Class 11

2017-18

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14. LIMITS, CONTINUITY AND DIFFERENTIABILITY

1. LIMITS

1.1 Intuitive Idea of Limits

(a) Suppose we are travelling from Kashmiri gate to Connaught Place by metro, which will reach Connaught place at 10 a.m. As the time gets closer and closer to 10 a.m. the distance of the train from Connaught place gets closer and closer to zero (here we assume that the train is not delayed). If we consider the time as an independent variable denoted by t and the distance remaining as a function of time, say f(t), then we say that f(t) approaches zero as t approaches zero. We can say that the limit of f(t) is zero as t approaches zero.

(b) Let a regular polygon be described in a circle of given radius. We notice the following points from the geometry.

(i) The area of the polygon cannot be greater than the area of the circle however large the number of sides may be.

(ii) As the number of sides of the polygon increases indefinitely, the area of the polygon continuously approaches the area of the circle.

(iii) Ultimately the difference between the area of the circle and the area of the polygon can be made as small as we please by sufficiently increasing the number of sides of the polygon.

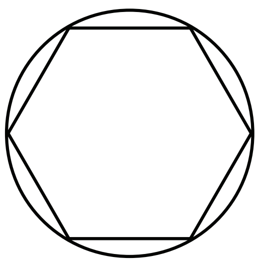


Figure 14.1

We can say that the limit of the area of the polygon inscribed in a circle is the area of the circle as the number of sides increases indefinitely.

1.2 Meaning of $x \to a$

Let x be a variable and a be constant. Since x is a variable; we can change its value at our pleasure. It can be changed in such a way that its value comes nearer and nearer to a. Then we say that x approaches a and it is denoted by $x \to a$:

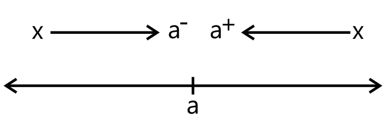


Figure 14.2

We know that $|x - a|$ is the distance between x and a on the real number line and $0 < |x - a|$ if $x \neq a$, "x tends to a" means

(a) $x \neq a$, i.e. $0 < |x - a|$,

(b) x takes up values nearer and nearer to a, i.e. the distance $|x - a|$ between x and a becomes smaller and smaller. One may ask "how much smaller"? The answer is, as much as we please. It may be less than 0.1 or 0.00001 or 0.0000001 and so on. In fact, we may choose any positive number δ. However small it may be, $|x - a|$ will always be less than δ. The above discussion leads up to the following definition of $x \to a$.

Let x be a variable and a be a constant.

Definition: Given a number $\delta > 0$ however small, if x takes up values, such that $0 < |x - a| < \delta$. Then x is said to tend to a, and is symbolically written as $x \to a$

Note. If x approaches a from values less than a, i.e. from the left side of a, we write $x \to a^-$. If x approaches a from values greater than a, i.e. from the right side of a, we write $x \to a^+$.

But $x \to a$ means both $x \to a^-$ and $x \to a^+$. So x approaches or tends to a means x approaches a from both sides right and left.

Neighbourhood of point a: The set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a. Neighbourhood of a = $(a - \delta, a + \delta)$; $x \in (a - \delta, a + \delta)$

1.3 Limit of a Function

Let us take some examples to find the limit of various functions:

Illustration 1: Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$. We investigate the behaviour of f(x) at the point x = 2 and near the point x =2. **(JEE MAIN)**

Sol: Here as $f(2) = 0$, therefore try to evaluate the value of f(x) when x is very near to 2.

$f(2) = \dfrac{4-4}{2-2} = \dfrac{0}{0}$, which is meaningless. Thus f(x) is not defined at x = 2.

Now we try to evaluate the value of f(x) when x is very near to 2 for some values of x less than 2 and then for x greater than 2.

$f(1.9) = \dfrac{(1.9)^2 - 4}{1.9 - 2} = \dfrac{-0.39}{-0.1} = 3.9$

$f(2.1) = \dfrac{(2.1)^2 - 4}{2.1 - 2} = \dfrac{0.41}{0.1} = 4.1$

$f(1.99) = \dfrac{(1.99)^2 - 4}{1.99 - 2} = \dfrac{-0.0399}{-0.01} = 3.99$

$f(2.01) = \dfrac{(2.01)^2 - 4}{2.01 - 2} = \dfrac{0.0401}{0.01} = 4.01$

$f(1.999) = \dfrac{(1.999)^2 - 4}{1.999 - 2} = \dfrac{-0.003999}{-0.001} = 3.999$

$f(2.001) = \dfrac{(2.001)^2 - 4}{2.001 - 2} = \dfrac{0.004001}{0.001} = 4.001$

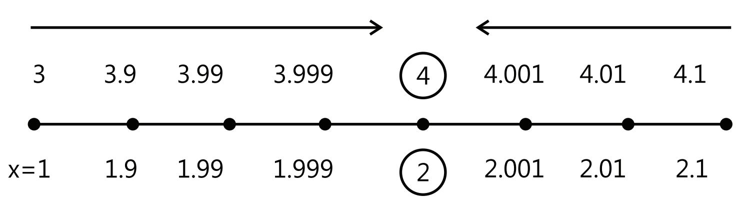


Figure 14.3

It is clear that as x gets nearer and nearer to 2 from either side, f(x) gets closer and closer to 4 from either side.

When x approaches 2 from the left hand side the function f(x) tends to a definite number 4. Thus we say that as x tends to 2 the left hand limit of the function f exists and equals to the definite number 4. Similarly, as x approaches 2 from the right hand side, the function f(x) tends to a definite number, 4.

Again we say that as x approaches 2 from the right hand side of 2, the right hand limit of f exists and equals to 4.

Illustration 2: Discuss the limit of the function $f(x)=\begin{cases}-1, \text{if } x<0\\ 1, \text{ if } x>0\end{cases}$ at x = 0 **(JEE MAIN)**

Sol: Sketch its graph when x is very near to 0 for range of x from – 1 to 1.

We have, $f(x)=\begin{cases}-1,\ x<0\\ +1,\ x>0\end{cases}$

Let us sketch its graph

x	-1	-0.5	-0.1	-0.01	-0.001	0.001	0.01	0.1	0.5	1
f(x)	-1	-1	-1	-1	-1	1	1	1	1	1

(i) As x approaches zero from the left of zero, f(x) remains at –1. And we say that the left hand limit of f exists at x = 0 and equals to –1. $\lim\limits_{x\to 0^-} f(x)=-1$

(ii) As x approaches zero from the right of zero, f(x) remains at 1. So we say that the right hand limit of f at x = 0 exists and equals to +1. $\lim\limits_{x\to 0^+} f(x)=+1$

(iii) Left hand limit of f(x) (at x = 0) ≠ Right hand limit of f(x) {at x = 0}.

So the $\lim\limits_{x\to 0} f(x)$ does not exist.

Illustration 3: Discuss the limit of the function $f(x)=\frac{1}{x}$ at x = 0 and its graph **(JEE MAIN)**

Sol: Same as above illustration.

We have, $f(x)=\frac{1}{x}$ Let us draw the graph of the given function $f(x)=\frac{1}{x}$.

x	-1	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1	1
f(x)	-1	-10	-100	-1000	-10000	10000	1000	100	10	1

(i) As x approaches zero from the left of zero the graph never approaches a finite number so we say that the left hand limit of f at x = 0 does not exist i.e. $\lim\limits_{x\to 0^-} f(x)$ does not exist

(ii) As x approaches zero from the right of zero, the graph again does not approach a finite number. Again we say that the right hand limit of *f* at x = 0 does not exist. $\lim\limits_{x\to 0^+} f(x)$ does not exist

(iii) At x=0 left hand limit of f ≠ right hand limit of f

Hence, the limit of f(x) does not exist.

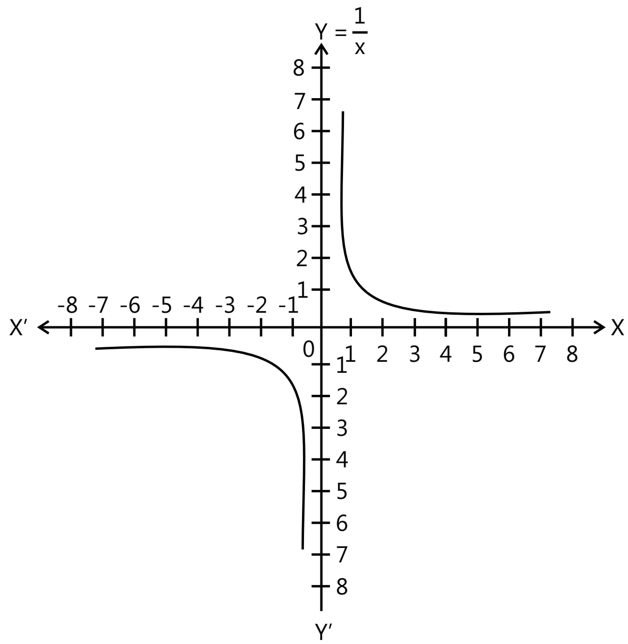


Figure 14.4

PLANCESS CONCEPTS

(a) The fact that the limit of f(x) exists at x = a means that the graph of f(x) approaches the same value from both sides of x = a.

(b) The fact that f(x) is continuous at x = a means that there is no break in the graph as x moves from a^- to a^+.

Vaibhav Krishnan (JEE 2009, AIR 22)

1.4 Different Cases of Limits

Right hand limit is the limit of the function as x approaches a from the positive side.

Left hand limit is the limit of the function as x approaches a from the negative side.

Note: A function will have a limiting value only if its right hand limit equals its left hand limit.

Illustration 4: Discuss the limits of f(x) = |x| at x = 0 and draw its graph. **(JEE MAIN)**

Sol: We have f(x) = |x|, therefore f(x) is equals to x for x > 0 and – x for x < 0.

Increasing x $\longrightarrow$ $\longleftarrow$ Decreasing x

x	-3	-2	-1	0	1	2	3
f(x)	3	2	1	0	1	2	3

Decreasing f(x) $\longrightarrow$ Limit $\longleftarrow$ Decreasing f(x)

We have f(x) = |x| Þ $f(x)=\begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

Let us draw its graph.

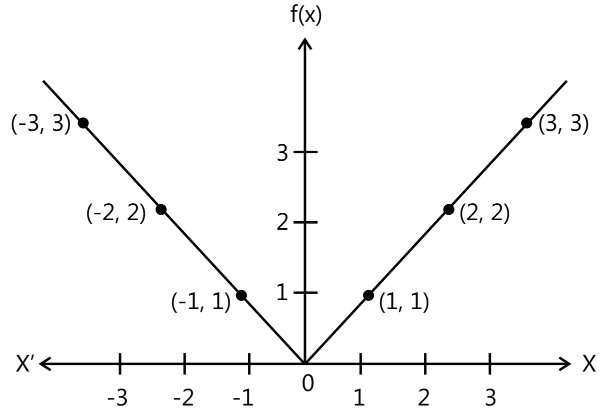


Figure 14.5

(i) As x approaches zero from left of zero, f(x) = 0. And we say that left hand limit of f(x) exists, and is equal to zero.

$$\lim_{x\to 0^-} f(x) = 0$$

(ii) As x approaches zero from the right of zero f(x) is equal to zero. So we say that the right hand limit of f at x = 0 exists, and is equal to zero. $\lim_{x\to 0^+} f(x) = 0$ Here $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$

Hence we say that the limit of *f* (x) at x = 0 exists and equals to zero.

Illustration 5: Evaluate: $\lim_{x\to 3} x+3$ **(JEE MAIN)**

Sol: Simply taking value of x very near to 3 we can obtain value of the function at these points.

Let us compute the value of function f(x) for x very near to 3. Some of the points near and to the left of 3 are 2.9, 2.99, 2.999.

Values of the function are given in the table below. Similarly, some of the numbers near and right of 3 are 3.001,3.01,3.1. Value of the function at these points are also given in the table.

Increasing x → ← Decreasing x

x	2.9	2.99	2.999	3	3.001	3.01	3.1
f(x)	5.9	5.99	5.999	6	6.001	6.01	6.1

Increasing f(x) → Limit ← Decreasing f(x)

From the table we deduce that the value of f(x) at x = 3 should be greater than 5.999 and less than 6.001.

It is reasonable to assume that the value of function f(x) at x = 3 from the left of 3 is 5.999.

$$\lim_{x\to 3^-} f(x) \approx 5.999 \quad \text{... (i)}$$

Similarly, when x approaches x = 3 from the right, f(x) should be 6.001

$$\lim_{x\to 3^+} f(x) \approx 6.001 \quad \text{... (ii)}$$

From (i) and (ii), we conclude that the limit is equal to 6.

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3} f(x) = 6$

Illustration 6: Evaluate: $\lim\limits_{x\to 2} \dfrac{3x^2 - x - 10}{x^2 - 4}$ **(JEE MAIN)**

Sol: Similar to above problem.

Let us compute the value of function f(x) for x very near to 2. Some of the points near and to the left of 2 are 1.9, 1.99, 1.999.

Values of the function are given in the table below. Similarly, some of the numbers near and to the right of 2 are 2.001, 2.01, 2.1. Values of the function at these points are also given in the table.

Increasing x → ← Decreasing x

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	2.743	2.749	2.7499	2.75	2.750	2.7506	2.756

Increasing f(x) → Limit ← Decreasing f(x)

From the table we deduce that the value of f(x) at x = 2 should be greater than 2.7499 and less than 2.750.

It is reasonable to assume that the value of function f(x) at x = 3 from the left of 2 is 2.7499.

$\therefore \quad \lim\limits_{x\to 2^-} f(x) \approx 2.7499$ .... (i)

Similarly, when x-approaches x = 2 from the right *f* (x) should be 2.750. $\lim\limits_{x\to 2^+} f(x) \approx 2.750$ ....(ii)

From (i) and (ii), we conclude that the limit is equal to 2.75. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2} f(x) = 2.75$

PLANCESS CONCEPTS

- If $\underset{x\to \bar{a}}{\text{Lim}}\, f\left(a-b\right) = \underset{x\to a^+}{\text{Lim}} + \left(a+b\right)$, then we can say that both the left hand limit and right hand limits exist at x = a, and irrespective of the value of the function at a, i.e f(a), the function does not have a limit at x = a, that is $\underset{x\to a}{\text{Lim}}\ f\left(x\right)$ does not exist
- If both the left hand limit and right hand limit of f(x) at x = a exist and at least one of them is not equal to f(a), then the limit of f at x = a does not exist.

Shrikant Nagori (JEE 2009, AIR 30)

1.5 Working Rule for Evaluation of Left and Right Hand Limits

Right hand limit of f(x), when $x \to a$ = $\lim\limits_{x\to a^+} f(x)$

Step I. Put x = a + h and replace a^+ by a.

Step II. Simplify $\lim\limits_{h\to 0} f(a+h)$.

Step III. The value obtained in step 2 is the right hand limit of f(x) at x = a.

Similarly for evaluating the left hand limit put $x = a - h$.

Evaluate the left-hand and right-hand limits of the following function at $x = 1$.

$$f(x) = \begin{cases} 5x-4, & \text{if } 0 < x \le 1 \\ 4x^2-3x, & \text{if } 1 < x < 2 \end{cases}$$

Illustration 7: Does $\lim\limits_{x\to 1} f(x)$ exist? **(JEE MAIN)**

Sol: By taking left hand limit and right hand limit we can conclude that the given limit exist or not.

Left hand limit $= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (5x-4)$ [$\because f(x) = 5x - 4$, if $0 < x \le 1$]

$= \lim\limits_{h\to 0}[5(1-h)-4] = \lim\limits_{h\to 0}\left[5-5h-4\right] = 5-4 = 1$ [Put $x = 1 - h$]

Right hand limit $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (4x^2-3x)$ [$\because f(x) = 4x^2 - 3x$, if $1 < x < 2$]

$= \lim\limits_{h\to 0}[4(1+h)^2-3(1+h)]$ [Put $x = 1 + h$]

$= \lim\limits_{h\to 0}[4+8h+4h^2-3-3h] = \lim\limits_{h\to 0}[1+5h+4h^2] = 1$

$\therefore$ At $x = 1$, Left hand limit = Right hand limit $\Rightarrow \lim\limits_{x\to 1} f(x)$ exists and it is equal to 1.

Illustration 8: Evaluate the left hand and right-hand limits of the following function at $x = 1$ **(JEE MAIN)**

$$f(x) = \begin{cases} 1+x^2, & \text{if } 0 \le x \le 1 \\ 2-x, & \text{if } x > 1 \end{cases} \qquad \text{does } \lim_{x\to 1} f(x) \text{ exist ?}$$

Sol: By putting $x = 1 - h$ and $1 + h$, we can conclude left hand limit and right hand limit respectively. If both are equal then the given limit exist otherwise not exist.

Left hand limit $= \lim\limits_{x\to 1^-} f(x)$

$= \lim\limits_{x\to 1^-} (1+x^2)$ [$\because f(x) = 1 + x^2$, if $0 \le x \le 1$]

$= \lim\limits_{h\to 0}[1+(1-h)^2] = 1+1 = 2$ [Put $x = 1 - h$]

Right hand limit $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2-x)$ [$\because f(x) = 2 - x$, if $x > 1$]

$= \lim\limits_{h\to 0}[2-(1+h)] = 2-1 = 1$ (Put $x = 1 + h$)

Therefore, At $x = 1$, Left hand limit $\ne$ Right hand limit $\Rightarrow \lim\limits_{x\to 1} f(x)$ does not exist.

Illustration 9: Evaluate the right hand limit of the function $f(x) = \begin{cases} \dfrac{|x-6|}{x-6}, & x \ne 6 \\ 0, & x = 6 \end{cases}$ at $x = 6$ **(JEE MAIN)**

Sol: Here Right hand limit of the given function $f(x)$ at $x = 6$ is $\lim\limits_{h\to 0} f(6+h)$.

Right hand limit of $f(x)$ at $x = 6$ = $\lim\limits_{x\to 6^+} f(x)$, $= \lim\limits_{h\to 0} f(6+h)$,

$= \lim\limits_{h\to 0}\dfrac{|6+h-6|}{6+h-6}, = \lim\limits_{h\to 0}\dfrac{|h|}{h} = \lim\limits_{h\to 0}\dfrac{h}{h}, = \lim\limits_{h\to 0} 1 = 1$

Illustration 10: Evaluate the left hand limit of the function: $f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$ at x = 4 **(JEE MAIN)**

Sol: Here Left hand limit of the given function f(x) at x = 4 is $\lim\limits_{h\to 0} f(4-h)$.

Left hand limit of f(x) at x = 4 = $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{h\to 0} f(4-h) = \lim\limits_{h\to 0}\dfrac{|4-h-4|}{4-h-4} = \lim\limits_{h\to 0}\dfrac{|-h|}{-h}$

$= \lim\limits_{h\to 0}\dfrac{h}{-h} = \lim\limits_{h\to 0} -1 = -1$

Illustration 11: Evaluate the left hand and right hand limits of the following function at x = 2: **(JEE MAIN)**

$f(x)=\begin{bmatrix}2x+3, & \text{if } x\le 2\\ x+5, & \text{if } x>2\end{bmatrix}$ Does $\lim\limits_{x\to 2} f(x)$ exist?

Sol: Similar to illustration 7.

Left hand limit = $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(2x+3)$ ($\because$ f(x) = 2x + 3, if $x \le 2$)

$= \lim\limits_{h\to 0}[2(2-h)+3]$ (Put x = 2 – h) = 4 + 3 = 7

Right hand limit = $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} f(x+5)$ ($\because$ f(x) = x + 5, if x > 2)

$= \lim\limits_{h\to 0}(2+h+5) = 7$ (put x = 2 + h)

Therefore, the left hand limit = right hand limit [at x = 2] $\Rightarrow \lim\limits_{x\to 2} f(x)$ exists and it is equal to 7.

Illustration 12: For what integers m and n does $\lim\limits_{x\to 0} f(x)$ exist, if $f(x)=\begin{cases}mx^2+n, & x<0\\ nx+m, & 0\le x\le 1\\ nx^2+m, & x>1\end{cases}$ **(JEE ADVANCED)**

Sol: As the given limit exist, therefore its Left hand limit must be equal to its Right hand limit.

Limit at x = 0 ; $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0-h}(mx^2+n) = \lim\limits_{h\to 0}\left[m(0-h)^2+n\right] = n$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to(0+h)}(nx+m) = \lim\limits_{h\to 0}[n(0+h)+m] = m$

Limit exists if $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) \Rightarrow n = m$

Illustration 13: Suppose $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$ and if $\lim\limits_{x\to 1} f(x) = f(1)$.What are possible values of a and b? **(JEE Advanced)**

Sol: Here $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} f(x)$.

We have, $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$ Left hand $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}[a+bx] = \lim\limits_{h\to 0}[a+b(1-h)] = a+b$

Right hand $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} [b-ax] = \lim\limits_{h\to0}[b-a(1+h)] = b-a$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1} f(x) \Rightarrow a + b = b - a = 4 \Rightarrow \quad a = 0, b = 4$

Illustration 14: If $f(x) = \begin{cases} \dfrac{x-|x|}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$ show that $\lim\limits_{x\to0} f(x)$ does not exist **(JEE ADVANCED)**

Sol: Here if left hand limit is not equal to the right hand limit of the given function then the $\lim\limits_{x\to0} f(x)$ does not exist.

Left hand limit of f at x = 0 = $\lim\limits_{x\to0^-} f(x) = \lim\limits_{h\to0} f(0-h) = \lim\limits_{h\to0} \dfrac{(0-h)-|0-h|}{(0-h)}$ [Putting x = 0 – h]

$$= \lim_{h\to0} \frac{-h-h}{-h} = \lim_{h\to0} \frac{-2h}{-h} = \lim_{h\to0} 2 = 2$$

Right hand limit of f at x = 0 = $\lim\limits_{x\to0^+} f(x) = \lim\limits_{h\to0} f(0+h) = \lim\limits_{h\to0} \dfrac{(0+h)-|0+h|}{(0+h)}$ [Putting x = 0 + h]

$$= \lim_{h\to0} \frac{h-|h|}{h} = \lim_{h\to0} \frac{h-h}{h} = \lim_{h\to0} \frac{0}{h} = \lim_{h\to0} 0 = 0$$

Here, left hand limit of f (at x = 0) ≠ right hand limit of f(at x = 0). Therefore $\lim\limits_{h\to0} f(x)$ does not exist at x = 0

PLANCESS CONCEPTS

If f(x) denotes the greatest integer function then $\lim\limits_{x\to0} f(x) = [0] = 0$ this representation is wrong.

The correct form is

$$\lim_{x\to0^+} f(x) = \lim_{h\to0} [0+h] = 0$$

$$\lim_{x\to0^-} f(x) = \lim_{h\to0} [0-h] = -1$$

Hence the limit doesn't exist. Remember that the limit must be applied only after complete simplification.

Nitish Jhawar (JEE 2009, AIR 7)

1.6 Value of a Function at a Point and Limit at a Point

Case I: $\lim\limits_{x\to a} f(x)$ and f(a) both exist but are not equal.

Example: $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$; $\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to1}(x+1) = 2$, f(x) exists at x = 1

f(1) = 0, value of f also exists at x = 1. But $\lim\limits_{x\to1} f(x) \neq f(1)$.

Case II: $\lim\limits_{x\to a} f(x)$ and f(a) both exist and are equal.

Example: $f(x) = x^2$; $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1}(x^2) = 1$ limit exists, and $f(1) = (1)^2 = 1$; $\Rightarrow$ value of f also exists. $\Rightarrow \lim\limits_{x \to 1} f(x) = f(1)$

1.7 Properties of Limits

Let f and g be two real functions with common domain D, then.

(a) $\lim\limits_{x \to a}(f + g)(x) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(b) $\lim\limits_{x \to a}(f - g)(x) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(c) $\lim\limits_{x \to a}(c \cdot f)(x) = c \lim\limits_{x \to a} f(x)$ [c is a constant]

(d) $\lim\limits_{x \to a}(fg)(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

(e) $\lim\limits_{x \to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

(f) $\lim\limits_{x \to a}\left(f(x)\right)^n = \left(\lim\limits_{x \to a} f(x)\right)^n$

(g) $\lim\limits_{x \to a}\left(f(x)\right)^{g(x)} = \left(\lim\limits_{x \to a} f(x)\right)^{\lim\limits_{x \to a} g(x)}$

(h) If f(x)g(x) xa then $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$

PLANCESS CONCEPTS

$\lim\limits_{x \to a} f(x) = \ell$, then $\lim\limits_{x \to a} |f(x)| = |\ell|$

The converse of this may not be true i.e.

$\lim\limits_{x \to a} |f(x)| = |\ell| \not\Rightarrow \lim\limits_{x \to a} f(x) = \ell$

$\lim\limits_{x \to a} f(x) = A > 0$ and $\lim\limits_{x \to a} g(x) = B$ then $\lim\limits_{x \to a} f(x)^{g(x)} = A^B$

Shivam Agarwal (JEE 2009, AIR 27)

1.8 Cancellation of Common Factor

Let $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$. If by substituting x = a, $\dfrac{f(x)}{g(x)}$ reduces to the form $\dfrac{0}{0}$, then (x–a) is a common factor of f(x) and g(x). So we first factorize f(x) and g(x) and then cancel out the common factor to evaluate the limit.

Working Rule:

To find out $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, where $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$.

Step I: Factorize f(x) and g(x).

Step II: Cancel the common factor (s).

Step 3: Use the substitution method to obtain the limit.

Important formulae for factorization

(a) $(a^2 - b^2) = (a - b)(a + b)$

(b) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(c) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(d) $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2)$

(e) If $f(\alpha) = 0$, then $x - \alpha$ is a factor of $f(x)$

Illustration 15: Evaluate $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$. **(JEE MAIN)**

Sol: Factorize the numerator and denominator.

If we put x = 1, the expression $\dfrac{x^3-1}{x-1}$ assumes the indeterminate form $\dfrac{0}{0}$. Therefore $(x - 1)$ is a common factor of $(x^3 - 1)$ and $(x - 1)$. Factorising the numerator and denominator, we have,

$$\lim_{x\to 1}\frac{x^3-1}{x-1}\ \left[\frac{0}{0}\text{Form}\right]$$

$$= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1}(x^2+x+1)\ \text{[After cancelling (x–1)]} = 1^2+1+1=3$$

Illustration 16: Evaluate $\lim\limits_{x\to \frac{1}{2}}\dfrac{4x^2-1}{2x-1}$ **(JEE MAIN)**

Sol: Similar to above illustration, By factorizing numerator and denominator we can evaluate given limit.

If we put $x=\dfrac{1}{2}$, the expression $\dfrac{4x^2-1}{2x-1}$ assumes the indeterminate from $\dfrac{0}{0}$.

Therefore $\left(x-\dfrac{1}{2}\right)$ i.e. $(2x - 1)$ is a common factor of $(4x^2 - 1)$ and $(2x - 1)$. Factorising the numerator and denominator, we have,

$$\lim_{x\to \frac{1}{2}}\frac{4x^2-1}{2x-1}\quad\left[\frac{0}{0}\text{Form}\right] = \lim_{x\to \frac{1}{2}}\frac{(2x-1)(2x+1)}{(2x-1)}\quad [\because a^2 - b^2 = (a - b)(a + b)]$$

$$= \lim_{x\to \frac{1}{2}}(2x+1)\ \text{[After cancelling (2x – 1)]} = 2\times\frac{1}{2}+1=1+1=2$$

Illustration 17: Evaluate $\lim\limits_{x\to 5}\dfrac{x^2-9x+20}{x^2-6x+5}$. **(JEE MAIN)**

Sol: Take $(x - 5)$ common from numerator and denominator to solve this problem.

If we put x = 5, the expression $\dfrac{x^2-9x+20}{x^2-6x+5}$ assumes the indeterminate form $\dfrac{0}{0}$.

Therefore $(x - 5)$ is a common factor of the numerator and denominator both. Factorising the numerator and denominator, we have $\left[\frac{0}{0}\text{Form}\right] = \lim\limits_{x\to5}\frac{(x-4)\,(x-5)}{(x-1)\,(x-5)} = \lim\limits_{x\to5}\frac{(x-4)}{(x-1)}$

[After cancelling (x – 5)] $= \frac{5-4}{5-1} = \frac{1}{4}$

Illustration 18: Evaluate $\lim\limits_{x\to2}\frac{x^3-7x^2+14x-8}{x^2+2x-8}$ **(JEE MAIN)**

Sol: Same as above illustration.

When x = 2 the expression $\frac{x^3-7x^2+14x-8}{x^2+2x-8}$ assumes the form $\frac{8-28+28-8}{4+4-8} = \frac{0}{0}$

Therefore (x – 2) is a common factor of the numerator and denominator.

Factorising the numerator and denominator, we get

$$\lim_{x\to2}\frac{(x-1)(x-2)(x-4)}{(x-2)(x+4)} = \lim_{x\to2}\frac{(x-1)(x-4)}{(x+4)} = \frac{(2-1)(2-4)}{(2+4)} = \frac{1\times(-2)}{6} = \frac{-2}{6} = -\frac{1}{3}$$

Illustration 19: Evaluate $\lim\limits_{x\to1}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$ **(JEE MAIN)**

Sol: Same as above illustration.

On putting x = 1 in $\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$, we get $\frac{0}{0}$.

It means (x – 1) is the common factor of the numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x\to1}\frac{(x-1)(x^3-2x^2-2x-2)}{(x-1)(x^2-4x-1)} = \lim_{x\to1}\frac{(x^3-2x^2-2x-2)}{x^2-4x-1}$$ [Cancellation of (x – 1)]

$= \frac{1^3-2\times1^2-2\times1-2}{1^2-4\times1-1}$ [Substitution method]

$= \frac{1-2-2-2}{1-4-1} = \frac{-5}{-4} = \frac{5}{4}$

Note: When the degree of the polynomial is higher, then it is difficult to factorize. So, we apply L'Hôpital's rule

1.9 L'Hôpital's Rule

L'Hôpital's rule states that for functions f and g which are differentiable:

If $\lim\limits_{x\to c}f(x) = \lim\limits_{x\to c}g(x) = 0$ or $\pm\infty$ and $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$ has a finite value then $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$

Note: Most common indeterminate forms are $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty$ and ∞^0

PLANCESS CONCEPTS

Evaluation of limits using L'Hôpital's rule is applicable only when $\frac{f(x)}{g(x)}$ becomes of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$ simplify the given expression till it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply the rule.

To apply L'Hôpital's rule differentiate the numerator and the denominator separately.

What you may overlook is the fact that the LH rule is applicable only when the modified limit (obtained by differentiating the numerator and denominator) also exists. Let's consider an example to illustrate this point. Consider the limit $\lim\limits_{x\to\infty}\frac{x^2+\sin x}{x^2}$. We see that the limit is of the indeterminate form $\frac{\infty}{\infty}$. Applying the LH rule two times in succession, we obtain: $\lim\limits_{x\to\infty}\frac{x^2+\sin x}{x^2}=\lim\limits_{x\to\infty}\frac{2x+\cos x}{2x}$ (again $\frac{\infty}{\infty}$ form)

$$=\lim_{x\to\infty}\frac{2-\sin x}{2}=1-\frac{1}{2}\lim_{x\to\infty}\sin x$$

Which does not exist. However, only a few moments of consideration are required to conclude that the limit must exist, because the numerator is x^2 + sinx, and since x tends to infinity, the term sinx can be ignored in comparison to x^2 (as sinx only ranges from -1 to 1); the denominator is x^2 and so the limit must be 1. Why did the LH rule go wrong?

Ravi Vooda (JEE 2009, AIR 71)

1.10 Theorems

1. Let n be any positive integer. Then. $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

First Proof: Putting x = a + h, we get $\frac{x^n-a^n}{x-a}=\frac{(a+h)^n-a^n}{a+h-a}=\frac{1}{h}\left[(a+h)^n-a^n\right]$

$$=\frac{1}{h}\left[(a^n+{}^nC_1a^{n-1}h+....+h^n)-a^n\right]\text{[Using the Binomial theorem]}$$

$$=\frac{1}{h}\left[{}^nC_1a^{n-1}h+{}^nC_2a^{n-2}h^2+....+h^n\right]={}^nC_1a^{n-1}+{}^nC_2a^{n-2}h+....+h^{n-1}$$

$$\therefore\quad\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\left[{}^nC_1a^{n-1}+{}^nC_2a^{n-2}h+....h^{n-1}\right]=na^{n-1}\qquad(\text{as } {}^nC_1=n)$$

Second Proof: We know that,

$x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+...a^{n-2}x+a^{n-1})$ Therefore, $\frac{x^n-a^n}{(x-a)}=x^{n-1}+ax^{n-2}+....a^{n-2}x+a^{n-1}$

$$\Rightarrow\lim_{x\to a}\frac{x^n-a^n}{(x-a)}=a^{n-1}+a(a^{n-2})+....a^{n-2}a+a^{n-1}=a^{n-1}+a^{n-1}+....a^{n-1}+a^{n-1}\text{(n terms)}=n\,a^{n-1}$$

Illustration 20: Evaluate: $\lim\limits_{x\to a}\frac{x^m-a^m}{x^n-a^n}$ **(JEE MAIN)**

Sol: By using L' Hospital rule, we can solve this problem.

We have, $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$

$$\left[\frac{0}{0}\text{ form}\right]=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\cdot\frac{x-a}{x^n-a^n}\right\}=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\div\frac{x^n-a^n}{x-a}\right\}=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\right\}\div\lim_{x\to a}\left\{\frac{x^n-a^n}{x-a}\right\}$$

$$=ma^{m-1}\div na^{n-1}=\frac{ma^{m-1}}{na^{n-1}}=\frac{m}{n}a^{m-1-n+1}=\frac{m}{n}a^{m-n}$$

Illustration 21: Evaluate: $\lim\limits_{x\to 2}\dfrac{x^5-32}{x^3-8}$ **(JEE MAIN)**

Sol: Dividing numerator and denominator by (x – 2) we can evaluate given limit.

When x = 2, the expression $\dfrac{x^5-32}{x^3-8}$ assumes the indeterminate form $\dfrac{0}{0}$.

Now $$\lim_{x\to 2}\frac{x^5-32}{x^3-8}=\lim_{x\to 2}\frac{x^5-2^5}{x^3-2^3}=\lim_{x\to 2}\frac{(x^5-2^5)/(x-2)}{(x^3-2^3)/(x-2)}=\lim_{x\to 2}\left\{\frac{x^5-2^5}{x-2}\right\}\div\lim_{x\to 2}\left\{\frac{x^3-2^3}{x-2}\right\}$$

$$=5\times 2^{5-1}\div 3\times 2^{3-1}\quad\left[\because\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=5\times 2^4\div 3\times 2^2=\frac{5\times 2^4}{3\times 2^2}=\frac{5}{3}\times 2^2=\frac{20}{3}$$

Illustration 22: Find all possible values of n, if $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$, $n\in N$ **(JEE MAIN)**

Sol: Simply using L'Hospital rule we can obtain value of n.

We have $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80 \Rightarrow n\cdot 2^{n-1}=80$ $\left[\therefore\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}\right]$

$\Rightarrow n\cdot 2^{n-1}=5\times 2^{5-1}\Rightarrow n=5$

Illustration 23: $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$ **(JEE ADVANCED)**

Sol: Put x + 2 = y and a + 2 = b and after that solve this by using L'hospital rule.

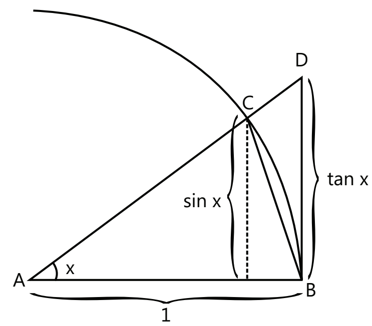
Putting x + 2 = y and a + 2 = b, we get

$$=\lim_{y\to b}\frac{y^{\frac{5}{2}}-b^{\frac{5}{2}}}{y-b}=\frac{5}{2}b^{\frac{5}{2}-1}=\frac{5}{2}b^{\frac{3}{2}}\quad\left[\because\lim_{y\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=\frac{5}{2}(a+2)^{\frac{3}{2}}$$

Illustration 24: Prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Sol: Proof by geometry



Draw a circle of radius unity and with its centre at O. Let $\angle AOB$ = x radians

Join AB. Draw AC $\perp$ OA at A. Produce OB to meet AC at C. Draw BD $\perp$ OA

From the figure

Area of DOAB < Area of sector OAB < Area of DOAC

$\Rightarrow \quad \frac{1}{2}(OA)(BD) < \left(\frac{x}{2\pi}\right)\pi(OA)^2 < \frac{1}{2}(OA)\cdot(AC)$

$\Rightarrow \quad \frac{1}{2}(OA)(OB)\sin x < \frac{1}{2}(OA)^2 x < \frac{1}{2}(OA)\cdot(OA)\tan x \qquad \left[\because \frac{BD}{OB} = \sin x, \tan x = \frac{AC}{OA}\right]$

$\Rightarrow \quad \frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x \qquad [\because OA = OB]$

$\Rightarrow$ sin x < x < tanx $\qquad \Rightarrow \quad \sin x < x < \frac{\sin x}{\cos x}$

$\Rightarrow \quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ (Dividing by sin x)

$\Rightarrow \quad \frac{x}{\sin x}$ lies between 1 and $\frac{1}{\cos x}$

When x $\to$ 0, cos x = 1

$\therefore$ When x $\to$ 0, $\frac{x}{\sin x}$ lies between 1 and 1

$\therefore \quad \lim_{x\to 0}\frac{x}{\sin x} = 1$ or $\lim_{x\to 0}\frac{\sin x}{x} = 1$ Proved

Proof by algebra

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(\frac{x-(x^3/3!)+(x^5/5!)-....}{x}\right) = \lim_{x\to 0}\frac{x\left(1-(x^2/3!)+(x^4/5!).....\right)}{x} = \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}.....\right) = 1$$

Illustration 25: Evaluate: $\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$ **(JEE MAIN)**

Sol: As we know $\lim_{x\to 0}\frac{\sin\theta}{\theta} = 1$, Therefore we can reduce given equation by multiplying and dividing by 25.

We have, $\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$ **=** $\lim_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{\sin 5x}{x} = \lim_{x\to 0}5\left(\frac{\sin 5x}{5x}\right)\times\lim_{x\to 0}5\left(\frac{\sin 5x}{5x}\right)$

= $5(1)\times 5(1) = 25$

Illustration 26: Evaluate: $\lim_{x\to 0}\frac{(a+x)^2\sin(a+x)-a^2\sin a}{x}$ **(JEE ADVANCED)**

Sol: By using algebra we can reduce given limit in the form of $\lim_{x\to 0}f(x)+\lim_{x\to 0}g(x)+\lim_{x\to 0}h(x)$, and then by solving we will get the result.

$$\lim_{x\to 0}\frac{(a+x)^2\sin(a+x)-a^2\sin a}{x} = \lim_{x\to 0}\frac{(a^2+2ax+x^2)\sin(a+x)-a^2\sin a}{x}$$

$= \lim\limits_{x\to 0}\dfrac{a^2\sin(a+x)+(2ax+x^2)\sin(a+x)-a^2\sin a}{x}$

$= \lim\limits_{x\to 0}\dfrac{a^2\left(\sin(a+x)-\sin a\right)}{x}+\lim\limits_{x\to 0}\dfrac{2ax\sin(a+x)}{x}+\lim\limits_{x\to 0}\dfrac{x^2\sin(a+x)}{x}$

$= a^2\lim\limits_{x\to 0}\dfrac{2\cos\left(a+(x/2)\right)\sin(x/2)}{x}+\lim\limits_{x\to 0}2a\sin(a+x)+\lim\limits_{x\to 0}x\sin(a+x)$

$= a^2\lim\limits_{x\to 0}\cos\left(a+\dfrac{x}{2}\right)\lim\limits_{x\to 0}\dfrac{\sin(x/2)}{(x/2)}+\lim\limits_{x\to 0}2a\sin(a+x)+\lim\limits_{x\to 0}x\sin(a+x)$

$= a^2\cos(a+0)+2a\sin(a+0)+0\sin(a+0)$

$= a^2\cos a+2a\sin a$

Illustration 27: Evaluate: $\lim\limits_{x\to 0}\dfrac{\cot 2x-\operatorname{cosec}2x}{x}$ **(JEE ADVANCED)**

Sol: By using trigonometric formulae, we can evaluate this problem.

We have, $\lim\limits_{x\to 0}\dfrac{\cot 2x-\operatorname{cosec}2x}{x} = \lim\limits_{x\to 0}\dfrac{\left((\cos 2x)/(\sin 2x)\right)-\left(1/(\sin 2x)\right)}{x}=\lim\limits_{x\to 0}\dfrac{\cos 2x-1}{x\cdot\sin 2x}$

$= \lim\limits_{x\to 0}\dfrac{-2\sin^2 x}{x\cdot 2\sin x\cdot\cos x} = -\lim\limits_{x\to 0}\dfrac{\sin x}{x\cdot\cos x}=-\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos x} = -1\cdot\dfrac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\cos x}=-1$

Illustration 28: Evaluate the following limits: $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$ **(JEE ADVANCED)**

Sol: Same as above problem.

We have, $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\left((\sin x)/(\cos x)\right)-\sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{\sin x-\sin x\cos x}{x^3\cos x}$

$= \lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{x^3\cdot\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{(1-\cos x)}{x^2}\cdot\dfrac{1}{\cos x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{2\sin^2(x/2)}{x^2}\cdot\lim\limits_{x\to 0}\cdot\dfrac{1}{\cos x}$

$= 1\cdot\lim\limits_{x\to 0}\dfrac{2}{4}\left(\dfrac{\sin(x/2)}{(x/2)}\right)^2\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos x} = (1)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{1}\right) = \dfrac{1}{2}$

Illustration 29: Evaluate: $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ **(JEE ADVANCED)**

Sol: Reduce given equation in the form of $\dfrac{\sin\theta}{\theta}$ by using trigonometric formula.

We have, $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2(x/2)}{x^2}=\dfrac{2}{4}\lim\limits_{x\to 0}\left(\dfrac{\sin(x/2)}{(x/2)}\right)^2=\dfrac{1}{2}$

Illustration 30: Prove that: $|\sin x|\le|x|$, holds for all x. **(JEE ADVANCED)**

Sol: We know that $\sin x < x < \tan x$ therefore by applying the cases, $0\le x\le 1<\dfrac{\pi}{2}$, $-1\le x\le 0$ and $|x|\ge 1$, we can prove this illustration.

We know that sin x < x < tanx

If $0 \le x \le 1 < \frac{\pi}{2} \Rightarrow |\sin x| = \sin x \le x \le |x|$

If $-1 \le x \le 0 \Rightarrow |\sin x| = -\sin x \Rightarrow \sin(-x) \le -x \le |x|$

If neither of the two cases hold, i.e. if $|x| \ge 1$ then $|\sin x| \le 1 \le |x| \Rightarrow |\sin x| \le |x|$

1.11 Trigonometric Limits

Now, we will learn to evaluate the trigonometric limits when the variable tends to a non-zero number.

Woking Rule:

Let the variable tend to a. ($x \to a$).

Step 1: Replace x by a + h, where $h \to 0$.

Step 2: Now the problem is transformed in h where $h \to 0$. Use the method already discussed in the previous exercise.

Illustration 31: Evaluate: $\lim\limits_{x\to\pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$ **(JEE MAIN)**

Sol: By replacing x by $\pi + h$.

We have, $\lim\limits_{x\to\pi} \frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}\lim\limits_{x\to\pi}\frac{\sin(\pi-x)}{(\pi-x)} = \frac{1}{\pi}\lim\limits_{h\to 0}\frac{\sinh}{h} = \frac{1}{\pi}$ $\left[\begin{array}{l}\because x\to\pi \\ \Rightarrow \pi - x \to 0\end{array}\right]$

Illustration 32: Evaluate: $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$ **(JEE MAIN)**

Sol: Simply replacing x by $\pi + h$ and using trigonometric limit we can solve above problem.

$\lim\limits_{x\to\pi}\frac{1+\cos x}{\tan^2 x} = \lim\limits_{h\to 0}\frac{1+\cos(\pi+h)}{\tan^2(\pi+h)}$ [Putting x = +h]

$= \lim\limits_{h\to 0}\frac{1-\cosh}{\tan^2 h}$ [$\because$ tan (+ h) = tan h] $= \lim\limits_{h\to 0}\frac{2\sin^2(h/2)}{\sin^2 h}\cos^2 h$

$= \lim\limits_{h\to 0}\frac{2\sin^2(h/2)}{\left(4\sin^2(h/2)\cos^2(h/2)\right)}\cos^2 h = \lim\limits_{h\to 0}\frac{\cos^2 h}{2\cos^2(h/2)} = \frac{1}{2}\frac{\times 1}{\times(1)} = \frac{1}{2}$

Illustration 33: Evaluate: $\lim\limits_{x\to a}\frac{\sin x - \sin a}{\sqrt{x}-\sqrt{a}}$ **(JEE ADVANCED)**

Sol: Same as above illustration replace x by a +h.

We have, $\lim\limits_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}} = \lim\limits_{h\to 0}\frac{\sin(a+h)-\sin a}{\sqrt{a+h}-\sqrt{a}}$ (Putting x = a + h)

$= \lim\limits_{h\to 0}\frac{2\cos\left(a+(h/2)\right)\sin(h/2)}{(a+h)-a}(\sqrt{a+h}+\sqrt{a}) = \lim\limits_{h\to 0} 2\cos\left(a+\frac{h}{2}\right)\frac{\sin(h/2)}{h}[\sqrt{a+h}+\sqrt{a}]$

$= 2\cos a \lim\limits_{\frac{h}{2}\to 0}\left(\frac{1}{2}\frac{\sin(h/2)}{(h/2)}\right)\lim\limits_{h\to 0}\left(\sqrt{a+h}+\sqrt{a}\right)$ $\left(\because h\to 0 \Rightarrow \frac{h}{2}\to 0\right)$

$= 2\cos a\left[\frac{1}{2}\right](1)[\sqrt{a+0}+\sqrt{a}] = 2\sqrt{a}\cos a$

Illustration 34: Evaluate: $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1+\cos(2x)}{(\pi-2x)^2}$ **(JEE ADVANCED)**

Sol: Replace x by $\dfrac{\pi}{2}+h$ and then by using trigonometric formulae's we can evaluate above problem.

We have, $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1+\cos(2x)}{(\pi-2x)^2}\ \lim\limits_{x\to\frac{\pi}{2}} \dfrac{1+\cos 2x}{(\pi-2x)^2} = \lim\limits_{h\to0} \dfrac{1+\cos 2\left((\pi/2)+h\right)}{\left[\pi-2\left((\pi/2)+h\right)\right]^2}$ $\left(\dfrac{0}{0}\text{form}\right)$ $\left[\text{Put}\, x=\dfrac{\pi}{2}+h\right]$

$= \lim\limits_{h\to0} \dfrac{1+\cos(\pi+2h)}{(\pi-\pi-2h)^2} = \lim\limits_{h\to0} \dfrac{1-\cos 2h}{4h^2}$ $\left(\dfrac{0}{0}\text{form}\right)$ $= \lim\limits_{h\to0} \dfrac{2\sin^2 h}{4h^2} = \dfrac{1}{2}\lim\limits_{h\to0}\left(\dfrac{\sinh}{h}\right)^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$

Illustration 35: Evaluate: $\lim\limits_{x\to a} \dfrac{\cos x-\cos a}{\cot x-\cot a}$ **(JEE ADVANCED)**

Sol: By using trigonometric limit method we can solve this.

We have, $\lim\limits_{x\to a} \dfrac{\cos x-\cos a}{\cot x-\cot a} = \lim\limits_{h\to0} \dfrac{\cos(a+h)-\cos a}{\cot(a+h)-\cot a} = \lim\limits_{h\to0} \dfrac{\cos(a+h)-\cos a}{\left((\cos(a+h)/(\sin(a+h)\right)-\left((\cos a)/(\sin a)\right)}$ [Put x = a+ h]

$= \lim\limits_{h\to0} \dfrac{2\sin\left((a-a-h)/2\right)\sin\left((a+a+h)/2\right)}{\cos(a+h)\sin a-\cos a\,\sin(a+h)}\sin(a+h)\sin a$

$= \lim\limits_{h\to0} \dfrac{2\sin\left(-(h/2)\right)\sin\left((2a+h)/2\right)}{\sin(a-(a+h))}\sin(a+h)\sin a$

$= \lim\limits_{h\to0} \dfrac{-2\sin\dfrac{h}{2}\sin\left((2a+h)/2\right)}{-\sinh}\sin(a+h)\sin a = \lim\limits_{h\to0} \dfrac{2\sin(h/2)\sin\left((2a+h)/2\right)}{2\sin(h/2)\cos(h/2)}\sin(a+h)\sin a$

$= \lim\limits_{h\to0} \dfrac{\sin\left((2a+h)/2\right)}{\cos(h/2)}\sin(a+h)\sin a = \dfrac{\sin\left((2a+0)/2\right)}{(1)}\sin(a+0)\sin a$

$= \sin a\cdot\sin a\cdot\sin a = (\sin a)^3 = \sin^3 a$

Illustration 36: Evaluate: $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$ **(JEE ADVANCED)**

Sol: Replace x by $\dfrac{\pi}{6}+h$.

We have, $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2} = \lim\limits_{h\to0} \dfrac{2-\sqrt{3}\cos\left((\pi/6)+h\right)-\sin\left((\pi/6)+h\right)}{\left[6\left((\pi/6)+h\right)-\pi\right]^2}$ $\left[\text{Put } x=\dfrac{\pi}{6}+h\right]$

$= \lim\limits_{h\to0} \dfrac{2-\sqrt{3}\left(\cos(\pi/6)\cosh-\sin(\pi/6)\sinh\right)-\left[\sin(\pi/6)\cosh+\cos(\pi/6)\sinh\right]}{\left[\pi+6h-\pi\right]^2}$

$= \lim\limits_{h\to0} \dfrac{2-\sqrt{3}\left((\sqrt{3}/2)\cosh-(1/2)\sinh\right)-\left((1/2)\cosh+(\sqrt{3}/2)\sinh\right)}{36h^2}$

$= \lim\limits_{h\to0} \dfrac{2-(3/2)\cosh+(\sqrt{3}/2)\sinh-(1/2)\cosh-(\sqrt{3}/2)\sinh}{36h^2}$

$$= \lim_{h\to 0} = \frac{2-2\cos h}{36h^2} = \lim_{h\to 0}\frac{1-\cos h}{18h^2} = \lim_{h\to 0}\frac{2\sin^2(h/2)}{18h^2} = \frac{1}{9}\lim_{h\to 0}\left(\frac{\sin(h/2)}{h}\right)^2$$

$$= \frac{1}{9\text{x}4}\left(\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}\right)^2 = \frac{1}{36}$$

Illustration 37: Evaluate: $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-(\pi/4)}$ **(JEE ADVANCED)**

Sol: Replace x by $\frac{\pi}{4}+h$.

We have, $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-(\pi/4)}$ $\begin{bmatrix}\text{put } x=(\pi/4)+h\\ \text{As } x\to(\pi/4)\Rightarrow h\to 0\end{bmatrix}$

$$= \lim_{h\to 0}\frac{\sin\left((\pi/4)+h\right)-\cos\left((\pi/4)+h\right)}{(\pi/4)+h-(\pi/4)} \qquad \begin{bmatrix}\cos\left((\pi/4)+h\right)\\ =\sin\left((\pi/4)+h+(\pi/2)\right)\\ =\sin\left(h+(3\pi/4)\right)\end{bmatrix}$$

$$= \lim_{h\to 0}\frac{\sin\left((\pi/4)+h\right)-\sin\left(h+(3\pi/4)\right)}{h} = \lim_{h\to 0}\frac{2\cos\left(h+(\pi/2)\right)\sin\left(-(\pi/4)\right)}{h}$$

$$= \lim_{h\to 0}\frac{(-\sin h)\left[\sin\left(-(\pi/4)\right)\right]}{h} \qquad = \lim_{h\to 0}2\sin\frac{\pi}{4}\left(\frac{\sin h}{h}\right) = 2\sin\frac{\pi}{4}(1) = 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

Note: All of the above illustrations can be solved using L'Hôpital's rule.

1.12 Infinite Functions

Now, we will discuss the evaluation of the limits of two functions

(a) Exponential function (b) Logarithmic function

(a) Exponential function

Consider the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+.....+\frac{1}{n!}+.....\infty$

This infinite series is denoted by e. The domain of a function f(x) = e^x, $x \in R$ is R and the range is the set of positive real numbers

(b) Logarithmic function

Let $e^y = x$ then it can be written as $\log_e x = y$

The domain of f(x) = $\log_e x$ is $R^{\oplus}$ and the range is R.

The graph of the logarithmic of a function is in the adjoining figure.

Some Important Functions

(i) $(1+x)^n = \left\{1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....\right\}$ $[|x| < 1]$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{3!} + ...$

(iii) $a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...$

(iv) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad (|x|<1)$

(v) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(ix) $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + ...$

(x) $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + ...(-1 < x < 1)$

(xi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} ..... \quad (-1 < x < 1)$

PLANCESS CONCEPTS

Using these expansions is helpful where the limits are in the indeterminate form. But selecting the number of terms to use in the expansion varies with problems.

Vaibhav Gupta (JEE 2009, AIR 54)

Theorem 1: Prove that $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$

Proof: We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \Rightarrow e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\Rightarrow \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ... \Rightarrow \lim_{x\to 0}\frac{e^x - 1}{x} = \lim_{x\to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ...\right) = 1$$

Hence proved.

Theorem 2: Prove that $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$

Proof: Let $\frac{\log_e(1+x)}{x} = y$. then,

$$\Rightarrow \log_e(1 + x) = xy \Rightarrow 1 + x = e^{xy} \Rightarrow e^{xy} - 1 = x \Rightarrow \frac{e^{xy} - 1}{x} = 1 \Rightarrow \frac{e^{xy} - 1}{xy} \cdot y = 1$$

Now taking limit, when $x \to 0$

$$\lim_{xy\to 0}\frac{e^{xy}-1}{xy}\cdot\lim_{x\to 0} y = 1 \qquad [\text{since } x \to 0 \Rightarrow xy \to 0]$$

$$\Rightarrow 1\cdot\lim_{x\to 0} y = 1 \quad \left[\because \lim_{xy\to 0}\frac{e^{xy}-1}{xy} = 1\right]$$

$$\Rightarrow \quad \lim_{x\to 0} y = 1 \;\Rightarrow\; \lim_{xy\to 0}\frac{\log_e(1+x)}{x} = 1$$

Hence proved

Note: If no base of log is mentioned, then it is taken for granted that the base is e.

$\therefore$ log a is same as $\log_e a$

Illustration 38: Evaluate: $\lim_{x\to 0}\frac{e^x - x - 1}{x}$ **(JEE MAIN)**

Sol: As we already prove that $\lim_{x\to 0}\frac{e^x-1}{x} = 1$, Therefore by writing given limit as $\lim_{x\to 0}\frac{e^x-1}{x} - 1$ we can easily solve above problem.

$$\lim_{x\to 0}\frac{e^x-x-1}{x} = \lim_{x\to 0}\frac{e^x-1}{x} - 1 = 1 - 1 = 0$$

Illustration 39: Evaluate: $\lim_{x\to 0}\frac{a^x - b^x}{x}$ **(JEE MAIN)**

Sol: Add and subtract 1 with numerator.

$$\lim_{x\to 0}\frac{a^x-b^x}{x} = \lim_{x\to 0}\frac{(a^x-1)-(b^x-1)}{x}$$

$$= \lim_{x\to 0}\left(\frac{a^x-1}{x} - \frac{b^x-1}{x}\right) = \lim_{x\to 0}\frac{a^x-1}{x} - \lim_{x\to 0}\frac{b^x-1}{x} = \log a - \log b = \log\frac{a}{b}$$

Illustration 40: Evaluate: $\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$ **(JEE ADVANCED)**

Sol: Reduce given limit in the form of $\lim_{x\to 0}\frac{e^x-1}{x}$.

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x} = \lim_{x\to 0}\frac{e^x - \frac{1}{e^x}}{x} = \lim_{x\to 0}\frac{e^{2x}-1}{x}\lim_{x\to 0}\frac{1}{e^x} = 2\lim_{x\to 0}\frac{e^{2x}-1}{2x}\frac{1}{e^0} = 2(1) = 2$$

Illustration 41: Evaluate $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$ **(JEE ADVANCED)**

Sol: By multiplying and dividing by $\sin x$.

$$\lim_{x\to 0}\frac{e^{\sin x}-1}{x} = \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\frac{\sin x}{x}$$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\lim_{x\to 0}\frac{\sin x}{x} = \lim_{\sin x\to 0}\frac{e^{\sin x}-1}{\sin x}\times 1 = 1$$

PLANCESS CONCEPTS

$\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$ then

$\lim\limits_{x\to a}\left[f(x)\right]^{g(x)} = e^{\lim\limits_{x\to a} g(x)\left[f(x)-1\right]}$

$\lim\limits_{n\to\infty} a^n = 0$, if $|a| < 1$ Does not exist if $|a| \geq 1$.

A common mistake made by students pertains to indeterminate limits. Consider a function f(x) = g(x) h(x). We are given that $\lim\limits_{x\to 0} g(x) = 0$. What is the value of $\lim\limits_{x\to 0} f(x)$? Many students would say that it is 0. However, the actual answer depends on $\lim\limits_{x\to 0} h(x) = 0$. If it is not finite, then the limit of f(x) is indeterminate. The point we are trying to make is that in calculating the limit of a function which is the product of two or more functions, if one of the function tends to 0, then that does not make it necessary for the entire limit to be 0 as well. Similar, remarks hold for other indeterminate forms. For example, if f(x) = $g(x)^{h(x)}$, and g(x) → 1 as x → a, it does not necessarily imply that f(x) → 1 as x → a, because if $\lim\limits_{x\to a} h(x)$ is not finite, then the limit on f(x) is indeterminate.

Akshat Kharaya (JEE 2009, AIR 235)

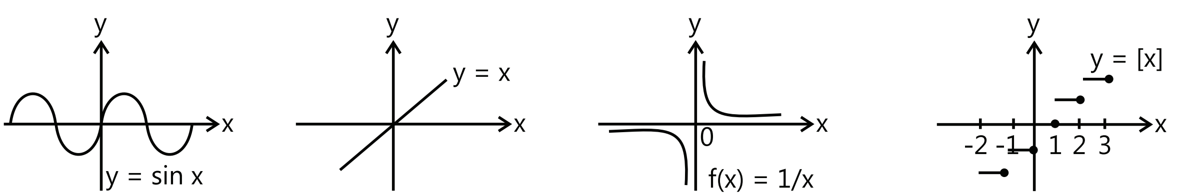
2. CONTINUITY

2.1 Introduction

The word 'continuous' means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be continuous. A function which is not continuous is called a discontinuous function.

Ex.

(i) (ii) (iii) (iv)



(Continuous function) (Continuous function) (Discontinuous at x = 0) (Discontinuous) At Every integer

Figure 14.6

Following are examples of some continuous functions:

(i) f(x) = x (Identity function)

(ii) f(x) = c (Constant function)

(iii) $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a^n$ (Polynomial function)

(iv) f(x) = sin x, cos x (Trigonometric function)

(v) $f(x) = a^x, e^x, e^{-x}$ (Exponential function)

(vi) f(x) = log x (Logarithmic function)

(vii) f(x) = sinh x, coshx, tanhx (Hyperbolic function)

(viii) f(x) = |x|, x + |x|, x – |x|, x |x| (Absolute value functions)

Following are example of some discontinuous functions:

No.	Functions	Points of discontinuity
(i)	[x]	Every Integer
(ii)	x – [x]	Every Integer
(iii)	$\frac{1}{x}$	x = 0
(iv)	tan x, sec x	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}.$ ....
(v)	cot x, cosec x	x= 0, ± π, ± 2π....
(vi)	$\sin\frac{1}{x}, \cos\frac{1}{x}$	x = 0
(vii)	$e^{1/x}$	x = 0
(viii)	coth x, cosech x	x = 0

2.2 Continuity of a Function at a Point

A function f(x) is said to be continuous at a point x = a, if

(a) (i) f(a) exists

(b) (ii) $\lim\limits_{x\to a} f(x)$ exists and is finite

So $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x)$

(c) (iii) $\lim\limits_{x\to a} f(x) = f(a)$ or

Function f(x) is continuous at x = a

If $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$

i.e. If right hand limit at 'a' = left hand limit at 'a' = value of the function at 'a'.

If $\lim\limits_{x\to a} f(x)$ does not exist or $\lim\limits_{x\to a} f(x) \neq f(a)$, then f(x) is said to be discontinuous at x = a

2.3 Continuity from Left and Right

Function f(x) is said to be

(i) Left continuous at x = a if, $\lim\limits_{x\to a^-} f(x) = f(a)$ i.e. $f(a^-) = f(a)$

(ii) Right continuous at x = a if, $\lim\limits_{x\to a^+} f(x) = f(a)$ i.e. $f(a^+) = f(a)$

Thus, a function f(x) is continuous at a point x = a, if it is left continuous as well as right continuous at x = a.

Illustration 42: $f(x) = \begin{cases} x + a\sqrt{2}\sin x & 0 \le x < \pi/4 \\ 2x\cot x + b & \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & \frac{\pi}{2} < x \le \pi \end{cases}$ is continuous in [0, p]. Find a and b. **(JEE MAIN)**

Sol: By checking left continuous and right continuous for x = $\frac{\pi}{4}$ and x = $\frac{\pi}{2}$ we can obtain value of a and b.

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + b;\ f\left(\frac{\pi}{4}^{-}\right) = \frac{\pi}{4} + a \Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow a - b = \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = b\ ;\ f\left(\frac{\pi}{2}^{+}\right) = -a - b \Rightarrow a - b = b \Rightarrow 2b = -a \Rightarrow \quad a = \frac{\pi}{6}, b = -\frac{\pi}{12}$$

Illustration 43: Examine the continuity of the function $f(x) = \begin{cases} 2x^2 + 2, & x \le 2 \\ 2x, & x > 0 \end{cases}$, at the point x = 2 **(JEE MAIN)**

Sol: By obtaining Left hand limit and right hand limit we will get to know that given function is continuous or not.

$f(2) = 2^2 + 2 = 6$ ... (i)

L.H.L. $f(2^-) = \lim_{h \to 0}((2-h)^2 + 1) = 5$ ... (ii)

R.H.L. $f(2^+) = \lim_{h \to 0} 2(2+h) = 4$ ... (iii)

$\because$ f(2 – 0) ≠ f(2 + 0) ≠ f(2)

$\therefore$ f(x) is not continuous at x = 2

Illustration 44: If $f(x) = \begin{cases} x + \lambda, & x < 3 \\ 4, & x = 3 \\ 3x - 5, & x > 3 \end{cases}$, is continuous at x = 3, then the value of λ is **(JEE MAIN)**

Sol: As given function is continuous at x = 3, hence its left hand limit will be equal to its right hand limit f(x) is continuous at x = 3

$$\therefore \quad f(3) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \Rightarrow \quad 4 = \lim_{h \to 0}((3-h) + \lambda) = 3 + \lambda \Rightarrow \lambda = 1$$

2.4 Continuity of a Function in an Interval

(a) A function f(x) is said to be continuous in an open interval (a, b), if it is continuous at every point in (a, b). For example, function y = sin x, y = cos x, y = e^x are continuous in (–∞, ∞)

(b) A function f(x) is said to be continuous in the closed interval [a, b], if it is:

  (i) Continuous at every point of the open interval (a, b)

  (ii) Right continuous at x = a, i.e. $RHL|_{x=a}$ = f(a)

  (iii) Left continuous at x = b, i.e. $LHL|_{x=b}$ = f(b)

2.5 Reasons of Discontinuity

(a) Limit does not exist i.e. $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\lim\limits_{x\to a} f(x) \neq f(a)$

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1, 2 and 3.

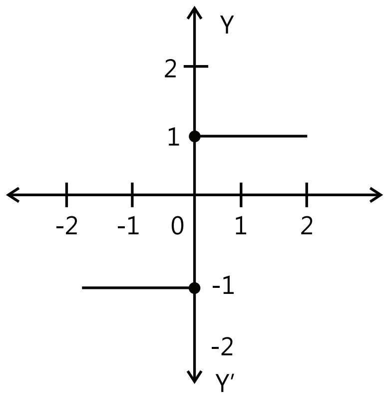


Figure 14.6

3 DIFFERENTIABILITY

3.1 Meaning of a Derivative

The instantaneous rate of change of a function with respect to the dependent variable is called the derivative. Let 'f' be a given function of one variable and let Dx denote a number (positive or negative) to be added to the number x. Let Df denote the corresponding change of 'f ', then

Df = f(x + Dx) – f(x) $\Rightarrow \dfrac{\Delta f}{\Delta x} = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

If $\dfrac{\Delta f}{\Delta x}$ approaches a finite value as Dx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is denoted by symbols such as

$$f'(x), \frac{df}{dx}, \frac{d}{dx}(f(x)) \text{ or if } y = f(x) \text{ by } \frac{dy}{dx} \text{ or } y' \Rightarrow \frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The process of finding derivative of a function is called differentiation. We also use the phrase differentiate f(x) with respect to x which means to find f '(x).

PLANCESS CONCEPTS

The fact that f(x) is differentiable at x = a means that the graph is smooth as x moves from a^- to a to a^+.

Derivative of an even function is always an odd function.

Anvit Tawar (JEE 2009, AIR 9)

3.2 Existence of Derivative at x = a

(a) Right hand derivative:

The right hand derivative of f(x) at x = a, denoted by f ' (a^+) is defined as:

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}, \text{ provided the limit exists \& is finite. (h>0)}$$

(b) Left hand derivative:

The left hand derivative of f(x) at x = a, denoted by f ' (a^-) is defined as:

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}, \text{ provided the limit exists \& is finite. (h > 0)}$$

Hence f(x) is said to be derivable or differentiable at x = a if

$f'(a^+) = f'(a^-)$ = finite quantity and it is denoted by $f'(a)$;

Right hand and left hand derivative at x = a is also denoted by $Rf'(a)$ and $Lf'(a)$ respectively.

PLANCESS CONCEPTS

- If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
- If a function is differentiable at a point, then it is also continuous at that point.
- If a function is continuous at point x = a, then nothing can be guaranteed about the differentiability of that function at that point.
- If a function f(x) is not differentiable at x = a, then it may or may not be continuous at x = a
- If a function f(x) is not continuous at x = a, then it is not differentiable at x = a
- If the left hand derivative and the right hand derivative of f(x) at x = a are finite (they may or may not be equal), then f(x) is continuous at x = a.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 45: $f(x) = \begin{cases} ax+b & x \le -1 \\ ax^3+x+2b & x > -1 \end{cases}$ is differentiable for xÎR find 'a' & 'b **(JEE ADVANCED)**

Sol: By equating left hand limit to its right hand limit we can obtain 'a' & 'b**.**

$$f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0}\frac{(a(-1-h)+b)-(b-a)}{-h} = \lim_{h\to 0}\frac{-ah}{-h} = a$$

$$f'(-1^+) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{(a(h-1)^3+(h-1)+2b)-(b-a)}{h}$$

$$= \lim_{h\to 0}\left(a\left(h^2-3h+3\right)+1+\frac{(-a-1+b+a)}{h}\right) = 3a+1+\lim_{h\to 0}\left(\frac{b-1}{h}\right)$$

For $f'(-1^+)$ to exist b = 1

Given f(x) is differentiable $\Rightarrow$ $3a+1 = a \Rightarrow a = -1/2$

3.3 Derivative Formula (Theorem)

If a function f(x) is derivable at x =a, then f(x) is continuous at x = a. i.e. every differentiable function is continuous.

Proof: $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists. Also $f(a + h) - f(a) = \frac{f(a+h)-f(a)}{h}h\,[h \ne 0]$

$$\therefore \lim_{h\to 0}[f(a+h)-f(a)] = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot h = f'(a).0 = 0$$

$$\Rightarrow \lim_{h\to 0}[f(a+h)-f(a)] = 0 \Rightarrow \lim_{h\to 0}f(a+h) = f(a) \Rightarrow f(x) \text{ is continuous at } x = a$$

PLANCESS CONCEPTS

(i) Converse of the theorem above is not true.

(ii) Every differentiable function is necessarily continuous but every continuous function is not necessarily differentiable i.e. Differentiability $\Rightarrow$ continuity

but continuity $\not\Rightarrow$ differentiability

(iii) All polynomial, trigonometric, logarithmic and exponential functions are continuous and differentiable in their domains.

(iv) If f(x) & g(x) are differentiable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x) g(x) will also be differentiable at x = a & if g(a) $\neq$ 0, then the function f(x) / g(x) will also be differentiable at x = a

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 46: Show that f(x) = |x – 3|, "x $\in$ R, is continuous but not differentiable at x = 3. **(JEE ADVANCED)**

Sol: If left hand limit and right hand limit of the function is equal then the function is continuous and if the left hand limit and right hand limit of the derivative of the function is equal then the function is differentiable otherwise not differentiable.

$$f(x) = |(x-3)| \Rightarrow f(x) = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}; \ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -(x-3) = 0$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x-3) = 0 \text{ and } f(3) = 3 - 3 = 0; \ \therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

So, f(x) is continuous at x = 3 $Lf'(3) = \lim_{x \to 3^-} \dfrac{f(x) - f(3)}{x-3} = \lim_{x \to 3^-} \dfrac{-(x-3) - 0}{x-3} = -1$

$$Rf'(3) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3} = \lim_{x \to 3^+} \frac{(x-3) - 0}{x-3} = 1; \quad Lf'(3) \neq Rf'(3)$$

So, f(x) is not differentiable at x = 3.

3.4 Differentiability in an Interval

(a) f(x) is said to be derivable over an open interval (a, b), if it is derivable at each & every point of the interval (a, b).

(b) A function f(x) is differentiable in a closed interval [a, b] if it is:

(i) Differentiable at every point of interval (a, b)

(ii) Right derivative exists at x = a

(iii) Left derivative exists at x = b.

PLANCESS CONCEPTS

If a function is said to be differentiable over an interval, it is differentiable at each and every point in the interval. If it is not so, even at a single point, then we cannot say that it is differentiable over the interval.

Rohit Kumar (JEE 2012, AIR 79)

3.5 Differentiable Functions and their Properties

A function is said to be a differentiable function, if it is differentiable at every point of its domain

(a) Examples of some differentiable functions:

(i) Every polynomial function

(ii) Exponential functions: a^x, e^x, e^{-x} ......

(iii) Logarithmic functions: $\log_a x$, $\log_e x$ ....

(iv) Trigonometric functions: sinx, cosx,

(v) Hyperbolic functions: sinhx, coshx, ...

(b) Examples of some non-differentiable functions:

(i) $|x|$ at x = 0

(ii) $x \pm |x|$ at x = 0

(iii) $[x]$, $x \pm [x]$ at every $n \in Z$

(iv) $x\sin\left(\frac{1}{x}\right)$, at x = 0

(v) $\cos\left(\frac{1}{x}\right)$, at x = 0

Illustration 46: Check the differentiability of the function $f(x) = \begin{cases} x+2, & x>3 \\ 5, & x=3 \\ 8-x, & x<3 \end{cases}$ at x=3 **(JEE MAIN)**

Sol: For the given function to be differentiable f'(3 + h) =f'(3 – h).

For function to be differentiable f'(3 + h) =f'(3 – h)

$$f'(3+h) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \Rightarrow \lim_{h\to 0}\frac{(3+h+2)-5}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$f'(3-h) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h} \Rightarrow \lim_{h\to 0}\frac{8-(3-h)-5}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$f'(3+h) \neq f'(3-h)$$

So function is not differentiable.

Illustration 47: Check the differentiability of the function $f(x) = \begin{cases} 1, & x<0 \\ 1+\sin x, & 0 \le x \le \pi/2 \\ 2+(x-\pi/2)^2, & \pi/2 < x \le \pi \end{cases}$, at x = π /2

(JEE ADVANCED)

Sol: Similar to above problem, Given function is differentiable if $f'\left(\frac{\pi}{2}+h\right) = f'\left(\frac{\pi}{2}-h\right)$.

$$\text{RHL } f'\left(\frac{\pi}{2}^{+}\right) = \frac{f(\pi/2+h)-f(\pi/2)}{h} = \lim_{h\to 0}\frac{2+(\pi/2+h-\pi/2)^2-(1+\sin\pi/2)}{h}$$

$$= \lim_{h\to 0}\frac{2+h^2-2}{h} = \lim_{h\to 0} h = 0\,;\ \text{LHL } f'\left(\frac{\pi}{2}^{-}\right) = \frac{f\big((\pi/2)-h\big)-f\big((\pi/2)\big)}{-h}$$

$$= \lim_{h\to 0}\frac{1+\sin\left((\pi/2)-h\right)-\left(1+\sin(\pi/2)\right)}{-h} = \lim_{h\to 0}\frac{1+\cosh-2}{-h} = \lim_{h\to 0}\frac{\cosh-1}{-h} = \lim_{h\to 0}\sinh = 0$$

So, the function is differentiable at $x = \frac{\pi}{2}$

Illustration 48: Check the differentiability of the function $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0 **(JEE ADVANCED)**

Sol: Here for the function to be differentiable $f'(0^+) = f'(0^-)$.

For the function to be differentiable

$$f'(0^+) = f'(0^-) = \frac{f'(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(1/h)-0}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

Which does not exist.

$$f'(0^-) = \frac{f'(0-h)-f(0)}{h} = \lim_{h\to 0}\frac{(-h)\sin\left(-(1/h)\right)-0}{-h} = \lim_{h\to 0}\sin\left(-\frac{1}{h}\right)$$

Which does not exist. So the function is not differentiable at x = 0

Illustration 49: A differentiable function f satisfies f(x +y) = f(x) f(y) ∀ x, yÎR find f(x) **(JEE ADVANCED)**

Sol: As given f(x +y) = f(x) f(y), By using this obtain f′(x).

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} = \lim_{h\to 0} f(x)\left(\frac{(f(h)-1)}{h}\right)$$

$$\Rightarrow \quad f'(x) = f(x)f'(0) \;;\; x = 0 \Rightarrow f'(0) = \left(f'(0)\right)^2 \Rightarrow f'(0) = 0 \text{ or } f'(0) = 1$$

$$f'(0) = 0 \Rightarrow f'(x) = 0 \Rightarrow f(x) \text{ is constant}$$

$$f'(0) = 1 \Rightarrow f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1 \Rightarrow \ln f(x) = x + c \Rightarrow f(x) = ae^x$$

$$\because\; f'(0) = 1 \Rightarrow f(x) = e^x$$

PLANCESS CONCEPTS

The sum, difference, product, quotient (denominator 0) and composite of two differentiable functions is also differentiable.

Chen Reddy Sandeep Reddy (JEE 2012, AIR 62)

3.6 Rolle's Theorem

If a function f defined on the closed interval [a, b], is:

(i) Continuous on [a, b],

(ii) Derivable on (a, b) and

(iii) f(a) = f(b), then there exists at least one real number c between a and b (a < c < b), such that f′ (c) = 0

Geometrical interpretation

Let, the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn.

The theorem states that between two points with equal ordinates on the continuous graph of f, there exists at least one point where the tangent is parallel to x-axis.

PLANCESS CONCEPTS

We can use this theorem to check if an equation has more than one root.

Consider an equation f(x)=0, where function f is continuous and differentiable and has more than 2 roots, then f satisfies all three conditions of Rolle's theorem and so we can say that the derivative f'(x) must be zero somewhere.

If the sign of the function doesn't change then the function can't have more than one root.

B Rajiv Reddy (JEE 2012, AIR 11)

3.7 Mean Value Theorem

If a function 'f' defined on the closed interval [a, b] is

(i) Continuous on [a, b] and

(ii) Derivable on (a, b), then there exists at least one real number c between a and b (a < c < b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation

The theorem states that between two points A and B on the graph of 'f' there exists at least one point, where the tangent is parallel to the chord AB.

Illustration 50: Verify Rolle's Theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2] **(JEE MAIN)**

Sol: By checking that the given function satisfies all the condition of Rolle's Theorem as mentioned above, we can verify Rolle's Theorem for the given function.

Given that f(x) is a polynomial function. So, it is always continuous and differentiable.

$f(0) = 0,\ f(2) = 2^3 - 3.(2)^2 + 2(2) = 0 \Rightarrow \quad f(0) = f(2)$

Thus, all the conditions of Rolle's Theorem are satisfied.

So, there must exist some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$$

Where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's Theorem is verified.

Illustration 51: Verify the mean value theorem for the function f(x) = x – 2 sin x, in the interval [–π, p]. **(JEE MAIN)**

Sol: Similar to above, by checking that the given function satisfies all the conditions of mean value Theorem as mentioned above, we can verify mean value Theorem for the given function.

Since x and sin x are everywhere continuous and differentiable, therefore f(x) is continuous on [–π, p] and differentiable on (–π, π). Thus, both the conditions of mean value theorem are satisfied. So, there must exist at least one $c \in (-\pi, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$

Now, f(x) = x – 2 sin x

$$\Rightarrow 1 - 2\cos c = \frac{\pi - (-\pi)}{\pi - (-\pi)} \Rightarrow 1 - 2\cos c = 1$$

$$\Rightarrow \cos c = 0 \Rightarrow c = \pm \pi/2$$

Thus, $c = \pm (\pi / 2) \in (-\pi, \pi)$; Hence the mean value theorem is verified

Illustration 52: Find c of the mean value theorem for the function $f(x) = 3x^2 + 5x+7$ in the interval [1, 3].

(JEE MAIN)

Sol: By using mean value theorem.

Given $f(x) = 3x^2 + 5x + 7$ ... (i)

$\Rightarrow f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$; $f'(x) = 6x + 5$

Now, from the mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 6c + 5 = \frac{f(3) - f(1)}{3 - 1} = \frac{49 - 15}{2} = 17 \Rightarrow c = 2$$

PROBLEM SOLVING TACTICS

Above we have discussed L'Hôpital's rule by an example. Let us consider the same example again

$$\lim_{x \to \infty} \frac{x^2 + \sin x}{x^2}$$

These kind of problems where oscillating functions are involved and $x \to \infty$ are solved using the sandwich theorem.

It states that Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have:

$$g(x) \le f(x) \le h(x)$$

and also suppose that: $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$

Then $\lim\limits_{x \to a} f(x) = L$

Now we know that $-1 \le \sin x \le 1 \Rightarrow \dfrac{-1}{x^2} \le \dfrac{\sin x}{x^2} \le \dfrac{1}{x^2} \Rightarrow \dfrac{x^2 - 1}{x^2} \le \dfrac{x^2 + \sin x}{x^2} \le \dfrac{x^2 + 1}{x^2}$

$$\lim_{x \to \infty} \frac{x^2 + 1}{x^2} = \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right) = 1 ; \lim_{x \to \infty} \frac{x^2 - 1}{x^2} = \lim_{x \to \infty} \left(1 - \frac{1}{x^2}\right) = 1 \Rightarrow \lim_{x \to \infty} \frac{x^2 + \sin x}{x^2} = 1$$

While examining the continuity and differentiability of a function f(x) at a point x = a, if you start with the differentiability and find that f(x) is differentiable then you can conclude that the function is also continuous. But if you find f(x) is not differentiable at x = a, you will also have to check the continuity separately. Instead, if you start with the continuity and find that the function is not continuous then you can conclude that the function is also non-differentiable. But if you find f(x) is continuous, you will also have to check the differentiability separately.

FORMULAE SHEET

	Let f and g be two real functions with a common domain D, then.
(i)	$\lim\limits_{x\to a}(f+g)(x)=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)$
(ii)	$\lim\limits_{x\to a}(f-g)(x)=\lim\limits_{x\to a}f(x)-\lim\limits_{x\to a}g(x)$
(iii)	$\lim\limits_{x\to a}(c\cdot f)(x)=c\lim\limits_{x\to a}f(x)$ [c is a constant]
(iv)	$\lim\limits_{x\to a}(fg)(x)=\lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x)$
(v)	$\lim\limits_{x\to a}\left(\frac{f}{g}\right)(x)=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$
(vi)	$\lim\limits_{x\to a}\left(f(x)\right)^n=\left(\lim\limits_{x\to a}f(x)\right)^n$
(vii)	If f(x) < = g(x), then $\lim\limits_{x\to a}f(x)\le\lim\limits_{x\to a}g(x)$
(viii)	$\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$
(ix)	$\lim\limits_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$
(x)	$\lim\limits_{x\to 0}\frac{\sin x}{x}=1$
(xi)	$\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$
(xii)	$\lim\limits_{x\to 0}\left(1+ax\right)^{1/x}=e^a=\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x$
(xiii)	if $\lim\limits_{x\to a}f(x)=1$ and $\lim\limits_{x\to a}g(x)=\infty$ then $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)}=e^{\lim\limits_{x\to a}g(x)\left[f(x)-1\right]}$ sin x < x < tanx
Expansions of Some Functions	
1.	$(1+x)^n=\left\{1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....\right\}$ [| x | < 1]

2.	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{3!} + ...$
3.	$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...$
4.	$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$
5.	$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......$
6.	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
7.	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
8.	$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$
9.	$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + ...$
10.	$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + ...(-1 < x < 1)$
11.	$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} .....\ (-1 < x < 1)$

L’Hôpital’s rule

For functions f and g which are differentiable: if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ has a finite value then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

Common Indeterminate Forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty$ and ∞^0

Differentiation

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Continuity

Function f(x) is continuous at x = a if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Differentiability	
f(x) is said to be derivable or differentiable at x = a if $f'(a^+) = f'(a^-)$ = finite quantity	
Rolle's theorem	
If a function f defined on the closed interval [a, b] is continuous on [a, b] and derivable on (a, b) and f(a) = f(b), then there exists at least one real number c between a and b (a < c < b), such that f ' (c) = 0	
Mean Value Theorem	
If a function f defined on the closed interval [a, b], is continuous on [a, b] and derivable on (a, b), then there exists at least one real number c between a and b (a < c < b), such that $f'(c) = \frac{f(b)-f(a)}{b-a}$	

Solved Examples

JEE Main/Boards

Example 1: $\lim\limits_{x\to 0}\frac{(\cos x)^{1/2}-(\cos x)^{1/3}}{\sin^2 x}$ is

Sol: Use L' Hôpital's rule to solve this problem.

$$=\lim_{x\to 0}\frac{-\frac{1}{2}(\cos x)^{-1/2}\sin x+\frac{1}{3}(\cos x)^{-2/3}\sin x}{2\sin x\cos x}$$

(using L'Hôpital's rule)

$$=\lim_{x\to 0}\frac{-\frac{1}{2}(\cos x)^{-1/2}+\frac{1}{3}(\cos x)^{-2/3}}{2\cos x}$$

$$=-\frac{1}{2}\left[\frac{1}{2}-\frac{1}{3}\right]=-\frac{1}{12}$$

Example 2: $\lim\limits_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+3\frac{x^4-1}{x^3-x^{-1}}\right]$ is

Sol: Simply using algebra, we can solve this problem.

$$\lim_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+3\frac{x^4-1}{x^3-x^{-1}}\right]$$

$$=\lim_{x\to 1}\left[\left(\frac{4x}{x^3-1}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+\frac{3x(x^4-1)}{x^4-1}\right]$$

$$=\lim_{x\to 1}\left[\left(\frac{4x+1-3x+x^2}{x^3-1}\right)^{-1}+3x\right]$$

$$=\lim_{x\to 1}[x-1+3x]=3$$

Example 3: $\lim\limits_{x\to \pi/3}\frac{2\sin(x-\pi/3)}{1-2\cos x}$ is

Sol: We can solve this problem using trigonometric formulae's.

$$\lim_{x\to \pi/3}\frac{2\sin(x-\pi/3)}{1-2\cos x}\quad\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to \pi/3}\frac{2\cos(x-\pi/3)}{2\sin x}$$

$$=\lim_{x\to \pi/3}\frac{2\cos(x-\pi/3)}{2\sin x}$$

$= \cos 0/\sin(\pi/3) = 2/\sqrt{3}$

Example 4: If $f(x)=\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$ then find $\lim\limits_{x\to\infty} f(x)$

Sol: Solve it by using 1^∞ form

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$$

$$=\lim_{x\to\infty}\left(\frac{1+(5/x)+(3/x^2)}{1+(1/x)+(2/x^2)}\right)^x\ \left\{1^\infty\text{form}\right\}=e^a\ \ldots\quad\text{(i)}$$

$$a = \lim_{x\to\infty} x\left(\frac{1+(5/x)+(3/x^2)}{1+(1/x)+(2/x^2)}-1\right)$$

$$= \lim_{x\to\infty}\frac{4+(1/x)}{1+(1/x)+(2/x^2)} = 4$$

$$\Rightarrow \lim_{x\to\infty} f(x) = e^4$$

Example 5: If a, b are chosen from {1, 2, 3, 4, 5, 6, 7} randomly with replacement. The probability that

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = 7 \text{ is}$$

Sol: Similar to above example.

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} \left[1^{\infty}\text{form}\right] = e^a$$

Where,

$$a = \lim_{x\to 0}\frac{2}{x}\left(\frac{a^x+b^x}{2}-1\right) = \lim_{x\to 0}\left(\frac{a^x+b^x-2}{x}\right)\left\{\frac{0}{0}\text{form}\right\}$$

$$= \lim_{x\to 0}\left(a^x\log a + b^x \log b\right) = \log ab$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = e^{\log ab} = ab$$

Given $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = 7$

Hence the favourable outcomes are (1, 7) and (7, 1). Therefore, the required probability is 2/49.

Example 6: Find $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$

Sol: Solve this by putting $x = \frac{\pi}{2}+h$

Put $x - \pi/2 = \theta$, so that

$$\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$$

$$= \lim_{\theta\to 0}\frac{\cot(\pi/2+\theta)-\cos(\pi/2+\theta)}{(-2\theta)^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\theta^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\frac{\sec^2\theta-\cos\theta}{3\theta^2}$$

$$= \frac{1}{24}\lim_{\theta\to 0}\frac{2\sec^2\theta\cdot\tan\theta+\sin\theta}{2\theta}$$

$$= \frac{1}{24}\left(\frac{2+1}{2}\right) = \frac{1}{16}$$

Example 7: Given function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(i) Evaluate h(g(2)) (ii) If $f(x) = \begin{cases} g(x); x \le 1 \\ h(x); x > 1\end{cases}$. Find 'a' so that f is continuous.

Sol: Equate left hand limit and right hand limit of f(x).

(i) $g(2) = \sqrt{6-4} = \sqrt{2}$

$h(g(2)) = h(\sqrt{2}) = 4 - 3\sqrt{2} + a$

(ii) $f(x) = \begin{cases} g(x);\ x \le 1 \\ h(x);\ x > 1\end{cases}$ $f(x) = \begin{cases} \sqrt{6-2x}, & x \le 1 \\ 2x^2-3x+a, & x > 1\end{cases}$

$f(1)=2$R.H.L. $|_{x=1} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x^2-3x+a)$

$= a - 1$ .... (i)

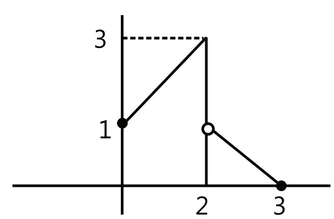
L.H.L. $|_{x=1} = \lim_{x\to 1^-}\sqrt{6-2x} = 2$

Since function is continuous

L.H.L. $|_{x=1}$ = R.H.L. $|_{x=1}$ = f(1) $\Rightarrow$ a = 3

Example 8: Let $f(x) = \begin{cases} 1+x\ ;\ 0 \le x \le 2 \\ 3-x\ ;\ 2 < x \le 3\end{cases}$.

Determine the form of g(x) = f[f(x)] & hence find the point of discontinuity of g (if any)



Graph of f(x)

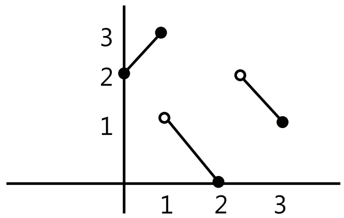
Sol: Sketch the graph of f(f(x)) to obtain the point of discontinuity.

$$f(x) = \begin{cases} 1+x\ ;\ 0 \le x \le 2 \\ 3-x\ ;\ 2 < x \le 3\end{cases}$$

$(fof)(x) = f(f(x))$

$$= \begin{cases} 1 + f(x)\ ;\ 0 \le f(x) \le 2 \\ 3 - f(x)\ ;\ 2 < f(x) \le 3 \end{cases}$$

Let f(x) = y



Graph of f(f(x))

$$f(f(x)) = \begin{cases} 1 + y & 0 \le y \le 2 \\ 3 - y & 2 < y \le 3 \end{cases}$$

$$= \begin{cases} 2 + x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \\ 4 - x & 2 < x \le 3 \end{cases}$$

Clearly from the graph we can see that at x=1 and 2 the function is discontinuous.

Example 9: Determine the value of a, b & c for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ;\ x < 0 \\ c & ;\ x = 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & ;\ x > 0 \end{cases}$$

is continuous at x = 0

Sol: Here the function is continuous at x = 0, therefore by equating Left hand limit and Right hand limit of the given function we can obtain required values.

$f(0) = c$; LHS $\big|_{x=0} \lim\limits_{x\to 0^-} \dfrac{\sin(a+1)x + \sin x}{x}$

$= \lim\limits_{x\to 0^-} (\cos(a+1)x)(a+1) + \cos x = (\cos 0)\ (a + 1)$

$+ \cos 0 = a+1 +1 = a+2$

$\text{RHS}\big|_{x=0} = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$

$$= \lim_{x\to 0^+} \frac{(1+bx)^{\frac{1}{2}} - 1}{bx} = \lim_{x\to 0^+} \frac{1}{2}(1+bx)^{-\frac{1}{2}} = \frac{1}{2}$$

Since the function is continuous

$$c = a + 2 = \frac{1}{2} \Rightarrow a = -\frac{3}{2}, c = \frac{1}{2} \text{ and } b \in R - \{0\}$$

Example 10: Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & ;\ x \le 1 \\ 3x & ;\ 1 < x < 2 \\ bx^2 - a & ;\ x \ge 2 \end{cases}$ is continuous at x = 1 but discontinuous at x = 2

Sol: Solve this example by using given condition.

at x =1 function should be continuous

$\Rightarrow$ a – b = 3 .....(i)

at x = 2 the function should be discontinuous

$\Rightarrow\ 6 \ne 4b - a$

$\Rightarrow\ 6 \ne 4b - b - 3 \quad \Rightarrow\ (a, b) \ne (6, 3)$

Hence the locus of (a,b) is y=x-3 and x6

Example 11: Let f: R [0, ∞) be such that $\lim\limits_{x\to 5} f(x)$ exists and $\lim\limits_{x\to 5} \dfrac{f(x)^2 - 9}{\sqrt{|x-5|}} = 0$.

Then $\lim\limits_{x\to 5} f(x)$ equals?

Sol: Here in the limit, denominator becomes zero but the limit has a finite value. So, numerator should also be zero (whether to apply L'Hôpital's rule or to factorize the common factor).Given,

$$\lim_{x\to 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$$

Here in the limit, the denominator becomes zero but the limit has a finite value. So the numerator should also be zero (whether to apply L'Hôpital's rule or to factorize the common factor).

Hence, $\lim\limits_{x\to 5}(f(x))^2 - 9 = 0 \Rightarrow \lim\limits_{x\to 5} f(x) = \pm 3$

Since the range of f is [0, ∞), $\lim\limits_{x\to 5} f(x) = 3$

JEE Advanced/Boards

Example 1: Evaluate

$$\lim_{x\to 0} \frac{\sqrt[3]{8+x} - \sqrt[3]{8+x^2-x^3}}{\sqrt[3]{8+x} - \sqrt[3]{8+x^2+x^3}}$$

Sol: $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8+x}-\sqrt[3]{8+x^2-x^3}}{\sqrt[3]{8+x}-\sqrt[3]{8+x^2+x^3}}\left[\dfrac{0}{0}\text{form}\right]$

$$=\lim_{x\to 0}\frac{\frac{1}{3}\left(\sqrt[3]{8+x}\right)^{-2}-\frac{1}{3}\left(\sqrt[3]{8+x^2-x^3}\right)^{-2}\left(2x-3x^2\right)}{\frac{1}{3}\left(\sqrt[3]{8+x}\right)^{-2}-\frac{1}{3}\left(\sqrt[3]{8+x^2+x^3}\right)^{-2}\left(2x+3x^2\right)}=1$$

Example 2: Evaluate $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

Sol: Multiply and divide to given limit by $\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)$.

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$$

$$=\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{1+(1/\sqrt{x})}}{\left\{\sqrt{1+\sqrt{(x+\sqrt{x})/x^2}}+1\right\}}=\frac{1}{1+1}=\frac{1}{2}$$

Example 3: Evaluate

$$\lim_{n\to\infty}\frac{1\cdot\sum_1^n r+2\cdot\sum_1^{n-1} r+3\cdot\sum_1^{n-2} r+.......n\cdot 1}{n^4}$$

Sol: By using summation formula of n numbers we can evaluate given limit.

consider $m\cdot\sum\limits_1^{n-m+1} r$

$$=m\cdot\frac{(n-m+1)(n-m+2)}{2}$$

$$=\frac{m}{2}\{n^2-(2m-3)n+(m-1)(m-2)\}$$

$$=\frac{n^2}{2}\cdot m-\frac{n}{2}m(2m-3)+\frac{m}{2}(m^2-3m+2)$$

$$=\frac{n^2}{2}\cdot m-n\cdot m^2+\frac{3n}{2}\cdot m+\frac{m^3}{2}-\frac{3}{2}m^2+m$$

$$=\left(\frac{n^2}{2}+\frac{3n}{2}+1\right)m-\left(n+\frac{3}{2}\right)m^2+\frac{1}{2}m^3$$

$$\Rightarrow\ \sum_1^n\left\{m\sum_1^{n-m+1} r\right\}$$

$$=\left(\frac{n^2}{2}+\frac{3n}{2}+1\right)\sum_1^n m-\left(n+\frac{3}{2}\right)\sum_1^n m^2+\frac{1}{2}\sum_1^n m^3$$

$$=\frac{n^2+3n+2}{2}\cdot\frac{n(n+1)}{2}$$

$$-\frac{2n+3}{2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\cdot\frac{n^2(n+1)^2}{4}$$

$$=\frac{n(n+1)^2(n+2)}{4}-\frac{n(n+1)(2n+1)(2n+3)}{12}+\frac{n^2(n+1)^2}{8}$$

$$\Rightarrow\ \lim_{n\to\infty}\frac{1\cdot\sum_1^n r+2\cdot\sum_1^{n-1} r+3\cdot\sum_1^{n-2} r+.......n\cdot 1}{n^4}$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\cdot n^4\left\{\begin{array}{l}\dfrac{\left((1/n)+1\right)^2\left((2/n)+1\right)}{4}\\[2ex]-\dfrac{\left(1+(1/n)\right)\left(2+(1/n)\right)\left(2+(3/n)\right)}{12}+\dfrac{\left(1+(1/n)\right)^2}{8}\end{array}\right\}$$

$$=\frac{1}{4}-\frac{4}{12}+\frac{1}{8}=\frac{1}{24}$$

Example 4: Evaluate $\lim\limits_{x\to-\infty}\dfrac{x^4\sin(1/x)+x^2}{1+|x|^3}$

Sol: Simply by putting x = –y in given limit, we can evaluate this.

Let x = –y. Then $y\to\infty$ when $x\to-\infty$

$$\Rightarrow\ \lim_{x\to-\infty}\frac{x^4\sin(1/x)+x^2}{1+|x|^3}=\lim_{y\to\infty}\frac{y^4\sin(-1/y)+y^2}{1+|-y|^3}$$

$$==\lim_{y\to\infty}\frac{-y^4\sin(1/y)+y^2}{1+y^3}\quad[\because\ y\text{ is positive}]$$

$$=\lim_{y\to\infty}\left\{\frac{\sin(1/y)}{(1/y)}\cdot\frac{-y^3}{1+y^3}+\frac{y^2}{1+y^3}\right\}$$

$$=\lim_{y\to\infty}\left\{\frac{\sin(1/y)}{(1/y)}\cdot\frac{-1}{(1/y^3)+1}+\frac{(1/y)}{\left((1/y^3)+1\right)}\right\}$$

$$=1\cdot\frac{-1}{1}+0=-1$$

Example 5: Without using expansions or using L'Hôpital's rule, Prove that $\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=\frac{1}{6}$.

Sol: Consider $\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=p$, and after that replace θ by 3θ to prove above problem.

Let the limit be p. then $p=\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}$

$$=\lim_{\theta\to 0}\frac{3\theta-\sin 3\theta}{(3\theta)^3}\quad \text{(replacing } \theta \text{ by } 3\theta)$$

$$=\lim_{\theta\to 0}\left[\frac{3\theta-3\sin\theta+4\sin^3\theta)}{(3\theta)^3}\right]$$

$$=\lim_{\theta\to 0}\left\{\frac{\theta-\sin\theta}{9\theta^3}+\frac{4}{27}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3\right\}$$

$$=\frac{1}{9}\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}+\frac{4}{27}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3=\frac{1}{9}p+\frac{4}{27}\cdot 1^3$$

$$\Rightarrow p-\frac{1}{9}p=\frac{4}{27}\Rightarrow \frac{8p}{9}=\frac{4}{27}\Rightarrow p=\frac{1}{6}$$

Example 6:

$$f(x)=\begin{cases}\dfrac{a^x-1}{x^n\sin x}\left(\dfrac{b\sin x-\sin bx}{\cos x-\cos bx}\right)^n & x>0\\[2ex] \dfrac{a^x\sin bx-b^x\sin ax}{\tan bx-\tan ax} & x<0\end{cases}$$

is continuous at x = 0 (a, b > 0, b ≠ 1, a ≠ b). Obtain f(0) and a relation between a, b and n.

Sol: Here the given function is continuous at x = 0, therefore $\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(0-h)$.

$$\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{a^h-1}{\sin h\cdot h^n}\left[\frac{b\sin h-\sin bh}{\cos h-\cos bh}\right]^n$$

$$=\lim_{h\to 0}\frac{a^h-1}{h}\cdot\frac{h}{\sin h}\left[\left(\frac{b\sin h-\sin bh}{h}\right)\frac{1}{\cos h-\cos bh}\right]^n$$

$$=\left(\lim_{h\to 0}\left[\left(\frac{b\sin h-\sin bh}{h^3}\right)\frac{h^2}{\cos h-\cos bh}\right]^n\right)\ln a$$

$$=\left(\begin{array}{l}\lim_{h\to 0}\left[\dfrac{b(\sin h-h)-(\sin bh-bh)}{h^3}\right]\\ \cdot\left[b^2\left(\dfrac{1-\cos bh}{b^2h^2}\right)-\left(\dfrac{1-\cos h}{h^2}\right)\right]^{-1}\end{array}\right)^n\ln a$$

$$=\left(\begin{array}{l}\lim_{h\to 0}\left[b\left(\dfrac{\sin h-h}{h^3}\right)-\left(\dfrac{\sin bh-bh}{b^3h^3}\right)b^3\right]\cdot\\ \left[\left(\dfrac{b^2}{2}-\dfrac{1}{2}\right)^{-1}\right]\end{array}\right)^n\ln a$$

$$=\left(\lim_{h\to 0}\left[b\left(-\frac{1}{6}\right)-b^3\left(-\frac{1}{6}\right)\right]\cdot\frac{2}{(b^2-1)}\right)^n\ln a$$

$$=\left(\lim_{h\to 0}\frac{b(b^2-1)}{6}\cdot\frac{2}{(b^2-1)}\right)^n\ln a=\ln a\left(\frac{b}{3}\right)^n$$

hence $f(0^+)=\ln a\cdot\left(\frac{b}{3}\right)^n$

$$f(0^-)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\frac{a^{-h}\sin(-bh)-b^{-h}\sin(-ah)}{\tan(-bh)-\tan(-ah)}$$

$$=\lim_{h\to 0}\frac{a^h\sin ah-b^h\sin bh}{a^h\cdot b^h[\tan ah-\tan bh]}$$

$$=\lim_{h\to 0}\frac{\dfrac{ah\cdot a^h\sin(ah)}{ah}-\dfrac{b^h\sin(bh)}{bh}\cdot bh}{ah\cdot\dfrac{\tan(ah)}{ah}-bh\dfrac{\tan(bh)}{bh}}=1$$

If f(x) is to be continuous at x = 0, then f(0) = 1 and

$$\ln a\cdot\left(\frac{b}{3}\right)^n=1$$

Example 7: Find a, b and c such that

$$\lim_{x\to 0}\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}=2$$

Sol: By using L' hôpital rule solve the given limit.

$$\text{Limit}=\lim_{x\to 0}\frac{a(e^x+xe^x)-b\cdot\left((1)/(1+x)\right)+c(e^{-x}-xe^{-x})}{2x\sin x+x^2\cos x}$$

{using L'Hôpital's rule} $=\frac{a-b+c}{0}$

But, the limit is given to be 2. So, the indeterminate form $\frac{0}{0}$ should continue.

So, a – b + c = 0 ....(i)

Then, limit =

$$\lim_{x\to 0}\frac{\left\{a(1+x)e^x+ae^x+(b/(1+x)^2)+c(-1)e^{-x}+c(1-x)e^{-x}\cdot(-1)\right\}}{2\sin x+2x\cos x+2x\cos x-x^2\sin x}$$

{using L' Hospital's rule}

$$= \lim_{x\to 0}\frac{a(2+x)e^{x}+(b/(1+x)^{2})-c(2-x)e^{-x}}{2\sin x+4x\cos x-x^{2}\sin x}$$

$$= \frac{2a+b-2c}{0}$$

But, the limit is given to be 2.

So, 2a + b – 2c = 0 ... (ii)

Then, limit =

$$\lim_{x\to 0}\frac{\left\{a(2+x)e^{x}+ae^{x}-\dfrac{2b}{(1+x)^{3}}+c(2-x)e^{-x}+ce^{-x}\right\}}{6\cos x-4x\sin x-2x\sin x-x^{2}\cos x}$$

$$= \frac{2a+a-2b+2c+c}{6} = 2\,(\text{given})$$

$\therefore$ 3a – 2b + 3c = 12 ... (iii)

Solving (i),(ii) and (iii) we get a = 3, b = 12, c= 9

Example 8: Consider the function

$$g(x)=\begin{cases}\dfrac{1-a^{x}+xa^{x}\ln a}{a^{x}x^{2}} & ;\ x<0\\[2ex] \dfrac{2^{x}a^{x}-x\ln 2-x\ln a-1}{x^{2}} & ;\ x>0\end{cases}$$

find the value of 'a' & g (0) so that the function g(x) is continuous at x = 0

Sol: By obtaining left hand limit and right hand limit and equating them we can easily solve given limit.

$$\text{L.H.L.}\,\big|_{x=0} = \lim_{x\to 0^-}\left(\frac{1-a^{x}+xa^{x}\ln a}{a^{x}x^{2}}\right)$$

Put x = 0 – h.= $\lim_{h\to 0}\dfrac{1-a^{-h}-ha^{-h}\ln a}{a^{-h}h^{2}}$

$$= \lim_{h\to 0}\frac{a^{h}-1-h\ln a}{h^{2}} = \lim_{h\to 0}\left(\frac{a^{h}\ln a-0-\ln a}{2h}\right)$$

$$= \lim_{h\to 0}\frac{a^{h}(\ln a)^{2}-0}{2} = \frac{(\ln a)^{2}}{2}$$

$$\text{R.H.L.}\,\big|_{x=0} = \lim_{x\to 0^+}\left(\frac{2^{x}a^{x}-x\ln 2-x\ln a-1}{x^{2}}\right)$$

Put x = 0 + h = $\lim_{h\to 0^+}\left(\dfrac{2^{h}a^{h}-h\ln 2-h\ln a-1}{h^{2}}\right)$

$$= \lim_{h\to 0}\frac{(2a)^{h}\ln 2a-\ln 2a}{2h}$$

$$= \lim_{h\to 0}\frac{(2a)^{h}(\ln 2a)^{2}-0}{2h} = \frac{(\ln 2a)^{2}}{2}$$

Since the function is continuous

g(0) = L.H.L.$|_{x-0}$ = R.H.L$|_{x-0}$

$$\frac{(\ln(2a))^{2}}{2} = \frac{(\ln a)^{2}}{2} \Rightarrow (\ln(2a)+\ln a)(\ln 2a-\ln a)=0$$

$(\ln(2a)\ \ln 2 = 0 \Rightarrow \ln(2a^2) = 0$

$$\Rightarrow 2a^{2}=1 \Rightarrow a=\pm\frac{1}{\sqrt{2}} \Rightarrow a=\frac{1}{\sqrt{2}}$$

$$g(0)=\frac{(\ln a)^{2}}{2}=\frac{(\ln 2^{-1/2})^{2}}{2}=\frac{1}{8}(\ln 2)^{2}$$

Example 9: Evaluate: $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}}$

Sol: Here the given limit is in the form of $1^{\infty} = e^{a}$.

$\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}}$ is of the form $1^{\infty} = e^{a}$

Where $a=\lim_{x\to 0}\left(\dfrac{(\sin x)/x}{1-((\sin x)/x)}\right)\left(\dfrac{\sin x}{x}-1\right)=-1$

$$\therefore \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}} = e^{-1}$$

Example 10: Let $f(x) = x^3 - x^2 - 3x - 1$, $g(x) = (x+1)a$ and $h(x)=\dfrac{f(x)}{g(x)}$ where h is a rational function such that

(a) It is continuous everywhere except when x = –1

(b) $\lim_{x\to\infty} h(x)=\infty$ and

(c) $\lim_{x\to -1} h(x)=\dfrac{1}{2}$ .Find $\lim_{x\to 0}(3h(x)+f(x)-2g(x))$

Sol: Simply by following given condition we can solve above example.

$$h(x)=\frac{f(x)}{g(x)}=\frac{x^{3}-x^{2}-3x-1}{(x+1)a} \qquad \text{... (i)}$$

Given that $\lim\limits_{x\to -1} h(x) = \frac{1}{2} \Rightarrow \lim\limits_{x\to -1} \frac{x^3 - x^2 - 3x - 1}{(x+1)a} = \frac{1}{2}$

$$\Rightarrow \lim_{x\to -1} \frac{(x^2 - 2x - 1)(x+1)}{(x+1)a} = \frac{1}{2}$$

$$\Rightarrow \frac{2}{a} = \frac{1}{2} \Rightarrow a = 4\,;\; h(x) = \frac{x^3 - x^2 - 3x - 1}{(x+1)4}$$

$g(x) = 4\,(x + 1)$; $\lim\limits_{x\to 0}(3h(x) + f(x) - 2g(x))$ ...(ii)

$$= \lim_{x\to 0}\left(\left(3\cdot\left(\frac{x^3 - x^2 - 3x - 1}{(x+1)4}\right)\right) + \left(x^3 - x^2 - 3x - 1\right) - 2\left(4\left(x+1\right)\right)\right)$$

$$= \frac{3(-1)}{4} - 1 - 8 = \frac{-3 - 4 - 32}{4} = \frac{-39}{4}$$

Example 11: Let

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0. \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, & x > 0 \end{cases}$$

If possible, find the value of a so that the function is continuous at x = 0.

Sol: f(x) will be continuous at x = 0 if $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(0-h) = f(0)$.

f(x) will be continuous at x = 0 if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0) \quad .... \text{(i)}$$

Now, $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} \dfrac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}} - 4}$

$$= \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}} - 4} = \lim_{h\to 0} \frac{\sqrt{h}\left\{\sqrt{16+\sqrt{h}} + 4\right\}}{16 + \sqrt{h} - 16}$$

$$= \lim_{h\to 0}\{\sqrt{16+\sqrt{h}} + 4\} = 8$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1 - \cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0} \frac{1-\cos 4h}{h^2} = \lim_{h\to 0} \frac{2\sin^2 2h}{h^2} = \lim_{h\to 0} 2\cdot\left(\frac{\sin 2h}{2h}\right)^2 \times 4 = 8$$

$\therefore$ if a = 8 functions will be continuous at x = 0.

Example 12: If $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$ examine the continuity of f(x) at x = 1.

Sol: In order to examine the continuity at x = 1, we are required to derive the definition of f(x) in the intervals x <1, x > 1 and at x = 1, i.e., on and around x=1.

In order to examine the continuity at x = 1, we are required to derive the definition of f(x) in the intervals x < 1, x > 1 and at x = 1, i.e., on and around x = 1.

Now, if 0 < x < 1,

$$f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$$

$$= \frac{\log(x+2) - 0\sin x}{0+1} = \log(x+2)$$

if x = 1, $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(x+2) - 1\cdot\sin x}{1+1} = \dfrac{\log(x+2) - \sin x}{2}$

if x > 1, $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(x+2) - \sin x}{1 + \frac{1}{x^{2n}}} = -\sin x$$

Thus, we have

$$f(x) = \begin{cases} \log\left(x+2\right) & 0 < x < 1 \\ \dfrac{\log\left(x+2\right) - \sin x}{2} & x = 1 \\ -\sin x & x > 1 \end{cases}$$

$$f(1^+) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\{-\sin(1+h)\} = -\sin 1$$

$$f(1^-) = \lim_{h\to 0} f(1-h)\ \lim_{h\to 0} \log(1-h+2) = \log 3$$

Clearly $f(1^+) \neq f(1^-)$

So f(x) is not continuous at x=1

Example 13: Is f(x) differentiable at x = 0 if

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}?$$

Sol: Here if $f'(0^+)$ is equal to the $f'(0^-)$ then only the given function is differentiable otherwise not differentiable.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{(0+h)/(1+e^{1/(0+h)})-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+e^{1/h}} = \frac{1}{1+e^{\infty}} = 0$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{(0-h)/(1+e^{1/(0-h)})-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$\therefore f'(0^+) \neq f'(0^-)$

Hence f(x) is not differentiable at x = 0

Example 14: Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max\{f(t), 0 \le t \le x\}, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}.$$

Discuss the continuity and differentiability of g(x) in the interval [0, 2]

Sol: Similar to above example.

Here, $f(x) = x^3 - x^2 + x + 1$

$\therefore f'(x) = 3x^2 - 2x + 1$ or $f'(x) = 3\left\{x^2 - \frac{2}{3}x\right\} + 1$

$$= 3\left\{x^2 - \frac{2}{3}x + \frac{1}{9}\right\} + 1 - \frac{1}{3} = 3\left\{x - \frac{1}{3}\right\}^2 + \frac{2}{3} > 0 \text{ for all } x$$

$\therefore$ f(x) is an increasing function of x (monotonic function)

$\therefore$ in $0 \le x \le 1$, max $\{f(t), 0 \le t \le x\} = f(x)$

$\therefore g(x) = f(x) = x^3 - x^2 + x + 1, 0 \le x \le 1$

$3 - x, 1 < x \le 2$

As polynomial functions are continuous and differentiable everywhere, g(x) is also continuous and differentiable everywhere except at the turning point of definition x = 1

Now, $g(1^+) = \lim_{h\to 0} g(1+h) = \lim_{h\to 0} \{3 - (1 + h)\} = 2$

$$g(1^-) = \lim_{h\to 0} g(1-h)$$

$$= \lim_{h\to 0}\{(1-h)^3 - (1-h)^2 + (1+h) + 1\} = 2$$

$g(1) = 1^3 - 1^2 + 1 + 1 = 2$

$\therefore \quad g(1^+) = g(1^-) = g(1)$

$\therefore \quad$ g(x) is continuous at x = 1

Next, $g'(1^+) = \lim_{h\to 0}\frac{g(1+h)-g(1)}{h} = \lim_{h\to 0}\frac{\{3-(1+h)\}-2}{h} = -1$

$$g'(1^-) = \lim_{h\to 0}\frac{g(1-h)-g(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^3-(1-h)^2+(1-h)+1-2}{-h}$$

$$= \lim_{h\to 0}\frac{-h^3+2h^2-2h}{-h} = \lim_{h\to 0}(h^2-2h+2) = 2$$

$\therefore g'(1+0) \neq g'(1-0)$;

So g(x) is not differentiable at x = 1.

$\therefore$ g(x) is continuous in [0, 2] and differentiable in [0, 2] except at the point x =1

Example 15: If $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x) \cdot G(x)$ where $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} G(x)$ exists. Prove that f(x) is continuous at all $x \in R$.

Sol: Simply by following the given condition we can solve this example. $\lim_{x\to 0} g(x) = 0$

$$\Rightarrow \lim_{h\to 0} g(0+h) = \lim_{h\to 0} g(0-h) = 0$$

$$\Rightarrow \lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h) = 0 \quad \text{.... (i)}$$

$\lim_{x\to 0} G(x)$ exists $\Rightarrow \lim_{h\to 0} G(0+h) = \lim_{h\to 0} G(0-h)$

$$\Rightarrow \lim_{h\to 0} G(h) = \lim_{h\to 0} G(-h) = \text{finite} \quad \text{.... (ii)}$$

Now, $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x)\cdot f(h) = f(x)\lim_{h\to 0} f(h)$

$\{\because f(x + y) = f(x) \cdot f(y)\}$

$= f(x)\cdot \lim_{h\to 0}\{1 + g(h)G(h)\}$, using the given relation

$$= f(x)\cdot\left\{1 + \lim_{h\to 0} g(h)\cdot \lim_{h\to 0} G(h)\right\}$$

$= f(x) \cdot \{1 + 0 \cdot \text{finite}\}$, using (i) and (ii) $= f(x)$ Also,

$$\lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x)\cdot f(-h) = f(x)\cdot \lim_{h\to 0} f(-h)$$

$= f(x)\cdot \lim_{h\to 0}\{1 + g(-h)\cdot G(-h)\}$, using the given relation

$$= f(x)\cdot\left\{1 + \lim_{h\to 0} g(-h)\cdot \lim_{h\to 0} G(-h)\right\}$$

$= f(x) \cdot \{1 + 0 \cdot \text{finite}\}$, using (i) and (ii) $= f(x)$

$$\therefore \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h) = f(x)$$

$\therefore$ f(x) is continuous everywhere

JEE Main/Boards

Exercise 1

Limits

Q.1 $\lim_{x\to -4} \frac{\sqrt{5+x}-1}{x^2+4x}$

Q.2 $\lim_{x\to 2}\left[\left(\frac{x^3-4x}{x^3-8}\right)^{-1}\right]$

Q.3 $\lim_{x\to 1} \frac{x^2 - x\ell nx + \ell nx - 1}{x-1}$

Q.4 $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

Q.5 $\lim_{x\to 2}\left[\frac{1}{x(x-2)^2} - \frac{1}{x^2-3x+2}\right]$

Q.6 (a) $\lim_{x\to 0} \tan^{-1}\frac{a}{x^2}$ where $a \in R$

(b) Plot the graph of the function

$f(x) = \lim_{t\to 0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

Q.7 $\lim_{x\to 1} \frac{\left(\sum_{K=1}^{100} x^k\right) - 100}{x-1}$

Q.8 $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

Q.9 $\lim_{x\to \frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x}$

Q.10 $\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x}$

Q.11 $\lim_{x\to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

Q.12 $\lim_{\theta\to \frac{\pi}{4}} \frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

Q.13 $\lim_{x\to 0} \frac{1-\cos x + 2\sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6\sin^2 x + x - 5x^3}$

Q.14 If $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite then find the value of 'a' and the limit

Q.15 $\lim_{x\to \pi/4} \tan 2x \tan(\pi/4 - x)$

Q.16 $\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$

Q.17 $\lim_{n\to\infty} n\cdot\cos\left(\frac{\pi}{4n}\right)\cdot\sin\left(\frac{\pi}{4n}\right)$

Q.18 Evaluate: $\lim_{x\to 2} \frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x-2}$

Q.19 $\lim_{x\to 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{2}-\sqrt{1+\cos x}}$

Q.20 $\lim_{x\to I} \frac{\sin\{x\}}{\{x\}}$ where $\{x\}$ is the fractional part function & I is any integer

Q.21 $\lim_{x\to 0}\left[\frac{n\sin x}{x}\right] + \lim_{x\to 0}\left[\frac{n\tan x}{x}\right]$ where [*] denotes the greater function and $n \in I - 0$

Q.22 ABC is an isosceles triangle inscribed in a circle of radius r. If AB = AC and h is the altitude from A to BC. Then in triangle ABC evaluate $\lim_{h\to 0}\frac{\Delta}{P^3}$, where Δ is area of the triangle and P is the perimeter.

Q.23 $\lim_{x\to 0} \frac{\ln(1+x+x^2)+\ln(1-x+x^2)}{x(e^x-1)}$

Q.24 $\lim_{n\to\infty} n^2\left(a^{\frac{1}{n}} - a^{\frac{1}{n+1}}\right)$, $a > 0$

Q.25 $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 4$ then find c

Continuity

Q.1 Find all possible values of a and b so that f (x) is continuous for all $x \in R$ if

$$f(x)=\begin{cases}|ax+3|, & x\le -1\\ |3x+a|, & -1<x\le 0\\ \dfrac{b\sin 2x}{x}-2b, & 0<x<\pi\\ \cos^2 x-3, & x\ge \pi\end{cases}$$

Q.2 The function

$$f(x)=\begin{cases}\left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5}}, & 0<x<\dfrac{\pi}{2}\\ b+2, & x=\dfrac{\pi}{2}\\ \left(1+|\cos x|\right)^{\left(\frac{|a\tan x|}{b}\right)}, & \dfrac{\pi}{2}<x<\pi\end{cases}$$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$

Q.3 Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and

$$h(x)=\begin{cases}\dfrac{f(x)}{x-3}, & x\ne 3\\ K, & x=3\end{cases}$$ then

(a) Find all zeros of f(x)

(b) Find the value of K that makes h continuous at x = 3.

(c) Using the value of K, determine whether h is an even function

Q.4 Let $y_n(x)=x^2+\dfrac{x^2}{1+x^2}+....+\dfrac{x^2}{(1+x^2)^{n-1}}$

and $y(x)=\lim\limits_{n\to\infty} y_n(x)$.

Discuss the continuity of $y_n(x)$ ($n \in N$) and y(x) at x = 0.

Q.5 Find the number of points of discontinuity of the function f(x) = [5x] + {3x} in [0, 5] where [y] denote greater integer function and {y} denote fractional part of y.

Q.6 Examine the continuity of $f(x)=\lim\limits_{n\to\infty}\dfrac{x}{(2\sin x)^{2n}+1}$ for $x \in R$.

Q.7 Let $f(x)=\begin{cases}\dfrac{\ln\cos x}{\sqrt[4]{1+x^2}-1}, & x>0\\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)}, & x<0\end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0? If yes, then what is the value of f(0), if not then indicate the nature of discontinuity.

Q.8 If $f(x+y)=f(x).f(y)$ for all $x,y\in R$ and $f(x)=1+g(x).G(x)$ where $\lim\limits_{x\to 0}g(x)=0$ and $\lim\limits_{x\to 0}G(x)$ exists, prove that f (x) is continuous at all $x \in R$.

Q.9 Find the number of ordered pair(s) (a, b) for which the function $f(x) = \text{sgn}((x^2 - ax + 1)(bx^2 - 2bx + 1))$ is discontinuous at exactly one point (where a, b are integers).

[Note: sgn(x) denotes signum function of x.]

Q.10 Let the equation $x^3 + 2x^2 + px + q = 0$ and $x^3 + x^2 + px + r = 0$ have two roots in common and the third root of each equation are represented by α and β respectively.

If $$f(x)=\begin{cases}e^{x\log_{\tan x}|\alpha+\beta|}, & -1<x<0\\ a, & x=0\\ b\dfrac{\ell n(e^{x^2}+\alpha\beta\sqrt{x}),}{\tan\sqrt{x}} & 0<x<1\end{cases}$$

is continuous at x = 0, then find the value of 2(a + b).

Q.11 A function f: R → R is defined as

$f(x)=\lim\limits_{n\to\infty}\dfrac{ax^2+bx+c+e^{nx}}{1+c\cdot e}$ where f is continuous on R.

Find the values of a, b and c.

Q.12 Let $f(x)=\dfrac{1-\cos 4x}{x^2}$, $x<0$, a, x = 0

$\dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, x>0$. If possible, find the value of a so that the function may be continuous at x = 0.

Q.13 Let $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n-1}+ax^3+bx^2}{x^{2n}+1}$.

If f(x) is continuous for all $x \in R$ find the bisector of angle between the lines $2x + y - 6 = 0$ and $2x - 4y + 7 = 0$ which contains the point (a, b).

Q.14 $f(x) = x^4, x^2 < 1$

$x, x^2 \geq 1$

Discuss the existence of limit at x = 1, -1.

Q.15 Let

$$f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x}, & \dfrac{-\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{|x|}}}{a e^{\frac{2}{x}} + b e^{\frac{3}{|x|}}}, & 0 < x < \dfrac{\pi}{2} \end{cases}$$

If f(x) is continuous at x = 0, then find the value of $(a^2 + b^2)$.

Differentiability

Q.1 Find the derivative of $\cos(x^2+1)$ w.r.t. x using the first principle.

Q.2 Find the derivative of $\tan\sqrt{x}$ w.r.t. x using first principle.

Q.3 Differentiae $e^{\sin x} + (\tan x)^x$ w.r.t.x

Q.4 Find the derivative of $\sin x^2$ w.r.t.x using first principle.

Q.5 Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t.x

Q.6 Differentiate w.r.t. x: $\tan^{-1}\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

Q.7 If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, then find $\dfrac{dy}{dx}$

Q.8 If $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$

Examine the continuity of f (x) at x = 1.

Q.9 Find from first principle, the derivative of $\sqrt{\cos x}$ w.r.t. x.

Q.10 Differentiate $\sqrt{\tan x}$ w.r.t. x from first principle

Q.11 Check differentiability of f(x)=|x| at x=0.

Q.12 If f(x) is differentiable at x = a and f′(a)=¼, then find $\lim\limits_{h\to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$.

Q.13 Let f be a real valued continuous function on R and satisfying $f(-x) - f(x) = 0 \ \forall\ x \in R$. If f(–5) = 5, f (–2) = 4, f(3) = –2 and f(0) = 0 then find the minimum number of zero's of the equation f(x) = 0.

Q.14 (a) If g: [a, b] onto [a,b] is continuous show that there is some c ∈ [a,b] such that g(c) = c.

(b) Let f be continuous on the interval [0, 1] to R such that f(0) = f(1). Prove that there exists a point c in $\left[0, \dfrac{1}{2}\right]$ such that $f(c) = f\left(c + \dfrac{1}{2}\right)$.

Exercise 2

Limits

Single Correct Choice Type

Q.1 C is a point on the circumference of a circle & D is the foot of the perpendicular from C on a fixed diameter AB. Then the limit of $\dfrac{CD^2}{DB}$ as C tends to B along the circumference

(A) Does not exist

(B) Equal to one

(C) Is equal to the length AB

(D) None

Q.2 $\lim\limits_{x\to 0}\left(1 + \log^2_{\cos\frac{x}{2}} \cos x\right)$

(A) Is equal to 4 (B) Is equal to 25

(C) Is equal to 289 (D) Is non existent

Q.3 Let α & β be the roots of the equation, $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$, then

$\lim\limits_{x\to m} \dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$, when:

(A) a > 0 & m > 1 (B) a < 0 & m < 1

(C) a < 0 & α < m < b (D) $\dfrac{|a|}{a} = 1$ & m > α

Q.4 $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{2}}-1}{(1+x)^{\frac{1}{3}}-1}$ is

(A) 1 (B) 0 (C) 3/2 (D) ∞

Q.5 $\lim\limits_{x\to 0}(\log x - x)$

(A) Equals ∞ (B) Equals e

(C) Equals $-\infty$ (D) Does not exist

Q.6 $\lim\limits_{x\to 0} x\tan\dfrac{1}{x}$

(A) Equals 0 (B) Equals 1 (C) Equals ∞
(D) Does not exist

Q.7 The limit of $\sqrt{x}\left(\sqrt{x+4}-\sqrt{x}\right)$ as $x\to\infty$

(A) Does not exist

(B) Exists and equals 0

(C) Exists and equals 1/2

(D) Exists and equals 2

Q.8 $\lim\limits_{x\to 0^+}\dfrac{\ell n(\sin 2x)}{\ell n(\sin x)}$ is equal to -

(A) 0 (B) 1 (C) 2 (D) 4

Q.9 Centre of circle is the limit of point of intersection of t lines 3x + 5 y = 1 and (2 + c) x + 5c^2 y = 1 as c tends to 1. If it passes through (2, 0) its radius is -

(A) $\dfrac{\sqrt{1601}}{25}$ (B) $\dfrac{41}{25}$ (C) $\dfrac{1601}{\sqrt{25}}$ (D) $\sqrt{\dfrac{1601}{25}}$

Q.10 $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{\sum\limits_{k=1}^{r}k}{\sum\limits_{k=1}^{r}k^3}$ is equal to

(A) 1 (B) 3 (C) 4 (D) 2

Q.11 $\lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{i=1}^{m}\sqrt[n]{(i-1)^n+i^n}}{m^2}\right)\right)$ is equal to

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5

Q.12 Let $f(x)=x\sin\left(\dfrac{1}{x}\right)$, then

(A) f(0) is not defined but $\lim\limits_{x\to 0}x\sin\left(\dfrac{1}{x}\right)=0$

(B) f(0) = 0 = $\lim\limits_{x\to 0}x\sin\left(\dfrac{1}{x}\right)$

(C) f(0) is defined but $\lim\limits_{x\to 0}x\sin\left(\dfrac{1}{x}\right)$ does not exist

(D) Both f(0) and $\lim\limits_{x\to 0}x\sin\left(\dfrac{1}{x}\right)$ are not defined

Continuity

Single Correct Choice Type

Q.1 Is f(x) differentiale at x = 0 if f(x) is defined as follows:

$f(x)=\dfrac{x}{1+e^{1/x}}$, $x\neq 0$

0, x= 0

(A) 0 (B) 1

(C) 2 (D) Not defferentiable

Q.2 Given f(x) = b ([x]^2 + [x]) + 1 for $x\geq -1$ = sin (π(x + a)) for x < –1

Where [x] denotes the integral part of x, then for what values of a, b the function is continuous at x = – 1 ?

(A) a = 2n + (3/2); b $\in$ R; n $\in$ I

(B) a = 4n + 2; b $\in$ R; n $\in$ I

(C) a = 4n + (3/2); b $\in$ R$^+$; n $\in$ I

(D) a = 4n + 1; b $\in$ R$^+$ n $\in$ I

Q.3 Given $f(x)=\dfrac{[|x|]e^{x^2}[x+|x|]}{e^{\frac{1}{x^2}}-1\,\mathrm{sgn}(\sin x)}$ for $x\neq 0 = 0$

for x = 0 where {x} is the fractional part function; [x] is the step up function and sgn (x) is the signum function of x then, f(x) -

(A) Is continuous at x = 0

(B) Is discontinuous at x = 0

(C) Has a removable discontinuity at x = 0

(D) Has an irremovable discontinuity at x = 0

Q.4 f(x) has an isolated point discontinuity at x = a, then,

(A) $\frac{1}{f(x)}$ necessarily has an isolated point discontinuity at x = a

(B) $\frac{1}{f(x)}$ can be continuous at x = a

(C) $\frac{1}{f(x)}$ will have non-removable discontinuity at x = a

(D) $\frac{1}{f(x)}$ may have missing point discontinuity at x = a

Q.5 The number of points at which the function, f(x) = | x – 0.5 | + | x – 1| + tanx does not have a derivative in the interval (0, 2) is -

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 Which of the following functions defined below are NOT differentiable at the indicated point ?

(A) $f(x)=\begin{cases} x^2 & \text{if } -1\le x<0 \\ -x^2 & \text{if } 0\le x\le 1\end{cases}$ at x = 0

(B) $g(x)=\begin{cases} x & \text{if } -1\le x<0 \\ \tan x & \text{if } 0\le x\le 1\end{cases}$ at x = 0

(C) $h(x)=\begin{cases} \sin 2x & \text{if } x\le 0 \\ 2x & \text{if } x>0\end{cases}$ at x = 0

(D) $k(x)=\begin{cases} x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } 1<x\le 2\end{cases}$ at x = 1

Q.7 If $f(x)=\begin{cases} e^x & \text{for } x<1 \\ a-bx & \text{for } x\ge 1\end{cases}$ is

differentiable for x ∈ R, then:

(A) a = 1, b = e – 1 (B) a = 0, b = e

(C) a = 0, b = – e (D) a = e, b =1

Q.8 $f(x)=\begin{cases} x^2+2x+3 & x\le 2 \\ \frac{a}{\pi}\sin(\pi x)+b & x>2\end{cases}$

If f(x) is derivable ∀ x ∈ R, then

(A) 2a + bπ = 7

(B) b + 2π = 3

(C) 2a + bπ = 13

(D) None of these

Q.9 The function f(x) is defined as follows

$f(x)=\begin{cases} x & \text{if } x<0 \\ x^2 & \text{if } 0\le x\le 1 \\ x^3-x+1 & \text{if } x>1\end{cases}$ then f(x) is

(A) Derivable and cont. at x = 0

(B) Derivable at x = 1 but not continuous at x = 1

(C) Neither derivable nor cont. at x = 1

(D) Not derivable at x = 0 but continuous at x = 1

Q.10 A function f defined as f(x) = x [x] for – 1 ≤ x ≤ 3 where [x] defines the greatest integer ≤ x is

(A) Continuous at all points in the domain of f but non-derivable at a finite number of points

(B) Discontinuous at all points & hence non-derivable at all points in the domain of f

(C) Discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) Discontinuous & also non-derivable at a finite number of points of f.

Q.11 Let $f(x)=\frac{|x|}{\sin x}$ for x ≠ 0 & f(0) = 1 then

(A) f(x) is continuous & differentiable at x = 0

(B) f(x) is continuous & not differentiable at x = 0

(C) f(x) is discontinuous & not differentiable at x = 0

(D) none of these

Q.12 Let f(x) = x^3 and g(x) = |x|, Then at x = 0, the composite functions

(A) gof is derivable but fog is not

(B) fog is derivable but gof is not

(C) gof and fog both are derivable

(D) neither gof nor fog is derivable

Q.13 [x] denotes the greatest integer less than or equal to x. If f(x) = [x] [sin π x] in (–1, 1) then f(x) is -

(A) Continuous at x = 0

(B) Continuous in (–1, 0) ∪ (0, 1)

(C) Differentiable in (–1, 1)

(D) None

Differentiability

Single Correct Choice Type

Q.1 Which of the following function is not differentiable at x = 0?

(A) $x|x|$ (B) x^3

(C) e^{-x} (D) $x + |x|$

Q.2 Which of the following is differentiable function?

(A) $x^2 \sin 1/x$ (B) $x|x|$

(C) cosh x (D) All of the above

Q.3 The function f(x) = sin |x| is

(A) Continuous for all x

(B) Continuous only at certain points

(C) Differentiable at all points

(D) None of these

Q.4 If f(x) = |x – 3| , then f is

(A) Discontinuous at x = 2

(B) Not differentiable at x =2

(C) Differentiable at x = 3

(D) Continuous but not differentiable at x = 3

Q.5 If $f(x) = \frac{|x-1|}{x-1}$, $x \neq 1$, and f(1) = 1, then the correct statements

(A) Discontinuous at x = 1

(B) Continuous at x = 1

(C) Differentiable at x = 1

(D) Discontinuous for x > 1

Q.6 If $f(x) = \begin{cases} x+1, & x>1 \\ 0, & x=1 \\ 7-3x, & x<1 \end{cases}$, then f′(0) equals

(A) 1 (B) 2 (C) 0 (D) –3

Q.7 The function f(x) = |x| + |x –1| is not differentiable at

(A) x = 0, 1 (B) x = 0, –1

(C) x = –1, 1 (D) x = 1, 2

Q.8 If $f(x) = \begin{cases} e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then which one is correct?

(A) f(x) is differentiable at x = 0

(B) f(x) is discontinuous at x = 0

(C) f(x) is continuous no where

(D) None of these

Q.9 Function [x] is not differentiable at

(A) Every rational number

(B) Every integer

(C) Origin

(D) Every where

Q.10 If $f(x) = \begin{cases} |x-3| & , \text{when} x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & , \text{when} x \geq 1 \end{cases}$, then correct statement is -

(A) f is discontinuous at x = 1

(B) f is discontinuous at x = 3

(C) f is differentiable at x = 1

(D) f is differentiable x = 3

Q.11 Function $f(x) = \frac{|x|}{x}$ is -

(A) Continuous everywhere

(B) Differentiable everywhere

(C) Differentiable everywhere except at x = 0

(D) None of these

Q.12 Let f(x) = |x – a| +|x–b|, then -

(A) f(x) is continuous for all $x \in R$

(B) f(x) is differential for $\forall\, x \in R$

(C) f(x) is continuous except at x = a and b

(D) None of these

Q.13 Function f(x) = |x – 1| + |x–2| is differentiable in [0, 3], except at -

(A) x = 0 and x = 3 (B) x = 1

(C) x = 2 (D) x = 1 and x = 2

Q.14 If $f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 1+\sin x, & \text{when } 0 \leq x \leq \pi/2 \end{cases}$

then at x = 0, f ′ (x) equals -

(A) 1 (B) 0 (C) ¥ (D) Does not exist

Q.15 If $f(x) = \begin{cases} \dfrac{x}{1+6^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

then the function f(x) is differentiable for -

(A) $x \in R^+$ (B) $x \in R$

(C) $x \in R_0$ (D) None of these

Q.16 If $f(x) = \begin{cases} x^{\alpha} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at x = 0, then -

(A) $\alpha > 0$ (B) $\alpha > 1$ (C) $\alpha \geq 1$ (D) $\alpha \geq 0$

Q.17 The function f(x) = x – | x | is not differentiable at -

(A) x = 1 (B) x = – 1 (C) x = 0 (D) No where

Previous Years' Questions

Q.1 For a real number y, let [y] denotes the greatest integer less than or equal to y. Then, the function

$f(x) = \dfrac{\tan[\pi(x-\pi)]}{1+[x]^2}$ is ***(1981)***

(A) Discontinuous at some x

(B) Continuous at all x, but the derivative f ′ (x) does not exist for some x

(C) f′(x) exists for all x, but the derivative f ′′ (x) does not exist for some x

(D) f ′(x) exists for all x

Q.2 There exists a function f(x) satisfying f(0) = 1. f′ (0) = –1, f (x) > 0 for all x and ***(1982)***

(A) f ″(x) < 0 for all x

(B) –1 < f ″ (x) < 0 for all x

(C) –2 ≤ f ′′ (x) ≤ – 1 for all x

(D) f ′′ (x) < – 2 for all x

Q.3 If $G(x) = -\sqrt{25-x^2}$, then $\lim\limits_{x\to 1} \dfrac{G(x)-G(1)}{x-1}$ has the value ***(1983)***

(A) $\dfrac{1}{\sqrt{24}}$ (B) $\dfrac{1}{5}$ (C) $-\sqrt{24}$ (D) None of these

Q.4 The function $f(x) = \dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0 so that it is continuous at x = 0, is ***(1983)***

(A) a – b (B) a + b

(C) log a + log b (D) None of these

Q.5 If f(a) = 2, f ′ (a) = 1, g(a) = –1, g′ (a) = 2, then the value of $\lim\limits_{x\to a} \dfrac{g(x)f(a)-g(a)f(x)}{x-a}$ is ***(1983)***

(A) –5 (B) $\dfrac{1}{5}$

(C) 5 (D) None of these

Q.6 $\lim\limits_{n\to\infty}\left(\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+....+\dfrac{n}{1-n^2}\right)$ is equal to ***(1984)***

(A) 0 (B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{2}$ (D) None of these

Q.7 If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

where [x] denotes the greatest integer less than or equal to x, ***(1985)***

(A) 1 (B) 0

(C) –1 (D) None of these

Q.8 If $f(x) = x\left(\sqrt{x}+\sqrt{(x+1)}\right)$, then ***(1985)***

(A) f(x) is continuous but not differentiable at x = 0

(B) f (x) is differentiable at x = 0

(C) f (x) is not differentiable at x = 0

(D) None of the above

Q.9 The set of all points, where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable, is ***(1987)***

(A) $(-\infty, \infty)$ (B) $[0, \infty)$

(C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

Q.10 If $y^2 = P(x)$ is a polynomial of degree 3, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right)$ equals ***(1988)***

(A) $P'''(x) + P'(x)$ (B) $P''(x) \cdot P'''(x)$

(C) $P(x)\,P'''(x)$ (D) A constant

Q.11 If $f(x) = \dfrac{1}{2}x - 1$, then on the interval $[0, p]$ ***(1989)***

(A) tan [f(x)] and 1/f(x) are both continuous

(B) tan [f(x)] and 1/f(x) are both discontinuous

(C) tan [f(x)] and $f^{-1}(x)$ are both continuous

(D) tan [f(x)] and 1/f is not continuous

Q.12 The value of $\lim\limits_{x\to 0}\dfrac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x}$ ***(1989)***

(A) 1 (B) –1 (C) 0 (D) None of these

Q.13 The function $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, $[\times]$ denotes the greatest integer function, is discontinuous at ***(1993)***

(A) All x (B) All integer points

(C) No x (D) x which is not an integer

Q.14 Let $[\times]$ denotes the greatest integer function and $f(x) = [\tan^2 x]$, then ***(1993)***

(A) $\lim\limits_{x\to 0} f(x)$ does not exist

(B) f(x) is continuous at x = 0

(C) f(x) is not differentiable at x = 0

(D) $f'(0) = 1$

Q.15 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & o \\ p & p^2 & p^3 \end{vmatrix}$,

where p is constant. Then $\dfrac{d^3}{dx^3}$ f(x) at x = 0 is ***(1997)***

(A) p (B) $p + p^2$

(C) $p + p^3$ (D) Independent of p

Q.16 Let $f(x) = \begin{cases} (x-1)\sin\left(\dfrac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$.

Then which one of the following is true? ***(2008)***

(A) f is neither differentiable at x = 0 nor at x = 1

(B) f is differentiable at $x = 0$ and at $x = 1$

(C) f is differentiable at $x = 0$ but not at $x = 1$

(D) f is differentiable at $x = 1$ but not at $x = 0$

Q.17 Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals ***(2009)***

(A) -1 (B) 1 (C) log 2 (D) –log 2

Q.18 Let $f : R \to R$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$.

Statement-I: $f(c) = \dfrac{1}{3}$, for some $c \in R$.

Statement-II: $0 < f(x) \leq \dfrac{1}{2\sqrt{2}}$, for all $x \in R$

(A) Statement-I is true, statement-II is true; statement-II is not the correct explanation for statement-I

(B) Statement-I is true, statement-II is false

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true; statement-II is the correct explanation for statement-I

Q.19 Let $f: R \to R$ be a positive increasing function with $\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)} = 1$. Then $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)} =$. ***(2010)***

(A) $\dfrac{2}{3}$ (B) $\dfrac{3}{2}$ (C) 3 (D) 1

Q.20 Let $f : (-1,1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$ Let $g(x) = [f(2f(x) + 2)]^2$

Then $g'(0) =$ ***(2010)***

(A)-4 (B) 0 (C) -2 (D) 4

Q.21 $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ ***(2011)***

(A) Equals $\sqrt{2}$ (B) Equals - $\sqrt{2}$

(C) Equals $\dfrac{1}{\sqrt{2}}$ (D) Does not exist

Q.22 The value of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q & x=0\\ \dfrac{\sqrt{x+x^2}}{x^{3/2}} & x>0\end{cases}$$

is continuous for all x in R, is ***(2011)***

(A) $p=\frac{5}{2},q=\frac{1}{2}$ (B) $p=-\frac{3}{2},q=\frac{1}{2}$

(C) $p=-\frac{1}{2},q=\frac{3}{2}$ (D) $p=-\frac{1}{2},q=-\frac{3}{2}$

Q.23 $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to ***(2013)***

(A) $-\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

Q.24 If $y=\sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x=1$ is equal to: ***(2013)***

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\sqrt{2}$

Q.25 $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$ is equal to ***(2014)***

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

Q.26 If f and g are differentiable functions in [0,1] satisfying $f(0)=2=g(1),g(0)=0$ and $f(1)=6$, then for some $c\in[0,1]$ ***(2014)***

(A) $f'(0)=2=g(1),g(0)=0$ (B) $f'(c)=2g'(c)$

(C) $2f'(c)=g'(c)$ (D) $2f'(c)=3g'(c)$

Q.27 $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to ***(2015)***

(A) 4 (B) 3 (C) 2 (D) $\frac{1}{2}$

Q.28 If the function $g(x)=\begin{cases}k\sqrt{x+1}\ , & 0\le x\le 3\\ k\sqrt{x+1}\ , & 3<x\le 5\end{cases}$

is differentiable, then the value of k + m is; ***(2015)***

(A) 2 (B) $\frac{16}{5}$ (C) $\frac{10}{3}$ (D) 4

Q.29 The normal to the curve, $x^2+2xy-3y^2=0$ at (1, 1) ***(2015)***

(A) Does not meet the curve again

(B) Meets the curve again in the second quadrant

(C) Meets the curve again in the third quadrant.

(D) Meets the curve again in the fourth quadrant.

Q.30 Let $p=\lim\limits_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to: ***(2016)***

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) 2

Q.31 $\lim\limits_{x\to\infty}\left(\dfrac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to: ***(2016)***

(A) $\frac{27}{e^2}$ (B) $\frac{9}{e^2}$ (C) $3\log 3-2$ (D) $\frac{18}{e^4}$

JEE Advanced/Boards

Exercise 1

Limits

Q.1 Find $\lim_{x\to 1}\frac{\left[\sum_{K=1}^{100} x^k\right]-100}{x-1}$

Q.2 Find the sum of an infinite geometric series whose first term is the limit of the function $f(x)=\frac{1-\tan x}{1-\sqrt{2}\sin x}$ as $x \to \pi/4$ and whose common ratio is the limit of the function $g(x)=\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}$ as $x \neq 1, \in R$.

Q.3 $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)p, q \in N$

Q.4 $\lim_{x\to\infty}(x-\ln\cosh x)$ where cosh $t=\frac{e^{+t}+e^{-t}}{2}$

Q.5 $\lim_{x\to 0}\frac{\sin^4(3\sqrt{x})}{1-\sqrt{\cos x}}$

Q.6 (a) $\lim_{x\to \frac{1}{\sqrt{2}}}\frac{\cos^{-1}(2x\sqrt{1-x^2})}{x-\frac{1}{\sqrt{2}}}$

(b) $\lim_{x\to\frac{\pi}{4}}\frac{1-\sqrt{\sin 2x}}{\pi-4x}$

(c) $\lim_{x\to -7}\frac{[x]^2+15[x]+56}{\sin(x+7)\sin(x+8)}$

Where [ ] denotes the greatest integer function

Q.7 Find $\lim_{x\to\frac{3\pi}{4}}\frac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$

Q.8 Find $\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

Q.9 Find

$$\lim_{h\to 0}\frac{\sin\left((\pi/3)+4h\right)-4\sin\left((\pi/3)+3h\right)+6\sin\left((\pi/3)+2h\right)-4\sin\left((\pi/3)+h\right)+\sin(\pi/3)}{h^4}$$

Q.10 Find $\lim_{x\to\infty}x^2\left(\frac{\sqrt{x+2}}{x}-\frac{\sqrt[3]{x+3}}{x}\right)$

Q.11 Find $\lim_{x\to -\infty}\frac{(3x^4+2x^2)\sin(1/x)+|x|^3+5}{|x|^3+|x|^2+|x|+1}$

Q.12 If $\ell=\lim_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1}-r\sin\frac{\pi}{r}\right)$ then find $\{\ell\}$. (where { } denotes the fractional part function)

Q.13 Find a & b if:

(i) $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$

(ii) $\lim_{x\to-\infty}\left(\sqrt{x^2-x+1}-ax-b\right)=0$

Q.14 $\lim_{x\to 0}[\ln(1+\sin^2 x)\cdot\cot(\ln^2(1+x))]$

Q.15 $\lim_{x\to 1}\frac{(\ln(1+x)-\ln 2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}]\cdot\sin(x-1)}$

Q.16 If $\lim_{x\to 0}\frac{e^{x^2}-3^{3x}}{\sin\left(\frac{x^2}{2}\right)-\sin x}=\ln K$ (where $K\in N$) find K.

Q.17 If $\lim_{x\to 3}\left(\frac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right)^{\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6}}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c $\in$ N, then find the least value of $(a^2+b^2+c^2)$.

Q.18 If the $\lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$ exists and has the value equal to ℓ, then find the value of $\frac{1}{a}-\frac{2}{\ell}+\frac{3}{b}$.

Q.19 Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

(i) $a_n + b_n + c_n = 2n + 1$

(ii) $a_nb_n + b_nc_n + c_na_n = 2n - 1$

(iii) $a_nb_nc_n = -1$

(iv) $a_n < b_n < c_n$

Then find the value of $\lim\limits_{n\to\infty}(na_n)$

Q.20 Let $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = x^2 + x - 2$

If $\lim\limits_{x\to 1}\frac{f(x)}{g(x)} = 1$ and $\lim\limits_{x\to -2}\frac{f(x)}{g(x)} = 4$, then find the value of $\frac{c^2+d^2}{a^2+b^2}$

Differentiability

Q.1 Discuss the continuity & differentiability of the functions, $f(x) = \sin x + \sin|x|$, $x \in R$.

Q.2 If the function f(x) defined as

$$f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$$

is continuous but not derivable at x = 0, then find the range of n.

Q.3 Let $g(y) = \lim\limits_{x\to y}\dfrac{\tan x - \tan y}{1 - \frac{x}{y} + \left(1 - \frac{x}{y}\right)\cdot \tan x \tan y}$

and $f(x) = x^2$. If h(x) = Min. (f(x), g(x)), find the number of points where h(x) is non-derivable-

Q.4 Let f(0) = 0 and f ′ (0) = 1. For a positive integer k, show that

$$\lim_{x\to 0}\frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + ...f\left(\frac{x}{k}\right)\right) =$$

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$$

Q.5 Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$;

$x \ne 0$, f(0) = 0, test the continuity & differentiability at x = 0 .

Q.6 If $f(x) = |x - 1| \cdot ([x] - [-x])$, then find $f'(1^+)$ & $f'(1^-)$. where [x] denotes greatest integer function.

Q.7 If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ \dfrac{-1}{|x|} & \text{if } |x| \ge 1 \end{cases}$

is derivable at x = 1. Find the values of a & b.

Q.8 Let $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$

and $g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 & \text{if } x \ge 0 \end{cases}$

If m, n and p are respectively the number of points where the functions f, g and gof are not derivable, find the value of (m + n + p)

Q.9 Let f(x) be defined in the interval [–2, 2] such that

$f(x) = \begin{cases} -1 & , -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$ & $g(x) = f(|x|) + |f(x)|$.

Test the differentiability of g(x) in (–2, 2).

Q.10 Examine for continuity & differentiability the points x = 1 & x = 2, the function f defined by

$f(x) = \begin{cases} x[x] & , 0 \le x < 2 \\ (x-1)[x] & , 2 \le x \le 3 \end{cases}$ where [x] = greatest integer less than or equal to x.

Q.11 Discuss the continuity & the derivability in [0, 2] of

$$f(x) = \begin{cases} |2x-3|\,[x] & \text{for } x \ge 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{cases}$$

where [ ] denotes greatest integer function

Q.12 Let f(x) = [3 + 4 sin x] (where [ ] denotes the greatest integer function). If sum of all the values of 'x' in [π, 2p] where f(x) fails to be differentiable, is $\frac{k\pi}{2}$, then find the value of k.

Q.13 The function

$$f(x) = \begin{cases} ax(x-1) + b & \text{when } x < 1 \\ x - 1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$$

Find the values of the constants a, b, p, q so that -

(i) f(x) is continuous for all x

(ii) f′ (1) does not exist

(iii) f′ (x) is continuous at x = 3

Q.14 Let a_1 and a_2 be two value of a for which

$$f(x) = \begin{cases} x.\dfrac{\ln(1+x)+\ln(1-x)}{\sec x - \cos x}, & x \in (-1,0) \\ (a^2-3a+1)x + x^2, & x \in [0,\infty) \end{cases}$$

is differentiable at x = 0, then find the value of $(a_1^2 + a_2^2)$

Exercise 2

Limits

Single Correct Choice Type

Q.1 $\lim\limits_{n\to\infty} \dfrac{1^2 n + 2^2(n-1) + 3^2(n-2) + \ldots\ldots\ldots +}{n^2 \cdot 1 \quad 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3}$ is equal to -

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{6}$

Q.2 If $\ell = \lim\limits_{x\to\infty}\left[\sqrt{x^2+2x} - x\right]$ & $m = \lim\limits_{x\to\infty}\left\{\sqrt{x^2-2x} + x\right\}$,

where $[\ell]$ & $\{\ell\}$ represent integral and fractional part respectively then ℓ + m is equal to -

(A) 0 (B) 1 (C) 2 (D) 3

Q.3 The limit of $x^3\sqrt{x^2+x^4+1} - x\sqrt{2}$ as $x\to\infty$

(A) Exists and equals $\dfrac{1}{2\sqrt{2}}$

(B) Exists and equals $\dfrac{1}{4\sqrt{2}}$

(C) Does not exist

(D) Exists and equals $\dfrac{3}{4\sqrt{2}}$

Q.4 Consider the function

$f(x) = \tan^{-1}\left(2\tan\dfrac{x}{2}\right)$ where $-\dfrac{\pi}{2} \le f(x) \le \dfrac{\pi}{2}$

($\lim\limits_{x\to\pi-0}$ means limit from the left at π and $\lim\limits_{x\to\pi+0}$ means limit from the right). Then

(A) $\lim\limits_{x\to\pi-0} f(x) = \dfrac{\pi}{2}, \lim\limits_{x\to\pi+0} f(x) = \dfrac{\pi}{2}$

(B) $\lim\limits_{x\to\pi-0} f(x) = -\dfrac{\pi}{2}, \lim\limits_{x\to\pi+0} f(x) = \dfrac{\pi}{2}$

(C) $\lim\limits_{x\to\pi} f(x) = \dfrac{\pi}{2}$

(D) $\lim\limits_{x\to\pi} f(x) = -\dfrac{\pi}{2}$

Q.5 If $x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n$ is a sequence of positive numbers such that

$x_n = x_{n-1} + x_{n-2}$. If $\lim\limits_{n\to\infty} \dfrac{x_n}{x_{n-1}}$ exists then it is equal to -

(A) $(\sqrt{5}+1)$ (B) $\sqrt{5}-1$ (C) $\dfrac{\sqrt{5}-1}{2}$ (D) $\dfrac{\sqrt{5}+1}{2}$

Q.6 If α and β be the roots of $ax^2 + bx + c = 0$, then

$\lim\limits_{x\to\alpha}(1+ax^2+bx+c)^{\frac{1}{(x-\alpha)}}$ is

(A) $\ln |a(\alpha-\beta)|$ (B) $e^{a(\alpha-\beta)}$

(C) $e^{a(\beta-\alpha)}$ (D) None of these

Multiple Correct Choice Type

Q.7 Identify the correct statement

(A) $\lim\limits_{x\to1} \dfrac{1-|\cos(x-1)|}{(x-1)^2} = \dfrac{1}{2}$ (B) $\lim\limits_{x\to0^+}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}} = 1$

(C) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 1$ (D) $\lim\limits_{x\to\infty} \dfrac{\tan x}{x} = 0$

Q.8 If $\ell = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x}}, m = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x^2}}$ and $n = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x^3}}$, then -

(A) ℓ =1 (B) m^{-2} = e

(C) ℓ and n are rational (D) m^2 = ln

Q.9 Which of the following limit(s) is are finite

(A) $\lim\limits_{x\to0^+}(\cot x)^{\frac{\sin x - x}{x^4}}$ (B) $\lim\limits_{x\to0^+}(\cot x)^{\frac{x}{\sin x - x}}$

(C) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$ (D) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x} - x\right)$

Q.10 Let $f(x) = \dfrac{x2^x - x}{1-\cos x}$ & $g(x) = 2^x \sin\left(\dfrac{\ln 2}{2^x}\right)$ then

(A) $\lim\limits_{x\to0} f(x) = \ln 2$ (B) $\lim\limits_{x\to\infty} g(x) = \ln 4$

(C) $\lim\limits_{x\to0} f(x) = \ln 4$ (D) $\lim\limits_{x\to\infty} g(x) = \ln 2$

Q.11 Which of the following limits is unity ?

(A) $\lim\limits_{t\to 0}\frac{\sin(\tan t)}{\sin t}$ (B) $\lim\limits_{x\to \pi/2}\frac{\sin(\cos x)}{\cos x}$

(C) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (D) $\lim\limits_{x\to 0}\frac{\sqrt{x^2}}{x}$

Q.12 Which of the following limits vanishes?

(A) $\lim\limits_{x\to\infty} x^{\frac{1}{4}}\sin\frac{1}{\sqrt{x}}$ (B) $\lim\limits_{x\to\pi/2}(1-\sin x)\cdot\tan x$

(C) $\lim\limits_{x\to\infty}\frac{2x^2+3}{x^2+x-5}\cdot\text{sgn}(x)$ (D) $\lim\limits_{x\to 3^+}\frac{[x]^2-9}{x^2-9}$

where [ ] denotes greatest integer function.

Q.13 Which of the following limits vanishes?

(A) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan x}\right)$ (B) $\lim\limits_{x\to\infty}\left(\frac{3x^2+1}{2x^2-1}\right)^{\frac{x^3}{1-x}}$

(C) $\lim\limits_{x\to\frac{\pi+}{4}}\left[\tan\left(x+\frac{\pi}{8}\right)\right]^{\tan 2x}$ (D) $\lim\limits_{x\to 1}\frac{x^4-2x^2+1}{x^3-1}$

Continuity

Single Correct Choice Type

Q.1 If $f(x)=\begin{cases} x/2-1 & , 0\le x<1 \\ 1/2 & , 1\le x<2 \end{cases}$,

g(x) = (2x + 1) (x – k) + 3, $0 \le x < \infty$, then g [f(x)], will be continuous at x = 1 if k is equal to -

(A) 1/2 (B) 1/6 (C) 11/6 (D) 13/6

Q.2 If function $f(x)=\frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$, is continuous function, then f(0) is equal to-

(A) 2 (B) 1/4 (C) 1/6 (D) 1/3

Q.3 If function $f(x)=\frac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/3}} (x\ne 0)$

is continuous at x = 0, then f(0) is equal to

(A) 2 (B) 4 (C) 6 (D) 2/3

Q.4 If

$$f(x)=\begin{cases} \frac{\log(1+2ax)-\log(1-bx)}{x} & ,x\ne 0 \\ k & ,x=0 \end{cases}$$

is continuous at x = 0, then k is equal to -

(A) 2a + b (B) 2a – b (C) b – 2a (D) a + b

Q.5 If $f(x)=\begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1} & , x\ne 0 \\ 1 & , x=0 \end{cases}$

then at x = 0, f(x) is

(A) Continuous (B) Left continuous

(C) Right continuous (D) None of these

Q.6 If the function

$$f(x)=\begin{cases} 1+\sin\frac{\pi}{2}x & , -\infty<x\le 1 \\ ax+b & , 1<x<3 \\ 6\tan\frac{\pi x}{12} & , 3\le x<6 \end{cases}$$

is continuous in the interval (–∞, 6), then the value of a and b are respectively

(A) 0, 2 (B) 1, 1 (C) 2, 0 (D) 2, 1

Q.7 If

$$f(x)=\begin{cases} \frac{\sin[x]}{[x]+1} & , x>0 \\ \frac{\cos \pi[x]/2}{[x]} & , x<0 \\ K & , x=0 \end{cases}$$

is a continuous function at x = 0, then the value of K ([·] denotes greatest integer function) is -

(A) 0 (B) 1 (C) –1 (D) None of these

Q.8 If $f(x)=\begin{cases} x, & \text{if } x \text{ rational} \\ -x, & \text{if } x \text{ irrational} \end{cases}$, then $\lim\limits_{x\to 0} f(x)$, is -

(A) 0 (B) 1

(C) –1 (D) Indeterminate

Q.9 Function f(x) = $[x]^2 - [x^2]$, where [x] greatest integer $\le$ x is discontinuous at -

(A) All integers

(B) All integers except 0 & 1

(C) At x = 1 only

(D) All integers except 1

Q.10 Function $f(x) = \begin{cases} x+2 & , \quad 1 \le x < 2 \\ 4 & , \quad x = 2 \\ 3x-2 & , \quad x > 2 \end{cases}$ is continuous -

(A) Only at x = 2 (B) For $x \le 2$

(C) For $x \ge 2$ (D) None of these

Q.11 If function $f(x) = \begin{cases} 5x-4 & , 0 < x \le 1 \\ 4x^2 + 3bx & , 1 < x < 2 \end{cases}$

is continuous at every point of its domain, then b is equal to -

(A) 0 (B) 1 (C) –1 (D) 13/13

Q.12 If function $f(x) = \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$

is continuous at every point of its domain, then f(0) is equal to -

(A) 1/3 (B) –1/3 (C) 2/3 (D) 2

Q.13 If

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & , \quad x < 0 \\ a & , \quad x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} & , \quad x > 0 \end{cases}$$

then correct statement is -

(A) f(x) is discontinuous at x = 0 for any value of a

(B) If f(x) is continuous at x = 0 when a = 8

(C) f(x) is continuous at x = 0 when a = 0

(D) None of these

Q.14 Function f(x) = 1 + |sin x| is -

(A) Continuous at all points

(B) Discontinuous at all points

(C) Continuous only at x = 0

(D) None of these

Q.15 The sum of two discontinuous functions

(A) Is always discontinuous

(B) May be continuous

(C) Is always continuous

(D) None of these

Q.16 If $f(x) = \begin{cases} x^{\alpha} \cos 1/x & , \ x \ne 0 \\ 0 & , \ x = 0 \end{cases}$

is continuous at x = 0, then

(A) $\alpha < 0$ (B) $\alpha > 0$

(C) $\alpha = 0$ (D) $\alpha \ge 0$

Q.17 If $f(x) = \begin{cases} \dfrac{1-\sin x}{\pi - 2x} & , x \ne \pi/2 \\ k & , x = \pi/2 \end{cases}$

is continuous at $x = \dfrac{\pi}{2}$, then k is equal to -

(A) 0 (B) 1 (C) –1 (D) 1/2

Q.18 If $f(x) = \begin{cases} x \sin 1/x & , x \ne 0 \\ k & , x = 0 \end{cases}$

is continuous at x = 0, then the value of k will be -

(A) 1 (B) –1

(C) 0 (D) None of these

Q.19 Function $f(x) = [x] \cos\left(\dfrac{2x-1}{2}\right)\pi$ is discontinuous at -

(A) Every x (B) No x

(C) Every integral point (D) Every non-integral point

Q.20 If

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x & , \quad 0 < x < \pi/4 \\ 2x \cot x + b & , \quad \pi/4 \le x \le \pi/2 \\ a\cos 2x - b \sin x & , \quad \pi/2 < x \le \pi \end{cases}$$

is continuous at x = π/4, then a –b is equal to -

(A) π/2 (B) 0 (C) 1/4 (D) π/4

Q.21 At origin, the function $f(x) = |x| + \dfrac{|x|}{x}$ is-

(A) Continuous

(B) Discontinuous because |x| is discontinuous there

(C) Discontinuous because $\dfrac{|x|}{x}$ is discontinuous there

(D) Discontinuous because |x| and $\dfrac{|x|}{x}$ both are discontinuous there

Q.22 If $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, then

(A) f(x) is discontinuous at x = 0

(B) f(x) is continuous at x = 0

(C) f(x) is continuous at x = 0 if f(0) = a^2

(D) Alternative (A) and (C)

Q.23 If $f(x) = \dfrac{1 - \sin x + \cos x}{1 + \sin x + \cos x}$,

is continuous at x = π, then f(π) is equal to -

(A) –1 (B) 2 (C) 1/4 (D) p

Q.24 If $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

is continuous function, then k is equal to -

(A) 1 (B) 3 (C) 1/3 (D) 0

Q.25 If $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

is continuous at x = 0, then k is equal to -

(A) ¼ (B) –1/2 (C) 0 (D) 1/2

Q.26 Consider $f(x) = \begin{bmatrix} x[x]^2 \log_{(1+x)} 2 & \text{for } -1 < x < 0 \\ \dfrac{\ln\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{bmatrix}$

where [*] & {*} are the greatest integer function & fractional part function respectively, then -

(A) f(0) = ln2 $\Rightarrow$ f is continuous at x = 0

(B) f(0) = 2 $\Rightarrow$ f is continuous at x = 0

(C) f(x) = e^2 $\Rightarrow$ f is continuous at x = 0

(D) f has an irremovable discontinuity at x = 0

Q.27 Let f(x) = [2+3 sin x] (where [] denotes the greatest integer function) x $\in$ (0, π). Then number of points at which f(x) is discontinuous is -

(A) 0 (B) 4 (C) 5 (D) Infinite

Q.28 y = f(x) is a continuous function such that its graph passes through (a, 0). Then

$\lim_{x \to a} \dfrac{\log_e(1 + 3f(x))}{2f(x)}$ is -

(A) 1 (B) 0 (C) 3/2 (D) 2/3

Q.29 'f' is a continuous function on the real line. Given that x^2 + (f(x)–2) x – $\sqrt{3}$.

f(x) + 2 $\sqrt{3}$ – 3 = 0 then the value of f $\sqrt{3}$ j.

(A) Cannot be determined (B) Is 2(1– $\sqrt{3}$)

(C) Is zero (D) Is $\dfrac{2(\sqrt{3} - 2)}{\sqrt{3}}$

Q.30 Let $f(x) = \begin{bmatrix} x^2 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{bmatrix}$ then

(A) f(x) is discontinuous for all x

(B) Discontinuous for all x except at x = 0

(C) Discontinuous for all except at x = 1 or –1

(D) None of these

Q.31 Consider $f(x) = \lim_{n \to \infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for x ? 0, x $\neq$ 1 f(1) = 0 then -

(A) f is continuous at x = 1

(B) f has a finite discontinuity at x = 1

(C) f has an infinite or oscillatory discontinuity at x = 1

(D) f has a removable type of discontinuity at x = 1

Q.32 If f(x) = a | sin x| + $be^{|x|}$ + c $|x|^3$ and if f(x) is differentiable at x = 0 then -

(A) b = 0, c = 0, a is any real

(B) a = 0, b = 0, c is any real

(C) c = 0, a = 0 b is any real

(D) None of these

Q.33 The number of points in (1, 3) where $f(x) = a^{[x^2]}$, a > 1 and [x] denote the greatest integer function is not differentiable is -

(A) 1 (B) 3 (C) 5 (D) 7

Q.34 A function f defined as $f(x) = x[x]$ for $-1 \le x \le 3$ where [x] defines the greatest integer $\le x$ is:

(A) continuous at all points in the domain of f but non-derivable at a finite number of points

(B) discontinuous at all points & hence non-derivable at all points in the domain of f.

(C) discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) discontinuous & also non-derivable at a finite number of points of f

Q.35 If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$

where {x} denotes the fractional part function, then:

(A) 'f ' is continuous & differentiable at x = 0

(B) 'f ' is continuous but not differentiable at x = 0

(C) 'f ' is continuous & differentiable at x =2

(D) None of these

Multiple Correct Choice Type

Q.36 If $f(x) = \begin{cases} \dfrac{x\ln(\cos x)}{\ln(1+x^2)} & x \ne 0 \\ 0 & x = 0 \end{cases}$ then

(A) f is continuous at x = 0

(B) f is continuous x = 0 but not differentiable at x = 0

(C) f is differentiable at x = 0

(D) f is not continuous at x = 0

Differentiability

Single Correct Choice Type

Q.1 If $f(x) = \begin{cases} e^x, & x \le 0 \\ |1-x|, & x > 0 \end{cases}$ then f(x) is -

(A) Continuous at x = 0

(B) Differentiable at x = 0

(C) Differentiable at x = 1

(D) Differentiable both at x = 0 and 1

Q.2 Which of the following function is not differentiable at x = 1

(A) $\sin^{-1} x$ (B) tan x (C) a^x (D) cos h x

Previous Years' Questions

Q.1 If $x + |y| = 2y$, then y as a function of x is ***(1984)***

(A) Defined for all real x

(B) Continuous at x = 0

(C) Differentiable for all x

(D) Such that $\dfrac{dy}{dx} = \dfrac{1}{3}$ for x < 0

Q.2 The function $f(x) = 1 + |\sin x|$ is ***(1986)***

(A) Continuous no where

(B) Continuous every where

(C) Differentiable at x = 0

(D) Not differentiable at infinite number of points

Q.3 Let [x] denote the greatest integer less than or equal to x. If f(x) = [x sin px], then f(x) is ***(1986)***

(A) Continuous at x = 0 (B) Continuous in (–1, 0)

(C) Differentiable at x = 1 (D) Differentiable in (–1, 1)

Q.4 The function

$f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is ***(1988)***

(A) Continuous at x = 1 (B) Differentiable at x = 1

(C) Discontinuous at x = 1 (D) Differentiable at x = 3

Q.5 The following functions are continuous on $(0, \pi)$ ***(1991, 2M)***

(A) tan x

(B) $\int_0^x t\sin\dfrac{1}{t}\,dt$

(C) $\begin{cases} 1, & 0 \le x \le \dfrac{3\pi}{4} \\ 2\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$

(D) $\begin{cases} x\sin x, & 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

Match the Columns

Match the conditions/expression is column I with statement in column II.

Q.6 ***(1992)***

	Column-I		Column-II
(A)	sin (p[x])	(p)	differentiable every where
(B)	sin (π(x-[x]))	(q)	nowhere differentiable
		(r)	not differentiable at 1 and –1

Q.7 In the following [x], denotes the greatest integer less than or equal to x. ***(2007)***

	Column-I		Column-II
(A)	x|x|	(p)	continuous in (–1, 1)
(B)	$\sqrt{\|x\|}$	(q)	differentiable in (–1, 1)
(C)	x + [x]	(r)	strictly increasing(–1, 1)
(D)	|x –1| + |x +1|	(s)	not differentiable at least at one point in (–1, 1)

Analytical and Descriptive Questions

Q.8 Evaluate the following limit

$$\lim_{x\to 1}\left(\frac{x-1}{2x^2-7x+5}\right)$$ ***(1978)***

Q.9 Evaluate $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$ ***(1978)***

Q.10 Differentiate from first principle $\sin(x^2+1)$ ***(1978, 3M)***

Q.11 If f(x) = x $\tan^{-1}x$, find f '(1) from first first principle ***(1978)***

Q.12 Find f ' (1) if

$$f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & \text{when } x\neq 1\\ -\dfrac{1}{3}, & \text{when } x=1\end{cases}$$ ***(1979, 3M)***

Q.13 Evaluate $\lim\limits_{x\to 0}\sqrt{\dfrac{x-\sin x}{x+\cos^2 x}}$ ***(1979)***

Q.14 Evaluate $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$ ***(1980)***

Q.15 Given $y=\dfrac{5x}{3\sqrt{(1-x)^2}}+\cos^2(2x+1)$,

find $\dfrac{dy}{dx}$ ***(1980)***

Q.16 Let f: $R\to R$ be a continuous function defined by

$f(x)=\dfrac{1}{e^x+2e^{-x}}$.

Statement-I: $f(c)=\dfrac{1}{3}$, for some $c\in R$. Statement-I:

$o\prec f(x)\le\dfrac{1}{2\sqrt{2}}$, for all $x\in R$ Statement-I:

$o\prec f(x)\le\dfrac{1}{2\sqrt{2}}$, for all $x\in R$.

(A) Statement-I is true, statement-II is true; statement-II is not the correct explanation for statement-I

(B) Statement-I is true, statement-II is false

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true; statement-II is the correct explanation for statement-I ***(2010)***

Q.17 Let $g(x)=\log(f(x))$ where $f(x)$ is a twice differentiable positive function on $(0,\infty)$ such that $f(x+1)=xf(x)$. Then, for $N=1,2,3,.$

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$ ***(2008)***

Paragraph 1:

Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ defined by

$$f(x)=\frac{x^2-ax+1}{x^2+ax+1}, 0<a<2.$$

Q.18 Which of the following is true?

(A) $(2+a)^2 f'(1)+(2-a)^2 f'(-1)=0$

(B) $(2+a)^2 f'(1)-(2-a)^2 f'(-1)=0$

(C) $f'(1)f'(-1)=(2-a)^2$

(D) $f'(1)f'(-1)=-(2-a)^2$ ***(2008)***

Q.19 Let

$$g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)};0<x<2, m$$

and n are integers, $m\neq 0, n>0$, and let p be the left hand derivative of $|x-1|$ at x = 1.

If $\lim\limits_{x\to 1^+} g(x)=p$, then ***(2008)***

(A) $n=1, m=1$ (B) $n=1, m=-1$

(C) $n=1, m=2$ (D) $n>1, m=n$

Q.20 Let f and g be real valued functions defined on interval $(-1,1)$ such that g″(x) is continuous $g(0)\neq 0, g'(0)=0, g''(0)\neq 0$ and $f(x)=g(x)\sin x$

And

Statement-II: $f(0)=g(0)$

(A) Statement-I is True, statement-II is True; statement-II is a correct explanation for statement-I

(B) Statement-I is True, statement-II is True; statement-II is NOT a correct explanation for statement-I

(C) Statement-I is True, statement-II is False

(D) Statement-I is False, statement-II is True ***(2008)***

Paragraph 1: Consider the functions defined implicitly by the equation $y^3-3y+x=0$ on various intervals in the real line. If $x\in(-\infty,-2)\cup(2,\infty)$, the equation implicitly defines a unique real valued differentiable function y = f(x). if $x\in(-2,2)$, the equation implicitly defines a unique real valued differentiable function y = g(x) satisfying g(0) = 0.

Q.21 If $f(-10\sqrt{2})=2\sqrt{2}$, then $f(-10\sqrt{2})=$ ***(2008)***

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

Q.22 For function $f(x)=x\cos\frac{1}{x}, x\geq 1$

(A) For atleast one x in interval $[1,\infty), f(x+2)-f(x)>2$

(B) $\lim\limits_{x\to\infty} f'(x)=1$

(C) For all x in the interval $[1,\infty), f(x+2)-f(x)>2$

(D) $f(x)$ is strictly decreasing in the interval $[1,\infty)$ ***(2009)***

Q.23 Let p(x) be a polynomial of degree 4 having extremum at x = 1,2 and

$\lim\limits_{x\to\infty} f\left(1+\frac{p(x)}{x^2}\right)=2$. Then the value of p(2) is ***(2009)***

Q.24 Let $L=\lim\limits_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}, a>0$. If finite, then ***(2009)***

(A) (1) $a=2$ (B) $a=1$ (C) $L=\frac{1}{64}$ (D) $L=\frac{1}{32}$

Q.25 Let f be a real-valued function defined on the interval $(0,\infty)$ by $f(x)=\ln x+\int_0^x\sqrt{1+\sin t}\,dt$.

Then which of the following statement (s) is (are) true?

(A) f″(x) exists for all $x\in(0,\infty)$

(B) f′(x) exists for all $f'(2^-)=\lim\limits_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

and f′ is continuous $(0,\infty)$, but not differentiable on $(0,\infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x\in(\alpha,\infty)$

(D) there exists $\beta > 1$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x\in(0,\infty)$ ***(2010)***

Q.26 If $\lim\limits_{x\to 0}\left[1+x\ln(1+b^2)\right]^{-1/x}=2b\sin^2\theta, b>0$

and $\theta\in(-\pi,\pi]$, then the value of θ is

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

(2011)

Q.27 If $f(x)=\begin{cases}-x-\dfrac{\pi}{2}, & x\le -\dfrac{\pi}{2}\\ -\cos x, & 0<x\le 1\\ \ln x, & x>1\end{cases}$ then, ***(2011)***

(A) f(x) is continuous at x = – π/2

(B) f(x) is not differentiable at x = 0

(C) f(x) is differentiable at x = 1

(D) f(x) is differentiable at x = -3/2

Q.28 Let y′(x) + y(x)g′(x) = g(x)g′(x), y(0) = 0, x ∈ R, where f′(x) denotes df(x) dx and g(x) is a given noncontact differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is ***(2011 (II))***

Q.29 Let f: R → R be a function such that f(x + y) = f(x) + f(y), ∀ x, y ∈ R. If f(x) is differentiable at x = 0, then ***(2011 (I))***

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous ∀ x ∈ R

(C) f′(x) is constant ∀ x ∈ R

(D) f(x) is differentiable except at finitely many points

Q.30 If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right)=4$, then ***(2012 (I))***

(A) a = 1, b = 4 (B) a = 1, b = –4

(C) a = 2, b = – 3 (D) a = 2, b = 3

Q.31 Let $f(x)=\begin{cases}x^2\left|\cos\dfrac{\pi}{x}\right| & x\ne 0\\ 0 & x=0\end{cases}$,x ∈ then f is

(A) Differentiable both at x = 0 and at x = 2

(B) Differentiable at x = 0 but not differentiable at x = 2

(C) Not differentiable at x = 0 but differentiable at x = 2

(D) Differentiable neither at x = 0 nor at x = 2 ***(2012)***

Q.32 For a ∈ R (the set of all real numbers), a ≠ –1,

$$\lim_{n\to\infty}\frac{\left(1^a+2^a+....+n^a\right)}{(n+1)^{a-1}\left[(na+1)+(na+2)+....+(na+n)\right]}=\frac{1}{60}$$

Then a = ***(2013)***

(A) 5 (B) 7 (C) $\dfrac{-15}{2}$ (D) $\dfrac{-17}{2}$

Q.33 Which of the following is true for 0 < x < 1 ? ***(2013)***

(A) $0<f(x)<\infty$ (B) $-\dfrac{1}{2}<f(x)<\dfrac{1}{2}$

(C) $-\dfrac{1}{4}<f(x)<1$ (D) $-\infty<f(x)<0$

Q.34 Let $f:\left[\dfrac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f(x)<2f(x)$ and $f\left(\dfrac{1}{2}\right)=1$. Then the value of $\int\limits_{1/2}^{1} f(x)dx$ lies in the interval ***(2013)***

(A) $(2-1,2e)$ (B) $(e-1,2e-1)$

(C) $\left(\dfrac{e-1}{2},e-1\right)$ (D) $\left(0,\dfrac{e-1}{2}\right)$

Q.35 Let $f:[0,2]\to R$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with f(0) = 1.

Let $F(x)=\int\limits_0^{x^2} f(\sqrt{t})dt$ for $x\in[0,2]$. If $F'(x)$ for all $x\in(0,2)$ with f(0) = 1. ***(2014)***

(A) e^2-1 (B) e^2-1 (C) $e-1$ (D) e^4

Q.36 The value of $g\left(\dfrac{1}{2}\right)$ is ***(2014)***

(A) $(-1,0)\cup(0,2)$ (B) 2π

$f'(x)-3g'(x)=0$

(C) $\dfrac{\pi}{2}$ (D) $\dfrac{\pi}{4}$

Q.37 The value of $g'\left(\dfrac{1}{2}\right)$ is ***(2014)***

(A) $\dfrac{\pi}{2}$ (B) π (C) $-\dfrac{\pi}{2}$ (D) 0

Q.38 Let $f(x)=\frac{19x^2}{2+\sin^4 \pi x}$ for all $x\in R$ with $f\left(\frac{1}{2}\right)=0$. If $m\le\int_{1/2}^{1} f(x)dx \le M$, then the possible values of *m* and *M* are ***(2015)***

(A) $m=13, M=24$ (B) $m=\frac{1}{4}, M=\frac{1}{2}$

(C) $m=-11, M=0$ (D) $m=1, M=12$

Q.39 Let $f,g:[-1,2]\to R$ be continuous functions which are twice differentiable on the interval $(-1,2)$ Let the values of f and g at the points $-1,0$ and 2 be as given in the following table:

In each of the intervals $(-1,0)$ and $(1,0)$ the function $(f-3g)^n$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x)-3g'(x)=0$ has exactly three solutions in $(-1,0)\cup(0,2)$

(B) $(-1,0)\cup(0,2)$ $f'(x)-3g'(x)=0$ has exactly one solution in $(-1,0)$

(C) $f'(x)-3g'(x)=0$ has exactly one solution in $(0,2)$

(D) $f'(x)-3g'(x)=0$ has exactly two solutions in $(-1,0)$ and exactly two solutions in (0, 2) ***(2015)***

Q.40 The correct statement (s) is (are)

(A) $f'(1)<0$

(B) $f(2)<0$

(C) $f'(x)\neq 0$ for any $x\in(1,3)$

(D) $f'(x)=0$ for some $x\in(1,3)$ ***(2015)***

Q.41 Let $g:R\to R$ be a differential function with $g(0)=0, g'(0)$ and $g(1)\neq f(x)=\begin{cases}\frac{x}{|x|}g(x) & x\neq 0\\ 0 & x=0\end{cases}$

And $h(x)=e^{|x|}$ $x\in R$ (f o h)(x) denote $f(h(x))$ denote $f(h(x))$ Then which of the following is (are) true?

(A) f is differentiable at $x=0$

(B) *h* is differentiable at $x=0$

(C) f o h is differentiable at $x=0$

(D) h o f is differentiable at $x=0$ ***(2015)***

Q.42 Let $a,b\in R$ and $R\to R$ be defined by $f(x)=a\cos(|x^3-x|)+b|x\sin(|x^3+x|)|$ Then f is

(A) Differentiable at $x=0$ if $a=0$ and $b=1$

(B) Differentiable at $x=1$ if $a=1$ and $b=0$

(C) NOT differentiable at $x=0$ if $a=1$ and $b=0$

(D) NOT differentiable at $x=1$ if $a=1$ and $b=1$ ***(2016)***

Q.43 Let $f(x)=\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)....\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)$,

for all x > 0. Then ***(2016)***

(A) $f\left(\frac{1}{2}\right)\ge f(1)$ (B) $f\left(\frac{1}{3}\right)\le f\left(\frac{2}{3}\right)$

(C) $f'(2)\le 0$ (D) $\frac{f'(3)}{f(3)}\ge\frac{f'(2)}{f(2)}$

Q.44 Let $f:\left[-\frac{1}{2},2\right]\to R$ and $g:\left[-\frac{1}{2},2\right]\to R$ be functions defined by $f(x)=[x^2-3]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where [y] denotes the greatest integer less than or equal to y For $y\in\mathbb{R}$. Then ***(2016)***

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

Q.45 let $f:(0,\infty)\to R$ be a differentiable function such that $f(x)=2-\frac{f(x)}{x}=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. Then

(A) $\lim_{x\to 0^+} x^2 f'\left(\frac{1}{x}\right)=1$ (B) $\lim_{x\to 0_+} x^2 f'\left(\frac{1}{x}\right)=2$

(C) $\lim_{x\to 0_+} x^2 f'(x)=2$

(D) $\left|f(x)\le 2\right|$ for all $x\in(0,2)$ ***(2016)***

Q.46 Let f: R $\to$ R, g: R $\to$ R and $h:R\to R$ be differentiable functions such that $f(x)=x^3+3x+2, g(f(x)=x)$ for all $x\in R$ ***(2016)***

(A) $g'(2)=\frac{1}{15}$ (B) $(1)=666$

(C) $h(0)=16$ (D) $h(g(3))=36$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Limits

Q.7 Q.13 Q.20 Q.22 Q.25

Continuity

Q.4 Q.9 Q.11 Q.13

Differentiability

Q.7 Q.8 Q.13 Q.14

Exercise 2

Limits

Q.1 Q.7 Q.9 Q.10

Continuity

Q.3 Q.4 Q.6 Q.9 Q.11 Q.14

Differentiability

Q.3 Q.7 Q.11 Q.15 Q.16

Previous Years' Questions

Q.1 Q.6 Q.10 Q.13 Q.15

JEE Advanced/Boards

Exercise 1

Limits

Q.6 Q.11 Q.17 Q.19 Q.20

Continuity

Q.4 Q.7 Q.12 Q.17 Q.19 Q22

Differentiability

Q.3 Q.6 Q.9 Q.11 Q.14

Exercise 2

Limits

Q.2 Q.4 Q.6 Q.8 Q.10

Continuity

Q.3 Q.6 Q.8 Q.10

Differentiability

Q.1

Previous Years' Questions

Q.1 Q.3 Q.7 Q.9 Q.14 Q.15 Q.16

Answer Key

JEE Main/Boards

Exercise 1

Limits

Q.1 $-\frac{1}{8}$

Q.2 $\frac{3}{2}$

Q.3 2

Q.4 $\frac{1}{2\sqrt{x}}$ if x > 0; ∞ if x = 0

Q.5 ∞

Q.6 (a) $\pi/2$ if a > 0 ; 0 if a = 0 and $-\pi/2$ if a <0 (b) f(x) = |x|

Q.7 5050

Q.8 $\frac{3}{2}$

Q.9 2

Q.10 $\sqrt{2}$

Q.11 –3

Q.12 $\frac{1}{16\sqrt{2}}$

Q.13 2

Q.14 a = 2; limit = 1

Q.15 0.5

Q.16 Does not exist

Q.17 $\frac{\pi}{4}$

Q.18 $\cos^2 \alpha/n \cos \alpha + \sin 2\alpha/n \sin a$

Q.19 $\sqrt{8}\, 2[\ell n3]^2$

Q.20 does not exist

Q.21 (2n–1)

Q.22 $\frac{1}{128r}$

Q.23 1

Q.24 ln a

Q.25 c = ln2

Continuity

Q.1 a = 0, b = 1

Q.2 a = 0, b = –1

Q.3 (a) –2, 2, 3 (b) K = 5 (c) even

Q.4 y_n (x) is continuous at x = 0 for all n and y(x) is discontinuous at x = 0

Q.5 30

Q.7 $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define

Q.9 6

Q.10 9

Q.11 c = 1, a, b ∈ R

Q.12 8

Q.13 6x – 2y – 5 =0

Q.14 Does not exist

Q.15 $e^2 + e^{-2}$

Differentiability

Q.1 $-2x \sin(x^2+1)$

Q.2 $-\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Q.3 $\frac{dy}{dx} = (\tan x)^x[\log(\tan x) + 2x \operatorname{cosec} 2x]$

Q.4 $2x \cos x^2$

Q.5 $\frac{1}{2}\left(\frac{1}{1+x^2}\right)$

Q.6 $\frac{1}{2}$

Q.7 $\frac{dy}{dx} = x^{\cos x}\left[-\sin x \cdot \log x + \frac{\cos x}{x}\right] + \cos x^{\sin x}\left[-\sin x \tan x + \cos x \cdot \log \cos x\right]$

Q.8 x = 1 **Q.9** $-\frac{\sin x}{2\sqrt{\cos x}}$ **Q.10** $\frac{1}{2\sqrt{\tan x}}\sec^2 x$

Q.11 Not differentiable **Q.12** 1 **Q.13** 5

Exercise 2

Single Correct Choice Type

Limits

Q.1 C	**Q.2** C	**Q.3** C	**Q.4** C	**Q.5** C	**Q.6** D
Q.7 D	**Q.8** B	**Q.9** A	**Q.10** D	**Q.11** A	**Q.12** A

Continuity

Q.1 D	**Q.2** A	**Q.3** A	**Q.4** D	**Q.5** C	
Q.6 D	**Q.7** C	**Q.8** D	**Q.9** D	**Q.10** D	**Q.11** A
Q.12 C	**Q.13** B				

Differentiability

Q.1 D	**Q.2** D	**Q.3** A	**Q.4** D	**Q.5** A	**Q.6** D
Q.7 A	**Q.8** B	**Q.9** B	**Q.10** C	**Q.11** C	**Q.12** A
Q.13 D	**Q.14** D	**Q.15** C	**Q.16** B	**Q.17** C	

Previous Years' Questions

Q.1 D	**Q.2** A	**Q.3** A	**Q.4** B	**Q.5** C	**Q.6** B
Q.7 D	**Q.8** C	**Q.9** A	**Q.10** C	**Q.11** B	**Q.12** D
Q.13 C	**Q.14** B	**Q.15** D	**Q.16** A	**Q.17** A	**Q.18** A
Q.19 D	**Q.20** A	**Q.21** D	**Q.22** B	**Q.23** D	**Q.24** A
Q.25 B	**Q.26** D	**Q.27** C	**Q.28** A	**Q.29** D	**Q.30** B
Q.31 A					

JEE Advanced/Boards

Exercise 1

Limits

Q.1 5050 **Q.2** $a = 2; r = \frac{1}{4}; S = \frac{8}{3};$ **Q.3** $\frac{p-q}{2}$

Q.4 ln2 **Q.5** 324

Q.6 (a) Does not exist (b) does not exist (c) 0 **Q.7** $-\frac{1}{3}$

Q.8 $\frac{1}{32}$ **Q.9** $\frac{\sqrt{3}}{2}$ **Q.10** 1/2

Q.11 –2 **Q.12** $\pi - 3$ **Q.13** (i) a = 1, b = –1 (ii) a = 1, $b = -\frac{1}{2}$

Q.14 1 **Q.15** $-\frac{9}{4}\ln\frac{4}{e}$ **Q.16** 27

Q.17 29 **Q.18** 72 **Q.19** –1/2 **Q.20** 16

Differentiability

Q.1 f(x) is continuous but not derivable at x = 0 **Q.2** $0 < n \le 1$

Q.3 2 **Q.4** f is cont. but not diff. at x = 0 **Q.5**

Q.6 $f'(1^+) = 3$, $f'(1^-) = -1$ **Q.7** a = 1/2, b = 3/2 **Q.8** 5

Q.9 Not derivable at x = 0 & x = 1

Q.10 Discontinuous & not derivable at x = 1, continuous but not derivable at x = 2

Q.11 f is conti. at x = 1, 3/2 & discount. at x = 2, f is not diff. at x = 1, 3/2, 2

Q.12 24 **Q.13** $a \neq 1, b = 0, p = \frac{1}{3}$ and q = –1 **Q.14** 5

Exercise 2

Limits

Single Correct Choice Type

Q.1 A **Q.2** A **Q.3** B **Q.4** A **Q.5** D **Q.6** B

Multiple Correct Choice Type

Q.7 A, B, C **Q.8** A, B, C **Q.9** A, B, C, D **Q.10** C, D **Q.11** A, B, C **Q.12** A, B, D

Q.13 A, B, C, D

Continuity

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** A	**Q.5** C	**Q.6** C
Q.7 A	**Q.8** A	**Q.9** D	**Q.10** C	**Q.11** C	**Q.12** A
Q.13 B	**Q.14** A	**Q.15** B	**Q.16** B	**Q.17** A	**Q.18** C
Q.19 B	**Q.20** D	**Q.21** C	**Q.22** A	**Q.23** A	**Q.24** B
Q.25 D	**Q.26** D	**Q.27** C	**Q.28** C	**Q.29** B	**Q.30** C
Q.31 B	**Q.32** A	**Q.33** D	**Q.34** D	**Q.35** D	

Multiple Correct Choice Type

Q.36 A, C

Differentiability

Single Correct Choice Type

Q.1 A **Q.2** A

Previous Years' Questions

Q.1 A, B, D	**Q.2** B, D	**Q.3** A, B, D	**Q.4** A, B	**Q.5** B, C	**Q.6** A $\rightarrow$ p; B $\rightarrow$ r
Q.7 A $\rightarrow$ p, q, r; B $\rightarrow$ p,s; C $\rightarrow$ r, s; D $\rightarrow$ p, q			**Q.16** A	**Q.17** A	**Q.18** A
Q.19 C	**Q.20** B	**Q.21** B	**Q.22** B, C, D	**Q.24** A, C	**Q.25** B, C
Q.26 D	**Q.27** A, B, C, D	**Q.29** B, C	**Q.30** B	**Q.31** B	**Q.31** B, D
Q.32 B, D	**Q.33** D	**Q.34** D	**Q.35** B	**Q.36** A	**Q.37** D
Q.38 D	**Q.39** B, C	**Q.40** A, B, D	**Q.41** A, B	**Q.42** A, D	**Q.43** A, B
Q.44 B, C	**Q.45** B, C	**Q.46** A			

Analytical and Descriptive Questions

Q.8 $-\frac{1}{3}$

Q.9 $\frac{2}{\pi}$

Q.10 $2x\cos(x^2+1)$

Q.11 $\frac{1}{2}+\frac{\pi}{4}$

Q.12 $f'(1)=-\frac{2}{9}$

Q.13 0

Q.14 $a^2\cos a + 2a\sin a$

Q.15. $$\begin{cases} \dfrac{5}{3(1-x)^2} - 2\sin(4x+2) & x \le 1 \\ -\dfrac{5}{3(x-1)^2} - 2\sin(4x+2) & x > 1 \end{cases}$$

Solutions

JEE Main/Boards

Exercise 1

Limits

Sol 1: $\lim\limits_{x\to -4}\dfrac{\sqrt{5+x}-1}{x^2+4x}=\lim\limits_{x\to -4}\dfrac{\left(\sqrt{5+x}-1\right)}{x(x+4)}\times\dfrac{\sqrt{5+x}+1}{\sqrt{5+x}+1}$

$=\lim\limits_{x\to -4}\dfrac{(5+x-1)}{x(x+4)\left(\sqrt{5+x}+1\right)}=\dfrac{1}{-4\times 2}=-\dfrac{1}{8}$

Sol 2: $\lim\limits_{x\to 2}\left\{\dfrac{x(x-2)(x+2)}{(x-2)\left(x^2+2x+4\right)}\right\}^{-1}$

$=\left\{\dfrac{2\times 4}{12}\right\}^{-1}=\left(\dfrac{2}{3}\right)^{-1}=\dfrac{3}{2}$

Sol 3: $\lim\limits_{x\to 1}\dfrac{\left(x^2-1\right)-(x-1)\ln x}{(x-1)}=2$

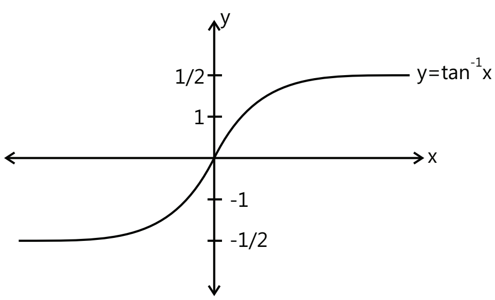
Sol 4: $\lim\limits_{x\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}=\lim\limits_{h\to 0}\dfrac{x+h-x}{h\left\{\sqrt{+h}+\sqrt{x}\right\}}=\dfrac{1}{2\sqrt{x}}$

Sol 5: $\lim\limits_{x\to 2}\left[\dfrac{1}{x(x-2)^2}-\dfrac{1}{x^{2-3x+2}}\right]$

$=\lim\limits_{x\to 2}\dfrac{(x-1)-x(x-2)}{(x-2)^2(x-1).x}=\lim\limits_{x\to 2}\dfrac{-x^2+3x-1}{x(x-2)^2(x-1)}=\infty$

Sol 6: (a) $\lim\limits_{x\to 0}\tan^{-1}\dfrac{a}{x^2}$

Graph of $y=\tan^{-1}x$



So in $\lim\limits_{x\to 0}\tan^{-1}\dfrac{a}{x^2}$

As $x\to 0$ $\dfrac{a}{x^2}\to\infty\Rightarrow\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

Depends upon value of a.

(b) $f(x)=\lim\limits_{t\to 0}\left(\dfrac{2x}{\pi}.\tan^{-1}\dfrac{x}{t^2}\right)$

$f(0)=-x=x$

$f(\infty)=\infty$

Sol 7: $\lim\limits_{x\to 1}\dfrac{\sum\limits_{k=1}^{100}x^k-100}{x-1}=\lim\limits_{x\to 1}\dfrac{\left(x+x^2+x^3+.....+x^{100}\right)}{(x-1)}$

$=\lim\limits_{x\to 1}\dfrac{(x-1)\left\{1+(1+x)+\left(1+x+x^2\right)+......\right\}}{(x-1)}$

$=1+2+....+100=\dfrac{100\times 101}{2}=5050$

Sol 8: Use Binomial Expansion.

Sol 9:

$\lim\limits_{x\to\frac{\pi}{4}}\dfrac{1-\tan x}{1-\sqrt{2}\sin x}\times\dfrac{1+\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{(1-\tan x)\left(1+\sqrt{2}\sin x\right)}{1-2\sin^2 x}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{(1-\tan x)\left(1+\sqrt{2}\sin x\right)}{\left(\dfrac{1-\tan^2 x}{1+\tan^2 x}\right)}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\left(1+\tan^2 x\right)\left(1+\sqrt{2}\sin x\right)}{(1+\tan x)}=2$

Sol 10: Use $1-\cos 2\theta=2\sin^2\theta$

Sol 11: $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{(2\sin x-1)(\sin x+1)}{(2\sin x-1)(\sin x-1)}=-3$

Sol 12:

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\left(\cos\theta+\sin\theta\right)}{16\left(\theta-\frac{\pi}{4}\right)^2} = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)}{16\left(\theta-\frac{\pi}{4}\right)^2}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\left\{2\sin^2\frac{\left(\theta-\frac{\pi}{4}\right)}{2}\right\}}{16.\left(\theta-\frac{\pi}{4}\right)^2} = \frac{1}{16\sqrt{2}}$$

Sol 13:

$$\lim_{x\to 0} \frac{2\sin^2\frac{x}{2}+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$$

$$= \lim_{x\to 0} \frac{\frac{\sin\frac{x}{2}}{x}.\sin\frac{x}{2}\times 2+\frac{2\sin x}{x}-\frac{2\sin x}{x}-x+3x^3}{\frac{\tan^3 x}{x}-\frac{6\sin^2 x}{x}+1-5x^2}$$

$= 2$

Sol 14: $\because \lim_{x\to 0}\frac{a\sin x-\sin 2x}{\tan^3 x}$ is finite

$\therefore \lim_{x\to 0} a\cos x-\left(\cos 2x\right).2=0$

$\to a-2=0$

$\to a=2$

Sol 15: $\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{1-\tan^2 x}\times\frac{1-\tan x}{1+\tan x}=\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{\left(1+\tan x\right)^2}$

$$=\frac{2}{\left(2\right)^2}=\frac{1}{2}$$

Sol 16: Does not exist.

Sol 17:

$$\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$$

$$=\lim_{n\to\infty}\cos\left(\frac{\pi}{4n}\right)\frac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\times\frac{\pi}{4}=\frac{\pi}{4}$$

Sol 18: Use L'Hospitals Rule

Sol 19: $\lim_{x\to 0}\frac{9^x\left(3^x-1\right)-1\left(3^x-1\right)}{\sqrt{2}\left\{1-\cos\frac{x}{2}\right\}}=\lim_{x\to 0}\frac{\left(3^x-1\right)\left(9^x-1\right)}{2\sqrt{2}\sin^2\frac{x}{4}}$

$$=\lim_{x\to 0}\frac{\frac{\left(3^x-1\right)}{x}\times\frac{9^x-1}{x}}{\frac{2\sqrt{2}\sin^2\frac{x}{4}}{\frac{x^2}{\left(4\right)^2}\times\left(4\right)^2}}=\frac{8}{\sqrt{2}}\log 3\times\log 9$$

$=8\sqrt{2}\left(\log 3\right)^2$

Sol 20: Does not exist.

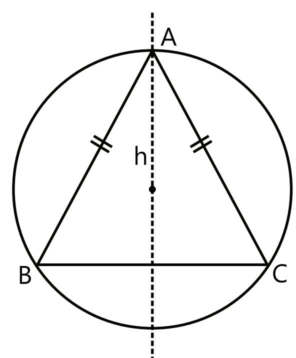
Hint: Use LHL and RHL.

Sol 21: $\lim_{x\to 0}\left[\frac{n\sin x}{x}\right]+\lim_{x\to 0}\left[\frac{n\tan x}{x}\right]$

$$=\lim_{x\to 0}\left[\frac{nx\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right\}}{x}\right]+\lim_{x\to 0}\left[\frac{nx\left\{1+\frac{x^2}{3}+\frac{2x^4}{15}\cdots\right\}}{x}\right]$$

$=n-1+n=2n-1$

Sol 22: From the diagram,



$h^2+\frac{a^2}{4}=b^2$

$\text{or}, h^2=b^2-\frac{a^2}{4}$

$$\lim_{h\to 0}\frac{\Delta}{p^3}=\lim_{h\to 0}\frac{\frac{ab^2}{4r}}{\left(a+2b\right)^3}$$

$$=\lim_{h\to\frac{a}{2}}\frac{a.\frac{a^2}{4}}{4r.8a^3}=\frac{1}{128r}$$

Sol 23: $\lim_{x\to 0}\frac{\log\left\{\left(1+x^2\right)^2-x^2\right\}}{x\left(e^x-1\right)}$

$$=\lim_{x\to 0}\frac{\log\left\{1+x^2+x^4\right\}}{\left(x^2+x^4\right)}\times\frac{\left(x^2+x^4\right)}{x^2\left(\frac{e^x-1}{x}\right)}=1$$

Sol 24: $= \lim_{n\to\infty} n^2\left\{\left(a^{\frac{1}{n}}-1\right)-\left(a^{\frac{1}{n+1}}-1\right)\right\}$

$$= \lim_{n\to\infty} n\left\{\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}\right\}-\frac{n^2}{(n+1)}\left\{\frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\right\}$$

$$= \lim_{n\to\infty} n\log a-\frac{n^2}{(n+1)}\log a=\log a\lim_{n\to\infty}\frac{n}{n+1}=\log a$$

Sol 25: $\lim_{x\to\infty}\left(\frac{(x+c)}{x-c}\right)^x = 4 \Rightarrow \lim_{x\to\infty}\left(1+\frac{2c}{x-c}\right)^x = 4$

Continuity

Sol 1: f(x) = $\begin{cases} |ax+3| & , \quad x\le -1 \\ |3x+a| & , \quad -1<x\le 0 \\ \frac{b\sin 2x}{x}-2b & , \quad 0<x<\pi \\ \cos^2 x-3 & , \quad x\ge\pi \end{cases}$

For x < –1, f(x) = |ax + 3| and it will be continuous.

At x = – 1,

Vf(x = –1) = |-a + 3| = |a – 3|

LHL = |a – 3|

RHL = $\lim_{x\to -1^+}|3x+a|$ = |a – 3|

At x = –1, function is continuous.

At x = 0,

LHL =VF(x = 0) = RHL

$\lim_{x\to 0^-}|3x+a|$ = |a| = $\lim_{x\to 0^+}\frac{b\sin 2x}{x}-2b$

LHL = f(0) = |a|

RHL = $\lim_{x\to 0^+}\frac{2.b\sin 2x}{2x}-2b$ = 2b – 2b = 0

$\therefore$|a| = 0 ⇒ a = 0

At x = p

LHL = f(π) = RHL

$\lim_{x\to\pi^-}\frac{b\sin 2x}{x}-2b$ = cos²π – 3 = $\lim_{x\to\pi^+}\cos^2 x-3$

⇒–2b = –2 = – 2 ⇒b = 1

$\therefore$ a = 0, b = 1

Sol 2: LHL = $\lim_{x\to\frac{\pi}{2}^-}\left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$

x = $\frac{\pi}{2}$ – h, as x → $\frac{\pi}{2}$; h → 0

$\therefore$LHL = $\lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan 6\left(\frac{\pi}{2}-h\right)}{\tan 5\left(\frac{\pi}{2}-h\right)}} = \lim_{h\to 0^+}\left(\frac{6}{5}\right)^{\frac{-\tan 6h}{\cot 5h}}$

= $\lim_{h\to 0^+}\left(\frac{6}{5}\right)^{-\tan 6h\,\tan 5h}$ = 1

f$\left(\frac{\pi}{2}\right)$ = b + 2 =LHL

⇒ b + 2 = 1

b = – 1

RHL = $\lim_{x\to\frac{\pi}{2}^+}(1+|\cos x)^{\left(\frac{|a\tan x|}{b}\right)}$

x = $\frac{\pi}{2}$ + h, as x → $\frac{\pi}{2}^+$, h → 0⁺

RHL = $\lim_{h\to 0^+}\left(1+\left|\cos\left(\frac{\pi}{2}+h\right)\right|\right)^{\frac{\left|a\tan\left(\frac{\pi}{2}+h\right)\right|}{b}}$

= $\lim_{h\to 0^+}(1+|\sinh|)^{\frac{|a\cot h|}{b}}$ = $\lim_{h\to 0^+}(1+|\sinh|)^{\frac{|a|\cot h}{b}}$

(as sin h > 0, for h → 0⁺ and cot h > 0, for h → 0⁺)

Now, this limit will be of the form equation for non zero values of a for a = 0,

RHL = $\lim_{h\to 0^+}(1+\sinh)^o$ = 1 = LHL = RHL = f$\left(\frac{\pi}{2}\right)$

Sol 3: (a) f(x) = x³ – 3x² – 4x + 12

= (x – 3) (x² – 4)

f(x) = (x – 2) (x + 2) (x – 3)

zeros : 2, –2, 3

(b) h(x) = $\begin{cases} \frac{(x-2)(x+2)(x-3)}{x-3} & , \quad x\ne 3 \\ k & ,, \quad x=3 \end{cases}$

Atx = 3, for the continuity

h(x = 3) = LHL = RHL

LHL = 1. 5 = 5 = k

(c) We have,

$h(x) = \dfrac{(-x-2)(-x+2)(-x-3)}{(-x-3)}$, $x \neq -3$

$= (x + 2)(x - 2)$, $x \neq 3 = h(x)$

hence even

At $x = -3$,

$h(x) = -1x - 5 = 5 = h(3)$

Hence, even function

Sol 4: $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + ... + \dfrac{x^2}{(1+x^2)^{n-1}}$

This is summation of a G P with n terms, $a = x^2$,

$r = \dfrac{1}{1+x^2}$

$\therefore y_n(x) = \dfrac{a(1-r^n)}{1-r} = \dfrac{x^2}{1-\dfrac{1}{1+x^2}}\left(1-\left(\dfrac{1}{1+x^2}\right)^n\right)$

$= \dfrac{x^2(1+x^2)}{1+x^2-1}\left[1-\left(\dfrac{1}{1+x^2}\right)^n\right]$

$y_n(x) = (1 + x^2)\left[1-\left(\dfrac{1}{1+x^2}\right)^n\right]$

For $y_n(x)$, at $x = 0$

$y_n(x) = 1\left(1-\left(\dfrac{1}{1}\right)^n\right) = 1$

LHL = RHL = 1

$\text{LHL} = \lim\limits_{x\to 0}(1+x^2)\left(1-\left(\dfrac{1}{1+x^2}\right)^n\right) = 1$

Therefore, $y_n(x)$ is continuous

LHL = RHL, as it is an even function

$y(x) = \lim\limits_{n\to\infty}(1+x^2)\left[1-\dfrac{1}{(1+x^2)^n}\right]$

$\lim\limits_{x\to 0} y(x) = \lim\limits_{x\to 0}\lim\limits_{n\to\infty}(1+x^2)\left[1-\left(\dfrac{1}{1+x^2}\right)^n\right]$

At $x = 0$

$y(x)$ will be indeterminate further LHL = RHL

$1 - \left(\dfrac{1}{1+x^2}\right)^n$, as $n \to \infty$, approaches

$\therefore \lim\limits_{x\to 0}\lim\limits_{n\to\infty} y_n(x) = 1$

Thus $Vf(x = 0) \neq$ LHL, therefore discontinuous

Sol 5: $f(x) = [5x] + \{3x\}$

at $x = 0$, $f(0) = 0$

$\text{RHL} = \lim\limits_{x\to 0}[5x] + \{3x\} = 0$

$\therefore$ continuous at $x = 0$

At $x = \dfrac{1}{5}$,

$f\left(\dfrac{1}{5}\right) = \left[5\times\dfrac{1}{5}\right] + \left\{3\times\dfrac{1}{5}\right\} = 1 + \dfrac{3}{5} = \dfrac{8}{5}$

$\text{LHL} = \lim\limits_{x\to \frac{1}{5}^-}[5x] + \{3x\}$

$= 0 + \dfrac{3}{5} = \dfrac{3}{5}$

$\text{LHL} \neq VF\left(x = \dfrac{1}{5}\right)$

Therefore discontinuous at $x = \dfrac{1}{5}$

Similarly discontinuous at $\dfrac{1}{5}$, i. $\in (1, 24)$, $i \neq 5, 10, 15, 20$

At $x = \dfrac{1}{3}$, $f\left(x = \dfrac{1}{3}\right) = \left[\dfrac{5}{3}\right] + \left\{3\times\dfrac{1}{3}\right\} = 0 + 0 = 0$

$\text{LHL} = \lim\limits_{x\to \frac{1}{3}^-}[5x] + \{3x\} = 1 \neq Vf\left(x = \dfrac{1}{3}\right)$

Therefore, discontinuous

Similarly discontinuous at $x = \dfrac{1}{3}$, $i \in (1, 14)$,

$i \neq 3, 6, 9, 12$

Now, at Total 10 points

$f(x = 1) = [5 \times 1] + \{3 \times 1\} = 5$

$\text{LHL} = \lim\limits_{x\to 1^-}[5x] + \{3x\} = 4 + 1 = 5$

$\text{RHL} = \lim\limits_{x\to 1^+}[5x] + \{3x\} = 5 + 0 = 5$

$\because$ LHL = RHL = $Vf(x = 1)$, continuous at $x = 1$

similarly continuous at x= 2, 3, 4

$\therefore$ D is continuous at total 20 + 10 = 30 points

Sol 6: If $|2\sin x| < 1, f(x) = \lim\limits_{n\to\infty}\dfrac{x}{(2\sin x)^{2n}+1} = \dfrac{x}{0+1} = x.$

If $\left|2\sin x\right| = 1, f(x) = \lim\limits_{n\to\infty} \dfrac{x}{(2\sin x)^{2n}+1} = \dfrac{x}{1+1} = \dfrac{1}{2}x.$

If $\left|2\sin x\right| > 1, f(x) = \lim\limits_{n\to\infty} \dfrac{x}{(2\sin x)^{2n}+1} = \dfrac{x}{\infty+1}0.$

But $\left|2\sin x\right| < 1 \Rightarrow \left|\sin x\right| < \dfrac{1}{2}$

$\Rightarrow -\dfrac{1}{2} < \sin x < \dfrac{1}{2} \Rightarrow n\pi - \dfrac{\pi}{6} < x < n\pi + \dfrac{\pi}{6}$

$\left|2\sin x\right| = 1 \Rightarrow \left|\sin x\right| = \dfrac{1}{2} \Rightarrow \sin x = \pm\dfrac{1}{2}$

$\Rightarrow x = n\pi + (-1)^n.\left(\pm\dfrac{\pi}{6}\right) = n\pi \pm \dfrac{\pi}{6}$

$\left|2\sin x\right| > 1 \Rightarrow \left|\sin x\right| > \dfrac{1}{2}$

$\Rightarrow \sin x > \dfrac{1}{2}$ or $\sin x < -\dfrac{1}{2}$

$\Rightarrow n\pi + \dfrac{\pi}{6} < x < n\pi + \dfrac{5\pi}{6}.$

Thus, we have $f(x) = x, n\pi - \dfrac{\pi}{6} < x < n\pi + \dfrac{\pi}{6}$

$\dfrac{1}{2}x, x = n\pi \pm \dfrac{\pi}{6}$

$0, n\pi + \dfrac{\pi}{6} < x < n\pi + \dfrac{5\pi}{6}$

As polynomial functions are continuous everywhere, only doubtful points are $x = n\pi \pm \dfrac{\pi}{6}$. Clearly,

$f\left(n\pi - \dfrac{\pi}{6} + 0\right) \neq f\left(n\pi - \dfrac{\pi}{6} - 0\right)$ because $n\pi - \dfrac{\pi}{6} \neq 0$

$f\left(n\pi + \dfrac{\pi}{6} + 0\right) \neq f\left(n\pi + \dfrac{\pi}{6} - 0\right)$ because $0 \neq n\pi + \dfrac{\pi}{6}$

$\therefore f(x)$ is not continuous at $x = n\pi \pm \dfrac{\pi}{6}$

$\therefore$ The function f (x) is continuous everywhere in R except the set of points $\left\{x \,\middle|\, x = n\pi \pm \dfrac{\pi}{6}, n \in Z\right\}$

Sol 7: f(x) = $\begin{cases} \dfrac{\ell n\cos x}{\sqrt[4]{1+x^2}-1}, & x > 0 \\ \dfrac{e^{\sin^4 x}-1}{\ell n(1+\tan 2x)}, & x < 0 \end{cases}$

LHL = $\lim\limits_{x\to 0^-} \dfrac{e^{\sin^4 x}-1}{\ell n(1+\tan 2x)}$

LHL= $\lim\limits_{x\to 0^-} \left(\dfrac{e^{\sin^4 x}-1}{\sin^4 x}\right).\left(\dfrac{1}{\dfrac{\sin(1+\tan 2x)}{\tan 2x}}\right).\dfrac{\sin^4 x}{\tan 2x}$

= $\lim\limits_{x\to 0^-} \dfrac{\sin 4x}{4x}.\dfrac{4x}{2x}.\dfrac{1}{\dfrac{\tan 2x}{2x}}$ = 2

RHL = $\lim\limits_{x\to 0^+} f(x)$ = $\lim\limits_{x\to 0^+} \dfrac{\ell n\cos x}{\sqrt[4]{1+\pi^2}-1}$, $\dfrac{0}{0}$ form

Using L Hospital rule.

RHL = $\lim\limits_{x\to 0^+} \dfrac{1}{\cos x}\dfrac{(-\sin x)}{\frac{1}{4}(1+x^2)^{-3/4}.2x}$

= $\lim\limits_{x\to 0} \dfrac{-2}{\cos}\left(\dfrac{\sin x}{x}\right)(1 + x^2)^{3/4} = -2\left(\because \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\right)$

$\because$ LHL $\neq$ RHL, it is not possible to make the function continuous. This is jump discontinuity.

Sol 8: $\lim\limits_{x\to 0} g(x), = 0 \Rightarrow \lim\limits_{h\to 0} g(0+h) = \lim\limits_{h\to 0} g(0-h) = 0$

$\Rightarrow \lim\limits_{h\to 0} g(h) = \lim\limits_{h\to 0} g(-h) = 0$ .....(i)

$\lim\limits_{x\to 0} G(x)$ exists $\Rightarrow \lim\limits_{h\to 0} G(0+h) = \lim\limits_{h\to 0} G(0-h)$

$\Rightarrow \lim\limits_{h\to 0} G(h) = \lim\limits_{h\to 0} G(-h) =$ finite ....(ii)

Now, $\lim\limits_{h\to 0} f(x+h) = \lim\limits_{h\to 0} f(x).f(h) = f(x)\lim\limits_{h\to 0} f(h)$

$\{\because f(x+y) = f(x).f(y)\}$

$= f(x).\lim\limits_{h\to 0} \{1 + g(h)G(h)\},$

Using given relation

$= f(x).\left\{1 + \lim\limits_{h\to 0} g(h).\lim\limits_{h\to 0} G(h)\right\}$

$= f(x).\{1 + 0.\text{ finite}\}$, using (1) and (2)

= f (x)

Also, $\lim\limits_{h\to 0} f(x-h) = \lim\limits_{h\to 0} f(x).f(-h) = f(x).\lim\limits_{h\to 0} f(-h)$

$= f(x).\lim\limits_{h\to 0}\{1 + g(-h).G(-h)\}$, using given relation

$= f(x).\left\{1 + \lim\limits_{h\to 0} g(-h).\lim\limits_{h\to 0} G(-h)\right\}$

$= f(x).\{1 + 0.\text{ finite}\}$, using (i) and (ii)

$= f(x)$

$\therefore \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h) = f(x).$

$\therefore f(x)$ is continuous everywhere.

Sol 9: f(x) = sgn((x^2 – ax + 1) (bx^2 – 2bx + 1))

This function will be discontinuous when (x^2 – ax + 1) (bx^2 – 2bx + 1) = 0

Therefore, for this to be discontinuous at exactly one point if

(x^2 – ax + 1) (bx^2 – 2bx + 1) has exactly one root.

For x^2 – ax + 1, D = a^2 – 4

$D \le 0 \Rightarrow a^2 \le 4$

$\Rightarrow$a = -2, –1, 0, 1, 2 ($\because$ a $\in$ z)

for bx^2 – 2bx + 1, D = 4(b^2 – b) $\le$ 0

$\Rightarrow$ b = 1, 0($\because$ b $\in$ z)

At b = 0, bx^2 – 2bx + 1 = 1, which has no root pairs (a, b) for which exactly root,

(–1, 1) , (0, 1) , (1, 1) , (–2, 0) , (2, 1) , (2, 0)

x = 1, x = 1, x = 1, x = –1, x = 1,x = 1

Total 6 ordered pairs.

Sol 10: Let the common roots be λ_1 and λ_2.

Then, we have from $x^3 + 2x^2 + px + q = 0$

$\alpha\lambda_1 + \alpha\lambda_2 + \lambda_1\lambda_2 = p$

$\alpha(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = p$ ... (i)

from $x^3 + x^2 + px + r = 0$

$\beta(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = p$ ... (ii)

from (i) and (ii)

$\alpha(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = \beta(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$

$\alpha(\lambda_1 + \lambda_2) = \beta(\lambda_1 + \lambda_2)$

$\Rightarrow\lambda_1 + \lambda_2 = 0$(for non zero α, β)

Now, from (a), $\alpha + \lambda_1 + \lambda_2 = -2 \Rightarrow \alpha = -2$

from (b), $\beta + \lambda_1 + \lambda_2 = -1$

$\beta = -1$

$\therefore|\alpha + \beta| = 3, \alpha\beta = 2$

Now, for f(x) at x = 0,

LHL = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{x\log_{1+x} 3} = \lim_{x\to 0^-} 3^{x\log_{1+x} e}$

$= \lim_{x\to 0^-} 3^{\frac{x}{\log(1+x)}} = 3 \left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$

RHL

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} b\frac{\ell n(e^{x^2} + \alpha\beta\sqrt{x})}{\tan\sqrt{x}}$

$= b \lim_{x\to 0^+} b\frac{\ell n\left(e^{x^2}\left(1 + \frac{\alpha\beta\sqrt{x}}{e^{x^2}}\right)\right)}{\tan\sqrt{x}}$

$= b\left[\lim_{x\to 0^+}\frac{x^2}{\tan\sqrt{x}} + \frac{\ell n\left(1 + \frac{\alpha\sqrt{x}}{e^{x^2}}\right)}{\frac{\alpha\beta\sqrt{x}}{e^{x^2}}}.\frac{\alpha\beta\sqrt{x}}{\tan\sqrt{x}}.\frac{1}{e^{x^2}}\right]$

$= b\left[\lim_{x\to 0^+}\left(\frac{x^{3/2}}{\frac{\tan\sqrt{x}}{\sqrt{x}}}\right) + \frac{\ell n\left(1 + \frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}\left(\frac{2}{\frac{\tan\sqrt{x}}{\sqrt{x}}}\right)\frac{1}{e^{x^2}}\right]$

= b(0/1 + 1. 2. 1) = 2b

Now, for continuity

LHL= RHL = Vf(x = 0)

$\therefore$ 3 = 2b = a $\quad\therefore$ a = 3, b = $\frac{3}{2}$

Hence, 2(a + b) = $2\left(3 + \frac{3}{2}\right)$ = 9

Sol 11: $f(x) = \lim_{x\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$

For x = 0

$f(x = 0) = \lim_{n\to\infty}\frac{c+1}{1+c}$

For x < 0,

LHL = $\lim_{x\to 0^-}\lim_{n\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + ce^{nx}}$

as $x \to 0^-$, $e^{nx} \to 0$

$\therefore$LHL = $\lim_{x\to 0^-}\lim_{n\to\infty}\frac{ax^2 + bx + c}{1 + 0} = c$

RHL = $\lim_{x\to 0^+}\lim_{n\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + ce^{nx}}$

$$= \lim_{x\to 0^+}\lim_{n\to\infty}\frac{\frac{ax^2}{e^{nx}}+\frac{bx}{e^{nx}}+\frac{c}{e^{nx}}+1}{\frac{1}{e^{nx}}+c} = \frac{1}{c}$$

For continuity, LHL = RHL $\Rightarrow \frac{1}{c} = c \Rightarrow c = \pm 1$

But, for c = –1,

f(x = 0) = indeterminate therefore, c = 1

Sol 12: f (x) will be continuous at x = 0 if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0) \quad \text{... (i)}$$

Now, $\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}$

$$= \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}\left\{\sqrt{16+\sqrt{h}}+4\right\}}{16+\sqrt{h}-16}$$

$$= \lim_{h\to 0}\left\{\sqrt{16+\sqrt{h}}+4\right\} = 8$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 4h}{h^2} = \lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$$

$$= \lim_{h\to 0} 2.\left(\frac{\sin 2h}{2h}\right)^2 \times 4 = 8$$

$f(0) = a \therefore$ by (1), 8 = 8 = a; a = 8.

Sol 13: We have to check continuity at x = ±1 only as function will be continuous for other points.

At x = 1,

LHL = $\lim_{x\to 1^-}\frac{x^{2n-1}+ax^3+bx^2}{x^{2n}+1}$

as $x \to 1^-$, $x^{2n-1} \to 0$, $x^{2n} \to 0$

LHL = $\lim_{x\to 1^-}\frac{ax^3+bx^2}{1}$ = a + b

Vf (x = 1) = $\frac{1+a+b}{1+1}$

for continuity, LHL = Vf(x = 1)

$\therefore$ a + b = $\frac{1+a+b}{2} \Rightarrow$ a + b = 1 ... (i)

At x = 1

RHL = $\lim_{x\to -1^+}\frac{x^{2n-1}+ax^3+bx^2}{x^{2n}+1}$

as $x \to -1^+$, x^{2n-1} and $x^{2n} \to 0$

$\therefore$ RHL = $\lim_{x\to -1^+}\frac{ax^3+b^2}{1}$ = –a + b

f(x = –1) = $\frac{-1-a+b}{1+1}$

for continuity RHL = Vf(x = –1)

$\therefore$ – a + b = $\frac{-1-a+b}{2}$ $\Rightarrow$ –2a + 2b – 1 – a + b

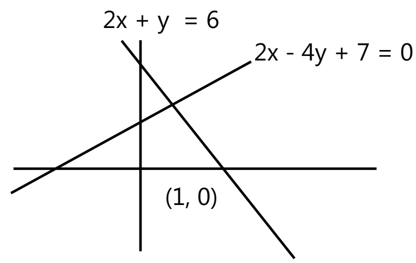
$\Rightarrow$ a – b = 1 ... (ii)

Solving (i) and (ii) , a = 1, b = 0

$\therefore$(a, b) = (1, 0)

Lines –2x – y + 6 = 0

2x – 4y + 7 = 0



We see that origin is on same side as (1, 0) .

$\therefore$ equation of angle bisector

$$\frac{2x-y+6}{\sqrt{5}} = \frac{2x-4y+7}{2\sqrt{5}}$$

–4x – 2y + 12 = 2x – 4y + 7

for 6x – 2y – 5 = 0

Sol 14: We have $x^2 < 1 \Rightarrow (x+1)(x-1) < 0$.

From the sign-scheme, $x \le -1$ or $x \ge 1$.

Thus, the function is $f(x) = x,\ x \le 1$

$x^4, -1 < x < 1$

$x, x \ge 1$

Now, $\lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\{1+h\} = 1$

$$\lim_{x\to 1-0} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^4 = 1$$

$\therefore \lim_{x\to 1} f(x)$ exists and it is equal to 1.

Next, $\lim\limits_{x\to 1+0} f(x) = \lim\limits_{h\to 0} f(-1+h) = \lim\limits_{h\to 0}(-1+h)^4 = 1$

$\lim\limits_{x\to 1-0} f(x) = \lim\limits_{h\to 0} f(-1-h) = \lim\limits_{h\to 0}\{-1-h\} = -1$

$\therefore \lim\limits_{x\to -1} f(x)$ does not exist.

Sol 15: At x = 0

LHL = $\lim\limits_{x\to 0^-}(\sin x+\cos x)^{\operatorname{cosec} x}$

$= \lim\limits_{x\to 0^-}(1+(\sin x+\cos x-1))^{\operatorname{cosec} x}$

$= e^{\lim\limits_{x\to 0}\operatorname{cosec} x(\sin x+\cos x-1)} = e^{1+\lim\limits_{x\to 0}\frac{\cos x-1}{\sin x}}$

$= e^{1+\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}} = e^{1-\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}}} = e$

RHL = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{|x|}}}{ae^{\frac{2}{x}}+be^{\frac{3}{|x|}}}$

$= \lim\limits_{x\to 0^+}\dfrac{e^{-\frac{1}{x}}+e^{-\frac{1}{x}}+1}{ae^{\frac{-1}{x}}+b} = \dfrac{1}{b}$

for continuity, e = a = $\dfrac{1}{b}$

$\therefore$ a = e, b = e^{-1} $\quad a^2 + b^2 = e^2 + e^{-2}$

Differentiability

Sol 1: $f(x) = \cos(x^2+1)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\left((x+h)^2+1\right)-\cos(x^2+1)}{h}$$

$$= \frac{\cos\left[x^2+h^2+2xh+1\right]-\cos(x^2+1)}{h}$$

$$= \frac{\cos\left[x^2+1+h^2+2xh\right]-\cos(x^2+1)}{h}$$

Apply $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\Rightarrow \lim_{h\to 0}\frac{\cos(x^2+1).\cos(h^2+2xh)-\sin(x^2+1).\sin(h^2+2xh)-\cos(x^2+1)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos(x^2+1)\left[\cos(h^2+2xh)-1\right]-\sin(x^2+1).\sin(h^2+2xh)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos(x^2+1).\left(\cos(h^2+2hx)-1\right)}{h(h+2x)}\times$$

$$(h+2x)-\frac{\sin(x^2+1).\sin(h^2+2xh)}{h(h+2x)}\times(h+2x)$$

$$\Rightarrow \lim_{h\to 0}\cos(x^2+1)(h+2x).\frac{\left[\cos(h^2+2hx)-1\right]}{h^2+2hx}-$$

$$\sin(x^2+1),(h+2x).\frac{\sin(h^2+2xh)}{h^2+2hx}$$

$\because \lim\limits_{x\to 0}\dfrac{\cos x-1}{x} = 0$ and $\because \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

$\cos(x^2+1)(2x)(0)-\sin(x^2+1).(2x)$

$\Rightarrow -2x.\sin(x^2+1)$

Sol 2: $f(x) = \tan\sqrt{x}$

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A+\tan B}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\tan\sqrt{x+h}-\tan\sqrt{x}}{h}$$

$$\lim_{h\to 0}\frac{\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right).\tan\left(\sqrt{x+h}-\sqrt{x}\right)}{h\left(\sqrt{x+h}-\sqrt{x}\right)}\times\left(\sqrt{x+h}-\sqrt{x}\right)$$

$$\Rightarrow \lim_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\lim_{h\to 0}-\frac{1}{2}\frac{\frac{1}{\sqrt{x+h}}}{1}$$

$\Rightarrow \left(1+\tan^2\sqrt{x}\right)\left(-\dfrac{1}{2\sqrt{x}}\right) \Rightarrow -\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Sol 3: $e^{\sin x} + (\tan x)^x = f(x)$

$\dfrac{df(x)}{dx} = \dfrac{de^{\sin x}}{dx} + \dfrac{d(\tan x)^x}{dx}$

↓↓

III

Part-I:

$\dfrac{de^{\sin x}}{dx} = \dfrac{de^{\sin x}}{d\sin x}.\dfrac{d\sin x}{dx} = \cos x\, e^{\sin x}$

Part-II:

$\dfrac{d(\tan x)^x}{dx}$

$\mu = (\tan x)^x$

$\log \mu = x \log(\tan x)$

$\therefore \dfrac{1}{\mu}\dfrac{d\mu}{dx} = \log(\tan x) + \dfrac{x}{\tan x}\times \sec^2 x$

$\therefore \dfrac{dy}{dx} = (\tan x)^x[\log(\tan x) + 2x\,\text{cosec}\,2x]$

Sol 4: $f(x) = \sin x^2$

$f'(x) = \lim\limits_{h\to 0}\dfrac{\sin(x+h)^2 - \sin x^2}{h}$

$= \lim\limits_{h\to 0}\dfrac{2\cos\left(\dfrac{x^2+h^2+2xh+x^2}{2}\right)\sin\left(\dfrac{h^2+2xh}{2}\right)}{h}$

$= 2\cos x^2 \lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{h^2}{2}+xh\right)}{\left(\dfrac{h^2}{2}+xh\right)}\times\dfrac{\left(\dfrac{h^2}{2}+xh\right)}{h} = 2x\cos x^2$

Sol 5: $f(x) = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

$f'(x) = \dfrac{d\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)}{d\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)} \times \dfrac{d\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)}{dx}$

$= \dfrac{1}{1+\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)^2}\left[\dfrac{\dfrac{2x^2}{2\sqrt{1+x^2}} - (\sqrt{1+x^2}-1)}{x^2}\right]$

$= \dfrac{x^2}{2\sqrt{1+x^2}(\sqrt{1+x^2}-1)}\,\dfrac{(x^2-1-x^2+\sqrt{1+x^2})}{x^2\sqrt{1+x^2}}$

$= \dfrac{1}{2(1+x^2)}$

Alternative

Put $x = \tan\theta$

Sol 6: $\tan^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = f(x)$

$\Rightarrow \tan^{-1}\left(\dfrac{\sqrt{\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)^2}+\sqrt{\left(\sin\dfrac{x}{2}-\cos\dfrac{x}{2}\right)^2}}{\sqrt{\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)^2}-\sqrt{\left(\sin\dfrac{x}{2}-\cos\dfrac{x}{2}\right)^2}}\right)$

$\Rightarrow \tan^{-1}\left(\dfrac{\sin\dfrac{x}{2}+\cos\dfrac{x}{2}+\sin\dfrac{x}{2}-\cos\dfrac{x}{2}}{\sin\dfrac{x}{2}+\cos\dfrac{x}{2}-\sin\dfrac{x}{2}+\cos\dfrac{x}{2}}\right)$

$\Rightarrow \tan^{-1}x\tan\dfrac{x}{2} = \dfrac{x}{2} = f(x)$

$\therefore f'(x) = \dfrac{d(x/2)}{dx} = \dfrac{1}{2}$

Sol 7: $y = (x)^{\cos x} + (\cos x)^{\sin x}$

$y = u(x) + v(x)$ where $u(x) = (x)^{\cos x}$

$\log u = \cos x \log x$

$\dfrac{1}{u}\dfrac{du}{dx} = \dfrac{\cos x}{x} - \sin x\log x$

$\therefore \dfrac{du}{dx} = x^{\cos x}\left(\dfrac{\cos x}{x} - \sin x\log x\right)$

$v(x) = (\cos x)^{\sin x}$

$\log v = \sin x \log\cos x$

$\dfrac{1}{v}\dfrac{dv}{dx} = \dfrac{\sin x}{\cos x}(-\sin x) + \cos x\log\cos x$

$\therefore \dfrac{dv}{dx} = (\cos x)^{\sin x}\left(\cos x\log\cos x - \dfrac{\sin^2 x}{\cos x}\right)$

$$\therefore \frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right)$$

$$+ (\cos x)^{\sin x}\left(\cos x \log \cos x - \frac{\sin^2 x}{\cos x}\right)$$

$$= \frac{1}{2\sqrt{\tan x}} \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h(\cos(x+h)\cos(x))}$$

$$= \frac{1}{2\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \lim_{h\to 0} \frac{\sin(x+h-x)}{h}$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}} \lim_{h\to 0} \frac{\sinh}{h} = \frac{s\sec^2 x}{2\sqrt{\tan x}}$$

Sol 8: In order to examine the continuity at x = 1, we are required to derive the definition of f (x) in the intervals x < 1, x > 1 and at x = 1, i.e., on and around x = 1.

Now, if $0<x<1$,

$$f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$$

$$= \frac{\log(x+2) - 0.\sin x}{0+1} = \log(x+2)$$

If $x=1$, $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - 1.\sin x}{1+1} = \frac{\log(x+2) - \sin x}{2}$

If $x>1, f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(x+2) - \sin x}{1 + \frac{1}{x^{2n}}} = -\sin x$$

Thus, we have

$$f(x) = \begin{cases} \log(x+2) & ; \quad 0<x<1 \\ \frac{1}{2}\{\log(x+2) - \sin x\} & ; \quad x=1 \\ -\sin x & ; \quad x>1 \end{cases}$$

$$\therefore f(1+0) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} f\{-\sin(1+h)\} = -\sin 1$$

$$f(1-0) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \log(1-h+2) = \log 3$$

$$\therefore f(1+0) \ne f(1-0)$$

So f (x) is not continuous at x = 1.

Sol 9: $f(x) = \sqrt{\cos x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\sqrt{\cos(h+x)} - \sqrt{\cos x}}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\cos(x+h) + \cos x}{h\left(\sqrt{\cos(x+h)} + \sqrt{\cos x}\right)} \quad (\because \text{Rationalise})$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{2\sqrt{\cos x}} . \frac{\cos x(\cosh - 1) - \sin x.\sinh}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{2\sqrt{\cos x}} . \left[\frac{\cos x(\cosh - 1)}{h} - \frac{\sin x.\sinh}{h}\right]$$

$$\Rightarrow -\frac{\sin x}{2\sqrt{\cos x}}$$

Sol 10: Here $f(x) = \sqrt{\tan x}$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} . \frac{\sqrt{\tan(x+h)} + \sqrt{\tan x}}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\tan(x+h) - \tan x}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\frac{\sin(x+h).\cos x - \sin x.\cos(x+h)}{\cos(x+h).\cos x}}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)}\left[\frac{\sin(x+h-x)}{h\cos(x+h).\cos x\left(\sqrt{\tan(x+h)}+\sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)}\left[\frac{\sin h}{h\cos(x+h).\cos x\left(\sqrt{\tan(x+h)}+\sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \frac{1}{\cos(x+0).\cos x\left(\sqrt{\tan(x+0)}+\sqrt{\tan x}\right)} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

$$\left[\because \lim_{(h\to 0)}\frac{\sinh}{h}=1\right]$$

Sol 11: $f(x)=\begin{cases}-x & : \quad x<0\\ 0 & : \quad x=0\\ x & : \quad x>0\end{cases}$

L.H.D. $f'(0^-)=-1$

R.H.D. $f'(0^+)=1$

Sol 12: $f'(a)=\frac{1}{4}$

$$\lim_{h\to 0}\frac{f(a+2h^2)-f(a-2h^2)}{h^2}$$

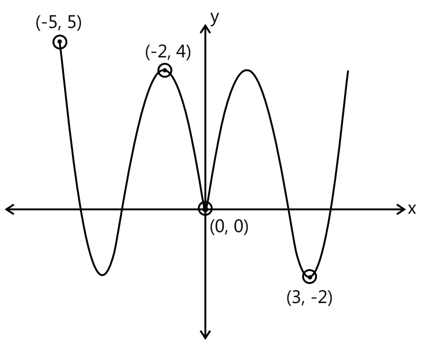
$$\lim_{h\to 0}\frac{f(a+2h^2)}{h^2}-\lim_{h\to 0}\frac{f(a-2h^2)}{h^2}$$

$$\Rightarrow \lim_{h\to 0}\frac{f'(a+2h^2).4h}{2h}-\lim_{h\to 0}\frac{f'(a-2h^2).(-4h)}{2h}$$

$$\Rightarrow \lim_{h\to 0}2.f'(a+2h^2)+\lim_{h\to 0}2.f'(a-2h^2)$$

$$\Rightarrow 2.f'(a)+2.f'(a) \quad \Rightarrow 2\times\frac{1}{4}+2\times\frac{1}{4} \quad \Rightarrow 1$$

Sol 13:



Sol 14: (a) Proof. We consider the function g (x) = f (x) – x . g is continuous on [a, b], and $g(a)=f(a)-a\geq 0$ because the range of f is [a, b]. By the same reason, $g(b)=f(b)-b\leq 0$. Now by the intermediate value theorem, there exists $c\in[a, b]$ such that $g(c)=0$ i.e., $f(c)=c$.

(b) Let f be continuous on the interval [0, 1] to $\mathbb{R}$ and such that f (0) = f (1). Prove that there exists a point $c\in\left[0,\frac{1}{2}\right]$ such that $f(c)=f\left(c+\frac{1}{2}\right)$.

Proof. (Paige Cudworth) Define

$h(x)=f(x)-f\left(x+\frac{1}{2}\right)$. Then $h(0)=f(0)-f\left(\frac{1}{2}\right)$ and $h\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)-f(1)=f\left(\frac{1}{2}\right)-f(0)$. So $h(0)=-h\left(\frac{1}{2}\right)$.

Case 1: $h(0)>0>h\left(\frac{1}{2}\right)$. Then by the Location of Roots Theorem, there exists $c\in\left(0,\frac{1}{2}\right)$ such that $h(c)=0$. So

$$0=f(c)=f\left(c+\frac{1}{2}\right)\Rightarrow f(c)=f\left(c+\frac{1}{2}\right)$$

Case 2: $h(0)<0<h\left(\frac{1}{2}\right)$. Similar to case 1.

Case 3: $h(0)=0=h\left(\frac{1}{2}\right)$. Then $0=f(0)=f\left(\frac{1}{2}\right)$, so $f(0)=f\left(0+\frac{1}{2}\right)$.

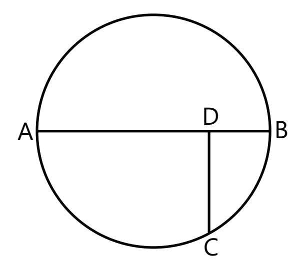
Exercise 2

Limits

Single Correct Choice Type

Sol 1: (C)

Hint : $CD^2=AD\times DB$



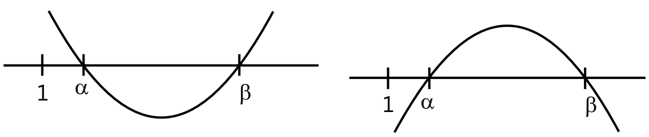
Sol 2: (C)

$$\lim_{x\to0}\left(1+\log^2_{\cos\frac{x}{2}}\cos x\right)^2$$

$$\lim_{x\to0}\log_{\cos\frac{x}{2}}\cos x=\lim_{x\to0}\frac{\log(\cos x)}{\log\left(\cos\frac{x}{2}\right)}$$

$$=\lim_{x\to0}\frac{\frac{-\sin x}{\cos x}}{\frac{1}{2}\frac{-\sin x/2}{\cos x/2}}=\lim_{x\to0}\frac{4\cos^2\frac{x}{2}}{\cos x}=4$$

Sol 3: (C)



Sol 4: (C) Use $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

Sol 5: (C) $\lim\limits_{x\to0}\ \log x-x=\infty$

Sol 6: (D)

$$\lim_{x\to0}\ x.\left\{\frac{\frac{1}{x}}{1}+\frac{\frac{1}{x}3}{3}+\frac{2\frac{1}{x}5}{15}+\ldots\ldots\right\}$$

$$=\lim_{x\to0}\ x\left\{\frac{1}{x}+\frac{1}{3x^3}+\frac{2}{15x^5}+\ldots\ldots\right\}$$

$$=\lim_{x\to0}\ 1+\frac{1}{3x^2}+\frac{2}{15x^5}\ldots\ldots$$

Hence, (D).

Sol 7: (D)

$$\lim_{x\to\infty}\ \sqrt{x}\frac{(\sqrt{x+4}-\sqrt{x})(\sqrt{x+4}+\sqrt{x})}{(\sqrt{x+4}+\sqrt{x})}$$

$$=\lim_{x\to\infty}\ \frac{\sqrt{x}(4)}{\sqrt{x+4}+\sqrt{x}}=\frac{4}{2}=2$$

Sol 8: (B)

$$\lim_{x\to0^+}\frac{\ln(\sin 2x)}{\ln(\sin x)}=\lim_{h\to0}\frac{\ln(\sin 2h)}{\ln(\sinh)}$$

Use L'Hospitals rule

Sol 9: (A)

On solving for x and y we get

$$x=\lim_{c\to1}\frac{1-C^2}{2+C-3C^2}\lim_{c\to1}\frac{(1-c)(1+c)}{(2+3c)(1-c)}=\frac{2}{5}$$

$$\therefore y=-\frac{1}{25}$$

Now, radius can be found by distance formula.

Sol 10: (D) $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{\sum\limits_{k=1}^{r}k}{\sum\limits_{k=1}^{r}k^3}\Rightarrow\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{r\frac{(r+1)}{2}}{\frac{r^4}{4}+\frac{r^3}{2}+\frac{r^2}{4}}$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{r}{2}(r+1)}{\frac{r^4+2r^3+r^2}{4}}\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r^2+2r}{r^4+2r^3+r^2}$$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r(r+1)}{r^2(r^2+2r+1)}\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r(r+1)}{r^2(r+1)^2}$$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2}{r(r+1)}\Rightarrow\lim_{n\to\infty}2.\sum_{r=1}^{n}\frac{1}{r(r+1)}$$

$$\Rightarrow\lim_{n\to\infty}2.\frac{n}{n+1}\Rightarrow\lim_{n\to\infty}2.\frac{1}{1+\frac{1}{n}}\Rightarrow2$$

Sol 11: (A) Take i^n common and use binomial theorem.

Sol 12: (A) $\lim\limits_{x\to0}x\sin\dfrac{1}{x}=\lim\limits_{x\to0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}\left(\because\left|\sin\frac{1}{x}\le1\right|\right)=0$

But f (0) is not defined

Hence, (A)

Continuity

Single Correct Choice Type

Sol 1: $f'(0+0)=\lim\limits_{h\to0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to0}\dfrac{\frac{0+h}{1+e^{\frac{1}{0+h}}}-0}{h}$

$$=\lim_{h\to0}\frac{1}{1+e^{1/h}}=\frac{1}{1+e^{\infty}}=0$$

$$f'(0-0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{\dfrac{0-h}{1+e^{\frac{1}{0-h}}}-0}{-h}$$

$$= \lim_{-h\to 0}\frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$\therefore f'(0+0) \neq f'(0-0)$.

Hence, f(x) is not differentiable at x = 0.

Sol 2: (A) At x = –1,

LHL = $\lim_{x\to -1} f(x) = \lim_{x\to -1^-} \sin(\pi(x+a))$

= $\sin(\pi(-1 + a)) = \sin(\pi - \pi a) = -\sin \pi a$

RHL = $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} b([x]^2+[x])+1$

= b(1 – 1) + 1 = 1

f(x = –1) = –b(1 – 1) + 1 = 1

For continuity, at

LHL = Vf(x = – 1) = RHL

$\Rightarrow -\sin \pi a = 1$

$\pi a = \left(2n+\frac{3}{2}\right)\pi$

$a = 2n + \frac{3}{2}$, $n \in I$, and $b \in R$

Sol 3: (A) $f(x) = \begin{cases} \dfrac{[|x|]e^{x^2}[x+|x|]}{e^{1/x^2}-1\,\text{sgn}(\sin x)} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

LHL = $\lim_{x\to 0^-} f(x)$

Let x = 0 – h, as $x \to 0^-$, $h \to 0^+$

$$\therefore \text{LHL} = \lim_{h\to 0^+}\frac{[|-h|]e^{(-h)^2}[-h+|-h|]}{e^{\frac{1}{(-h)^2}}-[x\,\text{sgn}(\sin(-h))]}$$

for $h \to 0^+$

Now, |-h| = h, [h] = 0,

sin(-h) = –sin h < 0 $\forall$ sgn |sin(-h)) | = –1

$$\therefore \text{LHL} = \lim_{h\to 0^+}\frac{0.[-h+h]}{e^{1/h^2}+1} = 0$$

$$\text{RHL} = \lim_{x\to 0^+}\frac{[|x|]e^{x^2}.[x+|x|]}{e^{\frac{1}{x^2}}-|x\,\text{sgn}(\sin x)} = \lim_{x\to 0^+}\frac{0.e^{x^2}.[x+x]}{e^{\frac{1}{x^2}}-1} = 0$$

LHL = Vf(x = 0) = RHL

f(x) is continuous at x = 0

Sol 4: (D) Consider a function

$f(x) = \begin{cases} x-3, & x \neq 3 \\ 6, & x = 3 \end{cases}$

This function has a isolated point continuity at x = 3.

Now, $g(x) = \dfrac{1}{f(x)} = \begin{cases} \dfrac{1}{x-3}, & x \neq 3 \\ \dfrac{1}{6}, & x = 3 \end{cases}$

Here, the function is not defined at x = 3. Therefore, g(x) has missing point discontinuity.

Sol 5: (C) f(x) = |x– 0. 5| + |x – 1| + tanx

We know |x – a| is not differentiable at x = a

$\therefore$ x = 0. 5 and x = 1 are two points of discontinues further in (0, 2) , tan x is not differentiable at $x = \frac{\pi}{2}$.

$\therefore$ Total three points at which function is not derivable.

Sol 6: (D) (A) for f(x) ,at x = 0,

LHD = $\left.\dfrac{d}{dx}x^2\right|_{x=0}$ = 2x = 0

RHD = $\dfrac{d}{dx}\left|-x^2\right|_{x=0}$ = –2x = 0

$\because$ LHD = RHD, f(x) is differentiable

(B) for g(x) , at x = 0

LHD = $\left.\dfrac{d}{dx}(x)\right|_{x=0}$ = 1

RHD = $\left.\dfrac{d}{dx}(\tan x)\right|_{x=0}$ = $\sec^2 x$ = 1

$\because$ LHD = RHD, g(x) is differentiable

(C) for h(x) at x = 0

LHD = $\left.\dfrac{d}{dx}(\sin 2x)\right|_{x=0}$ = $2\cos^2 x$ = 2

RHD = $\left.\dfrac{d}{dx}(2x)\right|_{x=0}$ = 2

∵ LHD = RHD, g(x) is differentiable

(D) for k(x), at x = 1

LHD = $\frac{d}{dx}(x)_{x=1} = 1$

RHD = $\frac{d}{dx}(2-x)_{x=1} = -1$

∵ LHD ≠ RHD, k(x) is not differentiable at x = 1

Sol 7: (C) It should also be continuous for continuity.

LHL = RHL= Vf(x = 1)

$\lim_{x\to1^-} e^x = \lim_{x\to1^+} a-bx = a - b \Rightarrow e^1 = a - b$ . . . (i)

for differentiability

LHD = RHD

$\frac{d}{dx}(e^x)\Big|_{x=1} = \frac{d}{dx}(a-bx)\Big|_{x=1} \Rightarrow e = -b$

Putting in (i), a = 0

Sol 8: (D) f(x) must also be continuous at x = 2

∴ LHL = RHL = Vf(x = 2)

$\therefore \lim_{x\to2^-} x^2+2x+3 = \lim_{x\to2^+} \frac{a}{\pi}\sin(\pi x)+b = 4 + 4 + 3$

$\Rightarrow \frac{a}{\pi}\sin 2\pi + b = 11 \Rightarrow b - 11$

for derivability,

LHD = RHD

$\frac{d}{dx}(x^2+2x+3)\Big|_{x=2} = \frac{d}{dx}\left(\frac{a}{\pi}\sin(\pi x)+b\right)\Big|_{x=2}$

$(2x + 2)\,|_{x=2} = (a\cos(\pi x))_{x=2} \Rightarrow 6 = a$

∴ b = 11, a = 6

∴ $2a + b\pi = 12 + 11\pi \neq 7$

$b + 2\pi = 11 + 2\pi \neq 3$

$2a + b\pi = 12 + 11\pi \neq 13$

∴ No correct option.

Sol 9: (D) $f(x) = \begin{cases} x & , \quad x<0 \\ x^2 & , \quad 0\le x\le1 \\ x^3-x+1 & , \quad x>0 \end{cases}$

At x = 0

LHL = $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} x = 0$

RHL = $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x^2 = 0$

$Vf(x = 0) = 0^2 = 0$

∵ LHL = RHL = Vf(x = 0), continuous at x = 0

for derivability LHD = $\frac{d}{dx}(x)\Big|_{x=0} = 1$

RHD = $\frac{d}{dx}(x^2)\Big|_{x=0} = 2x = 0$

∵ LHD ≠ RHD

f(x) is not derivable at x = 0

At x = 1,

LHL = $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1$

RHL = $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^3-x+1) = 1$

f(x = 1) = 1

LHL = Vf(x = 1) = RHL, function is continuous at x = 1

Now, LHD = $\frac{d}{dx}(x^2)\Big|_{x=1} = 2x|_{x=1}$

RHD = $\frac{d}{dx}(x^3-x+1)\Big|_{x=1} = (3x^2 - 1)_{x=1} = 2$

∵ LHD = RHD,

f(x) is derivable at x = 1

Sol 10: (D) Discontinuous & also non-derivable at a finite number of points of f.

Sol 11: (A) $f(x) = \begin{cases} \frac{|x|}{\sin x}, & x\neq0 \\ 1, & x=0 \end{cases}$

At x = 0,

LHL = $\lim_{x\to0^-} \frac{|x|}{\sin x}$

Let x = 0 – h, as $x \to 0^+$, $h \to 0^+$

$\therefore$ LHL = $\lim_{h\to0^+} \frac{|-h|}{-\sinh} = -\lim_{h\to0^+} \frac{|h|}{\sinh} = -1$

We see that LHL ≠ Vf(x = 0), therefore, f(x) is not continuous and hence not differentiable at x = 0

Sol 12: (C) $f(x) = x^3$, g(x) = |x|

$fog(x) = |x|^3$

gof = $|x|^3$

for gof, at x = 0,

gof (x) is continuous , gof(x = 0) = 0

$$\text{LHD} = \lim_{h\to0^+}\frac{gof(0-h)-gof(0)}{-h}$$

$$= \lim_{h\to0^+}\frac{|(-h)^3|-0}{-h} = \lim_{h\to0^+}\frac{|h^3|-0}{-h}$$

As $\lim_{h\to0^+}\frac{h^3}{-h} = 0$

$$\text{RHD} = \lim_{h\to0^+}\frac{gof(0+h)-gof(0)}{h} = \lim_{h\to0^+}\frac{|h^3|-0}{-h} = 0$$

Since LHD = RHD,

gof(x) is differentiable at x = 0

for fog(x) , at x = 0

$$\text{LHD} = \lim_{h\to0^+}\frac{fog(0-h)-fog(0)}{-h} = \lim_{h\to0^+}\frac{|-h|^3-0}{-h}$$

$$= \lim_{h\to0^+}\frac{|h|^3}{-h} = \lim_{h\to0^+}-h^2 = 0$$

$$\text{RHD} = \lim_{h\to0^+}\frac{fog(0+h)-fog(0)}{h} = \lim_{h\to0^+}\frac{|h^3|-0}{h}$$

$$= \lim_{h\to0^+}h^2 = 0$$

Since, LHD = RHD,

fog(x) is differentiable at x = 0

Sol 13: (B) Function can be discontinuous at x = 0

At x = 0

$\text{LHL} = \lim_{x\to0^-}[x][\sin\pi x]$

As $x \to 0^-$, $[x] \to -1$, $[\sin \pi x] \to -1$

$\therefore$ LHL = –1 × –1 = 1

$\text{RHL} = \lim_{x\to0^+}[x][\sin\pi x]$

as $x \to 0^+$, $[x] \to 0$

$\therefore$ RHL = 0

$\because$ LHL $\neq$ RHL, f(x) is discontinuous at x = 0.

It can also be discontinuous at $x = \frac{1}{2}$

At $x = \frac{1}{2}$

$\text{LHL} = \lim_{x\to\frac{1}{2}^-}[x][\sin\pi x] = 0$

$\text{RHL} = \lim_{x\to\frac{1}{2}^+}[x][\sin\pi x] = 0$

continuous at $x = \frac{1}{2}$ ($\because$ LHL = RHL)

At $x = \frac{-1}{2}$,

$\text{LHL} = \lim_{x\to\frac{1}{2}^-}[x][\sin\pi x]$

as $x \to \frac{-1}{2}$, [x] = –1,

$\sin \pi x \to -1^-$,

$\therefore$ $[\sin\pi x] = -1$

$\therefore$ LHL = – 1 × –1 = 1

$\text{RHL} = \lim_{x\to\frac{-1}{2}^+}[x][\sin\pi x]$

as $x \to \frac{1}{2}^+$, [x] = –1, $\sin \pi x \to -1\uparrow$

$[\sin \pi x] = -1$

$\therefore$ RHL = –1 × –1 = 1

$\because$ LHL = RHL, continuous at $x = \frac{-1}{2}$

$\therefore$ f(x) is continuous in $x \in (-1, 0) \cup (0, 1)$

Differentiability

Single Correct Choice Type

Sol 1: (D) Considering the case (A), $x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

this is continuous at x = 0

$\because$ RHL = LHL = 0

(B) x^3 is polynomial function which is continuous for $x \in R$.

(C) e^{-x} is exponential function which is continuous for $x \in R$

(D) $f(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}$

$$f'(0^+) = \lim_{h\to0}\frac{(2(\alpha+h))-0}{h} = \lim_{h\to0}\frac{2(0+h)}{2} = 2$$

$f'(0^-) = \lim_{h\to 0}\frac{2(0)}{-h} = 0$

$\because$ LHD $\neq$ RHD

$\therefore$ Function is not differentiable

Sol 2: (D) Considering the case A, $f(0^+)$

$$= \lim_{h\to 0}\frac{(x+h)^2\sin\left(\frac{1}{x+h}\right)-0}{h}$$

$$= \lim_{h\to 0}\frac{(x^2+2xh+h^2)\sin\left(\frac{1}{x+h}\right)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

$$f'(0^-) = \lim_{h\to 0}\frac{(-h)^2\sin\left(\frac{1}{-h}\right)}{-h} = \lim_{h\to 0} +h\sin\left(\frac{1}{h}\right) = 0$$

$\because$ LHD = RHD

$\therefore$ function is differentiable

(B) $x|x|$

As function changes the definition at x = 0. So we check differentiability at x = 0.

$$f(x) = \begin{cases} -x^2 & x<0 \\ x^2 & x\ge 0\end{cases}$$

$$f(0^+) = \lim_{h\to 0}\left[\frac{+(0+h)^2-0}{h}\right] = 0$$

$$f(0^-) = \lim_{h\to 0}\frac{-(0-h)^2-0}{-h} = \lim_{h\to 0} h = 0$$

$\because$ LHD = RHD

so function is differentiable.

(C) $f(x) = \cosh x = \frac{e^x+e^{-x}}{2}$

$$f(0^+) = \lim_{h\to 0}\frac{e^{+h}+e^{-h}-2}{-2h} = \lim_{h\to 0}\frac{e^h-1}{-2h}+\left[\frac{e^{-h}-1}{(-h)}\right]-\frac{1}{2} = 0$$

$$f(0^-) = \lim_{h\to 0}\frac{e^{-h}+e^h+2}{-2h} = \lim_{h\to 0}\left(\frac{e^{-h}-1}{-h}\right)+\left(-\frac{1}{2}\right)\frac{e^h-1}{h} = 0$$

$\because$ LHD = RHD

So, given function is differentiable

So, all the three functions are differentiable.

Sol 3: (A) $f(x) = \sin|x| = \begin{cases}\sin x & x\ge 0\\ -\sin x & x<0\end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(x)$$

$\therefore$ Function is continuous at x = 0

other than x = 0 the function is a trigonometric sin function which is continuous for $x \in R$.

$$f'(0^-) = \lim_{x\to 0}\frac{-\sin(-h)-0}{-h} = -1$$

$$f'(0^+) = \lim_{x\to 0}\frac{\sin(h)-0}{h} = 1$$

$\because$ LHD $\neq$ RHD

$\therefore$ Function is not derivable at x = 0

Sol 4: (D) $f(x) = |x-3| = \begin{cases} x-3 & x\ge 3\\ 3-x & x<3\end{cases}$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3-x = 0$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-3 = 0$$

$\therefore$ Function is continuous at x = 3

$$f'(3^-) = \lim_{h\to 0}\frac{(3-(3-h))}{-h} = -1$$

$$f'(3^+) = \lim_{h\to 0}\frac{(3+h)-3}{h} = 1$$

$\because$ LHD $\neq$ RHD

Function is continuous at x = 3 and not derivable at x = 3.

Sol 5: (A) We have, $f(x) = \frac{|x-1|}{x-1}, x\neq 1, f(1) = 1$

$$f(x) = \begin{cases} 1 & x>1\\ -1 & x<1\\ 1 & x=1\end{cases}$$

$$\lim_{x\to 1^-} f(x) = -1 \neq \lim_{x\to 1^+} f(x) = 1$$

$\therefore$ Function is discontinuous at x = 1

If function is discontinuous at x = a then it is also not differentiable at x = a.

Sol 6: (D) $f'(0) = \lim_{h\to 0}\frac{[7-3(0+h)]-7}{h}$

$$= \lim_{h\to 0}\frac{7-3h-7}{h} = -3$$

Sol 7: (A) $f(x) = |x| + |x-1| = \begin{cases} 2x-1 & x>1 \\ 1 & 0<x\le 1 \\ 1-2x & x\le 0 \end{cases}$

$f'(1^-) = \lim\limits_{h\to 0}\dfrac{1-1}{-h} = 0$

$f'(1^+) = \lim\limits_{h\to 0}\dfrac{2(1+h)-1-1}{h} = 2$

$f'(0^-) = \lim\limits_{h\to 0}\dfrac{[1-2(0-h)]-1}{h} = 2$

$f'(0^+) = \lim\limits_{h\to 0}\dfrac{1-1}{+h} = 0$

$\therefore$ Function is not differentiable at x = 0, 1.

Sol 8: (B) $f(x) = \begin{bmatrix} e^{1/x}, & x\ne 0 \\ 0 & x=0 \end{bmatrix}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{e^{\frac{1}{x+h}}-e^{\frac{1}{x}}}{h} = e^{\frac{1}{x}}\lim\limits_{x\to 0^+}\dfrac{e^{\frac{-h}{x(x+h)-1}}}{h}$

$= e^{\frac{1}{x}}\lim\limits_{x\to 0^+}\dfrac{e^{\frac{-h}{x(x+h)}}-1}{\frac{-h}{x(x+h)}}\times\dfrac{1}{x(x+h)}(-1) = \dfrac{-e^{\frac{1}{x}}}{x^2}$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\dfrac{e^{\frac{1}{x+h}}-e^{\frac{1}{x}}}{h} = \dfrac{-e^{\frac{1}{x}}}{x^2}$

LHL = RHL $\ne$ f(0) = 0

$\therefore$ Function is discontinuous at x = 0

Sol 9: (B) [x] = x – {x} = f(x)

$\left.\begin{matrix} \lim\limits_{x\to a^+}[x] = a \\ \lim\limits_{x\to a^-}[x] = a-1 \end{matrix}\right]a\in I$

$\therefore$ Function is discontinuous at every integer.

Sol 10: (C) $f(x) = \begin{bmatrix} |x-3| & x<1 \\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4} & x\ge 1 \end{bmatrix}$

$f'(1^-) = \lim\limits_{h\to 0}\dfrac{3-(1-h)-2}{-h} = -1$

$f'(1^+) = \lim\limits_{h\to 0}\left[\dfrac{\left(\dfrac{(1+h)^2}{4}-\dfrac{3(1+h)}{2}+\dfrac{13}{4}\right)-\left(\dfrac{1}{4}-\dfrac{3}{2}+\dfrac{13}{4}\right)}{h}\right]$

$= \lim\limits_{h\to 0}\dfrac{\dfrac{(1+h)^2}{4}-\dfrac{3(1+h)}{2}-\dfrac{1}{4}+\dfrac{3}{2}}{-h} = \lim\limits_{h\to 0}\dfrac{1}{4}\dfrac{h(2+h)}{h}-\dfrac{3}{2} = -1$

$\therefore$ LHD = RHD

So function is differentiable at x = 1

Sol 11: (C) $f(x) = \begin{cases} 1 & x\ge 0 \\ -1 & x<0 \end{cases}$

$\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$

$\therefore$ Function is discontinuous and not differentiable at x = 0

Sol 12: (A) $f(x) = \begin{cases} 2x-a-b & x\ge b \\ b-a & a<x<b \\ a+b-2x & x\le a \end{cases}$ let $b>a$

$\lim\limits_{x\to b^-}(2a-a-b) = b-a = \lim\limits_{x\to b^+}(b-a)$

$\therefore$ Function is discontinuous at x = b

$\lim\limits_{x\to a^-}(a+b-2a) = b-a$

$\lim\limits_{x\to a^+}(b-a) = b-a$

$\therefore$ function is continuous at x = a

$f'(b^+) = \lim\limits_{h\to 0}\dfrac{(2(b+h)-a-b)-(b-a)}{h} = +2$

$f'(b^-) = \lim\limits_{h\to 0}\dfrac{(b-a)-(b-a)}{-h} = 0$

$f'(a^+) = 0$

$f'(a^-) = \lim\limits_{h\to 0}\dfrac{[a+b-2(a-h)]-(b-a)}{-h} = -2$

$\therefore$ Function is not differentiable at x = b, a

Sol 13: (D) $f(x) = \begin{bmatrix} 2x-3 & x\ge 2 \\ 1 & 1<x<2 \\ 3-2x & x\le 1 \end{bmatrix}$

f(x) is not differentiable at x = 1 & 2

$f'(1^-) = -2$ and $f'(1^+) = 0$ and $f'(2^-) = 0 f'(2^+) = 2$

Sol 14: (D) $f'(0^-) = \lim\limits_{h\to 0}\dfrac{[1+\sin(-h)-1]}{-h} = 1$

$f'(0^+) = 0$

given function is not differentiable.

Sol 15: (C) $f'(0^-) = \lim\limits_{h\to 0}\dfrac{\dfrac{-h}{1+6^{\frac{-1}{h}}}}{-h} = \lim\limits_{h\to 0} +\dfrac{1}{1+\dfrac{1}{6^{\frac{1}{h}}}} = +1$

$f'(0^+) = \lim\limits_{h\to 0}\dfrac{\dfrac{h}{1+6^{\frac{1}{h}}}}{h} = \lim\limits_{h\to 0}\dfrac{1}{1+6^{\frac{1}{h}}} = 0$

$\therefore$ function is not differentiable at x = 0

Sol 16: (B) $f'(0^-) = \lim\limits_{h\to 0}\dfrac{(-h)^{\alpha}\sin\left(\dfrac{1}{-h}\right)}{(-h)}$

$f'(0^+) = \lim\limits_{h\to 0}\dfrac{(h)^{\alpha}\sin\left(\dfrac{1}{h}\right)}{(h)}$

$\therefore$ for function to be differentiable

$f'(0^-) = f'(0^+)$

$\Rightarrow -(-h)^{\alpha-1} = (h)^{\alpha-1}$

$\Rightarrow (-1)^{\alpha}(h)^{\alpha-1} = (h)^{\alpha-1}$

$\therefore \alpha$ = even $\Rightarrow \alpha > 1$

Sol 17: (C) $f(x) = \begin{Bmatrix} 0 & x\ge 0 \\ 2x & x<0 \end{Bmatrix}$

$f'(0^-) = \lim\limits_{h\to 0}\dfrac{2(-h)-0}{-h} = 2$

$f'(0^+) = 0$

$\therefore$ Function is not differentiable at x = 0.

for $x \ge 0$; f(x) = 0 which is derivable

for x < 0;f(x) = 2x which is a polynomial function

So, it is differentiable at x < 0

Previous Years' Questions

Sol 1: (D) Here $f(x) = \dfrac{\tan\pi[(x-\pi)]}{1+[x]^2}$

Since, we know $\pi[(x-\pi)]$ = nx and $\tan n\pi = 0$

$\because 1 + [x]^2 \neq 0$

$\therefore$ f(x) = 0 for all x

Thus, f(x) is a constant function

$\therefore$ f ′ (x) exists for all x.

Sol 2: (A) Since, f(x) is continuous and differentiable where f(0) = 1 thus f(x) is decreasing for x > 0 and concave down.

$\Rightarrow f''(x) < 0$

Therefore, (A) is answer.

Sol 3: (A) Given, $G(x) = -\sqrt{25-x^2}$

$\therefore \lim\limits_{x\to 1}\dfrac{G(x)-G(1)}{x-1} = \lim\limits_{x\to 1}\dfrac{G'(x)-0}{1-0}$

(using L' Hospital's rule)

$= G'(1) = \dfrac{1}{\sqrt{24}}$

$\left[\because G(x) = -\sqrt{25-x^2} \Rightarrow G'(x) = \dfrac{2x}{2\sqrt{25-x^2}}\right]$

Sol 4: (B) For f(x) to be continuous, we must have $f(0) = \lim\limits_{x\to 0} f(x)$

$= \lim\limits_{x\to 0}\dfrac{\log(1+ax)-\log(1-bx)}{x}$

$= \lim\limits_{x\to 0}\dfrac{a\log(1+ax)}{ax}+\dfrac{b\log(1-bx)}{-bx}$ = a · 1 + b · 1

{using $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = 1$)

= a+b $\therefore$ f(0) = (a + b)

Sol 5: Given f(a) = 2, f ′ (a) = 1,

g(a) = –1 g′ (a) = 2

$\therefore \lim\limits_{x\to a}\dfrac{g(x)f(a)-g(a)f(x)}{x-a} = \lim\limits_{x\to a}\dfrac{g'(x)f(a)-g(a)f'(x)}{1-0}$

(using L' Hospital's rule)

= g ′ (a) f(a) – g (a) f ′ (a) = 2(2) – (–1) (1) = 5

Sol 6: (B) $\lim\limits_{x\to\infty}\left(\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\ldots+\dfrac{n}{1-n^2}\right)$

$= \lim\limits_{x\to\infty}\dfrac{1+2+3+\ldots.+n}{(1-n^2)}$

$= \lim\limits_{x\to\infty}\dfrac{n(n+1)}{2(1-n)(1+n)} = \lim\limits_{x\to\infty}\dfrac{\pi}{2(1-n)} = -\dfrac{1}{2}$

Sol 7: (D) since, $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x]\neq 0 \\ 0, & [x]=0 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & x \in R - [0,1) \\ 0, & 0 \le x \lhd 1 \end{cases}$$

At x = 0

RHL = $\lim\limits_{x\to 0^+} 0 = 0$ and

LHL = $\lim\limits_{x\to 0^-} \dfrac{\sin[x]}{[x]} = \lim\limits_{h\to 0} \dfrac{\sin[0-h]}{[0-h]} = \lim\limits_{h\to 0} \dfrac{\sin(-1)}{-1} = \sin 1$

Since , RHL $\neq$ LHL

$\therefore$ Limit does not exist

Sol 8: (C) Given f(x) = $f(x) = x(\sqrt{x} + \sqrt{x+1})$

$\Rightarrow$ f(x) would exists when $x \ge 0$ and $x + 1 \ge 0$

$\Rightarrow$ f(x) would exist when $x \ge 0$

$\therefore$ f(x) is not continuous as x = 0, because LHL does not exist.

Hence, option (C) is correct

Sol 9: (A) Given $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1+x}, & x \ge 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$

$$\therefore\ f'(x) = \begin{cases} \dfrac{(1+x)1 \cdot -x \cdot 1}{(1+x)^2}, & x \ge 0 \\ \dfrac{(1-x)\cdot 1 - x(-1)}{(1-x)^2}, & x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x \ge 0 \\ \dfrac{1}{(1-x)^2}, & x < 0 \end{cases}$$

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$\therefore$ RHD at x = 0 $\lim\limits_{x\to 0} \dfrac{1}{(1+x)^2} = 1$

and LHD at x = 0

$\Rightarrow \lim\limits_{x\to 0} \dfrac{1}{(1-x)^2} = 1$

Hence f(x) is differentiable for all x.

Sol 10: (C) Since, $y^2 = P(x)$

On, differentiating both sides, we get$2yy_1 = P'(x)$,

Again, differentiating, we get

$2yy_2 + 2y_1^2 = P''(x)$

$\Rightarrow 2y^3y_2 + 2y^2y_1^2 = y^2P''(x)$

$\Rightarrow 2y^3y_2 = y^2 P''(x) - 2(yy_1)^2$

$\Rightarrow 2y^3y_2 = P(x)\cdot P''(x) - \dfrac{\{P'(x)\}^2}{2}$

Again, differentiating, we get

$2\dfrac{d}{dx}(y^3y_2) = P'(x) \Rightarrow P''(x) + P(x)\cdot P'''(x)$

$-\dfrac{2P'(x)\cdot P''(x)}{2}$

$\Rightarrow 2\dfrac{d}{dx}(y^3y_2) = P(x)\cdot P'''(x)$

$\Rightarrow 2\dfrac{d}{dx}\left(y^3 \cdot \dfrac{d^2y}{dx^2}\right) = P(x)\cdot P'''(x)$

Sol 11: (B) Given $f(x) = \dfrac{1}{2}x - 1$ for $0 \le x \le p$

$$\therefore\ [f(x)] = \begin{cases} -1, & 0 \le x < 2 \\ 0 & 2 \le x \le \pi \end{cases}$$

$$\tan[f(x)] \begin{cases} \tan(-1), & 0 \le x < 2 \\ \tan 0, & 2 \le x \le \pi \end{cases}$$

$\Rightarrow \lim\limits_{x\to 2^-} \tan[f(x)] = -\tan 1$

and $\lim\limits_{x\to 2^+} \tan[f(x)] = 0$

So, tan f(x) is not continuous at x = 2

Now, $f(x) = \dfrac{1}{2}x - 1$

$\Rightarrow f(x) = \dfrac{x-2}{2} \Rightarrow \dfrac{1}{f(x)} = \dfrac{2}{x-2}$

Clearly, 1/f(x) is not continuous at x = 2

So, tan[f(x)] and $\tan\left[\dfrac{1}{f(x)}\right]$ are both discontinuous at x = 2

Sol 12: (D)

$$\lim_{x\to 0} \frac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{2}} \cdot \frac{|\sin x|}{x}$$

Atx = 0

RHL $\lim\limits_{h\to 0} \dfrac{1}{\sqrt{2}} \dfrac{\sinh}{h} = \dfrac{1}{\sqrt{2}}$ and LHL = $\lim\limits_{h\to 0} \dfrac{1}{\sqrt{2}} \dfrac{\sinh}{-h} = -\dfrac{1}{\sqrt{2}}$

Here, RHL $\neq$ LHL

$\therefore$ Limit does not exist.

Sol 13: (C) Here, $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)\pi$

$$f(x) = \begin{cases} -\cos\left(\frac{2x-1}{2}\right)\pi & ; -1 \le x < 0 \\ 0 & ; 0 \le x < 1 \\ \cos\left(\frac{2x-1}{2}\right)\pi & ; 1 \le x < 2 \\ 2\cos\left(\frac{2x-1}{2}\right)\pi & ; 2 \le x < 3 \end{cases}$$

which shows

RHL = LHL at x = n $\in$ Integer as

if x = 1

$\Rightarrow \lim\limits_{x\to1^+} \cos\left(\frac{2x-1}{2}\right)\pi = 0$ and $\lim\limits_{x\to1^-} 0 = 0$

Also, f(1) = 0

$\therefore$ continuous at x = 1

Similarly, when x = 2

$\Rightarrow \lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^-} f(x) = 0$

Thus, function is discontinuous at no x

Hence, (c) is the correct answer

Sol 14: (B) Given, f(x) = [$\tan^2$x]

Now, –45° < x < 45°

$\Rightarrow$ tan (–45°) < tan x < tan (45°)

$\Rightarrow$ –tan 45° < tan x < tan (45°)

$\Rightarrow$ – 1 < tan x < 1

$\Rightarrow$ 0 < $\tan^2$x < 1 $\quad\Rightarrow$ [$\tan^2$ x] = 0

i.e f(x) is zero for all value of x from

x = –45° to 45°. Thus f(x) exists when x$\Rightarrow$0 and also it is continuous at x = 0.

Also, f(x) is differentiable at x = 0

and has a value of zero.

Sol 15: (D) Given, $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

On differentiating w.r.t.x, we get

$$f(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$+\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$$

$$\text{and } f'''(x) = \begin{vmatrix} 6 & -\cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$$

$$\therefore f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

Sol 16: (A) $f'(1) = \lim\limits_{h\to0} \frac{f(1+h) - f(1)}{h}$

$\Rightarrow f'(1) = \lim\limits_{h\to0} \sin\left(\frac{1}{h}\right)$

$\therefore$ f is not differentiable at $x = 1$

Similarly, $f'(0) = \lim\limits_{h\to0} \frac{f(h) - f(0)}{h}$

$$\Rightarrow f'(0) = \lim_{h\to0} \frac{(h-1)\sin\left(\frac{1}{h-1}\right) - \sin(1)}{h}$$

$\Rightarrow$ f is also not differentiable at $x = 0$

Sol 17: (A)

$x^{2x} - 2x^x \cot y - 1 = 0$ ... (i)

Now x = 1,

$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$

Now differentiating eq. (i) w.r.t. 'x'

$$2x^{2x}(1 + \log x) - 2\left[x^x\left(-\text{cosec}^2 y\right)\frac{dy}{dx} + \cot y\; x^x(1 + \log x)\right] = 0$$

Now at $\left(1, \frac{\pi}{2}\right)$

$$2\left(1+\log 1\right)-2\left(1\left(-1\right)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}+0\right)=0$$

$$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{\left(t,\frac{\pi}{2}\right)}=0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(t,\frac{\pi}{2}\right)}=-1$$

Sol 18: (A) $f\left(x\right)=\dfrac{1}{e^{x}+2e^{-x}}=\dfrac{e^{x}}{e^{2x}+2}$

$$f\left(x\right)=\frac{\left(e^{2x}+2\right)e^{x}-2e^{2x}.e^{x}}{\left(e^{2x+2}\right)^{2}}$$

$f'\left(x\right)=0 \qquad \Rightarrow e^{2x}+2=2e^{2x}$

$e^{2x}=2 \qquad \Rightarrow e^{2x}+2=2e^{2x}$

Maximum $f\left(x\right)=\dfrac{\sqrt{2}}{4}=\dfrac{1}{2\sqrt{2}}$

$0 \prec f\left(x\right) \le \dfrac{1}{2\sqrt{2}} \quad \Rightarrow$ for some $c \in R$

$f\left(c\right)=\dfrac{1}{3}$

Sol 19: (D) f(x) is a positive increasing function

$$\Rightarrow 0 \prec f\left(x\right) \prec f\left(2x\right) \prec f\left(3x\right)$$

$$\Rightarrow 0 \prec 1 \prec \frac{f\left(2x\right)}{f\left(x\right)} \prec \frac{f\left(3x\right)}{f\left(x\right)}$$

$$\Rightarrow \lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f\left(2x\right)}{f\left(x\right)} \le \lim_{x\to\infty}\frac{f\left(3x\right)}{f\left(x\right)}$$

By sandwich theorem,

$$\Rightarrow \lim_{x\to\infty}\frac{f\left(2x\right)}{f\left(x\right)}=1$$

Sol 20: (A)

$$g'\left(x\right)=2\left(f\left(2f\left(x\right)+2\right)\right)\left(\frac{d}{dx}\left(f\left(2f\left(x\right)+2\right)\right)\right)$$

$$=2f\left(2f\left(x\right)+2\right)f'\left(2f\left(x\right)+2\right).\left(2f'\left(x\right)\right)$$

$$\Rightarrow g'\left(0\right)=2f\left(2f\left(0\right)+2\right).f'\left(2f\left(0\right)+2\right).2\left(f'\left(0\right)=4f\left(0\right)f'\left(0\right)\right)$$

$$=4\left(-1\right)\left(1\right)=-4$$

Sol 21: (D) $\lim\limits_{x\to 2}\dfrac{\sqrt{2\sin^{2}\left(x-2\right)}}{x-2} \Rightarrow \lim\limits_{x\to 2}\dfrac{\sqrt{2}\left|\sin\left(x-2\right)\right|}{x-2}$

R.H.L $=\sqrt{2}$, L.H.L $= =-\sqrt{2}$

Limit does not exist.

Sol 22: (B) $\lim\limits_{x\to 0}\dfrac{\sin\left(p+1\right)+\sin x}{x}=q=\lim\limits_{x\to 0}\dfrac{\sqrt{x+x^{2}}-\sqrt{x}}{x^{3/2}}$

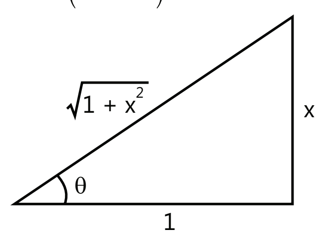
$$\lim_{x\to 0}\left(p+1\right)\cos\left(p+1\right)x+\cos x=q=\frac{1}{2}$$

$$\Rightarrow p+1+1=\frac{1}{2} \Rightarrow p=-\frac{3}{2}; q=\frac{1}{2}$$

Sol 23: (D) $I=\lim\limits_{x\to 0}\dfrac{\left(1-\cos 2x\right)}{x^{2}}\dfrac{\left(3+\cos x\right)}{1}.\dfrac{x}{\tan 4x}$

$$= I=\lim_{x\to 0}\frac{2\sin^{2}x}{x^{2}}\frac{3+\cos x}{1}.\frac{x}{\tan 4x} \;=\; 2.4.\frac{1}{4}=2$$

Sol 24: (A) $y=\sec\left(\tan^{-1}x\right)$



Let $\tan^{-1}x=\theta$

$y=\sec\theta$

$y=\sqrt{1+x^{2}}$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+x^{2}}}.2x$$

at $x=1$

$$\frac{dy}{dx}=\frac{1}{\sqrt{2}}$$

Sol 25: (B) $\lim\limits_{x\to 0}\dfrac{\sin\left(\pi\cos^{2}x\right)}{x^{2}}$

$$=\lim_{x\to 0}\frac{\sin\left(\pi(1-\sin^{2}x\right)}{x^{2}}=\lim_{x\to 0}\sin\frac{\left(\pi-\pi\sin^{2}x\right)}{x^{2}}$$

$$=\lim_{x\to 0}\sin\frac{\left(\pi\sin^{2}x\right)}{x^{2}} \qquad \left[\because \sin\left(\pi-\theta\right)=\sin\theta\right]$$

$$= \lim_{x\to 0} \sin\frac{(\pi \sin^2 x)}{(\pi \sin^2 x)} \times \frac{\pi \sin^2 x}{x^2} = \lim_{x\to 0} 1 \times \pi\left(\frac{\sin x}{x}\right)^2 = \pi$$

Sol 26: (D) Using, mean value theorem

$$f,(c) = \frac{f(1)-f(0)}{1-0} = 4$$

$$f'(c) = \frac{g(1)-g(0)}{1-0} = 2$$

So, $\boxed{f'(c) = g'(c)}$

Sol 27: (C)

$$\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2} \frac{(3+\cos x)}{\frac{\sin 4x}{4x}} \frac{\cos x}{4} = 2$$

Sol 28: (A)

$$g(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ k\sqrt{x+1} & 3 < x \le 5 \end{cases}$$

$$L(g'(3)) = \lim_{x\to 3^-} \frac{k\sqrt{x+1}-2k}{x-3} = \lim_{x+3^-} \left\{\frac{(x+1-4)}{(x-3)(\sqrt{x+1}+2)}\right\} = \frac{k}{4}$$

$$R(g'(3)) = \lim_{x\to 3^+} \frac{mx+2-2k}{x-3}$$

Since this limit exists

$3m+2-2k = 0 \Rightarrow 2k = 3m+2$ ....(i)

So $R(g'(3)) = m$ by L-Hospital rule

Since g(x) is differentiable k = 4m ...(ii)

Solving (i) & (ii)

$m = \frac{2}{5}, k = \frac{8}{5} \quad \Rightarrow k+m = 2$ ....(ii)

Sol 29: (D) $x^2 + 2xy - 3y^2 = 0$

$\Rightarrow 2x + 2xy' + 2y - 6yy' = 0$

$\Rightarrow y' = \frac{x+y}{3y-x}$

At $x = 1$, $y = 1$ we have $\frac{dy}{dx} = 1$

Equation of normal at (1, 1) is y – 1 = - (x - 1)

$\Rightarrow x + y = 2$

Solving with curve, $x^2 + 2x(2-x) - 3(2-x)^2 = 0$

$\Rightarrow x = 1, 3$

$\Rightarrow$ P (1, 1) and Q (3, -1)

So normal meets curve again at (3, -1) in fourth quadrant.

Sol 30: (B) $p = \lim_{x\to 0^+}\left(1+\tan^2\sqrt{x}^{\frac{1}{2x}}\right)$ then log p =

$$p = e^{\lim_{x\to 0^+}(1+\tan^2\sqrt{x}-1)\frac{1}{2x}} = e^{\lim_{x\to 0^+}\frac{(\tan\sqrt{x})^2}{2(\sqrt{x})^2} = e^{\frac{1}{2}}}$$

$$\log p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

Sol 31: (A) $p = \lim_{x\to\infty}\left(\frac{(n+1)(n+2)...(n+2n)}{n^{2n}}\right)$

$$\log p = \frac{1}{n}\left(\lim_{x\to\infty}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)\right)$$

$$\log p = \int_0^2 \log(1+x)dx$$

$$\log p = (x\log(1+x))_0^2 - \int_0^2 \frac{x}{1+x}dx$$

$$\log p = 2\log 3 - \int_0^2\left(1-\frac{1}{1+x}\right)dx$$

$$\log p = 2\log 3 - (x - \log(1+x))_0^2$$

$$\log p = 2\log 3 - (2 - \log 3)$$

$$\log p = 3\log 3 - 2 = \log\frac{27}{e^2}$$

$$p = \frac{27}{e^2}$$

JEE Advanced/Boards

Exercise 1

Limits

Sol 1: Use L'Hospitals rule

Sol 2:

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{-\sec^2 x}{-\sqrt{2}.\cos x}=\frac{(\sqrt{2})^3}{\sqrt{2}}=2$$

$\therefore a = 2$

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}=\lim_{x\to 1}\frac{-\frac{1}{2\sqrt{x}}}{2.\cos^{-1}x.\left(\frac{-1}{\sqrt{1-x^2}}\right)}$$

$$=\lim_{x\to 1}\frac{\sqrt{1-x^2}}{\sqrt{x}.\cos^{-1}x}=\lim_{x\to 1}\frac{\frac{1}{2\sqrt{1-x^2}}.(-2x)}{\sqrt{x}.\left(\frac{-1}{\sqrt{1-x^2}}\right)+\frac{\cos^{-1}x}{2\sqrt{x}}}$$

$$=\lim_{x\to 1}\frac{-x.2\sqrt{x}}{\sqrt{1-x^2}.\cos^{-1}x-2x}=\frac{1}{4}$$

$\therefore\ r=\frac{1}{4}$

Now, use the formula for sum of infinite terms in a GP.

Sol 3: $\lim_{x\to 1}\left\{\frac{p}{1-x^p}-\frac{q}{1-x^9}\right\}=\lim_{x\to 1}\frac{p(1-x^9)-q(1-x^p)}{(1-x^P)(1-x^9)}$

$$=\lim_{x\to 1}\frac{(P-q)+(9x^p-px^q)}{1-x^p-x^q+x^{P+q}}$$

$$=\lim_{x\to 1}\frac{(q.Px^{P-1}-P.qx^{q-1})}{-P.x^{P-1}-qx^{q-1}+(P+q)x^{P+q-1}}$$

$$=\lim_{x\to 1}\frac{Pq(x^{p-1}-x^{q-1})}{P.x^{P-1}(x^9-1)+9x^{9-1}(x^P-1)}$$

$$=\lim_{x\to 1}\frac{Pq.x^{q-1}(x^{P-q}-1)}{P.x^{P-1}(x^9-1)+q.x^{q-1}(x^P-1)}$$

$$=\lim_{x\to 1}\frac{Pq.x^{9-1}\frac{\left(x^{P-q}-1\right)}{(x-1)}}{P.x^{P-1}\left(\frac{x^q-1}{x-1}\right)+q.x^{q-1}\left(\frac{x^P-1}{x-1}\right)}$$

$$=\frac{pq(p-q)}{pq+pq}=\frac{p-q}{2}$$

Sol 4: $\lim_{x\to\infty}x-\ln\cosh x=\lim_{x\to\infty}x-\ln\left(\frac{e^x+e^{-x}}{2}\right)$

$$=\lim_{x\to\infty}x-\ln\left\{e^x\left(\frac{1+e^{-2x}}{2}\right)\right\}$$

$$=\lim_{x\to\infty}x-\ln e^x-\ln\left(\frac{1+e^{-2x}}{2}\right)=-\ln\left(\frac{1}{2}\right)=\ln 2$$

Sol 5: $\lim_{x\to 0}\frac{\sin^4 3\sqrt{x}}{1-\sqrt{\cos x}}=\lim_{x\to 0}\left\{\frac{\sin 3\sqrt{x}}{3\sqrt{x}}\right\}^4\frac{3^4x^2(1+\sqrt{\cos x})}{1-\cos x}$

$$=\lim_{x\to 0}\frac{3^4.x^2}{2.\sin^2\frac{x}{2}}\times(1+\sqrt{\cos x})\ =3^4\times 2^2=324$$

Sol 6: (a) $\lim_{x\to\frac{1}{\sqrt{2}}}\frac{\cos^{-1}(2x\sqrt{1-x^2})}{x-\frac{1}{\sqrt{2}}}$

Use L'hospitals rule.

(b) $\lim_{x\to\frac{\pi}{4}}\frac{1-\sqrt{\sin 2x}}{\pi-4x}=\lim_{x\to\frac{\pi}{4}}\frac{\frac{-1}{2\sqrt{\sin 2x}}.\cos 2x\times 2}{-4}=0$

(c) RHL $=\lim_{h\to 0}\frac{4a-105+56}{\sinh\sin(1+h)}=0$

$$\text{LHL}=\lim_{h\to\infty}\frac{64-120+56}{-\sinh\sin(1-h)}=0$$

Hence. Limit = 0

Sol 7: Using L'Hospitals rule

$$\lim_{x\to\frac{3\pi}{4}}\frac{\frac{1}{3(\tan x)^{2/3}}.\sec^2 x}{-2.2\cos x(-\sin x)}$$

$$=\frac{(-\sqrt{2})^2}{3.(-1)^2(-2).2.\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}=-\frac{1}{3}$$

Sol 8:

$$\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}.\cos\frac{x^2}{4}\right]$$

$$=\lim_{x\to 0}\frac{8\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)}{x^8}$$

Now, use $1-\cos 2\theta=2\sin^2\theta$ to get the answer.

Sol 9:

$$\lim_{h\to 0}\frac{2\sin\left(\frac{\pi}{3}+2h\right)\cos 2h-8\sin\left(\frac{\pi}{3}+2h\right)\cosh+6\sin\left(\frac{\pi}{3}+2h\right)}{h^4}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{\pi}{3}+2h\right)\{\cos 2h - 4\cosh + 3\}}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right)\{\cos^2 h - 2\cosh + 1\}}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right).(1-\cosh)^2}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right).4.\sin^4\frac{4}{2}}{\left(\frac{h}{2}\right)^4.2^4} = \frac{\sqrt{3}}{2}$$

Sol 10:

$$\lim_{x\to\infty} x^2\left\{\sqrt{\frac{x+2}{x}} - \sqrt[3]{\frac{x+3}{x}}\right\}$$

$$= \lim_{x\to\infty} x^2\left\{\left(1+\frac{2}{x}\right)^{\frac{1}{2}} - \left(1+\frac{3}{x}\right)^{\frac{1}{3}}\right\}$$

$$= \lim_{x\to\infty} x^2\left[\left\{1+\frac{1}{2}.\frac{2}{x}+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{2}{x}\right)^2}{21}+.....\right\}\right.$$

$$\left.-\left\{1+\frac{1}{3}\left(\frac{3}{x}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{3}{x}\right)^2}{21}+.....\right\}\right] = \frac{1}{2}$$

Sol 11: $$\lim_{x\to-\infty} \frac{x(3x^2+2).\frac{\sin\frac{1}{x}}{\frac{1}{x}} + |x|^3 + 5}{|x|^3+|x|^2+|x|+1}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{2}{x^2}+\frac{|x|^3}{x^3}+5}{\frac{|x|^3}{x^3}+\frac{|x|^2}{x^3}+\frac{|x|}{x^3}+\frac{1}{x^3}} = \frac{3-1}{-1} = -2$$

Sol 12:

$$I = \lim_{n\to\infty} (n+1)\sin\frac{\pi}{(n+1)} - 2$$

$$= \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n+1}\right)}{\left(\frac{\pi}{n+1}\right)} \times \pi - 2 = \pi - 2$$

$$\therefore \{I\} = \{\pi - 2\} = \pi - 3$$

Sol 13:

(i) $$\lim_{x\to\infty} \frac{x^2+1-ax^2-ax-bx-b}{x+1} = 0$$

$\because$ limit is o

$\therefore$ The numerator should not have terms containing x^2 and x

$\Rightarrow$ $a = 1$ and $b = -1$

(ii) $$\lim_{x\to-\infty} \sqrt{x^2-x+1} - ax - b = 0$$

$$\Rightarrow \lim_{x\to-\infty} \frac{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} - a}{\frac{1}{x}} = b$$

Use L'Hospitals Rule

$$a = 1,\ \ b = \frac{-1}{2}$$

Sol 14:

$$\lim_{x\to 0} \frac{\ln 1+\sin^2 x}{\tan(\ln^2 1+x)} = \lim_{x\to 0} \frac{\frac{\ln(1+\sin^2 x)}{\sin^2 x}\frac{\sin^2 x}{x^2}}{\frac{\tan(\ln^2(1+x))}{\ln^2(1+x)}\left(\frac{\ln(1+x)}{x}\right)^2} = 1$$

Sol 15:

$$\lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)(3.4^{x-1}-3x)}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)\sin(x-1)}$$

$$= \lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)3.\left[\frac{(4^{x-1}-1)}{x-1}+\frac{(1-x)}{x-1}\right]}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)\frac{\sin(x-1)}{x-1}}$$

$$= \lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)3(\ln 4-1)}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)}$$

$$= \lim_{x\to 1}\frac{\left(\frac{1}{1+x}\right)3(\ln 4-1)}{\frac{1}{3}(7+x)^{\frac{-2}{3}}-\frac{3}{2}(1+3x)^{\frac{-1}{2}}}=\frac{\left(\frac{1}{2}\right)3(\ln 4-1)}{\frac{1}{3}\left(\frac{1}{4}\right)-\frac{3}{2}\left(\frac{1}{2}\right)}$$

$$=\frac{-9}{4}\ln\frac{4}{e}$$

Sol 16:

$$\lim_{x\to 0}\frac{e^{x^2}-3^{3x}}{\sin\frac{x^2}{2}-\sin x}=\lim_{x\to 0}\frac{\left(\frac{e^{x^2}-1}{x^2}\right)x^2-\left(\frac{27^x-1}{x}\right)x}{\left(\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)\frac{x^2}{2}-\left(\frac{\sin x}{x}\right)x}$$

$$=\lim_{x\to 0}\frac{x^2-x\ln 27}{\frac{x^2}{2}-x}$$

$$\lim_{x\to 0}\frac{e^{x^2}-3^{3x}}{\sin\left(\frac{x^2}{2}\right)-\sin x}$$

Apply L.H. Rules

$$\lim_{x\to 0}\frac{e^{x^2}.2x-3^{3x}.(\ln 3).3}{\cos\left(\frac{x^2}{2}\right).x-\cos x}\Rightarrow\frac{0-1.3.\ln 3}{0-1}$$

$$\Rightarrow 3\ln 3\quad\Rightarrow \ln 3^3\quad\Rightarrow k=27$$

Sol 17: $\lim\limits_{x\to 3}\left(\dfrac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right)^{\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6}}$

$$=\lim_{x\to 3}\left[\frac{2x+3-x^2}{x+1-(x^2-2x+1)}.\frac{\sqrt{x+1}+x-1}{\sqrt{2x+3}+x}\right]^{\frac{(x^2-2x+1)-(x^2-5)}{(x^2-5x+6)((x-1)+\sqrt{x^2-5})}}$$

$$=\lim_{x\to 3}\left(\left(\frac{x+1}{x}\right)\frac{\sqrt{x+1}+x-1}{\sqrt{2x+3}+x}\right)^{\frac{-2}{(x-2)((x-1)+\sqrt{x^2-5})}}$$

$$=\left[\frac{4}{3}\left(\frac{2+3-1}{3+3}\right)\right]^{\frac{-2}{(1)(2+2)}}=\left(\frac{8}{9}\right)^{\frac{-1}{2}}=\sqrt{\frac{9}{8}}=\frac{3}{2\sqrt{2}}$$

$$=\frac{3}{4}\sqrt{2}=\frac{a\sqrt{6}}{c}$$

$$\therefore\ \left(a^2+b^2+c^2\right)_{\text{least}}=3^2+2^2+4^2=29$$

Sol 18:

$$\lim_{x\to 0}\frac{1}{x^3}\left[\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right]$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left[(1+x)^{\frac{-1}{2}}-(1+ax)(1+bx)^{-1}\right]$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left[\left(1-\frac{1}{2}x-\frac{\frac{1}{2}\left(\frac{-1}{2}-1\right)}{2}x^2-\frac{\frac{1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\right)-(1+ax)\left(1-bx+b^2x^2-b^3x^3\right)\right]$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left[\left(1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3\right)-\left[1+(a-b)x+b(b-a)x^2-b^2(b-a)x^3\right]\right]$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left[-x\left(a-b+\frac{1}{2}\right)+x^2\left(\frac{3}{8}-b^2+ab\right)+x^3\left(-\frac{5}{16}-b^3+ab^2\right)\right]$$

$$\Rightarrow\ a-b=\frac{-1}{2}\quad;\quad\frac{3}{8}-b(b-a)=0$$

$$\Rightarrow\ b=\frac{+3}{4},\qquad a=+\frac{1}{4}$$

$$l=\frac{-5}{16}-b^3+ab^2\ =\frac{-5}{16}+\frac{9}{16}\left[\frac{1}{2}\right]=\frac{-1}{32}$$

$$\frac{1}{a}-\frac{2}{l}+\frac{3}{b}=4-2.(-32)+3.\left(\frac{+4}{3}\right)=72$$

Sol 19: From the given condition we can write that a_n, b_n, c_n are roots of a cubic equation

$$f(n)=x^3-(2n+1)x^2+(2n-1)x+1$$

Clearly 1 is a root to this equation

$\therefore\ f(x)\equiv(x-1)(x^2-2nx-1)$

$\because\ a_n<b_n<c_n$

$$a_n=n-\sqrt{n^2+1},\ \ b_n=1,\ c_n=n+\sqrt{n^2+1}$$

$$\therefore\ \lim_{x\to\infty}na_n=\lim_{x\to\infty}n(n-\sqrt{n^2+1})$$

$$=\lim_{x\to\infty}n\frac{-1}{n+\sqrt{n^2+1}}=\lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{n_2}}}=\frac{-1}{2}$$

Sol 20: $\lim_{x\to 1}\frac{f(x)}{a(x)}=1$ & $\lim_{x\to -2}\frac{f(x)}{a(x)}=4$

$\Rightarrow$ $(x-1)$ & $(x+2)$ are factors of f(x)

$$\Rightarrow \lim_{x\to 1}\frac{a(x-1)(x+2)(x-k)}{(x-1)(x+2)}=1$$

$\Rightarrow$ $a(1-k)=$ ...(1)

$$\lim_{x\to -2}\frac{a(x-1)(x+2)(x-k)}{(x-1)(x+2)}=4$$

$\Rightarrow a(-2-k)=4$ ....(2)

$\Rightarrow k=2,\ a=-1$

$\Rightarrow f(x)=-(x-1)(x+2)(x-2)=-x^3+x^2+4x-4$

$$\therefore \frac{c^2+d^2}{a^2+b^2}=\frac{4^2+4^2}{1^2+1^2}=16$$

Differentiability

Sol 1: $f(x)=\begin{Bmatrix}2\sin x & x\ge 0\\ 0 & x<0\end{Bmatrix}$

$\lim_{x\to 0^+} f(x)=0;\quad \lim_{x\to 0^-} f(x)=0$

$\therefore$ Function is continuous at x = 0

$f'(0^-)=0$

$$f'(0^+)=\lim_{h\to 0}\frac{2\sin(0+h)}{h}=2$$

$\because f'(0^-)\ne f'(0^+)$

$\therefore$ Function is not differentiable at x = 0

Sol 2: $f(x)=\begin{bmatrix}-\frac{x^2}{2} & x\le 0\\ x^n\sin\frac{1}{x} & x>0\end{bmatrix}$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} x^n\sin\frac{1}{x}=0$ for n > 0

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(\frac{-x^2}{2}\right)=0$$

$$f'(0^-)=\lim_{h\to 0}\frac{-\frac{(0-h)}{2}-0}{-h}=\lim_{h\to 0}\frac{h}{2}=0$$

$$f'(0^+)=\lim_{h\to 0}\frac{h^n\sin\frac{1}{h}}{h}=\lim_{h\to 0}h^{n-1}\sin\frac{1}{h}=0$$

for function to be not differentiable

$n-1\le 0\ \therefore n\le 1$

Alson > 0 (for continuous)

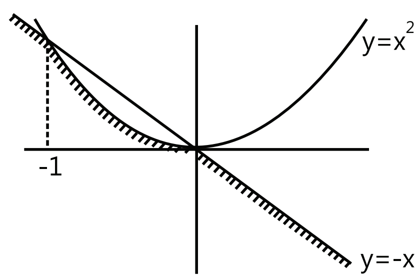
$\therefore 0<n\le 1$

Sol.3 $g(y)=\lim_{x\to y}\left(\frac{\tan x-\tan y}{1+\tan x\tan y}\right)\frac{1}{\left(1-\frac{x}{y}\right)}$

$$=\lim_{x\to y}-y\frac{\tan(x-y)}{(x-y)}=-x;\ f(x)=x^2$$

h(x) = min. (g(y), f(x))

$x^2=-x;\ x=0,-1$



$\therefore$ h(x) is not derivable at x = –1, 0

$$h(x)=\begin{Bmatrix}-x & x\le -1\\ x^2 & -1<x<0\\ -x & x\ge 0\end{Bmatrix}$$

$$f'(-1^+)=\lim_{h\to 0}\frac{-(-1+h)-1}{+h}=-1$$

$$f'(-1^-)=\lim_{h\to 0}\frac{(-1-h)^2-1}{-h}=2$$

$f'(0^+)=-1;\ f'(0^-)=2$

Sol 4: f(0) = 0, f′(0) = 1

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=1=\lim_{h\to 0}\frac{f(h)-f(0)}{h}\Rightarrow\lim_{h\to 0}\frac{f(h)}{h}=1$$

$$\therefore \lim_{h\to 0}\frac{f(h/2)}{(h/2)}\times\frac{1}{2}=\frac{1}{2}$$

$$\therefore \lim_{x\to 0}\left[\frac{f(x)}{x}+\frac{f(x/2)}{x/2}.\frac{1}{2}\ \frac{f(x/3)}{x/3}\times\frac{1}{3}...\frac{f(x/k)}{(x/k)}.\frac{1}{k}\right]$$

$=1+\frac{1}{2}+\frac{1}{3}\ ....\ .\frac{1}{k}$ hence proved.

Sol 5: $f(x)=\begin{Bmatrix}xe^{-\left\{\frac{2}{x}\right\}} & x>0\\ x & x<0\\ 0 & x=0\end{Bmatrix}$

$\lim\limits_{x\to 0^-} f(x) = 0 \Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} \frac{x}{e^{2/x}} = 0$

$\therefore$ Function is continuous at x = 0

$f(0^-) = \lim\limits_{h\to 0} \frac{(0-h)-0}{-h} = 1$

$f(0^+) = \lim\limits_{h\to 0} \frac{he^{-\frac{2}{h}}-0}{h} = \lim\limits_{h\to 0} \frac{1}{e^{2/h}} = 0$

$\therefore$ Function is not differentiable at x = 0.

Sol 6: f(x) = | x – 1 | ([x] – [–x])

$f(x) = \begin{bmatrix} (x-1)[1-(-2)] = 3(x-1) & x>1 \\ (1-x)[0-(-1)] = (1-x) & x<1 \\ 0 & x=1 \end{bmatrix}$

$f'(1^+) = \lim\limits_{h\to 0} \frac{3(1+h-1)-0}{h} = 3$

$f'(1^-) = \lim\limits_{h\to 0} \frac{1-(1-h)-0}{-h} = -1$

Sol 7: $f(1^+) = \lim\limits_{h\to 0} \frac{\left(-\frac{1}{1+h}\right)+1}{h} = \lim\limits_{h\to 0} \left(\frac{h}{1+h}\right)\frac{1}{h} = 1$

$f(1^-) = \lim\limits_{h\to 0} \frac{[a(1-h)^2-b]-[a-b]}{-h} = \lim\limits_{h\to 0} \left[\frac{ah^2-2ah}{-h}\right] = 2a$

$\therefore$ 2a = 1 (for $f(1^-) = f(1^+)$ i. e. function to be derivable.

$\therefore a = \frac{1}{2}$

Also function should be continuous at x = 1.

$\therefore a - b = -1; b = 1 + a; b = \frac{3}{2}$

Sol 8: $f(x) = \begin{bmatrix} x+1 & x<0 \\ 1-x & 0\le x<1 \\ x-1 & x\ge 1 \end{bmatrix}$

$f'(0^-) = 1$

$f'(0^+) = -1$

$f'(1^-) = -1$

$f'(1^+) = 1$

$\therefore$ f(x) is not derivable at x = 0, 1

$\therefore$ m = 2

$g(x) = \begin{bmatrix} x+1 & x<0 \\ (x-1)^2 & x\ge 0 \end{bmatrix}$

$g'(0^-) = 1$

$g'(0^+) = -2$

$\therefore$ g(x) is not differentiable at x = 0

$\therefore$ n = 1

$g \text{ of } = \begin{bmatrix} x+2 & x<0 \\ x^2 & 0\le x<1 \\ (x-2)^2 & x\ge 1 \end{bmatrix}$

$f'(0^-) = 1$

$f'(0^+) = 0$

$f'(1^-) = 2$

$f'(1^+) = -2$

gof is not differentiable at x=0,1

$\therefore$ p = 2

m + n + p = 1 + 2 + 2 = 5

Sol 9: $f(x) = \begin{bmatrix} -1 & -2\le x\le 0 \\ x-1 & 0<x\le 2 \end{bmatrix}$

g(x) = f|x| + |f(x)|

$f(x) = \begin{bmatrix} -x-1 & -2\le x\le 0 \\ x-1 & 0<x\le 2 \end{bmatrix}$

$|f(x)| = \begin{bmatrix} 1 & -2\le x\le 0 \\ 1-x & 0<x\le 1 \\ x-1 & 1<x\le 2 \end{bmatrix}$

$\therefore g(x) = \begin{bmatrix} -x & -2\le x\le 0 \\ 0 & 0<x\le 1 \\ 2x-2 & 1<x\le 2 \end{bmatrix}$

$g'(0) = -1$

$g'(0^+) = 0$

$g'(1^-) = 0$

$g'(1^+) = 2$

$\therefore$ g(x) is not differentiable at x = 0, 1

Sol 10: $f(x) = \begin{bmatrix} 0 & 0\le x<1 \\ x & 1\le x<2 \\ 2(x-1) & 2\le x<3 \\ 3(x-1) & x=3 \end{bmatrix}$

$f'(1^-) = 0$

$f'(1^+) = 1$

$f'(2^-) = \lim\limits_{h\to 0} \frac{(2-h)-2}{-h} = 1$

$f'(2^+) = \lim_{h\to 0}\frac{2(2+h-1)-2}{h} = 2$

$\therefore$ Function f(x) is not derivable at x = 1,2.

$\lim_{x\to 1^-} f(x) = 0 \ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

$\therefore$ Function is not continuous at x = 1

$\lim_{x\to 2^-} f(x) = 2$ and $\lim_{x\to 2^+} f(x) = 2$

$\therefore$ Function is continuous at x = 2.

Sol 11: $f(x) = \begin{bmatrix} 3-2x & 1\le x<3/2 \\ 2x-3 & 3/2\le x<2 \\ 2 & x=2 \\ \sin(\pi/2x) & x<1 \end{bmatrix}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\frac{\pi}{2}x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-2x) = 1$

$\therefore$ Function is continuous at x = 1

$\lim_{x\to \frac{3}{2}^-} f(x) = 3 - 2\times\frac{3}{2} = 0$

$\lim_{x\to \frac{3}{2}^+} f(x) = 2\times\frac{3}{2} - 3 = 0$

Function is continuous at $x = \frac{3}{2}$

$\lim_{x\to 2^-} f(x) = 2\times 2 - 3 = 1;\quad \lim_{x\to 2^+} f(x) = 2$

$\therefore$ Function is not continuous at x = 2

$f'(1^-) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$

$f'(1^+) = -2; f'(2^-) = 2; f'(2^0) = 0$

$f'\left(\frac{3^-}{2}\right) = -2; f'\left(\frac{3^+}{2}\right) = 2$

$\therefore$ Function is not derivable at x = 1, $\frac{3}{2}$, 2.

Sol 12: $f(x) = \begin{bmatrix} -1 & -1\le \sin x<\frac{-3}{4} \\ 0 & \frac{-3}{4}\le \sin x<\frac{-1}{2} \\ 1 & \frac{-1}{2}\le \sin x<\frac{-1}{4} \\ 2 & \frac{-1}{4}\le \sin x<0 \\ 3 & \sin x=0 \end{bmatrix}$

$\therefore$ Function is discontinuous at $\sin x = \frac{-3}{4}, \frac{-1}{2}, \frac{-1}{4}, 0$

and so is not differentiable at these points.

$\therefore x = \pi + \sin^{-1}\left(\frac{-3}{4}\right)$,

$2\pi - \sin^{-1}\left(\frac{-3}{4}\right), \pi + \sin^{-1}\left(\frac{-1}{2}\right)$,

$2\pi - \sin^{-1}\left(\frac{-1}{2}\right), \pi + \sin^{-1}\left(\frac{-1}{4}\right)$,

$2\pi - \sin^{-1}\left(\frac{-1}{4}\right)$

$\pi, 2\pi$

$\therefore$ Sum of all x = $12\pi = 24\frac{\pi}{2}$

Ans. = 24

Sol 13: $f(x) = \begin{bmatrix} ax(x-1)+b & x<1 \\ x-1 & 1\le x\le 3 \\ px^2+qx+2 & x>3 \end{bmatrix}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$a(1-1) + b = 1 - 1 \Rightarrow b = 0$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$3 - 1 = 9p + 3q + 2$

$3p + q = 0$

$f'(1^-) = 2ax - a\big|_{x=1} = a$

$f'(1^+) = 1 \qquad \therefore a \ne 1$

$f'(x) = \begin{cases} 2px+q & x>3 \\ 1 & 1\le x\le 3 \end{cases}$

$\therefore 2p\times 3 + q = 1$

$6p + q = 1$

$\therefore p = \frac{1}{3}$, q = –1, b = 0, $a \neq 1$

Sol 14: $f'(0) = \lim_{h\to 0} \frac{(-h)\left[\frac{\ell n(1-h)+\ell n(1+h)}{\sec(-h)-\cos(-h)}\right]}{(-h)}$

$= \lim_{h\to 0} \frac{\ell n(1-h)(1+h)}{(1-\cos^2 h)}\cosh$

$= \lim_{h\to 0} \frac{-\ell n(1-h^2)}{(-h^2)} \times \frac{\cosh}{\frac{(1-\cos^2 h)}{h^2}} = -1$

$f'(0^+) = a^2 - 3a + 1$

$\therefore a^2 - 3a + 2 = 0$

$(a - 2)(a - 1) = 0$

a = 1, 2

$a_1 = 1, a_2 = 2$

$\therefore a_1^2 + a_2^2 = 1 + 4 = 5$

Continuity

Sol 1: (A) $f(x) = \begin{cases} \frac{x}{2} - 1 & , \quad 0 \le x < 1 \\ \frac{1}{2} & , \quad 1 \le x < 2 \end{cases}$

$g(x) = (2x + 1)(x - k) + 3, 0 \le x < \infty$

$g[f(x)] = (2f(x) + 1)(f(x) - k) + 3,$

$0 \le f(x) < \infty$

for $0 \le x < 1$, $f(x) = \frac{x}{2} - 1$

$g(f(x)) = \left[2\left(\frac{x}{2}-1\right)+1\right]\left[\frac{x}{2}-1-k\right] + 3$

$= (x - 1)\left(\frac{x}{2}-(1+k)\right) + 3, 0 \le x < 1$

For $1 \le x < 2$, $f(x) = \frac{1}{2}$

$g(f(x)) = \left(2.\frac{1}{2}+1\right)\left(\frac{1}{2}-k\right) + 3 = 2\left(\frac{1}{2}-k\right) + 3$

For continuity at x = 1, LHL = RHL

$(1 - 1)\left[\frac{1}{2}-(1+k)\right]+3 = 2\left(\frac{1}{2}-k\right) + 3$

$0 = \frac{1}{2} - k \Rightarrow k = \frac{1}{2}$

Sol 2: (C) We have,

$f(x) = \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x} = \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$

This is $\frac{0}{0}$ form, using L'Hospital rule, limit becomes

$\lim_{x\to 0}\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{3}(1+x)^{-\frac{2}{3}} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Similar for LHL and RHL

$\therefore f(0) = \frac{1}{6}$

Sol 3: (A) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}}$

this is $\frac{0}{0}$ form

using L'Hospital rule

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\frac{1}{3}(27-2x)^{-2/3}(-2)}{-\frac{3}{5}(243+5x)^{-4/5}(5)}$

$= \frac{\frac{1}{3}\times\frac{-1}{9}\times(-2)}{\frac{3}{5}\times\frac{1}{81}\times(-5)} = \frac{1}{27} \times \frac{81\times 5}{3} \times \frac{2}{5} = 2$

Similar for LHL and RHL

$\therefore$ For function to be continuous,

f(0) = 2

Sol 4: (A) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\log(1+2ax)-\log(1-bx)}{x}$

$= \lim_{x\to 0}\frac{\log(1+2ax).2a}{2ax} - \frac{\log(1+(-bx).(-b)}{(-bx)}$

$= 2a + b \left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$

Similar for LHL and RHL

$\therefore$ For function to be continuous

$f(0) = \lim_{x\to 0} f(x) = 2a + b$

Sol 5: (C) LHL = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$x = 0 - h$, as $x \to 0^-$, $h \to 0^+$

$\therefore$ LHL = $\lim\limits_{h\to 0^+}\dfrac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1} = \lim\limits_{h\to 0^+}\dfrac{1-e^{1/h}}{1+e^{1/h}} = \lim\limits_{h\to 0^+}\dfrac{e^{1/h}-1}{e^{1/h}+1}$

This is $\frac{\infty}{\infty}$ form, using L Hospital

LHL = $-\lim\limits_{h\to 0}\dfrac{\frac{-1}{h^2}e^{\frac{1}{h}}}{\frac{-1}{h^2}e^{\frac{1}{h}}} = -1$

RHL = $\lim\limits_{x\to 0^+}\dfrac{e^{1/x}-1}{e^{1/x}+1}$

RHL = 1(using result of LHL)

Since, RHL = f(0) , this is right continuous

Sol 6: (C) At x = 1,

$f(x = 1) = 1 + \sin\frac{\pi}{2} = 2$

LHL = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 1+\sin\frac{\pi}{2}x$

RHL = $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} ax+b = a + b$

for continuity at x = 1,

$a + b = 2$ ... (i)

At x = 3,

$f(x = 1) = 6\tan\frac{\pi}{12}.63 = 6$

LHL = $\lim\limits_{x\to 3^-} ax+b = 3a + b$

RHL = $\lim\limits_{x\to 3^+} 6\tan\frac{\pi x}{12} = 6$

for continuity at x = 3, 3a + b = 6 ... (ii)

from (i) and (ii) , a = 2, b = 0

Sol 7: (A) LHL = $\lim\limits_{x\to 0^-} f(x) = 0$

$\lim\limits_{x\to 0^-}\dfrac{\cos\pi\frac{[x]}{2}}{[x]}$

at $x \to 0$, $[x] = -1$; $\therefore$ LHL = $\dfrac{\cos\left(-\frac{\pi}{2}\right)}{-1} = 0$

RHL = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{\sin[x]}{[x]+1}$

at $x \to 0^+$, $[x] = 0$

$\therefore$ RHL = $\dfrac{\sin[0]}{0+1} = 0$

$\therefore$ for continuity,

f(0) = LHL = RHL = 0 = k.

Sol 8: (A) f(0) = x = 0

In neighbourhood of x,

for rational x,

LHL = $\lim\limits_{x\to 0^-} x = 0$

RHL = $\lim\limits_{x\to 0^+} x = 0$

For irrational x,

LHL = $\lim\limits_{x\to 0^-} (-x) = 0$

RHL = $\lim\limits_{x\to 0^+} (-x) = 0$

Since, all limits are 0, $\lim\limits_{x\to 0} f(x) = 0$

Sol 9: (D) $f(x) = [x]^2 - [x^2]$

for x = 1,

$f(x = 1) = 1 - 1 = 0$

LHL = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} [x]^2 - [x^2] = 0 - 0 = 0$

RHL = $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1^+} [x^2] - [x^2] = 1 - 1 = 0$

$\because$ RHL = LHL = f(1)

continuous at x = 1

At x = N, $N \in Z$, $N \neq 1$

$f(N) = [N]^2 - [N^2] = 0$

RHL = $\lim\limits_{x\to N^+} [x]^2 - [x^2] = N^2 - N^2 = 0$

LHL = $\lim\limits_{x\to N^-} [x]^2 - [x^2] = (N - 1)^2 - (N^2 - 1)$

$(Q[(N^-)^2] = N^2 - 1$

$= N^2 + 1 - 2N - N^2 + 1 = 2(1 - N) \neq 0$, N for $N \neq 1$

$\therefore$ f(x) is discontinuous for x = N

Sol 10: (C) At x = 2,

$f(x = 2) = 4$

LHL = $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x+2) = 4$

RHL = $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (3x-2) = 4$

$\therefore$ LHL = Vf(x = 2) = RHL, continuous at x = 2.

For $x > 2$, continuous as $3x - 2$ is continuous .

$\therefore$ f(x) is continuous for $x \ge 2$.

Sol 11: (C) It can be discontinuous only at x = 1.

At x = 1

$f(x = 1) = 5(1) - 4 = 1$

$$LHL = \lim_{x\to1^-} f(x) = \lim_{x\to1} 5x - 4 = 1$$

$$RHL = \lim_{x\to1} f(x) = \lim_{x\to1^+} 4x^2 + 3bx = 4 + 3b$$

for continuity

LHL= RHL = Vf(x = 1)

$\Rightarrow 4 + 3b = 1 \Rightarrow b = -1$

Sol 12: (A) Discontinuity can arise only at x = 0

Now, for $x \ne 0$

$$f(x) = \frac{x\left(2-\frac{\sin^{-1}x}{x}\right)}{x\left(2+\frac{\tan^{-1}x}{x}\right)} = \frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}}$$

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}} = \frac{2-\lim\limits_{x\to0}\frac{\sin^{-1}x}{x}}{2+\lim\limits_{x\to0}\frac{\tan^{-1}x}{x}} = \frac{2-1}{2+1} = \frac{1}{3}$$

$\therefore$ For continuity, $f(0) = \frac{1}{3}$

Sol 13: (B) $LHL = \lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-}\frac{1-\cos^4 x}{x^2}$

$$= \lim_{x\to0^-}\frac{2\sin^2 2x}{x^2} = 2\lim_{x\to0^-}\left(\frac{\sin 2x}{2x^2}\right)^2 . 4 = 8$$

$$RHL = \lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to0^+}\frac{\sqrt{x}\left[\sqrt{16+\sqrt{x}}+4\right]}{16+\sqrt{x}-16} = 8$$

$\therefore$ For continuity,

f(0) = a = LHL = RHL = 8

Sol 14: (A) f(x) = 1 + |sin x|

We know, modulus function is continuous for all $x \in R$ and sin x is also continuous.

$\therefore$ |sin x| is also continuous for all $x \in R$

$\therefore$ 1 + |sin x| is continuous for all $x \in R$.

Sol 15: (B) Let f(x) = [x] and g(x) = {x}

we see that both f(x) and g(x) are discontinuous .

But, h(x) = x, which is a continuous function.

Thus, sum of two discontinuous functions may be continuous.

Sol 16: (B) We have, $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} x^{\alpha}\cos\frac{1}{x}$

Let α = n, n > 0

Then, $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} x^n\cos\frac{1}{x} = 0.\lim\limits_{x\to0}\cos\frac{1}{x} = 0$

$\therefore$ continuous

for $\alpha = 0$

$$\lim_{x\to0} f(x) = \lim_{x\to0}\cos\frac{1}{x},$$

which is indeterminate

for $\alpha = -n$, n > 0

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{\cos\frac{1}{x}}{x^n}$$

This is indeterminate

$\therefore$ For f(x) to be continuous , $\alpha > 0$

Sol 17: (A) $LHL = \lim\limits_{x\to\frac{\pi}{2}} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\pi-2x}$

Let $x = \frac{\pi}{2} - h$, as $x \to \frac{\pi}{2}$, $h \to 0^+$

$$\therefore LHL = \lim_{h\to0^+}\frac{1-\sin\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to0^+}\frac{1-\cosh}{2h}$$

$$= \lim_{h\to0^+}\frac{2\sin\frac{2h}{2}}{2h} = \lim_{h\to0^+}\frac{h}{4}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = 0$$

$$RHL = \lim_{x\to\frac{\pi}{2}^+}\frac{1-\sin x}{\pi-2x}$$

$x = \frac{\pi}{2} + h$, as $x \to \frac{\pi}{2}^+$, $h \to 0^+$

RHL = $\lim\limits_{x\to 0^+} \dfrac{1-\sin\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$ = $\lim\limits_{h\to 0^+} \dfrac{1-\cosh}{2h}$

= $\lim\limits_{h\to 0^+} \dfrac{-h}{4}\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$ = 0

$\therefore$ For continuity,

$f\left(\frac{\pi}{2}\right)$ = k = LHL = RHL = 0

Sol 18: (C) LHL = $\lim\limits_{x\to 0^+} x\sin\frac{1}{x}$ = 0. $\lim\limits_{x\to 0^+} \sin\frac{1}{x}$ = 0

RHL = $\lim\limits_{x\to 0^-} x\sin\frac{1}{x}$ = x $\lim\limits_{x\to 0^-} \sin\frac{1}{x}$ = 0

$\therefore$ For continuity,

f(0) = k = LHL= RHL = 0

Sol 19: (B) f(x) can be discontinuous at x = N, N $\in$ Z

At x = z, z is an integer

LHL = $\lim\limits_{x\to z^-} f(x)$ = $\lim\limits_{x\to z^-} [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ = (2 – 1) . 0 = 0

RHL = $\lim\limits_{x\to z^+} f(x)$ = $\lim\limits_{x\to z^+} [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ = z.0=0

f(x = 2) ′ = 0

$\because$ LHL = Vf(x = z) = RHL,

f(x) is continuous

Sol 20: (D) At = $\frac{\pi}{4}$

LHL = $\lim\limits_{x\to \frac{\pi}{4}^-} f(x)$ = $\lim\limits_{x\to \frac{\pi}{4}^-} x + a\sqrt{2}\sin x$

π = $\frac{\pi}{4}$ + $a\sqrt{2}.\frac{1}{\sqrt{2}}$

LHL = $\frac{\pi}{4}$ + a

$f\left(x=\frac{\pi}{4}\right)$ = 2 . $\frac{\pi}{4}\cot\frac{\pi}{4}$ + b = $\frac{\pi}{2}$ + b

RHL = $\lim\limits_{x\to \frac{\pi}{4}^+} f(x)$ = $\lim\limits_{x\to \frac{\pi}{4}^+} 2x\cot x$ + b = $\frac{\pi}{2}$ + b

Now, for continuity at x = $\frac{\pi}{4}$,

LHL = $Vf\left(x=\frac{\pi}{4}\right)$

$\therefore$ $\frac{\pi}{4}$ + a = $\frac{\pi}{2}$ + b $\Rightarrow$ a – b = $\frac{\pi}{4}$

Sol 21: (C) $\frac{|x|}{x}$ = sgn(x), which is discontinuous at x = 0.

Hence, f(x) = |x| + $\frac{|x|}{x}$, will be discontinuous

↓ ↓

Continuous at discontinuous

x = 0x = 0

at x = 0

Sol 22: (A) We have, at x = 0

RHL = $\lim\limits_{x\to 0^+} \dfrac{\sin^2 ax}{x}$ = $\lim\limits_{x\to 0^+} a^2x\left(\dfrac{\sin ax}{ax}\right)^2$ = a^2. 0 . 1 = 0

and f(x = 0) = 1

Now, since RHL $\neq$ Vf(x = 0) it is a discontinuous function

Sol 23: (A) At x = π,

LHL = $\lim\limits_{x\to \pi^+} f(x)$ = $\lim\limits_{x\to \pi^-} \dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$

Let x = π – h, as x $\to$ π^-, h $\to$ 0^+

$\therefore$ LHL = $\lim\limits_{h\to 0^+} \dfrac{1-\sin(\pi-h)+\cos(\pi-h)}{1+\sin(\pi-h)+\cos(\pi-h)}$

= $\lim\limits_{h\to 0^+} \dfrac{1-\sinh+\cosh}{1+\sinh-\cosh}$

This is $\frac{0}{0}$ form, using L hospital rule.

LHL = $\lim\limits_{h\to 0^+} \dfrac{\cosh+\sinh}{\cosh+\sinh}$ = –1

RHL = $\lim\limits_{x\to \pi^+} f(x)$ = $\lim\limits_{x\to \pi^+} \dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$

Let π = π + h, as x $\to$ π^+, h $\to$ $0^{+,}$

$\therefore$RHL = $\lim\limits_{x\to \pi^+} \dfrac{1-\sin(\pi+h)+\cos(\pi+h)}{1+\sin(\pi+h)+\cos(\pi+h)}$

= $\lim\limits_{h\to 0^+} \dfrac{1-\sinh-\cosh}{1+\sinh-\sinh}$ = –1 (same expression as in LHL)

Now, for continuity,

f(π) = LHL = RHL = –1

Sol 24: (B) At = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 3x}{\sin x} = \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{\frac{\sin x}{x}}\right).\left(\frac{3x}{x}\right)$$

$= 3 \times 1 \times 1 = 3$

for continuity, $f(0) = \lim_{x\to 0} f(x) = 3 = k$

Sol 25: (D) At x = 0, $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}\times 1 = \frac{1}{2}$$

Now, for continuity,

$f(x = 0) = k = \lim_{x\to 0} f(x) = \frac{1}{2}$

Sol 26: (D) At x = 0,

$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x[x]^2 \log_{(1+x)} 2$

At $x \to 0$, $[x] = -1 \Rightarrow [x]^2 = 1$

Let $x = 0 - h$, as $x \to 0^-$, $h \to 0^+$

$$\text{LHL} = \lim_{h\to 0^+} -h.\log_{(1-h)} 2 = \lim_{h\to 0^+}\frac{-h}{\log(1-h)}.\log 2$$

$$= \log 2 \lim_{h\to 0^+}\frac{(-h)}{\log(1+(-h))}. = \log 2$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\ln(e^{x^2}+2\sqrt{\{x\}})}{\tan\sqrt{x}}$$

for $x \to 0^+$, $\{x\} = p$

$$\therefore \text{RHL} = \lim_{x\to 0^+}\frac{\ln(e^{x^2}+2\sqrt{x})}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+}\frac{\ln\left(e^{x^2}\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)\right)}{\tan\sqrt{x}} = \lim_{x\to 0^+}\frac{\ell n e^{x^2} + \ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+}\frac{x^2}{\tan\sqrt{x}} + \frac{\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}.\frac{2\sqrt{x}}{e^{x^2}}.\frac{1}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+} x^{3/2}\left(\frac{\sqrt{x}}{\tan\sqrt{x}}\right) + \lim_{x\to 0^+}\frac{\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}$$

$$\lim_{x\to 0^+}\frac{2\sqrt{x}}{\tan\sqrt{x}}.\lim_{x\to 0^+}\frac{1}{e^{x^2}} = 0.(1) + 1\times 2\times 1 = 2$$

LHL ≠ RHL, this is an irremovable discontinuity.

Sol 27: (C) $f(x) = [2 + 3\sin x]$, $x \in [0, \pi]$

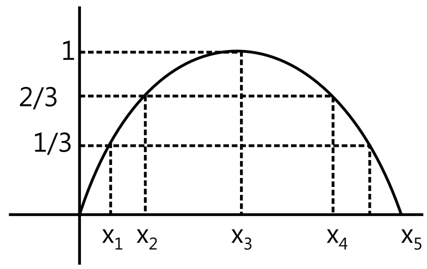
f(x) will be discontinuous at where $f(x) \in z$

we need to find x for which f(x) = z

Now, for $x \in (0, \pi)$, f(x) = z,

for $\sin x = 0, \frac{1}{3}, \frac{2}{3}, 1$

Discontinuous for 5 values of x marked in graph of sin x for $x \in (0, \pi)$



Sol 28: (C) We have $\lim_{x\to a} f(x) = 0$, as graph of f(x) passes through (a, 0)

Now, $L = \lim_{x\to a}\frac{\log_e(1+3f(x))}{2f(x)}$

Let f(x) = g, as $x \to a$, f(x) = 0

$$\therefore L = \lim_{y\to 0}\frac{\log_e(1+3y)}{2y\times 3}\times 3 = \frac{3}{2}\lim_{y\to 0}\frac{\log_e(1+3y)}{3y}$$

$L = \frac{3}{2}$

Sol 29: (B) We have, $X^2 + f(x) - 2x - \sqrt{3}f(x) + 2\sqrt{3} - 3 = 0$

$$f(x) = \frac{(x^2+2x+2\sqrt{3}-3)}{x-\sqrt{3}} = \frac{-(x-\sqrt{3})(x-(2-\sqrt{3}))}{(x-\sqrt{3})}$$

Now, since f(x) is continuous

$$f(\sqrt{3}) = \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}}\frac{(x-\sqrt{3})(x-2+\sqrt{3})}{(x-\sqrt{3})}$$

$= -(\sqrt{3} - 2 + \sqrt{3}) = 2(1 - \sqrt{3})$

Sol 30: (C) We have

$$f(x) = \begin{cases} x^2, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$$

for any real number k, in the neighbour of k, for irrational numbers.

LHL = RHL = $\lim\limits_{x\to k} x^2 = k^2$

For rational number,

LHL = RHL = $\lim\limits_{x\to k} 1 = 1$

f(x) = will be continuous if both of these limits are equal,

i. e. $k^2 = 1 \Rightarrow k = \pm 1$

Sol 31: (B) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n} = \lim\limits_{n\to\infty} \dfrac{1-\left(\dfrac{\sin x^n}{x^n}\right)}{1+\left(\dfrac{\sin x^n}{x^n}\right)}$

Now, at x = 1,

$$\text{LHL} = \lim_{x\to 1^-}\lim_{n\to\infty} \frac{1-\left(\dfrac{\sin x^n}{x^n}\right)}{1+\left(\dfrac{\sin x^n}{x^n}\right)}$$

As $x \to 1^-$ and $n \to \infty$, $x^n \to 0$

$$\lim_{x\to 1^-}\lim_{n\to\infty}\left(\frac{\sin x^n}{x^n}\right) = 1$$

$\therefore$ LHL = $\dfrac{1-1}{1+1}$ = 0

$$\text{RHL} = \lim_{x\to 1^+}\lim_{n\to\infty} \frac{1-\left(\dfrac{\sin x^n}{x^n}\right)}{1+\left(\dfrac{\sin x^n}{x^n}\right)}$$

Ax $x \to 1^+$ and $n \to \infty$, $x^n \to \infty$ and $\sin x^n \in [-1,1)$

$\therefore \lim\limits_{x\to 1^+}\lim\limits_{n\to\infty} \dfrac{\sin x^n}{x^n} = 0$

$\therefore$ RHL = $\dfrac{1-0}{1+0}$ = 1

$\because$ LHL $\neq$ RHL

The function has a finite discontinuity at x = 1

Sol 32: (A) |sin x| is not differentiable at x = 0. Therefore, for f(x) to be differentiable, a = 0

$e^{|x|}$ is not differentiable at x = 0. Therefore for f(x) to be differentiable at x = 0, b = 0.

$|x|^3$ is differentiable at x = 0. Therefore c can be any real number for f(x) to be different.

$\therefore$ a = 0, b = 0, $c \in R$

Sol 33: (D) $f(x) = a^{[x^2]}$, a > 1 will be differentiable at points where $[x^2]$ is not continuous . Now, for $x \in (1, 3)$,

$[x^2]$ will not be continuous at x = $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ (2) , $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, i. e. total 7 points. for which x^2 is an integer.

Sol 34: (D) Noting the definition of [x], f[x] becomes

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ 3x, & x = 3 \end{cases}$$

from def^n of f(x) , we note that f(x) is discontinuous at x = 1, 2, 3 and at these points only, the function will be non-differentiable.

Sol 35: (D) At x = 0,

LHL = $\lim\limits_{x\to 0^-} x + \{x\} + x\sin\{x\}$

at $x \to 0$, {x} = 1

LHL = $\lim\limits_{x\to 0^-} x + 1 + x\sin 1$ = 1

RHL = $\lim\limits_{x\to 0^+} x + \{x\} + x\sin\{x\}$

as $x \to 0^+$, {x} = 0

$\therefore$ RHL = $\lim\limits_{x\to 0^+} x = 0$

$\therefore$ LHL $\neq$ RHL

f(x) is not continuous at x = 0

At, x = 2

LHL = $\lim\limits_{x\to 2^-} x + \{x\} + x\sin\{x\}$

at as $x \to 2^-$, {x} = 1

$\therefore$ LHL = $\lim\limits_{x\to 2^-} x + 1 + x\sin 1$

= 3 + 2 sin 1

RHL = $\lim\limits_{x\to 2^+} x + \{x\} + x\sin\{x\}$

as $x \to 2^+$, {x} = 0

$\therefore$ RHL = $\lim\limits_{x\to 2^+} x + 0 + x\sin 0$ = 2

$\because$ RHL $\neq$ LHL, f(x) is not continuous at x = 0

Therefore, the function is not continuous at x = 0 and x = 2.

Multiple Correct Choice Type

Sol 36: (A, C) f(x) = $\begin{cases} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At x = 0

LHL = $\lim\limits_{x\to 0^-} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}$

x = 0 – h, as x → 0⁻, h → 0⁺

$\therefore$ LHL = $\lim\limits_{h\to 0^+} \dfrac{-h\ln(\cos(-h))}{\ln(1+h^2)}$

$$=-\lim_{h\to 0^+} \frac{\ln\left(1-\left(2\sin^2\frac{h}{2}\right)\right)}{\left(-2\sin^2\frac{h}{2}\right)}.\frac{1}{\frac{\ln(1+h^2)}{h^2}}.\frac{-2\sin^2\frac{h}{2}}{\frac{h^2}{4}}.\frac{h}{4}$$

$= -1 \times 1 \times -2 \times \dfrac{0}{4} = 0 \lim\limits_{h\to 0}\dfrac{\sinh}{h} = 1$

RHL = $\lim\limits_{x\to 0^+} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}$

Proceeding similar to LHL, we get RHL = 0

Since, LHL = Vf(x = 0) = RHL,

f(x) is continuous at x = 0

Now, let L = $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0}\dfrac{h\ln(\cosh)}{\ln(1+h^2)h} = \lim\limits_{h\to 0}\dfrac{\ln(\cosh)}{\ln(1+h^2)}$

$$= \lim_{h\to 0^+} \frac{\ln\left(1-2\sin^2\frac{h}{2}\right)}{-2\sin^2\frac{h}{2}}.\frac{1}{\frac{\ln(1+h^2)}{h^2}}.\frac{-2\left(\sin^2\frac{h}{2}\right)^2}{\left(\frac{h}{2}\right)^2}.\frac{1}{4}$$

$= 1 \times 1 \times (-2) \times \dfrac{1}{4}$

Since L exists, f(x) is differentiable at x = 0 and f ′(x) = $\dfrac{-1}{2}$

Exercise 2

Limits

Single Correct Choice Type

Sol.1: (A)

$$\lim_{n\to\infty}\frac{1^2n+2^2(n-1)+3^2(n-2)+....+n^2(n-(n-1))}{1^3+2^3+....+n^3}$$

$$=\lim_{n\to\infty}\frac{n(1^2+2^2+3^2+....+n^2)-(2^2+2.3^2+3.4^2+...+(n-1)n^2)}{\frac{n^2(n+1)^2}{4}}$$

$$=\lim_{n\to\infty}\frac{n\frac{n(n+1)(2n+1)}{6}-(1\;\;+2^3+3^3+....+n^3)(1+2^2+3^2+...+n^2)}{\frac{n^2(n+1)^2}{4}}$$

$$=\lim_{n\to\infty}\frac{n\sum n^2-\sum n^3+\sum n^2}{\sum n^3}$$

$$=\lim_{n\to\infty}\frac{(n+1)\frac{n(n+1)(2n+1)}{6}-\frac{n^2(n+1)^2}{4}}{\frac{n^2(n+1)^2}{4}}=\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{4}}=\frac{1}{3}.$$

Sol.2: (A) $l=\lim\limits_{n\to\infty}\left[(\sqrt{x^2+2x}-x)\left(\dfrac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)\right]$

$$=\lim_{n\to\infty}\left[\frac{2x}{\sqrt{x^2+2x}+x}\right]=1$$

$$m=\lim_{n\to\infty}\left\{(\sqrt{x^2-2x}+x)\left(\frac{\sqrt{x^2-2x}-x}{\sqrt{x^2-2x}-x}\right)\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{-2x}{\sqrt{x^2-2x}-x}\right\}=0$$

Sol 3: (B) $\lim\limits_{x\to\infty}x^3\left(\sqrt{x^2+\sqrt{x^4+1}}-\sqrt{2}x\right)$

$$=\lim_{x\to\infty}\frac{x^3\left(x^2+\sqrt{x^4+1}-2x^2\right)}{\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x}$$

$$=\lim_{x\to\infty}\frac{x^3\left(\sqrt{x^4+1}-x^2\right)}{\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x}$$

$$= \lim_{x\to\infty} \frac{x^3\left(x^4+1-x^4\right)}{\left(\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x\right)\left(\sqrt{x^4+1}+x^2\right)}$$

$$= \lim_{x\to\infty} \frac{x^3}{\left(\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x\right)\left(\sqrt{x^4+1}+x^2\right)}$$

$$= \lim_{x\to\infty} \frac{1}{\left(\sqrt{1+\sqrt{\frac{1}{x^4}}}+\sqrt{2}\right)\left(\sqrt{1+\frac{1}{x^4}}+1\right)} = \frac{1}{4\sqrt{2}}$$

Sol.4: (A)

$$\lim_{x\to\pi-0} \tan^{-1}\left(2\tan\frac{x}{2}\right) = \lim_{h\to 0} \tan^{-1}\left(2\tan\left(\frac{\pi-h}{2}\right)\right)$$

$$= \lim_{h\to 0} \tan^{-1}\left(2\cot\frac{h}{2}\right) = \frac{\pi}{2}$$

$$\lim_{x\to\pi+0} \tan^{-1}\left(2\tan\frac{x}{2}\right) = \lim_{h\to 0} \tan^{-1}\left(2\tan\left(\frac{\pi+h}{2}\right)\right)$$

$$= \lim_{h\to 0} \tan^{-1}\left(-2\cot\frac{h}{2}\right) = -\frac{\pi}{2}$$

Sol 5: (D) $x_n = x_{n-1} + x_{n-2}$

$$\Rightarrow \frac{x_n}{x_{n-1}} = 1 + \frac{x_{n-2}}{x_{n-1}} = 1 + \frac{1}{\frac{x_{n-1}}{x_{n-2}}}$$

let $\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = l \Rightarrow l = 1 + \frac{1}{l} \Rightarrow l^2 - l - 1 = 0$

$$\Rightarrow l = \frac{l \pm \sqrt{5}}{2} \Rightarrow l = \frac{\sqrt{5}+1}{2} \quad (\text{as } l > 0)$$

Sol 6: (B) $\lim_{x\to\alpha} (1+ax^2+bx+c)^{\frac{1}{x-\alpha}} = e^k$

$$k = \lim_{x\to\alpha} \frac{1}{x-\alpha}(1+ax^2+bx+c) = \lim_{x\to\alpha} \frac{a(x-\alpha)(x-\beta)}{x-\alpha} = a(\alpha-\beta)$$

Multiple Correct Choice Type

Sol 7: (A,B,C)

(a) $\lim_{x\to 1} \frac{1-|\cos(x-1)|}{(x-1)^2} = \lim_{x\to 1} \frac{1-\cos^2(x-1)}{(x-1)^2(1+|\cos(x-1)|)}$

$$= \lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2(1+|\cos(x-1)|)} = \frac{1}{2}$$

(b) $\lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^{1/x} = e^k; k = \lim_{x\to 0^+} \frac{1}{x}\left(\frac{\tan x - x}{x}\right)$

$$= \lim_{x\to 0^+} \frac{\sec^2 x - 1}{2x} = \lim_{x\to 0^+} \left(\frac{x}{2}\right)\left(\frac{\tan x}{x}\right)^2 = 0$$

(c) $\lim_{x\to\infty} \frac{\sin x}{x} = 0\times\sin\infty = 0\times \text{something defined}$

$= 0(\text{as } -1 \le \sin x \le 1)$

(d) $\lim_{x\to\infty} \frac{\tan x}{x} = 0\times\sin\infty = 0\times \text{something not defined}$

$= \text{not defined}$

Sol 8: (A,B,C) $l = \lim_{x\to 0^+} (\cos x)^{1/x} = e^K$

$$K = \lim_{x\to 0^+} \frac{1}{x}(\cos x - 1) = \lim_{x\to 0^+} \frac{-2\sin^2 x/2}{x^2}.x = 0$$

$$m = \lim_{x\to 0^+} (\cos x)^{1/x^2} = e^a$$

$$a = \lim_{x\to 0^+} \frac{1}{x^2}(\cos x - 1) = \lim_{x\to 0^+} \frac{-2\sin^2 x/2}{x^2}.x = -\frac{1}{2}$$

Sol 9: (A,B,C,D) (a) $\lim_{x\to 0^+} (\cot x)^{\frac{\sin x - x}{x^4}}$

$$\frac{\sin x - x}{x^4} = \frac{1}{x^4}\left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) - x\right]$$

$$= -\frac{1}{x}\left(\frac{1}{3!} - \frac{x^2}{5!} + ...\right)$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\sin x - x}{x^4} = -\infty; \ \lim_{x\to 0^+} (\cot x)^{\frac{\sin x - x}{x^4}} = \infty^{-\infty} = 0$$

(b) $\lim_{x\to 0^+} (\cot x)^{\frac{x^3}{\sin x - x}}$

$$\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + ...$$

$$\Rightarrow \lim_{x\to 0^+} \frac{x^3}{\sin x - x} = -6; \ \lim_{x\to 0^+} (\cot x)^{\frac{x^3}{\sin x - x}} = \infty^{-6} = 0$$

(c) $\lim\limits_{x\to\infty}\sqrt{x^2+1}-x=\lim\limits_{x\to\infty}\dfrac{x^2+1-x^2}{\sqrt{x^2+1}+x}$

$=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{1+1/x^2}+1}=0$

(d) $\lim\limits_{x\to\infty}\sqrt{x^2+2x}-x=\lim\limits_{x\to\infty}\dfrac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$

Sol 10: (C, D) $f(x)=\dfrac{x2^x-x}{1-\cos x}$

$$\lim_{x\to0}\frac{x2^x-x}{1-\cos x}=\lim_{x\to0}\frac{x\left[\left(1+x\ln2+\frac{x^2}{2!}(\ln2)^2+\ldots\right)-1\right]}{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right]}$$

$$=\lim_{x\to0}\frac{\ln2+\frac{x}{2!}(\ln2)^2+\ldots}{\frac{1}{2!}-\frac{x^2}{4!}+\ldots}=2\ln2$$

$g(x)=2x\sin\left(\dfrac{\ln2}{2^x}\right)$ as $x\to\infty\Rightarrow -x\to-\infty\Rightarrow 2^{-x}\to0$

$\therefore \lim\limits_{x\to\infty}2x\sin\left(\dfrac{\ln2}{2^x}\right)=\lim\limits_{y\to0}\dfrac{\sin(y\ln2)}{y}=\ln2$

Sol 11: (A,B,C)

(a) $\lim\limits_{t\to0}\dfrac{\sin(\tan t)}{\sin t}=\lim\limits_{t\to0}\dfrac{\sin(\tan t)}{\tan t}\times\dfrac{t}{\sin t}\times\dfrac{\tan t}{\sin t}=1$

(b) $\lim\limits_{x\to\pi/2}\dfrac{\sin(\cos x)}{\cos x}=1$ $\left[\because \cos x\to0 \text{ as } x\to\pi/2\right]$

(c) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim\limits_{x\to0}\dfrac{2}{\sqrt{1+x}-\sqrt{1-x}}=\dfrac{2}{2}=1$

(d) $\lim\limits_{x\to0}\dfrac{\sqrt{x^2}}{x}=\lim\limits_{x\to0}\dfrac{|x|}{x}$ (limit doesn't exists)

Sol 12: (A, B, D)

(a) $\lim\limits_{x\to\infty}x^{\frac14}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}\times\dfrac{1}{x}=\lim\limits_{x\to\infty}x^{\frac{-3}{4}}=0$

(b) $\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$=\lim_{x\to\frac{\pi}{2}}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2\frac{\sin x}{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right).\sin x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}=0$$

(c) $\lim\limits_{x\to\infty}\dfrac{2x^2+3}{x^2+x-5}.\text{sgn}(x)$

$=\lim\limits_{x\to\infty}\dfrac{2+\frac{3}{x^2}}{1+\frac{1}{x}-\frac{5}{x^2}}.\text{sgn}(x)=2.$

(d) $\lim\limits_{x\to0}\dfrac{[3+h]^2-9}{(3+h)^2-9}=0$

Hence A, B and D

Sol 13: (A, B, C, D)

(a) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{\tan x}\right)=\lim\limits_{x\to0}\dfrac{\tan x-x}{x\tan x}$

$=\lim\limits_{x\to0}\dfrac{\sec^2x-1}{\tan x+x.\sec^2x}=\lim\limits_{x\to0}\dfrac{\tan^2x}{\tan x+x\sec^2x}$

$=\lim\limits_{x\to0}\dfrac{\sin^2x}{\sin x.\cos x+x}=\lim\limits_{x\to0}\dfrac{\frac{\sin x}{x}.\sin x}{\frac{\sin x}{x}.\text{cox}+\frac{x}{x}}=0$

(b) $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+1}{2x^2-1}\right)^{x^3/1-x}=\left(\lim\limits_{x\to\infty}\dfrac{3x^2+1}{2x^2-1}\right)^{\lim\limits_{x\to\infty}\frac{x^3}{1-x}}$

$=\left(\dfrac{3}{2}\right)^{\lim\limits_{x\to\infty}\frac{x^3}{1-x}}=0$

(c) $\lim\limits_{x\to\frac{\pi}{4}^+}\left\{\tan\left(x+\dfrac{\pi}{8}\right)\right\}^{\tan2x}=\left(\tan\left(\dfrac{3\pi}{8}\right)\right)^{-\infty}=0$

(d) $\lim\limits_{x\to1}\dfrac{(x^2-1)^2}{(x-1)(x^2+x+1)}=\lim\limits_{x\to1}\dfrac{(x-1)^2(x+1)^2}{(x-1)(x^2+x+1)}=0$

Hence A, B, C and D.

Differentiability

Sol 1: (A) $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}f(1-x)=1$

$\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}e^x=e^0=1$

$\therefore$ function is continuous at $x=0$

for $x < 1$, $f(x) = 1 - x$

$x > 1$, $f(x) = x - 1$

$\therefore$ function is not differentiable at $x = 1$.

$$f'(0^-) = \lim_{h\to 0} \frac{e^{(-h)} - 1}{-h} = 1$$

$$f'(0^+) = \lim_{h\to 0} \frac{(1-h)-1}{h} = -1$$

$f'(0^-) \neq f'(0^+)$

$\therefore$ Function is not differentiable at $x = 0$.

Sol 2: (A) $f'(1^-) = \lim\limits_{h\to 0} \dfrac{\sin^{-1}(1-h) - \sin^{-1}1}{-h}$

$$f'(1^+) = \lim_{h\to 0} \frac{\sin^{-1}(1+h) - \sin^{-1}1}{h}$$

$\therefore$ Differentiation of $\sin^{-1}x$ is not defined for $x \geq 1$

$\therefore$ $\sin^{-1}x$ is not differentiable at $x = 1$

All other function are continuous at $x = 1$ and derivable at $x = 1$

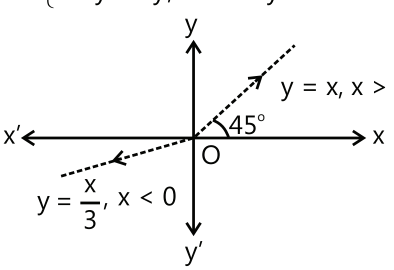
$$\frac{d\tan x}{dx} = \sec^2 x\ ;\ \frac{da^x}{dx} = a^x \log a$$

$$\frac{d\cosh x}{dx} = \frac{d\dfrac{e^x + e^{-x}}{2}}{dx} = \frac{e^x}{2} - \frac{e^{-x}}{2} = \frac{e^x - e^{-x}}{2}$$

Previous Years' Questions

Sol 1: (A, B, D) Since $x + |y| = 2y$

$$\Rightarrow \begin{cases} x + y = 2y, & \text{when } y > 0 \\ x - y = 2y, & \text{when } y < 0 \end{cases}$$



$$\Rightarrow \begin{cases} y = x, & \text{when } y > 0 \Rightarrow x > 0 \\ y = \dfrac{x}{3}, & \text{when } y < 0 \Rightarrow x < 0 \end{cases}$$

which could be plotted as,

Clearly, y is continuous for all x but not differentiable at $x = 0$,

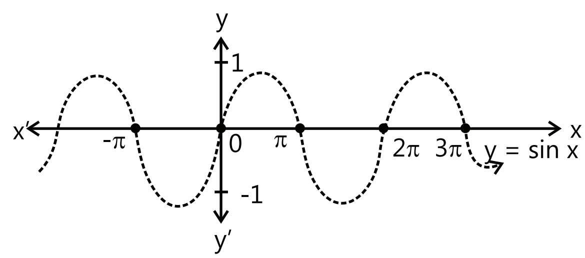
Also, $\dfrac{dy}{dx} = \begin{cases} 1, & x > 0 \\ 1/3, & x < 0 \end{cases}$

Thus, f(x) is defined for all x, continuous at $x = 0$, differentiable for all

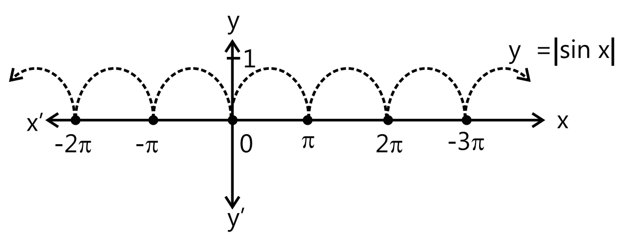
$x \in R - \{0\}$, $\dfrac{dy}{dx} = \dfrac{1}{3}$ for $x < 0$

Sol 2: (B, D) We know, $f(x) = 1 + |\sin x|$ could be plotted as,

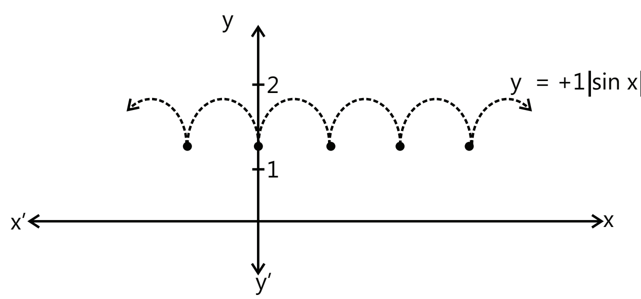
(i) $y = \sin x$ ...(i)



(ii) $y = |\sin x|$..(ii)



(iii) $y = 1 + |\sin x|$...(iii)|



Clearly, $y=1 + |\sin x|$ is continuous for all x, but not differentiable at infinite number of points

Sol 3: (A, B, D) We have, for $-1 < x < 1$

$\Rightarrow 0 \leq x \sin \pi x \leq 1/2$ $\therefore [x \sin \pi x] = 0$

Also, $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by definition of [x] $f(x) = [x \sin \pi x] = -1$, when $1 < x < 1 + h$

Thus, f(x) is constant and equal to 0 in the closed interval $(-1, 1)$ and so f(x) is continuous and differentiable in the open interval $(-1, 1)$

At x = 1, f(x) is discontinuous, since $\lim_{h\to 0}(1-h)=0$

and $\lim_{h\to 0}(1+h)=1$,

$\therefore$ f(x) is not differentiable at x = 1

Hence, (A) (B) and (D) are correct answers.

Sol 4: (A, B) Here, $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

$\therefore$ RHL at x = 1,

$\Rightarrow \lim_{h\to 0}|1+h-3|=2$

LHL at x = 1

$=\lim_{h\to 0}\frac{(1-h)^2}{4}-\frac{3(1-h)}{2}+\frac{13}{4}=\frac{1}{4}-\frac{3}{2}+\frac{13}{4}=\frac{14}{4}-\frac{3}{2}=2$

$\therefore$f(x) is continuous at x = 1

Again, $f(x)=\begin{cases}-(x-3), & 1\le x<3\\ (x-3), & x\ge 3\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

$\therefore\ f'(x)=\begin{cases}-1, & 1\le x<3\\ 1, & x\ge 3\\ \frac{x}{2}-\frac{3}{2}, & x<1\end{cases}$

$\left.\begin{array}{ll}\text{RHD at } x=1\Rightarrow & -1\\ \text{LHD at } x=1\Rightarrow & \frac{1}{2}-\frac{3}{2}=-1\end{array}\right]$

differentiable at x = 1

Again $\left.\begin{array}{ll}\text{RHD at } x=3 & \Rightarrow\ 1\\ \text{LHD at } x=3 & \Rightarrow\ -1\end{array}\right]$ not

differentiable at x = 3

Sol 5: (B, C) The function f(x) = tan x is not defined at $x=\frac{\pi}{2}$, so f(x) is not continuous on (0, π).

Since, g(x) = x sin $\frac{1}{x}$ is continuous on (0, π) and the integral function of a continuous function is continuous,

$\therefore\ f(x)=\int_0^x t\left(\sin\frac{1}{t}\right)dt$ is continuous on (0, π)

Also, $f(x)=\begin{cases}1, & 0<x\le\frac{3\pi}{4}\\ 2\sin\left(\frac{2x}{9}\right), & \frac{3\pi}{4}<x<\pi\end{cases}$

We have, $\lim_{x\to\frac{3\pi^-}{4}} f(x)=1$

$\lim_{x\to\frac{3\pi^+}{4}} f(x)=\lim_{x\to\frac{3\pi}{4}} 2\sin f\left(\frac{2x}{9}\right)=1$

So, f(x) is continuous at $x=\frac{3\pi}{4}$

$\Rightarrow$f(x) is continuous at all other points

Finally, $f(x)=\frac{\pi}{2}\sin(x+\pi)\ \Rightarrow\ f\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}$

$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0}\frac{\pi}{2}\sin\left(\frac{3\pi}{2}-h\right)=\frac{\pi}{2}$

and $\lim_{x\to\left(\frac{x}{2}\right)^+} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0}\frac{\pi}{2}\sin\left(\frac{3\pi}{2}+h\right)=\frac{\pi}{2}$

So, f(x) is not continuous at $x=\frac{\pi}{2}$

Sol 6: A $\to$ p; B $\to$ r

We know [x] $\in$ I, $\forall$ x $\in$ R. Therefore,

sin(π[x]) = 0, $\forall$ x $\Rightarrow$R, by theory we know that sin (π[x]) is differentiable everywhere, therefore (A) $\leftrightarrow$ (p).

Again, f(x) = sin (π(x– [x]))

Now, x – [x] = (x) then π (x – [x]) = π(x)

Which is not differentiable at x $\in$ I

Therefore, (B) $\leftrightarrow$ (r) is the answer

Sol 7: A $\to$ p,q,r; B $\to$ p,s; C $\to$ r,s; D $\to$ p,q

(A)x|x| is continuous, differentiable and strictlyincreasing in (–1, 1)

(B) $\sqrt{|x|}$ is continuous in (–1, 1) and not differentiable at x = 0

(C) x+[x] is strictly increased in (–1, 1) and discontinuous at x = 0

$\Rightarrow$ Not differentiable at x = 0

(D) $|x-1| + |x+1| = 2$ in $(-1, 1)$

$\Rightarrow$ The function is continuous and differentiable in $(-1, 1)$

Sol 8: $\lim\limits_{x\to 1}\left(\dfrac{x-1}{(x-1)\,(2x-5)}\right) = \lim\limits_{x\to 1}\dfrac{1}{(2x-5)} = -\dfrac{1}{3}$

Sol 9: $\lim\limits_{x\to 1}(1-x)\tan\left(\dfrac{\pi x}{2}\right)$

$= \lim\limits_{h\to 0} h\tan\dfrac{\pi}{2}(1-h) = \lim\limits_{h\to 0} h\cot\left(\dfrac{\pi h}{2}\right)$

$= \lim\limits_{h\to 0}\dfrac{\dfrac{\pi h}{2}\times\dfrac{2}{\pi}}{\tan\left(\dfrac{\pi h}{2}\right)} = \dfrac{2}{\pi}$

Sol 10: Let $y = \sin(x^2+1)$

$\therefore \dfrac{dy}{dx} = \lim\limits_{\delta x\to 0}\dfrac{\sin[(x+\delta x)^2+1]-\sin(x^2+1)}{\delta x}$

$= \lim\limits_{\delta x\to 0}\dfrac{2\cos\left[x^2+x\cdot\delta x+\dfrac{1}{2}(\delta x)^2+1\right]\sin\left[x\cdot\delta x+\dfrac{(\delta x)}{2}\right]}{\delta x}$

$= 2\cos(x^2+1)\lim\limits_{\delta x\to 0}\dfrac{\sin[x\cdot\delta x+\frac{1}{2}(\delta x)^2]}{x\cdot\delta x+\frac{1}{2}(\delta x)^2}\times\dfrac{x\cdot\delta x+\frac{1}{2}(\delta x)}{\delta x}$

$= 2\cos(x^2+1)\cdot 1\lim\limits_{\delta x\to 0}\dfrac{x\cdot\delta x+\frac{1}{2}(\delta x)^2}{\delta x} = 2x\cos(x^2+1)$

Sol 11: Given $f(x) = x\tan^{-1}x$

using first principle

$f'(1) = \lim\limits_{h\to 0}\left[\dfrac{f(1+h)-f(1)}{h}\right]$

$= \lim\limits_{h\to 0}\left[\dfrac{(1+h)\tan^{-1}(1+h)-\tan^{-1}(1)}{h}\right]$

$= \lim\limits_{h\to 0}\left[\dfrac{\tan^{-1}(1+h)-\tan^{-1}(1)}{h}+\dfrac{h\tan^{-1}(1+h)}{h}\right]$

$= \lim\limits_{h\to 0}\left[\dfrac{1}{h}\tan^{-1}\left(\dfrac{h}{2+h}\right)+\tan^{-1}(1+h)\right]$

$= \lim\limits_{h\to 0}\left(\dfrac{\tan^{-1}\left(\dfrac{h}{2+h}\right)}{(2+h)\cdot\dfrac{h}{2+h}}\right)+\dfrac{\pi}{4}$

$= \lim\limits_{h\to 0}\dfrac{1}{2+h}\left(\dfrac{\tan^{-1}\left(\dfrac{h}{2+h}\right)}{\dfrac{h}{(2+h)}}\right)+\dfrac{\pi}{4} = \dfrac{1}{2}+\dfrac{\pi}{4}$

Sol 12: Given that,

$$f(x) = \begin{cases}\dfrac{x-1}{2x^2-7x+5} & \text{, when } x\neq 1\\ -\dfrac{1}{3} & \text{, when } x = 1\end{cases}$$

RHD $= \lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0}\dfrac{\left[\dfrac{1+h-1}{2(1+h)^2-7(1+h)+5}-\left(-\dfrac{1}{3}\right)\right]}{h}$

$= \lim\limits_{h\to 0}\left[\dfrac{3h+2(1+h)^2-7(1+h)+5}{3h\{2(1+h)^2-7(1+h)+5\}}\right]$

$= \lim\limits_{h\to 0}\left(\dfrac{2h^2}{3h(-3h+2h^2)}\right) = -\dfrac{2}{9}$

LHD $= \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{\left[\dfrac{1-h-1}{2(1-h)^2-7(1-h)+5}-\left(-\dfrac{1}{3}\right)\right]}{-h}$

$= \lim\limits_{h\to 0}\dfrac{-3h+2(1+h^2-2h)-7(1-h)+5}{-3h[2(1-h)^2-7(1-h)+5]}$

$= \lim\limits_{h\to 0}\dfrac{2h^2}{-3h(2h^2+3h)} = -\dfrac{2}{9}$ $\quad\therefore$ LHD = RHD

Hence, required value of $f'(1)$ is $\dfrac{-2}{9}$

Sol 13: $\lim\limits_{x\to 0}\sqrt{\dfrac{x-\sin x}{x+\cos^2 x}} = \dfrac{\lim\limits_{x\to 0}(x-\sin x)^{1/2}}{\lim\limits_{x\to 0}(x+\cos^2 x)^{1/2}}$

$= \dfrac{\lim\limits_{x\to 0}\left[x\left(1-\dfrac{\sin x}{x}\right)\right]^{1/2}}{\lim\limits_{x\to 0}(0+1)^{1/2}} = \dfrac{0\cdot 0}{1} = 0$

Sol 14: Here, $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$= \lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]}{h} + \frac{h[2a\sin(a+h)+h\sin(a+h)]}{h}$

$= \lim_{h\to 0}\frac{a^2\cdot 2\cos\left(a+\frac{h}{2}\right)\cdot\sin\frac{h}{2}}{2\cdot\frac{h}{2}}+(2a+h)\sin(a+h)$

$= a^2\cos a + 2a\sin a$

Sol 15: Given, $y=\frac{5x}{3|1-x|}+\cos^2(2x+1)$

$$\Rightarrow y=\begin{cases}\frac{5x}{3(1-x)}+\cos^2(2x+1), & x<1\\ \frac{5x}{3(x-1)}+\cos^2(2x+1), & x>1\end{cases}$$

The function is not defined at x = 1

$$\Rightarrow \frac{dy}{dx}=\begin{cases}\frac{5}{3}\left\{\frac{(1-x)-x(-1)}{(1-x)^2}\right\}-2\sin(4x+2), & x\le 1\\ \frac{5}{3}\left\{\frac{(x-1)-x(1)}{(x-1)^2}\right\}-2\sin(4x+2), & x>1\end{cases}$$

$$\Rightarrow \frac{dy}{dx}=\begin{cases}\frac{5}{3(1-x)^2}-2\sin(4x+2), & x<1\\ -\frac{5}{3(x-1)^2}-2\sin(4x+2), & x>1\end{cases}$$

Clearly at x = 1, dy / dx is not defined

Sol 16: (A) $x^{2x}-2x^x\cot y-1=0$ ...(i)

Now $x=1$,

$1-2\cot y-1=0\Rightarrow\cot y=0\Rightarrow y=\frac{\pi}{2}$

Now differentiating eq. (i) w.r.t. 'x'

$2x^{2x}-(1+\log x)-2\left[x^x\left(-\operatorname{coses}^2 y\right)\frac{dy}{dx}+\operatorname{city} x^x(1+\log x)\right]=0$

Now at $\left(1\frac{\pi}{2}\right)$

$$2(1+\log 1)-2\left(1(-1)\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}+0\right)=0$$

$$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}=0\Rightarrow\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}=-1$$

Sol 17: (A) $g(x+1)=\log(f(x+1))=\log+\log|(f(x))$

$\Rightarrow g''(x+1)-g''(x)=\log x$

$g''\left(1+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4$

$g''\left(2+\frac{1}{2}\right)-g''\left(1+\frac{1}{2}\right)=-\frac{4}{9}$

......

......

$g''\left(N+\frac{1}{2}\right)-g''\left(N-\frac{1}{2}\right)=-\frac{4}{(2N-1)^2}$

Summing up all terms

Hence, $g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=4\left(1+\frac{1}{9}+.....+\frac{1}{(2N-1)^2}\right)$

Sol 18: (A)

$$f''(X)=\frac{4ax(x^2ax+1)^2-4ax(x^2-1)(2x+a)(x^2+ax+1)}{(x^2+ax+1)^4}$$

$f''(1)=\frac{4a}{(2+a)^2}$ $\qquad f'(-1)=\frac{-4a}{(2-a)^2}$

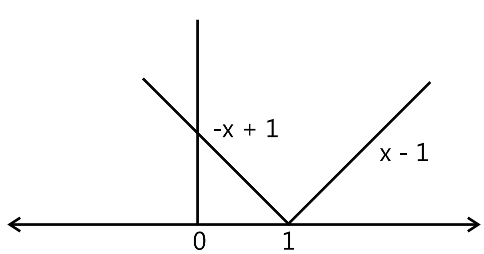
$(2+a)^2 f'(1)+(2-a)^2 f'(-1)=0$

Sol 19: (C) From graph, p = -1

$\Rightarrow \lim_{x\to 1^+} g(x)=-1$

$\Rightarrow \lim_{x\to 0} g(1+h)=-1$

$\Rightarrow \lim_{x\to 0}\left(\frac{h^n}{\log\cos^m h}\right)=-1$



$\Rightarrow \lim_{x\to 0}\frac{n.h^{n-1}}{(-\tan h)}=-\left(\frac{n}{m}\right)\lim_{h\to 0}\left(\frac{h^{n-1}}{\tan h}\right)=-1,$

which holds if $n=m=2$

Sol 20: (B) $f(x)=g(x)\cos x+\sin x.g'(x)$

$\Rightarrow f'(0)=g(0)$

$f'(x)=2g'(x)\cos x-g(x)\sin x+\sin x g''(x)$

$\Rightarrow f'(0)=2g'(0)=0$

But $\lim\limits_{x\to 0}\left[g(x)\cot x-g(0)\operatorname{cosec} x\right]=\lim\limits_{x\to 0}\dfrac{g(x)\cos x-g(0)}{\sin x}$

$=\lim\limits_{x\to 0}\dfrac{g'(x)\cos x-g(x)\sin x}{\cos x}=g'(0)=0=f'(0)$

Sol 21: (B) Differentiating the given equation, we get

$3y^2y'-3y+1=0$

$\Rightarrow y'\left(-10\sqrt{2}\right)=-\dfrac{1}{21}$

Differentiation again we get $6yy^2+3y^2y''-3y''=0$

$\Rightarrow f''\left(-10\sqrt{2}\right)=-\dfrac{6.2\sqrt{2}}{(21)^4}=-\dfrac{4\sqrt{2}}{7^33^2}$

Sol 22: (B, C, D) For $f(x)=x\cos\left(\dfrac{1}{x}\right), x\ge 1$

$F'f(x)=x\cos\left(\dfrac{1}{x}\right)+\dfrac{1}{x}\sin\left(\dfrac{1}{x}\right)\to 1$ for $x\to\infty$

Also $f'(x)=\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)-\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)-\dfrac{1}{x^3}-\cos\left(\dfrac{1}{x}\right)$

$=-\dfrac{1}{x^3}\cos\left(\dfrac{1}{x}\right)<0 \quad \text{for } x\ge 1$

$\Rightarrow (x)$ is decreasing for $[1,\infty)$

$\Rightarrow f'(x+2)<f'(x)$. Also,

$\lim\limits_{x\to\infty} f(x+2)-f(x)=\lim\limits_{x\to\infty}\left[(x+2)\cos\dfrac{1}{x+2}-x\cos\dfrac{1}{x}\right]=2$

$\therefore f(x+2)-f(x)>2\forall x\ge 1$

Sol 23: Let $P(x)=ax^4+bx^3+cx^2+dx+e$

$p'(1)=p'(2)=0$

$\lim\limits_{x\to 0}\left(\dfrac{x^2+p'(x)}{x^2}\right)=2$

$\Rightarrow p(0)=0\Rightarrow e=0$

$\lim\limits_{x\to 0}\left(\dfrac{2x+p'(x)}{2x}\right)=2$

$\Rightarrow p(0)=0\Rightarrow d=0$

$\lim\limits_{x\to 0}\left(\dfrac{2+p''(x)}{2}\right)=2$

$\Rightarrow c=1$

On solving, $a=1/4, b=-1$

So, $p(x)=\dfrac{x^4}{4}-x^3+x^2$

$\Rightarrow p(2)=0$

Sol 24: (A, C)

$L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}=\lim\limits_{x\to 0}\dfrac{1}{x^2\left(a+\sqrt{a^2-x^2}\right)}-\dfrac{1}{4x^2}$

$=\lim\limits_{x\to 0}\dfrac{(4-a)-\sqrt{a^2-x^2}}{4x^2\left(a+\sqrt{a^{2-x^2}}\right)}$

Numerator $\to 0$ if $a=2$ and then $L=\dfrac{1}{64}$

Sol 25: (B, C) $f(x)=\ln x+\int_0^2\sqrt{1+\sin t}\,dt$

$f'(x)=\dfrac{1}{x}+\sqrt{1+\sin x}$

$f''(x)=\dfrac{-1}{x^2}+\dfrac{\cos x}{2\sqrt{1+\sin x}}$

(A) f″ is not defined for $x=\dfrac{-\pi}{2}+n\pi, n\in I$, (A) is wrong

(B) f′(x) always exist for $x>0$

(C) $|f'|<|f|$

Since f′ > 0 and f > 0 f′ < f

$\dfrac{1}{x}+\sqrt{1+\sin x}<\ln x+\int_0^x\sqrt{1+\sin x}\,dx$

LHS is bounded RHS is increasing with range ∞ so there exist some α beyond which RHS is greater than LHS

(d) $|f|+|f'|\le b$ is wrong as f is M1 and its range is not bound while β is finite.

Sol 26: (D) $e^{\ln(1+b^2)} = 2b\sin^2\theta = \dfrac{1+b^2}{2b}$

$\Rightarrow \sin^2\theta = 1$ as $\dfrac{1+b^2}{2b} \geq 1$

$\theta = \pm\pi/2$.

Sol 27: (A, B, C, D) $\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = 0 = f(-\pi/2)$

$\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \cos\left(-\dfrac{\pi}{2}\right) = 0$

$$f'(x) = \begin{cases} -1 & x \leq -\pi/2 \\ \sin x, & -\pi/2 < x \leq 0 \\ 1 & 0 < x \leq 1 \\ ,1/x & x > 1 \end{cases}$$

Clearly, f (x) is not differentiable at x = 0 as f´(0-) = 0 and f´(0+) = 1. f (x) is differentiable at x = 1 as f¢ (1–) = f′ (1+) = 1.

Sol 28: $y'(x) + y(x)g'(x) = g(x)g'(x)$

$\Rightarrow e^{g(x)}y'(x) + e^{g(x)}g'(x) = e^{g(x)}g(x)g'(x)$

$\Rightarrow \dfrac{d}{dx}\left(y(x)e^{e(x)}\right) = e^{g(x)}g(x)g'(x)$

$\therefore y(x) = e^{g(x)} = \int e^{g(x)}g(x)g'(x)dx$

= $\int e^t tdt$, where g(x) = t

= (t – 1) et + c

$\therefore y(x)e^{g(x)} = (g(x)-1)e^{g(x)} + c$

Put x = 0 _ 0 = (0 – 1) . 1 + c ⇒ c = 1

Put x = 2 ⇒ y (2) . 1 = (0 – 1) . (1) + 1

Y(2)=0

Sol 29: (B, C) $\because f(0) = 0$

And $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x\_+h) - f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{f(h)}{h} = f'(0) = k$ (say)

$\Rightarrow f(x) = lx + x \Rightarrow f(x) = kx\,(\because f(0) = 0)$

Sol 30: (B) Given $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1} - ax - b\right) = 4$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{x^2+x+1-ax^2-ax-bx-b}{(x+1)} = 4$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)} = 4$

⇒ 1 – a = 0 and 1 – α – b = 4 b = –4, a = 1.

Sol 31: (B) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0}\dfrac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h} \quad = \lim\limits_{h\to 0} h\cos\left(\dfrac{\pi}{h}\right) = 0$

so, f(x) is differentiable at x = 0

$f'(2^+) = \lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h^2)\left|\cos\frac{\pi}{2+h}\right| - 0}{h} \quad = \lim\limits_{h\to 0}\dfrac{(2+h^2)\cos\left(\frac{\pi}{2+h}\right)}{h}$

$f'(2^+) = \lim\limits_{h\to 0}\dfrac{(2+h)^2}{h}\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{2+h}\right)$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{\pi h}\sin\left[\dfrac{\pi.h}{2(2+h)}\right]$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{\pi h}\sin\dfrac{\pi.h}{2(2+h)} \times \dfrac{\pi}{2(2+h)} = \pi$

Again $f'(2^-) = \lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{-h} \quad \lim\limits_{h\to 0}\dfrac{(2-h)^2\left|\cos\frac{\pi}{2-h}\right|}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h} \quad = \lim\limits_{h\to 0}\dfrac{(2-h)^2\sin\left[\frac{\pi}{2} - \frac{\pi}{2-h}\right]}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2}{\frac{\pi h}{2(2-h)}}.\sin\dfrac{\pi h}{2(2-h)} \times \dfrac{\pi}{2(2-h)} = -\pi$

Sol 32:(B, D) Required limit is

$$=\frac{\int_0^1 x^a dx}{\int_0^1 (a+x)dx}=\frac{2}{(2a+1)(a+1)}=\frac{2}{120}$$

$$\Rightarrow a=7 \text{ or} -\frac{17}{2}.$$

Sol 33: (D) Let g (x) = e–x f (x)

and g″ (x) > 1 > 0

So, g (x) is concave upward and g (0) = g (1) = 0

Hence, g (x) < 0 ∀ x ∈ (0, 1)

$$\Rightarrow e^{-x}f(x)<0$$

$$f(x)<0 \forall x\in(0,1)$$

Alternate solution

$$f(x)-2f'(x)+f(x)\ge e^x$$

$$\Rightarrow\left(f(x)e^{-x}-\frac{x^2}{2}\right)''\ge 0$$

Let $g(x)=f(x)e^{-x}-\frac{x^2}{2}$

$$g(0)=0, g(1)=-\frac{x^2}{2}$$

Since g is concave up so it will always lie below the chord joining the extremities which is y = $y=-\frac{x}{2}$

$$\Rightarrow f(x)e^{-x}-\frac{x^2}{2}<-\frac{x}{2}$$

$$\Rightarrow f(x)<\frac{(x^2-x)e^x}{2}<0\forall x\in(0,1)$$

Sol 34: (D) Given f′ (x) – 2f (x) < 0

$$\Rightarrow f(x)<ce^{2x}$$

Put $x=\frac{1}{2}\Rightarrow c>\frac{1}{e}$

Hence f (x) $< e^{2x-1}$

$$\Rightarrow 0<\int_{1/2}^{1} f(x)dx<\int_{1/2}^{1} e^{2x-1}dx$$

$$0<\int_{1/2}^{1} f(x)dx<\frac{e-1}{2}$$

Sol 35: (B) $F'(x)=2xf(x)=f(x)$

$$f(x)=e^{x^2+c}$$

$$f(x)=e^{x^2}\ (\because f(0)=1)$$

$$F(x)=\int_0^{x^2} e^x dx$$

$$F(x)=e^{x^2}-1(\because F(0)=0)$$

$$F(2)=e^4-1$$

Sol 36: (A) $g\left(\frac{1}{2}\right)=\lim_{h\to 0^+}\int_h^{1-h} t^{-1/2}dt$

$$=\int_0^1\frac{dt}{\sqrt{t-t^2}}=\int_0^1\frac{dt}{\sqrt{\frac{1}{4}-\left(t-\frac{1}{2}\right)^2}}=\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)_0^1$$

$$=\sin^{-1}1-\sin^{-1}(-1)=\pi.$$

Sol 37: (D) We have $g(a)=g(1-a)$ and g is differentiable

Hence $g'\left(\frac{1}{2}\right)=0$

Sol 38: (B, C) Let H (x) = f (x) – 3g (x)

H (-1) = H (0) = H (2) = 3.

Applying Rolle's Theorem in the interval $[-1,0]$

$H'(x)=f'(x)-3g'(x)=0$ for atleast one $c\in(-1,0)$

As H″(x) never vanishes in the interval

⇒ Exactly one $c\in(-1,0)$ for which H′(x) = 0

Similarly, apply Rolle's Theorem in the interval [0,2].

⇒ Exactly one $c\in(-1,0)$ for which H′(x) = 0

Similarly, apply Rolle's Theorem in the interval [0, 2].

$\Rightarrow H'(x)=0$ has exactly one solution in (0, 2)

Sol 39: (A, B, C) (A) $f(x)=F(x)+xF'(x)$

$$F'(1)=F(1)+F'(1)$$

$$f'(1)=F'(1)<0$$

$$f'(1)<0$$

(B) $f'(2)=2F(2)$

$f(x)$ is decreasing and $F(1)=0$

Hence $F(2)<0$

$\Rightarrow f(2)<0$

(C) $f(x)=F(x)+xF'(x)$

$F(x)<0\forall x\in(1,3)$

$F'(x)<0\forall x\in(1,3)$

Hence, $F(x)<0\forall x\in(1,3)$

Sol 40: (A, B) $p(\text{RedBall})=p(1).p(R\,|\,I)+p(II).p(R\,|\,II)$

$$p(II\,|\,R)=\frac{1}{3}=\frac{p(II).p(R\,|\,II)}{p(I).p(R\,|\,(R\,|\,I)+p(II).p(R\,|\,II))}$$

$$\frac{1}{3}=\frac{\frac{n_3}{n_3+n_3}}{\frac{n_1}{n_1+n_2}+\frac{n_3}{n_3+n_4}}$$

Of the given options, A and B satisfy above condition

Sol 41: (A, D) Differentiability of $f(x)$ at $x=0$

LHD $f(0^-)=\lim\limits_{\sigma\to 0}\left(\frac{f(0)-f(0-\sigma)}{\sigma}\right)=\lim\limits_{\sigma\to 0}\frac{0+g(-\sigma)}{\sigma}=0$

RHD $f'(0^+)=\lim\limits_{\sigma\to 0}\frac{f(0+\sigma)-f(0)}{\sigma}(\quad)=\lim\limits_{\sigma\to 0}\frac{g(\sigma)}{\sigma}=0$

$\Rightarrow f(x)$ differentiable at $x=0$

Differentiability of $h(x)x=0$

$h'(0)=1g\left(e^{|x|}\right)\forall\in R$

LHD

$$f(h(0^-))=\lim_{\sigma\to 0}\frac{f(h(0))-f(h(0-\sigma))}{\sigma}=\lim_{\sigma\to 0}\frac{g(1)-g(e^{\sigma})}{\sigma}=-g'(1)$$

RHD

$$f(h(0^+))=\lim_{\sigma\to 0}\frac{f(h(0+\sigma))-f(h(0))}{\sigma}=\lim_{\sigma\to 0}\frac{g(e^{\sigma})-g(1)}{\sigma}=-g'(1)$$

Since $g(1)\neq 0\Rightarrow f(h(x))$ is non diff. at x = 0

Differentiability of $h(f(x))$ at $x=0$

$$h(f(x))=\begin{cases}e^{|f(x)|} & x\neq 0\\ 1 & x=0\end{cases}$$

LHD $h'(f(0-\sigma))=\lim\limits_{\sigma\to 0}\frac{h(f(0))-h(f(0-\sigma))}{\sigma}$

$$=\lim_{\sigma\to 0}\frac{1-e^{|e-\sigma|}}{\left|g^{(\sigma)}\right|}.\frac{|g(-\sigma)|}{\sigma}=0$$

RHD $h'(f(0+\sigma))=\lim\limits_{\sigma\to 0}\frac{h(f(0+\sigma))-h(f(0))}{\sigma}$

$$=\lim_{\sigma\to 0}\frac{e^{|-\sigma|}-1}{\left|g^{(\sigma)}\right|}.\frac{|g(-\sigma)|}{\sigma}=0$$

Sol 42: (A, B) $f(x)=a\cos\left(\left|x^3-x\right|\right)+b|x|\sin\left(\left|x^3\right|+x\right)$

[A] If $a=0,b=1,\ f(x)=|x|\sin\left(\left|x^3=x\right|\right)$

$\Rightarrow f(x)=|x|\sin\left(\left|x^3+x\right|\right)$

Hence f(x) is differentiable.

[B] If $a=1,b=0\ f(x)=\cos\left(\left|x^3\right|-x\right)$

$\Rightarrow f(x)=\cos\left(x^3-x\right)$ Which is differentiable at x = 1 and x = 0.

Sol 43: (B, C)

$$\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{n}{r}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{n^2}{r^2}\right)}.\frac{1}{\prod_{r=1}^{n}\left(\frac{n}{r}\right)}\right]$$

$$\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{\frac{r}{n}}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{\left(\frac{r}{n}\right)^2}\right)}.\frac{1}{\prod_{r=1}^{n}\left(\frac{n}{r}\right)}\right]$$

$$=x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\ln\left(\frac{x\left(\frac{r}{n}\right)+1}{\left(x+\frac{r}{n}\right)^2+1}\right)$$

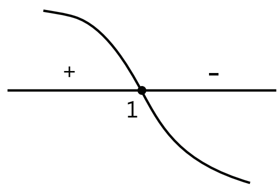
$$= x\int_0^1 \ln\left(\frac{1+tx}{1+t^2x^2}\right)dt$$

Put, tx = p, we get

$$\ln f(x) = \int_0^x \ln\left(\frac{1+p}{1+p^2}\right)dp$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$$

sign scheme of f′ (x)



Also, f′ (1) = 0

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1), f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right), f'(2) < 0$$

Also, $\frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ln\left(\frac{4}{10}\right) - \ell n\left(\frac{3}{5}\right)$

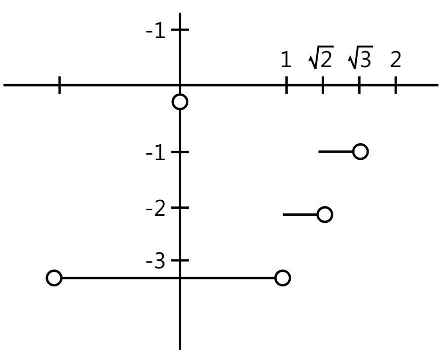
$$= \ln\left(\frac{4}{6}\right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

Sol 44: (B, C)

$$f(x) = \left[x^2 - 3\right] = \left[x^2\right] - 3$$

f(x) is discontinuous at x = 1, $\sqrt{2}$, $\sqrt{3}$, 2

$$g(x) = \left(|x| + |4-7|\right)\left(\left[x^2\right] - 3\right)$$



$$g(x) = \begin{cases} 15x - 21 & x < 0 \\ 9x - 21 & 0 \le x \le 1 \\ 6x - 14 & 1 \le x < \sqrt{2} \\ 3x - 7 & \sqrt{2} \le x < \sqrt{3} \\ 0 & \sqrt{3} \le x < 2 \\ 3 & x = 2 \end{cases}$$

$\therefore g(x)$ is not differentiable, at x = 0, 1, $\sqrt{2}$, $\sqrt{3}$

Sol 45: (A) $f'(x) + \frac{f(x)}{x} = 2$

$$\Rightarrow xf'(x) + f(x) = 2x \quad \Rightarrow \int d(x.f(x)) = \int 2x dx$$

$$\Rightarrow xf(x) = x^2 + c; \qquad f(x) = x + \frac{x}{c}\left(c \ne 0 \text{ as } f(1 \ne 1)\right)$$

For this function, only (A) is correct.

Sol 46: (B, C)

$$f(x) = x^3 + 3x + 2, \;\; f(1) = 6 \;\; g(6) = 1$$

$$g(f(x)) = x \Rightarrow g'f(x) \times f'(x) = 1$$

$$x = 0, \quad g'(f(0))f(0) = 1$$

$$g'(2) = \frac{1}{f(0)} = \frac{1}{3}$$

$$f(3) = 38$$

$$\therefore g(38) = 3$$

$$f(2) = 16 \Rightarrow g(16) = 2$$

$$\therefore h(g(g16)) = h(0)$$

$$\therefore 16 = h(g(g16)) = h(0)`$$

$$\therefore (c) \text{is correct}$$

$$f'(6) = 111, f(1) = 6 \Rightarrow g'(6)\frac{1}{2}$$

$$h(g(g(x))) = x$$

$$\Rightarrow h'(g(g(x))) \times g'(x) = 1$$

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