Dynamic resource allocation problems in communication networks:

Introduction and the Finite Horizon Restless Bandit problem

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LabSTICC (UMR CNRS 6285)

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IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

Lab-STICC

Acknowledgement

This course has been also elaborate during the project Ramonaas[1] (Regional Program STIC-AmSud):

- a STIC/AMSUD project between CAPES/BR (88881.694462/2022-01);
- Ministry for Europe and Foreign Affairs/FR;
- Campus France/FR and the National Agency for Research;
- Innovation/UY (MOV_CO_2022_1_1011515)

[1] Resource Allocation Methods for Optical Networks as a Service

IMT Atlantique : -Engineering institution under the tutelage of the French Ministry of Industry

- 3 campuses North-West of France
- Member of the Institut Mines-Télécom

Track MLA:

***where?* on Brest campus**

IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

when?

from 30 march to 2nd june 2023

What do in Brest?



Agenda of the course (Room B03-036)

- **Day I:** Resource allocation problem and Restless Bandit.
  1. Course, 9h - 12h, teacher: Alexandre Reiffers-Masson
  2. Lab, 13h30 - 16h30, teacher: Lucas Lopes, Alexandre Reiffers-Masson

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- **Day III:** Deep-learning and deep reinforcement learning applied to Resource Allocation Problems.
  1. Course, 9h - 12h, teacher: Alexandre Reiffers-Masson
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Agenda of Day I

Provably efficient heuristics for solving large-scale resource allocation problems.

- Introduction to resource allocation problem, Markov Decision Process, Restless Bandit in Finite Horizon and Infinite Horizon.
- Weakly coupled MDP and the resolving heuristic.
- Constrained Finite Horizon Stochastic Optimization Problems.

Objectives of the course

Provably efficient heuristics for solving large-scale resource allocation problems

1. Design heuristics and prove asymptotically optimal properties.
2. Code the heuristic in Python using *cvxpy*.

Machine Maintenance[2]

- **Scenario:** A collection of N machines which deteriorate under usage is maintained by a set of α repairmen. Maintenance interventions will improve a machine's condition and may preempt costly breakdowns.

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$$\begin{aligned} \mathbb{P}(S(t+1) = 0 | S(t) = s, A(t) = 1) &= 1 \\ \mathbb{P}(S(t+1) = s' | S(t) = s, A(t) = 0) &= I\{s' \geq s\} p_{ss'} \end{aligned} \tag{1}$$

Deadline Scheduling (for charging electric vehicles)[3]

- **Set-up:** Charging station has N charging spots and enough power to charge M vehicles at each round.

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  1. The charging station obtains a reward of $1 - c$ for each unit of electricity provided.
  2. If the station cannot fully charge the vehicle by the time it leaves, the station needs to pay a penalty proportional to the amount of the unfulfilled charge.

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Evolution: The (Markovian) evolution of the state is given by:

$$S_k(t+1) = \begin{cases} (T_k(t) - 1, [B_k(t) - a_k(t)]_+) & \text{if } T_k(t) > 1, \\ (T, B) \text{ with prob. } Q(T,B) & \text{otherwise,} \end{cases}$$

where $a_k(t)$ is the amount of electricity given to spot k at instant t.

Other applications

- Wireless Communication;
- Web Crawling;
- Congestion Control;
- Queuing Systems;
- Cluster and Cloud computing;
- Target Tracking;
- Clinical Trials.

A quick recall on Markov chain

Definition

A *Markov Chain* consists of a finite set $\mathcal{S}$ (called the state space) together with a countable family of random variables $S(0), S(1), S(2), \ldots$ with values in $\mathcal{S}$ such that for all $t \geq 0$:

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The matrix $P := [[p_{ss'}]]_{s,s'}$ is called the *transition matrix*.

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´ **Knowns parameters:** $\mathcal{S}$, $\mathcal{A}$, reward $R := [[r_s^a]]_{s,a}$, Horizon T, transition matrix $P^a := [[p_{s,s'}^a]]_{s,s'}$.

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2. When a randomized Markov policy π_t is used, the probability that the Markov process evolves to $S(t+1) = s'$ and action $A(t) = a$, knowing $S(t) = s$ is given by $p^a_{s,s'} \pi^a(s)$.

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The **Value function** is given by:

$$V_1^*(x^0, T) = \min_{\pi} V_1^{\pi}(x^0, T)$$

LP formulation

Let us define the following LP problem:

$$
\begin{aligned}
\min_{y \geq 0} \quad & \sum_{t=0}^{T-1} \sum_{s,a} R_s^a y_{a,s}(t) \\
\text{s.t.} \quad & y_{s,0}(t) + y_{s,1}(t) = x_s(t), \ \forall t \in [[0, T-1]], \ \forall s \in \mathcal{S}, \\
& x_s(t) = \sum_{s'} \sum_{a} y_{s',a}(t-1) p_{s',s}^a \ \forall t \in [[1, T-1]], \ \forall s \in \mathcal{S}, \\
& x_s(0) = x^0, \ \forall s \in \mathcal{S}
\end{aligned} \tag{3}
$$

Equivalence

Lemma: Let y^* be a solution of (3). If for all $0 \leq t \leq T-1$, for all $s \in \mathcal{S}$ and for all $a \in \mathcal{A}$, we define

$$\pi_t(a|s) = \begin{cases} y_s^a(t)/x_s^a(t), & \text{if } x_s^a(t) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$V_1^*(x^0, T) = V_1^\pi(x^0, T).$$

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´ **Knowns parameters:** $\mathcal{S}$, reward $R := [[r_s^a]]_{s,a}$, Horizon T, transition matrix $P^a := [[p_{s,s'}^a]]_{s,s'}$.

A new state representation

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- $Y_{s,a}^{(N)}(t) :=$ the fraction of arms in state s at time t for which decision a is taken. $Y^{(N)}(t) := [Y_{s,a}^{(N)}(t)]_{s\in\mathcal{S},a\in\{0,1\}}$ is the associated vector.

Mathematical Formulation

$$\min_{\pi} \quad \mathbb{E}\sum_{t=0}^{T-1}\sum_{s,a} r_s^a Y_{a,s}^{(N)}(t) := V_{opt}^{(N)}(m(0),T) \tag{4a}$$

s.t. Arms follow the Markovian evolution generated by $\Pi_n p^{a_n}_{s_n,s'_n}$, (4b)

$$Y_{0,s}^{(N)}(t) + Y_{1,s}^{(N)}(t) = M_s^{(N)}(t),\ \forall t \in [[0,T-1]],\ \forall s \in \mathcal{S}, \tag{4c}$$

$$\sum_s Y_{s,1}^{(N)}(t) \leq \alpha\ \forall t \in [[0,T-1]],, \tag{4d}$$

$$M_s^{(N)}(0) = m_s(0),\ \forall s \in \mathcal{S}, \tag{4e}$$

where $m_s(0) = \frac{1}{N}\sum_{k=1}^{N} I\{S_k(0)=s\}$, for all $s \in \mathcal{S}$.

Difficulty

The key difficulty of the N-Arms Restless Bandit problem is coming from:

$$\sum_s Y_{s,1}^{(N)}(t) \leq \alpha \; \forall t \in [[0, T-1]],$$

which couples all the arms together.

Challenge:

How to design an efficient heuristic to solve such problem?

Outline of the approach

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2. **Interpolation:** Construct a sequence of decision rules $\pi_t : \Delta^d \to \Delta^{2d}$ which is optimal for the relaxed problem.

Relaxed problem

$$
\begin{aligned}
\min_{\pi} \quad & \sum_{t=0}^{T-1} \mathbb{E} \sum_{s,a} r_s^a Y_{a,s}^{(N)}(t) =: V_{rel}^{(N)}(m(0), T) && \text{(5a)} \\
\text{s.t.} \quad & \text{Arms follow the Markovian evolution,} && \text{(5b)} \\
& Y_{0,s}^{(N)}(t) + Y_{1,s}^{(N)}(t) = M_s^{(N)}(t),\ \forall t \in [[0, T-1]],\ \forall s \in \mathcal{S}, && \text{(5c)} \\
& \sum_s \mathbb{E}[Y_{s,1}^{(N)}(t)] \leq \alpha \ \forall t \in [[0, T-1]],, && \text{(5d)} \\
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\end{aligned}
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LP formulation

Let us define the following LP problem:

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\begin{aligned}
\min_{y \geq 0} \quad & \sum_{t=0}^{T-1} \sum_{s,a} r_s^a y_{s,a}(t) =: V_{LP}(m(0), T) \\
\text{s.t.} \quad & y_{s,0}(t) + y_{s,1}(t) = m_s(t), \ \forall t \in [[0, T-1]], \ \forall s \in \mathcal{S}, \\
& m_s(t) = \sum_{s'} \sum_{a} y_{s',a}(t-1) p^a_{s',s} \ \forall t \in [[1, T-1]], \ \forall s \in \mathcal{S}, \\
& \sum_{s} y_{s,1}(t) \leq \alpha, \ \forall t \in [[0, T-1]], , \\
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We denote by $y^* := [[[y^*_{s,a}(t)]]]_{s,a,t}$ the optimal solution of (6) and we also define $m^* := [[m_s(t) := \sum_a y^*_{s,a}(t)]]_{s,t}$.

Equivalence

Lemma:

$$\begin{aligned} V_{rel}(m^0, T) &= V_{LP}(m^0, T), \\ V_{opt}^{(N)}(m(0), T) &\geq V_{LP}(m^0, T). \end{aligned}$$

Projection

We define the set of feasible control at time t by:

$$\mathcal{Y}(M^{(N)}(t)) := \left\{ y \in \mathbb{R}_+^{2d} | \sum_a y_{s,a} = M_s^{(N)}(t) \; \forall s \in \mathcal{S}; \; \sum_s y_{s,1} \leq \alpha \right\}$$

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We define the following **projection operator**:

$$\pi_t^{Proj}(M^{(N)}) := \mathsf{Proj}_t(M^{(N)}) := \mathsf{argmin}_{y \in \mathcal{Y}(M^{(N)}(t))} \|y - y^*(t)\|_2^2. \tag{7}$$

Algorithm

The Projection Policy

- **Input:** Initial system configuration vector $m(0)$ and time horizon T.
- **Solve** The LP to obtain y^*;
- **Set** $\hat{M}$:= m(0);
- **For** $t = 0, 2, \ldots, T-1$ **do:**
  1. *Projection step:* Compute $\hat{y}(t) := \mathrm{Proj}_t(\hat{M})$;
  2. *Rounding step:* For all $s \in \mathcal{S}$, set:

$$\hat{Y}_{s,a}^{(N)}(t) = \begin{cases} N^{-1}\lfloor N\hat{y}_{s,1}(t)\rfloor & \text{if } a = 1, \\ \hat{M}_s - N^{-1}\lfloor N\hat{y}_{s,1}(t)\rfloor & \text{otherwise.} \end{cases}$$

  3. Use control $\hat{Y}^{(N)}$ to advance to the next time-step ;
  4. Set $\hat{M}$:= current empirical distribution;

Policy construction scheme

A simple scheme can explain easily our algorithm:

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$$\hat{M}(t) \xrightarrow[\text{Proj. step}]{\mathsf{Proj}_t(\hat{M}(t))} \hat{y}(t) \xrightarrow[\text{Roun. step}]{} \hat{Y}^{(N)}_{s,a}(t) \xrightarrow[\text{Trans. step}]{} \hat{M}(t+1)$$

Remark: We will see in the next theorem that the projection step can be replaced by map $\pi(\cdot)$ such that:

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1. *Admissible policy:* $\pi_t(M^{(N)}(t)) \in \mathcal{Y}(M^{(N)}(t))$,
2. *LP-compatible policy:* $\pi_t(m^*(t)) = y^*(t)$.

Main result

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Let $\pi := \{\pi_t\}_{0 \leq t \leq T-1}$ *be a continuous an admissible and continuous policy then*

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$$\lim_{N \to +\infty} V^{(N)}_{opt,\pi}(m(0), T) = V_{LP}(m(0), T).$$

Infinite Restless Bandit

The initial Restless Bandit was defined as follows:

$$\min_{\pi \in \Pi} \quad \lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\sum_{t=0}^{T-1}\sum_{s,a} r_s^a Y_{a,s}^{(N)}(t) =: V_{opt}^{(N)}(\infty) \tag{8a}$$

$$\text{s.t.} \quad \text{Arms follow the Markovian evolution generated by } \Pi_n p_{s_n,s'_n}^{a_n}, \tag{8b}$$

$$Y_{0,s}^{(N)}(t) + Y_{1,s}^{(N)}(t) = M_s^{(N)}(t),\ \forall t \in [[0, T-1]],\ \forall s \in \mathcal{S}, \tag{8c}$$

$$\sum_s Y_{s,1}^{(N)}(t) \le \alpha\ \forall t \in [[0, T-1]],, \tag{8d}$$

$$M_s^{(N)}(0) = m_s(0),\ \forall s \in \mathcal{S}, \tag{8e}$$

where $m_s(0) = \frac{1}{N}\sum_{k=1}^{N} I\{S_k(0) = s\}$, for all $s \in \mathcal{S}$ and Π is the set of Markovian policy.

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We next relax the constraints $\sum_s Y_{s,1}^{(N)}(t) \leq \alpha, \; \forall t \in [[0, T-1]]$ into:

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By defining $y_{s,a} = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_s \mathbb{E}_\pi[Y_{s,a}^{(N)}(t)]$, for all a and s, we then obtain the following linear program:

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$$\begin{aligned}
\min_{y \geq 0} \quad & \sum_{s,a} r_s^a y_{s,a} =: V_{LP}(\infty) \\
\text{s.t.} \quad & y_{s,0} + y_{s,1} = \sum_{s'} \sum_a y_{s',a} p_{s',s}^a, \ \forall s \in \mathcal{S}, \\
& \sum_s y_{s,1} \leq \alpha, \sum_{s,a} y_{s,a} = 1.
\end{aligned} \tag{10}$$

Policies

As before we need to define a policy such that we can transfer the solution y^* of the LP to N-arms problems. Three solutions:

- LP-priority policy[4],
- LP-index policy[5],
- Whittle indices[6].

[4] Verloop M (2016) Asymptotically optimal priority policies for indexable and nonindexable restless bandits. Annals of Applied Probability 26(4):1947-1995.

[5] Gast, Nicolas, Bruno Gaujal, and Chen Yan. "Linear Program-Based Policies for Restless Bandits: Necessary and Sufficient Conditions for (Exponentially Fast) Asymptotic Optimality." Mathematics of Operations Research (2023).

[6] Weber RR, Weiss G (1990) On an index policy for restless bandits. Journal of Applied Probability 27(3):637-648, ISSN 00219002

LP-priority

We define the following four sets, which form a partition of S

$$\mathcal{S}^{+} = \{s \in \mathcal{S} | y^*_{s,1} > 0,\ y^*_{s,0} = 0\}, \tag{11}$$

$$\mathcal{S}^{0} = \{s \in \mathcal{S} | y^*_{s,1} > 0,\ y^*_{s,0} > 0\}, \tag{12}$$

$$\mathcal{S}^{-} = \{s \in \mathcal{S} | y^*_{s,1} = 0,\ y^*_{s,0} > 0\}, \tag{13}$$

$$\mathcal{S}^{\emptyset} = \{s \in \mathcal{S} | y^*_{s,1} = 0,\ y^*_{s,0} = 0\}. \tag{14}$$

Definition: The set of **LP-priorities** are defined as $\Sigma := \cup_{y^*} \Sigma(y^*)$, where $\Sigma(y^*)$ is the set of permutations $\sigma = \sigma_1 \ldots \sigma_d$ of the d states such that any state in $\mathcal{S}^+$ appears before any state in $\mathcal{S}^0$, and any state in $\mathcal{S}^0$ appears before any state in $\mathcal{S}^-$.

LP-indices

By strong duality, there exists Lagrange multiplier $\gamma^* \in \mathbb{R}$ such that y^* is also an optimal solution to the following linear program:

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\begin{aligned}
g(\gamma^*) = \min_{y \geq 0} \quad & \sum_{s,a} r_s^a y_{s,a} + \gamma^* \sum_s y_{s,1} \\
\text{s.t.} \quad & y_{s,0} + y_{s,1} = \sum_{s'} \sum_a y_{s',a} p_{s',s}^a, \ \forall s \in \mathcal{S}, \\
& \sum_{s,a} y_{s,a} = 1.
\end{aligned}
$$

LP-index policy

We can transform this LP into an MDP, with the value function $V^*(s)$ satisfies the Bellman equation:

$$g(\gamma^*) + V^*(s) = \min\{\underbrace{r_s^1 + \gamma^* r_s^1 + \sum_{s'} p_{s,s'}^1 V^*(s')}_{=:Q_s^1}, \underbrace{r_s^0 + \sum_{s'} p_{s,s'}^0 V^*(s')}_{=:Q_s^0}\}.$$

- The LP indices for the infinite horizon are defined as $I_s := Q_s^1 - Q_s^0$ for state s.
- The **LP-index policy** is the strict priority policy by using the values I_s as a priority order to rank states within $\mathcal{S}^+$, $\mathcal{S}^-$ and $\mathcal{S}^0$ at each decision epoch.

Whittle indices

- Let us define for each value $\gamma \in \mathbb{R}$, the value function $V_s(\gamma)$ for state s satisfies the Bellman equation:

$$g(\gamma) + V^*(s,\gamma) = \min\{\underbrace{r_s^1 + \gamma^* r_s^1 + \sum_{s'} p_{s,s'}^1 V^*(s',\gamma)}_{Q_s^1(\gamma)}, \underbrace{r_s^0 + \sum_{s'} p_{s,s'}^0 V^*(s',\gamma)}_{Q_s^0(\gamma)}\}.$$

- Let us also define the set for which the arg min of the parametrized Bellman equation:

$$\mathcal{S}(\gamma) := \{s \in \mathcal{S} | Q_s^1(\gamma) > Q_s^0(\gamma)\}.$$

Whittle indices (cont'd)

- We say that the Restless Bandit is **indexable** if $\mathcal{S}(\gamma)$ expands monotonically from to the full set $\mathcal{S}$ when γ is decreased from $+\infty$ to $-\infty$.
- The **Whittle index** γ_s for state s is defined to be the supremum value of γ for which s belongs to $\mathcal{S}(\gamma)$.
- **Whittle index policy** is the strict priority policy by using the values γ_s as a priority score to rank states within $\mathcal{S}^+$, $\mathcal{S}^-$ and $\mathcal{S}^0$ at each decision epoch.

Link between the policies

Theorem

Assume that the infinite horizon RB is unichain, so that $\mathcal{S}^{\emptyset} = \emptyset$. Then:

- $s \in \mathcal{S}^+ \Rightarrow I_s > 0$, $s \in \mathcal{S}^- \Rightarrow I_s < 0$, $s \in \mathcal{S}^0 \Rightarrow I_s = 0$.
- *If we assume furthermore that the infinite horizon RB is indexable in Whittle' s sense, then their Whittle indices* $\gamma(s)$ *satisfy:* $s \in \mathcal{S}^+ \Rightarrow \gamma(s) > \gamma^*$, $s \in \mathcal{S}^- \Rightarrow \gamma(s) < \gamma^*$, $s \in \mathcal{S}^0 \Rightarrow \gamma(s) = \gamma^*$.

Bibliography

- The proof of the main theorem and more advance theorem can be found here: Gast, Nicolas, Bruno Gaujal, and Chen Yan. "Linear Program-Based Policies for Restless Bandits: Necessary and Sufficient Conditions for (Exponentially Fast) Asymptotic Optimality." Mathematics of Operations Research (2023).
- If you want to find a lot of different applications, you can have a look at: Avrachenkov, Konstantin E., and Vivek S. Borkar. "Whittle index based Q-learning for restless bandits with average reward." Automatica 139 (2022): 110186.