Dynamic resource allocation problems in communication networks:

Weakly Coupled Markov decision processes

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Load balancing and service rate planning in parallel queue networks

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- **Scenario:** N queues are processing jobs. The rate of each queue can be controlled. Moreover, the scheduler can also decide to which queue a job can be sent.

- **Challenge:** In order to minimize the total load on the system:
  - to which queue should a job be *allocated* at each decision instant?
  - which queue should see its *service rate* increased or decreased at each decision instant?

Dynamic of the queue

- **Arrivals:** We consider that at every time slots, αN new jobs arrives in the system with probability $p \in (0,1)$. Let $T_n \in \mathbb{N}_+$ be the arrival time of the n-th batch new jobs.
- **Evolution of the queue length:** The length of the k-th queue, denoted by $S_k(T_{n+1})$ at instant T_{n+1} is given by:

$$S_k(T_{n+1}) = S_k(T_n) - D_k(T_{n+1} - T_n) + I\{S_k(T_n) < K\}A_k(T_n)$$

where:

  - K is the finite buffer size of a queue;
  - $D_k(T_n)$ the number of process jobs between T_n and T_{n+1}.
  - $A_k(T_n) \in \{0,1\}$ is equal to one if one job from n-th batch is sent to the queue k.

Job processing policies

Concurrent job service per slot, per queue (CJS): Each pending job at each queue can be served concurrently with other jobs. In this case, more than one job can be served per queue per time slot.

Single job service per slot, per queue (SJS): Only one job can be processed per queue at a given time slot. In particular, at each time slot, we assume that **head-of-line (HOL)** jobs across queues are candidates to be served.

Cost functions and constraints

Costs: We will assume that there are two instantaneous costs:

- *Energy cost:* $\sum_k C_s(S_k(t)) + \sum_k C_q(B_k(t))$, where $C_s(\cdot)$ and $C_q(\cdot)$ are convex increasing.
- *Job rejection cost:* $-\gamma \sum_k A_k(t)$, with $\gamma > 0$. This cost implies that we prefer to send jobs.

Constraints: We will also assume that there are two instantaneous constraints:

$$\sum_k A_k(t) \leq \alpha N, \tag{1}$$

$$\sum_k B_k(t) \leq \beta N. \tag{2}$$

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- We assume that the decision-maker has to respect the following resource allocation constraints:

$$\sum_k D_l(S_k(t), A_k(t)) \leq N\alpha_l, \; \forall l = 1, \ldots, L$$

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Discussion with respect to the constraints

We assume that all terms in $D_l(s, a)$ and α_l are non-negative numbers, and that $D(s, 0) = 0$.

This is a natural assumption under the resource allocation context in which $a = 0$ corresponds to a passive action that consumes no resources.

Implication: The later also implies that our resource constraint problem has at least a feasible solution by always choosing the passive action.

Mathematical Formulation

$$\min_{\pi} \quad \mathbb{E}\sum_{t=0}^{T-1}\sum_{s,a} r_s^a Y_{a,s}^{(N)}(t) := V_{opt}^{(N)}(m(0),T) \tag{3a}$$

s.t. Arms follow the Markovian evolution generated by $\Pi_n p_{s_n,s'_n}^{a_n}$, (3b)

$$\sum_a Y_{a,s}^{(N)}(t) = M_s^{(N)}(t),\ \forall t \in [[0,T-1]],\ \forall s \in \mathcal{S}, \tag{3c}$$

$$\sum_s D_l(s,a) Y_{s,a}^{(N)}(t) \leq \alpha_l\ \forall t \in [[0,T-1]],, \tag{3d}$$

$$M_s^{(N)}(0) = m_s(0),\ \forall s \in \mathcal{S}, \tag{3e}$$

where $m_s(0) = \frac{1}{N}\sum_{k=1}^{N} I\{S_k(0) = s\}$, for all $s \in \mathcal{S}$.

Difficulty

The key difficulty of Weakly Coupled Markov decision processes is coming from:

$$\sum_{s} D_l(s,a) Y_{s,a}^{(N)}(t) \leq \alpha_l \; \forall t \in [[0, T-1]],$$

which couples all the arms together.

Challenge of the day:

How to design an efficient heuristic to solve such problem?

A different one that the projection policy.

Outline of the approach

1. **Relaxation:** Classical approach is to relax this constraint and consider a problem where this constraint has to be satisfied only in expectation:

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2. **Interpolation:** Construct a sequence of decision rules $\pi_t : \Delta^d \to \Delta^{2d}$ which is optimal for the relaxed problem.

Relaxed problem

$$\min_{\pi} \quad \mathbb{E}[\sum_{t=0}^{T-1}\sum_{s,a} r_s^a Y_{a,s}^{(N)}(t)] =: V_{rel}^{(N)}(m(0),T) \tag{4a}$$

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LP formulation

Let us define the following LP problem:

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\begin{aligned}
\min_{y \geq 0} \quad & \sum_{t=0}^{T-1} \sum_{s,a} r_s^a y_{s,a}(t) =: V_{LP}(m(0), T) \\
\text{s.t.} \quad & \sum_{a} y_{s,a}(t) = m_s(t),\ \forall t \in [[0, T-1]],\ \forall s \in \mathcal{S}, \\
& m_s(t) = \sum_{s'} \sum_{a} y_{s',a}(t-1) p_{s',s}^a\ \forall t \in [[1, T-1]],\ \forall s \in \mathcal{S}, \\
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We denote by $y^* := [[[y^*_{s,a}(t)]]]_{s,a,t}$ the optimal solution of (6) and we also define $m^* := [[m_s(t) := \sum_a y^*_{s,a}(t)]]_{s,t}$.

Motivation

Weakly Coupled Markov decision processes

Construction of LP-Admissible Policy

Multistage Convex Stochastic Optimization Problems

Feasible control

We define the set of feasible control at time t by:

$$\mathcal{Y}(M^{(N)}(t)) := \Big\{ y \in \mathbb{R}_+^{2S} | \ \sum_a y_{s,a} = M_s^{(N)}(t), \ \forall s \in \mathcal{S},$$

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Resolving policy

We redefine the following LP:

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The solution of this LP is denoted by

$$y^{Res}(m(0), T-t) = [y_{t'}^{Res}(m(0), T-t)]_{0\leq t'\leq T-t-1}.$$

Algorithm to solve the LP

What could be a possible algorithm to solve this LP?

Solution 1: Simplex or Convex optimisation?

Solution 2: Dynamic programming. Observe that:

$$V_{LP}(m, T-t) = \min_{y \in \mathcal{Y}(m)} \sum_{s,a} r_s^a y_{s,a} + V_{LP}(\phi(m, y), T-t-1),$$

where $\phi_s(m, y) = \sum_{s'} \sum_a y_{s',a} p^a_{s',s}$ for all s.

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3. $y^*(t) = \pi_t^{Res}(m^*(t))$. (LP-Admissible Policy)

Algorithm

Resolving Policy

- **Input:** Initial system configuration vector $m(0)$ and time horizon T.
- **Set** $\hat{M}$:= m(0);
- **For** $t = 0, 2, \ldots, T-1$ **do:**
  1. **Compute** $y^{Res}(\hat{M}, T-t)$; Set $\hat{y}(t) = y_0^{Res}(\hat{M}, T-t)$
  2. *Rounding step:* For all $s \in \mathcal{S}$, set:

$$\hat{Y}^{(N)}_{s,a}(t) = \begin{cases} N^{-1}\lfloor N\hat{y}_{s,1}(t)\rfloor & \text{if } a = 1, \\ \hat{M}_s - N^{-1}\lfloor N\hat{y}_{s,1}(t)\rfloor & \text{otherwise.} \end{cases}$$

  3. Use control $\hat{Y}^{(N)}$ to advance to the next time-step ;
  4. Set $\hat{M}$:= current empirical distribution;

Certainty equivalent control

This policy is inspired from the *certainty equivalent control* (CEC).

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Principle of the CEC

Sub-optimal control that applies at each stage the control that would be optimal if some or all of the uncertain quantities were fixed at their expected values.

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3. How to handle continuous state?
4. Efficient algorithm to solve the LP when the parameters are unknown.

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Example: Access control and Utility Maximization

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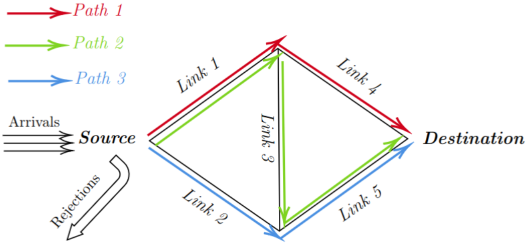
- **Scenario:** How bandwidth should I allocate on each path?
- **Challenge:** Maximize the total amount of Bandwidth sent into the network?



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$$0 \leq U_1(t) \leq W_1(t) \quad 0 \leq U_2(t) \leq W_2(t) \quad 0 \leq U_3(t) \leq W_3(t)$$

Bandwidth occupation

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- We assume that the evolution of $X_p(t)$ is given by:

$$X_p(t+1) = (X_p(t) + U_p(t)) \cdot q_p + \epsilon_p(t+1),$$
$$\text{for } 1 \leq p \leq 3 \text{ and } 1 \leq t \leq T,$$

where $\epsilon_p(t+1)$ is a r.v. with mean zero and support,

$$[-(X_p(t) + U_p(t)) \cdot q_p, (X_p(t) + U_p(t)) \cdot (1 - q_p)].$$

Capacity constraints

- Each directed edge of the graph is called a *link*, enumerated by $1 \leq l \leq 5$. Each link has a maximum bandwidth capacity, denoted as $c_l > 0$.

Capacity constraints

- Each directed edge of the graph is called a *link*, enumerated by $1 \leq l \leq 5$. Each link has a maximum bandwidth capacity, denoted as $c_l > 0$.
- The constraints that each link should satisfy are given by for all $1 \leq t \leq T$:

$$
\begin{aligned}
Y_1(t) &:= U_1(t) + X_1(t) + U_2(t) + X_2(t) \leq c_1 \\
Y_2(t) &:= U_3(t) + X_3(t) \leq c_2 \\
Y_3(t) &:= U_2(t) + X_2(t) \leq c_3 \\
Y_4(t) &:= U_1(t) + X_1(t) \leq c_4 \\
Y_5(t) &:= U_2(t) + X_2(t) + U_3(t) + X_3(t) \leq c_5.
\end{aligned}
$$

Utility funcion

The decision-maker aims to maximize the following α-fairness utility (with $\alpha > 0$)

$$\mathbb{E}\sum_{t=1}^{T}\sum_{p=1}^{3}\frac{(X_p(t)+U_p(t))^{1-\alpha}}{1-\alpha}$$

gained by allocating and rejecting the bandwidth demands over a finite horizon T, while respecting the dynamics and constraints described in the previous slides.

Constrained Finite Horizon Stochastic Optimization Problem

For each time-step $t = 1, \ldots, T$:

1. The decision-maker gets the current system state $X(t)$;

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$$g_{t,i}(X(t), W(t), U(t)) \leq 0,\ h_{t,j}(X(t), W(t), U(t)) = 0,$$

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4. The decision-maker collects a reward $R_t(X(t), W(t), U(t))$
5. The system evolves to the next state $(t+1)$ such that $X(t+1) \sim \phi\,(X(t), W(t), U(t)) + \epsilon(X(t), W(t), U(t))$.

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Objective: Maximize the expected total sum of rewards over the T time-steps.

Mathematical model

$$\begin{aligned}
\max_{[1,T]} \quad & \mathbb{E}\left[\sum_{t=1}^{T} R_t\left(X(t), W(t), U(t)\right)\right] =: V_{\text{opt}}(x(1), T) && \text{(8a)}\\
\text{s.t.} \quad & X(1) = x(1)\ a.s., && \text{(8b)}\\
& g_{t,i}(X(t), W(t), U(t)) \leq 0,\ \forall t \text{ and } \forall j, && \text{(8c)}\\
& h_{t,j}(X(t), W(t), U(t)) = 0,\ \forall t \text{ and } \forall j, && \text{(8d)}\\
& X(t+1) = \phi\left(X(t), W(t), U(t)\right) + \epsilon(X(t), W(t), U(t)),\ \forall t && \text{(8e)}
\end{aligned}$$

The Certainty Equivalent Control (CEC)

Based on CEC, we apply the following relaxation to the original problem: define $\mathbb{E}X(t) := x(t)$ and $\mathbb{E}U(t) = u(t)$ where the expectation is taken with the whole trajectory. By Jensen's inequality, we have

$$\mathbb{E}[\sum_{t=1}^{T} R_t\left(X(t), W(t), U(t)\right)] \leq \sum_{t=1}^{T} R_t\left(x(t), \mathbb{E}[w], u(t)\right). \tag{9}$$

$$\mathbb{E}[g_{t,i}(X(t), W(t), U(t))] \geq g_{t,i}(x(t), \mathbb{E}[w], u(t)),\ \forall t, i \tag{10}$$

$$\mathbb{E}[h_{t,j}(X(t), W(t), U(t))] = h_{t,j}(x(t), \mathbb{E}[w], u(t))\ \ \forall t, j \tag{11}$$

Relaxed mathematical program

All this consideration leads to the following relaxed mathematical program with decision variables $u(t)$:

$$\max_{u[1,T]} \quad \sum_{t=1}^{T} R_t\left(x(t), \overline{w}, u(t)\right) \tag{12a}$$

$$\text{s.t.} \quad x(1) = x, \tag{12b}$$

$$g_{t,i}(x(t), \overline{w}, u(t)) \leq 0,\ \forall t, i, \tag{12c}$$

$$h_{t,j}(x(t), \overline{w}, u(t)) = 0,\ \forall t, i, \tag{12d}$$

$$x(t+1) = \phi\left(x(t), \overline{w}, u(t)\right),\ \forall t, i \tag{12e}$$

Algorithms

You can apply the resolving algorithm or the projection policy in this case. You can even have a combination of both.

But we don't have the symmetry property. So our error will be this time controlled by the **variances** of the different variables.

Bibliography

- The proof of the main theorem and more advance theorem can be found here: Gast, Nicolas, Bruno Gaujal, and Chen Yan. "The LP-update policy for weakly coupled Markov decision processes." arXiv preprint arXiv:2211.01961 (2022).
- If you want to have a quick introduction to dynamic programming, please have a look to the lecture note of Nahum Shimkin: `https://webee.technion.ac.il/shimkin/LCS11/LCS11index.html`
- Yan, Chen, and Alexandre Reiffers-Masson. "Certainty Equivalence Control-Based Heuristics in Multi-Stage Convex Stochastic Optimization Problems." arXiv preprint arXiv:2308.13166 (2023).
- Application to load balancing and autoscaling. "Online Learning of Weakly Coupled MDP Policies for Load Balancing and Auto Scaling" `https://arxiv.org/html/2406.14141v1`