, exponentials, as well as sums, products, quotients and inverses of these functions).

This is not always possible, so sometimes, it is useful to solve an ODE numerically, which means to use an algorithm to generate a function which is *almost* solution.

Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a rectangular paddock for grazing. What is the maximal area of the paddock?

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- Let x, y be the side lengths of the rectangle in meters.
- Then $2(x + y) = 400$ and we wish to maximise $A = xy$.
- We find

$$A = x(200 - x);$$

we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.

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  we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.
- The answer is $A = 10000$ in m^2 (achieved by the square of side length 100m).

Bonus Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?

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Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?

The best shape is the circle, but why?