Risk Measurement and Management using the greeks

Quantitative Finance

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Section 1

Introduction

Motivation: solving valuation problem is not enough

- suppose we work at bank ABC which acts as market-maker
- customer S contacts us to buy 10,000 European call options on Air France-KLM which expire in 3 months from now and have 5 euro as strike price
  - assume that such European options are not traded on exchanges
- bank ABC steps in, to facilitate the needs of S, and offers to sell (write) the option
- using their models $10,000 \times 0.30 = 3,000$ euro is the no-arbitrage price for the options
- reasonable commerical price is: $3,000$ euro $+$ fee (to be received at $t = 0$)
- we have solved the valuation problem for bank ABC

Motivation: solving valuation problem is not enough

Suppose bank ABC takes no further action. Then at $t = 3/12$:

- if stock price at $t = 3/12$ is below 5 euro:
  - there are no additional cashflows [profit$=3,000 \times \exp(rT)$]
- if stock price at $t = 3/12$ exceeds 5 euro:
  - bank ABC needs to sell stocks to S for 5 euro; this leads to loss of $10,000 \times (S_{3/12} - 5) - 3,000$ euro
  - note that potential loss is unbounded (assuming GBM as model for S): for any $K > 0$: $\mathbb{P}(S_{3/12} > K) > 0$
- so selling the option yields risk. Bank ABC would like (and is enforced to do so by regulators) to *measure* the risk and bank ABC might want to *manage* the risk by hedging part of the risk (because of its own risk appetite or because of regulatory limits)
- we will discuss risk measurement and hedging using the greeks
- our discussion is very specific to option portfolios; see MSc in QFAS for wide-range of tools for risk measurement and management

Hedging

- Cambridge Dictionary describe **hedging** as

  'the activity of reducing the risk of losing money on shares, bonds, etc. that you own'

- In finance we have many concepts and techniques to reduce risk. Some examples from the point-of-view of an investor:
  - using *diversification* in investment portfolio (e.g. Markowitz portfolio theory)
  - buying specific contracts to eliminate or reduce risk(s) in your porfolio (e.g. buy put option in combination with stock)
- This topic takes point-of-view of (trading desk of) financial institution that sells (writes) (non-traded) European options
  - yields incoming cashflow (premia) at inception
  - yields liability at maturity/expiration date
  - liability at maturity yields risk
  - institution might want to hedge (part of) the risk

Agenda

These slides discuss hedging of option portfolios from point-of-view of (trading desk) of financial institution that sells (writes) (non-traded) European options

- we will use continuous-time model for financial market, but will acknowledge that we can only trade on discrete time-grid
- in this discussion we will focus on using/exploiting closed-form formulas for option prices and related objects

In the next section we first reconsider situation in which we assume that we can trade in continuous-time

Section 2

Hedging in continuous time

Replicating portfolio

For the Black-Scholes market we have obtained:

- if we have option with price process $C_t = F(t, S_t)$, then F satisfies

$$\frac{\delta F}{\delta t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0 \quad \forall s, t$$

- there is self-financing portfolio $(\phi_t, \psi_t)_{t\geq 0}$ that replicates payoff option, i.e. $C_T = V_T = \phi_T S_T + \psi_T B_T$, given by:

$$\begin{aligned} \phi_t &= \frac{\partial F}{\partial s}(t, S_t) \\ \psi_t &= \frac{F(t,S_t) - \phi(t,S_t)S_t}{B_t} \end{aligned}$$

Perfect hedge

- suppose financial institution sells option to customer
- receives price/premium option at $t = 0$
- needs to pay $C_T \geq 0$ at maturity
- if institution does not want to have any exposure to risk:
  - setup self-financing trading strategy (from previous sheet) at $t = 0$
  - costs C_0 at $t = 0$
  - the self-financing portfolio replicates payoff C_T at maturity $\Longrightarrow$ no net cashflow at time T
  - self-financing $\Longrightarrow$ no intermediate net cashflows
  - charging price C_0 + 'additional fee' yields positive cashflow without any risk (at $t = 0$)

Illustration

- Consider Black-Scholes market with $S_0 = 100$, $r = 1\%$, $\mu = 5\%$, $\sigma = 20\%$ and European option with maturity $T = 3$ and payoff $1,000 \times \max\{K - S_T, 0\}$ with $K = 90$ (i.e. 1,000 put options)
- a realization:

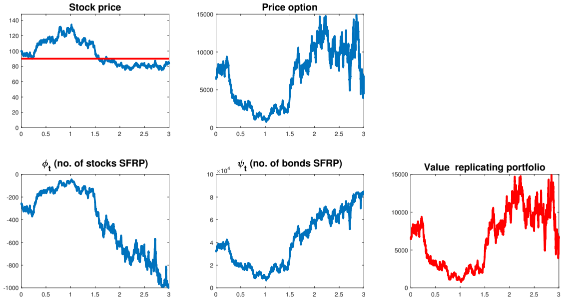


Illustration - remarks

- positions in self-financing porfolio look very wiggly
- we can show:

$$\phi_t = -1{,}000\Phi(-d_1)$$

with

$$d_1 = \frac{\log(S_t/K) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

- recall

$$S_t = S_0 \exp((\mu - 0.5\sigma^2)t + \sigma W_t^{\mathbb{P}})$$

so we see that ϕ_t is non-differentiable function of time

Reflection

The previous results strongly depend on the assumption that we can trade in continuous time...

How to proceed if we only trade on discrete time-grid (e.g. daily)?

Section 3

Hedging in discrete time and the greeks

Setup

- suppose we have sold (European) option with payoff C_T at maturity T
- price $C_t = F(t, S_t)$ at time t
- note that F will also depend on parameters
  - in case of European put/call and Black-Scholes market: $F(t, S_t) = F(t, S_t; T, K, r, \sigma)$
- we do not want to/cannot trade in continuous time, only on discrete-time grid (e.g. daily)
- we want to hedge exposure to risk; how to proceed?
- Itô suggests that change in option price over time interval $[t, t+\epsilon]$ is given by:

$$\begin{aligned} C_{t+\epsilon} - C_t \approx\ & F_S(t, S_t) \times \Delta S_{t+\epsilon} + F_t(t, S_t) \times \epsilon \\ & + \frac{1}{2} F_{SS}(t, S_t) \times (\Delta S_{t+\epsilon})^2 \end{aligned}$$

Delta

- we have

$$C_{t+\epsilon} - C_t \approx F_S(t, S_t) \times \Delta S_{t+\epsilon} + F_t(t, S_t) \times \epsilon$$
$$+ \frac{1}{2} F_{SS}(t, S_t) \times (\Delta S_{t+\epsilon})^2$$

- for any portfolio with price process $V_t = G(t, S_t)$ we define **Delta** by:

$$\Delta = \Delta_t = \frac{\partial}{\partial s} G(t, s)\Big|_{s=S_t} = G_S(t, S_t)$$

- portfolio for which Δ (at time t) is equal to 0 is called **delta neutral** (at time t)
- in case of small changes in stock price delta-neutral portfolios have limited (local) risk

Delta

- for Black-Scholes market the Delta of European call and put option are given by

$$\Delta_{\text{call}} = \Phi(d_1) \text{ and } \Delta_{\text{put}} = -\Phi(-d_1)$$

with

$$d_1 = \frac{\log(S_t/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- portfolio consisting of a options and b stocks (at time t) has Delta:

$$\Delta_{\text{portfolio}} = a\Delta_{\text{option}} + b$$

- to obtain delta-neutral portfolio at time t we set $b = -a\Delta_{\text{option}}$
- activity of obtaining delta-neutral portfolio is called **delta hedging**

Remark

- go back to continuous-time framework for the moment
- note that the Δ of the option corresponds to the position, in the stock, in the self-financing replicating portfolio!
- so in continuous-time we can perform delta-hedging using a self-financing portfolio
- only applying this hedge at discrete points-in-time destroys, in general, the self-financing property! Or: if we want to have self-financing portfolio in discrete time we are, in general, not able to replicate the payoff of the option without error (as we will see in simulations)!

Now back to trading on a discrete time grid.

Delta-hedging with self-financing portfolio

- supose we have a options, against price C_0
- take a position in stock and money market to obtain delta-neutral portfolio at $t = 0$ such that total portfolio is delta-neutral and has value 0
- at each point-in-time we rebalance the position in S such that the total portfolio is delta-neutral: for $t = \eta, 2\eta, \ldots, (n-1)\eta$, with $n\eta = T$,

$$\phi_t = -a\Delta_{\text{option},t}$$

  We want to rebalance positions in (S, B) in a budget-neutral way, i.e.

$$\psi_t = (\phi_{t-\eta} S_t + \psi_{t-\eta} B_t - \phi_t S_t)/B_t$$

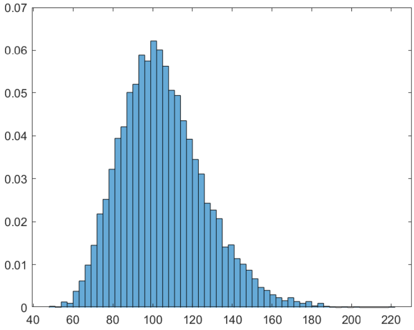
- at maturity the total portfolio has payoff

$$aC_T + \phi_{T-\eta} S_T + \psi_{T-\eta} B_T$$

  which in general wil differ from 0, i.e. we have replication error (due to rebalancing in discrete-time instead of continuous time)!

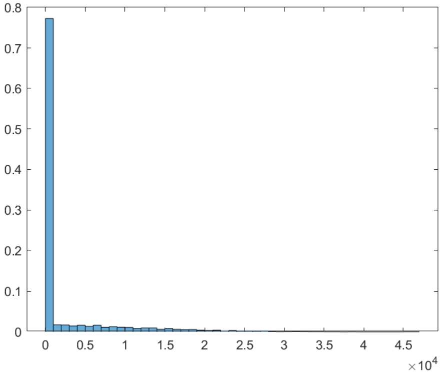
Illustration

- consider Black-Scholes market with $S_0 = 100$, $r = 1\%$, $\mu = 5\%$, $\sigma = 20\%$ and European option with maturity $T = 1$ and payoff $1,000 \times \max\{K - S_T, 0\}$ with $K = 90$ (i.e. 1,000 put options)
- no arbitrage price: 3,297 euro
- simulation-based approximation to distribution (real world) of S_T (10,000 replications)



Illustration

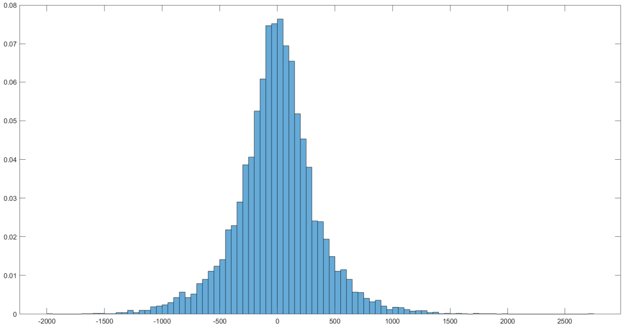
- simulation-based approximation to distribution (real world) of C_T (10,000 replications)



mean: 2,429 euro, 95%-quantile: 16,035 euro

Illustration

- simulation-based approximation to distribution (real world) of hedge/replication error at maturity T (10,000 replications)



So by using the "self-financing delta-neutral" hedge we have substantially reduced the risk at maturity T (without cost)

Gamma

- Recall that "delta" was motivated by approximation to change in portfolio value

$$V_{t+\epsilon} - V_t \approx \frac{\partial V}{\partial S_t}(t, S_t) \times \Delta S_{t+\epsilon} + \frac{\partial V}{\partial t} \times \epsilon + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2} \times (\Delta S_{t+\epsilon})^2$$

- higher order term might be relevant as well $\implies$ gamma

For any portfolio with price process $V_t = G(t, S_t)$ we define **Gamma** by:

$$\Gamma = \Gamma_t = \left.\frac{\partial^2}{\partial s^2} G(t, s)\right|_{s=S_t} = G_{SS}(t, S_t) = \frac{\partial}{\partial s}\, \Delta_t|_{s=S_t}$$

Note that gamma (also) measures direction of Δ. Portfolio for which Γ (at time t) is equal to 0 is called **gamma neutral** (at time t).

Gamma

- for Black-Scholes market the gamma of European call and put option are given by

$$\Gamma_{\text{call}} = \Gamma_{\text{put}} = \frac{1}{S_t \sigma \sqrt{T-t}} \phi(d_1)$$

  with

$$d_1 = \frac{\log(S_t/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- portfolio consisting of a options I, b stocks, and c options II (at time t) has Gamma:

$$\Gamma_{\text{portfolio}} = a\Gamma_{\text{option I}} + c\Gamma_{\text{option II}}$$

  $\Longrightarrow$ if we sell non-traded option, we can obtain **gamma-neutral** portfolio by taking position in *traded* option
- if we also take position in stock we can obtain **delta-gamma** neutral portfolio (i.e. delta and gamma are both equal to 0)
- this activity is called **delta-gamma hedging**

More Greeks

Rather specific to Black-Scholes market as underlying model:

- for any portfolio with price process $V_t = G(t, S_t; \sigma, r)$ we define **vega** by:

$$\nu = \frac{\partial}{\partial \sigma} G(t, S_t; \sigma, r)$$

- note that vega of a stock is 0
- if we want to hedge vega-risk then we need to add traded option to portfolio
  - leads to delta-gamma-vega hedging
- **rho** measures sensitivity of price with respect to interest rate:

$$\rho = \frac{\partial}{\partial r} G(t, S_t; \sigma, r)$$

And there are even more Greeks:

- for any portfolio with price process $V_t = G(t, S_t)$ we define **theta** by:

$$\Theta = \frac{\partial}{\partial t} G(t, S_t)$$

- Lambda, Vanna, Voma, Charm, etc.

Risk measurement

- we discussed Greeks in context of hedging
- the Greeks can also be used for risk measurement: for example, the delta and gamma of your portfolio indicate sensitivity with respect to change in stock price
  - institutions may have limits (and risk needs to be (partially) hedged if these limits are not met)
- Note that the greeks only provide local (in terms of time) information on the risk; in the assignment you will also consider the Value at Risk, which is a basic, but often used risk measure.