Chapter 13

Matrices

INTRODUCTION

The term matrix was apparently coined by Sylvester about 1850, but was introduced first by Cayley in 1860. By a 'matrix' we mean an "arrangement" or "rectangular array" of numbers. The elegant "short-hand" representation of an array of many numbers as a single object and perform calculations makes matrices very useful. Matrices (plural of matrix) find applications in solution of system of linear equations, probability, mathematical economics, quantum mechanics, electrical networks, curve fitting, transportation problems, frameworks in mechanics. Matrices are easily amenable for computers.

A brief revision of matrices, types, properties is presented.

A matrix is a rectangular array of $m \cdot n$ numbers (or functions) arranged in m rows (horizontal lines) and n columns l (vertical lines). These numbers known as elements or entries are enclosed in brackets [ ] or ( ) or || ||.

The order of such matrix is $m \times n$ and is said to be a rectangular matrix.

Notation

Elements of a matrix are located by the double subscript *ij* where *i* denotes the row and *j* the column.

Null or Zero matrix is a matrix with all elements zero.

Equality

Two matrices A and B are equal if they are of the same order and $a_{ij} = b_{ij}$, for every i, j.

Sum (difference)

$C = A \pm B$ where $c_{ij} = a_{ij} \pm b_{ij}$ (and A and B are conformable i.e., of the same order). Scalar multiplication: $C = kA$ where $c_{ij} = ka_{ij}$ i.e., every element of A is multiplied by constant k.

Matrix multiplication:

$$C_{m\times n} = A_{m\times p} B_{p\times n} \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$$

Transpose of a matrix $A_{m\times n}$ is denoted by $A^T_{n\times m}$ obtained by interchanging rows and columns.

Result: $(AB)^T = B^T A^T$.

Square matrix

$A : m = n$, when the number of rows equals to the number of columns, known as n-square matrix.

The elements a_{ii} are known as diagonal elements

Trace: $\sum_{i=1}^{n} a_{ii}$ = sum of the diagonal elements.

Singular matrix: if $|A| = 0$

Non-singular matrix: if $|A| \neq 0$

Upper triangular matrix A: $a_{ij} = 0$ for $i > j$

Lower triangular matrix A: $a_{ij} = 0$ for $i < j$

Diagonal matrix A: $a_{ij} = 0$ when $i \neq j$

Scalar matrix A: a diagonal matrix with $a_{ii} = k$ for every i and k is a constant.

Identity matrix: is a scalar matrix with $k = 1$

i.e.,
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: All the above definitions are only for square matrices.

Row matrix (vector) is a matrix having only one row.

Column matrix (vector) is a matrix having only one column.

Matrix addition and multiplication is associative but not (necessarily) commutative.

i.e., $A + (B + C) = (A + B) + C$ and $A(BC) = (AB)C$.

Distributive: $A(B + C) = AB = AC$.

Power of a matrix: A^n is a matrix obtained by multiplying A by itself n times.

13.1 INVERSE OF A MATRIX

Consider only square matrices.

Inverse of a n-square matrix A is denoted by A^{-1} and is defined such that

$$AA^{-1} = A^{-1}A = I$$

where I is $n \times n$ unit matrix.

Result 1: Inverse of A exists only if $|A| \neq 0$ i.e., is A is non-singular.

Result 2: Inverse of a matrix is unique.
If B, C are two inverses of A then $(CA)B = C(AB)$, $IB = CI$ i.e., $B = C$, so inverse is unique.

Result 3: Inverse of a product is the product of inverses in the reverse order
i.e., $(AB)^{-1} = B^{-1}A^{-1}$
since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$
$= AA^{-1} = I$.

Result 4: For a diagonal matrix D with d_{ii} as diagonal elements, D^{-1} is a diagonal matrix with reciprocals $1/d_{ii}$ as the diagonal elements.

Result 5: Transposition and inverse are commutative i.e.,
$(A^{-1})^T = (A^T)^{-1}$.

Taking transpose of $AA^{-1} = A^{-1}A = I_n$
$(A^{-1})^T\, A^T = A^T(A^{-1})^T = I^T = I$ i.e.,
$(A^{-1})^T$ is the inverse of A^T or $(A^{-1})^T = (A^T)^{-1}$.

Result 6: $(A^{-1})^{-1} = A$.

Taking inverse of $(AA^{-1}) = I$,
$(AA^{-1})^{-1} = (A^{-1})^{-1}A^{-1} = I^{-1} = I = A\,A^{-1}$.
Thus $A = (A^{-1})^{-1}$.

Inverse by Adjoint Matrix

Minor M_{ij} of an element a_{ij} of a $n \times n$ square matrix A is the determinant of the $(n - 1)$ square matrix of A obtained by deleting the ith row and jth column from A.
Cofactor A_{ij} of a_{ij} of A is a signed minor

$$\text{i.e.,}\quad A_{ij} = (-1)^{i+j} M_{ij}$$

Adjoint of a Matrix A

Adjoint of a matrix is denoted by adj A is the transpose of a n- square matrix $[A_{ij}]$ where the elements A_{ij} are the cofactors of a_{ij} of A.

$$\text{i.e.,}\quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \cdots A_{1n} \\ A_{21} & A_{22} & A_{23} \cdots A_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ A_{n1} & A_{n2} & A_{n3} \cdots A_{nn} \end{bmatrix}^T =$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \cdots A_{n1} \\ A_{12} & A_{22} & A_{32} \cdots A_{n2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ A_{1n} & A_{2n} & A_{3n} \cdots A_{nn} \end{bmatrix}$$

Result: adj $(AB) =$ (adj A)(adj B)

Inverse of a matrix can be calculated by several methods.

Inverse from the adjoint:

$$A^{-1} = \frac{\text{adj } A}{|A|}.$$

WORKED OUT EXAMPLES

Inverse of a matrix

Example: Find the adjoint and inverse of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

Solution:

$$\text{Adjoint of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

where A_{ij} are the cofactors of the element a_{ij}. Thus minors of a_{ij} are

$$M_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10, \quad M_{12} = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15$$

Similarly,

$$M_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5, \quad M_{21} = \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = 4,$$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4, \quad M_{23} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1,$$

$$M_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9, \quad M_{32} = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 14$$

$$M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6.$$

Cofactors $A_{ij} = (-1)^{i+j} M_{ij}$

$$\text{Adjoint of } A = \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & +14 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & +14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$|A| = 2(12-2) - 3(16-1) + 4(8-3)$$

$$= 20 - 45 + 20 = 40 - 45 = -5$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

or

$$A^{-1} = \frac{1}{5} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}.$$

EXERCISE

Inverse of a matrix

Find the inverse of the matrix A, by adjoint matrix:

1. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Ans. $|A| = -8$

$$\text{adj } A = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

Ans. $|A| = -2$

$$\text{adj } A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & -6 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

3. $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Ans. $|A| = 10$

$$\text{adj A} = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}, \ A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

4. $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 3 & 8 \end{bmatrix}$

Ans. $|A| = 130,$

$$\text{adj } A = \begin{bmatrix} 4 & -42 & 20 \\ 20 & 50 & -30 \\ -9 & -3 & 20 \end{bmatrix}$$

13.2 RANK OF A MATRIX

Let A be a rectangular matrix of order $m \times n$.

Submatrix

Submatrix of a matrix A is any matrix obtained from A by omitting some rows and columns in A.

A is a submatrix of itself (obtained by deleting zero rows and columns).

Rank

Rank of a matrix A is the positive integer r such that there exists at least one r-rowed square matrix with non-vanishing determinant while every $(r+1)$ or more rowed matrices have vanishing determinants.

Thus rank of a matrix is the largest order of a non-zero minor of matrix.

Rank of A is denoted by $r(A)$.

Result: Rank of A and A^T is same.

Note 1: Rank of a null matrix is zero.

Note 2: For a rectangular matrix A of order $m \times n$, rank of $A \leq \min(m, n)$ i.e., rank can not exceed the smaller of m and n.

Note 3: For a n-square matrix, if rank $= n$ then $|A| \neq 0$ i.e., A is non-singular.

Note 4: For any square matrix, if rank $< n$, then $|A| = 0$ i.e., A is singular.

Elementary Row Transformations (Operation) on a Matrix

1. R_{ij}: Interchange of the ith and jth rows.
2. $R_{i(k)}$: Multiplication of every element of ith row by a non-zero scalar k.
3. $R_{ij(k)}$: Addition to the elements of ith row, of k times the corresponding elements of the jth row.

In a similar way, elementary column transformations (operations) are denoted by $C_{ij}, C_{i(k)}, C_{ij(k)}$ where the row in the above definitions is replaced by column.

WORKED OUT EXAMPLES

Inverse by Gauss-Jordan

Example: Find the inverse of A by Gauss-Jordan method where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Solution: Consider $A|I$ and apply elementary row operations on both A and I until A gets transformed to I.

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_{21(-2)} \\ R_{31(-3)}}}{\sim} \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{R_{23} \\ R_{2(-1)} \\ R_{3(-1)}}}{\sim} \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\underset{\substack{R_{23(-3)} \\ R_{13(-3)}}}{\sim} \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\underset{R_{12(-2)}}{\sim} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] = [I|A^{-1}]$$

Thus $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

EXERCISE

By Gauss-Jordan elimination

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix} -23 & 29 & \frac{-64}{5} & \frac{-18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Ans. $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

Ans. $A^{-1} = -\frac{1}{35}\begin{bmatrix} -17 & 5 & 31 \\ 7 & 0 & -21 \\ 23 & -15 & -44 \end{bmatrix}$

5. $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

Ans. $A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}$

Equivalent matrices

Two matrices A and B are said to be equivalent, denoted by $A \sim B$, if one matrix say A can be obtained from B by a sequence of elementary transformations.

Row-equivalence

Two matrices A and B are said to be row-equivalent if A can be reduced to B by a sequence of elementary *row* transformations or vice versa.

Determination of Rank of a Matrix A

Let A be a rectangular matrix of order $m \times n$.

I. Enumeration: Evaluate all the minors such that a minor of r is non-zero and every minor of $(r+1)$ or more is zero:

Note: This is impracticable for matrices of higher order.

II. Apply only **elementary row operations** on A. Then the number of non-zero rows is the rank of A.

III. Normal form N of a matrix A of rank r is one of the froms

$$N = I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

where I_r is an identity matrix of order r. By the application of both elementary row and column operations, a matrix of rank r can be reduced to normal form. Then the rank of A is r.

IV. Echelon Form.[*] Row Reduced Echelon form: The number of non-zero rows in an Echelon form is the rank.

Result: Equivalent matrices have the same order and same rank because elementary transformations do not alter (effect) its order and rank.

WORKED OUT EXAMPLES

Rank of a matrix

Determine the rank of the following matrices:

Example 1: $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

Solution: Rank of $A \le 3$ since A is of 3rd order.
$|A| = 4(-6+6) - 2(-12+12) + 3(-8+8) = 0$

Since $|A| = 0$, rank of $A < 3$ i.e., $r(A) \le 2$

Consider the determinants of 2nd order submatrices

$$\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ -1 & -1.5 \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ -2 & -1.5 \end{vmatrix} = 0,$$

Since all 2nd order submatrices have zero determinants i.e., 2nd order minors are all zero. So $r(A) < 2$.
Since A is a non-zero matrix $r(A) > 0$.
Thus the rank of A is one.

Aliter: Apply elementary row operations on A

$$\begin{matrix} R_{21(-2)} \\ R_{31(\frac{1}{2})} \end{matrix} \sim \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows is one. So the rank of A is one.

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[*] A matrix $A = (a_{ij})$ is an echelon matrix or is said to be in echelon form, if the number of zeros preceding the first non-zero entry (known as distinguished elements) of a row increases row by row until only zero rows remain.
In row reduced echelon matrix, the distinguished elements are unity and are the only non-zero entry in their respective columns.

Example 2: $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find rank of A, rank of B, rank of $A+B$, rank of AB and rank of BA.

Solution: $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix} R_{31(-2)} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

$$R_{32(-1)} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of A is 2 since the number of non-zero rows is 2.

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{matrix} R_{21(-1)} \\ R_{31(-3)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\therefore \quad r(B) = 1$

$$A+B = \begin{bmatrix} 2 & 6 & 5 \\ 2 & 5 & 4 \\ 5 & 16 & 13 \end{bmatrix} \begin{matrix} R_{21(-2)} \\ R_{31(-\frac{5}{2})} \end{matrix} \sim \begin{bmatrix} 2 & 6 & 5 \\ 0 & -7 & -6 \\ 0 & -1 & \frac{1}{2} \end{bmatrix},$$

$r(A+B) = 3$

$$AB = \begin{bmatrix} 23 & 23 & 23 \\ 12 & 12 & 12 \\ 58 & 58 & 58 \end{bmatrix} \begin{matrix} R_{1(\frac{1}{23})} \\ R_{21(-12)} \\ R_{31(-58)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$r(AB) = 1$

$$BA = \begin{bmatrix} 3 & 21 & 16 \\ 6 & 42 & 32 \\ 9 & 63 & 48 \end{bmatrix} \begin{matrix} R_{21(-2)} \\ R_{31(-3)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$r(BA) = 1.$

Note: Rank of product $\leq$ rank of either.

Example 3:

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}_{4\times 4}$$

$$\begin{matrix} R_{21(-2)} \\ R_{31(1)} \\ R_{41(-2)} \end{matrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\begin{matrix} R_{32(1)} \\ R_{34} \\ R_{3(\frac{1}{3})} \end{matrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad r(A) = 4$

Example 4:

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}_{5\times 5}$$

$$\begin{matrix} R_{12(-1)} \\ R_{1(-1)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

$$\begin{matrix} R_{21(-4)} \\ R_{31(-5)} \\ R_{41(-10)} \\ R_{51(-15)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{matrix} R_{32(-1)} \\ R_{42(-1)} \\ R_{52(-1)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A is 2 since the number of non-zero rows is 2.

Example 5: Determine the values of b such that the rank of A is 3.

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

$$\begin{matrix} R_{21(-4)} \\ R_{31(-2)} \\ R_{41(-9)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 4 & 2 \\ 0 & 0 & b+9 & 3 \end{bmatrix}$$

$$\begin{matrix} R_{32(-4)} \\ R_{42(-3)} \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 0 & -2 \\ 0 & 0 & b+6 & 0 \end{bmatrix}$$

$$R_{43} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b+6 & 0 \\ b-2 & 0 & 0 & -2 \end{bmatrix}$$

Cases:

i. If $b = 2, |A| = 1 \cdot 0 \cdot 8 \cdot (-2) = 0$, rank of $A = 3$

ii. If $b = -6$, no. of non-zero rows is 3, rank of $A = 3$.

Echelon form

Example 6: Reduce A to Echelon form and then to its row canonical form (or row reduced Echelon form) where

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$$

Hence find the rank of A.

Solution: Applying elementary row operations on A

$$\begin{array}{c} R_{31(-2)} \\ R_{41(-4)} \end{array} \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

$$\begin{array}{c} R_{32(1)} \\ R_{42(1)} \\ \sim \end{array} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ This is Echelon form.}$$

$$\begin{array}{c} R_{2(\frac{1}{11})} \\ \sim \end{array} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{12(-3)} \begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row canonical or row reduced Echelon form.

Rank of A is 2 since there are two non-zero rows.

EXERCISE

Rank of a matrix

Find the rank of the matrix:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ *Ans.* 3

2. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ *Ans.* 2

3. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ *Ans.* 3

4. $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ *Ans.* 4

5. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ *Ans.* 2

6. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$ *Ans.* 2

7. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$ *Ans.* 3

8. $\begin{bmatrix} 1 & 2 & -3 & 4 & 9 \\ 1 & 0 & -1 & 1 & 1 \\ 3 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 2 & 9 \\ 3 & 1 & 0 & 3 & 9 \end{bmatrix}$ *Ans.* 4

9. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ *Ans.* 2

10. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ *Ans.* 2

11. $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ *Ans.* 2

12. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ *Ans.* 2

13. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ *Ans.* 3

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$ *Ans.* 2

15. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ *Ans.* 3

16. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ *Ans.* 3

Echelon form

Find the Echelon form and row reduced echelon form of the matrix and hence find the rank:

17. $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

Ans. $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{10}{7} \end{pmatrix}$, rank $= 3$

18. $\begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$

Ans. $\begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 7/9 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{pmatrix}$

rank $= 2$

19. $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & -3 & 4 \end{bmatrix}$

Ans. $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank $= 3$

20. $\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$

Ans. $\begin{pmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11 & 10 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 & \frac{4}{11} & \frac{5}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{10}{11} & \frac{15}{11} & -\frac{5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank $= 2$

13.3 NORMAL FORM

Procedure to Obtain Normal Form

Consider

$$A_{m\times n} = I_{m\times m}\, A_{m\times n}\, I_{n\times n}$$

Apply elementary row operations on A and on the prefactor $I_{m\times m}$ and apply elementary column operations on A and on the postfactor $I_{n\times n}$, such that A on the L.H.S. reduces to normal form. Then $I_{m\times m}$ reduces to $P_{m\times m}$ and $I_{n\times n}$ reduces to $Q_{n\times n}$; resulting in $N = PAQ$.

Here P and Q are non-singular matrices.

Thus for any matrix of rank r, there exist non-singular matrices P and Q such that

$$PAQ = N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

WORKED OUT EXAMPLES

Example 1: Find the non-singular matrices P and Q such that the normal form of A is PAQ where

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$$

Hence find its rank.

Solution: Consider $A_{3\times 4} = I_{3\times 3}\, A_{3\times 4}\, I_{4\times 4}$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A I_4$$

$$\begin{array}{l} R_{21(-1)} \\ R_{31(-1)} \\ \text{pre} \end{array} \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A I_4$$

$$\begin{array}{l} R_{32(-2)} \\ \text{pre} \end{array} \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A I_4$$

$$\begin{array}{l} C_{21(-3)} \\ C_{31(-6)} \\ C_{41(1)} \\ \text{post} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times$$

$$\times \quad A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} C_{32(1)} \\ C_{42(-2)} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times$$

$$\times \quad A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $I_2 = PAQ$ where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of A is 2.

Example 2: Find P and Q such that the normal form of

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

is PAQ.

Hence find the rank of A.

Solution: Consider

$$A_{3\times 3} = I_{3\times 3}\, A_{3\times 3}\, I_{3\times 3}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} C_{21(1)} \\ C_{31(1)} \\ \text{post} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_{21(-1)} \\ R_{31(-3)} \\ \text{pre} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_{2(\frac{1}{2})} \\ R_{3(\frac{1}{4})} \\ \text{pre} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_{32(-1)} \\ \text{pre} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} C_{32(-1)} \\ \text{post} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the L.H.S. is in the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. Hence

$$P_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \text{ and } Q_{3\times 3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of $A = 2$.

EXERCISE

Determine the non-singular matrices P and Q such that PAQ is in the normal form for A. Hence find the rank of A.

1. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

rank $= 2$

2. $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & \frac{4}{17} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$, rank = 2

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ 1 & 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -2 & -\frac{3}{5} \\ 0 & 1 & -\frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}$

rank $= 3$

4. $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank $= 3$

5. $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & \frac{-1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank = 2

6. $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

Ans. $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, rank = 3.

13.4 SYSTEM OF LINEAR NON-HOMOGENEOUS EQUATIONS

A system (or family) of m linear algebraic equations in n unknowns $x_1, x_2, \cdots x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

The numbers a_{ij} are known as coefficients and b_i are known as (R.H.S.) constants of the system.

(1) can be represented as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \; ; \; i = 1 \text{ to } m$$

Non-homogeneous system: When all b_i are not zero, i.e., at least one b_i is non-zero.

Homogeneous system: If $b_i = 0, i = 1$ to m (all R.H.S. constants are zero).

Solution of system (1) is a set of numbers $x_1, x_2, \cdots, x_n$ which satisfy (simultaneously) all the equations of the system (1).

Trivial solution is a solution where all x_i are zero i.e., $x_1 = x_2 = \cdots = x_n = 0$.

Matrix Representation

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

be two column vectors.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn} \end{bmatrix}_{m\times n}$$

Here

A = Coefficient matrix of the system (1)
B = (R.H.S.) constant vector
X = Solution (vector)

Then the system (1) can be represented as

$$A_{m\times n}\, X_{n\times 1} = B_{m\times 1}$$

Augmented matrix $[A|B]$ or $\tilde{A}$ of system (1) is obtained by augmenting A by the column B

$$\text{i.e., } \tilde{A} = [A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} & b_m \end{bmatrix}$$

Consistent: System is said to be consistent if (1) has at least one solution.
Inconsistent if system has no solution at all.

Solution of System of Linear Equations

We consider two methods of obtaining solution of system of n linear equations in n unknowns. They are

i. Cramer's rule **ii.** matrix inverse

Cramer's rule (Solution by determinants)

a. If A is non-singular i.e., D = determinant of $A = |A| \neq 0$. Then system (1) has a unique solution given by

$$x_i = \frac{Di}{D} \qquad \text{for} \quad i = 1, 2, \cdots n$$

where D_i is the determinant obtained from D by replacing the ith column in D by constant column vector B.

b. For homogeneous system with $D \neq 0$, only trivial solution exists.

c. For homogeneous system with $D = 0$, non-trivial solutions exists.

Note: Cramer's rule is not suitable for computations.

Matrix inversion method

Consider the system of n equations in n unknowns represented by

$$AX = B$$

where A is n-square non-singular matrix. Premultiplying by A^{-1} on both sides, we get

$$A^{-1}AX = A^{-1}B$$

or $$X = A^{-1}B$$

which is the required solution.

Here A^{-1}, the inverse of A is obtained by Gauss-Jordan method: (see Page 13.4)

Consider $A|I$

Apply *only* elementary row operations on both A and I such that A is reduced to an identity matrix I, then I gets transformed to A^{-1} i.e.,

$$A^{-1}\ A\big|A^{-1}I$$

$$I\big|A^{-1}$$

Consistency of System of Linear Equations

Consider m linear equations in n unknowns so that

$$A_{m\times n}\, X_{n\times 1} = B_{m\times 1}$$

Fundamental theorem

I. If rank of A and rank of the augmented matrix $\tilde{A}$ are equal, then the system is consistent.

a. If $r(A) = r(\tilde{A}) = n$

then unique solution exists.

b. If $r(A) = r(\tilde{A}) < n$

then infinitely many solutions exist in terms of $(n - r)$ arbitrary constants.

II. If rank of A is not equal to rank of $\tilde{A}$ then the system is inconsistent and has no solution at all.

Procedure

1. Determine $r(A)$ and $r(\tilde{A})$.

2. If $r(A) \neq r(\tilde{A})$, system inconsistent, no solutions.

3. If $r(A) = r(\tilde{A}) = n$

Then the unique solution may be obtained by Cramer's rule or matrix inversion method.

4. If $r(A) = r(\tilde{A}) < n$

Then rewrite $x_1, \cdots x_r$ variables (whose coefficient submatrix has rank r) in terms $(n-r)$ variables and solve by Gaussian elimination or Gauss-Jordan elimination method.

WORKED OUT EXAMPLES

Example 1: Solve by Cramer's rule

$$x + y + z = 11$$
$$2x - 6y - z = 0$$
$$3x + 4y + 2z = 0.$$

Solution:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 11, \quad D_1 = \begin{vmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{vmatrix} = -88,$$

$$D_2 = \begin{vmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix} = -77, \quad D_3 = \begin{vmatrix} 1 & 1 & 11 \\ 2 & -6 & 0 \\ 3 & 4 & 0 \end{vmatrix} = 286$$

The unique solution $x = \frac{D_1}{D} = \frac{-88}{11} = -8$

$$y = \frac{D_2}{D} = \frac{-77}{11} = -7, \quad z = \frac{D_3}{D} = \frac{286}{11} = 26$$

Thus $\quad x = -8, \quad y = -7, \quad z = 26.$

Example 2: Solve by calculating the inverse by adjoint method

$$x_1 + x_2 + 2x_3 = 4$$
$$2x_1 + 5x_2 - 2x_3 = 3$$
$$x_1 + 7x_2 - 7x_3 = 5.$$

Solution: The given system is written as $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Inverse by adjoint

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{vmatrix} = 9$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

where A_{ij} are cofactor of the element a_{ij}

$$A^{-1} = \frac{1}{9} \begin{bmatrix} -21 & 12 & 9 \\ 21 & -9 & -6 \\ -12 & 6 & 3 \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} -7 & 7 & -4 \\ 4 & -3 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

The solution to the given system is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}B = \frac{1}{3} \begin{bmatrix} -7 & 7 & -4 \\ 4 & -3 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -27 \\ 17 \\ 11 \end{bmatrix}$$

i.e., $x_1 = -27/3, x_2 = 17/3, \quad x_3 = 11/3.$

Example 3: Solve by calculating the inverse by elementary row operations

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 - x_4 = 4$$
$$x_1 + x_2 - x_3 + x_4 = -4$$
$$x_1 - x_2 + x_3 + x_4 = 2$$

Solution: The system is written as $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

Inverse by elementary row operations

$$[A|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_{21(-1)}, R_{31(-1)}, \\ R_{41(-1)} \text{ and} \\ R_{2(-1)}, R_{3(-1)}, \\ R_{4(-1)} \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R_{24} \\ R_{2(\frac{1}{2})} \\ R_{3(\frac{1}{2})} \\ R_{4(\frac{1}{2})} \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{array}\right] \begin{array}{l} R_{14(-1)} \\ R_{13(-1)} \\ R_{12(-1)} \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\text{Thus } A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

The required solution is

$$X = A^{-1}B = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

i.e., $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2.$

Example 4: Solve

$$\begin{aligned} 2x_1 - 2x_2 + 4x_3 + 3x_4 &= 9 \\ x_1 - x_2 + 2x_3 + 2x_4 &= 6 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 3 \\ x_1 - x_2 + x_4 &= 2 \end{aligned}$$

Solution: Apply elementary row operation on $[A|B]$

$$[A|B] = \left[\begin{array}{cccc|c} 2 & -2 & 4 & 3 & 9 \\ 1 & -1 & 2 & 2 & 6 \\ 2 & -2 & 1 & 2 & 3 \\ 1 & -1 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_{12} \\ R_{21(-2)} \\ R_{41(-1)} \\ R_{31(-2)} \\ R_{2(-1)} \\ R_{3(-1)} \\ R_{4(-1)} \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 9 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} R_{34(-1)} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} R_{32} \\ R_{43} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_{32(-2)} \\ R_{3(-1)} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_{43(-1)} \\ R_{4(-1)} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 6 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

rank of $(A) = 3 \neq 4 =$ rank of $[A|B]$

So the given system is inconsistent and therefore has no solution.

Example 5: Solve

$$\begin{aligned} 3x + 3y + 2z &= 1 \\ x + 2y &= 4 \\ 10y + 3z &= -2 \\ 2x - 3y - z &= 5. \end{aligned}$$

Solution:

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right] R_{12} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$$\begin{array}{c} R_{21(-3)} \\ \sim \\ R_{41(-2)} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$$\begin{array}{l} R_{2(-\frac{1}{3})} \\ R_{32(-10)} \\ R_{42(7)} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & \frac{29}{3} & \frac{-116}{3} \\ 0 & 0 & \frac{-17}{3} & \frac{68}{3} \end{array}\right]$$

$$\begin{array}{l} R_{3(\frac{3}{29})} \\ R_{43(\frac{17}{3})} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & \frac{-2}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -\frac{116}{29} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A) = 3 = r(A|B) = n =$ number of variables. The system is consistent and has unique solution. Solving

$$z = -\frac{116}{29} = -4$$

$$y - \frac{2}{3}z = \frac{11}{3} \quad \text{or} \quad y = \frac{11}{3} + \frac{2}{3}(-4) = 1$$

$$x + 2y + 0 = 4 \quad \text{or} \quad x = 4 - 2 = 2$$

i.e., $x = 2, y = 1, z = -4$.

Example 6: Solve

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 3$$
$$4x_1 + 2x_2 - 2x_3 = 2.$$

Solution: By applying elementary row operations

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right] \begin{array}{c} R_{21(-2)} \\ R_{31(-4)} \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{array}\right]$$

$$\begin{array}{c} R_{2(-\frac{1}{3})} \\ \sim \\ R_{3(-\frac{1}{2})} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$$\begin{array}{c} R_{32(-1)} \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A) = 2 = r(A|B) < 3 = n =$ number of variables.

The system is consistent but has infinite number of solutions in terms of $n - r = 3 - 2 = 1$ variable. Choose $x_3 = k =$ arbitrary constant.

Solving $x_2 - x_3 = -1$ or $x_2 = x_3 - 1 = k - 1$.

$$x_1 + x_2 - x_3 = 0 \quad \text{or} \quad x_1 = -x_2 + x_3 = -k + 1 + k = 1$$

Thus the solutions are

$$x_1 = 1, x_2 = k - 1, x_3 = k, \quad \text{where } k \text{ is arbitrary.}$$

Example 7: Determine the values of a and b for which the system

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

has (i) no solution (ii) unique solution (iii) infinite number of solutions. Find the solutions in case (ii) and (iii).

Solution:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right] \begin{array}{c} R_{21(-1)} \\ R_{31(-2)} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$$

$$R_{32(-1)} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$$

Case 1: $a = 8, b \neq 15, r(A) = 2 \neq 3 = r(A|B)$, inconsistent, no solution.

Case 2: $a \neq 8, b$ any value, $r(A) = 3 = r(A|B) = n =$ number of variables, unique solution, $z = \frac{b-15}{a-8}$,

$$y = (3a - 2b + 6)/(a - 8), x = z = (b - 15)/(a - 8).$$

Case 3: $a = 8, b = 15, r(A) = 2 = r(A|B) < 3 = n$, infinite solutions with $n - r = 3 - 2 = 1$ arbitrary variable. $x = k$, $y = 3 - 2k$, $z = k$, with k arbitrary.

EXERCISE

Solve the following:

1. $5x + 3y + 7z = 4,$
   $3x + 26y + 2z = 9,$
   $7x + 2y + 10z = 5.$

Ans. $x = (7 - 16k)/11$, $y = (3 + k)/11$, $z = k$, k arbitrary

2. $x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1,$
   $2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 2,$
   $3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3.$

Ans. $x_1 = 1, x_2 = 2a, x_3 = a, x_4 = -3b, x_5 = b$ where a and b are arbitrary constants

3. $x_1 + x_2 + 2x_3 + x_4 = 5,$
   $2x_1 + 3x_2 - x_3 - 2x_4 = 2,$
   $4x_1 + 5x_2 + 3x_3 = 7.$

Ans. No solution, system inconsistent

4. Using A^{-1} (inverse of the coefficient matrix)

$$2x_1 + x_2 + 5x_3 + x_4 = 5,$$
$$x_1 + x_2 - 3x_3 - 4x_4 = -1,$$
$$3x_1 + 6x_2 - 2x_3 + x_4 = 8,$$
$$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2.$$

Ans. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$, unique solution

Hint.

$$A^{-1} = \frac{1}{120}\begin{bmatrix} 120 & 120 & 0 & -120 \\ -69 & -73 & 17 & 80 \\ -15 & -35 & -5 & 40 \\ 24 & 8 & 8 & -40 \end{bmatrix}$$

5. $2x_1 + 3x_2 - x_3 = 1,$
$3x_1 - 4x_2 + 3x_3 = -1,$
$2x_1 - x_2 + 2x_3 = -3,$
$3x_1 + x_2 - 2x_3 = 4.$

Ans. Inconsistent, no solution

6. $3x_1 + 2x_2 + x_3 = 3,$
$2x_1 + x_2 + x_3 = 0,$
$6x_1 + 2x_2 + 4x_3 = 6.$

Ans. Inconsistent, no solution.

7. $-x_1 + x_2 + 2x_3 = 2,$
$3x_1 - x_2 + x_3 = 6,$
$-x_1 + 3x_2 + 4x_3 = 4 .$

Ans. $x_1 = 1, x_2 = -1, x_3 = 2$, unique solution

8. $7x + 16y - 7z = 4,$
$2x + 5y - 3z = -3,$
$x + y + 2z = 4.$

Ans. Inconsistent, no solution

9. $x + y + z = 4,$
$2x + 5y - 2z = 3,$
$x + 7y - 7z = 5.$

Ans. Inconsistent, no solution

10. $2x + y - z = 0,$
$2x + 5y + 7z = 52,$
$x + y + z = 9.$

Ans. unique solution $x = 1, y = 3, z = 5$

Find the values of a and b for which the system has (i) no solution (ii) unique solution (iii) infinitely many solutions for:

11. $2x + 3y + 5z = 9,$
$7x + 3y - 2z = 8,$
$2x + 3y + az = b.$

Ans. **i.** no solution of $a = 5, b \neq 9$;
ii. unique solution $a \neq 5$, b any value;
iii. infinitely many solutions $a = 5, b = 9$

12. $x + y + z = 6,$
$x + 2y + 3z = 10,$
$x + 2y + az = b.$

Ans. **i.** $a = 3, b \neq 10$ inconsistent
ii. $a \neq 3$, b any value, unique solution
iii. $a = 3, b = 10$ infinite solutions

13. Test for consistency
$-2x + y + z = a,$
$x - 2y + z = b,$
$x + y - 2z = c.$
where a, b, c are constants

Ans. i. If $a + b + c \neq 0$, inconsistent
ii. If $a + b + c = 0$, infinite solutions

14. Solve the system
$x + y + z = 6,$
$2x - 3y + 4z = 8,$
$x - y + 2z = 5$ by
i. Cramer's rule
ii. Matrix inversion
iii. Gauss-Jordan.

Ans. **i.** $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad \Delta = -1,$
$\Delta_1 = -1, \Delta_2 = -2, \Delta_3 = -3$

ii. $A^{-1} = \begin{bmatrix} 2 & 3 & -7 \\ 0 & -1 & 2 \\ -1 & -2 & 5 \end{bmatrix}$

iii. $\left[\begin{array}{rrrr} 1 & 1 & 1 & 6 \\ 2 & -3 & 4 & 8 \\ 1 & -1 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$

13.5 SYSTEM OF HOMOGENEOUS EQUATIONS

Solution to a System of m Homogeneous Equations in n Unknowns

Result 1: If $r < m$, omit $m - r$ equations such that the coefficient matrix of the remaining equations still has rank r. Rewrite r unknowns in terms of $n - r$ arbitrary unknowns and solve.

Result 2: If $m < n$, system has non-trivial solutions.

Result 3: If $m = n$, system has non-trivial solutions if its coefficient determinant is zero.

Note: A homogeneous system always has a trivial solution since $r[A|B] = r[A|O] = r[A]$ for any A.

WORKED OUT EXAMPLES

Example 1: Determine b such that the system of homogeneous equations

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

has (i) Trivial solution (ii) non-trivial solution.

Find the non-trivial solution.

Solution: The coefficient matrix A is

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix}$$

$$\begin{matrix} R_{21(-2)} \\ \sim R_{31(-4)} \end{matrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & b-12 \end{bmatrix}$$

$$R_{32(-1)} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & b-8 \end{bmatrix}$$

Case 1: If $b \neq 8$ then $r(A) = r(A|B) = 3 =$ number of variables. i.e., $|A| \neq 0$. System has only trivial solution $x = 0, y = 0; z = 0$.

Case 2: If $b = 8$ then $r(A) = r(A|B) = 2 < 3 = n$. System has non-trivial solutions in terms of $n - r = 3 - 2 = 1$ arbitrary variable. Solving the system

$$x + y + 3z = 0$$
$$y + 4z = 0$$

Choose z as arbitrary say $z = k =$ arbitrary constant. Then $y = -4z = -4k$ and $x = -y - 3z = 4k - 3k = k$.

Thus the infinite number of non-trivial solutions are obtained for different values of k as

$$x = k, y = -4k, z = k.$$

Example 2: Solve

$$x + y - 2z + 3w = 0$$
$$x - 2y + z - w = 0$$
$$4x + y - 5z + 8w = 0$$
$$5x - 7y + 2z - w = 0.$$

Solution: The coefficient matrix A is

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$\begin{matrix} R_{21(-1)} \\ R_{31(-4)} \\ R_{41(-5)} \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

$$\begin{matrix} R_{2(-1)} \\ R_{32(1)} \\ R_{42(4)} \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = r(A|B) = 2 < 4 = n =$ number of variables.

Non-trivial solutions exist in terms of $n - r = 4 - 2 = 2$ variables.

Choose $z = k_1$, and $w = k_2$. Then solving

$$x + y - 2z + 3w = 0$$
$$3y - 3z + 4w = 0$$

We get

$$y = \frac{1}{3}(3z - 4w) = z - \frac{4}{3}w = k_1 - \frac{4}{3}k_2$$
$$x = -y + 2z - 3w = -k_1 + \frac{4}{3}k_2 + 2k_1 - 3k_2$$
$$x = k_1 - \frac{5}{3}k_2$$

where k_1 and k_2 are arbitrary constants.

EXERCISE

Solve the system of homogeneous equations:

1. $x + 2y + 3z = 0,$
   $3x + 4y + 4z = 0,$

$7x + 10y + 12z = 0.$

Ans. Trivial solution $x = y = z = 0$ since $r(A) = 3 = n$

2. $4x + 2y + z + 3w = 0,$
$6x + 3y + 4z + 7w = 0,$
$2x + y + w = 0.$

Ans. $y = -2k_1 - k_2, z = -k_2, x = k_1, w = k_2$ where k_1 and k_2 are arbitrary constants, giving infinite number of solutions

3. $x + y - 3z + 2w = 0,$
$2x - y + 2z - 3w = 0,$
$3x - 2y + z - 4w = 0,$
$-4x + y - 3z + w = 0.$

Ans. Trivial solution $x = y = z = 0$, since $r(A) = 4 = n$

4. $x_1 + x_2 + x_3 + x_4 = 0,$
$x_1 + 3x_2 + 2x_3 + 4x_4 = 0,$
$2x_1 + x_3 - x_4 = 0.$

Ans. $x_1 = -\frac{1}{2}k_1 + \frac{1}{2}k_2, x_2 = -\frac{1}{2}k_1 \frac{-3}{2}k_2,$ $x_3 = k_1, x_4 = k_2$ where k_1 and k_2 are arbitrary constants giving infinite number of solutions: $r(A) = 2, n = 4$

5. $3x + 2y + z = 0,$
$2x + 3z = 0,$
$y + 5z = 0,$
$x + 2y + 3z = 0.$

Ans. $x = 0 = y = z$ is the only (trivial) solution since $r(A) = 3 = n$

6. $2x + 3y - 4z + w = 0,$
$x - y + z + 2w = 0,$
$5x - z + 7w = 0,$
$7x + 8y - 11z + 5w = 0.$

Ans. $z = k_1, w = k_2, x = (k_1 - 7k_2)/5,$ $y = (6k_1 + 3k_2)/5$ where k_1, k_2 are arbitrary constants

7. $x + 3y - 2z = 0,$
$2x - y + 4z = 0,$
$x - 11y + 14z = 0.$

Ans. $z = k, x = -10k/7, y = 8k/7, k$ arbitrary

8. $x_1 + 3x_2 + 2x_3 = 0,$
$2x_1 - x_2 + 3x_3 = 0,$
$3x_1 - 5x_2 + 4x_3 = 0,$
$x_1 + 17x_2 + 4x_3 = 0.$

Ans. $x_1 = 11k, x_2 = k, x_3 = -7k, k$ is arbitrary $r(A) = 2, n = 3$

9. Determine the values of b for which the system of equations has non-trivial solutions. Find them.

$$(b-1)x + (4b-2)y + (b+3)z = 0,$$
$$(b-1)x + (3b+1)y + 2bz = 0,$$
$$2x + (3b+1)y + 3(b-1)z = 0.$$

Ans. **i.** $b = 0, x = y = z$
ii. $b = 3, x = -5k_1 - 3k_2, y = k_1, z = k_2$ where k_1 and k_2 are arbitrary

10. Find the values of b for which the system has non-trivial solutions. Find them

$$2x + 3by + (3b+4)z = 0,$$
$$x + (b+4)y + (4b+2)z = 0,$$
$$x + 2(b+1)y + (3b+4)z = 0.$$

Ans. **i.** $b \neq \pm 2$, only trivial solution $x = y = z = 0$
ii. $b = 2, x = 0, z = k, y = -5k/3, k$ arbitrary
iii. $b = -2, x = 4k, y = z = k, k$ arbitrary.

13.6 GAUSSIAN ELIMINATION METHOD

Gaussian elimination method is an exact method which solves a given system of equations in n unknowns by transforming the coefficient matrix, into an upper triangular matrix and then solve for the unknowns by back substitution.

Consider a system of n equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = a_{1,n+1} \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = a_{2,n+1} \quad (2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = a_{n,n+1} \quad \text{(n)}$$

Eliminate the unknown x_1 from the $(n-1)$ equations namely (2), (3), $\ldots (n-1), (n)$ by subtracting the multiple $\frac{a_{i1}}{a_{11}}$ of the first equation from the ith equation, for $i = 2, 3, 4, \cdots, n$. Now eliminate x_2 from the $(n-2)$ equations of the resultant system. By this procedure, we arrive at a derived system as follows:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \quad (1)$$
$$a_{22}^{(1)} + x_2 + \cdots + a_{2n}^{(1)}x_n = a_{2,n+1}^{(2)} \quad (2^*)$$
$$a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n = a_{3,n+1}^{(2)} \quad (3^*)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{nn}^{(n-1)}x_n = a_{n,n+1}^{(n-1)} \quad (\text{n}^*)$$

In the forward elimination process, the coefficients are given by

$$\boxed{a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}a_{kj}^{(k-1)}}$$

where $k = 1, 2, \ldots, n-1$

$j = k+1, \ldots, n+1$

$i = k+1, \ldots, n$

and $a_{ij}^{(0)} = a_{ij}$

Back substitution

Now the drived system $(1), (2^*), (3^*) \cdots (n^*)$ is solved by back substitution. Solve equation (n^*) for the unknown x_n. Substituting this x_n in $(n^* - 1)$ equation, solve for x_{n-1}. Continuing this process, x_1 is solved from the first equation. Thus

$$\boxed{x_i = \frac{1}{a_{ii}^{(i-1)}}\left[a_{i,n+1}^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j\right]}$$

for $i = n, n-1, n-2, \ldots, 3, 2, 1.$

Check sum

Initially, for the given system, write row sum, the sum of the coefficients in each row, in the $(n+2)$nd column. Perform the same operations on the elements of this column also. Now in the absence of computational errors, at any stage, the row sum element in $(n+2)$nd row, will be equal to the sum of the elements of the corresponding transformed row.

Gauss-Jordan elimination method

Apply elementary row operations on both A and B such that A reduces to the normal form I_r. Then the solution is obtained (without the necessity of back substitution).

WORKED OUT EXAMPLES

Example 1: Solve by Gaussian elimination method, the following system of equations:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + 2x_4 &= 7 \\ -x_1 + 2x_2 + x_4 &= -2 \\ -3x_1 + x_2 + 2x_3 + x_4 &= -3 \\ -x_1 + 2x_4 &= 0. \end{aligned}$$

Solution: Arranging in tabular form, we get

Table 13.1

Row No.	x_1	x_2	x_3	x_4	b	Check Sum	Explanation
[1]	2	2	1	2	7	14	Equation 1
[2]	−1	2	0	1	−2	0	Equation 2
[3]	−3	1	2	1	−3	−2	Equation 3
[4]	−1	0	0	2	0	1	Equation 4
[5]	1	1	$\frac{1}{2}$	1	$\frac{7}{2}$	7	$R_1\left(\frac{1}{2}\right)$
[6]	0	3	$\frac{1}{2}$	2	$\frac{3}{2}$	7	$R_{25(1)}$
[7]	0	4	$\frac{7}{2}$	4	$\frac{15}{2}$	19	$R_{35(3)}$
[8]	0	1	$\frac{1}{2}$	3	$\frac{7}{2}$	8	$R_{45(1)}$
[9]	0	1	$\frac{1}{2}$	3	$\frac{7}{2}$	8	R_{86}
[10]	0	3	$\frac{1}{2}$	2	$\frac{3}{2}$	7	R_{78}
[11]	0	4	$\frac{7}{2}$	4	$\frac{15}{2}$	19	
[12]	0	0	−1	−7	−9	−17	$R_{10,9(-3)}$
[13]	0	0	$\frac{3}{2}$	−8	$\frac{-13}{2}$	−13	$R_{11.9(-4)}$
[14]	0	0	0	$\frac{-37}{2}$	−20	$\frac{-77}{2}$	$R_{13,12}\left(\frac{3}{2}\right)$

Here $R_{ij(k)}$ denotes a row operation in which the kth multiples of jth row are added to the corresponding elements of the ith row. Also, R_{ij}: interchange of ith and jth rows.

Check sum: The sum of the elements of any row must be equal to check sum (otherwise errors in operations). The given system of equations has reduced to an upper triangular matrix. Now using back substitution, solve [14] (row) equation

$$x_4 = \frac{40}{37} = 1.08.$$

Solve [13] equation

$$x_3 + 7x_4 = 9 \quad \text{or} \quad x_3 = 1.4324$$

Solve [9]: $x_2 = -0.4562$

Solve [5]: $x_1 = 2.1600$

The solution is ($x_1 = 2.16, x_2 = -0.4562, x_3 = 1.4324, x_4 = 1.08$).

Example 2: Solve the system by (i) Gaussian elimination method (ii) Gauss-Jordan method

$$\begin{aligned} 2x_1 + 5x_2 + 2x_3 - 3x_4 &= 3 \\ 3x_1 + 6x_2 + 5x_3 + 2x_4 &= 2 \\ 4x_1 + 5x_2 + 14x_3 + 14x_4 &= 11 \\ 5x_1 + 10x_2 + 8x_3 + 4x_4 &= 4 \end{aligned}$$

Solution: Consider the augmented matrix $[A|B]$

$$[A|B] = \left[\begin{array}{cccc|c} 2 & 5 & 2 & -3 & 3 \\ 3 & 6 & 5 & 2 & 2 \\ 4 & 5 & 14 & 14 & 11 \\ 5 & 10 & 8 & 4 & 4 \end{array}\right] \begin{array}{l} R_{21(-1)} \\ R_{32(-1)} \\ R_{43(-1)} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 5 & 2 & -3 & 3 \\ 1 & 1 & 3 & 5 & -1 \\ 1 & -1 & 9 & 12 & 9 \\ 1 & 5 & -6 & -10 & -7 \end{array}\right] \begin{array}{l} R_{41}, R_{23} \\ R_{21(-1)} \\ R_{31(-1)} \\ R_{41(-2)} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & -6 & -10 & -7 \\ 0 & -6 & 15 & 22 & 16 \\ 0 & -4 & 9 & 15 & 6 \\ 0 & -5 & 14 & 17 & 17 \end{array}\right] \begin{array}{ll} R_{24(-1)} & \\ R_{2(-1)}, & R_{32(-4)} \\ R_{3(-1)} & R_{42(-5)} \\ R_{4(-1)} & \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & -5 & 5 & -10 \\ 0 & 0 & 9 & -8 & 22 \end{array}\right] \begin{array}{l} R_{3(-\frac{1}{5})} \\ R_{43(-9)} \end{array}$$

$$[A|B] \sim \left[\begin{array}{cccc|c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

By back substitution: $x_4 = 4$

$$x_3 - x_4 = 2 \quad \text{or} \quad x_3 = 2 + x_4 = 2 + 4 = 6$$

$$x_2 - x_3 - 5x_4 = 1 \quad \text{or} \quad x_2 = 27$$

$$x_1 + 5x_2 - 6x_3 - 10x_4 = -7 \quad \text{or} \quad x_1 = -66$$

Thus $x_1 = -66, x_2 = 27, x_3 = 6, x_4 = 4$.

Gauss-Jordan method:

$$\left[\begin{array}{cccc|c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} R_{34(1)} \\ R_{24(5)} \\ R_{14(10)} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 5 & -6 & 0 & 33 \\ 0 & 1 & -1 & 0 & 21 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} R_{23(1)} \\ \sim \\ R_{13(6)} \end{array} \left[\begin{array}{cccc|c} 1 & 5 & 0 & 0 & 69 \\ 0 & 1 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{c} R_{12(-5)} \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -66 \\ 0 & 1 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore x_1 = -66, x_2 = 27, x_3 = 6, x_4 = 4.$

EXERCISE

Solve the following system of equations by Gaussian elimination method.

1. $x_1 + 2x_2 - x_3 = 3, \quad 3x_1 - x_2 + 2x_3 = 1,$
$2x_1 - 2x_2 + 3x_3 = 2, \quad x_1 - x_2 + x_3 = -1$

Ans. $x_1 = 1, x_2 = 4, x_3 = 4$

2. $2x_1 + x_2 + 3x_3 = 1, \quad 4x_1 + 4x_2 + 7x_3 = 1,$
$2x_1 + 5x_2 + 9x_3 = 3$

Ans. $x_1 = -\frac{1}{2}, x_2 = -1, x_3 = 1$

3. $2x_1 - 7x_2 + 4x_3 = 9, \quad x_1 + 9x_2 - 6x_3 = 1,$
$-3x_1 + 8x_2 + 5x_3 = 6$

Ans. $x_1 = 4, x_2 = 1, x_3 = 2$

4. $2x_1 + 2x_2 + 4x_3 = 18, \quad x_1 + 3x_2 + 2x_3 = 13,$
$3x_1 + x_2 + 3x_3 = 14$

Ans. $x_1 = 1, x_2 = 2, x_3 = 3$

5. $2x_1 + x_2 + x_3 = 10, \quad 3x_1 + 2x_2 + 3x_3 = 18,$
$x_1 + 4x_2 + 9x_3 = 16$

Ans. $x_1 = 7, x_2 = -9, x_3 = 5$

6. $2x_1 + x_2 + 4x_3 = 12, \quad 8x_1 - 3x_2 + 2x_3 = 20,$
$4x_1 + 11x_2 - x_3 = 33$

Ans. $x_1 = 3, x_2 = 2, x_3 = 1$

7. $x_1 + 4x_2 - x_3 = -5, \quad x_1 + x_2 - 6x_3 = -12,$

$3x_1 - x_2 - x_3 = 4$

Ans. $x_1 = \frac{117}{71}, x_2 = -\frac{81}{71}, x_3 = \frac{148}{71}$

8.
$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6,$$
$$-6x_1 + 8x_2 - x_3 - 4x_4 = 5,$$
$$3x_1 + x_2 + 4x_3 + 11x_4 = 2,$$
$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$

Ans. $x_1 = 5, x_2 = 4, x_3 = -7, x_4 = 1.$

13.7 LU-DECOMPOSITIONS

The Gaussian elimination with back substitution, Gauss-Jordan elimination, computing A^{-1} then $x = A^{-1}B$ and Cramer's rule are some of the direct (non-iterative) methods for solving system of linear equations. Gauss-Jordan elimination produces both solution for one or more R.H.S. vector B and also A^{-1}. Its principal weakness is (i) it requires all RHS $\overline{B}$ to be stored and manipulated and (ii) when A^{-1} is not required. The usefulness of Gaussian elimination with back substitution is primarily pedagogical. It stands between full elimination schemes such as Gauss-Jordan and triangular decomposition. *LU*-decomposition or triangular decomposition (triangular factorization) is a different approach in which the coefficient matrix A is factored into the product of a lower triangular matrix L and an upper triangular matrix U i.e.,

$$A = LU.$$

Since a matrix that is either upper triangular or lower triangular is called "triangular", so *LU*-decomposition is also referred to as triangular factorization. *LU* method can be easily adopted to solve a system with new R.H.S. B with great economy of effort.

It is popular because storage of space can be economized and accumulates sums in double precision (Example: LINPAK (1979) computer program of Argonne National Labs).

Solution of Linear System by LU-Decomposition

A non singular matrix A is said to have a triangular factorization or *LU*-decomposition if A can be expressed as the product of a lower triangular matrix L with ones on its main diagonal and an upper triangular matrix U. i.e.,

$$A = LU$$

For $n = 4$, we have $A_{4\times4} = L_{4\times4}U_{4\times4}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

The condition of non singularity of A implies that $u_{kk} \neq 0$ for all k. Now consider the system of equations

$$AX = B$$

or
$$LUX = B$$

Put $Y = UX$ then

$$LY = B$$

and
$$UX = Y$$

Solve first $LY = B$ for Y using forward substitution and then solve $UX = Y$ for X using backward substitution. Here X is the required solution vector.

LU-decomposition is also known as Doolittle's method. Another variation of *LU*-decomposition is crout's reduction or Cholosky's reduction in which the upper triangular matrix U has ones on its main diagonal (instead of L) in the triangular decomposition $A = LU$

Note that *LU* decompostion is not unique. Any matrix A with all non-zero diagonal elements (i.e., $a_{ii} \neq 0$ for $i = 1$ to n) can be factored in infinite number of ways.

Example 1:

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = LU$$

or

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{bmatrix} = LU$$

or

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$$

and so on.

Example 2:

$$A = LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The non-zero diagonal entries in L can be shifted to U.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{l_{21}}{l_{11}} & 1 & 0 \\ \frac{l_{31}}{l_{11}} & \frac{l_{32}}{l_{22}} & 1 \end{bmatrix} \begin{bmatrix} l_{11} & 0 & 0 \\ 0 & l_{22} & 0 \\ 0 & 0 & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{l_{21}}{l_{11}} & 1 & 0 \\ \frac{l_{31}}{l_{11}} & \frac{l_{32}}{l_{22}} & 1 \end{bmatrix} \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}l_{13} \\ 0 & l_{22} & l_{22}u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

which is another LU decomposition of A. However of the entire set of LU decompositions, choose the pair in which L has ones on its diagonal.

13.8 LU-DECOMPOSITION FROM GAUSSIAN ELIMINATION

Theorem: If A is a square matrix which can be reduced to echlon from U without using any row interchanges, then A has a LU decomposition and can be factored as $A = LU$ where L is a lower triangular matrix with ones on its main diagonal.

Explanation: In solving a system $AX = B$ of n equations in n unknowns, use the Gaussian elimination method to reduce A to an echlon form (upper triangular matrix) U. We assume that no row interchanges are necessary in this process. Then the multipliers l_{ij} used in the Gaussian elimination process will form the subdiagonal enteries in the lower triangular matrix L.

Step I: Use Gaussian elimination to reduce A to echolon form U, without using any row interchanges. Keep track of the multipliers used to introduce zeros below the leading diagonal elements of A.

Step II: In each position below the main diaginal (consisting of ones) of L, place the *negative* of the multiplier used to introduce zeros in that position in U.

The LU-decomposition can also be obtained by solving the equations in l_{ij} and u_{ij} as follows. Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = LU =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

From the first row elements.
$u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$
From 2nd row elements:
$l_{21}u_{11} = a_{21}, l_{21}u_{12} + u_{22} = a_{22},$

$$l_{21}u_{13} + u_{23} = a_{23}$$

From 3rd row elements

$$l_{31}u_{11} = a_{31}, l_{31}u_{12} + l_{32}u_{22} = a_{32},$$

$$l_{31}u_{13} = l_{32}u_{23} + u_{33} = a_{33}.$$

Solving we get u_{11}, u_{12}, u_{13}, then l_{21}, u_{22}, u_{23} followed by l_{31}, l_{32}, u_{33}.

LU-Decomposition by Gaussian Elimination

WORKED OUT EXAMPLES

Example 1: Solve $AX = B$ by LU-decomposition using Gaussian elimination where

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

and
(a) $B^T = (-4, 10, 5)$, (b) $B^T = (20, 49, 32)$

Solution: Since A has all non-zero diagonal elements, we can factor A as LU. Use Gaussian elimination to reduce A to echlon form U, without using any row interchanges.

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Step I: The multiplier $\left(-\frac{1}{2}\right)$ is used to reduce the element $a_{21} = 1$ to zero. The operation is $R_{21}\left(-\frac{1}{2}\right)$. So $m_{21} = -\frac{1}{2}$. Similarly the multiplier $\left(-\frac{1}{2}\right)$ is used to reduce the element $a_{31} = 1$ to zero i.e., $R_{31}\left(-\frac{1}{2}\right)$. So $m_{31} = -\frac{1}{2}$. This results

$$A \sim \begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 1 & 5 \end{pmatrix}$$

Use the multiplier $\left(-\frac{1}{3}\right)$ to reduce the element $a_{32} = 1$ to zero i.e., $R_{32}\left(-\frac{1}{3}\right)$. So the multiplier is $m_{32} = -\frac{1}{3}$. This yields the echelon form (or upper triangular matrix) U of A as

$$A \sim \begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

Step II: The lower triangular matrix L is obtained by simply placing the *negative* of the multipliers used in introducing zeros in that position in U i.e., $l_{21} = -m_{21} = -\left(-\frac{1}{2}\right) = +\frac{1}{2}$, $l_{31} = -m_{31} = \frac{1}{2}$ and $l_{32} = -m_{32} = \frac{1}{3}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \end{pmatrix}$$

Thus the LU (factorization) decomposition of A by Gaussian elimination is

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

Now to solve $AX = B$, $LUX = B$ put $UX = Y$ so $LY = B$. First we solve $LY = B$ for Y by using forward substitution

(a) $B^T = (-4, 10, 5)$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix}$$

By forward substitution

$$y_1 = -4$$

$$\frac{1}{2}y_1 + y_2 = 10 \text{ so } y_2 = 10 - \frac{1}{2}y_1 = 12$$

$\frac{1}{2}y_1 + \frac{1}{3}y_2 + y_3 = 5$ so $y_3 = 5 - \frac{1}{2}y_1, -\frac{1}{3}y_2 = 3.$

Thus $Y^T = (-4, 12, 3)$.

Now solve $UX = Y$ using backward substitution.

$$\begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \\ 3 \end{pmatrix}$$

So $3x_3 = 3$ or $x_3 = 1$

$$3x_2 + 6x_3 = 12 \text{ so } x_2 = \frac{12 - 6x_3}{3} = 2$$

$$2x_1 + 4x_2 - 6x_3 = -4 \text{ so } x_1 = \frac{-4 + 6x_3 - 4x_2}{2} = -3$$

Solution: $X^T = (-3, 2, 1)$

(b)$B^T = (20, 49, 32)$

$$LY = B$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 49 \\ 32 \end{pmatrix}$$

Solving $y_1 = 20$

$\frac{1}{2}y_1 + y_2 = 49$ so $y_2 = 49 - \frac{1}{2}y_1 = 39$

$$\frac{1}{2}y_1 + \frac{1}{3}y_2 + y_3 = 32 \text{ so } y_3 = 32 - \frac{1}{2}y_1 - \frac{1}{3}y_2 = 9$$

Thus $Y^T = (20, 39, 9)$
Now solve $UX = Y$

$$\begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 39 \\ 9 \end{pmatrix}$$

Solving $3x_3 = 9$ so $x_3 = 3$

$$3x_2 + 6x_3 = 39 \quad \text{so } x_2 = \frac{39 - 6x_3}{3} = 7$$

$$2x_1 + 4x_2 - 6x_3 = 20 \text{ so } x_1 = \frac{20 + 6x_3 - 4x_2}{2} = 5$$

Solution: $X^T = (5, 7, 3)$.

Example 2: Solve the system

$$3x_1 - 6x_2 - 3x_3 = -3$$
$$2x_1 \quad + 6x_3 = -22$$
$$-4x_1 + 7x_2 + 4x_3 = 3$$

Solution: $A = \begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}$

consider

$$A = \begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = LU =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

From first row: $u_{11} = 3, u_{12} = -6, u_{13} = -3$
First 2nd row: $l_{21}u_{11} = 2, l_{21} \cdot 3 = 2, l_{21} = \frac{2}{3}$
$l_{21}u_{12} + u_{22} = 0, u_{22} = -l_{21}u_{12} = -\frac{2}{3} \cdot (-6) = 4$
$l_{21}u_{13} + u_{23} = 6, u_{23} = 6 - l_{21}u_{13}$
$= 6 - \frac{2}{3}(-3) = 8$
From 3rd row: $l_{31}u_{11} = -4, l_{31} = -\frac{4}{3}$
$l_{31}u_{12} + l_{32}u_{22} = 7, l_{32} = \frac{7 - l_{31}u_{12}}{u_{22}} = \frac{7 - \left(-\frac{4}{3}\right)(-6)}{4}$

so $l_{32} = -\frac{1}{4}$
$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$
so $u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23}$
$u_{33} = 4 - \left(-\frac{4}{3}\right) \cdot (-3) - \left(\frac{-1}{4}\right) \cdot 8 = 0 + 2 = +2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{-4}{3} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & +2 \end{bmatrix}$$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}$$

Solving $y_1 = -3$, $\frac{2}{3}y_1 + y_2 = -22$

$$y_2 = -22 - \frac{2}{3}y_1 = -20$$

So $$-\frac{4}{3}y_1 - \frac{1}{4}y_2 + y_3 = 3$$

$$y_3 = 3 + \frac{4}{3}y_1 + \frac{1}{4}y_2 = 3 - 4 - 5 = -6$$

$$UX = Y$$

$$\begin{bmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -20 \\ -6 \end{bmatrix}$$

solving $2x_3 = -6$ or $x_3 = -3$
$4x_2 + 8x_3 = -20$ or $x_2 = \frac{-20 - 8x_3}{4} = 1$
$3x_1 - 6x_2 - 3x_3 = -3$ or $x_1 = \frac{-3 + 6x_2 + 3x_3}{3} = -2$
solution $X^T = [-2,\ 1,\ -3]$.

EXERCISE

1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a \neq 0$ find LU decomposition.

Ans. $\begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & \frac{ad - bc}{a} \end{bmatrix}$

2. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that A has no LU-decomposition.

Hint: Since $a_{11} = 0, a_{21} = 1$ can not be made zero.

Solve the following system of equations by *LU*-decomposition.

3. $3x_1 - 6x_2 = 0, -2x_1 + 5x_2 = 1$

Ans. $x_1 = 2, x_2 = 1$

Hint: $L = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & -6 \\ 0 & 1 \end{pmatrix}$

4. $2x_1 + 8x_2 = -2, -x_1 - x_2 = -2$

Ans. $x_1 = 3,\ x_2 = -1$

Hint: $L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 8 \\ 0 & 3 \end{pmatrix}$

5. $-5x_1 - 10x_2 = -10,\ 6x_1 + 5x_2 = 19$

Ans. $x_1 = 4, x_2 = -1$

Hint: $L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{5} & 1 \end{pmatrix}, U = \begin{pmatrix} -5 & -10 \\ 0 & -7 \end{pmatrix}$

6. $2x_1 - 2x_2 - 2x_3 = -4$
$-2x_2 + 2x_3 = -2$
$-x_1 + 5x_2 + 2x_3 = 6$

Ans. $x_1 = -1, x_2 = 1, x_3 = 0$

Hint: $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix},$

$U = \begin{pmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{pmatrix}$

7. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$

Ans. $x_1 = -3,\ x_2 = 1,\ x_3 = 2,\ x_4 = 1$

Hint: $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix},$

$U = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

8. Solve $\begin{bmatrix} 4 & 8 & 4 & 0 \\ 1 & 5 & 4 & -3 \\ 1 & 4 & 7 & 2 \\ 1 & 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

(a) $B^T = [8, -4, 10, -4]$

(b) $B^T = [28, 13, 23, 4]$

Ans. (a) $Y^T = [8, -6, 12, 2], X^T = [3, -1, 1, 2]$

(b) $Y^T = [28, 6, 12, 1], X^T = [3, 1, 2, 1]$

Hint: $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix},$

$U = \begin{bmatrix} 4 & 8 & 4 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find *LU* decomposition (triangular factorization) $A = LU$

9. $\begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$

Ans. $L = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix}$

10. $\begin{bmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{bmatrix}$

Ans. $L = \begin{bmatrix} 1 & 0 & 0 \\ -0.2 & 1 & 0 \\ -0.6 & 5.5 & 1 \end{bmatrix}$

$U = \begin{bmatrix} -5 & 2 & -1 \\ 0 & 0.4 & 2.8 \\ 0 & 0 & -10 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}$

$$Ans.\ \ L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -1.75 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -7.5 \end{bmatrix}$$

13.9 SOLUTION TO TRIDIAGONAL SYSTEMS

Band matrix is a $n \times n$ square matrix A with the property that $a_{ij} = 0$ whenever $i + p \le j$ or $j + q \le i$ for integers p and q with $p > 1$ and $q < n$. The *band width* of such matrix is defined to be $w = p + q - 1$.

Example:

$$A = \begin{bmatrix} 8 & 3 & 0 \\ 2 & 6 & -1 \\ 0 & 6 & -9 \end{bmatrix}$$

A is band matrix with $p = 2, q = 2$ and band width 3.

In band matrices, all the non-zero entries are concentrated about the diagonal.

Tridiagonal matrix is a band matrix of width 3 with $p = q = 2$. Thus *tridiagonal* matrices are those that have non-zero elements *only* on the diagonal a_{ii} or super diagonal $a_{i,i+1}$ or subdiagonal $a_{i+1,i}$. So $a_{ij} = -0$ if $|i - j| > 1$.

Example:

$$B = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix}$$

Note: Non-zero elements (entries) occur only on the diagonal and in the positions adjacent to the diagonal.

Most often tridiagonal matrices occur in cubic spline interpolation and numerical solution (crank-Nicolson method) of PDE involving heat equation.

13.10 CROUT REDUCTION FOR TRIDIAGONAL LINEAR SYSTEMS

Consider a tridiagonal linear system of n equations in n unknowns.

$$a_{11}x_1 + a_{12}x_2 \cdots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots = b_2$$

$$\ldots$$

$$a_{n-1,n-2}x_{n-2} + a_{n-1,n-1}x_{n-1} + a_{n-1,n}x^n = b_{n-1}$$

$$a_{n,n-1}x_{n-1} + a_{nn}x_n = b_n$$

with the tridiagonal coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}$$

In the case of tridiagonal matrix A having large number of zeros in regular patterns, the computational effort is reduced due to the structure of A. Using Crout or Doolittle factorization algorithm, A can be factored into L and U where L is lower triangular matrix and U is an upper triangular matrix with one's on its main diagonal.

Here

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & l_{n,n-1} & l_{nn} \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & u_{12} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Since A has only $(3n - 2)$ non-zero entries, there are only $(3n - 2)$ conditions to be applied to determine the entries of L and U. There are $(2n - 1)$ undetermined entries in L and $(n-1)$ undetermined entries of U, which totals the number of conditions $(3n-2)$.

Carrying out the multiplication LU, we get

$$a_{11} = l_{11}$$

$$a_{i,i-1} = l_{i,i-1} \quad \text{for} \quad i = 2, 3, \ldots n \quad (1)$$

$$a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} \quad \text{for} \quad i = 2, 3, \ldots n \quad (2)$$

$$a_{i,i+1} = l_{ii}u_{i,i+1} \quad \text{for} \quad i = 1, 2, \ldots n-1 \quad (3)$$

From (2), non-zero off-diagonal terms in L are calculated first. Then using (3) and (2) obtain alternately the remainder of the entries in U and L.

Thus the tridiagonal system can be solved by LU decomposition followed by forward and backward substitution. The LU decomposition can be obtained using Gaussian elimination as is done in the earlier section (without tedious calculations of l_{ii} and u_{ii} see W.E.2).

If A is a symmetric ($a_{ij} = a_{ji}$) and positive definite (i.e., $V^TAV > 0$ for all $V \neq 0$). Then by Cholesky decomposition we can factorize A as

$$A = LL^T$$

This factorization is sometimes referred to as taking the square root of the matrix A. Instead of seeking arbitrary lower triangular matrix L and upper triangular matrix U, Cholesky decomposition constructs a lower triangular matrix L whose transpose L^T can itself serves as upper triangular matrix U.

WORKED OUT EXAMPLES

Example 1: Solve the following tridiagonal system.

$$x_1 - x_2 = 0$$

$$-2x_1 + 4x_2 - 2x_3 = -1$$

$$-x_2 + 2x_3 = 1.5$$

Solution: The tridiagonal matrix is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

LU-decomposition:

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

From first column: $l_{11} = a_{11} = 1$, $l_{21} = a_{21} = -2$, $l_{31} = 0$

From 2nd column: $l_{11}u_{12} = -1$, $u_{12} = \frac{-1}{l_{11}} = \frac{-1}{1} = -1$

$l_{21}u_{12} + l_{22} = 4$, $l_{22} = 4 - l_{21}u_{12} = 4 - (-2)(-1) = 2$,

$l_{21}u_{13} + l_{22}u_{23} = -2$,

Also $l_{11}u_{13} = 0$ so $u_{13} = 0$

$$0 + 2 \cdot u_{23} = -2 \quad \text{so} \quad u_{23} = -1$$

From 3rd row: $l_{31} = 0$

$$l_{31}u_{12} + l_{32} = -1 \qquad \therefore \qquad l_{32} = -1$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$$

$$0 - 1 \cdot (-1) + l_{33} = 2 \qquad \therefore \qquad l_{33} = 1$$

Thus

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} = A = LU =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $AX = LUX = B$

Put $UX = Y$ so $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = B = \begin{bmatrix} 0 \\ -1 \\ \frac{3}{2} \end{bmatrix}$$

solving $y_1 = 0$, $y_2 = -\frac{1}{2}$, $y_3 = 1$

From $UX = Y$, we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

solving $x_3 = 1$, $x_2 = x_1 = \frac{1}{2}$

Example 2: Solve

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Also solve when $b_1 = 1$, $b_2 = 0$, $b_3 = 2$, $b_4 = 3$, $b_5 = -1$

Solution: Assume $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{bmatrix}$$

Now we reduce the tridiagonal matrix A to echolon form using Gaussian elimination method without any row interchanges.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{R_{21}\left(-\frac{1}{2}\right)}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_{32}\left(-\frac{2}{3}\right)}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{R_{43}\left(-\frac{3}{4}\right)}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_{54}\left(-\frac{4}{5}\right)}{\sim} U = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & \frac{6}{5} \end{bmatrix}$$

Then multipliers are $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5}$ so

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 1 \end{bmatrix}$$

Now solve by forward substitution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Solving $y_1 = b_1$, $y_2 = b_2 - \frac{1}{2}b_1$

$$y_3 = b_3 - \frac{2}{3}b_2 + \frac{1}{3}b_1$$

$$y_4 = b_4 - \frac{3}{4}b_3 + \frac{1}{2}b_2 - \frac{1}{4}b_1$$

$$y_5 = b_5 - \frac{4}{5}b_4 + \frac{3}{5}b_3 - \frac{2}{5}b_2 + \frac{1}{5}b_1$$

For $b_1 = 1,\ b_2 = 0,\ b_3 = 2,\ b_4 = 3,\ b_5 = -1,$

$$y_1 = 1,\ y_2 = -\frac{1}{2},\ y_3 = \frac{7}{3},\ y_4 = \frac{21}{4},\ y_5 = 0.$$

Now solve by backward substitution

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = B = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{7}{3} \\ \frac{21}{4} \\ 0 \end{bmatrix}$$

solving $x_5 = 0,\ x_4 = \dfrac{21}{5},\ x_3 = -\dfrac{13}{10},$ $x_2 = \dfrac{8}{15},\ x_1 = \dfrac{7}{30}$

Thus the solution is

$$X^T = \left[\frac{7}{30}, \frac{8}{15}, -\frac{13}{10}, \frac{21}{5}, 0\right]$$

Note the amount of simplification in calculation of L and U.

EXERCISE

Solve the tridiagonal systems.

1. $3x_1 + x_2 = -1$
   $2x_1 + 4x_2 + x_3 = 7$
   $2x_2 + 5x_3 = 9$

Ans. $x_1 = -0.999995$, $x_2 = 1.999999$, $x_3 = 1$