Convex Computation of the Reachable Set for Hybrid Systems with Parametric Uncertainty

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***Abstract*— To verify the correct operation of systems, engineers need to determine the set of configurations of a dynamical model that are able to safely reach a specified configuration under a control law. Unfortunately, constructing models for systems interacting in highly dynamic environments is difficult. This paper addresses this challenge by presenting a convex optimization method to efficiently compute the set of configurations of a polynomial hybrid dynamical system that are able to safely reach a user defined target set despite parametric uncertainty in the model. This class of models describes, for example, legged robots moving over uncertain terrains. The presented approach utilizes the notion of occupation measures to describe the evolution of trajectories of a nonlinear hybrid dynamical system with parametric uncertainty as a linear equation over measures whose supports coincide with the trajectories under investigation. This linear equation with user defined support constraints is approximated with vanishing conservatism using a hierarchy of semidefinite programs each of which is proven to compute an outer approximation to the set of initial conditions that can reach the user defined target set safely in spite of uncertainty. The efficacy of this method is illustrated on a pair of nonlinear systems with parametric uncertainty.**

I. Introduction

Computing the set of configurations that are able to safely reach a desired configuration is critical to ensuring the correct performance of a system in dynamic environments where deviations from planned behavior are to be expected. Many methods have been proposed to efficiently compute this set that is generally referred to as the *backwards reachable set* for deterministic systems. Unfortunately, the effect of intermittent contact with the world, especially in fluctuating environments, is demanding to model deterministically. A roboticist, for example, may be tasked with ensuring that a control for a legged robot beginning from an initial configuration is able to safely reach a desired goal; however, limitations in sensing or environment variability may render exact modeling of terrain height or friction impossible. The development of numerical tools to tractably compute the backwards reachable set of dynamical systems undergoing contact, or *hybrid dynamical systems*, with parametric uncertainty while providing systematic guarantees has been challenging due to the difficulty of efficiently accounting for the uncertainty.

Given its potential utility, many researchers have attempted to develop numerical tools to compute this *uncertain backwards reachable set*. Several researchers, for instance, have attempted to utilize this backwards reachable set while constructing controllers for legged robots that are able to walk over terrains of varying heights [1]–[4]. These approaches have relied upon discretizing the height of the terrain or selecting specific terrain profiles while constructing a safe controller across these specified heights, which verifies the performance of the controller only at those specific heights. Moreover, these approaches are unable to account for uncertainty associated with imperfect knowledge of terrain friction or parameters affecting the continuous dynamics.

Other researchers have developed tools to outer approximate the uncertain backwards reachable for linear systems with uncertain parameters using a variety of approaches [5], [6]. These methods can be extended to nonlinear hybrid systems, but can require the introduction of a large number of discrete states to represent the nonlinear behavior or require overly conservative estimates of potential uncertainty. More generally, Hamilton-Jacobi Bellman based approaches have also been applied to compute the uncertain backwards reachable set for nonlinear systems with arbitrary uncertainty affecting the state at any instance in time [7]. These approaches solve a more general problem, but rely on state space discretization which can be prohibitive for systems of dimension greater than four without relying upon specific system structure [8].

This paper leverages a method developed in several recent papers [9]–[11] that describe the evolution of trajectories of a deterministic hybrid dynamical system using measures, to describe the evolution of a hybrid dynamical system with parametric uncertainty as a linear equation over measures. As a result of this characterization, the set of configurations that are able to reach a target set despite parametric uncertainty, called the *uncertain backwards reachable set*, can be computed as the solution to an infinite dimensional linear program over the space of nonnegative measures. To compute an approximate solution to this infinite dimensional linear program, a sequence of finite dimensional relaxed semi-definite programs are constructed that satisfy an important property: each solution to this sequence of semi-definite programs is an outer approximation to the uncertain backwards reachable set with asymptotically vanishing conservatism. The approach is most comparable to those that check Lyapunov's criteria for stability via sums-of-squares programming to verify the safety of a system [12]. In contrast to these approaches, the algorithm described in this paper does not require solving a bilinear optimization problem that requires feasible initialization and allows for more general descriptions of the parametric uncertainty in the model.

The remainder of the paper is organized as follows: Section II introduces the notation used in the remainder of

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the paper, the class of systems under consideration, and the backwards reachable set problem under parametric uncertainty; Section III describes how the backwards reachable set under parametric uncertainty is the solution to an infinite dimensional linear program; Section IV constructs a sequence of finite dimensional semidefinite programs that outer approximate the infinite dimensional linear program with vanishing conservatism; Section V describes the performance of the approach on a pair of examples; and, Section VI concludes the paper.

II. Preliminaries

This section defines the notation, the class of systems, and problem considered throughout the paper.

A. Notation

In the remainder of this text the following notation is adopted: Sets are italicized and capitalized (ex. K). The boundary of a set K is denoted by ∂K. Finite truncations of the set of natural numbers are expressed as $\mathbb{N}_n := \{1, \ldots, n\}$. The set of continuous on a compact set K are denoted by $\mathcal{C}(K)$. The ring of polynomials in x is denoted by $\mathbb{R}[x]$, and the degree of a polynomial is equal to degree its largest multinomial; the degree of the multinomial x^α, $\alpha \in \mathbb{R}^n_{\geq 0}$ is $|\alpha| = \|\alpha\|_1$; and $\mathbb{R}_d[x]$ is the set of polynomials in x with maximum degree d.

The dual to $\mathcal{C}(K)$ is the set of Radon measures on K, denoted as $\mathcal{M}(K)$, and the pairing of $\mu \in \mathcal{M}(K)$ and $v \in \mathcal{C}(K)$ is:

$$\langle \mu, v \rangle = \int_K v(x)\, d\mu(x). \tag{1}$$

We denote the nonnegative Radon measures by $\mathcal{M}_+(K)$. The space of Radon probability measures on K is denoted by $\mathcal{P}(K)$. The Lebesgue measure is denoted by λ. Finally, supports of measures, μ, are identified as $supp(\mu)$.

B. Quasi-Uncertain Hybrid Systems

The class of uncertain systems considered in this study consists of hybrid systems that conform to the following definition and undergo executions as described by Alg. 1.

Definition 1 (Inspired by [13])**.** A 'quasi-uncertain' hybrid system is a tuple $\mathcal{H} = (\mathcal{J}, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}, \Gamma)$, where

- $\mathcal{J}$ is a finite set of indices of discrete states in of $\mathcal{H}$;
- $\mathcal{E} \subset \mathcal{J} \times \mathcal{J}$ is a set of two-tuples describing directed edges;
- $\mathcal{D} = \{D_j\}_{j \in \mathcal{J}}$ is the set of domains where each D_j is a compact n_j-dimensional manifold with boundary where $n_j \in \mathbb{N}$;
- $\Gamma = \{\mu_{\theta_j}\}_{j \in \mathcal{J}}$ where $\mu_{\theta_j} \in \mathcal{P}(\Theta_j)$ describes the uncertainty associated with discrete state $j \in \mathcal{J}$;
- $\mathcal{F} = \{\tilde{f}_j\}_{j \in \mathcal{J}}$ where $\tilde{f}_j : D_j \times \Theta_j \to D_j$ is a Lipschitz continuous function describing the dynamics on D_j;
- $\mathcal{G} = \{G_e\}_{e \in \mathcal{E}}$ is the set of guards where each $G_{(j,j')} \subset \partial D_j$ is a closed, embedded, co-dimension 1 submanifold with boundary, and is a guard in domain $j \in \mathcal{J}$ that defines a transition to mode $j' \in \mathcal{J}$;

Algorithm 1: Execution of $\mathcal{H}$

1 **Initialization:** $t = 0$, $j \in \mathcal{J}$, $(x_0, j) \in \mathcal{D}$, $x(0) = x_0$;
2 **while** *1* **do**
3 *Let* θ be drawn according to μ_{θ_j};
4 *Let* $\gamma\colon [t, T] \to D_j$, absolutely continuous st.
5   $\dot{\gamma}(s) = \tilde{f}(\gamma(s), \theta)$ λ_t^a-a.e., $s \in [t, T]$
6   $\gamma(t) = x(t)$;
7 $\Lambda_{(j,t)} := \{r \in [t, T] | \exists (j,k) \in \mathcal{E}$ st. $(\gamma(r), \theta) \in \mathcal{G}_{(j,k)}\}$;
8 **if** $\Lambda_{(j,t)} \neq \varnothing$ **then**
9   $t' := \min \Lambda_{(j,t)}$, k st. $\gamma(t') \in \mathcal{G}_{(j,k)}$
10    $x(s) \leftarrow \gamma(s)$, $\forall s \in [t, t')$
11    $t \leftarrow t'$, $x(t') \leftarrow R_{(j,k)}(\gamma(t'))$, $j \leftarrow k$
12 **else**
13   $x(s) = \gamma(s)$, $\forall s \in [t, T]$;
14   Stop;
15 **end**
16 **end**
17 [a]where λ_t is the Lebesgue measure on $[t, T]$

- $\mathcal{R} = \{R_e\}_{e \in \mathcal{E}}$ is the set of reset maps, where each map is a continuously differentiable injection $R_{(j,j')} : G_{(j,j')} \to D_{j'}$ and defines the transition from guard $G_{(j,j')}$.

The discrete states are sometimes referred to as *modes* of the system.

To avoid any ambiguity during transitions between discrete states, we assume the following:

Assumption 1. In each discrete state, the guards are mutually exclusive; i.e.

$$\mathcal{G}_{(i,j)} \cap \mathcal{G}_{(i,k)} = \varnothing, \quad \forall (i,j), (i,k) \in \mathcal{E}, \forall j \neq k \tag{2}$$

In addition, the systems are not allowed to undergo infinite mode transitions in any finite time-interval.

Assumption 2. $\mathcal{H}$ has no zeno execution.

Alg 1 describes the finite-time execution, $[0, T]$, of a hybrid system, $\mathcal{H}$, as in Defn. 1 as follows: Suppose that the system enters mode j at time t at location $x \in D_j$. Recall the dynamics in this domain, $\tilde{f}_j$, are a function of a random parameter drawn from the distribution μ_{θ_j}; let this random variable take the value θ. The trajectory of the hybrid system beginning at time t at x is then given by any absolutely continuous function that satisfies the differential equation $\tilde{f}_j$ with a fixed θ as described in Steps ⑤&⑥. This trajectory evolves until either the time evolution passes T or the trajectory arrives at a guard, whichever happens first. Steps ⑦–⑪ isolates the first hitting-time of a guard of mode j and resets $\mathcal{H}$ to a new mode whereafter the same procedure is repeated until $t = T$.

Note that the uncertainty does not evolve with time. The uncertainty only changes when the system mode resets. This class of systems is still quite rich as is illustrated by the two representative examples —a simple 1D pedagogical example, and a 2D representative of walking models—presented next.

Example 1 (1-D Quasi-Uncertain Linear System on a Circle)**.** Consider a quasi-uncertain linear system evolving on a circle with dynamics:

$$\dot{x} = -0.7x + 0.2\theta - 0.1, \tag{3}$$

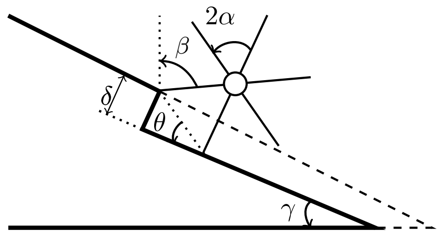

Fig. 1. Schematic of the rimless wheel with θ describing the effect of unknown terrain.

where $D_1 = [-1,1]$, $G_{(1,1)} = \{-1\}$, $R_{(1,1)}(x) = -x$. and $\theta \in [-1,1]$ is an unknown parameter affecting the dynamics. The uncertain parameter can be thought of as arising due to structural modeling errors, or as a result of reducing a singular-perturbed system.

Example 2 (Planar Rimless Wheel (PRW) with Uneven Terrain)**.** The planar rimless wheel—constituted by a massless axle to which n (angularly) equidistant spokes are connected—is a simple model of legged locomotion. Figure 1 presents a schematic of a rimless wheel—with spokes separated by an angle 2α—rolling down an inclined plane. The PRW is a hybrid system consisting of one mode; every time the spoke makes contact with the surface of the inclined plane, the system undergoes a reset. The continuous dynamics of the PRW are:

$$\begin{bmatrix}\dot{\beta} & \ddot{\beta}\end{bmatrix}' = \begin{bmatrix}\dot{\beta} & \sin(\beta)\end{bmatrix}', \tag{4}$$

where β is the angle between the vertical (which is defined as the line that is perpendicular to the base of the inclined plane) and the pivoting spoke. Once the swinging spoke makes contact with the terrain, the states are reset as:

$$R_{(1,1)}(\beta^-, \dot{\beta}^-) = \begin{bmatrix}2\gamma - \beta^- & \cos(2\alpha)\,\dot{\beta}^-\end{bmatrix}'. \tag{5}$$

For a PRW rolling down an inclined plane with flat terrain, at the instance when the swinging spoke makes contact with the ground $\beta = \gamma + \alpha$. To encode the uncertainty due to terrain height, suppose the PRW encounters a step of size δ, then if we let $\theta = \arcsin\left(\frac{\delta}{2l\sin\alpha}\right)$, the guard is defined as:

$$G_{(1,1)} = \{(\beta, \dot{\beta}) \mid \beta = \gamma + \alpha + \theta\}. \tag{6}$$

Observe that as the PRW continues to roll, *the terrain is allowed to change since the random variable θ is allowed to take a distinct value after each contact with the ground.*

C. Problem Description

The objective of this work is to estimate the largest set of initial conditions from which all state trajectories of $\mathcal{H}$, regardless of any encountered uncertainty, reach a terminal set by a pre-specified time, T. To formalize the definition of this *uncertain backwards reachable set*, we denote the terminal set as X_T and its projection into each mode by $X_{(T,j)}$, which we assume is compact. For convenience, we define $\mathcal{T} = [0,T]$. We define the uncertain backwards reachable set mode-wise:

$$X_{(0,j)} = \{x_0 \in D_j \mid \forall x : [0,T] \to \mathcal{D} \text{ constructed via Alg. 1 with } x(0) = x_0,\ x(T) \in X_T\} \tag{7}$$

The uncertain backwards reachable set is then defined as $X_0 = \{X_{(0,j)}\}_{j\in\mathcal{J}}$. Observe that by definition all initial conditions beginning from X_0 must reach X_T at time T regardless of mode transitions and uncertainty encountered along the way.

III. Problem Formulation

In this section, we present a pair of dual infinite dimensional linear programs that compute the uncertain backwards reachable set. Critically, note that despite the uncertainty being drawn from a distribution at the arrival into each mode, it remains constant throughout that mode. As a result, this unknown parameter can be appended to the dynamics of every mode j and treated as a portion of the state-space:

$$f_j = \begin{bmatrix}\tilde{f}_j' & \mathbf{0}_{n_{\theta_j}}'\end{bmatrix}'. \tag{8}$$

To address this problem, we rely on the notion of occupation measures, first introduced in [14], to transform the hybrid nonlinear dynamics of the system into a set linear dynamics over measures that can more readily be solved. Occupation measures can be interested as measuring the time a solution spends in a portion of the state-space. For instance, suppose the system enters mode j at τ_k with the states being initialized as $x(\tau_k) = x_0$ and $\theta(\tau_k) = \theta$. The occupation measure, $\mu_j(\cdot \mid \tau_k, x_0, \theta) \in \mathcal{M}_+(\mathcal{T} \times D_j \times \Theta_j)$, is defined as:

$$\mu_j(A \times B \times C | \tau_k, x_0, \theta) = \int_{\mathcal{T}} I_{A\times B\times C}(t, x(t|\tau_k, x_0, \theta), \theta)dt. \tag{9}$$

Note the follow relation between the Lebesgue measure on $\mathcal{T}$ and $\mu_j(\cdot \mid \tau_k, x_0, \theta)$ holds for all $v \in C(\mathcal{T} \times D_j \times \Theta_j)$:

$$\langle \mu_j(\cdot \mid \tau_k, x_0, \theta), v\rangle = \langle \lambda_t, v(t, x(t \mid \tau_k, x_0, \theta), \theta)\rangle, \tag{10}$$

The occupational measure as defined is a conditional measure – conditioned on the arrival-time and initial values of the states in that mode. To consider a set of possible arrival-times and initial conditions, we define the *average occupation measure* by integrating the conditional occupation measure against a measure on the set of possible initial conditions of the mode, $\mu_{s_j} \in \mathcal{M}_+(\mathcal{T} \times D_j \times \Theta_j)$:

$$\mu_j(A \times B \times C) = \int_{\mathcal{T}\times D_j\times\Theta} \mu_j(A \times B \times C \mid \tau_k, x_0, \theta)\, d\mu_{s_j}. \tag{11}$$

Observe that by definition, the uncertain variables are independent of the states' initial conditions; hence $\mu_{s_i} \in \mathcal{M}_+(\mathcal{T} \times D_j \times \Theta)$ is expressible as a product measure:

$$\mu_{s_j} = \bar{\mu}_{0_j} \otimes \mu_{\theta_j}, \tag{12}$$

where $\bar{\mu}_{0_j} \in \mathcal{M}_+(\mathcal{T} \times D_j)$ is a measure describing the set of initial conditions, and $\mu_{\theta_j} \in \mathcal{M}_+(\Theta_j)$ is as in the definition of $\mathcal{H}$.

Similarly, measures on terminals sets, $\mu_{T_j} \in \mathcal{M}_+(X_{(T,j)} \times \Theta_j)$:

$$\mu_{T_j}(A \times B) = \int_{\mathcal{T} \times X_{(T,j)} \times \Theta} I_{A\times B}(x(T \mid \tau_k, x_0, \theta), \theta)\, d\mu_{s_j}, \quad (13)$$

and guards, $\mu_{G_e} \in \mathcal{M}_+(\mathcal{T} \times G_{(j,k)})$:

$$\mu_{G_{(j,k)}}(A \times B \times C) = \int_{\mathcal{T} \times G_{(j,k)}} I_{A\times B\times C}(t, x(t \mid \tau_k, x_0, \theta), \theta)\, d\mu_{s_j} \quad (14)$$

for all $(j,k) \in \mathcal{E}$ can be defined. The measures $\mu_{G_{(j,k)}}$ are supported on the guards of mode j and should be interpreted as the hitting times of the guard. The *final measure* in each mode j can be defined as:

$$\mu_{f_j} = \delta_T \otimes \mu_{T_j} + \sum_{k \in \{l \mid (j,l) \in \mathcal{E}\}} \mu_{G_{(j,k)}}. \quad (15)$$

To compute X_0, we relate $\{\mu_{s_j}\}_{j\in\mathcal{J}}$ with $\{\mu_{f_j}\}_{j\in\mathcal{J}}$ using the dynamics of the system. As a first step, define linear operators $\mathcal{L}_{f_j} : \mathcal{C}^1(\mathcal{T} \times D_j \times \Theta_j) \to \mathcal{C}(\mathcal{T} \times D_j \times \Theta_j)$ as:

$$\mathcal{L}_{f_j} v = \frac{\partial v}{\partial t} + \left\langle \nabla_x v, \tilde{f}_j \right\rangle \quad (16)$$

where $v \in \mathcal{C}^1(\mathcal{T} \times D_j \times \Theta_j; \mathbb{R})$ is an arbitrary test function and $\nabla_x v$ computes the gradient of v in the D_j coordinates. Suppose the system transitioned to mode j at $t = \tau_{k-1}$ with the state taking value upon reset $x(\tau_{k-1})$ and θ. The value of v, evaluated along the flow of the system and at $t = \tau_k$ is computed using the Fundamental Theorem of Calculus:

$$\begin{aligned} v\big(\tau_k, x(\tau_k \mid x(\tau_{k-1}), \theta_{k-1})\big) &= v(\tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}) \\ &+ \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_f v(t, x(t \mid \tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}))\, dt. \end{aligned} \quad (17)$$

Using Eqn. (10), Eqn. (17) can be re-written as:

$$\begin{aligned} v\big(\tau_k, x(\tau_k \mid \tau_{k-1}, x(\tau_{k-1}), \theta_{k-1})\big) &= v(\tau_{k-1}, x(\tau_{k-1}), \theta_{k-1}) \\ &+ \langle \mu_j(\cdot \mid \tau_{k-1}, x(\tau_{k-1}, \theta_{k-1}), \mathcal{L}_f v \rangle, \end{aligned} \quad (18)$$

which can be simplified further by using Eqns. (11)–(15):

$$\left\langle \mu_{f_j}, v \right\rangle = \left\langle \mu_{s_j}, v \right\rangle + \left\langle \mu_j, \mathcal{L}_f v \right\rangle. \quad (19)$$

Alternatively, using the standard definition of adjoint operators[1], Eqn. (19) is re-written as:

$$\left\langle \mu_{f_j}, v \right\rangle = \left\langle \mu_{s_j}, v \right\rangle + \left\langle \mathcal{L}'_f \mu_j, v \right\rangle. \quad (20)$$

Eqn (20) defines a linear relation that initial and final measures evolving according to the hybrid dynamics must satsify.

During the execution of a hybrid system, any mode can be entered either at $t = 0$ or due to reset. The initial measure in the (t,x)-coordinate can be decomposed as:

$$\bar{\mu}_{0_j} = \delta_0 \otimes \mu_{0_j} + \pi_*^{(t,x)} \sigma_{0_j} \quad (21)$$

[1]A linear operator $\mathcal{L}$ and its adjoint, $\mathcal{L}'$, satisfy the following relation:

$$\left\langle \mathcal{L}' \mu, v \right\rangle = \langle \mu, \mathcal{L} v \rangle.$$

where $\mu_{0_j} \in \mathcal{M}_+(D_j)$ is the measure describing initial conditions to the system at $t = 0$, $\sigma_{0_j} \in \mathcal{M}_+(\mathcal{T} \times D_j \times \Theta_j)$ is a measure describing initial conditions arriving due to reset, and $\pi_*^{(t,x)}$ denotes the pushforward constructed by lifting the (t,x)-projection operator, $\pi^{(t,x)} : \mathcal{T} \times D_j \times \Theta_j \to \mathcal{T} \times D_j$, to measures (refer to Chapter 11 in [15] for an introduction to pushforwards). State resets occur when the state reaches a guard. As a result, we must formalize a relationship between $\mu_{G_{(i,j)}}$ and σ_{0_j}, $\forall (i,j) \in \mathcal{E}$. To formalize this relationship notice that σ_{0_j} can be decomposed into measures corresponding to the source of each arrival state:

$$\sigma_{0_j} = \sum_{i \in \{k \mid (k,j) \in \mathcal{E}\}} \sigma_{(i,j)} \otimes \mu_{\theta_j}, \quad (22)$$

where $\sigma_{(i,j)}$ is the measure describing initial conditions that are reset into mode j from guard $G_{(i,j)}$. Upon reaching the guard, the system transitions according to the reset map, $R_{(i,j)}$, which can be treated as a nonlinear transformation between D_i and D_j. By applying a change of variables formula, as in Lemma 1 in [11], we have:

$$\left\langle \sigma_{(i,j)}, w \right\rangle = \left\langle \pi_*^{(t,x)} \mu_{G_{(i,j)}}, w \circ R_{(i,j)} \right\rangle \quad (23)$$

where $w \in \mathcal{C}(\mathcal{T} \times \mathcal{D}_j)$. Note that since μ_{θ_j} is a probability distribution, and there is no evolution along the Θ_j coordinate:

$$\left\langle \pi_*^{(t,x)} \mu_{G_{(i,j)}}, s \right\rangle = \left\langle \mu_{G_{(i,j)}}, s \right\rangle, \quad (24)$$

for all $s \in \mathcal{C}(\mathcal{T} \times D_i)$. Essentially, $\sigma_{(i,j)}$ can be thought of as the push-forward measure of $\mu_{G_{(i,j)}}$.

A. The primal

The problem of computing the uncertain backwards reachable set of $\mathcal{H}$ can be formulated as an infinite-dimensional linear program that supremizes the *volume* of the set of initial condition:

$$\begin{aligned} \sup_{\Lambda} \quad & \sum_{j \in \mathcal{J}} \left\langle \mu_{0_j}, 1 \right\rangle & & (P) \\ \text{st.} \quad & \mu_{s_j} + \mathcal{L}'_f \mu_j = \mu_{f_j} & \forall j \in \mathcal{J} \quad & (25) \\ & \mu_{0_j} + \hat{\mu}_{0,j} = \lambda_j & \forall j \in \mathcal{J} \quad & (26) \\ & \sum_{j \in \mathcal{J}} \left\langle \mu_{T_j}, 1 \right\rangle = \sum_{j \in \mathcal{J}} \left\langle \mu_{0_j}, 1 \right\rangle & & (27) \end{aligned}$$

where λ_j is the Lebesgue measure supported on D_j, $\Lambda = \Big\{ \big(\{\mu_j\}_{j\in\mathcal{J}}, \{\mu_{0_j}\}_{j\in\mathcal{J}}, \{\mu_{T_j}\}_{j\in\mathcal{J}}, \{\hat{\mu}_{0,j}\}_{j\in\mathcal{J}}, \{\mu_e\}_{e\in\mathcal{E}} \big) \in \bigtimes_{j\in\mathcal{J}} \mathcal{M}_+(\mathcal{T} \times D_j \times \Theta_j) \bigtimes_{j\in\mathcal{J}} \mathcal{M}_+(D_j) \bigtimes_{j\in\mathcal{J}} \mathcal{M}_+(X_{(T,j)}) \bigtimes_{j\in\mathcal{J}} \mathcal{M}_+(D_j) \bigtimes_{e\in\mathcal{E}} \mathcal{M}_+(G_e) \Big\}$, and 1 denotes the function that takes value 1 everywhere. The $\hat{\mu}_{0_j} \in \mathcal{M}(D_j)$ are slack variables introduced to ensure that the mass of the μ_{0_j} are identical to the volume (under the Lebesgue measure) of the uncertain backwards reachable set, as we prove below. Eqn. (27) ensures that all trajectories that emanate $\cup_{j\in\mathcal{J}}\, supp(\mu_{0_j})$ reach X_T at $t = T$.

Lemma 1. *If $\{\mu_{0_j}\}_{j\in\mathcal{J}}$ are components of some $\mu \in \Lambda$ that optimizes (P), then $\mathcal{X}_{(0,j)} = supp(\mu_{0_j})$ for each $j \in \mathcal{J}$. In addition, the optimal value of (P) is equal to the sum of* volumes *of the uncertain backwards reachable set in each mode, i.e.* $\sum_{j\in\mathcal{J}} \lambda_j(X_{0_j})$.

Proof. Suppose $\sum_{j\in\mathcal{J}} \lambda_j(supp(\mu_{0_j})\backslash X_{(0,j)}) > 0$, then by Lemma 5 (in the appendix), there exist trajectories that begin in $\cup_{j\in\mathcal{J}}(supp(\mu_{0_j})\backslash X_{(0,j)})$ that reach X_T; this is a contradiction. Thus,

$$\bigcup_{j\in\mathcal{J}} supp(\mu_{0_j}) \subset \bigcup_{j\in\mathcal{J}} X_{(0,j)}, \tag{28}$$

$$\sum_{j\in\mathcal{J}} \lambda_j(supp(\mu_{0_j})) \leq \sum_{j\in\mathcal{J}} \lambda_j(X_{(0,j)}). \tag{29}$$

By definition of the BRS, all state trajectories that emanate from a subset of X_0 end in X_T. That is, for each $j \in \mathcal{J}$ and initial measure μ_{0_j}, if $supp(\mu_{0_j}) \subset X_{(0,j)}$, there exist measures μ_j and μ_{f_j} that satisfy Eqn. (25). Thus the following inequality is true:

$$\sum_{j\in\mathcal{J}} \lambda_j(spt(\mu_{0_j})) \geq \sum_{j\in\mathcal{J}} \lambda_j(X_{(0,j)}) \tag{30}$$

From Eqns. (29) and (30), $\cup_{j\in\mathcal{J}} spt(\mu_{0_j})$ is the BRS of the system. That the optimal value of (P) is the volume of the uncertain backward reachable set follows by noting that the slack variables ensure absolute continuity of each μ_{0_j} with respect to the Lebesgue measure and the observation that $\lambda_j|_{X_{(0,j)}}, \forall j \in \mathcal{J}$ is feasible in (P). □

B. The dual

The dual to (P) for a quasi-uncertain hybrid system $\mathcal{H}$ can be written as:

$$\begin{aligned}
&\inf_{F} \sum_{j\in\mathcal{J}} \langle \lambda_j, w_j \rangle && (D)\\
&\text{st. } w_j(x) \geq 0 && \forall x \in D_j\\
& && \forall j \in \mathcal{J},\\
&\quad v_j(T,x,\theta) + p \geq 0, && \forall (x,\theta) \in \Phi_j\\
& && \forall j \in \mathcal{J}, \quad (31)\\
&\quad -\mathcal{L}_f v_j(t,x,\theta) \geq 0, && \forall (t,x,\theta) \in \Omega_j\\
& && \forall j \in \mathcal{J}, \quad (32)\\
&\quad w_j(x) - \langle \mu_{\theta_j}, v_j(0,x,\theta) \rangle - p \geq 1, && \forall x \in D_j\\
& && \forall j \in \mathcal{J}, \quad (33)\\
&\quad v_j \geq \langle \mu_{\theta_k}, v_k \rangle \circ R_{(j,k)}, && \forall (x,\theta) \in \Upsilon_{(j,k)}\\
& && \forall (j,k) \in \mathcal{E}, \quad (34)
\end{aligned}$$

where $F = \Big\{ \big(\{v\}_{j\in\mathcal{J}}, \{w_j\}_{j\in\mathcal{J}}, p\big) \in \bigtimes_{j\in\mathcal{J}} C^1(\mathcal{T} \times D_j \times \Theta_j) \bigtimes_{j\in\mathcal{J}} C(D_j) \times \mathbb{R} \Big\}$, $\Phi_j = X_{(T,j)} \times \Theta_j$, $\Omega_j = \mathcal{T} \times D_j \times \Theta_j$, and $\Upsilon_{(j,k)} = G_{(j,k)} \times \Theta_j$. The solution to D can be used to determine the uncertain backwards reachable set:

Lemma 2. *If* $\big(\{v\}_{j\in\mathcal{J}}, \{w_j\}_{j\in\mathcal{J}}, p\big)$ *is a feasible point to D, then the super-level set:*

$$\bigcup_{j\in\mathcal{J}} \{x \in D_j \mid w_j(x) \geq 1\} \tag{35}$$

is an outer approximation of the uncertain backwards reachable set of $\mathcal{H}$. *Furthermore there is a sequence of feasible solutions to* (D) *such that for each* $j \in \mathcal{J}$, *the 1-super-level set of the feasible* w_j *converges from above to the indicator function on* $X_{(0,j)}$ *in the* L^1 *norm and almost uniformly.*

Proof. To prove this lemma we project the uncertain backwards reachable set into each mode and show that it is part of the 1-level set of w. Assume that the state trajectory terminates in $X_{(T,j_k)}$ for some j_k The state trajectory must have arrived in mode j_k through a finite sequence of mode-transitions (according to Assumption 2). Let this sequences of mode-transitions be of length k. Suppose the states entered mode j_k at time τ_k, then from the Fundamental Theorem of Calculus and the constraints in Eqns. (31)&(32), the following inequalities hold:

$$-q \leq v_{j_k}(T, x(T \mid x(\tau_k^+),\theta),\theta) \leq v_j(\tau_k, x(\tau_k^+),\theta) \tag{36}$$

$$\Rightarrow -q \leq \big\langle \mu_{\theta_{j_k}}, v_{j_k}(\tau_k, x(\tau_k^+),\theta) \big\rangle \tag{37}$$

By iterative application of the constraint in Eqn. (34) and finally Eqn. (33), it follows that:

$$\begin{aligned}
-q &\leq \big\langle \mu_{\theta_{j_k}}, v_{j_k}(t,x,\theta) \big\rangle \circ R_{(j_{k-1},j_k)}(\tau_k, x(\tau_k^-)) && (38)\\
&\leq v_{j_{k-1}}(\tau_k, x(\tau_k^- \mid x(\tau_{k-1}^+),\theta),\theta) && (39)\\
&\leq \big\langle \mu_{\theta_{j_{k-1}}}, v_{j_k}(\tau_k, x(\tau_{k-1}^+),\theta) \big\rangle && (40)\\
&\ \vdots\\
&\leq v_{j_0}(\tau_1, x(\tau_1^- \mid x_0,\theta),\theta) && (41)\\
&\leq v_{j_0}(0, x_0, \theta) && (42)\\
&\leq \big\langle \mu_{\theta_{j_0}}, v_{j_0}(0,x_0,\theta) \big\rangle && (43)\\
&\leq w_{j_0}(x_0) - q - 1. && (44)
\end{aligned}$$

The final inequality implies that the initial condition of every trajectory that ends in the terminal set belongs to the 1-superlevel set of w_j for some $j \in \mathcal{J}$. The remainder of the proof follows from a straightforward extension to [9, Theorem 2]. □

Finally, note that the value computed by either optimization problem is equal which follows from [16, Theorem 3.10] and is similar to [9, Theorem 2]:

Lemma 3. *There is not gap between* (P) *and* (D).

Remark 1. There are two key aspects of the presentation that deserve re-iteration: First, the uncertainties that influence the dynamics are drawn from the distribution each time a trajectory enters a new mode; Second, the uncertain backwards reachable set corresponds to the set of initial conditions for *all* trajectories that are able to reach the terminal set in spite of *all* possible sequences of "discrete" uncertainty that each have non-zero probability. Notice that the uncertain backwards reachable set is the intersection of the backwards reachable set for ever possible discrete uncertainty with non-zero probability.

IV. Numerical Implementation

In this section, a sequence of Semidefinite Programs (SDP)s that approximate the solution to the infinite dimensional primal and dual defined in Secs. III-A and III-B are introduced. This sequence of relaxations is constructed by characterizing each measure using a sequences of moments[2] and assuming the following:

Assumption 3. The vector field in each mode and reset map between modes is a polynomial. Moreover the domain, the value of uncertainties, the guard, and the target set in each mode is a semi-algebraic set.

Recall that polynomials are dense in the set of continuous functions by the Stone-Weierstrass Theorem so this assumption is made without too much loss of generality.

Under this assumption, given any finite d-degree truncation of the moment sequence of all measures in the primal (P), a primal relaxation, (P_d), can be formulated over the moments of measures to construct an SDP. The dual to (P_d), (D_d), can be expressed as a sums-of-squares (SOS) program by considering d-degree polynomials in place of the continuous variables in D.

To formalize this dual program, first note that a polynomial $p \in \mathbb{R}[x]$ is SOS or $p \in$ SOS if it can be written as $p(x) = \sum_{i=1}^{m} q_i^2(x)$ for a set of polynomials $\{q_i\}_{i=1}^m \subset \mathbb{R}[x]$. Note efficient tools exist to check whether a finite dimensional polynomial is SOS using SDPs [17]. Next, suppose we are given a semi-algebraic set $A = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0, h_i \in \mathbb{R}[x], \forall i \in \mathbb{N}_m\}$. We define the d-degree *quadratic module* of A as:

$$Q_d(A) = \left\{ q \in \mathbb{R}_d[x] \;\middle|\; \exists \{s_k\}_{k \in N_m \cup \{0\}} \subset \text{SOS s.t.} \right. \\ \left. q = s_0 + \sum_{k \in \mathbb{N}_m} h_k s_k \right\} \tag{45}$$

The d-degree relaxation of the dual, D_d, can be written as:

$$\begin{aligned}
&\inf_{\Xi_d} \sum_{j \in \mathcal{J}} \int_{D_j} w_j(x)\, d\lambda_j(x) && (D_d) \\
&\text{st. } w_j^d \in Q_d(X_{(T,j)}) && \forall j \in \mathcal{J} \\
&\quad v_j^d(T,\cdot) + p \in Q_d(D_j \times \Theta_j) && \forall j \in \mathcal{J} \\
&\quad -\mathcal{L}_{f_j} v_j^d \in Q_d(\mathcal{T} \times D_j \times \Theta_j) && \forall j \in \mathcal{J} \\
&\quad w_j^d - \left\langle \mu_{\theta_j}, v_j^d(0,\cdot) \right\rangle - p - 1 \in Q_d(D_j) && \forall j \in \mathcal{J} \\
&\quad v_j^d - \left\langle \mu_{\theta_k}, v_k^d \right\rangle \circ R_{(j,k)} \in Q_d(\mathcal{T} \times D_j \times \Theta_j) && \forall (j,k) \in \mathcal{E}
\end{aligned}$$

where $\Xi_d = \Big\{ \big(\{v_j^d\}_{j \in \mathcal{J}}, \{w_j^d\}_{j \in \mathcal{J}}, p \big) \in \times_{j \in \mathcal{J}} \mathbb{R}_d[t,x,\theta] \times_{j \in \mathcal{J}} \mathbb{R}_d[x] \times \mathbb{R} \Big\}$. A primal can similarly be constructed, but the solution to the dual can be used directly generate a sequence of outer approximations to the uncertain backwards reachable set:

Lemma 4. *For each $d \in \mathbb{N}$ and $j \in \mathcal{J}$, let w_{j_d} denote the j-slice of the w-component of the solution to D_d. Then $\mathcal{X}_{(0,j_d)} = \{x \in D_j \mid w_{j_d}(x) \geq 1\}$ is an outer approximation to $\mathcal{X}_{(0,j)}$ and $\lim_{d \to \infty} \lambda_{n_j}(\mathcal{X}_{(0,j_d)} \backslash \mathcal{X}_{(0,j)}) = 0$.*

Proof. The proof to this lemma is an extension of Theorems 5–7 in [11] given Lemma 2. □

[2]The nth moment of a measure (μ) is obtained by evaluating the following expression

$$y_{\mu,n} = \langle \mu, x^n \rangle .$$

V. Examples

In this section, the efficacy of the proposed method is evaluated through the two examples introduced in Sec.II. The relaxed problems were parsed using the SPOTLESS toolbox [18] and were numerically solved with MOSEK on a computer equipped with a Intel Xeon W3540 processor and 12GB of RAM. The following points on the examples considered are obligatory.

1) It is a characteristic trait of the problem formulation considered in this paper that the actual distribution of the uncertainty is immaterial. Consequently, in all examples, it is assumed that the disturbance, θ, is uniformly distributed. For notional convenience, $\theta \sim \mathcal{U}([a,b])$ is used to denote that θ is uniformly distributed in the interval $[a,b]$.
2) For reasons related to numerics, all problems are normalized such that the state-space is given by $[-1,1]^n$, for an appropriate value of n.

Also, since has been established that the solution of relaxed problems provides an outer approximation of the BRS, in this section, the qualifier 'approximate' is suppressed.

A. 1-D linear dynamics

As a review, we consider a (non-hybrid) 1-D linear dynamical system whose dynamics is given by

$$\dot{x}_1 = -0.7x_1 + 0.2\theta - 0.1, \tag{46}$$

where $\theta \in \mathcal{U}([0.2,1])$. Although this system is hybridizable as discussed in Ex. 1, in the version considered here, we do not hybridize its dynamics. The target set to which trajectories must reach in one second is set as $X_T = [0.2, 0.4]$. The BRS of the deterministic system which assumes that θ is a known constant is analytically computed to equal

$$BRS_\theta = \left[\left(0.2 - \frac{2\theta-1}{7}\right) e^{0.7} - \frac{2\theta-1}{7}, \left(0.4 - \frac{2\theta-1}{7}\right) e^{0.7} - \frac{2\theta-1}{7} \right]$$

Note that the expression for the BRS_θ is linear in θ and that the width of BRS_θ is constant for all values of θ. As the value of θ changes, BRS_θ slides along $\mathbb{R}$; the intersection of BRS_0 and BRS_1 is the BRS of the uncertain system, as defined in Eqn. (7).

Figure 2 presents the degree 12 approximation of the indicator function that is supported on the BRS of the uncertain system, w^{12}, for cases when θ takes a constant value and when it is drawn at random. The graphs in green and cyan correspond to the cases when θ takes a constant value of $\theta = 0$ and $\theta = 1$ respectively. The red trace is the polynomial solution to the uncertain problem. Observe that the BRS corresponding to uncertain case encloses the intersection of those of the deterministic of cases; this is the desired outcome.

B. Planar rimless wheel (PRW)

The rolling PRW, introduced in Ex.2, is a one mode hybrid system in which the guard is reached when the marching foot makes contact with the wedge. For a PRW rolling along a *smooth* wedge, an analytically computable stable limit cycle

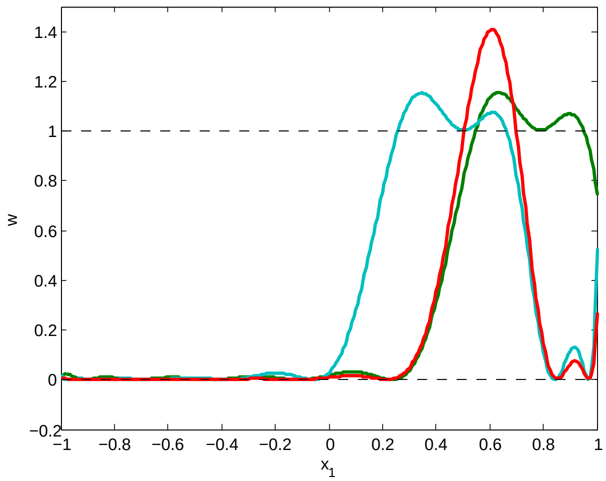

Fig. 2. Outer approximations of the BRS of the extreme deterministic cases and the stochastic case.(green) $\theta = 0$, (blue) $\theta = 1$ and (red) μ_θ

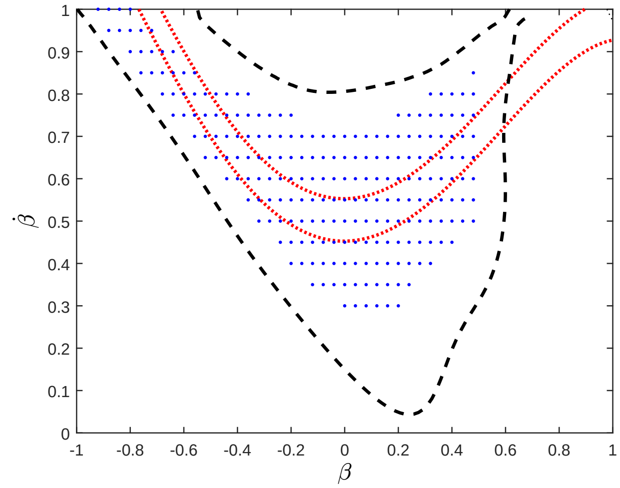

Fig. 3. Outer approximation and estimated BRS based on 100 iterations and T=4. Red band is the terminal set and the black outer is the boundary of the estimated BRS; the crosses correspond to results of MC simulation.

exits [19]; however, for the case considered in this example—with the wedge face having undulations—the definition of a limit cycle less clear. Consequently, a notion of *meta-stability*—when the system states arrive within ϵ of the stable limit cycle of the disturbance-free system—is adopted.

Figure 3 presents the polynomial degree 12 BRS (black dashed) for the rimless wheel (with $\alpha = 0.4$) which is tasked with arriving within the red band in $T = 4$ seconds, as it is rolling down a wedge with slope $\gamma = 0.2$ withstanding an a sequence of random changes to terrain drawn from $\theta \sim \mathcal{U}([-0.1, 0.1])$. The relative depths/height of the disturbance is about 25% the length of each spoke.

The BRS is validated by performing Monte Carlo simulations; the box I^2 is discretized into 51 points both ways and 100 independent trajectories are simulated (using MATLAB's *ode45* function) from each initial condition. The blue $\cdot$s depict the initial conditions that arrived within the terminal set at the desired time without violating any of the other constraints. Note that the set of points that succeeded in the MC simulation is entirely contained in the BRS.

At this juncture, a remark about the tightness of the BRS is warranted. Clearly, the BRS in Fig. 3 is not tight; and we attribute this to the set of basis functions with which are currently working–monomials; and the degree relaxation. As commented in [9], adopting an alternate basis set is likely to increase the rate of convergence and the tightness. As it stands, there are alternate ways to improve the tightness, primary amongst which is to create phantom modes using identity reset maps; this approach however, needs some care and is deferred for a future work.

VI. Conclusions

In this paper, a convex approximation of the reachable sets of a class of uncertain hybrid drift systems is presented. The presented method optimizes over the set of unsigned measures using converging moment relaxations and SDPs. A commentary on the accuracy and the adequacy of the proposed method is provided using examples. A future work will extend the work herein by synthesizing *cautious* feedback control laws that guarantee constraint satisfaction.

Appendix

Lemma 5 (Existence of solutions)**.** *Let $(\mu_{s_j}, \mu_{f_j}, \mu_j), j \in \mathcal{J}$ satisfy Eqn. (25). Then, there exists a family of absolutely continuous trajectories starting from μ_{s_j} such the occupation and final measures in each mode generated by this family of trajectories is equal to μ_j and μ_{f_j}.*

Proof (sketch). That the systems under consideration are uncertain is immaterial for, by definition, the uncertainty is constant in every mode and can be considered as a state in the augmented state vector; the augmented state vector in each mode has dynamics $\tilde{f}_j$. Hence, in the following presentation any dependence on θ is suppressed.

Adopting the approach used by the authors in proving [9, Lemma 3], it can be shown that there exists a solution to the continuity equation

$$\frac{d}{dt}\int_{\mathrm{M}_j} w(x)\, d\mu_j(\tilde{x} \mid t) = \int_{\mathrm{M}_j} \nabla_x \tilde{f}\, d\mu_j(x \mid t) \qquad \forall w \in \mathcal{C}(\mathrm{M}_j) \tag{47}$$

where $\mu_j(\cdot \mid t)$ is the stochastic kernel of μ_j given t.
By using test functions of the variety $v(t, \tilde{x}) = \varphi(t)\phi(\tilde{x})$ in Eqn. (25), we get the following

$$\varphi(T)a - \varphi(0)b = \int_{\mathcal{T}} \varphi(t)c(t) + \varphi(t)(d+e)(t)\, d\lambda_t(t) \tag{48}$$

where $a := \int_{X_{T,j}} \phi(x)\, d\mu_{f_j}$; $b := \int_{X_{0,j}} \phi(x)\, d\mu_{0_j}$;
$c(t) := \int_{\mathrm{M}_j} \phi(\tilde{x})\, d\mu_j(x \mid t)$; $d := \int_{\mathrm{M}_j} \nabla_x \phi(\tilde{x})\tilde{f}\, d\mu_j(x \mid t)$;
and $e(t) := \int_{\Sigma_j} \phi(\tilde{x})\, \sigma_j(x \mid t)$.
The $d\lambda_t$ a.e. unique solution to Eqn. (48) is shown to be

$$c(t) = b + \int_{\mathcal{T}} (d+e)(t)\, d\lambda_t, \tag{49}$$

and using the separability of $\mathcal{C}(\mathrm{M}_j)$ and the Riesz-representation theorem, that there is a representing measure μ_j $\square$

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