

Low-order Control Design Using a Novel Rank-constrained Optimization Approach

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Abstract—A recent equivalent representation of rank constraints is used to design a low-order controller with prescribed degree of stability. We solve an optimization problem involving linear matrix inequalities and rank constraints. We illustrate the potential of the proposed approach by comparing with similar approaches available in the literature.

I. INTRODUCTION

Rank-constrained optimization has gained increased attention in the last decades. Recent advances in convex optimization and the development of easy-to-use optimization software have helped to increase the usage of such software within the systems and control community. The success of nuclear norm, log-det and trace heuristics, see e.g. [7], [8], in some problems have motivated several researchers to formulate a large number of engineering problems in terms of optimization problems that include rank constraints. A classic example of such engineering problems arises in system identification, where the order of a rational system is equal to the rank of an infinite dimensional Hankel matrix [9]. Another example is Factor Analysis (see e.g. [5]), where the number of latent factors is equal to the rank of a covariance matrix.

Although heuristics such as the nuclear norm provide a convenient way to address rank constraints in optimization problems, there is an inherent loss of performance in the use of this heuristic [16]. Moreover, most heuristics consider the rank constraint as a soft constraint, i.e the obtained solution may violate the rank constraint. This approach to deal with rank constraints may be unsatisfactory in some applications. This has motivated the development of methods that consider the rank constraint as a hard constraint, see e.g. [14], [13], [4]. These methods are based on the notion of equivalent representations of a rank constraint. These equivalent representations are aimed at overcoming some of the undesirable features of the rank function, namely, non-linearity, non-smoothness and non-convexity of the rank function. In this paper, we focus on the rank-constraint representation described in [4] that allows to solve rank-constrained optimization problems within the framework of nonlinear programming.

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In the rank-constraint representation in [4] the value of the rank in the rank constraint, say r , is controlled by the value of the trace of an auxiliary matrix. These presents a major advantage over other rank-constraints representations where the value of r is controlled by the size of auxiliary matrices, see e.g. [15] or by including a constraint that is non-convex on r , see e.g. [14], [13]. This feature may be useful in problems where the value of the rank to be constrained is unknown *a priori*.

In this paper we apply the rank-constraint representation described in [4] to a Reduced Order Output Feedback stabilization problem and address variants of the same problem. We then perform a numerical comparison of the proposed approach against state-of-the-art alternative methods.

Notation and basic definitions: $\text{rank}\{A\}$ denotes the rank of a matrix A . $\lambda_i(A)$ denotes the i -th largest eigenvalue of a symmetric matrix A and $\sigma_i(A)$ denotes the i -th largest singular value of a matrix A . $A \succeq 0$ denotes that A is positive semidefinite, and $A \succeq B$ denotes that $A - B \succeq 0$. We denote the transpose of a given matrix A as A^T . \mathbb{S}^n denotes the set of symmetric matrices of size $n \times n$.

II. REDUCED ORDER OUTPUT FEEDBACK

To illustrate the features of the rank-constraint representation in [4] we apply it to a rank-constrained optimization problem. In particular, we focus in the problem of *Reduced order output feedback stabilization*. In this section, we describe the approach proposed in [11] that uses Linear Matrix Inequalities (LMI) to find a reduced order controller for the output feedback stabilization problem. A benefit of this formulation is that it allows us to define an optimization framework to solve the problem of interest.

Consider a continuous time, linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control signal and $y \in \mathbb{R}^p$ is the measured output. The controller is given by

$$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix} \quad (3)$$

where $K \in \mathbb{R}^{(n_c+m) \times (n_c+p)}$ and $x_c \in \mathbb{R}^{n_c}$ is the controller state.

Define $\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}$. The following lemma establishes necessary and

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sufficient conditions to make sure that the real part of the closed loop system ($\dot{\tilde{x}} = \mathcal{A}\tilde{x} = (\tilde{A} + \tilde{B}K\tilde{C})\tilde{x}$) poles are on the left of $s = -\alpha$.

Lemma 1: (see [11]) Let \mathcal{A} be a given square matrix and α be a given positive scalar. Then the following statements are equivalent:

- 1) The system $\dot{\tilde{x}} = \mathcal{A}\tilde{x}$ is α -stable (with prescribed degree of stability α).
- 2) There exists a matrix $\mathcal{Y} \succ 0$ such that $(\mathcal{A} + \alpha I)^T \mathcal{Y} + \mathcal{Y}(\mathcal{A} + \alpha I) \prec 0$.

Note that statement 2 of Lemma 1 involves a bilinear form of the two unknown matrices \mathcal{Y} and K (since the closed loop matrix \mathcal{A} depends on K). In [11], the unknown controller terms are eliminated from the bilinear form and necessary and sufficient conditions for the existence of an α -stabilizing controller of order n_c are found. In [11] the existence of a α -stabilizing controller of order n_c for a given $\alpha > 0$ can be tested by solving a set of LMI subject to rank constraint as described below.

Consider a system defined as in (1)-(2) and a given scalar $\alpha > 0$. Solving the following feasibility problem for $X \succ 0$ and $Y \succ 0$ assures that an α -stabilizing controller of order n_c exists (see [11]).

$$\begin{aligned} \mathcal{P}_0 : \quad & \text{Find } X, Y \in \mathcal{S}^n \\ \text{s.t.} \quad & -B^\perp (AX + XA^T + 2\alpha X) B^{\perp T} \succ 0 \\ & -C^{T\perp} (YA + A^T Y + 2\alpha Y) C^{T\perp T} \succ 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\ & \text{rank} \left\{ \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \right\} \leq n + n_c \end{aligned}$$

where B^\perp is a matrix of maximal rank such that its rows are orthogonal and $B^\perp B = 0$. Similar conditions hold for $C^{T\perp}$.

Solution matrices X and Y of \mathcal{P}_0 are related in the following way to statement 2 of Lemma 1 (see [11] for details):

$$\mathcal{Y} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{Y}^{-1} = \begin{bmatrix} X & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$$

where $X_{22}, Y_{22} \in \mathbb{R}^{n_c \times n_c}$. Thus, by solving \mathcal{P}_0 the unknown α -stabilizing controller K can be found by solving the LMI found in statement 2 of Lemma 1 as described below.

Consider the Matrix Inversion Lemma, and take the eigenvalue decomposition $X - Y^{-1} = V\Lambda V^T$, where Λ is a diagonal matrix whose entries are the eigenvalues ordered in decreasing order. Define $R = V(:, 1 : \lambda_{n_c}) \text{diag}(\lambda_1^{1/2}, \dots, \lambda_{n_c}^{1/2})$ and $\tilde{X} = \begin{bmatrix} X & R \\ R^T & I \end{bmatrix}$. A controller K that fulfills statement 2 of Lemma 1 is found by solving the following optimization problem:

$$\begin{aligned} \mathcal{P}_K : \quad & \max_{\gamma \in \mathbb{R}, K} \gamma \\ \text{s.t.} \quad & (\tilde{A} + \tilde{B}K\tilde{C})\tilde{X} + \tilde{X}(\tilde{A} + \tilde{B}K\tilde{C})^T + 2\gamma\tilde{X} \leq 0 \end{aligned} \quad (4)$$

Solution γ of problem \mathcal{P}_K represents a lower bound for the stability degree of the closed loop system $\dot{\tilde{x}} = \mathcal{A}\tilde{x}$ [18].

A Newton-like method to solve problems involving rank constrained linear matrix inequalities (LMI) is presented in [18]. In particular they use it to solve problem \mathcal{P}_0 . It is important to note the cited approach locally solves the problem. The algorithm is implemented in the LMIRank solver that is freely distributed by the authors.

A. Rank Minimization Approach

In this section we describe the method presented in [20] to solve a similar problem. This is mentioned for comparison purposes only. In [20] an iterative rank minimization procedure is presented and used for (locally) solving the similar problem of finding a stabilizing controller of a certain order (stability degree is not a constraint). The algorithm in [20] reduces the rank of a matrix constraint in a convex set. In [20], the problem of determining the existence of a low order controller is treated (for a system described as in (1)-(2)).

Lemma 2: (see [20],[10]) There exists a stabilizing output feedback law of order k if and only if the global minimum of the rank minimization problem is less than $n + k$.

$$\begin{aligned} \mathcal{P}_{rk} : \quad & \min_{W_1, W_2, \sigma} \text{rank} \left(\begin{bmatrix} W_1 & I \\ I & W_2 \end{bmatrix} \right) \\ \text{s.t.} \quad & AW_1 + W_1 A^T \prec \sigma B B^T \\ & A^T W_2 + W_2 A \prec \sigma C^T C \\ & \begin{bmatrix} W_1 & I \\ I & W_2 \end{bmatrix} \succeq 0 \\ & \sigma > 0 \end{aligned} \quad (5)$$

where $W_1, W_2 \in \mathbb{S}^n$ and $\sigma \in \mathbb{R}^+$.

Note that problem \mathcal{P}_{rk} represents a particular case of \mathcal{P}_1 . This can be seen by taking $\lim \alpha \rightarrow 0^+$ and by considering that $B^\perp B = 0$ (for further insight, see section 2.6 of [2]).

Note also that problem \mathcal{P}_{rk} must incorporate a stop condition. This is due to the existence of an infinite class of controllers satisfying the stated conditions. In the iterative rank minimization algorithm proposed in [20], the program is stopped once a desired order, i.e. rank, is achieved.

In this paper we implement the cited approach to solve problem \mathcal{P}_0 and compare its performance with other methods.

III. EQUIVALENCE BY RANK CONSTRAINT REPRESENTATION

In this section we use the approach presented in [4] to find an equivalent representation of problem \mathcal{P}_0 . The need of equivalent representations for rank constraints arise because the rank function has several features that are undesirable in optimization problems. In particular, the rank function is non-smooth, non-linear and non-convex. In the optimization literature, smoothness and convexity are widely exploited, and the lack of such features in the rank function limits the tools that can be used in the to solve the optimization problem. Thus, equivalent representations aim at overcoming

at least one of these undesirable features of the rank function. Recently, equivalent representations have been utilized to avoid the direct treatment of rank constraints ([15], [4], page 241 of [3]).

The following result describes the equivalent representation of a rank constraint presented in [4].

Lemma 3: Let $G \in \mathbb{R}^{m \times n}$, then the following expressions are equivalent

- (i) $\text{rank}\{G\} \leq r$
- (ii) $\exists W \in \Phi_{n,r}$, such that $GW = 0_{m \times n}$

where

$$\Phi_{n,r} = \{W \in \mathbb{S}^n, 0 \preceq W \preceq I, \text{trace}(W) = n - r\} \quad (7)$$

Proof: See [4]. ■

The result from Lemma 3 can be seen as a generalization of the one provided in [3] (the former can be used on rank constraints over real matrices of all sizes). An advantage of Lemma 3 is that it represents a rank-constraint in a form that can be used for optimization purposes. The constraints imposed by the set $\Phi_{n,r}$ can be handled by Semidefinite Programming. However, it is well known that computations that considers bilinear matrix constraints are fundamentally more difficult to those over linear matrix inequalities (see e.g. [22]). In this paper, the condition, $GW = 0$, will be addressed in the context of nonlinear programming by utilizing the optimization software BARON [19][21] which allows us to solve problems with this type of bilinear constraints and in addition to obtain a global solution.

This approach to deal with rank constraints has been applied in several framework including: Model Predictive control [1], Factor Analysis [5] and to nonlinear system identification [6].

The following result presents an equivalent representation of problem \mathcal{P}_0 .

Theorem 1: Let $W \in \mathbb{S}^{2n}$, $0 \preceq W \preceq I$, $\text{trace}(W) = n - n_c$, then feasibility problem \mathcal{P}_0 is equivalent (in the sense that has same global optimum) to the following problem

$$\begin{aligned} \mathcal{P}_1 : \quad & \text{Find } X, Y \in \mathbb{S}^n, W \in \mathbb{S}^{2n} \\ \text{s.t.} \quad & -B^\perp (AX + XA^T + 2\alpha X) B^{\perp T} \succeq 0 \\ & -C^{T\perp} (YA + A^T Y + 2\alpha Y) C^{T\perp T} \succeq 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\ & \text{trace}(W) = 2n - (n + n_c) \\ & 0 \preceq W \preceq I \end{aligned}$$

Proof: Problems \mathcal{P}_0 and \mathcal{P}_1 both have the same feasible set with respect to X and Y . Moreover, these conditions do not depend on W . Hence, to prove the equivalence between \mathcal{P}_0 and \mathcal{P}_1 it suffices with using Lemma 3, which proves the validity of the rank-constraint representation. ■

As stated in Theorem 1, problem \mathcal{P}_0 can be transformed into a problem, \mathcal{P}_1 , that does not explicitly include the rank constraint, but has same optimum as the original problem. This new formulation of the problem can be solved by

standard tools of nonlinear programming such as those provided by software BARON [21], [19].

IV. FURTHER EXTENSIONS OF THE APPROACH

Similar to Theorem 1 we can also formulate other problems of interest.

A. Minimization of Controller's Order

Lemma 3 relates the upper bound of a rank constraint with the trace of an auxiliary matrix W . This allows the formulation of a rank minimization problem by maximizing the trace of W . Thus, the problem of finding the α -stabilizing controller with minimum order can be formulated as follows (for a given α)

$$\begin{aligned} \mathcal{P}_2 : \quad & \min_{X, Y \in \mathbb{S}^n, W \in \mathbb{S}^{2n}} 2n - \text{trace}(W) \\ \text{s.t.} \quad & -B^\perp (AX + XA^T + 2\alpha X) B^{\perp T} \succ 0 \\ & -C^{T\perp} (YA + A^T Y + 2\alpha Y) C^{T\perp T} \succ 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\ & 0 \preceq W \preceq I \end{aligned}$$

Similar rank representations can be found in the literature. In [17], a representation involving two auxiliary matrices whose dimensions depend on the rank constraint's bound r is proposed. Due to the dependence of the auxiliary matrices dimensions with the rank bound, it is not plausible to use the representation in [17] to state problem \mathcal{P}_2 .

In [20], the reduced order controller is obtained by an iterative approach that minimizes the rank of a matrix. This approach however only assures local convergence, while the representation applied in this paper has same global optimum as the original problem and can be obtain utilizing nonlinear programming techniques.

Remark 1: It is important to note that when implementing problem \mathcal{P}_2 a stop mechanism must be incorporated. Suppose that for a given system, the minimum order possible for a stabilizing controller is n_c . Problem \mathcal{P}_2 is defined as a minimization problem, thus it will not stop once found a controller of order n_c , in fact it will continue searching in the infinite set of controllers, trying to find one of even smaller order. To fix this issue, we add a time constraint while solving \mathcal{P}_2 with the global optimization software BARON. Other authors such as [20] set a lower bound for the achieved controller's order which can be compared in each step of their iterative rank minimization approach.

B. Optimizing for controller order and α -stabilizing degree

Feasibility problem \mathcal{P}_0 stated before can be extended into a maximization problem, where the biggest value for parameter α is to be found (α is treated as a variable). Thus by maximizing α and maintaining the rank constraint, the resulting pair of matrices (X, Y) can be used to obtain the

fastest stabilizing controller of order n_c . This leads into the following optimization problem

$$\begin{aligned} \mathcal{P}_3 : \quad & \max_{\alpha \in \mathbb{R}_{>0}, X, Y \in \mathbb{S}^n, W \in \mathbb{S}^{2n}} \alpha \\ \text{s.t.} \quad & -B^\perp (AX + XA^T + 2\alpha X) B^{\perp T} \succ 0 \\ & -C^{T\perp} (YA + A^T Y + 2\alpha Y) C^{T\perp T} \succ 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\ & \text{trace}(W) = 2n - (n + n_c) \\ & 0 \preceq W \preceq I \end{aligned}$$

The flexibility of the approach presented in this paper can be also used to another where the stability degree and controller order are optimize at the same time by imposing a trade-off between them:

$$\begin{aligned} \mathcal{P}_4 : \quad & \min_{\alpha \in \mathbb{R}_{>0}, X, Y \in \mathbb{S}^n, W \in \mathbb{S}^{2n}} \text{trace}(W) - \alpha \\ \text{s.t.} \quad & -B^\perp (AX + XA^T + 2\alpha X) B^{\perp T} \succ 0 \\ & -C^{T\perp} (YA + A^T Y + 2\alpha Y) C^{T\perp T} \succ 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\ & 0 \preceq W \preceq I \end{aligned}$$

This problems are currently been studied in order to understand their solution space.

V. NUMERICAL COMPARISON

In this section we carry numerical examples in order to compare the performance and effectiveness of the approach.

We consider the reduced order feedback control problem used in [12] and [18]. The system has the following state-space matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T$$

A. α -stabilizing Controller

First, we search for α -stabilizing controllers of order $n_c = 2$ for given values of α (which imposes a lower bound for the stabilization degree). In Table I, the resulting stability degree $\hat{\alpha}$ of the closed loop system is shown. Note that that since we are solving a feasibility problem it is possible to obtain a closed loop with greater stabilizing degree. We use three different approaches to solve \mathcal{P}_0 : i) the approach presented in Orsi *et. al.* [18], ii) the approach in Sun *et. al.* [20] and iii) the one proposed in this paper. We constrain the computation-time and maximum number of iterations to compare the different approaches. Orsi's approach was limited to 20000 iterations while Sun's and our approach was limited to a 500[s] computation time.

TABLE I
ACHIEVED CLOSED LOOP α -STABILITY SOLVING \mathcal{P}_1

α	Orsi <i>et. al.</i>		Sun <i>et. al.</i>		Equivalent Representation	
	$\hat{\alpha}$	$T[s]$	$\hat{\alpha}$	$T[s]$	$\hat{\alpha}$	$T[s]$
0.2	0.203	1.4	0.2	2.4	0.305	151.2
0.42	0.420	3.0	-	-	0.506	440.8
0.46	0.467	172.2	-	-	0.521	380.3
0.5	-	-	-	-	0.500	125.4
0.502	-	-	-	-	0.529	451.7

Although our approach takes more time than the one shown in [18], we are able of finding controllers that result in a better closed loop stability degree $\hat{\alpha}$, or that others could not find. This is due to the nonlinear programming technique that the solver utilizes. Given that the goal is to find a static controller K , the solution time of the approach is not of much relevance¹. Note that the closed loop stability degree obtained is not necessarily the same as the required, and in some cases is far greater. This shows that the solver doesn't work in an iterative way (improving some parameter at each step), which might lead to local minimum.

Although the equivalent representation for rank constraints allows us to use nonlinear programming techniques, the problem is still computationally demanding. Complexity and computational load might increase for some particular problems. This is seen for example when solving \mathcal{P}_0 for higher closed loop stability degree (i.e. increasing α).

B. Reducing controller order

Next we solve problem \mathcal{P}_2 , where the objective is to minimize the controller's order for a user specified stabilizing degree α . Considering Remark 1 we add a time constraint into the solver of $t_{max} = 300[s]$. In Table II are shown the results such as achieved controller order \hat{n}_c and correspondent closed loop stabilizing degree.

TABLE II
CONTROLLER'S ORDER n_c FOR GIVEN STABILIZING DEGREE α ,
SOLUTION TIME $t_{max} = 300[s]$

α	\hat{n}_c	$\hat{\alpha}$
0.1	2	0.105
0.2	3	0.221
0.5	3	0.786
0.7	3	0.786
1	3	1.378

We note that in relatively the same time used for examples for problem \mathcal{P}_1 we have achieved a better closed loop stabilizing degree.

¹Note that, if time is a constraint, a suboptimal solution could be obtained by stopping the optimization procedure or by using local optimization procedure.

VI. CONCLUSIONS AND FUTURE WORK

In this paper we address the problem of designing a reduced order feedback control. We incorporate rank constraints in order to restrict the order of the unknown controller through an optimization problem. The resulting optimization framework gives us the possibility of formulating additional rank constrained problems for control design. We apply an equivalent rank constraint representation to reformulate the problem into another one that is equivalent in a global optimum sense. The resulting (global optimum-equivalent) problem can be solved by using nonlinear programming techniques. We also formulate two additional extensions of the original reduced order control problem: 1) maximization of the stability degree, given a controller's order, 2) given a certain stability degree, minimize the order of the controller. This shows the versatility of the rank constraint representation to solve different control design problems. Finally numerical examples are shown in order to illustrate the performance of the proposed method.

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