# 统计信号处理基础 第 01 次作业

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1. 设二元假设检验的观测信号模型为

$$\begin{cases} H_0:\ x=-1+n \\ H_1:\ x=1+n \end{cases} \qquad n \sim \mathcal{N}\left(0, \sigma_n^2=\frac{1}{2}\right)$$

若两种假设是等先验概率的，而代价因子为$C_{00}=1, C_{01}=8, C_{10}=4, C_{11}=2$，试求贝叶斯（最佳）表达式和平均代价$C$。

【解答】

似然函数分别为

$$p_0(x)=P(x|H_0)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{(x+1)^2}{2\sigma_n^2}\right)=\frac{1}{\sqrt{\pi}}\exp(-(x+1)^2)$$
$$p_1(x)=P(x|H_1)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{(x-1)^2}{2\sigma_n^2}\right)=\frac{1}{\sqrt{\pi}}\exp(-(x-1)^2)$$

则似然比为

$$\lambda(x)=\frac{p_1(x)}{p_0(x)}=\exp\big(-((x-1)^2-(x+1)^2)\big)=\exp(4x)$$

而

$$\xi=P(H_0)=\frac{1}{2}, \qquad \lambda_0=\frac{\xi(C_{10}-C_{00})}{(1-\xi)(C_{01}-C_{11})}=\frac{1}{2}$$

因此贝叶斯（最佳）表达式为

$$\lambda(x)\gtrless_{H_0}^{H_1}\lambda_0 \ \Rightarrow\ \frac{p_1(x)}{p_0(x)}\gtrless_{H_0}^{H_1}\frac{\xi(C_{10}-C_{00})}{(1-\xi)(C_{01}-C_{11})} \ \Rightarrow\ \exp(4x)\gtrless_{H_0}^{H_1}\frac{1}{2}$$

$$\Rightarrow\ x\gtrless_{H_0}^{H_1}\frac{1}{4}\ln\left(\frac{1}{2}\right)$$

记$V_T=\frac{1}{4}\ln(\frac{1}{2})$，则虚警概率和检测概率分别为

$$P_F=\int_{D_1}p_0(x)dx=\int_{V_T}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{(x+1)^2}{2\sigma_n^2}\right)dx=Q\left(\frac{V_T+1}{\sigma_n}\right)=Q\left(\sqrt{2}\left[\frac{1}{4}\ln\left(\frac{1}{2}\right)+1\right]\right)$$
$$P_D=\int_{D_1}p_1(x)dx=\int_{V_T}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{(x-1)^2}{2\sigma_n^2}\right)dx=Q\left(\frac{V_T-1}{\sigma_n}\right)=Q\left(\sqrt{2}\left[\frac{1}{4}\ln\left(\frac{1}{2}\right)-1\right]\right)$$

总的平均代价为

$$\begin{aligned}C&=C_{00}P(H_0)(1-P_F)+C_{10}P(H_0)P_F+C_{01}P(H_1)(1-P_D)+C_{11}P(H_1)P_D\\&=\frac{1}{2}[(1-P_F)+4P_F+8(1-P_D)+2P_D]\\&=\frac{9}{2}+\frac{3}{2}P_F-3P_D\\&=\frac{9}{2}+\frac{3}{2}Q\left(\sqrt{2}\left[\frac{1}{4}\ln\left(\frac{1}{2}\right)+1\right]\right)-3Q\left(\sqrt{2}\left[\frac{1}{4}\ln\left(\frac{1}{2}\right)-1\right]\right)\end{aligned}$$

2. 什么假设下代价函数曲线是上凸的?

【解答】

平均代价函数

$$
\begin{aligned}
C &= C_{00}P(H_0)P(D_0 \mid H_0) + C_{10}P(H_0)P(D_1 \mid H_0) + C_{01}P(H_1)P(D_0 \mid H_1) + C_{11}P(H_1)P(D_1 \mid H_1) \\
&= \xi C_{00}\int_{D_0} p_0(r)dr + \xi C_{10}\int_{D_1} p_0(r)dr + (1-\xi)C_{01}\int_{D_0} p_1(r)dr + (1-\xi)C_{11}\int_{D_1} p_1(r)dr \\
&= \xi C_{10} + (1-\xi)C_{11} + \int_{D_0}[\xi p_0(r)(C_{00}-C_{10}) + (1-\xi)p_1(r)(C_{01}-C_{11})]dr
\end{aligned}
$$

猜测先验概率为$P(H_0)=x$。假设正确判决的代价小于错误判决的代价，即$C_{10}-C_{00}>0$且$C_{01}-C_{11}>0$，则此时判决准则为

$$
\lambda(r) = \frac{p_1(r)}{p_0(r)} \gtrless_{H_0}^{H_1} \frac{x(C_{10}-C_{00})}{(1-x)(C_{01}-C_{11})} \triangleq \lambda_0(x)
$$

则平均代价为

$$
\begin{aligned}
C(\xi,x) &= \xi C_{00}(1-P_F(x)) + \xi C_{10}P_F(x) + (1-\xi)C_{01}P_M(x) + (1-\xi)C_{11}(1-P_M(x)) \\
&= \xi C_{10} + (1-\xi)C_{11} + \int_{D_0(x)}[\xi p_0(r)(C_{00}-C_{10}) + (1-\xi)p_1(r)(C_{01}-C_{11})]dr
\end{aligned}
$$

其中$D_0(x)=\{r|\lambda(r)<\lambda_0(x)\}$。注意$D_0(x)$有可能是分段的，取决于$\lambda(r)$的形式。

最小平均代价函数（对应贝叶斯准则的情形）为

$$
C_{min}(\xi) = C(\xi,x)|_{x=\xi}
$$

任意取定一个$x=\xi_1$，此时$C(\xi,\xi_1)$为一条直线（关于$\xi$的一次函数），且当$\xi\neq\xi_1$，即猜测的先验概率不等于实际的先验概率时，平均代价更大，则

$$
C(\xi,\xi_1) \geq C(\xi_1,\xi_1) = C_{min}(\xi_1)
$$

上式在$\xi=\xi_1$时取等。

又由于

$$
\frac{\partial C_{min}(\xi)}{\partial \xi} = C_{10}-C_{11} + \int_{D_0(\xi)}[p_0(r)(C_{00}-C_{10}) + p_1(r)(C_{11}-C_{01})]dr
$$

$$
\frac{\partial C(\xi,\xi_1)}{\partial \xi} = C_{10}-C_{11} + \int_{D_0(\xi_1)}[p_0(r)(C_{00}-C_{10}) + p_1(r)(C_{11}-C_{01})]dr = \left.\frac{\partial C_{min}(\xi)}{\partial \xi}\right|_{\xi=\xi_1}
$$

因此$C(\xi,\xi_1)$为$C_{min}(\xi)$在$\xi=\xi_1$处的切线。

记$f(r)=p_0(r)(C_{00}-C_{10})+p_1(r)(C_{11}-C_{01})$，则$f(r)$在$D_0(\xi)$上的积分可写为

$$
\int_{D_0(\xi)} f(r)dr = \int_{-\infty}^{\lambda_0(\xi)}\int_{\lambda(r)=u} f(r)dr\,du
$$

记$g(u)=\int_{\lambda(r)=u}f(r)dr=\int_{\lambda(r)=u}[p_0(r)(C_{00}-C_{10})+p_1(r)(C_{11}-C_{01})]dr$，则$\frac{\partial C_{min}(\xi)}{\partial \xi}$可化为

$$
\frac{\partial C_{min}(\xi)}{\partial \xi} = C_{10}-C_{11} + \int_{-\infty}^{\lambda_0(\xi)} g(u)du
$$

则$C_{min}(\xi)$的二阶导为

$$
\frac{\partial^2 C_{min}(\xi)}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(C_{10}-C_{11} + \int_{D_0(\xi)} f(r)dr\right) = \frac{\partial}{\partial \xi}\left(\int_{-\infty}^{\lambda_0(\xi)} g(u)du\right) = \lambda_0'(\xi)g(\lambda_0(\xi))
$$

其中

$$
\lambda_0'(\xi) = \frac{d}{d\xi}\left(\frac{\xi(C_{10}-C_{00})}{(1-\xi)(C_{01}-C_{11})}\right) = \frac{(C_{10}-C_{00})}{(1-\xi)^2(C_{01}-C_{11})} > 0
$$

$$
g(\lambda_0(\xi)) = \int_{\lambda(r)=\lambda_0(\xi)} f(r)dr = \int_{\lambda(r)=\lambda_0(\xi)}[p_0(r)(C_{00}-C_{10}) + p_1(r)(C_{11}-C_{01})]dr \leq 0
$$

（由于概率非负，即$p_0(r) \geq 0$且$p_1(r) \geq 0$，而$C_{00} - C_{10} < 0$且$C_{11} - C_{01} < 0$，因此恒有$f(r) \leq 0$，因此对$f(r)$的积分也将不大于 0，故上式成立。）

因此有

$$\frac{\partial^2 C_{min}(\xi)}{\partial \xi^2} = \lambda_0'(\xi) g(\lambda_0(\xi)) \leq 0$$

此式即满足$C_{min}(\xi)$上凸要求（上凸函数的二阶条件）。

综上所述，在正确判决的代价小于错误判决的代价的条件下（充分条件），$C_{min}(\xi)$恒为上凸函数。