CHAPTER 9

Multiple Integrals

Chapter Outline

9.1 INTRODUCTION

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called *double integration* and that of three variables is called *triple integration*. Sometimes, we have to change the variables to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the coordinate system (polar, spherical or cylindrical coordinates). Multiple integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions, etc.

9.2 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous function defined in a closed and bounded region R in the xy-plane. Divide the region R into small elementary rectangles by drawing lines parallel to coordinate axes. Let the total number of complete rectangles which lie inside the region R be n. Let δA_r be the area of r^{th} rectangle and (x_r, y_r) be any point in this rectangle.

Consider the sum $$S = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(1)$$

where $\delta A_r = \delta x_r \cdot \delta y_r$

If we increase the number of elementary rectangles then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by the Eq. (1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint\limits_R f(x, y)\mathrm{d}A$.

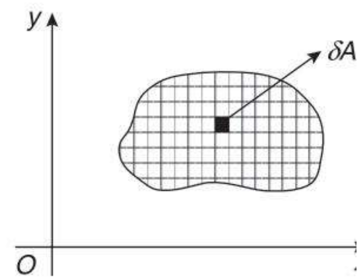

Fig. 9.1

Hence, $$\iint\limits_R f(x, y)\mathrm{d}A = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

9.2.1 Double Integrals over Rectangles and General Regions

Double integral of a function $f(x, y)$ over a region R can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

Method-I Let the region R, i.e., $PQRS$ be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$.

In the region $PQRS$, draw a vertical strip AB. Along the strip AB, y varies from y_1 to y_2 and x is fixed. Therefore, the double integral is integrated first w.r.t. y between the limits y_1 and y_2 treating x as constant.

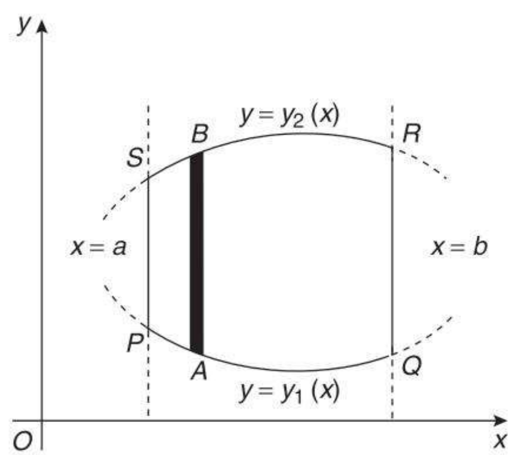

Fig. 9.2

Now, move the strip AB horizontally from PS (i.e., $x = a$) to QR (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. x between the limits a and b. Hence,

$$\iint\limits_R f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\right]\mathrm{d}x$$

Method-II Let the region R, i.e., $PQRS$ be bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$.

In the region $PQRS$, draw a horizontal strip AB. Along the strip AB, x varies from x_1 to x_2 and y is fixed. Therefore, the double integral is integrated first w.r.t. x between the limits x_1 and x_2 treating y as constant.

Now, move the strip AB vertically from PQ (i.e., $y = c$) to RS (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. y between the limits c and d.

Hence,

$$\iint_R f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\mathrm{d}x\right]\mathrm{d}y$$

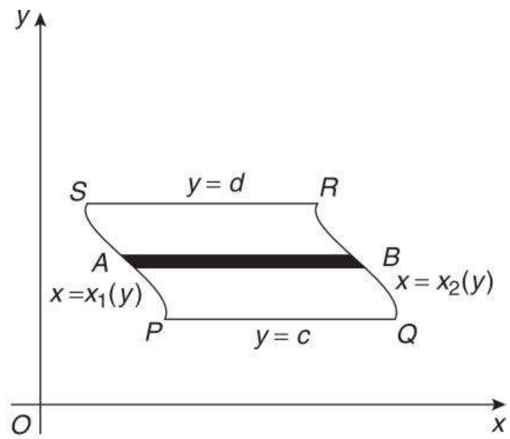

Fig. 9.3

Note:

(i) If all the four limits are constant then region of integration R is a rectangle. In this case, the function $f(x,y)$ can be integrated w.r.t. any variable first.

(ii) If all the four limits are constant and $f(x, y)$ is explicit then double integral can be written as product of two single integrals.

(iii) If inner limits depends on x then the function $f(x, y)$ is integrated first w.r.t. y and vice-versa.

9.2.2 Properties of Double Integrals

Various properties of double integrals are analogous to those for single integrals. For f and g continuous in region R with k as rational number,

(i) $\iint_R (f+g)\,\mathrm{d}x\mathrm{d}y = \iint_R f\,\mathrm{d}x\mathrm{d}y + \iint_R g\,\mathrm{d}x\mathrm{d}y$

(ii) $\iint_R k\,f\,\mathrm{d}x\mathrm{d}y = k\iint_R f\,\mathrm{d}x\mathrm{d}y$, where k is a constant.

For f continuous in region R, where $R = R_1 \cup R_2$ where R_1 and R_2 are non-overlapping regions whose union is R:

(iii) $\iint_R f\,\mathrm{d}x\mathrm{d}y = \iint_{R_1} f\,\mathrm{d}x\mathrm{d}y + \iint_{R_2} f\,\mathrm{d}x\mathrm{d}y$

9.2.3 Double Integrals as Volumes

If $z = f(x, y) \geq 0$ represents a surface and R is a rectangle in the xy-plane, then the double integral of $f(x, y)$ over R,

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y$$

represents the volume of the solid under the surface $z = f(x, y)$ and above the region R.

Example 1

Evaluate $\int_0^3 \int_0^1 (x^2 + 3y^2)\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\begin{aligned}\int_0^3 \int_0^1 (x^2 + 3y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^3 \left| x^2 y + y^3 \right|_0^1 \mathrm{d}x \\ &= \int_0^3 (x^2 + 1)\,\mathrm{d}x \\ &= \left| \frac{x^3}{3} + x \right|_0^3 \\ &= 12\end{aligned}$$

Example 2

Evaluate $\int_0^1 \int_0^2 (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x.$

Solution

$$\begin{aligned}\int_0^1 \int_0^2 (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \left| x^2 y + \frac{y^3}{3} \right|_0^2 \mathrm{d}x \\ &= \int_0^1 \left(2x^2 + \frac{8}{3} \right) \mathrm{d}x \\ &= \left| \frac{2x^3}{3} + \frac{8x}{3} \right|_0^1 \\ &= \frac{10}{3}\end{aligned}$$

Example 3

Evaluate $\int_{-1}^1 \int_0^2 (1 - 6x^2 y)\,\mathrm{d}x\,\mathrm{d}y.$ **[Winter 2016]**

Solution

$$\begin{aligned}\int_{-1}^1 \int_0^2 (1 - 6x^2 y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^1 \left[\int_0^2 (1 - 6x^2 y)\,\mathrm{d}x \right] \mathrm{d}y \\ &= \int_{-1}^1 \left| x - 6y \cdot \frac{x^3}{3} \right|_0^2 \mathrm{d}y\end{aligned}$$

$$= \int_{-1}^{1} \left| x - 2yx^3 \right|_0^2 \mathrm{d}y$$
$$= \int_{-1}^{1} (2 - 16y)\,\mathrm{d}y$$
$$= 2\int_0^1 2\,\mathrm{d}y$$
$$= 4$$

Example 4

Evaluate $\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy}$.

Solution

$$\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \int_2^a \left(\int_2^b \frac{\mathrm{d}x}{x} \right) \frac{\mathrm{d}y}{y} = \int_2^a \left| \log x \right|_2^b \frac{1}{y}\,\mathrm{d}y$$
$$= (\log b - \log 2) \int_2^a \frac{1}{y}\,\mathrm{d}y$$
$$= \log\left(\frac{b}{2}\right) \left| \log y \right|_2^a$$
$$= \log\left(\frac{b}{2}\right) (\log a - \log 2)$$
$$= \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$$

Another method: Since both the limits are constant and integrand (function) is explicit in x and y, the integral can be written as

$$\int_2^a \int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \int_2^a \frac{\mathrm{d}y}{y} \int_2^b \frac{\mathrm{d}x}{x}$$
$$= \left| \log y \right|_2^a \left| \log x \right|_2^b$$
$$= (\log a - \log 2)(\log b - \log 2)$$
$$= \log\left(\frac{a}{2}\right) \cdot \log\left(\frac{b}{2}\right)$$
$$= \log\left(\frac{b}{2}\right) \cdot \log\left(\frac{a}{2}\right)$$

Example 5

Evaluate $\int_0^1 \int_1^2 xy \, \mathrm{d}y \, \mathrm{d}x.$

Solution

$$\int_0^1 \int_1^2 xy \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \left\{ \int_1^2 y \, \mathrm{d}y \right\} x \, \mathrm{d}x$$

$$= \int_0^1 \left| \frac{y^2}{2} \right|_1^2 x \, \mathrm{d}x$$

$$= \int_0^1 \left(\frac{4}{2} - \frac{1}{2} \right) x \, \mathrm{d}x$$

$$= \frac{3}{2} \left| \frac{x^2}{2} \right|_0^1$$

$$= \frac{3}{2} \cdot \frac{1}{2}$$

$$= \frac{3}{4}$$

Another method: Since both the limits are constant and integrand (function) is explicit in x and y, the integral can be written as

$$\int_0^1 \int_1^2 xy \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 x \, \mathrm{d}x \cdot \int_1^2 y \, \mathrm{d}y$$

$$= \left| \frac{x^2}{2} \right|_0^1 \left| \frac{y^2}{2} \right|_1^2$$

$$= \frac{1}{2} \left(\frac{4}{2} - \frac{1}{2} \right)$$

$$= \frac{3}{4}$$

Example 6

Evaluate $\int_0^1 \int_0^x \mathrm{d}y \, \mathrm{d}x.$

Solution

$$\int_0^1 \int_0^x \mathrm{d}y \, \mathrm{d}x = \int_0^1 |y|_0^x \, \mathrm{d}x$$

$$= \int_0^1 x\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2}$$

Example 7

Evaluate $\int_0^1 \int_0^x e^{\frac{y}{x}}\,dy\,dx.$

Solution

$$\int_0^1 \int_0^x e^{\frac{y}{x}}\,dy\,dx = \int_0^1 \left| x e^{\frac{y}{x}} \right|_0^x dx$$

$$= \int_0^1 x(e-1)\,dx$$

$$= \left|\frac{x^2}{2}(e-1)\right|_0^1$$

$$= \frac{1}{2}(e-1)$$

Example 8

Evaluate $\int_0^1 \int_x^{x^2} xy\,dy\,dx.$

Solution

$$\int_0^1 \int_x^{x^2} xy\,dy\,dx = \int_0^1 \left\{ \int_x^{x^2} y\,dy \right\} x\,dx$$

$$= \int_0^1 \left|\frac{y^2}{2}\right|_x^{x^2} x\,dx$$

$$= \frac{1}{2}\int_0^1 \left[(x^2)^2 - x^2 \right] x\,dx$$

$$= \frac{1}{2}\int_0^1 (x^5 - x^3)\,dx$$

$$= \frac{1}{2}\left|\frac{x^6}{6} - \frac{x^4}{4}\right|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= -\frac{1}{24}$$

Example 9

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{\mathrm{d}x\,\mathrm{d}y}{1+x^2+y^2}$.

Solution

$$\int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{\mathrm{d}y}{\left(\sqrt{1+x^2}\right)^2+y^2}\right]\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left|\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right|_0^{\sqrt{1+x^2}} \mathrm{d}x$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0)\mathrm{d}x$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\mathrm{d}x$$

$$= \frac{\pi}{4}\left|\log\left(x+\sqrt{1+x^2}\right)\right|_0^1$$

$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Example 10

Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{1-x^2-y^2}}$.

Solution

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{1-x^2-y^2}} = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{\mathrm{d}x}{\sqrt{(1-y^2)-x^2}}\right]\mathrm{d}y$$

$$= \int_0^1 \left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{\frac{1-y^2}{2}}} \mathrm{d}y$$

$$= \int_0^1 \left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0\right)\mathrm{d}y$$

$$= \frac{\pi}{4}|y|_0^1$$

$$= \frac{\pi}{4}$$

Example 11

Sketch the region of integration and evaluate $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx$.

[Summer 2017]

Solution

1. Since the inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y: $y = 0$ to $y = \sqrt{x}$

   Limits of x: $x = 1$ to $x = 4$
3. The region is bounded by the line $x = 1$, $y = 0$, $x = 4$ and parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $x = 1$, $x = 4$ are (1, 1) (4, 2).

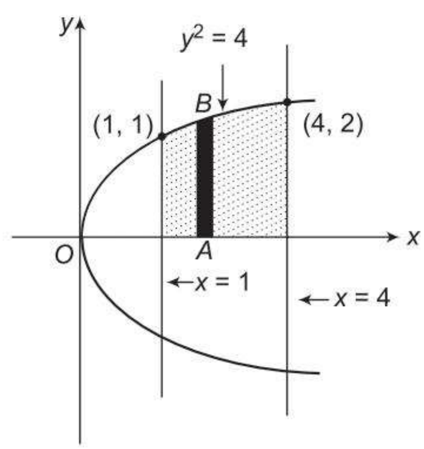

Fig. 9.4

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx = \int_1^4 \frac{3}{2}\left[\int_0^{\sqrt{x}} e^{\frac{y}{\sqrt{x}}} \, dy\right] dx$$

$$= \int_1^4 \frac{3}{2} \left| \frac{e^{\frac{y}{\sqrt{x}}}}{1/\sqrt{x}} \right|_0^{\sqrt{x}} dx$$

$$= \int_1^4 \frac{3}{2}\sqrt{x} \left| e^{\frac{4}{\sqrt{x}}} \right|_0^{\sqrt{x}} dx$$

$$= \int_1^4 \frac{3}{2}\sqrt{x}\,(e-1)dx$$

$$= \frac{3}{2}(e-1) \int_1^4 \sqrt{x} \, dx$$

$$= \frac{3}{2}(e-1) \left| \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^4$$

$$= \frac{3}{2}(e-1) \cdot \frac{2}{3} \left| x^{\frac{3}{2}} \right|_1^4$$

$$= (e-1)\left[4^{\frac{3}{2}} - 1\right]$$

$$= (e-1)\,[2^{2\cdot\frac{3}{2}} - 1]$$

$$= [e-1]\,[2^3 - 1]$$

$$= (e-1)\,[8-1]$$

$$= 7(e-1)$$

EXERCISE 9.1

Evaluate the following integrals:

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2 dx\, dy$

$\left[\textbf{Ans.:} \dfrac{856}{945}\right]$

2. $\int_0^1 \int_0^y xy\, e^{x-2} dx\, dy$

$\left[\textbf{Ans.:} \dfrac{1}{4e}\right]$

3. $\int_0^1 \int_0^x e^{x+y} dx\, dy$

$\left[\textbf{Ans.:} \dfrac{1}{2}(e-1)^2\right]$

4. $\int_{10}^1 \int_0^{\frac{1}{y}} ye^{xy} dx\, dy$

[**Ans.:** 9(1 – *e*)]

5. $\int_1^{\log 8} \int_0^{\log y} e^{x+y} dx\, dy$

[**Ans.:** 8(log 8 – 1)]

6. $\int_0^1 \int_{y^2}^y (1+xy^2) dx\, dy$

$\left[\textbf{Ans.:} \dfrac{41}{210}\right]$

7. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx$

$\left[\textbf{Ans.:} \dfrac{2a^4}{3}\right]$

9.2.4 Working Rule for Evaluation of Double Integrals Over a General Region

1. If the region is bounded by more than one curve then find the points of intersection of all the curves.
2. Draw all the curves and mark their point of intersection.
3. Identify the region of integration.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. The vertical strip starts from the lowest part of the region and terminates on the highest part of the region.
6. For vertical strip: (i) The lower limit of y is obtained from the curve where the vertical strip starts and the upper limit of y is obtained from the curve where it terminates.
   (ii) The lower limit of x is obtained from the leftmost point of the region and the upper limit of x is obtained from the rightmost point of the region.
7. The horizontal strip starts from the left part of the region and terminates on the right part of the region.
8. For horizontal strip: (i) The lower limit of x is obtained from the curve where the horizontal strip starts and upper limit is obtained from the curve where it terminates.
   (ii) The lower limit of y is obtained from the lowest point of the region and the upper limit of y is obtained from the highest point of the region.
9. If variation along the strip changes within the region then the region is divided into parts.

Example 1

Evaluate $\iint e^{ax+by}\,\mathrm{d}x\,\mathrm{d}y$, *over the triangle bounded by* $x = 0$, $y = 0$, $ax + by = 1$.

Solution

1. The region of integration is the ΔOPQ.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $ax + by = 1$.
3. Limits of y : $y = 0$ to $y = \dfrac{1-ax}{b}$

   Limits of x : $x = 0$ to $x = \dfrac{1}{a}$

$$I = \int_0^{\frac{1}{a}}\int_0^{\frac{1-ax}{b}} e^{ax+by}\,dx\,dy$$

$$= \int_0^{\frac{1}{a}} e^{ax}\int_0^{\frac{1-ax}{b}} e^{by}\,dy\,dx$$

$$= \int_0^{\frac{1}{a}} e^{ax}\left|\frac{e^{by}}{b}\right|_0^{\frac{1-ax}{b}}$$

$$= \frac{1}{b}\int_0^{\frac{1}{a}} e^{ax}\left[e^{(1-ax)}-1\right]dx$$

$$= \frac{1}{b}\int_0^{\frac{1}{a}}(e-e^{ax})dx$$

$$= \frac{1}{b}\left|ex-\frac{e^{ax}}{a}\right|_0^{\frac{1}{a}}$$

$$= \frac{1}{b}\left(\frac{e}{a}-\frac{e}{a}+\frac{1}{a}\right)$$

$$= \frac{1}{ab}$$

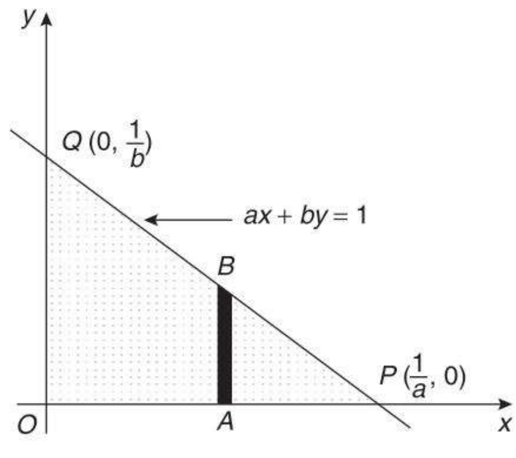

Fig. 9.5

Example 2

Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

Solution

1. The region of integration is *OPQ*.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to y-axis which starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.
3. Limits of $y: y = 0$ to $y = \sqrt{1-x^2}$

   Limits of $x: x = 0$ to $x = 1$

$$I = \int_0^1\int_0^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$$

$$= \int_0^1 x\int_0^{\sqrt{1-x^2}} -\frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)dy\,dx$$

$$= -\frac{1}{2}\int_0^1 x\left|2(1-y^2)^{\frac{1}{2}}\right|_0^{\sqrt{1-x^2}} dx$$

$$= -\frac{1}{2}\int_0^1 2x(x-1)dx$$

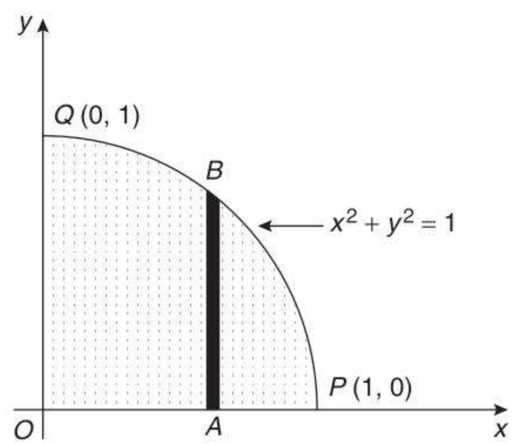

Fig. 9.6

$$\left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= -\left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1$$

$$= -\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{6}$$

Example 3

Evaluate $\iint (a-x)^2 \, dx \, dy$, *over the right half of the circle* $x^2 + y^2 = a^2$.

Solution

1. The region of integration is PQR.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the part of the circle $x^2 + y^2 = a^2$ below x-axis and terminates on the part of the circle $x^2 + y^2 = a^2$ above x-axis.
3. Limits of

$$y : y = -\sqrt{a^2 - x^2} \quad \text{to} \quad y = \sqrt{a^2 - x^2}$$

Limits of $x : x = 0$ to $x = a$

$$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a-x)^2 \, dx \, dy$$

$$= \int_0^a (a-x)^2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx$$

$$= \int_0^a (a-x)^2 \, |y|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \, dx$$

$$= \int_0^a (a^2 + x^2 - 2ax) \cdot 2\sqrt{a^2 - x^2} \, dx$$

$$= 2\int_0^a (a^2 + x^2 - 2ax)\sqrt{a^2 - x^2} \, dx$$

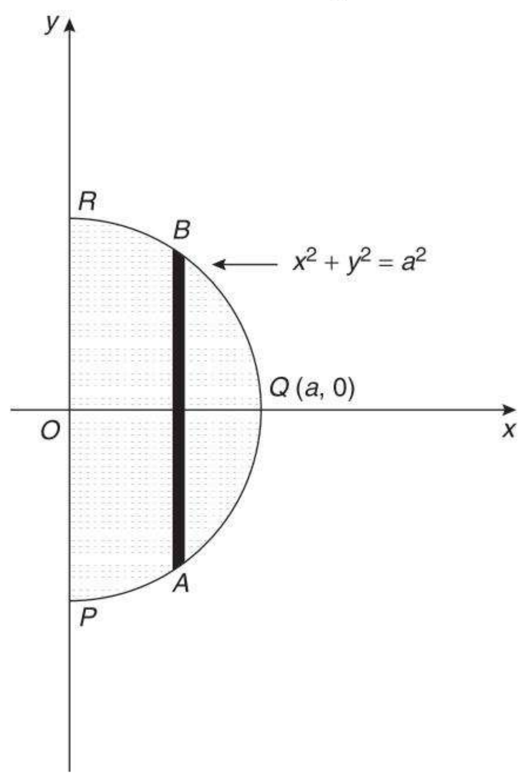

Fig. 9.7

Putting $x = a\sin\theta, \quad dx = a\cos\theta \, d\theta$

When $x = 0, \quad \theta = 0$

When $x = a, \quad \theta = \frac{\pi}{2}$

$$I = 2\int_0^{\frac{\pi}{2}} (a^2 + a^2\sin^2\theta - 2a^2\sin\theta) \cdot a\cos\theta \cdot a\cos\theta \, d\theta$$

$$= 2a^4 \int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta) d\theta$$

$$= a^4\left[\frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma 2}+\frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{\Gamma 3}-2\frac{\Gamma 1\,\Gamma\frac{3}{2}}{\Gamma\frac{5}{2}}\right] \qquad \left[\because 2\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,\mathrm{d}\theta = B\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}\right]$$

$$= a^4\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1}+\frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{2!}-2\frac{\Gamma\frac{3}{2}}{\frac{3}{2}\Gamma\frac{3}{2}}\right]$$

$$= a^4\left[\frac{\pi}{2}+\frac{\pi}{8}-\frac{4}{3}\right]$$

$$= a^4\left[\frac{5\pi}{8}-\frac{4}{3}\right]$$

Example 4

Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\mathrm{d}x\,\mathrm{d}y$, *over the first quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

1. The region of integration is OPQ.
2. The integration can be done w.r.t any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
3. Limits of y: $y=0$ to $y=b\sqrt{1-\frac{x^2}{a^2}}$

   Limits of x: $x=0$ to $x=a$

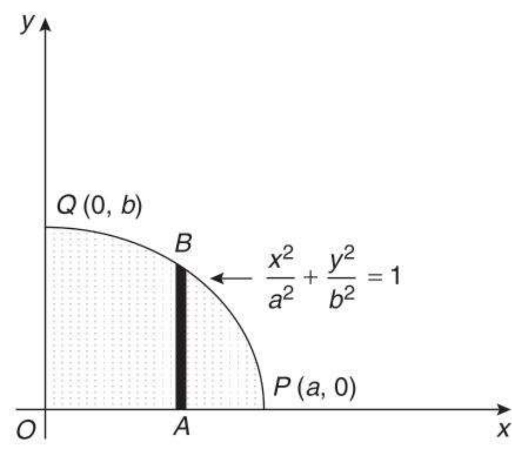

Fig. 9.8

$$\mathrm{I}=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^a x\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{b^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}\frac{2y}{b^2}\,\mathrm{d}y\,\mathrm{d}x$$

$$=\frac{b^2}{2}\int_0^a x\left|\frac{1}{\left(\frac{n}{2}+1\right)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}\right|_0^{b\sqrt{1-\frac{x^2}{a^2}}}\mathrm{d}x \qquad \left[\because \int[f(y)]^n f'(y)\,\mathrm{d}y=\frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= \frac{b^2}{n+2}\left|\frac{x^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{x^{n+4}}{n+4}\right|_0^a$$

$$= \frac{b^2}{n+2}\left|\frac{a^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{a^{n+4}}{n+4}\right|$$

$$= \frac{a^2b^2}{(n+2)} \cdot \frac{(n+2)}{2(n+4)}$$

$$= \frac{a^2b^2}{2(n+4)}$$

Example 5

Evaluate $\iint (x^2 + y^2)\mathrm{d}x\,\mathrm{d}y$ *over the ellipse* $2x^2 + y^2 = 1$.

Solution

1. The region of integration is *PQRS*, the ellipse $2x^2 + y^2 = 1$ or $\dfrac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \dfrac{y^2}{1^2} = 1$ with $\dfrac{1}{\sqrt{2}}$ and 1 as its axes.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to y-axis which starts from the part of the ellipse $2x^2 + y^2 = 1$ below x-axis and terminates on the part of the ellipse $2x^2 + y^2 = 1$ above x-axis.
3. Limits of

   $y : y = -\sqrt{1-2x^2}$ to $y = \sqrt{1-2x^2}$

   Limits of $x : x = -\dfrac{1}{\sqrt{2}}$ to $x = \dfrac{1}{\sqrt{2}}$

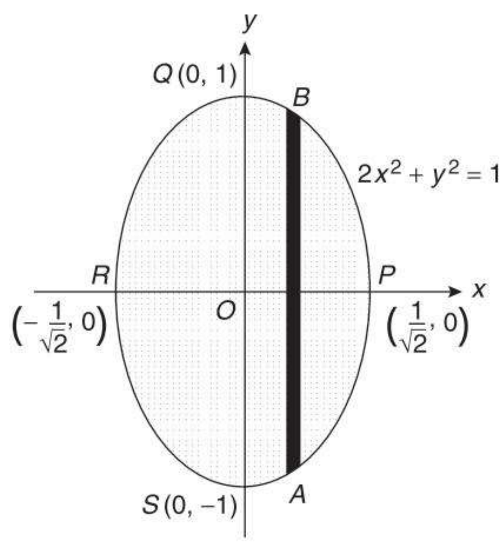

Fig. 9.9

$$\mathrm{I} = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (x^2 + y^2)\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left| x^2 y + \frac{y^3}{3} \right|_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \mathrm{d}x$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}}\right]\mathrm{d}x$$

$$= 4\int_0^{\frac{1}{\sqrt{2}}} \left[x^2\sqrt{1-2x^2} + \frac{1}{3}\left(1-2x^2\right)^{\frac{3}{2}}\right]\mathrm{d}x$$

Putting $2x^2 = t,\ x = \sqrt{\dfrac{t}{2}},\ \ \mathrm{d}x = \dfrac{1}{2\sqrt{2}\sqrt{t}}\,\mathrm{d}t$

When $x = 0$, $t = 0$

$$x = \frac{1}{\sqrt{2}},\ t = 1$$

$$\mathrm{I} = 4\int_0^1 \left[\frac{t}{2}\sqrt{1-t} + \frac{1}{3}(1-t)^{\frac{3}{2}}\right]\frac{1}{2\sqrt{2}\sqrt{t}}\,\mathrm{d}t$$

$$= \sqrt{2}\int_0^1 \left[\frac{1}{2}t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} + \frac{1}{3}t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}\right]\mathrm{d}t$$

$$= \sqrt{2}\left[\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right) + \frac{1}{3}B\left(\frac{1}{2},\frac{5}{2}\right)\right]$$

$$= \sqrt{2}\left[\frac{1}{2}\cdot\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3} + \frac{1}{3}\cdot\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma 3}\right]$$

$$= \sqrt{2}\left[\frac{1}{2}\cdot\frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{2} + \frac{1}{3}\cdot\frac{\Gamma\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2}\right]$$

$$= \sqrt{2}\left[\frac{1}{4}\cdot\frac{\pi}{4} + \frac{\pi}{8}\right]$$

$$= \frac{3\sqrt{2}p}{16}$$

Example 6

Evaluate $\iint (x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y$ *over the triangle with the vertices* (0, 1), (1, 1), (1, 2).

Solution

1. The region of integration is ΔPQR.
2. Equation of the line PQ is $y = 1$.

   Equation of the line PR is

$$y - 1 = \frac{2-1}{1-0}(x - 0) = x$$

$$y = x + 1$$

3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.
4. Limits of $y: y = 1$ to $y = x + 1$
   Limits of $x: x = 0$ to $x = 1$

$$I = \int_0^1 \int_1^{x+1} (x^2 - y^2)\,dy\,dx$$

$$= \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_1^{x+1} dx$$

$$= \int_0^1 \left[x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3} \right] dx$$

$$= \left| \frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3} \right|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12}$$

$$= -\frac{2}{3}$$

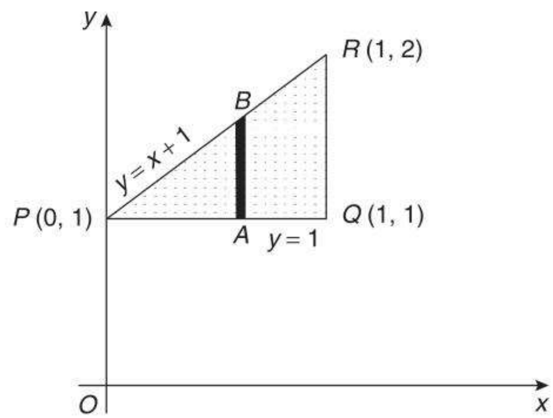

Fig. 9.10

Example 7

Evaluate $\iint e^{y^2}\,dx\,dy$ *over the region bounded by the triangle with vertices* (0, 0), (2, 1), (0, 1).

Solution

1. The region of integration is ΔOPQ.
2. Equation of the line OQ is $y = \dfrac{x}{2}$ or $x = 2y$.
3. Here, it is easier to integrate w.r.t. x first than y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $x = 2y$.
4. Limits of $x: x = 0$ to $x = 2y$
   Limits of $y: y = 0$ to $y = 1$

$$I = \int_0^1 \int_0^{2y} e^{y^2}\,dx\,dy$$

$$= \int_0^1 e^{y^2} \int_0^{2y} dx\,dy$$

$$= \int_0^1 e^{y^2} |x|_0^{2y}\,dy$$

$$= \int_0^1 e^{y^2} \cdot 2y\,dy$$

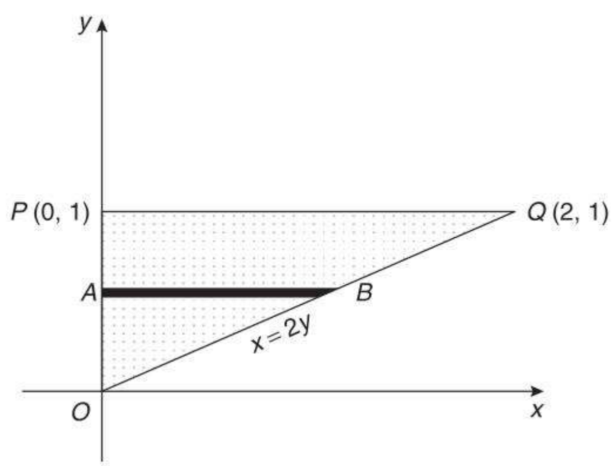

Fig. 9.11

$$= \left| e^{y^2} \right|_0^1 \qquad \left[\because \int e^{f(y)} f'(y) \mathrm{d}y = e^{f(y)} \right]$$

$$= e - 1$$

Example 8

Evaluate $\iint \dfrac{2xy^5}{\sqrt{1+x^2y^2-y^4}} \mathrm{d}x\,\mathrm{d}y$ *over the triangle having vertices* (0, 0), (1, 1) *and* (0, 1).

Solution

1. The region of integration is the ΔOPQ.
2. Equation of the line OP is $y = x$.
3. Here, it is easier to integrate w.r.t. x first than with y. Draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
4. Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = 1$

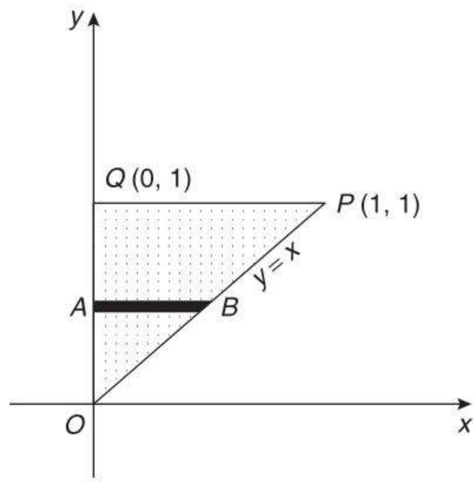

Fig. 9.12

$$I = \int_0^1 \int_0^y \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3 \int_0^y (1+x^2y^2-y^4)^{-\frac{1}{2}} \cdot 2xy^2 \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3 \left| 2(1+x^2y^2-y^4)^{\frac{1}{2}} \right|_0^y \mathrm{d}y \qquad \left[\because \int [f(x)]^n f'(x) \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \int_0^1 y^3 \cdot 2\left[1-(1-y^4)^{\frac{1}{2}} \right] \mathrm{d}y$$

$$= \int_0^1 2y^3 \mathrm{d}y - 2\int_0^1 (1-y^4)^{\frac{1}{2}} y^3 \mathrm{d}y$$

$$= 2\left| \frac{y^4}{4} \right|_0^1 - 2\int_0^1 (1-y^4)^{\frac{1}{2}} \frac{(-4y^3)}{-4} \mathrm{d}y$$

$$= \frac{1}{2} + \frac{1}{2} \left| \frac{2}{3}(1-y^4)^{\frac{3}{2}} \right|_0^1 \qquad \left[\because \int [f(x)]^n f'(x) \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

Example 9

Evaluate $\iint (x^2 + y^2)\,dx\,dy$ *over the region bounded by the lines* $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

Solution

1. The region of integration is *OPQR*.
2. The integration can be done w.r.t. any variable first. But in case of vertical strip we need to divide the region into three parts. Therefore, draw a horizontal strip *AB* parallel to x-axis which starts from the line $y = 4x$ and terminates on the line $x + y = 3$.

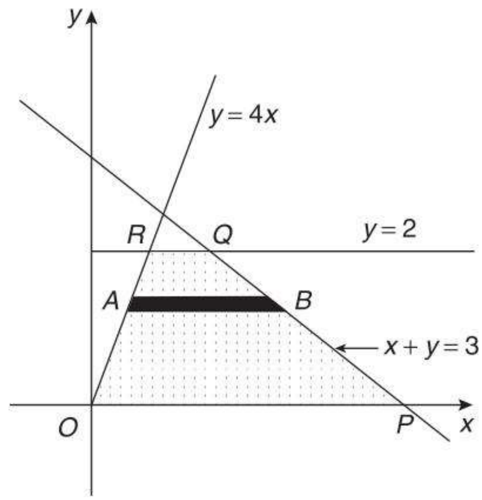

Fig. 9.13

3. Limits of $x : x = \frac{y}{4}$ to $x = 3 - y$

   Limits of y: $y = 0$ to $y = 2$

$$I = \int_0^2 \int_{\frac{y}{4}}^{3-y} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^2 \left| \frac{x^3}{3} + xy^2 \right|_{\frac{y}{4}}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3}\cdot\frac{y^3}{64} - \frac{y^3}{4} \right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192} y^3 \right] dy$$

$$= \left| \frac{1}{3}\cdot\frac{(3-y)^4}{-4} + 3\cdot\frac{y^3}{3} - \frac{241}{192}\cdot\frac{y^4}{4} \right|_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{192}\cdot 4 - \left(-\frac{27}{4}\right)$$

$$= \frac{463}{48}$$

Example 10

Evaluate $\iint_R (x+y)\,\mathrm{d}y\,\mathrm{d}x$, *where R is the region bounded by* $x = 0$, $x = 2$, $y = x$, $y = x + 2$. **[Summer 2016]**

Solution

1. The region of integration is $OPQR$.
2. The point of intersection of $x = 2$ and $y = x + 2$ is obtained as $y = 4$.

   The point of intersection is (2, 4).
3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the line $y = x$ and terminates on the line $y = x + 2$.

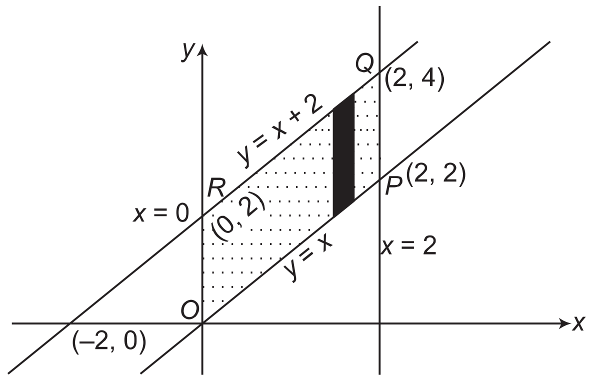

Fig. 9.14

4. Limits of y: $y = x$ to $y = x + 2$

   Limits of x: $x = 0$ to $x = 2$

$$I = \int_0^2 \int_x^{x+2} (x+y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \left[\int_x^{x+2} (x+y)\,\mathrm{d}y \right] \mathrm{d}x$$

$$= \int_0^2 \left| xy + \frac{y^2}{2} \right|_x^{x+2} \mathrm{d}x$$

$$= \int_0^2 \left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2} \right] \mathrm{d}x$$

$$= \int_0^2 \left[x^2 + 2x + \frac{x^2}{2} + 2x + 2 - x^2 - \frac{x^2}{2} \right] \mathrm{d}x$$

$$= \int_0^2 [4x + 2]\,\mathrm{d}x$$

$$= \left| 4\frac{x^2}{2} + 2x \right|_0^2$$

$$= \left| 2x^2 + 2x \right|_0^2$$

$$= 8 + 4 - 0$$

$$= 12$$

Example 11

$\iint\limits_R (2x - y^2)\,\mathrm{d}A$ *over the triangular region R enclosed between the lines* $y = -x + 1$, $y = x + 1$ *and* $y = 3$. **[Summer 2015]**

Solution

1. The region of integration is ΔPQR.
2. The points of intersection of
   (i) $y = -x + 1$ and $y = x + 1$ is obtained as
   $-x + 1 = x + 1$
   $2x = 0,\ x = 0$
   $y = 1$
   The points of intersection is $P(0, 1)$.
   (ii) $y = x + 1$ and $y = 3$ is obtained as
   $3 = x + 1$
   $x = 2,\ y = 3$
   The points of intersection is $Q(2, 3)$.

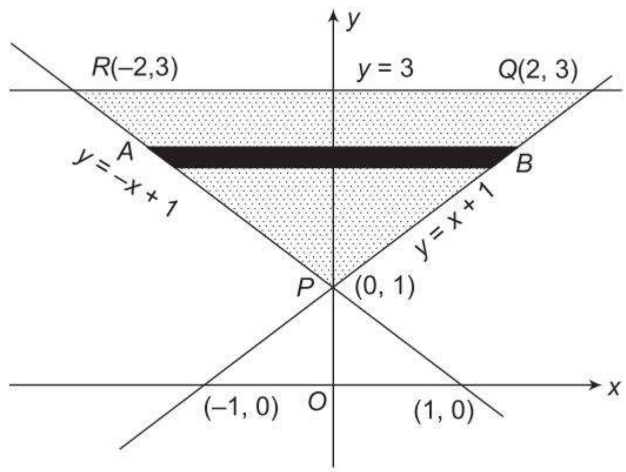

Fig. 9.15

   (iii) $y = -x + 1$ and $y = 3$ is obtained as
   $3 = -x + 1$
   $x = -2,\ y = 3$
   The points of intersection is $R(-2, 3)$.
3. The integration can be done w.r.t. any variable first. But in case of vertical strip, we need to divide the region into two parts. Therefore, draw a horizontal strip AB parallel to x-axis which starts from the line $y = -x + 1$ and terminates on the line $y = x + 1$.
4. Limits of x : $x = 1 - y$ to $x = y - 1$
   Limits of y : $y = 1$ to $y = 3$

$$I = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,\mathrm{d}A$$

$$= \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_1^3 \left| x^2 - xy^2 \right|_{1-y}^{y-1} \mathrm{d}y$$

$$= \int_1^3 \left[(y-1)^2 - y^2(y-1) - (1-y)^2 + y^2(1-y) \right] \mathrm{d}y$$

$$= \int_1^3 \left[(y-1)^2 - (1-y)^2 - 2y^3 + 2y^2 \right] dy$$

$$= \left| \frac{(y-1)^3}{3} - \frac{(1-y)^3}{-3} - 2\frac{y^4}{4} + 2\frac{y^3}{3} \right|_1^3$$

$$= \left[\frac{8}{3} - \frac{8}{3} - \frac{81}{2} + \frac{54}{3} + \frac{1}{2} - \frac{2}{3} \right]$$

$$= -\frac{68}{3}$$

Example 12

Evaluate $\iint \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$ *over the region bounded by the parabola* $y^2 = x$ *and the line* $y = x$.

Solution

1. The region of integration is OPQ.
2. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

$$x^2 = x$$
$$x(x-1) = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

The points of intersection are O (0, 0) and P (1, 1).

3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis, which starts from the line $y = x$ and terminates on the parabola $y^2 = x$.

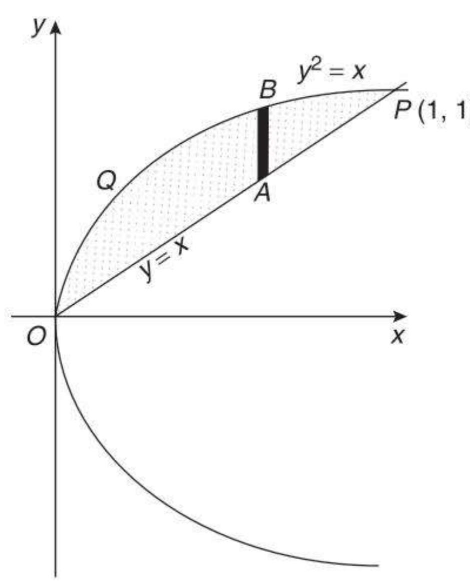

Fig. 9.16

4. Limits of y: $y = x$ to $y = \sqrt{x}$

Limits of x: $x = 0$ to $x = 1$

$$I = \int_0^1 \int_x^{\sqrt{x}} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^1 \frac{1}{(a-x)} \int_x^{\sqrt{x}} \left(-\frac{1}{2}\right)(ax-y^2)^{-\frac{1}{2}}(-2y)\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{(a-x)}\left|2(ax-y^2)^{\frac{1}{2}}\right|_x^{\sqrt{x}} \mathrm{d}x \qquad \left[\because \int [f(y)]^n f'(y)\mathrm{d}y = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= -\int_0^1 \frac{1}{(a-x)}\left[(ax-x)^{\frac{1}{2}} - (ax-x^2)^{\frac{1}{2}}\right]\mathrm{d}x$$

$$= -\int_0^1 \frac{\sqrt{x}}{a-x}\left(\sqrt{a-1} - \sqrt{a-x}\right)\mathrm{d}x$$

Putting $x = a\sin^2\theta$, $\mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta$

When $x = 0,\ \theta = 0$

When $x = 1, \theta = \sin^{-1}\dfrac{1}{\sqrt{a}}$

$$I = -\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sqrt{a}\sin\theta}{a\cos^2\theta}\left(\sqrt{a-1} - \sqrt{a}\cos\theta\right)2a\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sin^2\theta}{\cos\theta}\left(\sqrt{a-1} - \sqrt{a}\cos\theta\right)\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\sqrt{a-1} - \sqrt{a}\sin^2\theta\right]\mathrm{d}\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\sqrt{a-1}(\sec\theta - \cos\theta) - \frac{\sqrt{a}}{2}(1-\cos 2\theta)\right]\mathrm{d}\theta$$

$$= -2\sqrt{a}\left|\sqrt{a-1}[\log(\sec\theta + \tan\theta) - \sin\theta] - \frac{\sqrt{a}\theta}{2} + \frac{\sqrt{a}\sin 2\theta}{4}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left|\sqrt{a-1}\left[\log\left(\frac{1+\sin\theta}{\cos\theta}\right) - \sin\theta\right] - \frac{\sqrt{a}\theta}{2} + \frac{\sqrt{a}\sin\theta\cos\theta}{2}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left[\sqrt{a-1}\left(\log\frac{1+\frac{1}{\sqrt{a}}}{\sqrt{1-\frac{1}{a}}} - \frac{1}{\sqrt{a}}\right) - \frac{\sqrt{a}\sin^{-1}\frac{1}{\sqrt{a}}}{2} + \frac{\sqrt{a}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sqrt{1-\frac{1}{a}}\right]$$

$$= -2\sqrt{a(a-1)}\log\frac{\sqrt{a}+1}{\sqrt{a-1}} + \sqrt{a-1} + a\sin^{-1}\frac{1}{\sqrt{a}}$$

Example 13

Evaluate $\iint y\,\mathrm{d}x\,\mathrm{d}y$ *over the region enclosed by the parabola* $x^2 = y$ *and the line* $y = x + 2$.

Solution

1. The region of integration is POQ.
2. The points of intersection of $x^2 = y$ and $y = x + 2$ are obtained as

$$x^2 = x+2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$
$$\therefore y = 4, 1$$

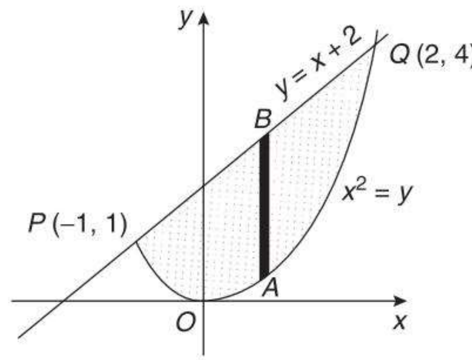

Fig. 9.17

The points of intersection are P (–1, 1) and Q (2, 4).

3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x + 2$.
4. Limits of y : $y = x^2$ to $y = x + 2$
   Limits of x : $x = -1$ to $x = 2$

$$I = \int_{-1}^{2}\int_{x^2}^{x+2} y\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_{-1}^{2}\left|\frac{y^2}{2}\right|_{x^2}^{x+2}\mathrm{d}x$$
$$= \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2 - x^4\right]\mathrm{d}x$$
$$= \frac{1}{2}\left|\frac{(x+2)^3}{3} - \frac{x^5}{5}\right|_{-1}^{2}$$
$$= \frac{1}{2}\left(\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right)$$
$$= \frac{36}{5}$$

Example 14

Evaluate $\iint xy(x+y)\,\mathrm{d}x\,\mathrm{d}y,$ *over the region enclosed by the parabolas* $x^2 = y$, $y^2 = -x$.

Solution

1. The region of integration is OPQ.

2. The points of intersection of the parabola $x^2 = y$, and $y^2 = -x$ are obtained as

$$y^4 = y$$
$$y = 0, 1$$
$$\therefore\ x = 0, -1.$$

The points of intersection are O (0, 0) and Q (–1, 1).

3. Here, it is easier to integrate w.r.t. x first. Draw a horizontal strip AB parallel to x-axis, which starts from the parabola $x^2 = y$ and terminates on the parabola $y^2 = -x$.

4. Limits of x : $x = -\sqrt{y}$ to $x = -y^2$

Limits of y : $y = 0$ to $y = 1$

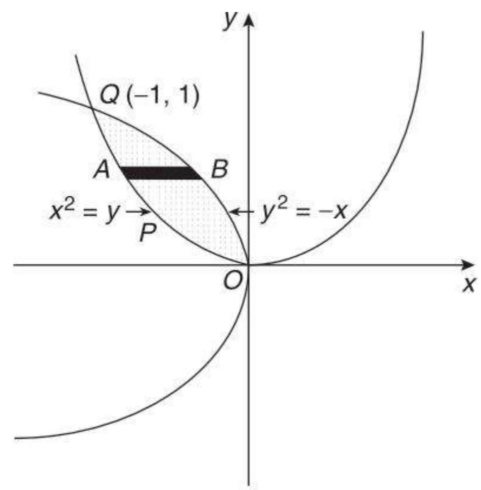

Fig. 9.18

$$I = \int_0^1 y \int_{-\sqrt{y}}^{-y^2} xy(x+y)\,dx\,dy$$

$$= \int_0^1 y \int_{-\sqrt{y}}^{-y^2} (x^2 + xy)\,dx\,dy$$

$$= \int_0^1 y \left| \frac{x^3}{3} + \frac{x^2 y}{2} \right|_{-\sqrt{y}}^{-y^2} dy$$

$$= \int_0^1 \left(\frac{-y^7}{3} + \frac{y^6}{2} + \frac{y^{\frac{5}{2}}}{3} - \frac{y^3}{2} \right) dy$$

$$= \left| -\frac{y^8}{24} + \frac{y^7}{14} + \frac{2y^{\frac{7}{2}}}{21} - \frac{y^4}{8} \right|_0^1$$

$$= 0$$

Example 15

Evaluate $\iint xy\,dx\,dy$ *over the region enclosed by the x-axis, the line* $x = 2a$ *and the parabola* $x^2 = 4ay$.

Solution

1. The region of integration is OPQ.
2. The point of intersection of the parabola $x^2 = 4ay$ and the line $x = 2a$ is obtained as

$$4a^2 = 4ay$$
$$y = a$$

The point of intersection is Q $(2a, a)$.

3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis, which starts from x-axis and terminates on the parabola $x^2 = 4ay$.

4. Limits of y : $y = 0$ to $y = \frac{x^2}{4a}$

   Limits of x : $x = 0$ to $x = 2a$

$$
\begin{aligned}
I &= \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dx\,dy \\
&= \int_0^{2a} x\int_0^{\frac{x^2}{4a}} y\,dy\,dx \\
&= \int_0^{2a} x\left|\frac{y^2}{2}\right|_0^{\frac{x^2}{4a}} dx \\
&= \int_0^{2a} x\cdot\frac{x^4}{32a^2}\,dx \\
&= \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} \\
&= \frac{1}{32a^2}\cdot\frac{64a^6}{6} \\
&= \frac{a^4}{3}
\end{aligned}
$$

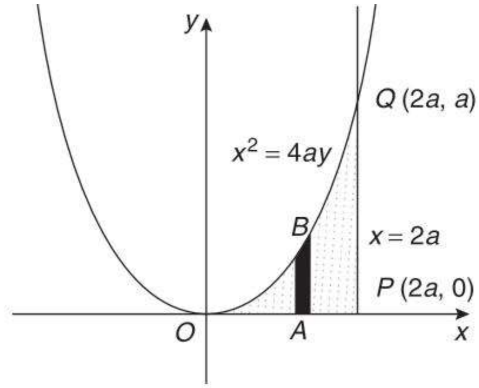

Fig. 9.19

Example 16

Evaluate $\iint xy\,dx\,dy$, *over the region enclosed by the circle* $x^2 + y^2 - 2x = 0$, *the parabola* $y^2 = 2x$ *and the line* $y = x$.

Solution

1. The region of integration is $OPQRO$.
2. (i) The points of intersection of the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as

$$x^2 + x^2 - 2x = 0$$
$$x = 0, 1$$
$$\therefore\; y = 0, 1$$

The points of intersection are O (0, 0) and P (1, 1).

(ii) The point of intersection of the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ is obtained as

$$x^2 + 2x - 2x = 0$$
$$x = 0$$
$$\therefore\; y = 0$$

The point of intersection is O (0, 0).

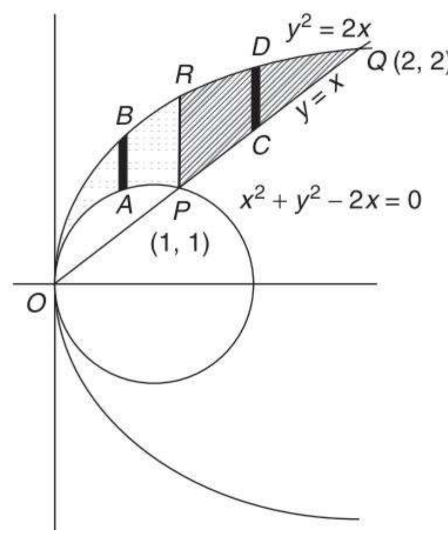

Fig. 9.20

(iii) The points of intersection of the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$

$$x = 0, 2$$

$$\therefore\ y = 0, 2$$

The points of intersection are O (0, 0) and Q (2, 2).

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y first we need to draw a vertical strip in the region. But one vertical strip does not cover the entire region, therefore, divide the region *OPQRO* into two subregions *OPR* and *RPQ* and draw one vertical strip in each subregion.
4. In the subregion *OPR,* strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$

   Limits of x : $x = 0$ to $x = 1$.
5. In the subregion *RPQ*, strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = x$ to $y = \sqrt{2x}$

   Limits of x : $x = 1$ to $x = 2$

$$\begin{aligned}
\mathrm{I} &= \int\int xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{OPR} xy\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{RPQ} xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^2\int_x^{\sqrt{2x}} xy\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 x\int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 x\int_x^{\sqrt{2x}} y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 x\left|\frac{y^2}{2}\right|_{\sqrt{2x-x^2}}^{\sqrt{2x}}\mathrm{d}x + \int_1^2 x\left|\frac{y^2}{2}\right|_x^{\sqrt{2x}}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 x(2x - 2x + x^2)\mathrm{d}x + \frac{1}{2}\int_1^2 x(2x - x^2)\mathrm{d}x \\
&= \frac{1}{2}\left|\frac{x^4}{4}\right|_0^1 + \frac{1}{2}\left|\frac{2x^3}{3} - \frac{x^4}{4}\right|_1^2 \\
&= \frac{1}{8} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{8} \\
&= \frac{7}{12}
\end{aligned}$$

Example 17

Evaluate $\int\int x^2\mathrm{d}x\mathrm{d}y$, *over the region in the first quadrant enclosed by the rectangular hyperbola* $xy = 16$, *the lines* $y = x$, $y = 0$ *and* $x = 8$. **[Winter 2014]**

Solution

1. The region of integration is *OPQR*.
2. (i) The points of intersection of the hyperbola $xy = 16$ and the line $y = x$ are obtained as

   $x^2 = 16, x = \pm 4$

   $\therefore\ y = \pm 4$

   Hence, R (4, 4) is the point of intersection in the first quadrant.

   (ii) The point of intersection of the hyperbola $xy = 16$ and line $x = 8$ is obtained as

   $8y = 16$

   $y = 2$

   The point of intersection is Q (8, 2).

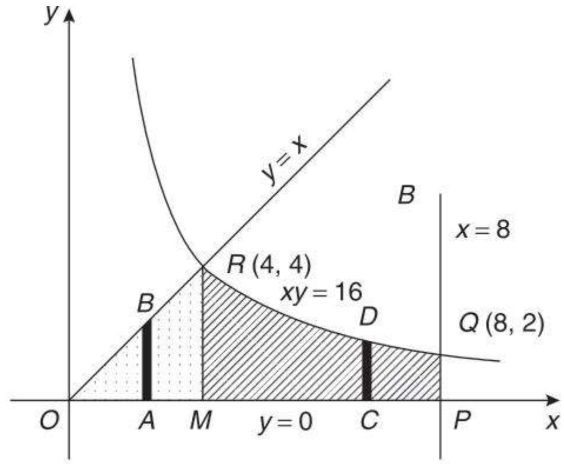

Fig. 9.21

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y first we need to draw a vertical strip in the region. But here one vertical strip cannot cover the entire region, and therefore divide the region *OPQR* into two subregions *OMR* and *RMPQ* and draw one vertical strip in each subregion.
4. In the subregion *OMR*, strip starts from x axis and terminates on the line $y = x$.

   Limits of y : $y = 0$ to $y = x$

   Limits of x : $x = 0$ to $x = 4$
5. In subregion *RMPQ*, strip starts from x axis and terminates on the rectangular hyperbola $xy = 16$

   Limits of y : $y = 0$ to $y = \dfrac{16}{x}$

   Limits of x : $x = 4$ to $x = 8$

$$
\begin{aligned}
\mathrm{I} &= \iint x^2 \mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{OMR} x^2 \mathrm{d}x\,\mathrm{d}y + \iint\limits_{RMPQ} x^2 \mathrm{d}x\,\mathrm{d}y \\
&= \int_0^4 \int_0^x x^2 \mathrm{d}y\,\mathrm{d}x + \int_4^8 \int_0^{\frac{16}{x}} x^2\ \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^4 x^2 \int_0^x \mathrm{d}y\,\mathrm{d}x + \int_4^8 x^2 \int_0^{\frac{16}{x}} \mathrm{d}y\,\mathrm{d}x \\
&= \int_0^4 x^2 \left|y\right|_0^x \mathrm{d}x + \int_4^8 x^2 \left|y\right|_0^{\frac{16}{x}} \mathrm{d}x \\
&= \int_0^4 x^3 \mathrm{d}x + \int_4^8 x^2 \cdot \frac{16}{x} \mathrm{d}x \\
&= \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 \\
&= 64 + 8(64 - 16) \\
&= 448
\end{aligned}
$$

Example 18

Evaluate $\iint \frac{dx\,dy}{x^4+y^2}$, *over the region bounded by the* $y \ge x^2$, $x \ge 1$.

Solution

1. The region of integration is bounded by $y \ge x^2$ (the region inside the parabola $x^2 = y$) and $x \ge 1$ (the region on the right of line $x = 1$).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$.
   The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of y : $y = x^2$ to $y \to \infty$
   Limits of x : $x = 1$ to $x \to \infty$

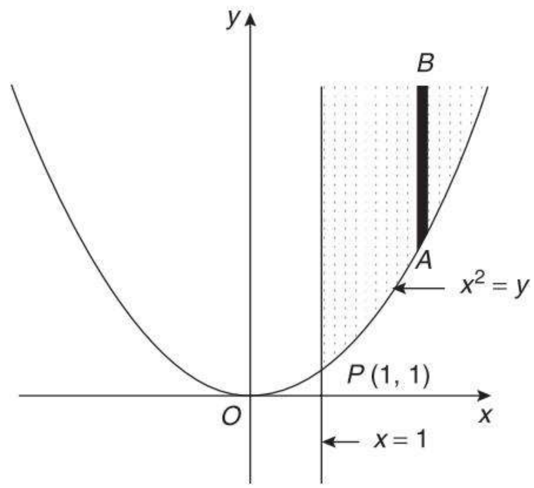

Fig. 9.22

$$I = \int_1^\infty \int_{x^2}^\infty \frac{1}{x^4+y^2}\,dy\,dx$$

$$= \int_1^\infty \left| \frac{1}{x^2}\tan^{-1}\frac{y}{x^2} \right|_{x^2}^{\infty} dx$$

$$= \int_1^\infty \frac{1}{x^2}(\tan^{-1}\infty - \tan^{-1}1)dx$$

$$= \int_1^\infty \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)dx$$

$$= \frac{\pi}{4}\left| -\frac{1}{x} \right|_1^\infty$$

$$= \frac{\pi}{4}$$

EXERCISE 9.2

Evaluate the following integrals:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \le x \le 2$, $1 \le y \le 2$.

[Ans.: $(\log 2)^2$**]**

2. $\iint \sin\pi(ax+by)\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$ and $ax + by = 1$.

$\left[\textbf{Ans.:}\ \frac{1}{\pi ab}\right]$

3. $\iint e^{3x+4y}\,dx\,dy,$ over the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{12}(3e^4 - 4e^3 + 1)\right]$$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\left[\textbf{Ans.:}\ \frac{16}{945}\right]$$

5. $\iint \sqrt{xy - y^2}\,dx\,dy,$ over the triangle having vertices (0, 0), (10, 1), (1, 1).

$[\textbf{Ans.:}\ 6]$

6. $\iint (x + y + a)\,dx\,dy,$ over the region bounded by the circle $x^2 + y^2 = a^2$.

$[\textbf{Ans.:}\ \pi a^3]$

7. $\iint xy\,dx\,dy,$ over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = \dfrac{x^2}{4a}$.

$$\left[\textbf{Ans.:}\ \frac{2048}{3}a^4\right]$$

8. $\iint 5 - 2x - y\,(dx)\,dy,$ over the region bounded by x-axis, the line $x + 2y = 3$ and the parabola $y^2 = x$.

$$\left[\textbf{Ans.:}\ \frac{217}{60}\right]$$

9. $\iint (4x^2 - y^2)^{\frac{1}{2}}\,dx\,dy,$ over the triangle bounded by x-axis, the line $y = x$ and $x = 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$$

10. $\iint xy(x + y)\,dx\,dy,$ over the region bounded by the parabola $y^2 = x$, $x^2 = y$.

$$\left[\textbf{Ans.:}\ \frac{3}{28}\right]$$

11. $\iint xy(x + y)\,dx\,dy,$ over the region bounded by the curve $x^2 = y$ and the line $x = y$.

$$\left[\textbf{Ans.:}\ \frac{3}{56}\right]$$

12. $\iint xy(x - 1)\,dx\,dy,$ over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y = 0$, $x = 1$, $x = 4$ and x-axis.

$[\textbf{Ans.:}\ 8(3 - \log 4)]$

9.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw a vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

Type I Change of Order of Integration

Example 1

Change the order of integration of $\int_0^1 \int_0^x f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

Solution

1. Since inner limits depends on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral
   $$= \int_0^1 \int_0^x f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$
2. Limits of y : $y = 0$ to $y = x$, along vertical strip $A'B'$
   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the lines $y = 0$, $y = x$, and $x = 1$.
4. The point of intersection of $y = x$ and $x = 1$ is $Q(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $y = x$ and terminates on the line $x = 1$.
   Limits of x : $x = y$ to $x = 1$
   Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_0^x f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_y^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

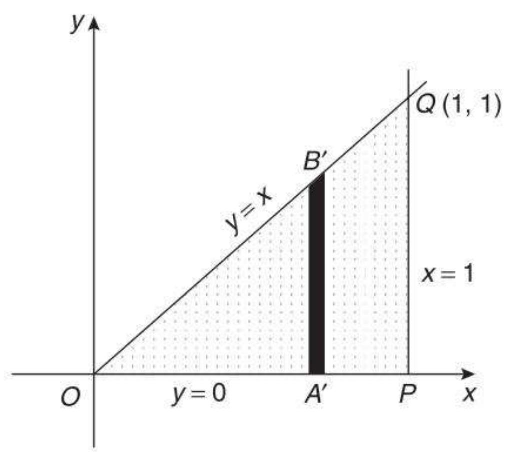

Fig. 9.23

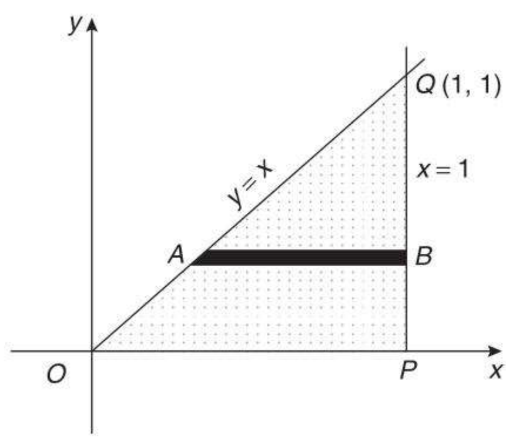

Fig. 9.24

Example 2

Change the order of integration of $\int_0^1 \int_0^y f(x, y)\,\mathrm{d}y\,\mathrm{d}x$.

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x and then w.r.t. y. The correct form of the integral
$= \int_0^1 \int_0^y f(x, y) \mathrm{d}x \, \mathrm{d}y.$
2. Limits of $x : x = 0$ to $x = y$, along horizontal strip $A'B'$
Limits of $y : y = 0$ to $y = 1$
3. The region is bounded by the lines $x = 0$, $x = y$, and $y = 1$.
4. The point of intersection of $x = y$ and $y = 1$ is $Q\,(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $x = y$ and terminates on the line $y = 1$.
Limits of $y : y = x$ to $y = 1$
Limits of $x : x = 0$ to $x = 1$

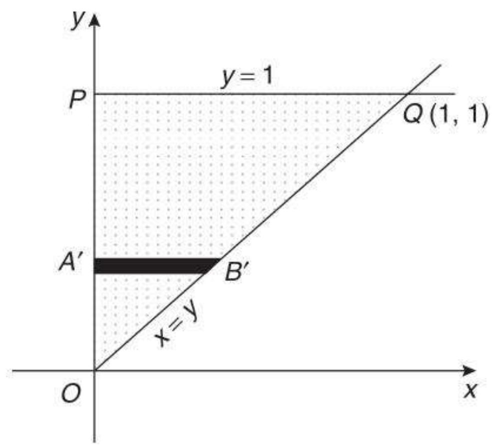

Fig. 9.25

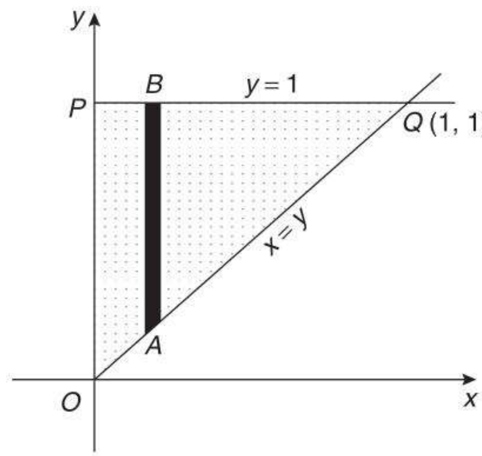

Fig. 9.26

Hence, the given integral after change of order is

$$\int_0^1 \int_0^y f(x, y) \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_x^1 f(x, y) \mathrm{d}y \, \mathrm{d}x$$

Example 3

Change the order of integration of $\int_0^a \int_x^a f(x, y) \mathrm{d}y \, \mathrm{d}x.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = x$ to $y = a$, along vertical strip $A'B'$.
Limits of $x : x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = x$ and $y = a$ is $Q\,(a, a)$.
5. To change the order of integration i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from

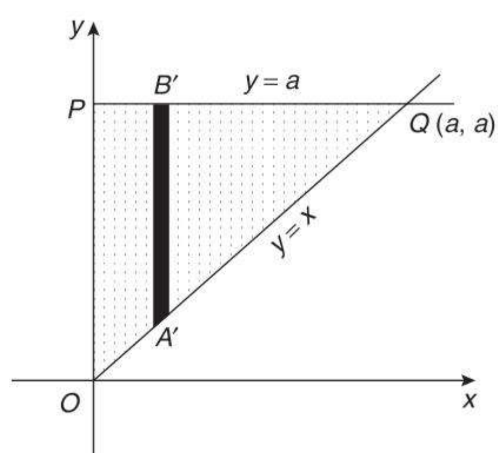

Fig. 9.27

the line $x = 0$ and terminates on the line $y = x$.

Limits of $x : x = 0$ to $x = y$

Limits of $y : y = 0$ to $y = a$

Hence, the given integral after change of order is

$$\int_0^a \int_x^a f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^a \int_0^y f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

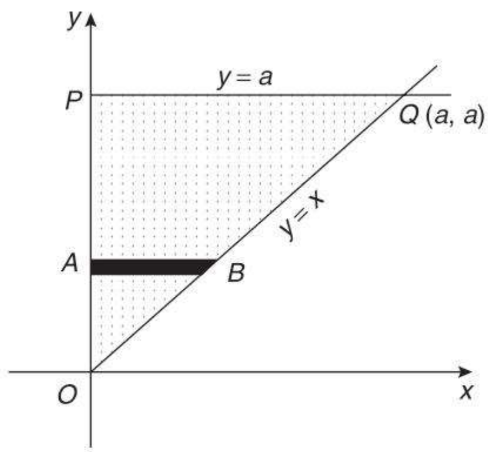

Fig. 9.28

Example 4

Change the order of integration of $\int_0^{\infty} \int_x^{\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral $= \int_0^{\infty} \int_x^{\infty} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$

2. Limits of $y : y = x$ to $y \to \infty$, along vertical strip

   Limits of $x : x = 0$ to $x \to \infty$

3. The region is bounded by the lines $y = x$ and $x = 0$.

4. Here the only point of intersection is origin O.

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $x = 0$ and terminates on the line $y = x$.

   Limits of $x : x = 0$ to $x = y$

   Limits of $y : y = 0$ to $y \to \infty$

Hence, the given integral after change of order is

$$\int_0^{\infty} \int_x^{\infty} f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\infty} \int_0^y f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

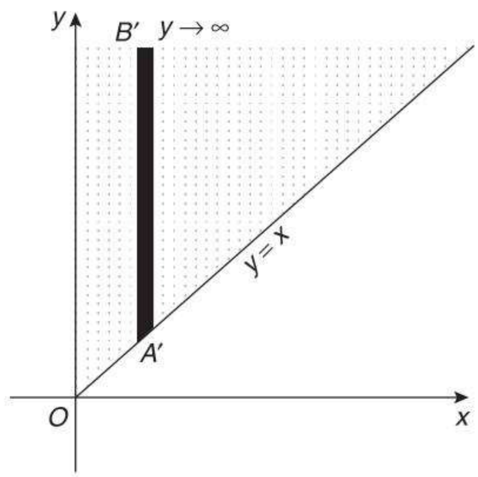

Fig. 9.29

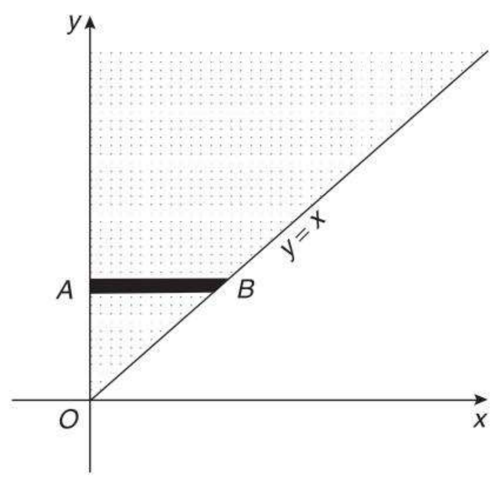

Fig. 9.30

Example 5

Change the order of integration of $\int_0^1 \int_x^{\sqrt{x}} f(x, y)\mathrm{d}y\, \mathrm{d}x.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the line $y = x$ and the parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

   $x^2 = x$

   $x = 0, 1$

   $\therefore\ y = 0, 1.$

   The points of intersection are O (0, 0) and Q (1, 1).
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = x$ and terminates on the line $y = x$.

   Limits of x : $x = y^2$ to $x = y$

   Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\sqrt{x}} f(x, y)\mathrm{d}y\, \mathrm{d}x = \int_0^1 \int_{y^2}^{y} f(x, y)\mathrm{d}x\, \mathrm{d}y$$

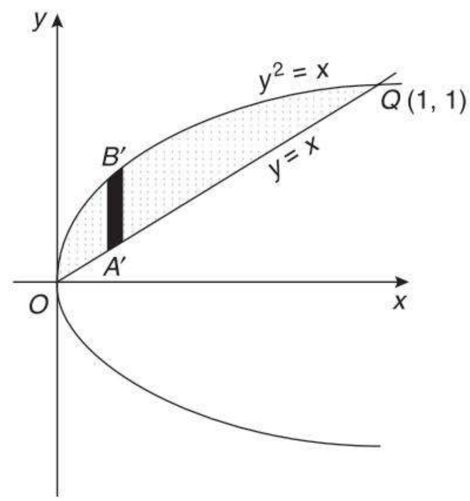

Fig. 9.31

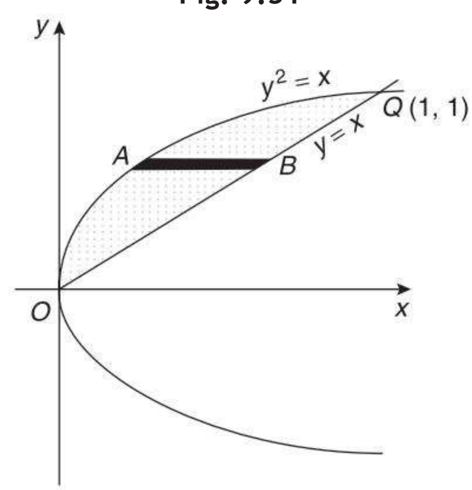

Fig. 9.32

Example 6

Change the order of integration of $\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\mathrm{d}x\, \mathrm{d}y.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.

2. Limits of x : $x = y^2$ to $x = y^{\frac{1}{3}}$

   Limits of y : $y = 0$ to $y = 1$
3. The region is bounded by the parabola $y^2 = x$ and the cubical parabola $y = x^3$.
4. The points of intersection of $y^2 = x$ and $y = x^3$ are obtained as

   $x^6 = x$

   $x = 0, 1$

   $\therefore\ y = 0, 1.$

   The points of intersection are O (0, 0) and Q (1, 1).

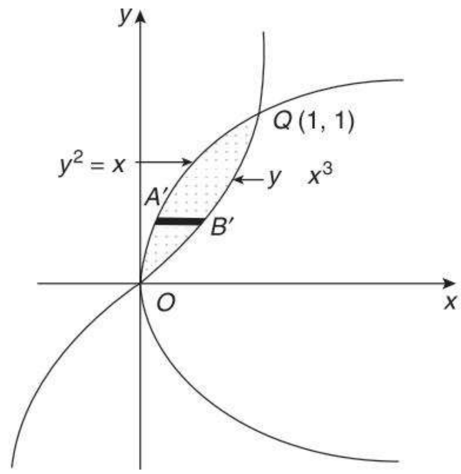

Fig. 9.33

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip parallel to y-axis which starts from the cubical parabola $y = x^3$ and terminates on the parabola $y^2 = x$.

   Limits of y : $y = x^3$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y)\mathrm{d}y\,\mathrm{d}x$$

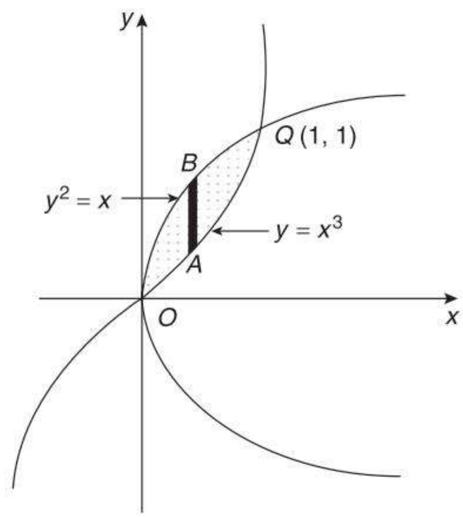

Fig. 9.34

Example 7

Change the order of integration of $\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\mathrm{d}x\,\mathrm{d}y.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = \dfrac{y-8}{4}$ to $x = \dfrac{y}{4}$

   Limits of y : $y = 0$ to $y = 8$
3. The region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and x-axis ($y = 0$).

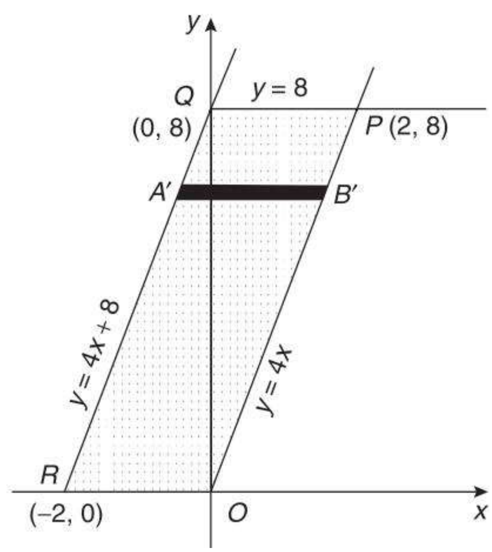

Fig. 9.35

4. The point of intersection of $y = 4x$ and $y = 8$ is obtained as
$$8 = 4x$$
$$x = 2.$$
The point of intersection is $P\,(2, 8)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region $OPQR$ into two subregions OQR and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

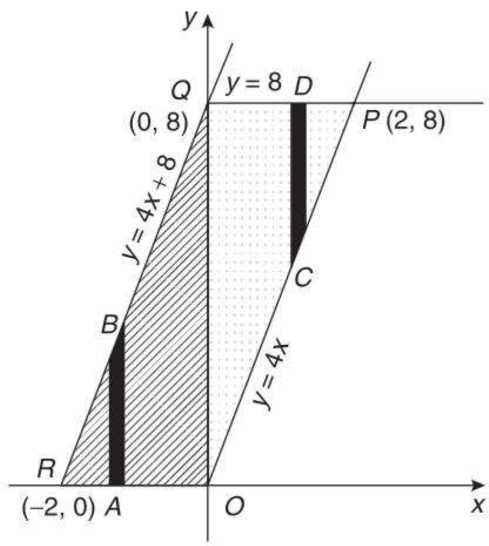

Fig. 9.36

(i) In subregion OQR, strip AB starts from x-axis and terminates on the line $y = 4x + 8$.
Limits of y : $y = 0$ to $y = 4x + 8$
Limits of x : $x = -2$ to $x = 0$

(ii) In subregion OPQ, strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.
Limits of y : $y = 4x$ to $y = 8$
Limits of x : $x = 0$ to $x = 2$

Hence, the given integral after change of order is

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy = \int_{-2}^{0} \int_0^{4x+8} f(x, y)\,dy\,dx + \int_0^2 \int_{4x}^{8} f(x, y)\,dy\,dx$$

Example 8

Change the order of integration of $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x, y)\,dx\,dy.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = 0$ to $x = \dfrac{y^2}{a}$

   Limits of y : $y = -a$ to $y = a$
3. The region is bounded by the y-axis, the parabola $y^2 = ax$, and the line $y = -a$, and $y = a$.
4. (i) The point of intersection of $y^2 = ax$ and $y = -a$ is obtained as
$$a^2 = ax$$
$$x = a$$
The point of intersection is $R\,(a, -a)$.

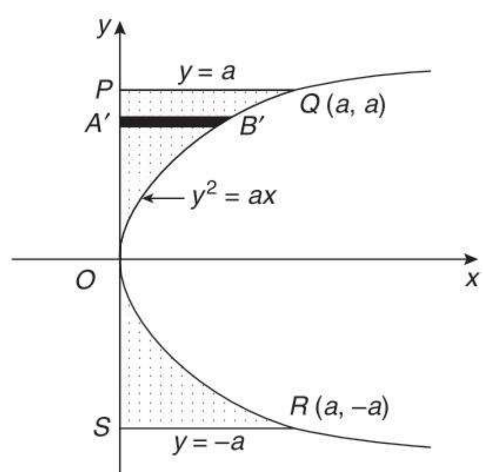

Fig. 9.37

(ii) The point of intersection of $y^2 = ax$ and $y = a$ is obtained as

$a^2 = ax$

$x = a$

The point of intersection is $Q\,(a, a)$.

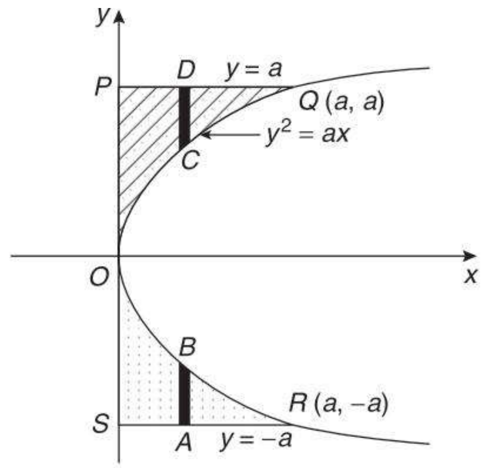

Fig. 9.38

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions ORS and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

(i) In subregion ORS, strip AB starts from the line $y = -a$ and terminates on the parabola $y^2 = ax$.

Limits of y : $y = -a$ to $y = -\sqrt{ax}$ (part of the parabola below x-axis)

Limits of x : $x = 0$ to $x = a$

(ii) In subregion OPQ, strip CD starts from the parabola $y^2 = ax$ and terminates on the line $y = a$.

Limits of y : $y = \sqrt{ax}$ to $y = a$

Limits of x : $x = 0$ to $x = a$

Hence, the given integral after change of order is

$$\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{0}^{a}\int_{-a}^{-\sqrt{ax}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{a}\int_{\sqrt{ax}}^{a} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

Example 9

Change the order of integration of $\displaystyle\int_{0}^{2}\int_{y}^{2+\sqrt{4-2y}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y$ to $x = 2 + \sqrt{4-2y}$

   Limits of y : $y = 0$ to $y = 2$
3. The region is bounded by the x-axis, the line $y = x$ and the parabola $(x-2)^2 = 2(2-y)$.
4. The points of intersection of $y = x$ and $(x-2)^2 = 2(2-y)$ are obtained as

   $(x-2)^2 = 2(2-x)$

   $x = 0, 2$

   $\therefore\ y = 0, 2.$

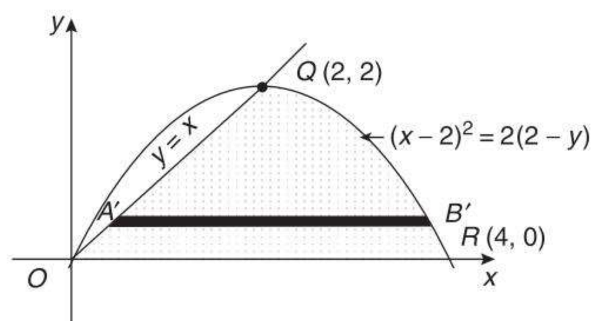

Fig. 9.39

The points of intersection are O (0, 0) and Q (2, 2).

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions OPQ and PQR. Draw a vertical strip parallel to y-axis in each subregion.

   (i) In subregion OPQ, strip AB starts from x-axis and terminates on the line $y = x$.
   Limits of $y : y = 0$ to $y = x$
   Limits of $x : x = 0$ to $x = 2$

   (ii) In subregion PQR, strip CD starts from x-axis and terminates on the parabola $(x - 2)^2 = 2(2 - y)$.

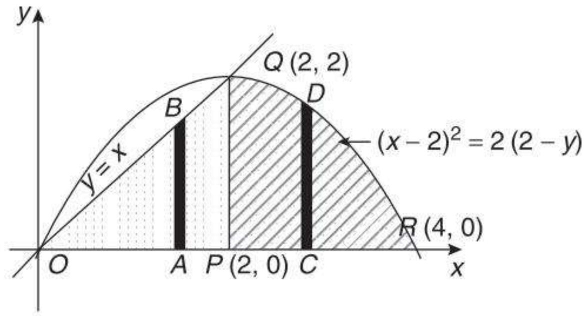

Fig. 9.40

   Limits of $y : y = 0$ to $y = 2x - \dfrac{x^2}{2}$
   Limits of $x : x = 2$ to $x = 4$

Hence, the given integral after change of order is

$$\int_0^2 \int_y^{2+\sqrt{4-2y}} f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_0^x f(x, y)\mathrm{d}y\,\mathrm{d}x + \int_2^4 \int_0^{2x-\frac{x^2}{2}} f(x, y)\mathrm{d}y\,\mathrm{d}x$$

Example 10

Change the order of integration of $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\mathrm{d}y\,\mathrm{d}x.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = x\tan\alpha$ to $y = \sqrt{a^2 - x^2}$
   Limits of $x : x = 0$ to $x = a\cos\alpha$
3. The region is bounded by the line $y = x\tan\alpha$, the circle $x^2 + y^2 = a^2$ and y-axis. Since given limits of x and y are positive, the region lies in the first quadrant.
4. The points of intersection of $y = x\tan\alpha$ and $x^2 + y^2 = a^2$ are obtained as
   $x^2 + x^2\tan^2\alpha = a^2$
   $x = \pm a\cos\alpha$
   $\therefore\ y = \pm a\sin\alpha.$
   The points of intersection are P $(a\cos\alpha, a\sin\alpha)$ and $P'(-a\cos\alpha, -a\sin\alpha)$.

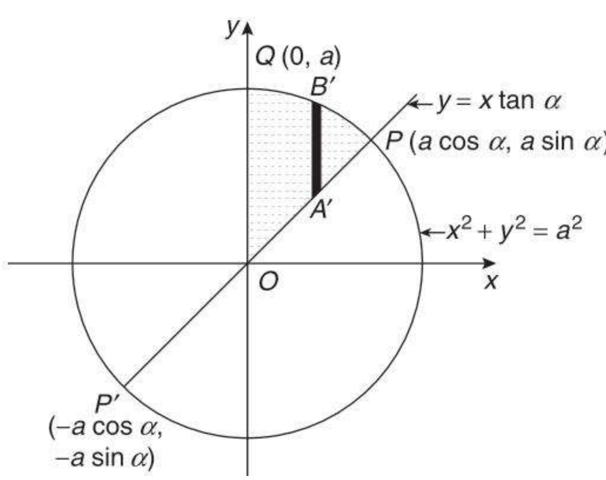

Fig. 9.41

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and PQR. Draw a horizontal strip in each subregion.

(i) In subregion OPR, strip AB starts from y-axis and terminates on the line $y = x \tan \alpha$.

Limits of x : $x = 0$ to $x = y \cot \alpha$

Limits of y : $y = 0$ to $y = a \sin \alpha$

(ii) In subregion PQR, strip CD starts from y-axis and terminates on the circle $x^2 + y^2 = a^2$.

Limits of x : $x = 0$ to $x = \sqrt{a^2 - y^2}$

Limits of y : $y = a \sin \alpha$ to $y = a$

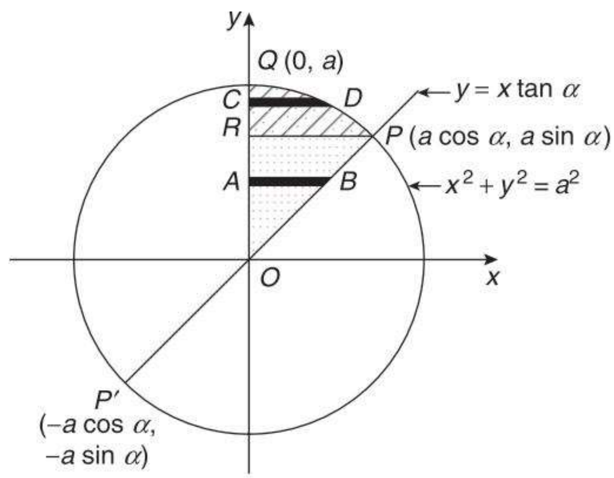

Fig. 9.42

Hence, given integral after change of order is

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\mathrm{d}y\,\mathrm{d}x = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\mathrm{d}x\,\mathrm{d}y + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\mathrm{d}x\,\mathrm{d}y$$

Example 11

Change the order of integration of $\int_0^4\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\mathrm{d}y\,\mathrm{d}x.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4x - x^2}$ to $y = \sqrt{4x}$

   Limits of x : $x = 0$ to $x = 4$.
3. The region is bounded by the circle $x^2 + y^2 - 4x = 0$, the parabola $y^2 = 4x$ and the line $x = 4$.
4. (i) The point of intersection of $x^2 + y^2 - 4x = 0$ and $y^2 = 4x$ is obtained as

   $x^2 = 0$

   $x = 0$

   $\therefore\ y = 0.$

   The point of intersection is O (0, 0).

   (ii) The points of intersection of $y^2 = 4x$ and $x = 4$ are obtained as

   $y^2 = 16$

   $y = \pm 4$

   The points of intersection are Q (4, 4) and Q' (4, −4).

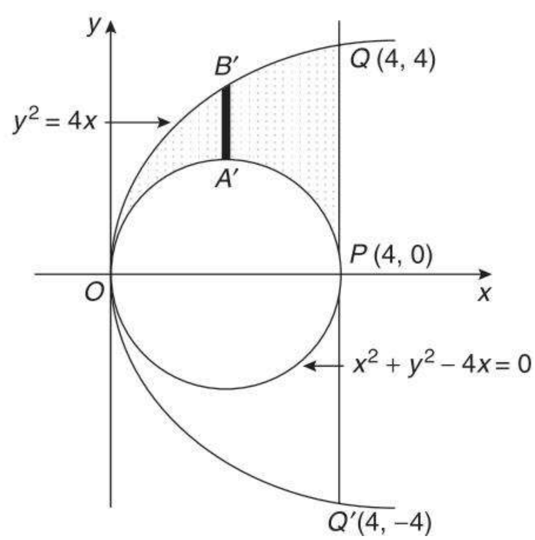

Fig. 9.43

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions ORT, TPS and RSQ. Draw a horizontal strip parallel to x-axis in each subregion.
   (i) In subregion ORT, strip AB starts from the parabola $y^2 = 4x$ and terminates on the circle $x^2 + y^2 - 4x = 0$.
   Limits of x :
   $$x = \frac{y^2}{4} \text{ to } x = 2 - \sqrt{4 - y^2}$$
   (Part of the circle where $x < 2$)
   Limits of y : $y = 0$ to $y = 2$
   (ii) In subregion TPS, strip CD starts from the circle $x^2 + y^2 - 4x = 0$ and terminates on the line $x = 4$.
   Limits of x : $x = 2 + \sqrt{4 - y^2}$
   (Part of circle where $x > 2$) to $x = 4$
   Limits of y : $y = 0$ to $y = 2$
   (iii) In subregion RSQ, strip EF starts from the parabola $y^2 = 4x$ and terminates on the line $x = 4$.
   Limits of x : $x = \frac{y^2}{4}$ to $x = 4$
   Limits of y : $y = 2$ to $y = 4$

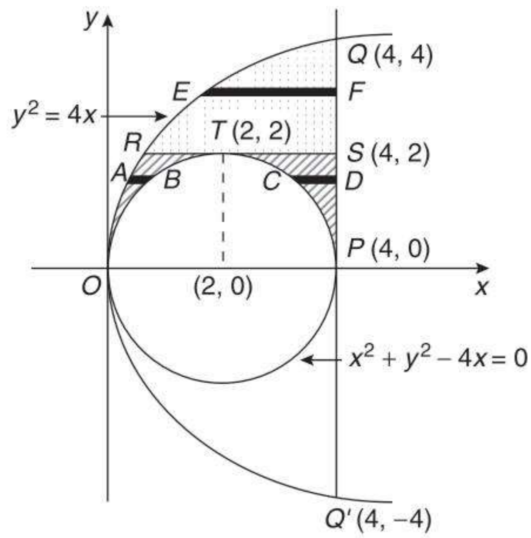

Fig. 9.44

Hence, given integral after change of order is

$$\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_{\frac{y^2}{4}}^{4} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Example 12

Change the order of integration of $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4-x}$ to $y = (4-x)^2$.
   Limits of x : $x = 0$ to $x = 2$
3. The region is enclosed by the parabolas $y^2 = 4 - x$, $y = (4-x)^2$, the lines $x = 0$ and $x = 2$.
4. (i) The points of intersection of $x = 2$ and $y^2 = 4 - x$ are obtained as

$y^2 = (4 - 2)$

$y = \pm\sqrt{2}.$

The points of intersection are $Q\,(2, \sqrt{2})$ and $Q'(2, -\sqrt{2})$.

(ii) The point of intersection of $x = 2$ and $y = (4 - x)^2$ is obtained as

$y = (4 - 2)^2 = 4.$

The point of intersection is $S\,(2, 4)$.

(iii) The points of intersection of $x = 0$ and $y^2 = 4 - x$ are obtained as

$y^2 = 4$

$y = \pm 2.$

The points of intersection are $P\,(0, 2)$ and $P'\,(0, -2)$.

(iv) The point of intersection of $x = 0$ and $y = (4 - x)^2$ is obtained as

$y = 16.$

The point of intersection is $U\,(0, 16)$.

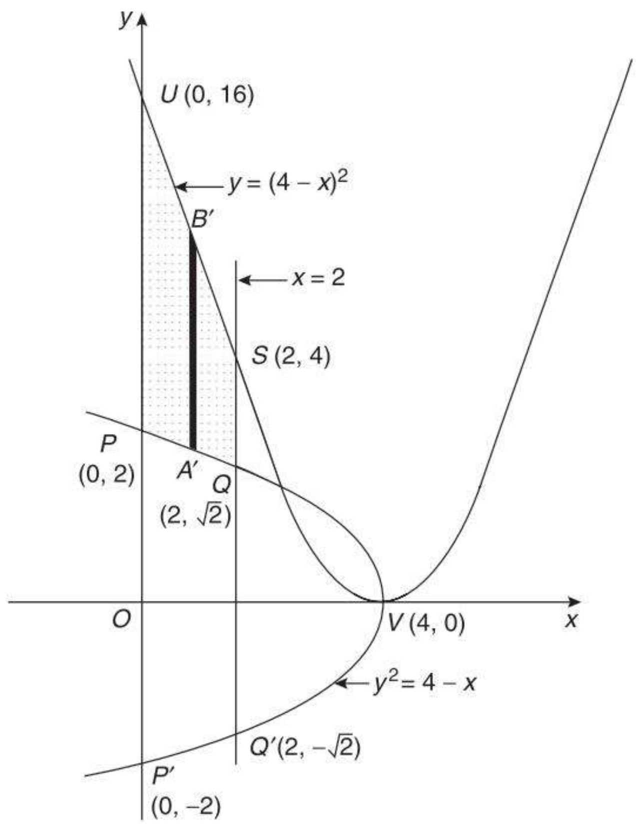

Fig. 9.45

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions PQR, $PRST$ and STU. Draw a horizontal strip in each subregion.

(i) In subregion PQR, strip AB starts from the parabola $y^2 = 4 - x$ and terminates on the line $x = 2$.

Limits of x :

$x = 4 - y^2 \quad \text{to} \quad x = 2$

Limits of y :

$y = \sqrt{2} \quad \text{to} \quad y = 2$

(ii) In subregion $PRST$, strip CD starts from y-axis and terminates on the line $x = 2$.

Limits of x : $x = 0 \;\text{ to }\; x = 2$

Limits of y : $y = 2 \;\text{ to }\; y = 4$

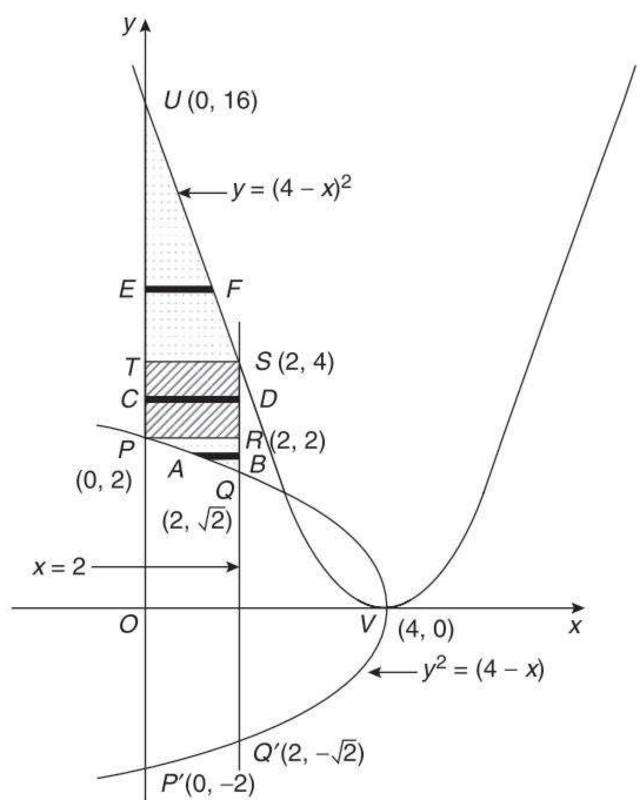

Fig. 9.46

(iii) In subregion *STU*, strip *EF* starts from *y*-axis and terminates on the parabola $y = (4-x)^2$.

Limits of $x : x = 0$ to $x = 4-\sqrt{y}$ (Part of the parabola where $x < 4$)

Limits of $y : y = 4$ to $y = 16$

Hence, the given integral after change of order is

$$\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx = \int_{\sqrt{2}}^2 \int_{4-y^2}^2 f(x, y)\,dx\,dy + \int_2^4 \int_0^2 f(x, y)\,dx\,dy$$
$$+ \int_4^{16} \int_0^{4-\sqrt{y}} f(x, y)\,dx\,dy$$

Type II Evaluation of Double Integrals by Changing the Order of Integration

Example 1

Change the order of integration and evaluate $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = x$ to $y = a$, along vertical strip

   Limits of $x : x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$
4. The point of intersection of $y = x$ and $y = a$ is $Q\,(a, a)$.
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip *AB* parallel to x-axis which starts from y-axis and terminates on the line $y = x$.

   Limits of $x : x = 0$ to $x = y$

   Limits of $y : y = 0$ to $y = a$

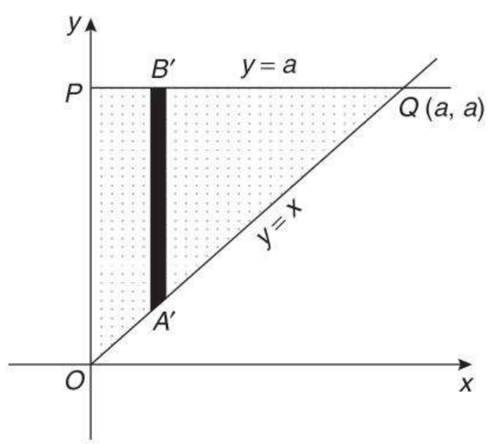

Fig. 9.47

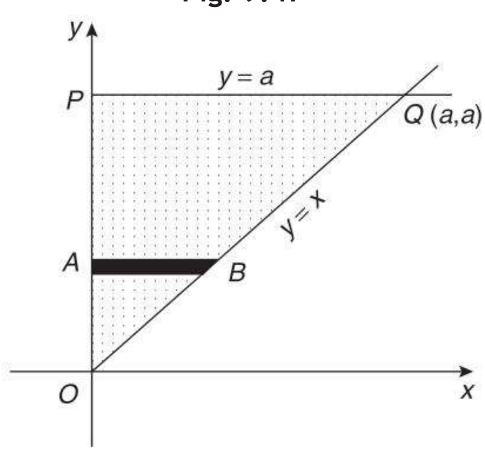

Fig. 9.48

Hence, the given integral after change of order is

$$\int_0^a \int_x^a (x^2+y^2)\,dy\,dx = \int_0^a \int_0^y (x^2+y^2)\,dx\,dy$$
$$= \int_0^a \left| \frac{x^3}{3} + y^2 x \right|_0^y dy$$
$$= \int_0^a \left(\frac{y^3}{3} + y^3 \right) dy$$

$$= \int_0^a \frac{4}{3} y^3 \, dy$$

$$= \frac{4}{3} \left| \frac{y^4}{4} \right|_0^a$$

$$= \frac{a^4}{3}$$

Example 2

Change the order of integration and evaluate $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$.

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = x$ to $y = \pi$
   Limits of $x : x = 0$ to $x = \pi$
3. The region is bounded by the line $y = x$, $y = \pi$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = \pi$ is $P\,(\pi, \pi)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = \pi$

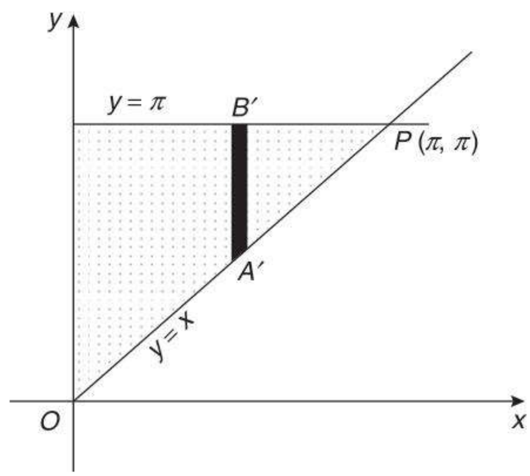

Fig. 9.49

Hence, the given integral after change of order is

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx = \int_0^{\pi} \frac{\sin y}{y} \int_0^{y} dx \, dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} |x|_0^y \, dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} \cdot y \, dy$$

$$= \int_0^{\pi} \sin y \, dy$$

$$= \left| -\cos y \right|_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= 2$$

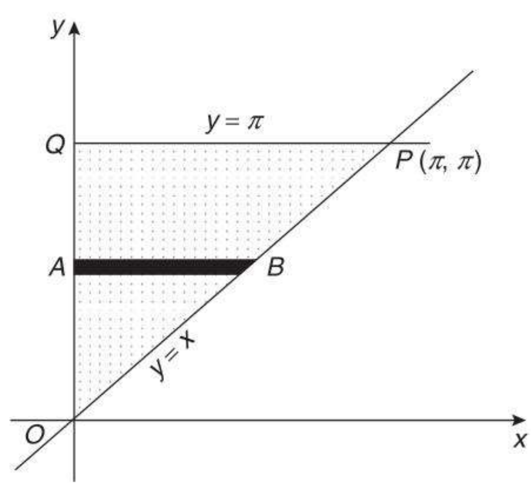

Fig. 9.50

Example 3

Evaluate the iterated integral $\int_0^1\int_x^1 \sin y^2 \,dy\,dx$. **[Winter 2013]**

Solution

1. Since the inner limits depend on x, the function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = 1$, along vertical strip
   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the lines $y = x$, $y = 1$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = 1$ is $P(1, 1)$.

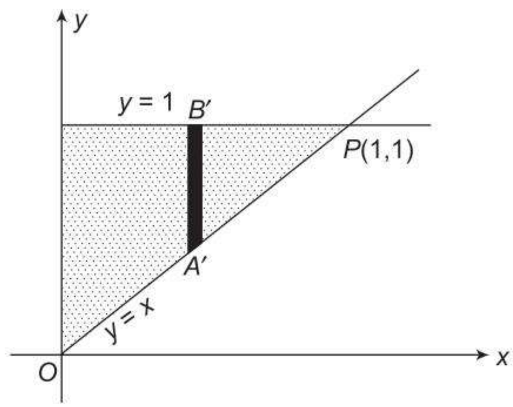

Fig. 9.51

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = 1$

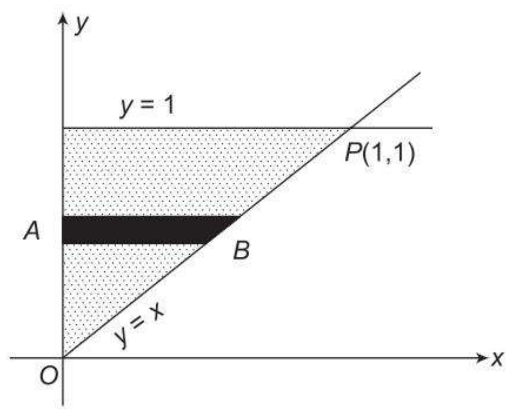

Fig. 9.52

Hence, the given integral after change of order is

$$\int_0^1\int_x^1 \sin y^2 \,dy\,dx = \int_0^1\int_0^y \sin y^2 \,dx\,dy$$

$$= \int_0^1 \sin y^2 \left|x\right|_0^y dy$$

$$= \int_0^1 \sin y^2 \cdot y\,dy$$

$$= \frac{1}{2}\int_0^1 \sin y^2 \cdot 2y\,dy$$

$$= \frac{1}{2}\left|-\cos y^2\right|_0^1 dy \qquad \left[\because \int \sin f(y)\cdot f'(y)\,dy = -\cos f(y)\right]$$

$$= \frac{1}{2}\left[-\cos 1 + \cos 0\right]$$

$$= \frac{1}{2}[1 - \cos 1]$$

Example 4

Change the order of integration and evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \mathrm{d}y\,\mathrm{d}x.$

[Winter 2015]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y \to \infty$, along vertical strip
   Limits of x : $x = 0$ to $x \to \infty$
3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here, the only point of intersection is origin.
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = \infty$

Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^\infty \left\{\int_0^y \mathrm{d}x\right\} \frac{e^{-y}}{y} \mathrm{d}y$$

$$= \int_0^\infty |x|_0^y \frac{e^{-y}}{y} \mathrm{d}y$$

$$= \int_0^\infty y \cdot \frac{e^{-y}}{y} \mathrm{d}y$$

$$= \int_0^\infty e^{-y} \mathrm{d}y$$

$$= \left|-e^{-y}\right|_0^\infty$$

$$= -(e^{-\infty} - e^0)$$

$$= 1$$

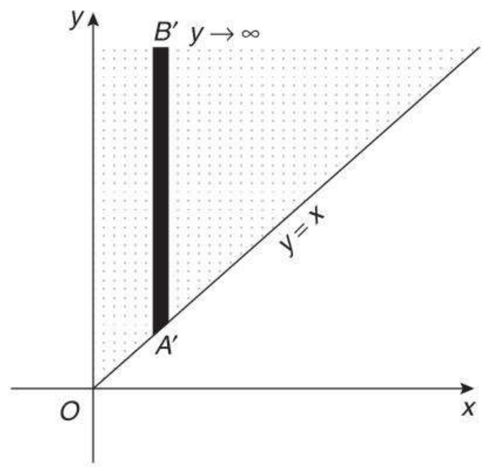

Fig. 9.53

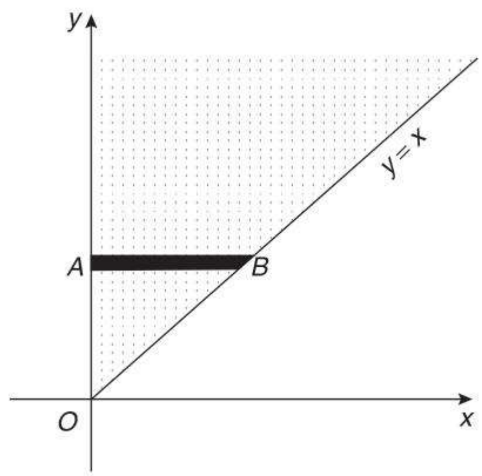

Fig. 9.54

Example 5

Evaluate the integral $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y$ *by changing the order of integration.*

[Summer 2017, 2015]

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = \dfrac{y}{2}$ to $x = 1$
   along horizontal strip $A'B'$
   Limits of $y : y = 0$ to $y = 2$
3. The region is bounded by the lines $y = 2x$, $x = 1$, $y = 2$, and $y = 0$.
4. The point of intersection of $y = 2x$ and $x = 1$ is $x = 1$, $y = 2$.
   The point of intersection is Q (1, 2).
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $y = 2x$.
   Limits of $y : y = 0$ to $y = 2x$
   Limits of $x : x = 0$ to $x = 1$

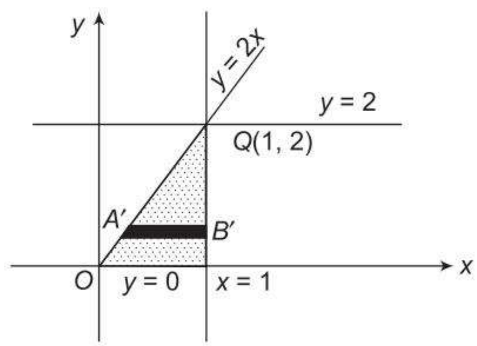

Fig. 9.55

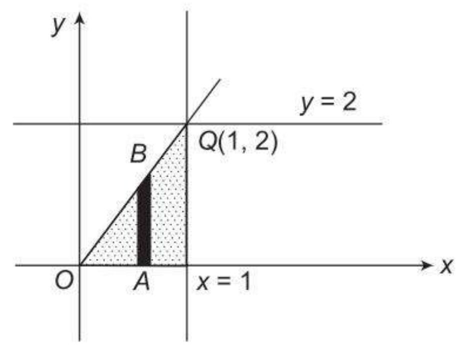

Fig. 9.56

Hence, the given integral after change of order is

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{2x} e^{x^2}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \left\{ \int_0^{2x} \mathrm{d}y \right\} e^{x^2}\,\mathrm{d}x$$

$$= \int_0^1 2x \cdot e^{x^2}\,\mathrm{d}x$$

$$= \left| e^{x^2} \right|_0^1 \qquad \left[\because \int e^{f(x)} f'(x)\,\mathrm{d}x = e^{f(x)} \right]$$

$$= e^1 - e^0$$

$$= e - 1$$

Example 6

Change the order of integration and evaluate $\int_0^{\infty}\int_0^{x} xe^{-\frac{x^2}{y}}\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = 0$ to $y = x$.
   Limits of $x : x = 0$ to $x \to \infty$.
3. The region is the part of the first quadrant bounded between the lines $y = x$ and $y = 0$.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from the line $y = x$ and extends up to infinity.
   Limits of $x : x = y$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$

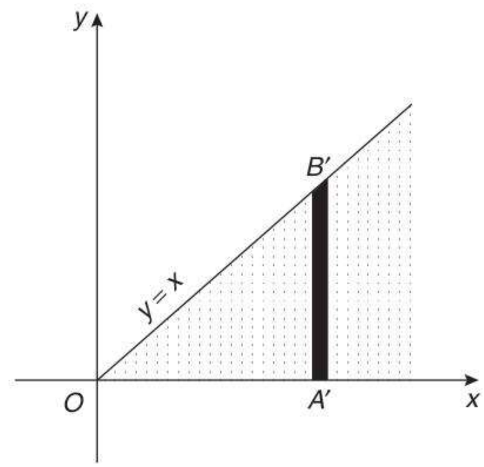

Fig. 9.57

Hence, the given integral after change of order is

$$\int_0^{\infty}\int_0^{x} xe^{-\frac{x^2}{y}}\,dy\,dx = \int_0^{\infty}\int_y^{\infty} xe^{-\frac{x^2}{y}}\,dx\,dy$$

$$= \int_0^{\infty}\left(-\frac{y}{2}\right)\int_y^{\infty} e^{-\frac{x^2}{y}}\left(-\frac{2x}{y}\right)dx\,dy$$

$$= -\frac{1}{2}\int_0^{\infty} y\left|e^{-\frac{x^2}{y}}\right|_y^{\infty} dy$$

$$\left[\because \int e^{f(x)} f'(x)dx = e^{f(x)}\right]$$

$$= -\frac{1}{2}\int_0^{\infty} y(0 - e^{-y})dy$$

$$= \frac{1}{2}\left|-ye^{-y} - e^{-y}\right|_0^{\infty}$$

$$= \frac{1}{2}$$

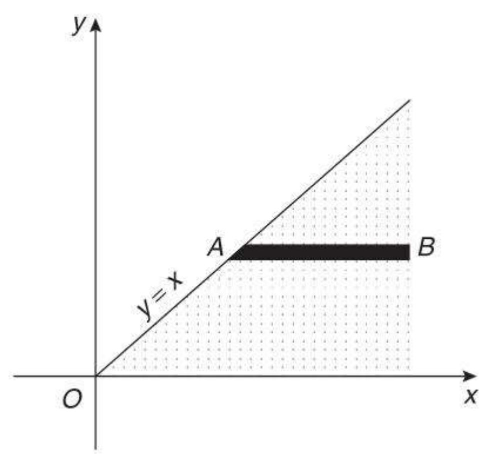

Fig. 9.58

Example 7

Change the order of integration and evaluate $\int_0^{a}\int_y^{a} \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy.$

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = y$ to $x = a$, along horizontal strip $A'B'$
   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the lines $y = x$, $x = a$ and $y = 0$.
4. The point of intersection of $y = x$ and $x = a$ is $x = a$, $y = a$.
   The point of intersection is $Q\ (a, a)$.
5. To change the order of integration, i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $y = x$.
   Limits of $y : y = 0$ to $y = x$
   Limits of x : $x = 0$ to $x = a$

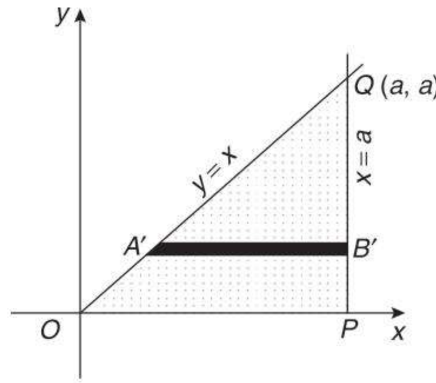

Fig. 9.59

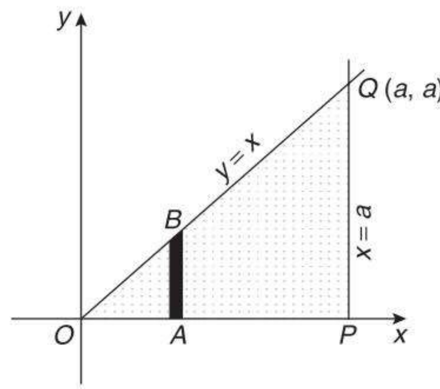

Fig. 9.60

Hence, the given integral after change of order is

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \int_0^x \frac{x^2}{\sqrt{x^2+y^2}}\,\mathrm{d}x$$

$$= \int_0^a \left\{ \int_0^x \frac{1}{\sqrt{x^2+y^2}}\,\mathrm{d}y \right\} x^2\,\mathrm{d}x$$

$$= \int_0^a \left| \log\left(y+\sqrt{y^2+x^2}\right) \right|_0^x x^2\,\mathrm{d}x$$

$$= \int_0^a \left[\log\left(x+\sqrt{2x^2}\right) - \log x \right] x^2\,\mathrm{d}x$$

$$= \int_0^a \left[\log \frac{x\left(1+\sqrt{2}\right)}{x} \right] x^2\,\mathrm{d}x$$

$$= \log\left(1+\sqrt{2}\right) \int_0^a x^2\,\mathrm{d}x$$

$$= \log\left(1+\sqrt{2}\right) \left| \frac{x^3}{3} \right|_0^a$$

$$= \log\left(1+\sqrt{2}\right) \frac{a^3}{3}$$

Example 8

Change the order of integration and evaluate **[Winter 2016]**

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx.$$

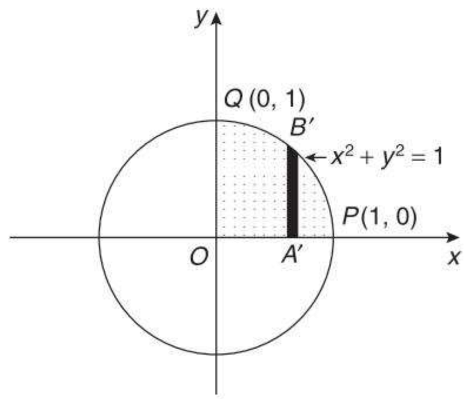

Fig. 9.61

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$

   Limits of x : $x = 0$ to $x = 1$
3. Since given limits of x and y are positive, the region is the part of circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y axis and terminates on the circle $x^2 + y^2 = 1$.

   Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$

   Limits of y : $y = 0$ to $y = 1$

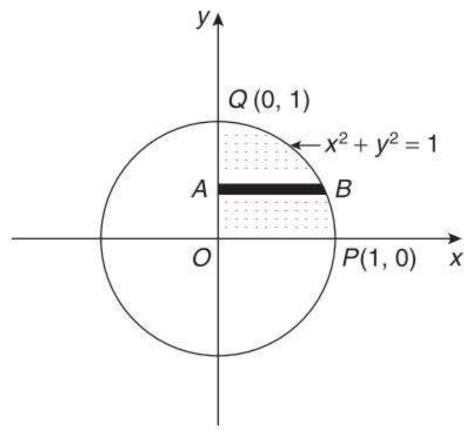

Fig. 9.62

Hence, the given integral after change of order is

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx = \int_0^1 \frac{e^y}{e^y+1}\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\,dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}(\sin^{-1}1 - \sin^{-1}0)\,dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\cdot\frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2}\left|\log(e^y+1)\right|_0^1 \qquad \left[\because \int \frac{f'(y)}{f(y)}\,dy = \log f(y)\right]$$

$$= \frac{\pi}{2}[\log(e+1) - \log 2]$$

$$= \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

Example 9

Change the order of integration and evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \mathrm{d}x\,\mathrm{d}y.$

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x.
2. Limits of $x : x = a-\sqrt{a^2-y^2}$ to $x = a+\sqrt{a^2-y^2}$, along horizontal strip $A'B'$

   Limits of $y : y = 0$ to $y = a$
3. The region is bounded by the circle $(x-a)^2 + y^2 = a^2$ and the line $y = 0$. Since limits of y are positive, the region is the part of the circle $(x-a)^2 + y^2 = a^2$ above x-axis.
4. The points of intersection of the circle with x-axis are O (0, 0) and Q ($2a$, 0).
5. To change the order of integration i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the circle

   $(x-a)^2 + y^2 = a^2$

   or $x^2 + y^2 - 2ax = 0$

   Limits of $y : y = 0$ to $y = \sqrt{2ax-x^2}$

   Limits of $x : x = 0$ to $x = 2a$

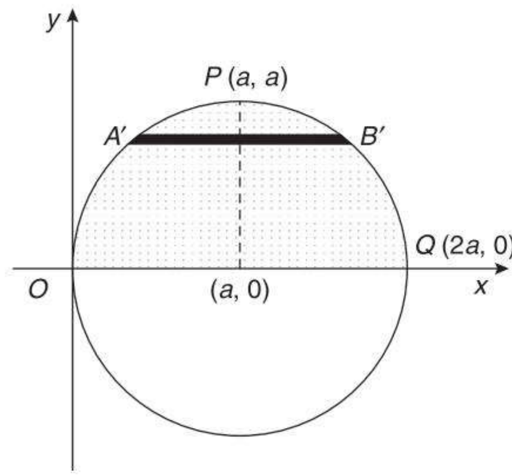

Fig. 9.63

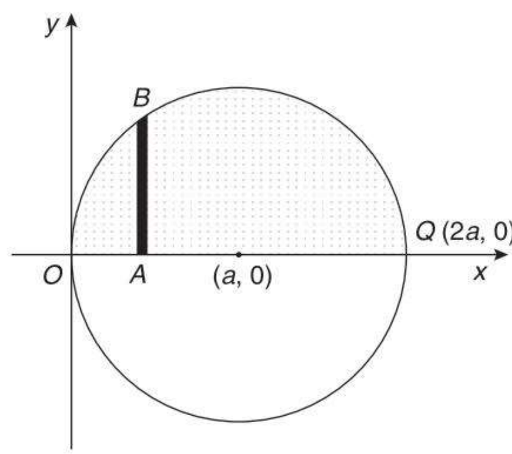

Fig. 9.64

Hence, the given integral after change of order is

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \mathrm{d}x\,\mathrm{d}y = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{2a} \left| y \right|_0^{\sqrt{2ax-x^2}} \mathrm{d}x$$

$$= \int_0^{2a} \sqrt{2ax-x^2}\,\mathrm{d}x$$

$$= \int_0^{2a} \sqrt{a^2-(x-a)^2}$$

$$= \left| \frac{(x-a)}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) \right|_0^{2a}$$

$$= \frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}1 - \frac{(0-a)}{2}\sqrt{0} - \frac{a^2}{2}\sin^{-1}(-1)$$

$$= a^2 \sin^{-1} 1 \quad [\because \sin^{-1}(-1) = -\sin^{-1}(1)]$$

$$= a^2 \frac{\pi}{2}$$

$$= \frac{\pi a^2}{2}$$

Example 10

Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ *by changing the order of integration.*

[Summer 2016]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = x$ to $y = \sqrt{2-x^2}$ along vertical setup $A'B'$
   Limits of $x : x = 0$ to $x = 1$
3. The region is bounded by y-axis, the line $y = x$ and the circle $x^2 + y^2 = 2$.
4. The point of intersection of the circle $y = \sqrt{2-x^2}$ and $y = x$ is obtained as

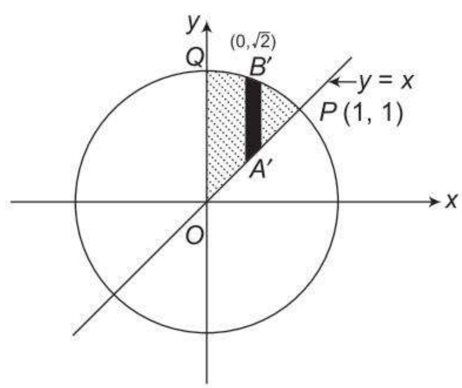

Fig. 9.65

$$x^2 = 2 - x^2$$
$$2x^2 = 2$$
$$x^2 = 1$$
$$x = \pm 1$$
$$\therefore \quad y = 1$$

Hence, $P(1, 1)$ is the point of intersection.

5. To change the order of integration, i.e. to integrate first w.r.t. x, divide the region into two subregions OPR and PQR. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In the subregion OPR, strip AB starts from y-axis and terminates on the line $y = x$.

   Limit of x: $x = 0$ to $x = y$

   Limit of y: $y = 0$ to $y = 1$

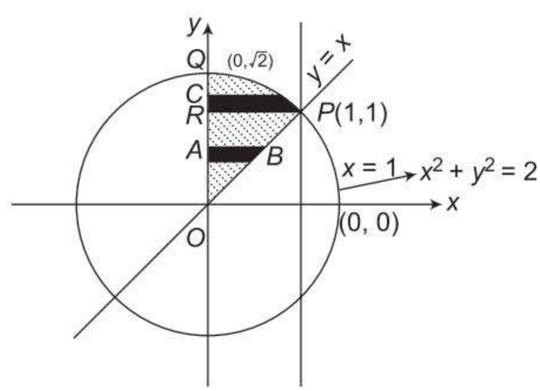

Fig. 9.66

(ii) In the subregion PQR, strip CD starts from y-axis and terminates on the circle $y=\sqrt{2-x^2}$.

Limit of x: $x=0$ to $x=\sqrt{2-y^2}$

Limit of y: $y=1$ to $y=\sqrt{2}$

$$\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\,\mathrm{d}x=\iint_{OPR}\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\,\mathrm{d}x+\iint_{PQR}\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\mathrm{d}x$$

$$=\int_0^1\int_0^y\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\,\mathrm{d}x+\int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\,\mathrm{d}x$$

$$=\int_0^1\frac{1}{2}\left[\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^y\mathrm{d}y+\int_1^{\sqrt{2}}\frac{1}{2}\left[\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\sqrt{2-y^2}}\mathrm{d}y$$

$$=\int_0^1\left[\sqrt{2}y-y\right]\mathrm{d}y+\int_1^{\sqrt{2}}\left[\sqrt{2}-y\right]\mathrm{d}y$$

$$=(\sqrt{2}-1)\left|\frac{y^2}{2}\right|_0^1+\left[\sqrt{2}y-\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$=(\sqrt{2}-1)\left(\frac{1}{2}-0\right)+\left(2-1-\sqrt{2}+\frac{1}{2}\right)$$

$$=\frac{\sqrt{2}}{2}-\frac{1}{2}+\frac{3}{2}-\sqrt{2}$$

$$=1-\sqrt{2}+\frac{1}{\sqrt{2}}$$

$$=1-\frac{1}{\sqrt{2}}$$

Example 11

Sketch the region of integration, reverse the order of integration and evaluate $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \mathrm{d}y\,\mathrm{d}x.$ **[Summer 2014]**

Solution

1. Since inner limit depends on x, the function is integrated first w.r.t. y.
2. Limits of y : $y = 0$ to $y = 4 - x^2$ along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 2$
3. The region is bounded by the parabola $x^2 = 4 - y$, x-axis and y-axis.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from y-axis and terminates on the parabola $x^2 = 4 - y$.

   Limits of x : $x = 0$ to $x = \sqrt{4-y}$

   Limits of y : $y = 0$ to $y = 4$

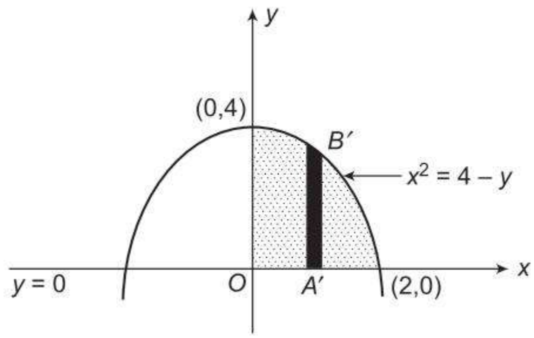

Fig. 9.67

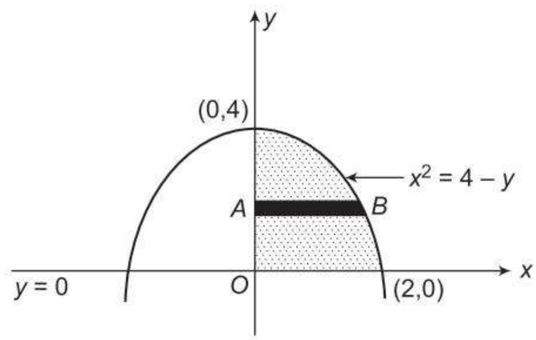

Fig. 9.68

Hence, the given integral after change of order is

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \mathrm{d}y\mathrm{d}x = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\int_0^{\sqrt{4-y}} x\,\mathrm{d}x \right] \mathrm{d}y$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left| \frac{x^2}{2} \right|_0^{\sqrt{4-y}} \mathrm{d}y$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{1}{2}(4-y) \right] \mathrm{d}y$$

$$= \int_0^4 \frac{e^{2y}}{2} \, dy$$

$$= \frac{1}{4} \left| e^{2y} \right|_0^4$$

$$= \frac{1}{4}(e^8 - 1)$$

Example 12

Change the order of integration and evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.

   The correct form of integral

   $= \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$
2. Limits of y : $y = x^2$ to $y = 2 - x$, along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by y-axis, the line $x + y = 2$ and the parabola $x^2 = y$. Since given limits of x and y are positive, the region lies in the first quadrant.
4. The points of intersection of $x + y = 2$ and $x^2 = y$ are obtained as

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$
$$x = 1, -2$$
$$y = 1, 4$$

   The points of intersection are P (1, 1) and P' (–2, 4).

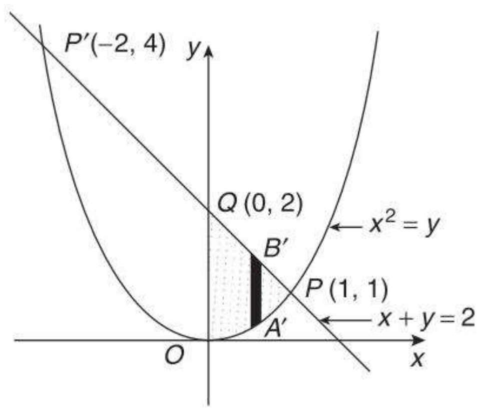

Fig. 9.69

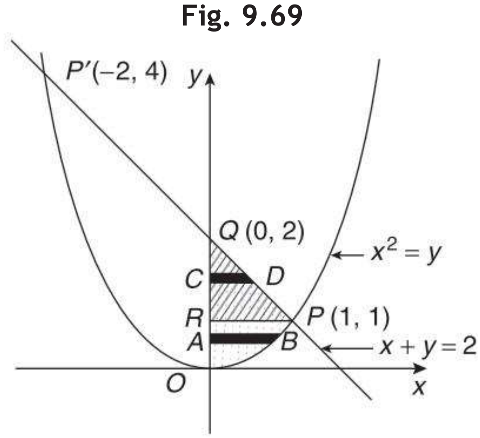

Fig. 9.70

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and RPQ. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In subregion OPR, strip AB starts from y-axis and terminates on the parabola $x^2 = y$.

   Limits of x : $x = 0$ to $x = \sqrt{y}$

   Limits of y : $y = 0$ to $y = 1$

(ii) In subregion RPQ, strip CD starts from y-axis and terminates on the line $x + y = 2$.

Limits of x : $x = 0$ to $x = 2 - y$

Limits of y : $y = 1$ to $y = 2$

Hence, the given integral after change of order is

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left| \frac{x^2}{2} \right|_0^{\sqrt{y}} y\,dy + \int_1^2 \left| \frac{x^2}{2} \right|_0^{2-y} y\,dy$$

$$= \frac{1}{2}\int_0^1 (y) y\,dy + \frac{1}{2}\int_1^2 (2-y)^2 y\,dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3) dy$$

$$= \frac{1}{2}\left| \frac{y^3}{3} \right|_0^1 + \frac{1}{2}\left| 4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4} \right|_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{9}{24}$$

$$= \frac{3}{8}$$

Example 13

Change the order of integration and evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$

[Winter 2014]

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$, along vertical strip $A'B'$
3. The region is bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.
4. The points of intersection of $x^2 = 4ay$ and $y^2 = 4ax$ are obtained as

$$x^4 = 16a^2y^2$$
$$= 16a^2 (4ax)$$

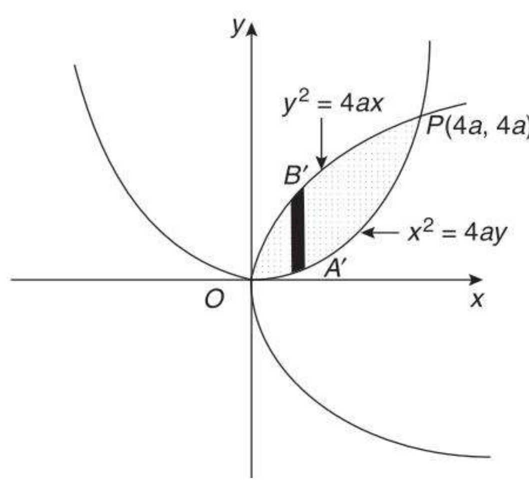

Fig. 9.71

$$x(x^3 - 64a^3) = 0$$
$$x = 0, x = 4a$$
$$\therefore\ y = 0, y = 4a$$

The points of intersection are O (0, 0) and P (4a, 4a).

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = 4ax$ and terminates on the parabola $x^2 = 4ay$.

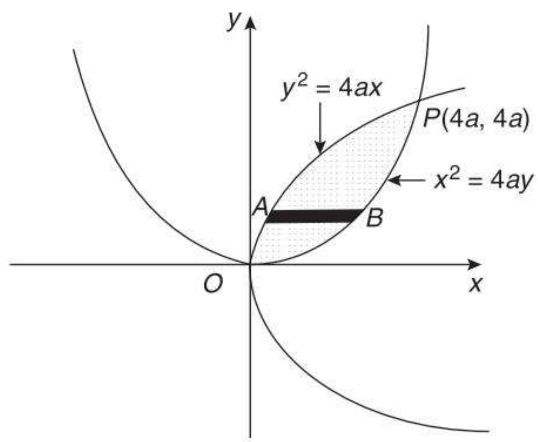

Fig. 9.72

Limits of $x : x = \dfrac{y^2}{4a}$ to $x = 2\sqrt{ay}$

Limits of $y : y = 0$ to $y = 4a$

Hence, the given integral after change of order is

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} \mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^{4a} \left|x\right|_{\frac{y^2}{4a}}^{2\sqrt{ay}} \mathrm{d}y$$
$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right)\mathrm{d}y$$
$$= 2\sqrt{a}\left|\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^{4a} - \frac{1}{4a}\left|\frac{y^3}{3}\right|_0^{4a}$$
$$= \frac{4}{3}\sqrt{a}(4)^{\frac{3}{2}}a^{\frac{3}{2}} - \frac{1}{12a}(64a^3)$$
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$
$$= \frac{16}{3}a^2$$

Example 14

Change the order of integration and evaluate

$$\int_0^a\int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\mathrm{d}x\,\mathrm{d}y.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.

2. Limits of x : $x = 0$ to $x = a - \sqrt{a^2 - y^2}$

   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the circle $(x - a)^2 + y^2 = a^2$, the lines $y = a$ and $x = 0$.
4. The point of intersection of $(x - a)^2 + y^2 = a^2$ and $y = a$ is obtained as

$$(x - a)^2 + a^2 = a^2$$
$$x = a$$

   The point of intersection is $P(a, a)$.

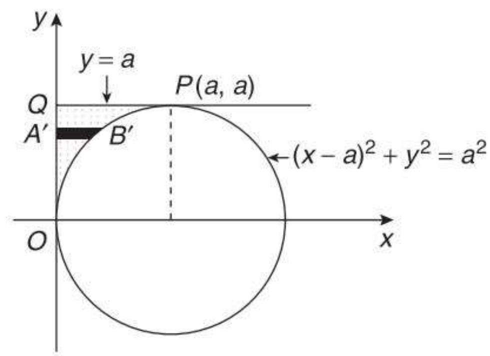

Fig. 9.73

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the circle $(x - a)^2 + y^2 = a^2$ and terminates on the line $y = a$.

   Limits of y : $y = \sqrt{2ax - x^2}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

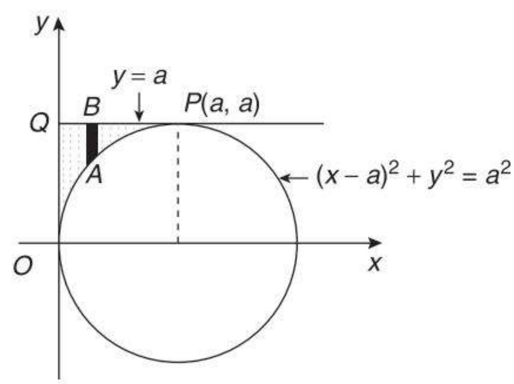

Fig. 9.74

Hence, the given integral after change of order is

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy = \int_0^a \int_{\sqrt{2ax-x^2}}^a \frac{x\log(x+a)}{(x-a)^2}\,y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left|\frac{y^2}{2}\right|_{\sqrt{2ax-x^2}}^a dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{a^2-2ax+x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\log(x+a)\right|_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{x+a}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\int_0^a \left\{(x-a) + \frac{a^2}{x+a}\right\}dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\left|\frac{x^2}{2} - ax + a^2\log(x+a)\right|_0^a\right]$$

$$= \frac{1}{4}\left(a^2\log 2a - \frac{a^2}{2} + a^2 - a^2\log 2a + a^2\log a\right)$$

$$= \frac{1}{4}\left(\frac{a^2}{2} + a^2 \log a\right)$$

$$= \frac{a^2}{8}(1 + 2\log a)$$

Example 15

Change the order of integration and evaluate

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-4y^2}$

   Limits of y : $y = 0$ to $y = \frac{1}{2}$
3. Since the limits of x and y are positive, the region is the part of the ellipse in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $x^2 + 4y^2 = 1$.

   Limits of y : $y = 0$ to $y = \frac{1}{2}\sqrt{1-x^2}$

   Limits of x : $x = 0$ to $x = 1$

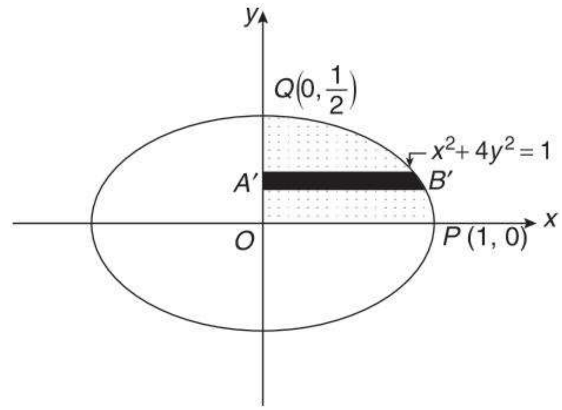

Fig. 9.75

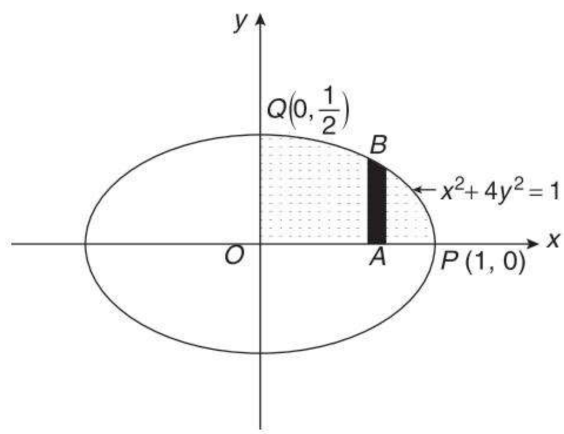

Fig. 9.76

Hence, the given integral after change of order is

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx dy = \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\, dy dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \left| \sin^{-1} \frac{y}{\sqrt{1-x^2}} \right|_0^{\frac{1}{2}\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \left(\sin^{-1} \frac{1}{2} - \sin^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{2-(1-x^2)}{\sqrt{1-x^2}} \cdot \frac{\pi}{6} dx$$

$$= \frac{\pi}{6} \int_0^1 \left(\frac{2}{\sqrt{1-x^2}} - \sqrt{1-x^2} \right) dx$$

$$= \frac{\pi}{6} \left| 2\sin^{-1} x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x \right|_0^1$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \frac{\pi}{6} \left(\frac{3}{2} \sin^{-1} 1 \right)$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

Example 16

Change the order of integration and evaluate

$$\int_0^a \int_0^y \frac{x\, dy\, dx}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = a$
3. The region is bounded by the line $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = a$ and $y = x$ is $P\,(a, a)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $y = x$ and terminates on the line $y = a$.

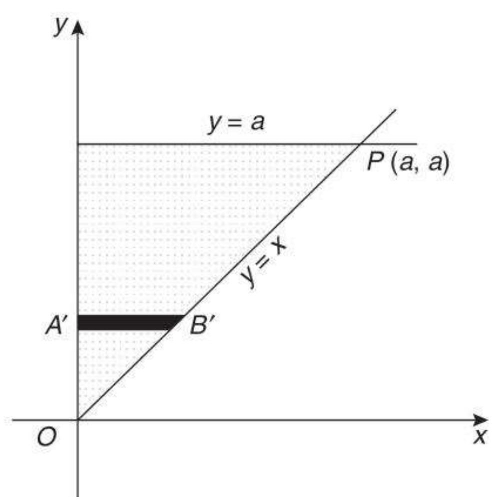

Fig. 9.77

Limits of y : $y = x$ to $y = a$

Limits of x : $x = 0$ to $x = a$

Hence, the given integral after change of order is

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

$$= \int_0^a \int_x^a \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

$$= \int_0^a \frac{x}{\sqrt{a^2-x^2}}\left[\int_x^a \frac{dy}{\sqrt{(a-y)(y-x)}}\right]dx$$

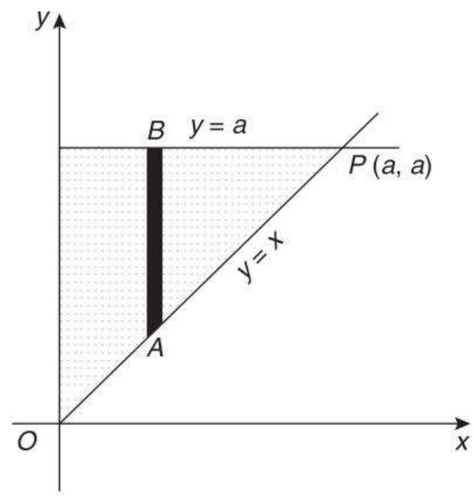

Fig. 9.78

Putting $y - x = t^2$, $dy = 2t\,dt$

When $y = x$, $t = 0$

When $y = a$, $t = \sqrt{a-x}$

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}} = \int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{2t\,dt}{\sqrt{(a-x-t^2)t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{dt}{\sqrt{(a-x)-t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \left|\sin^{-1}\frac{t}{\sqrt{a-x}}\right|_0^{\sqrt{a-x}} dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} (\sin^{-1} 1 - \sin^{-1} 0)\,dx$$

$$= 2\cdot\frac{\pi}{2}\int_0^a \left[-\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x)\right]dx$$

$$= -\frac{\pi}{2}\left|2(a^2-x^2)^{\frac{1}{2}}\right|_0^a \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \pi a$$

Example 17

Change the order of integration and evaluate

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx.$$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.

2. Limits of $y : y = x$ to $y = \dfrac{1}{x}$

   Limits of $x : x = 0$ to $x = 1$

3. The region is bounded by the rectangular hyperbola $xy = 1$, the line $y = x$ and y-axis in the first quadrant.
4. The point of intersection of $xy = 1$ and $y = x$ in the first quadrant is obtained as

   $x^2 = 1$

   $x = 1$

   $\therefore\ y = 1$

   The point of intersection is P (1, 1).

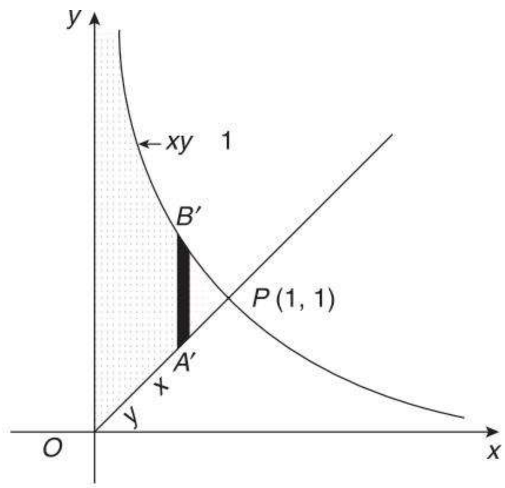

Fig. 9.79

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPQ and QPR. Draw a horizontal strip parallel to x-axis in each subregion.

(i) In subregion OPQ, strip AB starts from y-axis and terminates on the line $y = x$.

Limits of $x : x = 0$ to $x = y$

Limits of $y : y = 0$ to $y = 1$

(ii) In subregion QPR, strip CD starts from y-axis and terminates on the rectangular hyperbola $xy = 1$.

Limits of $x : x = 0$ to $x = \dfrac{1}{y}$

Limits of $y : y = 1$ to $y \to \infty$.

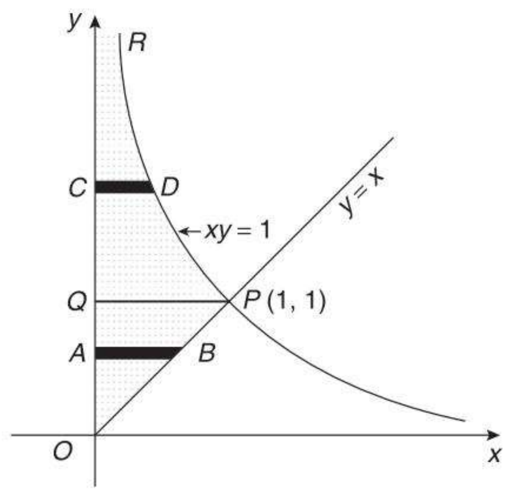

Fig. 9.80

Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^y \frac{y}{1+y^2}\,\frac{1}{(1+xy)^2}\,\mathrm{d}x\,\mathrm{d}y + \int_1^\infty \int_0^{\frac{1}{y}} \frac{y}{1+y^2}\,\frac{1}{(1+xy)^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^y \mathrm{d}y + \int_1^\infty \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^{\frac{1}{y}} \mathrm{d}y$$

$$= -\int_0^1 \frac{1}{1+y^2}\left(\frac{1}{1+y^2} - 1\right)\mathrm{d}y - \int_1^\infty \frac{1}{1+y^2}\left(\frac{1}{2} - 1\right)\mathrm{d}y$$

$$= -\int_0^1 \left[\frac{1}{(1+y^2)^2} - \frac{1}{1+y^2}\right]\mathrm{d}y + \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\,\mathrm{d}y$$

Putting $y = \tan\theta$ in the first term of first integral, $dy = \sec^2\theta\, d\theta$,

When $y = 0$, $\theta = 0$

When $y = 1$, $\theta = \dfrac{\pi}{4}$

$$\int_0^1\int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = -\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} + \left|\tan^{-1} y\right|_0^1 + \frac{1}{2}\left|\tan^{-1} y\right|_1^{\infty}$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(1+\cos 2\theta)}{2}\,d\theta + (\tan^{-1}1 - \tan^{-1}0) + \frac{1}{2}(\tan^{-1}\infty - \tan^{-1}1)$$

$$= -\frac{1}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + \frac{3\pi}{8}$$

$$= \frac{\pi - 1}{4}$$

Example 18

Change the order of integration and evaluate

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$
   Limits of y : $y = 0$ to $y = 1$
3. Since given limits of x and y are positive, the region is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis in the region. AB starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.
   Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$
   Limits of x : $x = 0$ to $x = 1$.

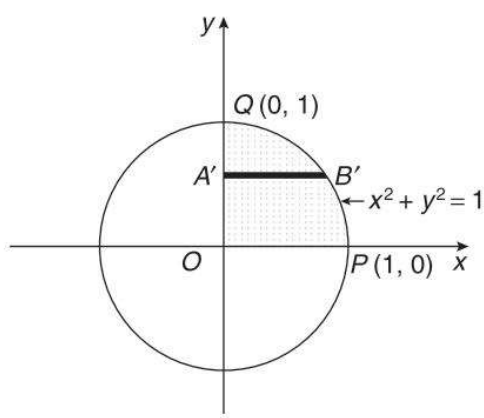

Fig. 9.81

Hence, the given integral after change of order is

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\frac{1}{\sqrt{(1-x^2)-y^2}}\,dy\,dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\left|\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}(\sin^{-1}1-\sin^{-1}0)\,dx$$

$$= -\frac{\pi}{2}\int_0^1 \cos^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= -\frac{\pi}{2}\left|\frac{(\cos^{-1}x)^2}{2}\right|_0^1 \qquad \left[\because \int f(x)f'(x)\,dx = \frac{[f(x)]^2}{2}\right]$$

$$= -\frac{\pi}{4}\left[(\cos^{-1}1)^2-(\cos^{-1}0)^2\right]$$

$$= -\frac{\pi}{4}\left[0-\left(\frac{\pi}{2}\right)^2\right]$$

$$= \frac{\pi^3}{16}$$

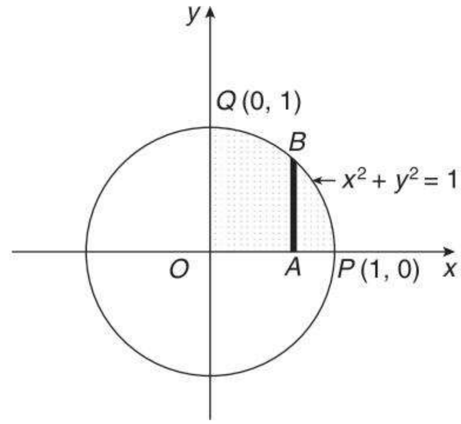

Fig. 9.82

EXERCISE 9.3

Change the order of integration of the following integrals:

1. $\int_0^6\int_{2-x}^{2+x} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_{-4}^{2}\int_{2-y}^{6} f(x, y)\,dy\,dx + \int_2^8\int_{y-2}^{6} f(x, y)\,dy\,dx\right]$$

2. $\int_0^1\int_x^{2x} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^1\int_{\frac{y}{2}}^{y} f(x, y)\,dx\,dy + \int_1^2\int_{\frac{y}{2}}^{1} f(x, y)\,dx\,dy\right]$$

3. $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.:} \int_{-1}^{1}\int_{x^2}^{1} f(x, y)\,dx\,dy\right]$$

4. $\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_0^a\int_{-a}^{-\sqrt{ax}} f(x, y)dy\,dx + \int_0^a\int_{\sqrt{ax}}^{a} f(x, y)dy\,dx\right]$$

5. $\int_0^a\int_{\frac{x^2}{a}}^{2a-x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_0^a\int_0^{\sqrt{ay}} f(x, y)dx\,dy + \int_a^{2a}\int_0^{2a-y} f(x, y)dx\,dy\right]$$

6. $\int_{-2}^{3}\int_{y^2-6}^{y} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_{-6}^{-2}\int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x, y)dy\,dx + \int_{-2}^{3}\int_{x}^{\sqrt{x+6}} f(x, y)dy\,dx\right]$$

7. $\int_0^1\int_{2y}^{2(1+\sqrt{1-y})} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.:} \int_0^2\int_0^{\frac{x}{2}} f(x, y)dy\,dx + \int_2^4\int_0^{\frac{4x-x^2}{4}} f(x, y)dy\,dx\right]$$

8. $\int_0^2\int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_1^2\int_0^{2\sqrt{y-1}} f(x, y)dx\,dy + \int_2^3\int_0^{6-2y} f(x, y)dx\,dy\right]$$

9. $\int_0^2\int_{\sqrt{4-x^2}}^{x+6a} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_0^2\int_{\sqrt{4-y^2}}^{2} f(x, y)dx\,dy + \int_2^{6a}\int_0^2 f(x, y)dx\,dy + \int_{6a}^{6a+2}\int_{y-6a}^{2} f(x, y)dx\,dy\right]$$

10. $\int_0^a\int_{\sqrt{a^2-y^2}}^{y+a} f(x, y)\,dxdy$

$$\left[\textbf{Ans.:} \int_0^a\int_{\sqrt{a^2-x^2}}^{a} f(x, y)dy\,dx + \int_a^{2a}\int_{x-a}^{a} f(x, y)dy\,dx\right]$$

11. $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)dx\,dy$

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \int_0^a\int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)dx\,dy + \int_0^a\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x, y)dx\,dy + \int_a^{2a}\int_{\frac{y^2}{2a}}^{2a} f(x, y)dx\,dy\end{array}\right]$$

12. $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_{\frac{a}{2}}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy\right]$$

13. $\int_0^a \int_x^{\frac{a^2}{x}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^a \int_0^y f(x, y)\,dx\,dy + \int_a^\infty \int_0^{\frac{a^2}{y}} f(x, y)\,dx\,dy\right]$$

14. $\int_a^b \int_{\frac{k}{x}}^{mx} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^{b} f(x, y)\,dx\,dy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x, y)\,dx\,dy + \int_{ma}^{mb} \int_{\frac{y}{m}}^{b} f(x, y)\,dx\,dy\right]$$

15. $\int_0^1 \int_1^{e^x} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_1^e \int_{\log y}^{1} f(x, y)\,dx\,dy\right]$$

16. $\int_0^2 \int_0^{x^3} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^8 \int_{\frac{1}{y^3}}^{2} f(x, y)\,dx\,dy\right]$$

17. $\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$$\left[\textbf{Ans.:} \int_0^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dy\,dx = \frac{2\pi}{3}\right]$$

18. $\int_0^2 \int_0^{\frac{x^2}{2}} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$

$$\left[\textbf{Ans.:} \int_0^2 \int_{\sqrt{2y}}^{2} \frac{x}{\sqrt{x^2+y^2+1}}\,dx\,dy = \frac{1}{4}(5\log 5 - 4)\right]$$

19. $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dy\,dx$

$$\left[\textbf{Ans.:} \frac{\pi a}{4}\right]$$

9.4 DOUBLE INTEGRALS IN POLAR COORDINATES

The integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\mathrm{d}r\mathrm{d}\theta$ represents the polar form of the double integration. This integral is first integrated w.r.t. r keeping θ constant and then the resulting expression is integrated w.r.t. θ.

Limits of Integration

If the limits of integration are not given then these limits are obtained from the equations of the given curves. Let the region be bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.

The region of integration is *PQRS*. Draw an elementary radius vector *AB* from origin which enters in the region from the curve $r = r_1(\theta)$ and leaves at the curve $r = r_2(\theta)$. Therefore, limits for r are $r_1(\theta)$ to $r_2(\theta)$.

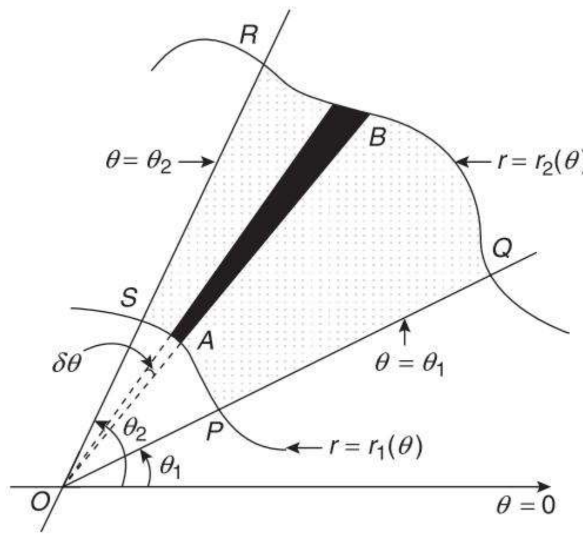

Fig. 9.83

To cover the entire region *PQRS*, rotate elementary radius vector *AB* from *PQ* to *RS*. Therefore, θ varies from θ_1 to θ_2.

$$\iint f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$

Type I Evaluation of Double Integrals in Polar Coordinates

Example 1

Evaluate $\int_0^{\frac{\pi}{4}}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta$.

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}}\left[\int_0^1 r\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}}\left|\frac{r^2}{2}\right|_0^1 \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}}\frac{1}{2}\,\mathrm{d}\theta$$

$$= \frac{1}{2}\left|\theta\right|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Another method: Since both the limits are constant and integrand (function) is explicit in r and θ, the integral can be written as

$$\int_0^{\frac{\pi}{4}} \int_0^1 r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \cdot \int_0^1 r \, \mathrm{d}r$$

$$= |\theta|_0^{\frac{\pi}{4}} \cdot \left| \frac{r^2}{2} \right|_0^1$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}$$

$$= \frac{\pi}{8}$$

Example 2

Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r \, \mathrm{d}r \, \mathrm{d}\theta.$

Solution

$$\int_0^{\pi} \int_0^{\sin\theta} r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{\pi} \left[\int_0^{\sin\theta} r \, \mathrm{d}r \right] \mathrm{d}\theta$$

$$= \int_0^{\pi} \left| \frac{r^2}{2} \right|_0^{\sin\theta} \mathrm{d}\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin^2 \theta \, \mathrm{d}\theta$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{2} \, \mathrm{d}\theta$$

$$= \frac{1}{4} \left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi}$$

$$= \frac{1}{4} \left(\pi - \frac{\sin 2\pi}{2} \right)$$

$$= \frac{\pi}{4}$$

Example 3

Evaluate the integral $\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta$. **[Summer 2014]**

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{1-\sin\theta} r^2 \, dr\right]\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{1-\sin\theta} \cos\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}}(1-\sin\theta)^3 \cos\theta \, d\theta$$

$$= \frac{1}{3}\left|\frac{(1-\sin\theta)^4}{4}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{3}\left[0-\frac{1}{4}\right]$$

$$= -\frac{1}{12}$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, d\theta \, dr$.

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{2a\cos\theta} r^2 dr\right]\sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{2a\cos\theta} \sin\theta \, d\theta$$

$$= \frac{8a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\sin\theta \, d\theta$$

$$= -\frac{8a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta(-\sin\theta) \, d\theta$$

$$= -\frac{8a^3}{3}\left|\frac{\cos^4\theta}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}, n \neq -1\right]$$

$$= -\frac{8a^3}{12}\left(\cos^4\frac{\pi}{2} - \cos^4 0\right)$$

$$= \frac{2}{3}a^3$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}\theta\,\mathrm{d}r.$

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{a(1+\sin\theta)} r^2\mathrm{d}r\right]\cos\theta\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{a(1+\sin\theta)}\cos\theta\mathrm{d}\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} a^3(1+\sin\theta)^3\cos\theta\,\mathrm{d}\theta$$

$$= \frac{a^3}{3}\left|\frac{(1+\sin\theta)^4}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}, \; n \neq -1\right]$$

$$= \frac{a^3}{12}\left[\left(1+\sin\frac{\pi}{2}\right)^4 - (1+\sin 0)^4\right]$$

$$= \frac{a^3}{12}\left[2^4 - 1\right]$$

$$= \frac{5}{4}a^3$$

Example 6

Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\mathrm{d}r\,\mathrm{d}\theta.$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{1}{2}\cdot\frac{2r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sqrt{\cos 2\theta}}(1+r^2)^{-2}\cdot 2r\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left|-(1+r^2)^{-1}\right|_0^{\sqrt{\cos 2\theta}}\mathrm{d}\theta \qquad \left[\begin{array}{c}\because \int [f(r)]^n f'(r)\,\mathrm{d}r = \dfrac{[f(r)]^{n+1}}{n+1} \\ n\neq -1\end{array}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2\theta}-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2\cos^2\theta}-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sec^2\theta-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\left|\frac{1}{2}\tan\theta-\theta\right|_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4}-\frac{\pi}{4}\right)$$

$$= \frac{1}{8}(\pi-2)$$

Example 7

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\mathrm{d}\theta\,\mathrm{d}\phi.$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\,\mathrm{d}\theta\right]\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\left|-\cos(\theta+\phi)\right|_0^{\frac{\pi}{2}}\mathrm{d}\phi$$

$$= -\int_0^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}+\phi\right)-\cos\phi\right]\mathrm{d}\phi$$

$$= -\int_0^{\frac{\pi}{2}}(-\sin\phi-\cos\phi)\,\mathrm{d}\phi$$

$$= -\left|\cos\phi - \sin\phi\right|_0^{\frac{\pi}{2}}$$

$$= -\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \cos 0 + \sin 0\right)$$

$$= 2$$

Type II Evaluation of Double Integrals Over a Given Region in Polar Coordinates

Example 1

Evaluate $\iint\limits_R r^3 \sin 2\theta \,\mathrm{d}r\,\mathrm{d}\theta$ *over the area bounded in the first quadrant between the circle* $r = 2$ *and* $r = 4$. **[Summer 2016]**

Solution

1. The region of integration is the interior of the circle between $r = 2$ to $r = 4$.
2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = 2$ and leaves at the circle $r = 4$.
3. Limits of r: $r = 2$ to $r = 4$

   Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

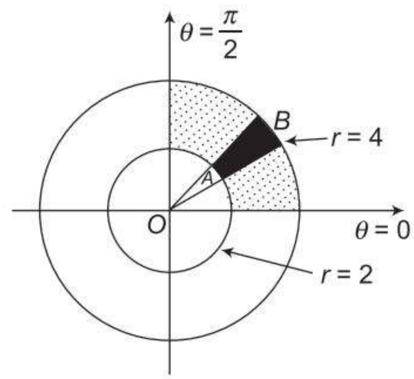

Fig. 9.84

$$I = \iint r^3 \sin 2\theta \,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_2^4 r^3 \sin 2\theta \,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \left|\frac{r^4}{4}\right|_2^4 \left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[(4)^4 - (2)^4\right]\left(\frac{1}{2}\right)(\cos\pi - \cos 0)$$

$$= \frac{1}{2}[256 - 16]\left(-\frac{1}{2}\right)(-2)$$

$$= \frac{1}{2}[240]$$

$$= 120$$

Example 2

Evaluate $\iint r\sqrt{a^2 - r^2}\,\mathrm{d}r\,\mathrm{d}\theta$ *over the upper half of the circle* $r = a\cos\theta$. **[Summer 2017]**

Solution

1. The region of integration is the upper half of the circle $r = a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a\cos\theta$.
3. Limits of $r : r = 0$ to $r = a\cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

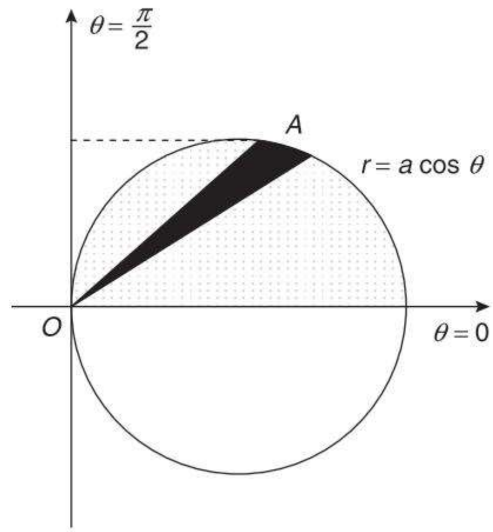

Fig. 9.85

$$
\begin{aligned}
I &= \iint r\sqrt{a^2 - r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)(a^2 - r^2)^{\frac{1}{2}}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left|\frac{2(a^2 - r^2)^{\frac{3}{2}}}{3}\right|_0^{a\cos\theta}\mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right] \\
&= -\frac{1}{3}\int_0^{\frac{\pi}{2}}(a^3\sin^3\theta - a^3)\,\mathrm{d}\theta \\
&= -\frac{a^3}{3}\int_0^{\frac{\pi}{2}}\left(\frac{3\sin\theta - \sin 3\theta}{4} - 1\right)\mathrm{d}\theta \\
&= -\frac{a^3}{3}\left|\frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right) - \theta\right|_0^{\frac{\pi}{2}} \\
&= -\frac{a^3}{3}\left(-\frac{3}{4}\cos\frac{\pi}{2} + \frac{1}{12}\cos\frac{3\pi}{2} - \frac{\pi}{2} + \frac{3}{4}\cos 0 - \frac{1}{12}\cos 0\right) \\
&= -\frac{a^3}{3}\left(-\frac{\pi}{2} + \frac{3}{4} - \frac{1}{12}\right) \\
&= -\frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)
\end{aligned}
$$

Example 3

Evaluate $\iint r^4 \cos^3\theta \,dr\,d\theta$ *over the interior of the circle* $r = 2a\cos\theta$.

Solution

1. The region of integration is the interior of the circle $r = 2a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2a\cos\theta$.
3. Limits of $r: r = 0$ to $r = 2a\cos\theta$

   Limits of $\theta: \theta = -\dfrac{\pi}{2}$ to $\theta = \dfrac{\pi}{2}$

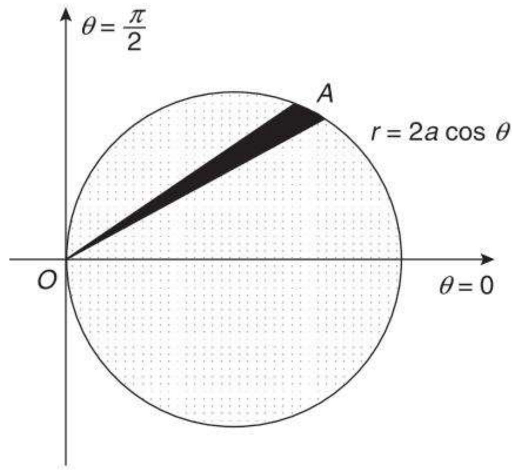

Fig. 9.86

$$I = \iint r^4 \cos^3\theta \,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \int_0^{2a\cos\theta} r^4 \,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \left| \frac{r^5}{5} \right|_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \,(2a\cos\theta)^5 \,d\theta$$

$$= \frac{32a^5}{5} \cdot 2\int_0^{\frac{\pi}{2}} \cos^8\theta \,d\theta$$

$$= \frac{32a^5}{5} \cdot B\left(\frac{9}{2}, \frac{1}{2}\right) \qquad \left[\because B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \,d\theta\right]$$

$$= \frac{32a^5}{5} \cdot \frac{\left\lceil \frac{9}{2} \right. \left\lceil \frac{1}{2} \right.}{\lceil 5}$$

$$= \frac{32a^5}{5} \cdot \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left\lceil \frac{1}{2} \right. \left\lceil \frac{1}{2} \right.}{24}$$

$$= \frac{7\pi}{4} a^5$$

Example 4

Evaluate $\iint r^2 \sin\theta \,dr\,d\theta$ *over the cardioid* $r = a\,(1 + \cos\theta)$ *above the initial line.*

Solution

1. The region of integration is the part of the cardioid $r = a(1 + \cos\theta)$ above the initial line ($\theta = 0$).
2. Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a(1 + \cos\theta)$.
3. Limits of r: $r = 0$ to $r = a(1 + \cos\theta)$

   Limits of θ : $\theta = 0$ to $\theta = \pi$

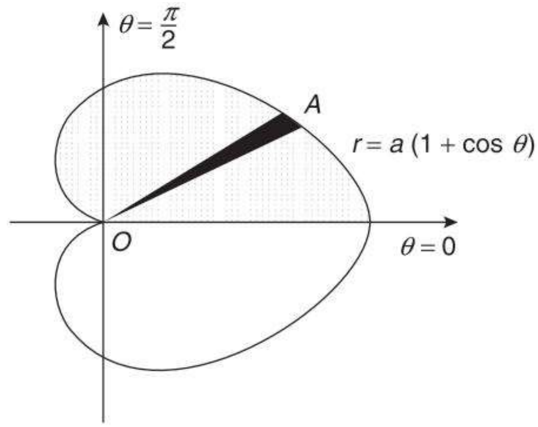

Fig. 9.87

$$I = \iint r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \cdot d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta) d\theta$$

$$= -\frac{a^3}{3}\left| \frac{(1+\cos\theta)^4}{4} \right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1} \right]$$

$$= -\frac{a^3}{12}\left[(1+\cos\pi)^4 - (1+\cos 0)^4 \right]$$

$$= -\frac{a^3}{12}(0 - 16)$$

$$= \frac{4}{3}a^2$$

Example 5

Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{r^2 + a^2}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Solution

1. The region of integration is one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ bounded between the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

2. Draw an elementary radius vector OA which starts from the origin and terminates on the lemniscate $r^2 = a^2 \cos 2\theta$.
3. Limits of r: $r = 0$ to $r = a\sqrt{\cos 2\theta}$

   Limits of $\theta : \theta = -\dfrac{\pi}{4}$ to $\theta = \dfrac{\pi}{4}$

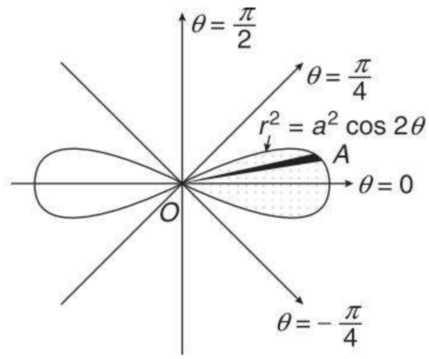

Fig. 9.88

$$I = \iint \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2+a^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2+a^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(r^2+a^2)^{-\frac{1}{2}}(2r)\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left| 2(r^2+a^2)^{\frac{1}{2}} \right|_0^{a\sqrt{\cos 2\theta}} \mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}, n \neq -1\right]$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a\left[(\cos 2\theta + 1)^{\frac{1}{2}} - 1\right]\mathrm{d}\theta$$

$$= a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta - 1\right)\mathrm{d}\theta$$

$$= a\left|\sqrt{2}\sin\theta - \theta\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= a\left[\sqrt{2}\sin\frac{\pi}{4} - \frac{\pi}{4} - \sqrt{2}\sin\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi}{4}\right)\right]$$

$$= a\left(2 - \frac{\pi}{2}\right)$$

$$= \frac{a}{2}(4-\pi)$$

Example 6

Evaluate $\iint r^2 \mathrm{d}r\,\mathrm{d}\theta$ *over the area between the circles* $r = a\sin\theta$ *and* $r = 2a\sin\theta$.

Solution

1. The region of integration is the area bounded between the circle $r = a \sin \theta$ and $r = 2a \sin \theta$.

2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = a \sin\theta$ and leaves at the circle $r = 2a \sin\theta$.
3. Limits of r : $r = a\sin\theta$ to $r = 2a\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \pi$

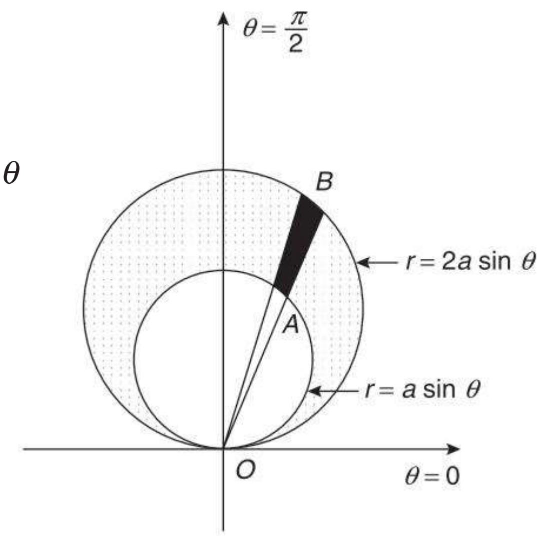

Fig. 9.89

$$
\begin{aligned}
I &= \iint r^2 \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi}\int_{a\sin\theta}^{2a\sin\theta} r^2 \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi} \left|\frac{r^3}{3}\right|_{a\sin\theta}^{2a\sin\theta} \mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{\pi} (8a^3\sin^3\theta - a^3\sin^3\theta)\mathrm{d}\theta \\
&= \frac{7a^3}{3}\int_0^{\pi} \sin^3\theta\,\mathrm{d}\theta \\
&= \frac{7a^3}{3}\int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4}\mathrm{d}\theta \\
&= \frac{7a^3}{12}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^{\pi} \\
&= \frac{7a^3}{12}\left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0)\right] \\
&= \frac{7a^3}{12}\left(\frac{16}{3}\right) \\
&= \frac{28}{9}a^3
\end{aligned}
$$

EXERCISE 9.4

Evaluate the following integrals:

1. $\iint re^{\frac{-r^2}{a^2}} \cos\theta \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta$ over the upper half of the circle $r = 2a\cos\theta$.

$$\left[\textbf{Ans.:}\ \frac{a^2}{16}\left(3 + \frac{1}{e^4}\right)\right]$$

2. $\iint r^3 \mathrm{d}r\,\mathrm{d}\theta$ over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\left[\textbf{Ans.:}\ \frac{45\pi}{2}\right]$$

3. $\iint r\sin\theta\, dA$ over the cardioid $r = a\ (1+\cos\theta)$ above the initial line.

$$\left[\text{Ans.}: \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2+4}}\, dr\, d\theta$ over one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$$\left[\text{Ans.}: (4-\pi)\right]$$

9.5 MULTIPLE INTEGRALS BY SUBSTITUTION

9.5.1 Change of Variables from Cartesian to Polar Coordinates

The double integral can be changed from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta$, $y = r\sin\theta$. Then $\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|J|\,dr\,d\theta$ where J is the Jacobian (functional determinant) defined as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence, $\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|r|\,dr\,d\theta$

$$= \iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 1

Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = 1$ is obtained as $r = 1$.
2. The region of integration is the part of the circle $r = 1$ in the first quadrant.

3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.
4. Limits of r : $r = 0$ to $r = 1$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

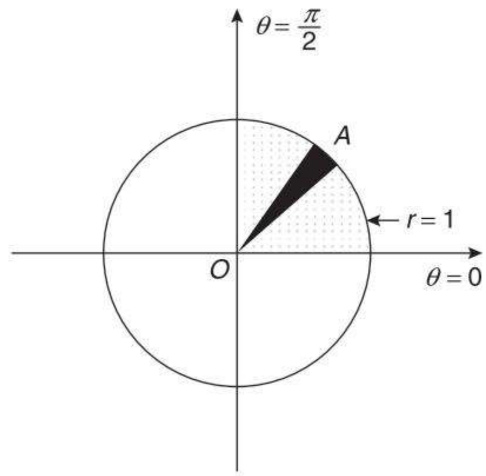

Fig. 9.90

Hence, the polar form of the given integral is

$$I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,\mathrm{d}r\,\mathrm{d}\theta$$

Putting $r^2 = \cos 2t$, $2r\,\mathrm{d}r = -2\sin 2t\,\mathrm{d}t$

When $r = 0, t = \frac{\pi}{4}$

When $r = 1, t = 0$

$$I = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}(-\sin 2t\,\mathrm{d}t)\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2 t}{2\cos^2 t}}\sin 2t\,\mathrm{d}t\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\cdot 2\sin t\cos t\,\mathrm{d}t\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} (1-\cos 2t)\mathrm{d}t$$

$$= \left|\theta\right|_0^{\frac{\pi}{2}} \left| t - \frac{\sin 2t}{2} \right|_0^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{8}(\pi - 2)$$

Example 2

Evaluate $\iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$ *over the region bounded by the circle* $x^2 + y^2 - x = 0$ *in the first quadrant.*

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - x = 0$ is $r^2 - r\cos\theta = 0$, $r = \cos\theta$.

2. The region of integration is the part of the circle $r = \cos\theta$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = \cos\theta$.
4. Limits of $r : r = 0$ to $r = \cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

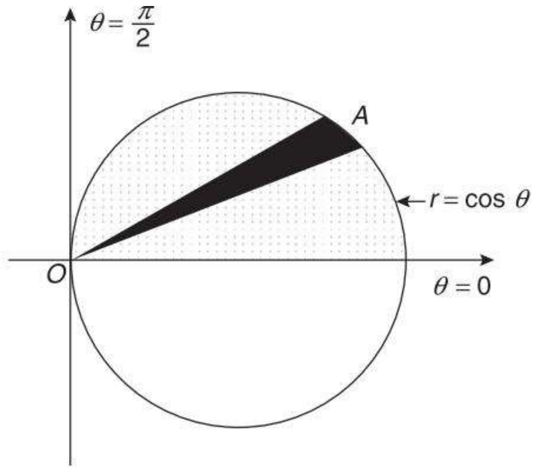

Fig. 9.91

Hence, the polar form of the given integral is

$$
\begin{aligned}
I &= \iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2}\, dx\, dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{4r^2 \cos\theta \sin\theta}{r^2} e^{-r^2} r\, dr\, d\theta \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left[\int_0^{\cos\theta} e^{-r^2} (-2r)\, dr \right] d\theta \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left| e^{-r^2} \right|_0^{\cos\theta} d\theta \qquad \left[\because \int e^{f(r)} f'(r) dr = e^{f(r)} \right] \\
&= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left(e^{-\cos^2\theta} - 1 \right) d\theta \\
&= -\int_0^{\frac{\pi}{2}} \left[e^{-\cos^2\theta} (2\cos\theta \sin\theta) - \sin 2\theta \right] d\theta \\
&= -\left| e^{-\cos^2\theta} + \frac{\cos 2\theta}{2} \right|_0^{\frac{\pi}{2}} \\
&= -\left(e^0 + \frac{\cos\pi}{2} - e^{-1} - \frac{\cos 0}{2} \right) \\
&= \frac{1}{e}
\end{aligned}
$$

Example 3

Evaluate $\iint \dfrac{x^2 y^2}{(x^2+y^2)} dx\, dy$ *over the region bounded by the circles* $x^2 + y^2 = a^2$ *and* $x^2 + y^2 = b^2$ $(a > b)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2 + y^2 = a^2$ is $r^2 = a^2$, $r = a$.
   (ii) the circle $x^2 + y^2 = b^2$ is $r^2 = b^2$, $r = b$.

2. The region of integration is the part bounded between the circles $r = a$ and $r = b$.
3. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = b$ and terminates on the circle $r = a$.
4. Limits of r : $r = b$ to $r = a$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

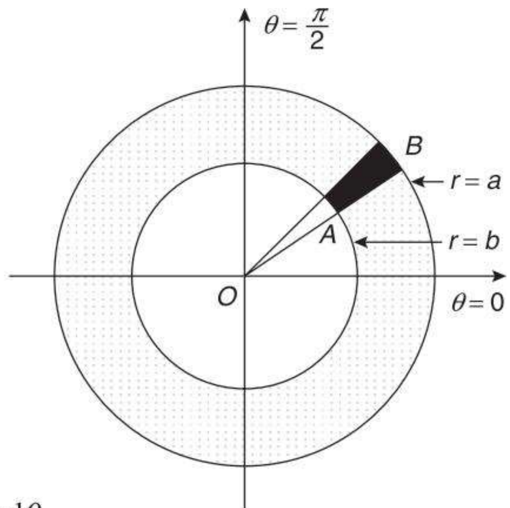

Fig. 9.92

Hence, the polar form of the given integral is

$$I = \iint \frac{x^2 y^2}{(x^2 + y^2)} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^{2\pi} \int_b^a \frac{r^4 \cos^2\theta \sin^2\theta}{r^2} \cdot r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{2\pi} \cos^2\theta \sin^2\theta \left| \frac{r^4}{4} \right|_b^a \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{\sin^2 2\theta}{4} \cdot \frac{(a^4 - b^4)}{4} \mathrm{d}\theta$$

$$= \frac{a^4 - b^4}{16} \int_0^{2\pi} \frac{(1 - \cos 4\theta)}{2} \mathrm{d}\theta$$

$$= \left(\frac{a^4 - b^4}{32} \right) \left| \theta - \frac{\sin 4\theta}{4} \right|_0^{2\pi}$$

$$= \left(\frac{a^4 - b^4}{32} \right) (2\pi)$$

$$= \frac{\pi}{16} (a^4 - b^4)$$

Example 4

Evaluate $\iint \frac{(x^2 + y^2)^2}{x^2 y^2} \mathrm{d}x\, \mathrm{d}y$ *over the region common to the circles* $x^2 + y^2 = ax$ *and* $x^2 + y^2 = by$ $(a, b > 0)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2 + y^2 = ax$ is $r^2 = ar\cos\theta$, $r = a\cos\theta$.
   (ii) the circle $x^2 + y^2 = by$ is $r^2 = br\sin\theta$, $r = b\sin\theta$.
2. The region of integration is the common part of the circles $r = a\cos\theta$ and $r = b\sin\theta$.

3. The point of intersection of the circle $r = a\cos\theta$ and $r = b\sin\theta$, is obtained as

$$b\sin\theta = a\cos\theta$$

$$\tan\theta = \frac{a}{b}$$

$$\theta = \tan^{-1}\frac{a}{b}$$

Hence, $\theta = \tan^{-1}\dfrac{a}{b}$ at P.

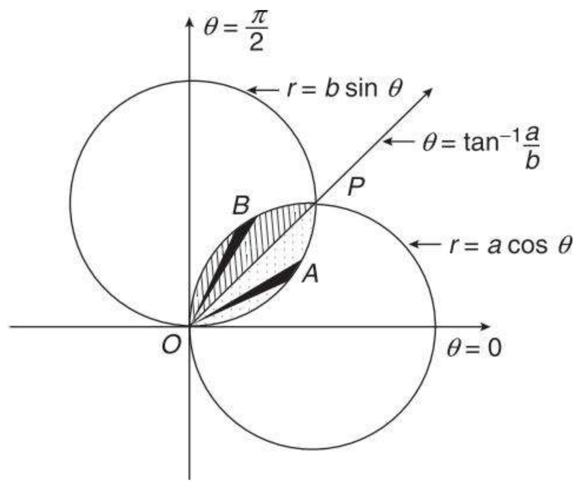

Fig. 9.93

4. Divide the region into two subregions OAP and OBP. Draw an elementary radius vector OA and OB in each subregion.
   (i) In subregion OAP, elementary radius vector OA starts from the origin and terminates on the circle $r = b\sin\theta$.
   Limits of r : $r = 0$ to $r = b\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \tan^{-1}\dfrac{a}{b}$
   (ii) In subregion OBP, elementary radius vector OB starts from the origin and terminates on the circle $r = a\cos\theta$.
   Limits of r : $r = 0$ to $r = a\cos\theta$
   Limits of θ : $\theta = \tan^{-1}\dfrac{a}{b}$ to $\theta = \dfrac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$= \int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{b\sin\theta} d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot b^2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot a^2\cos^2\theta\,d\theta$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}} \sec^2\theta\,d\theta + \frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{b^2}{2}\left|\tan\theta\right|_0^{\tan^{-1}\frac{a}{b}} + \frac{a^2}{2}\left|-\cot\theta\right|_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}$$

$$= \frac{b^2}{2}\left[\tan\tan^{-1}\left(\frac{a}{b}\right) - \tan 0\right] - \frac{a^2}{2}\left[\cot\frac{\pi}{2} - \cot\left(\tan^{-1}\frac{a}{b}\right)\right]$$

$$= \frac{b^2}{2}\left[\frac{a}{b} - 0\right] - \frac{a^2}{2}\left[0 - \frac{b}{a}\right]$$

$$= \frac{ab}{2} + \frac{ab}{2}$$

$$= ab$$

Example 5

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\,\mathrm{d}y.$

Solution

1. Limits of $x : x = 0$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$
2. The region of integration is the first quadrant.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the integral changes to polar form.
4. Draw an elementary radius vector which starts from the origin and extends up to infinity.
   Limits of $r : r = 0$ to $r \to \infty$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

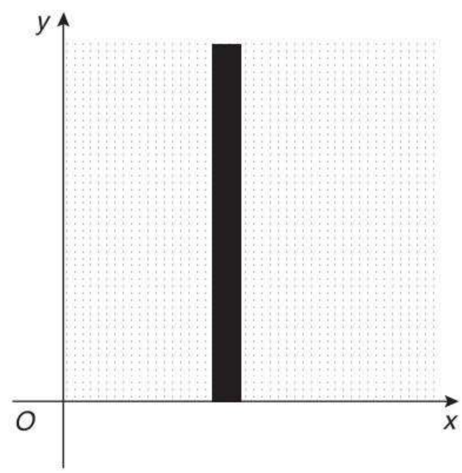

Fig. 9.94

Hence, the polar form of the given integral is

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}(-2r)\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left|e^{-r^2}\right|_0^\infty \mathrm{d}\theta \quad \left[\because \int e^{f(r)} f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0 - e^0)\mathrm{d}\theta$$

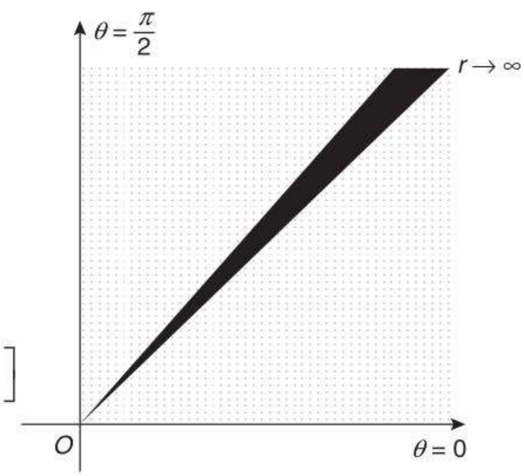

Fig. 9.95

$$= -\frac{1}{2}\left|-\theta\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

Example 6

Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^{\frac{3}{2}}}$.

Solution

1. Limits of $x : x \to -\infty$ to $x \to \infty$

   Limits of $y : y \to -\infty$ to $y \to \infty$

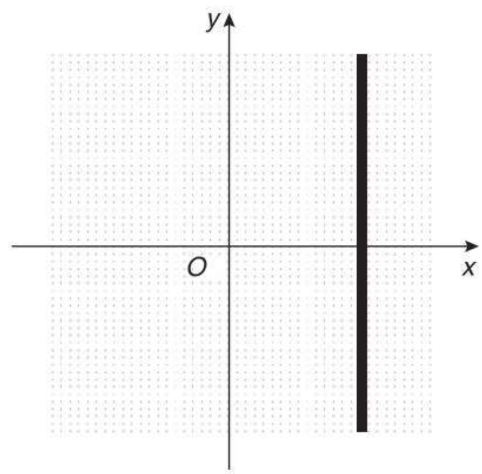

Fig. 9.96

2. The region of integration is the entire coordinate plane.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, integral changes to polar form.
4. Draw an elementary radius vector which starts from origin and extends up to ∞.

   Limits of r : $r = 0$ to $r \to \infty$

   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

Hence, the polar form of the given integral is

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^{\frac{3}{2}}}$$

$$= \int_0^{2\pi}\int_0^{\infty}\frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{(1+r^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}(1+r^2)^{-\frac{3}{2}}(2r)\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left|-2(1+r^2)^{-\frac{1}{2}}\right|_0^{\infty}\mathrm{d}\theta$$

$$\left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\int_0^{2\pi}\left|\frac{1}{\sqrt{1+r^2}}\right|_0^{\infty}\mathrm{d}\theta$$

$$= -\int_0^{2\pi}(0-1)\mathrm{d}\theta$$

$$= \left|\theta\right|_0^{2\pi}$$

$$= 2\pi$$

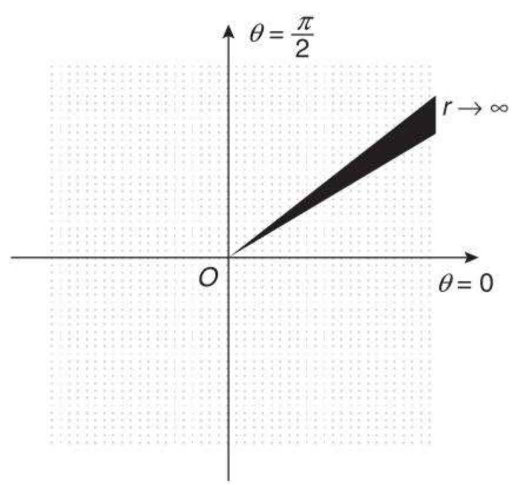

Fig. 9.97

Example 7

Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ *by transforming into polar coordinates.*

[Winter 2013]

Solution

1. Limits of $x : x = y$ to $x = a$
   Limits of $y : y = 0$ to $y = a$
2. The region of integration is bounded by the lines $y = x$, $x = a$ and $y = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1,\ \theta = \frac{\pi}{4}$.
   (ii) the line $x = a$ is $r\cos\theta = a$, $r = a\sec\theta$.
4. Draw an elementary radius vector OA which starts from the origin and terminates on the line $r = a\sec\theta$.
   Limits of $r : r = 0$ to $r = a\sec\theta$
   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{4}$

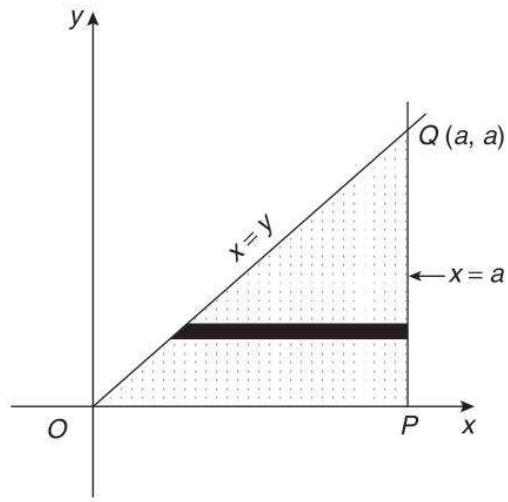

Fig. 9.98

Hence, the polar form of the given integral is

$$
\begin{aligned}
I &= \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy \\
&= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \cos\theta\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}} |r|_0^{a\sec\theta} \cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{4}} a\sec\theta\cos\theta\,d\theta \\
&= a\int_0^{\frac{\pi}{4}} d\theta \\
&= a|\theta|_0^{\frac{\pi}{4}} \\
&= a\cdot\frac{\pi}{4} \\
&= \frac{\pi a}{4}
\end{aligned}
$$

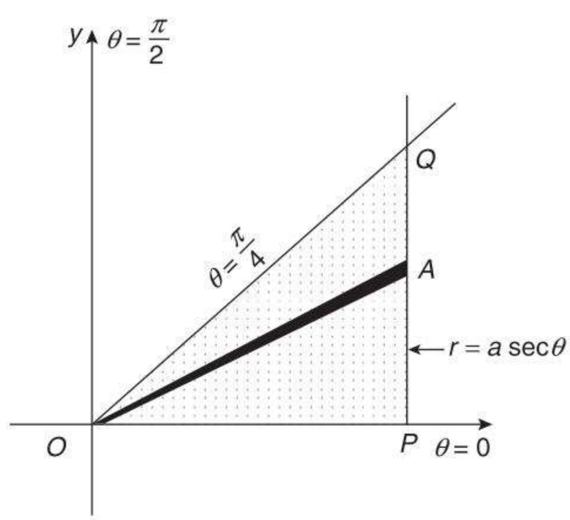

Fig. 9.99

Example 8

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$ *by transforming into polar coordinates.*

Solution

1. Limits of $y : y = 0$ to $y = \sqrt{2x - x^2}$
   Limits of $x : x = 0$ to $x = 2$
2. The region of integration is bounded by the circle $x^2 + y^2 - 2x = 0$ and the lines $y = 0, x = 0$. Since the limits of x and y are positive, the region of integration is the part of the circle in the first quadrant.

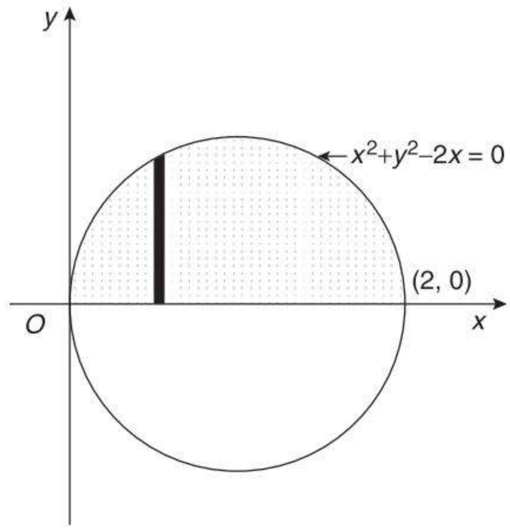

Fig. 9.100

3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - 2x = 0$ is

$$r^2 - 2r\cos\theta = 0$$
$$r = 2\cos\theta.$$

4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.

Limits of $r : r = 0$ to $r = 2\cos\theta$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

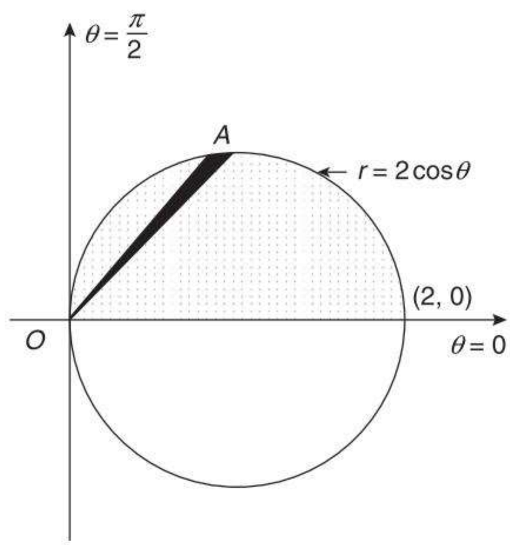

Fig. 9.101

Hence, the polar form of the given integral is

$$\begin{aligned}
I &= \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \cos\theta\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left| r \right|_0^{2\cos\theta} \cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta \\
&= \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}}
\end{aligned}$$

$$= \frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin 0$$
$$= \frac{\pi}{2}$$

Example 9

Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y.$

Solution

1. Limits of y : $y = x$ to $y = \sqrt{2x - x^2}$

   Limits of x : $x = 0$ to $x = 1$
2. The region of integration is bounded by the line $y = x$ and the circle $x^2 + y^2 - 2x = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is
   $$r\sin\theta = r\cos\theta,\ \tan\theta = 1,\ \theta = \frac{\pi}{4}.$$
   (ii) the circle $x^2 + y^2 - 2x = 0$ is
   $$r^2 - 2r\cos\theta = 0$$
   $$r = 2\cos\theta.$$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.

   Limits of r : $r = 0$ to $r = 2\cos\theta$

   Limits of θ : $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

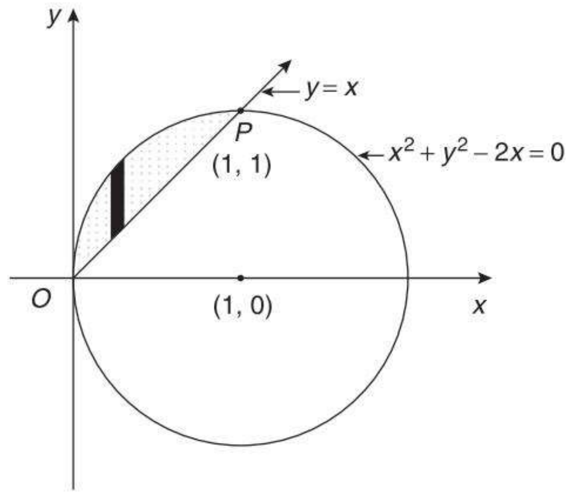

Fig. 9.102

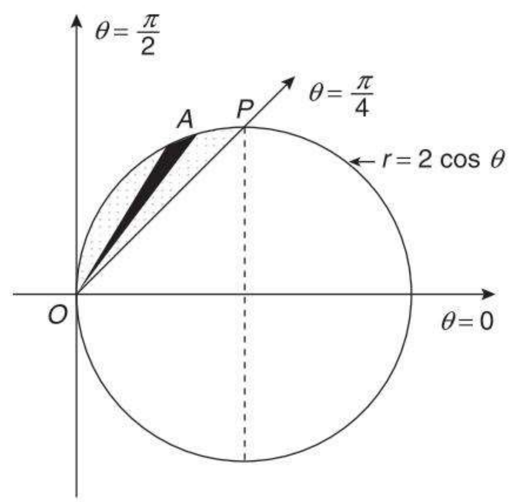

Fig. 9.103

Hence, the polar form of the given integral is

$$\mathrm{I} = \int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left| \frac{r^4}{4} \right|_0^{2\cos\theta} \mathrm{d}\theta$$
$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\,\mathrm{d}\theta$$

$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 \mathrm{d}\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+2\cos 2\theta + \cos^2 2\theta)\,\mathrm{d}\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\mathrm{d}\theta$$

$$= \left|\frac{3}{2}\theta + \frac{2\sin 2\theta}{2} + \frac{\sin 4\theta}{8}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{1}{8}(\sin 2\pi - \sin\pi)$$

$$= \frac{3\pi}{8} + 1$$

Example 10

Evaluate the integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$ *by changing into polar coordinates.* **[Summer 2014]**

Solution

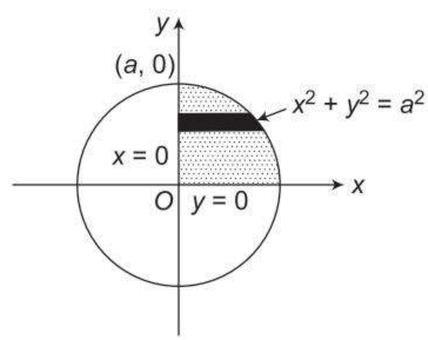

Fig. 9.104

1. Limits of x : $x = 0$ to $x = \sqrt{a^2 - y^2}$

   Limits of y : $y = 0$ to $y = a$
2. The region of integration is bounded by $x = 0$, $x = \sqrt{a^2 - y^2}$, $y = 0$ and $y = a$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = a^2$ is

   (i) $r^2 = a^2$

   $\therefore\ r = a$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a$.

   Limits of r: $r = 0$ to $r = a$

   Limit of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

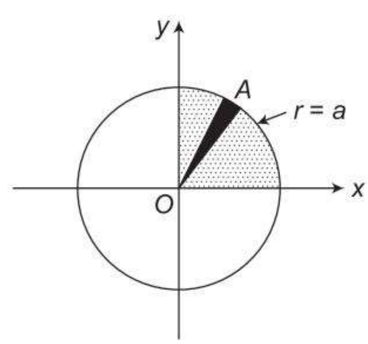

Fig. 9.105

Hence, the polar form of the given integral is

$$I = \int_0^{\frac{\pi}{2}}\int_0^a r^2 \sin^2\theta \cdot r \cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \left[\int_0^a r^4\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \left|\frac{r^5}{5}\right|_0^a \mathrm{d}\theta$$

$$= \frac{a^5}{5}\int_0^{\frac{\pi}{2}} \sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{a^5}{5}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{a^5\pi}{20}$$

Example 11

Evaluate $\int_0^1\int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y.$

Solution

1. Limits of $y : y=\sqrt{x-x^2}$ to $y=\sqrt{1-x^2}$

   Limits of $x : x=0$ to $x=1$
2. The region of integration is the part of the first quadrant bounded by the circles $x^2+y^2-x=0$ and $x^2+y^2=1$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2+y^2-x=0$ is $r^2 - r\cos\theta = 0$, $r = \cos\theta$.
   (ii) the circle $x^2+y^2=1$ is $r^2=1$, $r=1$.
4. Draw an elementary radius vector *OAB* from the origin which enters in the region from the circle $r=\cos\theta$ and terminates on the circle $r=1$.

   Limits of r : $r=\cos\theta$ to $r=1$

   Limits of θ : $\theta=0$ to $\theta=\frac{\pi}{2}$

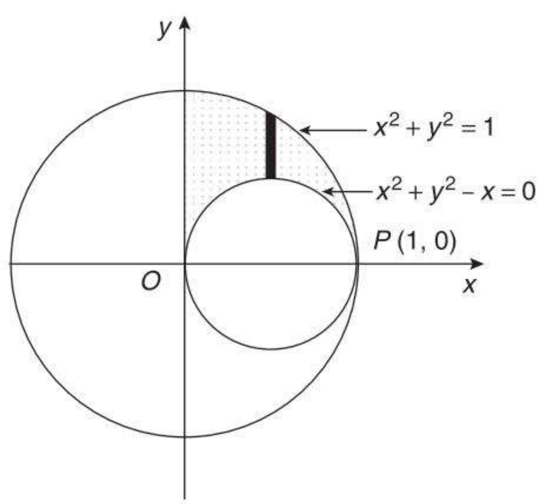

Fig. 9.106

Hence, the polar form of the given integral is

$$I = \int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_{\cos\theta}^{1} \frac{r^2\sin\theta\cos\theta e^{-r^2}}{r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \int_{\cos\theta}^{1} e^{-r^2}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left| e^{-r^2} \right|_{\cos\theta}^{1} \mathrm{d}\theta$$

$$\left[\because \int e^{f(r)} f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta(e^{-1} - e^{-\cos^2\theta})\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1}{e}\sin 2\theta - e^{-\cos^2\theta}\cdot 2\sin\theta\cos\theta\right)\mathrm{d}\theta$$

$$= -\frac{1}{4}\left|\frac{1}{e}\left(-\frac{\cos 2\theta}{2}\right) - e^{-\cos^2\theta}\right|_0^{\frac{\pi}{2}} \quad \left[\because \int e^{f(\theta)} f'(\theta)\mathrm{d}\theta = e^{f(\theta)}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}(\cos\pi - \cos 0) - e^{-\left(\cos^2\frac{\pi}{2}\right)} + e^{-\cos^2 0}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}(-2) - e^0 + e^{-1}\right]$$

$$= -\frac{1}{4}\left[\frac{1}{e} - 1 + \frac{1}{e}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2}{e}\right]$$

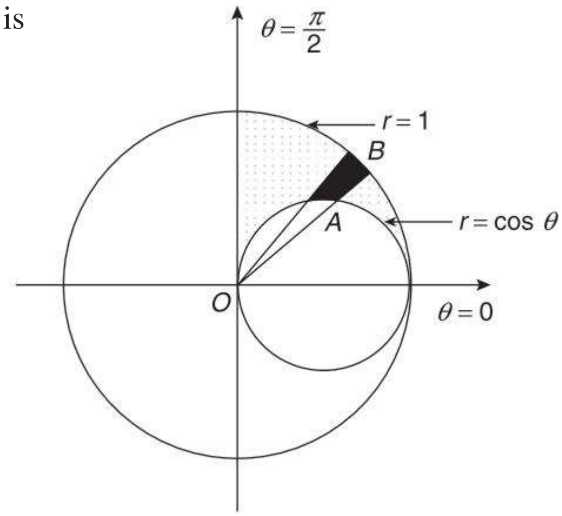

Fig. 9.107

Example 12

Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y.$ **[Winter 2013]**

Solution

1. Limits of $x: x = \frac{y^2}{4a}$ to $x = y$

   Limits of $y: y = 0$ to $y = 4a$.

2. The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1,\ \theta = \dfrac{\pi}{4}$.
   (ii) the parabola $y^2 = 4ax$ is $r^2\sin^2\theta = 4ar\cos\theta$, $r = 4a\cot\theta\operatorname{cosec}\theta$.
4. Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\operatorname{cosec}\theta$.

   Limits of $r: r = 0$ to $r = 4a\cot\theta\operatorname{cosec}\theta$

   Limits of $\theta: \theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{\pi}{2}$

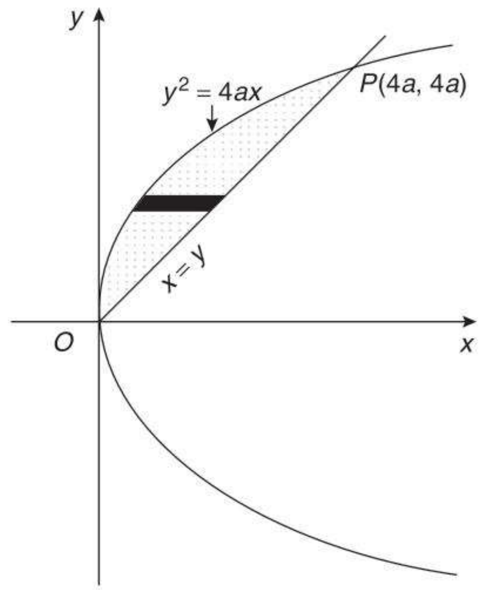

Fig. 9.108

Hence, the polar form of the given integral is

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\operatorname{cosec}\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\operatorname{cosec}\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)(4a)^2\cot^2\theta\operatorname{cosec}^2\theta\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\operatorname{cosec}^2\theta - 2\cot^2\theta)d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\left\{-(\cot^2\theta)(-\operatorname{cosec}^2\theta)\right\} - 2\operatorname{cosec}^2\theta + 2\right]d\theta$$

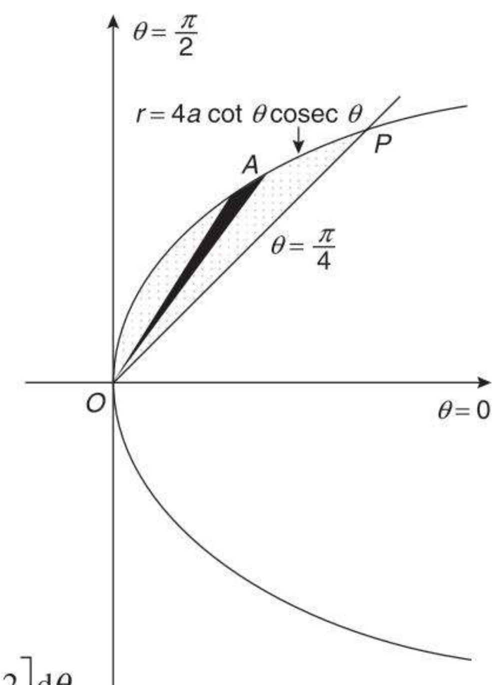

Fig. 9.109

$$= 8a^2\left|-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1)+2(-1)+2\cdot\frac{\pi}{4}\right]$$

$$= 8a^2\left[-\frac{5}{3}+\frac{\pi}{2}\right]$$

$$= 8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

Example 13

Evaluate $\displaystyle\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy.$

Solution

1. Limits of $y : y = 2\sqrt{ax}$ to
   $y = \sqrt{5ax - x^2}$
   Limits of x : $x = 0$ to $x = a$
2. Since the limits of x and y are positive, the region of integration is the part of the first quadrant bounded by the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 - 5ax = 0$
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of

   (i) the parabola $y^2 = 4ax$ is
   $$r^2\sin^2\theta = 4ar\cos\theta,$$
   $$r = 4a\cot\theta\,\text{cosec}\,\theta.$$
   (ii) the circle $x^2 + y^2 - 5ax = 0$ is
   $$r^2 - 5ar\cos\theta = 0,$$
   $$r = 5a\cos\theta.$$
4. The points of intersection of $r = 4a\cot\theta\,\text{cosec}\,\theta$ and $r = 5a\cos\theta$ are obtained as
   $$4a\cot\theta\,\text{cosec}\,\theta = 5a\cos\theta$$
   $$\sin^2\theta = \frac{4}{5}$$
   $$\theta = \pm\sin^{-1}\frac{2}{\sqrt{5}}$$

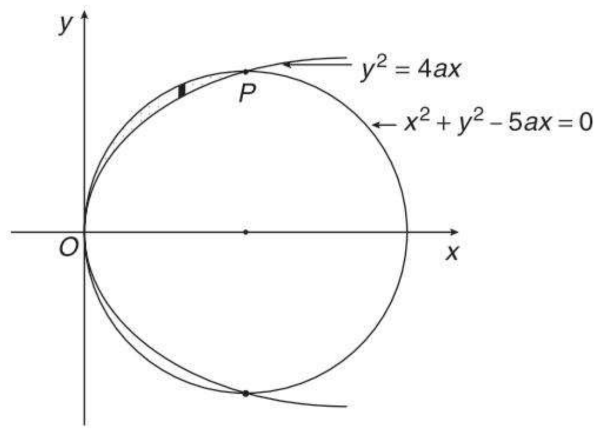

Fig. 9.110

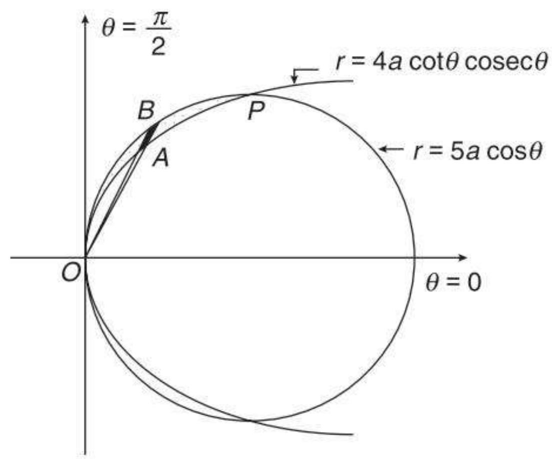

Fig. 9.111

Hence, $\theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$ at P.

5. Draw an elementary radius vector *OAB* from the origin which enters in the region from the parabola $r = 4a\cot\theta\,\text{cosec}\,\theta$ and terminates on the circle $r = 5a\cos\theta$.
   Limits of r : $r = 4a\cot\theta\,\text{cosec}\,\theta$ to $r = 5a\cos\theta$

   Limits of $\theta : \theta = \sin^{-1}\frac{2}{\sqrt{5}}$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dx\,dy$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \int_{4a\cot\theta\,\text{cosec}\,\theta}^{5a\cos\theta} \frac{r}{r^2\sin^2\theta}\,r\,dr\,d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \text{cosec}^2\theta \left| r \right|_{4a\cot\theta\,\text{cosec}\,\theta}^{5a\cos\theta} d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \text{cosec}^2\theta(5a\cos\theta - 4a\cot\theta\,\text{cosec}\,\theta)\,d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \left[5a\cot\theta\,\text{cosec}\,\theta + 4a\,\text{cosec}^2\theta(-\text{cosec}\,\theta\cot\theta)\right]d\theta$$

$$= \left| -5a\,\text{cosec}\,\theta + 4a\frac{\text{cosec}^3\theta}{3} \right|_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \left[-5a\,\text{cosec}\frac{\pi}{2} + 5a\,\text{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) + \frac{4a}{3}\text{cosec}^3\frac{\pi}{2} - \frac{4a}{3}\text{cosec}^3\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)\right]$$

$$= \left[-5a + 5a\frac{\sqrt{5}}{2} + \frac{4a}{3} - \frac{4a}{3}\left(\frac{\sqrt{5}}{2}\right)^3\right] \qquad \left[\begin{aligned}\because \text{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) &= \text{cosec}\left(\text{cosec}^{-1}\frac{\sqrt{5}}{2}\right)\\ &= \frac{\sqrt{5}}{2}\end{aligned}\right]$$

$$= \frac{a}{3}\left(5\sqrt{5} - 11\right)$$

Example 14

Evaluate $\iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\,dy$ *over the first quadrant of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution

Let $x = ar\cos\theta,\ y = br\cos\theta$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}r\,\mathrm{d}\theta = abr\,\mathrm{d}r\,\mathrm{d}\theta$$

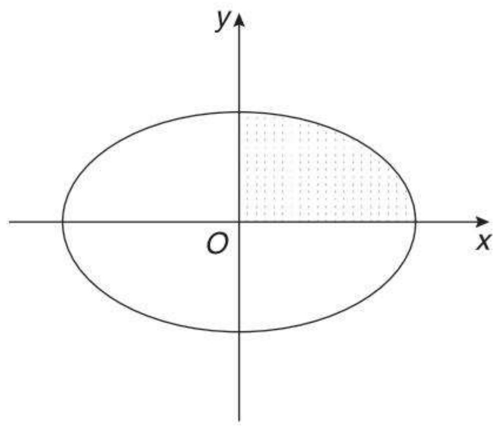

Fig. 9.112

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the xy-plane gets transformed to $r^2 = 1$ or $r = 1$, circle with centre (0, 0) and radius 1 in the $r\theta$-plane.

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.

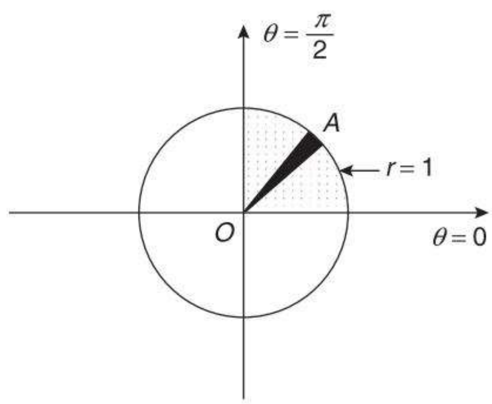

Fig. 9.113

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 abr^2\cos\theta\sin\theta(r^2)^{\frac{n}{2}}\,abr\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= a^2b^2\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\int_0^1 (r)^{n+3}\,\mathrm{d}r$$

$$= \frac{a^2b^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi + \cos 0)\cdot\frac{1}{n+4}$$

$$= \frac{a^2b^2}{2(n+4)}$$

EXERCISE 9.5

Change to polar coordinates and evaluate the following integrals:

1. $\iint \frac{1}{\sqrt{xy}}\,dx\,dy$ over the region bounded by the semicircle $x^2 + y^2 - x = 0$, $y \geq 0$.

$$\left[\textbf{Ans.:}\ \frac{\pi}{\sqrt{2}}\right]$$

2. $\iint y^2\,dx\,dy$ over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$.

$$\left[\textbf{Ans.:}\ \frac{15\pi a^4}{64}\right]$$

3. $\iint \sin(x^2 + y^2)\,dx\,dy$ over the circle $x^2 + y^2 = a^2$.

$$\left[\textbf{Ans.:}\ \pi(1 - \cos a^2)\right]$$

4. $\iint xy(x^2 + y^2)^{\frac{3}{2}}\,dx\,dy$ over the first quadrant of the circle $x^2 + y^2 = a^2$.

$$\left[\textbf{Ans.:}\ \frac{a^7}{14}\right]$$

5. $\int_0^3 \int_0^{\sqrt{3x}} \frac{dy\,dx}{\sqrt{x^2 + y^2}}$

$$\left[\textbf{Ans.:}\ \frac{3}{2}\log 3\right]$$

6. $\int_0^a \int_0^x \frac{x^3\,dx\,dy}{\sqrt{x^2 + y^2}}$

$$\left[\textbf{Ans.:}\ \frac{a^4}{4}\log\left(1 + \sqrt{2}\right)\right]$$

7. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sin\left[\frac{\pi}{a^2}(a^2 - x^2 - y^2)\right]dx\,dy$

$$\left[\textbf{Ans.:}\ \frac{a^2}{2}\right]$$

8. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-x^2 - y^2}\,dx\,dy$

$$\left[\textbf{Ans.:}\ \frac{\pi}{4}\left(1 - e^{-\alpha^2}\right)\right]$$

9. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{3\pi a^4}{4}\right]$

10. $\int_0^1\int_0^{\sqrt{x-x^2}}\frac{4xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{1}{e}\right]$

11. $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}}\log e(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{\pi}{4}a^2\left(\log a-\frac{1}{2}\right)\right]$

12. $\int_0^a\int_y^{a+\sqrt{a^2-y^2}}\frac{dx\,dy}{(4a^2+x^2+y^2)^3}$

$\left[\textbf{Ans.:}\ \frac{1}{8a^2}\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)\right]$

13. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

$\left[\textbf{Ans.:}\ a\right]$

14. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.:}\ \frac{1}{4a^2}\left[1-(1+a^2)e^{-a^2}\right]\right]$

15. $\int_0^1\int_{x^2}^{x}\frac{dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans.:}\ \sqrt{2}-1\right]$

9.5.2 Change of Variables from Cartesian to Other Coordinates

In some cases, evaluation of double integral becomes easier by changing the variables. Let the variables x, y be replaced by new variables u, v by the transformation $x = f_1(u,v), y = f_2(u,v)$, then

$$\iint f(x,y)\mathrm{d}x\,\mathrm{d}y = \iint f(f_1,f_2)\left|J\right|\mathrm{d}u\mathrm{d}v \qquad \ldots (1)$$

where

$$\text{Jacobian } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Using Eq. (1), the double integral can be transformed to new variables.

Example 1

Using the transformation $x - y = u, x + y = v$, *evaluate* $\iint \cos\left(\frac{x-y}{x+y}\right) dxdy$ *over the region bounded by the lines* $x = 0, y = 0, x + y = 1.$

Solution

$$x - y = u, x + y = v$$

$$x = \frac{u+v}{2}, y = \frac{v-u}{2}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$dxdy = |J|\, dudv = \frac{1}{2} dudv$$

The region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in xy-plane is a triangle OPQ.

Under the transformation $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$,

(i) the line $x = 0$ gets transformed to the line $u = -v$

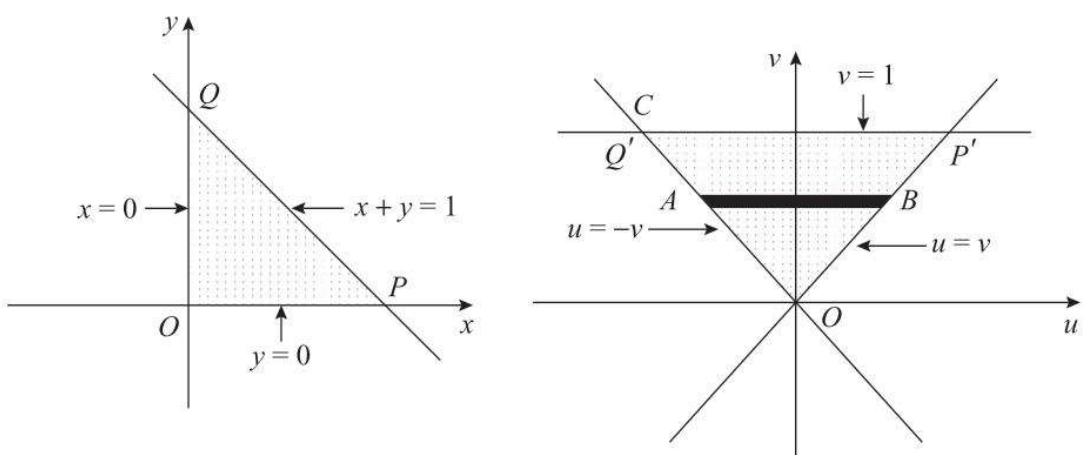

Fig. 9.114

(ii) the line $y = 0$ gets transformed to the line $u = v$

(iii) the line $x + y = 1$ gets transformed to the line $v = 1$

Thus, triangle OPQ in xy-plane gets transformed to triangle $OP'Q'$ in uv-plane bounded by the lines $u = v$, $u = -v$ and $v = 1$.

In the region, draw a horizontal strip AB parallel to u-axis which starts from the line $u = -v$ and terminates on the line $u = v$.

Limits of u : $u = -v$ to $u = v$

Limits of v : $v = 0$ to $v = 1$

$$\begin{aligned}
I &= \iint \cos\left(\frac{x-y}{x+y}\right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \frac{1}{2} \mathrm{d}u\, \mathrm{d}v \\
&= \frac{1}{2}\int_0^1 \left| v \sin\left(\frac{u}{v}\right)\right|_{-v}^{v} \mathrm{d}v \\
&= \frac{1}{2}\int_0^1 v\left[\sin 1 - \sin(-1)\right] \mathrm{d}v \\
&= \frac{1}{2}\cdot 2\sin 1 \left|\frac{v^2}{2}\right|_0^1 \\
&= \frac{1}{2}\sin 1
\end{aligned}$$

Example 2

Using the transformation $x^2 - y^2 = u$, $2xy = v$, *find* $\iint (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y$ *over the region in the first quadrant bounded by* $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$.

Solution

$$x^2 - y^2 = u,\ 2xy = v$$

It is difficult to express x and y in terms of u and v, therefore we write Jacobian of u, v in terms of x and y.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

$$\mathrm{d}u\mathrm{d}v = |J|\, \mathrm{d}x\mathrm{d}y = 4\left(x^2 + y^2\right)\mathrm{d}x\mathrm{d}y$$

$$\mathrm{d}x\mathrm{d}y = \frac{1}{4\left(x^2 + y^2\right)}\mathrm{d}u\mathrm{d}v$$

The region in xy-plane bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 4$, $xy = 2$ is transformed to a square in uv-plane bounded by the lines $u = 1$, $u = 2$, $v = 4$, $v = 8$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the line $v = 4$ and terminates on the line $v = 8$.

$$I = \iint (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y$$

$$= \int_1^2 \int_4^8 (x^2 + y^2) \frac{1}{4(x^2 + y^2)} \mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{4} \left| u \right|_1^2 \left| v \right|_4^8$$

$$= 1$$

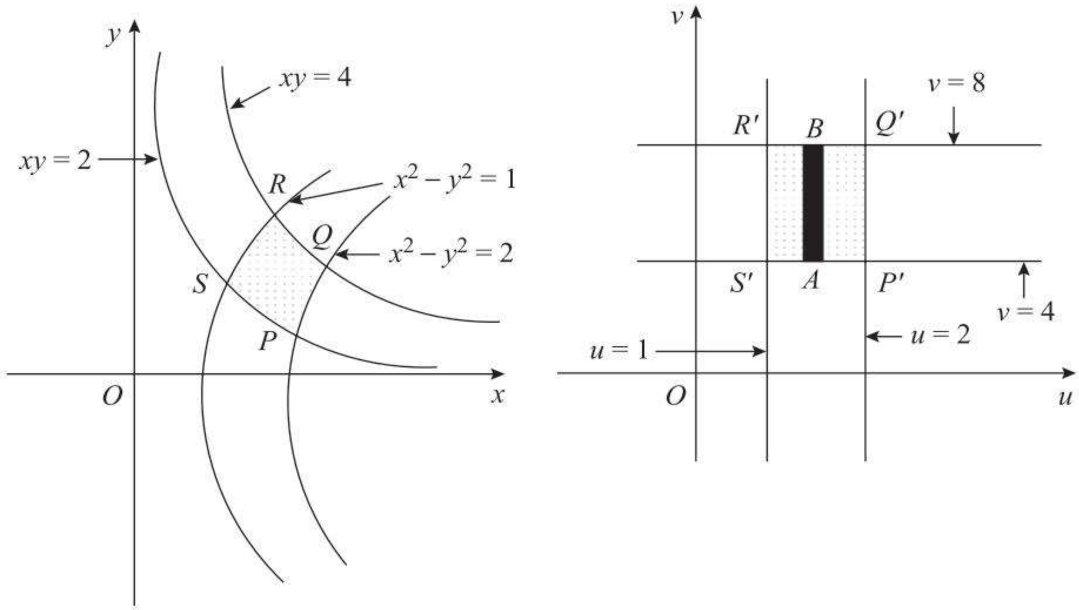

Fig. 9.115

Example 3

Evaluate $\iint\limits_R (x^2 + y^2)\, \mathrm{d}A$ *by changing the variables, where R is the region lying in the first quadrant and bounded by the hyperbola* $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ *and* $xy = 4$. **[Summer 2014]**

Solution

Let $u = x^2 - y^2 \qquad v = xy$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}$$

$$= 2x^2 + 2y^2 = 2(x^2 + y^2)$$

$$\mathrm{d}u\,\mathrm{d}v = |J|\,\mathrm{d}x\,\mathrm{d}y = 2(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$\mathrm{d}x\,\mathrm{d}y = \frac{1}{2(x^2 + y^2)} \cdot \mathrm{d}u\,\mathrm{d}v$$

The region in the xy-plane bounded by the curves $x^2 - y^2 - 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$ is transformed to a square in the uv-plane bounded by the lines $u = 1$, $u = 9$, $v = 2$, $v = 4$.

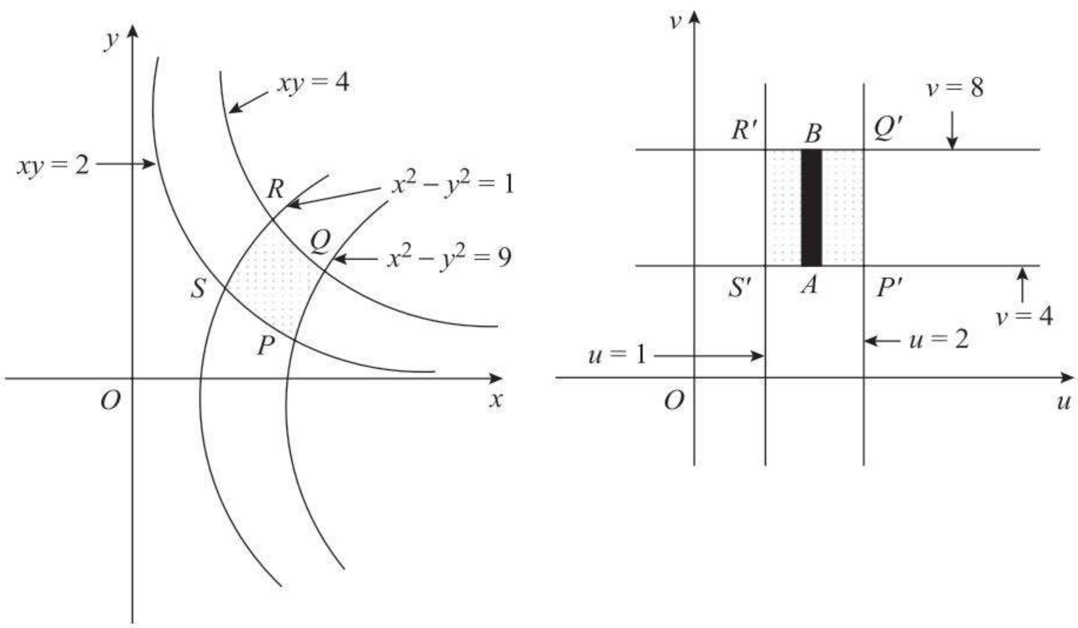

Fig. 9.116

$$
\begin{aligned}
I &= \iint\limits_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_R (x^2 + y^2)\cdot\frac{1}{2(x^2 + y^2)}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\iint \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\int_2^4\int_1^9 \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\left|v\right|_2^4 \quad \left|u\right|_1^9 \\
&= \frac{1}{2}(4-2)(9-1) \\
&= \frac{1}{2}(2)(8) \\
&= 8
\end{aligned}
$$

Example 4

Using the transformation $x + y = u$, $y = uv$, show that

$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\,(e-1).$$ **[Winter 2014]**

Solution

$$x+y=u,\ y=uv$$

$$x=u(1-v),\ y=uv$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=(1-v)u+uv=u$$

$$\mathrm{d}x\mathrm{d}y=|J|\,\mathrm{d}u\mathrm{d}v=u\mathrm{d}u\,\mathrm{d}v$$

Limits of y : $y=0$ to $y=1-x$

Limits of x : $x=0$ to $x=1$.

The region in xy-plane is the triangle OPQ bounded by the lines $x=0$, $y=0$ and $x+y=1$. Under the transformation $x=u(1-v)$ and $y=uv$,

(i) the line $x=0$ gets transformed to the line $u=0$ or $v=1$

(ii) the line $y=0$ gets transformed to the line $u=0$ or $v=0$

(iii) the line $x+y=1$ gets transformed to the line $u=1$

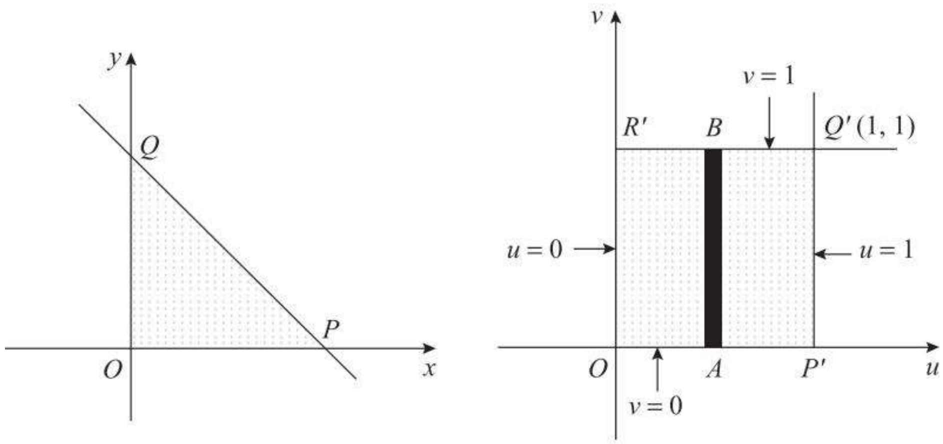

Fig. 9.117

Thus, the triangle OPQ in the xy-plane gets transformed to the square $OP'Q'R'$ in uv-plane bounded by the lines $u=0$, $v=0$, $u=1$ and $v=1$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the u-axis and terminates on the line $v=1$.

Limits of v : $v=0$ to $v=1$

Limits of u : $u=0$ to $u=1$

$$I=\int_0^1\int_0^{1-x}e^{\frac{y}{x+y}}\mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^1\int_0^1 e^v u\mathrm{d}u\,\mathrm{d}v$$

$$=\left|e^v\right|_0^1\left|\frac{u^2}{2}\right|_0^1$$

$$=(e^1-e^0)\cdot\frac{1}{2}$$

$$=\frac{1}{2}(e-1)$$

Example 5

Evaluate $\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,\mathrm{d}x\,\mathrm{d}y$ *by applying the transformations* $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$. *Draw both regions.* **[Winter 2015]**

Solution

1. The function is integrated first w.r.t. x.
2. Limits of x : $x = \frac{y}{2}$ to $x = \frac{y}{2}+1$

   Limits of y : $y = 0$ to $y = 4$.
3. The region is the parallelogram bounded by the lines $x = \frac{y}{2}$, $x = \frac{y}{2}+1$, $y = 0$ and $y = 4$ in xy-plane.

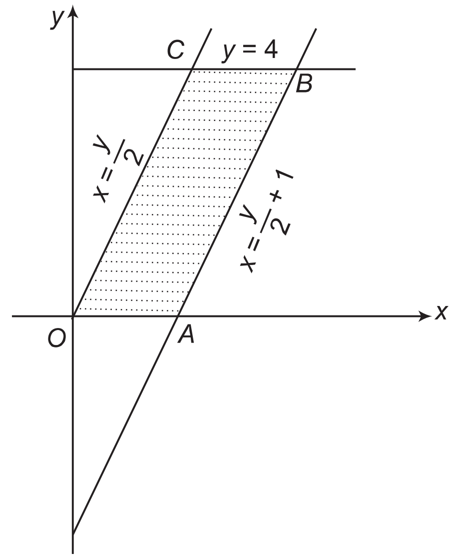

Fig. 9.118

   Applying the transformations

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}$$
$$= x - \frac{y}{2}$$

   (i) the line $x - \frac{y}{2} = 0$ mapped to the line $u = 0$

   (ii) the line $x - \frac{y}{2} = 1$ mapped to the line $u = 1$

   (iii) the line $y = 0$ mapped to the line $v = 0$

   (iv) the line $y = 4$ mapped to the line $v = 2$

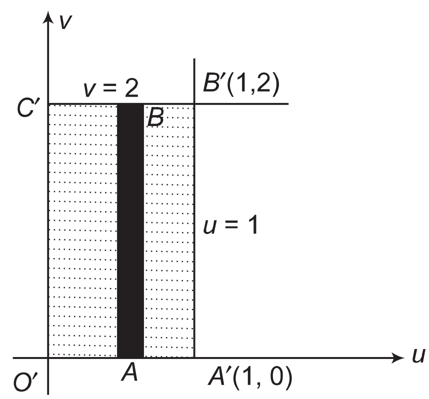

Fig. 9.119

Hence, the parallelogram $OABC$ in the xy-plane mapped to the rectangle $O'A'B'C'$ in uv-plane, bounded by the lines $u = 0$, $u = 1$, $v = 0$ and $v = 2$. In the region, draw a vertical strip AB parallel to the v-axis which starts from the u-axis and terminates on the line $v = 2$.

Limits of u : $u = 0$ to $u = 1$

Limits of v : $v = 0$ to $v = 2$

$$\mathrm{d}x\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$$

where Jacobian, $J = \frac{\partial(x,y)}{\partial(u,v)}$

$$J^* = \frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\dfrac{1}{2} \\ 0 & \dfrac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}$$

$$\therefore\ \mathrm{d}x\mathrm{d}y = \frac{1}{2}\mathrm{d}u\mathrm{d}v$$

Hence, the new form of the integral is

$$I = \int_0^4 \int_{\frac{y}{2}}^{y+1} \frac{2x-y}{2}\mathrm{d}x\mathrm{d}y$$

$$= \int_{v=0}^{2} \int_{u=0}^{1} u \cdot \frac{1}{2}\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{2}\int_0^2 \left|\frac{u^2}{2}\right|_0^1 \mathrm{d}v$$

$$= \frac{1}{4}\left|v\right|_0^2$$

$$= \frac{1}{4}(2)$$

$$= 2$$

Example 6

Using the transformation $x = u(1+v), y = v(1+u), u \geq 0, v \geq 0,$

evaluate $\int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y)+1\right]^{-\frac{1}{2}}\mathrm{d}y\mathrm{d}x.$

Solution

$$x = u(1+v), y = v(1+u)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$$

$$\mathrm{d}x\mathrm{d}v = |J|\,\mathrm{d}u\mathrm{d}v = (1+u+v)\,\mathrm{d}u\mathrm{d}v$$

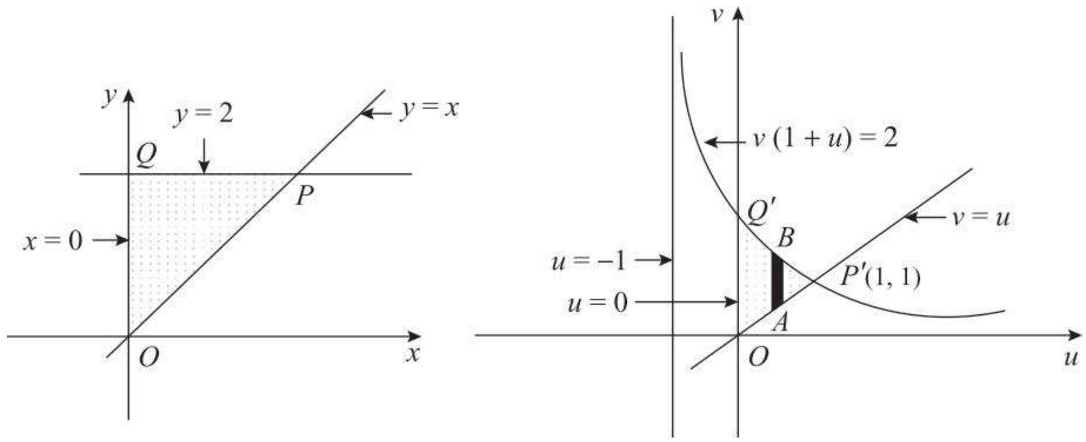

Fig. 9.120

Limits of x : $x = 0$ to $x = y$

Limits of y : $y = 0$ to $y = 2$.

The region in the xy-plane is the ΔOPQ bounded by the lines $x = 0$, $y = 2$ and $y = x$.

Under the transformation $x = u(1+v), y = v(1+u), u \geq 0, v \geq 0$

(i) the line $x = 0$ gets transformed to the line $u = 0$

(ii) the line $y = 2$ gets transformed to the curve $v(1 + u) = 2$

(iii) the line $y = x$ gets transformed to the line $u = v$

Thus, the triangle OPQ in the xy-plane gets transformed to the region $OP'Q'$ in uv plane bounded by the lines $u = 0$, $u = v$ and the curve $v(1 + u) = 2$.

The point of intersection of $u = v$ and $v(1 + u) = 2$ is obtained as $u^2 + u - 2 = 0$, $u = 1, -2$ and $v = 1, -2$.

The point of intersection is $P'(1,1)$.

In the region, draw a vertical strip AB parallel to the v-axis which starts from the line $u = v$ and terminates on the curve $v(1 + u) = 2$.

Limits of v : $v = u$ to $v = \dfrac{2}{1+u}$

Limits of u : $u = 0$ to $u = 1$

$$I = \int_0^2 \int_0^y \left[(x-y)^2 + 2(x+y) + 1\right]^{-\frac{1}{2}} \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} \left[(u-v)^2 + 2(u+v+2uv) + 1\right]^{-\frac{1}{2}} (1+u+v)\mathrm{d}u\, \mathrm{d}v$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} (1+u+v)^{-1}(1+u+v)\, \mathrm{d}v\, \mathrm{d}u$$

$$= \int_0^1 \int_u^{\frac{2}{1+u}} \mathrm{d}v\, \mathrm{d}u$$

$$= \int_0^1 \left| v \right|_u^{\frac{2}{1+u}} \mathrm{d}u$$

$$= \int_0^1 \left(\frac{2}{1+u} - u \right) \mathrm{d}u$$

$$= \left| 2\log(1+u) - \frac{u^2}{2} \right|_0^1$$

$$= 2\log 2 - \frac{1}{2}$$

Example 7

Evaluate $\iint xy\,\mathrm{d}x\mathrm{d}y$ *by changing the variables over the region in the first quadrant bounded by the hyperbolas* $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ *and the circles* $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ *with* $0 < a < b < c < d$.

Solution

Let $\quad x^2 - y^2 = u, x^2 + y^2 = v$

$$x^2 = \frac{u+v}{2}, y^2 = \frac{v-u}{2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4x} & \frac{1}{4x} \\ -\frac{1}{4y} & \frac{1}{4y} \end{vmatrix} = \frac{1}{8xy}$$

$$\mathrm{d}x\mathrm{d}y = |J|\,\mathrm{d}u\mathrm{d}v = \frac{1}{8xy}\mathrm{d}u\mathrm{d}v$$

$$xy\mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}u\mathrm{d}v}{8}$$

The region bounded by the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ and the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ in xy-plane is the curvilinear rectangle $PQRS$.

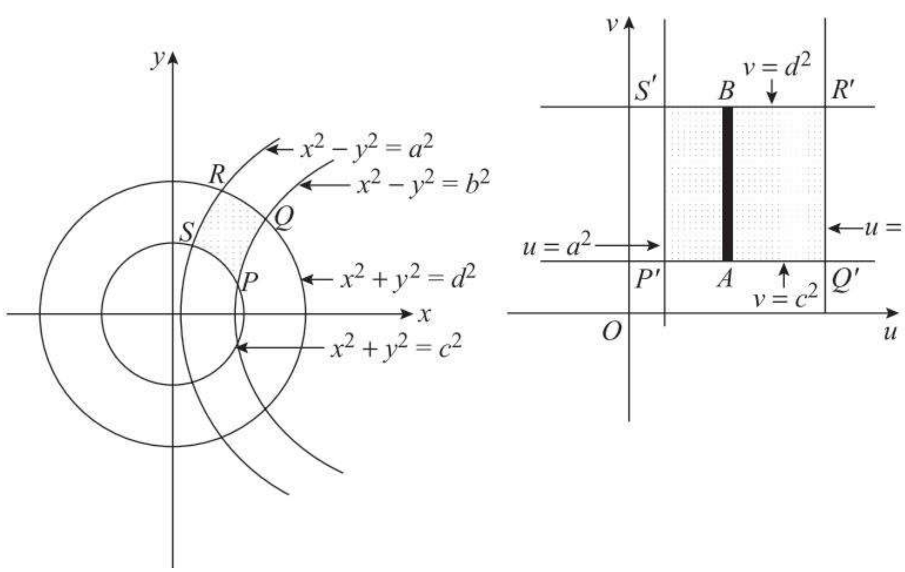

Fig. 9.121

Under the transformation $x^2 - y^2 = u$ and $x^2 + y^2 = v$,

(i) the hyperbolas $x^2 - y^2 = a^2$, $x^2 - y^2 = b^2$ get transformed to the lines $u = a^2$, $u = b^2$ respectively.

(ii) the circles $x^2 + y^2 = c^2$, $x^2 + y^2 = d^2$ get transformed to the lines $v = c^2$, $v = d^2$ respectively.

Thus, the curvilinear rectangle *PQRS* in the *xy*-plane gets transformed to the rectangle *P'Q'R'S'* in *uv*-plane bounded by the lines $u = a^2$, $u = b^2$, $v = c^2$ and $v = d^2$.

In the region, draw a vertical strip *AB* parallel to *v*-axis which starts from the line $v = c^2$ and terminates on the line $v = d^2$.

Limits of v : $v = c^2$ to $v = d^2$

Limits of u : $u = a^2$ to $u = b^2$

$$
\begin{aligned}
I &= \iint xy\,\mathrm{d}x\mathrm{d}y \\
&= \int_{u=a^2}^{b^2}\int_{v=c^2}^{d^2} \frac{1}{8}\,\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{8}\left|u\right|_{a^2}^{b^2}\left|v\right|_{c^2}^{d^2} \\
&= \frac{1}{8}(b^2 - a^2)(d^2 - c^2)
\end{aligned}
$$

Example 8

Evaluate $\iint (x+y)^2\,\mathrm{d}x\,\mathrm{d}y$, *by changing the variables over the parallelogram with vertices* (1, 0), (3, 1), (2, 2), (0, 1).

Solution

The region of integration in *xy*-plane is the parallelogram PQRS.

Equations of the sides of the parallelogram are obtained as

(i) $PQ: y - 0 = \dfrac{1-0}{3-1}(x-1)$

$$2y = x - 1$$
$$x - 2y = 1$$

(ii) $RS: y - 1 = \dfrac{2-1}{2-0}(x-0)$

$$2y - 2 = x$$
$$x - 2y = -2$$

(iii) $PS: y - 0 = \dfrac{1-0}{0-1}(x-1)$

$$x + y = 1$$

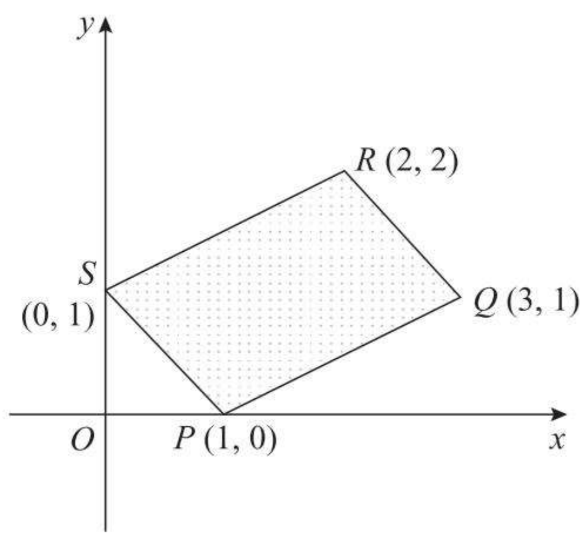

Fig. 9.122

(iv) $QR: y-1=\dfrac{2-1}{2-3}(x-3)$

$$y-1=-x+3$$

$$x+y=4$$

Let $x-2y=u, x+y=v$

$$x=\frac{u+2v}{3}, y=\frac{v-u}{3}$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{1}{3}\end{vmatrix}=\frac{1}{3}$$

$$dxdy=|J|\,dudv=\frac{1}{3}dudv$$

Under the transformation $x - 2y = u$, and $x + y = v$

(i) the lines $x - 2y = 1$, $x - 2y = -2$ get transformed to the lines $u = 1$, $u = -2$ respectively.

(ii) the lines $x + y = 1$, $x + y = 4$ get transformed to the lines $v = 1$, $v = 4$ respectively

Thus, the parallelogram $PQRS$ in the xy-plane gets transformed to a square $P'Q'R'S'$ in uv-plane bounded by the lines $u = 1$, $u = -2$, $v = 1$ and $v = 4$.

In the region, draw a vertical strip AB parallel to v-axis which starts from the line $v = 1$ and terminates on the line $v = 4$.

Limits of v : $v = 1$ to $v = 4$

Limits of u : $u = -2$ to $u = 1$

$$I=\iint (x+y)^2\,dxdy$$

$$=\int_{u=2}^{1}\int_{v=1}^{4} v^2\,\frac{1}{3}dudv$$

$$=\frac{1}{3}\left|u\right|_{-2}^{1}\left|\frac{v^3}{3}\right|_1^4$$

$$=21$$

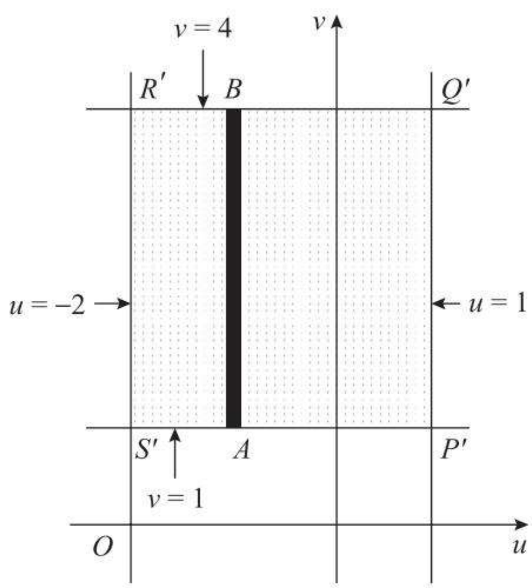

Fig. 9.123

Example 9

Evaluate $\iint xy\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^{\frac{n}{2}}$ *over the first quadrant of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution

Let $x = ar\cos\theta$, $y = br\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}r\,\mathrm{d}\theta = abr\,\mathrm{d}r\,\mathrm{d}\theta$$

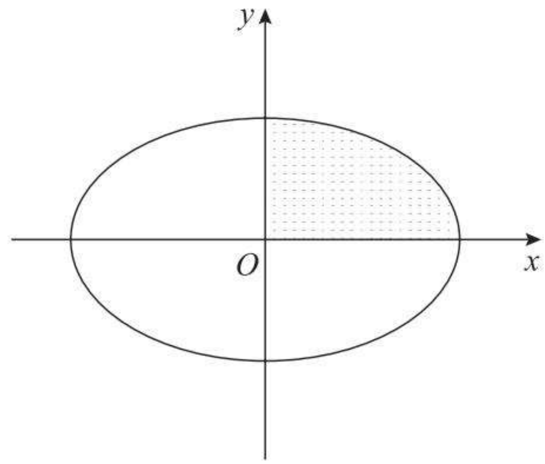

Fig. 9.124

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ in the xy-plane gets transformed to $r^2 = 1$ or $r =$ 1, circle with centre (0, 0) and radius 1 in the $r\theta$-plane.

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. In the region, draw an elementary radius vector OA from the pole which terminates on the circle $r = 1$.

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

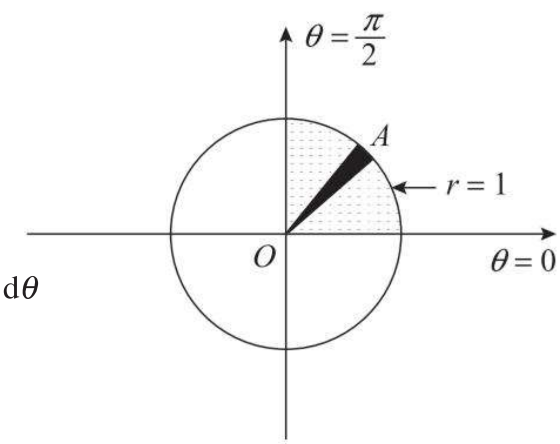

Fig. 9.125

$$I = \iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 abr^2 \cos\theta \sin\theta (r^2)^{\frac{n}{2}} abr\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= a^2b^2 \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} \int_0^1 (r)^{n+3}\mathrm{d}r$$

$$= \frac{a^2b^2}{2}\left| -\frac{\cos 2\theta}{2} \right|_0^{\frac{\pi}{2}} \left| \frac{r^{n+4}}{n+4} \right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi + \cos 0)\cdot\frac{1}{n+4}$$

$$= \frac{a^2b^2}{2(n+4)}$$

EXERCISE 9.6

1. Using the transformation $x + y = u$, $x - y = v$, evaluate $\iint e^{\frac{x-y}{x+y}} dxdy$ over the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\left[\textbf{Ans.: } \frac{1}{4}\left(e - \frac{1}{e}\right)\right]$$

2. Using the transformation $x^2 - y^2 = u$, $2xy = v$, evaluate $\iint (x^2 - y^2) dxdy$ over the region bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$.

[**Ans.:** 4]

3. Using the transformation $x + y = u$, $y = uv$, evaluate

$\int_0^\infty \int_0^\infty e^{-(x+y)} x^{p-1} y^{q-1} dx\, dy.$

$$\left[\textbf{Ans.: } \overline{|p}\,\overline{|q}\right]$$

4. Using the transformation $x = u$, $y = uv$, evaluate $\int_0^1 \int_0^x \sqrt{x^2 + y^2}\, dx\, dy.$

$$\left[\textbf{Ans.: } \frac{1}{3}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1 + \sqrt{2}\right)\right]\right]$$

5. Evaluate $\iint (x + y)^2 dx\, dy$ by changing the variables over the region bounded by the parallelogram with sides $x + y = 0$, $x + y = 2$, $3x - 2y = 0$ and $3x - 2y = 3$.

$$\left[\textbf{Ans. : } \frac{8}{5}\right]$$

6. Evaluate $\iint (x - y)^4 e^{x+y} dx\, dy$, by changing the variables over the region bounded by the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1).

$$\left[\textbf{Ans. : } \frac{e^3 - e}{5}\right]$$

7. Evaluate $\iint [xy(1 - x - y)]^{\frac{1}{2}} dx\, dy$, by changing the variables over the region bounded by the triangle with sides $x = 0$, $y = 0$, $x + y = 1$.

$$\left[\textbf{Ans.: } \frac{2\pi}{105}\right]$$

9.6 TRIPLE INTEGRALS

Let $f(x, y, z)$ be a continuous function defined in a closed and bounded region V in 3-dimensional space. Divide the region V into small elementary parallelopipeds by drawing planes parallel to the coordinate planes. Let the total number of complete parallelopipeds which lie inside the region V be n. Let δV_r be the volume of the r^{th} parallelopiped and (x_r, y_r, z_r) be any point in this parallelopiped. Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r \qquad \ldots(1)$$

where, $\delta V_r = \delta x_r \cdot \delta y_r \cdot \delta z_r$

If we increase the number of elementary parallelopipeds, n, then the volume of each parallelopiped decreases. Hence as $n \to \infty, \delta V_r \to 0$.

The limit of the sum given by Eq. (1), if it exists is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\iiint\limits_V f(x, y, z)\mathrm{d}V$

Hence, $$\iiint\limits_V f(x, y, z)\mathrm{d}V = \lim_{\substack{n\to\infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$$

where $\mathrm{d}V = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$

9.6.1 Triple Integrals in Cartesian Coordinates

Triple integral of a continuous function $f(x, y, z)$ over a region V can be evaluated by three successive integrations.

Let the region V be bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$. Let the projection of region V in xy-plane be R which be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$. Then the triple integral is defined as

$$I = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,\mathrm{d}z \right\} \mathrm{d}y \right] \mathrm{d}x$$

Note: The order of variables in $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ indicates the order of integration. In some cases this order is not maintained. Therefore, it is advisable to identify the order of integration with the help of the limits.

9.6.2 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates r, θ, z are used to evaluate the integral in the regions which are bounded by cylinders along z-axis, planes through z-axis, planes perpendicular to the z-axis.

Relations between Cartesian (rectangular) coordinates (x, y, z) and cylindrical coordinates (r, θ, ϕ) are given as $x = r\cos\theta$

$$y = r\sin\theta$$
$$z = z$$

Then $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$= \iiint f(r\cos\theta, r\sin\theta, z)\left|J\right|\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

where, $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta)$$

$$= r$$

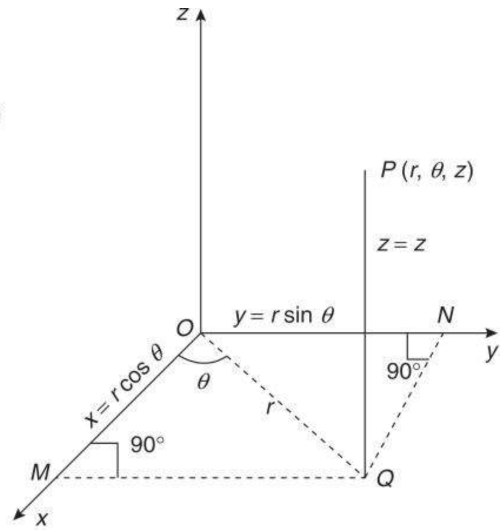

Fig. 9.126

Hence, $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

9.6.3 Triple Integrals in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are used to evaluate the integral in the regions which are bounded by the sphere with centre at the origin.

Relations between cartesian (rectangular) coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are given as

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Then

$$\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\left|J\right|\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

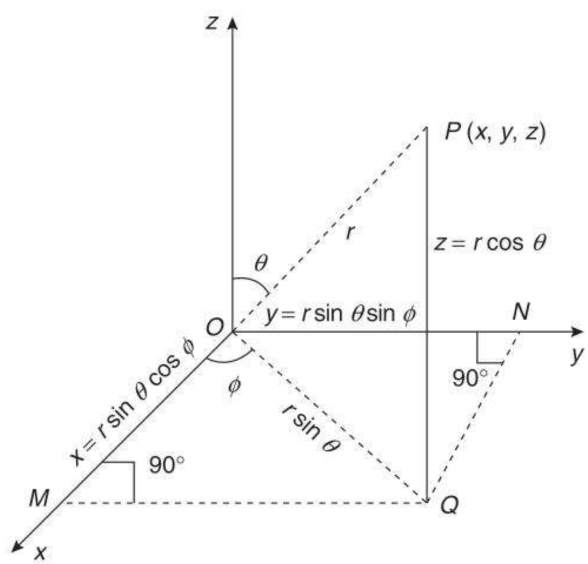

Fig. 9.127

where
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$
$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$
$$= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(-r\sin\theta\cos\phi\cos\theta)$$
$$- r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$
$$= r^2\sin\theta\cos^2\phi(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta$$

Hence, $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$

***Note*:** If the region of integration is a sphere $x^2 + y^2 + z^2 = a^2$ with centre at (0, 0, 0) and radius a, then limits of r, θ, ϕ are

(i) For positive octant of the sphere,
$$r : r = 0 \quad \text{to} \quad r = a$$
$$\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$
$$\phi : \phi = 0 \quad \text{to} \quad \phi = \frac{\pi}{2}$$

(ii) For hemisphere,
$$r : r = 0 \quad \text{to} \quad r = a$$
$$\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$
$$\phi : \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

(iii) For complete sphere,
$$r : r = 0 \quad \text{to} \quad r = a$$
$$\theta : \theta = 0 \quad \text{to} \quad \theta = \pi$$
$$\phi : \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

9.6.4 Change of Variables

In some cases, evaluation of a triple integral becomes easier by changing the variables. Let the variables x, y, z be replaced by new variables u, v, w by the transformation $x = f_1(u, v, w)$, $y = f_2(u, v, w)$, $z = f_3(u, v, w)$.

Then $$\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(f_1, f_2, f_3)\left|J\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

where, $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

9.6.5 Working Rule for Evaluation of Triple Integrals

1. Draw all the planes and surfaces and identify the region of integration.
2. Draw an elementary volume parallel to z (y or x) axis.
3. Find the variation of z (y or x) along the elementary volume.
4. Lower and upper limits of z (y or x) are obtained from the equation of the surface (or plane) where elementary volume starts and terminates respectively.
5. Find the projection of the region on xy (zx or yz) plane.
6. Draw the region of projection in xy (zx or yz) plane.
7. Follow the steps of double integration to find the limits of x and y (z and x or y and z).

Note: (1) If the region is bounded by the cylinders along the z-axis, planes through z-axis, the planes perpendicular to the z-axis, then the cylindrical coordinates are used.
(2) If the region is bounded by the sphere, then the spherical coordinates are used.

Type I Evaluation of Triple Integrals when Limits are Given

Example 1

Evaluate $\int_0^1\int_0^2\int_0^e \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$

Solution

$$\begin{aligned}\int_0^1\int_0^2\int_0^e \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z &= \int_0^1\int_0^2\left[\int_0^e \mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}z \\ &= \int_0^1\int_0^2 \left|y\right|_0^e \mathrm{d}x\,\mathrm{d}z \\ &= \int_0^1\left[\int_0^2 e\,\mathrm{d}x\right]\mathrm{d}z \\ &= e\int_0^1 \left|x\right|_0^2 \mathrm{d}z \\ &= e\int_0^1 2\,\mathrm{d}z\end{aligned}$$

$$= 2e|z|_0^1$$
$$= 2e$$

Another method

Since all the limits are constant and integrand (function) is explicit in x, y and z, the integral can be written as

$$\int_0^1\int_0^2\int_0^e \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \int_0^1 \mathrm{d}z \cdot \int_0^2 \mathrm{d}x \cdot \int_0^e \mathrm{d}y$$
$$= |z|_0^1 \cdot |x|_0^2 \cdot |y|_0^e$$
$$= 1 \cdot 2 \cdot e$$
$$= 2e.$$

Example 2

Evaluate $\int_0^2\int_1^3\int_1^2 xy^2z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Solution

Since all the limits are constant and integrand (function) is explicit in x, y and z, the integral can be written as

$$\int_0^2\int_1^3\int_1^2 xy^2z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 x\,\mathrm{d}x \cdot \int_1^3 y^2\mathrm{d}y \cdot \int_1^2 z\,\mathrm{d}z$$
$$= \left|\frac{x^2}{2}\right|_0^2 \cdot \left|\frac{y^3}{3}\right|_1^3 \cdot \left|\frac{z^2}{2}\right|_1^2$$
$$= 2 \cdot \frac{26}{3} \cdot \frac{3}{2}$$
$$= 26$$

Example 3

Evaluate $\int_0^1\int_0^\pi\int_0^\pi y\sin z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$ **[Winter 2013]**

Solution

Since all the limits are constant and integrand (function) is explicit in x, y, and z, the integral can be written as

$$\int_0^1\int_0^\pi\int_0^\pi y\sin z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \sin z\,\mathrm{d}z\int_0^\pi y\,\mathrm{d}y\int_0^\pi \mathrm{d}x$$
$$= |-\cos z|_0^1 \cdot \left|\frac{y^2}{2}\right|_0^\pi \cdot |x|_0^\pi$$

$$= (-\cos 1 + \cos 0)\left(\frac{\pi^2}{2}\right)(\pi)$$

$$= \frac{\pi^3}{2}(1 - \cos 1)$$

Example 4

Evaluate $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r\mathrm{d}\theta \mathrm{d}r\mathrm{d}z.$ **[Winter 2016]**

Solution

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r\mathrm{d}\theta \mathrm{d}r\mathrm{d}z = \int_0^1 \int_0^{\sqrt{z}} \left\{ \int_0^{2\pi} [r^2 \cos^2 \theta + z^2] \mathrm{d}\theta \right\} r\,\mathrm{d}r\mathrm{d}z$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{ \int_0^{2\pi} \left[r^2 \left(\frac{1 + \cos 2\theta}{2} \right) + z^2 \right] \mathrm{d}\theta \right\} r\,\mathrm{d}r\mathrm{d}z$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left\{ r \left[\frac{1}{2} r^2 \left(\theta + \frac{\sin 2\theta}{2} \right) + z^2 \theta \right]_0^{2\pi} \right\} \mathrm{d}r\mathrm{d}z$$

$$= 2\pi \int_0^1 \int_0^{\sqrt{z}} \left[\frac{1}{2} r^3 + z^2 r \right] \mathrm{d}r\mathrm{d}z$$

$$= 2\pi \int_0^1 \left| \frac{r^4}{8} + z^2 \frac{r^2}{2} \right|_0^{\sqrt{2}} \mathrm{d}z$$

$$= 2\pi \int_0^1 \left| \frac{z^2}{8} + \frac{z^3}{2} \right| \mathrm{d}z$$

$$= 2\pi \left[\frac{z^3}{24} + \frac{z^4}{8} \right]_0^1$$

$$= 2\pi \left[\frac{1}{24} + \frac{1}{8} \right]$$

$$= 2\pi \left[\frac{3 + 1}{24} \right]$$

$$= 2\pi \cdot \frac{4}{24}$$
$$= \frac{\pi}{3}$$

Example 5

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^{1-x} \left[\int_0^{x+y} \mathrm{d}z \right] \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} \left| z \right|_0^{x+y} \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} (x+y)\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \left| xy + \frac{y^2}{2} \right|_0^{1-x} \mathrm{d}x$$
$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^2}{2} \right] \mathrm{d}x$$
$$= \int_0^1 \left[x - x^2 + \frac{(1-x)^2}{2} \right] \mathrm{d}x$$
$$= \left| \frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{2} \cdot \frac{(1-x)^3}{(-3)} \right|_0^1$$
$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{6}$$
$$= \frac{1}{3}$$

Example 6

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^{1-x} \left[\int_0^{x+y} e^z \mathrm{d}z \right] \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^{1-x} \left| e^z \right|_0^{x+y} \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^1 \left[\int_0^{1-x} (e^{x+y} - e^0) \mathrm{d}y \right] \mathrm{d}x$$
$$= \int_0^1 \left| e^{x+y} - y \right|_0^{1-x} \mathrm{d}x$$
$$= \int_0^1 \left[(e - e^x) - (1-x) \right] \mathrm{d}x$$
$$= \left| (e-1)x + \frac{x^2}{2} - e^x \right|_0^1$$
$$= (e-1) + \frac{1}{2} - e + e^0$$
$$= \frac{1}{2}$$

Example 7

Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$

Solution

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x = \int_0^a e^x \int_0^x e^y \left[\int_0^{x+y} e^z \mathrm{d}z \right] \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^a e^x \int_0^x e^y \left| e^z \right|_0^{x+y} \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^a e^x \int_0^x e^y \left(e^{x+y} - e^0 \right) \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^a e^x \left[\int_0^x \left(e^x e^{2y} - e^y \right) \mathrm{d}y \right] \mathrm{d}x$$
$$= \int_0^a e^x \left| e^x \cdot \frac{e^{2y}}{2} - e^y \right|_0^x \mathrm{d}x$$
$$= \int_0^a e^x \left[\left(e^x \cdot \frac{e^{2x}}{2} - e^x \cdot \frac{1}{2} \right) - (e^x - e^0) \right] \mathrm{d}x$$
$$= \int_0^a \left(\frac{1}{2} e^{4x} - \frac{3}{2} e^{2x} + e^x \right) \mathrm{d}x$$
$$= \left| \frac{1}{2} \cdot \frac{e^{4x}}{4} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^x \right|_0^a$$
$$= \frac{1}{8} \left(e^{4a} - e^0 \right) - \frac{3}{4} \left(e^{2a} - e^0 \right) + \left(e^a - e^0 \right)$$
$$= \frac{1}{8} e^{4a} - \frac{3}{4} e^{2a} + e^a - \frac{3}{8}$$

Example 8

Evaluate the integral $\int_0^2 \int_1^2 \int_0^{yz} xyz \; dx\, dy\, dz.$ **[Summer 2014]**

Solution

$$\begin{aligned}
\int_0^2 \int_1^2 \int_0^{yz} xyz \, dx\, dy\, dz &= \int_0^2 \int_1^2 yz\left[\int_0^{yz} x \, dx\right] dy\, dz \\
&= \int_0^2 \int_1^2 yz \left|\frac{x^2}{2}\right|_0^{yz} dy\, dz \\
&= \int_0^2 \int_1^2 \frac{1}{2} y^3 z^3 dy\, dz \\
&= \frac{1}{2}\left|\frac{y^4}{4}\right|_1^2 \quad \left|\frac{z^4}{4}\right|_0^2 \\
&= \frac{1}{2}\left(\frac{1}{4}\right)[16-1]\frac{1}{4}[16-0] \\
&= \frac{15}{2}
\end{aligned}$$

Example 9

Evaluate $\int_1^3 \int_1^{\frac{1}{x}} \int_0^{\sqrt{xy}} xy \, dz\, dy\, dx.$

Solution

$$\begin{aligned}
\int_1^3 \int_1^{\frac{1}{x}} \int_0^{\sqrt{xy}} xy \, dz\, dy\, dx &= \int_1^3 \int_1^{\frac{1}{x}} \left[\int_0^{\sqrt{xy}} dz\right] xy \, dy\, dx \\
&= \int_1^3 \int_1^{\frac{1}{x}} |z|_0^{\sqrt{xy}} dy\, dx \\
&= \int_1^3 \left[\int_1^{\frac{1}{x}} \sqrt{xy}\, dy\right] dx \\
&= \int_1^3 \sqrt{x} \left|\frac{2y^{\frac{3}{2}}}{3}\right|_1^{\frac{1}{x}} dx \\
&= \frac{2}{3}\int_1^3 \sqrt{x}\left(\frac{1}{x^{\frac{3}{2}}} - 1\right) dx
\end{aligned}$$

$$= \frac{2}{3}\int_1^3\left(\frac{1}{x} - x^{\frac{1}{2}}\right)\mathrm{d}x$$

$$= \frac{2}{3}\left|\log x - \frac{2x^{\frac{3}{2}}}{3}\right|_1^3$$

$$= \frac{2}{3}\left[(\log 3 - \log 1) - \frac{2}{3}\left(3^{\frac{3}{2}} - 1\right)\right]$$

$$= \frac{2}{3}\left[\log 3 - 2\sqrt{3} + \frac{2}{3}\right]$$

Example 10

Evaluate $\int_0^2\int_1^z\int_0^{yz} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$ **[Summer 2017]**

Solution

The innermost limits depend on y and z. Hence, integrating first w.r.t. x,

$$\int_0^2\int_1^z\int_0^{yz} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2\int_1^z\left|\frac{x^2}{2}\right|_0^{yz} yz\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^2\int_1^z (y^2z^2)yz\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^2 z^3\left[\int_1^z y^3\,\mathrm{d}y\right]\mathrm{d}z$$

$$= \frac{1}{2}\int_0^2 z^3\left|\frac{y^4}{4}\right|_1^z\mathrm{d}z$$

$$= \frac{1}{8}\int_0^2 z^3(z^4 - 1)\mathrm{d}z$$

$$= \frac{1}{8}\left|\frac{z^8}{8} - \frac{z^4}{4}\right|_0^2$$

$$= \frac{1}{8}(32 - 4)$$

$$= \frac{7}{2}$$

Example 11

Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

The innermost limits depend on x and y. Hence, integrating first w.r.t. z,

$$\begin{aligned}
\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
&=\int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,\mathrm{d}z\right]\mathrm{d}y\,\mathrm{d}x\\
&=\int_0^1\int_0^{1-x}\left|\frac{1}{-2(x+y+z+1)^2}\right|_0^{1-x-y}\mathrm{d}y\,\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{\{x+y+(1-x-y)+1\}^2}-\frac{1}{(x+y+1)^2}\right]\mathrm{d}y\,\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\left[\int_0^{1-x}\left\{\frac{1}{4}-\frac{1}{(x+y+1)^2}\right\}\mathrm{d}y\right]\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\left|\frac{y}{4}+\frac{1}{x+y+1}\right|_0^{1-x}\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{x+(1-x)+1}-\frac{1}{x+1}\right]\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right)\mathrm{d}x\\
&=-\frac{1}{2}\left|\frac{x}{4}-\frac{x^2}{8}+\frac{x}{2}-\log(x+1)\right|_0^1\\
&=-\frac{1}{2}\left(\frac{5}{8}-\log 2\right)
\end{aligned}$$

Example 12

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$ **[Winter 2014]**

Solution

$$\int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\int_0^{\sqrt{1-x^2-y^2}} z\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y=\int_0^1\int_0^{\sqrt{1-x^2}} xy\left|\frac{z^2}{z}\right|_0^{\sqrt{1-x^2-y^2}}\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1-x^2}}\left[(1-x^2)y-y^3\right]\right]dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left|(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{(1-x^2)}{2}-\left(\frac{1-x^2)^2}{4}\right)\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x^2)^2}{4}\right]dx$$

$$= \frac{1}{8}\int_0^1 x(1-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^1 \{x(1-2x^2+x^4)\}\,dx$$

$$= \frac{1}{8}\int_0^1 (x-2x^3+x^5)\,dx$$

$$= \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)$$

$$= \frac{1}{48}$$

Example 13

Evaluate $\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dx\,dy\,dz.$ **[Summer 2016]**

Solution

The inner most limit depends on x and middle limit depends on y. Hence, integrating first w.r.t. z,

$$\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dx\,dy\,dz = \int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dx\,dy$$

$$= \int_1^e \int_1^{\log y} \left(\left| z \log z \right|_1^{e^x} - \int_1^{e^x} z \cdot \frac{1}{z} \mathrm{d}z \right) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_1^e \int_1^{\log y} \left(e^x \log e^x - \log 1 - \left| z \right|_1^{e^x} \right) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_1^e \left[\int_1^{\log y} \left(e^x x - e^x + 1 \right) \mathrm{d}x \right] \mathrm{d}y$$

$$= \int_1^e \left| xe^x - e^x - e^x + x \right|_1^{\log y} \mathrm{d}y$$

$$= \int_1^e \left[e^{\log y} (\log y - 2) + \log y - e(1-2) - 1 \right] \mathrm{d}y$$

$$= \int_1^e \left[y(\log y - 2) + \log y + e - 1 \right] \mathrm{d}y$$

$$= \int_1^e \left[(y+1) \log y - 2y + e - 1 \right] \mathrm{d}y$$

$$= \left| \log y \left(\frac{y^2}{2} + y \right) \right|_1^e - \int_1^e \frac{1}{y} \left(\frac{y^2}{2} + y \right) \mathrm{d}y - \left| y^2 \right|_1^e + \left| (e-1) y \right|_1^e$$

$$= \log e \left(\frac{e^2}{2} + e \right) - \log 1 \left(\frac{1}{2} + 1 \right) - \left| \frac{y^2}{4} + y \right|_1^e - (e^2 - 1) + \left[(e-1)(e-1) \right]$$

$$= \frac{e^2}{2} + e - \left[\frac{1}{4}(e^2 - 1) + (e-1) \right] - e^2 + 1 + e^2 - 2e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

Example 14

Evaluate $\displaystyle\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z}{(1 + x^2 + y^2 + z^2)^2}$.

Solution

1. It is difficult to integrate this integral in cartesian form. Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$ integral changes to spherical form.
2. Limits of $x : x = 0$ to $x \to \infty$

   Limits of $y : y = 0$ to $y \to \infty$

   Limits of $z : z = 0$ to $z \to \infty$

The region of integration is the positive octant of the plane.

Limits of $r : r = 0$ to $r \to \infty$

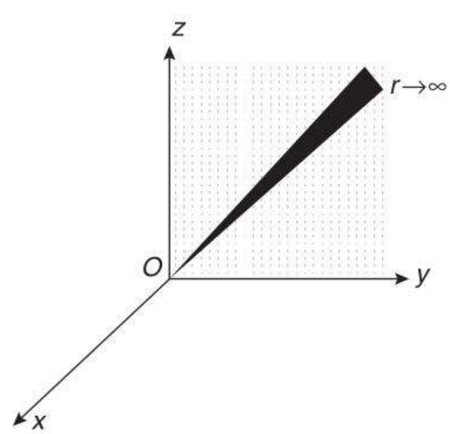

Fig. 9.128

$$\text{Limits of } \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$

$$\text{Limits of } \phi : \phi = 0 \quad \text{to} \quad \phi = \frac{\pi}{2}$$

Hence, the spherical form of the given integral is

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x^2+y^2+z^2)^2}$$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}{(1+r^2)^2}$$

$$= \int_0^\infty \frac{r^2\mathrm{d}r}{(1+r^2)^2} \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \mathrm{d}\phi$$

Putting $r = \tan t, dr = \sec^2 t \,\mathrm{d}t$

When $r = 0, t = 0$

When $r \to \infty, t = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sin\theta \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t} \sec^2 t \,\mathrm{d}t \int_0^{\frac{\pi}{2}} \mathrm{d}\phi$$

$$= \left| -\cos\theta \right|_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sin^2 t \,\mathrm{d}t \right) \left| \phi \right|_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} + \cos 0 \right) \cdot \left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2} \mathrm{d}t \right) \cdot \frac{\pi}{2}$$

$$= \frac{1}{2} \left| t - \frac{\sin 2t}{2} \right|_0^{\frac{\pi}{2}} \frac{\pi}{2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{1}{2} (\sin\pi - \sin 0) \right] \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

Example 15

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Solution

1. It is difficult to integrate this integral in cartesian form. Putting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ integral changes to spherical form.

2. Limits of $z : z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$

   Limits of $y : y = 0$ to $y = \sqrt{a^2 - x^2}$

   Limits of $x : x = 0$ to $x = a$

The region of integration is the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

   Limits of $r : r = 0$ to $r = a$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

   Limits of $\phi : \phi = 0$ to $\phi = \dfrac{\pi}{2}$

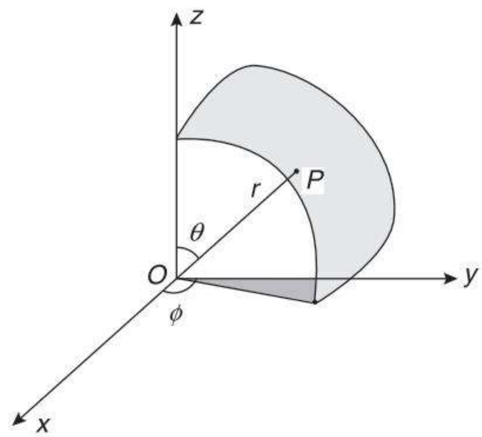

Fig. 9.129

Hence, the spherical form of the given integral is

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3 \sin^2\theta\cos\theta \cdot \cos\phi\sin\phi \cdot r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\phi}{2}\,\mathrm{d}\phi \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\,\mathrm{d}\theta \int_0^a r^5 \mathrm{d}r$$

$$= \frac{1}{2}\left|\frac{-\cos 2\phi}{2}\right|_0^{\frac{\pi}{2}} \cdot \left|\frac{\sin^4\theta}{4}\right|_0^{\frac{\pi}{2}} \cdot \left|\frac{r^6}{6}\right|_0^a \qquad \left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}(-\cos\pi + \cos 0)\cdot\frac{1}{4}\left(\sin\frac{\pi}{2} - \sin 0\right)\cdot\frac{a^6}{6}$$

$$= \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{1}{4}\cdot\frac{a^6}{6}$$

$$= \frac{a^6}{48}$$

Example 16

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}{\sqrt{x^2+y^2+z^2}}$ *by transforming into spherical polar coordinates.*

Solution

1. Limits of $z : z = \sqrt{x^2 + y^2}$ to $z = 1$

   Limits of $y : y = 0$ to $y = \sqrt{1 - x^2}$

   Limits of $x : x = 0$ to $x = 1$

2. The region of integration is the part of the cone $z^2 = x^2 + y^2$ bounded above by the plane $z = 1$ in the positive octant (since all three limits are positive).
3. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, spherical polar form of
   (i) the cone $z^2 = x^2 + y^2$ is

$$\begin{aligned} r^2\cos^2\theta &= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) \\ &= r^2\sin^2\theta \\ \cos\theta &= \sin\theta \\ \tan\theta &= 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

   (ii) the plane $z = 1$ is $r\cos\theta = 1$

$$r = \sec\theta$$

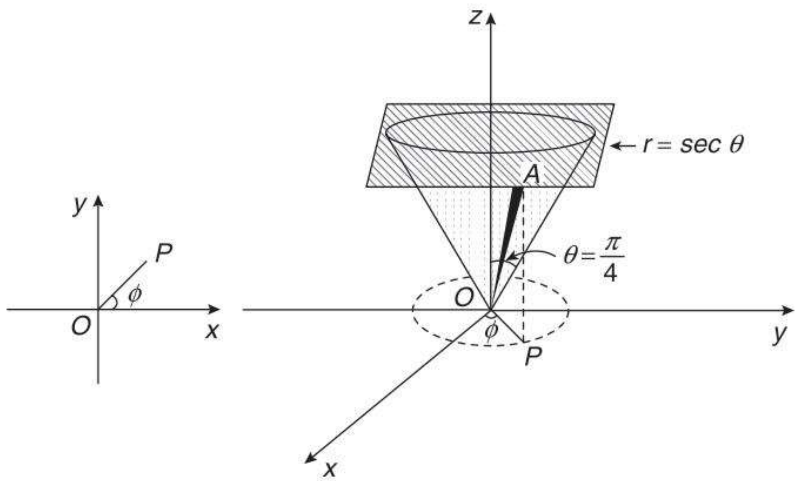

Fig. 9.130

4. Draw an elementary radius vector OA which starts from the origin and terminates on the plane $r = \sec\theta$.

   Limits of r : $r = 0$ to $r = \sec\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{4}$

   Limits of ϕ : $\phi = 0$ to $\phi = \frac{\pi}{2}$ (in positive octant)

Hence, the spherical form of the given integral is

$$\begin{aligned} I &= \int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}} \\ &= \int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\sec\theta} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{r} \\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sec\theta} r\,dr\right]\sin\theta\,d\theta\,d\phi \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left| \frac{r^2}{2} \right|_0^{\sec\theta} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{2} \sin\theta \, d\theta \right] d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \tan\theta \sec\theta \, d\theta$$

$$= \frac{1}{2} |\phi|_0^{\frac{\pi}{2}} |\sec\theta|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} \left(\sec\frac{\pi}{4} - \sec 0 \right)$$

$$= \frac{\pi}{4} \left(\sqrt{2} - 1 \right)$$

Type II Evaluation of Triple Integrals Over the Given Region

Example 1

Evaluate $\iiint x^2 yz \, dx \, dy \, dz$ *over the region bounded by the planes* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.

Solution

1. Draw an elementary volume AB parallel to z-axis in the region. AB starts from xy-plane and terminates on the plane $x + y + z = 1$.

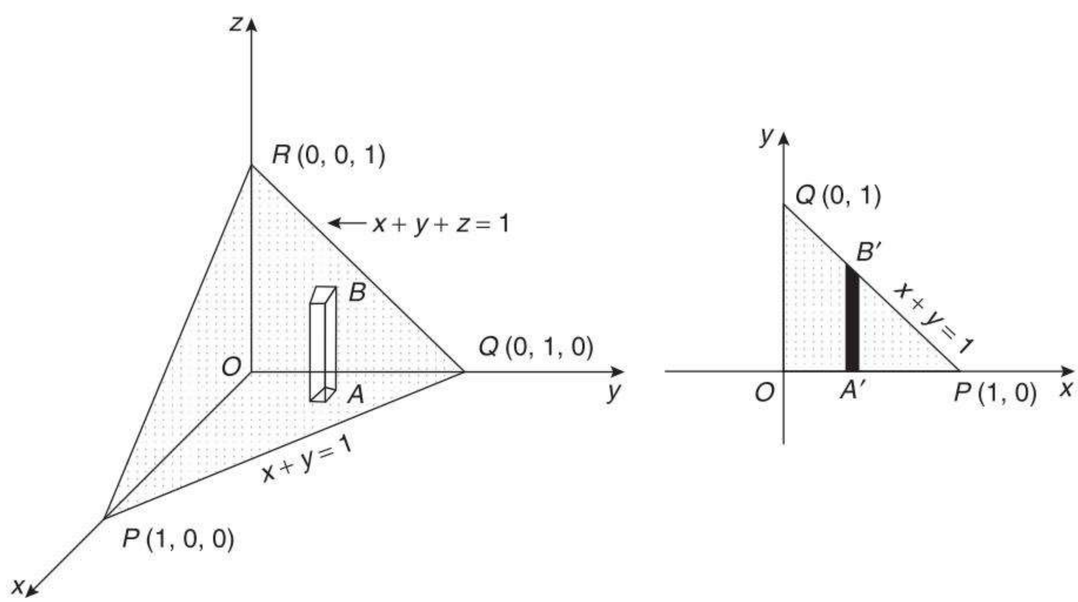

Fig. 9.131

Limits of z : $z = 0$ to $z = 1 - x - y$

2. Projection of the plane $x + y + z = 1$ in xy-plane is ΔOPQ. Putting $z = 0$ in $x + y + z = 1$, the equation of the line PQ is obtained as $x + y = 1$.
3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $x + y = 1$.

   Limits of y : $y = 0$ to $y = 1 - x$

   Limits of x : $x = 0$ to $x = 1$

$$
\begin{aligned}
I &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 yz \, dz \, dy \, dx \\
&= \int_0^1 \int_0^{1-x} x^2 y \left| \frac{z^2}{2} \right|_0^{1-x-y} dy \, dx \\
&= \frac{1}{2} \int_0^1 x^2 \left[\int_0^{1-x} y \left\{ (1-x)^2 + y^2 - 2y(1-x) \right\} dy \right] dx \\
&= \frac{1}{2} \int_0^1 x^2 \left[\int_0^{1-x} \left\{ y(1-x)^2 + y^3 - 2y^2(1-x) \right\} dy \right] dx \\
&= \frac{1}{2} \int_0^1 x^2 \left| (1-x)^2 \frac{y^2}{2} + \frac{y^4}{4} - 2(1-x)\frac{y^3}{3} \right|_0^{1-x} dx \\
&= \frac{1}{2} \int_0^1 x^2 \left[(1-x)^2 \cdot \frac{(1-x)^2}{2} + \frac{(1-x)^4}{4} - 2(1-x) \cdot \frac{(1-x)^3}{3} \right] dx \\
&= \frac{1}{2} \int_0^1 \frac{x^2}{12} (1-x)^4 \, dx \\
&= \frac{1}{24} \left| \frac{(1-x)^5}{-5} \cdot x^2 - \frac{(1-x)^6}{30} \cdot 2x + \frac{(1-x)^7}{-210} \cdot 2 \right|_0^1 \\
&= \frac{1}{24} \left(0 + \frac{1}{105} \right) \\
&= \frac{1}{2520}
\end{aligned}
$$

Example 2

Evaluate $\iiint\limits_E 2x \, dV$, *where* E *is the region under the plane* $2x + 3y + z = 6$ *that lies in the first octant.* **[Winter 2015]**

Solution

1. Draw an elementary volume AB parallel to z-axis in the region. AB starts from xy-plane and terminates on the plane $2x + 3y + z = 6$.

   Limits of z : $z = 0$ to $z = 6 - 2x - 3y$

2. Projection of the plane $2x + 3y + z = 6$ in xy-plane is ΔOPQ. Putting $z = 0$ in $2x + 3y + z = 6$, the equation of the line PQ is obtained as $2x + 3y = 6$.

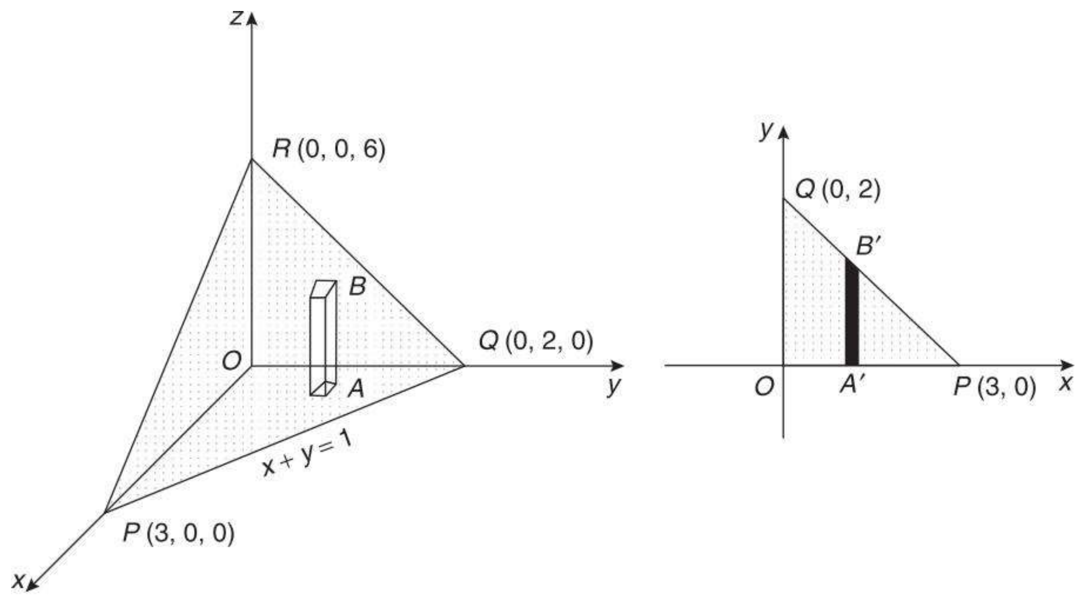

Fig. 9.132

3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $2x + 3y = 6$.

Limits of y : $y = 0$ to $y = \dfrac{6-2x}{3}$

Limits of x : $x = 0$ to $x = 3$

$$I = \iiint_E 2x \, dV$$

$$= \int_0^3 \int_0^{\frac{6-2x}{3}} \int_0^{6-2x-3y} 2x \, dz \, dy \, dx$$

$$= \int_0^3 \int_0^{\frac{6-2x}{3}} 2x \left| z \right|_0^{6-2x-3y} dy \, dx$$

$$= 2\int_0^3 \int_0^{\frac{6-2x}{3}} x(6-2x-3y) \, dy \, dx$$

$$= 2\int_0^3 x \left| (6-2x)y - \frac{3y^2}{2} \right|_0^{\frac{6-2x}{3}} dx$$

$$= 2\int_0^3 x \left[(6-2x)\frac{(6-2x)}{3} - \frac{3}{2}\left(\frac{6-2x}{3}\right)^2 \right] dx$$

$$= 2\int_0^3 x\frac{(6-2x)^2}{6}\,dx$$

$$= \frac{4}{3}\int_0^3 x(9+x^2-6x)dx$$

$$= \frac{4}{3}\int_0^3 (9x+x^3-6x^2)\,dx$$

$$= \frac{4}{3}\left|9\frac{x^2}{2}+\frac{x^4}{4}-6\frac{x^3}{3}\right|_0^3$$

$$= 9$$

Example 3

Evaluate $\iiint xyz\,dx\,dy\,dz$ *over the positive octant of the sphere* $x^2 + y^2 + z^2 = 4$.

Solution

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the equation of the sphere $x^2+y^2+z^2=4$ reduces to $r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = 4$, $r^2 = 4$, $r = 2$.

The region is the positive octant of the sphere $r = 2$.

Limits of r : $r = 0$ to $r = 2$

Limits of $\theta = 0$ to $\theta = \frac{\pi}{2}$

Limits of $\phi = 0$ to $\phi = \frac{\pi}{2}$

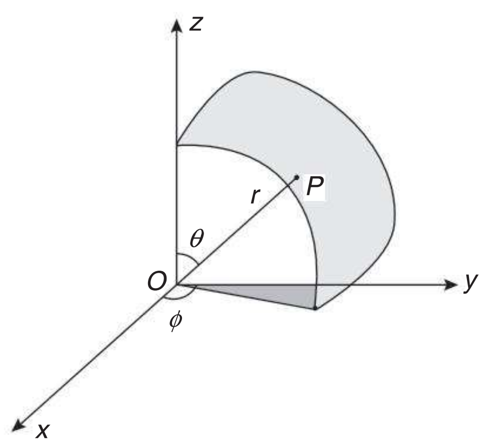

Fig. 9.133

Hence, the spherical form of the given integral is

$$I = \iiint xyz\,dx\,dy\,dz$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 (r^3\sin^2\theta\cos\theta\cos\phi\sin\phi)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\int_0^{\frac{\pi}{2}}\frac{\sin 2\phi}{2}d\phi\int_0^2 r^5 dr$$

$$= \left|\frac{\sin^4\theta}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{-\cos 2\phi}{4}\right|_0^{\frac{\pi}{2}}\left|\frac{r^6}{6}\right|_0^2 \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}, n \neq 1\right]$$

$$= \frac{1}{4}\left(\sin^4 \frac{\pi}{2} - \sin 0\right)\left[-\frac{1}{4}(\cos\pi - \cos 0)\right]\left(\frac{2^6}{6}\right)$$

$$= \frac{4}{3}$$

Example 4

Evaluate $\iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ *over the region bounded by the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution

1. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$.
2. For the complete sphere, limits of $r : r = 0$ to $r = a$
   limits of $\theta : \theta = 0$ to $\theta = \pi$
   limits of $\phi : \phi = 0$ to $\phi = 2\pi$

   Hence, the spherical form of the given integral is

$$I = \iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{a^2 - x^2 - y^2 - z^2}}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}{\sqrt{a^2 - r^2}}$$

$$= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{a}\frac{r^2 + a^2 - a^2}{\sqrt{a^2 - r^2}}\mathrm{d}r$$

$$= |\phi|_0^{2\pi} \cdot |-\cos\theta|_0^{\pi} \cdot \int_0^{a}\left(\frac{a^2}{\sqrt{a^2 - r^2}} - \sqrt{a^2 - r^2}\right)\mathrm{d}r$$

$$= (2\pi)(-\cos\pi + \cos 0)\left| a^2 \sin^{-1}\frac{r}{a} - \frac{r}{2}\sqrt{a^2 - r^2} - \frac{a^2}{2}\sin^{-1}\frac{r}{a}\right|_0^a$$

$$= 2\pi(2)\left(a^2\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}1\right)$$

$$= 4\pi\left(\frac{a^2}{2}\sin^{-1}1\right)$$

$$= 4\pi \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \pi^2 a^2$$

Example 5

Evaluate $\iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ *over the region bounded by the spheres* $x^2 + y^2 + z^2 = a^2$ *and* $x^2 + y^2 + z^2 = b^2$, $a > b > 0$.

Solution

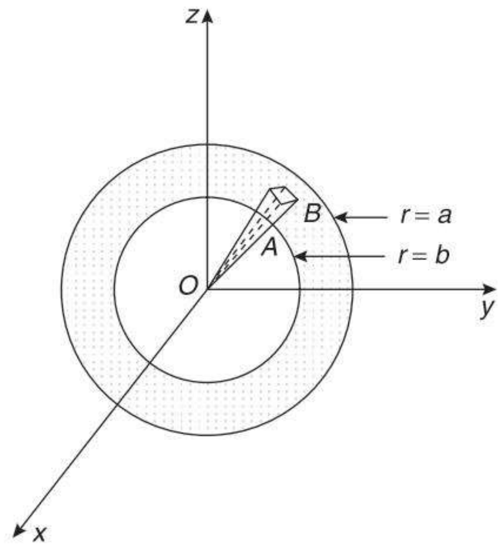

Fig. 9.134

1. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, equations of the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ reduce to $r = a$ and $r = b$ respectively.
2. Draw an elementary radius vector OAB from the origin in the region. This radius vector enters in the region from the sphere $r = b$ and terminates on the sphere $r = a$.
3. Limits of $r : r = b$ to $r = a$.
   For the complete sphere,
   limits of $\theta : \theta = 0$ to $\theta = \pi$
   limits of $\phi : \phi = 0$ to $\phi = 2\pi$
   Hence, the spherical form of the given integral is

$$I = \iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_b^{a} \frac{r^2\sin\theta}{r}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_b^{a} r\,\mathrm{d}r$$

$$= \left|\phi\right|_0^{2\pi}\cdot\left|-\cos\theta\right|_0^{\pi}\cdot\left|\frac{r^2}{2}\right|_b^{a}$$

$$= 2\pi(-\cos\pi + \cos 0)\frac{(a^2-b^2)}{2}$$

$$= 2\pi(a^2-b^2).$$

Example 6

Evaluate $\iiint z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the region common to the sphere* $x^2 + y^2 + z^2 = 4$ *and the cylinder* $x^2 + y^2 = 2x$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of
   (i) the sphere $x^2 + y^2 + z^2 = 4$ reduces to
   $$r^2 + z^2 = 4$$
   $$z^2 = 4 - r^2.$$
   (ii) the cylinder $x^2 + y^2 = 2x$ reduces to
   $r^2 = 2r\cos\theta$, $r = 2\cos\theta$.

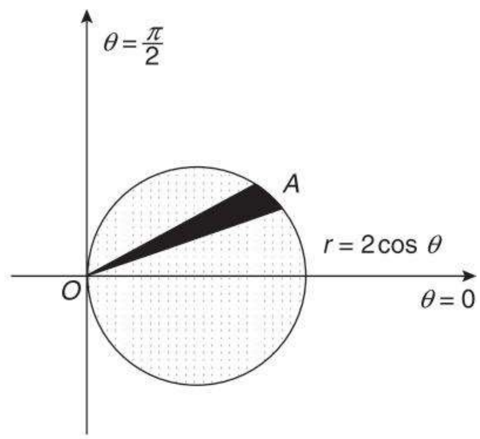

Fig. 9.135

2. Draw an elementary volume parallel to z-axis in the region. This elementary volume starts from the part of the sphere $z^2 = 4 - r^2$, below xy-plane and terminates on the part of the sphere $z^2 = 4 - r^2$, above xy-plane.

   Limits of $r : z = -\sqrt{4-r^2}$ to $z = \sqrt{4-r^2}$
3. Projection of the region in $r\theta$-plane is the circle $r = 2\cos\theta$.
4. Draw an elementary radius vector OA in the region ($r = 2\cos\theta$) which starts from the origin and terminates on the circle $r = 2\cos\theta$

   Limits of $r : r = 0$ to $r = 2\cos\theta$

   Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

   Hence, the cylindrical form of the given integral is

$$I = \iiint z^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2 r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left| \frac{z^3}{3} \right|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} 2(4-r^2)^{\frac{3}{2}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left[-(4-r^2)^{\frac{3}{2}} (-2r)\mathrm{d}r \right] \mathrm{d}\theta$$

$$= -\frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{2(4-r^2)^{\frac{5}{2}}}{5} \right|_0^{2\cos\theta} \mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1} \right]$$

$$= -\frac{2}{15} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[(4 - 4\cos^2\theta)^{\frac{5}{2}} - (4)^{\frac{5}{2}} \right] \mathrm{d}\theta$$

$$= -\frac{2}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2^5 \sin^5\theta - 2^5)\,d\theta$$

$$= -\frac{2}{15}\left[0 - 2^5\,|\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \qquad \left[\begin{array}{l} \because \int_{-a}^{a} f(\theta)d\theta = 0, \text{ if } f(-\theta) = -f(\theta) \\ \text{Here} \quad \sin^5(-\theta) = -\sin^5\theta \end{array}\right]$$

$$= \frac{2^6\pi}{15}$$

$$= \frac{64\pi}{15}$$

Example 7

Evaluate $\iiint xyz\,dx\,dy\,dz$ *over the region bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cylinder $x^2 + y^2 = 1$ reduces to $r^2 = 1$, $r = 1$.

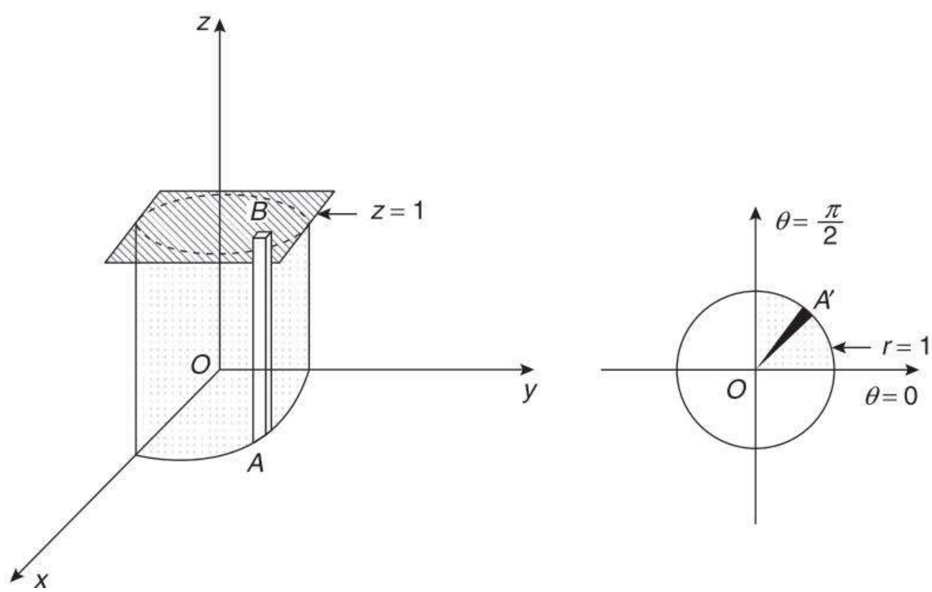

Fig. 9.136

2. Draw an elementary volume AB parallel to z-axis in the region. This elementary volume AB starts from xy-plane and terminates on the plane $z = 1$.
   Limits of z : $z = 0$ to $z = 1$

3. Projection of the region in $r\theta$-plane is the part of the circle $r = 1$ in the first quadrant.
4. Draw an elementary radius vector OA' in the region in the $r\theta$-plane which starts from the origin and terminates on the circle $r = 1$.

   Limits of $r : r = 0 \quad \text{to} \quad r = 1$

   Limits of $\theta : \theta = 0 \quad \text{to} \quad \theta = \dfrac{\pi}{2}$

   Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
I &= \iiint xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{z=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\cos\theta\sin\theta\cdot zr\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^1 z\,\mathrm{d}z\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\mathrm{d}\theta\int_0^1 r^3\,\mathrm{d}r \\
&= \left|\frac{z^2}{2}\right|_0^1 \left|-\frac{\cos 2\theta}{4}\right|_0^{\frac{\pi}{2}} \left|\frac{r^4}{4}\right|_0^1 \\
&= \frac{1}{16}
\end{aligned}
$$

Example 8

Evaluate $\iiint \sqrt{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the region bounded by the right circular cone* $x^2 + y^2 = z^2$, $z > 0$ *and the planes* $z = 0$ *and* $z = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the cone $x^2 + y^2 = z^2$ reduces to $r^2 = z^2$, $r = z$.
2. Draw an elementary volume AB parallel to z-axis in the region, which starts from the cone $r = z$ and terminates on the plane $z = 1$.

   Limits of $z : z = r \quad \text{to} \quad z = 1.$
3. Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and the plane $z = 1$ which is obtained as $r = 1$, a circle with centre at the origin and radius 1.
4. Draw an elementary radius vector OA' in the region which starts from the origin and terminates on the circle $r = 1$.

   Limits of $r : r = 0 \quad \text{to} \quad r = 1$

   Limits of $\theta : \theta = 0 \quad \text{to} \quad \theta = 2\pi$

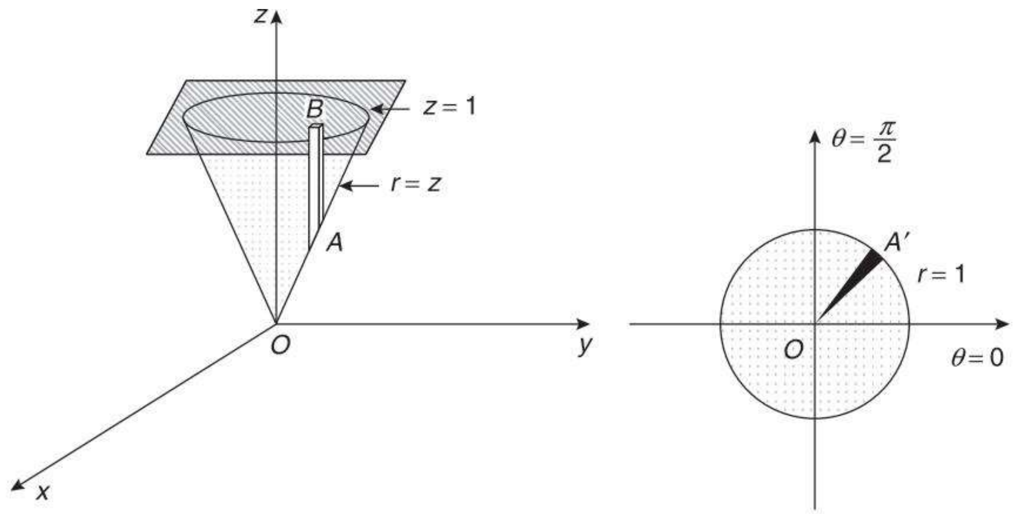

Fig. 9.137

Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
\mathrm{I} &= \iiint \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 r^2 \left|z\right|_r^1 \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\mathrm{d}\theta \int_0^1 r^2(1-r)\mathrm{d}r \\
&= \left|\theta\right|_0^{2\pi}\left|\frac{r^3}{3}-\frac{r^4}{4}\right|_0^1 \\
&= 2\pi\cdot\frac{1}{12} \\
&= \frac{\pi}{6}
\end{aligned}
$$

Example 9

Evaluate $\iiint (x^2+y^2)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the region bounded by the paraboloid* $x^2 + y^2 = 3z$ *and the plane* $z = 3$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the paraboloid $x^2 + y^2 = 3z$ reduces to $r^2 = 3z$.
2. Draw an elementary volume AB parallel to z-axis in the region which starts from the paraboloid $r^2 = 3z$ and terminates on the plane $z = 3$.

Limits of $z : z = \frac{r^2}{3}$ to $z = 3$

3. Projection of the region in $r\theta$-plane is the curve of intersection of the paraboloid $r^2 = 3z$ and the plane $z = 3$ which is obtained as $r^2 = 9$, $r = 3$, a circle with centre at the origin and radius 1.

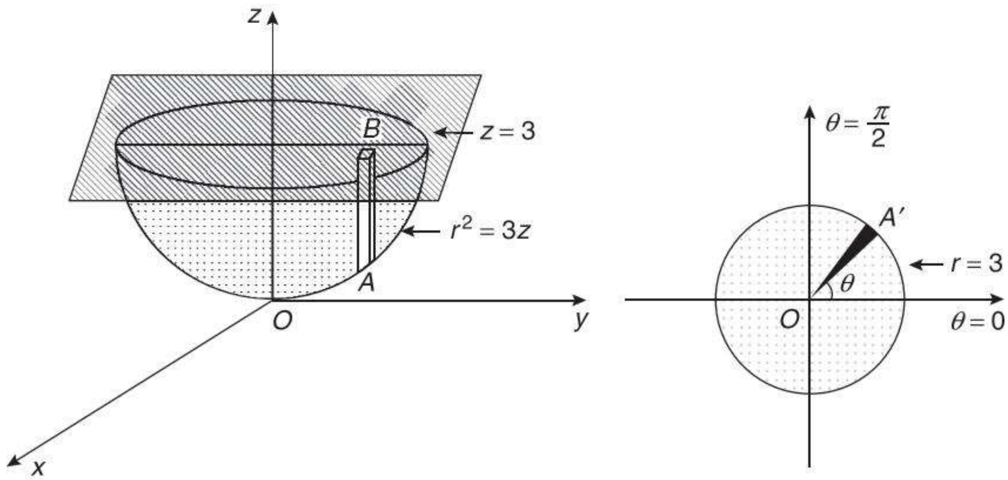

Fig. 9.138

4. Draw an elementary radius vector OA' in the region (circle $r = 3$) which starts from origin and terminates on the circle $r = 3$.

   Limits of $r : r = 0$ to $r = 3$

   Limits of $\theta : \theta = 0$ to $\theta = 2\pi$

Hence, the cylindrical form of the given integral is

$$
\begin{aligned}
I &= \iiint (x^2 + y^2)\,dx\,dy\,dz \\
&= \int_0^{2\pi} \int_0^3 \int_{\frac{r^2}{3}}^3 r^2 \cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^3 r^3 \left| z \right|_{\frac{r^2}{3}}^3 dr\,d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^3 r^3 \left(3 - \frac{r^2}{3} \right) dr \\
&= \left| \theta \right|_0^{2\pi} \left| \frac{3r^4}{4} - \frac{r^6}{18} \right|_0^3 \\
&= 2\pi \left(\frac{3^5}{4} - \frac{3^6}{18} \right) \\
&= \frac{81\pi}{2}
\end{aligned}
$$

Example 10

Evaluate $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where V is the volume of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution

It is difficult to integrate this integral in cartesian form. Therefore, transforming the ellipsoid into a sphere using following change of variables.

Putting $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w,$ equation of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ reduces to $u^2+v^2+w^2=1,$ which is a sphere of radius 1 and centre at the origin,

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

where,

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$$

Therefore, $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = abc\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$

New form of the integral is

$$I=\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$=\iiint\sqrt{1-u^2-v^2-w^2}\cdot abc\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Since in the new coordinate system u, v, w, the region of integration is a sphere, therefore using spherical coordinates $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ and $\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$, the equation of the sphere $u^2+v^2+w^2=1$ reduce to $r^2=1$, $r=1$.

For complete sphere limits of $r: r=0$ to $r=1$ (radius of sphere)

limits of $\theta:\theta=0$ to $\theta=\pi$

limits of $\phi:\phi=0$ to $\phi=2\pi$

Hence, the spherical form of the given integral is

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\sqrt{1-r^2}\,abc\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= abc\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{1}r^2\sqrt{1-r^2}\,\mathrm{d}r$$

Putting $r = \sin t$, $\quad \mathrm{d}r = \cos t\,\mathrm{d}t$

When $r = 0$, $\quad t = 0$

When $r = 1$, $\quad t = \dfrac{\pi}{2}$

$$\therefore I = abc\left|\phi\right|_0^{2\pi}\left|-\cos\theta\right|_0^{\pi}\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos t\cdot\cos t\,\mathrm{d}t$$

$$= abc(2\pi)(2)\cdot\frac{1}{2}B\left(\frac{3}{2},\ \frac{3}{2}\right)$$

$$= 2\pi\,abc\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= 2\pi\,abc\,\frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2}$$

$$= \frac{\pi^2 abc}{4}$$

Example 11

Evaluate $\iiint x^2y^2z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *over the region bounded by the surfaces* $xy = 4$, $xy = 9$, $yz = 1$, $yz = 4$, $zx = 25$, $zx = 49$.

Solution

Evaluation of integral becomes easier by changing the variables. Under the transformation $xy = u$, $yz = v$, $zx = w$, the surfaces get transformed to $u = 4$, $u = 9$, $v = 1$, $v = 4$, $w = 25$, $w = 49$.

These equations represent the planes parallel to vw, wu and uv planes in the new coordinate system.

It is easier to find partial derivatives of u, v, w w.r.t. x, y and z.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix}$$

$$= y(zx - 0) - x(0 - yz)$$

$$= 2xyz$$

$$du\, dv\, dw = |J|\, dx\, dy\, dz = 2xyz\, dx\, dy\, dz$$

$$dx\, dy\, dz = \frac{1}{2xyz}\, du\, dv\, dw$$

$$= \frac{1}{2\sqrt{uvw}}\, du\, dv\, dw \qquad [\because \quad x^2y^2z^2 = uvw]$$

Limits of u : $u = 4$ to $u = 9$
Limits of v : $v = 1$ to $v = 4$
Limits of w : $w = 25$ to $w = 49$

Hence, the new form of the integral is

$$I = \iiint x^2y^2z^2\, dx\, dy\, dz$$

$$= \int_{w=25}^{49}\int_{v=1}^{4}\int_{u=4}^{9} uvw \cdot \frac{1}{2\sqrt{uvw}}\, du\, dv\, dw$$

$$= \frac{1}{2}\int_{25}^{49} w^{\frac{1}{2}}\, dw \int_{1}^{4} v^{\frac{1}{2}}\, dv \int_{4}^{9} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left|\frac{2w^{\frac{3}{2}}}{3}\right|_{25}^{49}\left|\frac{2v^{\frac{3}{2}}}{3}\right|_{1}^{4}\left|\frac{2u^{\frac{3}{2}}}{3}\right|_{4}^{9}$$

$$= \frac{4}{27}(343 - 125)(8 - 1)(27 - 8)$$

$$= \frac{115976}{27}$$

EXERCISE 9.7

(I) Evaluate the following integrals:

1. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz\, dz$

$\left[\text{Ans.: } 1\right]$

2. $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$\left[\text{Ans.: } \frac{5}{8}\right]$

3. $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$\left[\textbf{Ans.:}\ \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)\right]$$

4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)}\int_0^{h} 2\left[1-\frac{r}{a(1+\cos\theta)}\right] r\,dz\,dr\,d\theta$

$$\left[\textbf{Ans.:}\ \frac{\pi a^2 h}{2}\right]$$

5. $\int_0^{4}\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$

$$\left[\textbf{Ans.:}\ 8\pi\right]$$

6. $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

$$\left[\textbf{Ans.:}\ \frac{5a^3}{64}\right]$$

7. $\int_0^{2}\int_0^{y}\int_{x-y}^{x+y} (x+y+z)\,dz\,dx\,dy$

$$\left[\textbf{Ans.:}\ 16\right]$$

8. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx.$

$$\left[\textbf{Ans.:}\ \frac{a^6}{48}\right]$$

(II) Evaluate the following integrals over the given region of integration:

1. $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{8}\right]$$

2. $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{2}\left(\log 2-\frac{5}{8}\right)\right]$$

3. $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

$$\left[\textbf{Ans.:}\ \frac{a^6}{48}\right]$$

4. $\iiint xyz(x^2+y^2+z^2)\,dx\,dy\,dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

$$\left[\textbf{Ans.:}\ \frac{a^8}{64}\right]$$

5. $\iiint (y^2z^2 + z^2x^2 + x^2y^2)\,dx\,dy\,dz$ over the sphere of radius a and centre at the origin.

$$\left[\textbf{Ans.:} \frac{4\pi a^7}{35}\right]$$

6. $\iiint \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$ over the sphere $x^2 + y^2 + z^2 = 2$.

$$\left[\textbf{Ans.:} \frac{8\pi\sqrt{2}}{9}\right]$$

7. $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ over the region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $a > b > 0$.

$$\left[\textbf{Ans.:} 4\pi\log\left(\frac{a}{b}\right)\right]$$

8. $\iiint z^2\,dx\,dy\,dz$ over the region common to the spheres $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.

$$\left[\textbf{Ans.:} \frac{2\pi a^5}{15}\right]$$

9. $\iiint (x^2+y^2)\,dx\,dy\,dz$ over the region bounded by the paraboloid $x^2 + y^2 = 2z$ and the plane $z = 2$.

$$\left[\textbf{Ans.:} \frac{16\pi}{3}\right]$$

10. $\iiint x^2yz\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$$\left[\textbf{Ans.:} \frac{a^3b^2c^2}{2520}\right]$$

11. $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

$$\left[\textbf{Ans.:} \frac{a^2b^2c^2}{48}\right]$$

12. $\iiint \sqrt{\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}}\,dx\,dy\,dz$ over the region bounded by the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=1$.

$$[\textbf{Ans.:} 8\pi]$$

9.7 AREA BY DOUBLE INTEGRALS

9.7.1 Area in Cartesian Coordinates

(i) The area A bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} \mathrm{d}y\, \mathrm{d}x$$

(ii) If equation of the curves are represented as $x = x_1(y)$ and $x = x_2(y)$ then

$$A = \int_b^d \int_{x_1(y)}^{x_2(y)} \mathrm{d}x\, \mathrm{d}y$$

Note: Consider the symmetricity of the region while calculating area.

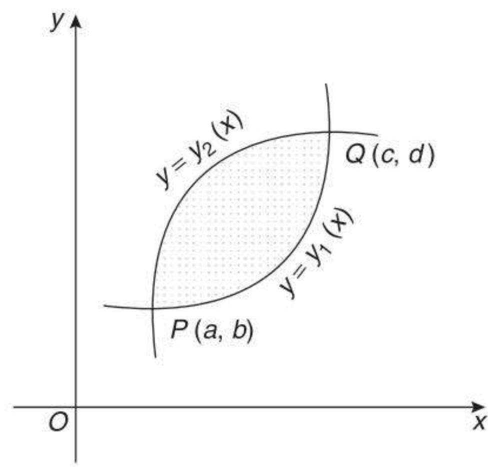

Fig. 9.139

Example 1

Find the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,$ *above x-axis.*

Solution

1. The region is symmetric about y-axis. Total area = 2 (area bounded by the ellipse in the first quadrant)
2. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the x-axis and terminates on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

   Limits of y : $y = 0$ to $y = b\sqrt{1 - \dfrac{x^2}{a^2}}$

   Limits of x : $x = 0$ to $x = a$

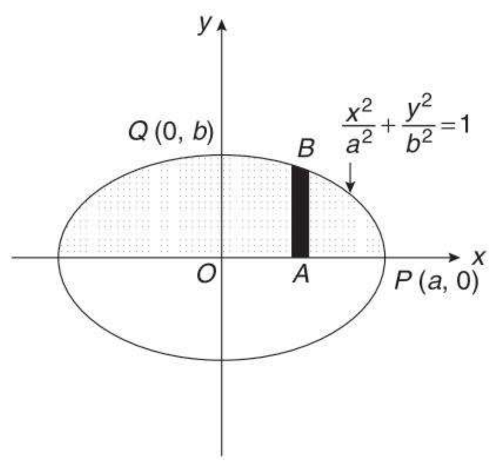

Fig. 9.140

$$A = 2\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}y\, \mathrm{d}x$$

$$= 2\int_0^a \left| y \right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}x$$

$$= 2\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\, \mathrm{d}x$$

$$= \frac{2b}{a}\int_0^a \sqrt{a^2 - x^2}\, \mathrm{d}x$$

$$= \frac{2b}{a}\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a$$
$$= \frac{2b}{a}\left(\frac{a^2}{2}\sin^{-1}1\right)$$
$$= \frac{2b}{a}\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right)$$
$$= \frac{\pi ab}{2}$$

Example 2

Find the area bounded by the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$.

Solution

1. The points of intersection of the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ are obtained as

$$\left(\frac{2x+4}{3}\right)^2 = 4x$$
$$(x+2)^2 = 9x$$
$$x^2 - 5x + 4 = 0$$
$$x = 1, 4$$
$$\therefore y = 2, 4$$

The points of intersection are P (1, 2) and Q (4, 4).

2. Draw a vertical strip AB which starts from the line $2x - 3y + 4 = 0$ and terminates on the parabola $y^2 = 4x$.

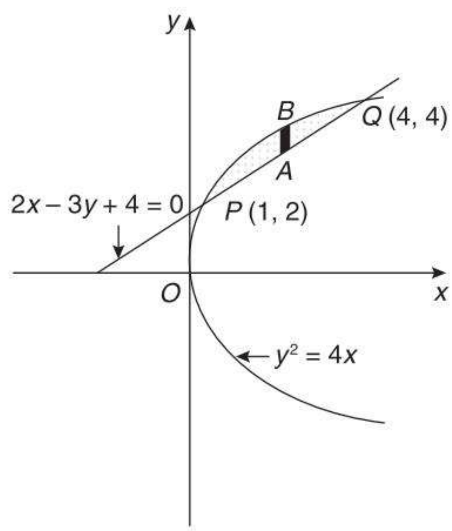

Fig. 9.141

Limits of y : $y = \dfrac{2x+4}{3}$ to $y = 2\sqrt{x}$

Limits of x : $x = 1$ to $x = 4$

$$A = 2\int_1^4 \int_{\frac{2x+4}{3}}^{2\sqrt{x}} \mathrm{d}y\mathrm{d}x$$
$$= \int_1^4 \left|y\right|_{\frac{2x+4}{3}}^{2\sqrt{x}} \mathrm{d}x$$
$$= \int_1^4 \left(2\sqrt{x} - \frac{2x+4}{3}\right)\mathrm{d}x$$
$$= \left|2\cdot 2\cdot\frac{x^{\frac{3}{2}}}{3} - \frac{x^2}{3} - \frac{4x}{3}\right|_1^4$$
$$= \frac{4}{3}(8-1) - \frac{1}{3}(16-1) - \frac{4}{3}(4-1)$$
$$= \frac{1}{3}$$

Example 3

Find the area enclosed by the curves $y = x^2$ and $y = x$.

Solution

1. The points of intersection of the parabola $y = x^2$ and the line $y = x$ are obtained as

$$x = x^2$$
$$x = 0, 1$$
$$\therefore \quad y = 0, 1$$

The points of intersection are O (0, 0) and P (1, 1).

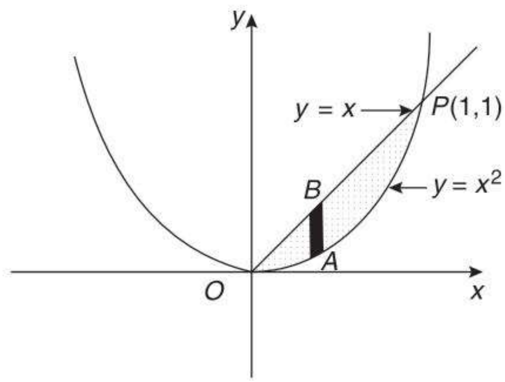

Fig. 9.142

2. Draw a vertical strip AB which starts from the parabola $y = x^2$ and terminates on the line $y = x$.

Limits of y : $y = x^2$ to $y = x$

Limits of x : $x = 0$ to $x = 1$

$$A = \int_0^1 \int_{x^2}^{x} \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^1 \left| y \right|_{x^2}^{x} \mathrm{d}x$$
$$= \int_0^1 (x - x^2)\, \mathrm{d}x$$
$$= \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

Example 4

Find the area enclosed by the parabola $y^2 = 4ax$ and the lines $x + y = 3a$, $y = 0$ in the first quadrant.

Solution

1. The points of intersection of the parabola $y^2 = 4ax$ and the line $x + y = 3a$ are obtained as

$$y^2 = 4a(3a - y)$$
$$y^2 + 4ay - 12a^2 = 0$$
$$y = 2a, -6a$$
$$\therefore \quad x = a, 9a$$

The point of intersection is Q $(a, 2a)$ which lies in the first quadrant.

2. Area enclosed in the first quadrant is OPQ.

Draw a horizontal strip AB which starts from the parabola $y^2 = 4ax$ and terminates on the line $x + y = 3a$.

Limits of $x : x = \dfrac{y^2}{4a}$ to $x = 3a - y$

Limits of $y : y = 0$ to $y = 2a$

$$A = \int_0^{2a} \int_{\frac{y^2}{4a}}^{3a-y} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^{2a} \left| x \right|_{\frac{y^2}{4a}}^{3a-y} \mathrm{d}y$$

$$= \int_0^{2a} \left(3a - y - \frac{y^2}{4a} \right) \mathrm{d}y$$

$$= \left| 3ay - \frac{y^2}{2} - \frac{1}{4a} \cdot \frac{y^3}{3} \right|_0^{2a}$$

$$= 6a^2 - 2a^2 - \frac{1}{4a} \cdot \frac{8a^3}{3}$$

$$= \frac{10}{3} a^2$$

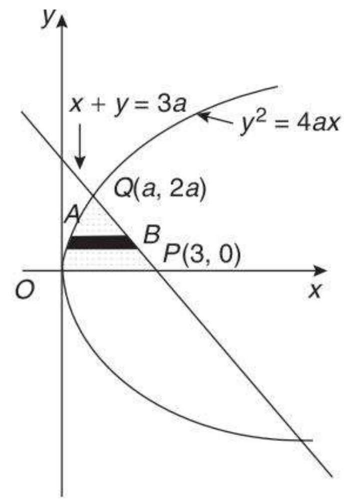

Fig. 9.143

Note: In case of vertical strip, two vertical strips are required to cover the entire region. Therefore one horizontal strip is preferred over vertical strip.

Example 5

Find the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution

1. The points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ are obtained as

$$\left(\frac{x^2}{4a} \right)^2 = 4ax$$

$$x^4 = 16a^2 (4ax)$$

$$x(x^3 - 64a^3) = 0$$

$$x = 0, \quad x = 4a$$

$$\therefore \; y = 0, \quad y = 4a$$

The points of intersection are O (0, 0) and P ($4a$, $4a$).

2. Draw a vertical strip AB which starts from the parabola $x^2 = 4ay$ and terminates on the parabola $y^2 = 4ax$.

Limits of $y : y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$

Limits of $x : x = 0$ to $x = 4a$

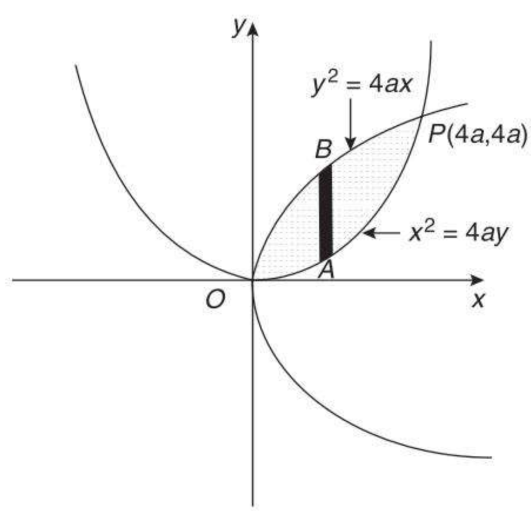

Fig. 9.144

$$A = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{4a} \left| y \right|_{\frac{x^2}{4a}}^{2\sqrt{ax}} \mathrm{d}x$$

$$= \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)\mathrm{d}x$$

$$= \left| 2\sqrt{a}\cdot\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

Example 6

Find the area enclosed by the curves $y = 2 - x$ and $y^2 = 2(2 - x)$.

Solution

1. The points of intersection of the line $y = 2 - x$ and the parabola $y^2 = 2(2 - x)$ are obtained as

$$(2 - x)^2 = 2(2 - x)$$
$$(2 - x)(2 - x - 2) = 0$$
$$(2 - x)(-x) = 0$$
$$x = 2, 0$$
$$y = 0, 2$$

The points of intersection are P (2, 0) and Q (0, 2).

2. Draw a vertical strip AB which starts from the line $y = 2 - x$ and terminates on the parabola $y^2 = 2\,(2 - x)$.

Limits of y : $y = 2 - x$ to $y = \sqrt{2(2-x)}$

Limits of x : $x = 0$ to $x = 2$

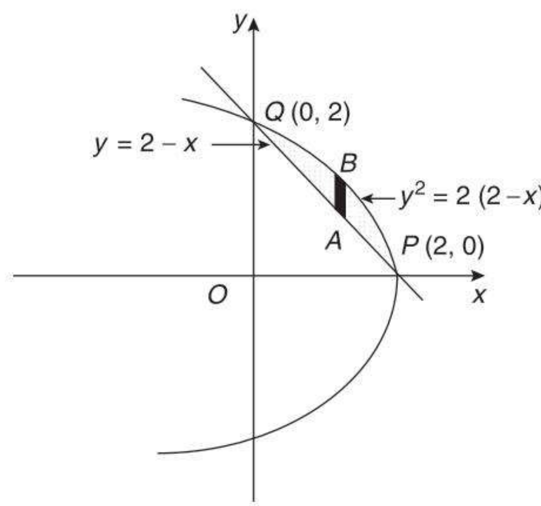

Fig. 9.145

$$A = \int_0^2\int_{2-x}^{\sqrt{2(2-x)}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \left| y \right|_{2-x}^{\sqrt{2(2-x)}} \mathrm{d}x$$

$$= \int_0^2\left[\sqrt{2}(2-x)^{\frac{1}{2}} - (2-x)\right]\mathrm{d}x$$

$$= \left| \sqrt{2} \cdot \frac{2(2-x)^{\frac{3}{2}}}{-3} - 2x + \frac{x^2}{2} \right|_0^2$$

$$= \left(0 + \frac{8}{3}\right) - 2(2-0) + \frac{1}{2}(4-0)$$

$$= \frac{2}{3}$$

Example 7

Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y-2a)$.

Solution

1. The parabola $x^2 = 4ay$ has vertex (0, 0) and the parabola $x^2 = -4a(y-2a)$ has vertex (0, 2a). Both the parabolas are symmetric about the y-axis.
2. The points of intersection of $x^2 = 4ay$ and $x^2 = -4a(y-2a)$ are obtained as

$$4ay = -4a(y-2a)$$
$$8ay = 8a^2$$
$$y = a$$
$$\therefore x = \pm 2a$$

   The points of intersection are $P(2a, a)$ and $R(-2a, a)$.

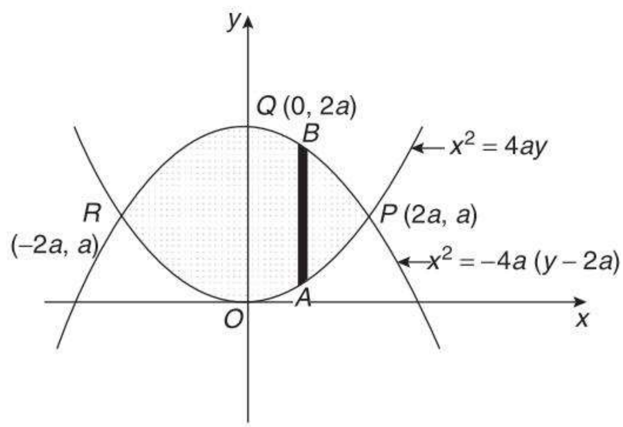

Fig. 9.146

3. The region is symmetric about y-axis.

   Total area = 2 (Area in the first quadrant)

4. Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the parabola $x^2 = 4ay$ and terminates on the parabola $x^2 = -4a(y-2a)$.

   Limit of y : $y = \frac{x^2}{4a}$ to $y = 2a - \frac{x^2}{4a}$

   Limits of x : $x = 0$ to $x = 2a$

$$A = 2\int_0^{2a} \int_{\frac{x^2}{4a}}^{2a-\frac{x^2}{4a}} dy\, dx$$

$$= 2\int_0^{2a} \left| y \right|_{\frac{x^2}{4a}}^{2a-\frac{x^2}{4a}} dx$$

$$= 2\int_0^{2a} \left(2a - \frac{x^2}{4a} - \frac{x^2}{4a}\right) dx$$

$$= 2\left|2ax - \frac{x^3}{6a}\right|_0^{2a}$$

$$= 2\left(4a^2 - \frac{4}{3}a^2\right)$$

$$= \frac{16}{3}a^2$$

Example 8

Find smaller of the area enclosed by the curves $y = 2 - x$ and $x^2 + y^2 = 4$.

Solution

1. The points of intersection of the line $y = 2 - x$ and the circle $x^2 + y^2 = 4$ are obtained as

$$x^2 + (2 - x)^2 = 4$$
$$x^2 + 4 - 4x + x^2 = 4$$
$$2x^2 = 4x$$
$$x = 2, 0$$
$$\therefore \quad y = 0, 2$$

The points of intersection are P (2, 0) and Q (0, 2).

2. Draw a vertical strip AB which starts from the line $y = 2 - x$ and terminates on the circle $x^2 + y^2 = 4$.

Limits of y : $y = 2 - x$ to $y = \sqrt{4 - x^2}$

Limits of x : $x = 0$ to $x = 2$

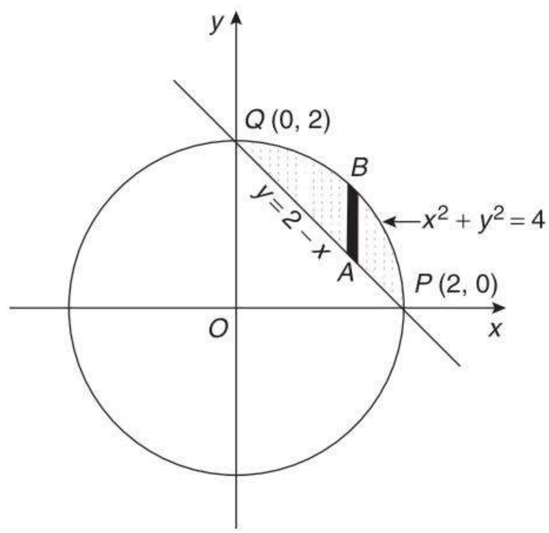

Fig. 9.147

$$A = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 |y|_{2-x}^{\sqrt{4-x^2}}\,\mathrm{d}x$$

$$= \int_0^2 \left[\sqrt{4-x^2} - (2-x)\right]\mathrm{d}x$$

$$= \left|\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right|_0^2$$

$$= 2\sin^{-1}1 - 2$$

$$= \pi - 2$$

Example 9

Find the area of the loop of the curve $x(x^2 + y^2) = a\,(x^2 - y^2)$.

Solution

The equation of the curve can be rewritten as $y^2 = x^2\left(\dfrac{a-x}{a+x}\right)$

1. The points of intersection of the curve with x-axis ($y = 0$) are obtained as

$$x^2(a-x) = 0$$
$$x = 0,\ x = a.$$

The loop of the curve lies between the points O (0, 0) and P (a, 0).

2. The region is symmetric about x-axis

Total area = 2 (Area above x-axis)

3. Draw a vertical strip AB in the region above x-axis. AB starts from x-axis and terminates on the curve $y^2 = x^2\left[\dfrac{a-x}{a+x}\right]$.

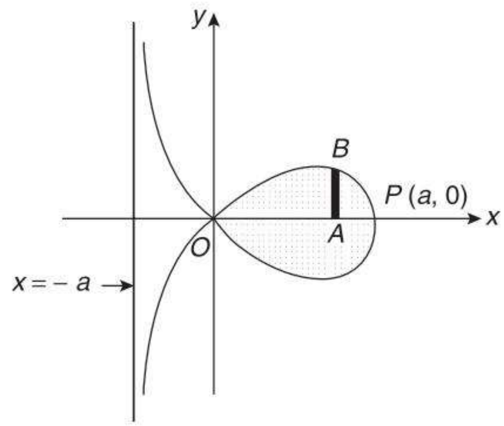

Fig. 9.148

Limits of y : $y = 0$ to $y = x\sqrt{\dfrac{a-x}{a+x}}$

Limits of x : $x = 0$ to $x = a$

$$\text{A} = 2\int_0^a \int_0^{x\sqrt{\frac{a-x}{a+x}}} \mathrm{d}y\,\mathrm{d}x$$

$$= 2\int_0^a \left|y\right|_0^{x\sqrt{\frac{a-x}{a+x}}} \mathrm{d}x$$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\mathrm{d}x$$

Putting $x = a\cos\theta,\ \mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta$

When $x = 0, \theta = \dfrac{\pi}{2}$

When $x = a,\ \theta = 0$

$$\text{A} = 2\int_{\frac{\pi}{2}}^0 a\cos\theta\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}(-a\sin\theta)\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\cdot\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} \cos\theta(1-\cos\theta)\,\mathrm{d}\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} \left(\cos\theta - \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= 2a^2 \left| \sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right|_0^{\frac{\pi}{2}}$$

$$= 2a^2 \left[\left(\sin\frac{\pi}{2} - \sin 0 \right) - \frac{1}{2}\left(\frac{\pi}{2} - 0 \right) - \frac{1}{4}(\sin\pi - \sin 0) \right]$$

$$= 2a^2 \left(1 - \frac{\pi}{4} \right)$$

Example 10

Find the area included between the curve $y^2(2a - x) = x^3$ and its asymptote. **[Summer 2017]**

Solution

The equation of the curve can be rewritten as

$$y^2 = \frac{x^3}{2a - x}$$

1. The point of intersection of the curve with x-axis ($y = 0$) is $x = 0$.
2. The region is symmetric about x-axis.
   Total area = 2 (Area above x-axis)
3. Draw a vertical strip AB in the region above x-axis. AB starts from x-axis and terminates on the curve $y^2 = \dfrac{x^3}{2a - x}$.

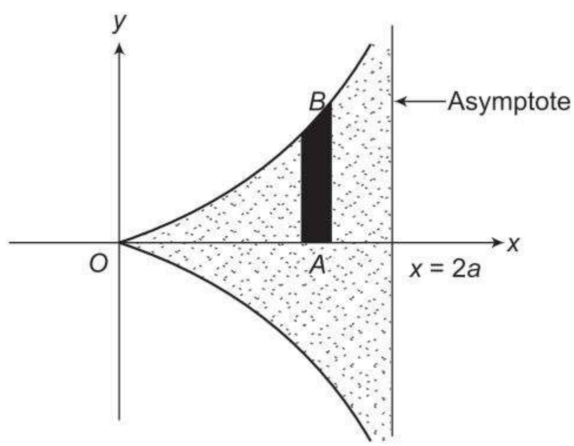

Fig. 9.149

Limits of y: $y = 0$ to $y = x\sqrt{\dfrac{x}{2a - x}}$

Limits of x: $x = 0$ to $x = 2a$

$$A = 2\int_0^{2a} \int_0^{x\sqrt{\frac{x}{2a-x}}} dy\, dx$$

$$= 2\int_0^{2a} |y|_0^{x\sqrt{\frac{x}{2a-x}}} dx$$

$$= 2\int_0^{2a} x\sqrt{\frac{x}{2a - x}}\, dx$$

Putting $\quad x = 2a\sin^2\theta,$

$\quad dx = 2a\,(2\sin\theta\cos\theta\,d\theta)$

When $\quad x = 0, \quad \theta = 0$

When $\quad x = 2a, \quad \theta = \dfrac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} 2a\sin^2\theta\,\sqrt{\frac{2a\sin^2\theta}{2a\cos^2\theta}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 2a\sin^2\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$$

$$= 16a^2\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{\pi}{4}$$

$$= 3\pi a^2$$

Example 11

Find the area between the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$.

Solution

1. The points of intersection of the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$ are obtained as

$$3x(6 - 12x) = 2$$

$$18x^2 - 9x + 1 = 0$$

$$x = \frac{1}{3}, \frac{1}{6}$$

$$\therefore y = 2, 4$$

The points of intersection are $P\left(\dfrac{1}{3}, 2\right)$ and $Q\left(\dfrac{1}{6}, 4\right)$.

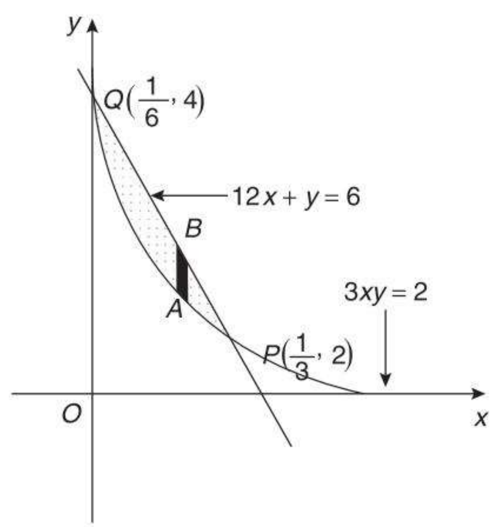

Fig. 9.150

2. Draw a vertical strip AB in the region which starts from the rectangular hyperbola $3xy = 2$ and terminates on the line $12x + y = 6$.

Limits of y : $y = \dfrac{2}{3x}$ to $y = 6 - 12x$

Limits of x : $x = \dfrac{1}{6}$ to $x = \dfrac{1}{3}$

$$A = \int_{\frac{1}{6}}^{\frac{1}{3}} \int_{\frac{2}{3x}}^{6-12x} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_{\frac{1}{6}}^{\frac{1}{3}} \left| y \right|_{\frac{2}{3x}}^{6-12x} \mathrm{d}x$$

$$= \int_{\frac{1}{6}}^{\frac{1}{3}} \left(6 - 12x - \frac{2}{3x} \right) \mathrm{d}x$$

$$= \left| 6x - 6x^2 - \frac{2}{3}\log x \right|_{\frac{1}{6}}^{\frac{1}{3}}$$

$$= (2-1) - 6\left(\frac{1}{9} - \frac{1}{36} \right) - \frac{2}{3}\left(\log\frac{1}{3} - \log\frac{1}{6} \right)$$

$$= \frac{1}{2} - \frac{2}{3}\log 2$$

Example 12

Find the area bounded by the hypocycloid $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

Solution

1. The hypocycloid is symmetric in all the quadrants.

   Total area = 4 (area in the first quadrant)

2. Draw a vertical strip AB parallel to y-axis in the region which lies in the first quadrant. AB starts from x-axis and terminates on the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

   Limits of y : $y = 0$ to $y = b\left[1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$

   Limits of x : $x = 0$ to $x = a$

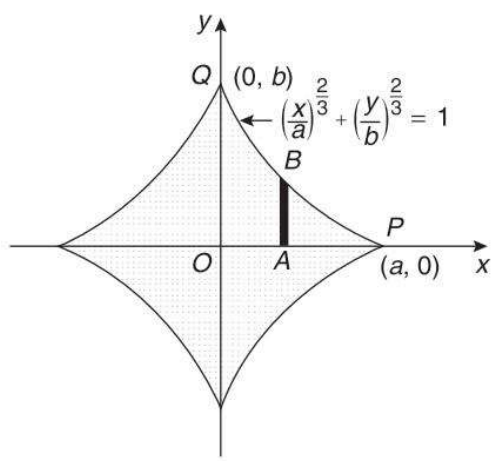

Fig. 9.151

$$A = 4\int_0^a \int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} \mathrm{d}y\,\mathrm{d}x$$

$$= 4\int_0^a \left| y \right|_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} \mathrm{d}x$$

$$= 4\int_0^a b\left[1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} \mathrm{d}x$$

Putting $x = a\cos^3 t,\ dx = 3a\cos^2 t\,(-\sin t)\,dt$

When $x = 0,\ t = \dfrac{\pi}{2}$

When $x = a,\ t = 0$

$$A = 4\int_{\frac{\pi}{2}}^{0} b(1-\cos^2 t)^{\frac{3}{2}}(-3a\cos^2 t\sin t)dt$$

$$= 12ab\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$$

$$= 12ab\frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= 6ab\frac{\left\lceil\frac{5}{2}\right.\left\lceil\frac{3}{2}\right.}{\lceil 4}$$

$$= 6ab\frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}{3!}$$

$$= \frac{3}{8}\pi ab$$

EXERCISE 9.8

1. Find the area bounded by y-axis, the line $y = 2x$ and the line $y = 4$.

[Ans. : 4**]**

2. Find the area bounded by the lines $y = 2 + x$, $y = 2 - x$ and $x = 5$.

[Ans. : 25**]**

3. Find the area bounded by the parabola $y^2 + x = 0$, and the line $y = x + 2$.

$$\left[\textbf{Ans. :} \frac{9}{2}\right]$$

4. Find the area bounded by the parabola $x = y - y^2$ and the line $x + y = 0$.

$$\left[\textbf{Ans. :} \frac{4}{3}\right]$$

5. Find the area bounded by the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.

$$\left[\textbf{Ans. :} \frac{1}{3}\right]$$

6. Find the area bounded by the parabola $y = x^2 - 3x$ and the line $y = 2x$.

$$\left[\textbf{Ans.}: \frac{125}{6}\right]$$

7. Find the area bounded by the parabolas $y^2 = x$, $x^2 = -8y$.

$$\left[\textbf{Ans.}: \frac{8}{3}\right]$$

8. Find the area bounded by the parabolas $y = ax^2$ and $y = 1 - \frac{x^2}{a}$, where $a > 0$.

$$\left[\textbf{Ans.}: \frac{4}{3}\sqrt{\frac{a}{a^2+1}}\right]$$

9. Find the area of the loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$

$$\left[\textbf{Ans.}: 2a^2\left(\frac{\pi}{4} - 1\right)\right]$$

10. Find the area of one of the loops of $x^4 + y^4 = 2a^2xy$.

$$\left[\textbf{Ans.}: \frac{\pi a^2}{4}\right]$$

11. Find the area enclosed by the curve $9xy = 4$ and the line $2x + y = 2$.

$$\left[\textbf{Ans.}: \frac{1}{3} - \frac{4}{9}\log 2\right]$$

12. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 9$ and a straight line $x = 3 - y$.

$$\left[\textbf{Ans.}: 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

13. Find the area bounded by the x-axis, circle $x^2 + y^2 = 16$ and the line $y = x$.

$$[\textbf{Ans.}: 2\pi]$$

14. Find the area bounded between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$.

$$\left[\textbf{Ans.}: \frac{45}{2}\right]$$

15. Find the area bounded by the asteroid $(x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}}$.

$$\left[\textbf{Ans.}: \frac{3}{8}\pi a^2\right]$$

9.7.2 Area in Polar Coordinates

The area A bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

Note: Consider the symmetricity of the region while calculating the area.

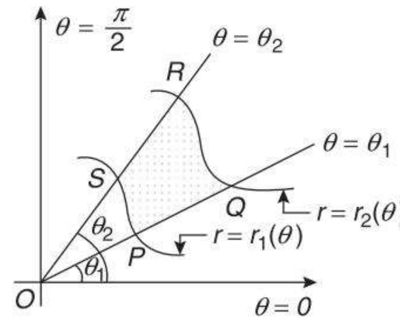

Fig. 9.152

Example 1

Find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution

1. The region is symmetric about the line $\theta = \dfrac{\pi}{2}$.

   Total area = 2 (area in the first quadrant)

2. Draw an elementary radius vector OAB from the origin in the region which lies in the first quadrant. OAB enters in the region from the circle $r = 2\sin\theta$ and terminates on at the circle $r = 4\sin\theta$.

   Limits of $r : r = 2\sin\theta$ to $r = 4\sin\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

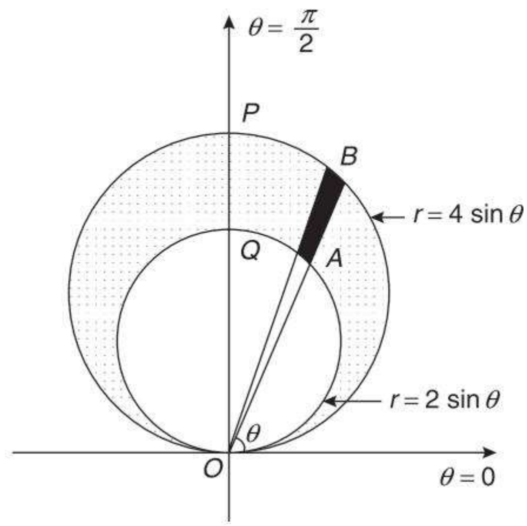

Fig. 9.153

$$A = 2\int_0^{\frac{\pi}{2}} \int_{2\sin\theta}^{4\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_{2\sin\theta}^{4\sin\theta} \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} (16\sin^2\theta - 4\sin^2\theta)\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 12\sin^2\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 6(1 - \cos 2\theta)\,\mathrm{d}\theta$$

$$= 6\left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}} \mathrm{d}\theta$$

$$= 6\left(\frac{\pi}{2} - \frac{\sin\pi - \sin 0}{2} \right)$$

$$= 3\pi$$

Example 2

Use double integral in polar form to find the area enclosed by the three petalled rose $r = \sin 3\theta$. **[Winter 2015]**

Solution

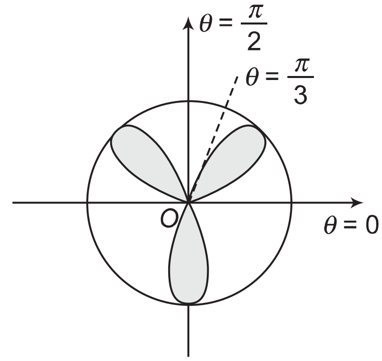

Fig. 9.154

1. This curve consists of three similar loops. Total area = 3 (area of the loop in the first quadrant)
2. When $r = 0$, $\sin 3\theta = 0$

$$3\theta = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$$

   Since, in the first quadrant,

$$r = 0 \text{ at } \theta = 0, \frac{\pi}{3}$$

   loop exists between $\theta = 0$ and $\theta = \frac{\pi}{3}$.

3. Draw an elementary radius vector *OA* from the origin in the loop which lies in the first quadrant. *OA* starts from the origin and terminates on the curve = sin θ.

   Limits of r : $r = 0$ to $r = \sin 3\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{3}$

$$A = 3\int_0^{\frac{\pi}{3}}\int_0^{\sin 3\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{3}} \left|\frac{r^2}{2}\right|_0^{\sin 3\theta} \mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta\,\mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 6\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3}{4}\left|\theta - \frac{\sin 6\theta}{6}\right|_0^{\frac{\pi}{3}}$$

$$= \frac{3}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin 2\pi\right)$$

$$= \frac{3}{4}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{4}$$

Example 3

Find the area of the crescent bounded by the circles $r=\sqrt{3}$ *and* $r = 2\cos\theta$.

Solution

1. The points of intersection of $r=\sqrt{3}$ and $r = 2\cos\theta$ are obtained as

$$\sqrt{3} = 2\cos\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{6}$$

Hence, $\theta = \frac{\pi}{6}$ at P.

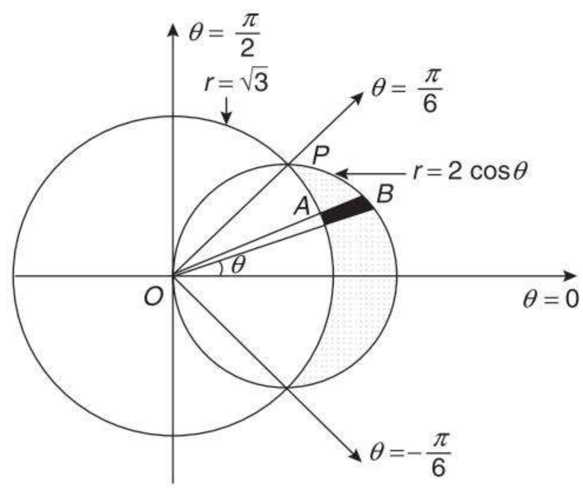

Fig. 9.155

2. The region is symmetric about the initial line, $\theta = 0$.
   Area of the crescent = 2 (area above the initial line, $\theta = 0$)
3. Draw an elementary radius vector OAB from the origin in the region above the initial line. OAB enters in the region from the circle $r=\sqrt{3}$ and terminates on at the circle $r = 2\cos\theta$.
   Limits of $r: r=\sqrt{3}$ to $r = 2\cos\theta$

   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{6}$

$$A = 2\int_0^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \left|\frac{r^2}{2}\right|_{\sqrt{3}}^{2\cos\theta} \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} (4\cos^2\theta - 3)\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} [2(1+\cos 2\theta) - 3]\mathrm{d}\theta$$

$$= \left|2\frac{\sin 2\theta}{2} - \theta\right|_0^{\frac{\pi}{6}}$$

$$= \sin\frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Example 4

Find the area which lies inside the circle $r = 3a \cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.

Solution

1. The points of intersection of the circle $r = 3a \cos\theta$ and the cardioid $r = a(1 + \cos\theta)$ are obtained as

$$3a\cos\theta = a(1+\cos\theta)$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \pm\frac{\pi}{3}$$

Hence, $\theta = \frac{\pi}{3}$ at R.

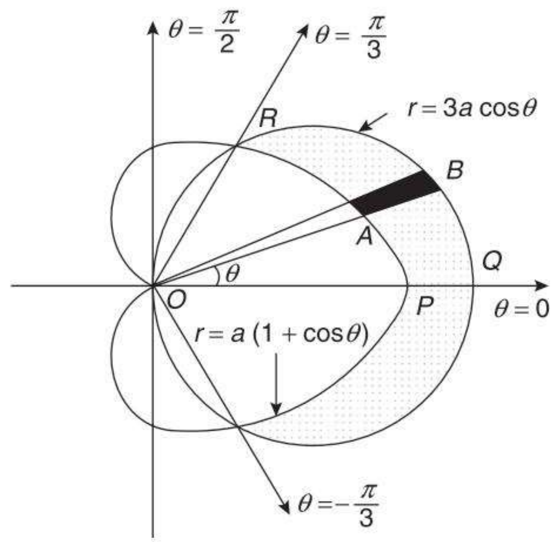

Fig. 9.156

2. The region is symmetric about the initial line $\theta = 0$.

   Total area = 2 (area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line. OAB enters in the region from the cardioid $r = a(1 + \cos\theta)$ and terminates on the circle $r = 3a\cos\theta$.

   Limits of $r: r = a(1 + \cos\theta)$ to $r = 3a\cos\theta$

   Limits of $\theta: \theta = 0$ to $\theta = \frac{\pi}{3}$

$$A = 2\int_0^{\frac{\pi}{3}}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\left|\frac{r^2}{2}\right|_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[8a^2\cos^2\theta - a^2 - 2a^2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\frac{\pi}{3}}\left[4(1+\cos 2\theta) - 1 - 2\cos\theta\right]d\theta$$

$$= a^2\left|3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right|_0^{\frac{\pi}{3}}$$

$$= a^2\left(3\frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right)$$

$$= \pi a^2$$

Example 5

Find the area lying inside the circle $r = a \sin \theta$ *and outside the cardioid* $r = a(1 - \cos\theta)$. **[Summer 2016]**

Solution

1. The points of intersection of circle $r = a\sin\theta$ and the cardioid $r = a(1 - \cos\theta)$ are obtained as

$$a\sin\theta = a(1-\cos\theta)$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$$

$$\sin\frac{\theta}{2} = 0,\ \tan\frac{\theta}{2} = 1$$

$$\frac{\theta}{2} = 0,\ \frac{\theta}{2} = \frac{\pi}{4}$$

$$\theta = 0,\ \theta = \frac{\pi}{2}$$

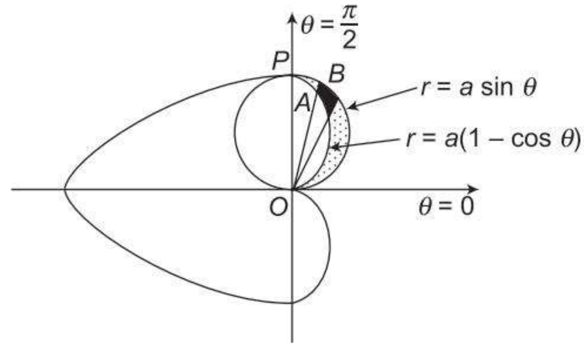

Fig. 9.157

Hence, $\theta = 0$ at origin and $\theta = \dfrac{\pi}{2}$ at P.

2. Draw an elementary radius vector OAB from origin in the region. OAB enters in the region from the cardioid $r = a(1 - \cos\theta)$ and terminates on the circle $r = a\sin\theta$.

Limit of r: $r = a(1 - \cos\theta)$ to $r = a\sin\theta$

Limit of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_{a(1-\cos\theta)}^{a\sin\theta}\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right]\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left[\sin^2\theta - (1 - 2\cos\theta + \cos^2\theta)\right]\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left[\sin^2\theta - \cos^2\theta + 2\cos\theta - 1\right]\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\frac{\pi}{2}}(-\cos 2\theta)\mathrm{d}\theta + 2\int_0^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\right]$$

$$= \frac{a^2}{2}\left[-\left|\frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} + 2\left|\sin\theta\right|_0^{\frac{\pi}{2}} - \left|\theta\right|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{a^2}{2}\left[-\frac{1}{2}(\sin\pi - \sin\theta) + 2(\sin\frac{\pi}{2} - \sin 0) - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$$

$$= a^2\left(1 - \frac{\pi}{4}\right)$$

Example 6

Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.

Solution

1. The points of intersection of the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ are obtained as

$$a(1+\cos\theta) = a(1-\cos\theta)$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

Hence, $\theta = \frac{\pi}{2}$ at P.

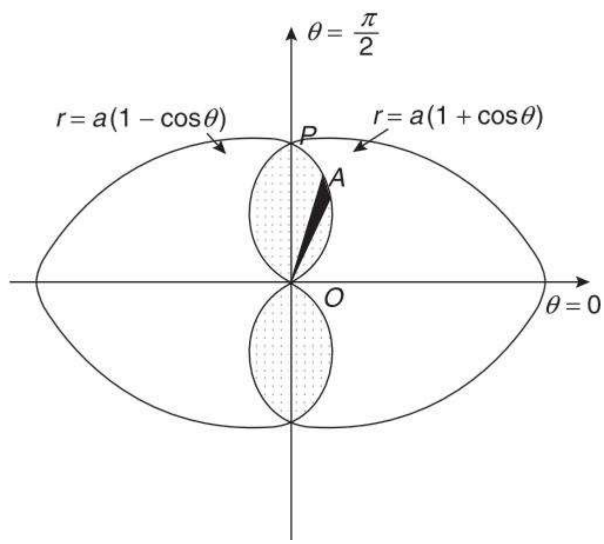

Fig. 9.158

2. The region is symmetric in all the quadrants

Total area = 4 (area in the first quadrant)

3. Draw an elementary radius vector OA from the origin in the region which lies in the first quadrant. OA starts from the origin and terminates on the cardioid $r = a(1 - \cos\theta)$.

Limits of $r : r = 0$ to $r = a(1-\cos\theta)$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}} \int_0^{a(1-\cos\theta)} r \, dr \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} a^2 (1-\cos\theta)^2 \, d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= 2a^2 \left| \frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4} \right|_0^{\frac{\pi}{2}}$$

$$= 2a^2 \left(\frac{3\pi}{4} - 2 \right)$$

Example 7

Find the area inside the cardioid $r = 3(1+\cos\theta)$ *and outside the parabola* $r = \dfrac{3}{1+\cos\theta}$.

Solution

1. The points of intersection of the cardioid $r = 3(1+\cos\theta)$ and the parabola $r = \dfrac{3}{1+\cos\theta}$ are obtained as

$$3(1+\cos\theta) = \frac{3}{(1+\cos\theta)}$$

$$(1+\cos\theta)^2 = 1$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

Hence, $\theta = \frac{\pi}{2}$ at P.

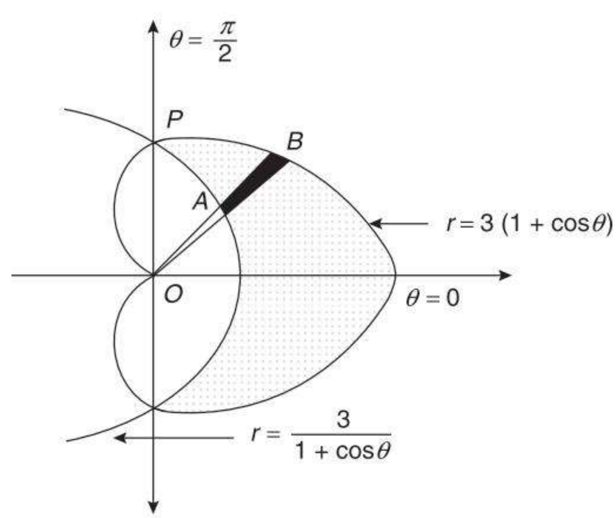

Fig. 9.159

2. The region is symmetric about the initial line $\theta = 0$.

   Total area = 2 (area above the initial line)

3. Draw an elementary radius vector OAB from the origin in the region above the initial line $\theta = 0$. OAB enters in the region from the parabola $r = \dfrac{3}{1+\cos\theta}$ and terminates on the cardioid $r = 3(1+\cos\theta)$.

   Limits of $r : r = \dfrac{3}{1+\cos\theta}$ to $r = 3(1+\cos\theta)$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} \int_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_{\frac{3}{1+\cos\theta}}^{3(1+\cos\theta)} \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 9\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right]\mathrm{d}\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} - \frac{1}{4}\sec^2\frac{\theta}{2} - \frac{1}{2}\cdot\tan^2\frac{\theta}{2}\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)\right]\mathrm{d}\theta$$

$$= 9\left| \frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} - \frac{1}{4}\cdot 2\tan\frac{\theta}{2} - \frac{1}{2}\,\frac{\tan^3\frac{\theta}{2}}{3} \right|_0^{\frac{\pi}{2}}$$

$$\left[\because \int [f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 9\left(\frac{3\pi}{4} + 2\sin\frac{\pi}{2} + \frac{\sin\pi}{4} - \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{6}\tan^3\frac{\pi}{4}\right)$$

$$= 9\left(\frac{3\pi}{4} + \frac{4}{3}\right)$$

Example 8

Find the area common to both the circles $r = \cos\theta$ and $r = \sin\theta$.

[Winter 2013]

Solution

1. So the point of intersection of the circles $r = \cos\theta$ and $r = \sin\theta$ is obtained as

$$\cos\theta = \sin\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Hence, $\theta = \frac{\pi}{4}$ is the point of intersection

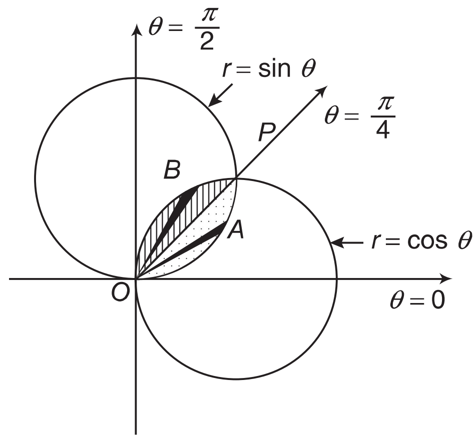

Fig. 9.160

2. The point of intersection divides the region into two subregions *OAP* and *OBP*.
3. Draw an elementary radius in each subregion.
   (i) In subregion *OAP*, radius vector *OA* starts from the origin and terminates on the circle $r = \sin\theta$.
   Limit of r: $r = 0$ to $r = \sin\theta$
   Limit of θ: $\theta = 0$ to $\theta = \frac{\pi}{4}$
   (ii) In the subregion *OBP*, the radius vector *OB* starts from the origin and terminates on the circle $r = \cos\theta$.
   Limit of r: $r = 0$ to $r = \cos\theta$
   Limit of θ: $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \left|\frac{r^2}{2}\right|_0^{\sin\theta} \mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_0^{\cos\theta} \mathrm{d}\theta$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \sin^2\theta\,\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta\right]$$

$$= \frac{1}{4}\left[\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\,\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,\mathrm{d}\theta\right]$$

$$= \frac{1}{4}\left[\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)+\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{2}-1\right)$$

$$= \frac{\pi}{8}-\frac{1}{4}$$

Example 9

Find the area common to the circles $r=\cos\theta$ *and* $r=\sqrt{3}\sin\theta$.

Solution

1. The point of intersection of the circles $r=\cos\theta$ and $r=\sqrt{3}\sin\theta$ is obtained as

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

Hence, $\theta = \frac{\pi}{6}$ at P.

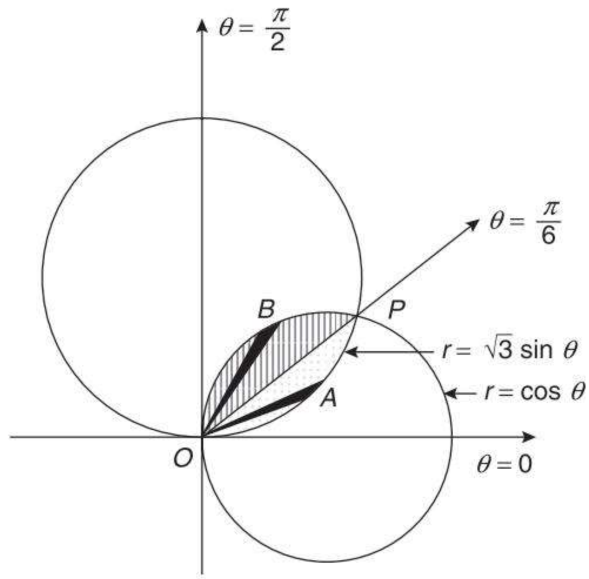

Fig. 9.161

2. Divide the region *OAPBO* into two subregions *OAP* and *OBP*. Draw an elementary radius vector in each subregion.

   (i) In subregion *OAP*, radius vector *OA* starts from the origin and terminates on the circle $r=\sqrt{3}\sin\theta$.

   Limits of $r: r=0$ to $r=\sqrt{3}\sin\theta$

   Limits of $\theta: \theta=0$ to $\theta=\frac{\pi}{6}$

(ii) In the subregion *OBP*, the radius vector *OB* starts from the origin and terminates on the circle $r = \cos\theta$.

Limits of $r : r = 0$ to $r = \cos\theta$

Limits of $\theta : \theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{6}} \int_0^{\sqrt{3}\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} \left|\frac{r^2}{2}\right|_0^{\sqrt{3}\sin\theta} \mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_0^{\cos\theta} \mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 3\sin^2\theta\,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{3}{4}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{6}} + \frac{1}{4}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3}{4}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) + \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{6} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin\frac{\pi}{3}\right)$$

$$= \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$$

Example 10

Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$.

Solution

1. The points of intersection of the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$ are obtained as

$$a = a(1+\cos\theta)$$

$$\cos\theta = 0$$

$$\theta = \pm\frac{\pi}{2}$$

Hence, $\theta = \dfrac{\pi}{2}$ at *Q*.

2. The region is symmetric about the initial line $\theta = 0$.

Total area = 2 (area above the initial line)

3. Divide the region *OPQO* above the initial line into two subregions *OPQ* and *OBQ*. Draw an elementary radius vector in each subregion.
   (i) In the subregion *OPQ* the radius vector *OA* starts from the origin and terminates on the circle $r = a$.
   Limits of $r : r = 0 \quad \text{to} \quad r = a$

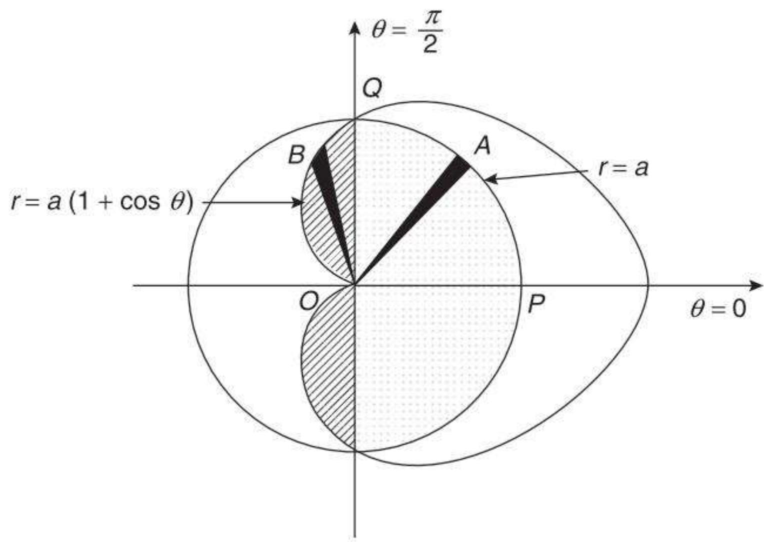

Fig. 9.162

   Limits of $\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$

   (ii) In the subregion *OBQ*, radius vector *OB* starts from the origin and terminates on the cardioid $r = a\,(1 + \cos\theta)$.
   Limits of $r : r = 0 \quad \text{to} \quad r = a(1+\cos\theta)$

   Limits of $\theta : \theta = \frac{\pi}{2} \quad \text{to} \quad \theta = \pi$

$$A = 2\left(\int_0^{\frac{\pi}{2}}\int_0^a r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi}\int_0^{a(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta\right)$$

$$= 2\left(\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^a \mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi}\left|\frac{r^2}{2}\right|_0^{a(1+\cos\theta)} \mathrm{d}\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} a^2\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi} a^2(1+\cos\theta)^2\,\mathrm{d}\theta$$

$$= a^2 \left|\theta\right|_0^{\frac{\pi}{2}} + a^2 \int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) \mathrm{d}\theta$$

$$= a^2 . \frac{\pi}{2} + a^2 \left| \frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} \right|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi a^2}{2} + \frac{3a^2}{2}\left(\pi - \frac{\pi}{2}\right) + 2a^2\left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{a^2}{4}(\sin 2\pi - \sin\pi)$$

$$= a^2\left(\frac{5\pi}{4} - 2\right)$$

Example 11

Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote $r = a \sec \theta$.

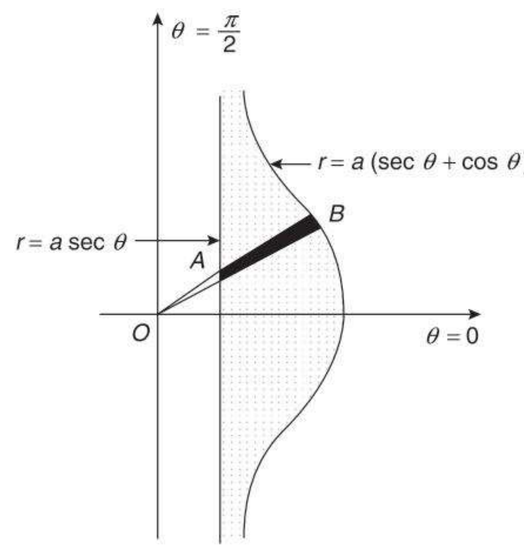

Fig. 9.163

Solution

1. The region is symmetric about the initial line $\theta = 0$
   Total area = 2(area above the initial line)
2. Draw an elementary radius vector *OAB* in the region above the initial line.
   OAB enters in the region from the line $r = a \sec \theta$ and terminates on the curve $r = a(\sec\theta + \cos\theta)$.
   Limits of r : $r = a \sec \theta$ to $r = a(\sec\theta + \cos\theta)$
   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_{a\sec\theta}^{a(\sec\theta+\cos\theta)} \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} [a^2(\sec\theta + \cos\theta)^2 - a^2\sec^2\theta]\mathrm{d}\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} (\cos^2\theta + 2)\mathrm{d}\theta$$

$$= a^2\left[\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right)+\left|2\theta\right|_0^{\frac{\pi}{2}}\right]$$

$$= a^2\left[\frac{1}{2}\frac{\overline{|\tfrac{3}{2}}\,\overline{|\tfrac{1}{2}}}{\overline{|2}}+\frac{2\pi}{2}\right]$$

$$= \frac{5\pi}{4}a^2$$

Example 12

Find the area of the loop of the curve $x^4 + y^4 = 8xy$.

Solution

1. The equation of the curve in polar form is $r^4(\cos^4\theta+\sin^4\theta) = 8r^2\cos\theta\sin\theta$

$$r^2 = \frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$$

2. Draw an elementary radius vector *OA* from the origin in the region which lies in the first quadrant. *OA* starts from the origin and terminates on the curve $r^2 = \dfrac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$.

   Limits of

   $r : r = 0$ to $r = \sqrt{\dfrac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

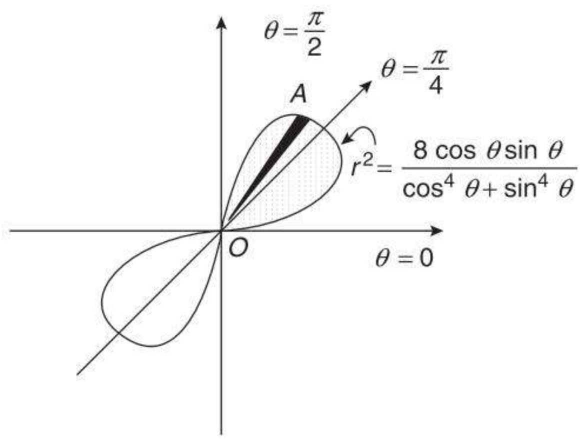

Fig. 9.164

$$A = \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_0^{\sqrt{\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}}}\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{8\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right)\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{\tan\theta\sec^2\theta}{1+\tan^4\theta}\right)\mathrm{d}\theta$$

Putting $\tan^2\theta = t,\ 2\tan\theta\sec^2\theta\, d\theta = dt$

When $\theta = 0, t = 0$

When $\theta = \dfrac{\pi}{2}, t \to \infty$

$$A = 2\int_0^\infty \frac{dt}{1+t^2}$$

$$= 2\left|\tan^{-1} t\right|_0^\infty$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

EXERCISE 9.9

1. Find the area common to the circles $r = a$ and $r = 2a\cos\theta$.

$$\left[\textbf{Ans.:}\ a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

2. Find the area of the crescent bounded by the circles $r = \sqrt{2}$ and $r = 2\cos\theta$.

$$[\textbf{Ans.:}\ 1]$$

3. Find the area which lies inside the cardioid $r = 2a\,(1 + \cos\theta)$ and outside the parabola $r = \dfrac{2a}{1+\cos\theta}$.

$$\left[\textbf{Ans.:}\ 3\pi a^2 + \frac{16a^2}{3}\right]$$

4. Find the area bounded between the circles $r = 2a\sin\theta$, $r = 2b\sin\theta$ $(b > a)$.

$$\left[\textbf{Ans.:}\ \pi(b^2 - a^2)\right]$$

5. Find the area outside the circle $r = a$ and inside the cardioid $r = a(1 + \cos\theta)$.

$$\left[\textbf{Ans.:}\ \frac{a^2}{4}(\pi + 8)\right]$$

Points to Remember

Double Integrals

The double integral of a function $f(x, y)$ over the region R is denoted by $\iint\limits_R f(x, y)\,dx\,dy$.

Double Integrals over Rectangles and General Regions

Method-I: When the region R is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$,

$$\iint\limits_R f(x, y)\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\right] dx$$

Method-II: When the region R is bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and $y = c$, $y = d$,

$$\iint\limits_R f(x, y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,dx\right] dy$$

If all the four limits are constant and $f(x, y)$ is explicit, then the $f(x, y)$ can be integrated w.r.t. any variable first and also can be written as product of two single integrals.

Change of Order of Integration

Sometimes evaluation of double integral becomes easier by changing the order of integration. To change the order of integration,

1. Draw the region of integration with the help of the given limits.
2. Draw vertical or horizontal strip as per the required order of integration
3. Find the limits of integration

$$\int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\right] dx = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,dx\right] dy$$

Double Integrals in Polar Coordinates

Putting $x = r\cos\theta$, $y = r\sin\theta$,

$$\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

where Jacobian, $J = r$

Hence, $$\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Triple Integrals

The triple integral of a continuous function $f(x, y, z)$ over a region V is denoted by $\iiint\limits_V f(x, y, z)\,dx\,dy\,dz$.

Triple Integrals in Cartesian Coordinates

If the region V is bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$ and if the projection of region V in xy-plane is R which is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a$, $x = b$ then

$$\iiint_V f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\mathrm{d}z \right\} \mathrm{d}y \right] \mathrm{d}x$$

Triple Integrals in Cylindrical Coordinates

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\cos\theta, r\sin\theta, z)\left|J\right|\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

where Jacobian, $J = r$

Hence, $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

Triple Integrals in Spherical Coordinates

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\left|J\right|\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

where Jacobian, $J = r^2\sin\theta$

Hence, $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$= \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Area by Double Integrals

Area in Cartesian Coordinates

(i) The area bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} \mathrm{d}y\,\mathrm{d}x$$

(ii) The area bounded by the curves $x = x_1(y)$ and $x = x_2(y)$ and intersecting at the points $P(a, b)$ and $Q(c, d)$ is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} \mathrm{d}y\,\mathrm{d}x$$

Area in Polar Coordinates

The area bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. To evaluate $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}}\,dx\,dy$ by change of order of integration, the lower limit for the variable x is equal to

(a) y^2 (b) 0 (c) ∞ (d) y

2. $\int_0^4 \int_0^3 \int_0^2 dx\,dy\,dz =$

(a) 9 (b) 24 (c) 1 (d) 0

3. By changing the order of integration, the integral $\int_0^2 \int_1^{e^x} dy\,dx$ is equivalent to the double integral_______.

(a) $\int_1^e \int_{\log y}^2 dx\,dy$ (b) $\int_1^{e^2} \int_{\log y}^2 dx\,dy$

(c) $\int_{e^2}^1 \int_2^{\log y} dx\,dy$ (d) $\int_1^{e^2} \int_2^{\log y} dx\,dy$

4. By changing to spherical polar co-ordinates, $\iiint_R dy\,dx\,dz,$ where R is the region of hemisphere $x^2 + y^2 + z^2 = a^2$ is equivalent to triple integral ______

(a) $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$ (b) $\int_0^{2\pi} \int_0^{\pi} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$

(c) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$ (d) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \cos\theta\, dr\, d\theta\, d\phi$

5. $\int_0^a \int_0^x \int_0^y xyz\, dz\, dy\, dx =$

(a) $\dfrac{a^6}{24}$ (b) $\dfrac{a^6}{48}$ (c) $\dfrac{a^4}{48}$ (d) $\dfrac{a^4}{24}$

6. In evaluating $\iint xy(x+y)\,dx\,dy$ over the region between $y = x^2$ and $y = x$, the limits are

(a) $x = 0$ to 1, $xy = 0$ to 1 (b) $x = 0$ to 1, $y = 0$ to x

(c) $x = 0$ to 1, $y = 0$ to x^2 (d) $x = 0$ to 1, $y = x^2$ to x

7. $\int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2 - x^2 - y^2)\,dx\,dy =$

(a) $\dfrac{\pi a^4}{8}$ (b) $\dfrac{\pi a^4}{4}$ (c) $\dfrac{\pi a^4}{2}$ (d) πa^4

8. After transforming to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\, \mathrm{d}y =$

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} \mathrm{d}r\, \mathrm{d}\theta$ (b) $\int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} r\, \mathrm{d}r\, \mathrm{d}\theta$

(c) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\, \mathrm{d}r\, \mathrm{d}\theta$ (d) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-y^2} r\, \mathrm{d}r\, \mathrm{d}\theta$

9. $\int_0^\pi \int_0^{a\cos\theta} r \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta =$

(a) $\frac{a^2}{4}$ (b) $\frac{a^2}{3}$ (c) $\frac{a^2}{2}$ (d) $\frac{a^2}{6}$

10. $\int_0^1 \int_0^2 xy^2\, \mathrm{d}y\, \mathrm{d}x =$

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

11. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\, \mathrm{d}x\, \mathrm{d}y$ is

(a) 0 (b) π (c) $\frac{\pi}{2}$ (d) 2

12. The value of the integral $\iint xy\, \mathrm{d}x\, \mathrm{d}y$ over the region bounded by the x-axis, ordinate at $x = 2a$ and the parabola $x^2 = 4ay$ is

(a) $\frac{a^4}{3}$ (b) $\frac{a^4}{5}$ (c) $\frac{a^4}{7}$ (d) $\frac{a^4}{9}$

13. The triple integral $\iiint_R \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ gives

(a) volume (b) area (c) surface area (d) density

14. The value of $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$ is

(a) $\frac{4}{35}$ (b) $\frac{3}{35}$ (c) $\frac{8}{35}$ (d) $\frac{6}{35}$

15. The value of the integral $\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

16. The value of the integral $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} x\, \mathrm{d}x\, \mathrm{d}y$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

17. Changing the order of integration in the double integral $\int_0^\infty \int_{\frac{x}{4}}^{2} f(x, y)\, dy\, dx$ leads to $\int_r^s \int_p^q f(x, y)\, dx\, dy$, then q is

(a) $4y$ (b) $16y^2$ (c) x (d) 8

18. The limits of integration of $\iint (x^2 + y^2)\, dx\, dy$ over the domain bounded by $y = x^2$ and $y^2 = x$ are

(a) $x = 0$ to 1, $y = x^2$ to $\sqrt{x}$ (b) $x = 0$ to 1, $y = 0$ to 1

(c) $x = y^2$ to $\sqrt{y}$, $y = 0$ to 1 (d) $x = 0$ to y, $y = \sqrt{x}$ to x^2

19. $\iint \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$ is

(a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) $\frac{5}{3}$

20. $\iint r^3\, dr\, d\theta$ over the region included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ is

(a) $\int_0^\pi \int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$ (b) $\int_0^{\frac{\pi}{2}} \int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$

(c) $\int_{-\pi}^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$ (d) $\int_0^{\frac{\pi}{2}} \int_{\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$

21. On converting into polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)\, dy\, dx =$

(a) $\int_0^a \int_0^{\frac{\pi}{2}} r^2\, dr\, d\theta$ (b) $\int_0^a \int_0^{\frac{\pi}{2}} r^3\, dr\, d\theta$

(c) $\int_0^a \int_0^{\frac{\pi}{4}} r^3\, dr\, d\theta$ (d) $\int_0^a \int_0^{\frac{\pi}{4}} r^2\, dr\, d\theta$

22. In spherical co-ordinates, $dx\, dy\, dz$ is equal to

(a) $r\, d\theta\, d\phi\, dr$ (d) $r \sin\theta\, d\theta\, d\phi\, dr$ (c) $r^2 \sin\theta\, d\theta\, d\phi\, dr$ (d) $r^2\, d\theta\, d\phi\, dr$

23. The value of the integral $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$ is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 3

24. The value of $\int_0^a dx \int_0^{\sqrt{a^2 - x^2}} dy \int_0^{\sqrt{a^2 - x^2 - y^2}} dz$ is

(a) $4\pi a^2$ (b) $\frac{\pi a^3}{6}$ (c) $4\pi a^3$ (d) $\frac{\pi}{3} a^2$

25. The transformations $x + y = u$, $y = uv$ transform the area element $\mathrm{d}y\,\mathrm{d}x$ into $|J|\,\mathrm{d}u\,\mathrm{d}v$, where $|J|$ is equal to

(a) 1 (b) u (c) -1 (d) u^2

26. The value of $\iint_R x^3 y\,\mathrm{d}x\,\mathrm{d}y$, where R is region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant is

(a) $\frac{b^2a^4}{24}$ (b) $\frac{b^3a^4}{24}$ (c) $\frac{ba^4}{24}$ (d) $\frac{b^2a^2}{24}$

27. By changing the order of integration, $\int_0^1\int_0^x f(x,y)\mathrm{d}y\,\mathrm{d}x =$

(a) $\int_1^0\int_1^y f(x,y)\mathrm{d}x\,\mathrm{d}y$ (b) $\int_1^0\int_y^1 f(x,y)\mathrm{d}x\,\mathrm{d}y$

(c) $\int_0^1\int_1^y f(x,y)\mathrm{d}x\,\mathrm{d}y$ (d) $\int_0^1\int_y^1 f(x,y)\mathrm{d}x\,\mathrm{d}y$

28. $\int_0^{2\pi}\mathrm{d}\theta\int_0^1 e^{2r}\,\mathrm{d}r$ is equal to

(a) $e^2 - 1$ (b) $\frac{\pi}{2}(e^2-1)$ (c) $\pi(e^2 - 1)$ (d) $2\pi(e^2 - 1)$

29. The value of $\iint_R x^2y^3\,\mathrm{d}x\,\mathrm{d}y$, where R is the region bounded by the rectangle $0\le x\le 1$ and $0\le y\le 3$ is

(a) $\frac{27}{4}$ (b) $\frac{27}{8}$ (c) $\frac{29}{4}$ (d) $\frac{29}{8}$

30. The value of $\iint 3y\,\mathrm{d}x\,\mathrm{d}y$ over the triangle with vertices (–1, 1), (0, 0) and (1, 1) is **[Winter 2015]**

(a) 0 (b) 1 (c) 2 (d) 3

31. The area of the curve $y = x^2 + 1$ bounded by the x-axis and the line $x = 1$ and $x = 2$ is **[Summer 2014]**

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$ (c) 6 (d) $\frac{1}{6}$

32. The equation of a cylindrical surface $x^2 + y^2 = 9$ becomes ____ when converted to cylindrical polar coordinates. **[Summer 2016]**

(a) $r = 9$ (b) $r^2 = 9$ (c) $r = \pm 3$ (d) $r = 3$

33. $\int_0^2\int_0^{x^2} e^{\frac{y}{x}}\,\mathrm{d}y\,\mathrm{d}x$ is equal to **[Summer 2016]**

(a) $e^2 - 1$ (b) e^2 (c) $e^2 + 1$ (d) e^{-2}

34. $\int_1^2 \int_1^2 \frac{1}{xy} \mathrm{d}x\mathrm{d}y =$ **[Winter 2016]**

(a) 0 (b) $(\log 2)^2$ (c) 1 (d) $\log 2$

35. The region of $\int_1^4 \int_2^6 \mathrm{d}x\,\mathrm{d}y$ represents **[Winter 2016]**

(a) rectangle (b) square (c) circle (d) triangle

36. The region $\int_1^2 \int_1^2 \mathrm{d}x\,\mathrm{d}y$ represents **[Summer 2017]**

(a) rectangle (b) square (c) circle (d) triangle

37. The value of $\int_0^1 \int_0^1 (3x^2 - 2y^2)\,\mathrm{d}x\,\mathrm{d}y$ is **[Summer 2017]**

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{3}$

Answers

1. (d)	2. (b)	3. (b)	4. (a)	5. (b)	6. (d)	7. (a)	8. (c)	9. (b)
10. (d)	11. (d)	12. (a)	13. (a)	14. (a)	15. (a)	16. (c)	17. (a)	18. (a)
19. (a)	20. (a)	21. (b)	22. (c)	23. (a)	24. (b)	25. (b)	26. (a)	27. (d)
28. (c)	29. (a)	30. (c)	31. (b)	32. (d)	33. (a)	34. (b)	35. (a)	36. (a)
37. (d)								