Convex Computation of the Reachable Set for Hybrid Systems with Parametric Uncertainty

Shankar Mohan and Ram Vasudevan

***Abstract*— To verify the correct operation of systems, engineers need to determine the set of configurations of a dynamical model that are able to safely reach a specified configuration under a control law. Unfortunately, constructing models for systems interacting in highly dynamic environments is difficult. This paper addresses this challenge by presenting a convex optimization method to efficiently compute the set of configurations of a polynomial hybrid dynamical system that are able to safely reach a user defined target set despite parametric uncertainty in the model. This class of models describes, for example, legged robots moving over uncertain terrains. The presented approach utilizes the notion of occupation measures to describe the evolution of trajectories of a nonlinear hybrid dynamical system with parametric uncertainty as a linear equation over measures whose supports coincide with the trajectories under investigation. This linear equation with user defined support constraints is approximated with vanishing conservatism using a hierarchy of semidefinite programs each of which is proven to compute an outer approximation to the set of initial conditions that can reach the user defined target set safely in spite of uncertainty. The efficacy of this method is illustrated on a pair of nonlinear systems with parametric uncertainty.**

I. Introduction

Computing the set of configurations that are able to safely reach a desired configuration is critical to ensuring the correct performance of a system in dynamic environments where deviations from planned behavior are to be expected. Many methods have been proposed to efficiently compute this set that is generally referred to as the *backwards reachable set* for deterministic systems. Unfortunately, the effect of intermittent contact with the world, especially in fluctuating environments, is demanding to model deterministically. A roboticist, for example, may be tasked with ensuring that a control for a legged robot beginning from an initial configuration is able to safely reach a desired goal; however, limitations in sensing or environment variability may render exact modeling of terrain height or friction impossible. The development of numerical tools to tractably compute the backwards reachable set of dynamical systems undergoing contact, or *hybrid dynamical systems*, with parametric uncertainty while providing systematic guarantees has been challenging due to the difficulty of efficiently accounting for the uncertainty.

Given its potential utility, many researchers have attempted to develop numerical tools to compute this *uncertain backwards reachable set*. Several researchers, for instance, have attempted to utilize this backwards reachable set while constructing controllers for legged robots that are able to walk over terrains of varying heights [1]–[4]. These approaches have relied upon discretizing the height of the terrain or selecting specific terrain profiles while constructing a safe controller across these specified heights, which verifies the performance of the controller only at those specific heights. Moreover, these approaches are unable to account for uncertainty associated with imperfect knowledge of terrain friction or parameters affecting the continuous dynamics.

Other researchers have developed tools to outer approximate the uncertain backwards reachable for linear systems with uncertain parameters using a variety of approaches [5], [6]. These methods can be extended to nonlinear hybrid systems, but can require the introduction of a large number of discrete states to represent the nonlinear behavior or require overly conservative estimates of potential uncertainty. More generally, Hamilton-Jacobi Bellman based approaches have also been applied to compute the uncertain backwards reachable set for nonlinear systems with arbitrary uncertainty affecting the state at any instance in time [7]. These approaches solve a more general problem, but rely on state space discretization which can be prohibitive for systems of dimension greater than four without relying upon specific system structure [8].

This paper leverages a method developed in several recent papers [9]–[11] that describe the evolution of trajectories of a deterministic hybrid dynamical system using measures to describe the evolution of a hybrid dynamical system with parametric uncertainty as a linear equation over measures. As a result of this characterization, the set of configurations that are able to reach a target set despite parametric uncertainty, called the *uncertain backwards reachable set*, can be computed as the solution to an infinite dimensional linear program over the space of nonnegative measures. To compute an approximate solution to this infinite dimensional linear program, a sequence of finite dimensional relaxations semidefinite programs are constructed that satisfy an important property: each solution to this sequence of semidefinite programs is an outer approximation to the uncertain backwards reachable set with asymptotically vanishing conservatism. The approach is most comparable to those that check Lyapunov's criteria for stability via sums-of-squares programming to verify the safety of a system [12]. In contrast to these approaches, the algorithm described in this paper does note require solving a bilinear optimization problem that requires feasible initialization and allows for more general descriptions of the parametric uncertainty in the model.

The remainder of the paper is organized as follows: Section II introduces the notation used in the remainder of

S. Mohan is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109 `elemsn@umich.edu`

R. Vasudevan is with the Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109 `ramv@umich.edu`

the paper, the class of systems under consideration, and the backwards reachable set problem under parametric uncertainty; Section III describes how the backwards reachable set under parametric uncertainty is the solution to an infinite dimensional linear program; Section IV constructs a sequence of finite dimensional semidefinite programs that outer approximate the infinite dimensional linear program with vanishing conservatism; Section V describes the performance of the approach on a pair of examples; and, Section VI concludes the paper.

II. Preliminaries

This section defines the notation, the class of systems, and problem considered throughout the paper.

A. Notations

In the remainder of this text the following notation is adopted: Sets are italicized and capitalized (ex. K). The disjoint union of sets is defined as: $\coprod_{i\in I} K_i = \cup_{i\in I} K_i \times \{i\}$. Finite truncations of the set of natural numbers are expressed as $\mathbb{N}_n := \{1,\ldots,n\}$. The set of continuous functions and absolutely-continuous functions supported on K are represented as $\mathcal{C}(K)$ and $\mathcal{C}_{ab}(K)$. The ring of polynomials in x is denoted by $\mathbb{R}[x]$, and the degree of a polynomial is equal to degree its largest multinomial; the degree of the multinomial x^α, $\alpha \in \mathbb{R}^n_{\geq 0}$ is $|\alpha| = \|\alpha\|_1$; and $\mathbb{R}_d[x]$ is the set of polynomials in x with maximum degree d. The dual to $\mathcal{C}(K)$ is the set of measures on K, denoted as $\mathcal{M}(K)$, and the pairing of $\mu \in \mathcal{M}(K)$ and $v \in \mathcal{C}(K)$ is:

$$\langle \mu, v\rangle = \int_K v(x)\, d\mu(x). \tag{1}$$

The space of probability measures on K is denoted by $\mathcal{P}(K)$. The Lebesgue measure is denoted by λ. Finally, supports of measures, μ, are identified as $supp(\mu)$.

B. System class description

The class of uncertain systems considered in this study consists of hybrid systems that conform to the following definition and undergo executions as described by Alg. 1.

Definition 1 (Inspired by [13]). A 'quasi-uncertain' hybrid system is a tuple $\mathcal{H} = (\mathcal{J}, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}, \Gamma)$, where

- $\mathcal{J}$ is a finite set of indices of discrete states in of $\mathcal{H}$;
- $\mathcal{E} \subset \mathcal{J} \times \mathcal{J}$ is a set of two-tuples describing directed edges;
- $\mathcal{D} := \coprod_{j\in\mathcal{J}} \mathrm{M}_j$ is a disjoint union of domains;
- $\mu_{\theta_j} \in \mathcal{P}(\Theta_j)$ with Θ_j (compact) being the manifold from which the uncertainty associated with state j takes values, $\Gamma := \coprod_{j\in\mathcal{J}} \mu_{\theta_j}$ is the disjoint union of probability distributions,
- $\mathcal{F} := \{\tilde{f}_j\}_{j\in\mathcal{J}}$ where $\tilde{f}_j \in (\mathcal{C}(\mathrm{M}_j \times \Theta_j; \mathbb{R}))^{n_{x_j}}$ is a tangent vector to M_j at (x,θ),
- $\mathcal{G} := \coprod_{p\in\mathcal{E}} \mathcal{G}_p$ is the disjoint union of guards; $\mathcal{G}_{(i,j)} := \{(x,\theta) \in \mathrm{M}_j \times \Theta_j \mid \text{algebraic constraints}\}$,
- $\mathcal{R}$ is the set of reset maps associated with each edge in $\mathcal{E}$; $R_{(i,j)} \colon \mathcal{T} \times \pi_x \mathcal{G}_{(i,j)} \to \mathcal{T} \times \mathrm{M}_j$ is a continuously differentiable injection; $R_{(i,j)} \in \mathcal{C}(\mathcal{T} \times \mathrm{M}_j)$ and denotes the transformation accompanying state transition.

A further qualification of the systems under consideration is warranted. It is assumed that upon reaching a guard, there is no ambiguity in into which discrete state the system transitions; this can be achieved by enforcing the next assumption.

Assumption 1. In each discrete state, the guards are mutually exclusive; i.e.

$$\mathcal{G}_{(i,j)} \cap \mathcal{G}_{(i,k)} = \emptyset, \;\; \forall (i,j),(i,k) \in \mathcal{E}, \forall j \neq k \tag{2}$$

In line with standard definition in literature related to switched systems, the discrete states are alternatively referred to as *modes* of the system. In addition, the systems considered are not allowed to undergo infinite mode transitions in a finite time-interval.

Assumption 2. $\mathcal{H}$ has no zeno execution.

To complete the characterization of systems in $\mathfrak{U}$, a description of how the components in Defn. 1 are related is warranted. Algorithm 1 describes the finite-time execution ($t \in [0,T]$) of a hybrid system $\mathcal{H}$ as defined by Defn. 1 and whose states are denoted by x. The sequence of steps undertaken as the states evolve in accordance with Alg. 1 is briefly elaborated below.

Suppose, without loss of generality (wlog.), that the system enters mode j at time t. As a reminder, the dynamics of this system, $\tilde{f}_j$, is a function of a random parameter drawn from the distribution μ_{θ_j}; let this random variable take the value θ. Consider a (non-hybrid) system, Σ, with states denoted by γ whose dynamics is identical to that of x in mode j, $\tilde{f}_j$; and let γ have identical initial conditions as x in mode j. The trajectory of the states of Σ is given by an absolutely continuous function that is the solution to the ODE in Steps 5&6. If $\gamma(s)$, $s \in [t,T]$, does not satisfies any of the constraints that define the guards of mode j of $\mathcal{H}$, then the trajectory of x remains in mode j and is identical to that of γ, and the execution is terminated; otherwise, $\mathcal{H}$ undergoes a mode transition. Steps 7–11 isolates the first hitting-time of a guard of mode j and resets $\mathcal{H}$ to a new mode whereafter the same procedure is repeated until $t = T$.

Of key note in the system execution is the fact that the uncertainty does not evolve with time; changes to the value that the uncertainty takes is triggered with system mode resets. In spite of this peculiar requirement, $\mathfrak{U}$ is quite rich and includes many physical systems; to better elucidate the properties of systems in this class, two representative examples—a simple 1D pedagogical example, and a 2D representative of walking models—are presented hereafter.

Example 1 (1-D linear dynamics). One of the simplest linear examples in $\mathfrak{U}$ has dynamics described by

$$\dot{x} = -0.7x + 0.2\theta - 0.1, \tag{3}$$

where $x(t) \in [-1,1]$ and $\theta \in [-1,1]$ is an uncertain, unknown parameter; the uncertain parameter can be thought of as having arisen due to structural modeling errors or as a result of reducing a singular-perturbed system. Note that this

Algorithm 1: Execution of $\mathcal{H}$

Initialization: $t = 0$, $j \in \mathcal{J}$, $(x_0, j) \in \mathcal{D}$, $x(0) = x_0$;
while *1* **do**
  Let θ be drawn according to μ_{θ_j};
  Let $\gamma\colon [t, T] \to \mathrm{M}_j$, abs. ct. st.
    $\dot{\gamma}(s) = \tilde{f}(\gamma(s), \theta)$ λ_t^{a}-a.e., $s \in [t, T]$
    $\gamma(t) = x(t)$;
  $\Lambda_{(j,t)} := \{r \in [t, T] | \exists (j, k) \in \mathcal{E}$ st. $(\gamma(r), \theta) \in \mathcal{G}_{(j,k)}\}$;
  if $\Lambda_{(j,t)} \neq \emptyset$ **then**
    $t' := \min \Lambda_{(j,t)}$, k st. $\gamma(t') \in \pi_x \mathcal{G}_{(j,k)}$
      $x(s) \leftarrow \gamma(s)$, $\forall s \in [t, t')$
      $t \leftarrow t'$, $x(t') \leftarrow R_{(j,k)}(\gamma(t'))$, $j \leftarrow k$
  else
    $x(s) = \gamma(s)$, $\forall s \in [t, T]$;
    Stop;
  end
end
[a]where λ_t is the Lebesgue measure on $[t, T]$

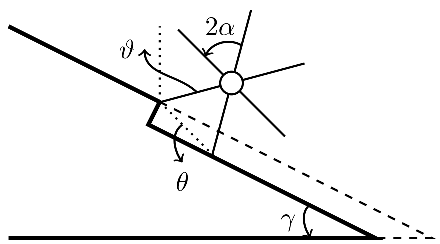

Fig. 1. Schematic of the rimless wheel with θ being the disturbance.

system is not a hybrid system; however, it can be hybridized. One way to achieve this is by setting $n_m = 1$ and using placing a guard $\mathcal{G}_{(1,1)}$ at $x(t) = -1$ with a corresponding reset map $x \mapsto -x$.

Example 2 (Planar rimless wheel (PRW))**.** The planar rimless wheel—constituted by a massless axle to which n (angularly) equidistant spokes are connected—is one of the simplest models of legged locomotion. Figure 1 presents a schematic of a rimless wheel—with spokes separated by an angle 2α—rolling down an infinite wedge. The PRW is, by definition, a hybrid system consisting of one mode; every-time the spoke make contact with the surface of wedge, the system undergoes a rest as the pivoting leg, and the origin of the local generalized coordinates changes. Between resets, the dynamics of the PRW is described by

$$\begin{bmatrix}\dot{\vartheta} & \ddot{\vartheta}\end{bmatrix}' = \begin{bmatrix}\dot{\vartheta} & \sin(\vartheta)\end{bmatrix}', \tag{4}$$

where ϑ is the angle between the pivoted spoke and the vertical located at the stationary leg. Once the marching spoke makes contact with the terrain, the states are reset using the reset map

$$R_{(1,1)}(t, \vartheta^-, \dot{\vartheta}^-) = \begin{bmatrix}t & 2\gamma - \vartheta^- & \cos(2\alpha)\,\dot{\vartheta}^-\end{bmatrix}'. \tag{5}$$

For a PRW rolling down a flat (constant slope) wedge, at the instance when the marching spoke makes contact with the wedge, and the system undergoes a reset, the states of the system satisfy the following condition

$$\vartheta = \gamma + \alpha. \tag{6}$$

For a PRW rolling down a wedge with an uneven ramp with the relative slope between the pivoted leg and the contact point of the marching leg, θ, the guard, $\mathcal{G}_{(1,1)}$ is defined as follows

$$\mathcal{G}_{(1,1)} = \{x \mid x = \gamma + \alpha + \theta\}. \tag{7}$$

Observe that as the PRW continues to roll, the undulations in the surface can change and hence the random variable, θ, will likely take different values as the system resets.

C. Problem description

The objective of this work is to estimate the *largest set of initial conditions* from which all state trajectories of a hybrid quasi-uncertain system reach the terminal set X_T in a pre-specified time, T.

Depending on $\mathcal{E}$, there may be more than one trunk through which state trajectories can reach the terminal set. Consequently, the problem can be re-stated, with specificity, as wanting to find the largest set of initial conditions in each mode, $X_{(0,j)}$, $\forall j \in \mathcal{J}$, that can reach X_T where $X_{(0,j)}$ is given by

$$\begin{aligned} X_{(0,j)} := \{x_0 \in \mathrm{M}_j \mid \forall x = \mathcal{H}(x_0, [0, T]) \\ x(T) \in X_T\} \end{aligned} \tag{8}$$

Observe that, by definition, for systems from $\mathfrak{U}$, if $X_{(0,j)}$ is the largest non-empty BRS in mode j, then all initial conditions from $X_{(0,j)}$, must reach X_T at time T, regardless of the number of mode transitions that may occur in the interim and for every possible concomitant sequence of parametric uncertainty.

For convenience, hereafter the times at which the system's state is relevant is denoted by the set $\mathcal{T} := [0, T]$, and the projections of X_T onto every mode, j, is denoted by $X_{(T,j)}$.

III. Problem formulation

The critical idea of the ensuing presentation—related to the definition of members of $\mathfrak{U}$—is the following: the uncertainty takes a constant value in each mode, although its value is drawn from a distribution; so, technically, the uncertainty is an unknown parameter of the dynamics which can be added to and used to extend the state-space. That is, the dynamics in each mode j can be represented as

$$f_j = \begin{bmatrix}\tilde{f}_j' & \mathbf{0}_{n_{\theta_j}}'\end{bmatrix}'. \tag{9}$$

The object of interest is a set from which trajectories (piece-wise absolutely continuous functions) that emanate, and are governed by the dynamics of the system, reach another pre-specified set. Given the problem structure, one might be better served to formulate the problem as one based in an appropriate functional space; and use measures defined on the sets of interest as surrogates, and hence variables to be determined.

Since the free variables in the ensuing problem formulation are measures on sets associated with a dynamical system, it is helpful to use occupation measures as a template. The occupation measure, first introduced in [14], is to be interpreted as measuring the time the solution trajectories spend in a particular region of the state-space. For instance, suppose the system enters mode j at τ_k with states taking initial values $x(\tau_k) = x_0$ and $\theta(\tau_k) = \theta$, the occupation measure, $\mu_j(\cdot \mid \tau_k, x_0, \theta) \in \mathcal{M}(\mathcal{T} \times \mathrm{M}_j \times \Theta_j)$, is defined as

$$\mu_j(A \times B \times C|\tau_k, x_0, \theta) = \int_0^T I_{A\times B\times C}(t, x(t|\tau_k, x_0, \theta), \theta)dt. \tag{10}$$

From the above, the follow relation between the Lesbegue measure on $\mathcal{T}$ and $\mu_j(\cdot \mid \tau_k, x_0, \theta)$ holds by definition.

$$\langle \mu_j(\cdot \mid \tau_k, x_0, \theta), v\rangle = \langle \lambda_t, v(t, x(t \mid \tau_k, x_0, \theta), \theta)\rangle\,, \tag{11}$$

Note that in its definition, the occupational measure is a conditional measure – conditioned on the arrival-time and initial values of the states in that mode. When considering a set of possible arrival-times and initial conditions, the *average occupation measure* is defined by *averaging* the occupation measure wrt. to a measure on the set of possible initial conditions of the mode ($\breve{\mu}_{0_j}$); i.e.

$$\mu_j(A \times B \times C) = \int\limits_{\mathcal{T}\times \mathrm{M}_j\times\Theta} \mu_j(A \times B \times C \mid \tau_k, x_0, \theta)\, d\breve{\mu}_{0_j}. \tag{12}$$

Observe that by definition, the uncertain variables are independent of the states' initial conditions; hence $\breve{\mu}_{0_i} \in \mathcal{M}(\mathcal{T} \times \mathrm{M}_j \times \Theta)$ is expressible as a product measure:

$$\mu_{s_j} = \bar{\mu}_{0_j} \otimes \mu_{\theta_j}, \tag{13}$$

where $\bar{\mu}_{0_j} \in \mathcal{M}(\mathcal{T}\times\mathrm{M}_j)$ is the measure on the set of initial conditions, and $\mu_{\theta_j} \in \mathcal{M}(\Theta)$ is provided by in the definition of $\mathcal{H}$.

Similarly, measures on terminals sets $X_{(T,j)}$, $\mu_{T_j} \in \mathcal{M}(X_{(T,j)} \times \Theta)$

$$\mu_{T_j}(A \times B) = \int\limits_{\mathcal{T}\times X_{(T,j)}\times\Theta} I_{A\times B}(x(T \mid \tau_k, x_0, \theta), \theta)\, d\breve{\mu}_{0_j}, \tag{14}$$

and guards, $\mu_{\mathcal{G}_{(j,k)}} \in \mathcal{M}(\mathcal{T} \times \mathcal{G}_{(j,k)}), \forall (j, k) \in \mathcal{E}$

$$\mu_{\mathcal{G}_{(j,k)}}(A \times B{\times}C) = \int\limits_{\mathcal{T}\times\mathcal{G}_{(j,k)}} I_{A\times B\times C}(t, x(t \mid \tau_k, x_0, \theta), \theta)\, d\breve{\mu}_{0_j}, \tag{15}$$

are defined. While measure μ_{T_j}—supported on the terminal set at the final time—has an obvious interpretation, measures $\mu_{\mathcal{G}_{(j,k)}}$, $\forall (j, k) \in \mathcal{E}$ are supported on the guards of mode j and should be interpreted as the hitting times of the guard. For convenience, the *final measure* for each mode j is defined as

$$\mu_{f_j} = \delta_T \otimes \mu_{T_j} + \sum_{k\in\{l|(j,l)\in\mathcal{E}\}} \mu_{\mathcal{G}_{(j,k)}}. \tag{16}$$

Given a set of initial conditions X_0, the dynamics of the system—under appropriate assumptions—defines a bundle of trajectories of the system states. It is of interest to ensure that this bundle terminates in the desired set X_T, making X_0 a subset of the BRS; stated differently, it is necessary to relate $\coprod_{j\in\mathcal{J}} \mu_{s_j}$ with $\coprod_{j\in\mathcal{J}} \mu_{f_i}$ and the dynamics of the system. As a first step towards deducing said relation, linear operators $\mathcal{L}_{f_j} : \mathcal{C}^1(\mathcal{T} \times \mathrm{M}_j \times \Theta_j) \to \mathcal{C}(\mathcal{T} \times \mathrm{M}_j \times \Theta_j)$ satisfying Eqn. (17) in which $v \in \mathcal{C}^1(\mathcal{T} \times \mathrm{M}_j \times \Theta_j; \mathbb{R})$ is an arbitrary test function, are defined.

$$\mathcal{L}_{f_j} v = \frac{\partial v}{\partial t} + \left\langle \nabla_x v, \tilde{f}_j \right\rangle \tag{17}$$

Suppose the system transitioned to mode j at $t = \tau_{k-1}$ with the states taking initial values $x(\tau_{k-1})$ and θ; the value of v, evaluated along the flow of the system states and at $t = \tau_k$ is computed using the fundamental theorem of calculus according to Eqn. (18).

$$\begin{aligned} v\left(\tau_k, x\left(\tau_k \mid x(\tau_{k-1}), \theta_{k-1}\right)\right) &= v(\tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}) \\ &+ \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_f v(t, x(t \mid \tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}))\, dt. \end{aligned} \tag{18}$$

Using Eqn. (11), Eqn. (18) can be re-written as

$$\begin{aligned} v\left(\tau_k, x\left(\tau_k \mid \tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}\right)\right) &= v(\tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}) \\ &+ \langle \mu_j(\cdot \mid \tau_{k-1}, x(\tau_{k-1}, \theta_{k-1}), \mathcal{L}_f v\rangle \end{aligned} \tag{19}$$

which can be simplified by *averaging* wrt. to the set of initial conditions $x(\tau_{k-1})$ and θ using Eqns. (12)–(16) to arrive at

$$\left\langle \mu_{f_j}, v\right\rangle = \left\langle \mu_{s_j}, v\right\rangle + \langle \mu_j, \mathcal{L}_f v\rangle\,. \tag{20}$$

Alternatively, using the standard definition of adjoint operators[1], Eqn. (20) is re-written as

$$\left\langle \mu_{f_j}, v\right\rangle = \left\langle \mu_{s_j}, v\right\rangle + \left\langle \mathcal{L}'_f \mu_j, v\right\rangle \tag{21}$$

Eqn (21) is the desired equation that relates the dynamics of the state to the initial and final measures in each mode of the system.

In the execution of system $\mathcal{H}$, each mode can be entered in two ways – at $t = 0$; and because of a reset map, at any time $t \in \mathcal{T}\backslash\{0, T\}$; hence the initial measure in the (t, x)-projection can be decomposed as

$$\bar{\mu}_{0_j} = \delta_0 \otimes \mu_{0_j} + \pi_{t,x}\sigma_{0_j} \tag{22}$$

with $\mu_{0_j} \in \mathcal{M}(\mathrm{M}_j)$ is the measure supported on the initial conditions to the system at $t = 0$, and $\sigma_{0_j} \in \mathcal{M}(\mathcal{T} \times \mathrm{M}_j \times \Theta_j)$ is the measure on initial conditions after the first reset. State resets occur when the states reach the guard and, unless the solution terminates at the guard, for every solution terminating in the support of $\mu_{\mathcal{G}_{(i,j)}}$, $\forall (i, j) \in \mathcal{E}$, there must exist a trajectory originating in the support of σ_{0_j}, $\forall j \in \mathcal{J}$; that is, $\mu_{\mathcal{G}_{(i,j)}}, \forall (i, j) \in \mathcal{E}$, and $\sigma_{0_j}, \forall j \in \mathcal{J}$, are related.

[1]A linear operator $\mathcal{L}$ and its adjoint, $\mathcal{L}'$, satisfy the following relation:

$$\left\langle \mathcal{L}'\mu, v\right\rangle = \langle \mu, \mathcal{L}v\rangle = \int\limits_{\mathcal{X}} \mathcal{L}v\, d\mu.$$

To see this relation, σ_{0_j} is first decomposed into measures corresponding to the source of each arrival state; i.e.

$$\sigma_{0_j} = \sum_{i \in \{k | (k,j) \in \mathcal{E}\}} \sigma_{(i,j)} \otimes \mu_{\theta_j}, \tag{23}$$

where $\sigma_{(i,j)}$ is the measure on initial conditions post reset for all trajectories arriving at mode j from guard $\mathcal{G}_{(i,j)}$ of mode i. Upon reaching the guard, the system transitions according to the reset map; in essence, viewing $R_{(j,k)}$ as a nonlinear transformation of the state-space, the relation in Eqn. (24) between $\sigma_{(i,j)}$ and $\mu_{\mathcal{G}_{(i,j)}}$ is established.

$$\left\langle \sigma_{(i,j)}, w \right\rangle = \left\langle \pi_{t,x} \mu_{\mathcal{G}_{(i,j)}}, w \circ R_{(i,j)} \right\rangle \tag{24}$$

where $w \in \mathcal{C}(\mathcal{T} \times \mathcal{X}_j)$ and

$$\left\langle \pi_{t,x} \mu_{\mathcal{G}_{(i,j)}}, s \right\rangle = \left\langle \mu_{\mathcal{G}_{(i,j)}}, s \right\rangle, \forall s \in \mathcal{C}(\mathcal{T} \times \mathrm{M}_i);$$

essentially, $\sigma_{(i,j)}$ is a push-forward measure of $\mu_{\mathcal{G}_{(i,j)}}$.

A. The primal

With the constraints expressed in terms of measures, the problem of approximating the BRS is formulated as an infinite-dimensional Linear Program that supremizes the *volume* of the set of initial condition.

$$\sup_{\Lambda} \quad \sum_{j=1}^{n_m} \left\langle \mu_{0_j}, 1 \right\rangle \tag{P}$$

st.

$$\mu_{s_j} + \mathcal{L}'_f \mu_j = \mu_{f_j} \qquad \forall j \in \mathbb{N}_{n_m} \tag{25}$$

$$\mu_{0_j} + \hat{\mu}_{0,j} = \lambda_j \qquad \forall j \in \mathbb{N}_{n_m} \tag{26}$$

$$\sum_{j=1}^{n_m} \left\langle \mu_{T_j}, 1 \right\rangle = \sum_{j=1}^{n_m} \left\langle \mu_{0_j}, 1 \right\rangle \tag{27}$$

where λ_j is the Lebesgue measure supported on M_j.

$$\Lambda = \{\mu_j, \mu_{0_j}, \mu_{T_j}, \hat{\mu}_{0_j}, \mu_{\mathcal{G}_{(j,k)}} \geq 0, \forall j \in \mathbb{N}_{n_m}, (j,k) \in \mathcal{E}\}.$$

Variables $\hat{\mu}_{0_j} \in \mathcal{M}(\mathrm{M}_j)$ are slack variables introduced to enforce a stronger constraint than absolute continuity of μ_{0_j} wrt. to λ_j

$$\mu_{0_j}(A) \leq \lambda_j(A) \qquad \forall A \subset \mathrm{M}_j \tag{28}$$

The constraint in Eqn. (27) ensures that all trajectories that emanate $\cup_{j \in \mathcal{J}} spt(\mu_{0_j})$ reach X_T at $t = T$, and is not *stuck* at any of the guards.

Lemma 1. *If $\mu_{0_j}, \forall j \in \mathcal{J}$ is part of the optimal solution of (P) then $\bigcup_{j \in \mathcal{J}} spt(\mu_{0_j})$ is the BRS of the system. In addition, the optimal value of (P) is equal to $\sum_{j \in \mathcal{J}} \lambda_j(X_{0_j})$, the sum of* volumes *of the BRSs in each mode.*

Proof. Suppose $\sum_{j \in \mathcal{J}} \lambda_j(spt(\mu_{0_j}) \backslash X_{(0,j)}) > 0$, then by Lemma 5, there exist trajectories that begin in $\cup_{j \in \mathcal{J}}(spt(\mu_{0_j}) \backslash X_{(0,j)})$ that reach X_T; this is a contradiction. Thus,

$$\bigcup_{j \in \mathcal{J}} spt(\mu_{0_j}) \subset \bigcup_{j \in \mathcal{J}} X_{(0,j)}, \tag{29}$$

$$\sum_{j \in \mathcal{J}} \lambda_j(spt(\mu_{0_j})) \leq \sum_{j \in \mathcal{J}} \lambda_j(X_{(0,j)}). \tag{30}$$

By definition of the BRS, all state trajectories that emanate from a subset of X_0 end in X_T. That is, for each $j \in \mathcal{J}$ and initial measure μ_{0_j}, if $spt(\mu_{0_j}) \subset X_{(0,j)}$, there exist measures μ_j and $\mu_{f,j}$ that satisfy Eqn. (25). Thus the following inequality is true.

$$\sum_{j \in \mathcal{J}} \lambda_j(spt(\mu_{0_j})) \geq \sum_{j \in \mathcal{J}} \lambda_j(X_{(0,j)}) \tag{31}$$

From Eqns. (30)&(31), $\cup_{j \in \mathcal{J}} spt(\mu_{0_j})$ is the BRS of the system.

That the optimal value of (P) is volume of the BRS follows from Eqn. (28) and the observation that $\lambda_j|_{X_{(0,j)}}, \forall j \in \mathcal{J}$ is feasible in (P). ☐

B. The dual

The dual corresponding to (P) is derived using standard techniques and is presented below.

$$\inf \sum_{j \in \mathbb{N}_{n_m}} \langle \lambda_j, w_j \rangle \tag{D}$$

st.

$$w_j \geq 0 \qquad \forall (x,j) \in \mathcal{D} \tag{32}$$

$$v_j(T, \cdot) + q \geq 0, \qquad \forall (x,j,\theta) \in \mathcal{X}_T \times \Theta \tag{33}$$

$$-\mathcal{L}_{\tilde{f}} v_j \geq 0, \qquad \forall (t,x,j,\theta) \in \mathcal{T} \times \mathcal{D} \times \Theta \tag{34}$$

$$w_j - \left\langle \mu_{\theta_j}, v(0, \cdot) \right\rangle - q \geq 1, \qquad \forall (x,j) \in \mathcal{D} \tag{35}$$

$$v_j \geq \langle \mu_{\theta_k}, v_k \rangle \circ R_{(j,k)}, \qquad \forall (t,x,\theta,(j,k)), \in \mathcal{T} \times \mathcal{G} \times \mathcal{E} \tag{36}$$

where $q \in \mathbb{R}$, $v_j \in C^1(\mathcal{T} \times \mathrm{M}_j \times \Theta_j)$ and $w_j \in C(\mathrm{M}_j)$.

Lemma 2. *If (w, v, q) is the solution to (D), then the super-level set*

$$\bigcup_{j \in \mathcal{J}} \{x \mid w_j(x) \geq 1\} \tag{37}$$

is an outer approximation of the BRS of the system whose dynamics is described by Alg. 1.

Proof. The approach we adopt to prove this lemma is to construct the projection of the BRS on any mode and show that it is a 1-level set of the appropriate function. To assist in constructing the arguments, assume wlog., that the state trajectory terminates in $X_{(T,j_k)}$ for some j_k. The state trajectory must have arrived in mode j_k through a finite sequence of mode-transitions (according to Assumption 2); wlog., let this sequences of mode-transitions be of length k. Suppose the states entered mode j_k at time τ_k, then, from the fundamental theorem of calculus and the constraints in Eqns. (33)&(34), the following inequalities follow.

$$-q \leq v_{j_k}(T, x(T \mid x(\tau_k^+), \theta), \theta) \leq v_j(\tau_k, x(\tau_k^+), \theta) \tag{38}$$

$$\Rightarrow -q \leq \left\langle \mu_{\theta_{j_k}}, v_{j_k}(\tau_k, x(\tau_k^+), \theta) \right\rangle \tag{39}$$

By iterative application of the constraint in Eqn. (36) and finally Eqn. (35), it follows that

$$
\begin{aligned}
-q &\leq \left\langle \mu_{\theta_{j_k}}, v_{j_k}(t,x,\theta) \right\rangle \circ R_{(j_{k-1},j_k)}(\tau_k, x(\tau_k^-)) && (40)\\
&\leq v_{j_{k-1}}(\tau_k, x(\tau_k^- \mid x(\tau_{k-1}^+),\theta),\theta) && (41)\\
&\leq \left\langle \mu_{\theta_{j_{k-1}}}, v_{j_k}(\tau_k, x(\tau_{k-1}^+),\theta) \right\rangle && (42)\\
&\ \ \vdots \\
&\leq v_{j_0}(\tau_1, x(\tau_1^- \mid x_0,\theta),\theta) && (43)\\
&\leq v_{j_0}(0,x_0,\theta) && (44)\\
&\leq \left\langle \mu_{\theta_{j_0}}, v_{j_0}(0,x_0,\theta) \right\rangle && (45)\\
&\leq w_{j_0}(x_0) - q - 1. && (46)
\end{aligned}
$$

The final inequality implies that for every trajectory that ends in $X_{(t,j_k)}$, $x(t) = x_0 \in \mathrm{M}_{j_0}$ satisfies the condition $w_{x_0} \geq 1$. Thus the set of initial conditions that begin in mode j_0 and reach the terminal set projected into mode j_k is given by the super-level set

$$I_{(j_0,j_k)} = \{x \mid w_{j_0} \geq 1\}. \quad (47)$$

Note that the definition of I_{j_0,j_k} does not depend on the mode in which the terminal set is reached; thus $I_{(j_0,\mathcal{J})} = X_{(0,j_0)}$. Finally, by observing that j_0 is an arbitrary element of $\mathcal{J}$, we deduce the stated result. □

Lemma 3. *There is not gap between (P) and (D).*

Proof. The proof follows from [15, Theorem 3.10], and is similar to [9, Theorem 2]; it is not presented for brevity. □

Remark 1. There are two key aspects of the presentation in this section that deserve re-iteration: (1) by definition, the uncertainties that influence the dynamics can be visualized as a discrete random process with updates to the instantiation of the uncertainty occurring upon entering a new mode; (2) the estimated BRS is the set of initial conditions from which *all* trajectories that emanate reach the terminal set for *all* possible discrete sequence of uncertainty. As a direct implication of the second point, the solution of the problem is the intersection of the BRS of every possible sequence of uncertainty.

IV. Numerical Implementation

The infinite-dimensional problems described in Secs. III-A and III-B are hard to implement and solve directly. In this section, a sequence of *relaxed* SDPs—that contains a sub-sequence whose optimal values converges to the optimal value of the problems introduced in Secs. III-A and III-B—is introduced.

The fundamental idea behind this sequence of relaxations is that measures supported on a compact can be characterized by their moments[2]. Similar to Taylor approximations of functions, longer sequence of moments (higher the order of moments considered) provide a finer approximation of the measure.

[2]The nth moment of a measure (μ) is obtained by evaluating the following expression

$$y_{\mu,n} = \langle \mu, x^n \rangle .$$

By this definition, the mean of a probability distribution (read probability measure) is y_μ^1 and its variance is $y_\mu^2 - (y_\mu^1)^2$.

Since polynomials are dense in set of continuous functions, we restrict our focus on members of $\mathfrak{U}$ that have a polynomial vector-field in each mode and have domains defined by semi-algebraic sets. For such systems, given any finite d-degree truncation of the moment sequence of all measures in the primal P, a new problem, P_d, can be formulated over the moments of measures. This new problem is a Semi-definite Program (SDP). The dual to P_d, D_d, can be expressed as a sub-of-squares program (SOS program) by considering d-degree polynomials in place of the continuous variables in D.

For each semi-algebraic set A, let (h_{a_i}) be a collection of polynomials that define the set. For every semi-algebraic set A, let its d-degree *quadratic module* be defined as follows

$$Q_d(A) = \left\{ q \in \mathbb{R}_d[x] \,\middle|\, \exists s_k \in \mathbb{R}_{\leq d}[x],\ \text{SOS}\ , k \in \mathbb{N}_{n_a \cup \{0\}},\ q = s_0 + \sum_{j \in \mathbb{N}_{n_a}} h_{a_j} s_j \right\} \quad (48)$$

Using the above notation, the d-degree relaxation of the dual, D_d, is presented below.

$$\inf_{\Xi_d} \sum_{j \in \mathcal{J}} \int_{\mathrm{M}_j} w_j \, d\lambda_j \qquad (D_d)$$

st.

$$
\begin{aligned}
& w_j^d \in Q_d(X_{(T,j)}) && \forall j \in \mathcal{J} \quad (49)\\
& v_j^d(T,\cdot) + p \in Q_d(\mathrm{M}_j) && \forall j \in \mathcal{J} \quad (50)\\
& -\mathcal{L}_{f_j} v_j^d \in Q_d(\mathcal{T} \cup \mathrm{M}_j) && \forall j \in \mathcal{J} \quad (51)\\
& w_j^d - \left\langle \mu_{\theta_j}, v_j^d(0,\cdot) \right\rangle - p - 1 \in Q_d(\mathrm{M}_j) && \forall j \in \mathcal{J} \quad (52)\\
& v_j^d - \left\langle \mu_{\theta_k}, v_k^d \right\rangle \circ R_{(j,k)} \in Q_d(\mathcal{T}, \mathrm{M}_j) && \forall (j,k) \in \Upsilon \quad (53)
\end{aligned}
$$

where $\Xi_d = \{v_j^d, w_j^d, p\} \in (\mathbb{R}_d[t,x,\theta])^{n_m} \times (\mathbb{R}_d[x])^{n_m} \times \mathbb{R}$, $\Upsilon = \{(a,b) \mid a \in \mathcal{J},\ (a,b) \in \mathcal{E}\}$ and the other variables are from the given hybrid system $\mathcal{H}$.

Lemma 4. *The sequence* $(\cup_{j\in\mathcal{J}}\{x \mid w_j^d \geq 1\})_d$ *has a convergent sub-sequence of outer approximations of the BRS.*

Proof. The proof to this lemma is a compilation of proofs to Thms. 5–7 in [11]; it is not reproduced for brevity. □

V. Examples

In this section, the efficacy of the proposed method is evaluated through the two examples introduced in Sec.II. The relaxed problems were parsed using the SPOTLESS toolbox [16] and were numerically solved with MOSEK on a computer equipped with a Intel Xeon W3540 processor and 12GB of RAM. The following points on the examples considered are obligatory.

1) It is a characteristic trait of the problem formulation considered in this paper that the actual distribution of the uncertainty is immaterial. Consequently, in all examples, it is assumed that the disturbance, θ, is uniformly distributed. For notional convenience, $\theta \sim$

$\mathcal{U}([a,b])$ is used to denote that θ is uniformly distributed in the interval $[a,b]$.

2) For reasons related to numerics, all problems are normalized such that the state-space is given by $[-1,1]^n$, for an appropriate value of n.

Also, since has been established that the solution of relaxed problems provides an outer approximation of the BRS, in this section, the qualifier 'approximate' is suppressed.

A. 1-D linear dynamics

As a review, we consider a (non-hybrid) 1-D linear dynamical system whose dynamics is given by

$$\dot{x}_1 = -0.7x_1 + 0.2\theta - 0.1, \tag{54}$$

where $\theta \in \mathcal{U}([0.2,1])$. Although this system is hybridizable as discussed in Ex. 1, in the version considered here, we do not hybridize its dynamics. The target set to which trajectories must reach in one second is set as $X_T = [0.2, 0.4]$. The BRS of the deterministic system which assumes that θ is a known constant is analytically computed to equal

$$BRS_\theta = \left[\left(0.2 - \frac{2\theta-1}{7}\right)e^{0.7} - \frac{2\theta-1}{7}, \left(0.4 - \frac{2\theta-1}{7}\right)e^{0.7} - \frac{2\theta-1}{7}\right]$$

Note that the expression for the BRS_θ is linear in θ and that the width of BRS_θ is constant for all values of θ. As the value of θ changes, BRS_θ slides along $\mathbb{R}$; the intersection of BRS_0 and BRS_1 is the BRS of the uncertain system, as defined in Eqn. (8).

Figure 2 presents the degree 12 approximation of the indicator function that is supported on the BRS of the uncertain system, w^{12}, for cases when θ takes a constant value and when it is drawn at random. The graphs in green and cyan correspond to the cases when θ takes a constant value of $\theta = 0$ and $\theta = 1$ respectively. The red trace is the polynomial solution to the uncertain problem. Observe that the BRS corresponding to uncertain case encloses the intersection of those of the deterministic of cases; this is the desired outcome.

B. Planar rimless wheel (PRW)

The rolling PRW, introduced in Ex.2, is a one mode hybrid system in which the guard is reached when the marching foot makes contact with the wedge. For a PRW rolling along a *smooth* wedge, an analytically computable stable limit cycle exits [17]; however, for the case considered in this example—with the wedge face having undulations—the definition of a limit cycle less clear. Consequently, a notion of *meta-stability*—when the system states arrive within ϵ of the stable limit cycle of the disturbance-free system—is adopted.

Figure 3 presents the polynomial degree 12 BRS (black dashed) for the rimless wheel (with $\alpha = 0.4$) which is tasked with arriving within the red band in $T = 4$ seconds, as it is rolling down a wedge with slope $\gamma = 0.2$ withstanding an a sequence of random changes to terrain drawn from $\theta \sim \mathcal{U}([-0.1, 0.1])$. The relative depths/height of the disturbance is about 25% the length of each spoke.

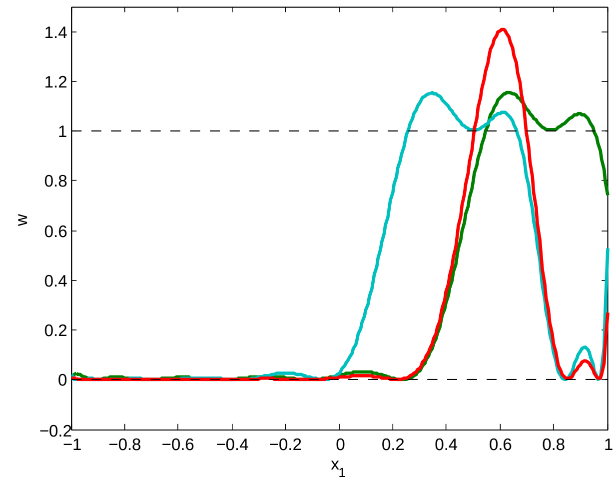

Fig. 2. Outer approximations of the BRS of the extreme deterministic cases and the stochastic case.(green) $\theta = 0$, (blue) $\theta = 1$ and (red) μ_θ

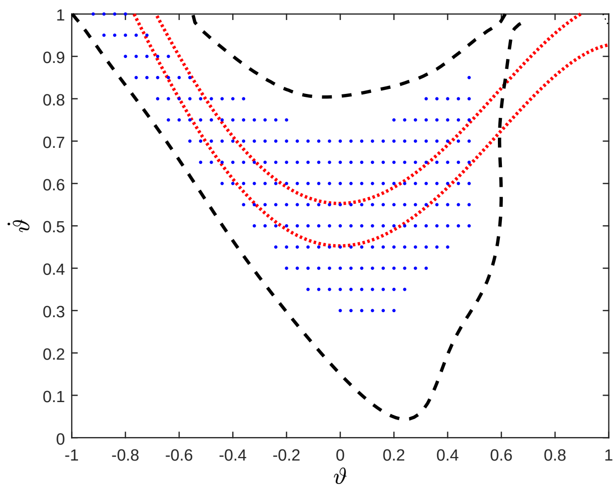

Fig. 3. Outer approximation and estimated BRS based on 100 iterations and T=4. Red band is the terminal set and the black outer is the boundary of the estimated BRS; the crosses correspond to results of MC simulation.

The BRS is validated by performing Monte Carlo simulations; the box I^2 is discretized into 51 points both ways and 100 independent trajectories are simulated (using MATLAB's *ode45* function) from each initial condition. The blue $\cdot$s depict the initial conditions that arrived within the terminal set at the desired time without violating any of the other constraints. Note that the set of points that succeeded in the MC simulation is entirely contained in the BRS.

At this juncture, a remark about the tightness of the BRS is warranted. Clearly, the BRS in Fig. 3 is not tight; and we attribute this to the set of basis functions with which are currently working–monomials; and the degree relaxation. As commented in [9], adopting an alternate basis set is likely to increase the rate of convergence and the tightness. As it stands, there are alternate ways to improve the tightness, primary amongst which is to create phantom modes using identity reset maps; this approach however, needs some care and is deferred for a future work.

VI. Conclusions

In this paper, a convex approximation of the reachable sets of a class of uncertain hybrid drift systems is presented. The presented method optimizes over the set of unsigned measures using converging moment relaxations and SDPs. A commentary on the accuracy and the adequacy of the proposed method is provided using examples. A future work will extend the work herein by synthesizing *cautious* feedback control laws that guarantee constraint satisfaction.

Appendix

Lemma 5 (Existence of solutions)**.** *Let* $(\mu_{s_j}, \mu_{f_j}, \mu_j), j \in \mathcal{J}$ *satisfy Eqn. (25). Then, there exists a family of absolutely continuous trajectories starting from* μ_{s_j} *such the occupation and final measures in each mode generated by this family of trajectories is equal to* μ_j *and* μ_{f_j}*.*

References

[1] K. Byl, *Metastable legged-robot locomotion*. PhD thesis, Massachusetts Institute of Technology, 2008.

[2] H. Dai and R. Tedrake, "Optimizing robust limit cycles for legged locomotion on unknown terrain," in *2012 IEEE 51st Annual Conference on Decision and Control*, pp. 1207–1213, IEEE, 2012.

[3] B. Griffin and J. Grizzle, "Walking gait optimization for accomodation of unknown terrain height variations," in *American Control Conference 2015*, 2015.

[4] C. O. Saglam and K. Byl, "Switching policies for metastable walking," in *2013 IEEE 52nd Annual Conference on Decision and Control (CDC)*, pp. 977–983, IEEE, 2013.

[5] A. Girard, "Reachability of uncertain linear systems using zonotopes," in *Hybrid Systems: Computation and Control*, pp. 291–305, Springer, 2005.

[6] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization," in *47th IEEE Conference on Decision and Control*, pp. 4042–4048, IEEE, 2008.

[7] C. J. Tomlin, I. Mitchell, A. M. Bayen, and M. Oishi, "Computational techniques for the verification of hybrid systems," *Proceedings of the IEEE*, vol. 91, no. 7, pp. 986–1001, 2003.

[8] J. N. Maidens, S. Kaynama, I. M. Mitchell, M. M. Oishi, and G. A. Dumont, "Lagrangian methods for approximating the viability kernel in high-dimensional systems," *Automatica*, vol. 49, no. 7, pp. 2017–2029, 2013.

[9] D. Henrion and M. Korda, "Convex computation of the region of attraction of polynomial control systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 297–312, 2014.

[10] A. Majumdar, R. Vasudevan, M. M. Tobenkin, and R. Tedrake, "Convex optimization of nonlinear feedback controllers via occupation measures," *The International Journal of Robotics Research*, p. 0278364914528059, 2014.

[11] V. Shia, R. Vasudevan, R. Bajcsy, and R. Tedrake, "Convex computation of the reachable set for controlled polynomial hybrid systems," in *2014 IEEE 53rd Annual Conference on Decision and Control (CDC)*, pp. 1499–1506, IEEE, 2014.

[12] S. Prajna and A. Jadbabaie, "Safety verification of hybrid systems using barrier certificates," in *Hybrid Systems: Computation and Control*, pp. 477–492, Springer, 2004.

[13] S. Burden, H. Gonzalez, R. Vasudevan, R. Bajcsy, and S. Shankar Sastry, "Metrization and Simulation of Controlled Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2307–2320, 2015.

[14] J. W. Pitman, "Occupation measures for markov chains," *Advances in Applied Probability*, vol. 9, no. 1, pp. pp. 69–86, 1977.

[15] E. Anderson and P. Nash, *Linear programming in infinite-dimensional spaces: theory and applications*. Wiley-Interscience series in discrete mathematics and optimization, Wiley, 1987.

[16] "Spotless."

[17] M. J. Coleman, *A stability study of a three-dimensional passive-dynamic model of human gait*. Cornell University, May, 1998.