Bayes' Theorem

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Topics to be covered:

- Joint Probability
- Marginal Probability
- Conditional Probability
- Two rules of probability
  - Sum Rule
  - Product Rule
- Bayes' theorem

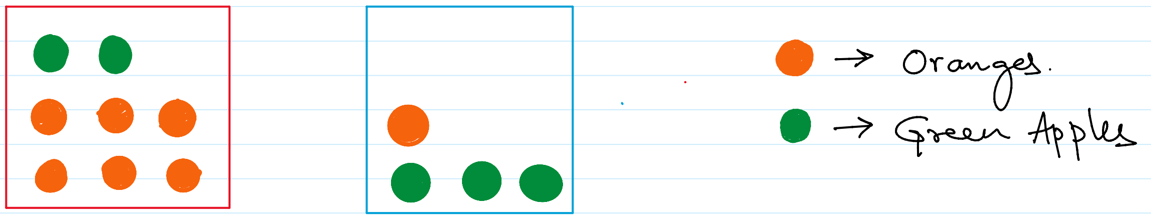
$S = \{H, T\}$

$H : 1$ $\quad S = \{1, 0\}$

$T : 0$

$P(X) = 3/10$

Let us begin by considering a simple example.



RED Basket   Blue Basket

Red basket has → 6 'O' + 2 'A'

Blue basket has → 1 'O' + 3 'A'.

Following are some assumptions.

- All fruits within a basket are equally available. (Equi-probable).

- Selection of baskets are NOT equi-probable.

  $P(\text{Red basket}) = 40\% \Rightarrow P(B = r) = 0.4$

  $P(\text{Blue basket}) = 60\% \Rightarrow P(B = b) = 0.6$

Random Variables:

B : which basket we pick

F : which fruit we pick

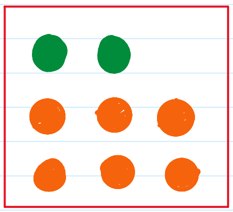
Sample Space:

$B: \quad S_B = \{b, r\}$

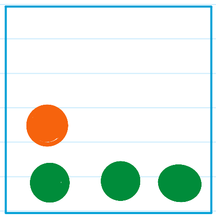
$F: \quad S_F = \{O, A\}$

Let $N = 100$ total # trials.

	F = O	F = A	Total
B = red	(r, o) 30 $= n_{11}$	(r, A) 10 $= n_{12}$	40
B = Blue	(b, o) 15 $= n_{21}$	(b, A) 45 $= n_{22}$	60
Total	45	55	N = 100



RED Basket



Blue Basket

$O \rightarrow \frac{6}{8} \times 40 = 30$ $\qquad \frac{1}{4} \times 60 = 15$

$A \rightarrow \frac{2}{8} \times 40 = 10$ $\qquad \frac{3}{4} \times 60 = 45$

	F = O	F = A	
B = r	0.3	0.1	0.4
B = b	0.15	0.45	0.6
	0.45	0.55	1

Joint-Prob. Table.

$$P(B=r,\ F=O) = 0.3 = \frac{n_{11}}{N}$$

$$P(B=r,\ F=A) = 0.1 = \frac{n_{12}}{N}$$

$$P(B=b,\ F=O) = 0.15 = n_{21}/N$$

$$P(B=b,\ F=A) = 0.45 = n_{22}/N$$

$m = 2$
$n = 2$

In general,

$X : x_i : i = 1, 2, 3, ----, m.$

$Y : y_j : j = 1, 2, 3, ----, n.$

$m = 5$
$n = 3$

c_i

n_{11}	n_{12}			
			n_{ij}	

$\} r_j$

$\uparrow y_j$

$i = $ # columns.
$j = $ # rows.

$x_i \longrightarrow$

Joint - Prob. $\Rightarrow$ $P(X = x_i, Y = y_j)$ n_{ij}

$$P(B = r, F = o) = \frac{30}{100} = 0.3$$

$$P(B = b, F = A) = \frac{45}{100} = 0.45$$

$$\boxed{P(X = x_i, Y = y_j) = \frac{n_{ij}}{N}} \longrightarrow \text{Eq}^n \; 1$$

where N = Total # of trials.

Sum Rule.

$$P(B = r) \quad \text{or} \quad P(F = o) = ? \quad \frac{45}{100} = 0.45$$

$$P(X = x_i) = \underbrace{\frac{c_i}{N}}_{\text{Marginal Probability.}} = \frac{\sum_j n_{ij}}{N} = \sum_j \frac{n_{ij}}{N}$$

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

This is the Sum Rule of Probability.

Conditional Prob.

$$P(B = r \mid F = o) = \frac{30}{45} =$$

$$P(Y = y_j \mid X = x_i) = \frac{n_{ij}}{c_i}$$

$$P(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Consider this

n_{ij} (n_{ij}) (c_i)

$$\frac{n_{ij}}{N} = \underbrace{\left(\frac{n_{ij}}{c_i}\right)}_{\text{Condn. Prob.}} \cdot \underbrace{\left(\frac{c_i}{N}\right)}_{\text{Marginal Prob.}}$$

$$P(B=r, F=o) = 0.3$$

$$= \frac{30}{45} \cdot \frac{45}{100} = 0.3$$

$$\boxed{P(X=x_i, Y=y_j) = P(Y=y_j \mid X=x_i) \cdot P(X=x_i)}$$

This is the product rule.

Sum Rule : $P(X) = \sum_{Y} P(X, Y)$

Product Rule : $P(X, Y) = P(Y \mid X) \cdot P(X)$.

Similarly, $P(Y, X) = P(X \mid Y) \cdot P(Y)$

Since, $P(X, Y) = P(Y, X)$

$$\Rightarrow P(Y \mid X) \cdot P(X) = P(X \mid Y) \cdot P(Y)$$

$$\boxed{P(Y \mid X) = \frac{P(X \mid Y) \cdot P(Y)}{P(X)}}$$

This is the Bayes' theorem.

We can write the other way also.

$$P(X \mid Y) = \frac{P(Y \mid X) \cdot P(X)}{P(Y)}.$$