Sketch of proof of important calculus rule for quadratic covariations

QF*

Calculus rule

If X is a solution to the SDE $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t$, with $W_1 \sim \mathrm{N}(0,1)$ and Y is another process with continuous sample paths then we have $\mathrm{d}[X,Y]_t = \sigma_t\,\mathrm{d}[W,Y]_t$. In particular, $\mathrm{d}[X,X]_t = \sigma_t^2\,\mathrm{d}[W,W]_t = \sigma_t^2\,\mathrm{d}t$.

Sketch of proof

Fix $T > 0$.

Step 1

Introduce the processes m and Z by $m_t = \int_0^t \mu_u\,\mathrm{d}u$ and $Z_t = \int_0^t \sigma_u\,\mathrm{d}W_u$. Since m is of Bounded variation we have $[X,Y]_T = [m+Z,Y]_T = [Z,Y]_T$.

Step 2

First we consider simple processes σ_t. This means that there exists $M \in \mathbb{N}$ and $0 = s_0 < \cdots < s_M = T$ such that $\sigma_t = \sum_{j=1}^M \sigma^j 1_{[s_{j-1},s_j)}(t) + \sigma^M 1_{\{T\}}(t)$, with σ^j $\mathcal{F}_{s_{j-1}}$-measurable. Next we consider partitions $0 = s_0 = t_0 < \cdots < t_m = s_1 < \cdots < t_{2m} = s_2 < \cdots < t_{Mm} = s_M = T$, $n = Mm$. See Figure **??** for an illustration. Now notice that

$$
\begin{aligned}
[Z,Y]_T &= \mathrm{plim}_{m\to\infty} \sum_{k=1}^n (Z_{t_k} - Z_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) = \mathrm{plim}_{m\to\infty} \sum_{j=1}^M \sum_{k=(j-1)m+1}^{jm} (Z_{t_k} - Z_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) \\
&= \mathrm{plim}_{m\to\infty} \sum_{j=1}^M \sum_{k=(j-1)m+1}^{jm} \sigma^j (W_{t_k} - W_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) \\
&= \sum_{j=1}^M \sigma^j\, \mathrm{plim}_{m\to\infty} \sum_{k=(j-1)m+1}^{jm} (W_{t_k} - W_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) \\
&= \sum_{j=1}^M \sigma^j \left([W,Y]_{s_j} - [W,Y]_{s_{j-1}}\right) = \int_0^T \sigma_u\,\mathrm{d}[W,Y]_u.
\end{aligned}
$$

So the theorem holds for simple processes.

Step 3

Next we consider general σ_t. There exists a sequence of simple processes $\sigma^{(n)}$ with $\sigma_t^{(n)}(\omega) \to \sigma_t(\omega)$ for

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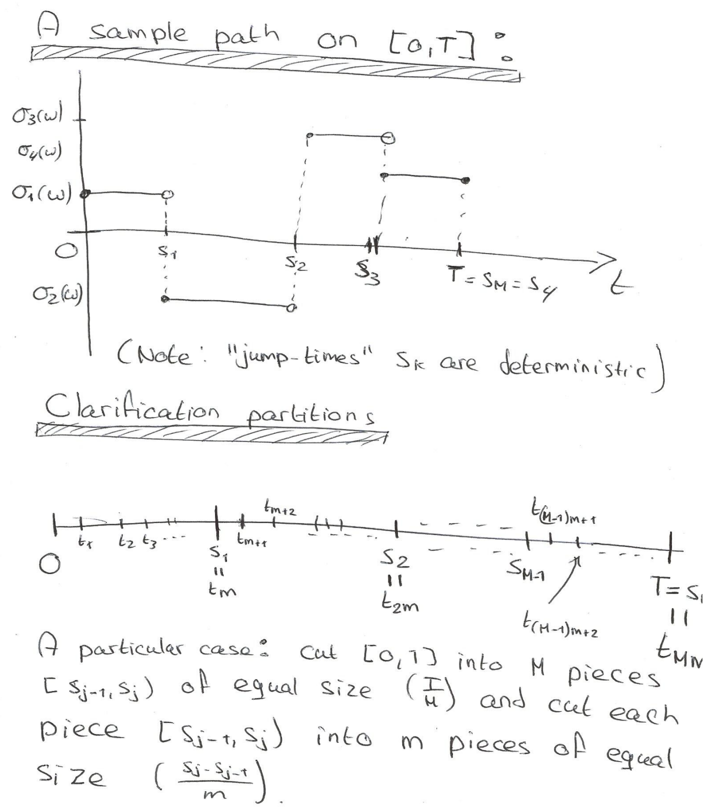

Figure 1: Illustration of 'simple processes' and the partitions.

almost all (t, ω) (this is a standard construction technique from measure theory). Now define processes $Z^{(n)}$ by $Z_t^{(n)} = \int_0^t \sigma_u^{(n)} \, \mathrm{d}W_u$. Now we have

$$\begin{aligned}
[X, Y]_T = [Z, Y]_T = [\operatorname{plim}_{n\to\infty} Z^{(n)}, Y]_T &\stackrel{?}{=} \operatorname{plim}_{n\to\infty} [Z^{(n)}, Y]_T \\
&= \operatorname{plim}_{n\to\infty} \int_0^T \sigma_u^{(n)} \, \mathrm{d}[W, Y]_u \stackrel{?}{=} \int_0^T \operatorname{plim}_{n\to\infty} \sigma_u^{(n)} \, \mathrm{d}[W, Y]_u = \int_0^T \sigma_u \, \mathrm{d}[W, Y]_u,
\end{aligned}$$

which concludes the proof. Notice that we interchanged the order of operations (limits and integrals) several times. To justify this conditions have to be satisfied, which we do not discuss in the sketch of the proof (and in the formulation of the result).