A Rank-Constrained Optimization approach: Application to Factor Analysis

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Motivation

why include rank constraints?

- Complexity of a model (e.g rank of a Hankel matrix)
- Low-Rank Decomposition (Principal Components Analysis)
- Recent interest on sparse representations.

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Some Difficulties with rank constraints

$$\mathrm{rank}\,\{X\} \leq r$$

- Non-differentiable
- Nonlinear
- Combinatorial nature in optimization

Find a differentiable representation for the rank constraints, and hopefully reduce the number of nonlinearities.

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Some existing representations
By construction

- $\text{rank}\{X\} \leq r \iff X = A \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} B$.

- $\text{rank}\{X\} \leq r \iff X = UV$.

where $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{m\times r}, V \in \mathbb{R}^{r\times n}$.

Disadvantage

Already assign a structure to X.

Some existing representations
Using the Characteristic Polynomial

Consider that $c_i(X)$, for $i = 1, \ldots, n$ are the coefficients of the characteristic polynomial of X.

- $X \in \mathbb{S}_+^n$, [Helmersson 2009]
  $\operatorname{rank}\{X\} \leq r \iff c_{n-r-1}(-X) = 0$.

- $X \in \mathbb{S}_+^n$, [d'Aspremont 2003]
  $\operatorname{rank}\{X\} = \min_{v \in \mathbb{R}^n} \sum_{i=1}^n v_i$
  s.t. $c_i(G)(1 - v_i) = 0, \quad v_i \geq 0$ for $i = 1, \ldots, n$.

Disadvantages

- Only valid for $X \in \mathbb{S}_+^n$.
- $c_i(X)$ is, in general, nonlinear.

Some existing representations used in optimization

Closely related to our representation

I. Markovsky $\operatorname{rank}\{X\} \leq r \iff \exists R \in \mathbb{R}^{(m-r)\times m}$ such that
$RX = 0_{m-r\times m}$ and R is full row rank

$X \in \mathbb{R}^{m\times n}$, but a rank constraint is now imposed on an auxiliary matrix.

J. Dattorro $X \in \mathbb{S}^n_+$,
$\operatorname{rank}\{X\} \leq r \iff \exists W \in \Phi_{n,r}$ such that $\operatorname{trace}(WX) = 0$.

where $\Phi_{n,r} = \{W \in \mathbb{S}^n,\ 0 \preceq W \preceq I, \operatorname{trace}(W) = n - r\}$

Only valid for $X \in \mathbb{S}^n_+$.

Main result[1]

Theorem

Let $X \in \mathbb{R}^{m\times n}$ *then*

$$\operatorname{rank}\{X\} \leq r \iff \exists\, W \in \Phi_{n,r},\ \textit{such that}\ GX = 0_{m\times n}$$

where

$$\Phi_{n,r} = \{W \in \mathbb{S}^n,\ 0 \preceq W \preceq I, \operatorname{trace}(W) = n - r\}$$

[1] submitted for publication

Advantages

- Differentiable
- Freedom to impose a desired structure on X (e.g. Hankel).
- Generalisation of Dattorro's result
- Avoid some difficulties in Markovsky and d'Aspremont's results.

Disadvantage

We still have a bilinear condition $WX = 0_{m\times n}$.

Rank-Constrained Optimisation

An equivalent representation for rank-constrained optimization problems

$$
\begin{aligned}
\mathcal{P}_{rco} : \quad & \min_{\theta \in \mathbb{R}^p} \; f(\theta) \\
\text{s.t.} \quad & \theta \in \Omega \\
& \operatorname{rank}\{X(\theta)\} \le r
\end{aligned}
\qquad \equiv \qquad
\begin{aligned}
\mathcal{P}_{bi} : \quad & \min_{\theta \in \mathbb{R}^p} \min_{W \in \mathbb{S}^n} \; f(\theta) \\
\text{s.t.} \quad & \theta \in \Omega \\
& X(\theta) W = 0_{m \times n} \\
& W \in \Phi_{n,r}
\end{aligned}
$$

Rank-Constrained Optimisation

We have developed

- a *local optimisation* method
- For the case $X(\theta) \in \mathbb{S}^n_+$, a *global optimisation* method.

The bilinear constraint can be imposed in several ways:

$$X(\theta) \in \mathbb{R}^{m\times n} \qquad \|X(\theta)W\| = 0 \iff X(\theta)W = 0_{m\times n}$$

$$X(\theta) \in \mathbb{S}^n_+ \qquad \operatorname{trace}(X(\theta)W) = 0 \iff X(\theta)W = 0_{n\times n}$$

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Factor Analysis

Consider a measured output $y_k \in \mathbb{R}^N$, factors $f_k \in \mathbb{R}^r$, idiosyncratic noise $v_k \in \mathbb{R}^N$, and a model:

$$y_k = Af_k + v_k \tag{1}$$

where $A \in \mathbb{R}^{N \times n}$ is a tall matrix.

$$f_k \sim \mathcal{N}(0, \Phi) \tag{2}$$
$$v_k \sim \mathcal{N}(0, \Psi) \tag{3}$$

Then, $y_k \sim \mathcal{N}(0, \Sigma)$, where Σ is given by

$$\Sigma = A\Phi A^\top + \Psi$$

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Sparse Noise Covariance

- Existing approaches require Ψ diagonal.
  - $\Psi = \sigma^2 I$, then there is a closed-form solution, e.g. PCA.
  - Ψ diagonal, e.g. Maximum Likelihood (EM algorithm).
- Considering the advances on sparse representations. We propose to assume that Ψ is sparse.

$$
\begin{aligned}
\mathcal{P}_{rcofa}: \quad & \min_{\Psi \in \mathbb{S}^N} && \|\Psi\|_1 \\
& \text{subject to} && \operatorname{rank}\{\Sigma - \Psi\} \leq r \\
& && \Psi \succeq 0 \\
& && \Sigma - \Psi \succeq 0
\end{aligned}
$$

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Numerical Example
Local Optimization method

Consider $r = 3$ factors, $N = 20$ measured outputs and $T = 100$ Samples, $\Psi_{ij} = (0.7)^{|i-j|}$. and the performance index

$$d(P_m) = 1 - \frac{\mathrm{trace}(AP_mA^\top)}{\mathrm{trace}(AA^\top)}$$

Method	$d(\cdot)$	Total Time [s]
PCA	0.1992	0.0136
RCO	0.1002	274.6732
EM	0.1330	0.0198

Table: Mean value over $N_{mc} = 100$ Monte Carlo simulations of $d(\cdot)$.

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Conclusions

- We have developed a new representation of rank constraints.
  - Second-order differentiable.
  - Avoid several nonlinearities, excepting a bilinear constraint.
- We have developed two optimization algorithms.
  - Local Optimisation.
  - Global optimisation.
- We have applied the method:
  - Factor Analysis with Correlated Errors.
  - Sparse Control (tomorrow FrA04.2).

Thanks for your attention!

Any questions?

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Global Optimization Example

Consider $r = 1$ factor, $N = 3$ measured outputs and $T = 100$ Samples.

	$d(\cdot)$	$\|\widehat{\Psi}\|_1$	time [s]
RCO	0.066	4.46	23.83
RCO-G	0.066	$\underline{\mathbf{4.46}}$	310.19

	$d(\cdot)$	$\|vec(\widehat{\Psi})\|_1$	time [s]
RCO	0.165	11.12	43.20
RCO-G	0.218	$\underline{\mathbf{9.17}}$	1768.31

Solving the Optimisation Problem

Proposed method

Solve $\mathcal{P}_{bi}$ iteratively.
In each iteration, solves a feasibility problem that deals with the bilinear constraint.

Given the estimate $\theta^m \in \Omega$, at iteration m

$$\begin{aligned} \min_{\theta \in \mathbb{R}^p} \min_{W \in \mathbb{S}^n} \quad & \|X(\theta)W\| \\ \text{subject to} \quad & f(\theta) \leq f(\theta^m) - \eta_m \\ & \theta \in \Omega \\ & W \in \Phi_{n,r} \end{aligned}$$

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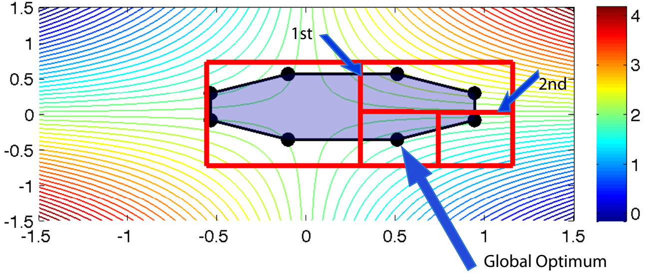
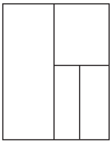
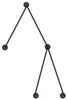
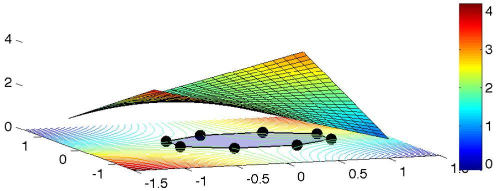
Branch and Bound









ℓ_1-norm

- Computationally efficient.
- Unintuitive to chose the sparse parameters
- Poor handling of group-constraints

ℓ_0-norm

- Computationally more expensive.
- ℓ_0-norm constraints have a clear interpretation.
- Can handle Group constraints.

ℓ_0-norm equivalence

$$\mathcal{P}_{\ell_0}: \quad \begin{aligned} \min_{\theta\in\mathbb{R}^p} \ & f(\theta) \\ \text{s.t.} \ & \theta \in \Omega \\ & \|\theta\|_0 \le r \end{aligned} \quad \equiv \quad \mathcal{P}_{eq}: \quad \begin{aligned} \min_{\theta\in\mathbb{R}^p} \min_{w\in\mathbb{R}^p} \ & f(\theta) \\ \text{s.t.} \ & \theta \in \Omega \\ & \theta \circ w = \mathbf{0} \\ & \mathbf{0} \le w \le \mathbf{1} \\ & \mathbf{1}^\top w = p - r \end{aligned}$$

Delgado, Agüero & Goodwin 2014 (IFAC 2014).

Delgado, Agüero & Goodwin submitted to Automatica.

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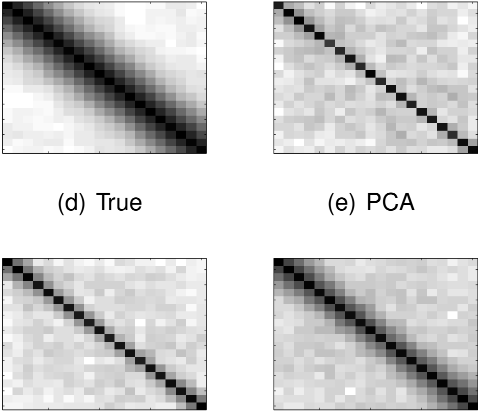
Local Optimization

Ψ



(d) True

(e) PCA

(f) EM

(g) RCO

Concave Minimisation

Concave minimization may lead to a combinatorial problem. (e.g. matrix-rank function is quasi-concave on the Positive Semidefinite Cone, thus leads to Concave-minimisation and “reverse-convex” constraints)