EIGENVALUES AND EIGENVECTORS

Eigen Vector: Any nonzero vector x is said to be a characteristic vector or eigenvector of a matrix A if there exists a number λ such that $Ax = \lambda x$ where $A = [a_{ij}]_{n\times n}$ is an $n-$ rowed square matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a non - zero column vector. Also, λ is said to be characteristic root or characteristic value or eigenvalue of the matrix A.

Now, $Ax = \lambda x = \lambda \, \mathrm{I} \, x \Rightarrow (A - \lambda I)x = 0$

Note1: The matrix $A - \lambda I$ is called the ***characteristic matrix*** of A where I is the unit matrix of order n.

Note2: $\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & -a_{nn}\lambda \end{vmatrix}$ which is an ordinary polynomial in λ of degree n, is called the ***characteristic polynomial*** of A.

Note3: The equation $\det(A - \lambda I) = 0$ is called the ***characteristic equation*** $of A$

Note4: The roots this equation is called the eigenvalues of the matrix A.

Note5: The set of all eigenvectors is called the ***eigenspace*** of A corresponding to λ.

Note6: The set of all eigenvalues of A is called the ***spectrum*** of A.

Note7: The characteristic equation of the matrix A of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1 = sum$ of principal diagonal elements and

S_2 =determinant A

Note8: The characteristic equation of the matrix A of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Where $S_1 = sum$ of principal diagonal elements,

$S_2 = sum$ of minors of principal diagonal elements and

S_3 =determinant A

Note9: The sum of the eigenvalues of a matrix is the sum of its principal diagonal elements.

Note10: The product of the eigenvalues of a matrix is the determinant of the matrix.

PROPERTIES OF EEIGENVALUE

Property 1: If λ is an eigenvalue of the matrix A then λ is also an eigenvalue of A^T.

Proof: Let λ be an eigenvalue of the matrix A.

The characteristic equation of A is $\det(A - \lambda I) = 0.$

The characteristic equation of A^T is $\det(A^T - \lambda I) = 0.$

The determinant value does not change by the interchange of rows and columns.

$$\det(A - \lambda I) = \det(A^T - \lambda I)$$

The characteristic equations are same for both A and A^T

Hence, λ is also an eigenvalue of A^T

Property 2: If λ is an eigenvalue of the non - singular matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Proof: Let λ be an eigenvalue of the non - singular matrix A.

$$Ax = \lambda x \ldots (1)$$

Pre-multiplying, both sides of Eq. (1) by A^{-1},

$$A^{-1}Ax = A^{-1}\lambda x$$

$$Ix = \lambda A^{-1}x$$

$$x = \lambda A^{-1}x$$

$$A^{-1}x = \frac{1}{\lambda}x$$

Hence, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Property 3: If λ is an eigenvalue of the matrix A then λ^k is an eigenvalue of A^k.

Proof: Let λ be an eigenvalue of the matrix A.

$$Ax = \lambda x \cdots (1)$$

Pre-multiplying both sides of Eq. (1) by A,

$$AAx = A\lambda x$$

$$A^2x = \lambda(Ax)$$

$$= \lambda(\lambda x) \; = \lambda^2 x$$

Similarly, $A^3x = \lambda^3 x$

In general, $A^k x = \lambda^k x$

Hence, λ^k is an eigenvalue of A^k.

Property 4: If λ is an eigenvalue of the matrix A then $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

Proof: Let λ be an eigenvalue of the matrix A.

$$Ax = \lambda x \cdots (1)$$

Adding kIx on both sides of Eq. (1),

$$Ax + kIx = \lambda x + kIx$$

$$(A + kI)x = \lambda x + kx$$

$$= (\lambda + k)x$$

Similarly, $(A - kI)x = (\lambda - k)x$

In general, $(A \pm kI)x = (\lambda \pm k)x$

Hence, $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

Property 5: If λ is an eigenvalue of the matrix A then $k\lambda$ is an eigenvalue of kA.

Proof: Let λ be an eigenvalue of the matrix A.

$$Ax = \lambda x \cdots (1)$$

Multiplying both sides of Eq. (1) by the scalar k,

$$kAx = k\lambda x$$

Hence, $k\lambda$ is an eigenvalue of kA.

Property 6: The eigenvalues of a triangular matrix are the diagonal elements of matrix.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ 0 & a_{22} & a_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 0 & 0 & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} a_{11} & -\lambda & a_{12} & \dots. & a_{1n} \\ 0 & 0 & a_{22} - \lambda & \dots. & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots. & a_{nn} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$$\lambda = a_{11}, a_{22}, \dots, a_{nn}$$

Hence, the eigenvalues of A are $a_{11}, a_{22}, \dots, a_{nn}$ which are the diagonal elements of the matrix.

Property 7: The eigenvalues of a real symmetric matrix are real.

Proof: Let λ be an eigenvalue of the real symmetric matrix.

$$Ax = \lambda x \text{....(1)}$$

Pre-multiplying both sides of Eq. (1) by $(\bar{x})^T$,

$$(\bar{x})^T Ax = \lambda(\bar{x})^T x \text{...(2)}$$

Taking complex conjugate on both sides of Eq. (2),

$$x^T \overline{A} \overline{x} = \overline{\lambda} x^T \overline{x}$$

$x^T A \overline{x} = \overline{\lambda} x^T \overline{x}$ $(Since\ \overline{A} = A$ for real matrix).....(3)

Taking transpose on both sides of Eq. (3),

$$\overline{x} A^T x = \overline{\lambda} \overline{x} x$$

$\overline{x} A x = \overline{\lambda} \overline{x}$ ($A^T = A$ for symmetric matrix).... (4)

From Eqs (2) and (4), we have $\lambda \overline{x} x = \overline{\lambda} \overline{x}$

$$(\lambda - \overline{\lambda}) \overline{x} x = 0$$

$\overline{x} x$ is a 1×1 matrix, i.e., a single element which is positive, $\lambda - \overline{\lambda} = 0$

i.e., λ is real.

Hence, the eigenvalues of a real symmetric matrix are real.

Property 8: The eigen values of a Hermitian matrix are real numbers.

Proof: Let A be a Hermitian matrix. That is, $\overline{A^T}$ = A

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$\Rightarrow \quad AX = \lambda X$

$\Rightarrow \quad (AX)^T = (\lambda X)^T$

$\Rightarrow \quad X^T \, A^T = \lambda \, X^T$

$\Rightarrow \quad \overline{X^T} \, \overline{A^T} = \overline{\lambda} \, \overline{X^T}$

$\Rightarrow \quad \overline{X^T} \, A = \overline{\lambda} \, \overline{X^T}$

$\Rightarrow \quad \overline{X^T} \, A \, X = \overline{\lambda} \, \overline{X^T} \, X$

$\Rightarrow \quad \overline{X^T} \, \lambda X = \overline{\lambda} \, \overline{X^T} \, X$

$\Rightarrow \quad \lambda \, \overline{X^T} X = \overline{\lambda} \, \overline{X^T} \, X$

$\Rightarrow \quad \lambda \, \overline{X^T} X - \overline{\lambda} \, \overline{X^T} \, X = 0$

$\Rightarrow \quad \left(\lambda - \overline{\lambda}\right) \overline{X^T} \, X = 0$

Since $\overline{X^T} \, X \neq 0$, we have $\lambda - \overline{\lambda} = 0$

$\Rightarrow \lambda = \overline{\lambda}$

$\Rightarrow \lambda$ is a real number.

Hence The eigen values of a Hermitian matrix are real numbers.

Property 9: The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Proof: Let A be a skew- Hermitian matrix. That is, $\overline{A^T} = -A$

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$\Rightarrow \quad AX = \lambda X$

$\Rightarrow \quad (AX)^T = (\lambda X)^T$

$\Rightarrow \quad X^T \ A^T = \lambda \ X^T$

$\Rightarrow \quad \overline{X^T} \ \overline{A^T} = \overline{\lambda} \ \overline{X^T}$

$\Rightarrow \quad -\overline{X^T} \ A = \overline{\lambda} \ \overline{X^T}$

$\Rightarrow -\overline{X^T} \ A \ X = \overline{\lambda} \ \overline{X^T} \ X$

$\Rightarrow \quad -\overline{X^T} \ \lambda X = \overline{\lambda} \ \overline{X^T} \ X$

$\Rightarrow \quad -\lambda \ \overline{X^T} X = \overline{\lambda} \ \overline{X^T} \ X$

$\Rightarrow \quad \lambda \ \overline{X^T} X + \overline{\lambda} \ \overline{X^T} \ X = 0$

$\Rightarrow \quad \left(\lambda + \overline{\lambda}\right) \overline{X^T} \ X = 0$

Since $\overline{X^T} \ X \neq 0$, we have $\lambda + \overline{\lambda} = 0$

$\operatorname{Re} \lambda = 0$ where $\operatorname{Re} \lambda$ denotes the real part of λ

$\Rightarrow \lambda = 0$ or λ is purely imaginary.

Hence The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Property 10: The eigen values of a Unitary matrix have absolute value one.

Proof: Let A be a Unitary matrix. That is, $A \ \overline{A^T} = \overline{A^T} \ A = I$

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$\Rightarrow \quad AX = \lambda X$

$\Rightarrow \quad (AX)^T = (\lambda X)^T$

$\Rightarrow \quad X^T \, A^T = \lambda \, X^T$

$\Rightarrow \quad \overline{X^T} \, \overline{A^T} = \overline{\lambda} \, \overline{X^T}$

$\Rightarrow \quad \overline{X^T} \, \overline{A^T} \quad A = \overline{\lambda} \, \overline{X^T} \, A$

$\Rightarrow \quad \overline{X^T} \, I = \overline{\lambda} \, \overline{X^T} \, A$

$\Rightarrow \quad \overline{X^T} = \overline{\lambda} \, \overline{X^T} \, A$

$\Rightarrow \quad \overline{X^T} \, X = \overline{\lambda} \, \overline{X^T} \, AX$

$\Rightarrow \quad \overline{X^T} \, X = \overline{\lambda} \, \overline{X^T} \, \lambda X$

$\Rightarrow \quad \overline{X^T} \, X = \lambda \, \overline{\lambda} \, \overline{X^T} \, X$

$\Rightarrow \quad \lambda \, \overline{\lambda} \, \overline{X^T} \, X - \overline{X^T} \, X = 0$

$\Rightarrow \quad \left(\lambda \, \overline{\lambda} - 1\right) \overline{X^T} \, X = 0$

Since $\overline{X^T} \, X \neq 0$, we have $\lambda \, \overline{\lambda} - 1 = 0$

$\Rightarrow \lambda \, \overline{\lambda} = 1$

$\Rightarrow |\lambda|^2 = 1$

$\Rightarrow |\lambda| = 1$

Hence The eigen values of a Unitary matrix have absolute value one.

Problem1: Find the sum and product of the eigenvalues of the matrix $\mathrm{A} = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$.

Solution: Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$= 2 - 2 = 0$$

Product of the eigenvalues of A =Determinant of A

$$= \begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} = -4 + 12 = 8$$

Problem2: Find the sum and product of the eigenvalues of the matrix $\mathrm{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{bmatrix}$

Solution: Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$= 1 + 3 + 7 = 11$$

Product of the eigenvalues of A =Determinant of A

$$= \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{vmatrix}$$

$$= 1(21 - 24) - 2(14 - 12) + 5(12 - 9) = -3 - 4 + 15 = 8$$

Problem3: The product of two eigenvalues of the matrix $\mathrm{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16.

Find the third eigenvalue.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 \lambda_2 = 16$$

Product of the eigenvalues of A =Determinant of A

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 - 2 & 2 \\ -2 & 3 - 1 \\ 2 & -13 \end{vmatrix} \Rightarrow 16\lambda_3 = 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$$

$$= 48 - 8 - 8 \Rightarrow \lambda_3 = 2$$

Problem4: Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and are $\frac{1}{5}$ times to the third.

Find the eigenvalues.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = \lambda_2, \lambda_1 = \frac{1}{5}\lambda_3, \lambda_2 = \frac{1}{5}\lambda_3$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

$$\lambda_1 = \lambda_2 = 1$$

Hence, the eigenvalues of A are 1, 1, 5.

Problem5: If 2 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, find the other two eigen values.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 2$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$2 + \lambda_2 + \lambda_3 = 2 + 1 - 1 \Rightarrow \lambda_2 + \lambda_3 = 0\text{......(1)}$$

Product of the eigenvalues of A =Determinant of A

$$2\lambda_2\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= 2(-1-3) + 2(-1-1) + 2(3-1) = -8 - 4 + 4 = -8$$

$$\lambda_2\lambda_3 = -4 \ldots\ldots(2)$$

Solving Eqs (1) and (2), $\lambda_2 = 2, \lambda_3 = -2$

Hence, the other two eigenvalues are 2, -2.

Problem6: For a given matrix A of order 3, $|A| = 32$ and two of its eigenvalues are 8 and 2. Find the sum of the eigenvalues.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 8, \lambda_2 = 2$$

Product of the eigenvalues of A =Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A| \Rightarrow 8 \times 2 \times \lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

Hence, the sum of the eigenvalues $= 8 + 2 + 2 = 12$

Problem7: For a singular matrix of order three, 2 and 3 are the eigenvalues. Find its third eigenvalue.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$\lambda_1 = 2, \lambda_2 = 3$

For a singular matrix, $|A| = 0$

Product of the eigenvalues of A =Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A| \Rightarrow 2 \times 3 \times \lambda_3 = 0 \Rightarrow \lambda_3 = 0$$

Hence, the third eigenvalue $= 0$

Problem8: If 2, 3 are the eigenvalues of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, find the value of a.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 2, \lambda_2 = 3$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$2 + 3 + \lambda_3 = 2 + 2 + 2 \Rightarrow \lambda_3 = 1$$

Product of the eigenvalues of A =Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$2 \times 3 \times 1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$

$$6 = 2(4-0) - 0(0-0) + 1(0-2a) = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

Problem9: If 3 and 15 are the two eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find $|A|$ without expanding the determinant.

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = 15$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$3 + 15 + \lambda_3 = 8 + 7 + 3 \Rightarrow \lambda_3 = 0$$

Product of the eigenvalues of A =Determinant of A

$$3 \times 15 \times 0 = |A| \Rightarrow |A| = 0$$

Problem10: Find a and b such that $A = \begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and -2 as eigenvalues.

Solution: Let λ_1 and λ_2 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = -2$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$3 - 2 = a + b \Rightarrow a + b = 1.....(1)$$

Product of the eigenvalues of A =Determinant of A

$$\lambda_1 \lambda_2 = |A|$$

$(3)\ (-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix} = ab - 4 \ \Rightarrow ab = -2 \ ....(2)$

From(1), $a = 1 - b$

Substituting in Eq. (2), $(1 - b)b = -2 \Rightarrow b - b^2 = -2 \Rightarrow b^2 - b - 2 = 0$

$b = 2$ or $b = -1$

$\therefore a = -1$ or $a = 2$

Problem11: Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are 1 and 1. Find the eigenvalues of A^{-1}

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = \lambda_2 = 1$$

Sum of the eigenvalues of $A = Sum$ of principal diagonal elements of A

$$1 + 1 + \lambda_3 = 2 + 3 + 2 \Rightarrow \lambda_3 = 5$$

Hence, the eigenvalues of A are 1, 1, 5.

The eigenvalues of A^{-1} are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$, i.e., 1, 1, $\frac{1}{5}$.

Problem12: Two of the eigenvalues of the matrix $\mathrm{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigenvalues of $\mathrm{A}^{-1}$and A^3

Solution: Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A.

$$\lambda_1 = 3, \lambda_2 = 6$$

Sum of the eigenvalues of $A = Sum$ of the principal diagonal elements of A

$$3 + 6 + \lambda_3 = 3 + 5 + 3 \Rightarrow \lambda_3 = 2$$

The eigenvalues of A are 3, 6, 2.

Hence, the eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$, and the eigenvalues $of\, A^3$ are $3^3, 6^3, 2^3$, i.e., 27, 216, 8.

Problem13: Form the matrix whose eigenvalues are $\alpha - 5, \beta - 5, \gamma - 5$ where α, β, γ are the eigenvalues of $\mathrm{A} = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ -8 & 7 & 9 \end{bmatrix}$

Solution: If λ_1, λ_2 and λ_3 are eigenvalues of the matrix A then $\lambda_1 - k$, $\lambda_2 - k$ and $\lambda_3 - k$ are the eigenvalues of $A - kI$.

$$\text{Required matrix} = A - 5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ -8 & 7 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$$

Problem14: If α and β are the eigenvalues of $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$, form a matrix whose eigenvalues are α^3 and β^3

Solution: If λ_1 and λ_2 are the eigenvalues of the matrix A then λ_1^k and λ_2^k are the eigenvalues of A^k.

Let $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

Required matrix $= A^3 = A^2A = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$

Problem15: If $\mathrm{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find the eigenvalues for the following matrices.

(i) A (ii) A^{T} (iii) A^{-1} (iv) $4\mathrm{A}^{-1}$ (v) A^2 (vi) $\mathrm{A}^2 - 2\mathrm{A} + \mathrm{I}$ (vii) $\mathrm{A}^3 + 2\mathrm{I}$

Solution: $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristics equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= (15 - 1) + (9 - 1) + (15 - 1) = 14 + 8 + 14 = 36$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3(15 - 1) + 1(-3 + 1) + 1(1 - 5) = 42 - 2 - 4 = 36$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2,3,6$$

(i) Eigenvalues of $A = \lambda$:2, 3, 6

(ii) Eigenvalues of $A^T = \lambda$:2, 3, 6

(iii) Eigenvalues of $A^{-1} = \lambda^{-1}$: $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(iv) Eigenvalues of $4A^{-1} = 4\lambda^{-1}$: $2, \frac{4}{3}, \frac{2}{3}$

(v) Eigenvalues of $A^2 = \lambda^2$: 4, 9, 36

(vi) Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$: 1, 4, 25

(vii) Eigenvalues of $A^3 + 2I = \lambda^3 + 2$: 10, 29, 218

Properties of Eigen Vectors

Theorem-1: If X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to an eigen value λ then $X_1 + X_2$ is also an eigen vector of A corresponding to λ.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to an eigen value λ. Then $AX_1 = \lambda X_1 \rightarrow$ (1)

$AX_2 = \lambda X_2 \rightarrow$ (2)

$\Rightarrow A(X_1 + X_2) = AX_1 + AX_2$

$= \lambda X_1 + \lambda X_2$

$= \lambda(X_1 + X_2)$

$\Rightarrow A(X_1 + X_2) = \lambda(X_1 + X_2)$

$\Rightarrow X_1 + X_2$ is an eigen vector of A corresponding to λ.

Theorem-2: If k is a constant and X is an eigen vector of a square matrix A corresponding to eigen value λ then kX is also eigen vector of A corresponding to λ.

Proof: Suppose that k is a constant and X is an eigen vector of a square matrix A corresponding to eigen value λ. Then $AX = \lambda X \Rightarrow kAX = k\lambda X$

$\Rightarrow A(kX) = \lambda(kX)$

$\Rightarrow kX$ is eigen vector of A corresponding to λ.

Theorem-3: Two eigen vectors of a square matrix A corresponding to two different eigen values are linearly independent.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to two different eigen values λ_1 and λ_2 respectively.

$\Rightarrow AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$

Suppose that $c_1X_1 + c_2X_2 = O \rightarrow$ (1) where c_1 and c_2 are constants

$\Rightarrow A(c_1X_1 + c_2X_2) = O$

$\Rightarrow c_1(AX_1) + c_2(AX_2) = O$

$\Rightarrow c_1(\lambda_1 X_1) + c_2(\lambda_2 X_2) = O \rightarrow$ (2)

Solving (1) and (2)

$c_1(\lambda_1 - \lambda_2) X_1 = O$

$\Rightarrow c_1 = 0$

Similarly, $c_2(\lambda_2 - \lambda_1) X_2 = O$

$\Rightarrow \; c_2 = 0$

Hence X_1 and X_2 are linearly independent.

Note: If two or more eigen values are equal then the corresponding eigen vectors may or may not be linearly independent**.**

Definition: Two eigen vectors X_1 and X_2 are said to be orthogonal if

$X_1^T X_2 = 0$ or $X_2^T X_1 = 0$.

Theorem-4: Two eigen vectors of a symmetric matrix A corresponding to two different eigen values are orthogonal.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a symmetric matrix A corresponding to two different eigen values λ_1 and λ_2 respectively.

$\Rightarrow \; AX_1 = \lambda_1 X_1 \quad$ and $\quad AX_2 = \lambda_2 X_2$

$\Rightarrow \; X_2^T AX_1 = X_2^T \lambda_1 X_1 \quad$ and $\quad AX_2 = \lambda_2 X_2$

$\Rightarrow \; \left(X_2^T AX_1\right)^T = \left(X_2^T \lambda_1 X_1\right)^T \quad$ and $\quad X_1^T AX_2 = X_1^T \lambda_2 X_2$

$\Rightarrow \; X_1^T A^T X_2 = \lambda_1 X_1^T X_2 \quad$ and $\quad X_1^T AX_2 = \lambda_2 X_1^T X_2$

Since A is symmetric, $A^T = A$.

Consider $\quad X_1^T A^T X_2 = \lambda_1 X_1^T X_2 \quad$ and $\quad X_1^T AX_2 = \lambda_2 X_1^T X_2$

$\Rightarrow \; X_1^T AX_2 = \lambda_1 X_1^T X_2 \quad$ and $\quad X_1^T AX_2 = \lambda_2 X_1^T X_2$

$\Rightarrow \; \lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$

$\Rightarrow \; \lambda_1 X_1^T X_2 - \lambda_2 X_1^T X_2 = 0$

$$\Rightarrow (\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad or \quad X_1^T X_2 = 0$$

Since $\lambda_1 \neq \lambda_2$, $\lambda_1 - \lambda_2 \neq 0$

$$\Rightarrow X_1^T X_2 = 0$$

$\Rightarrow X_1$ and X_2 are orthogonal.

Working Rule for Finding the Eigenvalues and Eigen Vectors

(i) Write characteristic equation $\det(A - \lambda I) = 0$ for the given square matrix.

(ii) Find the eigenvalues of the matrix by solving characteristic equation.

(iii) Find eigenvectors corresponding to each eigen values from the equation $[A - \lambda I]x = 0$.

Problem1: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda 1) = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 5 + 2 = 7$

$$S_2 = \det(A) = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

Hence, the characteristic equation is $\lambda^2 - 7\lambda^2 + 6 = 0 \Rightarrow \lambda = 6,1$

(a) For $\lambda = 6$,

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-x + 4y = 0$$

Let $y = t \Rightarrow x = 4t$

$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{Bmatrix} 4t \\ t \end{Bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 6$.

(b) For $\lambda = 1$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$x + y = 0$$

Let $y = t \Rightarrow x = -t$

$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = tx_2$ where x_2 is an eigenvector corresponding to $\lambda = 1$.

Problem2: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$.

Solution: Let A = $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 4 + 3 - 3 = 4$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-9 + 8) + (-12 + 6) + (12 - 6) = -1 - 6 + 6 = -1$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix}$$

$$= 4(-9 + 8) - 6(-3 + 2) + 6(-4 + 3) = -4 + 6 - 6 = -4$$

Hence, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0 \Rightarrow \lambda = -1,1,4$$

(a) For $\lambda = -1$,

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$5x + 6y + 6z = 0 \text{ and}$$

$$x + 4y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} = t$$

$$\frac{x}{-12} = \frac{y}{-4} = \frac{Z}{14} = t \Rightarrow \frac{x}{-6} = \frac{y}{-2} = \frac{z}{7} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6t \\ -2t \\ 7t \end{bmatrix} = t\begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = -1$.

(b) For $\lambda = 1$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$x + 2y + 2z = 0$$

$$-x - 4y - 4z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{Z}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = t$$

$$\frac{x}{0} = \frac{y}{2} = \frac{Z}{-2} = t$$

$$\frac{x}{0} = \frac{y}{1} = \frac{Z}{-1} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = tx_2$ where x_2 is an eigenvector corresponding to $\lambda = 1$.

(c) For $\lambda = 4$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{Bmatrix} x \\ y \\ Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$0x + 6y + 6z = 0$$

$$x - y + 2z = 0$$

$$x - 4y - 7z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}} = t$$

$$\frac{x}{18} = \frac{y}{6} = \frac{Z}{-6} = t$$

$$\frac{x}{3} = \frac{y}{1} = \frac{Z}{-1} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = tx_3$ where x_3 is an eigenvector corresponding to $\lambda = 4$.

Problem3: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda l) = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = -2 + 1 + 0 = -1$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (0 - 12) + (0 - 3) + (-2 - 4) = -12 - 3 - 6 = -21$$

$$S_3 = \det(A) = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= (-2)(0 - 12) - 2(0 - 6) - 3(-4 + 1) = 24 + 12 + 9 = 45$$

Hence, the characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow \lambda = 5, -3, -3$$

(a) For $\lambda = 5$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-7x + 2y - 3z = 0$$

$$2x - 4y - 6z = 0$$

$$-x - 2y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}} = t$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{Z}{24} = t \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{Z}{-1} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 5$.

(b) For $\lambda = -3$, $[A - \lambda l]x = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$x + 2y - 3z = 0$$

Let $y = t_1$ and $z = t_2 \Rightarrow x = -2t_1 + 3t_2$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 + 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = t_1 x_2 + t_2 x_3$$

where x_2 and x_3 are linearly independent eigenvectors corresponding to $\lambda = -3$.

Problem4: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 0 + 0 + 3 = 3$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= (0 + 3) + (0) + (0) = 3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix}$$

$$= 0 - 1(0 - 1) + 0 = 1$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow \lambda = 1,1,1$$

(a) For $\lambda = 1$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + 0z = 0$$

$$0x - y + z = 0$$

$$x - 3y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = t$$

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 1$.

Hence, there is only one eigen vector corresponding to repeated eigen value λ = 1

Problem5: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 8 + 7 + 3 = 18$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= (21 - 16) + (24 - 4) + (56 - 36) = 5 + 20 + 20 = 45$

$$S_3 = \det(A) = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$$

Hence, the characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0 \Rightarrow \lambda = 0,3,15$$

(a) For $\lambda = 0$,

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$8x - 6y + 2z = 0$$

$$-6x + 7y - 4z = 0$$

$$2x - 4y + 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} = t$$

$$\frac{x}{10} = \frac{y}{20} = \frac{Z}{20} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{Z}{2} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 3$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$5x - 6y + 2z = 0$$

$$-6x + 4y - 4z = 0$$

$$2x - 4y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-6 & 2\\ 4 & -4\end{vmatrix}} = -\frac{y}{\begin{vmatrix}2 & 5\\ -6 & -4\end{vmatrix}} = \frac{z}{\begin{vmatrix}5 & -6\\ -6 & 4\end{vmatrix}} = t$$

$$\frac{x}{16} = \frac{y}{8} = \frac{Z}{-16} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{Z}{-2} = t$$

$x = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2t\\ t\\ -2t\end{bmatrix} = t\begin{bmatrix}2\\ 1\\ -2\end{bmatrix} = tx_2$ where x_2 is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 15$, $[A - \lambda I]x = 0$

$$\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

$$-7x - 6y + 2z = 0$$

$$-6x - 8y - 4z = 0$$

$$2x - 4y - 12z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = \frac{y}{\begin{vmatrix}2 & -7\\ -6 & -4\end{vmatrix}} = \frac{z}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}} = t$$

$$\frac{x}{40} = -\frac{y}{40} = \frac{Z}{20} = t$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{Z}{1} = t$$

$x = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2t\\ -2t\\ t\end{bmatrix} = t\begin{bmatrix}2\\ -2\\ 1\end{bmatrix} = tx_3$ where x_3 is an eigenvector corresponding to $\lambda = 15$.

Note: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal which can be verified with this example.

$$x_1^T x_2 = [1\ 2\ 2]\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = [0] = 0\ ;\ \ x_2^T x_3 = [2\ 1\ -2]\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = [0] = 0;\ \ x_3^T x_1 = [2\ -2\ 1]\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Thus, x_1, x_2 and x_3 are orthogonal to each other.

Problem 6: Find orthogonal eigenvectors for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda l) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 1 + 4 + 9 = 14$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= (36 - 36) + (9 - 9) + (4 - 4) = 0$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix}$$

$$= 1(36 - 36) - 2(18 - 18) + 3(12 - 12) = 0$$

Hence, the characteristic equation is

$$\lambda^3 - 14\lambda^2 = 0 \Rightarrow \lambda = 0,0,14$$

(a) For $\lambda = 14$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-13x + 2y + 3z = 0$$

$$2x - 10y + 6z = 0$$

$$3x + 6y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{Z}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}} = t$$

$$\frac{x}{42} = \frac{y}{84} = \frac{Z}{126} = t \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{Z}{3} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 14$

(b) For $\lambda = 0$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 3z = 0$$

Let $y = t_1$ and $z = t_2 \Rightarrow x = -2t_1 - 3t_2$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = t_1x_2 + t_2x_3$$

where x_2 and x_3 are linearly independent eigenvectors corresponding to $\lambda = 0$

Since x_2 and x_3 are not orthogonal, we must choose x_3 such that x_1, x_2, x_3 are orthogonal.

Let $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

For x_1 and x_3 to be orthogonal $x_1^T x_3 = 0$

$$\Rightarrow [1\ 2\ 3]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow l + 2m + 3n = 0 \cdots (1)$$

For x_2 and x_3 to be orthogonal, $x_2^T x_3 = 0$

$$\Rightarrow [-2\ 1\ 0]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow\ -2l+m=0\cdots(2)$$

Solving Eqs (1) and (2) by Cramer's rule,

$$\frac{l}{\begin{vmatrix}2 & 3\\ 1 & 0\end{vmatrix}}=-\frac{m}{\begin{vmatrix}1 & 3\\ -2 & 0\end{vmatrix}}=\frac{n}{\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}}=t$$

$$\frac{l}{-3}=\frac{m}{-6}=\frac{n}{5}=t\Rightarrow\frac{l}{3}=\frac{m}{6}=\frac{n}{5}=t$$

$x=\begin{bmatrix}l\\ m\\ n\end{bmatrix}=\begin{bmatrix}3t\\ 6t\\ 5t\end{bmatrix}=t\begin{bmatrix}3\\ 6\\ 5\end{bmatrix}=tx_3$ where x_3 is an eigenvector corresponding to $\lambda=14$

EXERCISE

1. Find the sum and product of the eigenvalues of the following matrices:

(i) $\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$ (ii) $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ [Ans.: $(i)-3,4\ \ (ii)-1,45$]

2. The matrix $A=\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & -2\\ 1 & 2 & 3\end{bmatrix}$ is singular. One of its eigenvalues is 2. Find the other two eigenvalues. [Ans.: $1+\sqrt{5},1-\sqrt{5}$]

3. If two of the eigenvalues of $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ are 2 and 8, find the third [Ans.: 2]

4. Find the eigenvalues of the matrix $\begin{bmatrix}1 & -2\\ -5 & 4\end{bmatrix}$. Hence, form the matrix whose eigenvalues are $\frac{1}{6}$ and -1. $[Ans.:6,-1,-\frac{1}{6}\begin{bmatrix}4 & 2\\ 5 & 1\end{bmatrix}]$

5. If 2 and 3 are the eigenvalues of $A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$) find the eigenvalues of A^{-1} and A^3

$[Ans:(i)\frac{1}{2},\frac{1}{2},\frac{1}{3}$ (ii) $2^3,2^3,3^3]$

6. Two eigenvalues of $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ are equal and are double the third. Find the eigenvalues of A^2 [Ans.: 1, 4, 4]

7. If $A = \begin{bmatrix} 1 & 2-3 \\ 0 & 3 \quad 2 \\ 0 & 0-2 \end{bmatrix}$, find the eigenvalues of $3A^3 + 5A^2 - 6A + 2I$ [Ans.: 4, 110, 10]

8. If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigenvalues of the following matrices:

(i) $A^3 + I$ (ii) A^{-1} (iii) $A^2 - 2A + I$ (iv) $A^3 - 3A^2 + A$

[$Ans.$: (i) 2, 2, 126 (ii)1, 1, $\frac{1}{5}$ (iii) 0,0,16 (iv)-1, $-1, 55$]

9. Find the eigenvalues and eigenvectors for the following matrices:

(i) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ [Ans: 1, 2, 3; $\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 2 \\ 2 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$]

(ii) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$ [Ans: 3, 6, 9; $\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 2 \\ -2 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 2 \\ 1 \\ -2 \end{Bmatrix}$]

(iii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ [Ans: 1, 3, 3; $\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$]

(iV) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ [Ans: 1, 1, 1; $\begin{Bmatrix} -3 \\ 1 \\ 1 \end{Bmatrix}$

(V) If $A = \begin{bmatrix} 3-1 & 1 \\ 3-1 & -1 \\ 3 \quad 1 & -1 \end{bmatrix}$ then check whether eigenvectors of A are orthogonal.

(V) If $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ then verify whether eigenvectors are linearly independent or not.

CAYLEY - HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation

Problem1: Apply Cayley – Hamilton theorem to $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $\mathrm{A}^8 = 625\mathrm{I}$

Solution: $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The characteristic equation is

$$\det\,(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 1 - 1 = 0$

$$S_2 = \det\,(A) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$$

Hence, the characteristic equation is $\lambda^2 - 5 = 0$

By Cayley - Hamilton theorem, the matrix A satisfies its own characteristic equation.

$$A^2 - 5I = 0 \Rightarrow A^2 = 5I \Rightarrow A^4 = 25I \Rightarrow A^8 = 625I$$

Problem2: Verify Cayley –Hamilton theorem for the following matrix and hence,

find A^{-1} and A^4 Where $\mathrm{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation is $\det\,(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-1) + (4-1) = 9$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 2(4-1) + 1(-2+1) + 1(1-2) = 6 - 1 - 1 = 4$

Hence, the characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \ldots\ldots(1)$$

The matrix A satisfies its own characteristic equation. Hence, Cayley - Hamilton theorem is verified.

Pre-multiplying Eq. (1) by A^{-1},

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = (A^2 - 6A + 9I)$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1-1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Multiplying Eq. (1) by A,

$$A(A^3 - 6A^2 + 9A - 41) = 0$$

$$A^4 - 6A^3 + 9A^2 - 4A = 0$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix} == \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

Problem3: Show that matrix $\mathrm{A} = \begin{bmatrix} 0 & \mathrm{c} & -\mathrm{b} \\ -\mathrm{c} & 0 & \mathrm{a} \\ \mathrm{b} & -\mathrm{a} & 0 \end{bmatrix}$ satisfies Cayley – Hamilton theorem

and hence find A^{-1}, if it exists.

Solution: $A = \begin{bmatrix} 0 & \mathrm{c} & -\mathrm{b} \\ -\mathrm{c} & 0 & \mathrm{a} \\ \mathrm{b} & -\mathrm{a} & 0 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 0$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$$

$$= (0 + a^2) + (0 + b^2) + (0 + c^2) = a^2 + b^2 + c^2$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

$= 0 - c(0 - ab) - b(ac - 0) = abc - abc = 0$

Hence, the characteristic equation is $\lambda^3 + (a^2 + b^2 + c^2)\lambda = 0$

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-cb^2-ca^2 & b^3+bc^2+ba^2 \\ c^3+ca^2+cb^2 & 0 & -a^2b-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)A$$

$$A^3 + (a^2+b^2+c^2)A = 0$$

The matrix A satisfies its own characteristic equation. Hence, Cayley - Hamilton

theorem is verified.

$$\det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0-ab) - b(ac-0)$$

$$= abc - abc = 0$$

Hence, A^{-1} does not exist.

Problem 4: Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - \text{A} - 10\,\text{I}$ as a linear polynomial in A.

Solution: Given $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation $of A$ is $|A - \lambda I| = 0 \Rightarrow \lambda^2 - S_1\lambda + S_2 = 0$

where $S_1 = 1 + 3 = 4, S_2 = |A| = 3 - 8 = -5$

the characteristic equation is $\lambda^2 - 4\lambda - 5 = 0 \ldots\ldots (1)$

By Cayley - Hamilton theorem, A satisfies (1)

$$\therefore A^2 - 4A - 5I = 0 \ldots\ldots(2)$$

We shall now verify this by direct computations.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow A^2 - 4A - 5I = 0$

Hence, the theorem is verified.

To find A^{-1}: We have $5I = A^2 - 4A$

Multiply by A^{-1}, we get $5A^{-1} = A^{-1}A^2 - 4A^{-1}A$

$$5A^{-1} = A - 4I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1-4 & 4-0 \\ 2-0 & 3-4 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Finally, to find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$:

Consider the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$……….(3)

Divide the polynomial (3) by $\lambda^2 - 4\lambda - 5$.

Division is indicated below.

$$
\begin{array}{r|l}
 & \lambda^3 - 2\lambda + 3 \\
\hline
\lambda^2 - 4\lambda - 5 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \underline{\lambda^5 - 4\lambda^4 - 5\lambda^3} \\
 & \qquad\quad -2\lambda^3 + 11\lambda^2 - \lambda \\
 & \qquad\quad \underline{-2\lambda^3 + 8\lambda^2 \quad + 10\lambda} \\
 & \qquad\qquad\qquad 3\lambda^2 - 11\lambda - 10 \\
 & \qquad\qquad\qquad \underline{3\lambda^2 - 12\lambda - 15} \\
 & \qquad\qquad\qquad\qquad\quad \lambda + 5
\end{array}
$$

We get the quotient $\lambda^3 - 2\lambda + 3$ and remainder $\lambda + 5$.

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

Replace λ by A, we get

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I$$

$$= 0 + A + 5I = A + 5I \qquad \text{[using (2)]}$$

which is a linear polynomial in A.

EXERCISE

1. Verify Cayley - Hamilton theorem for the matrix A and hence, find A^{-1} and A^4.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ [Ans: $\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}, \begin{pmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{pmatrix}$]

2. Find the characteristic equation of the matrix $\mathrm{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $\mathrm{A}^8 - 5\mathrm{A}^7 + 7\mathrm{A}^6 - 3\mathrm{A}^5 + \mathrm{A}^4 - 5\mathrm{A}^3 + 8\mathrm{A}^2 - 2\mathrm{A} + \mathrm{I}$.

[Ans: $A^2 + A + I = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$]

SIMILARITY TRANSFORMATION

If A and B are two square matrices of order n then B is said to be similar to A, ifthere exists a nonsingular matrix P such that $B = P^{-1}AP$

Note:1 Similarity of matrices is an equivalence relation.

Note:2 Similar matrices have the same determinant.

Note:3 Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If x is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}x$ is an eigenvector of B corresponding to the eigenvalue λ where $B = P^{-1}AP$.

DIAGONALIZATION OF A MATRIX

A matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix.

A matrix A is diagonalizable ifthere exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix, also known as *spectral matrix*. The matrix P is then said to diagonalize A or transform A to a diagonal form. P is known as the **modal matrix**.

Note:1 An $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

Note:2 If the eigen values of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

Note:3 If A is similar to a diagonal matrix D, the diagonal elements $of D$ are the eigen values $of A$.

Note:4: The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the algebraic multiplicity of each of its eigen values is equal to the geometric multiplicity.

Note:5: For a distinct eigen value, algebraic multiplicity is always equal to its geometric multiplicity.

Note:6: If λ is an eigen value ofthe characteristic equation $|A - \lambda l| = 0$ repeated n times, then n is called the **algebraic multiplicity** of λ. The number of linearly independent solutions of $[A - \lambda l]X = O$ is called the **geometric multiplicity** of λ.

Orthogonally Similar Matrices

If A and B are two square matrices of order n then B is said to be orthogonally similar to A if there exists an orthogonal matrix P such that $B = P^{-1}AP$

Since P is orthogonal, $P^{-1} = P^T \Rightarrow B = P^{-1}AP = P^TAP$

Note:1 Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

Note:2 A real symmetric matrix of order n has n mutually orthogonal real eigenvectors.

Note:3 Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

Note:4 To find the orthogonal matrix P, each element ofthe eigenvector is divided by its norm (length).

Working Rule for Diagonalization of Square Matrix A

(i) Find the eigenvalues of the square matrix A.

(ii) Find the eigenvectors corresponding to each eigenvalue.

(iii) Find the modal matrix P having the normalized eigenvectors as its colunm vectors.

(iv) Find the diagonal matrix $D = P^TAP$. The diagonal matrix D has eigenvalues as its

diagonal elements.

Problem1: Show that the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Solution: Let A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda l) = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements $of A = 2 + 2 + 2 = 6$

$S_2 = Sum$ of the minors ofprincipal diagonal elements $of A$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= (4-0) + (4-0) + (4-0) = 12$$

$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix}$

$= 2(4-0) - 1(0-0) + 0 = 8 - 0 + 0 = 8$

Hence, the characteristic equation is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \Rightarrow \lambda = 2,2,2$

For $\lambda = 2$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + y + 0z = 0$$

$$0x + 0y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = t$$

$$\frac{x}{1} = \frac{y}{0} = \frac{Z}{0} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 2$.

Since matrix A has only one linearly independent eigenvector which is less than its order 3, matrix A is not diagonalizable.

Note: algebraic multiplicity = 3 and geometric multiplicity = 1

So, the matrix is not diagonalizable

Problem2: Show that the matrix $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ is not diagonalizable.

Solution: Let A = $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

The characteristic equation is $\det(A-\lambda l)=0$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & 1-\lambda & 2 \\ 2 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements $of A = 1+2+3 = 6$

$S_2 = Sum$ of the minors of principal diagonal elements $of A$

$$= \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1-2 \\ 12 \end{vmatrix}$$

$$= (6-4) + (3-0) + (2+2) = 9$$

$$S_3 = \det(A) = \begin{vmatrix} 1-2 & 0 \\ 12 & 2 \\ 12 & 3 \end{vmatrix}$$

$$= 1(6-4) + 2(3-2) + 0 = 4$$

Hence, the characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \Rightarrow \lambda = 1,1,4$

(a) For $\lambda = 1$, $[A-\lambda I]x = 0$

$$\begin{bmatrix} 0-2 & 0 \\ 1 \quad 1 & 2 \\ 0 \quad 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$0x - 2y + 0z = 0$$

$$x + y + 2z = 0$$

$$x + 2y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}} = \frac{Z}{\begin{vmatrix} 0-2 \\ 11 \end{vmatrix}} = t$$

$$\frac{x}{-4} = \frac{y}{0} = \frac{Z}{2} = t$$

$$\frac{x}{2} = \frac{y}{0} = \frac{Z}{-1} = t$$

$$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = tx_1 \text{ where } x_1 \text{ is an eigenvector corresponding to } \lambda = 1.$$

(b) For $\lambda = 4$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -3 & -2 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-3x - 2y + 0z = 0$$

$$x - 2y + 2z = 0$$

$$x + 2y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 0 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 - 2 \\ 1 - 2 \end{vmatrix}} = t$$

$$\frac{x}{-4} = \frac{y}{6} = \frac{Z}{8} = t$$

$$\frac{x}{-2} = \frac{y}{3} = \frac{z}{4} = t$$

$$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} -2t \\ 3t \\ 4t \end{bmatrix} = t\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} = tx_2 \text{ where } x_2 \text{ is an eigenvector corresponding to } \lambda = 4.$$

Since the matrix A has two linearly independent eigenvectors which is less than its order 3, matrix A is not diagonalizable.

Note: (b) For $\lambda = 4$, algebraic multiplicity $=$ geometric multiplicity $= 1$.

(b) For $\lambda = 1$, $[A - \lambda I]X = O$

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow R_{13}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow R_2 - R_1\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow R_3 - 2R_2\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix $= 2$. And Number of unknowns $= 3$.

Number of linearly independent solutions $= 3 - 2 = 1$.

Hence, geometric multiplicity is 1. Since the eigen value 1 is repeated twice, its algebraic multiplicity is 2.

Thus, algebraic multiplicity $\neq$ geometric multiplicity.

Hence, matrix A is not diagonalizable.

Problem3: Determine a diagonal matrix orthogonally similar to the real symmetric matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. Also find the modal matrix.

Solution: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements $of A = 3 + 5 + 3 = 11$

$S_2 = Sum$ of the minors ofprincipal diagonal elements $of A$

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1) = 36$$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4 = 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0 \Rightarrow \lambda = 2,3,6$$

(a) For $\lambda = 2$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$x - y + z = 0$$

$$-x + 3y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}} = t$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{Z}{2} = t$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{Z}{1} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = 3$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$0x - y + z = 0$$

$$-x + 2y - z = 0$$

$$x - y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}} = t$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{Z}{-1} = t$$

$$\frac{x}{1} = \frac{y}{1} = \frac{Z}{1} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = tx_2$ where x_2 is an eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 6$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - y + z = 0$$

$$x + y - z = 0$$

$$x - y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 - 1 \\ -1 - 1 \end{vmatrix}} = t$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{Z}{2} = t$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{Z}{1} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = tx_3$ where x_3 is an eigenvector corresponding to $\lambda = 6$.

Since matrix A has three linearly independent eigenvectors which is same as its order, the matrix A is diagonalizable.

Length of eigenvector $x_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

Length of eigenvector $x_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of eigenvector $x_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

The normalized eigenvectors are

$$x_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, x_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, x_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$\boldsymbol{p} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \mathbf{0} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \mathbf{0} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Problem4: Determine a diagonal matrix orthogonally similar to the real symmetric matrix

$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Also find the modal matrix.

Solution: Let A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements $of A = 6 + 3 + 3 = 12$

$S_2 = Sum$ of the minors ofprincipal diagonal elements $of A$

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14 = 36$$

$$S_3 = \det(A) = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8 = 32$$

Hence, the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow \lambda = 2{,}2{,}8$

(a) For $\lambda = 8$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + 2z = 0$$

$$-2x - 5y - z = 0$$

$$2x - y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}} = t$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{Z}{6} = t$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{Z}{1} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = 8$.

Since matrix A has three linearly independent eigenvectors which is same as its order, matrix A is diagonalizable.

(b) For $\lambda = 2$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 2y + 2z = 0$$

Let $y = t_1$ and $z = t_2 \Rightarrow x = \frac{1}{2}t_1 - \frac{1}{2}t_2$

$$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 - \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = t_1 x_2 + t_2 x_3$$

where x_2 and x_3 are linearly independent eigenvectors corresponding to $\lambda = 2$.

The orthogonal matrix P has mutually orthogonal eigenvectors. Since x_2 and x_3 are not orthogonal, we must choose x_3 such that x_1, x_2, x_3 are orthogonal.

Let $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

For orthogonality of eigenvectors, $x_1^T x_3 = 0$ and $x_2^T x_3 = 0$

$[2 \; -1 \; 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ and $[\frac{1}{2} \; 1 \; 0] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$2l - m + n = 0$ and $\frac{1}{2}l + m + 0n = 0$

By Cramer's rule,

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 2 & 1 \\ \frac{1}{2} & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}} = t$$

$$\frac{l}{-1} = \frac{m}{\frac{1}{2}} = \frac{n}{\frac{5}{2}} = t$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{5} = t$$

$x = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 5t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = tx_3$ where x_3 is an eigenvector corresponding to $\lambda = 2$.

Length of eigenvector $x_1 = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$

Length of eigenvector $x_2 = \sqrt{(\frac{1}{2})^2 + 1^2 + 0^2} = \sqrt{\frac{5}{2}}$

Length of eigenvector $x_3 = \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{30}$

The normalized eigenvectors are

$$x_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, x_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & \mathbf{0} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \mathbf{0} \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$D = P^T A P = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \mathbf{0} \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & \mathbf{0} & \frac{5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

Problem5: For a symmetric matrix A, the eigenvectors are $[1, 1, 1]^T$, $[1, -2, 1]^T$ corresponding to $\lambda_1 = 6$ and $\lambda_2 = 12$. Find the eigenvector corresponding to $\lambda_3 = 6$ and find the matrix A.

Solution: Let $x_3 = [x_3, y_3, z_3]^T$ be the eigenvector corresponding to $\lambda_3 = 6$.

$x_1 = [1,1,1]^T, x_2 = [1,-2,1]^T$

Since A is real symmetric matrix, x_1, x_2 and x_3 are orffiogonal.

i.e., $x_1^T x_3 = 0$ and $x_2^T x_3 = 0$

$[1, 1, 1]\begin{bmatrix} x_3 \\ y_3 \\ Z_3 \end{bmatrix} = 0$ and $[1, -2, 1]\begin{bmatrix} x_3 \\ y_3 \\ Z_3 \end{bmatrix} = 0$

$$x_3 + y_3 + z_3 = 0$$

$$x_3 + 2y_3 + z_3 = 0$$

By Cramer's rule,

$$\frac{x_3}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y_3}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{Z_3}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = t$$

$$\frac{x_3}{3} = -\frac{y_3}{0} = \frac{Z_3}{-3} = t$$

$$\frac{x_3}{1} = \frac{y_3}{0} = \frac{Z_3}{-1} = t$$

$x = \begin{bmatrix} x_3 \\ y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = tx_3$ where x is an eigenvector corresponding to $\lambda = 8$.

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Length of eigenvector $x_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of eigenvector $x_2 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

Length of eigenvector $x_3 = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

The normalized eigenvectors are

$$x_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, x_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, x_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} = P^T AP$$

Now $A = IAI$

$= PP^T \, A \, PP^T$

$= P(P^T AP)P^T$

$= PDP^T$

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

Type-II: Non-Real Symmetric Matrix:

Problem1: Determine the diagonal matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = Sum$ of the principal diagonal elements of $A = 4 + 3 - 3 = 4$

$S_2 = Sum$ of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-9 + 8) + (-12 + 6) + (12 - 6) = -1 - 6 + 6 = -1$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix}$$

$$= 4(-9 + 8) - 6(-3 + 2) + 6(-4 + 3) = -4 + 6 - 6 = -4$$

Hence, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0 \Rightarrow \lambda = -1,1,4$$

(a) For $\lambda = -1$,

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x + 6y + 6z = 0 \text{ and}$$

$$x + 4y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} = t$$

$$\frac{x}{-12} = \frac{y}{-4} = \frac{Z}{14} = t \Rightarrow \frac{x}{-6} = \frac{y}{-2} = \frac{z}{7} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} -6t \\ -2t \\ 7t \end{bmatrix} = t \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix} = tx_1$ where x_1 is an eigenvector corresponding to $\lambda = -1$.

(b) For $\lambda = 1$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$x + 2y + 2z = 0$$

$$-x - 4y - 4z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{Z}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = t$$

$$\frac{x}{0} = \frac{y}{2} = \frac{Z}{-2} = t$$

$$\frac{x}{0} = \frac{y}{1} = \frac{Z}{-1} = t$$

$x = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = tx_2$ where x_2 is an eigenvector corresponding to $\lambda = 1$.

(c) For $\lambda = 4$, $[A - \lambda I]x = 0$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 6y + 6z = 0$$

$$x - y + 2z = 0$$

$$x - 4y - 7z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}} = t$$

$$\frac{x}{18} = \frac{y}{6} = \frac{Z}{-6} = t$$

$$\frac{x}{3} = \frac{y}{1} = \frac{Z}{-1} = t$$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = tx_3$ where x_3 is an eigenvector corresponding to $\lambda = 4$.

$$p = \begin{bmatrix} -6 & 0 & 3 \\ -2 & 1 & 1 \\ 7 & -1 & -1 \end{bmatrix}$$

$$p^{-1} = -\frac{1}{15}\begin{bmatrix} 0 & -3 & -3 \\ 5 & -15 & 0 \\ -5 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

$$p^{-1}AP = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} -6 & 0 & 3 \\ -2 & 1 & 1 \\ 7 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{D}$$

QUADRATIC FORM

A homogeneous polynomial of second degree in n variables is called a quadratic form. An expression of the form$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = 1$ where $a_{ij} = a_{ji}$ are all real, is called a quadratic form in n variables $x_1, x_2 \dots\dots x_n$.

Matrix of a Quadratic Form: The quadratic form corresponding to a symmetric matrix A can be written as $Q = X^T AX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = 1$…………(1)

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{n1}\cdots & a_{n2}\cdots & a_{nn} \end{bmatrix}$

Coefficient of $x_i x_j$ in Eq. (1) $= a_{ij} + a_{ji} = 2a_{ij} = 2a_{ji}$

Coefficient of x_i^2 in Eq. (1) $= a_{ii}$

Linear Transformation: Let $Q = X^T AX$ be a quadratic form and $X = PY$ be a non - singular linear transformation.

$$X = PY$$

$$X^T = (PY)^T = Y^T P^T$$

$$Q = X^T AX = Y^T PAPY$$

$$= Y^T BY \qquad\qquad \text{where } B = P^T AP$$

The form $Y^T BY$ is called linear transformation of the quadratic form $X^T AX$ under a non - singular transformation $X = PY$ and P is called the matrix of the transformation.

Further, $B^T = (P^T AP)^T = P^T A^T P = P^T AP = B$ [$since\ A$ is symmetric]

Hence, matrix B is also symmetric.

Rank of Quadratic Form: The rank of the coefficient matrix A is called the rank ofthe quadratic form $X^T AX$. The number of non - zero eigen values $of A$ also gives the rank of the quadratic form of A.

If $\rho(A) < n$ (order $of A$), i.e., $|A| = 0$, then the quadratic form is singular, otherwise it is non - singular.

Canonical or Normal Form: Let $Q = X^\Gamma AX$ be a quadratic form of rank r. An orthogonal transformation $X = PY$ which diagonalize A, i.e., $P^T AP = D$, transforms the quadratic form Q to $\sum_{i=1}^{r} \lambda_i y_i^2 i = 1$ (i.e.,sum of r squares) or in matrix form $Y^T DY$ in new variables. This new quadratic form containing only the squares of y_i is called the canonical form or sum of squares form of the given quadratic form.

Index: The number of positive terms in the canonical form is called the index of the quadratic form and is denoted by P.

Signature: The difference between the number of positive and negative terms in the canonical form is called the signature of the quadratic form. If index is P and total terms are r, then signature $= P - (r - P) = 2p - r$.

The signature of a quadratic form is invariant for all normal reductions.

Criteria for the Value Class of a Quadratic form in Terms of the Nature of Eigen Values

Value Class	**Nature of Eigen Values**
1. Positive definite	positive eigen values
2. Positive semidefinite	positive eigen values and at least one is zero
3. Negative definite	negative eigen values
4. Negative semidefinite	negative eigen values and at least one is zero
5. Indefinite	positive as well as negative eigen values

Methods to Reduce Quadratic Form to Canonical Form:

Orthogonal Transformation

If $Q = X^T AX$ is a quadratic form, then there exists a real orthogonal transformation

$X = PY$ (Where P is an orthogonal matrix) which transforms the given quadratic form

$X^T AX$ to

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \dots \dots \dots \dots \dots + \lambda_r y_r^2$$

where $\lambda_1, \lambda_2 \dots \lambda_r$ are the r non - zero eigen values of matrix A

Problem1: Express the following quadratic forms in matrix notation:

(i) $x^2 - 6xy + y^2$

(ii) $2x^2 + 3y^2 - 5z^2 - 2xy + 6xz - 10yz$

(iii) $x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 + 8x_3x_4$

Solution: $(i)\ X^T AX = [x\ y]\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $X^T AX = [x\ y\ z]\begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -5 \\ 3 & -5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ Z \end{bmatrix}$

(iii) $X^T AX = [x_1\ x_2\ x_3\ x_4]\begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & 3 & 4 \\ -1 & -3 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Problem2: Write down the quadratic forms corresponding to the following matrices:

(i) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

Solution: (i) $Q = 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 + 10x_1x_3 - 4x_2x_3$

(ii) $Q = 2x_2^2 + 4x_3^2 + 6x_4^2 + 2x_1x_2 + 4x_1x_3 + 6x_1x_4 + 6x_{23} + 8x_2x_4 + 10x_3x_4$

Problem3: Determine the nature (value class), index and signature of the following quadratic forms:

(i) $x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

(ii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

(iii) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$

(iv) $-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

Solution: (i) $Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

$$Q = X^TAX = [x_1\ x_2x_3]\begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Let $A = \begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix}1-\lambda & 1 & 3\\1 & 5-\lambda & 1\\3 & 1 & 1-\lambda\end{vmatrix} = 0$$

$\lambda^3 - 7\lambda^2 + 36 = 0 \Rightarrow \lambda = -2,3,6$

Since there are positive as well as negative eigen values, value class of quadratic form is indefinite.

Index $=$ Number ofpositive eigen values $= 2$

Signature $=$ Difference between the number of positive and negative eigen values $= 2 - 1 = 1$

(ii) $Q = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

$$Q = X^TAX = [x_1\ x_2\ x_3]\begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Let $A = \begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}$

The characteristic $e$quation is $|A - \lambda I| = 0$

$$\begin{vmatrix}6-\lambda & -2 & 2\\-2 & 3-\lambda & -1\\2 & -1 & 3-\lambda\end{vmatrix} = 0$$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow \lambda = 8,2,2$

Since all the eigen values $of A$ are positive, value class of quadratic form is positive definite.

Index=Number ofpositive eigen values $= 3$

Signature=Difference between the number of positive and negative eigen values $= 3 - 0 = 3$

(iii) $Q = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$

$$Q = X^{\Gamma}AX = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda l| = 0$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 6\lambda^2 = 0 \Rightarrow \lambda = 0,0,6$

Since the eigen values $of A$ are positive and two eigen values are zero, value class of quadratic form is positive semidefinite.

Index $=$ Number ofpositive eigen values $= 1$.

Signature$=$Difference between the number of positive and negative eigen values $= 1 - 0 = 1$

(iv) $Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

$$Q = X^{\Gamma}AX = [x_1\ x_2\ x_3]\begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

The characteristic equation is $|A - \lambda l| = 0$

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = 0$$

$\lambda^3 + 9\lambda^2 + 24\lambda + 16 = 0 \Rightarrow \lambda = -1, -4, -4$

Since all the eigen values of A are negative, the quadratic form is negative definite.

Index $=$ Number ofpositive eigen values $= 0$

Signature $=$ Difference between the number of positive and negative eigen values $= 0 - 3 = -3$

Problem4: Reduce the following quadratic forms to canonical forms by orthogonal transformation. Also find the rank, index, signature and value class (nature) of the quadratic forms.

(i) $Q = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3$

(ii) $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$

(iii) $Q = 2x^2 + 2y^2 - z^2 - 4yz + 4xz - 8xy$.

Solution: (i) $Q = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3$

$$Q = X^TAX = [x_1\ x_2\ x_3]\begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$$

The characteristic equation is $|A - \lambda l| = 0$

$$\begin{vmatrix}2-\lambda & 0 & 1\\0 & 2-\lambda & 0\\1 & 0 & 2-\lambda\end{vmatrix} = 0$$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow \lambda = 1,2,3$

(a) For $\lambda = 1$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Taking $x_1 = 1, x_3 = -1$

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

$$X_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

(b) For $\lambda = 2$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$x_3 = 0$$

$$x_1 = 0$$

Taking $x_2 = 1$

$$X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

(c) For $\lambda = 3$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix}-1 & 0 & 1\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Taking $x_1 = 1, x_3 = 1$

$$-x_1 + x_3 = 0$$

$$x_2 = 0$$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Length of vector $X_1 = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

Length of vector $X_2 = \sqrt{0^2 + 1^2 + 0^2} = 1$

Length of vector $X_3 = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

Normalized eigen vectors are,

$\overline{X_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \overline{X_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

P is orthogonal matrix, i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to canonical form.

$$Q = Y^T(P^TAP)Y = Y^TDY$$

$$= [y_1 y_2 y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 3y_3^2$$

Rank $r = Number$ of non - zero terms in canonical form $= 3$

Index $P =$Number of positive terms in canonical form $= 3$

Signature=Difference between the number ofpositive and negative terms in canonical form $= 3 - 0 = 3$

Since only positive terms occur in the canonical form, value class of quadratic

form is positive definite.

(ii) $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$

$$Q = X^T AX = [x_1\ x_2\ x_3]\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda l| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0 \Rightarrow \lambda = 2,3,6$

(a) For $\lambda = 2$, $[A - \lambda_1 1]X = O$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 - 1 \\ -1 \quad 3 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) For $\lambda = 3$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) For $\lambda = 6$, $[A - \lambda_3 I]X = O$

$$\begin{bmatrix} -3 - 1 & 1 \\ -1 - 1 - 1 \\ 1 - 1 - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 - 1 \\ -1 - 1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Length of vector $X_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

Length of vector $X_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of vector $X_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

The normalized eigen vectors are

$$\overline{X_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}}, \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \overline{X_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \overline{X_3} = \begin{bmatrix} \frac{1}{\sqrt{6}}, \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}, & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

P is orthogonal matrix i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to canonical form.

$$Q = Y^T(P^TAP)Y = Y^TDY$$

$$= [\gamma_1 y_2 y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Rank r =Number of non - zero terms in canonical form $= 3$

Index P =Number of positive terms in canonical form $= 3$

Signature=Difference between the number ofpositive and negative terms in canonical form $= 3 - 0 = 3$

Since only positive terms occur in the canonical form, value class of quadratic form is positive definite.

(iii) $Q = 2x^2 + 2y^2 - z^2 - 8xy + 4xz - 4yz$

$$= X^\Gamma AX = [x\ y\ z] \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda l| = 0$$

$$\begin{vmatrix} 2-\lambda & -4 & 2 \\ -4 & 2-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 - 24\lambda - 28 = 0$$

$$\lambda = -2, -2, 7$$

(a) For $\lambda = -2$, $[A - \lambda_1 I]X = O$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1,\ R_3 - \frac{1}{2}R_1$

$$\begin{bmatrix} 4-4 & 2 \\ 0 \quad 0 & 0 \\ 0 \quad 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 4y - 2z = 0$$

Rank of coefficient matrix $= 1$

Number of unknowns $= 3$

Number of linearly independent solutions $= 3 - 1 = 2$

Taking $z = 2$ and $y = 1, x = 2$

$z = -2$ and $y = 2, x = 1$

$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

(b) For $\lambda = 7$, $[A - \lambda_2 I]X = O$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x - 4y + 2z = 0$$

$$-4x - 5y - 2z = 0$$

$$2x - 2y - 8z = 0$$

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ -5 & -2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 2-5 \\ -4-2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5-4 \\ -4-5 \end{vmatrix}}$$

$$\frac{x}{18} = -\frac{y}{18} = \frac{z}{9}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{Z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Length of vector $X_1 = \sqrt{2^2 + 1^2 + 2^2} = 3$

Length of vector $X_2 = \sqrt{1^2 + 2^2 + (-2)^2} = 3$

Length of vector $X_3 = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

The normalized eigen vectors are

$$\overline{X_1} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \overline{X_2} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix}, \overline{X_3} = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

P is orthogonal matrix, i.e., $P^{-1} = P^T$

Let $X = PY$ be the orthogonal transformation which transforms Q to cannonical form.

$$Q = Y^{\Gamma}(P^T AP)Y = Y^T DY$$

$$= [\gamma_1 y_2 y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2y_1^2 - 2y_2^2 + 7y_3^2$$

Rank r =Number of non - zero terms in canonical form = 3

Index P =Number of positive terms in canonical form $= 1$

Signature=Difference between the number ofpositive and negative terms in canonical form $= 1 - 2 = -1$

Since both positive and negative terms occur in canonical form, value class of quadratic form is indefinite.