

# NOTES ON ROA PROJECT

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## 1. EQUATIONS OF MOTION FOR DICE

## 2. CONSTRAINT EQUATION

Let  $\psi(t), \nu(t) \in L^2(M)$  for  $t \in [0, T]$ . Then observe that

$$\begin{aligned}\langle \psi(T), \nu(T) \rangle_{L^2} &= \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \frac{d}{dt} \langle \psi(t), \nu(t) \rangle dt \\ &= \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \langle \partial_t \psi, \nu \rangle_{L^2} + \langle \psi, \partial_t \nu \rangle_{L^2} dt\end{aligned}$$

Let  $A$  be an anti-symmetric operator on  $L^2(M)$ . Then  $\psi(t)$  satisfies

$$(1) \quad \partial_t \psi + A[\psi] = 0$$

if and only if

$$\langle \partial_t \psi + A[\psi], \nu \rangle_{L^2} = 0$$

for all  $\nu \in L^2(M)$ . Invoking the anti-symmetry of  $A$  we find that the above equation holds if and only if

$$\langle \partial_t \psi, \nu \rangle = \langle \psi, A[\nu] \rangle_{L^2}$$

for all  $\nu$ .

Thus  $\psi$  satisfies (1) if and only if

$$\langle \psi(T), \nu(T) \rangle_{L^2} = \langle \psi(0), \nu(0) \rangle_{L^2} + \int_0^T \langle \psi, \partial_t \nu + A[\nu] \rangle_{L^2} dt$$

for all  $\nu(t) \in L^2(M)$  with  $t \in [0, 1]$ .

In our case  $A[\psi](x) := X^i(x) \partial_i \psi(x) + \frac{1}{2} \partial_i X^i(x) \psi(x)$  and we use weak-derivatives if needed.

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## 3. CONTROLS

We wish to solve for the region of attraction under the controlled dynamics

$$\dot{x} = X_0(x) + \sum_{k=1}^c u^k X_k(x).$$

where  $X_0, X_1, \dots, X_c$  are vector fields. Define the unbounded Hermetian operators  $\hat{H}_i \in \text{Herm}(L^2(M))$  by

$$\hat{H}_i \cdot \psi := i \sum_{\alpha=1}^d \frac{1}{2} \partial_\alpha X_i^\alpha \psi + X_i^\alpha \partial_\alpha \psi,$$

for  $i = 0, 1, \dots, c$ . Then the dynamics of a half-density evolve under the controlled Schrodinger equation

$$\dot{\psi} = i \hat{H} \cdot \psi + i \sum_{k=1}^c u^k \hat{H}_k \cdot \psi = i \hat{H}(u) \cdot \psi.$$

where  $\hat{H}(u) = \hat{H}_0 + u^k \hat{H}_k$ . Let  $\pi_M : U \times M \rightarrow M$  be the cartesian projection onto  $M$ . We can define the operator  $\hat{J} : L^2(M) \rightarrow L^2(M \times U)$  given by  $(\hat{J} \cdot \psi)(x, u) = (\hat{H}(u) \cdot \psi)(x) \sqrt{du}$  or more concisely  $\hat{J} = \hat{H}(\cdot) \otimes \sqrt{u}$ . From here we may compute the dual operator  $\hat{J}^\dagger : L^2(U \times M) \rightarrow L^2(M)$ . Explicitly  $\hat{J}^\dagger$  takes the form

$$(\hat{J}^\dagger \cdot \phi)(x) := \int_U \hat{H}(u) \phi(x, u) du$$

Inspired by Henrion-Korda we should be able to obtain the ROA by solving the primal QP

$$p^* = \sup_{\substack{\delta_T \otimes \psi_T - \delta_0 \otimes \psi_0 = \int_0^T \hat{J}^\dagger \cdot \phi dt \\ -1 \leq \psi_T \leq 1}} \|\psi\|_{L^2(M)}^2$$

where the decision variables are  $\phi \in H^1([0, T]; L^2(U \times M))$ , and  $\psi_T \in L^2(M)$ .

## 4. FUNCTION ADVECTION

Why does the following idea not work? We consider the optimization problem

$$\sup \int f_T(x) dx$$

subject to decision variables,  $f_T(x), f(x), \mu(x, u)$  with the constraints

$$f_T(x) - f_0(x) = \int \mathcal{L}_{X_u}[f](x) \mu(x, u) du dt$$

$$\mu(x, u) \geq 0$$

where  $f_0$  is the indicator function on  $X_0$