**Matrices**

**Definition:**A *matrix* is a rectangular arrangement of elements in rows and columns. If a matrix  has  rows and columns then we say that the *order* of the matrix is . If  is a matrix of order then it can be written as



**Definition:** If  is a matrix of order  , then

1.  is called a *rectangular matrix* if .
2.  is called a *square matrix* if .
3.  is called a *row matrix* if .
4.  is called a *column matrix* if .

**Definition:** If all the entries of a matrix  are real numbers then it is called a *real matrix.* If at least one entry of a matrix  is a complex number then  is called a *complex matrix*.

**Definition:** A matrix  is called the zero matrix if  for all  and . The zero matrix is denoted by the symbol .

**Definition:** If is a real or complex matrix of order  and if  is a constant then



**Definition:** If  and  are any two real or complex matrices of same order then the sum  is also an  matrix, given by



**Remarks:** Let  be the set of all real matrices of order . Then  satisfies the following properties with respect to the addition of matrices

1. If  and  then . That is if  and  are any two real matrices of order  then  is also a real matrix of the same order . (*Closure property*)
2. If  are any three real matrices of order  then     (*Associative law*)
3. If  is a real matrix of order  then , where  is the zero matrix of order . (*Existence of Identity*)
4. If is a real matrix of order  then  is also a real matrix of order  and . The matrix  is called the additive inverse of . (*Existence of Inverse*)
5. If  and  are any two real matrices of order then .

*(Commutative law)*

**Definition:** If  and are any two real or complex matrices then the *multiplication* of  and  is defined to be the matrix where

.

**Definition:** If  is any real or complex matrix of order  then the *transpose* matrix of  is defined by



**Remarks:**

1. If  and  are any two matrices of the same order then 
2. If  is a constant and is a matrix of order then .
3. If is a matrix of order  and  is a matrix of order then 
4. If is a matrix of order then 

**Definition:** If  is a square matrix of order  such that 

Then the elements  are called *principal diagonal* entries.

The *trace* of  is defined to be the number  .

**Remarks:** If  and  are any two square matrices of same order then

1. 
2.  where  is a constant
3. 

**Definition:** A real square matrix  is said to be

1. *Symmetric* if 
2. *Skew - symmetric* if or  where  is the zero matrix
3. *Orthogonal* if  where  is the identity matrix given by , 

**Theorem:** Every real square matrix can be expressed as a sum of a symmetric matrix and a skew– symmetric matrix in a unique way.

**Proof:** Let  be a real square matrix. Let  and Then 







Now we prove that  is symmetric and  is skew - symmetric

We have 



 is symmetric

And also, 



Hence , where  is symmetric and  is skew -symmetric

Now we prove that this representation is unique. Suppose that  where  is symmetric and  is skew – symmetric

Since  is symmetric, 

Since  is skew – symmetric, 

Now 



Similarly, 









Thus every real square matrix can be written as a sum of a symmetric matrix and a skew – symmetric matrix in a unique way.

**Remarks:**

1. If  and  are any two symmetric matrices then  and  are also symmetric
2. If  and  are any two skew- symmetric matrices then  and  are also skew – symmetric
3. If  is a positive real number and if  is symmetric then  is also symmetric
4. If  is a positive real number and if  is skew - symmetric then  is also skew – symmetric
5. If  and  are orthogonal matrices then  is also an orthogonal matrix

**Problem:** Resolve the matrix  into a symmetric matrix and a skew – symmetric matrix.

**Solution:** Given 



Then  and 

Clearly , where  is symmetric and  is skew – symmetric. The matrix  is called the symmetric part of  and  is called the skew – symmetric part of .

**Definition:** If  is a complex matrix then its *conjugate matrix* is defined by  where  denotes the conjugate of . The *conjugate transpose* of  is defined by .

**Remarks:** If  and  are any two complex matrices then

1. 
2.  where  is a real or complex number
3. 

**Definition:** A complex square matrix  is said to be

1. Hermitian if 
2. *Skew-Hermitian* if 
3. *Unitary* if  where  is the identity matrix

**Theorem:** Every complex square matrix can be expressed as a sum of a Hermitian matrix and a Skew-Hermitian matrix in a unique way.

**Proof:** Let  be a complex square matrix. Let and Then







Now we prove that  isHermitian and  is Skew-Hermitian

Consider 







 is Hermitian

Now 

 





 is skew- Hermitian

Hence , where  is Hermitian and  is Skew-Hermitian

Now we prove that this representation is unique. Suppose that where is Hermitian and  is Skew-Hermitian

Since  is Hermitian, 

Since  is Skew-Hermitian, 

Now 





Similarly, 









Thus every complex square matrix can be written as a sum of a Hermitian matrix and a Skew-Hermitian matrix in a unique way.

**Definition:**If  is a square matrix then its *determinant* is defined as  where  is called the *co-factor* of the element  and it is given by  and  is called the *minor* of . The minor  of  is the determinant of the  matrix that survives where the row and column containing  are struck out.

**Definition:** A square matrix is said to be

1. *Singular* if 
2. *Non – singular* or invertible, if .

**Definition:** If  is a square matrix of order  then it said to be

1. *Upper triangular* if  for all 
2. *Lower triangular* if  for all 
3. *Diagonal* if  for all 

**Remarks:**

1. If any row or columns of is modified by adding  times the corresponding elements of another row or column to it, yielding a new matrix , then  .
2. If any rows or columns of  are interchanged, yielding a new matrix , then 
3. If  is triangular or diagonal, then  is simply the product of the diagonal elements, 
4. If all the elements of any row or column are zero, then 
5. 
6. 
7. 
8. 

**Definition:** If  is any square matrix then the *adjoint* of  is defined to be the transpose of the cofactor matrix of . That is  where  and  is called the *minor* of . The minor  of  is the determinant of the  matrix that survives where the row and column containing  are struck out.

**Definition:** If  is a non-singular matrix then its multiplicative inverse  is defined by . Clearly,  where  is the identity matrix.

**Remark:**

Let  be any square matrix of order . Suppose that  is non-singular to find the inverse  of , we use the Gauss – Jordan method. In this method, we write  where  is the identity matrix of order . Then we apply row operations to  on the LHS and to the matrix  on the RHS until the equation becomes  then the matrix  is the inverse of  i.e., . This method is useful if the order of  is large.

**Exercise**

1. Resolve the following matrices as a sum of symmetric and skew-symmetric parts.
2.  b. c. d. 
3. Find the determinant of the following matrices.
4. b. c.d.
5. Find the inverse of the following matrices by finding their adjoints.
6. b. c. 
7. Find the inverse of the following matrices by using elementary transformations.
8. b. c. 

d. 

1. Verify whether the following matrices are orthogonal or not.
2.  b.  c.  d. 
3. Express the following complex matrices as a sum of Hermitian and skew- Hermitian matrices
4.   b. 

c.  d. 

7. Show that  is skew- Hermitian.

8. If  then show that  is Hermitian and is skew- Hermitian.

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**Rank of a Matrix**

**Definition:**We say that the *rank* of a matrix  is  if

1. There exists a at least one non-zero minor of order .
2. All the minors of order  and above are zero.

The rank of a matrix  is denoted by  or .

**Remarks:**

1. If  is a rectangular matrix of order then .
2. If  is any matrix then .
3. The rank of a zero matrix is zero.
4. The rank of a non – zero matrix is positive.
5. If  is a square matrix of order  and  then .
6. If  is a square matrix of order  and  then .
7. Elementary row operations do not alter the rank of a matrix.

**Echelon Form:**

A matrix is said to be in Echelon from if it has the following properties.

1. The zero rows must follow non- zero rows
2. The number of zero rows before the first non-zero element of a row is less than the number of such zeros in the next row.

The rank of a matrix  is the number of non-zero rows in an Echelon form of .

**Problem:** Find the rank of  using elementary row operations.

**Solution:** Given matrix is 

Applying  and . Then we get



Applying  . Then we obtain



Applying . Then we obtain



The number of non-zero rows in the echelon form of  is 3.

Hence the rank of  is .

**Problem:** Find the rank of the matrix  using elementary row operations.

**Solution:**Given matrix is 

Applying  ,  and 



Applying  ,  and 



Applying and 



The number of non-zero rows in the echelon form of  is 2.

Hence the rank of  is .

**Normal form:**

A normal form of a matrix  is given by  where  is some positive integer. If a matrix  is converted by using elementary operations into the normal form then the rank of  is .

**Problem:**

Find the rank of a matrix  by using normal form.

**Solution:** Given matrix is 

Apply  and 



Apply 



Apply 



Apply 



Apply 



Apply 



 The rank of  is .

**Procedure:** To find two non-singular matrices  and  in such a way that the matrix  is in the normal form, we adapt the following procedure.

1. Suppose that  is a rectangular matrix of order 
2. Write  where  and  are identity matrices of orders  and  respectively
3. Apply row operations to  on the LHS and to  on the RHS
4. Apply column operations to  on the LHS and to  on the RHS
5. Convert the matrix  on the LHS of  to normal form. Then  takes the form , where  is the normal form of . Observe that the matrices  and  are non- singular.

**Remark:** If  is a non- singular matrix then we can find two non- singular matrices  and  such that  is in the normal form. Then .

**Problem:** Obtain two non-singular matrices  and  such that  is in the normal form, where  here .

**Solution:**Given matrix is 

Write 



Apply 



Apply 



Apply 



Apply 



 where   and 

Here  and  are non- singular matrices and  is in the normal form. The rank of  is .

**Exercise**

1. Find the Rank of the following matrices by using Echelon form.
2.  b. 

c.  d. 

e. 

1. Find the Rank of the following matrices by reducing into normal form.
2.  b. 

c.d.

e. 

3. Obtain two non – singular matrices  and such that  is in the

Normal Form for the following matrices

a. b. 

c. d.

d. 

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**System of linear equations**

A system of liner equations in unknowns can be written as







…………………………………………………………………….

.…………………………………………………………………….



We restrict the values of  and  to be finite and ,  to be real numbers. This system of liner equations in unknowns can be represented as a matrix equation , where

     and

A value of  which satisfies the matrix equation  is called a solution of thegiven system of  liner equations in  unknowns. If a system of linear equations  has a solution then it is said to be *consistent*. If a system  of linear equations has no solution the it is said to be *inconsistent*. The augmented matrix for the given system of liner equations with unknowns is given by



1. If  then the given system of linear equations has no solution or the system is inconsistent.
2. If  then the given system of linear equations has atleast one solution or the system is consistent.
3. If , where  is the number of unknowns then the system of linear equations has a unique solution or the system has exactly one solution.
4. If , where  is the number of unknowns then the system of linear equations has infinitely many solutions.
5. If the number of equations is less than the number of unknowns in a system and if the system is consistent then it cannot have a unique solution.

**Definition:** A system  of  liner equations in  unknowns is said to be a *homogeneous system* if  where the  is the zero matrix.

If  is a non-zero matrix then the system  is said to be a

*non- homogeneous system.*

**Remarks:**

1. A homogeneous system  is always consistent.
2. The zero matrix  is always a solution for  and it is called the trivial solution.
3. If  is a non-singular matrix then the system  has exactly one solution and it is the trivial solution only.
4. If is singular matrix then the system  has infinitely many solutions.

**Gauss elimination method**

Consider a system  of  liner equations in  unknowns.

1. Verify whether the given system of linear equations is consistent or not
2. If the given system of linear equations is consistent then convert the augmented matrix as an upper triangular matrix by using elementary row operations
3. First find  and by using the value of  find  thus find all the unknowns by the method of substitution.

**Problem:** Verify whether the following system of linear equations is consistent or not.



**Solution:**The given system of linear equations is



This system can be represented as , where



The augmented matrix is 

Consider

 and 

Apply  and 

and 

Apply 

 and 

  and 

 

The given system is inconsistent.

**Problem:** Verify whether the following system of linear equations is consistent or not.



**Solution:**The given system of linear equations is



This system can be represented as , where



The augmented matrix is 

Consider

 and

Apply ; and

and

Apply  and

 and

Apply 

 and

  and 



 The given system is consistent and , the number of unknowns.

 The given system has a unique solution

**Problem:** Solve the following system of linear equations



**Solution:**The given system of linear equations is



This system can be represented as , where

    

The augmented matrix is 

Consider

 and

Apply and

 and

Apply 

and

  and 



 The given system is consistent and , the number of unknowns.

 The given system has a unique solution

Now consider 

By using the last row of , we get 

By using the second row of , we get







By using the first row of , we get



 is a unique solution of the given system.

**Problem:** Solve the following system of linear equations



**Solution:**The given system of linear equations is



This system can be represented as , where



The augmented matrix is

Consider

 and

Apply  and

and

Apply

 and

Apply

and 

  and 



 The given system is consistent and , the numberof unknowns

The given system of linear equations has infinitely many solutions

Consider



Put , where  is a constant.

From the second row of ,







From the first row of ,





 is a general solution of the given system of linear equations.

Put . Then  is a solution of the given system of liner equations.

**Problem:** Solve the following homogeneous system of linear equations



**Solution:** The given system of linear equations is



This system can be represented as, where



Consider



Apply and



Apply 



Apply



 , the number of unknowns

 the given homogeneous system has infinitely many solutions

Consider 

Put , where  is a constant.

From the second row of , we have

From the first row of, we have 

 is a general solution of the given system of linear equations.

Put . Then  is a solution of the given homogeneous system of linear equations.

**Gauss – Seidel iteration method**

Consider the following system of linear equations







This system of linear equations is said to be *diagonally dominant* , if

;  and 

Suppose that the given system of linear equations is diagonally dominant. Consider the given system













Now we choose initial approximations  and  for the values of and. By substituting these initial approximations in eq (1) and we find  as follows



By using the values of  and  , we find  as follows



By using  and  , we find  as follows



Now we get the first approximation where







Next we find the second approximation  to a solution of the given system of linear equations, where







The third approximation  can be obtained by the following







We continue this process and we get a sequence of approximations, , ,….

From this sequence of approximations we take the desired solution of the given system of linear equations.

**Problem:** Find the first three approximations to a solution of the following system of linear equations by using Gauss- Seidel iteration method









**Solution:** Given system of linear equations is

















By taking  as an initial approximation, we get 

By using  and  we get 



By using  and , we get





By using  and  , we get



Then the first approximation is given by 

The second approximation  is obtained as follows











The third approximation  is obtained as follows



**Exercise**

1. Verify whether the following systems are consistent or inconsistent
2.  b. c. 

d. e. f.

1. Verify the consistency and hence solve the following systems of linear equations
2.  b.  c. 

d.e.

3. Solve the following homogeneous systems of linear equations

a.  b. c. 

d. e.

4. Find the first four approximations to a solution of the following systems of

linear equations byusing Gauss-Seidel

Method

a. b. 

c.  d. 