Q) $1+3+9+27+\dots+3^n = \frac{3^{n+1}-1}{2}$ for all $n \geq 0$

Solution:

$$1+3+9+27+\dots+3^n = \frac{3^{n+1}-1}{2}$$

The above equation can also be written as

$$3^0+3^1+3^2+3^3+\dots+3^n = \frac{3^{n+1}-1}{2} \rightarrow (1)$$

$$3^0+3^{(1+2+3+\dots+n)} = \frac{3^{n+1}-1}{2}$$

$$3^0+3^{n(n+1)/2} = \frac{3^{n+1}-1}{2} \rightarrow (2)$$

$n=1$ in the above equation, we have

$$1+3^{1(1+1)/2} = 3^{1+1}-1/2$$

$$1+3^{2/2} = 3^2-1/2$$

$$4=4$$

Hence the result is true for $n=1$

$n=k$ in the equation (1), we get,

$$3^0+3^1+3^2+3^3+\dots+3^k = \frac{3^{k+1}-1}{2} \rightarrow (3)$$

$n=k+1$ in the equation (1), we get,

$$\underbrace{3^0+3^1+3^2+3^3+\dots+3^k}+3^{k+1} = \frac{3^{(k+1)+1}-1}{2}$$

$\hookrightarrow$ from eq $\rightarrow$ (3)

$$\frac{3^{k+1}-1}{2}+3^{k+1} = \frac{3^{(k+1)+1}-1}{2}$$

$$\frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} = \frac{3^{k+2}-1}{2}$$

$$\frac{3^{k+1}(1+2)-1}{2} = \frac{3^{k+2}-1}{2}$$

$$\frac{3^{k+1}(3)-1}{2} = \frac{3^{k+2}-1}{2}$$

$$\frac{3^{k+1+1}-1}{2} = \frac{3^{k+2}-1}{2}$$

$$\frac{3^{k+2}-1}{2} = \frac{3^{k+2}-1}{2}$$

Hence the result is true for $n=k+1$
$\therefore$ for all $n \geq 0$, the result is correct.
$\therefore$ by, mathematical induction, the equation $1+3+9+27+\cdots+3^n = \frac{3^{n+1}-1}{2}$ is correct for all $n \geq 0$

Proof of the equation

Let $S = 1+3+9+27+\cdots+3^n$

$S-1 = 3+9+27+\cdots+3^n \rightarrow (1)$

Multiply '3' on both sides

$3(S-1) = 3(3+9+27+\cdots+3^n)$

$3(S-1) = 9+27+81+\cdots+3^{n+1} \rightarrow (2)$

From $eq^n(1)$

From eq ...

$s-1 = 3+9+27+\cdots+3^n$

Subtract '3' on both sides

$s-1-3 = (\cancel{3}+9+27+\cdots+3^n)-\cancel{3}$

$(s-4) = 9+27+\cdots+3^n \rightarrow (3)$

Subtract eq $\rightarrow$ (2) & eq $\rightarrow$ (3)

$3(s-1)-(s-4) = (9+27+\cancel{+}\cdots+3^n+3^{n+1}) - (9+27+\cancel{+}\cdots 3^n)$

$3s-3-s+4 = 3^{n+1}$

$2s+1 = 3^{n+1}$

$2s = 3^{n+1}-1$

$\boxed{s = \frac{3^{n+1}-1}{2}}$ for all $n \geq 0$