# Knowledge Representation and Reasoning with Propositional Logic (CP0)

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# Knowledge Based Systems with CP0

Using the logic CP0 for building knowledge based systems:

- Knowledge of the system: Set of sentences related to what we believe is always true
- Evidences from the outside: Set of sentences related to what can be true or false on the environment

Sentences are build using atomic propositions and logical connectives

The vocabulary of atomic propositions must include symbols for each possible property (either internal to the system or about the environment) we would like to reason about !

# Knowledge Based Systems with CP0

A gentle reminder: The logic of CP0

Syntax: What can we express with this logic ?

- Atomic Propositions ($P, \; Q, \; R, \ldots$),
- Propositional sentences ($P \rightarrow Q, \; P \wedge Q, \; P \rightarrow (Q \vee R), \ldots$)

Semantics: What is the meaning of the sentences ?

- A propositional interpretation I assigns to every atomic proposition P a truth value I(P)
- The truth value for sentences is obtained from the semantics of the logical connectives:
    1. $I(A \wedge B) = T$ iff $I(A) = T$ and $I(B) = T$
    2. $I(A \vee B) = T$ iff $I(A) = T$ or $I(B) = T$
    3. $I(A \rightarrow B) = T$ iff $I(A) = F$ or $I(B) = T$
    4. $I(\neg A) = T$ iff $I(A) = F$

# Knowledge Based Systems with CP0

A propositional formula Γ, a set of sentences, is satisfiable if there is at least one interpretation that makes true all the sentences from Γ

Basic Questions (to ask with algorithms !):

- Is a formula Γ satisfiable ? (Has at least one model?)
- If Γ is true, is a given sentence B necessarily true ? (True in all the models of Γ)

Beware ! Check the lecture notes about computational logic on the campus !

# Knowledge Based Systems with CP0

A key concept is the set of models of a formula $\Gamma$

- Every complete assignment of truth values to all his atomic propositions (propositional interpretation) that satisfies all the sentences in $\Gamma$ represents a state that is consistent with the knowledge of the system


So, if $\Gamma$ is the formula that contains the current knowledge of the system, each model of the formula is associated with a possible state (of the world) in which the system believes it can be

# Knowledge Based Systems with CP0

Example: Curiosity Mars Rover

The vocabulary of atomic propositions represents basic properties about the environment of the Rover:

1. OF $\equiv$ "There is an obstacle in front of the Rover"
2. ROF $\equiv$ "There is a rock in front of the Rover"
3. ML $\equiv$ "There is a Martian on the left of the Rover"
4. BR $\equiv$ "Jose Luis Barcenas in on the right of the Rover"
5. DL $\equiv$ "There is a danger on the left of the Rover"
6. DR $\equiv$ "There is a danger on the right of the Rover"

# Knowledge Based Systems with CP0

Example: Curiosity Mars Rover

We also have a vocabulary of atomic propositions that represent possible actions the Rover may be asked to perform:

1. AV $\equiv$ “Move Forward”
2. TL $\equiv$ “Turn Left”
3. TR $\equiv$ “Turn Right”

# Knowledge Based Systems with CP0

Example: Curiosity Mars Rover

Consider the Rover has the following knowledge at the initial state:

$$
\begin{aligned}
\Gamma \quad = \quad & \{\mathrm{ROF} \rightarrow \mathrm{OF}, && (1) \\
& (\mathrm{AV} \wedge \mathrm{OF}) \rightarrow (\mathrm{TL} \vee \mathrm{TR}), && (2) \\
& \mathrm{DL} \rightarrow \neg \mathrm{TL}, && (3) \\
& \mathrm{DR} \rightarrow \neg \mathrm{TR}, && (4) \\
& (\mathrm{ML} \wedge \mathrm{BR}) \rightarrow (\mathrm{DL} \vee \mathrm{DR})\} && (5)
\end{aligned}
$$

Suppose he next discovers, using his sensors, some evidence E about its environment such that the following variables become true:

$$\mathrm{E} = \{\mathrm{ROF}, \mathrm{ML}, \mathrm{BR}\}$$

What can we infer from $\Gamma$ and this new information ?

# Knowledge Based Systems with CP0

We have some sentences that are true in all the models of $\Gamma \cup E$. They are logical consequences of $\Gamma \cup E$:

1. $\Gamma \cup E \models OF$
2. $\Gamma \cup E \models (DL \vee DR)$
3. $\Gamma \cup E \models (\neg TL \vee \neg TR)$

Any of these logical consequences $\Gamma \cup E \models S$ can be checked by deriving a contradiction from $\Gamma \cup E \cup \{\neg S\}$

In other words, determining if $\Gamma \cup E \cup \{\neg S\}$ is unsatisfiable

Warning: We consider that $\Gamma \cup E$ is satisfiable for any E. That is, $\Gamma$ and E are not contradictory sources of information !

# Knowledge Based Systems with CP0

Derivation of a contradiction ($\square$) from sentences in $\Gamma \cup \mathrm{E}$ with $\neg\mathrm{OF}$:

$$\mathrm{ROF} \rightarrow \mathrm{OF}, \mathrm{ROF} \vdash \mathrm{OF} \tag{6}$$

$$\mathrm{OF}, \neg\mathrm{OF} \vdash \square \tag{7}$$

So, OF is true in all the models of $\Gamma \cup \mathrm{E}$

# Knowledge Based Systems with CP0

A similar derivation obtains a contradiction from $\Gamma \cup \mathrm{E}$ with $\neg(\mathrm{DL} \vee \mathrm{DR}) \equiv \neg\mathrm{DL} \wedge \neg\mathrm{DR}$:

$$(\mathrm{ML} \wedge \mathrm{BR}) \rightarrow (\mathrm{DL} \vee \mathrm{DR}), \mathrm{ML}, \mathrm{BR} \vdash (\mathrm{DL} \vee \mathrm{DR}) \tag{8}$$

$$(\mathrm{DL} \vee \mathrm{DR}), \neg\mathrm{DL} \vdash \mathrm{DR} \tag{9}$$

$$\mathrm{DR}, \neg\mathrm{DR} \vdash \square \tag{10}$$

So, $(\mathrm{DL} \vee \mathrm{DR})$ is true in all the models of $\Gamma \cup \mathrm{E}$

# Knowledge Based Systems with CP0

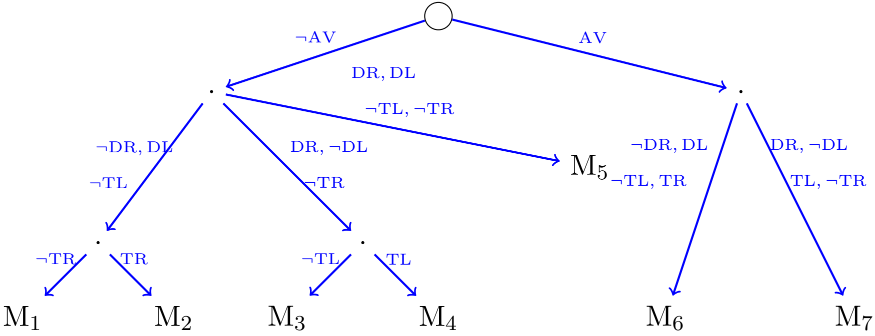
And also a contradiction from $\Gamma \cup E$ with $\neg(\neg TL \vee \neg TR) \equiv TL \wedge TR$:

$$(ML \wedge BR) \rightarrow (DL \vee DR), ML, BR \vdash (DL \vee DR) \tag{11}$$
$$(DL \vee DR), DL \rightarrow \neg TL \vdash (\neg TL \vee DR) \tag{12}$$
$$(DL \vee DR), DR \rightarrow \neg TR \vdash (\neg TR \vee DL) \tag{13}$$
$$TL, DL \rightarrow \neg TL \vdash \neg DL \tag{14}$$
$$TR, (\neg TR \vee DL) \vdash DL \tag{15}$$
$$\neg DL, DL \vdash \Box \tag{16}$$

So, $(\neg TL \vee \neg TR)$ is true in all the models of $\Gamma \cup E$

# Knowledge Based Systems with CP0 - Models

Models of $\Gamma \cup$ E:



# Knowledge Based Systems with CP0

Suppose the Rover wants to check what would happen now if he would like to move forward.

So, he would set the variable AV to true

In this case, he would derive from $\Gamma \cup \mathrm{E} \cup \{\mathrm{AV}\}$ that

$$(\mathrm{TL} \vee \mathrm{TR})$$

is a logical consequence. Exercise: Prove it !

$$\Rightarrow$$

The Rover would discover that he would have to turn left or right if he wants to continue moving forward

# Knowledge Based Systems with CP0

Suppose more information is added to the evidence:

$$\mathrm{E}' = \mathrm{E} \cup \{\mathrm{DR}\}$$

How does this change the previous consequences ?

As $\mathrm{E}'$ does not change the truth value of sentences on the previous evidence, assuming again that $\Gamma \cup \mathrm{E}'$ is satisfiable, all the previous consequences will still be true

With this augmented evidence and still assuming AV true, we have that:

$$\Gamma \cup \mathrm{E}' \cup \{\mathrm{AV}\} \models \neg \mathrm{TR} \wedge \mathrm{TL}$$

# The Barcenas World

The Barcenas World:

- An $n \times n$ grid world ($n$ rows, $n$ columns)
- The Barcenas, a terrible monster, lives at some unknown cell $(b_i, b_j)$ but different to position $(1, 1)$
- We must move an agent A trough the world, starting at position $(1, 1)$, to discover where Barcenas hides
- At any position $(i, j)$ we arrive, an smell sensor will give us two possible signals:
  1. it smells (like hell): that means that Barcenas is at current position $(i, j)$, or at some of the surrounding positions: $S(i, j) = \{(i, j-1), (i, j+1), (i+1, j), (i-1, j)\}$
  2. it does not smell: that means Barcenas is not at any of the positions $(i, j) \cup S(i, j)$

# The Barcenas World

Example: $3 \times 3$ world. Our agent: A, Barcenas: B

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| | B | |
| A | | |

Some possible sequences of positions:

1. $(1,1),(1,2),(1,3)$
2. $(1,1),(2,1),(3,1)$

Both of them allow to discover the position of Barcenas with 100% confidence. Why ?

How does the uncertainty about the location of Barcenas changes as we move on ?

# The Barcenas World

Let's analyze the sequence: $(1,1),(1,2),(1,3)$
Initial set of possible locations:

$$(b_i, b_j) \in B_p = \{(i,j) \mid \forall i, \forall j, 1 \leq i, \leq 3, 1 \leq j, \leq 3\} \setminus (1,1)$$

1. It does not smell at $(1,1)$, so

$$(b_i, b_j) \notin \{(2,1),(1,1),(1,2)\}$$

2. It smells at $(1,2)$, so

$$(b_i, b_j) \in \{(1,1),(1,2),(1,3),(2,2)\}$$

3. It does not smell at $(1,3)$, so

$$(b_i, b_j) \notin \{(1,2),(1,3),(2,3)\}$$

Summing up:

$$(b_i, b_j) \in \{(1,1),(1,2),(1,3),(2,2)\} \setminus \{(1,1),(1,2),(1,3)\} = \{(2,2)\}$$

# The Barcenas World

Example: $3 \times 3$ world. Our agent: A, Barcenas: B

| | | |
|---|---|---|
| | | B |
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| A | | |

Consider these two possible sequences of positions:

1. $(1,1),(1,2),(1,3)$
2. $(1,1),(2,1),(2,2)$

Exercise: Which one discovers more information ?

# The Barcenas World with CP0 - First Solution

We will write a set of sentences $\Gamma$ with variables:

1. $b_{i,j}$: meaning "Barcenas is at $(i, j)$"
2. $s_{i,j}$: meaning "It smells at $(i, j)$"

such that every model of $\Gamma$ encodes a possible position for Barcenas combined with its consistent reading from the smell sensor at all positions:

Some models for the $3 \times 3$ case:

1. $b_{1,3} = 1$, $b_{i,j} = 0$ for the other positions, and $s_{1,1} = 0, s_{1,2} = 1, s_{1,3} = 1, s_{2,1} = 0, s_{2,2} = 0, s_{2,3} = 1, s_{3,1} = 0, s_{3,2} = 0, s_{3,3} = 0$
2. $b_{2,2} = 1$, $b_{i,j} = 0$ for the other positions, and $s_{1,1} = 0, s_{1,2} = 1, s_{1,3} = 0, s_{2,1} = 1, s_{2,2} = 1, s_{2,3} = 1, s_{3,1} = 0, s_{3,2} = 1, s_{3,3} = 0$

...

# The Barcenas World with CP0 - First Solution

When we know the reading from the smell sensor at some locations, this evidence E together with $\Gamma$ will allow us to automatically discover positions for B consistent with E

That is, the models for

$$\Gamma \cup E$$

will contain the assignments that are consistent with all the sensor readings at E

So, the models for $\Gamma \cup E$ will contain all the possible locations for Barcenas that are allowed by our evidence E

# The Barcenas World with CP0 - First Solution

## The sentences of $\Gamma$:

1. A sentence for ensuring that Barcenas is at some location, except at $(1, 1)$:

$$(b_{1,2} \vee b_{1,3} \vee b_{2,1} \vee \ldots \vee b_{n,n})$$

2. For every position $(i, j)$ two different sets of sentences:
   1. A sentence for ensuring that if it smells at $(i, j)$, then barcenas cannot be at positions $(i', j')$ outside the scope of the smell sensor:

   $$\forall (i', j') \notin \text{scope}(i, j) \;\; (s_{i,j} \rightarrow \neg b_{i',j'})$$

   2. A sentence for ensuring that if it does not smell at $(i, j)$, then barcenas cannot be at positions within the scope of the smell sensor:

   $$\forall (i', j') \in \text{scope}(i, j) \;\; (\neg s_{i,j} \rightarrow \neg b_{i',j'})$$

3. A sentence for ensuring that Barcenas is never at position $(1, 1)$:

$$(\neg b_{1,1})$$

# The Barcenas World with CP0 - First Solution

Some remarks about this CP0 formula encoding for the Barcenas world:

- The set $\mathrm{scope}(i,j)$ contains the possible locations $(i',j')$ where Barcenas must be if it smells at $(i,j)$
- This formula will allow the agent to reason about the set of possible locations of Barcenas given a set of readings from its sensor obtained at different locations.

# The Barcenas World with CP0 - First Solution

Once our agent obtains the readings from some sensors, collecting some evidence $E = E_p \wedge E_n$ of this form:

- $E_p$: A set of positions ps where it smells, so their variables must be set to true:

$$E_p = \bigwedge_{(i,j) \in ps} s_{i,j}$$

- $E_n$: A set of positions ns where it does not smell, so their variables must be set to false:

$$E_n = \bigwedge_{(i,j) \in ns} \neg s_{i,j}$$

We assume that the sensor works well, so we never obtain a sequence of readings that is not consistent with the true position of Barcenas

# The Barcenas World with CP0 - First Solution

What questions can the agent try to answer using $\Gamma$ and E ?

1. Given the evidence E we have from the sensor readings, is a given position $(i, j)$ absolutely discarded as a possible location for Barcenas ? That is:

$$\Gamma \cup E \models \neg b_{i,j} \quad ?$$

2. Given a particular set of positions p, is Barcenas located in one of these positions ? That is:

$$\Gamma \cup E \models \bigvee_{(i,j) \in p} b_{i,j} \quad ?$$

Remark: Once we know the answer for the first question for all the locations $(i, j)$, we know the whole subset of possible positions for Barcenas, given E

So, asking the first question for all the positions makes unnecessary to ask the second class of questions

# The Barcenas World with CP0 - First Solution

Problem: Suppose the agent wants to know if Barcenas could be located at a particular position $(i, j)$, given E

How can he answer this question with $\Gamma$ and E ?

Using the same first question:

$$\Gamma \cup E \models \neg b_{i,j} \quad ?$$

If $\neg b_{i,j}$ is not a logical consequence, that means that $b_{i,j}$ is true at some model of $\Gamma \cup E$, so Barcenas could be located at $(i, j)$

# The Barcenas World with CP0 - Second Solution

With the solution we have presented for modeling the Barcenas World, we have variables that represent the sensor readings obtained in all the previous steps of the agent

So, every time we get a new reading $E_t$, we expand the current evidence $E_{t-1}$, to obtain a new one $E = E_{t-1} \cup E_t$.

To speed up the new questions, we could store all the previous logical consequences obtained with $E_{t-1}$, because assuming the sensor works well, $E_{t-1} \cup E_t$ will give, at least, the same previous logical consequences (and possibly some others)

# The Barcenas World with CP0 - Second Solution

A second solution for modeling the Barcenas World. Use three sets of variables:

1. $V_{t-1}$: Variables $b_{i,j}^{t-1}$ for modeling the previous state, just before we get a new sensor reading (time $t-1$)
2. $V_t$: Variables $s_{i,j}^{t}$ for modeling the new readings we obtain at the current time (time $t$)
3. $V_{t+1}$: Variables $b_{i,j}^{t+1}$ for modeling the new state resulting after we use the information from the new readings (time $t+1$)

The time steps do not need to be associated with predefined real times, they just indicate an ordering of two states (previous, next) around the event associated with new information arriving to the system

# The Barcenas World with CP0 - Second Solution

Our knowledge formula $\Gamma$ will contain the same sentences as before, but:

1. Sentences that talk about properties that must hold always, independently of sensor readings, will be written two times: once with the variables $V_{t-1}$ and once with the variables $V_{t+1}$
2. Sentences that talk about properties that become true when certain evidence is obtained at time step t, will be written with variables from $V_t \cup V_{t+1}$
3. A new set of sentences is needed: sentences that indicate that once a certain property holds true at current time step $t-1$, it must hold true from now on.

# The Barcenas World with CP0 - Second Solution

So, $\Gamma$ contains:

1. $$(b^{t-1}_{1,2} \vee b^{t-1}_{1,3} \vee b^{t-1}_{2,1} \vee \ldots \vee b^{t-1}_{n,n}) \wedge (b^{t+1}_{1,2} \vee b^{t+1}_{1,3} \vee b^{t+1}_{2,1} \vee \ldots \vee b^{t+1}_{n,n})$$
2. For every position $(i, j)$ two different sets of sentences:
    1. $$\forall (i', j') \notin \mathrm{scope}(i, j) \;\; (s^t_{i,j} \rightarrow \neg b^{t+1}_{i',j'})$$
    2. $$\forall (i', j') \in \mathrm{scope}(i, j) \;\; (\neg s^t_{i,j} \rightarrow \neg b^{t+1}_{i',j'})$$
3. $$(\neg b^{t-1}_{1,1}) \wedge (\neg b^{t+1}_{1,1})$$
4. $$\forall (i, j)(\neg b^{t-1}_{i,j} \rightarrow \neg b^{t+1}_{i,j})$$

# The Barcenas World with CP0 - Second Solution

How many variables does Γ contain?

$$|\text{Variables}| = n^2 + n^2 + n^2$$

How many clauses does Γ contain?

- 2 long clauses
- $O(n^2 \cdot n^2)$ clauses
- 2 unit clauses
- $O(n^2)$ clauses

So:

$$|\text{Clauses}| = O(4 + n^2 + n^4) = O(n^4)$$

In case you do not know what is the big-O notation:
https://en.wikipedia.org/wiki/Big_O_notation

# The Barcenas World with CP0 - Second Solution

## Life-cycle of the Agent:

At every time step t, starting from previous knowledge $\Gamma_{t-1}$:

1. Collect all the information from outside sources (e.g. sensors) to obtain new evidence $E_t$.
2. Infer all the relevant logical consequences with variables in $V_{t+1}$ obtained from $\Gamma_{t-1} \cup E_t$:

$$\Gamma_{t-1} \cup E_t \models \neg b_{i,j}^{t+1}$$

3. Replace all the logical consequences obtained on the previous step as sentences over their corresponding variables over $V_{t-1}$, and extend $\Gamma_{t-1}$

The third step simulates the passing of time: time step $t+1$ becomes the time step $t-1$ for the next iteration

# The Barcenas World with CP0 - Second Solution

Suppose the following sequence of steps in the $3 \times 3$ world:

$$[(1,1),(1,2),(2,2),(2,3)]$$

and that Barcenas is at position $(3,3)$
Then, next we show the evolution of the knowledge formula $\Gamma$ of the agent, considering the reading from the smell sensor obtained in each position. The consequences that are new, the ones that hold at time $t+1$ but not at time $t-1$, are shown in blue.

# The Barcenas World with CP0 - Second Solution

$[(1,1),(1,2),(2,2),(2,3)]$

Time 1:

- Starting formula:

$$\Gamma_0 = \Gamma$$

- New consequences:

$$\Gamma_0 \cup \{\neg s^{t}_{1,1}\} \models \{\neg b^{t+1}_{1,1}, \neg b^{t+1}_{1,2}, \neg b^{t+1}_{2,1}\}$$

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# The Barcenas World with CP0 - Second Solution

$[(1,1), (1,2), (2,2), (2,3)]$

Time 2:

- Update formula with previous consequences:

$$\Gamma_1 = \Gamma_0 \cup \{\neg b_{1,1}^{t-1}, \neg b_{1,2}^{t-1}, \neg b_{2,1}^{t-1}\}$$

- New consequences:

$$\Gamma_1 \cup \{\neg s_{1,2}^{t}\} \models \{\neg b_{1,1}^{t+1}, \neg b_{1,2}^{t+1}, \neg b_{2,1}^{t+1}, \neg b_{2,2}^{t+1}, \neg b_{1,3}^{t+1}\}$$

| | | |
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| | | B |
| | | |
| | A | |

# The Barcenas World with CP0 - Second Solution

$[(1,1),(1,2),(2,2),(2,3)]$

Time 3:

- Update formula with previous consequences:

$$\Gamma_2 = \Gamma_1 \cup \{\neg b^{t-1}_{2,2}, \neg b^{t-1}_{1,3}\}$$

- New consequences:

$$\Gamma_2 \cup \{\neg s^{t}_{2,2}\} \models \{\neg b^{t+1}_{1,1}, \neg b^{t+1}_{1,2}, \neg b^{t+1}_{2,1}, \neg b^{t+1}_{2,2}, \neg b^{t+1}_{1,3}, \neg b^{t+1}_{2,3}, \neg b^{t+1}_{3,2}\}$$

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# The Barcenas World with CP0 - Second Solution

$[(1,1),(1,2),(2,2),(2,3)]$

Time 4:

- Update formula with previous consequences:

$$\Gamma_3 = \Gamma_2 \cup \{\neg b^{t-1}_{2,3}, \neg b^{t-1}_{3,2}\}$$

- New consequences:

$$\Gamma_3 \cup \{s^t_{2,3}\} \models \{\neg b^{t+1}_{1,1}, \neg b^{t+1}_{1,2}, \neg b^{t+1}_{2,1}, \neg b^{t+1}_{2,2}, \neg b^{t+1}_{1,3}, \neg b^{t+1}_{2,3}, \neg b^{t+1}_{3,2}, \neg b^{t+1}_{3,1}\}$$

So, at time 4, we can discover that $(3,3)$ is the unique possible location for Barcenas

| | | |
|---|---|---|
| | | B |
| | | A |
| | | |