Valuation of Options - part 1

Quantitative Finance

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Section 1

Introduction

Options (quick recap)

Call option

- contract between buyer (option holder) and seller (option writer)
- contract offers buyer *right* (but *not obligation*!) to *buy* specified security (or other *underlying financial asset*) at *specific* date (*maturity/expiration date*) or during a specified period of time at *agreed upon price* (*strike price*)

Put option

- analogous definition; but put offers buyer right to *sell*

Remarks

- As a put or call provides its buyer a right and not an obligation, the contract has a (positive) price.
- Please note that the seller (writer) of the contract has the obligation to sell (call) or buy (put) in case the option is exercised by the buyer.

Options (quick recap)

Underlying asset:

- stock, index, ETF, interest rate(s), real estate, etc.

Exercising the option:

- *European (style) options*: option can only be exercised *at expiration date*
- *American*: option can be exercised *at any time between purchase and expiration date*
- (there also exist *Asian*, *Bermudan* and *Canary* versions; all these names have nothing to do with geographical locations)
- typically: options on indices and interest rates are European style, while options on (individual) stocks and ETFs are American style

Further terminology:

- *strike price* is a.k.a. *exercise price*
- final date for exercising option: *maturity* or *expiration date*

Options (quick recap)

Remarks:

- options are traded on exchanges, in Over-The-Counter markets, and in bespoke contracts (investment banks)
- this course focuses on European style options (on stocks, indices, ETFs); see MSc QFAS for American style options and options on interest rates
- options are examples of *financial derivatives* which are contracts that are defined in terms of underlying asset which is traded on financial market

Options (quick recap)

Exercise

Check the following formulas for the payoff at maturity T.

- European call option with strike price K:
  $C_T = \max\{S_T - K, 0\}$
- European put option with strike price K:
  $C_T = \max\{K - S_T, 0\}$

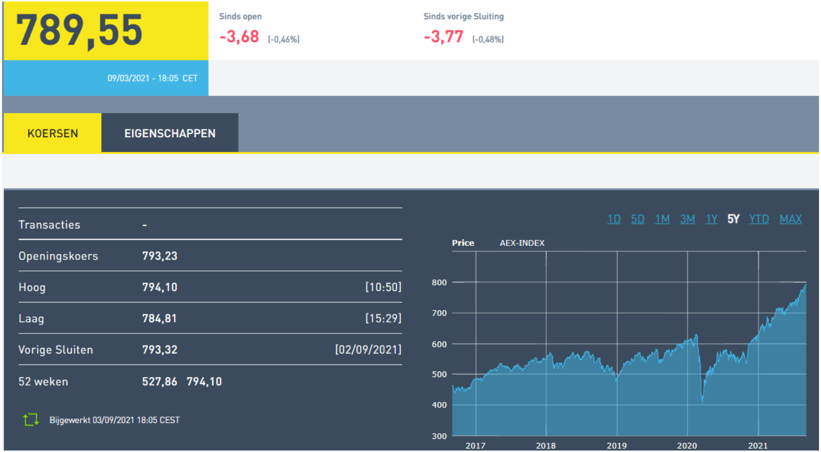
We will often use the following options that are defined by their pay-off:

- European digital (binary) call option with strike price K:
  $C_T = 1\{S_T - K \geq 0\}$
- European digital (binary) put option with strike price K:
  $C_T = 1\{K - S_T \geq 0\}$

Please recall that $1\{x \leq K\} = 1$ if $x \leq K$ and 0 otherwise.

Note that can write all these payoffs as a function of S_T, i.e. $C_T = g(S_T)$.

As an example, we will consider European call and put options on the AEX-index at September 3, 2021.



Options (quick recap)

As an example, we will consider European call and put options, on the AEX-index at September 3, 2021, with expiration date in December 2022.

AEX-INDEX

EURONEXT DERIVATIVES AMSTERDAM
INDEX OPTION

AEX

Aantal 03/09/2021 17:29	Openstaande positie 02/09/2021	Valuta	Tijdzone
24 732	287 683	EUR	CET

DECEMBER 2022 KOERSEN - 03/09/21

SETTL.	LAATSTE	BIED	LAAT		UITOEFENPRIJS		BIED	LAATSTE	SETTL.
230.19	-	225.25	238.95	C	550.00	P	13.60	14.10	14.03
185.35	185.00	183.00	188.50	C	600.00	P	19.10	18.90	19.56
142.83	-	141.00	144.00	C	650.00	P	26.80	27.20	27.40
103.35	100.00	102.00	104.00	C	700.00	P	37.50	38.20	38.29
68.30	68.20	67.60	68.90	C	750.00	P	52.70	-	53.62
40.26	41.55	39.50	40.70	C	800.00	P	74.10	76.80	75.94
20.67	20.45	20.30	20.90	C	850.00	P	-	106.50	106.72
9.39	9.50	9.05	9.65	C	900.00	P	140.00	-	145.81
4.16	3.90	3.85	4.40	C	950.00	P	-	-	190.95
1.89	2.00	1.70	2.15	C	1000.00	P	-	-	239.05
0.53	0.63	0.25	0.80	C	1100.00	P	-	-	338.42

Main problems

Valuation/pricing
What is the 'fair' price of an option and how does it evolve over time? This is an important question for:

- investors/traders
  - to identify that you don't pay too much
  - to make money in case you think the market price is wrong
- financial institutions
  - who need to calculate price for selling OTC-options (or to price 'embedded' options)
  - who need to perform risk management on portfolios that contain options

Remark: later on we will also discuss **hedging**.

Main problems - valuation

- we are interested in (fair) price C_t, for $t \in [0, T)$, of (European) option with payoff $C_T = f(T, S_T)$ at maturity T
- we will learn three methods to determine 'fair price' (using specified continuous-time model for financial market):
  - Black-Scholes Partial Differential Equation
  - risk-neutral pricing
  - pricing kernel
- in particular: you will derive the famous Black-Scholes price for European put option with maturity T and strike K:

$$C_t = p(S_t, T - t, K, r, \sigma) = K\, \mathrm{e}^{-r(T-t)}\, \Phi(-d_2) - S_t \Phi(-d_1)$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t},$$

  - does not depend on μ!

Black-Scholes formula

- in 1973 Fisher Black and Myron Scholes published paper ‘The Pricing of Options and Corporate Liabilities’ and Robert Merton published ‘Theory of Rational Option Pricing’
- Merton and Scholes received Nobel Memorial Prize in Economic Sciences in 1997 (Black died in 1995)

Arbitrage

All pricing methods we discuss rely on insisting that arbitrage opportunities do/should not exist.

Definition

If it is possible to find a self-financing trading strategy with value process V such that, for some $T > 0$, we have $V_0 \leq 0$ and

$$\mathbb{P}(V_T < 0) = 0 \text{ and } \mathbb{P}(V_T > 0) > 0,$$

then we say that an arbitrage opportunity exists.

Section 2

Toy model

Motivation

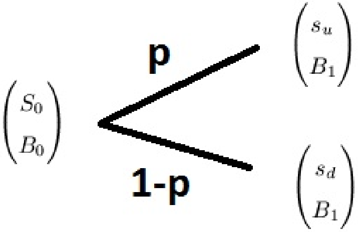
First we discuss a very simple discrete time model and discuss pricing of options under assumption that arbitrage opportunities do not exist. We discuss two methods:

- pricing by replication
- risk-neutral pricing

Later on we will discuss their continuous-time analogues. Although the math gets more complicated, the underlying ideas are the same!

Model for financial market

- two states-of-the-world: $\Omega = \{\text{“up”}, \text{“down”}\}$
- two assets: risky asset S and riskless asset B
    - $B_1(\text{“up”}) = B_1(\text{“down”}) = B_1 > 0$
    - for
    $$S_1(\omega) = \begin{cases} s_u, & \text{if } \omega = \text{“up”} \\ s_d, & \text{if } \omega = \text{“down”} \end{cases}$$
    we assume $0 < s_d < s_u$
- probability of state “up” is $p \in (0,1)$, i.e. $\mathbb{P}(\text{“up”}) = p$



No arbitrage

Theorem There are no arbitrage opportunities in our financial market if and only if

$$0 < \frac{s_d}{s_0} < \frac{B_1}{B_0} < \frac{s_u}{s_0}.$$

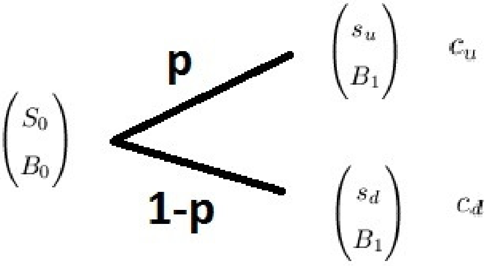
Proof
'$\Rightarrow$': exercise
'$\Leftarrow$': this will follow from our proofs of other results (verify!)

What's the price of an option in this market?

- now consider European option with payoff

$$C_1(\omega) = \begin{cases} c_u, & \text{if } \omega = \text{“up”} \\ c_d, & \text{if } \omega = \text{“down”} \end{cases}$$

  - for call: $c_u = \max\{s_u - K, 0\}$ and $c_d = \max\{s_d - K, 0\}$
- what is price at $t = 0$ for call?



Method 1: pricing by replication

- construct portfolio V with $V_1(\omega) = C_1(\omega)$ for all ω
  - as portfolio generates exactly the same pay-offs as option does (for all states-of-the-world), the portfolio is called *replicating* portfolio
  - Why? As there are no intermediate cashflows, we must have $C_0 = \phi S_0 + \psi B_0$. Otherwise we would create arbitrage opportunities. So via the replicating portfolio we find no-arbitrage price of option
- notice that we have to determine ϕ and ψ such that

$$c_u = \phi s_u + \psi B_1$$
$$c_d = \phi s_d + \psi B_1$$

which yields

$$\phi = \frac{c_u - c_d}{s_u - s_d} \text{ and } \psi = \frac{c_d s_u - s_d c_u}{B_1(s_u - s_d)}$$

- price of option (at $t = 0$):

$$C_0 = \phi S_0 + \psi B_0$$

Method 2: risk-neutral pricing

Motivation

- people would like to have pricing formula of the form:

$$\frac{A_0}{B_0} = \mathbb{E}\left[\frac{A_1}{B_1}\right]$$

  for all assets A, i.e. price equals expected discounted cashflows

- note that

$$\frac{A_0}{B_0} = \mathbb{E}\left[\frac{A_1}{B_1}\right] \quad \Longleftrightarrow \quad \mathbb{E}\left[\frac{A_1}{A_0}\right] = \frac{B_1}{B_0}$$

- however, already for $A = S$ this almost never holds true in real financial markets:

$$\mathbb{E}\left[\frac{S_1}{S_0}\right] = \frac{B_1}{B_0},$$

  would mean that expected (gross) return on stock is same as return on riskless asset

Intermezzo: a biased coin can save the day

- Suppose we throw a *fair* coin, $\Omega = \{H, T\}$, and you get pay-off $X = 1$ if $\omega = H$ and $X = 0$ if $\omega = T$
- the price of entering this game is 0.3 (given)
- (for some reason) we want to have the identity

$$\text{price} = \mathbb{E}[X] = \text{Prob}(\{H\})$$

- as the coin is fair the equation does not hold true if we use the true, "real-world", probability measure ($0.3 \neq 0.5$)
- however: we can just create *artificial* probability measure $\mathbb{Q}$ with $\mathbb{Q}(\{H\}) = 0.3$, which does lead to

$$\text{price} = \mathbb{E}_{\mathbb{Q}}[X]$$

Method 2: risk-neutral pricing

- people would like to have pricing formula of the form:

$$\frac{A_0}{B_0} = \mathbb{E}\left[\frac{A_1}{B_1}\right] \quad \Longleftrightarrow \quad \mathbb{E}\left[\frac{A_1}{A_0}\right] = \frac{B_1}{B_0} \quad (\star)$$

  for all assets A
- if we use "real-world" probabilities then this identity will typically not hold for $A = S$
- could we repair identity by using artificial probability measure $\mathbb{Q}$ instead of "real-world" measure $\mathbb{P}$?
- solving

$$\frac{S_0}{B_0} = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{B_1}\right] = q\frac{s_u}{B_1} + (1-q)\frac{s_d}{B_1}$$

  yields

$$q = \frac{\frac{B_1}{B_0} - \frac{s_d}{s_0}}{\frac{s_u}{s_0} - \frac{s_d}{s_0}}$$

- if $q \in [0, 1]$ then we indeed have found artificial probability measure for which $(\star)$ is true by construction (for $A = B$ the identity is trivial)

Method 2: risk-neutral pricing

Recall earlier theorem: there is no arbitrage if and only if

$$0 < \frac{s_d}{s_0} < \frac{B_1}{B_0} < \frac{s_u}{s_0}$$

which occurs exactly if $q \in (0, 1)$. So we have proved:

Intermediate result
The market is free of arbitrage opportunities if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S/B is a martingale under $\mathbb{Q}$, i.e.

$$\frac{S_0}{B_0} = \mathbb{E}_{\mathbb{Q}}[\frac{S_1}{B_1}].$$

Method 2: risk-neutral pricing

Terminology:

- As S/B is martingale under $\mathbb{Q}$ we have

$$\frac{s_0}{B_0} = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{B_1}\right],$$

  which yields

$$\frac{B_1}{B_0} = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{S_0}\right],$$

  i.e. the expected (gross) return on the risky asset S, under $\mathbb{Q}$, is equal to the return/interest on the riskless asset B.
- $\mathbb{Q}$ is often called "risk-neutral measure"
  - Please note that $\mathbb{Q}$ is *artificial* measure and does not describe actual probabilities!
- $\mathbb{Q}$ is also called *equivalent martingale measure*

Method 2: risk-neutral pricing

So we have repaired the formula $S_0/B_0 = \mathbb{E}[S_1/B_1]$, but it was quite a discussion. What is the true benefit?

- introduce, as before, an option to the market (with payoff C_1 at $t = 1$)
- we already know that there is a replicating portfolio (ϕ, ψ) such that $V_1(\omega) = \phi S_1(\omega) + \psi B_1 = C_1(\omega)$ for all ω
- and, if there are no arbitrage opportunities, $C_0 = \phi S_0 + \psi B_0$
- now note that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{C_1}{B_1}\right] = \phi \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{B_1}\right] + \psi = \phi\frac{S_0}{B_0} + \psi = \frac{1}{B_0}(\phi S_0 + \psi B_0) = \frac{C_0}{B_0},$$

  so we can calculate for any option (or asset) its price at $t = 0$ via the above identity as soon as we know $\mathbb{Q}$! There is no need to calculate the replicating portfolios.

Method 2: risk-neutral pricing

First fundamental theorem of asset pricing (for the toy model)
The market is free of arbitrage opportunities if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that A/B is a martingale, under $\mathbb{Q}$, for all assets A.

Application We can price options by applying this theorem twice:

- Write down model for B and S (without arbitrage opportunities).
- Apply theorem (with $A = B$ and $A = S$; note that for $A = B$ the 'martingale-property' is trivial). This yields unique $\mathbb{Q}$. $(\star)$
- Now introduce an option to market with payoff C_1. We want to set price such that we do not create arbitrage opportunity.
- Apply theorem again: A/B should be a martingale for $A = B, S, C$. For $A = S$ we obtain unique $\mathbb{Q}$ from previous step $(\star)$. And $A = C$ yields

$$C_0 = B_0 \mathbb{E}_{\mathbb{Q}}[C_1/B_1].$$

Proof

'$\Rightarrow$': exercise - check that we have proved this implication (via explicit construction of $\mathbb{Q}$)

'$\Leftarrow$':

Consider portfolio V with $V_1 \geq 0$ and $V_1(\omega) > 0$ for at least one state-of-the-world. If we prove that $V_0 > 0$, then we have shown that arbitrage opportunities do not exist.

Note that we have (why?!)

$$
\begin{aligned}
0 < \mathbb{E}_{\mathbb{Q}}\left[\frac{V_1}{B_1}\right] &= \phi\mathbb{E}_{\mathbb{Q}}\left[\frac{S_1}{B_1}\right] + \psi \\
&= \phi\frac{S_0}{B_0} + \psi = \frac{1}{B_0}(\phi S_0 + \psi B_0) = \frac{V_0}{B_0},
\end{aligned}
$$

which yields $V_0 > 0$.

Toy model - concluding remarks

For the toy model we have seen that we can price options by

- constructing a replicating portfolio
- by using risk-neutral pricing

(and it would also be possible to introduce pricing kernels as a third method).

For the continuous time case we will see that exactly the same ideas are used. Unfortunately (?), the math gets a bit more difficult.