Solution of a Simple Poisson Equation in a Rectangle

1 Equation

As an example, I chose a simple equation defined on a rectangle $[0,1] \times [0,1]$ with an analytical solution.

$$\begin{cases} \Delta u = -2\pi^2 \sin \pi x \sin \pi y, \\ u(\text{boundary}) = 0. \end{cases} \tag{1}$$

The analytical solution is:

$$u = \sin(\pi x) \sin(\pi y). \tag{2}$$

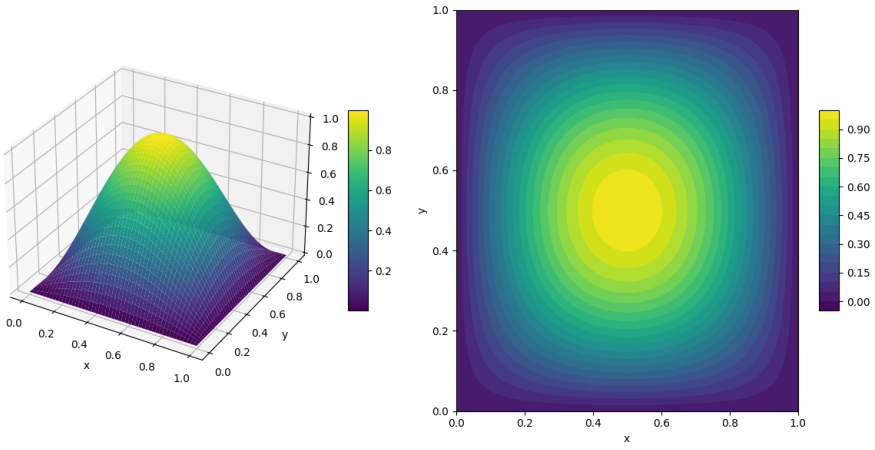

Figure 1: Analytical solution

2 Architecture Selection

The equation is very simple with smooth boundaries and no discontinuities. Therefore, I chose the simplest model — a Physics-Informed Neural Network (PINN). This is sufficient to achieve a good approximation. The activation function used is Tanh(), as it is smooth.

3 Data Selection

There are two types of points: points inside the rectangle and points on its boundary. The points inside the rectangle are randomly selected using **torch.rand**. It is worth noting that this is where the advantage of PINN becomes apparent — no mesh is needed. The points on the boundary also do not require special selection.

4 Training

The training process consists of two stages. First, training is performed using the first-order optimizer Adam, followed by the second-order optimizer L-BFGS[1]. Training is done on a CPU. There is no need to fine-tune the optimizer parameters.

5 Result

Final loss	L2 Relative Error (L2RE)
$2.2 \cdot 10^{-5}$	$1.5 \cdot 10^{-4}$

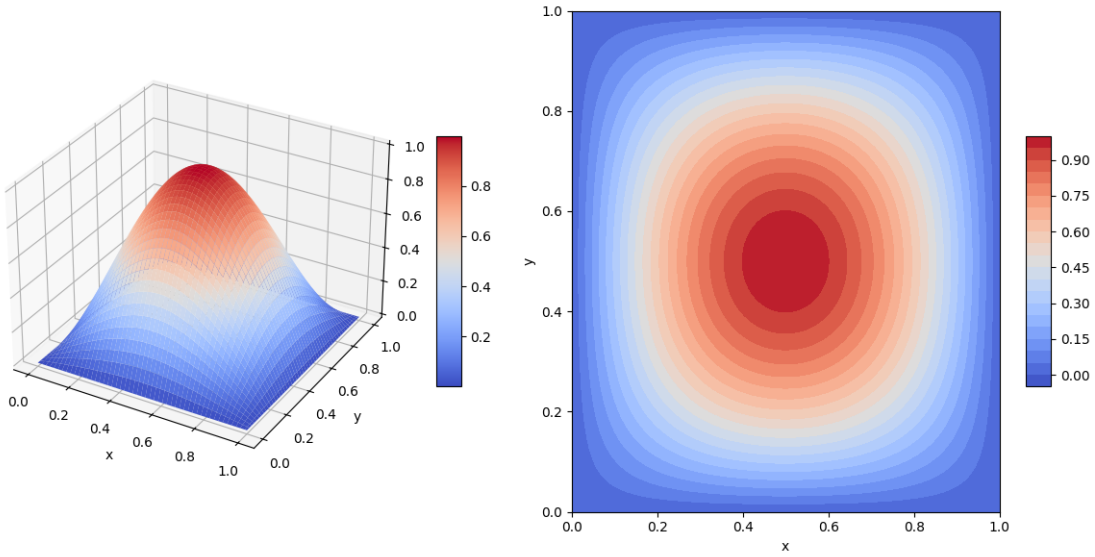

Figure 2: Solution obtained using PINN

[1]https://arxiv.org/abs/2402.01868