Control of Inverted Pendulum using Kalman Filter

CS530 Machine Learning Project

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Abstract

This project investigates the control of an inverted pendulum system mounted on a cart using Kalman filtering and Linear Quadratic Regulator (LQR) techniques. The Kalman filter is employed to estimate the full state from noisy measurements of system variables, while LQR is utilized to generate optimal control signals for stabilizing the inverted pendulum. The study involves rigorous simulations conducted in Python to explore the dynamic behavior, stability, controllability, and observability of the system. By examining the interplay between these fundamental aspects of control theory, insights are gained into the system's response to external influences, its inherent robustness, and susceptibility to disturbances. The analysis of controllability and observability sheds light on the system's manipulability and our ability to accurately observe its states. Overall, this research aims to deepen understanding of inverted pendulum systems and provide a foundation for the development of effective control strategies.

1 Introduction

In the realm of control systems, the inverted pendulum has long been emblematic of intricate challenges, demanding precise manipulation for stability maintenance. Its control not only holds theoretical significance but also finds practical applications across domains such as robotics, aerospace, and transportation. In this endeavor, we delve into the dynamics, stability, controllability, and observability of an inverted pendulum system mounted on a cart. Leveraging Python simulations, we explore the nuanced interplay of these foundational aspects of control theory. By scrutinizing the system's dynamics, we unravel insights into its behaviors and responses to external stimuli. Evaluation of stability unveils the inherent robustness and susceptibility to disturbances. Additionally, analysis of controllability and observability elucidates the system's manipulability and our capacity to accurately perceive its states. Through this holistic inquiry, we aim to deepen comprehension of inverted pendulum systems, laying the groundwork for effective control strategies.

2 Control System

Control systems play a pivotal role in regulating the behavior of dynamic systems, such as the inverted pendulum mounted on a cart. Two primary paradigms of control systems are closed-loop and open-loop control.

2.1 Open-loop Control

Open-loop control, also known as feedforward control, is a straightforward approach where the control action is determined solely based on the system's input, without considering the system's output or the effect of disturbances. In an open-loop control system, a predefined control signal is applied to the system, expecting a desired response.

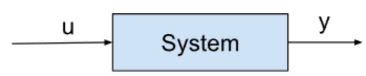

Figure 2.1: Open loop control system

2.2 Closed-loop Control

Closed-loop control, also known as feedback control, is a more sophisticated approach that utilizes feedback from the system's output to adjust the control action dynamically. In a closed-loop control system, the system's output is continuously monitored, and this information is fed back to the controller to compute corrective actions. By continuously comparing the desired output (setpoint) with the actual output (feedback), closed-loop control systems can effectively regulate system behavior, compensating for disturbances and uncertainties.

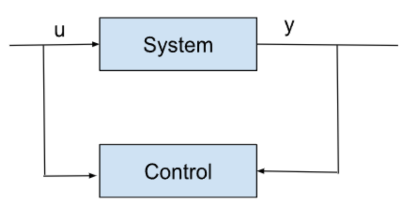

Figure 2.2: Closed loop control system

2.3 Benefits of feedback control systems

Following are the benefits of feedback control systems:

- It is used to stabilize the unstable system.
- It is used to mitigate external disturbances.
- It is used to correct model uncertainty.

3 Linearized dynamic systems

Linearized dynamic systems are mathematical approximations of nonlinear systems around a specific operating point. Nonlinear systems often exhibit complex behaviors that are challenging to analyze directly. Linearization simplifies the analysis by approximating the nonlinear system with a linear one in the vicinity of the operating point.

This linearization process involves calculating the first-order derivatives of the nonlinear equations and evaluating them at the operating point. The resulting linearized equations are typically represented in state-space form, comprising state equations and output equations.

The inverted pendulum is a nonlinear and inherently unstable system. The dynamics of the pendulum involve complex interactions between them. Analyzing these dynamics directly can be challenging due to their nonlinear nature. To simplify the analysis and design of control strategies for the inverted pendulum, linearization is often employed. This process involves approximating the nonlinear equations governing the pendulum's behavior with linear equations around a specific operating point, such as the upright position.

The equations for linearized dynamic systems are typically represented in state-space form, which comprises state equations and output equations. For a generic nonlinear system described by state variables x, control inputs u, and system dynamics f(x,u), the linearized state-space equation is

$$\dot{x} = Ax + Bu \tag{1}$$

$\dot{x}$ represents the rate of change of the state vector x with respect to time.

x is the state vector, which comprises all the system's state variables.

A is the system matrix, which describes how the state variables evolve over time.

B is the input matrix, which relates the control inputs to their effect on the state variables..

u represents the control input vector.

The system dynamics of the inverted pendulum system is as follows:

$$D = m \cdot L \cdot L \cdot (M + m \cdot (1 - \cos^2(y_3)))$$

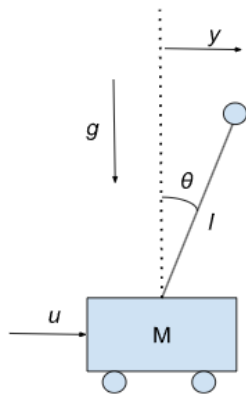

Figure 3.1: Inverted Pendulum

D represents a key term in the equations of motion, influencing the system's behavior by accounting for the combined effects of mass distribution, geometry, and external forces.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad u = \begin{bmatrix} u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -d & \frac{(m+M)mg}{D} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{(mL^2)}{D} & -d\frac{mL\cos(y_3)}{D} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{mL^2}{D} \\ -mL\cos(y_3)\frac{1}{D} \end{bmatrix}$$

The state variables y_1, y_2, y_3, and y_4 represent position of the cart, velocity of the cart, angular position of the pendulum and the angular velocity of the pendulum respectively.

The system dynamics are influenced by parameters such as gravitational acceleration g, pendulum arm length L, damping d, mass of the pendulum m, and mass of the cart M.

The eigenvalues of matrix A determine the stability of the system. In this case, with specific choices of parameters, the system has unstable eigenvalues, indicating an unstable saddle-type fixed point.

However, we can employ feedback control to stabilize the system. By applying a control law of the form $u = -Kx$, where K is a gain matrix, we can modify the system's dynamics. The closed-loop system becomes:

$$\dot{x} = (A - BK)x$$

By appropriately choosing the gain matrix K, we can place the eigenvalues of the closed-loop system where we desire. In this example, choosing K as a specific matrix leads to stable eigenvalues for the closed-loop system, ensuring stability despite the inherent instability of the inverted pendulum. This demonstrates the power of feedback control in modifying system behavior to achieve desired performance.

4 Controllability and Observability

Controllability and observability are fundamental concepts in control theory, particularly relevant when dealing with dynamic systems like the inverted pendulum.

4.1 Controllability

Controllability refers to the ability to steer the system from any initial state to any desired state using appropriate control inputs within a finite time. For the inverted pendulum, controllability means being able to stabilize the pendulum at any desired angle or position by applying appropriate torques to the pendulum arm.

The controllability matrix $\mathbf{Ctb}$ is given by:

$$\mathbf{Ctb} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Where $\mathbf{A}$ is the system matrix, $\mathbf{B}$ is the control input matrix, and n is the dimension of the state vector $\mathbf{x}$.

The system is said to be controllable if the controllability matrix $\mathbf{C}$ has full rank, meaning its columns are linearly independent. In other words, the system is controllable if it's possible to reach any state from any initial state within a finite time by applying appropriate control inputs.

4.2 Observability

Observability refers to the ability to infer the complete internal state of the system based on its outputs or measurements. In the context of the inverted pendulum, observability means being able to accurately estimate the pendulum's angle and angular velocity based on measurements such as the position of the cart and the pendulum. Observability is crucial for feedback control strategies because it allows the controller to make informed decisions based on the system's current state.

For a linear system represented by the output equation:

$$y = \mathbf{C}x$$

where y is the output vector and $\mathbf{C}$ is the observation matrix.

The observability matrix $\mathbf{O}$ is given by:

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{(n-1)} \end{bmatrix}$$

where n is the dimension of the state vector x.

The system is said to be observable if the observability matrix $\mathbf{O}$ has full rank, meaning its rows are linearly independent. In other words, the system is observable if it's possible to reconstruct the complete state of the system from its outputs over a finite time horizon.

5 Linear Quadratic Regulator (LQR)

The Linear Quadratic Regulator (LQR) is a control strategy used to stabilize linear systems by minimizing a quadratic cost function. This controller is particularly effective for systems with full-state measurements or when a reliable estimate of the full state is available.

Objective: The main goal of the LQR controller is to regulate the state of the system to a desired value, typically zero, by adjusting the control inputs.

Cost Function: The control strategy is based on minimizing a cost function, $J(t)$, defined as the integral of the squared deviations of the state variables from zero and the squared control inputs. The cost function is given by:

$$J(t) = \int_0^t (x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)) d\tau$$

where $x(\tau)$ represents the state vector, $u(\tau)$ represents the control input vector, and Q and R are positive semi-definite and positive definite matrices, respectively, used to weigh the state deviations and control efforts.

Optimal Controller Design: The optimal control law is derived by minimizing the cost function $J(t)$ over an infinite time horizon. The resulting controller gains, K_r, are calculated by solving the algebraic Riccati equation:

$$K_r = R^{-1} B^T X$$

where X is the solution to the Riccati equation:

$$A^T X + XA - XBR^{-1} B^T X + Q = 0$$

Implementation: The Riccati equation can be solved numerically to obtain the optimal controller gains. In MATLAB, for example, the LQR controller gains can be computed using the `lqr` function:

$$K_r = \mathrm{lqr}(A, B, Q, R)$$

Application: Once the controller gains are obtained, the control law $u = -K_r x$ is applied to the system, where x is the state vector. This control law adjusts the control inputs to stabilize the system and minimize the cost function over time.

Overall, the LQR controller provides an optimal balance between stabilizing the system and minimizing control effort, making it a widely used strategy in control engineering.

6 Kalman Filter

The Kalman filter is a powerful tool for estimating the full state of a system based on limited and noisy measurements. It's particularly useful when obtaining full-state measurements is difficult or expensive.

Objective: The goal of the Kalman filter is to estimate the full state x of a system from noisy measurements y, even when only partial information is available.

Assumptions: The Kalman filter assumes that both the disturbances w_d and the measurement noise w_n are zero-mean Gaussian processes with known covariances.

Estimation Process: The filter works by recursively updating its estimate of the state based on the measurements and the system dynamics. The estimate evolves over time according to:

$$\begin{aligned}\frac{d}{dt}\hat{x} &= A\hat{x} + Bu + K_f(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du\end{aligned}$$

where A, B, C, and D are matrices obtained from the system model, K_f is the filter gain, and $\hat{x}$ and $\hat{y}$ are the estimated state and output, respectively.

Optimal Estimation: The filter gain K_f is determined to minimize a cost function that balances the accuracy of the estimation and the noise attenuation. The solution to this optimization problem is obtained by solving an algebraic Riccati equation.

Convergence: When the system is observable, the eigenvalues of $A - K_f C$ can be arbitrarily placed by choosing K_f. Stable eigenvalues ensure that the estimate $\hat{x}$ converges to the true state x asymptotically.

Overall, the Kalman filter is a versatile tool for estimating the full state of a system from noisy measurements, making it widely used in various fields such as control engineering, navigation, and signal processing.

7 State Estimation and Control System

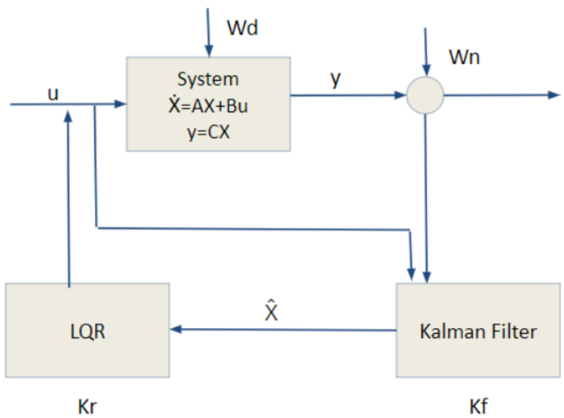

Figure 7.1: this block diagram illustrates the integration of the Kalman filter for state estimation and the LQR controller for optimal control to stabilize and regulate the system's behavior.

System Block: This represents the physical system under control. It consists of dynamics described by differential equations, typically represented by the state-space model:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the control input, and $\mathbf{y}$ is the output. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ define the system dynamics.

Kalman Filter Block: The Kalman filter is used to estimate the full state of the system based on noisy measurements. It takes the measurements $\mathbf{y}$ as input and produces an estimate $\hat{\mathbf{x}}$ of the full state $\mathbf{x}$. The filter works by recursively updating the state estimate based on the measurements and the system dynamics. It incorporates information about the system's dynamics and the measurement noise to produce an optimal estimate of the state.

LQR Block: The Linear Quadratic Regulator (LQR) is a control strategy used to stabilize the system by minimizing a quadratic cost function. It takes the estimated state $\hat{\mathbf{x}}$ as input and computes the control input $\mathbf{u}$ needed to stabilize the system. The LQR controller adjusts the control inputs based on the estimated state to regulate the state of the system to a desired value, typically zero, while minimizing control effort.

Feedback Loop: The estimated state $\hat{\mathbf{x}}$ from the Kalman filter is fed back to the LQR controller, which uses it to compute the control input $\mathbf{u}$. This feedback loop allows the controller to continuously adjust the control inputs based on the estimated state, ensuring that the system remains stable and achieves the desired performance.

Overall, this block diagram illustrates the integration of the Kalman filter for state estimation and the LQR controller for optimal control to stabilize and regulate the system's behavior. The combined use of estimation and control allows for robust and efficient control of the system, even in the presence of noise and uncertainties.

8 Results

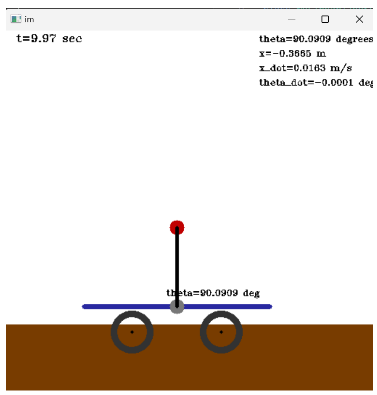

Figure 8.1: Simulation of balancing of inverted pendulum

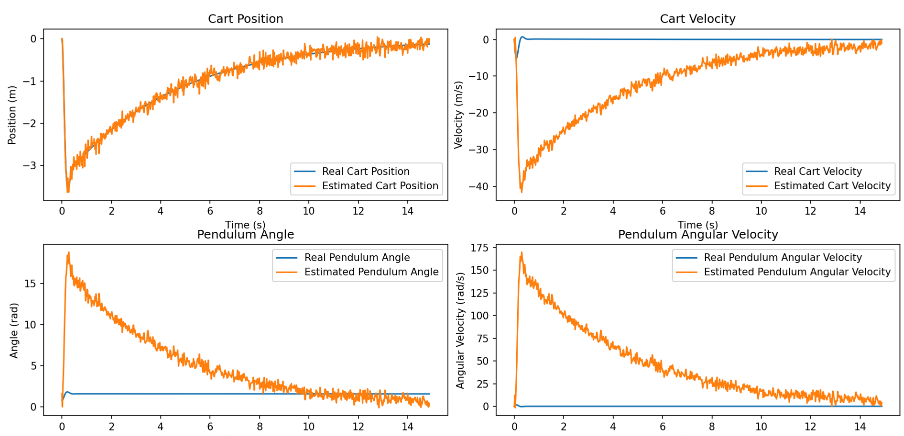

Figure 8.2: Graph of estimated values vs real values for all the state variables

9 Reference

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