2.1. Introduction

Logic is concerned with the study of the principles and techniques of reasoning. The Greek philosopher and scientist Aristotle (381–322 BC) is said to be the first person to have studied logical reasoning. Logical reasoning is the essence of mathematics and is therefore an important starting point for study of discrete mathematics. Logic, among other things, have provided the theoretical basis for many areas of computer science such as digital logic design, automata theory and computability, artificial intelligence etc. One component of logic is **proposition calculus** which deals with statements with values true and false and is concerned with analysis of propositions. And the other part is **predicate calculus** which deals with the predicates which are propositions containing variables. In this chapter, we discuss a few of the basic ideas and define some of the logical concepts that are useful in computer science.

2.2. Proposition (Statement)

A **Proposition** or **Statement** is a declarative sentence that is either true or false, but not both. For example, "Three plus three equals six." and "Three plus three equals seven" are both statements, the first because it is true and the second because it is false. Similarly "$x + y > 1$" is not a statement because for some values of x and y the sentence is true, whereas for others it is false. For instance, if $x = 1$ and $y = 2$, the sentence is true, if $x = -3$ and $y = 1$, this is false. The truth or falsity of a statement is called its **truth value.** Since only two possible truth values are admitted this logic is sometimes called **two – valued logic.** Questions, exclamations and commands are not propositions. For example, consider the following sentences:

(*a*) The sun rises in the west.

(*b*) $2 + 4 = 6$

(*c*) $(5, 6) \subset (7, 6, 5)$

(*d*) Do you speak Hindi?

(*e*) $4 - x = 8$

(*f*) Close the door.

(*g*) What a hot day!

(*h*) We shall have chicken for dinner.

The sentences (*a*), (*b*) and (*c*) are statements, the first is false and second and third are true.

(*d*) is a question, not a declarative sentence, hence it is not a statement.

(*e*) is a declarative sentence, but not a statement, since it is true or false depends on the value of x.

(*f*) is not a statement, it is a command.

(*g*) is not a statement, it is exclamation.

(*h*) is a statement since it is either true or false but not both, although one has to wait until dinner to find out if it is true or false.

It is customary to represent simple propositions by letters p, q, r........... known as **proposition variables** (Note that usually a real variable is represented by the symbol x. This means that x is not a real number but can take a real value. Similarly, a propositional variable is not a proposition but can be replaced by a proposition) Propositional variables can only assume two values: 'true' denoted by T or 1 and 'false' denoted by F or 0. If p denotes the proposition "The sun sets in the east", then instead of saying the proposition is false, one can simply say the value of p is F.

2.3. Compound Proposition

A proposition consisting of only a single propositional variable or a single propositional constant is called an **atomic (primary, primitive)** proposition or simply proposition; that is they can not be further subdivided. A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition is referred to **molecular or composite or compound proposition.**

Connectives

The words and phrases (or symbols) used to form compound propositions are called **connectives.** There are five basic connectives called Negation, Conjunction, Disjunction, Conditional and Biconditional. The following symbols are used to represent connectives.

Symbol used	Connective word	Nature of the compound statement formed by the connective	Symbolic from	Negation
$\sim, \neg, \rceil. N$	not	Negation	$\sim p$	$\sim(\sim p) = p$
$\wedge$	and	Conjuction	$p \wedge q$	$(\sim p) \vee (\sim q)$
$\vee$	or	Disjunction	$p \vee q$	$(\sim p) \wedge (\sim q)$
$\rightarrow$	if..........then	Conditional	$p \rightarrow q$	$p \wedge (\sim q)$
$<->$	if and only if	Bi-conditional	$p \leftrightarrow q$	$[p \wedge (\sim q)] \vee [q \wedge (\sim p)]$

Negation

If p is any proposition, the negation of p, denoted by $\sim p$ and read as not p, is a proposition which is false when p is true and true when p is false. Consider the statement

p : Paris is in France.

Then the negation of p is the statement

$\sim p$: It is not the case that Paris is in France.

Normally it is written as

$\sim p$: Paris is not in France.

Strictly speaking, not is not a connective, since it does not join two statements and $\sim p$ is not really a compound statement. However, not is a unary operation for the collection of statements, and $\sim p$ is a statement if p is considered a statement.

Note: 1. The following propositions all have the same meaning:

p : All people are intelligent.
q : Every person is intelligent.
r : Each person is intelligent.
s : Any person is intelligent.

2. The negation of the proposition

p : All students are intelligent.

is

$\sim p$: Some students are not intelligent.
$\sim p$: There exists a student who is not intelligent.
$\sim p$: At least one student is not intelligent.

The negation of

q : No student is intelligent.

is

$\sim q$: Some students are intelligent.

Note that "No student is intelligent" is *not* the negation of p; "All students are intelligent" is *not* the negation of q.

Conjunction

If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. The truth values of $p \wedge q$ are given in the truth table shown in Table 2.1 (a).

Example 1. Form the conjunction of p and q for each of the following.

(a) p : Ram is healthy $\quad$ q : He has blue eyes

(b) p : It is cold $\quad$ q : It is raining

(c) p : $5x + 6 = 26$ $\quad$ q : $x > 3$.

Solution: (a) $p \wedge q$: Ram is healthy and he has blue eyes.

(b) $p \wedge q$: It is cold and raining.

(c) $p \wedge q$: $5x + 6 = 26$ and $x > 3$.

Remark

In logic we may combine any two sentences to form a conjunction, there is no requirement that the two sentences be related in content or subject matter. Any combinations, however absurd, are permitted, of course, we are usually not interested in sentences like 'Tapas loves Rini, and 4 is divisible by 2'.

Disjunction

If p and q are two statements, the disjunction of p and q is the compound statement denoted by $p \vee q$ and read as "p or q". The statement $p \vee q$ is true if at least one of p or q is true. It is false when both p and q are false. The truth of $p \vee q$ are given in the truth table shown in Table 2.1 (b).

Example 2. Assign a truth value to each of the following statements.

(i) $5 < 5 \vee 5 < 6$

(ii) $5 \times 4 = 21 \vee 9 + 7 = 17$

(iii) $6 + 4 = 10 \vee 0 > 2$.

Solution: (i) True, since one of its components *viz.* $5 < 6$ is true.

(ii) False, since both of its components are false.

(iii) True, since one of its components *viz.* $6 + 4 = 10$ is true.

Example 3. If p : It is cold $\quad$ and $\quad$ q : It is raining.

Write simple verbal sentence which describes each of the following statements

(a) $\sim p$ $\quad$ (b) $p \wedge q$ $\quad$ (c) $p \vee q$ $\quad$ (d) $p \vee \sim q$.

Solution. (a) $\sim p$: It is not cold.

(b) $p \wedge q$: It is cold and raining

(c) $p \vee q$: It is cold or raining.

(d) $p \vee \sim q$: It is cold or it is not raining.

2.4. Propositions and Truth Tables

A truth table is a table that shows the truth value of a compound proposition for all possible cases.

For example, consider the conjunction of any two propositions p and q. The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. There are four possible cases.

1. p is true and q is true.
2. p is true and q is false.
3. p is false and q is true.
4. p is false and q is false.

These four cases are listed in the first two coloumns and the truth values of $p \wedge q$ are shown in the third column of Table 2.1 (*a*). The truth tables for the other two connectives disjunction and negation discussed above are shown in Table 2.1 (*b*) and 2.1 (*c*).

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*)

p	q	$(q \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

(*b*)

p	$\sim p$
T	F
F	T

(*c*)

Table 2.1 Truth tables for the three propositional connectives

The first columns of the table are for the variables p, q, ... and the number of rows depends on the number of variables. For 2 variables, 4 rows are necessary ; for 3 variables, 8 rows are necessary ; and in general, for n variables, 2^n rows are required. There is then a column for each elementary stage of the construction of the proposition. The truth value at each step is determined from the previous stages by the definition of connectives. The truth value of the proposition appears in the last column.

Example 4. Construct a truth table for each compound proposition.

(*i*) $p \wedge (\sim q \vee q)$ (*ii*) $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

Solution. (*i*) Make columns labeled p, q, $\sim q$, $(\sim q \vee q)$ and $p \wedge (\sim q \vee q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim q$ and $\sim q \vee q$ columns with the appropriate truth values. Complete the table by considering the truth values of $p \wedge (\sim q \vee q)$.

p	q	$\sim q$	$\sim q \vee q$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

Table 2.2 Truth table for $p \wedge (\sim q \vee q)$

(*ii*) Set up columns labeled p, q, $\sim q$ $p \vee q$, $\sim (p \vee q)$, $(\sim p \wedge - q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim p$, $\sim q$, $p \vee q$, $\sim (p \vee q)$, $(\sim p \wedge \sim q)$ columns with appropriate truth values. Complete the table by considering the truth values of $\sim (p \vee q) \vee (\sim p \wedge \sim q)$.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$(\sim p \wedge \sim q)$	$\sim (p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

Table 2.3 Truth table for $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

2.5. Logical Equivalence

If two propositions $P(p, q, \ldots\ldots)$ and $Q(p, q, \ldots\ldots)$ where $p, q, \ldots\ldots$ are propositional variables have the same truth values in every possible case or $P \leftrightarrow Q$ is a tautology, then the propositions are called logically equivalent or simply equivalent, and denoted by

$$P(p, q, \ldots\ldots) \equiv Q(p, q, \ldots\ldots) \text{ or } P(p, q, \ldots\ldots) \Leftrightarrow Q(p, q, \ldots\ldots)$$

It is always permissible, and sometimes desirable to replace a given proposition by an equivalent one.

To test whether two propositions P and Q are logically equivalent the following steps are followed.

1. Construct the truth table for P.
2. Construct the truth table for Q using the same propositional variables.
3. Check each combinations of truth values of the propositional variables to see whether the value of P is the same as the truth value of Q. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

2.6. Algebra of Propositions

Propositions satisfy various laws which are listed in Table 2.4. These laws are useful to simplify expression. Note that, with the exception of the Involution law, all the laws of Table come in pairs, called **dual pairs.** For each expression, one finds the dual by replacing all T by F and all F by T and replacing all ^ by ∨ and all ∨ by ^.

Idempotent laws	
(1*a*) $p \vee p \equiv p$	(1*b*) $p \wedge p \equiv p$
Associative laws	
(2*a*) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2*b*) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3*a*) $p \vee q \equiv q \vee p$	(3*b*) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4*a*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4*b*) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5*a*) $p \vee \mathrm{F} \equiv p$	(5*b*) $p \wedge \mathrm{T} \equiv p$
(6*a*) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6*b*) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	
(7*a*) $p \vee \sim p \equiv \mathrm{T}$	(7*b*) $p \wedge \sim p \equiv \mathrm{F}$
(8*a*) $\sim \mathrm{T} \equiv \mathrm{F}$	(8*b*) $\sim \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\sim(\sim p) \equiv p$	
DeMorgan's laws	
(10*a*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$	(10*b*) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Table 2.4 Laws of the algebra of propositions

From the laws given in Table 2.4 one can derive further laws. Of particular importance are the **absorption laws,** which are

$$p \vee (p \wedge q) \equiv p \qquad 11\ (a)$$
$$p \wedge (p \vee q) \equiv p \qquad 11\ (b)$$

Equivalence in 11 (*a*) can be proved as follows:

$$\begin{aligned} p \vee (p \wedge q) &\equiv (p \wedge \mathrm{T}) \vee (p \wedge q) \\ &\equiv (p \wedge (\mathrm{T} \vee q) \\ &\equiv p \wedge \mathrm{T} && \text{using Identity law} \\ &\equiv p && \text{using Identity law} \end{aligned}$$

The proof of 11 (*b*) is similar. The absorption laws are very useful when expressions need to be simplified.

All the laws given in Table 2.4 can be proved with the help of truth table.

Example 5. Use truth table to prove the distributive law

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Solution. The truth table of the compound propositions is shown in the table. Since the entries in the 5th and last column of the table are the same, the two propositions are logically equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 2.5

2.7. Conditional Proposition

If p and q are proposition, the compound proposition " if p then q" denoted by $p \rightarrow q$ is called a **conditional proposition** and the connective is the **conditional connective.** The proposition p is called **antecedent** or **hypothesis,** and the proposition q is called the **consequent** or **conclusion.**

The only circumstances under which $p \rightarrow q$ is false when p is true and q is false.

Examples include

1. If tomorrow is Sunday then today is Saturday.
2. If it rains then I will carry an umbrella.

Here p : Tomorrow is Sunday is antecedent.
q : Today is Saturday is consequent.
and p : It rains is antecedent.
q : I will carry an umbrella is consequent.

The connective if .... then can also be read as follows:

1. p is sufficient for q.
2. p only if q.
3. q is necessary for p.
4. q if p.
5. q follows from p.
7. q is consequence of p.

The truth table of $p \rightarrow q$ is given in Table 2.6

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.6 Truth table of $p \rightarrow q$.

Note: Some authors call $p \rightarrow q$ as an implication.