Valuation of Options - part 2

Quantitative Finance

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Assumptions Black-Scholes market (recap)

In class we will almost exclusively work with the 'Black-Scholes market'.

Assumptions on price processes:

Asset price:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \quad S_0 = s_0, \ \mathrm{var}(W_1) = 1$$

Money Market Account:

$$\mathrm{d}B_t = rB_t \,\mathrm{d}t, \quad B_0 = 1$$

Assumptions on market:

- frictionless trading
  - no transaction costs
  - trading in continuous-time possible
  - no restrictions on short sales and fractional positions
- borrowing rate = lending rate

Agenda

- we are interested in price C_t of (European) option with payoff $C_T = h(S_T)$
- we will discuss three methods to determine fair price (using continuous-time model for financial market):
  - Black-Scholes Partial Differential Equation
  - risk-neutral pricing
  - pricing kernel

  These slides discuss the Black-Scholes Partial Differential Equation approach.

Section 1

The Black-Scholes Partial Differential Equation

The problem

Setup:

- Black-Scholes market (1 risky asset)
- we restrict to Markovian trading strategies:

$$\phi_t = \tilde{\phi}(B_t, S_t) = \phi(t, S_t) \text{ and } \psi_t = \tilde{\psi}(B_t, S_t) = \psi(t, S_t)$$

  So we can write, for some function F,

$$V_t = \phi(t, S_t)S_t + \psi_t(t, S_t)B_t = F(t, S_t)$$

Question:

Which F's correspond to **self-financing** trading strategies?

Why interesting?

- consider option with payoff $h(S_T)$ at maturity
- if we can find self-financing portfolio with $F(T, S_T) = h(S_T)$ then no-arbitrage price of option at time $0 \leq t < T$ is given by

$$C_t = F(t, S_t)$$

Black-Scholes Partial Differential Equation:

F corresponds to a self-financing trading strategy if and only if F is a solution to the Partial Differential Equation (equation with function G as variable)

$$\frac{\partial G}{\delta t}(t,s)+rs\frac{\partial G}{\partial s}(t,s)+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 G}{\partial s^2}(t,s)-rG(t,s)=0\ \forall s>0, t\in[0,T),$$

and in that case the positions in the self-financing trading strategy are given by:

$$\phi(t,s)=\frac{\partial F}{\partial s}(t,s),$$

and

$$\psi(t,s)=\frac{F(t,s)-\phi(t,s)s}{\exp(rt)}.$$

Remarks:

- μ does not play a role!
- using different models (i.e. SDEs) for B and S leads, in general, to different PDE!

Derivation of the PDE

Which F are possible (when using self-financing strategies $(\phi(t,S_t),\psi(t,S_t))$)? We have

(1) $V_t = F(t,S_t)$

(2) $\mathrm{d}V_t = \phi(t,S_t)\,\mathrm{d}S_t + \psi(t,S_t)\,\mathrm{d}B_t$

- apply Itô to (1) and insert $\mathrm{d}S_t \implies \mathrm{d}V_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}W_t$
- insert $\mathrm{d}S_t$ and $\mathrm{d}B_t$ in RHS of (2) $\implies \mathrm{d}V_t = c_t\,\mathrm{d}t + d_t\,\mathrm{d}W_t$
- obtain system of equations

$$\begin{cases} a_t = c_t \\ b_t = d_t \end{cases}$$

- solving $\implies$ conditions on F (PDE)

Derivation

We have

(1) $V_t = F(t, S_t)$

(2) $\mathrm{d}V_t = \phi(t, S_t)\,\mathrm{d}S_t + \psi(t, S_t)\,\mathrm{d}B_t$

- apply Itô to (1) and insert $\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \implies$

$$\begin{aligned}\mathrm{d}V_t &= F_S\,\mathrm{d}S_t + F_t\,\mathrm{d}t + \frac{1}{2}F_{SS}\,\mathrm{d}[S,S]_t \\ &= \left(F_S\mu S_t + F_t + \frac{1}{2}F_{SS}\sigma^2 S_t^2\right)\mathrm{d}t + F_S\sigma S_t\,\mathrm{d}W_t\end{aligned}$$

- insert $\mathrm{d}S_t$ and $\mathrm{d}B_t = rB_t\,\mathrm{d}t$ in RHS of (2) $\implies$

$$\mathrm{d}V_t = (\phi_t\mu S_t + r\psi_t B_t)\,\mathrm{d}t + \phi_t\sigma S_t\,\mathrm{d}W_t$$

- hence $\phi_t = \phi(t, S_t) = F_S(t, S_t) = F_S$
- and

$$F_S\mu S_t + F_t + \frac{1}{2}\sigma^2 F_{SS}S_t^2 = \phi_t\mu S_t + r\psi_t B_t$$

Derivation

We need to prove

$$\frac{\delta F}{\delta t}(t,s)+rs\frac{\partial F}{\partial s}(t,s)+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 F}{\partial s^2}(t,s)-rF(t,s)=0 \quad \forall s>0,\ t\in[0,T)$$

On the previous slide we obtained

$$F_t+\frac{1}{2}\sigma^2 F_{SS}S_t^2=r\psi_t B_t.$$

As $\psi_t B_t = F - \phi_t S_t = F - F_S S_t$ we obtain the result.

Alternative Derivation

- sell (write) and hold 1 option with price process $C_t = F(t, S_t)$ (assumption!)
- let ϕ_t be number of stocks at time t and ψ_t the position in the MMA
- yields portfolio value $V_t = -C_t + \phi_t S_t + \psi_t B_t$
- this portfolio is self-financing if

$$\mathrm{d}V_t = -\,\mathrm{d}C_t + \phi_t\,\mathrm{d}S_t + \psi_t\,\mathrm{d}B_t$$

- using Itô:

$$\mathrm{d}V_t = -F_S\,\mathrm{d}S_t - F_t\,\mathrm{d}t - \frac{1}{2}F_{SS}\,\mathrm{d}[S,S]_t + \phi_t\,\mathrm{d}S_t + \psi_t r B_t\,\mathrm{d}t$$

- can only eliminate local risk ($\mathrm{d}W_t$) for $\phi_t = F_S(t, S_t)$!
- yields $\mathrm{d}V_t = \cdots \mathrm{d}t$ which has (locally) no risk, so rate of return is same as on $B \implies \mathrm{d}V_t = rV_t\,\mathrm{d}t$
- hence

$$-F_t - \frac{1}{2}F_{SS}\sigma^2 S_t^2 + r\psi_t B_t = r\left(-F + F_S S_t + \psi_t B_t\right)$$

which yields Black-Scholes PDE

Application

Given is a European option with payoff $h(S_T)$ at expiration date/maturity T. How can we use the PDE to obtain the price of this option?

- Solve the PDE,

$$\frac{\delta G}{\delta t}(t,s) + rs\frac{\partial G}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 G}{\partial s^2}(t,s) - rG(t,s) = 0,$$

  for all $s > 0$, $t \in [0, T)$, under the boundary condition

$$G(T,s) = h(s) \text{ for all } s > 0.$$

- If F is solution to PDE satisfying the boundary condition, then no-arbitrage implies that the price of the option at time $t \in [0, T)$ is given by $F(t, S_t)$.
- If you are lucky: 'closed-form' solution can be found
  - there is relation to well studied heat equation
- unlucky: use numerical techniques; see notebook

Examples: closed-form solutions

- $h(S_T) = S_T$

$$F(t, s) = s$$

- European call option: $h(S_T) = \max\{S_T - K, 0\}$

$$F(t, s) = s\Phi(d_1) - \exp(-r(T - t))K\Phi(d_2),$$

with $d_1 = d_2 + \sigma\sqrt{T - t}$ and

$$d_2 = \frac{\log(s/K) + (r - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

Examples: European digital call option

- payoff: $h(S_T) = 1\{S_T \geq K\}$
- the B-S PDE can be solved explictly:

$$F(t,s) = \exp(-r(T-t))\Phi(d_2),$$

with

$$d = \frac{\log(s/K) + (r - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

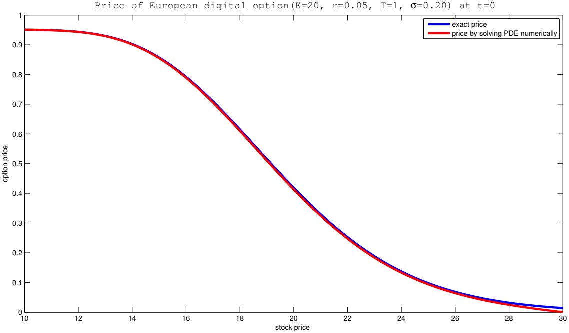
- Probability that option ends in-the-money:

$$\mathbb{P}\{S_T \geq K\} = \Phi\left(\frac{\log(s/K) + (\mu - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

In the exercise set you will be asked to verify that F is a solution to the B-S PDE.

Examples: European digital put option

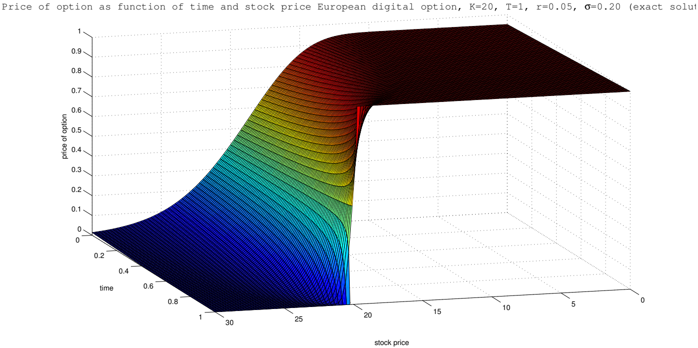
Price of the option (at $t = 0$) $F(s, 0)$ - numerical solution compared to exact solution:



The Group Assignment might contain an exercise on solving the PDE numerically.

Examples: European digital put option

Value function $F(t, s)$:



Feynman-Kac Theorem

Let $\alpha(t,x)$, $\beta(t,x)$, and $k(t,x)$ be functions satisfying 'regularity conditions'. Consider the PDE

$$\frac{\partial G}{\delta t}(t,x)+\alpha(t,x)\frac{\partial G}{\partial s}(t,x)+\frac{1}{2}\beta^2(t,x)\frac{\partial^2 G}{\partial s^2}(t,x)-k(t,x)G(t,x)=0,$$

for all x and $t \in [0, T)$ with boundary condition $G(T,x) = h(x)$ for all x.

Then the solution is given by:

$$G(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T k(s,X_s)\,\mathrm{d}s\right) h(X_T) \mid X_t = x\right],$$

where X is a stochastic process, on the time interval $[t, T]$, defined via the SDE

$$\mathrm{d}X_u = \alpha(u,X_u)\,\mathrm{d}u + \beta(u,X_u)\,\mathrm{d}W_u$$

and with starting value $X_t = x$, and where W is a standard Brownian motion.

Black-Scholes PDE reconsidered

The no-arbitrage price, at time $t \in [0, T)$, of a European option with payoff $h(S_T)$ at expiration date $T > 0$ is given by $F(t, S_t)$ where F satisfies

$$\frac{\delta F}{\delta t}(t,s)+rs\frac{\partial F}{\partial s}(t,s)+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 F}{\partial s^2}(t,s)-rF(t,s)=0 \quad \forall s>0,\ t\in[0,T)$$

Feynman-Kac tells us that we have

$$F(t,s)=\mathbb{E}\left[\mathrm{e}^{-r(T-t)}\,h(S_T)\mid S_t=s\right],$$

where S is defined via the SDE

$$\mathrm{d}S_u=rS_u\,\mathrm{d}u+\sigma S_u\,\mathrm{d}W_u,\quad S_t=s,$$

where W is a standard Brownian motion.