Differential Equations

INTRODUCTION: Differential equations are very important in engineering mathematics.

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders.

It provides the medium for the interaction between mathematics and various branches

of science and engineering. Most common differential equations are radioactive decay,

chemical reactions, Newtons law of cooling, series RL, RC and RLC circuits, simple

harmonic motions, etc.

DIFFERENTIAL EQUATION: A differential equation is an equation which involves variables (dependent and independent) and their derivatives, $e.g.$,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cdots (1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - [(\frac{dy}{dx})^2 + 1]^3 = 0 \cdots (2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \cdots (3)$$

Equations (1) and (2) involve ordinary derivatives and hence called "ordinary

differential equations" whereas Eq. (3) involves partial derivatives and hence called

'partial differential equation'

Order: Order of a differential equation is the order of the highest derivative present in the

equation, e.g., the order of Eqs. (1) and (2) is 2.

Degree: Degree of a differential equation is the power of the highest order derivative after

clearing the radical sign and fraction, e.g., the degree of Eq (1) is 1 and the degree of Eq (2) is 2.

Solution or Primitive: Solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

Solution of a differential equation is not always unique. It may have more than one

solution or sometimes no solution.

General solution of a differential equation of order n contains n arbitrary constants.

Particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

Note

(1) Solve a differential equation means finding the general solution

(2) The general solution does not mean that it includes all possible solutions of the differential equation

There may exist other solutions which cannot be deduced from the general solution (or not included in the general solution). Such solutions not containing arbitrary constants, are called Singular solutions.

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first order and first-degree derivative of y (dependent variable) and known functions of x (independent variable) and y is known as ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F(x, y, \frac{dy}{dx}) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable

(ii) Homogeneous differential equations

(iii) Non homogeneous differential equations

(iv) Exact differential equations

(v) Non - exact differential equations reducible to exact form

(vi) Linear differential equations

(vii) Non - linear differential equations reducible to linear form

Variable Separable

A differential equation of the form $M(x)dx + N(y)dy = 0 \cdots$ (1)

where $M(x)$ is the function of x only and $N(y)$ is the function of y only, is called a differential equation with variables separable as in Eq. (1) function of x and function of y can be separated easily.

Integrating Eq. (1) we get the solution as

$$\int M\,(x)dx + \int N\,(y)dy = c$$

or

$$\int g\,(y)dy = \int f\,(x)dx + c$$

where c is the arbitrary constant.

Problem1: Solve $y(1+x^2)^{\frac{1}{2}}dy + x\sqrt{1+y^2}dx = 0.$

Solution: $y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}dx$

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx + c$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx + c$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c\ [\,Since \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Problem2: Solve $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0.$

Solution: $\sec^2 x \tan y dx = -\sec^2 y \tan x dy$

$$\int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy + c$$

$$\log\ \tan x = -\log\ \tan y + c \qquad [\,Since \int \frac{f'(x)}{f(x)}dx = \log f(x)]$$

$$\log\ \tan x + \log\ \tan y = c$$

$$\log\ (\text{tanx}\ \tan y) = c$$

$$\tan x \tan y = e^c = k$$

$$\tan x \tan y = k$$

Homogeneous Differential Equation A differential equation of the form

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \cdots (1)$$

is called a homogeneous equation $if M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, i.e., degree of the R.H.S. of Eq. (1) is zero.

Equation (1) can be reduced to variable separable form by putting $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (1) reduces to

$$v + x\frac{dv}{dx} = \frac{M(x, vx)}{N(x, vx)} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Above equation is in variable separable form and can be solved by integrating

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (1).

Note: Homogeneous functions: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z)\ ,$$

where n is a real number.

Problem1: Solve $x(x-y)dy + y^2dx = 0$.

Solution: $\frac{dy}{dx} = \frac{-y^2}{x^2-xy} = \frac{M(x,y)}{N(x,y)}$

The equation is homogeneous since M and N are ofthe same degree 2.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2(1-v)} = \frac{-v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1-v} - v = \frac{-v}{1-v}$$

$$(\frac{v-1}{v})dv = \frac{dx}{x}$$

$$(1-\frac{1}{v})dv = \frac{dx}{x}$$

Integrating both the sides,

$$\int(1-\frac{1}{v})dv = \int\frac{dx}{x}$$

$$v - \log v = \log x + \log c$$

$$v = \log v + \log cx = \log cxv$$

$$\frac{y}{x} = \log cy$$

$$y = x\log cy$$

Non-Homogeneous Differential Equations: A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \cdots (1)$$

is called non - homogeneous equation where $a_1, b_1, c_1, a_2, b_2, c_2$ are all constants. These equations are classified into two parts and can be solved by following methods:

Case I: If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$

$$a_1 = a_2 m, b_1 = b_2 m,$$

then Eq. (1) reduces to

$$\frac{dy}{dx} = \frac{m(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2} \cdots (2)$$

Putting $a_2x + b_2y = t$, $a_2 + b_2\frac{dy}{dx} = \frac{dt}{dx}$, Eq. (2) reduces to variable - separable form and

can be solved using the method of variable - separable equation.

Case II: If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substituting

$x = X + h, y = Y + k$ in the Eq. (1),

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{(a_1X+b_1Y)+(a_1h+b_1k+c_1)}{(a_2X+b_2Y)+(a_2h+b_2k+c_2)} \cdots (3)$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0, a_2h + b_2k + c_2 = 0,$$

then Eq. (3) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally substituting $X = x - h, Y = y - k$, we get the solution of Eq. (1).

Problems based on Case I: $\mathbf{\frac{a_1}{b_2} = \frac{b_1}{b_2}}$

Problem1: Solve $(x + y - 1)dx + (2x + 2y - 3)dy = 0.$

Solution: $\frac{dy}{dx} = -\frac{x+y-1}{2x+2y-3} = \frac{-x-y+1}{2x+2y-3} \cdots (1)$

The equation is non - homogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = -\frac{1}{2}$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}, \frac{dy}{dx} = \frac{dt}{dx} - 1$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 1 = \frac{-t+1}{2t-3}$$

$$\frac{dt}{dx} = \frac{-t+1}{2t-3} + 1 = \frac{-t+1+2t-3}{2t-3} = \frac{t-2}{2t-3}$$

$$(\frac{2t-3}{t-2})dt = dx$$

$$(2 + \frac{1}{t-2})dt = dx$$

Integrating both the sides,

$$\int (2 + \frac{1}{t-2})dt = \int d\,x$$

$$2t + \log(t-2) = x + c$$

$$2(x+y) + \log(x+y-2) = x + c$$

$$x + 2y + \log(x+y-2) = c$$

Problem Based on Case II: $\frac{\mathbf{a_1}}{\mathbf{a_2}} \neq \frac{\mathbf{b_1}}{\mathbf{b_2}}$

Problem 1: Solve $(x+2y)dx + (y-1)dy = 0$.

Solution: $\frac{dy}{dx} = \frac{-x-2y}{y-1}$......(1)

The equation is non - homogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X + h, y = Y + k$

$dx = dX, dy = dY$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Substituting in Eq. (1),

$$\frac{dY}{dX} = \frac{-(X+h) - 2(Y+k)}{(Y+k) - 1} = \frac{(-X-2Y) + (-h-2k)}{Y + (k-1)} \ldots\ldots (2)$$

Choosing h, k such that

$$-h - 2k = 0, k - 1 = 0 \ldots\ldots (3)$$

Solving these equations,

$$k = 1, h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{dY}{dX} = \frac{-X - 2Y}{Y} \dots \dots \dots \dots (4)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v + X\frac{dv}{dX} = \frac{-X - 2vX}{vX} = \frac{-1 - 2v}{v}$$

$$X\frac{dv}{dX} = \frac{-1 - 2v}{v} - v = \frac{-1 - 2v - v^2}{v} = \frac{-(v + 1)^2}{v}$$

$$\frac{v}{(v + 1)^2}dv = -\frac{dX}{X}$$

$$[\frac{1}{v + 1} - \frac{1}{(v + 1)^2}]dv = -\frac{dX}{X}$$

Integrating both the sides,

$$\int \frac{1}{v + 1}dv - \int \frac{1}{(v + 1)^2}dv = -\int \frac{dX}{X}$$

$$\log{(v + 1)} + \frac{1}{v + 1} = -\log X + c$$

$$\log{(\frac{Y}{X} + 1)} + \frac{1}{\frac{Y}{X} + 1} = -\log X + c$$

$$\log{(\frac{Y + X}{X})} + \frac{X}{Y + X} = -\log X + c$$

$$\log{(Y + X)} - \log X + \frac{X}{Y + X} = -\log X + c$$

$$\log{(Y + X)} + \frac{X}{Y + X} = c$$

Now,

Hence, solution is

$$X = x - h = x + 2$$

$$Y = y - k = y - 1$$

$$\log(x + y + 1) + (\frac{x+2}{x+y+1}) = c$$

Exact Differential Equation:

Any first order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If f$(x, y) = c$ is the general solution,

then

$$df = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$M(x, y)dx + N(x, y)dy = 0 \cdots (1)$$

represents an exact differential equation

where $M(x, y) = \frac{\partial f}{\partial x}, N(x, y) = \frac{\partial f}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of Eq. (1) can be written as

$\int_{y\ constant} M\,(x,y)dx + \int($ terms of N not containing $x)\ dy = c$

Note: Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case solution can be written as

$\int$(terms of M not containing y) $dx + \int_{x\ constant} N\,(x,y)dy = c$

Problem 1: Solve $(y^2 - x^2)dx + 2\phi dy = 0.$

Solution: $M = y^2 - x^2, N = 2xy$

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$\int Mdx \qquad + \quad \int Ndy = c$

y constant terms not containing x

$$\int (y^2 - x^2)dx + \int 0\,dy = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Problem 2: Solve $(x\sqrt{1 - x^2y^2} - y)dy + (x + y\sqrt{1 - x^2y^2})dx = 0.$

Solution: $N = x\sqrt{1 - x^2y^2} - y,$ $M = x + y\sqrt{1 - x^2y^2}$

$\frac{\partial N}{\partial x} = \sqrt{1 - x^2y^2} + x\left[\frac{-2xy^2}{2\sqrt{1-x^2y^2}}\right]$ $\frac{\partial M}{\partial y} = \sqrt{1 - x^2y^2} + y[\frac{-2x^2y}{2\sqrt{1-x^2y^2}}]$

$= \sqrt{1 - x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}}$ $= \sqrt{1 - x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$\int M dx + \int \text{ Ndy} = c$

y constant terms not containing x

$$\int (X + y\sqrt{1 - x^2y^2})dx + \int (-y)dy = C$$

$$\frac{x^2}{2} + y^2 \int (\sqrt{\frac{1}{y^2} - x^2})dx - \frac{y^2}{2} = c$$

$$\frac{x^2}{2} + y^2[|\frac{x}{2}\sqrt{\frac{1}{y^2} - x^2}| + \frac{1}{2y^2}\sin^{-1}(\frac{x}{\frac{1}{y}})] - \frac{y^2}{2} = c$$

$$\frac{x^2 - y^2}{2} + \frac{xy}{2}\sqrt{1 - x^2y^2} + \frac{1}{2}\sin^{-1}(\text{xy}) = c$$

$$x^2 - y^2 + \text{xy}\sqrt{1 - x^2y^2} + \sin^{-1}(xy) = 2c = k$$

Problem 3: Solve $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0.$

Solution: $M = 2xy\cos x^2 - 2xy + 1,\qquad N = \sin x^2 - x^2$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x,\qquad \frac{\partial N}{\partial x} = (\cos x^2)(2x) - 2x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is $\int M\, dx \quad + \quad \int \text{ Ndy } = c$

y constant terms not containing x

$$\int (2xy\cos x^2 - 2xy + 1)dx + \int 0\, dy = c$$

$$y\sin x^2 - x^2y + x = c \quad [\,since\ \int\{\cos f(x)\}f'(x)dx = \sin f(x)]$$

Problem 4: Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Solution: $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

$M = y\cos x + \sin y + y,\qquad N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1,\qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is $\int M\,dx + \int \text{Ndy} = c$

y constant terms not containing x

$$\int (y\cos x + \sin y + y)dx + \int 0\,dy = c$$

$$y\sin x + x(\sin y + y) = c$$

Problem 5: Solve $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy = 0,\, y(0) = 4.$

Solution: $M = 1 + e^{\frac{x}{y}},$ $N = e^{\frac{x}{y}}(1 - \frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}(-\frac{x}{y^2}), \qquad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}(\frac{1}{y})(1 - \frac{x}{y}) + e^{\frac{x}{y}}(-\frac{1}{y})$$

$$= -\frac{x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int M\,dx + \int N\,dy = C$$

y constant terms not containing x

$$\int (1 + e^{\frac{x}{y}})\,dx + \int 0\,dy = c$$

$$x + \frac{e}{\frac{1}{y}} = c$$

$$x + ye^{\frac{x}{y}} = c \ldots\ldots\ldots..(1)$$

Given $y(0) = 4$

Substituting in Eq. (1),

$$0 + 4e^0 = c$$

$$4 = c$$

Hence, solution is

$$x + ye^{\frac{x}{y}} = 4$$

Problem 6: Solve $[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}]dx+\frac{2xy}{x^2+y^2}\,dy=0.$

Solution: $M=\left[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right]$ $\quad N=\frac{2xy}{x^2+y^2}$

$$\frac{\partial M}{\partial y}=\frac{1}{x^2+y^2}\cdot 2y-\frac{2x^2}{(x^2+y^2)^2}\cdot 2y=\frac{2y}{x^2+y^2}-\frac{4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x}=\frac{2y}{x^2+y^2}-\frac{2xy}{(x^2+y^2)^2}\cdot 2x=\frac{2y}{x^2+y^2}-\frac{4x^2y}{(x^2+y^2)^2}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is $\int M\,dx \quad + \quad \int N\,dy=C$

terms not containing y $\quad$ x constant

$$\int 0\,dx+\int\frac{2xy}{x^2+y^2}\,dy=c$$

$$\frac{1}{2}x\log(x^2+y^2)=c$$

INTEGRATING FACTORS

Sometimes the equation $Mdx+Ndy=0$ may not be exact, but it can be made exact by multiplying it by a suitable function $\mu(x,y)$. Such a function is called an integrating factor (I.F).

So, $\mu(Mdx+Ndy)=0$ is an exact differential equation.

Though there are standard techniques of finding the integrating factors of $Mdx+Ndy=0$, in some cases it is possible to find an I.F by inspection, after grouping the terms suitably.

The following list of exact differentials will be useful to recognize an integrating factor.

(1) $xdy+ydx=d(xy)$ $\quad$ **(2)** $\frac{ydx-xdy}{y^2}=d(\frac{x}{y})$

This means that $ydx-xdy=0$ becomes exact by multiplying by $\frac{1}{y^2}$.

So, $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

Similarly,

(3) $\frac{xdy-ydx}{x^2} = d(\frac{y}{x})$ **(4)** $\frac{xdy+ydx}{xy} = d(\log_e xy)$

[Integrating factor of $xdy + ydx = 0$ is $\frac{1}{xy}$]

(5) $xdx + ydy = \frac{1}{2}d(x^2 + y^2)$

(6) $\frac{ydx-xdy}{x^2+y^2} = d(\tan^{-1}\frac{x}{y})$

(7) $\frac{ydx-xdy}{xy} = d(\log_e \frac{x}{y})$

(8) $\frac{xdy-ydx}{x^2-y^2} = \frac{1}{2}d[\log_e \frac{x+y}{x-y}]$

(9) $\frac{xdy+ydx}{x^2y^2} = d(-\frac{1}{xy})$

Note From this list, it is seen that the simple differential equation $ydx - xdy = 0$ has $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{x^2+y^2}, \frac{1}{xy}$ as integrating factors and so the equation can be solved in different ways.

Hence, I.F is not unique. If $\mu(x, y)$ is an I.F, then $k\mu(x, y)$ is also an I.F for any non - zero constant k.

Problem1: Solve $ydx - xdy + 3x^2y^2e^{x^3} = 0$.

Solution: The given equation is $ydx - xdy + 3x^2y^2e^{x^3} = 0$

Dividing by y^2,

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3} = 0 \Rightarrow d(\frac{x}{y}) + d(e^{x^3}) = 0$$

Integrating, we get

$$\int d\,(\frac{x}{y}) + \int d\,(e^{x^3}) = 0 \Rightarrow \frac{x}{y} + e^{x^3} = C \Rightarrow x + ye^{x^3} = Cy$$

which is the general solution

Problem2: Solve $xdx + ydy = \frac{a^2(xdy-ydx)}{x^2+y^2}$.

Solution: The given equation is

$$xdx + ydy = \frac{a^2(xdx - ydx)}{x^2 + y^2} \Rightarrow xdx + ydy = a^2 d[\tan^{-1}(\frac{x}{y})]$$

Integrating, we get

$$\int x\,dx + \int y\,dy = a^2 \int d\,(\tan^{-1}\frac{x}{y})$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = a^2\tan^{-1}(\frac{x}{y}) + C \Rightarrow x^2 + y^2 = 2a^2\tan^{-1}(\frac{x}{y}) + 2C$$

the general solution is

$x^2 + y^2 = 2a^2\tan^{-1}(\frac{x}{y}) + C'$, where $C' = 2C$

Problem3: Solve $x\cos(\frac{y}{x})[ydx + xdy] = y\sin(\frac{y}{x})[xdy - ydx]$.

Solution: The given equation is

$$x\cos(\frac{y}{x})[ydx + xdy] = y\sin(\frac{y}{x})[xdy - ydx]$$

Dividing by x^2y, we get

$$\frac{x\cos(\frac{y}{x})}{x^2y}d(xy) = \sin(\frac{y}{x})[\frac{xdy - ydx}{x^2}]$$

$$\Rightarrow \cos(\frac{y}{x})\frac{d(xy)}{xy} = \sin(\frac{y}{x})d(\frac{y}{x})$$

$$\Rightarrow \frac{d(xy)}{xy} = \frac{\sin\frac{y}{x}}{\cos\frac{y}{x}}d(\frac{y}{x})$$

Integrating, we get

$$\int \frac{d(xy)}{xy} = \int \frac{\sin\frac{y}{x}}{\cos\frac{y}{x}}d(\frac{y}{x})$$

$$\Rightarrow \log_e xy = -\log_e\cos(\frac{y}{x}) + \log_e C$$

$$\Rightarrow \log_e xy + \log_e\cos\frac{y}{x} = \log_e C \Rightarrow \log_e xycos(\frac{y}{x}) = \log_e C \Rightarrow xy\cos(\frac{y}{x}) = C$$

which is the general solution of the given equation.

Problem4: Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2ydy = 0.$

Solution: The given equation is

$$(xy^2 - e^{\frac{1}{x^3}})dx - x^2ydy = 0$$

$$\Rightarrow xy^2dx - x^2ydy - e^{\frac{1}{x^3}}dx = 0$$

$$\Rightarrow xy[ydx - xdy] - e^{\frac{1}{x^3}}dx = 0$$

$$\Rightarrow x^3y\frac{ydx - xdy}{x^2} - e^{\frac{1}{x^3}}dx = 0$$

$$\Rightarrow -x^3y[\frac{xdy - ydx}{x^2}] - e^{\frac{1}{x^3}}dx = 0$$

$$\Rightarrow x^3yd(\frac{y}{x}) + e^{\frac{1}{x^3}}dx = 0$$

$\Rightarrow x^4(\frac{y}{x})d(\frac{y}{x}) + e^{\frac{1}{x^3}}dx = 0 \;\Rightarrow\; (\frac{y}{x})d(\frac{y}{x}) + \frac{1}{x^4}e^{\frac{1}{x^3}}dx = 0$ [dividing by x^4]

Integrating, we get

$$\int \frac{y}{x}d(\frac{y}{x}) + \int e^{\frac{1}{x^3}} \cdot \frac{1}{x^4}dx = 0 \;\Rightarrow\; \frac{(\frac{y}{x})^2}{2} + \int e^{\frac{1}{x^3}}\frac{1}{x^4}dx = 0$$

Let $I = \int e^{\frac{1}{x^3}}\frac{1}{x^4}dx$

Put $t = \frac{1}{x^3}\, dt = -\frac{3}{x^4}dx \Rightarrow -\frac{1}{3}dt = \frac{1}{x^4}dx$

$$I = -\frac{1}{3}\int e^t\, dt = -\frac{1}{3}e^t = -\frac{1}{3}e^{\frac{1}{x^3}}$$

$$\frac{1}{2}(\frac{y}{x})^2 - \frac{1}{3}e^{\frac{1}{x^3}} = c \;\Rightarrow\; 3y^2 - 2x^2e^{\frac{1}{x^3}} = 6cx^2$$

which is the general solution of the given equation.

Rules for Finding the Integrating Factor for Non - Exact Differential Equation

$$\boldsymbol{Mdx + Ndy = 0}$$

Rule 1. If the equation $Mdx + Ndy = 0$ is homogeneous,

that is M and N are homogeneous functions in x and y of the same degree, then $\frac{1}{Mx+Ny}$ is an integrating factor if $Mx + Ny \neq 0$

Rule 2. If $Mdx + Ndy = 0$ is of the form $f_1(x, y)ydx + f_2(x, y)xdy = 0$,

that is $M = f_1(x, y)y$, $N = f_2(x, y)x$, then $\frac{1}{Mx-Ny}$ is an I.F if $Mx - Ny \neq 0$

Rule 3. If $Mdx + Ndy = 0$ is such that $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is a function of x, say $f(x)$, then $e^{\int f(x)dx}$ is an I.F

Rule 4. If $Mdx + Ndy = 0$ is such that $\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is a function ofy, say $g(y)$, then $e^{\int g(y)dy}$ is an I.F.

Problems using Rule 1

Problem1: Show that the equation $(2x - y)dy + (2y + x)dx = 0$ can be made exact by the integrating factor $\frac{1}{x^2+y^2}$ and hence, solve the equation.

Solution: Given $(2x - y)dx + (2y + x)dy = 0$ … … …. (1)

It is of the form $Mdx + Ndy = 0$.

Here $M = 2x - y, N = 2y + x$, which are homogeneous functions ofthe same degree 1.

Now $Mx + Ny = (2x - y)x + (2y + x)y = 2x^2 + 2y^2 - xy + xy = 2(x^2 + y^2) \neq 0$

$\therefore \frac{1}{Mx+Ny} = \frac{1}{2(x^2+y^2)}$ is an integrating factor.

Hence, $\frac{1}{(x^2+y^2)}$ is an integrating factor omitting the constant factor 2, by Rule 1.

Multiplying (1) by $\frac{1}{(x^2+y^2)}$, it will become exact.

$\therefore \frac{2x-y}{x^2+y^2}dx + \frac{2y+x}{x^2+y^2}dy = 0$ is exact.

Here $M_1 = \frac{2x-y}{x^2+y^2}$ and $N_1 = \frac{2y+x}{x^2+y^2}$

To find the solution, integrate M $w.r.$ to x, treating y as constant.

$$F = \int M\,dx = \int \frac{2x - y}{x^2 + y^2}dx = \int \frac{2x}{x^2 + y^2}dx - y\int \frac{dx}{x^2 + y^2}$$

$$= \log_e(x^2 + y^2) - y \cdot \frac{1}{y}\tan^{-1}(\frac{x}{y}) = \log_e(x^2 + y^2) - \tan^{-1}(\frac{x}{y})$$

In $N = \frac{2y+x}{x^2+y^2}$, there is no term without x and there is no constant term. $\therefore G = 0$

$\therefore$ the general solution is $F + G = C \Rightarrow \log_e(x^2 + y^2) - \tan^{-1}(\frac{x}{y}) = C$.

Problem2: Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$.

Solution: The given equation is

$$(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0 \dots\dots\dots \quad (1)$$

It is of the form $Mdx + Ndy = 0$, where $M = y^3 - 2yx^2$ and $N = 2xy^2 - x^3$ are homogenous functions of degree 3.

Now $Mx + Ny = (y^3 - 2yx^2)x + (2xy^2 - x^3)y$

$$= xy^3 - 2x^3y + 2xy^3 - x^3y = 3xy^3 - 3x^3y = 3xy(y^2 - x^2) \neq 0 \; (if y \cdot x \neq 0)$$

$\frac{1}{Mx+Ny} = \frac{1}{3xy(y^2-x^2)}$ is an integrating factor by Rule 1.

Hence, $\frac{1}{xy(y^2-x^2)}$ is an integrating factor.

Multiplying (1) by $\frac{1}{xy(y^2-x^2)}$ it will be exact.

$\frac{(y^3-2yx^2)}{xy(y^2-x^2)}dx + \frac{(2xy^2-x^3)}{xy(y^2-x^2)}dy = 0$ is exact.

$$\Rightarrow \frac{y(y^2-2x^2)}{xy(y^2-x^2)}dx + \frac{x(2y^2-x^2)}{xy(y^2-x^2)}dy = 0 \;\Rightarrow\; \frac{y^2-2x^2}{x(y^2-x^2)}dx + \frac{2y^2-x^2}{y(y^2-x^2)}dy = 0$$

For this exact equation $M_1 = \frac{y^2-2x^2}{x(y^2-x^2)}$ and $N_1 = \frac{2y^2-x^2}{y(y^2-x^2)}$

To find the solution, integrate M w.r.to x, treating y as constant.

$$F = \int M\,dx = \int \frac{(y^2-2x^2)}{x(y^2-x^2)}dx = \int \frac{(y^2-x^2)-x^2}{x(y^2-x^2)}dx$$

$$= \int \left(\frac{1}{x} - \frac{x}{y^2-x^2}\right)dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{y^2-x^2}dx$$

$$= \int \frac{1}{x}dx + \frac{1}{2}\int \frac{-2x}{y^2-x^2}dx$$

$$= \log_e x + \frac{1}{2}\log_e(y^2-x^2) = \log_e x + \log_e\sqrt{y^2-x^2}$$

But $N = \frac{y^2+(y^2-x^2)}{y(y^2-x^2)} = \frac{y}{y^2-x^2} + \frac{1}{y}$,

which does not contain a constant term.

So, integrating the terms $of N$ not containing x w.r.to y, we get

$$G = \int \frac{1}{y} dy = \log_e y$$

the general solution is $F + G = C$

$$\Rightarrow \log_e x + \log_e \sqrt{y^2 - x^2} + \log_e y = \log_e C'$$

$$\Rightarrow \log xy\sqrt{y^2 - x^2} = \log_e C' \Rightarrow xy\sqrt{y^2 - x^2} = C'$$

Problem3: Solve $(x^4 + y^4)dx - xy^3 dy = 0$.

Solution: $M = x^4 + y^4, N = -xy^3$

The differential equation is homogeneous as each term is of degree 4.

$$IF = \frac{1}{Mx+Ny} = \frac{1}{x^5+xy^4-xy^4} = \frac{1}{x^5}$$

Multiplying the DE by the IF,

$$\frac{1}{x^5}(x^4 + y^4)dx - \frac{1}{x^5}(xy^3)dy = 0$$

$$(\frac{1}{x} + \frac{y^4}{x^5})dx - \frac{y^3}{x^4}dy = 0$$

$$M_1 = \frac{1}{x} + \frac{y^4}{x^5}, N_1 = -\frac{y^3}{x^4}$$

$$\frac{\partial M_1}{\partial y} = \frac{4y^3}{x^5}, \frac{\partial N_1}{\partial x} = \frac{4y^3}{x^5}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$\int_{y\ constant} M_1\, dx + \int($ terms of N_1 not containing x) $dy = c$

$$\int (\frac{1}{x} + \frac{y^4}{x^5})dx + \int 0\, dy = c$$

$$\log x - \frac{y^4}{4x^4} = c$$

Problem4: Solve $x^2ydx - (x^3 + xy)^2dy = 0$.

Solution: $M = x^2y, N = -x^3 - xy^2$

The differential equation is homogeneous as each term is of degree 3.

$$\text{IF} = \frac{1}{Mx+Ny} = \frac{1}{x^3y - x^3y - xy^3} = -\frac{1}{xy^3}$$

Multiplying the DE by the IF,

$$-\frac{1}{xy^3}(x^2y)dx - \left(-\frac{1}{xy^3}\right)(x^3 + xy^2)dy = 0$$

$$-\frac{x}{y^2}dx + \left(\frac{x^2}{y^3} + \frac{1}{y}\right)dy = 0$$

$$M_1 = -\frac{x}{y^2},$$

$$\frac{\partial M_1}{\partial y} = \frac{2x}{y^3},$$

$$N_1 = \frac{x^2}{3} + \frac{1}{y}$$

$$\frac{\partial N_1}{\partial x} = \frac{2x}{y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\ constant} M_1\, dx + \int (\text{ terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int -\frac{x}{y^2}dx + \int \frac{1}{y}dy = c$$

$$-\frac{x^2}{2y^2} + \log y = c$$

Problem5: Solve $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0.$

Solution: $M = xy - 2y^2, N = -x^2 + 3xy$

The differential equation is homogeneous as each term is of degree 2.

$$\text{IF} = \frac{1}{Mx+Ny} = \frac{1}{x^2y - 2xy^2 - x^2y + 3xy^2} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(xy - 2y^2)dx - \frac{1}{xy^2}(x^2 - 3xy)dy = 0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+(-\frac{x}{y^2}+\frac{3}{y})dy=0$$

$$M_1=\frac{1}{y}-\frac{2}{x}, N_1=-\frac{x}{y^2}+\frac{3}{y}$$

$\frac{\partial M_1}{\partial y}=-\frac{1}{y^2}, \frac{\partial N_1}{\partial x}=-\frac{1}{y^2}$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\ constant} M_1\, dx+\int(\text{ terms of } N_1 \text{ not containing } x)\, dy=c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$\frac{x}{y}-2\log x+3\log y=c$$

Problem6: Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$.

Solution: $M=x^2y-2xy^2$ $N=-x^3+3x^2y$

The differential equation is homogeneous as each term is of degree 3.

$$\text{IF}=\frac{1}{Mx+Ny}=\frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2}=\frac{1}{x^2y^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2y^2}(x^2y-2xy^2)dx-\frac{1}{x^2y^2}(x^3-3x^2y)dy=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-(\frac{x}{y^2}-\frac{3}{y})dy=0$$

$$M_1=\frac{1}{y}-\frac{2}{x}, N_1=-\frac{x}{y^2}+\frac{3}{y}$$

$$\frac{\partial M_1}{\partial y}=-\frac{1}{y^2}, \frac{\partial N_1}{\partial x}=-\frac{1}{y^2}$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\ constant} M_1\, dx + \int(\text{ terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int (\frac{1}{y} - \frac{2}{x})\, dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log \frac{y^3}{x^2} = c$$

Problems using Rule 2

If differential equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$, then

I.F. $= \frac{1}{Mx - Ny}$, where $M = f_1(xy)y$, $N = f_2(xy)x$; provided $Mx - Ny \neq 0$

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Problem1: Solve $y(1 + xy + x^2y^2)dx + x(1 - xy + x^2y^2)dy = 0$.

Solution: Equation is of the form

$$f_1(xy)ydx + f_2(xy)xdy = 0$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying D.E. by I.F.,

$$\frac{y}{2x^2y^2}(1 + xy + x^2y^2)dx + \frac{x}{2x^2y^2}(1 - xy + x^2y^2)dy = 0$$

$$(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2})dx + (\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2})dy = 0$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \Rightarrow \frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2},$$

$$N_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2} \Rightarrow \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant   terms not containing x

$$\int (\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2})dx + \int -\frac{1}{2y}dy = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$-\frac{1}{2xy} + \frac{xy}{2} + \frac{1}{2}\log \frac{x}{y} = c$$

Problem2: Solve $(xy\sin\phi + \cos xy)ydx + (xy\sin\phi - \cos xy)xdy = 0.$

Solution: $M = xy^2\sin xy + y\cos xy, \quad N = x^2y\sin xy - x\cos xy$

The equation is in the form

$$f_1(xy)ydx + f_2(xy)xdy = 0$$

I.F. $= \frac{1}{Mx - Ny}$

$$= \frac{1}{x^2y^2\sin \mathrm{xy} + xy\cos xy - x^2y^2\sin xy + xy\cos \mathrm{xy}} = \frac{1}{2xy\cos \mathrm{xy}}$$

Multiplying D.E. by I.F.,

$$\frac{1}{2xy\cos \mathrm{xy}}(xy\sin \mathrm{xy} + \cos xy)ydx + \frac{1}{2xy\cos xy}(xy\sin xy - \cos xy)xdy = 0$$

$$(\frac{y\tan \mathrm{xy}}{2} + \frac{1}{2x})dx + (\frac{x\tan xy}{2} - \frac{1}{2y})dy = 0$$

$$M_1 = \frac{y\tan xy}{2} + \frac{1}{2x} \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\tan xy + xy\sec^2 xy}{2}$$

$$N_1 = \frac{x\tan xy}{2} - \frac{1}{2y} \Rightarrow \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy\sec^2 xy}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant   terms not containing x

$$\frac{1}{2}\int (\, y \tan xy + \frac{1}{x})dx + \int -\frac{1}{2y}dy = c$$

$$\frac{1}{2}(\frac{y}{y}\log\ \sec \mathrm{xy} + \ \log x) - \frac{1}{2}\log y = c$$

$$\log\,(x \sec \mathrm{xy}) - \ \log y = 2c$$

$$\log\,(\frac{x}{y}\sec xy) = 2c$$

$$\frac{x}{y}\sec xy = e^c = k, \qquad \frac{x}{y}\sec xy = k$$

Problems using Rule 3

If $Mdx + Ndy = 0$ is such that $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is a function of x, say $f(x)$, then $e^{\int f(x)dx}$ is an I.F

Problem1: Example 1: Solve $(x^2 + y^2 + 1)dx - 2xydy = 0.$

Solution:

$$M = x^2 + y^2 + 1, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{2y - (-2y)}{-2xy} = -\frac{2}{x}$$

I.F. $= e^{\int \frac{-2}{x}dx} = e^{-2\log x} = e^{\,\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(x^2 + y^2 + 1)dx - \frac{1}{x^2}2xydy = 0$$

$$\left(1 + \frac{y^2 + 1}{x^2}\right)dx - \frac{-2y}{x}dy = 0$$

$$M_1 = 1 + \frac{y^2+1}{x^2}, \quad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

$$y \text{ constant} \qquad \text{terms not containing } x$$

$$\int (1 + \frac{y^2+1}{x^2}) dx + \int \mathbf{0}\, dy = c$$

$$x - \frac{y^2+1}{x} = c \;\Rightarrow\; x^2 - y^2 - 1 = cx$$

Problem2: Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2y dy = 0$.

Solution:

$$M = xy^2 - e^{\frac{1}{x^3}}, \qquad N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = -2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x}$$

I.F. $= e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = e^{\log x^{-4}} = x^{-4} = \frac{1}{x^4}$

Multiplying D.E. by I.F.,

$$\frac{1}{x^4}(xy^2 - e^{\frac{1}{x^3}})dx - \frac{1}{x^4}(x^2y)dy = 0$$

$$\left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$$

$$M_1 = \frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}, \qquad N_1 = -\frac{y}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^3}, \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant $\qquad$ terms not containing x

$$\int (\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}) dx + \int \mathbf{0}\, dy = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int e^{\frac{1}{x^3}} (-\frac{3}{x^4}) dx = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c \ \ [\text{since} \int e^{f(x)} f'(x) dx = e^{f(x)} + c]$$

Problem3: Solve $(2x \log x - xy)dy + 2ydx = 0$.

Solution: $2ydx + (2x \log x - xy)dy = 0$

$$M = 2y, \qquad N = 2x \log x - xy$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2 \log x + 2 - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{2 - (2 \log x + 2 - y)}{2x \log x - xy}$$

$$= \frac{-(2 \log x - y)}{x(2 \log x - y)} = -\frac{1}{x}$$

I.F. $= e^{\int \frac{-1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

Multiplying D.E. by I.F.,

$$\frac{1}{x}(2y)dx + \frac{1}{x}(2x \log x - xy)dy = 0$$

$$\frac{2y}{x}dx + (2\log x - y)dy = 0$$

$$M_1 = \frac{2\mathrm{y}}{\mathrm{x}}, \qquad N_1 = 2\log x - y$$

$$\frac{\partial M}{\partial y} = \frac{2}{\mathrm{x}}, \qquad \frac{\partial N}{\partial x} = \frac{2}{\mathrm{x}}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant terms not containing x

$$\frac{1}{2}\int (\frac{2\mathrm{y}}{\mathrm{x}})dx + \int -ydy = c$$

$$2y\log x - \frac{y^2}{2} = c$$

Problem4: Solve $x\sin x\frac{dy}{dx} + y(x\cos x - \sin x) = 2.$

Solution: $x\sin xdy + (xy\cos x - y\sin x - 2)dx = 0$

$$(xy\cos x - y\sin x - 2)dx + x\sin xdy = 0$$

$$M = xy\cos x - y\sin x - 2, \qquad N = x\sin x$$

$$\frac{\partial M}{\partial y} = x\cos x - \sin x, \qquad \frac{\partial N}{\partial x} = \sin x + x\cos x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x}$$

I.F. $= e^{\int\frac{-2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(xy\cos x - y\sin x - 2)dx + \frac{1}{x^2}(x\sin x)dy = 0$$

$$(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2})dx + \frac{1}{x}\sin xdy = 0$$

$$M_1 = \frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}, \qquad N_1 = \frac{sinx}{\text{x}}$$

$$\frac{\partial M_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant   terms not containing x

$$\frac{1}{2}\int -\frac{2}{x^2}dx + \int \frac{\sin x}{x}dy = c$$

$$\frac{2}{x} + (\frac{\sin x}{x})y = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

Problems using Rule 4

If $Mdx + Ndy = 0$ is such that $\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is a function ofy, say $g(\gamma)$, then $e^{\int g(y)dy}$ is an I.F.

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Problem1: Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

Solution: $M = y^4 + 2y,\ N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{y^3 - 4 - (4y^3 + 2)}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}$$

I.F. $= e^{\int \frac{-3}{y}dy} = e^{-3\log y} = e^{\log y^3} = y^{-3} = \frac{1}{y^3}$

Multiplying D.E. by I.F,

$$\frac{1}{y^3}(y^4+2y)dx+\frac{1}{y^3}(xy^3+2y^4-4x)dy=0$$

$$(y+\frac{2}{y^2})dx+(x+2y-\frac{4x}{y^3})dy=0$$

$$M_1=y+\frac{2}{y^2},\qquad N_1=x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y}=1-\frac{4}{y^3},\qquad \frac{\partial N_1}{\partial x}=1-\frac{4}{y^3}$$

Since, $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\,dx+\int N_1\,dy=c$$

y constant terms not containing x

$$\int(y+\frac{2}{y^2})dx+\int \mathbf{2y}\,dy=c$$

$$(y+\frac{2}{y^2})x+y^2=c$$

Problem2: Solve $(2xy^4e^y+2xy^3+y)dx+(x^2y^4e^y-x^2y^2-3x)dy=0.$

Solution: $M=2xy^4e^y+2xy^3+y,\quad N=x^2y^4e^y-x^2y^2-3x$

$$\frac{\partial M}{\partial y}=2x(y^4e^y+4y^3e^y+3y^2)+1,\qquad \frac{\partial N}{\partial x}=2xy^4e^y-2xy^2-3$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{M}(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})=\frac{(2xy^4e^y-2xy^2-3)-(2xy^4e^y+8xy^3e^y+6xy^2+1)}{2xy^4e^y+2xy^3+y}$$

$$=\frac{-4(2xy^3e^y+2xy^2+1)}{y(2xy^3e^y+2xy^2+1)}=-\frac{4}{y}$$

I.F. $=e^{\int\frac{-4}{y}dy}=e^{-4\log y}=e^{\log y^{-4}}=y^{-4}=\frac{1}{y^4}$

Multiplying D.E. by I.F.,

$$\frac{1}{y^4}(2xy^4e^y+2xy^3+y)dx+\frac{1}{y^4}(x^2y^4e^y-x^2y^2-3x)dy=0$$

$$(2xe^y + \frac{2x}{y} + \frac{1}{y^3})dx + (x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4})dy = 0$$

$$M_1 = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}, \qquad N_1 = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$$

$$\frac{\partial M_1}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}, \qquad \frac{\partial N_1}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant terms not containing x

$$\int (2xe^y + \frac{2x}{y} + \frac{1}{y^3})dx + \int \mathbf{0}\, dy = c$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Problem3: Solve $xe^x(dx - dy) + e^x dx + ye^y dy = 0$.

Solution: $(xe^x + e^x)dx + (ye^y - xe^x)dy = 0$

$M = xe^x + e^x, \;\; N = ye^y - xe^x$

$\frac{\partial M}{\partial y} = 0, \;\; \frac{\partial N}{\partial x} = -e^x - xe^x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

I.F. $= e^{\int dy} = e^{-y}$

Multiplying D.E. by I.F.,

$$e^{-y}(xe^x + e^x)dx + e^{-y}(ye^y - xe^x) = 0$$

$$M_1 = e^{-y}(xe^x + e^x), \qquad N_1 = y - xe^{x-y}$$

$$\frac{\partial M_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial N_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int M_1\, dx + \int N_1\, dy = c$$

y constant   terms not containing x

$$\int \boldsymbol{e^{-y}}\,(xe^x + e^x)dx + \int \boldsymbol{y}\, dy = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Problem4: Solve $(\frac{y}{x}\sec y - \tan y)dx + (\sec y \log x - x)dy = 0$.

Solution: $M = \frac{y}{x}\sec y - \tan y$, $N = \sec y \log x - x$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\frac{\sec y}{x} - \frac{\sec y}{x} - \frac{y}{x}\, secy\, tany + secy^2}{\frac{y}{x}secy - tany}$$

$$\frac{\frac{-y}{x}\, secy\, tany + tany^2}{\frac{y}{x}secy - tany} = -tany$$

I.F. $= e^{\int -\tan y dy} = e^{-\log\ \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$

Multiplying D.E. by I.F.,

$$\cos y(\frac{y}{x}\sec y - \tan y)dx + \cos y(\sec y \log x - x)dy = 0$$

$$(\frac{y}{x} - \sin y)dx + (\log x - x\cos y)dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x} - \cos y, \qquad N_1 = \log x - x\cos y$$

$$M_1 = \frac{y}{x} - siny, \quad \frac{\partial N_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

$$\int M_1\,dx + \int N_1\,dy = c$$

y constant terms not containing x

$$\int \left(\frac{y}{x} - siny\right)dx + \int \mathbf{0}\,dy = c$$

$$y \log x - x \sin y = c$$

Type IV Linear Differential equation:

The general form of the first order linear differential equation in the dependent variable y is

$$a_0(x)\frac{dy}{dx} + a_1(x)y = a_2(x) \ldots \ldots \ldots \quad (1)$$

where $a_0(x) \neq 0$

Dividing by $a_0(x)$, we get

$$\frac{dy}{dx} + \frac{a_1(x)}{a_0(x)}y = \frac{a_2(x)}{a_0(x)}$$

$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x) \ldots \ldots \ldots$ (2)

where $P(x) = \frac{a_1(x)}{a_0(x)}$ and $Q(x) = \frac{a_2(x)}{a_0(x)}$

The equation (2) is the standard form of the linear differential equation in y or Leibnitz's linear equation.

Solution of the linear differential equation (2)

Equation (2) is $\frac{dy}{dx} + Py = Q$

Multiply (2) by $e^{\int pdx}$, we get

$$e^{\int pdx}\frac{dy}{dx} + e^{\int pdx}py = Qe^{\int pdx}$$

$$\Rightarrow \frac{d}{dx}\,(y\,.e^{\int pdx}) = Q \cdot e^{\int pdx}$$

Integrating $w.r.$ to x, we get

This is the general solution of (2).

$$y \cdot e^{\int p dx} = \int Q \cdot e^{\int p dx} dx + C$$

i.e $\quad y \cdot (I.F) = \int Q \cdot (I.F) dx + C$

This is the general solution of (2).

Problem1: Solve $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} \sin \mathrm{x} = \mathrm{e}^{\cos \mathrm{x}}$

Solution: The equation is linear in y.

$$P = \sin x, Q = e^{\cos x}$$

$\mathrm{IF} = e^{\int \sin \mathrm{x}\, \mathrm{dx}} = e^{-\cos x}$

Hence, the general solution is $y \cdot (I.F) = \int Q \cdot (I.F) dx + C$

$$e^{-\cos x} y = \int e^{-\cos x} \cdot e^{\cos x} dx + c$$

$$= \int e^{0}\, dx + c$$

$$= \int d\, x + c$$

$$= x + c$$

$$y = (x + c) e^{\cos x}$$

Problem2: Solve $\frac{\mathrm{dy}}{\mathrm{dx}} + \frac{3\mathrm{y}}{\mathrm{x}} = \frac{\sin \mathrm{x}}{\mathrm{x}^3}$.

Solution: The equation is linear in y.

$$P = \frac{3}{x}, Q = \frac{\sin x}{x^3}$$

$\mathrm{IF} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3} = x^3$ [Since $e^{\log x} = x$]

Hence, the general solution is

$$x^3 y = \int x^3 \frac{\sin x}{x^3} dx + c$$

$$= \int \sin x dx + c$$

$$= -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Problem3: Solve the differential equation $(1-x^2)\frac{dy}{dx}-xy=1$.

Solution: The given differential equation is $(1-x^2)\frac{dy}{dx}-xy=1$

Dividing by $(1-x^2)$, we get

$$\frac{dy}{dx}-\frac{x}{1-x^2}y=\frac{1}{1-x^2}$$

This is linear in y. Here $P=-\frac{x}{1-x^2}$ and $Q=\frac{1}{1-x^2}$.

the general solution is

$$ye^{\int pdx}=\int Q\cdot e^{\int pdx}dx+C$$

Now

$$\int p\,dx=\int -\frac{x}{1-x^2}dx=\frac{1}{2}\log_e(1-x^2)=\log_e\sqrt{1-x^2}$$

$e^{\int pdx}=e^{log\sqrt{1-x^2}}=\sqrt{1-x^2}$ [Since $e^{\log x}=x$]

and

$$\int Q\cdot e^{\int pdx}dx=\int\frac{1}{1-x^2}\sqrt{1-x^2}dx=\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}x$$

the solution is

$$y\sqrt{1-x^2}=\sin^{-1}x+C$$

Problem4: Solve $\frac{\mathrm{dy}}{\mathrm{dx}}+2\mathrm{y}\tan\mathrm{x}=\sin\mathrm{x}$.

Solution: The equation is linear in y.

$$P=2\tan x, Q=\sin x$$

IF $=e^{\int 2\tan xdx}=e^{2\log\ \sec x}=e^{1\mathrm{ogsec}^2x}=\sec^2x$

Hence, the general solution is

$$(\sec^2 x)y = \int \sec^2 x \sin x dx + c$$

$$y\sec^2 x = \int \sec x \frac{\sin x}{\cos x} dx + c$$

$$= \int \sec x \tan x dx + c$$

$$= \sec x + c$$

Problem5: Solve $(\mathrm{x}+1)\frac{\mathrm{dy}}{\mathrm{dx}} - \mathrm{y} = \mathrm{e}^{3\mathrm{x}}(\mathrm{x}+1)^2$

Solution: Rewriting the equation,

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

The equation is linear in y.

$$P = -\frac{1}{x+1}, Q = e^{3x}(x+1)$$

$$\text{IF} = e^{\int -\frac{1}{x+1}dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Hence, the general solution is

$$(\frac{1}{x+1})y = \int (\frac{1}{x+1}) e^{3x}(x+1)dx + c$$

$$= \int e^{3x}\, dx + c$$

$$= \frac{e^{3x}}{3} + c$$

$$y = (x+1)(\frac{e^{3x}}{3} + c)$$

Problem6: Solve $\frac{dy}{dx} + \frac{4x}{1+x^2} y = \frac{1}{(x^2+1)^3}$.

Solution: The equation is linear in y.

$$P = \frac{4x}{1+x^2}, Q = \frac{1}{(x^2+1)^3}$$

$$\text{IF} = e^{\int \frac{4x}{1+x^2}dx} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, the general solution is

$$(1+x^2)^2y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3}\,dx + c$$

$$= \int \frac{1}{x^2+1}\,dx + c$$

$$= \tan^{-1}x + c$$

Problem7: Solve $(1+x+xy^2)dy + (y+y^3)dx = 0$.

Solution: Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)} \quad \ldots\ldots\ldots.(1)$$

The equation is linear in x.

$$P = \frac{1}{y}, Q = -\frac{1}{y(1+y^2)}$$

IF $= e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Hence, the general solution is

$$yx = \int y\,[-\frac{1}{y(1+y^2)}]dy + c$$

$$= -\int \frac{1}{1+y^2}dy + c$$

$$= -\tan^{-1}y + c$$

$$xy = c - \tan^{-1}y$$

Problem8: Solve $(1+\sin y)dx = (2y\cos y - x\sec y - x\tan y)dy$.

Solution: Rewriting the equation,

$$(1+\sin y)\frac{dx}{dy} = 2y\cos y - (\sec y + \tan y)x$$

$$(1+\sin y)\frac{dx}{dy}+(\frac{1+\sin y}{\cos y})x=2y\cos y$$

$$\frac{dx}{dy}+(\frac{1}{\cos y})x=\frac{2y\cos y}{1+\sin y}$$

The equation is linear in x.

$$P=\frac{1}{\cos y},Q=\frac{2y\cos y}{1+\sin y}$$

IF $= e^{\int\frac{1}{\cos y}dy} = e^{\int\sec y dy} = e^{\log(\sec y+\tan y)} = \sec y+\tan y$

Hence, the general solution is

$$(\sec y+\tan y)x=\int(\sec y+\tan y)(\frac{2y\cos y}{1+\sin y})dy+c$$

$$=2\int(\frac{1+\sin y}{\cos y})(\frac{y\cos y}{1+\sin y})dy+c$$

$$=2\int y\,dy+c$$

$$(\sec y+\tan y)x=y^2+c$$

Problem9: Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$.

Solution: Rewriting the equation,

$$(1+y^2)\frac{dx}{dy}=\tan^{-1}y-x$$

$$\frac{dx}{dy}+(\frac{1}{1+y^2})x=\frac{\tan^{-1}y}{1+y^2}$$

The equation is linear in x.

$$P=\frac{1}{1+y^2},Q=\frac{\tan^{-1}y}{1+y^2}$$

IF $= e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Hence, the general solution is

$$(e^{\tan^{-1}y})x=\int e^{\tan^{-1}y}(\frac{\tan^{-1}y}{1+y^2})dy+c$$

Let $\tan^{-1}y=t$

$$\frac{1}{1+y^2}dy = dt$$

$$(e^{\tan^{-1}y})x = \int e^t\, tdt + c$$

$$= te^t - e^t + c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$$

Problem10: Solve $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$.

Solution: Rewriting the equation,

$$\frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$P = 2\cot\theta, Q = -\sin 2\theta$$

IF $= e^{\int 2\cot\theta d\theta} = e^{2\log\,\sin\theta} = e^{\,\log\sin^2\theta} = \sin^2\theta$

Hence, the general solution is

$$\sin^2\theta\cdot r = \int \sin^2\theta(-\sin 2\theta)d\theta + c$$

$$= -2\int \sin^3\theta\cos\theta d\theta + c$$

$$= -2\frac{\sin^4\theta}{4} + c \qquad [Since \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}]$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

Problem11: Solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$.

Solution:

$$\frac{dy}{dx} - \frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{(x-1)}$$

The equation is linear in y.

$$P=-\frac{x-2}{x(x-1)},Q=\frac{x^2(2x-1)}{x-1}$$

$$=-(\frac{2}{x}-\frac{1}{x-1})$$

IF $= e^{\int(-\frac{2}{X}+\frac{1}{x-1})dx} = e^{-2\log x+\log(x-1)} = e^{\log(\frac{x-1}{x^2})} = \frac{x-1}{x^2}$

Hence, the general solution is

$$(\frac{x-1}{x^2})\cdot y=\int(\frac{x-1}{x^2})\cdot x^2(\frac{2x-1}{x-1})dx+c$$

$$=x^2-x+c$$

$$y=\frac{x^3(x-1)}{x-1}+\frac{cx^2}{x-1}$$

$$\boldsymbol{y=x^3+\frac{cx^2}{x-1}}$$

Nonlinear Differential Equations Reducible to Linear Form

Type 1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \quad \text{..........(1)}$$

where P and Q are functions of x or constants is a nonlinear equation, known as Bernoulli's equation. This equation can be made linear using the following method:

Dividing Eq. (1) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \quad \text{.......... (2)}$$

Let $\frac{1}{y^{n-1}}=v \Rightarrow y^{-(n-1)}=v \Rightarrow$ D.w.r.to x we get, $-(n-1)y^{-(n-1)-1}\frac{dy}{dx}=\frac{dv}{dx}$

$$\frac{(1-n)dy}{y^n dx}=\frac{dv}{dx}$$

$$\frac{1dy}{y^n dx}=\frac{1}{(1-n)}\cdot\frac{dv}{dx}$$

Substituting in Eq. (2),

$$\frac{1}{1-n} \cdot \frac{dv}{dx} + Pv = Q$$

$$\frac{dv}{dx} + (1-n)Pv = Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = \frac{1}{y^{n-1}}$, we get the solution of Eq. (1).

Problem1: Solve $\frac{dy}{dx} + \frac{2y}{x} = y^2x^2$

Solution: The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} \cdot \frac{2}{x} = x^2 \cdots (1)$$

Let $\frac{1}{y} = v, -\frac{1 dy}{y^2 dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx} + \left(\frac{2}{x}\right)v = x^2$$

$$\frac{dv}{dx} - \left(\frac{2}{x}\right)v = -x^2 \text{ .... (2)}$$

The equation is linear in v.

$$P = -\frac{2}{x}, Q = -x^2$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\frac{1}{x^2}v = \int \frac{1}{x^2}(-x^2)dx + c$$

$$= \int -dx + c$$

$$= -x + c$$

$$v = -x^3 + cx^2$$

Hence, $\frac{1}{y} = -x^3 + cx^2$

Problem2: Solve $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$.

Solution: The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \cos x - \sin x \dots\dots\dots (1)$$

Let $\frac{1}{y} = -\frac{1dy}{y^2dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx} + v = \cos x - \sin x$$

$$\frac{dv}{dx} - v = -\cos x + \sin x \dots\dots\dots.(2)$$

The equation is linear in v.

$$P = -1, Q = -\cos x + \sin x$$

IF $= e^{\int -dx} = e^{-x}$

The general solution of Eq. (2) is

$$e^{-x} \cdot v = \int e^{-x}(-\cos x + \sin x)dx + c$$

$$= -\int e^{-x} \cos xdx + \int e^{-x} \sin xdx + c$$

$$= -[\frac{e^{-x}}{2}(-\cos x + \sin x)] + [\frac{e^{-x}}{2}(-\sin x - \cos x)] + c$$

$$e^{-x}v = -e^{-x}\sin x + c$$

$$v = -\sin x + ce^{x}$$

Hence, $\frac{1}{y} = -\sin x + ce^{X}$

Problem3: Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$.

Solution: Rewriting the equation, $\frac{dx}{dy} = xy + x^2y^3$

$$\frac{dx}{dy} - xy = x^2y^3$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{dx}{dy} - \left(\frac{1}{x}\right)y = \mathcal{Y}^3 \ldots\ldots..(1)$$

Let $-\frac{1}{x} = v$, $\frac{1dx}{x^2dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{dv}{dy} + vy = y^3 \ldots\ldots.(2)$$

The equation is linear in v.

$$P = y, Q = y^3$$

$\text{IF} = e^{\int y dy} = e^{\frac{y^2}{2}}$

The general solution of Eq. (2) is

$$e^{\frac{y^2}{2}} \cdot v = \int e^{\frac{y^2}{2}} y^3 dy + c$$

Putting $\frac{y^2}{2} = t, ydy = dt$

$$e^{\frac{y^2}{2}} \cdot v = \int e^t \cdot 2tdt + c$$

$$= 2(e^t t - e^t) + c$$

$$= 2e^t(t-1) + c$$

$$= 2e^{\frac{y^2}{2}}(\frac{y^2}{2} - 1) + c$$

$$v = y^2 - 2 + ce^{\frac{y^2}{2}}$$

Hence,

$-\frac{1}{x} = y^2 - 2 + ce^{\frac{-y^2}{2}}$

Problem4: Solve $\frac{dr}{d\theta} = r\tan\theta - \frac{r^2}{\cos\theta}$.

Solution: Rewriting the equation, $\frac{dr}{d\theta} - r\tan\theta = -\frac{r^2}{\cos\theta}$

The equation is in Bernoulli's form, where r is a dependent variable.

Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \dots\dots (1)$$

Let $-\frac{1}{r} = v,\ \frac{1}{r^2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$

Substituting in Eq. (1),

$$\frac{dv}{d\theta} + v\tan\theta = -\frac{1}{\cos\theta} \dots\dots..(2)$$

The equation is linear in v.

$$P = \tan\theta, Q = -\frac{1}{\cos\theta}$$

IF $= e^{\int \tan\theta d\theta} = e^{\log \sec\theta} = \sec\theta$

The general solution of Eq. (2) is

$$\sec\theta \cdot v = \int \sec\theta\left(-\frac{1}{\cos\theta}\right)d\theta + c$$

$$= \int -\sec^2\theta d\theta + c$$

$$= -\tan\theta + c$$

Hence,

$$\sec\theta\left(-\frac{1}{r}\right) = -\tan\theta + c$$

$$\frac{sec\theta}{r} = \tan\theta - c$$

TYPE II

The equation of the form $\mathrm{f}'(\mathrm{y})\frac{dy}{dx} + P\mathrm{f}(\mathrm{x}) = \mathrm{Q}$ ....(1)

where P and Q are functions of x or constants, can be reduced to the linear form by putting

$\mathrm{f}(\mathrm{y}) = \mathrm{v},\ \mathrm{f}'(\mathrm{y})\frac{dy}{dx} = \frac{dv}{dx}$ in eq.(1)

$\frac{dv}{dx} + Pv = Q$ ....(2)

Equation (2) is linear in v and can be solved using the method of linear differential equation .Finally,

substituting ▯ $= f(y)$, we get the solution of Eq.(1).

Problem1: Solve $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Solution

Given differential equation is $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Dividing the equation by $\cos^2 y$,

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2\sin y \cos y}{\cos^2 y} = x^3$$

$sec^2 y \frac{dy}{dx} + 2\tan y\,.\,x = x^3$ ....(1)

Let tan ▯ = ▯, $\quad sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$

Sustituting in eq.(1)

$\frac{dv}{dx} + 2vx = x^3$ ....(2)

The equation is linear in ▯.

$▯ = 2▯,\ ▯ = x^3$

IF = ▯$^{\int 2x▯x}$ = e^{x^2}

Solution of Eq.(2) is

$$e^{x^2}v = \int e^{x^2}x^3 dx + c$$

Putting $x^2 = t,\ 2xdx = dt,\ xdx = \frac{dt}{2}$

$$e^{x^2}v = \int e^t t\frac{dt}{2} + c$$

$$e^{x^2}v = \frac{1}{2}(e^t t - e^t) + c$$

$$e^{x^2}v = \frac{1}{2}e^{x^2}(x^2 - 1) + c$$

$$v = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Hence, $\tan y = = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

Problem2: Solve $\frac{dy}{dx} - \frac{\tan y}{1+▯} = (1 + ▯)e^x \sec y$

Solution

Given differential equation is $\frac{dy}{dx} - \frac{\tan y}{1+▯} = (1 + ▯)e^x \sec y$

Dividing the equation by sec▯,

We get cos▯ $\frac{dy}{dx} - \frac{\sin y}{1+x} = (1 + x)e^x$ ……………………………..(1)

Let $\sin y = ▯$, $\cos y\frac{dy}{dx} = \frac{dv}{dx}$

Sustituting in eq.(1)

$\frac{dv}{dx} + \frac{v}{1+x} = (1+x)e^{x}$ ……………………………….(2)

The equation is linear in □.

□ $= \frac{-1}{1+□}$, □ $= (1+x)e^{x}$

IF = □$^{\int \frac{-1}{1+x}□x}$ $= e^{-\log_e(1+x)} = \frac{1}{1+x}$

Solution of Eq.(2) is

$\frac{v}{1+x} = \int (1+x)e^{x}\frac{1}{1+x}dx + c$

$\frac{v}{1+x} = \int e^{x}dx + c$

$$\frac{v}{1+x} = e^{x} + c$$

□ $= (1+x)(e^{x} + c)$

Hence $\sin y = (1+x)(e^{x} + c)$

Applications of first ODE

Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of another family at right angles.

Working Rules

1. Cartesian curve $\boldsymbol{f(x,y,c) = 0}$

(i) Obtain the differential equation $F(x, y, \frac{dy}{dx}) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above differentialequation to obtain the differential

equation of the family of orthogonal trajectories as $F(x, y, -\frac{dx}{dy}) = 0$.

(iii) Solve the differential equation $F(x, y, -\frac{dx}{dy}) = 0$ to obtain the equation of

the family of orthogonal trajectories.

2. Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F(r, \theta, \frac{dr}{d\theta}) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as

$$F(r, \theta, -r^2\frac{d\theta}{dr}) = 0.$$

(iii) Solve the differential equation $F(r, \theta, -r^2\frac{d\theta}{dr}) = 0$ to obtain the equation

of the family of orthogonal trajectories.

Problem1: Find the orthogonal trajectories of the family of semi cubical parabolas $\mathrm{ay}^2 = \mathrm{x}^3$

Solution: The equation of the family of curves is

$$ay^2 = x^3 \quad (1)$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y\frac{dy}{dx} = 3x^2$$

Substituting $a = \frac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y\frac{dy}{dx} = 3x^2$$

$$\frac{2x}{y}\frac{dy}{dx} = 3 \ldots\ldots\ldots (2)$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \dots\dots\dots (3)$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int -2x dx = \int 3\, y dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

Problem2: Find the orthogonal trajectories of the family of curves $\frac{X^2}{\alpha^2} + \frac{Y^2}{b^2+\lambda} = 1$, where λ is a parameter.

Sol. The equation of the family of given curves is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{b^2+\lambda} = 1 \dots. (1)$$

Differentiating (1) $w.r.t.\ x$

$$\frac{2x}{\alpha^2} + \frac{2y}{b^2+\lambda}.\frac{dy}{dx} = 0 \Rightarrow \frac{x}{\alpha^2} + \frac{y}{b^2+\lambda}.\frac{dy}{dx} = 0 \dots\dots\dots. \quad (2)$$

To eliminate the parameter λ, we equate the values of $b^2 + \lambda$ from (1) and (2).

From (1), $\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{\alpha^2} = \frac{\alpha^2 - x^2}{\alpha^2} \Rightarrow b^2 + \lambda = \frac{\alpha^2 y^2}{\alpha^2 - x^2}$

From (2), $b^2 + \lambda = -\frac{\alpha^2 y}{x}.\frac{dy}{dx}$

$$\therefore \frac{\alpha^2 y^2}{\alpha^2 - x^2} = -\frac{\alpha^2 y}{x}.\frac{dy}{dx}$$

$\Rightarrow \frac{y}{\alpha^2 - x^2} = -\frac{1}{x}\frac{dy}{dx}$ or $\frac{dy}{dx} + \frac{xy}{\alpha^2 - x^2} = 0 \dots\dots (3)$

which is the differential equation of the given family (1).

Now to find the orthogonal trajectories

(i) Replace $\frac{dy}{dx}$ by— $\frac{dx}{dy}$ to get the differential equation of the orthogonal trajectory.

(ii) Solve the above equation to get the equation of the required orthogonal trajectory.

So, replacing $\frac{dy}{dx}$ by— $\frac{dx}{dy}$ in equation (3), we get

$$\frac{xy}{\alpha^2-x^2} - \frac{dx}{dy} = 0 \Rightarrow ydy - \left(\frac{\alpha^2-x^2}{x}\right)dx = 0 \ldots\ldots \quad (4)$$

which is the differential equation of the orthogonal trajectories.

Integrating (4), we get

$$\int y\, dy - \int \left(\frac{\alpha^2}{x} - x\right)dx = C \text{ or } \frac{y^2}{2} - a^2 \log x + \frac{x^2}{2} = C$$

$$\Rightarrow x^2 + y^2 = 2a^2 \log x + C$$

which is the equation of the required orthogonal trajectories of (1).

Problem3: Find the orthogonal trajectories of the family of parabolas $y = \alpha x^2$.

Sol. The equation of the family of given parabolas is $y = ax^2 \ldots\ldots (1)$

Differentiating (1) $w.r.t.\, x\ \frac{dy}{dx} = 2ax \ldots\ldots (2)$

Eliminating 'a' between (1) and (2), we have

$$\frac{dy}{dx} = 2\left(\frac{y}{x^2}\right)x \ or\ \frac{dy}{dx} = 2.\frac{y}{x}$$

$$\frac{dy}{dx} - \frac{2}{x}.y = 0 \ldots\ldots (3)$$

which is the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by — $\frac{dx}{dy}$ in (3), we get

$$-\ \frac{dx}{dy} - \frac{2}{x}.y = 0 \Rightarrow \frac{dx}{dy} = -\frac{2y}{x} \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \ldots\ldots (4)$$

$$2y.dy = -xdx.$$

Integrating, we get $y^2 = -\frac{x^2}{2} + C \Rightarrow x^2 + 2y^2 = C.$

Problem4: Prove that the system of confocal and coaxial parabolas $\mathrm{y}^2 = 4\alpha(\mathrm{x} + \alpha)$ is self - orthogonal

Sol. The equation of the family of given parabolas is

$$y^2 = 4\alpha(x + \alpha) \dots \dots \quad (1)$$

Differentiating equation (1) $w.r.t.\, x$

$$2y\frac{dy}{dx} = 4\alpha \text{ or } y\frac{dy}{dx} = 2\alpha \dots \dots \quad (2)$$

Eliminating a between (1) and (2), we have

$$y^2 = 2y\frac{dy}{dx}(x + \frac{y}{2}.\frac{dy}{dx}) \text{ or } y^2 = 2xy\frac{dy}{dx} + y^2(\frac{dy}{dx})^2$$

$\Rightarrow y(\frac{dy}{dx})^2 + 2x.\frac{dy}{dx} - y = 0 \dots \dots (3)$

which is the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$y(\frac{dx}{dy})^2 - 2x(\frac{dx}{dy}) - y = 0 \;\Rightarrow y - 2x.\frac{dy}{dx} - y(\frac{dy}{dx})^2 = 0 \Rightarrow y(\frac{dy}{dx})^2 + 2x.\frac{dy}{dx} - y = 0 \dots \dots (4)$

which is the differential equation of the orthogonal trajectories.

Since equation (4) is the same as (3), the system of confocal and coaxial parabolas is self-orthogonal, i.e., each member of (1) cuts every other member orthogonally, at right angles

Problem5: Find the orthogonal trajectory of the cardioids $r = a(l - cos\Theta)$.

Sol. The equation of the family of given cardioids is $r = a(1 - \cos\Theta) \cdots (1)$

Differentiating (1) w.r.t. Θ, $\frac{dr}{d\Theta} = a \sin\Theta \dots \dots (2)$

Dividing (2) by (1) [to eliminate a]

$$\frac{1}{r}.\frac{dr}{d\Theta} = \frac{\sin\Theta}{1 - \cos\Theta} \quad \Rightarrow \quad \frac{1}{r}.\frac{dr}{d\Theta} = \frac{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}{2\sin^2\frac{\Theta}{2}} = \cot\frac{\Theta}{2} \dots \dots \quad (3)$$

which is the differential equation of the given family (1).

Replacing $\frac{dr}{d\Theta}\, by - r^2\frac{d\Theta}{dr}$ in (3), we get

$$\frac{1}{r}(-r^2\frac{d\Theta}{dr}) = \cot\frac{\Theta}{2} \quad \Rightarrow \quad r\frac{d\Theta}{dr} + \cot\frac{\Theta}{2} = 0 \text{ or } \frac{dr}{r} + \tan\frac{\Theta}{2} = 0 \dots \dots \quad (4)$$

which is the differential equation of the family of orthogonal trajectories.

Integrating (4), $\log r - 2\log(\cos\frac{\Theta}{2}) = \log C$

$\Rightarrow \log r = \log C \cos 2(\frac{\Theta}{2})$ or $r = C.\ \cos 2\frac{\Theta}{2} = \frac{C}{2}(1 + \cos\Theta)$

$\Rightarrow r = C(1 + \cos\Theta)$

which is the required equation of orthogonal trajectories of (1).

Problem6: Find the orthogonal trajectories of $r^2 = asin2\Theta$.

Sol. $r^2 = a\sin 2\Theta \ldots \ldots (1)$

Differentiating (1) $w.r$.t. Θ $2r\frac{\partial r}{\partial\Theta} = 2a\cos 2\Theta \ldots \ldots \ldots (2)$

Dividing (2) by (1) to eliminate 'a', we get $\frac{2}{r}\frac{\partial r}{\partial\Theta} = 2\cot 2\Theta \ldots \ldots$ (3)

which is the D.E. of the given family (1).

Replacing $\frac{\partial r}{\partial\Theta} by -r^2\frac{\partial\Theta}{\partial r}$ in (3), we get

$$\frac{2}{r}(-r^2\frac{\partial\Theta}{\partial r}) = 2\cot 2\Theta \text{ or } -2r\frac{\partial\Theta}{\partial r} = 2\cot 2\Theta$$

$$\frac{-1}{2r}\frac{dr}{d\Theta} = \frac{1}{2}\tan 2\Theta \text{ or } \frac{1}{2r}\frac{dr}{d\Theta} + \frac{1}{2}\tan 2\Theta = 0 \ldots \ldots \quad (4)$$

which is the D.E. of the family of orthogonal trajectories.

Integrating (4), $\frac{1}{2}\log r - \frac{1}{2}\log\ \cos 2\Theta = \log C$

$$\log r = \log c^2 + \log\ \cos 2\Theta$$

$$\log r = \log c^2\cos 2\Theta \Rightarrow r = c^2\cos 2\Theta.$$

Problem7: Find the orthogonal trajectories of the family of curves $r^n \sin n\theta = a^n$

Solution:

The family of curves is given by the equation

$$r^n \sin n\theta = a^n \cdots(1)$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{dr}{d\theta}\sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{dr}{d\theta} = -rcot\theta \ldots\ldots(2)$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = -r\cot n\theta$$

$$r\frac{d\theta}{dr} = \cot n\theta \text{ .....(3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int tan\, n\theta d\theta = \int \frac{dr}{r}$$

$$\frac{\log\ \sec n\theta}{n} = \log r + \log c$$

$$\log\ \sec n\theta = n\log rc$$

$$= \log (rc)^n$$

$$\sec n\theta = c^n r^n$$

$r^n \cos n\theta = k$ where $k = \frac{1}{c^n}$

which is the equation of the required orthogonal trajectories.

Newton's Law of Cooling:

It states that rate of change of temperature of a body is directly proportional to the difference between the temperature of the body and that of the surrounding medium.

If T is the temperature of the body and T_0 is the temperature of the surrounding medium at any time t then its differential equation is

$\frac{dT}{dt} = -k(T - T_0)$ where k is a constant.

Problem1: If the air temperature is 30^oC and the substance cools from 100^oC to 70^oC in 15 minutes, find when the temperature will be 40^oC.

Sol: By Newton's law of cooling $\frac{dT}{dt} = -k$ (T—30) gives the temperature T of the substance at any instant t, where k is the constant.

$$\Rightarrow \frac{dT}{T-30} = -kdt$$

Integrating $\log$ (T—30) $= -kt + C$, where C is a constant………(1)

Initially, when $t = 0, T = 100$

From (1), $C = \log 70 \Rightarrow \log$ (T—30) $= -kt + \log 70$

or $kt = \log 70 - \log$ (T—30)…….(2)

When, $t = 15, T = 70$

From (2), $15k = \log 70 - \log 40 \dots \dots \dots \dots..(3)$

Dividing equation (2) by (3), we have

$$\frac{t}{15} = \frac{\log 70 - \log (T-30)}{\log 70 - \log 40} \dots \dots \dots (4)$$

Now, when $T = 40$, from equation (4), we have

$$\frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40} = \frac{\log 7}{\log 7/4} = \frac{\log 7}{\log 1.75} = 3.4773$$

$\Rightarrow t = 52.16$ minutes.

Hence the temperature will be $40^o C$ after 52.16 minutes.

Problem2: A body is heated to 110^0C and is placed in air at 10^o C. After 1 hour, its temperature is 60^oC. How much additional time is required for it to cool to 30^o C.

Sol. Let the unit of time be a minute and T the temperature of the body at any instance. Thus by Newton's Law of cooling we have

$\frac{dT}{dt} = -k$ (T—10) where k is a constant.

$\Rightarrow \frac{dT}{T-10} = -kdt \ or \ \log (T-10) = -kt + C$……………..(1)

Initially when $t = 0, T = 110, C = \log 100$

Putting the value of C in (1), we have

$\log$ (T—10) $= -kt + \log 100$ or $kt = \log 100 - \log$ (T—10)

Also when $t = 60, T = 60^o$……………..(2)

$$60k = \log 100 - \log 50 \dots \dots \dots \dots (3)$$

Dividing (2) by (3) we get

$$\frac{t}{60} = \frac{\log l00 - \log (T-10)}{\log 100 - \log 50} \; or \; \frac{t}{60} = \frac{\log 100 - \log (T-10)}{\log 2}$$

Now, when $T = 30^o$

$$\frac{t}{60} = \frac{\log 100 - \log 20}{\log 2} = \frac{\log 5}{\log 2}$$

or $t = 60 \times \frac{\log 5}{\log 2} = \frac{2.2513}{L8718} \times 60$ or $t = 72.16 \text{ min}$.

Problem3: A body originally at 80^{o}C cools down to 60^{o}C in 20 minutes, the temperature of the air being 40^{o}C. What will be the temperature of the body after 40 minutes from the original.

Sol. Let the unit of time be a minute and T the temperature of the body at any instant t. Then by Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - 40)$, where k is a constant.

$\Rightarrow \frac{dT}{T-40} = -kdt$

On integrating, $\log$ (T—40) $= -kt + C$ where C is a constant……….. (1)

Initially, when $t = 0, T = 80$

From (1), $C = \log 40$

Substituting the value of C in (1), we have

$\log$ (T—40) $= -kt + \log 40$ or $kt = \log 40 - \log$ (T—40)………..(2)

Also when $t = 20, T = 60$

From (2), 20 $k = \log 40 - \log 20 \ldots..(3)$

Dividing equation (2) by (3), we have

$$\frac{t}{20} = \frac{\log 40 - \log (T - 40)}{\log 40 - \log 20} \ldots \ldots \ldots \ldots (4)$$

Now, when $t = 40$ we have from equation (4)

$$\frac{40}{20} = \frac{\log (\frac{40}{T - 40})}{\log 2} \Rightarrow 2 \log 2 = \log (\frac{40}{T - 40})$$

or $\log 2^2 = \log (\frac{40}{T-40})$ *i.e.*, $\frac{40}{T-40} = 4$ or T—40 $= 10$

$$T = 50$$

Hence, the temperature of the body after 40 minutes will be 50^oC.

Problem4: If the air is maintained at 30^oC and the temperature of the body cools from 80^oC to 60^oC in 12 minutes, find the temperature of the body after 24 minutes.

Sol. By Newton's law of cooling, we have

$\frac{dT}{dt} = -k$ (T—30), where k is a constant.

$$\Rightarrow \frac{dT}{T-30} = -kdt$$

$$\log\ (T—30) = -kt + C \dots \dots \dots \dots\ (1)$$

Initially when $t = 0, T = 80$

$$C = \log 50$$

$\Rightarrow \log$ (T—30) $= -kt + \log 50$ or $kt = \log 50 - \log$ (T—30)…………(2)

Also when $t = 12, T = 60$

$$12k = \log 50 - \log 30 \dots \dots .. (3)$$

Dividing (2) by (3), we have

$$\frac{t}{12} = \frac{\log 50 - \log (T-30)}{\log 50 - \log 30} \dots \dots \dots \dots \dots\ (4)$$

Now, when $t = 24$,

$2 = \frac{\log (\frac{50}{T-30})}{\log (\frac{50}{30})} \Rightarrow 2 \log (\frac{50}{30}) = \log (\frac{50}{T-30}) \Rightarrow \frac{25}{9} = \frac{50}{T-30} \Rightarrow 25T - 750 = 450$

$T = 48$. Hence temperature will be 48^oC.

Rate of Growth or Decay:

If the rate of change of a quantity y at any instant t is directly proportional to the quantity present at that time, then the differential equation of

(i) growth is

$$\frac{dy}{dt} = ky$$

(ii) decay is

$$\frac{dy}{dt} = -ky$$

Problem5: In a culture of yeast, at each instant, the time rate of change of active ferment is proportional to the amount present. If the active ferment doubles in two hours, how much can be expected at the end of 8 hours at the same rate of growth. Find also, how much time will elapse, before the active ferment grows to eight times its initial value.

Solution: Let y quantity of active ferment be present at any time t.

The equation of fermentation of yeast is $\frac{dy}{dt} = ky$, where k is a constant

Separating the variables and integrating,

$$\int \frac{dy}{y} = \int k\, dt$$

$$\log y = kt + c$$

Let at $t = 0, y = y_0$

Hence,

$$\log y_0 = c$$

$$\log y = kt + \log y_0$$

$$\log\left(\frac{y}{y_0}\right) = kt \ldots\ldots\ldots\ldots\ldots.(1)$$

The active ferment doubles in two hours.

Therefore, at $t = 2, y = 2y_0$

$$\log\left(\frac{2y_0}{y_0}\right) = k(2)$$

$k = \frac{1}{2}\log 2$

Substituting in Eq. (1), $\log\left(\frac{y}{y_0}\right) = \frac{t}{2}\log 2 \Rightarrow y = y_0 e^{\frac{t}{2}\log 2}$

(i) When $t = 8$

$$y = y_0 e^{4\log 2} = y_0 e^{\log 2^4} = y_0 \cdot 2^4$$

$$y = 16y_0$$

Hence, active ferment grows 16 times of its initial value at the end of 8 hours.

(ii) When $y = 8y_0$

$$8\, y_0 = y_0 e^{\frac{t}{2}\log 2}$$

$$\log 8 = \frac{t}{2}\log 2$$

$$\log 2^3 = \frac{t}{2}\log 2$$

$$3\log 2 = \frac{t}{2}\log 2$$

$t = 6$ hours

Hence, active ferment grows 8 times its initial value at the end of 6 hours.

Problem6: Find the half - life of uranium, which disintegrates at a rate proportional to the amount present at any instant. Given that m_1 and m_2 grams of uranium are present at time t_1 and t_2 respectively.

Solution: Let m grams of uranium be present at any time t. The equation of disintegration of uranium is

$\frac{dm}{dt}$ =- km where k is a constant

$$\frac{dm}{m} = -kdt$$

Integrating both the sides,

$$\log m = -kt + c$$

At $t = 0, m = m_0$

$$\log m_0 = c$$

Hence,

$$\log m = -kt + \log m_0$$

$$kt = \log m_0 - \log m \ldots\ldots\ldots..(1)$$

At $t = t_1, m = m_1$ and at $t = t_2, m = m_2$

$$kt_1 = \log m_0 - \log m_1 \ldots\ldots\ldots..(2)$$

$$kt_2 = \log m_0 - \log m_2 \ldots\ldots\ldots..(3)$$

Subtracting Eqs. (2) from (3),

$$k(t_2 - t_1) = \log m_1 - \log m_2$$

$$k = \frac{\log\left(\frac{m_1}{m_2}\right)}{t_2 - t_1}$$

Let T be the half - life of uranium, i.e., at $t = T$, $m = \frac{1}{2} m_0$

From Eq. (1),

$$kT = \log m_0 - \log \frac{m_0}{2} = \log 2$$

$$T = \frac{\log 2}{k} = \frac{(t_2 - t_1)\log 2}{\log\left(\frac{m_l}{m_2}\right)}$$