HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1} y}{dx^{n-1}}+a_2\frac{d^{n-2} y}{dx^{n-2}}+\cdots+a_n y=0 \quad .....(1)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is known as a homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since the degree of the dependent variable y and all its differential coefficients is one.

Equation (1) can also be written as

$$(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots+a_n)y=0$$

$$f(D)y=0$$

where $f(D)=a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots+a_n$.

Here, $D\equiv\frac{d}{dx}$ is known as the differential operator.

The operator D obeys the laws of algebra.

General Solution of a Homogeneous Linear Differential Equation

The homogeneous equation

$$f(D)y=0 \quad (1)$$

can be solved by replacing D by m in $f(D)$ and solving the auxiliary equation (AE)

$$f(m)=0 \quad (2)$$

The general solution of Eq. (1) depends upon the nature of the roots of the auxiliary Eq. (2).

If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the auxiliary equation, the following cases arise:

Case I Real and distinct roots: If roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then the solution of Eq. (1) is given as

$$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

Case II: Real and repeated roots: If two roots m_1, m_2 are real and equal, and the remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are all real and distinct then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Note: If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and remaining $(n - r)$ roots m_{r+1}, $m_{r+2}, \ldots, m_n$ are all real and distinct then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_r x^{r-1})e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$$

Case III: Complex roots: If two roots m_1, m_2 are complex say, $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ (Conjugate pair) and remaining $(n - 2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct then the solution of Eq. (1) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pairs of complex roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$ and remaining $(n - 4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct then the solution of Eq. (1) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

Remark

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Problem1: Solve $(D^2 + 2D - 1)y = 0$.

Solution: The auxiliary equation is

$$m^2 + 2m - 1 = 0$$

$m = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$ (real and distinct)

Hence, the general solution is

$$y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$$

Problem2: Solve $2D^2 y + Dy - 6y = 0$.

Solution: The equation can be written as

$$(2D^2 + D - 6)y = 0$$

The auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

Hence, the general solution is

$m = -2, \frac{3}{2}$ (real and distinct)

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Problem3: Solve $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0.$

Solution: $(D^2 + 6D + 9)x = 0$

The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$m = -3, -3$ (real and repeated)

Hence, the general solution is

$$x = (c_1 + c_2 x)e^{-3x}$$

Problem4: Solve $y'' + 4y' + 4y = 0, y(O) = 1, y^{'}(O) = 1.$

Solution: $(D^2 + 4D + 4)y = 0$

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

$$(m + 2)^2 = 0$$

$m = -2, -2$ (real and repeated)

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-2x} \text{ .... (1)}$$

Differentiating Eq. (1),

$$y^{'} = -2(c_1 + c_2 x)e^{-2x} + e^{-2x}c_2 \text{ ... (2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$

$$c_1 = 1 \text{ .... (3)}$$

$$y'(0) = -2c_1 + c_2$$

$$1 = -2c_1 + c_2$$

$$1 = -2 + c_2$$

$$c_2 = 3 \; .... \; (4)$$

Hence, the particular solution is

$$y = (1 + 3x)e^{-2x}$$

Problem5: Solve $y''' - 6y'' + 11y' - 6y = 0$.

Solution: $(D^3 - 6D^2 + 11D - 6)y = 0$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$(m-1)(m^2 - 5m + 6) = 0$$

$$(m-1)(m-2)(m-3) = 0$$

$m = 1,2,3$ (real and distinct)

Hence, the general solution is

$$y = c_1e^X + c_2e^{2x} + c_3e^{3x}$$

Problem6: Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution: The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m-1)(m^2 - 4m + 4) = 0$$

$$(m-1)(m-2)^2 = 0$$

$m = 1$ (real and distinct), $m = 2,2$ (real and repeated)

Hence, the general solution is

$$y = c_1e^x + (c_2 + c_3x)e^{2x}$$

Problem7: Solve $(D^3 + 1)y = 0$.

Solution: The auxiliary equation is

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m+1=0, m^2-m+1=0$$

$m=-1$ (real and distinct), $m=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ (complex)

Hence, the general solution is

$$y=c_1e^{-x}+e^{\frac{1}{2}x}(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x)$$

Problem8: Solve $(\mathrm{D}^4-2\mathrm{D}^3+\mathrm{D}^2)\mathrm{y}=0.$

Solution: The auxiliary equation is

$$m^4-2m^3+m^2=0$$

$$m^2(m^2-2m+1)=0$$

$$m^2(m-1)^2=0$$

$m=0,0,1,1$ (real and repeated)

Hence, the general solution is

$$y=(c_1+c_2x)e^{0x}+(c_3+c_4x)e^x$$

$$=c_1+c_2x+(c_3+c_4x)e^x$$

Problem9: Solve $(\mathrm{D}^4-1)\mathrm{y}=0.$

Solution: The auxiliary equation is

$$m^4-1=0\Rightarrow(m^2-1)(m^2+1)=0\Rightarrow m^2=1, m^2=-1$$

$m=\pm1$ ((real and distinct), $m=\pm i$ (complex)

Hence, the general solution is

$$y=c_1e^x+c_2e^{-x}+e^{0x}(c_3\cos x+c_4\sin x)$$

$$=c_1e^X+c_2e^{-x}+c_3\cos x+c_4\sin x$$

Problem10: Solve $(\mathrm{D}^4+4\mathrm{D}^2)\mathrm{y}=0.$

Solution:

The auxiliary equation is

$$m^4+4m^2=0$$

$$m^2(m^2+4)=0$$

$m=0,0$ (real and distinct), $m=\pm2i$ (complex)

Hence, the general solution is

$$y = (c_1 + c_2x)e^{0x} + c_3 \cos 2x + c_4 \sin 2x$$
$$= c_1 + c_2x + c_3 \cos 2x + c_4 \sin 2x$$

Problem11: Solve $(\mathrm{D}^4 + 4)\mathrm{y} = 0$.

Solution: The auxiliary equation is

$$m^4 + 4 = 0$$
$$m^4 + 4 + 4m^2 - 4m^2 = 0$$
$$(m^2 + 2)^2 - (2m)^2 = 0$$
$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$
$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

$m = -1 \pm i$ and $m = 1 \pm i$ (complex)

Hence, the general solution is

$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q(x) \cdots (1)$$

where a_0, a_1, a_2, a_n are constants and Q is a function of x, is known as a *nonhomogeneous linear differential equation with constant coefficients*.

Equation (1) can also be written as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = Q(x) \quad (3.24)$$
$$f(D)y = Q(x)$$

where $f(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n$

General Solution of a Nonhomogeneous Linear Differential Equation

A general solution of Eq. (1) is obtained in two parts as

General solution=Complementary function+Particular integral

$$y = CF + PI$$

The complementary function (CF) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (1).

The particular integral (PI) is any particular solution of the nonhomogeneous Eq. (1) and contains no arbitrary constants.

Direct (Short - cut) Method of Obtaining Particular Integrals

This method depends on the nature of $Q(x)$ in Eq. (1). The particular integral by this method can be obtained when $Q(x)$ has the following forms:

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax + b)$ or $\cos(ax + b)$

(iii) $Q(x) = x^m$ or polynomial in x

(iv) $Q(x) = e^{ax}v(x)$

(v) $Q(x) = xv(x)$

Case I: $Q(x) = e^{ax+b}$

Now $f(D)y = e^{ax+b}$ put D = a

PI $= \frac{1}{f(a)} e^{ax+b}$ if $f(a) \neq 0$

Note: If $f(a) = 0$ then $\frac{1}{f(D)} e^{ax+b} = x \cdot \frac{1}{f'(a)} e^{ax+b}$ where $f'(a) \neq 0$

If $f'(a) = 0$ then repeating the above process,

General form of P.I $= x^r \frac{1}{f^{(r)}(a)} e^{ax+b}$

Problem1: Solve $(4D^2 - 4D + 1)y = 4$.

Solution: The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

Hence, the general solution is

$m = \frac{1}{2}, \frac{1}{2}$ (real and repeated)

$\text{CF} = (c_1 + c_2 x)e^{\frac{x}{2}}$

$\text{PI} = \frac{1}{4D^2-4D+1} 4e^{0x} = 4 \cdot \frac{1}{4(0)-4(0)+1} e^{0x} = 4$

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + 4$$

Problem2: Solve $(D^2 + 5\mathrm{D} + 6)\mathrm{y} = \mathrm{e}^{\mathrm{x}}$

Solution: The auxiliary equation is

$$m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0$$

$m = -2, -3$ (real and distinct)

$\text{CF} = c_1 e^{-2x} + c_2 e^{-3x}$

$\text{PI} = \frac{1}{D^2+5D+6} e^x = \frac{1}{1^2+5(1)+6} e^x = \frac{1}{12} e^x$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^X$$

Problem3: Solve $(\mathrm{D}^2 + 1)\mathrm{y} = \mathrm{e}^{-\mathrm{x}}$

Solution: The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m^2 = -1$$

$m = \pm i$ (complex)

$\text{CF} = c_1 \cos x + c_2 \sin x$

$$PI = \frac{1}{D^2+1} e^{-x} = \frac{1}{(-1)^2+1} e^{-x} = \frac{1}{2} e^{-x}$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x}$$

Problem4: Solve $(\mathrm{D}^2 + 2\mathrm{D} + 1)\mathrm{y} = \mathrm{e}^{-\mathrm{x}}$

Solution: The auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$$

$m = -1, -1$ (real and repeated)

CF $= (c_1 + c_2 x)e^{-x}$

PI $= \frac{1}{D^2+2D+1} e^{-x}$ put D = a = -1, method fails, then

PI $= x \frac{1}{2D+2} e^{-x}$ put D = a = -1, again method fails, then

PI $= x^2 \frac{1}{2} e^{-x} = \frac{1}{2} x^2 e^{-x}$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

Problem5: Solve $(\mathrm{D}^2 - 2\mathrm{D} + 1)\mathrm{y} = 10\mathrm{e}^{\mathrm{x}}$

Solution: The auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$$

$m = 1,1$ (real and repeated)

CF $= (c_1 + c_2 x)e^{x}$

PI $= \frac{1}{D^2-2D+1} 10e^{x} = \frac{1}{(D-1)^2} 10e^{x}$ put D = a = 1, method fails, then

PI$= \frac{x}{2(D-1)} 10e^{x}$ put D = a = 1, again method fails, then

PI $= \frac{x^2}{2}(10e^{x}) = 5x^2 e^{x}$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{x} + 5x^2 e^{x}$$

Problem6: Solve $(4\mathrm{D}^2 - 4\mathrm{D} + 1)\mathrm{y} = \mathrm{e}^{\frac{\mathrm{x}}{2}}$

Solution: The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$m = \frac{1}{2}, \frac{1}{2}$ (real and repeated)

CF $= (c_1 + c_2 x)e^{\frac{x}{2}}$

P.I $= \frac{1}{4D^2-4D+1} e^{\frac{x}{2}}$ put D = a = $\frac{1}{2}$ method fails, then

P.I $= x \cdot \frac{1}{8D-4} e^{\frac{x}{2}}$ put D = a = $\frac{1}{2}$ again method fails, then

P.I $= x^2 \frac{1}{8} e^{\frac{x}{2}} = \frac{x^2}{8} e^{\frac{x}{2}}$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + \frac{x^2}{8} e^{\frac{x}{2}}$$

Problem7: Solve $(\mathrm{D}^2 - 4)\mathrm{y} = \mathrm{e}^{2\mathrm{x}} + \mathrm{e}^{-4\mathrm{x}}$

Solution: The auxiliary equation is

$$m^2 - 4 = 0 \Rightarrow (m-2)(m+2) = 0$$

$m = 2, -2$ (real and distinct)

CF $= c_1 e^{2x} + c_2 e^{-2x}$

PI $= \frac{1}{D^2-4}(e^{2x} + e^{-4x})$

$$= \frac{1}{D^2 - 4} e^{2x} + \frac{1}{D^2 - 4} e^{-4x}$$

$$= x \cdot \frac{1}{2D} e^{2x} + \frac{1}{(-4)^2 - 4} e^{-4x}$$

$$= x \cdot \frac{1}{2(2)} e^{2x} + \frac{1}{12} e^{-4x}$$

$$= \frac{x}{4} e^{2x} + \frac{1}{12} e^{-4x}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4} e^{2x} + \frac{1}{12} e^{-4x}$$

Problem8: Solve $(\mathrm{D}^2 + 4\mathrm{D} + 5)\mathrm{y} = -2 \cosh \mathrm{x}.$

Solution: The auxiliary equation is

$$m^2 + 4m + 5 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i \text{ (complex)}$$

CF $= e^{-2x}(c_1 \cos x + c_2 \sin x)$

PI $= \frac{1}{D^2+4D+5}(-2\cosh x)$

$$= -2\frac{1}{D^2+4D+5}(\frac{e^x+e^{-x}}{2})$$

$$= -\frac{1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x}$$

$$= -\frac{1}{(1)^2+4(1)+5}e^x - \frac{1}{(-1)^2+4(-1)+5}e^{-x}$$

$$= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Hence, the general solution is

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^X - \frac{1}{2}e^{-x}$$

Problem9: Solve $(\mathrm{D}^2 + 6\mathrm{D} + 9)\mathrm{y} = 5^{\mathrm{x}} - \log 2.$

Solution: The auxiliary equation is

$$m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0$$

$m = -3, -3$ (real and repeated)

CF $= (c_1 + c_2 x)e^{-3x}$

PI $= \frac{1}{D^2+6D+9}(5^x - \log 2)$

$$= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x}$$

$$= \frac{1}{(\log 5+3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x}$$

$$= \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-3x} + \frac{5^x}{(\log 5+3)^2}\frac{\log 2}{9}$$

Problem10: Solve $(\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4)\mathrm{y} = \mathrm{e}^{2\mathrm{x}} + 2\mathrm{e}^{\mathrm{x}} + 3\mathrm{e}^{-\mathrm{x}} + 2.$

Solution: The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0 \Rightarrow (m-1)(m^2 - 4m + 4) = 0 \Rightarrow (m-1)(m-2)^2 = 0$$

$m = 1$ (real and distinct), $m = 2{,}2$ (real and repeated)

$$CF = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

PI $= \frac{1}{D^3-5D^2+8D-4}(e^{2x} + 2e^x + 3e^{-x} + 2e^{0x})$

$$= \frac{1}{D^3 - 5D^2 + 8D - 4}e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4}2e^x + \frac{1}{D^3 - 5D^2 + 8D - 4}3e^{-x}$$
$$+ \frac{1}{D^3 - 5D^2 + 8D - 4}2e^{0x}$$

$$= x \cdot \frac{1}{3D^2 - 10D + 8}e^{2x} + x \cdot \frac{1}{3D^2 - 10D + 8}2e^x + \frac{1}{-1 - 5 - 8 - 4}3e^{-x} + \frac{1}{-4}2e^{0x}$$

$$= x^2 \cdot \frac{1}{6D - 10}e^{2x} + x\frac{1}{3 - 10 + 8}2e^x - \frac{1}{18} \cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2 \frac{1}{12 - 10}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

$$= \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Hence, the general solution is

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x} + \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Method II:

If $Q(x) = \sin(ax + b)$ or $\cos(ax + b)$

then Eq. (1) reduces to $f(D)y = \sin(ax + b)$ or $\cos(ax + b)$

PI $= \frac{1}{f(D)} \sin(ax + b)$ or $\cos(ax + b)$

$= \frac{1}{\psi(D^2)} \sin(ax + b)$ or $\cos(ax + b)$ put $D^2 = -a^2$

$= \frac{1}{\psi(-a^2)} \sin(ax + b)$ or $\cos(ax + b)$, if $\psi(-a^2) \neq 0$

If $\psi(-a^2) = 0$

PI = $x \cdot \frac{1}{\varphi'(D^2)} \sin(ax + b)$ or $\cos(ax + b)$ if $\varphi'(D^2) \neq 0$

Continuing like this as in the first model

Problem1: Solve $(D^2 + 9)y = \cos 4x$.

Solution:

The auxiliary equation is $m^2 + 9 = 0$

$m = \pm 3i$ (complex)

CF $= c_1 \cos 3x + c_2 \sin 3x$

PI $= \frac{1}{D^2+9} \sin 4x.$ put $D^2 = -a^2$

PI $= \frac{1}{-4^2+9} \cos 4x = \frac{1}{-7} \cos 4x$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7} \cos 4x$$

Problem2: Solve$(D^2 + 1)y = \sin^2 x.$

Solution:

The auxiliary equation is $m^2 + 1 = 0$

$m = \pm i$ (complex)

CF $= c_1 \cos x + c_2 \sin x$

$$\text{P.I} = \frac{1}{D^2+1} \sin^2 x \qquad \text{put } D^2 = -a^2$$

P.I $= \frac{1}{D^2+1} (\frac{1-\cos 2x}{2})$

$$= (\frac{1}{D^2+1} \cdot \frac{1}{2}) - (\frac{1}{2} \cdot \frac{1}{D^2+1}) \cos 2x$$

$$= (\frac{1}{2} \cdot \frac{1}{D^2+1} e^{0x}) - (\frac{1}{2} \cdot \frac{1}{-2^2+1} \cos 2x)$$

$$= \frac{1}{2} \cdot \frac{1}{0+1} e^{0x} - \frac{1}{2} \cdot \frac{1}{-3} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{6} \cos 2x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6} \cos 2x$$

Problem3: Solve $(D^2 + 3D + 2)y = \sin 2x.$

Solution: The auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$ (real and distinct)

CF $= c_1 e^{-x} + c_2 e^{-2x}$

PI $= \frac{1}{D^2+3D+2} \sin 2x$ put $D^2 = -a^2$

$= \frac{1}{-4+3D+2} \sin 2x = \frac{1}{3D-2} \sin 2x$

$= \frac{1}{(3D-2)} \cdot \frac{(3D+2)}{(3D+2)} \sin 2x = \frac{(3D+2)}{9D^2-4} \sin 2x$

$= \frac{3D+2}{9(-2^2)-4} \sin 2x = \frac{3D+2}{-40} \sin 2x$

$= -\frac{3}{40}(D \sin 2x) - \frac{1}{20} \sin 2x = -\frac{3}{40} \cdot 2 \cos 2x - \frac{1}{20} \sin 2x$

$$= -\frac{1}{20}(3 \cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{20}(3 \cos 2x + \sin 2x)$$

Problem4: Solve $(D^2 + 9)y = 2 \sin 3x + \cos 3x$.

Solution: The auxiliary equation is $m^2 + 9 = 0$

$m = \pm 3i$ (complex)

CF $= c_1 \cos 3x + c_2 \sin 3x$

PI $= \frac{1}{D^2+9}(2 \sin 3x + \cos 3x)$

$$= 2\frac{1}{D^2+9} \sin 3x + \frac{1}{D^2+9} \cos 3x \quad \text{put } D^2 = -a^2$$

$$= 2\frac{x}{2D} \sin 3x + \frac{x}{2D} \cos 3x$$

$$= x\int \sin 3x dx + \frac{x}{2}\int \cos 3x dx$$

$$= -\frac{x}{3} \cos 3x + \frac{x}{2}(\frac{\sin 3x}{3})$$

$$= -\frac{x}{3} \cos 3x + \frac{x}{6} \sin 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3} \cos 3x + \frac{x}{6} \sin 3x$$

Problem5: Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution: The auxiliary equation is $m^2 - 4m + 3 = 0$

$\Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1,3$ (realand distinct)

CF $= c_1 e^X + c_2 e^{3x}$

PI $= \frac{1}{D^2-4D+3}(\sin 3x \cos 2x)$

$$= \frac{1}{D^2 - 4D + 32} \cdot \frac{1}{2}(\sin 5x + \sin x)$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3} \sin 5x + \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3} \sin x$$

$$= \frac{1}{2} \cdot \frac{1}{-5^2 - 4D + 3} \sin 5x + \frac{1}{2} \cdot \frac{1}{-1^2 - 4D + 3} \sin x$$

$$= \frac{1}{2} \cdot \frac{1}{-4D - 22} \sin 5x + \frac{1}{2} \cdot \frac{1}{2 - 4D} \sin x$$

$$= -\frac{1}{4} \cdot \frac{1}{2D + 11} \cdot \frac{2D - 11}{2D - 11} \sin 5x + \frac{1}{4} \cdot \frac{1}{1 - 2D} \cdot \frac{1 + 2D}{1 + 2D} \sin x$$

$$= -\frac{1}{4} \cdot \frac{2D - 11}{4D^2 - 121} \sin 5x + \frac{1}{4} \cdot \frac{1 + 2D}{1 - 4D^2} \sin x$$

$$= -\frac{1}{4} \cdot \frac{2D - 11}{4(-5^2) - 121} \sin 5x + \frac{1}{4} \cdot \frac{1 + 2D}{1 - 4(-1^2)} \sin x$$

$$= \frac{2}{884}(D \sin 5x) - \frac{11}{884} \sin 5x + \frac{1}{20} \sin x + \frac{2}{20}(D \sin x)$$

$$= \frac{10}{884} \cos 5x - \frac{11}{884} \sin 5x + \frac{1}{20} \sin x + \frac{1}{10} \cos x$$

Hence, the general solution is

$$y = c_1 e^X + c_2 e^{3x} + \frac{10}{884} \cos 5x - \frac{11}{884} \sin 5x + \frac{1}{20} \sin x + \frac{1}{10} \cos x$$

Problem6: Solve $(D^2 - 4D + 4) = e^{2x} + \cos 2x$.

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2,2$ (real and repeated)

CF $= (c_1 + c_2 x)e^{2x}$

PI $= \frac{1}{D^2-4D+4}(e^{2x} + \cos 2x)$

$$= \frac{1}{D^2 - 4D + 4}e^{2x} + \frac{1}{D^2 - 4D + 4} \cos 2x$$

$$= x\frac{1}{2D - 4}e^{2x} + \frac{1}{-2^2 - 4D + 4} \cos 2x$$

$$= x^2\frac{1}{2}e^{2x} + \frac{1}{-4D} \cos 2x$$

$$= \frac{x^2}{2}e^{2x} - \frac{1}{4}\int \cos 2x dx$$

$$= \frac{x^2}{2}e^{2x} - \frac{1}{4}\frac{\sin 2x}{2}$$

$$= \frac{x^2}{2}e^{2x} - \frac{1}{8} \sin 2x$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{x^2}{2}e^{2x} - \frac{1}{8} \sin 2x$$

Problem7: Solve $(D^2 - 3D + 2)y = 2 \cos (2x + 3) + 2e^X$

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow (m - 2)(m - 1) = 0 \Rightarrow m = 2,1$ (real and distinct)

CF $= c_1 e^x + c_1 e^{2x}$

PI $= \frac{1}{D^2-3D+2}[2 \cos (2x + 3) + 2e^X]$

$$= 2\frac{1}{D^2 - 3D + 2} \cos (2x + 3) + 2\frac{1}{D^2 - 3D + 2}e^X$$

$$= 2\frac{1}{-2^2 - 3D + 2} \cos (2x + 3) + 2\frac{1}{(D - 1)(D - 2)}e^X$$

$$= 2\frac{1}{-2 - 3D} \cos (2x + 3) + 2\frac{1}{D - 1}\frac{1}{(1 - 2)}e^x$$

$$= -2\frac{3D - 2}{9D^2 - 4} \cos (2x + 3) - 2x\frac{1}{1}e^X$$

$$= \frac{-2[3D \cos (2x + 3) - 2 \cos (2x + 3)]}{9(-2^2) - 4} - 2xe^x$$

$$= \frac{-2[-3 \sin(2x+3)(2) - 2\cos(2x+3)]}{-36-4} - 2xe^x$$

$$= \frac{1}{20}[-6 \sin(2x+3) - 2\cos(2x+3)] - 2xe^X$$

$$= -\frac{1}{10}[3 \sin(2x+3) + \cos(2x+3)] - 2xe^X$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{10}[3 \sin(2x+3) + \cos(2x+3)] - 2xe^x$$

Problem8: Solve $(D^4 + 2a^2 D^2 + a^4)y = 8\cos ax$.

Solution: The auxiliary equation is $m^4 + 2a^2 m^2 + a^4 = 0$

$\Rightarrow (m^2 + a^2)^2 = 0 \Rightarrow m = \pm ia, \pm ia$ (complex and repeated)

CF $= (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax$

PI $= \frac{1}{D^4 + 2a^2D^2 + a^4} 8\cos ax$

$$= x \cdot \frac{1}{4D^3 + 4a^2 D} 8\cos ax$$

$$= x^2 \cdot \frac{1}{12D^2 + 4a^2} 8\cos ax$$

$$= x^2 \cdot \frac{1}{12(-a^2) + 4a^2} 8\cos ax$$

$$-\frac{x^2}{a^2}\cos ax$$

Hence, the general solution is

$$y = (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Problem9: Solve $(D-1)^2(D^2+1)y = e^x + \sin^2\frac{x}{2}$.

Solution: The auxiliary equation is $(m-1)^2(m^2+1) = 0$

$\Rightarrow (m-1)^2 = 0, m^2 + 1 = 0 \Rightarrow m = 1,1$ (real and repeated), $m = \pm i$ (complex)

$$CF = (c_1 + c_2 x)e^X + c_3 \cos x + c_4 \sin x$$

PI $= \frac{1}{(D-1)^2(D^2+1)}(e^x + \sin^2\frac{x}{2})$

$$= \frac{1}{(D-1)^2(D^2+1)}(e^x + \frac{1- \cos x}{2})$$

$$= \frac{1}{(D-1)^2(D^2+1)}(e^x + \frac{e^{0x}}{2} - \frac{\cos x}{2})$$

$$= \frac{1}{(D-1)^2} \cdot \frac{1}{(1^2+1)} e^x + \frac{1}{(0-1)^2(0+1)} \cdot \frac{e^{0x}}{2} - \frac{1}{(D^2+1)(D^2-2D+1)} \cdot \frac{\cos x}{2}$$

$$= x \cdot \frac{1}{2(D-1)} \cdot \frac{e^x}{2} + \frac{1}{2} - \frac{1}{(D^2+1)(-1^2-2D+1)} \cdot \frac{\cos x}{2}$$

$$= \frac{x^2}{2} \cdot \frac{e^x}{2} + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)} . \frac{1}{D} \cos x$$

$$= \frac{x^2 e^x}{4} + \frac{1}{2} + \frac{1}{4} \frac{1}{(D^2+1)} \int \cos x dx$$

$$= \frac{x^2 e^X}{4} + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{D^2+1} \sin x$$

$$= \frac{x^2 e^x}{4} + \frac{1}{2} + \frac{1}{4} x \frac{1}{2D} \sin x$$

$$= \frac{x^2 e^X}{4} + \frac{1}{2} + \frac{x}{8} \int \sin x dx$$

$$= \frac{x^2 e^X}{4} + \frac{1}{2} + \frac{x}{8}(- \cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^x + c_3 \cos x + c_4 \sin x + \frac{x^2 e^x}{4} + \frac{1}{2} - \frac{x \cos x}{8}$$

Case III $\mathbf{Q(x) = x^m}$

In this case, Eq. (1) reduces to $f(D)y = x^m$

Hence, PI $= \frac{1}{f(D)} x^m$

$$= [f(D)]^{-1} x^m$$

$$= [1 + \psi(D)]^{-1} x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since

$D^n x^m = 0$ when $n > m$,

$$\text{PI} = (a_0 + a_1 D + a_2 D^2 + \cdots + a_m D^m) x^m$$

Problem1: Solve $(D^2 + 2D + 1)y = x$.

Solution: The auxiliary equation is $m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$ (real and repeated)

CF $= (c_1 + c_2 x)e^{-x}$

PI $= \frac{1}{D^2+2D+1} x$

$$= \frac{1}{(1+D)^2} x$$

$$= (1+D)^{-2} x$$

$$= (1 - 2D + 3D^2)x$$

$$= x - 2Dx + 3D^2 x$$

$$= x - 2 + 0$$

$$= x - 2$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + x - 2$$

Problem2: Solve $(D^2 + D)y = x^2 + 2x + 4$.

Solution: The auxiliary equation is $m^2 + m = 0$

$\Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1$ (real and distinct)

CF $= c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$

PI $= \frac{1}{D^2+D}(x^2 + 2x + 4)$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1+D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - D^3 + \cdots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}[(x^2 + 2x + 4) - D(x^2 + 2x + 4) + D^2(x^2 + 2x + 4) - D^3(x^2 + 2x + 4) + \cdots]$$

$$= \frac{1}{D}[(x^2 + 2x + 4) - (2x + 2) + 2 - 0]$$

$$= \frac{1}{D}(x^2 + 4)$$

$$= \int (x^2 + 4)dx$$

$$= \frac{x^3}{3} + 4x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

Problem3: Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$.

Solution: The auxiliary equation is $m^2 + 2 = 0$,

$\Rightarrow m = \pm i\sqrt{2}$ (imaginary)

CF $= c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$

$$PI = \frac{1}{D^2 + 2}(x^3 + x^2 + e^{-2x} + \cos 3x)$$

$$= \frac{1}{2(1 + \frac{D^2}{2})}(x^3 + x^2) + \frac{1}{D^2 + 2}e^{-2x} + \frac{1}{D^2 + 2}\cos 3x$$

$$= \frac{1}{2}(1 + \frac{D^2}{2})^{-1}(x^3 + x^2) + \frac{1}{4 + 2}e^{-2x} + \frac{1}{-3^2 + 2}\cos 3x$$

$$= \frac{1}{2}(1 - \frac{D^2}{2} + \frac{D^4}{4} - \cdots)(x^3 + x^2) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= [\frac{1}{2}(x^3 + x^2) - \frac{1}{4}D^2(x^3 + x^2) + \frac{D^4}{8}(x^3 + x^2\,] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= [\frac{1}{2}(x^3 + x^2) - \frac{1}{4}(6x + 2) + 0] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + \frac{1}{2}(x^3 + x^2 - 3x - 1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Problem4: Solve $(D^3 + 8)y = x^4 + 2x + 1$.

Solution: The auxiliary equation is $m^3 + 8 = 0$

$\Rightarrow m = -2$ (real), $m = 1 \pm i\sqrt{3}$ (imaginary)

$$\text{CF} = c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$$

PI $= \frac{1}{D^3+8}(x^4 + 2x + 1)$

$$= \frac{1}{8(1+\frac{D^3}{8})}(x^4 + 2x + 1)$$

$$= \frac{1}{8}(1+\frac{D^3}{8})^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}(1 - \frac{D^3}{8} + \frac{D^6}{64} - \cdots)(x^4 + 2x + 1)$$

$= \frac{1}{8}[(x^4 + 2x + 1) - \frac{1}{8}D^3(x^4 + 2x + 1) + \frac{1}{64}D^6(x^4 + 2x + 1\,]$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x)$$

$$= \frac{1}{8}(x^4 - x + 1)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^{X}(c_2 \cos\sqrt{3x} + c_3 \sin\sqrt{3}x) + \frac{1}{8}(x^4 - x + 1)$$

Problem5: Solve $(D^3 - D^2 - 6D)y = 1 + x^2$

Solution: The auxiliary equation is $m^3 - m^2 - 6m = 0$

$\Rightarrow m(m^2 - m - 6) = 0 \Rightarrow m(m-3)(m+2) = 0$

$m = 0, 3, -2$ (real and distinct)

CF $= c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x} = c_1 + c_2 e^{3x} + c_3 e^{-2x}$

PI $= \frac{1}{D^3 - D^2 - 6D}(1 + x^2)$

$$= \frac{1}{-6D[1 - \frac{D^2 - D}{6}]}(1 + x^2)$$

$$= -\frac{1}{6D}[1 - (\frac{D^2 - D}{6}1]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}[1 + (\frac{D^2 - D}{6}1 + (\frac{D^2 - D}{6}1^2 + \cdots](1 + x^2)$$

$$= -\frac{1}{6D}[1 + \frac{D^2 - D}{6} + \frac{D^4 - 2D^3 + D^2}{36} + \cdots](1 + x^2)$$

$$= -\frac{1}{6D}[1 - \frac{D}{6} + \frac{7D^2}{36} - \frac{D^3}{18} + \cdots](1 + x^2)$$

$$= -\frac{1}{6D}[(1 + x^2) - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2) - \frac{1}{18}D^3(1 + x^2) + \cdots]$$

$$= -\frac{1}{6D}[1 + x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2) - 0]$$

$$= -\frac{1}{6D}[x^2 - \frac{x}{3} + \frac{25}{18}]$$

$$= -\frac{1}{6}\int (x^2 - \frac{x}{3} + \frac{25}{18})dx$$

$$= -\frac{1}{6}(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x)$$

Hence, the general solution is

$$\mathcal{Y} = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}(x^3 - \frac{x^2}{2} + \frac{25}{6}x)$$

Problem6: Solve $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x}$

Solution: The auxiliary equation is $m^4 + 2m^3 - 3m^2 = 0$

$\Rightarrow m^2(m^2 + 2m - 3) = 0 \Rightarrow m^2(m - 1)(m + 3) = 0$

$m = 0,0$ (real and repeated), $m = 1, -3$ (real and distinct)

$$\text{CF} = (c_1 + c_2x)e^{0x} + c_3e^x + c_4e^{-3x} = c_1 + c_2x + c_3e^X + c_4e^{-3x}$$

PI $= \frac{1}{D^4+2D^3-3D^2}(x^2 + 3e^{2x})$

$$= \frac{1}{D^4 + 2D^3 - 3D^2}x^2 + \frac{1}{D^4 + 2D^3 - 3D^2}3e^{2x}$$

$$= \frac{1}{-3D^2(1 - \frac{D^2 + 2D}{3})}x^2 + \frac{1}{16 + 16 - 12}3e^{2x}$$

$$= -\frac{1}{3D^2}(1 - \frac{D^2 + 2D}{3})^{-1}x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}[1 + \frac{D^2 + 2D}{3} + (\frac{D^2 + 2D}{3})^2 + \cdots]x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}(1 + \frac{D^2 + 2D}{3} + \frac{D^4 + 4D^2 + 4D^3}{9} + \cdots)x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}(x^2 + \frac{2}{3}Dx^2 + \frac{7}{9}D^2x^2 + \frac{4}{9}D^3x^2 + \cdots) + \frac{3}{20}e^{2x}$$

$$= -\frac{1}{3D^2}[x^2 + \frac{2}{3}(2x) + \frac{7}{9}(2) + 0] + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D}[\int (x^2 + \frac{4}{3}x + \frac{14}{9})dx] + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D}(\frac{x^3}{3} + \frac{4}{3}\frac{x^2}{2} + \frac{14}{9}x) + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int (\frac{x^3}{3} + \frac{2}{3}x^2 + \frac{14}{9}x)dx + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}(\frac{x^4}{12} + \frac{2x^3}{9} + \frac{7x^2}{9}1 + \frac{3e^{2x}}{20}$$

Hence, the general solution is

$$\mathcal{Y} = c_1 + c_2x + c_3e^{\chi} + c_4e^{-3x} - \frac{x^2}{9}(\frac{x^2}{4} + \frac{2x}{3} + \frac{7}{3}) + \frac{3e^{2x}}{20}$$

Case IV: $Q = e^{ax}V$, where V is a function of x.

In this case, Eq. (3.24) reduces to $f(D)y = e^{ax}V$.

Hence PI $= \frac{1}{f(D)} \cdot e^{ax}V \ = e^{ax} \cdot \frac{1}{f(D+a)}V$

Problem1: Solve $(D + 2)^2y = e^{-2x}\sin x$.

Solution: The auxiliary equation is $(m + 2)^2 = 0 \Rightarrow m = -2, -2$ (realand repeated)

CF $= (c_1 + c_2x)e^{-2x}$

PI $= \frac{1}{(D+2)^2}e^{-2x}\sin x$

$$= e^{-2x}\frac{1}{(D - 2 + 2)^2}\sin x$$

$$= e^{-2x}\frac{1}{D^2}\sin x$$

$$= e^{-2x}\frac{1}{-1^2}\sin x$$

$$= -e^{-2x}\sin x$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{-2x} - e^{-2x}\sin x$$

$$= (c_1 + c_2x - \sin x)e^{-2x}$$

Problem2: Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x.$

Solution: $(D^3 - 2D + 4)y = e^x \cos x$

The auxiliary equation is $m^3 - 2m + 4 = 0$

$$\Rightarrow m^2(m+2) - 2m(m+2) + 2(m+2) = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$m = -2$ (real), $m = 1 \pm i$ (complex)

$$\text{CF} = c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x)$$

$\text{PI} = \frac{1}{D^3-2D+4}e^x \cos x$

$$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4}\cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3}\cos x$$

$$= e^x[x\frac{1}{3D^2 + 6D + 1}\cos x]$$

$$= e^x x \frac{1}{3(-1)^2+6D+1}\cos x \quad [since\ D^3 + 3D^2 + D + 3 = 0\ at D^2 = -1^2 = -1]$$

$$= e^x x \frac{1}{6D-2}\cos x$$

$$= e^x x \frac{1}{2(3D-1)} \cdot \frac{(3D+1)}{(3D+1)}\cos x$$

$$= e^x x \frac{3D+1}{2(9D^2-1)}\cos x$$

$$= e^x x \frac{(3D+1)\cos x}{2[9(-1^2)-1]}$$

$$= -\frac{e^x x}{20}(3D\cos x + \cos x)$$

$$= -\frac{e^x x}{20}(-3\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{e^x x}{20}(3\sin x - \cos x)$$

Problem3: Solve $(D^2 - 2D + 2)y = e^X x^2 + 5 + e^{-2x}$

Solution: The auxiliary equation is $m^2 - 2m + 2 = 0$

$\Rightarrow m = 1 \pm i$ (complex)

CF $= e^X(c_1 \cos x + c_2 \sin x)$

PI $= \frac{1}{D^2-2D+2}(e^x x^2 + 5 + e^{-2x})$

$$= \frac{1}{D^2 - 2D + 2} e^X x^2 + \frac{1}{D^2 - 2D + 2} 5 + \frac{1}{D^2 - 2D + 2} e^{-2x}$$

$$= e^X \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2 + \frac{1}{D^2 - 2D + 2} 5e^{0x} + \frac{1}{4 - 2(-2) + 2} e^{-2x}$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2} x^2 + \frac{1}{0 - 0 + 2} 5e^{0x} + \frac{1}{10} e^{-2x}$$

$$= e^x \frac{1}{D^2 + 1} x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^X (1 + D^2)^{-1} x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x (1 - D^2 + D^4) x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^X [x^2 - D^2(x^2) + D^4(x^2) - \cdots] + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x (x^2 - 2) + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

Hence, the general solution is

$$y = e^X(c_1 \cos x + c_2 \sin x) + e^X(x^2 - 2) + \frac{5}{2} + \frac{1}{10} e^{-2}$$

Problem4: Solve $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$

Solution: The auxiliary equation is $m^3 + 3m^2 - 4m - 12 = 0$

$$\Rightarrow m^2(m+3) - 4(m+3) = 0 \Rightarrow (m+3)(m^2 - 4) = 0$$

$m = -3, -2, 2$ (real and distinct)

CF $= c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$

PI $= \frac{1}{(D+3)(D+2)(D-2)} 12xe^{-2x}$

$$= 12e^{-2x} \frac{1}{(D-2+3)(D-2+2)(D-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(D+1)D(D-4)} x$$

$$= 12e^{-2x} \frac{1}{D(D^2-3D-4)} x$$

$$= 12e^{-2x} \frac{1}{-4D(1+\frac{3D-D^2}{4})} x$$

$$= -3e^{-2x} \frac{1}{D}(1+\frac{3D-D^2}{4})^{-1} x$$

$$= -3e^{-2x} \frac{1}{D}(1-\frac{3D-D^2}{4}+\cdots) x$$

$$= -3e^{-2x} \frac{1}{D}[x - \frac{3}{4}D(x) + \frac{1}{4}D^2(x) + \cdots]$$

$$= -3e^{-2x} \frac{1}{D}(x - \frac{3}{4} + 0)$$

$$= -3e^{-2x} \int (x - \frac{3}{4}) dx$$

$$= -3e^{-2x}(\frac{x^2}{2} - \frac{3}{4}x)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x} - 3e^{-2x}(\frac{x^2}{2} - \frac{3}{4}x)$$

Case V: $Q = xV$, where V is a function of x.

In this case Eq. (3.24) reduces to $f(D)y = xV$.

Hence, PI $= \frac{1}{f(D)} xV \ = \ x\frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

Problem1: Solve $(D^2 - 5D + 6)y = x \cos 2x$.

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$

$$\Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2{,}3 \text{ (real and distinct)}$$

CF $= c_1 e^{2x} + c_2 e^{3x}$

PI $= \frac{1}{D^2-5D+6} x \cos 2x$

$$= x \frac{1}{D^2 - 5D + 6} \cos 2x - \frac{2D-5}{(D^2-5D+6)^2} \cos 2x$$

$$= x \frac{1}{-2^2 - 5D + 6} \cos 2x - \frac{2D-5}{(-2^2-5D+6)^2} \cos 2x$$

$$= x \frac{1}{(2-5D)} \cdot \frac{(2+5D)}{(2+5D)} \cos 2x - \frac{2D-5}{(4-20D+25D^2)} \cos 2x$$

$$= x \frac{(2+5D)}{4-25D^2} \cos 2x - \frac{2D-5}{[4-20D+25(-2^2)]} \cos 2x$$

$$= x \frac{(2+5D)}{4+100} \cos 2x + \frac{2D-5}{4(5D+24)} \cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2D-5}{4(5D+24)} \cdot \frac{(5D-24)}{(5D-24)} \cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(25D^2 - 576)} \cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(-100-576)} \cos 2x$$

$$= \frac{1}{52} x(\cos 2x - 5\sin 2x) - \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{52} x(\cos 2x - 5\sin 2x) - \frac{1}{1352}(40\cos 2x + 73\sin 2x)$$

Problem2: Solve $(D^2 + 3D + 2)y = xe^x \sin x$.

Solution: The auxiliary equation is $m^2 + 3m + 2 = 0$

$$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2 \text{ (real and distinct)}$$

CF $= c_1 e^{-x} + c_2 e^{-2x}$

PI $= \frac{1}{(D+1)(D+2)} xe^x \sin x$

$$= e^X \frac{1}{(D+1+1)(D+1+2)} x \sin x$$

$$= e^x \frac{1}{(D+2)(D+3)} x \sin x$$

$$= e^X \frac{1}{D^2+5D+6} x \sin x$$

$$= e^x [x \frac{1}{D^2+5D+6} \sin x - \frac{2D+5}{(D^2+5D+6)^2} \sin x]$$

$$= e^X [x \frac{1}{-1^2+5D+6} \sin x - \frac{2D+5}{(-1^2+5D+6)^2} \sin x]$$

$$= e^x [x \frac{1}{5(D+1)} \cdot \frac{(D-1)}{(D-1)} \sin x - \frac{2D+5}{25(D^2+2D+1)} \sin x]$$

$$= e^x [\frac{x}{5} \cdot \frac{(D-1)}{(D^2-1)} \sin x - \frac{2D+5}{25(-1^2+2D+1)} \sin x]$$

$$= e^X [\frac{x}{5} \cdot \frac{(D-1)}{(-1^2-1)} \sin x - \frac{2D+5}{25(2D)} \sin x]$$

$$= e^x [-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} (1 + \frac{5}{2D}) \sin x]$$

$$= e^X [-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} (\sin x + \frac{5}{2} \int \sin x dx)]$$

$$= e^x [-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} (\sin x - \frac{5}{2} \cos x)]$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{5} e^x [\frac{x}{2} (\cos x - \sin x) + \frac{1}{5} (\sin x - \frac{5}{2} \cos x)]$$

Problem3: Solve $(D^2-1)y = \sin x + e^X + x^2 e^X$.

Solution: The auxiliary equation is $m^2 - 1 = 0$

$m = 1, -1$ (real and distinct)

CF $= c_1 e^X + c_2 e^{-x}$

PI $= \frac{1}{D^2-1} (x \sin x + e^X + x^2 e^X)$

$$= \frac{1}{D^2-1} x \sin x + \frac{1}{D^2-1} e^x + \frac{1}{D^2-1} x^2 e^x$$

$$= [x \frac{1}{D^2-1} \sin x - \frac{2D}{(D^2-1)^2} \sin x] + x \frac{1}{2D} e^X + e^X \frac{1}{(D+1)^2-1} x^2$$

$$= [x \frac{1}{-1^2-1} \sin x - \frac{2D}{(-1^2-1)^2} \sin x] + \frac{x}{2(1)} e^X + e^X \frac{1}{D^2+2D} x^2$$

$$= [-\frac{x \sin x}{2} - \frac{2D \sin x}{4}] + \frac{xe^X}{2} + e^x \frac{1}{2D(1+\frac{D}{2})} x^2$$

$$= [-\frac{x \sin x}{2} - \frac{\cos x}{2}] + \frac{xe^X}{2} + e^X \frac{1}{2D} (1+\frac{D}{2})^{-1} x^2$$

$$= -[\frac{x \sin x}{2} + \frac{\cos x}{2}] \frac{xe^x}{2} + e^x \frac{1}{2D} (1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \cdots) x^2$$

$$= -[\frac{x \sin x}{2} + \frac{\cos x}{2}] + \frac{xe^X}{2} + e^x \frac{1}{2D} (x^2 - \frac{2x}{2} + \frac{2}{4} - 0)$$

$$= -[\frac{x \sin x}{2} + \frac{\cos x}{2}] + \frac{xe^X}{2} + e^x \frac{1}{2} \int (x^2 - x + \frac{1}{2}) dx$$

$$= -[\frac{x \sin x}{2} + \frac{\cos x}{2}] + \frac{xe^x}{2} + \frac{e^x}{2} (\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2} x)$$

$$= -[\frac{x \sin x}{2} + \frac{\cos x}{2}] + \frac{1}{2} e^x (\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4})$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - [\frac{x \sin x}{2} + \frac{\cos x}{2}] + \frac{1}{2} e^x (\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4})$$

Important problems

Problem1: Solve $(D^2+1)y = x^2 sin2x$.

Sol. A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

C.F. $= C_1 \cos x + C_2 \sin x$

P.I. $= \frac{1}{D^2+1} x^2 \sin 2x$ = Imaginary part of $\frac{1}{D^2+1} x^2 e^{2ix}$

$= I.P.$ of $e^{2ix} \frac{1}{(D+2i)^2+1} . x^2 = I.P.$ of $e^{2ix} \frac{1}{D^2+4iD-3} x^2$

$$= I.P.\text{ of } e^{2ix}\frac{1}{-3(1-\frac{4}{3}iD-\frac{D^2}{3})}.x^2 = I.P.\text{ of } \frac{e^{2ix}}{-3}[1-(\frac{4iD+D^2}{3})]^{-1}.x^2$$

$= I.P.$ of - $\frac{1}{3}e^{2ix}[1+(\frac{4iD+D^2}{3})+(\frac{4iD+D^2}{3})^2+x^2$

$= I.P.$ of - $\frac{1}{3}e^{2ix}[1+\frac{4iD}{3}+(\frac{1}{3}-\frac{16}{9})D^2+x^2$

$= I.P.$ of - $\frac{1}{3}e^{2ix}[x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)]$

$= I.P.$ of - $\frac{1}{3}(\cos 2x+i\sin 2x)[(x^2-\frac{26}{9})+(\frac{8x}{3})i]$

$$= -\frac{1}{3}[\frac{8x}{3}\cos 2x+(x^2-\frac{26}{9})\sin 2x] = -\frac{1}{27}[24x\cos 2x+(9x^2-26)\sin 2x]$$

Hence the complete solution is $y = C_1\cos x + C_2\sin x - \frac{1}{27}[24x\cos 2x+(9x^2-26)\sin 2x]$.

Problem2: Solve: $(D^4+2D^2+1)y = x^2 cosx.$

Sol. Auxiliary equation is $(D^2+1)^2 = 0 \Rightarrow D = \pm i, \pm i$

C.F. $= (C_1x+C_2)\cos x+(C_3x+C_4)\sin x$

P.I. $= \frac{1}{(D^2+1)^2}x^2\cos x =$ Real part of $\frac{1}{(D^2+1)^2}x^2(\cos x+i\sin x)$

$$= R.P.\text{ of } \frac{1}{(D^2+1)^2}x^2e^{ix} = R.P.\text{ of } e^{ix}\frac{1}{[(D+i)^2+1]^2}.x^2$$

$$= R.P.\text{ of } e^{ix}\frac{1}{(D^2+2iD)^2}.x^2 = R.P.\text{ of } e^{ix}\frac{1}{[2iD(1+\frac{D}{2i})]^2}.x^2$$

$$= R.P.\text{ of } e^{ix}\frac{1}{-4D^2(1-\frac{iD}{2})^2}.x^2 = R.P.\text{ of } \frac{e^{ix}}{-4}.\frac{1}{D^2}(1-\frac{iD}{2})^{-2}.x^2$$

$= R.P.$ of - $\frac{1}{4}e^{ix}.\frac{1}{D^2}[1+2(\frac{iD}{2})+3(\frac{iD}{2})^2+x^2$

$= R.P.$ of - $\frac{1}{4}e^{ix}.\frac{1}{D^2}[x^2+i(2x)-\frac{3}{4}(2)] = R.P.$ of - $\frac{1}{4}e^{ix}.\frac{1}{D}[\frac{x^3}{3}+ix^2-\frac{3}{2}x]$

$= R.P.$ of - $\frac{1}{4}e^{ix}[\frac{x^4}{12}+i\frac{x^3}{3}-\frac{3x^2}{4}]$

$= R.P.$ of - $\frac{1}{48}(\cos x+i\sin x)[(x^4-9x^2)+(4x^3)i]$

P.I. $= -\frac{1}{48}[(x^4-9x^2)\cos x-4x^3\sin x]$

Hence complete solution is

$$y = (C_1x+C_2)\cos x+(C_3x+C_4)\sin x-\frac{1}{48}[(x^4-9x^2)\cos x-4x^3\sin x].$$

Problem3: Solve $(D^2 - 4D + 4)y = 8x^2e^{2x}sin2x.$

Sol. The auxiliary equation is $m^2 - 4m + 4 = 0 \Rightarrow m = 2,2$

$C.F. = (C_1 + C_2x)e^{2x}$

$$\text{P.I.} = \frac{1}{D^2-4D+4}(8x^2e^{2x}\sin 2x) = \frac{1}{(D-2)^2}(8x^2e^{2x}\sin 2x)$$

$$= 8.e^{2x}\frac{1}{(D+2-2)^2}(x^2\sin 2x) = 8e^{2x}\ \frac{1}{D^2}(x^2\sin 2x) = 8e^{2x}.\frac{1}{D}\int x^2\ \sin 2xdx$$

$$\text{P.I.} = 8e^{2x}.\frac{1}{D}[x^2(-\frac{\cos 2x}{2}) - \int 2\,x.(-\frac{\cos 2x}{2})dx]$$

$$= 8e^{2x}\ \frac{1}{D}[-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1.\frac{\sin 2x}{2}dx]$$

$$= 8e^{2x}\ \frac{1}{D}[-\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}]$$

$$= 8e^{2x}.\frac{x^2}{2}1\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx + \int\frac{x\sin 2x}{2}dx + \frac{\sin 2x}{8}]$$

$$= 8e^{2x}[-\frac{x^2}{4}\sin 2x + \frac{\sin 2x}{8} + \int x\ \sin 2xdx]$$

$$= 8e^{2x}[(\frac{1}{8} - \frac{x^2}{4})\sin 2x + x(\frac{-\cos 2x}{2}1 - \int 1.(\frac{-\cos 2x}{2})dx]$$

$$= 8e^{2x}[(\frac{1}{8} - \frac{x^2}{4})\sin 2x - \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}]$$

$$= 8e^{2x}[(\frac{3}{8} - \frac{x^2}{4})\sin 2x - \frac{x}{2}\cos 2x] = e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

Hence the complete solution is

$$y = C.F. + P.I. \text{ or } y = e^{2x}[C_1 + C_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$$

where C_1 and C_2 are arbitrary constants of integration.

METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if the complementary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complementary function and, hence, is known as variation of parameters method.

Consider a linear nonhomogeneous differential equation of second order with constant coefficients.

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = Q(x) \quad ..... (1)$$

Let the complementary function be

$$\text{CF} = c_1y_1 + c_2y_2 \quad ..... (2)$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \quad ..... (3)$$

Let the particular integral be

$$y = u(x)y_1 + v(x)y_2 \quad ..... (4)$$

where u and v are unknown functions of x.

Here u, v can be obtained by using the equations

$$u = \int -\frac{y_2Q}{y_1y_2 - y_1y_2}dx = \int -\frac{y_2Q}{W}dx$$

$$v = \int \frac{y_1Q}{y_1y_2 - y_1y_2}dx = \int \frac{y_1Q}{W}dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as the Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (3.26) is

$$y = CF + PI$$

$$= c_1y_1 + c_2y_2 + uy_1 + vy_2$$

Note: The above method can also be extended to third - order differential equation.

Working Rule:

1. Find the complementary function as CF $= c_1y_1 + c_2y_2$.

2. Find the Wronskian of y_1, y_2 as $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

3. Assume the particular integral as PI $= u(x)y_1 + v(x)y_2$.

4. Find u and v by evaluating the integrals $u = \int -\frac{y_2Q}{W}dx, v = \int \frac{y_1Q}{W}dx$.

5. Substitute u and v in PI and write the general solution as $y = CF + PI$.

Problem1: Solve $\frac{d^2y}{dx^2} + y = \sin x$.

Solution: $(D^2 + 1)y = \sin x$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$ (complex)

CF $= c_1 \cos x + c_2 \sin x$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \text{cosx} & \text{sinx} \\ -\text{sinx} & \text{cosx} \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let PI $= u \cos x + v \sin x \cdots$(1)

$u = \int -\frac{y_2 Q}{W} dx = \int -\frac{\sin x \sin x}{1} dx = -\int \frac{(1-\cos 2x)}{2} dx = -\frac{1}{2}(x - \frac{\sin 2x}{2})$

$v = \int \frac{y_1 Q}{W} dx = \int \frac{\cos x \sin x}{1} dx = \int \frac{\sin 2x}{2} dx = \frac{1}{2}(-\frac{\cos 2x}{2}) = -\frac{1}{4} \cos 2x$

Substituting u and v in Eq. (1),

$$\text{PI} = -\frac{1}{2}(x - \frac{\sin 2x}{2}) \cos x - \frac{1}{4} \cos 2x \sin x$$

$$= -\frac{1}{2}x + \frac{1}{4}(\sin 2x \cos x - \cos 2x \sin x)$$

$$= -\frac{1}{2}x + \frac{1}{4} \sin (2x - x) = -\frac{1}{2}x + \frac{1}{4} \sin x$$

Hence, the general solution is $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x + \frac{1}{4} \sin x$

Problem2: Solve $(D^2 + 1)y = \text{cosec}\, x$.

Solution: The auxiliary equation is $m^2 = -1$

$m = \pm i$ (complex)

CF $= c_1 \cos x + c_2 \sin x$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \text{cosx} & \text{sinx} \\ -\text{sinx} & \text{cosx} \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let PI $= u \cos x + v \sin x \cdots$(1)

$u = \int -\frac{y_2 Q}{W} dx = -\int \frac{\sin x \,\text{cosec}\, x}{1} dx = -\int d\,x = -x$

And $v = \int \frac{y_1 Q}{W} dx = \int \frac{\cos x \,\text{cosec}\, x}{1} dx = \int \cot x dx = \log \sin x$

Substituting u and v in Eq. (1),

PI $= -x\cos x + (\log \sin x)\sin x$

Hence, the general solution is $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log \sin x$

Problem3: Solve $y'' + a^2 y = \tan ax$.

Solution: $(D^2 + a^2)y = \tan ax$

The auxiliary equation is $m^2 + a^2 = 0$

$m = \pm ai$ (complex)

CF $= c_1 \cos ax + c_2 \sin ax$

$$y_1 = \cos ax, \qquad y_2 = \sin ax$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \text{cosax} & \text{sinax} \\ -\text{a sinax} & \text{a cosax} \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$$

$$1'1$$

PI $= u\cos ax + v\sin x$ ....(1)

$u = \int -\frac{y_2 Q}{W}dx = \int -\frac{\sin ax \tan ax}{a}dx = -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}dx$

$= -\frac{1}{a}\int(\sec ax - \cos ax)dx = -\frac{1}{a}\cdot\frac{1}{a}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$

$$= \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

$v = \int \frac{y_1 Q}{W}dx = \int \frac{\cos ax \tan ax}{a}dx = \frac{1}{a}\int \sin ax\, dx = \frac{1}{a}(-\frac{1}{a}\cos ax) = -\frac{1}{a}\cos ax$

Substituting u and v in Eq. (1),

$$PI = \frac{1}{a^2}\sin ax \cos ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax) - \frac{1}{a^2}\sin ax \cos ax$$

$$= -\frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Problem4: Solve $(D^2 + 1) = \operatorname{cosec} x \cot x$.

Solution: The auxiliary equation is $m^2 + 1 = 0$

$m = \pm i$ (complex)

$\text{CF} = c_1 \cos x + c_2 \sin x$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\text{PI} = u \cos x + v \sin x \cdots (1)$

$u = \int -\frac{y_2 Q}{W} dx = \int -\frac{\sin x \operatorname{cosec} x \cot x}{1} dx = -\int cot\, x dx = -\log(\sin x)$

$v = \int \frac{y_1 Q}{W} dx = \int \frac{\cos x \operatorname{cosec} x \cot x}{1} dx = \int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1) dx = -\cot x - x$

Substituting u and v in Eq. (1),

$\text{PI} = -\log(\sin x) \cos x + (-\cot x - x) \sin x$

$$= -\cos x \log(\sin x) - (\cot x + x) \sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sin x) - (\cot x + x) \sin x$$

Problem5: Solve $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

Solution: The auxiliary equation is $m^2 - 1 = 0$

$m = \pm 1$ (real and distinct)

$\text{CF} = c_1 e^x + c_2 e^{-x}$

$$y_1 = e^x, \qquad y_2 = e^{-x}$$

$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

Let $\text{PI} = u e^x + v e^{-x} \cdots (1)$

where $u = \int -\frac{y_2 Q}{W} dx$

$$= -\int \frac{e^{-x}[e^{-x} \sin(e^{-x}) + \cos(e^{-x})]}{-2} dx$$

Let $e^{-x} = t, -e^{-x} dx = dt$,

$u = -\frac{1}{2} \int (t \sin t + \cos t) dt = -\frac{1}{2}[t(-\cos t) - (-\sin t) + \sin t)]$

$= \frac{1}{2} t \cos t - \sin t = \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{e^X[e^{-x} \sin (e^{-x}) + \cos (e^{-x})]}{-2} dx$$

$$= \int \frac{e^X[\cos (e^{-x}) + e^{-x} \sin (e^{-x})]}{-2} dx$$

$= -\frac{1}{2} e^x \cos (e^{-x})$ $\quad [\int e^x \{f(x) + f'(x)\} dx = e^x f(x)$ Here $f(x) = \cos e^{-x}]$

Substituting u and v in Eq. (1),

PI $= \frac{1}{2} \cos (e^{-x}) - e^x \sin (e^{-x}) - \frac{1}{2} \cos (e^{-x}) = -e^x \sin (e^{-x})$

Hence, the general solution is $y = c_1 e^x + c_2 e^{-x} - e^x \sin (e^{-x})$

Problem6: Solve $(D^2 - 2D + 2)y = e^x \tan x$.

Solution: The auxiliary equation is $m^2 - 2m + 2 = 0$

$m = 1 \pm i$ (complex)

CF $= e^x(c_1 \cos x + c_2 \sin x)$

$y_1 = e^x \cos x,\ y_2 = e^x \sin x$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x cosx & e^x sinx \\ e^x cosx - e^x sinx & e^x sinx + e^x cosx \end{vmatrix}$$

$$= e^X \cos x(e^X \sin x + e^X \cos x) - e^X \sin x(e^X \cos x - e^X \sin x)$$

$$= e^{2x} \cos x \sin x + e^{2x}\cos^2 x - e^{2x} \cos x \sin x + e^{2x}\sin^2 x$$

$$= e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$$

Let PI $= ue^x \cos x + ve^x \sin x \cdots(1)$

$$u = \int -\frac{y_2 Q}{W} dx$$

$$= \int -\frac{e^X \sin x \cdot e^X \tan x}{e^{2x}} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= -\int \sec x dx + \int \cos x dx$$

$$= -\log (\sec x + \tan x) + \sin x$$

and

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{e^X \cos x \cdot e^X \tan x}{e^{2x}} dx$$

$$= \int \sin x dx$$

$$= -\cos x$$

Substituting u and v in Eq. (1),

$$PI = [-\log(\sec x + \tan x) + \sin x] \cdot e^X \cos x + (-\cos x) \cdot e^X \sin x$$

$$= -e^x \cos x \cdot \log(\sec x + \tan x)$$

Hence, the general solution is $y = e^X(c_1 \cos x + c_2 \sin x) - e^X \cos x \cdot \log(\sec x + \tan x)$