Asymptotic Analysis of Nearly Unstable INAR(1) Models

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Outline

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Motivation

- basic, popular time-series model is AR(1):

$$Y_0 = 0, \quad Y_t = \mu + \theta Y_{t-1} + u_t, \quad u_t \text{ i.i.d. } \mathrm{N}(0, \sigma^2)$$

- in economics/medicine: often encounter count data
  - number of restaurants in Brussels
  - number of patients in hospital
  - number of transactions in a stock
- AR(1) model not suitable for count data
- AL-OSH & ALZAID (1987) introduced *Nonnegative* **IN**teger-valued analogue of **AR**(1) processes: **INAR**(1)

INteger-valued AutoRegressive process of order 1

Start in state 0:

$$X_0 = 0$$

Given $X_0, \ldots, X_{t-1}$ draw

$$\theta \circ X_{t-1} \sim \text{Binomial}(X_{t-1}, \theta)$$

and independent ε_t from distribution G on $\mathbb{Z}_+$.

Now X_t defined by:

$$X_t = \underbrace{\theta \circ X_{t-1}}_{\text{survivors during } (t-1,t]} + \underbrace{\varepsilon_t}_{\text{immigration during } (t-1,t]}$$

- all variables defined on probability space, $(\Omega, \mathcal{F}, \mathbb{P}_{\theta,G})$

Elementary properties

- $$\mathbb{E}\left[X_t \mid X_{t-1}\right] = \mu_G + \theta X_{t-1}$$

  $\Longrightarrow$ same Auto-Regression function as AR(1) process
- $$\operatorname{var}\left[X_t \mid X_{t-1}\right] = \sigma_G^2 + \theta(1-\theta) X_{t-1}$$

  $\Longrightarrow$ INAR process has ARCH(1) structure
- INAR process has same auto-correlation structure as AR(1) for $\theta \in [0, 1]$

  $\Longrightarrow$ only positive auto-correlation possible

Elementary properties

- if $\theta = 1$ AR(1) process is random walk with drift

$$Y_t = \mu + \theta Y_{t-1} + u_t = \mu + Y_{t-1} + u_t$$

- under $\mathbb{P}_1$ INAR process X is also random walk with drift

$$X_t = \theta \circ X_{t-1} + \varepsilon_t = \mu_G + X_{t-1} + (\varepsilon_t - \mu_G)$$

  - but $X_t \geq X_{t-1}$

Transition probabilities

- X is discrete-time Markov chain on $\mathbb{Z}_+$
- given X_{t-1}: ε_t and $\theta \circ X_{t-1}$ independent
- given X_{t-1}: $X_{t-1} - \theta \circ X_{t-1}$, 'number of deaths in $(t-1, t]$', has Binomial$(X_{t-1}, 1-\theta)$ distribution

So

$$\begin{aligned} P^{\theta}_{x_{t-1},x_t} &= \mathbb{P}_\theta \left\{ X_t = x_t \mid X_{t-1} = x_{t-1} \right\} \\ &= \sum_{k=-\Delta x_t \vee 0}^{x_{t-1}} \mathrm{b}_{x_{t-1},1-\theta}(k) g(\Delta x_t + k), \end{aligned}$$

- $\mathrm{b}_{m,p}(\cdot)$ mass function of Binomial(m, p)
- $\Longrightarrow$ likelihood quite intractable

Relation to literature

ISPÁNY, PAP & VAN ZUIJLEN (2003):

- observations $X_0, \ldots, X_n$ from $\mathbb{P}_{1-\frac{h}{n}}$
- $h \geq 0$ unknown, G known
- estimate h by OLS
- If $\mathbb{E}\varepsilon_1^3 < \infty$:

$$\sqrt{n}(\widehat{h}_n - h) \xrightarrow{d} \mathrm{N}(0, \sigma_h^2) \text{ under } \mathbb{P}_{1-h/n}$$

We show under $\mathbb{P}_{1-h/n^2}$:

$$|\widehat{h}_n| \xrightarrow{p} \infty.$$

Available:

- observations $X_0, \ldots, X_n$ from $\mathbb{P}_{1-\frac{h}{n^2}}$
- $h \geq 0$ unknown, G known
- rate n^2 becomes apparent later on

Remark:

- later: semiparametric model

Yields sequence of experiments:

$$\mathcal{E}_n(G) = \left(\mathbb{Z}_+^{n+1}, 2^{\mathbb{Z}_+^{n+1}}, \left(\mathbb{P}^{(n)}_{1-\frac{h}{n^2}} \mid h \geq 0\right)\right)$$

where $\mathbb{P}^{(n)}_{1-h/n^2}$ denotes law of $(X_0, \ldots, X_n)$ under $\mathbb{P}_{1-h/n^2}$.

The main goal

Determine limit experiment of $(\mathcal{E}_n(G))_{n\geq 1}$.

Recall:

$(\mathcal{E}_n(G))_{n\geq 1}$ converges (in weak Le Cam topology) to limit experiment $(\mathcal{X}, \mathcal{A}, (\mathbb{Q}_h \mid h \geq 0))$ if:

for every finite subset $I \subset \mathbb{R}_+$ and every $h_0 \in \mathbb{R}_+$:

$$\left(\frac{\mathrm{d}\mathbb{P}^{(n)}_{1-\frac{h}{n^2}}}{\mathrm{d}\mathbb{P}^{(n)}_{1-\frac{h_0}{n^2}}} \right)_{h\in I} \xrightarrow{d} \left(\frac{\mathrm{d}\mathbb{Q}_h}{\mathrm{d}\mathbb{Q}_{h_0}} \right)_{h\in I} \text{ under } \mathbb{P}^{(n)}_{1-\frac{h_0}{n^2}}.$$

Denote the observation from limit experiment by Z.

Why interested in limit experiment?

- determine asymptotic structure of *experiments* themselves
- hope: analysis limit experiment easier than difficult experiment $\mathcal{E}_n(G)$
- in some sense: limit experiment gives 'lower-bound to precision' inference procedures in sequence $\mathcal{E}_n(G)$
  - *Le Cam-Van der Vaart Asymptotic Representation Theorem*: if $T_n = t_n(X_0, \ldots, X_n)$ estimator of h s.t. $\mathcal{L}\left(T_n \mid \mathbb{P}_{1-h/n^2}\right) \to Z_h$ for all $h \geq 0$ then there exists randomized estimator $T = t(Z, U)$ in limit experiment s.t. $\mathcal{L}(T \mid \mathbb{Q}_h) = Z_h$ for all $h \geq 0$.

A special case to identify the rate

In case $g(k) = 2^{-(k+1)}$, $k \geq 0$, we have

$$\frac{\mathrm{d}\mathbb{P}^{(n)}_{1-\frac{h}{r_n}}}{\mathrm{d}\mathbb{P}^{(n)}_{1}} \xrightarrow{p} \begin{cases} 0 & \text{if } \frac{r_n}{n^2} \to 0, \\ \exp\left(-\frac{hg(0)\mu_G}{2}\right) & \text{if } \frac{r_n}{n^2} \to 1, \\ 1 & \text{if } \frac{r_n}{n^2} \to \infty, \end{cases} \qquad \text{under } \mathbb{P}_1.$$

Intuition:

- $r_n/n^2 \to \infty$: $\mathbb{P}^{(n)}_{1-h/r_n} \approx \mathbb{P}^{(n)}_1$
- $r_n/n^2 \to 0$: $\mathbb{P}^{(n)}_{1-h/r_n}$ and $\mathbb{P}^{(n)}_1$ become orthogonal
- n^2 indeed proper localizing rate

The limit experiment

If

- $\sigma_G^2 < \infty$
- support(G) $= \{0, \ldots, M\}$ or support(G) $= \mathbb{Z}_+$ and G is eventually decreasing

then $(\mathcal{E}_n(G))_{n \in \mathbb{N}}$ converges to the experiment

$$\mathcal{E}(G) = \left(\mathbb{Z}_+, 2^{\mathbb{Z}_+}, \left(\mathbb{Q}_h = \text{Poisson} \left(\frac{hg(0)\mu_G}{2} \right) \mid h \geq 0 \right) \right),$$

i.e. one draw Z from a Poisson$(hg(0)\mu_G/2)$ distribution.

Remark:

For nearly unstable Gaussian AR(1) model with $\mu \neq 0$ known:

- localizing rate $n^{3/2}$
- limit experiment: one draw from $\mathrm{N}(h, \tau(\mu))$

Some remarks on the proof

Usually:

- use differentiability arguments to make quadratic expansion of log-likelihood ratio
- first-order term (the score) is a martingale

But:

$$\mathbb{E}_1\left[\frac{\partial}{\partial\theta}\log P^{\theta}_{X_{t-1},X_t}\Big|_{\theta=1}\,\Big|\,X_{t-1}\right]\neq 0$$

$\Longrightarrow$ score is not a martingale under $\mathbb{P}_1$

$\Longrightarrow$ we see 'something non-standard is going on'

$\Longrightarrow$ we manipulate LR directly

- rather long, tedious proof
- main trick: split LLR into parts: part with $\Delta X_t \geq 0$, $\Delta X_t < 0$

Some remarks on the proof

One of important steps is to determine limit behavior of $\sum_t 1\{\Delta X_t < 0\}$.

A Poisson law of small numbers:

$$\sum_t 1\{\Delta X_t < 0\} \approx \sum_t 1\{\Delta X_t = -1, \varepsilon_t = 0\}$$

$$\xrightarrow{d} \text{Poisson}\left(\frac{g(0)\mu_G h_0}{2}\right) \text{ under } \mathbb{P}_{1-\frac{h_0}{n^2}}$$

Some remarks on the proof

In the end we find:

$$\log \frac{\mathrm{d}\mathbb{P}^{(n)}_{1-h/n^2}}{\mathrm{d}\mathbb{P}^{(n)}_{1-h_0/n^2}} \approx -\frac{(h-h_0)g(0)\mu_G}{2} + \log\left[\frac{h}{h_0}\right]\sum_t 1\{\Delta X_t < 0\}$$

$$\xrightarrow{d} \log \frac{\mathrm{d}\mathbb{Q}_h}{\mathrm{d}\mathbb{Q}_{h_0}}(Z) \text{ under } \mathbb{P}_{1-h_0/n^2}$$

$$\text{where } Z \sim \mathbb{Q}_{h_0} = \text{Poisson}\left(\frac{h_0 g(0)\mu_G}{2}\right).$$

Which yields the result.

Heuristic interpretation:
the statistic $\sum_t 1\{\Delta X_t < 0\}$ is 'asymptotically sufficient'

Some remarks on the proof

It is easy to show that the sequences of experiments

$$\mathcal{E}_n^0 = \left(\text{Bin}\left(n, \frac{h}{n}\right) \mid h \geq 0\right) \text{ and } \mathcal{E}_n^1 = \left(\text{Bin}\left(n, 1 - \frac{h}{n}\right) \mid h \geq 0\right)$$

both converge to the experiment

$$\mathcal{E} = \left(\mathbb{Z}_+, 2^{\mathbb{Z}_+}, (\text{Poisson}(h) \mid h \geq 0)\right)$$

- you might be tempted to 'derive' $\mathcal{E}_n(G) \to \mathcal{E}(G)$ from $\mathcal{E}_n^1 \to \mathcal{E}$
- however

$$\left(\mathbb{Z}_+^{n+1}, 2^{\mathbb{Z}_+^{n+1}}, \left(\mathbb{P}^{(n)}_{\frac{h}{\sqrt{n}}} \mid h \geq 0\right)\right)$$

converges to the usual normal location experiment: one draw Z from $\mathrm{N}(h, \tau)$.

Efficient estimation of h in case G is known

- Suppose
  - T_n estimator of h in $\mathcal{E}_n(G)$
  - $\mathcal{L}(T_n|\mathbb{P}_{1-h/n^2}) \to Z_h$ for $h \geq 0$

  By Asympt. Repr. Thm. exists $T(Z,U)$ such that

$$Z_h = \mathcal{L}(T(Z,U) \mid \mathbb{Q}_h) \text{ for all } h \geq 0.$$

- What's optimal in Poisson limit experiment?
  from Lehmann-Scheffé:
  if $T(Z,U)$ satisfies $\mathbb{E}_h T(Z,U) = h$ for all $h \geq 0$ then

$$\operatorname{var}_h T(Z,U) \geq \frac{2h}{g(0)\mu_G}.$$

Efficient estimation of h in case G is known

A lower-bound on asymptotic variance:

if T_n estimator of h s.t.

- $\mathcal{L}(T_n \mid \mathbb{P}_{1-h/n^2}) \to Z_h$ for $h \geq 0$
- $\int z \,\mathrm{d}Z_h = h$

then:

$$\int (z-h)^2 \,\mathrm{d}Z_h \geq \frac{2h}{g(0)\mu_G}$$

Efficient estimator:

$$T_n = \frac{2\sum_t 1\{\Delta X_t < 0\}}{g(0)\mu_G}$$

is asymptotically unbiased and attains variance-bound.

RECALL:

$\sum_t 1\{\Delta X_t < 0\}$ is approximately asymptotic sufficient.

Efficient estimation of h in case G is unknown

Now $h \geq 0$ and G unknown:

- Assumptions on G:
  - either support(G) $= \{0, \ldots, M\}$ or support$(G) = \mathbb{Z}_+$ and G eventually decreasing and $\sigma_G^2 < \infty$
- obvious idea: use consistent estimators of $g(0)$ and μ_G and estimate h by

$$T_n = \frac{2 \sum_t 1\{\Delta X_t < 0\}}{\widehat{g}(0)\widehat{\mu}_G}$$

- we construct these consistent estimators and prove that T_n is asymptotically unbiased and achieves variance-bound for case G known
  - $\Longrightarrow$ adaptive estimation of h!!!

Dickey-Fuller test for a unit root (in $\mathcal{E}_n(G)$)

- $\mathrm{H}_0 : h = 0$ versus $\mathrm{H}_1 : h > 0$ in the model $\mathcal{E}_n(G)$
- Dickey-Fuller test (size α):
  reject H_0 if and only if

$$\tau_n = \frac{\widehat{\theta}_n - 1}{\text{S.E.}(\widehat{\theta}_n)} < \Phi^{-1}(\alpha),$$

  where
  - $\widehat{\theta}_n$ is the OLS estimator of $\theta_n = 1 - h/n^2$
  - S.E.$(\widehat{\theta}_n)$ is the *standard* standard error of the OLS estimator
- if $\mathbb{E}\varepsilon_1^3 < \infty$ then

$$\tau_n \xrightarrow{d} \mathrm{N}(0,1) \text{ under } \mathbb{P}_{1-h/n^2} \text{ for all } h \geq 0$$

  Hence Dickey-Fuller test has no power

Efficient test for a unit root

In the Poisson limit experiment:

- rejecting H_0 if $X \geq 1$ and with prob. α if $X = 0$ is the UMP test for $\mathrm{H}_0 : h = 0$ versus $\mathrm{H}_1 : h > 0$

Using the asymptotic representation theorem for tests we find that

- rejecting H_0 if $\sum_t 1\{\Delta X_t < 0\} \geq 1$ and with prob. α if $\sum_t 1\{\Delta X_t < 0\} = 0$

is efficient (also in semiparametric model) for testing $\mathrm{H}_0 : h = 0$ versus $\mathrm{H}_1 : h > 0$.

Summary

- proper localizing rate for nearly unstable INAR(1) model is n^2 instead of n
- nearly unstable INAR(1) model yields Poisson limit experiment
- adaptive estimator for semiparametric nearly unstable INAR(1) model
- standard Dickey-Fuller test for unit root has no power
- the intuitively obvious test is efficient

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The final slide

Thank you very much for your attention !