Output feedback control synthesis for a class of polynomial systems

Full State-feedback : Primal problem ($\mathbf{P}_f$)

supp $\mu_0(X)$
subject to

$$
\begin{aligned}
\mathcal{L}'_f\mu + \mathcal{L}'_g(\sigma^+ - \sigma^-) + \delta_0 \times \mu_0 &= \delta_T \times \mu_T \\
[\sigma^+]_k + [\sigma^-]_k + [\hat{\sigma}]_k &= \mu && \forall k \in \{1, \ldots, m\} \\
\mu_0 + \hat{\mu}_0 &= \lambda \\
[\sigma^+]_k, [\sigma^-]_k, [\hat{\sigma}]_k &\geq 0 && \forall k \in \{1, \ldots, m\} \\
\mu, \mu_0, \hat{\mu}, \mu_T &\geq 0 \\
\text{supp } (\mu_T) &\subset X_T \\
\text{supp } (\mu_0) &\subset X_1 \times X_2 \\
\text{supp } (\hat{\mu}_0) &\subset X_1 \times X_2 \\
\text{supp } (\mu) &\subset T \times X_1 \times X_2
\end{aligned}
$$

Partial State-feedback : Primal problem ($\mathbf{P}_p$)

Let $\Gamma_m = \mathbb{N}_m$ and $\Lambda = \{\bar{\mu}, \mu_0, \mu_T, \hat{\mu}_0, [\bar{\sigma}^+]_\Gamma, [\bar{\sigma}^-]_\Gamma, [\hat{\bar{\sigma}}]_\Gamma\}$

$\sup\limits_{\Lambda} \mu_0(X)$
subject to

$$
\begin{aligned}
\mathcal{L}'_f\mu + \mathcal{L}'_g(\sigma^+ - \sigma^-) + \delta_0 \times \mu_0 &= \delta_T \times \mu_T \\
[\sigma^+]_k + [\sigma^-]_k + [\hat{\sigma}]_k &= \mu && \forall k \in \{1, \ldots, m\} \\
d\nu^*(x_2|t, x_1)d\bar{\mu} &= d\mu \\
d\nu^*(x_2|t, x_1)d[\bar{\sigma}^+]_k &= d[\sigma^+]_k && \forall k \in \{1, \ldots, m\} \\
d\nu^*(x_2|t, x_1)d[\bar{\sigma}^-]_k &= d[\sigma^-]_k && \forall k \in \{1, \ldots, m\} \\
d\nu^*(x_2|t, x_1)d[\hat{\bar{\sigma}}]_k &= d[\hat{\sigma}]_k && \forall k \in \{1, \ldots, m\} \\
\mu_0 + \hat{\mu}_0 &= \lambda \\
[\bar{\sigma}^+]_k, [\bar{\sigma}^-]_k, [\hat{\bar{\sigma}}]_k &\geq 0 && \forall k \in \{1, \ldots, m\} \\
\bar{\mu}, \mu_0, \hat{\mu}, \mu_T &\geq 0 \\
\text{supp } (\mu_T) &\subset X_T \\
\text{supp } (\mu_0) &\subset X_1 \times X_2 \\
\text{supp } (\hat{\mu}_0) &\subset X_1 \times X_2 \\
\text{supp } (\bar{\mu}) &\subset T \times X_1 \\
\text{supp } (\bar{\sigma}^\pm) &\subset (T \times X_1)^m \\
\text{supp } (\hat{\bar{\sigma}}) &\subset (T \times X_1)^m
\end{aligned}
$$

where μ^* is the optimal solution to $\mathbf{P}_f$ and $d\mu^* = d\nu^*(x_2|t, x_1)d\pi_{t,x_1}\mu^*(t, x_1)$.

Partial State-feedback : Dual problem ($\mathbf{D}_p$)
In the dual formulation below, let ϕ: $C_c(T \times X_1 \times X_2) \to C_c(T \times X_1)$ be the operator representation of integrating-out the conditional $\nu^*(x_2|t, x_1)$; i.e.

$$v(t, x_1, x_2) \mapsto \int_{X_2} v(t, x_1, x_2)\, d\nu^*(x_2|t, x_1). \qquad \forall v \in C_c(T \times X_1 \times X_2)$$

The dual:

$\inf \langle \lambda, w \rangle$
subject to

$$\begin{aligned} \phi \circ \mathcal{L}_f v + \sum_{k=1}^{m} \phi \circ [p]_k \; &\leq 0 \\ w - v(0, \cdot) - 1 &\geq 0 \\ v(T, \cdot) &\geq 0 \\ w &\geq 0 \\ \phi \circ [p]_k &\geq 0, \quad \phi \circ [p]_k \geq \; |\phi \circ [\mathcal{L}_g v]_k| \qquad \forall k \in \{1, \ldots, m\} \end{aligned}$$

where $w \in C(X_1 \times X_2)$, $v \in C^1(T \times X_1 \times X_2)$, $p_k \in C(T \times X_1 \times X_2)$, $q \in C(T \times X_1 \times X_2)$ and '$\circ$' is used to denote the action of operator ϕ on a function.

Definition 1 ([3, Definition 1.3.4])**.** *A countably additive measure μ on a space (X, Σ) is a probability measure if $\mu \geq 0$ and $\mu(X) = 1$.*

Definition 2. *Suppose μ is a measure on the space (X, Σ). The Jordan-Hahn decomposition of μ is unique and given by*

$$\mu = \mu^+ - \mu^-,$$

where μ^+, $\mu^- \geq 0$. The total variation of μ is defined as $|\mu| = \mu^+ + \mu^-$. Clearly, if $\mu \geq 0$, then $|\mu| = \mu$.

Fact 1 ([4, Theorem 10.4.14])**.** *(Check!!) Let μ be an unsigned measure defined on $(X \times Y, \mathcal{B})$ where $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $\mathcal{B}$ is the Borel σ-algebra, and $\pi_y \mu$ be the projection (marginal) onto Y. Then, $\forall y \in Y$, there exists an unsigned measure $\nu(\cdot, y)$ on $\mathcal{B}$ such that*

$$d\mu = d\nu(x_1|x_2)\, d\pi_{x_2}\mu.$$

Fact 2 ([4, Remark 10.4.4])**.** *(Check!!) If σ is a signed measure on $(X \times Y, \mathcal{B})$ possessing as regular conditional measures, unsigned measures $\nu_\sigma(\cdot, y)$, $\forall y \in Y$, $|\sigma|$ admits $\nu_\sigma(\cdot, y)$ as its regular conditional measure.*

Conversely, if $\nu(\cdot, y)$ is the regular conditional measure for $\forall y \in Y$ and it is unsigned, then the regular conditional measures of μ^+ and μ^- are $\nu(\cdot, y)$.

Lemma 1 (Sufficiency)**.** *Let μ and σ be radon measures on the product space $X_1 \times X_2$ such that $\sigma \ll \mu$, where X_1 and X_2 are Polish spaces. Then each measure can be decomposed as follows*

$$\begin{aligned} d\mu &= d\nu_\mu(x_2|x_1) d\pi_{x_1}\mu(x_1) \\ d\sigma &= d\nu_\sigma(x_2|x_1) d\pi_{x_1}\sigma(x_1).' \end{aligned} \tag{1}$$

where $\pi_{x_1}\mu$ is the x_1 marginal of μ. If $d\nu_\sigma(x_2|x_1) = d\nu_\mu(x_2|x_1)$, then the Radon-Nikodym derivative $\dfrac{d\sigma}{d\mu}$ is not a function of x_2.

Additionally, if σ and μ satisfy the above conditions with $\mu \geq 0$, $\exists\, \hat{\sigma} \geq 0$ such that

$$\sigma + \hat{\sigma} = \mu \tag{2}$$

and $d\hat{\sigma} = d\nu_\mu(x_2|x_1)\, d\hat{\hat{\sigma}}$. That is, the regular conditional of $\hat{\sigma}$ given x_1 is identical to that of μ and σ.

Proof. That each measure can be decomposed as in Eqn. (1) follows from a standard result in measure theory ([4]) and hence we concentrate on the Radon-Nikodym derivative. Since $\sigma \ll \mu$, from the definition of marginals, $\pi_{x_1}\sigma \ll \pi_{x_1}\mu$; let $d\pi_{x_1}\sigma = \phi(x_1)\, d\pi_{x_1}\mu$.

Consider an arbitrary test function $v(x_1, x_2)$; then the following equalities holds by definition.

$$g(x_1) := \int_{X_2} v(x_1, x_2)\, d\nu_\sigma(x_2|x_1) = \int_{X_2} v(x_1, x_2)\, d\nu_\mu(x_2|x_1)$$

$$\begin{aligned}\int_{X_1\times X_2} v(x_1, x_2)\, d\sigma &= \int_{X_1} g(x_1)\, d\pi_{x_1}\sigma(x_1) \\ &= \int_{X_1\times X_2} \phi(x_1)\, g(x_1)\, d\pi_{x_1}\mu(x_1) \\ &= \int_{X_1\times X_2} \phi(x_1)\, v(x_1, x_2)\, d\mu\end{aligned}$$

Thus $d\sigma = \phi(x_1)\, d\mu$ and the Radon-Nikodym derivative is not a function of x_2.

The existence of $\hat{\sigma} \geq 0$ such that Eqn. (2) holds since $\sigma \ll \mu$. Additionally, as σ and μ share the same conditional with respect to x_1, σ admits the following decomposition.

$$\begin{aligned}d\hat{\sigma} &= d\nu(x_2|x_1)\, d(\pi_{x_1}\mu - \pi_{x_1}\sigma)(x_1) \\ &= d\nu(x_2|x_1)\, d\hat{\bar{\sigma}}(x_1)\end{aligned}$$

□

Lemma 2. *The solution to the primal problem characterizes the BRS when using partial state-feedback control.*

Proof. In this proof, we consider the special case of single input control; a more general version follows naturally. For convenience, denote by $\mathcal{U}_s$ and $\mathcal{U}_p$ respectively, the set of all admissible state-feedback and partial-state-feedback control laws; by definition, $\mathcal{U}_p \subseteq \mathcal{U}_s$. Similarly, let $\mathcal{X}_s$ and $\mathcal{X}_p$ represent the collection of admissible state trajectories; $\mathcal{X}_p \subseteq \mathcal{X}_s$.

By definition, for any initial condition $x_0 \in X_0^o$, there exists a partial state-feedback control law $u(t, x_1)$ such that the resulting state trajectory is admissible. Thus, for any initial measure μ_0, there exist measures μ, μ_T, σ which satisfy the following constraints

$$\delta_T \times \mu_T = \delta_0 \times \mu_0 + \mathcal{L}_f'\mu + \mathcal{L}_g'\sigma$$

such that μ_0, μ_T, $\mu \geq 0$, supp $(\mu_0) \subset \mathcal{X}$, supp $(\mu_T) \subset X_T$ and supp $(\mu) \subset T \times X_1 \times X_2$. This assertion follows from applying the result in [1, Lemma 1] to the state-feedback problem posed in [2] for any feasible trajectory and reviewing the relation between $\mathcal{U}_s$ and $\mathcal{U}_p$, and $\mathcal{X}_s$ and $\mathcal{X}_p$.

Since the Euclidean space is Polish and hence Souslin, and separable, it follows that μ can be decomposed as follows [4, Corollary 10.4.13]

$$d\mu = d\nu(x_2|t, x_1)\, d\pi_{t,x_1}\mu(t, x_1).$$

Now, since the occupation measure can be interpreted as the time spent in the region of the product space, naturally, the (t, x_1) marginal $(\pi_{t,x_1}\mu)$, is the total time spent by all feasible state trajectories in a slice of the product space; and $\nu(x_2|t, x_1)$ can be interpreted as measure of how well the trajectories are distributed along the x_2 direction at every time instant (??). Naturally, if $\nu^*(A|t, x_1) = 0$, then $\nu(A|t, x_1) = 0\ \forall (t, x_1) \in T \times X_1$ ($\because \mathcal{X}_p \subseteq \mathcal{X}_s$); and hence $\nu(\cdot|t, x_1) \ll \nu^*(\cdot|t, x_1)$. Thus, using [5, Theorem 58], $\exists \phi(t, x_1)$, a $\pi_{t,x_1}\mu$ measurable function such that

$$\begin{aligned}d\mu &= \phi(t, x_1)\, d\nu^*(x_2|t, x_1)\, d\pi_{t,x_1}\mu(t, x_1), \\ &= d\nu^*(x_2|t, x_1)\, d\bar{\mu}(t, x_1),\end{aligned}$$

where the final equality is obtained by appropriately defining $\bar{\mu}$.

Since σ and μ are solutions to the Liouville equation with R-N derivative $u = u(t, x_1)$, the following relations hold

$$\begin{aligned} d\sigma &= u\, d\mu, \\ &= u d\nu^*(x_2|t, x_1)\, d\bar{\mu}(t, x_1), \\ &= d\nu^*(x_2|t, x_1)\, d\bar{\sigma}(t, x_1), \end{aligned}$$

with $d\bar{\sigma} = u\, d\bar{\mu}$.

Define σ^+, $\sigma^- \geq 0$ such that the following holds (Jordan decomposition)

$$\sigma = \sigma^+ - \sigma^-.$$

The regular conditional measures of σ^+ and σ^- $\forall y \in Y$ are both $\nu(\cdot|t, x_1)$ (Facts 1 and 2). Consequently, $\sigma^\pm$ can be decomposed as follows

$$\begin{aligned} d\sigma^+ &= d\nu^*(x_2|t, x_1)\, d\bar{\sigma}^+(t, x_1) \\ d\sigma^- &= d\nu^*(x_2|t, x_1)\, d\bar{\sigma}^-(t, x_1). \end{aligned}$$

Finally, to see that $\hat{\sigma}$ shares the same conditional as μ, use Fact 2 and Lemma 1.

Thus, it follows that for every initial condition in the BRS, there exists a feasible solution to $\mathbf{P}_p$ and hence the optimization problem $\mathbf{P}_p$ is stronger that any other any problem that exactly identifies the BRS. Thus if q^* is the optimal value of the cost, then $q^* \geq \lambda(X_0^o)$.

Next, we show that $q^* \leq \lambda(X_0^o)$ by contradiction. Let $(\mu_0,\, \mu_T,\, \mu,\, \sigma^+,\, \sigma^-,\, \hat{\sigma})$ be a feasible solution to $\mathbf{P}_p$ and suppose $\lambda\,(A := \text{supp }(\mu_0)\backslash X_0^o) \neq 0$. By definition, $\exists u = u(t, x_1)$ such that $d\sigma = u\, d\mu$; re-define the dynamics of the system by subsuming this control law into the drift vector field; i.e.

$$\dot{x} = \bar{f}(t, x_1, x_2) = f(t, x_1, x_2) + u(t, x_1) g(t, x_1, x_2).$$

It is evident that the tuple $(\mu_0,\, \mu,\, \mu_T)$ is a feasible solution to the optimization problem

$$\delta_0 \times \mu_0 = \delta_T \times \mu_T + \mathcal{L}'_{\bar{f}} \mu.$$

Using [1, Lemma 3], it then follows that there exists a family of admissible trajectories emanating from supp (μ_0) and terminating in supp (μ_T). This is a contradiction since trajectories starting from supp $(\mu)\backslash X_0^o$ cannot be admissible; i.e. $\lambda(X_0^o) \geq \lambda(\text{supp }(\mu_0))$. Thus, $\lambda(X_0^o) = \lambda(\text{supp }(\mu_0))$. □

The next Lemma is a variation of [1, Lemma 2] in that it establishes that the BRS can be identified as the interior of the super-level set $\{(x_1, x_2) \mid w(x_1, x_2) \geq 1\}$.

Lemma 3. *Let $(v, w, p_{1,\dots,m})$ be a tuple of feasible solutions to the dual $\mathbf{D}_p$. Then $v(0, \cdot, \cdot) \geq 0$ and $w \geq 1$ on X_0 (:= supp (μ_0)).*

Proof. By the fundamental theorem of calculus and the constraints of $\mathbf{D}_p$,

$$\begin{aligned} 0 \leq v(T, x(T)) &= v(0, x(0)) + \int_0^T \mathcal{L}_f v + \mathcal{L}_g v\, u(t)\, dt \\ &= v(0, x(0)) + \int_{T \times X_1 \times X_2} (\mathcal{L}_f v + \mathcal{L}_g v\, u(t, x_1, x_2)\,)\, d\mu \\ &= v(0, x(0)) + \int_{T \times X_1} (\phi \circ \mathcal{L}_f v + \phi \circ (\mathcal{L}_g v\, u(t, x_1, x_2))\,)\, d\bar{\mu} \\ &= v(0, x(0)) + \int_{T \times X_1} \left(\phi \circ \mathcal{L}_f v + \sum_{i=1}^m \phi \circ [p]_i + \phi \circ (\mathcal{L}_g v\, u(t, x)) - \sum_{i=1}^m \phi \circ [p]_i \right) d\bar{\mu} \\ &\leq v(0, x(0)) \leq w(x(0)) - 1 \end{aligned}$$

where $x(t) = (x_1(t), x_2(t))$. □

Notes / todo

1. Check to ensure that all text in red make sense.
2. Why is it okay to optimize over just $\bar{\mu} \in \mathcal{M}(T \times X_1)$? Use result in [4, Theorem 10.7.2].
3. Using the regular conditional from an unsigned measure and a positive 'marginal', we can construct another unsigned measure?
4. Evaluating output-feedback controllability using inner approximations
5. Change notations to ensure there is no confusions between the final time T and the interval $T = [0, T]$.
6. Having changed the primal problem formulation, need to show that the $\hat{\sigma}$ has the same conditional as μ. (Lemma 2)
7. Check to see if the assumption of controllability is required and the kind of dynamical systems can be entertained [1, Assumption 2].

References

[1] Henrion, D.; Korda, M., "Convex Computation of the Region of Attraction of Polynomial Control Systems," Automatic Control, IEEE Transactions on , vol.59, no.2, pp.297,312, Feb. 2014 doi: 10.1109/TAC.2013.2283095

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