15. $\begin{vmatrix} b^2c^2+a^2 & bc+a & 1 \\ c^2a^2+b^2 & ca+b & 1 \\ a^2b^2+c^2 & ab+c & 1 \end{vmatrix}$ **16.** $\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ bcd & cda & dab & abc \end{vmatrix}$

17. If $a+b+c=0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ *(Andhra, 1999)*

18. Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0.$

19. Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = 4a^2b^2c^2.$

2.4 MATRICES

(1) Definition. *A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n* **matrix** ; *which is written as m × n matrix.* A matrix is also denoted by a single capital letter.

Thus
$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots a_{1j} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2j} & \ldots a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots a_{ij} & \ldots a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{mj} & \ldots a_{mn} \end{bmatrix}$$

is a matrix of *order mn*. It has *m rows* and *n columns*. Each of the *mn* numbers is called an element of the matrix.

To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes which respectively specify the rows and columns. Thus a_{ij} is the element in the i-th row and j-th column of A. In this notation, the matrix A is denoted by $[a_{ij}]$.

A matrix should be treated as a single entity with a number of components, rather than a collection of numbers. For example, the coordinates of a point in solid geometry, are given by a set of three numbers which can be represented by the matrix $[x, y, z]$. Unlike a determinant, a matrix cannot reduce to a single number and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is brought out by the fact that an interchange of rows and columns does not alter the determinant but gives an entirely different matrix.

(2) Special matrices

Row and column matrices. *A matrix having a single row is called a row matrix, e.g.,* $\begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$.

A matrix having a single column is called a column matrix, e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Row and column matrices are sometimes called *row vectors* and *column vectors*.

Square matrix. *A matrix having n rows and n columns is called a square matrix of order n.*

The determinant having the same elements as the square matrix A is called the *determinant of the matrix* and is denoted by the symbol $|A|$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad \text{then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

The diagonal of this matrix containing the elements 1, 3, 5 is called the *leading or principal diagonal. The sum of the diagonal elements of a square matrix A is called the* **trace** *of A.*

A square matrix is said to be **singular** *if its determinant is zero otherwise* **non-singular.**

Diagonal matrix. *A square matrix all of whose elements except those in the leading diagonal, are zero is called a diagonal matrix.*

A diagonal matrix whose all the leading diagonal elements are equal is called a *scalar matrix.* For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

Unit matrix. *A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order n* and is denoted by I_n. For example, unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix. *If all the elements of a matrix are zero, it is called a null or zero matrix* and is denoted by '0' ; *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix.}$$

Symmetric and skew-symmetric matrices. *A square matrix* $A = [a_{ij}]$ *is said to be symmetric when* $a_{ij} = a_{ji}$ *for all i and j.*

If $a_{ij} = -a_{ji}$ *for all i and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix.* Examples of symmetric and skew-symmetric matrices are

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix} \text{ respectively.}$$

Triangular matrix. A square matrix all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix.* A square matrix all of whose elements above the leading diagonal are zero, is called a *lower triangular matrix.* Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are upper and lower triangular matrices respectively.

2.5 MATRICES OPERATIONS

(1) Equality of Matrices

Two matrices A and B are said to equal if and only if

(i) they are of the same order

and (ii) each element of A is equal to the corresponding element of B.

(2) Addition and subtraction of matrices. *If A, B be two matrices of the same order, then their sum A + B is defined as the matrix each element of which is the sum of the corresponding elements of A and B.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly, *A – B is defined as a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

Obs. 1. Only matrices of the same order can be added or subtracted.

2. Addition of matrices is *commutative*,

i.e., $A + B = B + A.$

3. Addition and subtraction of matrices is *associative*.

i.e. $(A + B) - C = A + (B - C) = B + (A - C)$.

(3) Multiplication of matrix by a scalar. *The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.*

Thus,
$$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, *i.e.*, $k(A + B) = kA + kB$.

Obs. All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Example 2.12. *Find x, y, z and w given that*
$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

Solution. We have $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$

Equating the corresponding elements, we get

$$3x = x + 6,\ 3y = 5 + x + y,\ 3z = -1 + z + w,\ 3w = 2w + 5.$$

or
$$2x = 6,\ 2y = 5 + x,\ 2z = w - 1,\ w = 5$$

Hence $x = 3, y = 4, z = 2, w = 5$.

Example 2.13. *Express* $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix}$ *as the sum of a lower triangular matrix with zero leading diagonal and an upper triangular matrix.*

Solution. Let $L = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$ be the lower triangular matrix with zero leading diagonal.

and $U = \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ be the upper triangular matrix.

Then
$$\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} + \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$$

Equating corresponding elements from both sides, we obtain $3 = l, 5 = m, -7 = n, -8 = a, 11 = p, 4 = q, 13 = b, -14 = c, 6 = r$.

Hence
$$L = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 0 & 0 \\ 13 & -14 & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 11 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

(4) Multiplication of matrices. *Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be* **conformable.**

For instance, the product
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$$

is defined as the matrix
$$\begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$$

In general, if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$

be two $m \times n$ and $n \times p$ conformable matrices, then their product is defined as the $m \times p$ matrix

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{in} b_{nj}$, *i.e.*, the element in the ith row and the jth column of the matrix AB is obtained by weaving the ith row of A with jth column of B. The expression for c_{ij} is known as the *inner product* of the ith row with the jth column.

Post-multiplication and Pre-multiplication. In the product AB, the matrix A is said to be *post-multiplied* by the matrix B. Whereas in BA, the matrix A is said to be *pre-multiplied* by B. In one case the product may exist and in the other case it may not. Also the product in both cases may exist yet may or may not be equal.

Obs. 1. *Multiplication of matrices is associative. i.e.,* $(AB)C = A(BC)$
provided A, B are conformable for the product AB and B, C are conformable for the product BC. (Ex. 2.16).

Obs. 2. *Multiplication of matrices is distributive. i.e.,* $A(B + C) = AB + AC$.
provided A, B are conformable for the product AB and A, C are conformable for the product AC.

Obs. 3. Power of a matrix. If A be a square matrix, then the product AA is defined as A^2. Similarly, we define higher powers of A. *i.e.*, $A \cdot A^2 = A^3$, $A^2 \cdot A^2 = A^4$ etc.

If $A^2 = A$, then the matrix A is called idempotent.

Example 2.14. *If* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, *form the product of AB. Is BA defined?*

Solution. Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB is defined and

$$= \begin{bmatrix} 0.1 + 1.-1 + 2.2, & 0.-2 + 1.0 + 2.-1 \\ 1.1 + 2.-1 + 3.2, & 1.-2 + 2.0 + 3.-1 \\ 2.1 + 3.-1 + 4.2, & 2.-2 + 3.0 + 4.-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Again since the number of columns of $B \neq$ the number of rows of A.

$\therefore$ The product BA is not possible.

Example 2.15. *If* $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, *compute AB and BA and show that* $AB \neq BA$.

Solution. Considering rows of A and columns of B, we have

$$AB = \begin{bmatrix} 1.2 + 3.1 + 0.-1, & 1.3 + 3.2 + 0.1, & 1.4 + 3.3 + 0.2 \\ -1.2 + 2.1 + 1.-1, & -1.3 + 2.1 + 1.1, & -1.4 + 2.3 + 1.2 \\ 0.2 + 0.1 + 2.-1, & 0.3 + 0.2 + 2.1, & 0.4 + 0.3 + 2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

Again considering the rows of B and columns of A, we have

$$BA = \begin{bmatrix} 2.1 + 3.-1 + 4.0, & 2.3 + 3.2 + 4.0 & 2.0 + 3.1 + 4.2 \\ 1.1 + 2.-1 + 3.0, & 1.3 + 2.2 + 3.0 & 1.0 + 2.1 + 3.2 \\ -1.1 + 1.-1 + 2.0, & -1.3 + 1.2 + 2.0 & -1.0 + 1.1 + 2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

Evidently $AB \neq BA$.

Example 2.16. *If* $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, *find the matrix B such that* $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. *(Mumbai, 2005)*

Solution. Let $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}\begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix}$

$$= \begin{bmatrix} 3l+2p+2u & 3m+2q+2v & 3n+2r+2w \\ l+3p+u & m+3q+v & n+3r+w \\ 5l+3p+4u & 5m+3q+4v & 5n+3r+4w \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix} \quad \text{(given)}$$

Equating corresponding elements, we get

$$3l + 2p + 2u = 3, \quad l + 3p + u = 1, \quad 5l + 3p + 4u = 5 \qquad ...(i)$$

$$3m + 2q + 2v = 4, \quad m + 3q + v = 6, \quad 5m + 3q + 4v = 6 \qquad ...(ii)$$

$$3n + 2r + 2w = 2, \quad n + 3r + w = 1, \quad 5n + 3r + 4w = 4 \qquad ...(iii)$$

Solving the equations (*i*), we get $l = 1, p = 0, u = 0$

Similarly equations (*ii*) give $m = 0, q = 2, v = 0$

and equations (*iii*) give $n = 0, r = 0, w = 1$

Thus, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 2.17. *Prove that* $A^3 - 4A^2 - 3A + 11I = 0$, *where* $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution. $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$A^3 - 4A^2 - 3A + 11I$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6-0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Example 2.18. *By mathematical induction, prove that if*

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}.$$

Solution. When $n = 1$, A^n gives $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$ ...(*i*)

Let us assume that the result is true for any positive integer k, so that

$$A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

$$\therefore \quad A^{k+1} = A^k \,.\, A^1 = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11(1+10k)-100k & -25(1+10k)+225k \\ 44k+4(1-10k) & -100k-9(1-10k) \end{bmatrix}$$

$$= \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

This is true for $n = k + 1$ ...(*ii*)

We have seen in (*i*) that the result is true for $n = 1$.

$\therefore$ It is true for $n = 1 + 1 = 2$ [by (*ii*)]

Similarly, it is true for $n = 2 + 1 = 3$ and so on.

Hence by mathematical induction, the result is true for all positive integers n.

Example 2.19. *Prove that (AB)C = A(BC), where A, B, C are matrices conformable for the products.* (*J.N.T.U., 2002 S*)

Solution. Let $A = [a_{ij}]$ be of order $m \times n$, $B = [b_{ij}]$ be of order $n \times p$ and $C = [c_{ij}]$ be of order of $p \times q$.

Then $\quad AB = [a_{ik}]\,[b_{kj}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

$$\therefore \quad (AB)C = \left[\sum_{k=1}^{n} a_{ik}\, b_{kl}\right] . [c_{lj}] = \left[\sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik}\, b_{kl}\right) c_{lj}\right] = \left[\sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right]$$

Similarly, $\quad BC = [b_{kl}] \,.\, [c_{lj}] = \sum_{l=1}^{p} b_{kl}\, c_{lj}$

$$\therefore \quad A(BC) = [a_{ik}] \left[\sum_{l=1}^{p} b_{kl}\, c_{lj}\right] = \left[\sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl}\, c_{lj}\right)\right] = \left[\sum_{k=1}^{n} \left(\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right)\right]$$

Hence $\quad (AB)C = A(BC)$.

PROBLEMS 2.2

1. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?
2. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.
3. Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exist. Find x and y.
4. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, calculate the product AB.
5. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, find AB or BA, whichever exists.
6. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)C = A(BC)$ and $A(B + C) = AB + AC$.
7. Evaluate (*i*) $[x, y, z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -6 & 4 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$; (*iii*) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times [3 \;\; 2]$

8. Prove that the product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ and ϕ differ by an odd multiple of $\pi/2$.

9. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

10. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, find the value of $A^2 - 6A + 8I$, where I is a unit matrix of second order. (B.P.T.U., 2006)

11. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, and I is the unit matrix of order 3, evaluate $A^2 - 3A + 9I$.

12. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$, verify the result $(A + B)^2 = A^2 + BA + AB + B^2$.

13. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + FE^2 = E$.

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer.

15. Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular and U is upper triangular matrix.

2.6 RELATED MATRICES

(1) Transpose of a matrix. *The matrix obtained from any given matrix A, by interchanging rows and columns is called the* **transpose** *of A and is denoted by A′.*

Thus the transposed matrix of $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ is $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

Clearly, the transpose of an $m \times n$ matrix is an $n \times m$ matrix. Also the transpose of the transpose of a matrix coincides with itself, *i.e.*, $(A')' = A$.

For a symmetric matrix, $A' = A$ and for a skew-symmetric matrix, $A' = -A$.

Obs. 1. *The transpose of the product of the two matrices is the product of their transposes taken in the reverse order i.e.,* $(\mathbf{AB})' = \mathbf{B'A'}$.

For, the element in the ith row and jth col. of $(AB)'$

= element in the jth row and ith col. of AB = inner product of jth row of A with ith col. of B

= inner product of jth col. of A' with ith row of B' = element in the ith row and jth col. of $B'A'$

Hence $(AB)' = B'A'$.

Obs. 2. *Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.* (J.N.T.U., 2001)

Let A be the given square matrix, then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

$\therefore$ $B' = \left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}(A' + A) = B$, *i.e.*, $B = \frac{1}{2}(A + A')$ is a symmetric matrix.

Again, $C' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A) = -C$, *i.e.*, $C = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Hence A can be expressed as the sum of a symmetric and a skew-symmetric matrix.

To prove the uniqueness, assume that P is a symmetric matrix and Q is a skew-symmetric matrix such that $A = P + Q$.

Then $A' = (P + Q)' = P' + Q' = P - Q$

Thus, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

which shows that there is one and only one way of expressing A as the sum of a symmetric and skew-symmetric matrix.

Example 2.20. *Express the matrix A as the sum of a symmetric and a skew-symmetric matrix where*

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

Solution. We have $A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

Then $A + A' = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$ and $A - A' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$\therefore$ $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}.$$

(2) Adjoint of a square matrix. The determinant of the square matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in Δ is

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$. Then the transpose of this matrix, *i.e.*, $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

is called the *adjoint of the matrix A* and is written as *Adj. A*.

Thus the **adjoint** *of A is the transposed matrix of cofactors of A.*

(3) Inverse of a matrix. *If A be any matrix, then a matrix B if it exists, such that AB = BA = I, is called the* **Inverse** *of A* which is denoted by A^{-1} so that $AA^{-1} = I$.

Also $$\mathbf{A^{-1} = \frac{Adj.A}{|A|}}$$

For $$A\,(Adj\,A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$A \cdot \frac{Adj.A}{|A|} = I \quad [\because \ |A| \neq 0] \quad \text{or} \quad \frac{Adj.A}{|A|} \text{ is the inverse of } A.$$

Obs. 1. *Inverse of a matrix, is unique.*

If possible, let the two inverses of the matrix A be B and C,

then $AB = BA = I$ and $AC = CA = I$

$\therefore$ $CAB = (CA)B = IB = B$ and $CAB = C(AB) = CI = C$

Thus, $B = C$.

Obs. 2. *The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.,* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ *(Assam, 1999)*

If A, B be two matrices, then the reciprocal of their product is $(AB)^{-1}$.

Clearly, $(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by Associative law]

$$= AIA^{-1} = AA^{-1} = I.$$

Similarly, $(B^{-1}A^{-1}) \cdot (AB) = I$

Hence $B^{-1}A^{-1}$ is the reciprocal of AB.

Obs. 3. Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra.

i.e., if $[A][B] = [C][D]$, then $[A]^{-1}[A][B] = [A^{-1}][C][D]$

or $B = A^{-1}[C][D]$, *i.e.*, $\dfrac{[C][D]}{[A]} = A^{-1}[C][D]$.

Example 2.21. *Find the inverse of* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution. The determinant of the given matrix A is

$$\Delta = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{(say)}$$

If $A_1, A_2, \ldots\ldots$ be the cofactors of $a_1, a_2 \ldots$ in Δ, then $A_1 = -24, A_2 = -8, A_3 = -12$; $B_1 = 10, B_2 = 2, B_3 = 6$; $C_1 = 2, C_2 = 2, C_3 = 2$.

Thus $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8.$

and $$adj\ A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}.$$

Hence the inverse of the given matrix A

$$= \frac{adj\ A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Note. For other methods see Examples 2.25 ; 2.28 and 2.46.

Example 2.22. *Find the matrix A if* $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$ *(Mumbai, 2008)*

Solution. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = B, \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = C$ and $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} = D$, then

$$BAC = D \quad \text{or} \quad AC = B^{-1}D$$

$\therefore$ $$A = B^{-1}DC^{-1}$$

Now, $$B^{-1} = \frac{adj\ B}{|B|} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Similarly, $$C^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Hence, $$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 14 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}.$$

PROBLEMS 2.3

1. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I = A'A$, where I is the unit matrix.

2. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix :

   (i) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$

3. If A is a non-singular matrix of order n, prove that $A\ adj\ A = |\ A\ |I$. (*Mumbai, 2006*)

   Verify that $A\ (adj\ A) = (adj\ A)\ A = |\ A\ |\ I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$.

4. Find the inverse of the matrix (i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (*Mumbai, 2009*) (ii) $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ (*B.P.T.U., 2005*)

5. If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, compute $adj\ A$ and A^{-1}. Also find B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. (*Mumbai, 2008*)

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, (i) find A^{-1} ; (ii) show that $A^3 = A^{-1}$.

7. Find the inverse of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and if } A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix},$$

   show that SAS^{-1} is a diagonal matrix dig (2, 3, 1). (*Mumbai, 2007*)

8. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, prove that $A^{-1} = A'$.

9. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$.

10. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$, where A' is the transpose of A.

11. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

12. If A is a square matrix, show that (i) $A + A'$ is symmetric, and (ii) $A - A'$ is skew-symmetric. (*P.T.U., 1999*)

13. If $D = \text{diag}\ [d_1, d_2, d_3]$, $d_1, d_2, d_3 \neq 0$, prove that $D^{-1} = \text{diag}\ [d_1^{-1}, d_2^{-1}, d_3^{-1}]$.

14. If A and B are square matrices of the same order and A is symmetrical, show that $B'AB$ is also symmetrical. [**Hint.** Show that $(B'AB)' = B'AB$]

15. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

2.7 (1) RANK OF A MATRIX

If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the *minor of A of order r*. Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

Def. *A matrix is said to be of rank r when*

(i) *it has at least one non-zero minor of order r,*

and (ii) *every minor of order higher than r vanishes.*

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

If a matrix has a non-zero minor of order r, its rank is $\geq r$.

If all minors of a matrix of order $r + 1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

(2) Elementary transformation of a matrix. The following operations, three of which refer to rows and three to columns are known as *elementary transformations :*

I. The interchange of any two rows (columns).

II. The multiplication of any row (column) by a non-zero number.

III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Notation. The *elementary row transformations* will be denoted by the following symbols :

(*i*) R_{ij} for the interchange of the ith and jth rows.

(*ii*) kR_i for multiplication of the ith row by k.

(*iii*) $R_i + pR_j$ for addition to the ith row, p times the jth row.

The corresponding column transformation will be denoted by writing C in place of R.

Elementary transformations do not change either the order or rank of a matrix. While the value of the minors may get changed by the transformation I and II, their zero or non-zero character remains unaffected.

(3) Equivalent matrix. *Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.* Two equivalent matrices have the same order and *the same rank.* The symbol ~ is used for equivalence.

Example 2.23. *Determine the rank of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (V.T.U., 2011)

Solution. (*i*) Operate $R_2 - R_1$ and $R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$.

$\therefore$ $\rho(A) = 2$. Hence the rank of the given matrix is 2.

(*ii*) Given matrix

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[Operating $C_3 - C_1, C_4 - C_1$] [Operating $R_3 - R_1, R_4 - R_1$]

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ (say)}$$

[Operating $R_3 - 3R_2, R_4 - R_2$] [Operating $C_3 + 3C_2, C_4 + C_2$]

Obviously, the 4th order minor of A is zero. Also every 3rd order minor of A is zero. But, of all the 2nd order minors, only $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. $\therefore$ $\rho(A) = 2$.

Hence the rank of the given matrix is 2.

(4) Elementary matrices. *An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.*

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \text{ are } R_{23} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix} = C_{23}\,;\ kR_2 = \begin{bmatrix}1 & 0 & 0\\0 & k & 0\\0 & 0 & 1\end{bmatrix}\,;\ R_1 + pR_2 = \begin{bmatrix}1 & p & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

(5) Theorem. *Elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrices.*

Consider the matrix $A = \begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix}$

Then $$R_{23} \times A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix} \times \begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix} = \begin{bmatrix}a_1 & b_1 & c_1\\a_3 & b_3 & c_3\\a_2 & b_2 & c_2\end{bmatrix}$$

So a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of A. Similarly, pre-multiplication by kR_2 will multiply the 2nd row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of A to its 1st row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A. It can easily be seen that post mutiplication will perform the elementary column transformations.

(6) Gauss-Jordan method of finding the inverse*. *Those elementary row transformations which reduce a given square matrix A to the unit matrix, when applied to unit matrix I give the inverse of A.*

Let the successive row transformations which reduce A to I result from pre-multiplication by the elementary matrices $R_1, R_2, \ldots R_i$ so that

$$R_i R_{i-1} \ldots R_2 R_1 A = I$$

$\therefore$ $$R_i R_{i-1} \ldots R_2 R_1 A A^{-1} = I A^{-1}$$

or $$R_i R_{i-1} \ldots R_2 R_1 I = A^{-1} \qquad [\because\ AA^{-1} = I]$$

Hence the result.

Working rule to evaluate A^{-1}. *Write the two matrices A and I side by side. Then perform the same row transformations on both. As soon as A is reduced to I, the other matrix represents A^{-1}.*

Example 2.24. *Using the Gauss-Jordan method, find the inverse of the matrix*

$$\begin{bmatrix}1 & 1 & 3\\1 & 3 & -3\\-2 & -4 & -4\end{bmatrix}.$$ *(Kurukshetra, 2006)*

Solution. Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{rrr:rrr}1 & 1 & 3 & 1 & 0 & 0\\1 & 3 & -3 & 0 & 1 & 0\\-2 & -4 & -4 & 0 & 0 & 1\end{array}\right] \qquad \text{(Operate } R_2 - R_1 \text{ and } R_3 + 2R_1\text{)}$$

$$\sim \left[\begin{array}{rrr:rrr}1 & 1 & 3 & 1 & 0 & 0\\0 & 2 & -6 & -1 & 1 & 0\\0 & -2 & 2 & 2 & 0 & 1\end{array}\right] \qquad \text{(Operate } \tfrac{1}{2}R_2 \text{ and } \tfrac{1}{2}R_3\text{)}$$

$$\sim \left[\begin{array}{rrr:rrr}1 & 1 & 3 & 1 & 0 & 0\\0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0\\0 & -1 & 1 & 1 & 0 & \frac{1}{2}\end{array}\right] \qquad \text{(Operate } R_1 - R_2 \text{ and } R_3 + R_2\text{)}$$

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*Named after the great German mathematician *Carl Friedrich Gauss* (1777–1855) who made his first great discovery as a student at Gottingen. His important contributions are to algebra, number theory, mechanics, complex analysis, differential equations, differential geometry, non-Euclidean geometry, numerical analysis, astronomy and electromagnetism. He became director of the observatory at Gottingen in 1807.

Name after another German mathematician and geodesist *Wilhelm Jordan* (1842–1899).

$$\sim \begin{bmatrix} 1 & 0 & 6 : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 : & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \left[\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } \left(-\frac{1}{2}\right)R_2\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 : & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 : & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ [cf. Example 2.21]

(7) Normal form of a matrix. *Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called the } \textbf{normal form} \text{ of } A. \qquad ...(i)$$

Cor. 1. *The rank of a matrix A is r if and only if it can be reduced to the normal form (i).*

Cor. 2. *Since each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result :*

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that PAQ equals (i).

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n respectively.

Example 2.25. *Reduce the following matrix into its normal form and hence find its rank.*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}. \qquad \text{(U.P.T.U., 2005)}$$

Solution.

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \qquad [\text{By } R_{12}]$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } R_4 - R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } R_2 - R_3\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } R_3 - 4R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } C_3 + 6C_2, C_4 + 3C_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\text{By } \frac{1}{33} C_3\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } C_4 - 22C_3\text{]}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$.

Example 2.26. *For the matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$,

find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

(Kurukshetra, 2005)

Solution. We write $A = IAI$, *i.e.*, $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (post-factor) of A to the same.

Operate $C_2 - C_1, C_3 - 2C_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $C_3 - C_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_3 + R_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **and** $\rho(A) = 2$.

PROBLEMS 2.4

Determine the rank of the following matrices (1–4) :

1. $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ *(P.T.U., 2005)*

2. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ *(W.B.T.U., 2005)*

3. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ *(Kottayam, 2005)*

4. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ *(Rohtak, 2004)*

5. $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$ *(Bhopal, 2008)*

6. Determine the values of p such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$ is 3. *(Mumbai, 2007)*

7. Use Gauss-Jordan method to find the inverse of the following matrices :

   (*i*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ *(Mumbai, 2008)*

   (*iii*) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ *(B.P.T.U., 2006)* (*iv*) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *(Kurukshetra, 2006)*

8. Find the non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ is reduced to normal form. Also find its rank. *(S.V.T.U., 2009 ; Mumbai, 2007)*

9. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Also find two non-signular matrices P and Q such that $PAQ = I$, where I is the unit matrix and verify that $A^{-1} = QP$.

10. Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices :

    (*i*) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ *(Rohtak, 2004)* (*ii*) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

11. Reduce each of the following matrices to normal form and hence find their ranks :

    (*i*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ *(Kurukshetra, 2005)* (*ii*) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ *(Bhopal 2009)*

    (*iii*) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ *(Mumbai, 2008)* (*iv*) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ *(U.T.U., 2010)*

2.8 PARTITION METHOD OF FINDING THE INVERSE

According to this method of finding the inverse, if the inverse of a matrix A_n of order n is known, then the inverse of the matrix A_{n+1} can easily be obtained by adding $(n+1)$th row and $(n+1)$th column to A_n.

Let $$A = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix}$$

where A_2, X_2 are column vectors and A_3', X_3' are row vectors (being transposes of column vectors A_3, X_3) and α, x are ordinary numbers. We also assume that A_1^{-1} is known.

Then, $AA^{-1} = I_{n+1}$, *i.e.*, $$\begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

gives
$$A_1X_1 + A_2X_3' = I_n \qquad ...(i)$$
$$A_1X_2 + A_2x = 0 \qquad ...(ii)$$
$$A_3'X_1 + \alpha X_3' = 0 \qquad ...(iii)$$
$$A_3'X_2 + \alpha x = 1 \qquad ...(iv)$$

From (*ii*), $X_2 = -A_1^{-1}A_2x$ and using this, (*iv*) gives $x = (\alpha - A_3'A_1^{-1}A_2)^{-1}$

Hence x and then X_2 are given.

Also from (*i*), $X_1 = A_1^{-1}(I_n - A_2X_3')$

and using this, (*iii*) gives $X_3' = -A_3'A_1^{-1}(\alpha - A_3'A_1^{-1}A_2)^{-1} = -A_3'A_1^{-1}x$

Then X_1 is determined and hence A^{-1} is computed.

Obs. This is also known as the *'Escalator method'*. For evaluation of A^{-1} we only need to determine two inverse matrices A_1^{-1} and $(\alpha - A_3'A_1^{-1}A_2)^{-1}$.

Example 2.27. *Using the partition method, find the inverse of* $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

Solution. Let $$A = \begin{bmatrix} 1 & 1 & : & 1 \\ 4 & 3 & : & -1 \\ \cdots & \cdots & : & \cdots \\ 3 & 5 & : & 3 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix}$$

so that $$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let $$A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix} \text{ so that } AA^{-1} = I.$$

$$\alpha - A_3'A_1^{-1}A_2 = 3 + [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$$\therefore \qquad x = (\alpha - A_3'A_1^{-1}A_2)^{-1} = -\frac{1}{10}$$

Also, $$X_2 = -A_1^{-1}A_2x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10}\begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then $$X_3' = -A_3'A_1^{-1}x = [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10}[-11 \quad 2]$$

Finally, $$X_1 = A_1^{-1}(I - A_2X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-11 \quad 2]$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}.$$

Example 2.28. *If A and C are non-singular matrices, then show that* $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Hence find inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$. *(Mumbai, 2005)*

Solution. Let the given matrix be $M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ and its inverse be $M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ both in the partioned form where A, B, C, P, Q, R, S are all matrices.

$\therefore$ $$MM^{-1} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I$$

or $$\begin{bmatrix} AP + OR & AQ + OS \\ BP + CR & BQ + CS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$\therefore$ *Equating corresponding elements, we have*

$$AP + OR = I, AQ + OS = 0, BP + CR = 0, BQ + CS = I.$$

Second relation gives $AQ = 0$, *i.e.,* $Q = 0$ as A is non-singular.

First relation gives $AP = I$, *i.e.,* $P = A^{-1}$.

From third equation, $BP + CR = 0$, *i.e.,* $CR = -BP = -BA^{-1}$

$\therefore$ $C^{-1}CR = -C^{-1}BA^{-1}$ or $IR = -C^{-1}BA^{-1}$ or $R = -C^{-1}BA^{-1}$

From fourth equation, $BQ + CS = I$, or $CS = I$ or $S = C^{-1}$

Hence $$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$

(*ii*) Let $$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$$

Whence $$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ $$-C^{-1}(BA^{-1}) = -\frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\left\{\frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= -\frac{1}{24}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{24}\begin{bmatrix} 18 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence, $$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 \\ 0 & -1/6 & 0 & 1/3 \end{bmatrix}.$$

PROBLEMS 2.5

Find the inverse of each of the following matrices using the partition method :

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (*Nagpur, 1997*)

2. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$.

2.9 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

(1) Method of determinants—Cramer's* rule

Consider the equations $\left\{\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}\right\}$ ...(*i*)

If the determinant of coefficient be $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $$x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Operate } C_1 + yC_2 + zC_3]$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{By } (i)]$$

Thus $$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ provided } \Delta \neq 0. \qquad ...(ii)$$

Similarly, $$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad ...(iii)$$

and $$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad ...(iv)$$

Equation (*ii*), (*iii*) and (*iv*) *giving the values of x, y, z constitute the* **Cramer's rule,** which reduces the solution of the linear equations (*i*) to a problem in evaluation of determinants.

(2) Matrix inversion method

If $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then the equations (*i*) are equivalent to the matrix equation $AX = D$ ...(*v*)

where A is the *coefficient matrix.*

Multiplying both sides of (*v*) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX = A^{-1}D \quad \text{or} \quad IX = A^{-1}D \qquad [\because A^{-1}A = I]$$

**Gabriel Cramer* (1704–1752), a Swiss mathematician.

or $$X = A^{-1}D \quad i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad ...(vi)$$

where A_1, B_1 etc. are the cofactors of a_1, b_1 etc. in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(\Delta \neq 0)$

Hence equating the values of x, y, z to the corresponding elements in the product on the right side of (vi), we get the desired solutions.

Obs. *When A is a singular matrix, i.e., $\Delta = 0$, the above methods fail. These also fail when the number of equations and the number of unknowns are unequal. Matrices can, however, be usefully applied to deal with such equations as will be seen in § 2.10(2).*

Example 2.29. *Solve the equations $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$ by (i) determinants (ii) matrices.*

Solution. (*i*) *By determinants :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 2(1-4) + (-1+6) = 8 \quad [\text{Expanding by } C_1]$$

$\therefore$ $$x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad [\text{Expand by } C_1]$$
$$= \frac{1}{8}[3(-3+2) + 3(1-4) + 4(-1+6)] = 1$$

Similarly, $$y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2 \quad \text{and} \quad z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

Hence $x = 1, y = 2, z = -1$.

Note. The use of Cramer's rule involves a lot of labour when the number of equations exceeds four. In such and other cases, the numerical methods given in § 28.4 to 28.6 are perferable.

(*ii*) *By matrices :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\text{say}).$$

Then $A_1 = -1, A_2 = 3, A_3 = 5$; $B_1 = -3, B_2 = 1, B_3 = 7$; $C_1 = 7, C_2 = -5, C_3 = -11$.

Also $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = 8$.

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
$$= \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1, y = 2, z = -1$.

Example 2.30. *Solve the equations $x_1 - x_2 + x_3 + x_4 = 2$; $x_1 + x_2 - x_3 + x_4 = -4$; $x_1 + x_2 + x_3 - x_4 = 4$; $x_1 + x_2 + x_3 + x_4 = 0$, by finding the inverse by elementary row operations.*

Solution. Given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

To find A^{-1}, we write

$$[A:I]=\left[\begin{array}{cccc:cccc}1&-1&1&1&1&0&0&0\\1&1&-1&1&0&1&0&0\\1&1&1&-1&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right]\begin{bmatrix}R_2-R_1\\R_3+R_1\\R_4+R_1\end{bmatrix}$$

$$=\left[\begin{array}{cccc:cccc}1&-1&1&1&1&0&0&0\\0&2&-2&0&-1&1&0&0\\2&0&2&0&1&0&1&0\\2&0&2&2&1&0&0&1\end{array}\right]\begin{bmatrix}\frac{1}{2}R_2\\\frac{1}{2}R_3\\\frac{1}{2}R_4\end{bmatrix}$$

$$=\left[\begin{array}{cccc:cccc}1&-1&1&1&1&0&0&0\\0&1&-1&0&-1/2&1/2&0&0\\1&0&1&0&1/2&0&1/2&0\\1&0&1&1&1/2&0&0&1/2\end{array}\right]\begin{bmatrix}R_3-R_2\\R_4-R_3\end{bmatrix}$$

$$=\left[\begin{array}{cccc:cccc}1&0&0&1&1/2&1/2&0&0\\0&1&-1&0&-1/2&1/2&0&0\\1&0&1&0&1/2&0&1/2&0\\0&0&0&1&0&0&-1/2&1/2\end{array}\right]\begin{bmatrix}R_1-R_4\\R_2+R_3\end{bmatrix}$$

$$=\left[\begin{array}{cccc:cccc}1&0&0&0&1/2&1/2&+1/2&-1/2\\1&1&0&0&0&1/2&1/2&0\\1&0&1&0&1/2&0&1/2&0\\0&0&0&1&0&0&-1/2&1/2\end{array}\right]\begin{bmatrix}R_2-R_1\\R_3-R_1\end{bmatrix}$$

$$=\left[\begin{array}{cccc:cccc}1&0&0&0&1/2&1/2&1/2&-1/2\\0&1&0&0&-1/2&0&0&1/2\\0&0&1&0&0&-1/2&0&1/2\\0&0&0&1&0&0&-1/2&1/2\end{array}\right]$$

Thus, $$A^{-1}=\begin{bmatrix}1/2&1/2&1/2&-1/2\\-1/2&0&0&1/2\\0&-1/2&0&1/2\\0&0&-1/2&1/2\end{bmatrix}$$

Hence, $$X=A^{-1}B=\begin{bmatrix}1/2&1/2&1/2&-1/2\\-1/2&0&0&1/2\\0&-1/2&0&1/2\\0&0&-1/2&1/2\end{bmatrix}\begin{bmatrix}2\\-4\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix}$$

i.e., $$x_1=1, x_2=-1, x_3=2, x_4=-2.$$

PROBLEMS 2.6

Solve the following equations with the help of determinants (1 to 4) :

1. $x+y+z=4$; $x-y+z=0$; $2x+y+z=5$. *(Osmania, 2003)*
2. $x+3y+6z=2$; $3x-y+4z=9$; $x-4y+2z=7$.
3. $x+y+z=6.6$; $x-y+z=2.2$; $x+2y+3z=15.2$.
4. $x^2z^3/y=e^8$; $y^2z/x=e^4$; $x^3y/z^4=1$.
5. $2vw-wu+uv=3uvw$; $3vw+2wu+4uv=19uvw$; $6vw+7wu-uv=17uvw$.

Solve the following system of equations by matrix method (6 to 8) :

6. $x_1+x_2+x_3=1, x_1+2x_2+3x_3=6, x_1+3x_2+4x_3=6$. *(P.T.U., 2006)*
7. $x+y+z=3$; $x+2y+3z=4$; $x+4y+9z=6$. *(Bhopal, 2003)*
8. $2x-3y+4z=-4, x+z=0, -y+4z=2$. *(W.B.T.U., 2005)*
9. $2x-y+3z=8$; $x-2y-z=-4$; $3x+y-4z=0$. *(Mumbai, 2005)*
10. $2x_1+x_2+2x_3+x_4=6, 4x_1+3x_2+3x_3-3x_4=-1, 6x_1-6x_2+6x_3+12x_4=36, 2x_1+2x_2-x_3+x_4=10$. *(U.P.T.U., 2001)*

11. By finding A^{-1}, solve the linear equation $AX = B$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

12. In a given electrical network, the equations for the currents i_1, i_2, i_3 are
$3i_1 + i_2 + i_3 = 8$; $2i_1 - 3i_2 - 2i_3 = -5$; $7i_1 + 2i_2 - 5i_3 = 0$.
Calculate i_1 and i_3 by Cramer's rule.

13. Using the loop current method on a circuit, the follwoing equations are obtained :
$7i_1 - 4i_2 = 12, -4i_1 + 12i_2 - 6i_3 = 0, -6i_2 + 14i_3 = 0$.
By matrix method, solve for i_1, i_2 and i_3.

14. Solve the following equations by calculating the inverse by elementary row operations :
$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$; $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $x_1 + x_2 - 3x_3 - 4x_4 = -1$; $2x_1 + x_2 + 5x_3 + x_4 = 5$.

2.10 (1) CONSISTENCY OF LINEAR SYSTEM OF EQUATIONS

Consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= k_m \end{aligned}\right\} \quad \dots(i)$$

containing the n unknowns $x_1, x_2, \dots, x_n$. To determine whether the equations (i) are consistent (*i.e.*, possess a solution) or not, we consider the ranks of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

A is the *co-efficient matrix* and K is called the *augmented matrix* of the equations (i).

(2) **Rouche's theorem.** *The system of equations (i) is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank otherwise the system is inconsistent.*

Proof. We consider the following two possible cases :

I. Rank of A = *rank of* $K = r$ ($r \le$ the smaller of the numbers m and n). The equations (i) can, be suitable row operations, be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1 \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r \end{aligned}\right\} \quad \dots(ii)$$

and the remaining $m - r$ equations being all of the form $0.x_1 + 0.x_2 + \dots\dots + 0.x_n = 0$.

The equations (ii) will have a solution, though $n - r$ of the unknowns may be chosen arbitrarily. The solution, will be unique only when $r = n$. Hence the equations (i) are consistent.

II. Rank of A (*i.e.*, r) < *rank of* K. In particular, let the rank of K be $r + 1$. In this case, the equations (i) will reduce, by suitable row operations, to

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1, \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&, \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r, \\ 0.x_1 + 0.x_2 + \dots + 0.x_n &= l_{r+1}, \end{aligned}$$

and the remaining $m - (r + 1)$ equations are of the form $0.x_1 + 0.x_2 + \dots + 0.x_n = 0$.

Clearly, the $(r + 1)$th equation cannot be satisfied by any set of values for the unknowns. Hence the equations (i) are inconsistent.

[Procedure to test the consistency of a system of equations in n unknowns :

Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of K be r'.

(*i*) *If* $r \neq r'$, *the equations are inconsistent, i.e., there is no solution.*

(*ii*) *If* $r = r' = n$, *the equations are consitent and there is a unique solution.*

(*iii*) *If* $r = r' < n$, *the equations are consistent and there are infinite number of solutions.* Giving arbitrary values to $n - r$ of the unknowns, we may express the other r unknowns in terms of these.]

Example 2.31. *Test for consistency and solve*

$$5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5.$$

(*Bhopal, 2008 ; J.N.T.U., 2005 ; P.T.U., 2005*)

Solution. We have $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

Operate $3R_1, 5R_2$, $\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$

Operate $\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2$, $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$

Operate $R_3 - R_1 + R_2, \frac{1}{7}R_1$, $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

The ranks of coefficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4,\ 11y - z = 3, \quad \therefore \quad y = \frac{3}{11} + \frac{z}{11} \quad \text{and} \quad x = \frac{7}{11} - \frac{16}{11}z$$

where z is a parameter.

Hence $x = \frac{7}{11}, y = \frac{3}{11}$ and $z = 0$, is a particular solution.

Obs. In the above solution, the coefficient matrix is reduced to an upper triangular matrix by row-transformations.

Example 2.32. *Investigate the values of* λ *and* μ *so that the equations*

$$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu,$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

(*Mumbai, 2007 ; V.T.U., 2007*)

Solution. We have $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

The system admits of unique solution if, and only if, the coefficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have no solution for those values of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But A is of rank 2 and K is not of rank 2 unless $\mu = 9$. Thus if $\lambda = 5$ and $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

Example 2.33. *Test for consistency the following equations and solve them if consistent :* $x - 2y + 3t = 2$, $2x + y + z + t = -4$; $4x - 3y + z + 7t = 8$. *(Mumbai, 2008)*

Solution. Given equation can be written as

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$,
$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, rank of the coefficient matrix is 2 and the rank of augmented matrix is also 2. Hence the given equations are consistent. But the rank $2 < 4$, the number of unknowns.

$\therefore$ The number of parameters is $4 - 2 = 2$

Thus the equations have doubly infinite solutions. Now putting $t = k_1$ and $z = k_2$ in

$$x - 2y + 3t = 2 \quad \text{and} \quad 5y + z - 5t = 0,$$

we get $x - 2y + 3k_1 = 2$ and $5y + k_2 - 5k_1 = 0$

Hence $y = k_1 - k_2/5$

and
$$\begin{aligned} x &= 2 + 2y - 3k_1 \\ &= 2 + 2(k_1 - k_2/5) - 3k_1 \\ &= 2 - k_1 - \frac{2}{5} k_2 \end{aligned}$$

(3) System of linear homogeneous equations. Consider the homogeneous linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad \ldots(iii)$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary row operations.

I. *If $r = n$, the equations (iii) have only a trivial zero solution*

$$x_1 = x_2 = \ldots = x_n = 0$$

If $r < n$, the equation (iii) have $(n - r)$ linearly independent solutions.

The number of linearly independent solutions is $(n - r)$ means, if arbitrary values are assigned to $(n - r)$ of the variables, the values of the remaining variables can be uniquely found.

Thus the equations (*iii*) will have an infinite number of solutions.

II. *When $m < n$* (*i.e.*, the number of equations is less than the number of variables), *the solution is always other than $x_1 = x_2 = \ldots = x_n = 0$.* The number of solutions is infinite.

III. *When $m = n$* (*i.e.*, the number of equations = the number of variables), *the necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots = x_n = 0$, is that the determinant of the coefficient matrix is zero.* In this case the equations are said to be *consistent* and such a solution is called *non-trivial solution. The determinant is called the* **eliminant** *of the equations.*

Example 2.34. *Solve the equations*

(i) $x + 2y + 3z = 0,\ 3x + 4y + 4z = 0,\ 7x + 10y + 12z = 0$

(ii) $4x + 2y + z + 3w = 0,\ 6x + 3y + 4z + 7w = 0,\ 2x + y + w = 0.$

Solution. (*i*) Rank of the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \qquad \text{[Operating } R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{[Operating } R_3 - 7R_1 - 2R_2]$$

is 3 which = the number of variables (*i.e.*, $r = n$)

$\therefore$ The equations have only a trivial solution : $x = y = z = 0$.

(*ii*) Rank of the coefficient matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \qquad \text{[Operating } R_2 - \tfrac{3}{2} R_1, R_3 - \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[Operating } R_3 + \tfrac{1}{5} R_2$$

is 2 which < the number of variable (*i.e.*, $r < n$)

$\therefore$ Number of independent solutions = 4 – 2 = 2. Given system is equivalent to

$$4x + 2y + z + 3w = 0,\ z + w = 0.$$

$\therefore$ We have $z = -w$ and $y = -2x - w$

which give an infinite number of non-trivial solutions, x and w being the parameters.

Example 2.35. *Find the values of k for which the system of equations* $(3k-8)x + 3y + 3z = 0,\ 3x + (3k-8)y + 3z = 0,\ 3x + 3y + (3k-8)z = 0$ *has a non-trivial solution.* (*U.P.T.U., 2006*)

Solution. For the given system of equations to have a non-trivial solution, the determinant of the coefficient matrix should be zero.

$$\text{i.e.,} \quad \begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0 \qquad \text{[Operating } C_1 + (C_2 + C_3)]$$

$$\text{or} \quad (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \quad \text{or} \quad (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0 \qquad \text{[Operating } R_2 - R_1, R_3 - R_1]$$

$$\text{or} \quad (3k-2)(3k-11)^2 = 0 \text{ whence } k = 2/3,\ 11/3,\ 11/3.$$

Example 2.36. *If the following system has non-trivial solution, prove that* $a + b + c = 0$ *or* $a = b = c$ *:* $ax + by + cz = 0,\ bx + cy + az = 0,\ cx + ay + bz = 0.$ (*Mumbai, 2006*)

Solution. For the given system of equations to have non-trivial solution, the determinant of the coefficient matrix is zero.

$$\text{i.e.,} \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \qquad \text{[Operating } R_1 + R_2 + R_3]$$

$$\text{or} \quad (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad \text{or} \quad (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0 \qquad \text{[Operating } C_2 - C_1, C_3 - C_1]$$

or $\quad (a+b+c)\,[(c-b)(b-c)-(a-c)(a-b)]=0$

or $\quad (a+b+c)(-a^2-b^2-c^2+ab+bc+ca)=0$

i.e., $\quad a+b+c=0 \quad \text{or} \quad a^2+b^2+c^2-ab-bc-ca=0$

or $\quad a+b+c=0 \quad \text{or} \quad \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$

or $\quad a+b+c=0\,;\ a=b,\ b=c,\ c=a.$

Hence the given system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.

Example 2.37. *Find the values of λ for which the equations*

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$
$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$
$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

are consistent, and find the ratios of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greatest of these values. (Kurukshetra, 2006 ; Delhi, 2002)

Solution. The given equations will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } R_2-R_1]$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } C_3+C_2]$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix}=0 \qquad [\text{Expand by } R_2]$$

or if, $$(\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda+1) \end{vmatrix}=0 \quad \text{or if, } 2(\lambda-3)\,[(\lambda-1)(3\lambda-1)-(5\lambda+1)]=0$$

or if, $$6\lambda(\lambda-3)^2=0 \quad \text{or if,} \quad \lambda=0 \quad \text{or} \quad 3.$$

(*a*) *When* $\lambda=0$, the equations become $-x+y=0$ ...(*i*)

$$-x-2y+3z=0 \qquad ...(ii)$$
$$2x+y-3z=0 \qquad ...(iii)$$

Solving (*ii*) and (*iii*), we get $\dfrac{x}{6-3}=\dfrac{y}{6-3}=\dfrac{z}{-1+4}$. Hence $x=y=z$.

(*b*) *When* $\lambda=3$, equations becomes identical.

PROBLEMS 2.7

1. Investigate for consistency of the following equations and if possible find the solutions :
$$4x-2y+6z=8,\ x+y-3z=-1,\ 15x-3y+9z=21.$$
2. For what values of k the equations $x+y+z=1,\ 2x+y+4z=k,\ 4x+y+10z=k^2$ have a solution and solve them completely in each case. (Bhopal, 2008 ; Mumbai, 2008 ; V.T.U., 2006)
3. Investigate for what values of λ and μ the simultaneous equations
$$x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu,$$
have (*i*) no solution, (*ii*) a unique solution, (*iii*) an infinite number of solutions. (Mumbai, 2007 ; U.P.T.U., 2006 ; Rohtak, 2004)
4. Test for consistency and solve,
(*i*) $2x-3y+7z=5,\ 3x+y-3z=13,\ 2x+19y-47z=32.$ (Bhopal, 2009 ; Kurukshetra, 2005 ; Raipur, 2005)
(*ii*) $x+2y+z=3,\ 2x+3y+2z=5,\ 3x-5y+5z=2,\ 3x+9y-z=4.$ (Bhilai, 2005 ; Madras, 2002)
(*iii*) $2x+6y+11=0,\ 6x+20y-6z+3=0,\ 6y-18z+1=0.$ (Rajasthan, 2005)
(*iv*) $3x+3y+2z=1,\ x+2y=4,\ 10y+3z=-2,\ 2x-3y-z=5.$ (U.T.U., 2010 ; Nagarjuna, 2008)

5. Find the values of a and b for which the equations

$$x + ay + z = 3, x + 2y + 2z = b, x + 5y + 3z = 9$$

are consistent. When will these equations have a unique solution ? *(Kurukshetra, 2005 ; Madras, 2003)*

6. Show that if $\lambda \neq -5$, the system of equations

$$3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3,$$

have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

7. Show that the equations

$$3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$$

do not have a solution unless $a + c = 2b$. *(Raipur, 2004 ; Nagpur, 2001)*

8. Prove that the equations $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$, $39x + 43y + 45z = c$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$, where t is any arbitrary quantity.

9. Find the values of λ for which the equations $(2 - \lambda)x + 2y + 3 = 0$, $2x + (4 - \lambda)y + 7 = 0$, $2x + 5y + (6 - \lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ.

10. Show that there are three real values of λ for which the equations $(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

11. Determine the values of k for which the following system of equations has non-trivial solutions and find them :

$$(k - 1)x + (4k - 2)y + (k + 3)z = 0, (k - 1)x + (3k + 1)y + 2kz = 0, 2x + (3k + 1)y + 3(k - 1)z = 0.$$

(Mumbai, 2005)

12. Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can possess a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

13. Determine the values of λ for which the following set of equations may possess non-trivial solution :

$$3x_1 + x_2 - \lambda x_3 = 0, 4x_1 - 2x_2 - 3x_3 = 0, 2\lambda x_1 + 4x_2 + \lambda x_3 = 0.$$

For each permissible value of λ, determine the general solution. *(Kurukshetra, 2006)*

14. Solve completely the system of equations

(*i*) $x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$; $5x - 7y + 2z - w = 0$.

(*ii*) $3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$; $x + 3y + 13z + 3w = 0$. *(J.N.T.U., 2002 S)*

2.11 (1) LINEAR TRANSFORMATIONS

Let (x, y) be the co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ given by

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad ...(i)$$

A more general transformation than (*i*) is

$$\left.\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \end{aligned}\right\} \quad ...(ii)$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Such transformations as (*i*) and (*ii*), are called *linear transformations* in two dimensions.

Similarly, the relations of the type
$$\left.\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \end{aligned}\right\} \quad ...(iii)$$

give a *linear transformation* from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation $Y = AX$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & k_n \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ ...(*iv*) give *linear transformation* from n variables $x_1, x_2, ..., x_n$ to the variables $y_1, y_2, ..., y_n$ *i.e.*, the transformation of the vector X to the vector Y.

This transformation is called linear because the linear relations $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$, hold for this transformation.

If the transformation matrix A is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation $Y = AX$, we can also write the inverse transformation $X = A^{-1} Y$. A non-singular transformation is also called a *regular* transformation.

Cor. If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by $Y = AX$ and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by $Z = BY$, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by

$$Z = BY = B(AX) = (BA)X.$$

(2) Orthogonal transformation. *The linear transformation (iv), i.e., $Y = AX$, is said to be* **orthogonal** *if, it transforms*

$$y_1^2 + y_2^2 + ... + y_n^2 \text{ into } x_1^2 + x_2^2 + ... + x_n^2$$

The matrix of an orthogonal transformation is called an **orthogonal matrix.**

We have $$X'X = [x_1 x_2 ...... x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + ... + x_n^2$$

and similarly, $$Y'Y = y_1^2 + y_2^2 + ... + y_n^2.$$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = Y'Y = (AX)'(AX) = X'A'AX \text{ which is possible only if } A'A = I.$$

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence a *square matrix A is said to be orthogonal if $AA' = A'A = I$.*

Obs. 1. *If A is orthogonal, A′ and A^{-1} are also orthogonal.*

Since A is orthogonal, $A' = A^{-1}$.

$\therefore$ $(A')' = (A^{-1})' = (A')^{-1}$, *i.e.*, $B' = B^{-1}$ where $B = A'$

Hence B (*i.e.*, A') is orthogonal. As $A' = A^{-1}$, A^{-1} is also orthogonal.

Obs. 2. *If A is orthogonal, then* $|A| = \pm 1$.

Since $AA' = A'A = I$ $\therefore$ $|A||A'| = |I|$ *(Mumbai, 2006)*

But $|A'| = |A|$, $\therefore$ $|A||A| = |1|$

or $|A|^2 = 1$ *i.e.*, $|A| = \pm 1$.

Example 2.38. *Show that the transformation*

$$y_1 = 2x_1 + x_2 + x_3,\ y_2 = x_1 + x_2 + 2x_3,\ y_3 = x_1 - 2x_3$$

is regular. Write down the inverse transformation.

Solution. The given transformation may be written as

$$Y = AX$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Now $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1$$

Thus the matrix A is non-singular and hence the transformatiuon is regular.

$\therefore$ The inverse transformation is given by

$$X = A^{-1} Y$$

where $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

Thus $x_1 = 2y_1 - 2y_2 - y_3$; $x_2 = -4y_1 + 5y_2 + 3y_3$; $x_3 = y_1 - y_2 - y_3$
is the inverse transformation.

Example 2.39. *Prove that the following matrix is orthogonal :*

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

(Kurukshetra, 2005)

Solution. We have $AA' = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$= \begin{bmatrix} 4/9+1/9+4/9 & -4/9+2/9+2/9 & -2/9-2/9+4/9 \\ -4/9+2/9+2/9 & 4/9+4/9+1/9 & 2/9-4/9+2/9 \\ -2/9-2/9+4/9 & 2/9-4/9+2/9 & 1/9+4/9+4/9 \end{bmatrix} = I.$$

Hence the matrix is orthogonal.

Example 2.40. *If* $A = \dfrac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ *is orthogonal, find a, b, c and* A^{-1}. *(Mumbai, 2006)*

Solution. As A is orthogonal, $AA' = I$

$$\therefore \quad \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1+4+a^2 & 2+2+ab & 2-4+ac \\ 2+2+ab & 4+1+b^2 & 4-2+bc \\ 2-4+ac & 4-2+bc & 4+4+c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore \quad 5 + a^2 = 9,\ 5 + b^2 = 9,\ 8 + c^2 = 9, \quad i.e., a^2 = 4, b^2 = 4, c^2 = 1$$

Thus $a = 2, b = 2, c = 1$.

Since A is orthogonal, $A^{-1} = A' = \dfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$.

2.12 (1) VECTORS

Any quantity having n-components is called a *vector of order n*. Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any n numbers $x_1, x_2, \ldots, x_n$ written in a particular order, constitute a vector $\mathbf{x}$.

(2) Linear dependence. *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are said to be* **linearly dependent,** *if there exist numbers* $\lambda_1, \lambda_2, \ldots, \lambda_r$ *not all zero, such that*

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_r\mathbf{x}_r = \mathbf{0}. \qquad \ldots(i)$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent.** If $\lambda_1 \neq 0$, transposing $\lambda_1\mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (i) in the form

$\mathbf{x}_1 = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 + \ldots + \mu_r\mathbf{x}_r$.

Then the vector $\mathbf{x}_1$ is said to be a linear combination of the vectors $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_r$.

Example 2.41. *Are the vectors* $\mathbf{x}_1 = (1, 3, 4, 2)$, $\mathbf{x}_2 = (3, -5, 2, 2)$ *and* $\mathbf{x}_3 = (2, -1, 3, 2)$ *linearly dependent ? If so express one of these as a linear combination of the others.*

Solution. The relation $\qquad \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 = \mathbf{0}.$

i.e., $\qquad \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0,\ 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0,$
$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0,\ 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$

As these are satisfied by the values $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$ which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = \mathbf{0}$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

Obs. Applying elementary row operations to the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of B being 2, the rank of A is also 2. Moreover $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent and $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ $\left[\because\ \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)\right]$. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors are remaining vectors (if any) can be expressed as a linear combination of these vectors.

PROBLEMS 2.8

1. Represent each of the transformations
$$x_1 = 3y_1 + 2y_2, y_1 = z_1 + 2z_2 \text{ and } x_2 = -y_1 + 4y_2, y_2 = 3z_1$$
by the use of matrices and find the composite transformation which express x_1, x_2 in terms of z_1, z_2.
2. If $\xi = x \cos \alpha - y \sin \alpha$, $\eta = x \sin \alpha + y \cos \alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.
3. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$ Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.
4. Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = 2x_1 + 4x_2 + 11x_3$; $y_3 = -x_2 + 2x_3$.
5. Verify that the following matrix is orthogonal :
(*i*) $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ (*Hissar, 2005 S ; P.T.U., 2003*) (*ii*) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ (*Kurukshetra, 2005*)
6. Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal ? (*Mumbai, 2005 S*)
7. Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = 2/7, m = 3/7, n = 6/7$.
8. If A and B are orthogonal matrices, prove that AB is also orthogonal. (*Anna, 2005*)
9. Are the following vectors linearly dependent. If so, find the relation between them :
(*i*) (2, 1, 1), (2, 0, – 1), (4, 2, 1). (*Mumbai, 2009*)
(*ii*) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9).
(*iii*) $\mathbf{x}_1 = (1, 2, 4), \mathbf{x}_2 = (2, -1, 3), \mathbf{x}_3 = (0, 1, 2), \mathbf{x}_4 = (-3, 7, 2)$. (*U.P.T.U., 2003 ; Nagpur, 2001*)

2.13 (1) EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth order unit matrix. The determinant of this matrix equated to zero,

i.e.,
$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the *characteristic equation of A*. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n = 0,$$

where k's are expressible in terms of the elements a_{ij}. The roots of this equation are called the *eigenvalues* or *latent roots* or *characteristic roots* of the matrix A.

(2) Eigen vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then the linear transformation $Y = AX$ ...(i)

carries the column vector X into the column vector Y by means of the square-matrix A. In practice, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (*i*).

Then $\lambda X = AX$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$ ...(ii)

This matrix equation represents n homogeneous linear equations

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad \text{...(iii)}$$

which will have a non-trivial solution only if the coefficient matrix is singular, *i.e.*, if $|A - \lambda I| = 0$.

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A. It has n roots and corresponding to each root, the equation (*ii*) [or (*iii*)] will have a non-zero solution.

$X = [x_1, x_2, \ldots\ldots, x_n]'$, which is known as the *eigen vector* or *latent vector*.

Obs. 1. Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Obs. 2. If X_i is a solution for a eigen value λ_i then it follows from (*ii*) that cX_i is also a solution, where c is arbitrary constant. Thus *the eigen vector corresponding to a eigen value is not unique* but may be any one of the vectors cX_i.

Example 2.42. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. (*Bhopal, 2008*)

Solution. The characteristic equation is $[A - \lambda I] = 0$

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 6 = 0$

or $(\lambda - 6)(\lambda - 1) = 0$ $\therefore$ $\lambda = 6, 1.$

Thus the eigen values are 6 and 1.

If x, y be the components of an eigen vector corresponding to the eigen value λ, then

$$[A - \lambda I]\, X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $-x + 4y = 0$

$\therefore$ $\dfrac{x}{4} = \dfrac{y}{1}$ giving the eigen vector $(4, 1)$.