苏弘源 2019010448 贝叶斯 HW2

1、

$P(y|\theta) = \theta e^{-\theta y}$, $\log P(y|\theta) = \log\theta - \theta y$

$d^2 \log P(y|\theta) / d\theta^2 = -\frac{1}{\theta^2}$

$\therefore J(\theta) = -E\left[d^2 \log P(y|\theta) / d\theta^2 \,|\, \theta \right] \propto \frac{1}{\theta^2}$

$p(\theta) \propto J(\theta)^{1/2} \propto \frac{1}{\theta}$

2.

a)

$p(\theta_1, \theta_2 | y) \propto \theta_1^{\alpha_1+y_1-1} \cdot \theta_2^{\alpha_2+y_2-1} \cdot (1-\theta_1-\theta_2)^{\sum_{i=3}^{J}\alpha_i+y_i-1}$

$\alpha = \frac{\theta_1}{\theta_1+\theta_2}$, $\beta = \theta_1+\theta_2$

$\left|\frac{\partial(\alpha,\beta)}{\partial(\theta_1,\theta_2)}\right| = \left|\begin{matrix} \frac{\theta_2}{(\theta_1+\theta_2)^2} & 1 \\ -\frac{\theta_1}{(\theta_1+\theta_2)^2} & 1 \end{matrix}\right| = \frac{1}{\theta_1+\theta_2} = \frac{1}{\beta}$

$\therefore p(\alpha,\beta|y) \propto \beta(\alpha\beta)^{\alpha_1+y_1-1} \cdot [(1-\alpha)\beta]^{\alpha_2+y_2-1} \cdot (1-\beta)^{\sum_{i=3}^{J}(\alpha_i+y_i)-1}$

$= \alpha^{\alpha_1+y_1-1} \cdot (1-\alpha)^{\alpha_2+y_2-1} \cdot \beta^{\alpha_1+y_1+\alpha_2+y_2-1} (1-\beta)^{\sum_{i=3}^{J}(\alpha_i+y_i)-1}$

$\propto \mathrm{Beta}(\alpha|\alpha_1+y_1, \alpha_2+y_2) \cdot \mathrm{Beta}(\beta|\alpha_1+y_1+\alpha_2+y_2, \sum_{i=3}^{J}\alpha_i+y_i)$

$\therefore p(\alpha|y) \propto \mathrm{Beta}(\alpha_1+y_1, \alpha_2+y_2)$

b) 对于先验 $\mathrm{Beta}(\alpha_1, \alpha_2)$

$p(\alpha|y) \propto p(y|\alpha) \cdot p(\alpha)$

$\propto \alpha^{y_1} \cdot (1-\alpha)^{y_2} \cdot \alpha^{\alpha_1-1} \cdot (1-\alpha)^{\alpha_2-1}$

$= \alpha^{y_1+\alpha_1-1}(1-\alpha)^{y_2+\alpha_2-1} \propto \mathrm{Beta}(\alpha_1+y_1, \alpha_2+y_2)$

3.

$P(\mu,\sigma^2|y) \propto P(y|\mu,\sigma^2) \cdot p(\mu,\sigma^2)$

$\propto (\sigma^2)^{-n/2} \exp\left\{-\frac{(n-1)s^2+n(\mu-\bar{y})^2}{2\sigma^2}\right\}$

$\cdot \sigma^{-1}(\sigma^2)^{-(\nu_0/2+1)} \exp\left\{\frac{1}{2\sigma^2}[\nu_0\sigma_0^2+k_0(\mu-\mu_0)^2]\right\}$

$\propto N\text{-}\mathrm{Inv}\text{-}\chi^2\left(\frac{\mu_0 k_0+n\bar{y}}{n+k_0}, \frac{\sigma_n^2}{n+k_0}; n+\nu_0, \sigma_n^2\right)$

其中 $\sigma_n^2 = \dfrac{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0(\bar{y}-k_0)^2}{n+k_0}}{n+\nu_0}$

对于 noninformative prior

$P(\mu,\sigma^2) \propto (\sigma^2)^{-1}$

$\Rightarrow \nu_0=-1, \sigma_0=0, k_0=0$

$P(\mu,\sigma^2) \propto (\sigma^2)^{-1}$

$P(\mu,\sigma^2|y) \propto (\sigma^2)^{-(n/2+1)} \exp\left\{-\frac{(n-1)s^2+n(\mu-\bar{y})^2}{2\sigma^2}\right\}$

$\therefore \sigma^2|\mu,y \sim \mathrm{Inv}\text{-}\chi^2\left(n, \frac{(n-1)s^2+n(\mu-\bar{y})^2}{n}\right)$