Valuation of Options - part 3

Quantitative Finance

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Assumptions Black-Scholes market (recap)

In class we will almost exclusively work with the 'Black-Scholes market'.

Assumptions on price processes:

Asset price:

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 = s_0, \ \text{var}(W_1) = 1$$

Money Market Account:

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \quad B_0 = 1$$

Assumptions on market:

- frictionless trading
  - no transaction costs
  - trading in continuous-time is possible
  - no restrictions on short sales and fractional positions
- no credit/counterparty risk: borrowing rate = lending rate

Agenda

- we are interested in the price C_t, for $t \in [0, T)$, of (European) option with payoff $h(S_T)$ at expiration date/maturity T
- we will discuss three methods to determine fair price (using continuous-time model for financial market):
  - Black-Scholes Partial Differential Equation
  - risk-neutral pricing
  - pricing kernel

  These slides discuss risk neutral pricing and the pricing kernel approach.

Section 1

Risk-neutral pricing

Motivation

We have proved the First Fundamental Theorem of asset pricing for the toy model (see part 1). Now we will state the continuous-time version.

The First Fundamental Theorem of asset pricing

Theorem

Under regularity conditions: absence of arbitrage holds if and only if, for some numéraire N, there exists a probability measure $\mathbb{Q} = \mathbb{Q}_N$ such that:

(1) *$\mathbb{Q}$ is equivalent to $\mathbb{P}$;*
   - *i.e. $\mathbb{Q}(A) > 0$ iff $\mathbb{P}(A) > 0$ and $\mathbb{Q}(A) = 0$ iff $\mathbb{P}(A) = 0$;*

(2) *for any asset A in the market: the discounted price process A/N is a $\mathbb{Q}$-martingale, i.e.*

$$\frac{A_t}{N_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{A_{t+h}}{N_{t+h}} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{A_T}{N_T} \mid \mathcal{F}_t\right].$$

Remarks:

- any traded asset A with $\mathbb{P}(A_t > 0) = 1$ for all t can be used as numéraire
- typically the Money Market Account B is used as numéraire

Proof

We will only give a full proof for the Black-Scholes market. To prepare for the proof we first notice the following:

- Consider Black-Scholes market
- Use $N = B$
- If V is self-financing portfolio then

$$\begin{aligned} \mathrm{d}\frac{V_t}{B_t} &= \frac{1}{B_t}\,\mathrm{d}V_t - r\frac{V_t}{B_t}\,\mathrm{d}t = \frac{1}{B_t}(\phi_t\,\mathrm{d}S_t + \psi_t\,\mathrm{d}B_t) - r\frac{V_t}{B_t} \\ &= \cdots = \phi_t\,\mathrm{d}\frac{S_t}{B_t} \end{aligned}$$

Hence: if S/B is $\mathbb{Q}$-martingale, then V/B is $\mathbb{Q}$-martingale as well!

First Fundamental Theorem of asset pricing

Proof.

'$\Leftarrow$' (general proof; not only for Black-Scholes market)

- let V be the value process corresponding to a self-financing trading strategy
- suppose V satisfies $\mathbb{P}(V_T < 0) = 0$ and $\mathbb{P}(V_T > 0) > 0$
  - if $\mathbb{P}(V_0 \leq 0) = 1$ we would have an arbitrage opportunity
- equivalence yields: $\mathbb{Q}(V_T < 0) = 0$ and $\mathbb{Q}(V_T > 0) > 0$
- we have (use previous slide!)

$$V_0 = N_0 \mathbb{E}_{\mathbb{Q}}\left[\frac{V_T}{N_T} \mid \mathcal{F}_0\right] = N_0 \mathbb{E}_{\mathbb{Q}}\left[\frac{V_T}{N_T}\right] > 0$$

- consequently $\mathbb{P}(V_0 > 0) > 0 \implies$ strategy does not yield arbitrage opportunity

'$\Rightarrow$' We do not prove this implication in general. For the Black-Scholes market you will see an 'explicit' construction. $\square$

Using the first fundamental theorem to price options

(a) Construct model for basic assets (for example, S and B).

(b) Apply FFT: If your model does not allow for arbitrage opportunities, FFT says that you should be able to find $\mathbb{Q} \sim \mathbb{P}$ such that S/B is $\mathbb{Q}$-martingale (assuming that you use B as numéraire, note that B/B then yields a martingale). Use this to determine $\mathbb{Q}$. We will (typically) find unique solution.

(c) Now introduce an option (on S) to the market. We know its payoff $h(S_T)$ at maturity and want to determine its no-arbitrage price C_t for $t < T$.

(d) Apply FFT: B/N, S/N, and C/N should be $\mathbb{Q}$-martingales. From step (b) you already obtained unique $\mathbb{Q}$. This implies that we must set price of option C_t, for $t \in [0, T)$, via

$$C_t = N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{C_T}{N_T} \mid \mathcal{F}_t\right] = N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{h(S_T)}{N_T} \mid \mathcal{F}_t\right].$$

Using the first fundamental theorem to price option

- price processes basic assets in standard Black-Scholes market given; in particular:

$$S_T = S_t \exp\left((\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^{\mathbb{P}} - W_t^{\mathbb{P}}) \right),$$

  where $W^{\mathbb{P}}$ is a standard $\mathbb{P}$-Brownian motion
- use these to solve for $\mathbb{Q}$
  - **how?**
- want to price European option with payoff $C_T = h(S_T)$:

$$C_t = N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{C_T}{N_T} \mid \mathcal{F}_t \right] = N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{h(S_T)}{N_T} \mid \mathcal{F}_t \right]$$

  - in case $N_t = B_t = \exp(rt)$:

$$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[C_T \mid \mathcal{F}_t] = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[h(S_T) \mid \mathcal{F}_t]$$

- $\Longrightarrow$ **we do not need to know $\mathbb{Q}$, but 'only' need to know the distribution of $W^{\mathbb{P}}$, and hence S_T, under $\mathbb{Q}$!**

Intermezzo: Girsanov

Theorem (Girsanov)

Suppose the process $W^{\mathbb{P}}$ is a standard Brownian motion under $\mathbb{P}$ and let $T < \infty$. Consider a process $(\gamma_t)_{t \geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $W^{\mathbb{P}}$ and satisfying appropriate integrability conditions. Define $\mathbb{Q}$ by, for events $A \in \mathcal{F}_T$,

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\left[1_A \exp\left(-\frac{1}{2}\int_0^T \gamma_t^2 \,\mathrm{d}t - \int_0^T \gamma_t \,\mathrm{d}W_t^{\mathbb{P}}\right) 1_A\right].$$

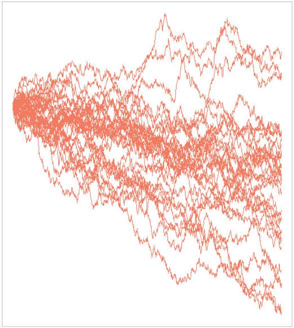
Then we have:

- *$\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$,*
- *and the process $W^{\mathbb{Q}}$ defined by*

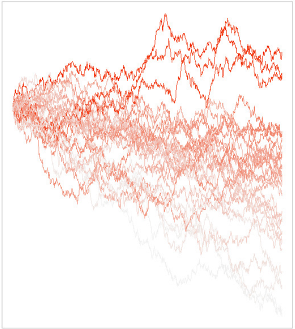
$$W_0^{\mathbb{Q}} = 0, \quad \mathrm{d}W_t^{\mathbb{Q}} = \gamma_t \,\mathrm{d}t + \mathrm{d}W_t^{\mathbb{P}},$$

or equivalently $W_t^{\mathbb{Q}} = \int_0^t \gamma_u \,\mathrm{d}u + W_t^{\mathbb{P}}$, is a standard Brownian motion under $\mathbb{Q}$.

Intermezzo: Girsanov



30 paths of a Brownian motion with negative drift



The same paths reweighted according to the Girsanov formula

- source: Wikipedia
- under $\mathbb{Q}$ and $\mathbb{P}$ same sample paths possible (equivalence), but with different likelihoods

Sketch proof Girsanov

- given the integrability conditions, it is easy to show that $\mathbb{Q}$ is probability measure equivalent to $\mathbb{P}$
- to prove that $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$ we need to show, for any $0 = t_0 < t_1 < \cdots < t_n \leq T$, that

$$(W_{t_1}^{\mathbb{Q}} - W_{t_0}^{\mathbb{Q}}, \ldots, W_{t_n}^{\mathbb{Q}} - W_{t_{n-1}}^{\mathbb{Q}}) \sim N(0_n, \Sigma)$$

  where $\Sigma_{ij} = 0$ for $i \neq j$ and $\Sigma_{ii} = t_i - t_{i-1}$
- we only consider case in which $u \mapsto \gamma_u$ is deterministic and $n = 2$ with $t_2 = T$
- it is sufficient to show (MGF determines the probability distribution)

$$\mathbb{E}_{\mathbb{Q}}\left[e^{a_1(W_{t_1}^{\mathbb{Q}} - W_{t_0}^{\mathbb{Q}}) + a_2(W_{t_2}^{\mathbb{Q}} - W_{t_1}^{\mathbb{Q}})}\right] = e^{\frac{1}{2}\left(a_1^2(t_1 - t_0) + a_2^2(t_2 - t_1)\right)}$$

Sketch proof Girsanov

We have

$$\mathbb{E}_{\mathbb{Q}}\left[e^{a_1(W^{\mathbb{Q}}_{t_1}-W^{\mathbb{Q}}_{t_0})+a_2(W^{\mathbb{Q}}_{t_2}-W^{\mathbb{Q}}_{t_1})}\right]=\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,e^{a_1(W^{\mathbb{Q}}_{t_1}-W^{\mathbb{Q}}_{t_0})+a_2(W^{\mathbb{Q}}_{t_2}-W^{\mathbb{Q}}_{t_1})}\right]$$

$$=\mathbb{E}_{\mathbb{P}}\left[e^{a_1(W^{\mathbb{Q}}_{t_1}-W^{\mathbb{Q}}_{t_0})+a_2(W^{\mathbb{Q}}_{t_2}-W^{\mathbb{Q}}_{t_1})}\,e^{-\frac{1}{2}\int_0^T\gamma_u^2\,\mathrm{d}u-\int_0^T\gamma_u\,\mathrm{d}W^{\mathbb{P}}_u}\right]$$

$$=e^{a_1\int_{t_0}^{t_1}\gamma_u\,\mathrm{d}u+a_2\int_{t_1}^{t_2}\gamma_u\,\mathrm{d}u-\frac{1}{2}\int_{t_0}^{t_2}\gamma_u^2\,\mathrm{d}u}$$

$$\times\,\mathbb{E}_{\mathbb{P}}\left[e^{-\int_{t_0}^{t_1}(\gamma_u-a_1)\,\mathrm{d}W^{\mathbb{P}}_u-\int_{t_1}^{t_2}(\gamma_u-a_2)\,\mathrm{d}W^{\mathbb{P}}_u}\right]$$

Exploiting independent increments and $\int_a^b f_u\,\mathrm{d}W^{\mathbb{P}}_u\sim N(0,\int_a^b f_u^2\,\mathrm{d}u)$ for deterministic functions f:

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\int_{t_0}^{t_1}(\gamma_u-a_1)\,\mathrm{d}W^{\mathbb{P}}_u-\int_{t_1}^{t_2}(\gamma_u-a_2)\,\mathrm{d}W^{\mathbb{P}}_u}\right]$$

$$=\mathbb{E}_{\mathbb{P}}\left[e^{-\int_{t_0}^{t_1}(\gamma_u-a_1)\,\mathrm{d}W^{\mathbb{P}}_u}\right]\mathbb{E}_{\mathbb{P}}\left[e^{-\int_{t_1}^{t_2}(\gamma_u-a_2)\,\mathrm{d}W^{\mathbb{P}}_u}\right]$$

$$=e^{\frac{1}{2}\int_{t_0}^{t_1}(\gamma_u-a_1)^2\,\mathrm{d}u}\,e^{\frac{1}{2}\int_{t_1}^{t_2}(\gamma_u-a_2)^2\,\mathrm{d}u}=e^{\frac{1}{2}\int_{t_0}^{t_2}\gamma_u^2\,\mathrm{d}u-a_1\int_{t_0}^{t_1}\gamma_u\,\mathrm{d}u-a_2\int_{t_1}^{t_2}\gamma_u\,\mathrm{d}u}$$

$$\times\,e^{\frac{1}{2}\left(a_1^2(t_1-t_0)+a_2^2(t_2-t_0)\right)}$$ which yields the result

Remarks

- Girsanov: for

$$W_0^{\mathbb{Q}} = 0, \quad \mathrm{d}W_t^{\mathbb{Q}} = \gamma_t \,\mathrm{d}t + \mathrm{d}W_t^{\mathbb{P}} \quad (W^{\mathbb{P}} \text{ standard BM under } \mathbb{P})$$

  we can find $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $W^{\mathbb{Q}}$ is standard Brownian motion under $\mathbb{Q}$
- Let $0 < a \neq 1$. Suppose that we can find probability measure $\mathbb{Q}$ such that

$$Z_0^{\mathbb{Q}} = 0, \quad \mathrm{d}Z_t^{\mathbb{Q}} = \gamma_t \,\mathrm{d}t + a \,\mathrm{d}W_t^{\mathbb{P}} \quad (W^{\mathbb{P}} \text{ standard BM under } \mathbb{P}),$$

  is standard Brownian motion under $\mathbb{Q}$. Then $\mathbb{Q}$ is *not* equivalent to $\mathbb{P}$.

Back to valuation: exploiting Girsanov

Theorem

Under regularity conditions: absence of arbitrage holds if and only if, for some numéraire N, there exists a probability measure $\mathbb{Q} = \mathbb{Q}_N$ such that:

(1) $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$*;*
   - *i.e.* $\mathbb{Q}(A) > 0$ *iff* $\mathbb{P}(A) > 0$ *and* $\mathbb{Q}(A) = 0$ *iff* $\mathbb{P}(A) = 0$*;*

(2) *for any price process A in the market, the discounted price process A/N is a $\mathbb{Q}$-martingale, i.e.*

$$\frac{A_t}{N_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{A_{t+h}}{N_{t+h}} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{A_T}{N_T} \mid \mathcal{F}_t\right].$$

Remarks:

- any asset A with $\mathbb{P}(A_t > 0) = 1$ for all t can be used as numéraire
- typically Money Market Account B is used as numéraire

Using FFT and Girsanov

- setup: standard Black-Scholes market ($W = W^{\mathbb{P}}$ is standard Brownian motion):

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \quad \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{P}}, \quad S_0 = s_0,\ B_0 = b_0$$

- use B as numéraire
- Easy exercise (Itô) to show that

$$\mathrm{d}\frac{S_t}{B_t} = (\mu - r)\frac{S_t}{B_t}\,\mathrm{d}t + \sigma\frac{S_t}{B_t}\,\mathrm{d}W_t^{\mathbb{P}}$$

- FFT $\implies$ we need to find $\mathbb{Q}$ such that S_t/B_t is $\mathbb{Q}$-martingale
  - note that B/B is automatically a martingale

Using FFT and Girsanov

$$\mathrm{d}\frac{S_t}{B_t} = (\mu - r)\frac{S_t}{B_t}\,\mathrm{d}t + \sigma\frac{S_t}{B_t}\,\mathrm{d}W_t^{\mathbb{P}}$$

- for
  $$\mathrm{d}W_t^{\mathbb{Q}} = \gamma_t\,\mathrm{d}t + \mathrm{d}W_t^{\mathbb{P}}$$
  we have $\mathrm{d}W_t^{\mathbb{P}} = -\gamma_t\,\mathrm{d}t + \mathrm{d}W_t^{\mathbb{Q}}$ which yields
  $$\mathrm{d}\frac{S_t}{B_t} = ((\mu - r) - \sigma\gamma_t)\,\frac{S_t}{B_t}\,\mathrm{d}t + \sigma\frac{S_t}{B_t}\,\mathrm{d}W_t^{\mathbb{Q}}$$
- application of Girsanov's theorem with
  $$\gamma_t = \frac{\mu - r}{\sigma} \quad \textit{market price of risk}$$
  yields $\mathbb{Q} \sim \mathbb{P}$ such that $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-standard Brownian motion
- we have
  $$\mathrm{d}\frac{S_t}{B_t} = \sigma\frac{S_t}{B_t}\,\mathrm{d}W_t^{\mathbb{Q}}$$
  so S/B is martingale under $\mathbb{Q}$ (not under $\mathbb{P}$!)

Remark:

Under $\mathbb{Q}$ we have

$$\mathrm{d}\frac{S_t}{B_t} = \sigma \frac{S_t}{B_t}\,\mathrm{d}W_t^{\mathbb{Q}}$$

$\implies$ (use Itô)

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}$$

- under $\mathbb{P}$ we have

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{P}}$$

- under $\mathbb{Q}$ stock has same drift as bond, but with risk $\implies$ **risk-neutral (world)**

Using FFT and Girsanov

- the SDE
$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}$$
is GBM
- so, under $\mathbb{Q}$,
$$S_T = S_t\,\mathrm{e}^{(r-0.5\sigma^2)(T-t)+\sigma(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}})}$$
- option with payoff $h(S_T)$
- FFT yields
$$\begin{aligned} C_t &= \mathrm{e}^{-r(T-t)}\,\mathbb{E}_{\mathbb{Q}}\left[h\left(S_t\,\mathrm{e}^{(r-0.5\sigma^2)(T-t)+\sigma(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}})}\right) \mid \mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(T-t)}\,\mathbb{E}_{\mathbb{Q}}\left[h\left(s_t\,\mathrm{e}^{(r-0.5\sigma^2)(T-t)+\sigma(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}})}\right)\right] \quad (s_t \text{ is constant}) \end{aligned}$$
- using $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}} \sim \sqrt{T-t}Z$ (under $\mathbb{Q}$) with $Z \sim \mathrm{N}(0,1)$:
$$C_t = \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} h\left(s_t\,\mathrm{e}^{(r-0.5\sigma^2)(T-t)+\sigma\sqrt{T-t}z}\right)\phi(z)\,\mathrm{d}z$$

Example - digital put

- digital European put: $h(S_T) = 1\{S_T \leq K\}$; what is the price at $t = 0$?
- we have to calculate

$$C_0 = \exp(-rT)\mathbb{E}_{\mathbb{Q}}\left[h\left(S_T\right)\right] = \exp(-rT)\mathbb{Q}\left(S_T \leq K\right)$$

- using

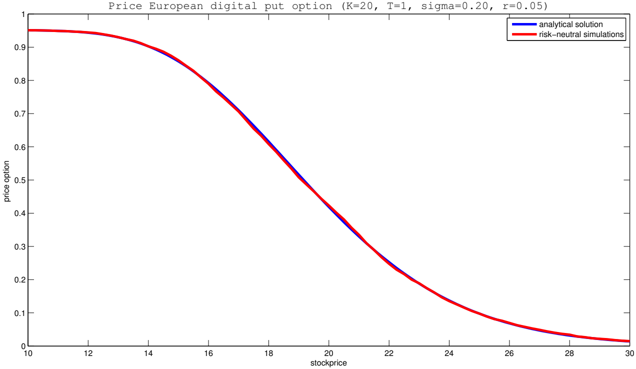
$$\begin{aligned} S_T = s_0 \exp\left((r - 0.5\sigma^2)T + \sigma W_T^{\mathbb{Q}}\right) \leq K \\ \Leftrightarrow \ \sigma W_T^{\mathbb{Q}} \leq \log\frac{K}{s_0} - (r - 0.5\sigma^2)T \end{aligned}$$

we obtain

$$\begin{aligned} C_0 &= \exp(-rT)\mathbb{Q}\left(\sigma W_T^{\mathbb{Q}} \leq \log\frac{K}{s_0} - (r - 0.5\sigma^2)T\right) \\ &= \exp(-rT)\Phi\left(\frac{\log\frac{K}{s_0} - (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Example digital put

The blue line plots C_0 as function of S_0 for $K = 20$, $T = 1$, $\sigma = 20\%$, and $r = 5\%$.



Later on we will discuss how we can use Monte Carlo techniques to obtain a numerical approximation. The red line is such an approximation based on $5,000$ simulations.

Example: put on Philips

historical data yields estimated $\sigma = 0.29$

Example: put on Philips

DECEMBER 2021 PRICES - 17/07/20

	SETTL.	LAST	BID	ASK		STRIKE		BID	LAST	SETTL.
+	18.02	-	17.70	18.00	C	25.00	P	0.46	-	0.52
+	13.63	-	13.30	13.60	C	30.00	P	1.12	-	1.17
+	9.75	-	9.45	9.75	C	35.00	P	2.29	-	2.29
+	6.48	-	6.20	6.50	C	40.00	P	4.07	-	4.06
+	3.98	-	3.81	4.00	C	45.00	P	6.60	-	6.60
+	2.24	-	2.14	2.29	C	50.00	P	9.95	-	9.93
+	0.60	-	0.52	0.67	C	60.00	P	18.40	-	18.37

Updated 17/07/2020 14:11:54

Help

- $T = 17.5/12$, $r = 0.01$, $K = 45$, $\sigma = 0.29$ and $S_0 = 43.145$ yields 6.70 as B-S price for the put, where $\sigma = 0.29$ is result from estimation on historical data on S.

Section 2

Pricing Kernels

Motivation

- suppose we work in standard Black-Scholes market with B as numéraire
- Girsanov yields probability measure $\mathbb{Q}$ defined by $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}} 1_A Z_T$ where

$$Z_t = \exp\left(-t(\mu - r)^2/(2\sigma^2) - ((\mu - r)/\sigma) W_t^{\mathbb{P}}\right)$$

- we know $\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[X Z_T]$
- it can be shown that $\mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[X(Z_T/Z_t) | \mathcal{F}_t]$
- we thus have

$$A_t = \mathbb{E}_{\mathbb{Q}}\left[\frac{N_t}{N_T} A_T \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{N_t}{N_T} \frac{Z_T}{Z_t} A_T \mid \mathcal{F}_t\right]$$

  which yields, with $K_t = Z_t / N_t$,

$$K_t A_t = \mathbb{E}_{\mathbb{P}}[K_T A_T \mid \mathcal{F}_t],$$

  i.e. KA is a $\mathbb{P}$-martingale

The first fundamental theorem of asset pricing (pricing kernel version)

We proved, for the special case of the Black-Scholes market, the following theorem.

Theorem

Absence of arbitrage holds if and only if there exists a process K, with $K_t > 0$, such that for any asset with price A_t the process $K_t A_t$ is a $\mathbb{P}$-martingale

Application:

- determine K (using your models for the basic assets)
- (no arbitrage) price of a derivative, with payoff C_T, based on these assets is given by

$$C_t = \frac{1}{K_t} \mathbb{E}_{\mathbb{P}}[K_T C_T \mid \mathcal{F}_t]$$

The pricing kernel for the standard Black-Scholes world

- try K of the form $\mathrm{d}K_t = \mu_t^K \,\mathrm{d}t + \sigma_t^K \,\mathrm{d}W_t$
- Itô's product rule yields

$$\begin{aligned}\mathrm{d}K_t B_t &= K_t \,\mathrm{d}B_t + B_t \,\mathrm{d}K_t + \mathrm{d}[K,B]_t \\ &= (rK_t B_t + B_t \mu_t^K)\,\mathrm{d}t + \ldots \mathrm{d}W_t\end{aligned}$$

$$K_t B_t \text{ martingale} \implies \text{no drift} \implies \mu_t^K = -rK_t$$

- Itô's product rule yields

$$\begin{aligned}\mathrm{d}K_t S_t &= K_t \,\mathrm{d}S_t + S_t \,\mathrm{d}K_t + \mathrm{d}[S,K]_t \\ &= \left(K_t \mu S_t + S_t \mu_t^K + \sigma \sigma_t^K S_t\right)\mathrm{d}t + \ldots \mathrm{d}W_t\end{aligned}$$

$$K_t S_t \text{ martingale} \implies \text{no drift} \implies \sigma_t^K = -K_t(\mu - r)/\sigma$$

So

$$K_0 = 1, \quad \mathrm{d}K_t = -rK_t \,\mathrm{d}t - \frac{\mu - r}{\sigma} K_t \,\mathrm{d}W_t$$

determines the pricing kernel process in the B-S model.

The pricing kernel for the standard Black-Scholes world

$$K_0 = 1, \quad \mathrm{d}K_t = -rK_t\,\mathrm{d}t - \frac{\mu - r}{\sigma}K_t\,\mathrm{d}W_t$$

has as solution

$$K_t = \exp\left\{\left(-r - 0.5\left(\frac{\mu - r}{\sigma}\right)^2\right)t - \frac{\mu - r}{\sigma}W_t\right\}$$

Fair price of a derivative, with payoff $C_T = F(S_T)$, based on these assets is given by

$$C_0 = \frac{1}{K_0}\mathbb{E}_{\mathbb{P}}[K_T C_T]$$

- $K_T C_T = g(W_T)f(S_T) = g(W_T)\tilde{f}(W_T) = h(W_T)$, so price determined by expectation of functional of $\mathrm{N}(0, T)$ variable
- price seems to depend on μ, however it does not!

Remark

- you can always set $K_0 = 1$
- which yields
$$C_0 = \mathbb{E}_{\mathbb{P}}[K_T C_T]$$
- suppose $C_T = 1\{\omega \in E\}$, i.e. the pay-off is 1 euro in case event E occurs and 0 otherwise
- then
$$C_0 = \mathbb{E}_{\mathbb{P}}[1_E K_T],$$
so $K_T(\omega)$ kind of measures how expensive it is to get a pay-off in state ω
- For the Black-Scholes market we have (assume $\mu > r$)
$$K_T = \exp\left\{\left(-r - 0.5\left(\frac{\mu - r}{\sigma}\right)^2\right) T - \frac{\mu - r}{\sigma} W_T\right\},$$
showing that K_T takes large values for large, negative values of W_T. And such state-of-the-world correspond to small values of $S_T(\omega)$

Section 3

Second Fundamental Theorem of asset pricing

In our discussion we have assumed that it is possible to find a self-financing portfolio for every option pay-off C_T. This property is referred to as 'completeness'.

Definition

A financial market is said to be **complete** if it is possible to find, for each $T > 0$, and all random variables C_T that are $\mathcal{F}_T$-measurable, a self-financing portfolio V (satisfying regularity conditions) that satisfies $V_T = C_T$ (a.s.)

Theorem

Under regularity conditions we have: a financial market is complete if and only if there is a *unique* measure $\mathbb{Q} \sim \mathbb{P}$ that makes all deflated asset prices A/N $\mathbb{Q}$-martingales (where N is chosen numéraire.

Examples

- For our Black-Scholes market $\mathbb{Q}$ was unique, hence the market is complete.
- Consider the Black-Scholes market. Now we also add an additional risk factor to the market: an independent Brownian motion. This has impact on $\mathcal{F}_T$! And now claims could depend on both Brownian motions. No additional assets are available. This leads to an incomplete market. (Why is $\mathbb{Q}$ not unique?)

Remark In the MSc in QFAS we discuss valuation (and hedging) in incomplete markets (which you often encounter in case there is financial risk as well as actuarial risk).