苏泓源 2019010448 HW3

1.

Maxwell equations:

$$\begin{cases}\nabla\times\vec{E} = -j\omega\vec{B} & \text{-- ①}\\ \nabla\cdot\vec{D} = \rho & \text{-- ②}\\ \nabla\times\vec{H} = \vec{J}+j\omega\vec{D} & \text{-- ③}\\ \nabla\cdot\vec{B} = 0 & \text{-- ④}\end{cases}$$

由①、④可延伸出

$$\begin{cases}\vec{B} = \nabla\times\vec{A}\\ \vec{E}+j\omega\vec{A} = -\nabla\phi\end{cases}$$ 代入③中有

$$\frac{1}{\mu}\nabla\times(\nabla\times\vec{A}) = \vec{J}-j\omega\varepsilon(\nabla\phi+j\omega\vec{A})$$

又 $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$

$$\Rightarrow \nabla^2\vec{A}+\omega^2\mu\varepsilon\vec{A} = -\mu\vec{J}+\nabla(\nabla\cdot\vec{A}+j\omega\mu\varepsilon\phi)$$

引入洛伦兹规范使得

$\nabla\cdot\vec{A} = -j\omega\mu\varepsilon\phi$，代入上式即有

$$\nabla^2\vec{A}+\omega^2\mu\varepsilon\vec{A} = -\mu\vec{J}$$

同理 代入②中有

$$-\nabla\cdot\varepsilon(\nabla\phi+j\omega\vec{A}) = \rho$$

$$\Rightarrow \nabla^2\phi+\omega^2\mu\varepsilon\phi = -\rho/\varepsilon - j\omega(\nabla\cdot\vec{A}+j\omega\mu\varepsilon\phi)$$

引入洛伦兹规范

$$\nabla^2\phi+\omega^2\mu\varepsilon\phi = -\rho/\varepsilon$$

2.

通解：$\nabla^2 g(r)+\beta^2 g(r) = 0$ -- ① $(r\neq 0)$

记 $g(r) = u(r)/r$，在球坐标下有

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\frac{du(r)}{dr}r-u(r)}{r^2}\right)+\beta^2\frac{u(r)}{r} = 0$$

$$\Rightarrow \frac{d^2u(r)}{dr^2}+\beta^2 u(r) = 0$$

特征方程为 $x^2+\beta^2 = 0$，$x = e^{\pm j\beta r}$

∴独立解为

$$g_1(r) = C_1 e^{-j\beta r}/r,\quad g_2(r) = C_2 e^{j\beta r}/r$$

$r=0$ 处，

$$\nabla^2 g+\beta^2 g = -\delta(0)$$

$$\iiint(\nabla^2 g+\beta^2 g)\,dv = -1$$

又 $\nabla^2 g = \nabla\cdot\nabla g$

$$\iiint\nabla^2 g\,dv = \iiint\nabla\cdot(\nabla g)\,dv = \iint\nabla g\cdot d\vec{S}$$

代入 $g_1(r)$ 有

$$\iint\nabla g\cdot d\vec{S} = C_1\iint\frac{-j\beta r e^{-j\beta r}-e^{-j\beta r}}{r^2}\cdot d\vec{S},\quad r\to 0$$

$$= -C_1\cdot 4\pi = -1 \qquad \therefore C_1 = \frac{1}{4\pi}$$

$$\therefore g_1(r) = \frac{e^{-j\beta r}}{4\pi r}$$

同理 $g_2(r) = \dfrac{e^{j\beta r}}{4\pi r}$

为方程两个独立解.

3.

$\vec{D} = -\nabla\times\vec{F} = \varepsilon\vec{E}$ $\quad\therefore \vec{E} = -\frac{1}{\varepsilon}\nabla\times\vec{F}$

$\nabla\times\vec{H} = j\omega\vec{D} = -j\omega\nabla\times\vec{F}$

$\therefore \nabla\times(\vec{H}+j\omega\vec{F}) = 0 \Rightarrow \vec{B}+j\omega\vec{F} = \nabla\phi'$

$$\begin{cases}\vec{D} = -\nabla\times\vec{F}\\ \vec{H}+j\omega\vec{F} = \nabla\phi'\end{cases}\qquad \begin{cases}\vec{E} = -\frac{1}{\varepsilon}\nabla\times\vec{F}\\ \vec{B}+j\omega\mu\vec{F} = \nabla\phi'\end{cases}$$

场方程

$$\begin{matrix}\nabla\times\vec{E} = -j\omega\vec{B}\\ \nabla\times\vec{H} = j\omega\vec{D}\end{matrix} \Rightarrow [\nabla^2+\mu\varepsilon\omega^2]\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix} = 0$$

$$\begin{cases}\nabla^2\vec{E}+\mu\varepsilon\omega^2\vec{E} = 0\\ \nabla^2\vec{H}+\mu\varepsilon\omega^2\vec{H} = 0\end{cases}$$

仿照电流辐射场推导有

$$g(\vec{r},\vec{r}') = e^{-jk|\vec{r}-\vec{r}'|} / |\vec{r}-\vec{r}'| \quad , \; k^2 = \mu\varepsilon\omega^2$$

$$\phi'(\vec{r}) = \int -\frac{\rho_m}{\mu} g(\vec{r},\vec{r}')\,dv' \qquad \vec{F}(\vec{r}) = \int -\varepsilon \vec{M}(\vec{r}') g(\vec{r},\vec{r}')\,dv'$$

$$\vec{D} = -\nabla \times \vec{F}$$

$$\vec{H} = \nabla\phi'(\vec{r}) - j\omega\vec{F}$$