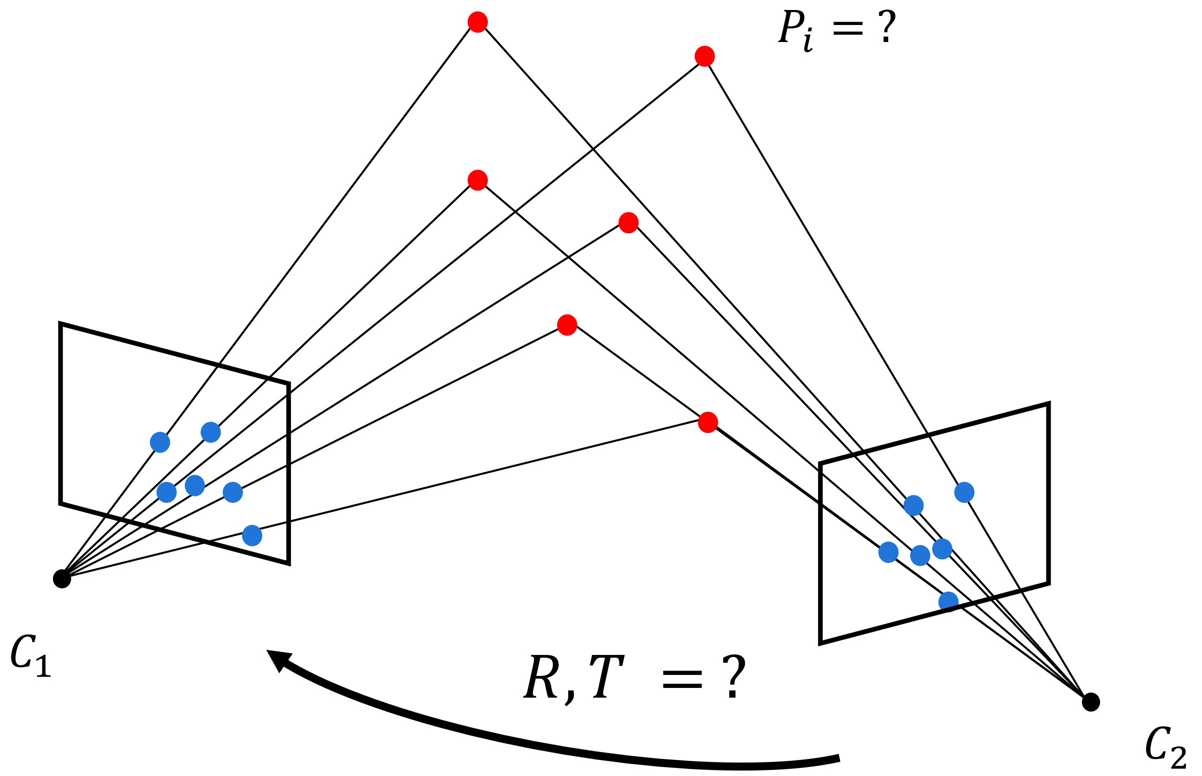
Structure from Motion | definition

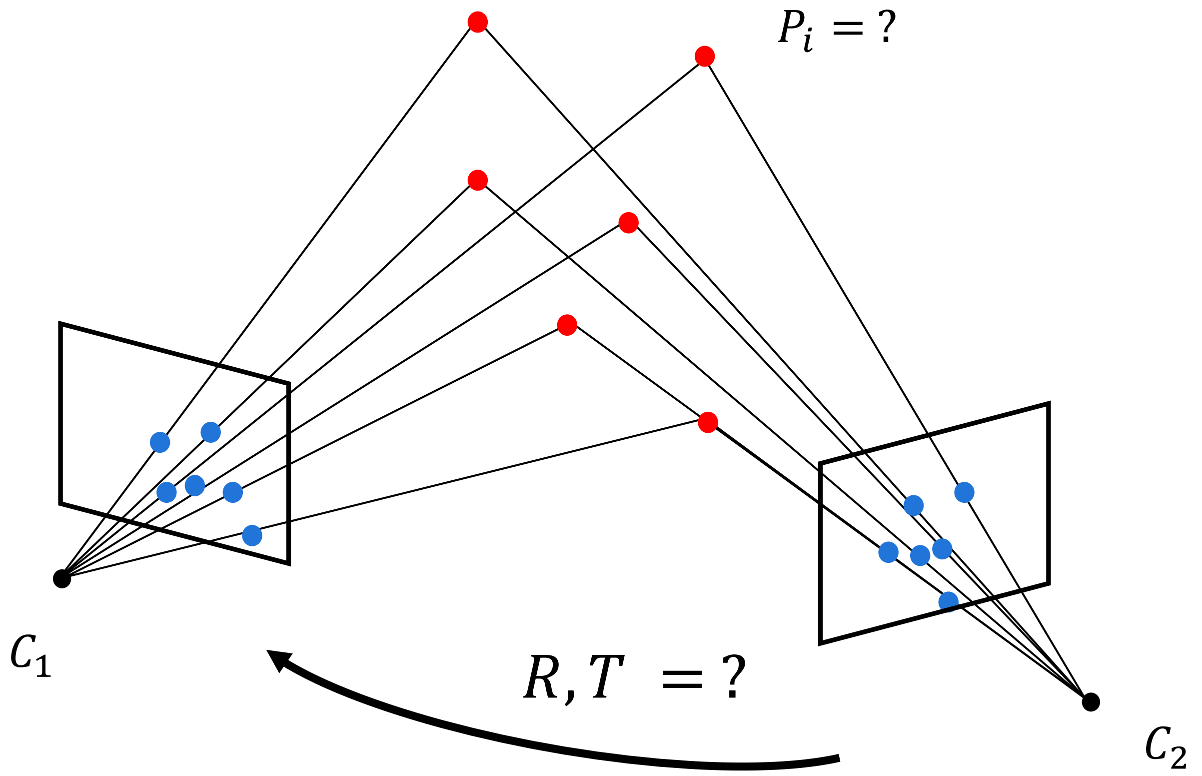
- **Goal:** given many image point correspondences, compute simultaneously the 3D structure and the relative pose



Structure from Motion | definition

- **Problem formulation:** Given many points *correspondence* between two images, $\{(u^i{}_1, v^i{}_1),\ (u^i{}_2, v^i{}_2)\}$, simultaneously compute the 3D location $\boldsymbol{P}_i$, the camera relative-motion parameters $(\boldsymbol{R}, \boldsymbol{t})$, and camera intrinsic $\boldsymbol{K}_{1,2}$ that satisfy

$$
\begin{cases}
\lambda_1 \begin{bmatrix} u^i{}_1 \\ v^i{}_1 \\ 1 \end{bmatrix} = K_1 [0|0] \cdot \begin{bmatrix} X^i{}_w \\ Y^i{}_w \\ Z^i{}_w \\ 1 \end{bmatrix} \\
\lambda_2 \begin{bmatrix} u^i{}_2 \\ v^i{}_2 \\ 1 \end{bmatrix} = K_2 [R|T] \cdot \begin{bmatrix} X^i{}_w \\ Y^i{}_w \\ Z^i{}_w \\ 1 \end{bmatrix}
\end{cases}
$$



Structure from Motion | definition

- We study the case in which the camera is «calibrated» (K is known)
- Thus, we want to find R, T, P_i that satisfy

$$\begin{cases} \lambda_1 \begin{bmatrix} \bar{u}^i{}_1 \\ \bar{v}^i{}_1 \\ 1 \end{bmatrix} = [I|0] \cdot \begin{bmatrix} X^i{}_w \\ Y^i{}_w \\ Z^i{}_w \\ 1 \end{bmatrix} \\ \lambda_2 \begin{bmatrix} \bar{u}^i{}_2 \\ \bar{v}^i{}_2 \\ 1 \end{bmatrix} = [R|T] \cdot \begin{bmatrix} X^i{}_w \\ Y^i{}_w \\ Z^i{}_w \\ 1 \end{bmatrix} \end{cases}$$

Structure from Motion | how many points?

How many knowns and unknowns?

- $\boldsymbol{4n}$ **knowns:**
  - n correspondences; each one $(u^i{}_1, v^i{}_1)$ and $(u^i{}_2, v^i{}_2),\ i = 1 \dots n$
- $\boldsymbol{5 + 3n}$ **unknowns**
  - 5 for the motion up to a scale (rotation $\mapsto$ 3, translation $\mapsto$ 2)
  - $3n =$ number of coordinates of the n 3D points

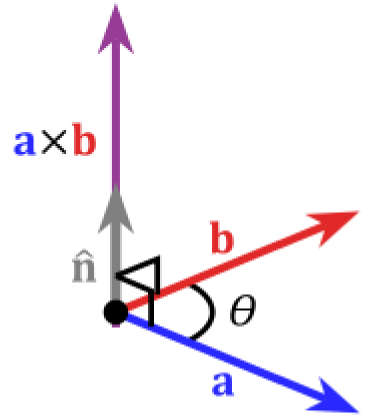
Does a solution exist?

- Yes, if and only if the number of independent equations $\geq$ number of unknowns

$$\Rightarrow 4n \geq 5 + 3n \Rightarrow \boxed{\mathbf{n \geq 5}}$$

Cross Product (or Vector Product) | review

$$\vec{a} \times \vec{b} = \vec{c}, \quad \|\vec{c}\| = \|\vec{a}\|\|\vec{b}\| \sin(\theta) \cdot \vec{n}$$



- Vector cross product takes two vectors and returns a third vector that is perpendicular to both inputs

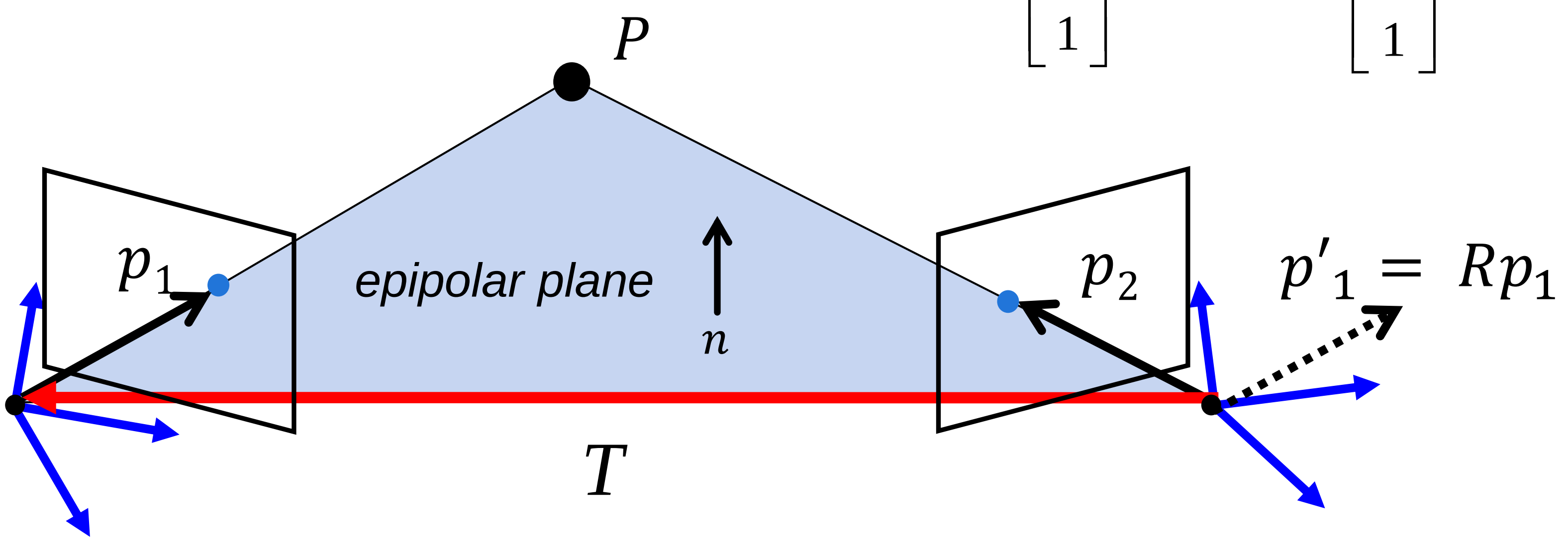
$$\vec{a} \cdot \vec{c} = 0$$
$$\vec{b} \cdot \vec{c} = 0$$

- The cross product of two parallel vectors = 0
- The vector cross product also can be expressed as the product of a skew-symmetric matrix and a vector

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Epipolar Geometry

$$p_1 = \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ 1 \end{bmatrix}$$



p_1, p_2, T are coplanar:

$$p_2^T \cdot n = 0 \Rightarrow p_2^T \cdot (T \times p_1') = 0 \Rightarrow p_2^T \cdot (T \times (Rp_1)) = 0$$

$$\Rightarrow p_2^T [T]_\times R\, p_1 = 0 \quad \Rightarrow \boxed{p_2^T\, E\, p_1 = 0 \quad \textit{epipolar constraint}}$$

$$\boxed{\mathrm{E} = [T]_\times R \quad \textit{essential matrix}}$$

Epipolar Geometry

- The Essential Matrix can be computed from 5 image correspondences [Kruppa, 1913]. The more points, the higher accuracy
- The Essential Matrix can be decomposed into R and T by recalling that $\mathrm{E} = [T]_{\times} R$
  Two distinct solutions for R and T are possible (i.e., 4-fold ambiguity)

$$p_1 = \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ 1 \end{bmatrix} \quad p_2 = \begin{bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ 1 \end{bmatrix} \quad \textit{Normalized image coordinates}$$

$$\boxed{\begin{array}{ll} p_2^T \, E \, p_1 = 0 & \textit{Epipolar constraint} \\ \\ \mathrm{E} = [T]_{\times} R & \textit{Essential matrix} \end{array}}$$

How to compute the Essential Matrix?

- The Essential Matrix can be computed from 5 image correspondences [Kruppa, 1913]. However, this solution is not simple. It took almost one century until an efficient solution was found! [Nister, CVPR'2004]

- The first popular solution uses 8 points and is called 8-point algorithm [Longuet Higgins, 1981]

The 8-point algorithm

$$\boldsymbol{p_1} = (\bar{u}_1, \bar{v}_1, 1)^T, \quad \boldsymbol{p_2} = (\bar{u}_2, \bar{v}_2, 1) \qquad p_2^T \; E \; p_1 = 0$$

$$\begin{bmatrix} \bar{u}_2 & \bar{v}_2 & 1 \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \underbrace{\begin{bmatrix} u_2 u_1 & u_2 v_1 & u_2 & v_2 u_1 & v_2 v_1 & v_2 & u_1 & v_1 & 1 \end{bmatrix}}_{\text{Q (this matrix is \textbf{known})}} \underbrace{\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{21} \\ e_{22} \\ e_{23} \\ e_{31} \\ e_{32} \\ e_{33} \end{bmatrix}}_{\text{E (this matrix is \textbf{unknown})}} = 0$$

Minimize:

$$\boxed{\sum_{i=1}^{N} (p_1^{iT} \boldsymbol{E} \, p_1^{i})^2 :}$$

under the constraint $\|\boldsymbol{E}\|^2 = 1$

- A linear least-square solution is given through Singular Value Decomposition by the eigenvector of Q corresponding to its smallest eigenvalue (which is the unit vector that minimizes $|Q \cdot E|^2$)

The 8-point algorithm

```
function F = calibrated_eightpoint(p1, p2)

p1 = p1'; % 3xN vector; each column = [u;v;1]
p2 = p2'; % 3xN vector; each column = [u;v;1]

Q = [p1(:,1).*p2(:,1) , ...
     p1(:,2).*p2(:,1) , ...
     p1(:,3).*p2(:,1) , ...
     p1(:,1).*p2(:,2) , ...
     p1(:,2).*p2(:,2) , ...
     p1(:,3).*p2(:,2) , ...
     p1(:,1).*p2(:,3) , ...
     p1(:,2).*p2(:,3) , ...
     p1(:,3).*p2(:,3) ] ;

[U,S,V] = svd(Q);
E = V(:,9);
E = reshape(V(:,9),3,3)';
```