Analysis of the Stability in Nonlinear Ordinary Differential Equations

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Overview

Ordinary Differential Equations

Differential equations, which involve functions and their derivatives, are widely applied in physics, engineering, and biology. Mathematicians are interested in finding and characterizing the solutions (equilibrium points) to these equations.

A system of n^{th} order Autonomous Ordinary Differential Equation (ODE) can be represented as:

$$\dot{X} = F(X) \tag{1}$$

for some vector valued function F. Here we assume F is a collection of continuous differentiable functions.

Equilibrium points of an Ordinary Differential Equation

It is clear that for a system to achieve equilibrium the rate of change $\dot{X}$ must be 0.

Definition

An equilibrium of the n dimensional system $\dot{X} = F(X)$ is a vector $X^* \in \mathbb{R}^n$ such that $F(X^*) = 0$.

Stable, unstable and asymptotically stable systems

Any equilibrium point can be shifted to the origin through a variable transformation.

Definition

The equilibrium point $X = 0$ of $\dot{X} = F(X)$ is

- **stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|X(0)\| < \delta \implies \|X(t)\| < \epsilon, \forall t \geq 0.$$

- **unstable** if it is not stable.
- **asymptotically stable** if it is stable *and* δ can be chosen such that

$$\|X(0)\| < \delta \implies \lim_{t \to \infty} X(t) = 0.$$

Here, $\|\|$ represents the L_2 norm.

Stability of nonlinear systems by linearization

We can check the stability of nonlinear systems by linearizing the system in the vicinity of equilibrium points. After that, we can use a similar theorem as before to arrive at conclusions regarding the nonlinear system.

Recall that the Jacobian matrix of F evaluated at $X = 0$ can be represented as,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(0) & \frac{\partial f_1}{\partial x_2}(0) & \dots & \frac{\partial f_1}{\partial x_n}(0) \\ \frac{\partial f_2}{\partial x_1}(0) & \frac{\partial f_2}{\partial x_2}(0) & \dots & \frac{\partial f_2}{\partial x_n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(0) & \frac{\partial f_n}{\partial x_2}(0) & \dots & \frac{\partial f_n}{\partial x_n}(0) \end{bmatrix}.$$

Stability of nonlinear systems by linearization

Theorem

Let $X = 0$ be an equilibrium point for the nonlinear system

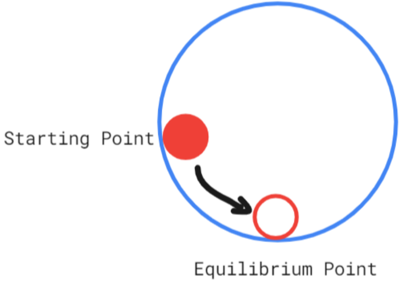
$$\dot{X} = F(X)$$

where all the functions $f_1, f_2, \ldots, f_n$ in F are continuously differentiable in the neighbourhood of the origin. Let A be the Jacobian matrix of F evaluated at $X = 0$. Then,

- *The origin is asymptotically stable if all eigenvalues λ_i of A satisfy $Re(\lambda_i) < 0$.*
- *The origin is unstable if $Re(\lambda_i) > 0$ for one or more eigenvalues of A.*

Lyapunov's Stability Theorem

- We can deduce the stability of the system by observing what happens to the energy of the system over time.
- If the energy of the system remains constant or reduces over time, then the system is stable.
- If the energy increases over time then the system is not stable.
- By utilizing the concept of energy, Lyapunov's Stability Theorem can be formulated to deduce the stability of a system.



Lyapunov's Stability Theorem

Theorem

Let $X = 0$ *be an equilibrium point for the system* $\dot{X} = F(X)$ *and* $D \subset \mathbb{R}^n$ *be a domain containing* $X = 0$*. Let* $V : D \to \mathbb{R}$ *be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(X) > 0 \text{ for } X \in D \setminus \{0\}.$$

If

$$\frac{dV(X)}{dt} \leq 0 \text{ for } X \in D,$$

then $X = 0$ *is stable. Moreover, if*

$$\frac{dV(X)}{dt} < 0 \text{ for } X \in D \setminus \{0\}$$

then $X = 0$ *is asymptotically stable.*

Here $V(X)$ is called the *Lyapunov's function*.

Lotka-Volterra Predator vs. Prey Model

The Lotka–Volterra model describes the dynamics of interacting predator and prey species with a set of two nonlinear differential equations.

$$\dot{x_1} = ax_1 - bx_1x_2$$
$$\dot{x_2} = -cx_2 + dx_1x_2.$$

- x_1, x_2 are the population densities of pray and predators respectively.
- $\dot{x_1}, \dot{x_2}$ are the instantaneous growth rates of the two populations and t represents time.
- a and b represent, respectively, the highest per capita growth rate of prey and the impact of predator presence on the rate of prey growth.
- c and d, respectively, characterize the per capita death rate of predators and the influence of prey presence on the predator's growth rate.
- All parameters a, b, c, d are positive and real.

Equilibrium points

We can represent this system as

$$\dot{X} = F(X). \tag{2}$$

This system has two equilibrium points:

- $(0, 0)$ (trivial equilibrium point)
- $(\frac{c}{d}, \frac{a}{b})$ (nontrivial equilibrium point)

Trivial equilibrium point

We can linearize the system in a neighbourhood of the equilibrium point to analyse the stability. We will find the eigenvalues of the Jacobian of F at $X = 0$.

$$A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}.$$

Eigenvalues are $\lambda_1 = a$ and $\lambda_2 = -c$. Since we get a positive eigenvalue, the system is **unstable** at the trivial equilibrium point.

Trivial equilibrium point numerical simulation

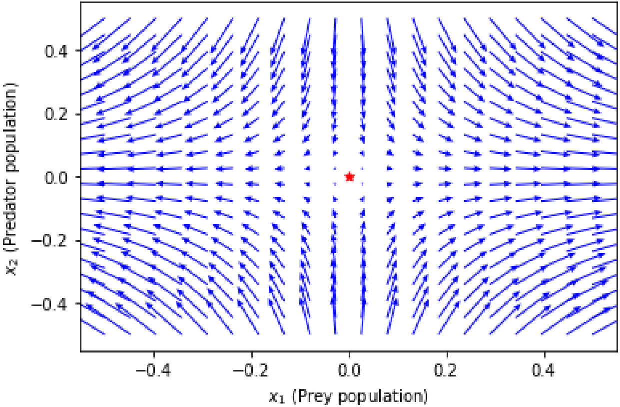


Figure: Flow lines near the equilibrium point at the origin.

Nontrivial equilibrium point

Stability analysis by linearization is not conclusive at this point since the eigenvalues of the Jacobian matrix becomes purely imaginary ($Re(\lambda) = 0$).

Therefore, we resolve to conduct stability analysis using the Lyapunov's method. Consider the function

$$V(y_1, y_2) = dy_1 - c \ln\left(\frac{dy_1 + c}{d}\right) + by_2 - a \ln\left(\frac{by_2 + a}{b}\right) + K, \tag{3}$$

where a, b, c, d are the Lotka-Volterra parameters and K is a constant. Then, by differentiating we see that

$$\frac{V(Y)}{dt} = 0.$$

Thus, by Lyapunov's theorem we can conclude that the above system is **stable** at the equilibrium point $Y = (0, 0)$.

Nontrivial equilibrium point numerical simulation

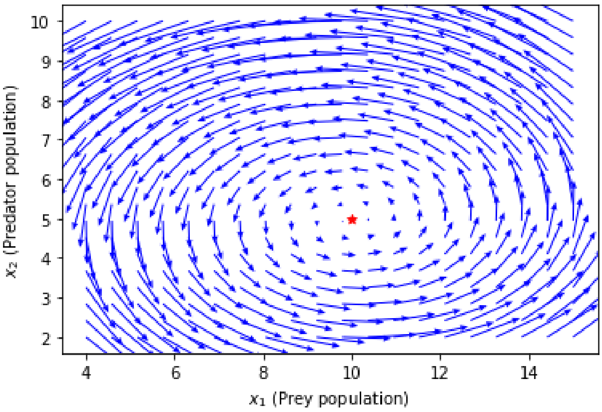


Figure: Flow lines near the nontrivial equilibrium point.

References

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Thank You