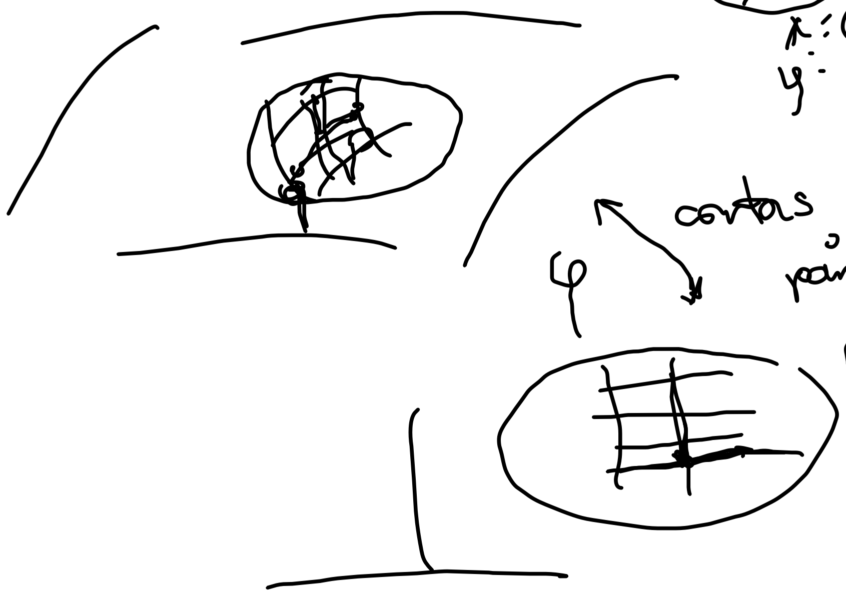
# Métrica Hiperbólica $H^+ \subset \mathbb{R}^2$

Definir en $H^2$ una métrica Riemanniana

$S \subset \mathbb{R}^n$

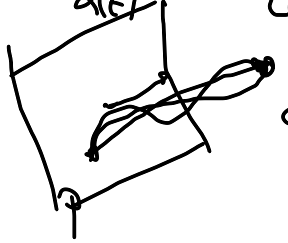
$x: \varphi(u) \to \mathbb{R}$

$y:$




cartas, coordenadas, parametrizaciones

$\alpha'(t) \in T_{\alpha(t)} S$




$d(p,q) = \inf\{\ell(\alpha) \mid \alpha \text{ curva que une } p \text{ con } q\}$

$\alpha : [a,b] \longrightarrow S$

$\alpha(a) = p \quad \alpha(b) = q$

$$\ell(\alpha) = \int_a^b \|\alpha'(t)\|\, dt$$

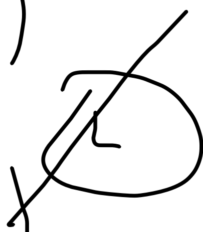
$v, u \quad T_p S = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \quad p \in S$

$$u \cdot v = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}$$

definido positivo

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$



$H^1 \subset \mathbb{R}^2$

$\mathbb{H}^2 = H^1 \subset \mathbb{C}$

$z^k$

$v, u \in T_z \mathbb{H}^2 = \mathbb{R}^2 \qquad z = x+iy$

$$u \cdot_{H^+} v = \frac{u \cdot_E v}{y^2}$$ <u>Producto interior hiperbólico</u>

- Toma valores reales
- Es positivo definido

$$u \cdot u \geq 0 \quad \forall u$$

$$u \cdot u = 0 \Rightarrow u = 0$$

- Simétrico

$$u \cdot_H v = v \cdot_H u$$

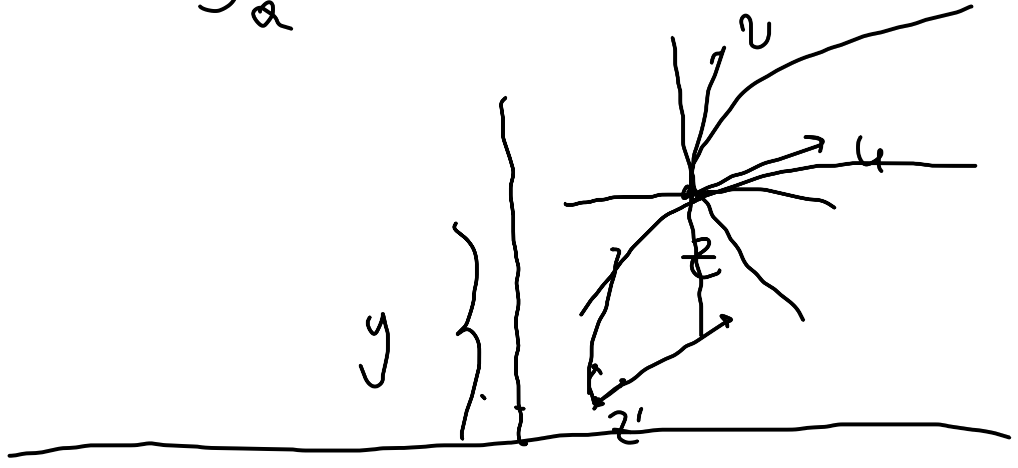
## Norma hiperbólica

$$\|u\|_H = \sqrt{u \cdot_H u}$$

## Long Hiperbólica

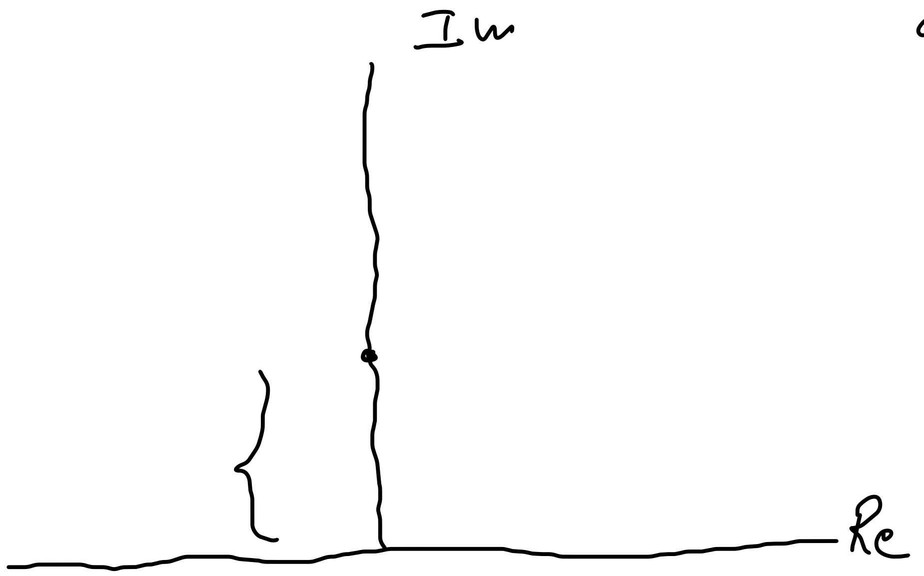
$$\alpha : [a,b] \longrightarrow \mathbb{H}^2 = H^+$$
$[a,b] \subset \mathbb{R}$

$$\ell(\alpha) = \int_a^b \|\alpha'(t)\|_H \, dt$$



## Afirmación

1. Rectas hiperbólicas tienen long infinita.
2. Las rectas hiperbólicas son geodésicas (minimizan la distancia entre sus puntos)




$$\alpha(t) = ti = (0,t)$$

$$\alpha : (0,1]$$

$$\ell(\alpha) = \lim_{a \to 0} \int_0^1 \|\alpha'(t)\|_H \, dt$$

$$\alpha'(t) = (0,1)$$

$$\|\alpha'(t)\| = \frac{1}{t}$$

$$\ell(\alpha) = \ln(1) - \lim_{a \to 0} \ln(a)$$

$$= 0 - \lim_{a \to 0} \ln(a) = \infty$$

$$0 \to -1$$
$$\infty \to 1$$
$$i \to i$$

$$\frac{z-1}{z+1} = f(z)$$

$\alpha(t) = it \qquad f(\alpha(t)) = \beta(t)$

$$f(\alpha(t)) = \frac{it-1}{it+1} = \left(\frac{-1+t^2}{1+t^2}, \frac{2t}{1+t^2}\right) = \beta(t)$$

$$Re(\alpha(t)) = \frac{1}{2}\left(\frac{it-1}{it+1} + \frac{-it-1}{-it+1}\right)$$

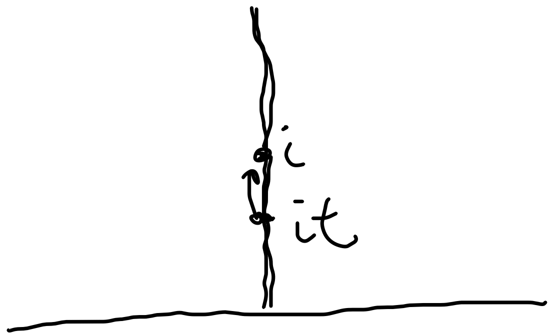
$$= \frac{-1+t^2}{1+t^2}$$

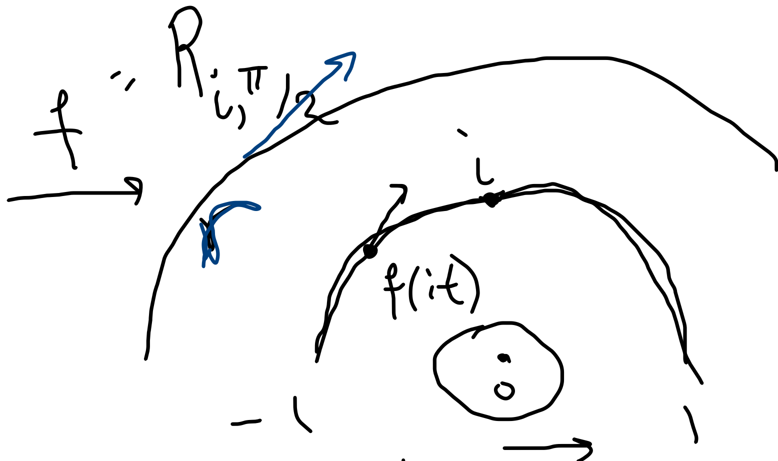
$$Im(\alpha(t)) = \frac{2t}{1+t^2}$$

$$\sqrt{\beta'(t) \cdot_H \beta'(t)} = \sqrt{\frac{\left(\frac{4t}{(1+t^2)^2}, \frac{2(1-t)^2}{(1+t^2)^2}\right) \cdot \left(\quad\right)}{4t^2/(1+t^2)^2}}$$

Trabajos de ejercicio (1)

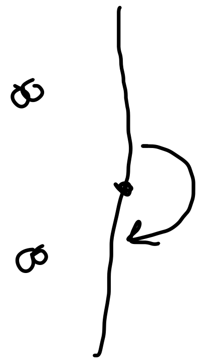
$$\|\beta'(t)\|_H = \frac{1}{t}$$




Conclusión: f es una isometría $x \in \mathbb{R}$

$$f^2 = R_{i,\pi}$$




(2) ejercicio Traslador 0 a $x \in \mathbb{R}$

(3) ejercicio radio $r > 0$ también es isometría

Teorema Toda $f \in PSL(2, \mathbb{R})$ es una isometría de $H^+$

$$G^+_{H^+} \subset Iso^+(\mathbb{H}^2) \qquad \square$$