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Quadratic Model Predictive Control Including Input Cardinality Constraints

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***Abstract*—This note addresses the problem of feedback control with a constrained number of active inputs. This problem is known as sparse control. Specifically, we describe a novel quadratic model predictive control strategy that guarantees sparsity by bounding directly the ℓ_0-norm of the control input vector at each control horizon instant. Besides this sparsity constraint, bounded constraints are also imposed on both control input and system state. Under this scenario, we provide sufficient conditions for guaranteeing practical stability of the closed-loop. We transform the combinatorial optimization problem into an equivalent optimization problem that does not consider relaxation in the cardinality constraints. The equivalent optimization problem can be solved utilizing standard non-linear programming toolboxes that provides the input control sequence corresponding to the global optimum.**

***Index Terms*—Model predictive control, constrained control, sparse control, ℓ_0 optimization, practical stability**

I. Introduction

Classical control theory considers the full control action vector to govern a process [1]. However, in the latest years the control community has been attracted to study the so-called sparse control [2], where the goal is to control a process employing a reduced number of inputs, see e.g. [3], [4]. Decreasing the number of active control inputs can benefit the operation of control systems. For instance, sparse control has been proposed in [5] to alleviate the traffic information when dealing with limited communication resources. This can also be useful to reduce the power budget when transmitting through self-powered devices [6]. On the other hand, sparse control was utilized in [7] to minimize propellant ejection and to accommodate the minimal impulse constraint in the spacecraft rendezvous problem.

Promoting sparsity (having a zero value on most of the elements of the decision variable) has also called the attention in other research fields with an increasing number of interesting applications in system identification [8], [9], state estimation [10], compressive sampling [11], [12], power grids [13], and over-actuated control systems [14] among others.

The inherent characterization of sparsity is through the ℓ_0-norm (number of non-zero elements of a vector), which represents the cardinality of a vector. However, explicitly including ℓ_0-norm constraints in the control decision problem leads to an NP-hard combinatorial problem [15]. Mainly, three approaches have been proposed in optimal control problems to avoid the computational burden: i) a greedy algorithm known as Orthogonal Matching Pursuit (OMP) [5], ii) a ℓ_1-norm relaxation [3], [16] and more recently iii) an algorithm based on coordinate descent type methods [17].

OMP algorithms [18] rely on computing suboptimal solutions satisfying ℓ_0 constraints. Even if it is computationally inexpensive, adding further constraints into the optimization problem (as states and control inputs belonging to convex sets) is not a simple question. Approaches based on a ℓ_1-norm relaxation do offer enough flexibility to introduce these kind of constraints. Despite the fact that in [8] the authors proposed an approach to choose the regularization parameter of a modified ℓ_1-norm regularization algorithm, the ℓ_1-norm has no clear meaning in most applications (as it just represents the sum of the absolute values). On the other hand, coordinate descent type methods [19], where one decision variable is updated at each iteration using some selection rule, handle the ℓ_0-norm but provide only local minima [17].

Works such as [4], [7], [16] and [5] have introduced sparsity constraints on the control inputs when dealing with model predictive control (MPC). While in [4], [7], [16] the authors also included extra convex constraints in the optimization problem, in [5] this issue is not clearly established. Still, neither of them consider ℓ_0-norm restrictions to limit the number of active control actions at each control horizon instant.

In the current work, we develop a receding horizon technique for quadratic MPC controllers with explicit ℓ_0-constraints on each control horizon instant. The contribution of the current work is twofold: i) we establish sufficient conditions to guarantee asymptotic and practical stability of the closed loop system considering that the input sequence satisfies a combination of a non-convex (but closed) and a cardinality constraints, and ii) we re-write the corresponding optimization problem into an equivalent form that, in the simulation study in section VI performs better than alternative formulations available in the existent literature [20]. Also, as another novelty, we address the chattering phenomenon

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(infinite frequency of switching of active control inputs [21]) when the system is close to the origin by using a dual-MPC strategy. This work extends our recent paper [22] by: i) including extra bounded constraints (on the states and control actions) in the optimization problem, ii) guaranteeing practical stability instead of exponential stability, which is more difficult to achieve (due to the additional constraints), iii) considering the chattering problem and iv) improving the ℓ_0 optimization algorithm to obtain a global optimum instead of a local one.

The remainder of this note is organized as follows: Section II introduces some preliminary definitions on practical stability. In Section III we describe the MPC problem with ℓ_0 constraints. The ℓ_0 optimization algorithm is discussed in Section IV and in Section V we address the stability issues. Numerical studies are included in Section VI and Section VII draws conclusions.

Notation and Basic Definitions: Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and non-negative real number sets. The difference between two given sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$ is denoted by $\mathcal{A}\backslash\mathcal{B} = \{x \in \mathbb{R}^n : x \in \mathcal{A}, x \notin \mathcal{B}\}$. We represent the transpose of a given matrix A and a vector x via $(Ax)' = x'A'$. The Euclidean norm is denoted via $|\cdot|$ while the weighted Euclidean norm (squared) is denoted by $|x|_P^2 = x'Px$. Additionally, the induced norm of a given matrix A is its largest singular value. The maximum and minimum eigenvalues of a given matrix A are represented via $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ respectively. $\mathcal{I}$ denotes an identity matrix of appropriate dimension. $\vec{0}_m$, and $\vec{1}_m$ denote vectors with only zero or one entries respectively. The operator $\text{diag}_m(x)$ transforms $x \in \mathbb{R}^n$ into a diagonal matrix $A \in \mathbb{R}^{n\times n}$.

Definition 1. *A function σ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}$-function if it is continuous, strictly increasing and $\sigma(0) = 0$; σ is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$-function and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$); a function β: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if it is continuous and if, for each $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each $s \geq 0$ the function $\beta(s, \cdot)$ is non-increasing and satisfies $\beta(s, t) \to 0$ as $t \to \infty$.*

II. Preliminaries: Practical Stability

In this section, main aspects on practical stability for discrete-time systems are given. These concepts are based on the regional input-to-state practical stability framework presented in [23], [24]. The term *regional* is related to the fact that stability properties hold only in a specific region, which is often the case when system constraints are present, see [25]. The term *practical* is used to emphasize that, in some cases, only stability of a neighborhood of the origin can be guaranteed, see e.g., [26].

Consider a discrete-time system described by:

$$x_{k+1} = f(x_k), \quad f(0) = 0, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state and $f(\cdot)$ is not necessarily continuous.

Definition 1 (Positively Invariant Set)**.** A set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be a *positively invariant* (PI) set for the system (1) if $f(x) \in \mathcal{A}$, for all $x \in \mathcal{A}$.

Definition 2 (UpAS)**.** The system (1) is said to be *Uniformly practically Asymptotically Stable* (UpAS) in $\mathcal{A} \subseteq \mathbb{R}^n$ if $\mathcal{A}$ is a PI set for (1) and if there exist a $\mathcal{KL}$-function β, and a nonnegative constant $\delta \geq 0$ such that

$$|x_k| \leq \beta(|x_0|, k) + \delta, \quad \forall x_0 \in \mathcal{A}, k \in \mathbb{N}.$$

Particularly, if $\delta = 0$ then, system (1) is said to be UAS. If $\mathcal{A} \triangleq \mathbb{R}^n$ then, system (1) is said to be *globally* UpAS.

Definition 3 (Practical-Lyapunov function)**.** A (not necessarily continuous) function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a *practical-Lyapunov function* in $\mathcal{A}$ for the system (1) if $\mathcal{A}$ is a PI set and if there exist a compact set, $\Omega \subseteq \mathcal{A}$, neighborhood of the origin, $x = 0$, some $\mathcal{K}_\infty$-functions α_1, α_2, and α_3, and a constants $d \geq 0$, such that

$$V(|x|) \geq \alpha_1(|x|), \qquad \forall x \in \mathcal{A}, \tag{2}$$

$$V(|x|) \leq \alpha_2(|x|), \qquad \forall x \in \Omega, \tag{3}$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|) + d, \quad \forall x \in \mathcal{A}. \tag{4}$$

If $\mathcal{A} \triangleq \mathbb{R}^n$ then, the function, V, is said to be a *global* practical-Lyapunov function.

Theorem 1 ([24])**.** *If* (1) *admits a practical-Lyapunov function in $\mathcal{A}$, then it is UpAS in $\mathcal{A}$.*

III. Problem Description

Consider the following discrete-time linear time-invariant system:

$$x_{k+1} = Ax_k + Bu_k, \tag{5}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state, $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input vector. Here, both $\mathbb{X}$ and $\mathbb{U}$ are assumed to be compact sets which contain the origin in their interior. Moreover, convexity is only assumed for $\mathbb{X}$. The pair (A, B) is assumed to be stabilizable where the matrix A is not necessarily Schur stable. In this case, we seek to control system (5), if possible, by using an MPC with a reduced number of active inputs γ, i.e., $\gamma \in \{0, \ldots, m\}$. To this end, one needs to design a controller which can provide the best possible actuation considering only γ active inputs while the remaining $m - \gamma$ inactive inputs will take a null value. For this problem, we denote by $\sigma \in \mathbb{R}^m$ the binary vector which indicates the active and inactive inputs, i.e., the $i-th$ component of σ_k is given by:

$$e_i'\sigma_k = \begin{cases} 1, & \text{if } e_i'u_k \text{ is active,} \\ 0, & \text{otherwise } (e_i'u_k = 0), \end{cases}$$

for all $i \in \{1, \ldots, m\}$, where e_i is the i-th column of the identity matrix. Thus, the number of non-zero elements of vector σ_k (ℓ_0-norm) is $|\sigma_k|_0 = \gamma$. To formulate the MPC optimal problem, we first consider the following cost function

$$V_N(x, \vec{u}) = \sum_{j=0}^{N-1} \ell(\hat{x}_j, \hat{u}_j) + V_f(\hat{x}_N), \tag{6}$$

where N is the prediction horizon, and $\ell(\hat{x}, \hat{u}) = |\hat{x}|_Q^2 + |\hat{u}|_R^2$ is the stage cost with Q and R positive definite matrices, while the term $V_f(\hat{x}) = |\hat{x}|_P^2$, in which P is positive definite,

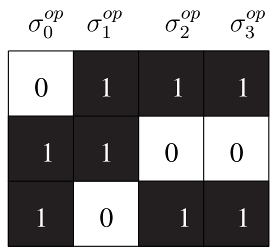

Fig. 1. Illustration of an optimal active input sequence, $\vec{\sigma}^{op}$, for a prediction horizon $N = 4$.

represents the terminal cost. The vector $\vec{u}$ in (6) contains the tentative control actions over the prediction horizon, i.e.,

$$\vec{u} = \left[\hat{u}_0', \ldots, \hat{u}_{N-1}'\right]' \in \mathbb{R}^{Nm}.$$

The MPC optimization of interest for the current state, $x_k = x$, is given as

$$\mathbb{P}_N(x): \quad V_N^{op}(x) = \min_{\vec{u}}\{V_N(x, \vec{u})\}, \tag{7}$$

subject to:

$$\begin{aligned} \hat{x}_{j+1} &= A\hat{x}_j + B\hat{u}_j, && (8)\\ \hat{u}_j &\in \mathbb{U}, && (9)\\ \hat{x}_j &\in \mathbb{X}, && (10)\\ |\hat{u}_j|_0 &\leq \gamma, && (11)\\ \hat{x}_N &\in \mathbb{X}_f \subseteq \mathbb{X}, && (12) \end{aligned}$$

for all $j \in \{0, \ldots, N-1\}$, where $\hat{x}_0 = x_k$ and $\gamma \leq m$.

Here, (9) and (10) take into account the system bounded constraints, where $\mathbb{U}$ is not necessarily convex. Constraint (11) encompasses the number of active inputs (sparse) constraint over the prediction horizon. Constraint (12) is the, so-called, terminal constraint. Similarly to convex MPC formulations, the terminal region $\mathbb{X}_f$ and matrix P can be designed to guarantee stability of the resulting closed-loop [27]. Their design will be considered in the stability analysis presented in Section V. We define the set $\mathcal{U}(x)$ to represent all the input sequences sequences, $\vec{u}$, which satisfy constraints (9)-(12).

Consequently, the optimal input sequence, $\vec{u}^{op}(x)$, is the one which minimizes the cost function, i.e.,

$$\vec{u}^{op}(x) \triangleq \arg\left\{\min_{\vec{u}\in\mathcal{U}(x)} V_N(x, \vec{u})\right\}.$$

Thus, the resulting optimal solution is the, so-called, optimal input control sequence

$$\vec{u}^{op}(x) = \left[(\hat{u}_0^{op})', \ldots, (\hat{u}_{N-1}^{op})'\right]', \tag{13}$$

while the resulting optimal state sequence is:

$$\vec{x}^{op}(x) = [x', (\hat{x}_1^{op})', \ldots, (\hat{x}_N^{op})']'.$$

Additionally, for this particular problem, we also obtain the resulting optimal active input sequence, given by:

$$\vec{\sigma}^{op}(x) = \left[(\sigma_0^{op})', \ldots, (\sigma_{N-1}^{op})'\right]'.$$

Notice that the elements of $\vec{\sigma}^{op}(x)$ may differ from each other. However, $|\sigma_j^{op}| \leq \gamma$ for all $j \in \{0, \ldots, N-1\}$. For example, if $N = 4$, $m = 3$ and $\gamma = 2$ a possible $\vec{\sigma}^{op}(x)$ is shown in Fig. 1. We also denote the domain of attraction of the cost function, $V_N(x)$, via

$$\mathrm{X}_N \triangleq \{x \in \mathbb{X} : \mathcal{U}(x) \neq \emptyset\}. \tag{14}$$

Therefore, X_N contains all $x \in \mathbb{X}$ such that there exists a control sequence $\vec{u} \in \mathcal{U}(x)$ satisfying conditions (9)-(12).

Finally, we use a *receding horizon* policy, i.e., only the first element of $\vec{u}^{op}(x)$ is applied to the system at each sampling instant (see, e.g. [27]). The solution of the optimal problem, $\mathbb{P}_N(x)$ in (7), yields the sparse MPC law, $\kappa_N(\cdot): \mathrm{X}_N \to \mathbb{U}$,

$$\kappa_N(x) \triangleq \hat{u}_0^{op}. \tag{15}$$

Consequently, the resulting sparse MPC loop can be represented via

$$x_{k+1} = Ax_k + B\kappa_N(x_k). \tag{16}$$

In the following section, we present a general method to solve an optimization problem subject to ℓ_0-norm constraints. This solution is then used to solve the sparse quadratic MPC problem in (7)-(12).

IV. ℓ_0-CONSTRAINED BASED SOLUTION

Consider the following ℓ_0-constrained optimization problem

$$\mathbb{P}_0: \quad \min_{x\in\mathbb{R}^p} f(x), \tag{17}$$

subject to:

$$\begin{aligned} x &\in \Omega,\\ |x|_0 &\leq \gamma. \end{aligned}$$

A way of handling cardinality constraints is through the following mixed-integer programming formulation [20]:

$$\mathbb{P}_{0,MIP}: \quad \min_{x\in\mathbb{R}^p, z\in\{0,1\}^p} f(x), \tag{18}$$

subject to:

$$\begin{aligned} &x \in \Omega,\\ -Le_i'z \leq e_i'x &\leq Le_i'z, \quad \forall i \in \{1, \ldots, p\}, && (19)\\ \vec{1}_m'z &= \gamma, && (20) \end{aligned}$$

where each entry of vector z is set to be binary, and e_i is the i-th column of the identity matrix. By means of constraint (19), a semi-continuous behavior is induced on variable x_i.

To solve the problem $\mathbb{P}_{0,MIP}$, standard Mixed-Integer Quadratic Programming (MIQP) solvers such as CPLEX or BARON [28] can be used. However, in this work the input is restricted to belong to a compact set that may be non-convex. Therefore, it cannot be handled by CPLEX [20].

Also, consider the following optimization problem involving bilinear constraints

$$\mathbb{P}_{0equiv}: \quad \min_{x,w\in\mathbb{R}^p} f(x), \tag{21}$$

subject to:

$$\begin{aligned} &x \in \Omega,\\ (e_i'x)(e_i'w) &= 0, \quad \forall i \in \{1, \ldots, p\}, && (22)\\ \vec{0}_p \leq w &\leq \vec{1}_p, && (23)\\ \vec{1}_m'w &= p - \gamma, && (24) \end{aligned}$$

where $\Omega \subset \mathbb{R}^p$ is a constraining set, $f : \mathbb{R}^p \to \mathbb{R}$ is the objective function, and e_i is the i-th column of the identity matrix. The following result, that was independently obtained in [29]–[32], shows that problems $\mathbb{P}_0$ in (17) and $\mathbb{P}_{0equiv}$ in (21)-(24) are equivalent.

Theorem 2 ([29]–[32])**.** *A vector $x^* \in \mathbb{R}^p$ is a global solution of $\mathbb{P}_0$ if and only if there exists a vector $w^* \in \mathbb{R}^p$ such that the pair (x^*, w^*) is a global solution of $\mathbb{P}_{0equiv}$.*

Results in [29] and [30] are similar. However, [30] have been obtained in a more general framework where constraints on the rank of a matrix are utilized. A key observation is that $\mathbb{P}_{0equiv}$ can be solved by using standard tools of nonlinear programming. In particular, we obtain a global solution of $\mathbb{P}_{0equiv}$ by using the optimization software BARON [28].

In problem $\mathbb{P}_{0equiv}$, the auxiliary variable w in (22)-(24), at the optimum is a binary variable taking the value $e_i'w = 1$ for those elements corresponding to $e_i'x = 0$. Additional constraints over w can be included in the optimization problem to manage how the zero and non-zero elements of x interact. These interactions are difficult to handle by relaxation methods such as the ℓ_1-norm heuristic. Moreover, our approach obtains a solution in less time than the corresponding binary non-linear programming (i.e., $e_i'w \in \{0, 1\}$) for the simulation study in Section VI.

Remark 1. The proposed approach can easily handle ℓ_0-norm constraints over a selection in the vector, i.e. $|\text{diag}_m(a_i)\vec{u}|_0 \leq \gamma$, where a_i is a given vector with entries $\{0, 1\}$. We use this approach latter in the note to solve problem (7)-(12), where ℓ_0-norm constraints are imposed on several selections of vector $\vec{u}$. In addition, we can also minimize the ℓ_0-norm of the whole optimal input vector, i.e., $|\vec{u}|_0 \leq \gamma$.

Therefore, a comparison between both approaches is done using the optimization software BARON.

A. Application to Sparse Quadratic MPC

The quadratic MPC with ℓ_0-input constraint described by (7)-(12) can be equivalently formulated as the following optimization problem

$$\mathbb{P}_{equiv,N}(x): \quad V^{op}_{equiv,N}(x) = \min_{\vec{u},\vec{w}} \{V_N(x, \vec{u})\}, \tag{25}$$

subject to:
$$\begin{aligned} \hat{x}_{j+1} &= A\hat{x}_j + \hat{u}_j, & (26)\\ \hat{u}_j &\in \mathbb{U}, & (27)\\ \hat{x}_j &\in \mathbb{X}, & (28)\\ \text{diag}_m(w_j)u_j &= \vec{0}_m, & (29)\\ \vec{0}_m &\leq w_j \leq \vec{1}_m, & (30)\\ w_j'\vec{1}_m &= m - \gamma, & (31)\\ \hat{x}_N &\in \mathbb{X}_f \subseteq \mathbb{X}, & (32) \end{aligned}$$

for all $j \in \{0, \dots, N-1\}$, where $\hat{x}_0 = x_k$ and $\gamma \leq m$. Note that in this case the set $\mathcal{U}(x)$ represents all the input sequences, $\vec{u}$, that satisfy constraints (27)-(32).

The ℓ_0-norm constraint (11), in problem $\mathbb{P}_N(x)$ in (7), is substituted by (29)-(31) in problem $\mathbb{P}_{equiv,N}(x)$ as per (25). This substitution allows us to obtain a global solution of $\mathbb{P}_N(x)$ by using standard tools in nonlinear programming over $\mathbb{P}_{equiv,N}(x)$. Note that the equivalence between $\mathbb{P}_N(x)$ and $\mathbb{P}_{equiv,N}(x)$ holds in the global optimum (see [31], [32]).

V. Stability Analysis

In this section, sufficient conditions to guarantee stability of the sparse quadratic MPC loop in (16) are established.

Firstly, we define the predicted state sequence as

$$\vec{x}_{[1:N]} = [\hat{x}_1', \dots, \hat{x}_N']'.$$

Considering an initial system state $\hat{x}_0 = x$, from (8), we obtain

$$\vec{x}_{[1:N]} = \Lambda x + \Phi\vec{u},$$

where

$$\Lambda \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Phi \triangleq \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}.$$

Thus, the cost function (6) can be re-written as

$$V_N(x, \vec{u}) = \nu(x) + \vec{u}'H_N\vec{u} + 2\vec{u}'F_Nx,$$

where the term $\nu(x)$ is independent of $\vec{u}$ and $H_N \triangleq \Phi'Q_N\Phi + R_N \in \mathbb{R}^{Nm\times Nm}$, $F_N \triangleq \Phi'Q_N\Lambda \in \mathbb{R}^{Nm\times n}$, with $Q_N \triangleq \text{diag}\{Q, \dots, Q, P\} \in \mathbb{R}^{Nn\times Nn}$, $R_N \triangleq \text{diag}\{R, \dots, R\} \in \mathbb{R}^{Nm\times Nm}$. Notice that, since Q, R, and P are positive definite, so is H_N. Based on this representation, the following unconstrained optimal input, $\vec{u}^{op}_{uc}(x)$, can be defined, see [27].

Lemma 1 (Unconstrained Solution)**.** *If for the optimal problem $\mathbb{P}_N(x)$ in (7), constraints (9)-(12) are not taken into account, i.e., $\mathbb{U} \triangleq \mathbb{R}^m$, $\mathbb{X} = \mathbb{X}_f \triangleq \mathbb{R}^n$, and $\gamma = m$, then $V_N(x, \vec{u})$ is minimized when*

$$\vec{u}^{op}_{uc}(x) \triangleq \arg\left\{\min_{\vec{u}\in\mathbb{R}^{Nm}} V_N(x, \vec{u})\right\} \triangleq -H_N^{-1}F_Nx. \tag{33}$$

for all $x \in \mathbb{R}^n$.

A. Sparse Local Controller

We propose to prove stability of ℓ_0-input constrained MPC loop in (16) by examining properties of a feasible local controller based on the optimal nominal solution presented in (33) with prediction horizon $N = 1$; cf. [26]. To take into account the ℓ_o-input constraint, for a given $\gamma = |\sigma_f|$, we consider the following sparse matrix

$$L_\sigma = \text{diag}_m\{\sigma_f\} \in \mathbb{R}^{m\times m}. \tag{34}$$

Thus, the proposed feasible sparse local controller is given by

$$\kappa_f(x) = K_\sigma x = (K + \Delta_\sigma)x, \quad \Delta_\sigma = (L_\sigma - \mathcal{I})K, \tag{35}$$

where

$$K = -H_1^{-1}F_1 = (B'PB + R)^{-1}B'PA.$$

Thus, based on the nominal sparse local controller, we chose the terminal region in (12) as:

$$\mathbb{X}_f \triangleq \{x'Px \leq \varphi_x : x \in \mathbb{X}, K_\sigma x \in \mathbb{U}\}. \tag{36}$$

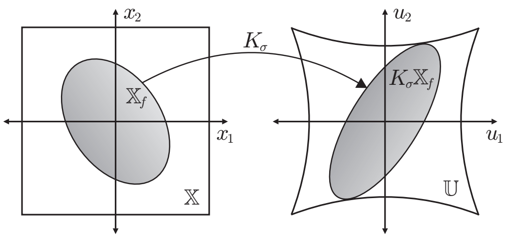

Fig. 2. Illustration of the terminal set $\mathbb{X}_f \subset \mathbb{X} \subset \mathbb{R}^2$ for the case when $\mathbb{U} \subset \mathbb{R}^2$ is non-convex.

Here, $\varphi_x \in \mathbb{R}_{\geq 0}$ is designed to obtain the largest ellipsoid which guarantees that for all $x \in \mathbb{X}, K_\sigma x \in \mathbb{U}$. Notice that since the origin belongs to $\mathbb{X}$ and $\mathbb{U}$ then, $\mathbb{X}_f \neq \emptyset$. Therefore, both the proposed local controller and terminal region provide that for all $x \in \mathbb{X}_f$, $\kappa_f(x) \in \mathbb{U}$. It is important to emphasize that in this work the compact set $\mathbb{U}$ is not restricted to be convex. However, the local controller, $\kappa_f(x)$, maps the states in $\mathbb{X}_f$ to the convex set $K_\sigma \mathbb{X}_f$, which is contained by $\mathbb{U}$, i.e., $K_\sigma \mathbb{X}_f \subseteq \mathbb{U}$. This is illustrated in Fig. 2.

On the other hand, the closed-loop expression for system (5) governed by the local controller (35) is given by

$$x_{k+1} = A_{K_\sigma} x_k = (A_K + B\Delta_\sigma)x_k, \quad \forall x_k \in \mathbb{X}_f, \tag{37}$$

where $A_K = A+BK$, and the term $B\Delta_\sigma$ represents the sparse control effect on the "nominal system", $x_{k+1} = A_K x_k$.

Theorem 3. *Suppose that the terminal cost, $V_f(x)$, in* (6) *is designed such that the matrix P is chosen to be the solution to the algebraic Riccati equation*

$$A_K' P A_K - P + Q^* = 0, \quad Q^* \triangleq Q + K'RK. \tag{38}$$

If γ in (11) *is chosen such that there exist a σ_f in* (34) *which satisfies that $\gamma = |\sigma_f|$ and*

$$\begin{aligned} Q^* - \Psi_\sigma &\succ 0, \\ \Psi_\sigma &= (2A_K + B\Delta_\sigma)' P B \Delta_\sigma. \end{aligned} \tag{39}$$

Then, κ_f in (35) *is a uniformly exponentially stabilizing sparse local controller in $\mathbb{X}_f$ for the system* (5).

Proof. We first consider the terminal cost, $V_f(x)$ in (6), as a candidate Lyapunov function. Therefore, we apply Theorem 1 with $\alpha_1(s) = a_1 s^2$ and $\alpha_2(s) = a_2 s^2$, where $a_1 \triangleq \lambda_{\min}(P)$, $\quad a_2 \triangleq \lambda_{\max}(P)$. Direct calculations give that:

$$\begin{aligned} V_f(Ax + B\kappa_f(x)) - V_f(x) &= x'\big(A_{K_\sigma}' P A_{K_\sigma} - P\big)x \\ &= x'\big(A_K' P A_K - P + 2A_K' P B \Delta_\sigma + B'\Delta_\sigma' P \Delta_\sigma B\big)x. \end{aligned}$$

Since matrix P is chosen according to (38), it follows that

$$V_f(Ax+B\kappa_f(x)) - V_f(x) = -x'(Q^* - (2A_K + \Delta_\sigma)' P \Delta_\sigma)x.$$

Then, considering the proposed stabilizing condition (39), property (4) holds with $\alpha_3(s) = a_3 s^2$, where $a_3 = \lambda_{\min}(Q^* - \Psi) > 0$. Therefore, it follows that

$$\Delta V_f(x_k) = V_f(x_{k+1}) - V_f(x_k) \leq -a_3 |x_k|^2, \tag{40}$$

for all $x_k \in \mathbb{X}_f$. This allows us to establish the following relationship

$$V_f(x_{k+1}) \leq \rho V_f(x_k), \quad \forall x_k \in \mathbb{X}_f. \tag{41}$$

Taking into account inequality (40), it follows that

$$0 \leq V_f(x_{k+1}) \leq V_f(x_k) - a_3|x_k|^2 \leq (a_2 - a_3)|x_k|^2.$$

Hence, $0 < a_3 \leq a_2$, which implies that $\rho = 1 - \frac{a_3}{a_2} \in [0, 1)$.

Therefore, considering (36) and (41), we have that $\mathbb{X}_f$ is a PI set for (37). Moreover, for all $x \in \mathbb{X}_f$, $\kappa_f(x) \in \mathbb{U}$. By iterating (41), it is possible to exponentially bound the system state evolution via:

$$|x_k|^2 \leq \frac{a_2}{a_1} \rho^k |x_0|^2, \quad \forall k > 0, x_0 \in \mathbb{X}_f.$$

Thus, $\limsup_{k\to\infty} |x_k| = 0$, provided that $x_0 \in \mathbb{X}_f$.

Consequently, the proposed sparse local controller, $\kappa_f(x)$ in (35), is a stabilizing controller for (5) for all $x \in \mathbb{X}_f$. More precisely, the local sparse MPC loop (37) is uniformly exponentially stable. $\square$

Remark 2. Since $Q, R \succ 0$ and the pair (A, B) is stabilizable then, there exists a unique solution of the discrete algebraic Ricatti equation (38), i.e., $P > 0$. Moreover, A_K in (37) is Schur stable; see [33].

B. Multi-Step Sparse MPC Stability Analysis

Based on the proposed stabilizing local controller, $\kappa_f(x)$, we next establish sufficient conditions for practical stability for the ℓ_0-input constrained multi-step MPC loop in (16).

Theorem 4. *Consider the positive constants $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P + W_\sigma)$, and $c_3 = \lambda_{\min}(Q)$, where*

$$\begin{aligned} W_\sigma &= F' H_N^{-1}(\mathcal{L}_\sigma - \mathcal{I}) H_N (\mathcal{L}_\sigma - \mathcal{I}) H_N^{-1} F \in \mathbb{R}^{n\times n}, \\ \mathcal{L}_\sigma &= diag\{L_\sigma, \ldots, L_\sigma\} \in \mathbb{R}^{Nm\times Nm}. \end{aligned}$$

Suppose that $x_0 \in X_N$ and matrix P, in $V_f(x)$, is designed as per (38). *If the proposed sparse local controller, $\kappa_f(x)$, in* (35) *satisfies both* (39) *and*

$$|G_\sigma| < a_1 \left(\frac{c_3}{c_2}\right), \quad G_\sigma = \Delta_\sigma' H_1 \Delta_\sigma. \tag{42}$$

Then, the sparse MPC closed-loop system (16) *is UpAS for all $x \in X_N$, with*

$$\mathcal{D}_\delta \triangleq \left\{ x \in \mathbb{X}_f : x'Px \leq \delta = \frac{c_3 \varphi_x}{c_2 a_1} |G_\sigma| \right\} \tag{43}$$

as an ultimately invariant set.

Proof. To prove this theorem, we verify conditions presented in Definition 3. Since matrix P in (6) satisfies (38), the unconstrained solution, $\vec{u}_{uc}^{op}(x)$ in (33) can be expressed via:

$$\vec{u}_{uc}^{op}(x) = \begin{bmatrix} (K\hat{x})' & (K\hat{x}_1)' & \ldots & (K\hat{x}_{N-1})' \end{bmatrix}'.$$

Now, $V_N^{op}(x)$, with $x_k = x$, can be rewritten as:

$$\begin{aligned} V_N^{op}(x) &= V_N^{op}(x, \vec{u}^{op}(x)) \\ &= x'Px + \left(\vec{u}^{op}(x) - \vec{u}_{uc}^{op}(x)\right)' H_N \left(\vec{u}^{op}(x) - \vec{u}_{uc}^{op}(x)\right). \end{aligned} \tag{44}$$

Note that when constraints (9)-(12) are not considered, i.e., $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$, $\mathbb{X}_f = \mathbb{R}^n$, and $\gamma = m$, we have that

$\vec{u}^{op}(x) = \vec{u}^{op}_{uc}(x)$. Hence, $V^{op}_{uc}(x) = x'Px$, implying that

$$V^{op}_N(x) \geq c_1|x|^2, \quad \forall x \in \mathrm{X}_N,$$

Thus, property (2) holds with $\alpha_1(s) = c_1 s^2$. Then, we obtain an upper bound for the cost function for the case when $\gamma \leq m$. To do this, we use the following suboptimal solution[1] based on the proposed sparse local controller, $\kappa_f(x)$ in (35):

$$\tilde{u} = \mathcal{L}_\sigma \vec{u}^{op}_{uc}(x) = [(L_\sigma K\hat{x})' \quad (L_\sigma K\hat{x}_1)' \quad \ldots \quad (L_\sigma K\hat{x}_{N-1})']',$$

for all $x \in \mathbb{X}_f$. Thus, the optimal cost function satisfies that:

$$\begin{aligned} V^{op}_N(x) &\leq V_N(x, \tilde{u}) \\ &= x'Px + \vec{u}^{op}_{uc}(x)'(\mathcal{L}_\sigma - I)' H_N (\mathcal{L}_\sigma - I)\vec{u}^{op}_{uc}(x) \end{aligned}$$

Then, we obtain that

$$V^{op}_N(x) \leq c_2|x|^2, \quad \forall x \in \mathbb{X}_f. \tag{45}$$

Thus, property (3) holds with $a_2(s) = c_2 s^2$. Based on (13) and considering the proposed stabilizing local controller, $\kappa_f(x)$, we adopt and use the following shifted sequence (see [27]):

$$\tilde{u}(x_{k+1}) = [(\hat{u}^{op}_1)', \ldots, (\hat{u}^{op}_{N-1})', \kappa_f(\hat{x}_N)']'. \tag{46}$$

This generates the following state sequence:

$$\tilde{x}(x_{k+1}) = [(\hat{x}^{op}_1)', \ldots, (\hat{x}^{op}_N)', (\hat{x}_{N+1})']'.$$

Notice that by constraint (12), $\hat{x}_N \in \mathbb{X}_f$. Therefore, since $\kappa_f(x)$ satisfies (39), we have that $\hat{x}_{N+1} = A_{K\sigma}\hat{x}_N \in \mathbb{X}_f$.

By optimality, we obtain the bound

$$V^{op}_N(x_{k+1}) \leq V^{op}_N(x_{k+1}, \tilde{u}(x_{k+1}))$$

Comparing (44) with (45), we obtain that

$$\begin{aligned} \Delta V^{op}_N(x) &= V^{op}_N(x_{k+1}) - V^{op}_N(x_k) \\ &\leq -|x|^2_Q + |\hat{x}_{N+1}|^2_P - |\hat{x}_N|^2_P + |\hat{x}_N|^2_Q + |\kappa_f(\hat{x}_N)|^2_R \\ &= -|x|^2_Q + |(A_K + B\Delta_\sigma)\hat{x}_N|^2_P - |\hat{x}_N|^2_P + |\hat{x}_N|^2_Q \\ &\quad + |(K + \Delta_\sigma)\hat{x}_N|^2_R \\ &= -|x|^2_Q + \hat{x}'_N \Delta'_\sigma (B'PB + R)\Delta_\sigma \hat{x}_N \\ &\quad + \hat{x}'_N (A'_K P A_K - P + Q^* + 2(A'_K PB + K'R)\Delta_\sigma)\hat{x}_N \end{aligned}$$

Since matrix P is chosen according to (38), we have that

$$A'_K PB + K'R = A'PB + K'(B'PB + R) = 0.$$

Then, we obtain that

$$\Delta V^{op}_N(x) \leq -|x|^2_Q + \hat{x}'_N G_\sigma \hat{x}_N.$$

Considering that $\hat{x}_N \in \mathbb{X}_f$, and considering from (36) that

$$a_1|x|^2 \leq x'Px \leq \varphi_x, \quad \forall x \in \mathbb{X}_f,$$

it follows that

$$\Delta V^{op}_N(x) \leq -c_3|x|^2 + d, \quad \forall x \in \mathrm{X}_N, \tag{47}$$

Thus, condition (4) holds with $\alpha_3(s) = c_3 s^2$ and $d = \frac{\varphi_x}{a_1}|G_\sigma|$.

Now, suppose that for an instant $t > 0$, $x \in \mathbb{X}_f$. Then, using (45) and (47), it is possible to establish that

$$\begin{aligned} V^{op}_N(x_{k+1}) &\leq V^{op}_N(x_k) - c_3|x_k|^2 + d \\ &\leq V^{op}_N(x_k) - \frac{c_3}{c_2}V^{op}_N(x_k) + d \\ &\leq \rho_n V^{op}_N(x_k) + d \end{aligned} \tag{48}$$

where $\rho_n = 1 - \frac{c_3}{c_2} \in [0, 1)$. Therefore, by iterating (48), the optimal cost function will be exponentially bounded by

$$V^{op}_N(x_k) \leq \rho^k_n V^{op}_N(x_t) + \left(\frac{1 - \rho^k_n}{1 - \rho_n}\right) d, \quad \forall k \geq t, x \in \mathbb{X}_f.$$

From (44), we have that $|x|^2_P = x'Px \leq V^{op}_N(x)$. Consequently, considering (45), the system state evolution will be exponentially bounded via

$$|x_k|^2_P \leq c_2\rho^k|x_t| + \left(\frac{1 - \rho^k_n}{1 - \rho_n}\right) d, \quad \forall k \geq t, x \in \mathbb{X}_f,$$

Finally, the system state will be ultimately bounded by

$$\limsup_{k \to \infty} |x_k|^2_P \leq \frac{d}{1 - \rho_n} = \delta.$$

By considering (42), we obtain that $\delta < \varphi_x$. Hence, $\mathcal{D}_\delta \subset \mathbb{X}_f$. Consequently, by Theorem 1, the multi-step sparse MPC loop (16) is UpAS for all $x_0 \in \mathrm{X}_N \backslash \mathbb{X}_f$ and practically exponentially stable for $k > t$. □

Theorem 4 establishes that for all $x_0 \in \mathrm{X}_N \subseteq \mathbb{X}$, the system state will be steered by the multi-step sparse MPC, $\kappa_N(x)$ in (15), towards the terminal region $\mathbb{X}_f \subset \mathrm{X}_N$ and then (with the same controller) into the ultimately bounded set $\mathcal{D}_\delta \subset \mathbb{X}_f$.

Remark 3. Notice that decay rate ρ_n in Theorem 4 depends on the binary variable σ_f. Thus, one can use the results of this theorem to reduce the number of active inputs to guarantee stability of the closed-loop while obtaining a desired performance in terms of the decay rate ρ_n.

Dual-Mode Sparse MPC Formulation : By using the local sparse controller in (35), it is possible to define a dual-mode sparse MPC strategy as follows:

$$\kappa_{DM}(x) = \begin{cases} \kappa_N(x), & x \in X_N \backslash \mathbb{X}_f \\ \kappa_f(x), & x \in \mathbb{X}_f \end{cases}$$

The resulting dual-mode sparse MPC loop is expressed via:

$$x_{k+1} = Ax_k + B\kappa_{DM}(x_k), \quad \forall x_k \in X_N. \tag{49}$$

Theorem 5 (Stability of Dual-Mode Sparse MPC)**.** *Suppose that the matrix P in the terminal cost, $V_f(x)$, satisfies* (38)*, and the proposed sparse local controller, $\kappa_f(x)$ in* (35)*, satisfies both* (39) *and* (42)*, then* (49) *is UAS, i.e.,* $\limsup_{k\to\infty}|x_k| = 0$ *for all $x_0 \in X_N$.*

Proof. The proof can be derived based on the proofs of Lemma 1 and Theorem 4. □

The proposed dual-mode sparse MPC, $\kappa_{DM}(x)$, allows the system state to achieve the origin by relying on the local sparse controller, $\kappa_f(x)$. Thus, potential infinite number of switches of the control signal on a finite-time interval, i.e., chattering effects (see [21] for further details) can be avoided.

[1] It is important to highlight that this suboptimal input sequence is only used to facilitate the stability analysis. The actual optimal input sequence, $\vec{u}^{op}(x)$, may present sparse elements which differ from each other, see e.g. Fig. 1.

VI. Simulation study

Here, we illustrate the benefits of the proposed sparse MPC strategy. Consider the system (5) with

$$\begin{aligned} A &= \begin{bmatrix} 0.0721 & 0.6583 & -0.4689 & 0.2238 \\ -0.1881 & 0.5344 & 0.2543 & -0.6755 \\ 0.6522 & 0.3096 & 0.5503 & 0.1500 \\ -0.4926 & 0.1645 & 0.5091 & 1.0681 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.2138 & 0.3385 & -0.1888 \\ 0.4112 & -0.0666 & -0.2024 \\ 0.6095 & 0.1967 & 0.2353 \\ -0.2627 & -0.0707 & 0.3762 \end{bmatrix}, \end{aligned} \tag{50}$$

where $x_k \in \mathbb{R}^4$, $u_k \in \mathbb{R}^3$. Matrix A has 2 unstable eigenvalues, and the pair (A, B) is controllable. The sparse constraint over the input is set as $|u_k|_0 \leq \gamma$, with $\gamma = 2$. Additionally, a convex constraint is imposed to the system as $|x_k| \leq \delta_x = 3$.

The sparse MPC strategy (16) was implemented with parameters $N = 4$, $Q = \mathcal{I}_{4\times4}$, and $R = 3 \cdot \mathcal{I}_{3\times3}$. The terminal cost, $V_f = |x|_P^2$, is chosen to satisfy condition (38), yielding:

$$P = \begin{bmatrix} 2.5820 & -0.2209 & -0.3899 & -0.8719 \\ -0.2209 & 3.1023 & 1.2104 & 1.0563 \\ -0.3899 & 1.2104 & 5.2290 & 2.7497 \\ -0.8719 & 1.0563 & 2.7497 & 6.3644 \end{bmatrix}.$$

In order to illustrate the benefit of the proposed approach, we introduce a non-convex constraint over each vector input $\hat{u}_j$ over the prediction horizon. These constraints are as follows

$$\hat{u}_j' Q_1 \hat{u}_j \leq 1, \qquad \hat{u}_j' Q_2 \hat{u}_j + f_2 \hat{u}_j + \rho_2 \geq 1,$$

with $Q_1 = 0.3472 \cdot \mathcal{I}_{3\times3}$, $Q_2 = 3.125 \cdot \mathcal{I}_{3\times3}$, $\rho_2 = 9$, and $f_2 = \begin{bmatrix} -10.6066 & 0 & 0 \end{bmatrix}$. Based on the proposed design, the terminal region is chosen as:

$$\mathbb{X}_f \triangleq \left\{ x_f' P x_f \leq \varphi_x = 4.1223 \right\}.$$

This value of φ_x assures that the terminal region satisfies that $\mathbb{X}_f \subset \mathbb{X}$, and that $K_\sigma x \in \mathbb{U}$ (definition of terminal region in (36)). For this example, the vector of active inputs $\sigma_f = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}'$ satisfies condition (39). Consequently, by Theorem 3, system (5) with (50) governed by the proposed sparse MPC is UpAS. Starting from the initial state $x_0 = \begin{bmatrix} 1 & -1.5 & -1 & 2 \end{bmatrix}'$, the proposed sparse MPC strategy is implemented using the solver BARON [28].

An exhaustive search method (i.e., evaluating all possibilities and then selecting the optimal one) was implemented using BARON by fixing zeros in the standard MPC problem and solving the resulting quadratic programming (QP) problem. This approach proved to be impractical for this particular example due to the big amount of time required for some solutions. This is due to a resulting complex optimization problem when forcing some variables to be zero.

Solution of the resulting MPC problem using the proposed approach ($\mathbb{P}_{equiv,N}$) is obtained using BARON optimization software. At each sampling instant, the optimization algorithm is initialized using the feasible suboptimal solution in (46). For comparison purposes, the same ℓ_0-input constrained MPC problem is formulated using a mixed-integer approach and solved also by utilizing BARON.

Note that this particular system can be controlled using a fixed active input set, e.g., $\kappa_f(x)$ in (35) with $\sigma_f = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}'$. However, this is in general a suboptimal solution of the multi-step sparse MPC, $\kappa_N(x)$. Moreover, $\kappa_f(x)$ provides a region of attraction, $\mathbb{X}_f$, smaller than the one obtained by $\kappa_N(x)$, i.e., $\mathbb{X}_f \subset \mathrm{X}_N$. On the other hand, when the system state approaches the origin, the multi-step sparse MPC still may provide optimal inputs with alternating active inputs, σ_k^{op}, which have unnoticeable effect over the system state. This is referred to as chattering effect. Therefore, we fix $\sigma_k = \sigma_f$ by commuting from $\kappa_N(x)$ to $\kappa_f(x)$ only when the system state is close enough to the origin (dual-mode operation).

The results of the simulations of the two different approaches are shown in Figs. 3 and 4. Here, u`_bilinear` and u`_mixed_integer` represent the optimal sparse input obtained by the proposed sparse MPC strategy and the mixed-integer approach. These inputs lead to the corresponding system state trajectories denoted by x`_bilinear` and x`_mixed_integer` respectively. From Figs. 3, 4 and 5, it can be noticed that the system constraints are satisfied, and that the system is led to the origin by using only 2 active inputs. Moreover, in Figs. 4 and 5 we note that the optimal inputs obtained with the two strategies are practically the same. Only a slightly differences arises when the state is near the origin, which could be due to numerical problems. Some chattering can be observed before commuting to the local controller at the simulation step 13 (specially in input u_2).

Finally, an important matter to analyze is the execution time carried out for each optimization approach. The computing time of the proposed approach was 46.2 seconds, while the mixed-integer formulation took 83.6 seconds, thus being slower. However, a more comprehensive study is needed for the general case, in order to derive further conclusions.

VII. Conclusion

In this note, we address the problem of sparse feedback control utilizing a quadratic MPC technique for deterministic time-invariant linear systems written in state-space form. The proposed control strategy considers only some of the available inputs as "active" at each control horizon instant. This condition is imposed by utilizing an ℓ_0-norm constraint. The resulting optimization problem is then rewritten into an equivalent one, which can be solved utilizing a non-linear programming optimization toolbox (e.g. BARON). Sufficient conditions are given to ensure stability of the feedback system. Finally, we propose a solution for the potential chattering effect that might happen when the state approaches the origin.

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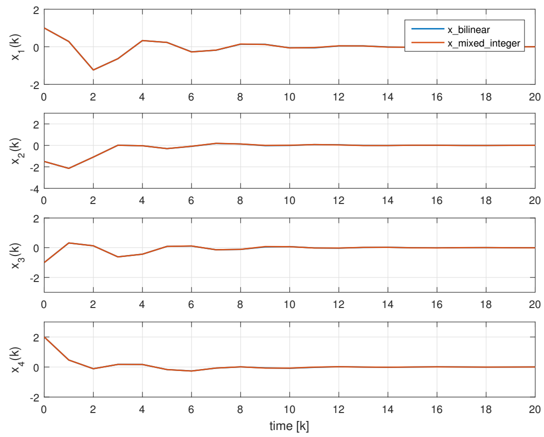

Fig. 3. System state trajectory at each time step.

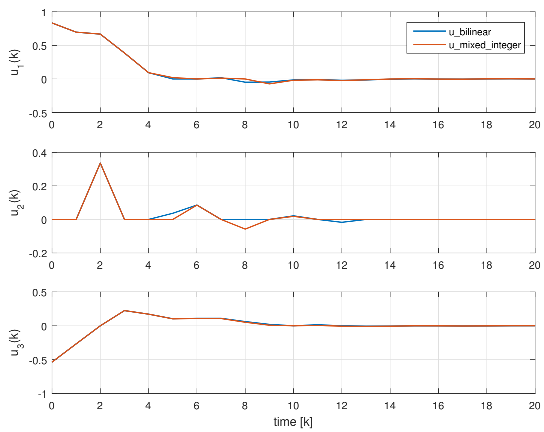

Fig. 4. Sparse optimal input at each time step.

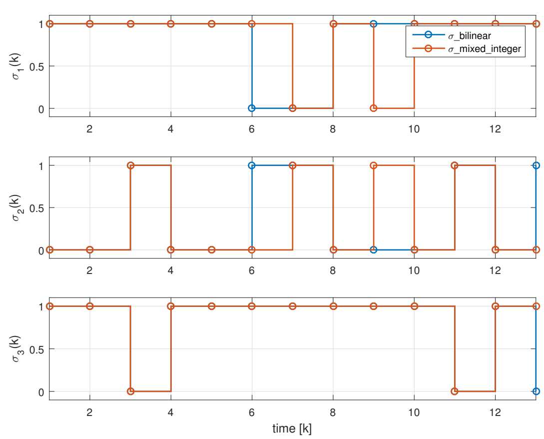

Fig. 5. Optimal active input sequence $\vec{\sigma}^{op}$ for each simulation time k.

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