12. $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$ 5(ii) or method of proportions
Ans: $(y-x)^3 = C(y+x-2)$

13. $\dfrac{dy}{dx} = \dfrac{ax+by-a}{bx+ay-b}$ q 5(ii) or method of proportions
Ans: $(y-x+1)^{a+b}(y+x-1)^{a-b} = C$

14. $\dfrac{dy}{dx} = \dfrac{2x+y-2}{3x+y-3}$ 5(ii)
Ans: $\dfrac{2+\sqrt{3}}{2\sqrt{3}}\log\left(\dfrac{y}{x-1}+1-\sqrt{3}\right) - \dfrac{2-\sqrt{3}}{2\sqrt{3}}\log\left(\dfrac{y}{x-1}+1+\sqrt{3}\right) + \log(x+1) = C$

15. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$ 5(i) or method of proportions
Ans: $y = x + \log(x+y) + C$

16. $(2x^2+3y^2-7)xdx + (3x^2+2y^2-8)ydy = 0$ 4
[Hint: Put $x^2 = X, y^2 = Y$ and group the terms $(2X-7)dX + 3\,(YdX+XdY) + (2Y-8)dY = 0$]
Ans: $X^2+3XY+Y^2-7X-8Y+C=0$; $x^4+3x^2y^2+y^4-7x^2-8y^2+C=0$

17. $\dfrac{dy}{dx} = \dfrac{(2e^x-e^y+1)e^x}{(e^x+2e^y)e^y}$ 4
Ans: $e^{2x}+e^x = C+e^{x+y}+e^{2y}$;

18. $\dfrac{dy}{dx} = \dfrac{y-x+2}{3y-3x+6}$
Ans: $3y = x + C$

1.3.4 Exact Equations

Definition A differential equation which is obtained from its primitive by mere differentiation without any further operation is called an exact equation.

The equations $xdy+ydx=0, xdx+ydy=0$ and $\dfrac{xdy-ydx}{x^2}=0$ are exact since these can be written as $d(xy)=0, \dfrac{1}{2}d(x^2+y^2)=0$ and $d\left(\dfrac{y}{x}\right)=0$, respectively.

The following theorem gives a criterion for exact equations.

Theorem 1.1 If $M(x,y)$ and $N(x,y)$ are real-valued functions having continuous first partial derivatives on some rectangle $R : |x-x_0| \leq a, |y-y_0| \leq b$, then a necessary and sufficient condition for the equation

$$Mdx + Ndy = 0 \tag{1.7}$$

to be exact in R is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{in } R. \tag{1.8}$$

Proof: (i) **The condition is necessary:** Suppose Eq. (1.7) is exact. Then there exists a function $u(x,y)$ such that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$= Mdx + Ndy \tag{1.9}$$
$$\Rightarrow M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y} \tag{1.10}$$

Differentiating M and N partially with respect to y and x, respectively

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \tag{1.11}$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} \tag{1.12}$$

From Eqs. (1.11) and (1.12) it follows that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left(\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right) \tag{1.13}$$

(ii) **The condition is sufficient:** Assume that Eq. (1.8) holds.

Let $\quad v(x,y) = \displaystyle\int_{y\ \text{constant}} Mdx$

where the integral is evaluated partially with respect to x treating y as constant so that $\dfrac{\partial v}{\partial x} = M$.

$$\text{Now, } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \tag{1.14}$$

$$= \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(N - \frac{\partial v}{\partial y}\right) = 0$$

$$\Rightarrow N - \frac{\partial v}{\partial y} = \phi(y), \text{a function of } y \text{ alone}$$

$$\Rightarrow N = \frac{\partial v}{\partial y} + \phi(y)$$

Hence the equation becomes

$$0 = Mdx + Ndy = \frac{\partial v}{\partial x}dx + \left(\frac{\partial v}{\partial y} + \phi(y)\right) dy$$

$$= d\left(v + \int \phi(y)dy\right) \tag{1.15}$$

which proves that the equation is exact.

Procedure for solving an exact equation

(1) Test if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

(2) Compute $v = \displaystyle\int_{y \text{ constant}} Mdx$

(or) $u = \displaystyle\int_{x \text{ constant}} Ndy$.

(3) Compute $\displaystyle\int \left(N - \frac{\partial v}{\partial y}\right) dy$

(or) $\displaystyle\int \left(M - \frac{\partial u}{\partial x}\right) dx$.

(4) The general solution is

$$\int_{y \text{ constant}} Mdx + \int \left(N - \frac{\partial v}{\partial y}\right)dy = c$$

(or) $$\int_{x \text{ constant}} Ndy + \int \left(M - \frac{\partial u}{\partial x}\right) dx = c.$$

Note 1

In most cases, collecting and grouping terms which are exact differentials will not only prove the exactness of the equation but also yield the general solution, on integration, as the illustrative examples will show.

Note 2

The following formula

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c \tag{1.16}$$

though in many cases, gives the correct solution, fails in some cases and is therefore not advisable.

(Example 1.31 shows the failure of this method to give the correct solution.)

Example 1.31

Solve $y \sin 2x\, dx = (y^2 + \cos^2 x)dy$.

Solution Here

$$M = y \sin 2x; \quad N = -y^2 - \cos^2 x$$

$$\frac{\partial M}{\partial y} = \sin 2x; \quad \frac{\partial N}{\partial x} = -2\cos x(-\sin x) = \sin 2x$$

The equation is exact, since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\text{Now } v = \int_{y \text{ constant}} Mdx = \int_{y \text{ constant}} y \sin 2x dx = -\frac{y}{2}\cos 2x$$

$$\left(N - \frac{\partial v}{\partial y}\right) = -y^2 - \cos^2 x + \frac{1}{2}\cos 2x$$

$$= -y^2 - \cos^2 x + \frac{1}{2}\left(2\cos^2 x - 1\right)$$

$$= -y^2 - \frac{1}{2}$$

$$\int \left(N - \frac{\partial v}{\partial y}\right) dy = -\frac{1}{3}y^3 - \frac{1}{2}y$$

Hence the general solution is

$$-\frac{y}{2}\cos 2x - \frac{1}{3}y^3 - \frac{1}{2}y = \text{constant}$$

$$\Rightarrow y\cos^2 x + \frac{1}{3}y^3 = c$$

$$\Rightarrow \frac{y}{2}(\cos 2x + 1) + \frac{1}{3}y^3 = c.$$

Note

If we compute the solution using Eq. (1.16) we get

$$-\frac{y}{2}\cos 2x - \int y^2 dy = \text{constant}$$

$$\Rightarrow \frac{y}{2}\cos 2x + \frac{y^3}{3} = c$$

in which the term $\frac{1}{2}y$ is missing.

Example 1.32

Solve $\dfrac{dy}{dx} + \dfrac{3x - 2y + 5}{-2x + 4y + 1} = 0.$

Solution In the standard form, the equation is

$$(3x - 2y + 5)dx + (-2x + 4y + 1)dy = 0$$
$$M = 3x - 2y + 5; \quad N = -2x + 4y + 1$$
$$M_y = -2 = N_x = -2$$

The equation is exact since $\partial M/\partial y = \partial N/\partial x$

$$v = \int_{y\text{ constant}} Mdx = \int \left(3x - 2y + 5\right)dx$$
$$= \frac{3}{2}x^2 - 2xy + 5x$$
$$\frac{\partial v}{\partial x} = -2x$$
$$N - \frac{\partial v}{\partial y} = \left(-2x + 4y + 1\right) - (-2x)$$
$$= 4y + 1$$
$$\int \left(N - \frac{\partial v}{\partial y}\right)dy = 2y^2 + y$$

The general solution is

$$v + \int \left(N - \frac{\partial v}{\partial y}\right)dy = \frac{3}{2}x^2 - 2xy + 5x + 2y^2 + y$$
$$= \text{Constant}$$
$$\Rightarrow 3x^2 - 4xy + 4y^2 + 10x + 2y + c = 0.$$

Alternative method: Grouping the terms

$$(3x + 5)dx - 2(ydx + xdy) + (4y + 1)dy = 0$$

$$\text{or } d\left(\frac{3}{2}x^2 + 5x\right) - 2d(xy) + d(2y^2 + y) = 0$$

This shows that the equation is exact. Integrating it, we get

$$\frac{3}{2}x^2 + 5x - 2xy + 2y^2 + y = \text{Constant}$$

or $3x^2 - 4xy + 4y^2 + 10x + 2y + c = 0.$

Example 1.33

Solve $\dfrac{dy}{dx} = \dfrac{3x^2 - 2xy - 5}{x^2 + y^2 - 2y}.$

Solution Writing the equation in the standard form

$$l(3x^2 - 2xy - 5)dx - (x^2 + y^2 - 2y)dy = 0$$
$$M = 3x^2 - 2xy - 5; \quad N = -x^2 - y^2 + 2y$$
$$M_y = -2x = N_x = -2x$$

The equation is exact.

$$v = \int_{y\text{ constant}} Mdx = x^3 - x^2y - 5x;$$

$$\frac{\partial v}{\partial y} = -x^2$$

$$\int \left(N - \frac{\partial v}{\partial y}\right)dy = \int \left(-y^2 + 2y\right)dy = \frac{1}{3}y^3 + y^2$$

The general solution is,

$$x^3 - x^2y - 5x - \frac{1}{3}y^3 + y^2 + \text{constant} = 0$$
$$\Rightarrow 3x^3 - y^3 - 3x^2y + 3y^2 - 15x + c = 0.$$

Alternative method: Grouping the terms

$$(3x^2 - 5)dx + (2y - y^2)dy - (2xydx + x^2dy) = 0$$
$$\Rightarrow d(x^3 - 5x) + d\left(y^2 - \frac{1}{3}y^3\right) - d(x^2y) = 0$$

The equation is exact. Integrating and multiplying by 3, we get the general solution as

$$3x^3 - 3x^2y - y^3 + 3y^2 - 15x + c = 0.$$

Example 1.34
Solve $\left(1+e^{\frac{x}{y}}\right)dx+\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}dy=0.$
[JNTU 2000]

Solution Here

$$M=1+e^{\frac{x}{y}};\quad N=\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}$$

$$\begin{aligned}M_y &= e^{\frac{x}{y}}\left(-\frac{x}{y}\right)\\ &= N_x=\left[\left(1-\frac{x}{y}\right)\frac{1}{y}-\frac{1}{y}\right]e^{\frac{x}{y}}\\ &= -\frac{x}{y^2}e^{\frac{x}{y}}\end{aligned}$$

The equation is exact.

$$v=\int\limits_{y\text{ constant}} Mdx=x+y\mathrm{e}^{\frac{x}{y}};$$

$$\frac{\partial v}{\partial y}=1\cdot e^{\frac{x}{y}}+y\left(-\frac{x}{y^2}\right)e^{\frac{x}{y}}=\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}$$

$$N-\frac{\partial v}{\partial y}=\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}-\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}=0$$

The general solution is $x+ye^{\frac{x}{y}}=c$.

Alternative method: Grouping the terms

$$dx+e^{\frac{x}{y}}dy+\left(dx-\frac{x}{y}dy\right)e^{\frac{x}{y}}=0$$

$$\Rightarrow\quad dx+e^{\frac{x}{y}}dy+y\cdot e^{\frac{x}{y}}d\left(\frac{x}{y}\right)=0$$

$$\Rightarrow\quad dx+d\left(ye^{\frac{x}{y}}\right)=0$$

The equation is exact.
The general solution is $x+ye^{\frac{x}{y}}=c$.

Example 1.35
Solve $(y^2-2xy)dx=(x^2-2xy)dy$.
[JNTU 1995]

Solution

$$M=y^2-2xy;\quad N=2xy-x^2$$

$$M_y=2y-2x=N_x=2y-2x$$

The equation is exact.

$$v=\int\limits_{y\text{ constant}} Mdx=xy^2-x^2y;$$

$$\frac{\partial v}{\partial y}=2xy-x^2$$

$$N-\frac{\partial v}{\partial y}=-x^2+2xy-2xy+x^2=0$$

The general solution is $xy^2-x^2y=c$.

Alternative method: Grouping the terms

$$y^2dx+2xydy=x^2dy+2xydx$$

$$\Rightarrow\quad d\left(xy^2\right)-d\left(x^2y\right)=0$$

(The equation is exact.)

The general solution is $xy^2-x^2y=c$.

Example 1.36
Solve $xdx+ydy=\dfrac{xdy-ydx}{x^2+y^2}$.

Solution

$$\text{LHS}=\frac{1}{2}d(x^2+y^2)$$

$$\text{RHS}=\frac{\left(\frac{xdy-ydx}{x^2}\right)}{\left(\frac{x^2+y^2}{x}\right)}$$

(dividing the numerator and denominator by x^2)

$$=\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=d\left(\tan^{-1}\frac{y}{x}\right).$$

From the above results, we observe that the equation is exact. Its general solution is

$$x^2+y^2=2\tan^{-1}\frac{y}{x}+c.$$

Example 1.37
Solve $(e^y+1)\cos x\,dx+e^y\sin\,xdy=0$.

Solution

$$\left(e^y\cos xdx+e^y\sin xdy\right)+\cos xdx=0$$

$$\Rightarrow\quad d\left(e^y\sin x\right)+d(\sin x)=0$$

(The equation is exact.)

Integrating, we get the general solution as

$$(e^y + 1)\sin x = c.$$

EXERCISE 1.4

Solve:

1. $(2x + 3y + 4)dx + (3x - 6y + 2)dy = 0.$
Ans: $x^2 + 3xy - 3y^2 + 4x + 2y + c = 0$

2. $(3x^2 - 9x^2y^2 + 2xy)dx + (6y^2 - 6x^3y^2 + x^2)dy = 0.$
Ans: $x^3 + 2y^3 - 3x^3y^2 + x^2y = c$

3. $e^x(\sin x + \cos x)\sec y\, dx + e^x \sin x \sec y \tan y\, dy = 0.$
Ans: $e^x \sin x \sec y = c$

4. $(\cos x \cos y - \cot x)\, dx = \sin x \sin y\, dy.$
Ans: $\sin x \cos y = \log | c \sin x |$

5. $\left\{y\left(1 + \frac{1}{x}\right) + \sin y\right\} dx + \{x + \log x + x \cos y\}\, dy = 0.$
Ans: $xy + y \log x + x \sin y = c$

6. $4y \sin 2x dx + (2y + 3 - 4\cos^2 x)dy = 0.$
Ans: $y^2 + y - 2y \cos 2x = c$

7. $xdx + ydy + \dfrac{xdy - ydx}{x^2 - y^2} = 0.$
Ans: $(x^2 + y^2)e^{\frac{x+y}{x-y}} = c$

8. $(x^2 - ay)dx = (ax - y^2)dy.$
Ans: $x^3 + y^3 = 3axy + c$

9. $\dfrac{xdx + ydy}{x^2 + y^2} + \dfrac{xdy - ydx}{x^2 - y^2} = 0.$
Ans: $(x^2 + y^2)\left(\frac{x+y}{x-y}\right) = c$

10. $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$
Ans: $xy + y \sin x + x \sin y = c$

11. $\dfrac{y(xy + e^x)dx - e^x dy}{y^2} = 0.$
$\left[\text{Hint: } xdx + \dfrac{ye^x dx - e^x dy}{y^2} = 0.\right]$
Ans: $\frac{x^2}{2} + \frac{e^x}{y} = c$

12. $\dfrac{ydx - xdy}{y^2} + xe^x dx = 0.$
Ans: $\frac{x}{y} + (x - 1)e^x = c$

1.3.5 Inexact Equation—Reducible to Exact Equation by Integrating Factors

Integrating factor (I.F.)

If the differential equation $Mdx + Ndy = 0$ becomes exact when we multiply it by a function $\mu(x, y)$ then $\mu(x, y)$ is called an integrating factor of the equation.

Consider the equation $ydx - xdy = 0$

Here

$$M = y, \quad N = -x \Rightarrow \frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$$

The equation is not exact. If we multiply it by $\frac{1}{y^2}$ we get

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0.$$

Now $M = \frac{1}{y}, N = -\frac{x}{y^2}$ so that

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}.$$

So, the equation becomes exact. $\frac{1}{y^2}$ is an integrating factor. We can easily check that $\frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$ are also integrating factors for the equation.

Integrating factor can be found by inspection, after grouping of terms. Table 1.1 gives the list of integrating factors.

Example 1.38
Solve the following differential equations after finding the integrating factor in each case:

Differential Equation	Integrating Factor
(i) $ydx - xdy + y^2xe^x dx = 0$	$\frac{1}{y^2}$

Table 1.1

S. No.	Group of terms	Integrating factor	Exact differential
1.	$xdy + ydx$	1	$d(xy)$
2.	$xdy + ydx$	$\frac{1}{xy}$	$\frac{xdy+ydx}{xy} = d(\log xy)$
3.	$xdy + ydx$	$\frac{1}{(xy)^a}$	$\frac{xdy+ydx}{(xy)^a} = d\left(\frac{(xy)^{1-a}}{1-a}\right) \quad (a \neq 1)$
4.	$xdx + ydy$	$\frac{2}{x^2+y^2}$	$\frac{2xdx+2ydy}{x^2+y^2} = d[\log(x^2+y^2)]$
5.	$xdx + ydy$	$\frac{2}{(x^2+y^2)^a}$	$\frac{2xdx+2ydy}{(x^2+y^2)^a} = d\left[\frac{(x^2+y^2)^{1-a}}{1-a}\right] \quad (a \neq 1)$
6.	$xdx + ydy$	2	$d(x^2+y^2)$
7.	$xdy - ydx$	$\frac{1}{x^2}$	$\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$
8.	$xdy - ydx$	$-\frac{1}{y^2}$	$\frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right)$
9.	$xdy - ydx$	$\frac{1}{xy}$	$\frac{dy}{y} - \frac{dx}{x} = d\left(\log\frac{y}{x}\right)$
10.	$xdy - ydx$	$\frac{1}{x^2+y^2}$	$\frac{xdy-ydx}{x^2+y^2} = \frac{\frac{xdy-ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2} = d(\tan^{-1}\frac{y}{x})$
11.	$xdy - ydx$	$-\frac{1}{x^2+y^2}$	$\frac{\frac{ydx-xdy}{y^2}}{1+\left(\frac{x}{y}\right)^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
12.	$2xdy - ydx$	$\frac{y}{x^2}$	$\frac{2xydy-y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$
13.	$xdy - 2ydx$	$-\frac{x}{y^2}$	$\frac{2xydx-x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$
14.	$xdy - ydx$	$\frac{2}{x^2-y^2}$	$\frac{2xdy-2ydx}{x^2-y^2} = \frac{2\left(\frac{xdy-ydx}{x^2}\right)}{1-\left(\frac{y}{x}\right)^2} = d\left(\log\frac{x+y}{x-y}\right)$

(ii) $ydx - xdy + 2x^2y\sin x^2dx = 0$ $\qquad \frac{1}{xy}$

(iii) $ydx - xdy + 2x^2ye^{y^2}dy = 0$ $\qquad \frac{1}{x^2}$

(iv) $ydx - xdy + (x^2+y^2)\frac{dx}{\sqrt{1-x^2}} = 0$ $\qquad \frac{1}{x^2+y^2}$

(v) $xdx + ydy + (x^2+y^2)\tan xdx = 0$ $\qquad \frac{1}{x^2+y^2}$

(vi) $y(1+xy)dx + x(1-xy)dy = 0$ $\qquad \frac{1}{(xy)^2}$

Solution The suitable integrating factor in each case is shown against each differential equation. After multiplying by the integrating factor, we can derive the general solution as follows:

(i) $$\frac{ydx - xdy}{y^2} + xe^x dx = 0$$

$$\Rightarrow \int d\left(\frac{x}{y}\right) + \int xe^x dx = c$$

$$\Rightarrow \frac{x}{y} + (x-1)\,e^x = c.$$

(ii) $$\frac{ydx - xdy}{xy} + 2x\sin x^2 dx = 0$$

$$\Rightarrow \quad \frac{dx}{x} - \frac{dy}{y} + \sin\left(x^2\right) d\left(x^2\right) = 0$$
$$\Rightarrow \quad \log\frac{x}{y} = \cos x^2 + c.$$

(iii) $$\frac{ydx - xdy}{x^2} + e^{y^2} dy^2 = 0$$
$$\Rightarrow \quad -d\left(\frac{y}{x}\right) + d\left(e^{y^2}\right) = 0$$
$$\Rightarrow \quad -\frac{y}{x} + e^{y} = c$$
$$\Rightarrow \quad e^{y} = c + \frac{y}{x}.$$

(iv) $$\frac{ydx - xdy}{x^2+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$
$$\Rightarrow \quad \frac{\frac{ydx-xdy}{x^2}}{1+\left(\frac{y}{x}\right)^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$
$$\Rightarrow \quad \frac{-d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$
$$\Rightarrow \quad \sin^{-1} x = \tan^{-1}\frac{y}{x} + c.$$

(v) $$\frac{2xdx + 2ydy}{x^2+y^2} + \tan x dx = 0$$
$$\Rightarrow \quad d\left(\log\left(x^2+y^2\right)\right) + d\left(\log(\sec x)\right) = 0$$
$$\Rightarrow \quad \left(x^2+y^2\right)(\sec x) = c.$$

(vi) $$ydx + xdy + xy\left(ydx - xdy\right) = 0$$
$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$
$$\Rightarrow \quad -\frac{1}{xy} + \log\frac{x}{y} = c.$$

EXERCISE 1.5

Find the integrating factors by inspection and solve the following.

1. $xdy - \left(y - 3x^2\right) dx = 0$
Ans: $\frac{y}{x} + 3x = c$

2. $ye^{-x} + e^{-x}dy = 2xy^2 dx$
Ans: $\frac{e^{-x}}{y} = x^2 + c$

3. $xdy + 2ydx = 2y^2xdy$
Ans: $xy^2 = ce^{y^2}$

4. $(x^2+y^2)(xdy - ydx) = (x^2 - y^2)(xdx + ydy)$
Ans: $(x+y) = c(x-y)(x^2+y^2)$

5. $2xydy - y^2dx + x^3e^xdx = 0$
Ans: $y^2 + x(x-1)e^x + cx = 0$

Given below are the rules for finding integrating factors of $Mdx + Ndy = 0$ depending on the nature of functions M and N.

Rule 1
If $Mdx + Ndy = 0$ is a homogeneous differential equation and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of the equation.

Note 1
Recall that we have already solved homogeneous differential equations reducing them to variables separable form by putting $y = vx$ or $x = vy$. **Rule** 1 provides an alternative method for its solution.

Note 2
If $Mx + Ny = 0$, then $\frac{M}{y} = \frac{N}{-x}$ and the equation becomes $ydx - xdy = 0$ whose general solution is $\frac{x}{y} = c$.

Rule 2
If $Mdx + Ndy = 0$ is of the form

$$f_1(xy)ydx + f_2(xy)xdy = 0$$

and $Mx - Ny \neq 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor of the equation.

Rule 3
If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} / N = p(x)$, purely a function of x alone, then $e^{\int p(x)dx}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Note that the linear equation $\frac{dy}{dx} + Py = Q$ where P, Q are functions of x alone may be written in the form $Mdx + Ndy = 0$ where $M = Py - Q$ and $N = 1$ so that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P-0}{1}$$
$$= P,$$

purely a function of x and the integrating factor in this case is $e^{\int p(x)dx}$, as we will see in linear equations discussed below.

Rule 4

If $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) / M = q(y)$, purely a function of y alone then $e^{\int q(y)dy}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Rule 5

If $Mdx + Ndy = 0$ is expressible in the form

$$x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$$

where a, b, c, d, m, n, p, q are all constants such that $mq - np \neq 0$, then $x^h y^k$ is an integrating factor of the equation for some suitable constants h, k to be determined from the two equations $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ and $\frac{c+h+1}{p} = \frac{d+k+1}{q}$.

Example 1.39

Solve $(x^2+y^2)dx - xydy = 0$ by finding an integrating factor.

Solution This is a homogeneous differential equation.

Here,

$$M = x^2 + y^2, \qquad N = -xy;$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -y$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact

$$Mx + Ny = (x^2 + y^2)x - (xy)y = x^3 \neq 0$$

Multiplying by the integrating factor,

$$\frac{1}{Mx + Ny} = \frac{1}{x^3}$$

$$\left(\frac{1}{x} + \frac{y^2}{x^3}\right) dx - \frac{y}{x^2} dy = 0$$

$$v = \int\limits_{y \text{ constant}} Mdx = \int \left(\frac{1}{x} + \frac{y^2}{x^3}\right) dx$$

$$= \log x - \frac{y^2}{2x^2}$$

$$\frac{\partial v}{\partial y} = -\frac{y}{x^2}; \quad N - \frac{\partial v}{\partial y} = 0$$

$\therefore$ The general solution is $\log x = \frac{y^2}{2x^2} + c$.

Example 1.40

Find an integrating factor and solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2 + y^2}{x^2}$.

Solution Putting the equation in the standard form $\left(xy + x^2 + y^2\right) dx - x^2 dy = 0$

$$M = xy + x^2 + y^2, \quad N = -x^2$$

$$\frac{\partial M}{\partial y} = x + 2y, \qquad \frac{\partial N}{\partial x} = -2x$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact

$$Mx + Ny = x\left(xy + x^2 + y^2\right) - x^2 y$$
$$= x\left(x^2 + y^2\right) \neq 0$$

The given equation is a homogeneous differential equation.

$$\therefore \quad \text{Integrating factor} = \frac{1}{Mx + Ny} = \frac{1}{x\left(x^2 + y^2\right)}$$

Multiplying the equation by the integrating factor

$$\frac{dx}{x} = \frac{xdy - ydx}{x^2 + y^2}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{\frac{xdy - ydx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \quad \log x = \tan^{-1}\frac{y}{x} + c.$$

Example 1.41

Solve $\left(xy + 1\right) ydx + \left(-xy + 1\right) xdy = 0$.

Solution This equation is of the form

$$f_1\left(xy\right) ydx + f_2\left(xy\right) xdy = 0$$

By **Rule** 2, the integrating factor is $\dfrac{1}{Mx - Ny}$ if $Mx - Ny \neq 0$.

$Mx - Ny = \left(xy + 1\right) yx - \left(-xy + 1\right) yx = 2x^2y^2 \neq 0$

Multiplying by the integrating factor $= \dfrac{1}{x^2y^2}$, the equation can be written as

$$\frac{dx}{x} - \frac{dy}{y} + \frac{d(xy)}{(xy)^2} = 0 \Rightarrow \log x/y = \frac{1}{xy} + c.$$

Example 1.42
Solve $\left(x^2y^2+xy+1\right)ydx+\left(x^2y^2-xy+1\right)xdy=0.$

Solution The equation is of the form
$f_1\left(xy\right)ydx+f_2\left(xy\right)xdy=0$

$$\begin{aligned} Mx-Ny &= x^3y^3+x^2y^2+xy-\left(x^3y^3-x^2y^2+xy\right) \\ &= 2x^2y^2 \neq 0 \end{aligned}$$

Integrating factor $= \dfrac{1}{Mx-Ny} = \dfrac{1}{2x^2y^2}$

Multiplying the equation by $\dfrac{1}{x^2y^2}$ and omitting the constant,

$$\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right)ydx+\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)xdy = 0$$

$$\left(1+\frac{1}{x^2y^2}\right)(ydx+xdy)+\frac{1}{xy}(ydx-xdy) = 0$$

$$\left(1+\frac{1}{x^2y^2}\right)d(xy)+\frac{1}{x}dx-\frac{1}{y}dy = 0$$

$$\left(1+\frac{1}{\left(xy\right)^2}\right)d\left(xy\right)+d\left[\log\left(\frac{x}{y}\right)\right] = 0$$

The general solution is $xy-\dfrac{1}{xy}+\log\dfrac{x}{y}=c.$

Example 1.43
Solve $\left(x-y\right)dx-dy=0.$

Solution

$$M=x-y,\quad N=-1,$$

$$\frac{\partial M}{\partial y}=-1,\quad \frac{\partial N}{\partial x}=0;$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-1-0}{-1}=1=p\left(x\right)$$

(Constant can be considered as a function of x.)

By **Rule** 3

Integrating factor $= e^{\int p(x)dx} = e^{\int 1dx} = e^x$

Multiplying by the integrating factor $= e^x$, the equation can be written as

$$xe^xdx=\left(ye^xdx+e^xdy\right)\Rightarrow (x-1)\,e^x=ye^x+c$$
or $\quad x=y+1+ce^{-x}.$

Example 1.44
Solve $\left(3xy-2y^2\right)dx+\left(x^2-2xy\right)dy=0.$

Solution

$$M=3xy-2y^2;\quad N=x^2-2xy$$

$$\frac{\partial M}{\partial y}=3x-4y\qquad \frac{\partial N}{\partial x}=2x-2y$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{\left(3x-4y\right)-\left(2x-2y\right)}{x\left(x-2y\right)}$$

$$=\frac{x-2y}{x\left(x-2y\right)}=\frac{1}{x}=p\left(x\right),\text{ a function of } x$$

By **Rule** 3

Integrating factor $=e^{\int p(x)dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

Multiplying by integrating factor $=x$, the equation becomes

$$\left(3x^2y-2xy^2\right)dx+\left(x^3-2x^2y\right)dy=0$$

$$d\left(x^3y\right)=d\left(x^2y^2\right)\Rightarrow x^3y=x^2y^2+c$$

is the general solution of the differential equation.

Example 1.45
Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y\right)dy=0.$

Solution

$$M=xy^3+y\qquad N=2x^2y^2+2x+2y$$

$$\frac{\partial M}{\partial y}=3xy^2+1\qquad \frac{\partial N}{\partial x}=4xy^2+2$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{xy^2+1}{\left(xy^2+1\right)y}=\frac{1}{y}=q\left(y\right),$$

a function of y alone.

By **Rule** 4

$$\text{Integrating factor}=e^{\int q\left(y\right)dy}=e^{\int \frac{1}{y}dy}=y$$

Multiplying by integrating factor, the equation can be written as

$$\left(xy^4dx+2x^2y^3dy\right)+\left(y^2dx+2xydy\right)+2ydy=0$$

$$\frac{1}{2}d\left(x^2y^4\right)+d\left(xy^2\right)+d\left(y^2\right)=0$$

General solution is

$$\frac{1}{2}x^2y^4+xy^2+y^2=c.$$

Example 1.46
Solve $\left(y+xy^2\right)dx - xdy = 0.$

Solution

$$M = y + xy^2; \quad N = -x$$
$$\frac{\partial M}{\partial y} = 1 + 2xy; \ \frac{\partial N}{\partial x} = -1$$
$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y\left(1+xy\right)}\left(-1-1-2xy\right)$$
$$= \frac{-2\left(1+xy\right)}{y\left(1+xy\right)} = -\frac{2}{y} = q\left(y\right)$$
$$(xy \neq -1)$$

By **Rule** 4, integrating factor is

$$e^{\int q(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2},$$

a pure function of y.

Multiplying the equation by $\frac{1}{y^2}$ we can write the equation as $d\left(x^2\right) + 2\left(\frac{ydx - xdy}{y^2}\right) = 0$
whose general solution is
$x^2 + \frac{2x}{y} = c.$

Example 1.47
Solve $xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0$

Solution The given differential equation is

$$xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0 \quad (1)$$
$$\text{Here } M = xy^4 + 3y \qquad N = 2x^2y^3 + 5x$$
$$M_y = 4xy^3 + 3 \qquad N_x = 4xy^3 + 5$$

Since $M_y \neq N_x$, the equation is not exact. We can easily verify that Rules 1–4 are not applicable here. So, we try to find an I.F. of the form $x^h y^k$, by applying Rule 5.

Comparing Eq. (1) with the standard form

$$x^a y^b(mydx + nxdy) + x^c y^d(pydx + qydy) = 0 \quad (2)$$

We have

$$a = 1, \ b = 3, \ m = 1, \ n = 2$$
$$c = 0, \ d = 0, \ p = 3, \ q = 5 \quad (3)$$
$$\text{Also,} \quad mp - nq = 1.3 - 2.5 = -7 \neq 0 \quad (4)$$

The constants h, k are determined from

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{and}$$
$$\frac{c+h+1}{p} = \frac{d+k+1}{q}$$
$$\Rightarrow \quad \frac{1+h+1}{1} = \frac{3+k+1}{2} \quad \text{and}$$
$$\frac{0+h+1}{3} = \frac{0+k+1}{5}$$
$$\Rightarrow \quad 2h+4 = k+4 \quad \text{and} \quad 5h+5 = 3k+3$$
$$\Rightarrow \quad 2h = k \quad \text{and} \quad 5h+2 = 3k \quad (5)$$
$$\Rightarrow \quad (h,k) = (2,4) \quad (6)$$

An integrating factor is x^2y^4. Multiplying Eq. (1) by this integrating factor, we have

$$x^3 - y^8dx + 2x^4y^7dy$$
$$+3x^2y^5dx + 5x^3y^4dy = 0 \quad (7)$$

which, on regrouping and expressing as exact differentials, yields

$$\frac{1}{4}d(x^4y^8) + d(x^3y^5) = 0$$
$$\Rightarrow \quad x^4y^8 + 4x^3y^5 = c$$
$$\Rightarrow \quad x^3y^5(xy^3+4) = c \quad (8)$$

which is the required solution.

Example 1.48
Solve $x(3ydx + 2xdy) + 8y^4(ydx + xdy) = 0.$

Solution The given differential equation is

$$x(3ydx + 2xdy) + 8y^4(ydx + xdy) = 0 \quad (1)$$

We observe that an integrating factor for this equation is of the form $x^h y^k$ for some h, k. Multiplying Eq. (1) by $x^h y^k$, it can be written in the form

$$Mdx + Ndy = 0 \quad (2)$$

$$\text{where } M = 3x^{h+1}y^{k+1} + 8x^h y^{k+5},$$
$$N = 2x^{h+2}y^k + 8x^{h+1}y^{k+4}$$
$$\therefore \quad M_y = 3(k+1)x^{h+1}y^k$$
$$+8(k+5)x^h y^{k+4} \quad (3)$$
$$N_x = 2(h+2)x^{h+1}y^k + 8(h+1)x^h y^{k+4}$$

Exactness condition is

$$M_y = N_x \Rightarrow 3k = 2h + 1, k + 4 = h$$
$$\Rightarrow (h, k) = (13, 9) \qquad (4)$$

Now the left-hand side expression of Eq. (1) is exact

$$\Rightarrow \; Mdx + Ndy = \frac{3}{15} d(x^{15}y^{10}) + \frac{8}{14} d(x^{14}y^{14}) = 0$$
$$\Rightarrow \; x^{14}y^{10}(7x + 20y^4) = c$$

which is the required solution.

Example 1.49
Solve $2x^2(ydx + xdy) + y(ydx - xdy) = 0$.

Solution The given differential equation is

$$2x^2(ydx + xdy) + y(ydx - xdy) = 0 \qquad (1)$$

Here $\quad M = 2x^2y + y^2 \quad N = 2x^3 - xy$

$\therefore \quad M_y = 2x^2 + 2y \quad N_x = 6x^2 - y$

Since $M_y \neq N_x$, the equation is not exact.
To find an integrating factor of the form x^hy^k, we multiply Eq. (1) by x^hy^k so that new values of

$$M = 2x^{h+2}y^{k+1} + x^{hy\ k+2} \quad \text{and}$$
$$N = 2x^{h+3}y^k - x^{h+1}y^{k+1}$$
$$M_y = 2(k+1)x^{h+2}y^k + (k+2)x^hy^{k+1}$$
$$N_x = 2(h+3)x^{h+2}y^k - (h+1)x^hy^{k+1}$$

Equating $M_y = N_x$, we have

$$k = h + 2 \quad k + 3 = -h$$
$$(h, k) = (-5/2, -1/2)$$
$$\therefore Mdx + Ndy = \left(2x^{-1/2}y^{1/2} + x^{-5/2}y^{3/2}\right) dx + \left(2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right) dy = 0$$
$$\Rightarrow 4d(x^{1/2}y^{1/2}) - \frac{2}{3} d(x^{-3/2}y^{3/2}) = 0$$

Integrating, we get

$$4x^{1/2}y^{1/2} - \frac{2}{3}x^{-3/2}y^{3/2} = \text{Constant}$$
$$\Rightarrow 6x^{1/2}y^{1/2} - x^{-3/2}y^{3/2} = c$$

which is the required solution.

Example 1.50
Solve $xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0$.

Solution Multiplying the equation by an integrating factor x^hy^k, we obtain

$$Mdx + Ndy = \left(x^{h+1}y^{k+2} + 2x^{h+2}y^{k+3}\right) dx + \left(x^{h+2}y^{k+1} - x^{h+3}y^{k+2}\right) dy = 0 \qquad (1)$$
$$M_y = N_x$$

$$\Rightarrow (k+2)x^{h+1}y^{k+1} + 2(k+3)x^{h+2}y^{(k+2)} = (h+2)x^{(h+1)}y^{(k+1)} - (h+3)x^{h+2}y^{k+2}$$

Equating the coefficients of like powers

$$k + 2 = h + 2, \qquad 2k + 6 = -h - 3$$
$$\Rightarrow (h, k) = (-3, -3) \qquad (2)$$

Now Eq.(1) becomes

$$(x^{-2}y^{-1} + 2x^{-1})dx + (x^{-1}y^{-2} - y^{-1})dy = 0$$
$$-d(x^{-1}y^{-1}) + d(2\log x) - \log y = 0$$

Integrating, we get

$$-\frac{1}{xy} + \frac{\log x^2}{y} = c$$

the required general solution.

EXERCISE 1.6

Solve the following equations by finding integrating factors:

1. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$
(Madras 1975, Karnataka 1971, Calcutta Hon. 1975)
Ans: $(x/y) + \log\left(y^3/x^2\right) = c$

2. $(x^2 + y^2)dx - 2xydy = 0$
Ans: $x^2 - y^2 = cx$

3. $x^2ydx - (x^3 + y^3)dy = 0$
Ans: $y = ce^{x^3/\left(3y^3\right)}$

4. $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0$
(Rajasthan 1969, Kanpur 1974, Karnataka 1971, Punjab 1971)
Ans: $\log(x^2/y) = \dfrac{1}{xy} + c$

5. $(x^3y^3 + x^2y^2 + xy + 1)ydx$
$+ (x^3y^3 - x^2y^2 - xy + 1)xdy = 0$
(Kanpur 1980, Gorakhpur 1972)
Ans: $xy = c + \dfrac{1}{xy} + \log y^2$

6. $(xy\sin xy + \cos xy)ydx$
$+ (xy\sin xy - \cos xy)xdy = 0$
(Gorakhpur 1974, Kanpur 1977, Lucknow 1975, Rajasthan 1976, Marathawada 1974)
Ans: $(x/y)\sec xy = c$

7. $(x^2 + y^2 + 2x)dx + 2ydy = 0$
(Srivenkateswara 1984, Calicut 1983)
Ans: $e^x\left(x^3 + y^3\right) = c$

8. $(y + \log x)dx - xdy = 0$
(Marathwada 1994)
Ans: $ex + y + \log x + 1 = 0$

9. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$
(Calcutta Hon. 1952, 1954; Utkal 1980)
Ans: $x^3y^3 + \dfrac{1}{x^2}$

10. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
Ans: $3x^2y^4 + 6xy^2 + 2y^6 = c$

11. $x(4ydx + 2xdy) + y^3(3ydx + 5xdy) = 0$
Ans: $x^4y^2 + x^3y^3 = c$

12. $xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0$
Ans: $\frac{1}{4}x^4y^8 + x^3y^5 = c$

1.3.6 Linear Equations

A differential equation of the form

$$p_0(x)\frac{dy}{dx} + p_1(x)\,y = p_2(x) \quad (p_0(x) \neq 0) \qquad (1.17)$$

where p_0, p_1 and p_2 are continuous functions of x in some interval I, is called a first order linear differential equation in y. Dividing Eq. (1.17) by p_0, we can write it as

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad (1.18)$$

which is taken as the standard form of the first order linear equation. Here the dependent variable y and its derivative $\dfrac{dy}{dx}$ occur separately and to a first degree. For example,

(i) $(1+x)\dfrac{dy}{dx} + 2xy = 3x^2$

(ii) $\cos x\dfrac{dy}{dx} + (\sin x)y = \tan x.$

If $Q(x) \not\equiv 0$, then Eq. (1.17) is called a non-homogeneous linear equation. If $Q(x) \equiv 0$, then it is called a homogeneous or reduced linear equation.

Writing Eq. (1.17) in the form

$$\left(Py - Q\right)dx + dy = 0 \qquad (1.19)$$

and comparing it with

$$Mdx + Ndy = 0 \qquad (1.20)$$

we have $M = Py - Q,\ N = 1$

Since

$$\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = P(x) \qquad (1.21)$$

a function of x alone $e^{\int pdx}$ is an integrating factor of Eq. (1.18). Multiplying Eq. (1.18) by the integrating factor $e^{\int pdx}$, we have

$$\frac{dy}{dx}e^{\int pdx} + ye^{\int pdx} = \frac{d}{dx}\left(ye^{\int pdx}\right) = Qe^{\int pdx}$$

Separating the variables and integrating, we get the general solution of Eq. (1.18) as

$$ye^{\int pdx} = c + \int Qe^{\int pdx}dx \qquad (1.22)$$

where c is an arbitrary constant.

Method of solving a linear equation

(1) Write the equation in the standard form $\dfrac{dy}{dx} + Py = Q$.

(2) Find the integrating factor $e^{\int pdx}$ and multiply the equation by the integrating factor.

(3) Write the general solution of the differential equation as $y\left(\text{integrating factor}\right) = c + \int Q\left(\text{integrating factor}\right) dx$, after evaluating the integral.

Note 1
The words homogeneous and non-homogeneous used here are not to be confused with similar words used earlier.

Note 2
Sometimes a linear equation may be written in the form

$$P_o(x)\frac{dy}{dx} + P(x)\,y = Q(x) \tag{1.23}$$

$$\text{where} \quad P_o' = P \tag{1.24}$$

so that Eq. (1.23) can be written as $d(P_o y) = Qdx$ which, on integration, yields the general solution

$$P_o y = c + \int Qdx. \tag{1.25}$$

Example 1.51
Solve $\left(1 + x^2\right)\frac{dy}{dx} + 2xy = \cot x.$

Solution This can be written as

$$\frac{d}{dx}\left[\left(1 + x^2\right)y\right] = \cot x$$

Separating the variables and integrating

$$\left(1 + x^2\right)y = \log \sin x + c$$

where c is an arbitrary constant.

Note 3
An equation which is not linear in y may be linear in the variable x.

Example 1.52
Solve $y^2dx + \left(xy - 1\right)dy = 0.$

Solution Rewriting this equation as

$$\frac{dx}{dy} + \frac{1}{y}x = \frac{1}{y^2},$$

which is linear in x.

$$\text{Integrating factor} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

$$x\left(\text{integrating factor}\right) = c + \int \frac{1}{y^2}$$

Integrating factor dy

$$\Rightarrow xy = c + \int \frac{1}{y}dy = c + \log y.$$

Example 1.53
Solve $\frac{dy}{dx} + \frac{1}{x}y = e^x + \sin x.$

Solution The equation is linear in y

Here $\quad P(x) = \frac{1}{x}, \quad Q(x) = e^x + \sin x$

$$\begin{aligned}\text{Integrating factor} &= e^{\int pdx}\\ &= e^{\int \frac{1}{x}dx}\\ &= e^{\log x} = x\end{aligned}$$

Multiplying the equation by x, we have

$$x\frac{dy}{dx} + y = x\left(e^x + \sin x\right)$$

$$\Rightarrow \frac{d}{dx}\left(xy\right) = x\left(e^x + \sin x\right)$$

Integrating $\quad xy = c + x\left(e^x - \cos x\right)$

$$= c + x\left(e^x - \cos x\right) - e^x + \cos x$$

$$xy = c + (x - 1)\left(e^x - \cos x\right).$$

Example 1.54
Solve $x \log x\frac{dy}{dx} + y = 2\log x.$

Solution Writing the equation in the standard form

$$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$$

Here $\quad P(x) = \frac{1}{x\log x}, \quad Q(x) = \frac{2}{x}$

This is a linear equation in y

$$\int P dx = \int \frac{1}{x \log x} dx$$
$$= \int \frac{1}{\log x} d(\log x)$$
$$= \log(\log x)$$
$$\text{Integrating factor} = e^{\int P dx} = e^{\log(\log x)}$$
$$= \log x$$

The general solution is

$$y(\log x) = c + \int \frac{2}{x} \log x dx$$
$$= c + 2\int \log x d(\log x)$$
$$= c + (\log x)^2$$

$\Big[$Hint: put $\log x = t$,

$\int \frac{2}{x} \log x dx \frac{1}{x} dx = dt = \int 2t dt = t^2\Big]$

Example 1.55

Solve $(x + 2y^3)\dfrac{dy}{dx} = y.$

Solution This equation can be written as

$$y\frac{dx}{dy} - x = 2y^3 \quad \text{or} \quad \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2$$

Here $\quad P(y) = -\dfrac{1}{y}, \quad Q(y) = 2y^2$

The equation is linear in x

$$\int P(y) dy = \int -\frac{1}{y} dy$$
$$= -\log y = \log \frac{1}{y}$$

$$\text{Integrating factor} = e^{\int P(y) dy} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

The general solution is

$$x \cdot \frac{1}{y} = c + \int 2y^2 \cdot \frac{1}{y} dy$$
$$= c + y^2$$
$$\text{or} \quad \frac{x}{y} = c + y^2$$

Example 1.56

Solve $(1 + y^2)\dfrac{dx}{dy} + x = e^{\tan^{-1} y}$ [JNTU 2001S].

Solution Writing the equation as

$$\frac{dx}{dy} + \frac{1}{1 + y^2} x = \frac{1}{1 + y^2} e^{\tan^{-1} y}$$

Here, $P(y) = \dfrac{1}{1 + y^2}, \quad Q(y) = \dfrac{1}{1 + y^2} e^{\tan^{-1} y}$

The equation is linear in x

$$\int P(y) dy = \int \frac{1}{1 + y^2} dy = \tan^{-1} y$$
$$\text{Integrating factor} = e^{\int P(y) dy} = e^{\tan^{-1} y}$$

The general solution is

$$x e^{\tan^{-1} y} = c + \int \frac{1}{1 + y^2} e^{2\tan^{-1} y} dy$$

$$\text{Put} \quad \tan^{-1} y = t, \quad \frac{1}{1 + y^2} dy = dt$$
$$= c + \int e^{2t} dt$$
$$= c + \frac{1}{2} e^{2t}$$
$$x e^{\tan^{-1} y} = c + \frac{1}{2} e^{2(\tan^{-1} y)}.$$

Example 1.57

Solve $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$ [JNTU 1999S]

Solution Writing the equation as

$$\frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x$$

Here $P(x) = \sec^2 x, \quad Q(x) = \tan x \sec^2 x$

This is linear in y. Also,

$$\int P(x) dx = \int \sec^2 x\, dx = \tan x$$
$$\text{Integrating factor} = e^{\tan x}$$

The general solution is

$$ye^{\tan x} = c + \int \tan x \sec^2 x e^{\tan x} dx$$

Put $\tan x = t, \quad \sec^2 x dx = dt$

$$= c + \int t\, e^t dt$$

$$= c + (t-1)\, e^t$$

Integration by parts

$$y = ce^{-\tan x} + (\tan x - 1)\,.$$

Example 1.58

Solve $xy' + y + 4 = 0.$ [JNTU 2001]

Solution Writing the equation in the form

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{4}{x},$$

we note that this is a linear equation in y with

$$P(x) = \frac{1}{x}, \quad Q(x) = -\frac{4}{x}$$

But, as it is, the equation can be written as

$$d(xy) + 4dx = 0$$

which, on integration, yields the general solution

$$xy + 4x + c = 0.$$

Example 1.59

Solve $\left(x^2-1\right)\dfrac{dy}{dx} + 2xy = 1.$ [JNTU 1999]

Solution This is a linear equation in y with

$$P(x) = \frac{2x}{x-1}, \quad Q(x) = \frac{1}{x^2-1}$$

But, as it is, the equation can be written as

$$\frac{d}{dx}\left[(x^2-1)y\right] = 1$$

The general solution is

$$\left(x^2-1\right)y = x + c.$$

Example 1.60

Solve $x\dfrac{dy}{dx} + y = \log x.$ [JNTU 1996*S*]

Solution This is a linear equation in y with

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{\log x}{x}$$

But, as it is, it can be written as

$$d\left(xy\right) = \log x\, dx$$

$$\int \log x\, dx = x \log x - x$$

Integrating, the general solution is

$$xy = c + (x-1)\log x.$$

Example 1.61

Solve $x(x-1)\dfrac{dy}{dx} - y = x^2(x-1)^3\,.$

Solution The equation can be put in the standard form

$$\frac{dy}{dx} - \frac{1}{x(x-1)}\cdot y = x(x-1)^2$$

This is linear in y with

$$P(x) = -\frac{1}{x(x-1)}, \quad Q(x) = x(x-1)^2$$

$$\text{Now,} \quad \int P(x)\,dx = \int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx$$

$$= \log\left(\frac{x}{x-1}\right)$$

$$\text{Integrating factor} = e^{\int P(x)dx}$$

$$= e^{\log x/(x-1)}$$

$$= \frac{x}{x-1}$$

The general solution is

$$y\left(\frac{x}{x-1}\right) = c + \int (x-1)^2 x\frac{x}{x-1}dx$$

$$\frac{xy}{x-1} = c + \int\left(x^3 - x^2\right)dx$$

$$= c + \frac{1}{4}x^4 - \frac{1}{3}x^3.$$

Example 1.62

Solve $\dfrac{dy}{dx} - \dfrac{2}{x}y = \dfrac{5x^2}{(2+x)(3-2x)}.$

[JNTU 2004 (Set 3)]

Solution Divide by x^2

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \frac{5}{(2+x)(3-2x)}$$

$$\Rightarrow \int d\left(\frac{1}{x^2}y\right) = 5\int\left(\frac{1}{2+x} + \frac{2}{3-2x}\right)dx$$

$$\Rightarrow \frac{y}{x^2} = 5\left[\log(2+x) - \log(3-2x)\right] + c.$$

Example 1.63

Solve $(x+1)\dfrac{dy}{dx} - ny = e^x(x+1)^{n+1}$.

[JNTU 2004 (Set 2)]

Solution Divide by $(x+1)^{n+1}$

$$(x+1)^{-n}\frac{dy}{dx} - n(x+1)^{-(n+1)}y = e^x$$

$$\Rightarrow \int d\left[(x+1)^{-n}y\right] = \int e^x dx + c$$

$$\Rightarrow (x+1)^{-n}y = c + e^x.$$

EXERCISE 1.7

Solve the following:

1. $x\log x\dfrac{dy}{dx} + y = x\sin 2x$.

   [JNTU 2003, 2004]

   Ans: $y\log x = C - \dfrac{1}{2}\cos 2x$

2. $x\cos x\dfrac{dy}{dx} + y(x\sin x + \cos x) = 1$.

   Ans: $xy\sec x = \tan x + C$

3. $\dfrac{dy}{dx} + 2y = e^x(3\sin 2x + 2\cos 2x)$.

   Ans: $y = Ce^{-2x} + e^x\sin 2x$

4. $y' + y\tan x = \sin 2x, y(0) = 1$.

   [Hint :$y\sec x = \sin^2 x + c$

   or $y = \cos x - \cos^3 x + c\cos x$

   $y(0) = 1 \Rightarrow 1 = c$

   Ans: $y = 2\cos x - \cos^3 x$

5. $dx = (x + y + 1)\,dy$.

   Ans: $x + y + 2 = ce^y$

6. $y' + y\cot x = 2x\ \mathrm{cosec}\ x$.

   Ans: $y = (x^2 + c)\ \mathrm{cosec}\ x$

7. $x\dfrac{dy}{dx} + 2y = x^2\log x$.

   Ans: $16x^2y + x^4(1 - 4\log x) = c$

8. $\dfrac{dy}{dx} + y\tan x = \cos^3 x$.

   Ans: $4y\sec x = 2x + \sin 2x + c$

9. $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$.

   Ans: $y = \sqrt{1 - x^2} + (1 - x^2)$

10. $x(1 - x^2)\dfrac{dy}{dx} + (2x^2 - 1)y = x^3$.

    Ans: $y = x + cx\sqrt{1 - x^2}$

11. $\dfrac{dx}{dy} = \sec y - x\tan y$.

    Ans: $x\sec y = \tan y + c$

12. $ye^y dx = (y^3 + 2xe^y dy)$.

    Ans: $\dfrac{x}{y^2} + e^{-y} = c$

13. $\dfrac{dy}{dx} + y\cot x = 4x\ \mathrm{cosec}\ x\ y(\pi/2) = 0$.

    Ans: $y\sin x = 2x^2 - \dfrac{\pi^2}{2}$

14. $\dfrac{dy}{dx} = y/(2y\log y + y - x)$.

    Ans: $x = \dfrac{c}{y} + y\log y$

15. $\sqrt{1 - y^2}dx = (\sin^{-1} y - x)\,dy$.

    Ans: $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$

16. $\sin x\cos x\dfrac{dy}{dx} + y = xe^x\cos^2 x$

    $\left[\text{Hint: } \tan x\dfrac{dy}{dx} + y\sec^2 x = xe^x,\right.$

    on multiplication by $\sec^2 x$

    $\Rightarrow \displaystyle\int d(y\tan x) = \int xe^x dx$

    Ans: $y\tan x = c + (x - 1)e^x$

1.3.7 Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + Py = Qy^a \quad (a \in R, a \neq 1) \qquad (1.26)$$

is called a Bernoulli's equation.

A Bernoulli's equation is reducible to the linear form by the substitution $z = y^{1-a}$.

Multiply Eq. (1.26) by $(1-a)y^{-a}$, it becomes

$$(1-a)\,y^{-a}\frac{dy}{dx}+P\,(1-a)\,y^{1-a} = Q\,(1-a) \quad (1.27)$$

$$\text{or} \quad \frac{dz}{dx} + P_1 z = Q_1 \quad (1.28)$$

$$P_1 = (1-a)P, \quad Q_1 = (1-a)Q$$
$$z = y^{1-a}, \quad dz = (1-a)y^{-a}$$

which is linear in z. Its general solution is

$$ze^{\int (1-a)Pdx} = y^{1-a}e^{\int (1-a)Pdx}$$
$$= c + \int (1-a)\,Qe^{\int (1-a)Pdx} \quad (1.29)$$

Note

There are also equations that are not of the Bernoulli type shown in Eq. (1.26). These equations are reducible to linear form by appropriate substitution. (See Example 1.67)

Example 1.64

Solve $\frac{dy}{dx} - y\tan x = -y^2 \sec x$.

Solution This is Bernoulli's type equation.

$$P = -\tan x, \quad Q = -\sec x, \quad a = 2.$$

Multiplying the equation by $-\frac{1}{y^2}$, we have

$$-\frac{1}{y^2}\frac{dy}{dx} + (\tan x)\frac{1}{y} = \sec x$$

$$\text{Put } \frac{1}{y} = z \;\Rightarrow\; -\frac{1}{y^2}dy = dz$$

$$\frac{dz}{dx} + (\tan x)z = \sec x$$

(This is a linear equation.)

Integrating factor

$$e^{\int Pdx} = e^{\int \tan x} = e^{\log\sec x} = \sec x$$

The general solution is

$$z\sec x = \frac{1}{y}\sec x = c + \int \sec^2 x dx = c + \tan x.$$

Example 1.65

Solve $y(2xy + e^x)dx = e^x dy$.

Solution

$$\frac{dy}{dx} - y = 2xe^{-x}\cdot y^2$$

This is Bernoulli's equation.

$$P = -1, \quad Q = 2xe^{-x}, \quad a = 2$$

Multiplying the equation by $-\frac{1}{y^2}$, we have

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = -2xe^{-x}$$
$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = -2xe^{-x}$$

This is a linear equation in $\frac{1}{y} = z$

Integrating factor $= e^{\int 1\cdot dx} = e^x$

The general solution is

$$ze^x = \frac{1}{y}e^x + c$$

$$\Rightarrow \int 2xe^{-x}e^x dx = c - x^2$$

$$\Rightarrow \frac{e^x}{y} = c - x^2.$$

Example 1.66

Solve $xy\left(1 + xy^2\right)\frac{dy}{dx} = 1.$

Solution

$$\frac{dx}{dy} - y\cdot x = y^3x^2$$

This is linear in $\frac{1}{x}$.

Multiplying by $-\frac{1}{x^2}$, we have

$$-\frac{1}{x^2}\frac{dx}{dy} + y\cdot\frac{1}{x} = -y^3$$

$$\text{Put } \frac{1}{x} = z \;\Rightarrow\; \frac{dz}{dy} + yz = -y^3$$

$$e^{\int ydy} = e^{+\int \frac{y^2}{2}}$$

The general solution is

$$ze^{\frac{y^2}{2}} = \frac{1}{x}e^{\frac{y^2}{2}}$$
$$= c - \int y^3 e^{\frac{y^2}{2}}\, dy$$

Put $\dfrac{y^2}{2} = u; \quad ydy = du$

$$= c - \int 2ue^u du$$
$$= c - 2(u-1)e^u$$

$$\frac{1}{x}e^{\frac{y^2}{2}} = c - \left(y^2 - 2\right)e^{\frac{y^2}{2}}$$
$$\frac{1}{x} = ce^{-\frac{y^2}{2}} + 2 - y^2.$$

Example 1.67

Solve $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y.$

Solution This is not strictly in the form of Bernoulli equation (1.26). But we can write it as

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = z \Rightarrow \sec^2 ydy = dz$

$$\frac{dz}{dx} + 2xz = x^3,$$

which is a Bernoulli type Eq. (1.26).

$$\text{Integrating factor } = e^{\int 2xdx} = e^{x^2}$$
$$ze^{x^2} = \tan y e^{x^2}$$
$$= c + \int x^3 e^{x^2} dx$$
$$= c + \frac{1}{2}\int te^t dt$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$= c + \frac{1}{2}(t-1)e^t$$

The general solution is

$$\tan y\, e^{x^2} = c + \frac{1}{2}\left(x^2 - 1\right)e^{x^2}$$

or $$\tan y = ce^{-\frac{x^2}{2}} + \frac{1}{2}\left(x^2 - 1\right).$$

EXERCISE 1.8

Solve the following:

1. $x\dfrac{dy}{dx} + y = x^3y^6.$ [JNTU 1995, 2002, 2004]
   Ans: $(5 + cx^2)x^3y^5 = 2$
2. $xy\left(1 + xy^2\right)\dfrac{dy}{dx} = 1.$
   Ans: $\frac{1}{x} = 2 - y^2 + ce^{-\frac{y^2}{2}}$
3. $\dfrac{dy}{dx} + \dfrac{y}{x}\log y = \dfrac{y}{x}\left(\log y\right)^2.$
   Ans: $(\log y)^{-1} = 1 + cx$
4. $2xy\dfrac{dy}{dx} = \left(x^2 + y^2 + 1\right).$
   Ans: $y^2 - x^2 = cx - 1$
5. $x\left(x - y\right)dy + y^2dx = 0.$
   Ans: $\frac{y}{x} = \log y + c$
6. $\tan y\dfrac{dy}{dx} + \tan x = \cos y\cos^2 x.$
   Ans: $\cos y = \cos x\,(\sin x + c)$
7. $y = \cos x\dfrac{dy}{dx} + y^2(1 - \sin x)\cos x,$
   $y(0) = 2.$
   Ans: $2(\tan x + \sec x) = y(2\sin x + 1)$
8. $y\left(2xy + e^x\right)dx = e^x dy.$
   Ans: $e^x = y\left(c - x^2\right)$
9. $\tan y\dfrac{dy}{dx} + \tan x = \cos y\cos^2 x.$
   Ans: $\sec y = (c + \sin x)\cos x$
10. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x\sec y.$
    Ans: $\sin y = (1+x)(e^x + c)$

1.4 APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

Newton's law of cooling

Physical experiments show that the rate of change of temperature T with respect to time t, dT/dt, of a body is proportional to the difference between the temperature of the body (T) and that of the surrounding medium (T_0).

This principle is known as Newton's Law of Cooling and is expressed through the following first order and first degree differential equation:

$$\frac{dT}{dt} = -k(T - T_0) \qquad (k > 0) \tag{1.30}$$

Separating the variables

$$\frac{dT}{(T - T_0)} = -kdt \tag{1.31}$$

Integrating, we get

$$\begin{aligned} \log(T - T_0) &= -kt + \log c \\ \Rightarrow (T - T_0) &= ce^{-kt} \end{aligned} \tag{1.32}$$

If T_i is the initial temperature of the body when $t = 0$, we get from Eq. (1.32)

$$T_i - T_0 = c \tag{1.33}$$

Eliminating c between Eqs. (1.32) and (1.33), we get

$$\begin{aligned} (T - T_0) &= (T_i - T_0)\, e^{-kt} \\ \text{or} \quad T(t) &= T_0 + (T_i - T_0)\, e^{-kt}. \end{aligned} \tag{1.34}$$

Method of solving the problem of Newton's law of cooling

(1) Identify T_0, the temperature of the surrounding medium. Then the general solution is given by Eq. (1.34).

(2) Use two given conditions and find the constant of integration c and the proportionality constant k.

(3) Substitute c and k obtained from step 2 in Eq. (1.34). We can determine (i) the value of T for a given time t or (ii) the value of t for a given temperature T from Eq. (1.34).

Law of natural growth or decay

If the rate of change of a quantity y at any time t is proportional to y, then

$$\frac{dy}{dt} \propto y \tag{1.35}$$

If k is the constant of proportionality, then the required differential equation is

$$\frac{dy}{dt} = ky \tag{1.36}$$

where k is a real constant.

For growth, $k > 0$ and the differential equation is

$$\frac{dy}{dt} = ky \quad (k > 0) \tag{1.37}$$

For decay, the differential equation is

$$\frac{dy}{dt} = -ky \quad (k > 0). \tag{1.38}$$

Example 1.68

The temperature of a body initially at 80°C reduces to 60°C in 12 min. If the temperature of the surrounding air is 30°C, find the temperature of the body after 24 min.

Solution Let T be the temperature of the body at time t.

By Newton's law of cooling

$$\frac{dT}{dt} = -k\,(T - T_0) \Rightarrow \frac{dT}{T - T_0} = -kdt$$

$$\Rightarrow \quad \log(T - T_0) = -kt + \log c$$

$$\Rightarrow \quad T - T_0 = ce^{-kt} \tag{1}$$

Temperature of the surrounding medium $T_0 = 30$

$$T = T_0 + ce^{-kt} = 30 + ce^{-kt} \tag{2}$$

Initial temperature $T = T_i = 80$ when $t = 0$

$$\begin{aligned} T_i - T_0 &= 80 - 30 = ce^0 \\ &\Rightarrow c = 50 \qquad (3) \\ \therefore \quad T &= T_0 + ce^{-kt} = 30 + 50e^{-kt} \\ \text{When } t = 12, T &= 60 \Rightarrow 60 = 30 + 50e^{-k \cdot 12} \\ \Rightarrow k &= \frac{1}{12} \log\left(\frac{5}{3}\right) \\ \text{When } t = 24, T &= 30 + 50e^{-\frac{1}{12}\left(\log \frac{5}{3}\right) \cdot 24} \\ &= 30 + 50\left(\frac{3}{5}\right)^2 \\ &= 30 + 50 \times \frac{9}{25} = 48. \end{aligned}$$

Example 1.69
A body is heated to 105°C and placed in air at 15°C. After 1 hr its temperature is 60°C. How much additional time is required for it to cool to $37\frac{1}{2}$°C?

Solution Let T be the temperature of the body at time t.

By Newton's law of cooling

$$\frac{dT}{dt} = -k(T - T_0)$$

$$\Rightarrow \frac{dT}{T - T_0} = -kdt$$

$$\Rightarrow \log(T - T_0) = -kt + \log c$$

$$\Rightarrow T - T_0 = ce^{-kt} \tag{1}$$

Temperature of the surrounding medium $T_0 = 15$

$$T = T_0 + ce^{-kt} = 15 + ce^{-kt} \tag{2}$$

Initial temperature ($t = 0$)

$$T = T_i = 105 \quad \Rightarrow \quad 105 = 15 + ce^0$$

$$\Rightarrow c = 90 \tag{3}$$

When $t = 1, T = 60 \Rightarrow 60 = 15 + 90e^{-k\cdot 1}$

$$\Rightarrow e^{-k} = \frac{1}{2}$$

When $T = 37\frac{1}{2}, 37\frac{1}{2} = 15 + 90e^{-kt}$

$$= 15 + 90\left(\frac{1}{2}\right)^t$$

$$\Rightarrow \frac{22\frac{1}{2}}{90} = \left(\frac{1}{2}\right)^t \Rightarrow t = 2$$

Additional time required = 2 hr −1 hr = 1 hr.

Example 1.70
The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in 1 hr What was the value of N after $\frac{3}{2}$ hrs. [JNTU 1996S, 2003]

Solution Here N is a natural number and is therefore discrete. But in view of its largeness N will be treated as a continuous variable which is a differentiable function of time t. The differential equation to be solved is

$$\frac{dN}{dt} = kN \Rightarrow \frac{dN}{N} = kdt$$

$$\Rightarrow \log N = kt + \log c \Rightarrow N = ce^{kt} \tag{1}$$

When $t = 0, \quad N = 100$

$$\Rightarrow 100 = ce^0 \Rightarrow N = 100\, e^{kt}$$

$$t = 1\text{hr} = 3600\,\text{sec}, \quad N = 332 = 100e^{kt} \tag{2}$$

$$\Rightarrow e^{k.1} = \frac{332}{100} \tag{3}$$

When $t = \frac{3}{2}$hr $\quad N = 100\left(e^k\right)^{\frac{3}{2}} = 100\left(\frac{332}{100}\right)^{\frac{3}{2}}$

$$\Rightarrow \frac{N^2}{10000} = \left(\frac{332}{100}\right)^3$$

$$\Rightarrow (10N)^2 = 332^3 \Rightarrow N = 604.9.$$

Example 1.71
A radioactive substance disintegrates at a rate proportional to its mass. When the mass is 10 mg the rate of disintegration is 0.051 mg per day. How long will it take for the mass of 10 mg to reduce to its half. [JNTU 1995]

Solution The governing differential equation is

$$\frac{dm}{dt} = -km \quad (k > 0) \tag{1}$$

where m is the mass of the substance.

Separating variables and integrating

$$\log m = kt + \log c \Rightarrow m = ce^{-kt} \tag{2}$$

When mass $m = 10$ mg

$$\frac{dm}{dt} = -0.051$$

negative sign is to be taken since $\frac{dm}{dt}$ is decreasing rate.

$$\therefore \quad -0.051 = -k \times 10$$

$$\Rightarrow k = \frac{0.051}{10} = 0.0051 \tag{3}$$

Eq. (2) becomes

$$m = ce^{-(0.0051)t} \tag{4}$$

When $t = 0 \quad m = 10$ mg

$$\therefore \quad 10 = ce^{-(0.0051)10} \Rightarrow c = 10 \tag{5}$$

$$\therefore \quad m = 10e^{-(0.0051)t}$$

We have to find t when $m = 5$ mg

$$5 = 10e^{-(0.0051)t}$$

$$\Rightarrow e^{(0.0051)t} = 2 \quad \Rightarrow \quad 0.0051\, t = \ln 2$$

$$\Rightarrow \qquad t = \frac{0.6931}{0.0051} = 135.9 \text{ days.}$$

EXERCISE 1.9

1. A body initially at 80°C cools down to 60ºC in 20 min. The temperature of the air is 40°C. Find the temperature of the body after 40 min.

Ans: 50°C

2. The air temperature is 20°C. A body cools from 140°C to 80°C in 20 min. How much time will it take to reach a temperature of 35°C?

Ans: 60 min

3. If the temperature of the air is 20°C and the temperature of a body drops from 100°C to 75°C in 10 min. what will be its temperature after 30 min? When will the temperature be 30°C?

Ans: 46°C, 55.5 min

4. Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 gms of uranium are present at time t_1 and t_2, respectively. Show that the half-life of uranium is $[(t_2 - t_1)\log 2] / \log (m_1/m_2)$.

5. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hrs, in how many hours will it triple? [JNTU 1987, 2000]

Ans: $\dfrac{2\log 3}{\log 2}$ hrs = 3.17 hrs

6. The rate of decay of radium varies as its mass at a given time. Given that half-life of radium is 1600 yrs, find out the percentage of the mass of radium it will disintegrate in 200 yrs.

Ans: 8.3% approximately

1.4.1 Geometrical Applications

Orthogonal trajectories of a family of curves

Definition A curve which cuts every member of a given family of curves at a right angle is called an orthogonal trajectory of the given family of curves.

A family of curves which are orthogonal to themselves are called self-orthogonal.

(1) In the electrical field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves.

(2) In fluid mechanics, the stream lines and the equipotential lines are orthogonal trajectories of one another.

(3) In thermodynamics, the lines of heat flow are perpendicular to isothermal curves.

Method for finding orthogonal trajectories

(1) Cartesian coordinates

$$\text{Let} \quad f(x, y, c) = 0 \tag{1.39}$$

represent the equation of a given family of curves with single parameter 'c'.

Differentiating Eq. (1.39) with respect to 'x' and eliminating 'c' from the equation thus obtained and Eq. (1.39) we obtain a differential equation of the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0 \tag{1.40}$$

for the given family of curves.

Suppose there passes a curve of the given family and a member of the orthogonal trajectories through a point $P(x, y)$. Let m_1 be the slope of the curve of the given family and m_2 the slope of a orthogonal trajectory. Since the curves cut at right angles we have $m_1 m_2 = -1$ and $m_1 = \dfrac{dy}{dx}$ so that

$$m_2 = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{dx}{dy}$$

Therefore, if we replace $\dfrac{dy}{dx}$ in Eq. (1.40) by $-\dfrac{dx}{dy}$, then we get the differential equation for the orthogonal trajectories as

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0 \tag{1.41}$$

Integrating Eq. (1.41), we obtain the equation for the orthogonal trajectories.

Example 1.72
Find the orthogonal trajectories of the rectangular hyperbolas $x^2 - y^2 = c$ where c is a parameter.

Solution Equation of the given curves

$$x^2 - y^2 = c \tag{1}$$

Differentiating Eq. (1) with respect to x, we get the differential equation of the given curves as

$$x = y\frac{dy}{dx} \tag{2}$$

Replacing $\frac{dy}{dx}$ in Eq. (2) by $-\frac{dx}{dy}$ the differential equation for the orthogonal trajectories is obtained as

$$x - y\left(-\frac{dx}{dy}\right) = 0 \Rightarrow x + y\frac{dx}{dy} = 0 \tag{3}$$

Separating the variables, we get

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, the equation for orthogonal trajectories is

$$\log x + \log y = \log k \quad \text{or} \quad xy = k. \tag{4}$$

Example 1.73
Find the orthogonal trajectories of the family of parabolas through the origin and focii on the y-axis.

Solution Equation of the family of parabolas

$$x^2 = 4ay \tag{1}$$

Differentiating Eq. (1) with respect to x,

$$2x = 4a\frac{dy}{dx} \tag{2}$$

Eliminating 'a' between Eqs. (1) and (2), the differential equation for the given curves is

$$x^2\frac{dy}{dx} = 2xy \quad \text{or} \quad x\frac{dy}{dx} = 2y \tag{3}$$

Replacing $\frac{dy}{dx}$ in Eq. (3) by $-\frac{dx}{dy}$, the differential equation for orthogonal trajectories is

$$x\left(-\frac{dx}{dy}\right) = 2y \quad \text{or} \quad 2y\frac{dy}{dx} + x = 0 \tag{4}$$

Separating the variables and integrating, we get the equation for orthogonal trajectories as

$$2y^2 + x^2 = c$$

where 'c' is a constant.

Example 1.74
Find the orthogonal trajectories of the family of semi-cubical parabolas $ay^2 = x^3$ where 'a' is a parameter.

Solution The equation of the given curves is

$$ay^2 = x^3 \tag{1}$$

Differentiating Eq. (1) with respect to 'x', we get

$$2ay\frac{dy}{dx} = 3x^2 \tag{2}$$

Eliminating 'a' between Eqs. (1) and (2), the differential equation is obtained as

$$2x\frac{dy}{dx} = 3y \tag{3}$$

Replacing $\frac{dy}{dx}$ in Eq. (3) by $-\frac{dx}{dy}$, the differential equation for the orthogonal trajectories is obtained as

$$3y = 2x\left(-\frac{dx}{dy}\right) \quad \text{or} \quad 3y\frac{dy}{dx} + 2x = 0 \tag{4}$$

Separating the variables and integrating, we obtain the equation for orthogonal trajectories as

$$3y^2 - 2x^2 = c$$

where 'c' is an arbitrary constant.

Example 1.75
Show that the system of confocal ellipses $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ (λ, parameter) is self-orthogonal.

Solution The equation of the system of ellipses is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \tag{1}$$

Differentiating Eq. (1) with respect to 'x'

$$\frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0 \qquad (2)$$

$$\Rightarrow \lambda\left(x+yy'\right)+\left(a^2yy'+b^2x\right)=0$$

$$\Rightarrow \lambda=\frac{-\left(a^2yy'+b^2x\right)}{\left(x+yy'\right)} \quad \left(y'=\frac{dy}{dx}\right)$$

$$\left.\begin{aligned} a^2+\lambda &= a^2-\frac{a^2yy'+b^2x}{x+yy'}=\frac{\left(a^2-b^2\right)x}{x+yy'} \\ b^2+\lambda &= b^2-\frac{a^2yy'+b^2x}{x+yy'}=\frac{\left(a^2-b^2\right)yy'}{x+yy'} \end{aligned}\right\} \qquad (3)$$

Eliminating λ from Eq. (1) using Eq.(3)

$$\frac{\left(x+yy'\right)x^2}{\left(a^2-b^2\right)x}-\frac{\left(x+yy'\right)y^2}{\left(a^2-b^2\right)yy'}=1$$

$$\Rightarrow \left(x+yy'\right)\left(x-\frac{y}{y'}\right)=a^2-b^2$$

It is clear that if we replace y' by $-\frac{1}{y'}$, the same equation is obtained.

Example 1.76

Show that the system of confocal and coaxial parabolas $y^2=4a(x+a)$ is self-orthogonal.

Solution The equation of the given parabolas

$$y^2=4a(x+a) \qquad (1)$$

Differentiating Eq. (1) with respect to 'x', we have

$$2yy_1=4a \quad \left(y_1=\frac{dy}{dx}\right) \qquad (2)$$

Eliminating 'a' from Eqs. (1) and (2), we get

$$y^2=2yy_1\left(x+\frac{yy_1}{2}\right) \quad \text{or} \quad y=2xy_1+yy_1^2 \qquad (3)$$

Replacing y_1 by $-\frac{1}{y_1}$ in the above differential equation for the given curves, we obtain the same equation for the orthogonal trajectories.

(2) Polar coordinates

$$\text{Let } f(r,\theta,a)=0 \qquad (1.42)$$

be the equation of the given system of curves in polar coordinates where 'a' is a parameter.

Differentiating Eq. (1.42) with respect to 'θ' we get another equation. Between these two equations, we eliminate the parameter 'a' and obtain the differential equation for the given system of curves as

$$F\left(r,\theta,\frac{dr}{d\theta}\right)=0 \qquad (1.43)$$

If we replace $\frac{dr}{d\theta}$ in Eq. (1.43) by $-r^2\frac{d\theta}{dr}$ (which amounts to interchanging the role of the polar subnormal and the polar sub-tangent), we get the differential equation for the orthogonal trajectories as

$$F\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0 \qquad (1.44)$$

Solving this differential equation, we get the equation for the orthogonal trajectories.
The following solved examples illustrate the procedure.

Example 1.77

Find the orthogonal trajectories of the family of cardioids $r=a(1-\cos\theta)$ where 'a' is a parameter.

Solution The equation of the given cardioids is

$$r=a\left(1-\cos\theta\right) \qquad (1)$$

Differentiating Eq. (1) with respect to 'θ', we have

$$\frac{dr}{d\theta}=a\sin\theta \qquad (2)$$

Eliminating 'a' between Eqs. (1) and (2), we get

$$\frac{r}{\left(\frac{dr}{d\theta}\right)}=\frac{1-\cos\theta}{\sin\theta}$$

$$=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

The differential equation for the given curves

$$\frac{dr}{d\theta}=r\cot\frac{\theta}{2}$$

Replacing $\frac{dr}{d\theta}$ by $\frac{-r^2 d\theta}{dr}$ we obtain from Eq. (3) the differential equation for orthogonal trajectories as

$$\frac{-r^2 d\theta}{dr} = r\cot\frac{\theta}{2} \qquad (3)$$

$$\Rightarrow \frac{dr}{r} = -\tan\frac{\theta}{2}d\theta$$

Integrating we get (4)

$$\log r = 2\log\cos\frac{\theta}{2} + \log 2c$$

$$\Rightarrow r = 2c\cos^2\frac{\theta}{2} = c\,(1+\cos\theta) \qquad (5)$$

which is the equation for the orthogonal trajectories.

Example 1.78

Find the equation of the system of orthogonal trajectories of the family of curves $r^n \sin n\theta = a^n$ where a^n is a parameter.

Solution The given curves are

$$r^n \sin n\theta = a^n \qquad (1)$$

By logarithmic differentiation of Eq. (1) with respect to 'θ'

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{n\cos n\theta}{\sin n\theta} = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} + \cot n\theta = 0 \qquad (2)$$

which is the differential equation for the given curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2), we get the differential equation for the orthogonal system as

$$-\frac{1}{r}\cdot r^2\frac{d\theta}{dr} + \frac{\cos n\theta}{\sin n\theta} = 0$$

$$\Rightarrow \quad \frac{n\sin n\theta}{\cos n\theta} = n\frac{dr}{r} \qquad (3)$$

By integrating, we obtain the equation for orthogonal trajectories as

$$\log r^n = -\log\cos n\theta + n\log c \qquad (4)$$

$$\Rightarrow \quad r^n\cos n\theta = c^n. \qquad (5)$$

EXERCISE 1.10

Find the orthogonal trajectories of the family of curves given in Table 1.2

Table 1.2

S. No.	Curves	Parameter	Orthogonal trajectories
	Cartesian form		
1.	$y^2 = 4ax$ (parabolas)	a	$2x^2 + y^2 = c$ (ellipses)
2.	$x^2 + y^2 + 2gx + c = 0$ (coaxial circles)	g	$x^2 + y^2 + 2fy - c' = 0$ (circles)
3.	$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (astroids)	a	$x^{\frac{4}{3}} - y^{\frac{4}{3}} = c^{\frac{4}{3}}$
4.	$x^2 - y^2 = cx$	c	$y(y^2 + 3x^2) = a$
	Polar form		
5.	$r\theta = a$	a	$r^2 = c\exp\left(\frac{\theta^2}{2}\right)$
6.	$r = \frac{2a}{(1+\cos\theta)}$	a	$r(1-\cos\theta) = 2c$
7.	$r^n = a^n\cos n\theta$	a	$r^n = c^n\sin n\theta$
8.	$r^n\cos n\theta = a^n$	a	$r^n\sin n\theta = c^n$