MATH7501 Practical 2, Semester 1-2021

Topic: Matrices

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Date: 05-03-2021

Pre-Tutorial Activity

Resources

- https://reference.wolfram.com/language/guide/MatrixOperations.html
- https://reference.wolfram.com/language/ref/Table.html
- https://reference.wolfram.com/language/ref/RandomInteger.html

Q1. Matrix Operations

- (Task) Create matrices *A, B* and *L*, where *A* is *2-by-3,* B is 3-by-2 and *L* is 3-by-3. You can choose the elements of these matrices to be real numbers. Refer to Practical 1 (week 1) handout for creating matrices.

```
ClearAll[A, B, L]
```

In[∘]:= $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$; $B = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 1 \end{pmatrix}$; $L = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 1 \\ 6 & 0 & 2 \end{pmatrix}$;

- (Task) Compute A+2A
- (Task) Compute AB (how about BA?) (Use either the **Dot[ ]** function or the **'.'** operator for **matrix multiplication**). The output from the Dot[ ] function and the ',' operator is a list of lists. To visualise the output as a matrix, use either **MatrixForm[ ]** function or the postfix function operator (**//**).

In[∘]:=
```
AB = A.B
```

Out[∘]= {{10, 8}, {19, 8}}

In[∘]:=
```
AB // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 10 & 8 \\ 19 & 8 \end{pmatrix}$$

In[ ]:=
```
Dot[A, B] // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 10 & 8 \\ 19 & 8 \end{pmatrix}$$

- (Task) Compute L^2, L^3 (Use **MatrixPower[ ]** function to compute powers of matrices)

In[ ]:=
```
Dot[L, L] // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 37 & 10 & 17 \\ 21 & 22 & 21 \\ 18 & 12 & 34 \end{pmatrix}$$

In[ ]:=
```
MatrixForm[MatrixPower[L, 2]]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 37 & 10 & 17 \\ 21 & 22 & 21 \\ 18 & 12 & 34 \end{pmatrix}$$

In[ ]:=
```
MatrixForm[MatrixPower[L, 3]]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 169 & 114 & 229 \\ 213 & 130 & 169 \\ 258 & 84 & 170 \end{pmatrix}$$

- (Task) Compute A^T, B^T (Use the **Transpose[ ]** function to compute transpose of a matrix Or ‘**Esc+tr+Esc**’)

In[ ]:=
```
Atr = MatrixForm[ Transpose[A]];
Btr = MatrixForm[Transpose[B]];
```

In[ ]:=
```
Btr
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

- (Task) Is $B^T A^T$ and compare the result with $(AB)^T$. Does the result hold in general?

In[ ]:=
```
MatrixForm[(A.B)ᵀ]
MatrixForm[Bᵀ.Aᵀ]
```

$$\begin{pmatrix} 10 & 19 \\ 8 & 8 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 10 & 19 \\ 8 & 8 \end{pmatrix}$$

Show $(AB)^T = B^T A^T$

Let A be an m×n matrix and B be an n×p matrix.
Let C = AB. Then C is an m×p matrix. The ijth element of C is given by:

$$c_{ji} = \sum_{k=1}^{n} a_{ik} \times b_{kj}, \quad i \in \{1, \ldots, m\}, \; j \in \{1, \ldots, p\}$$

Let D = $(AB)^T$, then D is a p×m matrix, whose jith element is given by:

$$d_{ji} = \sum_{k=1}^{n} a_{ik} \times b_{kj}, \quad j \in \{1, \ldots, p\}, \; i \in \{1, \ldots, m\}$$

Let $E = B^T A^T$. Then E is a p×m matrix, whose jith element is given by:

$$e_{ji} = \sum_{k=1}^{n} b_{kj} \times a_{ik}\ ,\ j \in \{1,\ \ldots,\ p\},\ i \in \{1,\ \ldots,\ m\}$$

$$= \sum_{k=1}^{n} a_{ik} \times b_{kj}$$

$$= d_{ji}$$

Thus, $(AB)^T = B^T A^T$.

- (Task) Compute the determinant and inverse of L. Check if the computed determinant is the same, if you calculate it by hand

```
Det[L];
Inverse[L];
```

Out[•]= -112

Compute determinant of a 3×3 matrix using cofactor method.

Suppose Y is as given below.

$$Y = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Cofactor of each element of Y is computed using $(-1)^{i+j}$, where i,j $\in$ {1,2,3}. So expanding along the first row, you get:

$$\det(Y) = (-1)^2 \times a \times \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} + (-1)^3 \times b \times \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + (-1)^4 \times c \times \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a\,(ei - fh) - b\,(di - fg) + c\,(dh - eg)$$

- So **det(Y) = a(ei - fh) - b(di - fg)+ c(dh-eg)**

Q2. Crating a Matrix Using a given Rule

- (Task) Create an n-by-n matrix, A, whose elements are given by: $a_{ij} = i + j$. You can choose a number for n.
- (**Hint**: first use **Table[ ]** function to create a list of elements and then **MatrixForm[ ]** to visualise the matrix)

```
Clear[A]
A = Table[i + j, {i, 1, 2}, {j, 1, 2}];
```

In[•]:= `A // MatrixForm`

Out[•]//MatrixForm=

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

- (Task) Find a formula for the entries B_{ij}, where B = AA and A is as described above.

- (**Hint**: Note B_{ij} is given by $\sum_{k=1}^{n}(i+k)(k+j)$. Expand the brackets inside the sum and simplify. You may use the following sums (see below).

$$
\begin{aligned}
B_{ij} &= \sum_{k=1}^{n} a_{ik} \times a_{kj}\ ,\ i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n\} \\
&= \sum_{k=1}^{n} (i+k)(k+j) \\
&= \sum_{k=1}^{n} \left(ik + ij + k^2 + kj\right) \\
&= \sum_{k=1}^{n} ik + \sum_{k=1}^{n} ij + \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} kj \\
&= i \sum_{k=1}^{n} k + ij \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} k^2 + j \sum_{k=1}^{n} k \\
&= (i+j) \sum_{k=1}^{n} k + nij + \sum_{k=1}^{n} k^2 \\
&= (i+j)\left[\frac{1}{2} n (n+1)\right] + nij + \left[\frac{1}{6} n (n+1)(2n+1)\right]
\end{aligned}
$$

In[254]:=
```
S1 = Sum[k^2, {k, 1, n}]
```
Out[254]= $\frac{1}{6} n (1+n)(1+2n)$

```
S2 = Sum[i j, {k, 1, n}];
```

In[255]:=
```
S3 = Sum[i k, {k, 1, n}]
```
Out[255]= $\frac{1}{2} i n (1+n)$

In[•]:=
```
S4 = Sum[k j, {k, 1, n}];
```

In[•]:=
```
S1 + S2 + S3 + S4
```
Out[•]= $i j n + \frac{1}{2} i n (1+n) + \frac{1}{2} j n (1+n) + \frac{1}{6} n (1+n)(1+2n)$

Q3. Generating Random Matrices

- (Task) Generate two random 2-by-2 matrix whose entries are drawn from the set {1, 2, 3} and check if $(AB)^T = A^T B^T$. (Use **RandomInteger[ ]** function).

checking if the property $(AB)^T = A^T B^T$ holds for two matrices. Note, this property does not hold in general, so you should expect the value of $AB)^T == A^T B^T$ be False mostly.

In[251]:=
```
A = RandomInteger[{1, 3}, {2, 2}]; B = RandomInteger[{1, 3}, {2, 2}];
(A.B)ᵀ == Aᵀ.Bᵀ
Clear[A, B]
```

Out[252]= False

- (Task)Write a function to draw 100,000 random pairs of matrices from the randM[ ] function. For each pair, check if (**AB^T)=A^{T}B^T** holds. Out of the 100,000 simulations, estimate the proportion of times this property holds. Note, this property does not holds in genera, so you should expect the proportion to be close to 0.

Step1:First write a function to generate an n-by-n matrix whose elements are uniformly drawn from a given set of numbers.

The function randM[ ], generates a random n-by-n matrix whose elements are uniformly drawn from the set {1,2,..,r}

In[194]:=
```
randM[n_, r_] := RandomInteger[{1, r}, {n, n}]
```

In[184]:=
```
A = randM[3, 3] (*test the function randM[ ],with n=3, r=3*)
```

Out[184]= {{1, 1, 2}, {3, 3, 2}, {2, 1, 1}}

In[186]:=
```
A // MatrixForm
```

Out[186]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

In[187]:=
```
Clear[A]
```

In[218]:=
```
Clear[transposeAB]
```

Step2:Use a Do[ ] loop within Module[ ] function to estimate the above mentioned proportion

In[239]:=
```
transposeAB[size_, range_, nrun_] :=
 Module[{A, B, n = size, r = range, count = 0, f = 0, run = nrun},
  Do[
          A = randM[n, r];
          B = randM[n, r];
          If[(A.B)ᵀ == Aᵀ.Bᵀ, count++, f++],
      (* Print[MatrixForm[(A.B)ᵀ]]; Print[MatrixForm[Aᵀ.Bᵀ]],*)
   {run}
  ];
  Print[{count, f}];
  N[count / run]
 ]
```

In[242]:=
```
transposeAB[3, 3, 100000]
```

{15, 99985}

Out[242]= 0.00015

Q4. Trigonometric Identities

For any angle θ in $[0, 2\pi]$, consider the reflection matrix:

$$A_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

- You may use the following trigonometric identities for the following tasks:
  - (T1) $\cos^2\theta + \sin^2\theta = 1$
  - (T2) $\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$
  - (T3) $\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$
- (Task) Find the determinant of A_θ

$$\begin{aligned} \det(A_\theta) &= -\cos(2\theta)\cos(2\theta) - \sin(2\theta)\sin(2\theta) \\ &= -\left[\cos^2(x) + \sin^2(x)\right], \text{ assuming } x = 2\theta \\ &= -1 \qquad\qquad , \text{ using (T1)} \end{aligned}$$

$$A_{\theta_1} A_{\theta_2} = \begin{pmatrix} \cos(2\theta_1) & \sin(2\theta_1) \\ \sin(2\theta_1) & -\cos(2\theta_1) \end{pmatrix} \begin{pmatrix} \cos(2\theta_2) & \sin(2\theta_2) \\ \sin(2\theta_2) & -\cos(2\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta_1)\cos(2\theta_2) + \sin(2\theta_1)\sin(2\theta_2) & \cos(2\theta_1)\sin(2\theta_2) - \sin(2\theta_1 \\ \sin(2\theta_1)\cos(2\theta_2) - \cos(2\theta_1)\sin(2\theta_2) & \sin(2\theta_1)\sin(2\theta_2) + \cos(2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(x)\cos(y) + \sin(x)\sin(y) & \cos(x)\sin(y) - \sin(x)\cos(y) \\ \sin(x)\cos(y) - \cos(x)\sin(y) & \sin(x)\sin(y) + \cos(x)\cos(y) \end{pmatrix},$$

assuming $x = 2\theta_1$ and $y = 2\theta_2$

$$= \begin{pmatrix} \cos(x-y) & \sin(y-x) \\ \sin(x-y) & \cos(x-y) \end{pmatrix}, \text{ using (T1) and (T2)}$$

$$= \begin{pmatrix} \cos(2(\theta_1-\theta_2)) & \sin(2(\theta_2-\theta_1)) \\ \sin(2(\theta_1-\theta_2)) & \cos(2(\theta_1-\theta_2)) \end{pmatrix}$$

- (Task) Given some A_θ, , find the inverse matrix of A_θ

$$\begin{aligned} (A_\theta)^{-1} &= \frac{1}{\det(A_\theta)} \begin{pmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\ &= -1 \begin{pmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \end{aligned}$$