NOTES ON ROA PROJECT

HENRY O. JACOBS

1. Equations of motion for dice

2. Constraint equation

Let $\psi(t), \nu(t) \in L^2(M)$ for $t \in [0,T]$. Then observe that

$$\begin{aligned}\langle \psi(T), \nu(T) \rangle_{L^2} &= \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \frac{d}{dt} \langle \psi(t), \nu(t) \rangle dt \\ &= \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \langle \partial_t \psi, \nu \rangle_{L^2} + \langle \psi, \partial_t \nu \rangle_{L^2} dt\end{aligned}$$

Let A be an anti-symmetric operator on $L^2(M)$. Then $\psi(t)$ satisfies

$$\partial_t \psi + A[\psi] = 0 \tag{1}$$

if and only if

$$\langle \partial_t \psi + A[\psi], \nu \rangle_{L^2} = 0$$

for all $\nu \in L^2(M)$. Invoking the anti-symmetry of A we find that the above equation holds if and only if

$$\langle \partial_t \psi, \nu \rangle = \langle \psi, A[\nu] \rangle_{L^2}$$

for all ν.

Thus ψ satisfies (1) if and only if

$$\langle \psi(T), \nu(T) \rangle_{L^2} = \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \langle \psi, \partial_t \nu + A[\nu] \rangle_{L^2} dt$$

for all $\nu(t) \in L^2(M)$ with $t \in [0,1]$.

In our case $A[\psi](x) := X^i(x) \partial_i \psi(x) + \frac{1}{2} \partial_i X^i(x) \psi(x)$ and we use weak-derivatives if needed.

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3. Controls

We wish to solve for the region of attraction under the controlled dynamics

$$\dot{x} = X_0(x) + \sum_{k=1}^{c} u^k X_k(x).$$

where $X_0, X_1, \ldots, X_c$ are vector fields. Define the unbounded Hermetian operators $\hat{H}_i \in \mathrm{Herm}(L^2(M))$ by

$$\hat{H}_i \cdot \psi := i \sum_{\alpha=1}^{d} \frac{1}{2} \partial_\alpha X_i^\alpha \psi + X_i^\alpha \partial_\alpha \psi,$$

for $i = 0, 1, \ldots, c$. Then the dynamics of a half-density evolve under the controlled Schrodinger equation

$$\dot{\psi} = i\hat{H} \cdot \psi + i \sum_{k=1}^{c} u^k \hat{H}_k \cdot \psi = i\hat{H}(u) \cdot \psi.$$

where $\hat{H}(u) = \hat{H}_0 + u^k \hat{H}_k$. Let $\pi_M : U \times M \to M$ be the cartesian projection onto M. We can define the operator $\hat{J} : L^2(M) \to L^2(M \times U)$ given by $(\hat{J} \cdot \psi)(x, u) = (\hat{H}(u) \cdot \psi)(x)\sqrt{du}$ or more concisely $\hat{J} = \hat{H}(\cdot) \otimes \sqrt{u}$. From here we may compute the dual operator $\hat{J}^\dagger : L^2(U \times M) \to L^2(M)$. Explicitly $\hat{J}^\dagger$ takes the the form

$$(\hat{J}^\dagger \cdot \phi)(x) := \int_U \hat{H}(u)\phi(x, u)du$$

Inspired by Henrion-Korda we should be able to obtain the ROA by solving the primal QP

$$p^* = \sup_{\substack{\delta_T \otimes \psi_T - \delta_0 \otimes \psi_0 = \int_0^T \hat{J}^\dagger \cdot \phi dt \\ -1 \leq \psi_T \leq 1}} \|\psi\|^2_{L^2(M)}$$

where the decision variables are $\phi \in H^1([0, T]; L^2(U \times M))$, and $\psi_T \in L^2(M)$.