Class 12

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16. MATRICES

1. INTRODUCTION

A rectangular array of m x n numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m × n matrix. Such an array is enclosed by [ ] or ( ) or || ||. An m × n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$$

In brief, the above matrix is represented by $A = [a_{ij}]_{m\times n}$. The number a_{11}, a_{12}, ... etc., are known as the elements of the matrix A, where a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

2. ORDER OF A MATRIX

A matrix which has m rows and n columns is called a matrix of order m × n E.g. the order of $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ matrix is 2 × 3.

Note: (a) The difference between a determinant and a matrix is that a determinant has a certain value, while the matrix has none. The matrix is just an arrangement of certain quantities.

(a) The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, then the matrix is called a real matrix.

(a) An m × n matrix has m.n elements.

Illustration 1: Construct a 3×4 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = 2i + 3j$. **(JEE MAIN)**

Sol: In this problem, i and j are the number of rows and columns respectively. By substituting the respective values of rows and columns in $a_{ij} = 2i + 3j$ we can construct the required matrix.

We have $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$; $\therefore a_{11} = 2 \times 1 + 3 \times 1 = 5$**;** $a_{12} = 2 \times 1 + 3 \times 2 = 8$

Similarly, $a_{13} = 11$, $a_{14} = 14$, $a_{21} = 7$, $a_{22} = 10$, $a_{23} = 13$, $a_{24} = 16$, $a_{31} = 9$, $a_{32} = 12$, $a_{33} = 15$, $a_{34} = 18$

$$\therefore A = \begin{bmatrix} 5 & 8 & 11 & 14 \\ 7 & 10 & 13 & 16 \\ 9 & 12 & 15 & 18 \end{bmatrix}$$

Illustration 2: Construct a 3 × 4 matrix, whose elements are given by: $a_{ij} = \frac{1}{2}|-3i + j|$ **(JEE MAIN)**

Sol: Method for solving this problem is the same as in the above problem.

Since $a_{ij} = \frac{1}{2}|-3i + j|$ we have

$a_{11} = \frac{1}{2}|-3(1) + 1| = \frac{1}{2}|-3 + 1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$

$a_{12} = \frac{1}{2}|-3(1) + 2| = \frac{1}{2}|-3 + 2| = \frac{1}{2}|-1| = \frac{1}{2}$

$a_{13} = \frac{1}{2}|-3(1) + 3| = \frac{1}{2}|-3 + 3| = \frac{1}{2}(0) = 0$

$a_{14} = \frac{1}{2}|-3(1) + 4| = \frac{1}{2}|-3 + 4| = \frac{1}{2}$; $a_{21} = \frac{1}{2}|-3(2) + 1| = \frac{1}{2}|-6 + 1| = \frac{5}{2}$

$a_{22} = \frac{1}{2}|-3(2) + 2| = \frac{1}{2}|-6 + 2| = \frac{4}{2} = 2$; $a_{23} = \frac{1}{2}|-3(2) + 3| = \frac{1}{2}|-6 + 3| = \frac{3}{2}$

$a_{24} = \frac{1}{2}|-3(2) + 4| = \frac{1}{2}|-6 + 4| = \frac{2}{2} = 1$; Similarly $a_{31} = 4$, $a_{32} = \frac{7}{2}$, $a_{33} = 3$, $a_{34} = \frac{5}{2}$

Hence, the required matrix is given by $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

3. TYPES OF MATRICES

3.1 Row Matrix

A matrix having only one row is called a row matrix. Thus $A = [a_{ij}]_{m\times n}$ is a row matrix if m = 1;

E.g. A = [1 2 4 5] is row matrix of order 1 × 4.

3.2 Column Matrix

A matrix having only one column is called a column matrix. Thus $A = [a_{ij}]_{m\times n}$ is a column matrix if

n = 1; E.g. $A = \begin{bmatrix} -1 \\ 2 \\ -4 \\ 5 \end{bmatrix}$ is column matrix of order 4 × 1.

3.3 Zero or Null Matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by 0.Thus, $A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j; E.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 × 3.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3 × 2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3 × 3 null matrix.

3.4 Singleton Matrix

If in a matrix there is only one element then it is called singleton matrix. Thus, A = $[a_{ij}]_{m\times n}$ is a singleton matrix if m = n = 1. E.g. [2], [3], [a], [–3] are singleton matrices.

3.5 Horizontal Matrix

A matrix of order m × n is a horizontal matrix if n > m; E.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

3.6 Vertical Matrix

A matrix of order m × n is a vertical matrix if m > n; E.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

3.7 Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

Thus, A = $[a_{ij}]_{m\times n}$ is a square matrix if m = n; E.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 × 3.

The sum of the diagonal elements in a square matrix A is called the trace of matrix A, and which is denoted by tr(A); $\quad \text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots\ldots a_{nn.}$

3.8 Diagonal Matrix

If all the elements, except the principal diagonal, in a square matrix are zero, it is called a diagonal matrix.

Thus, a square matrix A = $[a_{ij}]$ is a diagonal matrix if a_{ij} = 0, when i ≠ j; E.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3 × 3, which can also be denoted by diagonal [2 3 4].

3.9 Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix. Thus, a square matrix A = $[a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases}$ where k is a constant.

E.g. $\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ is a scalar Matrix.

3.10 Unit Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, then it is called a unit matrix. A unit matrix of order n is denoted by $I_{n.}$ Thus, a square matrix A = $[a_{ij}]_{n\times n}$ is a unit matrix if

$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad$ E.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: Every unit matrix is a scalar matrix.

3.11 Triangular Matrix

A square matrix is said to be a triangular matrix if the elements above or below the principal diagonal are zero. There are two types:

3.11.1 Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an upper triangular matrix, If $a_{ij} = 0$, when $i > j$.

E.g. $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper uriangular matrix of order 3×3.

3.11.2 Lower Triangular Matrix

A square matrix is called a lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3×3.

3.12 Singular Matrix

Matrix A is said to be a singular matrix if its determinant $|A| = 0$, otherwise a non-singular matrix, i.e.

If $\det |A| = 0 \Rightarrow$ Singular and $\det |A| \neq 0 \Rightarrow$ non-singular

PLANCESS CONCEPTS

- A triangular matrix $A = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ij} = 0\ \forall\ i=j$
- The multiplication of two triangular matrices is a triangular matrix.
- Every row matrix is also a horizontal matrix but not the converse. Similarly every column matrix is also a vertical matrix but not the converse.

Vaibhav Gupta (JEE 2009 AIR 54)

3.13 Symmetric and Skew Symmetric Matrices

Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i,j values;

E.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}$ is symmetric, because $a_{12} = 2 = a_{21}$, $a_{31} = 3 = a_{13}$ etc.

Note: A is symmetric $\Leftrightarrow A = A'$ (where A′ is the transpose of matrix)

Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$, for all values of i,j.

$\because\ a_{ij} = -a_{ji}$, for all i,j. $\Rightarrow a_{ii} = -a_{ii}$, [putting j = i] $\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

Thus, in a skew-symmetric matrix all diagonal elements are zero; E.g. $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Note: A square matrix A is a skew-symmetric matrix $\Leftrightarrow A' = -A$.

Few results:

(a) If A is any square matrix, then A + A′ is a symmetric matrix and A – A′ is a skew-symmetric matrix.

(b) Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(B + C)$, where B is symmetric and C is a skew symmetric matrix.

(c) If A and B are symmetric matrices, then AB is symmetric $\Leftrightarrow$ AB = BA, i.e. A & B commute.

(d) The matrix B′AB is symmetric or skew-symmetric in correspondence if A is symmetric or skew-symmetric.

(e) All positive integral powers of a symmetric matrix are symmetric.

(f) Positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

PLANCESS CONCEPTS

Elements of the main diagonal of a skew-symmetric matrix are zero because by definition $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$ for all values of i.

Trace of a skew symmetric matrix is always 0. The sum of symmetric matrices is symmetric.

Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(B + C)$, where B is symmetric and C is a skew symmetric matrix.

If A and B are symmetric matrices, then AB is symmetric $\Leftrightarrow$ AB = BA, i.e. A & B commute. The matrix B′AB is symmetric or skew-symmetric accordingly when A is symmetric or skew symmetric. All positive integral powers of a symmetric matrix are symmetric. Positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

Chen Reddy Sandeep Reddy (JEE 2012 AIR 62)

3.14 Hermitian and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be a Hermitian matrix if $a_{ij} = \overline{a}_{ji}\ \forall$ i, j; i.e. $A = A^{\theta}$

E.g. $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ are Hermitian matrices

Note: (a) If A is a Hermitian matrix then $a_{ii} = \overline{a}_{ii} \Rightarrow a_{ii}$ is real $\forall$ i, thus every diagonal element of a Hermitian matrix must be real.

(b) If a Hermitian matrix over the set of real numbers is actually a real symmetric matrix; and A a square matrix, $A = [a_{ij}]$ is said to be a skew-Hermitian if $a_{ij} = -\overline{a}_{ji}$, $\forall$ i, j;

i.e. $A^{\theta} = -A$; E.g. $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}, \begin{pmatrix} 3i & -3+2i & -1-i \\ 3-2i & -2i & -2-4i \\ 1+i & 2+4i & 0 \end{pmatrix}$ are skew-Hermitian matrices.

(c) If A is a skew-Hermitian matrix then $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii} = 0$

i.e. a_{ii} must be purely imaginary or zero.

(d) A skew-Hermitian matrix over the set of real numbers is actually is real skew-symmetric matrix.

4. TRACE OF A MATRIX

Let A = $[a_{ij}]_{n\times n}$ and B = $[b_{ij}]_{n\times n}$ and λ be a scalar,

(i) $tr(\lambda A) = \lambda\, tr(A)$ (ii) $tr(A + B) = tr(A) + tr(B)$ (iii) $tr(AB) = tr(BA)$

Square Matrix

- Triangular Matrix
  - Upper triangular Matrix
    If $a_{ij} = 0\ \forall\ i > j$
    $A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$,
  - Lower triangular Matrix
    If $a_{ij} = 0\ \forall\ i < j$
    $A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$
- Diagonal Matrix
  Atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$
  $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$
  Abbreviated as dia $(d_1, d_2, d_3, \ldots\ldots d_n)$
  - Scalar Matrix
    If $d_1=d_2=d_3 \ldots= a\neq 0$
    $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
  - Unit Matrix
    If $d_1=d_2=d_3 \ldots= 1$
    $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

5. TRANSPOSE OF A MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called the transpose of matrix A and is denoted by A^T or A′. From the definition it is obvious that if the order of A is m × n, then the order of A^T becomes n × m; E.g. transpose of matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times 3} \text{ is } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times 2}$$

5.1 Properties of Transpose of Matrix

(i) $(A^T)^T = A$ (ii) $(A \pm B)^T = A^T \pm B^T$ (iii) $(AB)^T = B^TA^T$ (iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3\ldots\ldots A_{n-1}A_n)^T = A_n^T\,A_{n-1}^T \ldots\ldots A_3^T\,A_2^T\,A_1^T$ (vi) $I^T = I$ (vii) $tr(A) = tr(A^T)$

Illustration 3: If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$ then prove that $(AB)^T = B^TA^T$. **(JEE MAIN)**

Sol: By obtaining the transpose of AB i.e. $(AB)^T$ and multiplying B^T and A^T we can easily get the result.

Here, $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1(1)-2(-1)+3(2) & 1(3)-2(0)+3(4) \\ -4(1)+2(-1)+5(2) & -4(3)+2(0)+5(4) \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 4 & 8 \end{bmatrix}$

$\therefore\ (AB)^T = \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix}$; $B^TA^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1(1)-1(-2)+2(3) & 1(-4)-1(2)+2(5) \\ 3(1)+0(-2)+4(3) & 3(-4)+0(2)+4(5) \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix} = (AB)^T$

Illustration 4: If A = $\begin{bmatrix} 5 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}$ then what is (AB′)′ is equal to? **(JEE MAIN)**

Sol: In this problem, we use the properties of the transpose of matrix to get the required result.

We have $(AB')' = (B')'\,A' = BA' = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 18 & 7 \end{bmatrix}$

Illustration 5: If the matrix A = $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is a singular matrix then find x. Verify whether $AA^T = I$ for that value of x. **(JEE ADVANCED)**

Sol: Using the condition of singular matrix, i.e. | A | = 0, we get the value of x and then substituting the value of x in matrix A and multiplying it to its transpose we will obtain the required result.

Here, A is a singular matrix if | A | = 0, i.e., $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$

or $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ 0 & -x & -x \end{vmatrix} = 0$, using $R_3 \to R_3 + R_2$ or $\begin{vmatrix} 3-x & 0 & 2 \\ 2 & 3-x & 1 \\ 0 & 0 & -x \end{vmatrix} = 0$, using $C_2 \to C_2 - C_3$

or $-x(3-x)^2 = 0, \ \therefore\ x = 0, 3.$

When x = 0, A = $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$; $\therefore\ AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & -4 \\ 2 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 17 & 16 & -16 \\ 16 & 21 & -21 \\ -16 & -21 & 21 \end{bmatrix} \neq I$

When x = 3,A = $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$ $\therefore\ AA^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -4 \\ 2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -16 \\ 4 & 6 & -12 \\ -16 & -12 & 36 \end{bmatrix} \neq I$

Note: A simple way to solve is that if A is a singular matrix then |A| = 0 and $|A^T| = 0$. But |I| is 1. Hence, $AA^T \neq I$ if |A| = 0.

Illustration 6: If the matrix A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c, are positive real numbers such that abc = 1 and $A^TA = I$ then find the value of $a^3 + b^3 + c^3$. **(JEE ADVANCED)**

Sol: Here, A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. So, $A^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, interchanging rows and columns.

$\therefore A^TA = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}^2 = A^2 \therefore |A^TA| = |A^2|$; But $A^TA = I$ (given). $\therefore |I| = |A|^2 \Rightarrow 1 = |A|^2$

Now, $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$, $R_1 \to R_1 + R_2 + R_3$

$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$, $\begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$

$= (a+b+c)\{(c-b)(b-c) - (a-b)(a-c)\}$ $\quad = (a+b+c)(-b^2 - c^2 + 2bc - a^2 + ac + ab - bc)$

$= -(a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab)$ $\quad = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a^3 + b^3 + c^3 - 3)$ ($\because abc = 1$) $\quad \therefore |A|^2 = 1 \Rightarrow (a^3 + b^3 + c^3 - 3)^2 = 1$ ... (i)

As a, b, c, are positive, $\frac{a^3 + b^3 + c^3}{3} > \sqrt[3]{a^3b^3c^3}$ ($\because abc = 1$); $\quad \therefore a^3 + b^3 + c^3 > 3$

$\therefore$ (i) $\Rightarrow a^3 + b^3 + c^3 - 3 = 1$ $\quad \therefore a^3 + b^3 + c^3 = 4$

6. MATRIX OPERATIONS

6.1 Equality of Matrices

Two matrices A and B are said to be equal if they are of the same order and their corresponding elements are equal, i.e. Two matrices $A = [a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{r\times s}$ are equal if

(a) m = r i.e. the number of rows in A = the number of rows in B.

(b) n = s, i.e. the number of columns in A = the number of columns in B

(c) $a_{ij} = b_{ij}$, for i = 1, 2, ..., m and j = 1, 2, ..., n, i.e. the corresponding elements are equal;

E.g. Matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not equal because their orders are not the same.

E.g. If $A = \begin{bmatrix} 1 & 6 & 3 \\ 5 & 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ are equal matrices then,

$a_1 = 1, a_2 = 6, a_3 = 3, b_1 = 5, b_2 = 2, b_3 = 1$.

6.2 Addition of Matrices

If $A[a_{ij}]_{m\times n}$ and $B[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A + B is a matrix, and each element of that matrix is the sum of the corresponding elements. i.e. $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Properties of Matrix Addition: If A, B and C are matrices of same order, then

(a) A + B = B + A (Commutative law),

(b) (A + B) + C = A + (B + C) (Associative law),

(c) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix,

(d) $A + (-A) = 0 = (-A) + A$, where $(-A)$ is obtained by changing the sign of every element of A which is additive inverse of the matrix,

(e) $\left.\begin{matrix} A+B=A+C \\ B+A=C+A \end{matrix}\right\} \Rightarrow B = C$

(f) $tr(A \pm B) = tr(A) \pm tr(B)$

(g) **Additive Inverse:** If $A + B = 0 = B + A$, then B is called additive inverse of A and also A is called the additive inverse of A.

(h) **Existence of Additive Identity:** Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, O is the additive identity for matrix addition.

6.3 Subtraction of Matrices

If A and B are two matrices of the same order, then we define $A - B = A + (-B)$.

6.4 Scalar Multiplication of Matrices

If $A=[a_{ij}]_{m\times n}$ is a matrix and k any number, then the matrix which is obtained by multiplying the elements of A by k is called the scalar multiplication of A by k and it is denoted by k A thus if $A = [a_{ij}]_{m\times n}$

Then $kA_{m\times n} = A_{m\times n}k = [ka_{i\times j}]$

Properties of Scalar Multiplication: If A, B are matrices of the same order and λ, μ are any two scalars then

(a) $\lambda (A + B) = \lambda A + \lambda B$ (b) $(\lambda + \mu)A = \lambda A + \mu A$ (c) $\lambda(\mu A) = (\lambda \mu A) = \mu(\lambda A)$

(d) $(-\lambda A) = -(\lambda A) = \lambda(-A)$ (e) $tr(kA) = k\, tr(A)$

6.5 Multiplication of Matrices

If A and B be any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B. If $A[a_{ij}]_{m\times n}$ and $B[b_{ij}]_{n\times p}$ then their product $AB = C = [c_{ij}]$, will be a matrix of order $m \times p$, where $(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}$

Proof: Let $A=[a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the $m \times p$ matrix $C = [c_{ij}]$ is called the product if $C_{ij}=A_iB_j$ Where A_i is the i^{th} row of A and B_j is the j^{th} column of B. Thus the product AB is obtained as following:

$$A = m \times n \qquad\qquad B = n \times p$$

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ R_i \rightarrow \\ R_m \rightarrow \end{array} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & & & & & & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \times \begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & & C_j & & C_p \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \cdots & & & & & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{ip} \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{npn} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} R_1C_1 & R_1C_2 & \dots\dots\dots & R_1C_j & R_1C_p \\ R_2C_1 & R_2C_2 & \dots\dots\dots & R_2C_j & R_2C_p \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ R_iC_1 & R_iC_2 & \dots\dots\dots & R_iC_j & R_iC_p \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ R_mC_1 & R_mC_2 & \dots\dots\dots & R_mC_j & R_mC_p \end{pmatrix} \text{ Thus } (AB)_{ij} = A_iB_j$$

$$(AB)_{ij} = [a_i1\ a_i2\ \dots\dots\ a_{ij}\dots\dots\ a_in] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{ij} \\ \dots \\ b_{nj} \end{bmatrix} = [a_i1\ b1_j + a_i2\ b2_j + \dots\dots + a_inbn_j]\ ; \qquad (AB)_{ij} = \sum_{r=1}^{n}(a_{ir}.b_{rj})$$

Properties of matrix multiplication:

(a) Matrix multiplication is not commutative in general, i.e. in general $AB \neq BA$.

(b) Matrix multiplication is associative, i.e. $(AB)C = A(BC)$.

(c) Matrix multiplication is distributive over matrix addition, i.e. $A.(B + C) = A.B + A.C$ and $(A + B)C = AC + BC$.

(d) If A is an $m \times n$ matrix, then $I_mA = A = AI_n$.

(e) The product of two matrices can be a null matrix while neither of them is null, i.e. if $AB = 0$, it is not necessary that either $A = 0$ or $B = 0$.

(f) If A is an $m \times n$ matrix and O is a null matrix then $A_{m\times n}.\ O_{n\times p} = O_{m\times p}$, i.e. the product of the matrix with a null matrix is always a null matrix.

(g) If $AB = 0$ (It does not mean that $A = 0$ or $B = 0$, again the product of two non-zero matrices may be a zero matrix).

(h) If $AB = AC \Rightarrow B \neq C$ (Cancellation Law is not applicable).

(i) $tr(AB) = tr(BA)$.

Illustration 7: If $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$ find AB and BA if possible **(JEE MAIN)**

Sol: Using matrix multiplication. Here, A is a 3×3 matrix and B is a 3×2 matrix, therefore, A and B are conformable for the product AB and it is of the order 3×2 such that

$$(AB)_{11} = \text{(First row of A) (First column of B)} = [2\ 1\ 3] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2 \times 1 + 1 \times 2 + 3 \times 4 = 16$$

$$(AB)_{12} = \text{(First row of A) (Second column of B)} = [2\ 1\ 3] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = 2 \times (-2) + 1 \times 1 + 3 \times (-3) = -12$$

$(AB)_{21}$ = (Second row of A) (First column of B)= $[3\ -2\ 1]\begin{bmatrix}1\\2\\4\end{bmatrix} = 3 \times 1 + (-2) \times 2 + 1 \times 4 = 3$

Similarly $(AB)_{22} = -11$, $(AB)_{31} = 3$ and $(AB)_{32} = -1$; $\therefore\ AB = \begin{bmatrix}16 & -12\\3 & -11\\3 & -1\end{bmatrix}$

BA is not possible since number of columns of B ≠ number of rows of A.

Illustration 8: Find the value of x and y if $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$ **(JEE MAIN)**

Sol: Using the method of multiplication and addition of matrices, then equating the corresponding elements of L.H.S. and R.H.S., we can easily get the required values of x and y.

We have, $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix} \Rightarrow \begin{bmatrix}2+y & 6+0\\0+1 & 2x+2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

Equating the corresponding elements, a_{11} and a_{22}, we get

$2 + y = 5 \Rightarrow\ y = 3;\quad 2x + 2 = 8 \quad \Rightarrow \quad 2x = 6 \quad \Rightarrow x = 3;$

Hence   x = 3 and y = 3.

Illustration 9: Find the value of a, b, c and d, if $\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix} = \begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$ **(JEE MAIN)**

Sol: As the two matrices are equal, their corresponding elements are equal. Therefore, by equating the corresponding elements of given matrices we will obtain the value of a, b, c and d.

$\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix} = \begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$ (given)

$a - b = -1$ ... (i)

$2a + c = 5$ ... (ii)

$2a - b = 0$ ... (iii)

$3c + d = 13$ ... (iv)

Subtracting equation (i) from (iii), we have a = 1;

Putting the value of a in equation (i), we have $1 - b = -1 \Rightarrow b = 2$;

Putting the value of a in equation (ii), we have $2 + c = 5 \Rightarrow c = 3$;

Putting the value of c in equation (iv), we find $9 + d = 13 \Rightarrow d =$

Hence a = 1, b = 2, c = 3, d = 4.

Illustration 10: Find x and y, if $2x + 3y = \begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}$ and $3x + 2y = \begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}$ **(JEE MAIN)**

Sol: Solving the given equations simultaneously, we will obtain the values of x and y.

We have $2x + 3y = \begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}$ ... (i)

$3x + 2y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$ ... (ii)

Multiplying (i) by 3 and (ii) by 2, we get $6x + 9y = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix}$ ... (iii)

$6x + 4y = \begin{bmatrix} 4 & -4 \\ -2 & 10 \end{bmatrix}$ ... (iv)

Subtracting (iv) from (iii), we get $5y = \begin{bmatrix} 6-4 & 9+4 \\ 12+2 & 0-10 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix}$

$\Rightarrow y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & \frac{-10}{5} \end{bmatrix} \quad \Rightarrow y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

Putting the value of y in (iii), we get $2x + 3\begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$

$\Rightarrow 2x = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix} = \begin{bmatrix} 2-\frac{6}{5} & 3-\frac{39}{5} \\ 4-\frac{42}{5} & 0+6 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix} \quad \Rightarrow x = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$

Hence $\quad x = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$ and $y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

Illustration 11: If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+z \\ 2b+4 & -21 & 0 \end{bmatrix}$ then find the values of a, b, c, x, y and z.

(JEE ADVANCED)

Sol: As the two matrices are equal, their corresponding elements are also equal. Therefore, by equating the corresponding elements of the given matrices, we will obtain the values of a, b, c, x, y and z.

$$\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+z \\ 2b+4 & -21 & 0 \end{bmatrix}$$

Comparing both sides, we get $x + 3 = 0 \quad \Rightarrow x = -3$ ... (i)

and $z + 4 = 6 \quad \Rightarrow z = 6 - 4 \quad \Rightarrow z = 2$ ... (ii)

and $2y - 7 = 3y - 2 \quad \Rightarrow 2y - 3y = -2 + 7 \quad \Rightarrow -y = 5 \quad \Rightarrow y = -5$ ... (iii)

and $a - 1 = -3 \quad \Rightarrow a = -3 + 1 \quad \Rightarrow a = -2$ ... (iv)

and $b - 3 = 2b + 4 \quad \Rightarrow b - 2b = 4 + 3 \quad \Rightarrow -b = 7 \quad \Rightarrow b = -7$ ... (v)

and $2c + 2 = 0 \quad \Rightarrow 2c + 2 = 0 \quad \Rightarrow 2c = -2 \quad \Rightarrow c = \frac{-2}{2} \quad \Rightarrow c = -1$ ... (vi)

[from(2)] Thus; a = –2, b = –7, c = –1, x =–3, y = –5 and z = 2

7. RANK OF A MATRIX

If $A = (a_{ij})_{m\times n}$ is a matrix, and B is its sub-matrix of order r, then | B |, the determinant is called r-rowed minor of A.

Definition: Let $A = (a_{ij})_{m\times n}$ be a matrix. A positive integer r is said to be a rank of A if

(a) A possesses at least one r-rowed minor which is different from zero; and

(b) Every (r + 1) rowed minor of A is zero.

From (ii), it automatically follows that all minors of higher order are zeros. We denote rank of A by $\rho(A)$

Note: The rank of a matrix does not change when the following elementary row operations are applied to the matrix:

(a) Two rows are interchanged ($R_i \leftrightarrow R_j$);

(b) A row is multiplied by a non-zero constant, ($R_i \rightarrow kR_i$, with $k \neq 0$);

(c) A constant multiple of another row is added to a given row ($R_i \rightarrow R_i, + kR_j$) where $i \neq j$.

Note: The arrow $\rightarrow$ means "replaced by".

Note that the application of these elementary row operations does not change a singular matrix to a non-singular matrix nor does a non-singular matrix change to a singular matrix. Therefore, the order of the largest non-singular square sub-matrix is not affected by the application of any of the elementary row operations. Thus, the rank of a matrix does not change by the application of any of the elementary row operations. A matrix obtained from a given matrix by applying any of the elementary row operations is said to be equivalent to it. If A and B are two equivalent matrices, we write A ~ B. Note that if A ~ B, then $\rho(A) = \rho(B)$

By using the elementary row operations, we shall try to transform the given matrix in the following

form $\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0... & * \end{pmatrix}$

Where * stands for zero or non-zero element. That is, we shall try to make a_{ii} as 1 and all the elements below a_{ij} as zero.

PLANCESS CONCEPTS

A non zero matrix A is said to have rank r, if

- Every square sub-matrix of order (r + 1) or more is singular;
- There exists at least one square sub-matrix or order r which is non singular.

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Illustration 12: For what values of x does the matrix $\begin{bmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{bmatrix}$ have the rank 2? **(JEE ADVANCED)**

Sol: The given matrix has only one 3[rd]-order minor. In order that the rank arrive at 2, we must bring about its determinant to zero. Hence, by applying the invariance method we can obtain values of x.

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0 \qquad \text{... (i)}$$

Now, using $R_1 \to R_1 - R_3$

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = \begin{vmatrix} 1+x & 0 & -1-x \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix}; \text{ using } C_3 \to C_3 + C_1 = \begin{vmatrix} 1+x & 0 & 0 \\ 1 & 7+x & 7 \\ 2 & 5 & 5+x \end{vmatrix}$$

$$= (1+x)\begin{vmatrix} 7+x & 7 \\ 5 & 5+x \end{vmatrix} = (1+x)\,[(7+x(5+x) - 35] = (1+x)(x^2+12x) = x(1+x)(x+12)$$

$\therefore$ (i) holds for $x = 0, -1, -12$

When $x = 0$, the matrix $= \begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} \neq 0$, So, the rank = 2

When $x = -1$, the matrix $= \begin{bmatrix} 2 & 5 & 2 \\ 1 & 6 & 6 \\ 2 & 5 & 2 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \neq 0$, So, the rank = 2

When $x = -12$, the matrix $= \begin{bmatrix} -9 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 5 & -9 \end{bmatrix}$ Clearly, a minor $\begin{bmatrix} -9 & 5 \\ 1 & -5 \end{bmatrix} \neq 0$, So, the rank = 2

$\therefore$ The matrix has the rank 2 if $x = 0, -1, -12$.

8. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then, $A^2 = A.A$; $A^3 = A.A.A = A^2A$. Also for any positive integers m, n:

(a) $A^mA^n = A^{m+n}$ **(b)** $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^n = I$, $I^m = I$ **(d)** $A^0 = I_n$

Matrix polynomial: If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_nx^0$, then we define a matrix polynomial a,b

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI_n$ where A is the given square matrix. If f(A) is a null matrix, then A is called the zero or root of the matrix polynomial f(A)

9. SPECIAL MATRICES

(a) **Idempotent Matrix:** A square matrix is idempotent, provided $A^2 = A$. For an idempotent matrix A, $A^n = A \; \forall \; n>2$, $n \in N \Rightarrow A^n = A$, $n \geq 2$.

For an idempotent matrix A, det A = 0 or 1 A^2, $|A|^2 = |A|$).

(b) **Nilpotent Matrix:** A nilpotent matrix is said to be nilpotent of index p, ($p \in N$), if $A^p = O$, $A^{p-1} \neq O$, i.e. if p is the least positive integer for which $A^p = O$, then A is said to be nilpotent of index p.

(c) Periodic Matrix: A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K, then A is periodic with period K, i.e. if K is the least positive integer for which $A^{K+1} = A$, and A is said to be periodic with period K. If K =1 then A is called idempotent.

E.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ has the period 1.

Note: (i) Period of a square null matrix is not defined. (ii) Period of an idempotent matrix is 1.

(d) Involutary Matrix: If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix its own inverse

E.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

PLANCESS CONCEPTS

Two matrices cannot be added if they are of different order

If A is an involutary matrix, then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $(I + A)(I - A) = 0$

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Illustration 13: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}$ and $f(x) = x^2 - 5x + 6I_3$. Find f(A). **(JEE MAIN)**

Sol: By using methods of multiplication and addition of matrices we will obtain the required result. Here $f(A) = A^2 - 5A + 6I_3$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ -5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+0\times2+1(-1) & 2\times0+0\times1+1(-1) & 2\times1+0\times3+1\times0 \\ 2\times2+1\times2+3(-1) & 2\times0+1\times1+3(-1) & 2\times1+1\times3+3\times0 \\ (-1)2+(-1)2+0(-1) & (-1)0+(-1)1+0(-1) & (-1)1+(-1)3+0\times0 \end{bmatrix} + \begin{bmatrix} 6-10 & 0-0 & 0-5 \\ 0-10 & 6-5 & 0-15 \\ 0-(-5) & 0-(-15) & 6-0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 3 & -2 & 5 \\ -4 & -1 & -4 \end{bmatrix} + \begin{bmatrix} -4 & 0 & -5 \\ -10 & 1 & -15 \\ 5 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3-4 & -1+0 & 2-5 \\ 3-10 & -2+1 & 5-15 \\ -4+5 & -1+5 & -4+6 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -7 & -1 & -10 \\ 1 & 4 & 2 \end{bmatrix}$$

Illustration 14: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such that $A^3 = 0$, but $A \neq 0$, then **(JEE MAIN)**

Sol: (a) As $A^3 = 0$, we get $|A^3| = 0$; $|A^3| = 0 \Rightarrow |A| = 0 \Rightarrow ad - bc = 0$

In this problem, $A^3 = 0$ means $|A|$ also is equal to 0; therefore, by calculating A^2 we can obtain the result.

(a) $A^2 = 0$ (b) $A^2 = A$ (c) $A^2 = I - A$ (d) None of these

Also, $A^2 = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{pmatrix} = \begin{pmatrix} a^2+ad & (a+d)b \\ (a+d)c & ad+d^2 \end{pmatrix} = (a+d)A$

If $a + d = 0$, we get $A^2 = 0$. But, if $a + d \neq 0$, then $A^3 = A^2A = (a + d)A^2 \Rightarrow 0 = (a + d)A^2 \Rightarrow A^2 = 0$

Illustration 15: If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction. **(JEE MAIN)**

(a) $A^n = nA + (n-1)I$ (b) $A^n = 2^{n-1}A + (n-1)I$

(c) $A^n = nA - (n-1)I$ (d) $A^n = 2^{n-1}A - (n-1)I$

Sol: By substituting $n = 2$ we can determine the correct answer.

For $n = 2$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ For $n = 2$, RHS of (a) $= 2A + I = 3\begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \neq A^2$

For $n = 2$, RHS of (b) $= 2A + I \neq A^2$ So possible answer is (c) or (d)

In fact $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ which equals $nA - (n-1)I$;

Alternatively. Write $A = I + B$ Where $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

As $B^2 = 0$, we get $B^r = 0\ \forall r \geq 2$

By the binomial theorem, $A^n = I + nB = I + n(A - I) = nA - (n-1)I$

10. ADJOINT OF A MATRIX

Let the determinant of a square matrix A be $|A|$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The matrix formed by the cofactors of the elements in $|A|$ is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Where $A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{23}.\,a_{32}$

$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -a_{21}.\,a_{33} + a_{23}.\,a_{31}$; $A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{22}\,a_{31}$;

$A_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -a_{12}\,a_{33} + a_{13}.\,a_{32}$; $A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\,a_{33} - a_{13}.\,a_{31}$;

$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -a_{11}\, a_{32} + a_{12}.\, a_{31};\ A_{31} = (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\, a_{23} - a_{13}.\, a_{22};$

$A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -a_{11}\, a_{23} + a_{13}\, a_{21};\ A_{33} = (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}.\, a_{21};$

Then the transpose of the matrix of co-factors is called the adjoint of the matrix A and is written as adj A.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant A.

Let A be a square matrix, then (Adjoint A). A = A. (Adjoint A) = | A |. I

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$$A.\,(\text{adj. } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I.$$

Illustration 16: If $A = \begin{bmatrix} x & 3 & 2 \\ -3 & y & -7 \\ -2 & 7 & 0 \end{bmatrix}$ and A = – A′, then x + y is equal to

(a) 2 (b) –1 (c) 0 (d) 12 **(JEE MAIN)**

Sol: (c) A = – A′; $\Leftrightarrow$ A is skew-symmetric matrix; $\Rightarrow$ diagonal elements of A are zeros

$\Rightarrow x = 0, y = 0;$ $\therefore$ $x + y = 0$

Illustration 17: If A and B are two skew-symmetric matrices of order n, then, **(JEE MAIN)**

(a) AB is a skew-symmetric matrix (b) AB is a symmetric matrix

(c) AB is a symmetric matrix if A and B commute (d) None of these

Sol: (c) We are given A′ = – A and B′ = – B; Now, (AB)′ = B′A′ = (–B) (–A) = BA = AB if A and B commute.

Illustration 18: Let A and B be two matrices such that AB′ + BA′ = O. If A is skew symmetric ,then BA **(JEE MAIN)**

(a) Symmetric (b) Skew symmetric (c) Invertible (d) None of these

Sol: (c) we have, (BA)′ = A′B′ = – AB′ [$\because$ A is skew symmetric]; = BA′ = B(–A) = – BA $\Rightarrow$ BA is skew symmetric.

Illustration 19: Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$, then the co-factors of elements of A are given by - **(JEE MAIN)**

Sol: Co-factors of the elements of any matrix are obtain by eliminating all the elements of the same row and column and calculating the determinant of the remaining elements.

$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 4 = -7$

$A_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1, \quad A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1; \qquad A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1; \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$\therefore$ Adj A = $\begin{vmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix}$

Illustration 20: Which of the following statements are false – **(JEE MAIN)**

(a) If | A | = 0, then | adj A | = 0;

(b) Adjoint of a diagonal matrix of order 3 × 3 is a diagonal matrix;

(c) Product of two upper triangular matrices is a upper triangular matrix;

(d) adj (AB) = adj (A) adj (B);

Sol: (d) We have, adj (AB) = adj (B) adj (A) and not adj (AB) = adj (A) adj (B)

11. INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix)

Then B is called the inverse of A, i.e. B = A^{-1} and A is the inverse of B. Condition for a square matrix A to possess an inverse is that the matrix A is non-singular, i.e., | A | ≠ 0. If A is a square matrix and B is its inverse then AB = I. Taking determinant of both sides | AB | = | I | or | A | | B | = I. From this relation it is clear that | A | ≠ 0, i.e. the matrix A is non-singular.

To find the inverse of matrix by using adjoint matrix:

We know that, A. (Adj A) = | A | I or $\frac{A.(\text{Adj}A)}{|A|} = I$ (Provided | A | ≠ 0)

and A. A^{-1} = I; $\therefore$ $A^{-1} = \frac{1}{|A|}$ (Adj. A)

Illustration 21: Let A = $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$. What is inverse of A ? **(JEE MAIN)**

Sol: By using the formula $A^{-1} = \frac{\text{adj}A}{|A|}$ we can obtain the value of A^{-1}.

We have $A_{11} = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} = 2$ $A_{12} = -\begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix} = 21$

And similarly $A_{13} = -18, A_{31} = 4, A_{32} = -8, A_{33} = 4, A_{21} = +6, A_{22} = -7, A_{23} = 6$

$\therefore$ adj $A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$ Also $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4\times(-7) - (-6)\times 5 - 3\times(-6)\}$

$= -28 + 30 + 18 = 20$ $\therefore$ $A^{-1} = \dfrac{\text{adj} A}{|A|} = \dfrac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Illustration 22: If the product of a matrix A and $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ is the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then A^{-1} is given by: **(JEE MAIN)**

(a) $\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -2 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$ (d) None of these

Sol: (a) We know if AB = C, then $B^{-1}A^{-1} = C^{-1} \Rightarrow A^{-1} = BC^{-1}$ by using this formula we will get value of A^{-1} in the above problem.

Here, $A\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$

Illustration 23: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Prove that $(AB)^{-1} = B^{-1}A^{-1}$. **(JEE ADVANCED)**

Sol: By obtaining | AB | and adj AB we can obtain $(AB)^{-1}$ by using the formula $(AB)^{-1} = \dfrac{\text{adj}AB}{|AB|}$. Similarly we can also obtain the values of B^{-1} and A^{-1}.Then by multiplying B^{-1} and A^{-1} we can prove the given problem.

Here, $AB = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2+1 & 4+3-1 & 10+1-1 \\ 0+2+0 & 0+3+0 & 0+1+0 \\ 1+6+1 & 2+9-1 & 5+3-1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{bmatrix}$

Now, $|AB| = \begin{vmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{vmatrix} = 5(21-10) - 6(14-8) + 10(20-24) = 55 - 36 - 40 = -21.$

The matrix of cofactors of | AB | is $= \begin{bmatrix} 3(7)-1(10) & -\{2(7)-8(1)\} & 2(10)-3(8) \\ -\{6(7)-10(10)\} & 5(7)-8(10) & -\{5(10-6(8)\} \\ 6(1)-10(3) & -\{5(1)-2(10)\} & 5(3)-6(2) \end{bmatrix} = \begin{bmatrix} 11 & -6 & -4 \\ 58 & -45 & -2 \\ -24 & 15 & 3 \end{bmatrix}$

$\therefore$ adj $AB = \begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$; $\therefore$ $(AB)^{-1} = \dfrac{\text{adj}AB}{|AB|} = \dfrac{-1}{21}\begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$

Next, $|B| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21$

$\therefore B^{-1} = \dfrac{\text{adj } B}{|B|} = \dfrac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$; $\quad |A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 1(-2+1) = -1$

$\therefore A^{-1} = \dfrac{\text{adj} A}{|A|} = \dfrac{1}{-1}\begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$; $\therefore B^{-1}A^{-1} = -\dfrac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$

$= -\dfrac{1}{21}\begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$ $\qquad$ Thus, $(AB)^{-1} = B^{-1}A^{-1}$

Illustration 24: If $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$, then **(JEE ADVANCED)**

(a) $x = \pm 1/\sqrt{6}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$ $\qquad$ (b) $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$

(c) $x = \pm 1/\sqrt{6}$, $y = \pm 1/\sqrt{2}$, $z = \pm 1/\sqrt{3}$ $\qquad$ (d) $x = \pm 1/\sqrt{2}$, $y = \pm 1/3$, $z = \pm 1/\sqrt{2}$

Sol: (b) Given that $A' = A^{-1}$ and we know that $AA^{-1} = I$ and therefore $AA' = I$. Using the multiplication method we can obtain values of x, y and z.

$A' = A^{-1} \Leftrightarrow AA' = I$

Now, $AA' = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix}$

Thus, $AA' = I$ $\qquad \Rightarrow 4y^2 + z^2 = 1$, $2y^2 - z^2 = 0$, $\qquad x^2 + y^2 + z^2 = 1$, $x^2 - y^2 - z^2 = 0$

$\therefore \quad x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$

Illustration 25: If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then **(JEE ADVANCED)**

(a) $x = 1$, $y = -1$ $\qquad$ (b) $x = -1$, $y = 1$

(c) $x = 2$, $y = -1/2$ $\qquad$ (d) $x = 1/2$, $y = \dfrac{1}{2}$

Sol: (a) We know $AA^{-1} = I$, hence by solving it we can obtain the values of x and y.

We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & y+1 \\ 0 & 1 & 2(y+1) \\ 4(1-x) & 3(x-1) & 2+xy \end{bmatrix}$$

$\Rightarrow \quad 1 - x = 0$, $x - 1 = 0$; $y + 1 = 0$, $y + 1 = 0$, $2 + xy = 1$; $\qquad \therefore \quad x = 1$, $y = -1$

12. SYSTEM OF LINEAR EQUATIONS

Let the equations be

$a_1x + a_2y + a_3z = d_1$

$b_1x + b_2y + b_3z = d_2$

$c_1x + c_2y + c_3z = d_3$

We write the above equations in the matrix form as follows

$$\begin{bmatrix} a_1x+a_2y+a_3z \\ b_1x+b_2y+b_3z \\ c_1x+c_2y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \Rightarrow AX = B \quad \text{... (i)}$$

Where, $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Multiplying (i) by A^{-1}, we get $A^{-1}AX = A^{-1}B \Rightarrow I.\ X = A^{-1}B \Rightarrow X = A^{-1}B$

12.1 Solution to a System of Equations

A set of values of x, y, z which simultaneously satisfy all the equations is called a solution to the system of equations.

Consider, $x + y + z = 9$ $\quad 2x - y + z = 5$ $\quad 4x + y - z = 7$

Here, the set of values – $x = 2, y = 3, z = 4$, is a solution to the system of linear equations.

Because, $2 + 3 + 4 = 9$ $\quad 4 - 3 + 4 = 5$ $\quad 8 + 3 - 4 = 7$

12.2 Consistent Equations

If the system of equations has one or more solution, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations. For example, the system of linear equations $x + 3y = 5$ $x - y = 1$ is consistent, because $x = 2, y = 1$ is a solution to it. However, the system of linear equations $x + 3y = 5$ $2x + 6y = 8$ is inconsistent, because there is no set of values of x and y which may satisfy the two equations simultaneously.

Condition for consistency of a system of linear equation $AX = B$

(a) If $|A| \neq 0$, then the system is consistent and has a unique solution, given by $X = A^{-1}B$

(b) If $|A| = 0$, and $(\text{Adj } A)\, B \neq 0$ then the system is inconsistent.

(c) If $|A| = 0$, and $(\text{Adj } A)\, B = 0$, then the system is consistent and has infinitely many solutions.

Note, $AX = 0$ is known as homogeneous system of linear equations, here $B = 0$. A system of homogeneous equations is always consistent.

The system has non-trivial solution (non-zero solution), if $|A| = 0$

Theorem 1: Let $AX = B$ be a system of linear equations, where A is the coefficient matrix. If A is invertible then the system has a unique solution, given by $X = A^{-1}B$

Proof: $AX = B$; Multiplying both sides by A^{-1}. Since A^{-1} exists $\Rightarrow |A| \neq 0$

$\Rightarrow A^{-1}AX = A^{-1}B \quad \Rightarrow IX = A^{-1}B \quad \Rightarrow X = A^{-1}B$

Thus, the system of equations $AX = B$ has a solution given by $X = A^{-1}B$

Uniqueness: If $AX = B$ has two sets of solutions X_1 and X_2, then

$AX_1 = B$ and $AX_2 = B$ (Each equal to B) $\Rightarrow AX_1 = AX_2$

By cancellation law, A being invertible $\Rightarrow X_1 = X_2$

Hence, the given system AX = B has a unique solution. **Proved**

Note: A homogeneous system of equations is always consistent.

Illustration 26: Let A = $\begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix}$, B = $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and C = $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. If AB = C. Then find the matrix A^2 **(JEE MAIN)**

Sol: By solving AB = C we get the values of x and y. Then by substituting these values in A we obtain A^2.

Here $\begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2(x+y)-y \\ 2x.2-(x-y) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow 2(x + y) - y = 3$ and $4x - (x - y) = 2$

$\Rightarrow 2x + y = 3$ and $3x + y = 2$ Subtracting the two equations, we get, x = –1, So, y = 5.

$\therefore A = \begin{bmatrix} -1+5 & 5 \\ 2(-1) & -1-5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}$; $\therefore A^2 = \begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 4\times4+5(-2) & 4\times5+5(-6) \\ -2\times4+(-6)(-2) & -2\times5+(-6)(-6) \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 4 & 26 \end{bmatrix}$

Illustration 27: Solve the following equations by matrix inversion

$2x + y + 2z = 0 \qquad 2x - y + z = 10 \qquad x + 3y - z = 5$ **(JEE ADVANCED)**

Sol: The given equation can be written in a matrix form as AX = D and then by obtaining A^{-1} and multiplying it on both sides we can solve the given problem.

$\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$ $\therefore$ AX = D where $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$\Rightarrow A^{-1}(AX) = A^{-1}D \Rightarrow (A^{-1}A)X = A^{-1}D \qquad \Rightarrow IX = A^{-1}D \Rightarrow X = A^{-1}D$ ...(i)

Now $A^{-1} = \dfrac{\text{adj}A}{|A|}$; $|A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1 - 3) - 1(-2 - 1) + 2(6 + 1) = 13$

The matrix of co-factors of | A | is $\begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & -4 \end{bmatrix}$. So, adj A = $\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$. $\therefore$ from (i), $X = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}$; $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix} \Rightarrow x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$

Illustration 28: If $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then matrix A equals:

(a) $\begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 1 \\ 34 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 3 \\ 13 & 8 \end{bmatrix}$ **(JEE ADVANCED)**

Sol: (a) We know that if XAY = I, then $A = X^{-1} Y^{-1} = (YX)^{-1}$.

In this case $YX = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}$; $\therefore A = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$

Illustration 29: The system of equations $\begin{pmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$ has no solution if a and b are

(a) $a = -3, b \neq 1/3$ (b) $a = 2/3, b \neq 1/3$

(c) $a \neq 1/4, b = 1/3$ (d) $a \neq -3, b \neq 1/3$ **(JEE ADVANCED)**

Sol: By applying row operation in the given matrices and comparing them we can obtain the required result.

(a) The augmented matrix is given by $(A|B) = \left(\begin{array}{ccc|c} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right)$

Applying $R_1 \to 2R_1 - R_2$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 4 & -7 & 2b-3 \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right)$

Applying $R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 4 & -7 & 2b-3 \\ 0 & -28 & 44 & 18-10b \\ 0 & -7 & a+14 & 5-4b \end{array}\right)$

The system of equations will have no solution if $\frac{-28}{-7} = \frac{44}{a+14} \neq \frac{18-10b}{5-4b}$

$\Rightarrow a + 14 = 11$ and $20 - 16b \neq 18 - 10b$

$\Rightarrow a = -3$ and $b \neq -1/3$.

Illustration 30: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and

$Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ equals:

(a) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **(JEE ADVANCED)**

Sol: (c) Adding Au_1 and Au_2 we get $A(u_1 + u_2)$. Then using the invariance method we obtain $u_1 + u_2$.

By adding, we have $A(u_1 + u_2) = Au_1 + Au_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

We then solve the above equation for $u_1 + u_2$, if we consider the augmented matrix $(A|B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right)$

Applying $R_3 \to R_3 - 2R_2 + R_1$ and $R_2 \to R_2 - 2R_1$, we get $(A|B) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow u_1 + u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

PROBLEM-SOLVING TACTICS

If A, B are square matrices of order n, and I_n is a corresponding unit matrix, then

(a) $A(\text{adj}.A) = |A| I_n = (\text{adj } A) A$

(b) $|\text{adj } A| = |A|^{n-1}$ (Thus A (adj A) is always a scalar matrix)

(c) $\text{adj}(\text{adj}.A) = |A|^{n-2} A$

(d) $|\text{adj}(\text{adj}.A)| = |A|^{(n-1)^2}$

(e) $\text{adj}(A^T) = (\text{adj } A)^T$

(f) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(g) $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in N$

(h) $\text{adj}(kA) = k^{n-1}(\text{adj}.A)$, $k \in R$

(i) $\text{adj}(I_n) = I_n$

(j) $\text{adj } 0 = 0$

(k) A is symmetric $\Rightarrow$ adj A is also symmetric

(l) A is diagonal $\Rightarrow$ adj A is also diagonal

(m) A is triangular $\Rightarrow$ adj A is also triangular

(n) A is singular $\Rightarrow |\text{adj } A| = 0$

FORMULAE SHEET

(a) Types of matrix:

(i) Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i,j.

(ii) Skew-Symmetric Matrix: when $a_{ij} = -a_{ji}$

(iii) Hermitian and skew – Hermitian Matrix: $A = A^{\theta}$ (Hermitian matrix)

$A^{\theta} = -A$ (skew-Hermitian matrix)

(iv) Orthogonal matrix: if $AA^T = I_n = A^TA$

(v) Idempotent matrix: if $A^2 = A$

(vi) Involuntary matrix: if $A^2 = I$ or $A^{-1} = A$

(vii) Nilpotent matrix: if $\exists\ p \in N$ such that $A^P = 0$

(b) Trace of matrix:

(i) $tr(\lambda A) = \lambda\ tr(A)$

(ii) $tr(A + B) = tr(A) + tr(B)$

(iii) $tr(AB) = tr(BA)$

(c) Transpose of matrix:

(i) $(A^T)^T = A$ **(ii)** $(A \pm B)^T = A^T \pm B^T$ **(iii)** $(AB)^T = B^TA^T$ **(iv)** $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 \ldots\ldots\ldots A_{n-1}A_n)^T = A_n^T\ A_{n-1}^T \ldots\ldots\ A_3^T\ A_2^T\ A_1^T$ **(vi)** $I^T = I$ **(vii)** $tr(A) = tr(A^T)$

(d) Properties of multiplication:

(i) $AB \neq BA$ **(ii)** $(AB)C = A(BC)$ **(iii)** $A.(B + C) = A.B + A.C$

(e) Adjoint of a Matrix:

(i) $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$ **(ii)** $|\text{adj } A| = |A|^{n-1}$

(iii) $(\text{adj } AB) = (\text{adj } B)\ (\text{adj } A)$ **(iv)** $\text{adj } (\text{adj } A) = |A|^{n-2}$

(v) $(\text{adj } KA) = K^{n-1}(\text{adj } A)$

(e) Inverse of a matrix: A^{-1} exists if A is non singular i.e. $|A| \neq 0$

(i) $A^{-1} = \dfrac{1}{|A|}$ (Adj. A) **(ii)** $A^{-1}A = I_n = AA^{-1}$

(iii) $(A^T)^{-1} = (A^{-1})^T$ **(iv)** $(A^{-1})^{-1} = A$

(v) $|A^{-1}| = |A|^{-1} = \dfrac{1}{|A|}$

Solved Examples

JEE Main/Boards

Example 1: If $\begin{bmatrix} x-y & 2x+z \\ 3x+y & 3z+4w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & 25 \end{bmatrix}$,

find x, y, z, w.

Sol. We know that in equal matrices the corresponding elements are equal. Therefore, by equating the elements of these two matrices which have the same number of rows and columns, we get the value of x, y, z and w.

Given $\begin{bmatrix} x-y & 2x+z \\ 3x+y & 3z+4w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & 25 \end{bmatrix}$

$x - y = -1$,

$2x + z = 5$;

$3x + y = 5$,

$3z + 4w = 25$

By solving these equations, we get

$x = 1, y = 2, z = 3, w = 4$

Example 2: Show that the matrix

$\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3

Sol: Value of the index at which all elements of the matrix become 0, i.e. null matrix, is called the nilpotent matrix of that index. Here we calculate the n^{th} power of the matrix, where n =1, 2, 3, ... The value of n at which the matrix becomes null matrix is the index value.

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$\Rightarrow A^2 = A \times A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} =$

$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

Similarly,

$\therefore A^3 = A^2 . A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} =$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow A^3 = 0$.

i.e. $A^k = 0$

Here k = 3

Hence, A is nilpotent matrix of index 3

Example 3: Solve the following system of homogeneous equations:

$2x + 3y - z = 0$, $x - y - 2z = 0$ and

$3x + y + 3z = 0$

Sol: In this problem we can write the given homogeneous equations in a matrix form, i.e. [A][X] = [O] and then by calculating the determinant of matrix A we can find if that given system has a trivial solution or not.

The given system can be written as

$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or AX = O

Where, $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}$

$= 2(-3 + 2) - 3(3 + 6) - 1(1 + 3)$

$= -2 - 27 - 4 = -33 \neq 0$

Thus $|A| \neq 0$.

So the given system has only the trivial solution given by $x = y = z = 0$

Example 4: Find x, y, z and a for which

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Sol: We know that, in equal matrices the corresponding elements are equal. Therefore by equating the elements of these two matrices which have the same number of rows and columns we get the values of x, y, z and w.

Given, $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

We know, that for equal matrices the corresponding elements are equal, therefore

$$x+3=0;$$
$$2y+x=-7;$$
$$z-1=3;$$
$$4a-6=2a;$$

By solving these equations, we get

$\therefore$ x = – 3, z = 4, y = – 2, a = 3.

Example 5: Compute the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$$

Sol: For this problem, we use the formula to get the co-factors of all the elements of matrix A. Then by taking the transpose of the co-factor matrix we can get the adjoint of matrix A.

Consider C_{ij} be a co-factor of a_{ij} in matrix A.

Then the co-factors of the elements of A are given by

$$C_{11} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = 0 - 6 = -6,$$

$$C_{12} = -\begin{vmatrix} 3 & 6 \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{13} = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$$

$$C_{21} = -\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} = -(0-5) = 5$$

$$C_{22} = \begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{23} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -(1-0) = -1,$$

$$C_{31} = \begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = (24-10) = 14,$$

$$C_{32} = -\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = -(6-15) = 9,$$

$$C_{33} = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = (2-12) = -10,$$

$$\therefore \text{adj } A = \begin{bmatrix} -6 & 0 & 3 \\ 5 & 0 & -1 \\ 14 & 9 & -10 \end{bmatrix}^T = \begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix}$$

Example 6: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, Then find $\lim\limits_{n\to\infty} \frac{1}{n} A^n$.

Sol: For this problem, we first have to calculate the n^{th} power of matrix A, i.e. A^n, and multiply the matrix A^n by $\frac{1}{n}$.

Then, by with the given limit we can find the solution of this problem.

Given $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^n = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

$$\Rightarrow \frac{1}{n} A^n = \begin{bmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{bmatrix}$$

But $-1 \le \cos n\theta, \sin n\theta \le 1$;

$$\therefore \lim_{n\to\infty} \frac{\cos n\theta}{n} = 0,$$

$$\lim_{n\to\infty} \frac{\sin n\theta}{n} = 0,$$

$$\therefore \lim_{n\to\infty} \frac{1}{n} A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 7: A trust fund has Rs. 50,000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix multiplication determine how to divide by Rs, 50,000 among the two types of bonds so as to obtain an annual total interest of Rs. 2,780.

Sol: In this problem, investment amounts can be written in the form of a row matrix and interest amounts can be written in the form of column matrix. By multiplying these two matrix we will get the equation for annual interest rates. By equating this to the given annual interest value we will get the required answer.

Consider investment of first type of bond = Rs. x

And second type of bond = Rs. 50,000 – x

These amounts can be written in the form of a row matrix A which is given by

$$A = \begin{bmatrix} x & 50000 - x \end{bmatrix}_{1\times2}$$

The interest amounts per rupee, per year from the two bonds are Rs. $\frac{5}{100}$ and $\frac{6}{100}$ which can be written in the form of a column matrix B which is given by

$$B = \begin{bmatrix} \frac{5}{100} \\ \frac{6}{100} \end{bmatrix}_{2\times1}$$

∴ The total interest per year is given by

$$A.B = [x\ 50,000 - x] \times \begin{bmatrix} \frac{5}{100} \\ \frac{6}{100} \end{bmatrix}$$

= [x. 5/100 + (50,000 – x). 6/100]

= [3000 – x/100]

Since the required total annual interest is

= Rs.2,780. ∴ [3000 – x/100] = [2780]

⇒ 3000 – x/100 = 2780

⇒ x = 100(3000 – 2780) = 22,000

Hence the required amounts to be invested in the two bonds are Rs. 22,000 and Rs. (50,000 – 22,000), i.e. Rs. 22,000 and Rs. 28,000 respectively.

Example 8: If $f(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and if α, β, γ are the angles of a triangle, then prove that f(α). f(β) , f(γ) = $-I_2$

Sol: In this problem, by the methods of substitution and multiplication of matrices we can easily prove the given equation.

Given that $f(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore f(\beta) = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$ and

$$f(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

$$f(\alpha)f(\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

Similarly f(α) f(β) f(γ)

$$= \begin{bmatrix} \cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) \\ -\sin(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) \end{bmatrix}$$

$= \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix}$ and as α + β + γ = p

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_2$$

Example 9: If $M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$M(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

then prove that $[M(\alpha)\ M(\beta)]^{-1} = M(-\beta)\,M(-\alpha)$

Sol: In this problem, by finding the inverse of the matrix we can easily get the required answer.

$[M(\alpha)\ M(\beta)]^{-1} = M(\beta)^{-1}\ M(\alpha)^{-1}$

Given $M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$M(\alpha)^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can also write this in the form

$$\begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = M(-\alpha)$$

Similarly,

$$M(\beta)^{-1} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$= \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix} = M(-\beta)$$

$\therefore\ [M(\alpha)\ M(\beta)]^{-1} = M(-\beta)\ M(-\alpha)$

Example 10: Show that the homogeneous system of equations $x - 2y + z = 0$, $x + y - z = 0$, $3x + 6y - 5z = 0$ has a non-trivial solution, Also, find the solution.

Sol: In this problem we can write the given homogeneous equations in a matrix form, i.e. [A][X] = [O] and then by calculating the determinant of matrix A we can find if that given system has a non- trivial solution or not.

The given equations are

$x - 2y + z = 0,$

$x + y - z = 0,$

$3x + 6y - 5z = 0,$

We can write these equations in the form of matrices as shown below

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } AX = O, \text{ where}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{vmatrix}$$

$= 1(-5 + 6) + 2(-5 + 3) + 1(6 - 3) = 0$

Thus, $|A| = 0$

Hence, the given system of equations has a non-trivial solution.

To find the solution, we take $z = k$ in the first two equations and write them as follows:

$x - 2y = -k$ and $x + y = k$

$$\text{or } \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix} \quad \text{or } AX = B,$$

$$\text{where } A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} -k \\ k \end{bmatrix} \quad \text{Now, } |A| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0.$$

$$\text{So } A^{-1} \text{ exists;} \quad \text{We have, adj } A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Now $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} k/3 \\ 2k/3 \end{bmatrix}$$

$\Rightarrow x = k/3,\ y = 2k/3$

These values of x, y and z also satisfy the third equation. Hence $x = k/3$, $y = 2k/3$ and $z = k$, where k is any real number and which satisfy the given system of equations.

JEE Advanced/Boards

Example 1: Let A and B be symmetric matrices of the same order. Then show that

(i) A + B is symmetric

(ii) AB – BA is skew-symmetric

(iii) AB + BA is symmetric

Sol: In this problem, by using the conditions for symmetric and skew-symmetric matrices we can get the required result.

As given, A and B are symmetric.

$\therefore\ A' = A$ and $B' = B$

(i) $(A + B)' = A' + B' = A + B$

$\therefore$ A + B is symmetric

(ii) $(AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B'$ [by reversal law]

$= BA - AB$ [$A' = A$, $B' = B$]

$\therefore$ AB – BA is skew-symmetric

(iii) $(AB + BA)' = (AB)' + (BA)'$

$= B'A' + A'B' = BA + AB = AB + BA$

$\therefore$ AB + BA is symmetric.

Example 2: Solve the following equations:

$2x - 3y + z = 9$ $x + y + z = 6$ $x - y + z = 2$

Sol: In this problem, we can write the given homogeneous equations in a matrix form, i.e. [A][X] = [O]. Then, by calculating the determinant of matrix A and adjoint of A, we get an inverse of matrix A, i.e. A^{-1}. By multiplying this into [A][X] = [O] we get the required values of x, y, and z.

We can also find if that given system has a trivial solution or not.

As given, $2x - 3y + z = 9$

$x + y + z = 6$

$x - y + z = 2$

This system can be written as AX = B,

Where, $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$|A|=2(2)+3(0)+1(-2) = 2$

$$\text{Adj } A = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & -1 \\ -2 & -1 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj } A$$

Now, $X = A^{-1}B$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & -1 \\ -2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 22 \\ 4 \\ -14 \end{bmatrix}$$

$\therefore$ x =11, y=2, z=-7 is the solution.

Example 3: Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$,

prove that $A^2 - 4A - 5I = 0$, hence obtain A^{-1}:

Sol: In this problem, by using a simple multiplication method we can get the matrix A^2, then by substituting these in the given equation we will easily obtain the required result.

$$A^2 = A.\,A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

Now $A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0 [Here 0 is the zero matrix]

Thus $A^2 - 4A - 5I = O$

$\therefore A^{-1} A^2 - 4A^{-1} A - 5A^{-1}I = A^{-1}O = O$

or $(A^{-1}A)A - 4(A^{-1}A) - 5A^{-1}I = O$;

or $IA - 4I - 5A^{-1} = O$; $\therefore 5A^{-1} = A - 4I$

$$= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

Example 4: Find the product of two matrices

A and B where $A = \begin{pmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ and use it for solving the equations

$x + y + 2z = 1$, $3x + 2y + z = 7$ and $2x + y + 3z = 2$

Sol: As the given system of equations is in the form BX = C, multiplying it by B^{-1}, which is obtained by the multiplication of AB, we can get the required result.

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Also the given system of equations in matrix form is BX = C ... (ii)

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$

From (ii), $X = B^{-1}C$

[Multiplying both sides of (ii) by B^{-1}

$\therefore B^{-1} B = I$

From (1), $AB = 4I_3 \therefore \frac{A}{4}. B = I_3$

$$\therefore B^{-1} = \frac{A}{4} = \begin{bmatrix} -5/4 & 1/4 & 3/4 \\ 7/4 & 1/4 & -5/4 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = B^{-1}C$$

$$= \begin{bmatrix} -5/4 & 1/4 & 3/4 \\ 7/4 & 1/4 & -5/4 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \begin{pmatrix} -\frac{5}{4}+\frac{7}{4}+\frac{6}{4} \\ \frac{7}{4}+\frac{7}{4}-\frac{10}{4} \\ \frac{1}{4}-\frac{7}{4}+\frac{2}{4} \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \therefore x = 2, y = 1, z = -1$$

Example 5: Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Sol: Pre-multiplying both sides by B^{-1} and Post-multiplying both sides by A^{-1} in

$BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ we can find P.

Given $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$B^{-1}BPA\,A^{-1} = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}A^{-1}$$

$$\Rightarrow IPI = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}A^{-1}$$

$$\Rightarrow P = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}A^{-1} \quad ...(i)$$

To find B^{-1}, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

$$|B| = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1 \neq 0$$

Let C be the matrix of co-factors of elements in | B |;

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$\therefore C_{11} = 4\; C_{12} = -3\; C_{21} = -3\; C_{22} = 2$

$$\therefore C = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{adjB}{|B|} = \frac{C'}{-1} = -C'$$

$$= -\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \quad ... (ii)$$

To Find A^{-1}, Since $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$\therefore |A| = 1(4-3) - 1(2-2) + 1(6-8)$

$= 1 - 0 - 2 = -1 \neq 0$

Let C be the matrix of co-factors of elements in | A |;

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ -3 & 1 & 2 \end{bmatrix}; \quad \therefore C' = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj} A}{|A|} = -\text{Adj}, A$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \qquad \text{... (iii)}$$

Substituting eq. (ii) and (iii) in eq. (i), we get

$$P = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 4+0-8 & 8+3-4 & -12-3+8 \\ -3-0+6 & -6-2+3 & 9+2-6 \end{bmatrix}$$

$$P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$$

Example 6: If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that F(x). F(y) = F(x+ y).

Hence, prove that $[F(x)]^{-1} = F(-x)$.

Sol: By substituting x and y in place of α in given matrices we will get F(x) and F(y) respectively and then by multiplying them we will get the required result.

$$F(x)\,F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y)$$

i.e. F(x).F(y) = F(x +y) ... (i)

2nd part.

As we know that $F(x)\,[F(x)]^{-1} = I$ ... (ii)

Replacing y by –x in (i),

we get F(x). F(–x) = F(x – x) = F(0)

$$= \begin{bmatrix} \cos 0 & -\sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. F(x) F(–x) = I ... (iii)

therefore from (ii) and (iii)

$\Rightarrow [F(x)]^{-1} = F(-x)$.

Example 7: Show that every square matrix A can be uniquely expressed as P + iQ where P and Q are Hermitian matrices.

Sol: By considering $P = \frac{1}{2}(A + A^\theta)$

And $Q = \frac{1}{2i}(A - A^\theta)$ we get A = P + iQ

Then, using the property of a Hermitian matrix we can prove the above problem.

$$\text{Now } P^\theta = \left\{\frac{1}{2}(A + A^\theta)\right\}^\theta = \frac{1}{2}(A + A^\theta)^q$$

$$= \frac{1}{2}\{A^\theta + (A^\theta)^q\} = \frac{1}{2}(A^\theta + A) = \frac{1}{2}(A + A^\theta) = P$$

$\therefore P = P^\theta$, hence P is a Hermitian matrix.

Similarly

$$Q^\theta = \left\{\frac{1}{2i}(A - A^\theta)\right\}^\theta = \left\{\frac{1}{2i}\right\}(A - A^\theta)^q$$

$$= -\frac{1}{2i}\{A^\theta - (A^\theta)^q\} = -\frac{1}{2i}(A^\theta - A) = \frac{1}{2i}(A - A^\theta) = Q$$

$\therefore$ Q is also Hermitian matrix, Therefore A can be expressed as P + iQ ,where P and Q are Hermitian matrices.

Let A = R + iS where R and S are both Hermitian matrices

We have $A^\theta = (R + iS)^\theta = R^\theta + (iS)^q$

$= R^\theta + iS^\theta = R^\theta - iS^\theta = R - iS$

(since R and S are both Hermitian)

$\therefore A + A^\theta = (R + iS) + (R - iS) = 2R$

$\Rightarrow R = \frac{1}{2}(A + A^\theta) = P$

Also $A - A^\theta = (R + iS) - (R - iS) = 2iS$

$\Rightarrow S = \frac{1}{2i}(A - A^\theta) = Q$

Hence expression (1) for A is unique

Example 8: If A is Hermitian such that $A^2 = 0$, show that A = 0,

Sol: As A is a Hermitian matrix therefore $A^\theta = A$. By considering $A = [a_{ij}]_{n\times n}$ to be a Hermitian matrix of order n and as given $A^2 = 0$, we can solve given problem as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix} \text{ and}$$

$$A^\theta = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \ldots\ldots & \bar{a}_{n1} \\ \bar{a}_{12} & \bar{a}_{22} & \ldots\ldots & \bar{a}_{n2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \bar{a}_{1n} & \bar{a}_{2n} & \ldots\ldots & \bar{a}_{nn} \end{bmatrix}$$

Since $A^2 = 0$;

Let $AA^\theta = [b_{ij}]_{n\times n} \Rightarrow AA^\theta = 0$

Then each element of AA^θ is zero and so all the principal diagonal elements of AA^θ are zero

$\therefore b_{ii} = 0$ for all i = 1, 2, ..., n

Now, $b_{ii} = a_{i1}\,\bar{a}_{i1} + a_{i2}\,\bar{a}_{i2} + \ldots + a_{in}\,\bar{a}_{in}$

$= |a_{i1}|^2 + |a_{i2}|^2 + \ldots\ldots |a_{in}|^2 \quad \therefore b_{ii} = 0$

$\Rightarrow |a_{i1}|^2 + |a_{i2}|^2 + \ldots\ldots + |a_{in}|^2 = 0$

$\Rightarrow |a_{i1}| = |a_{i2}| = \ldots\ldots = |a_{in}| = 0$

$\Rightarrow a_{i1} = a_{i2} = \ldots\ldots = a_{in} = 0$

$\Rightarrow$ each element of the i^{th} row of A is zero, but $b_{ii} = 0\ \forall$ i = 1, ... n

$\therefore$ Each element of each row of A is zero. Hence, A = O

Example 9: If the non-singular matrix A is symmetric, then prove that A^{-1} is also symmetric.

Sol: By using the conditions of non-singular and symmetric matrix we can easily find the required result.

As given matrix A is a non-singular symmetric matrix.

$\therefore |A| \neq 0$ and $A^T = A$,

So, A^{-1} exists

Now, $AA^{-1} = I = A^{-1}A$

$\Rightarrow (AA^{-1})^T = (I)^T = (A^{-1}A)^T$

$\Rightarrow (A^{-1})^T A^T = I = A^T(A^{-1})^T$

$\Rightarrow (A^{-1})^T A = I = A(A^{-1})^T \quad [A^T = A]$

$\Rightarrow A^{-1} = (A^{-1})^T$

$\Rightarrow A^{-1}$ is symmetric.

Example 10: Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol: given

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\begin{bmatrix} R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1 \\ \text{and } R_4 \to R_4 - 6R_1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

[Applying $R_4 \to R_4 - R_2 - R_3$]

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_2 \to R_2 - R_3$]

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_3 \to R_3 - 4R_2$]

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_3 \to 1/11\ R_3$]

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the equivalent matrix is in echelon form having three non-zero rows. Hence, r(A) = 3

JEE Main/Boards

Exercise 1

Q.1 Find x and y, if $\begin{pmatrix} 2x-1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ x+y \end{pmatrix}$

Q.2 A matrix has 2 rows and 3 columns. How many elements a matrix has? Find the number of elements of a matrix if it has 3 rows and 2 columns.

Q.3 Order of matrix A is 2 × 2 and order of matrix B is 2 × 3. Find the order of AB and BA, if defined.

Q.4 Given a matrix $A = [a_{ij}]$, $1 \le i \le 3$ and $1 \le j \le 3$, where $a_{ij} = i + 2j$. Write the element

(i) a_{11} (ii) a_{32} (iii) a_{23} (iv) a_{34}

Q.5 A matrix has 18 elements. Write the possible orders of matrix.

Q.6 Give an example of a diagonal matrix, which is not a scalar matrix. Also give an example of a scalar matrix.

Q.7 For the matrix A, show that $A + A^T$ is a symmetric matrix.

Q.8 For the matrix A, Show that $A - A^T$ is a skew-symmetric matrix.

Q.9 The total number of elements in a matrix represents a prime number. How many possible orders a matrix can have ?

Q.10 Find x and y, if $\begin{pmatrix} x \\ 2y \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

Q.11 If $f(x) = 3x^2 - 9x + 7$, then for a square matrix A, write f(A).

Q.12 If A, B and AB are symmetric matrices, then what is the relation between AB and BA?

Q.13 If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ B, [2 –2 4], find AB.

Q.14 Are the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ equal? Give reasons.

Q.15 Given a matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$.

Find matrix kA, where $k = -\frac{1}{2}$

Q.16 Simplify: $\tan\theta \begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & -\sec\theta \end{bmatrix} + \sec\theta \begin{bmatrix} -\tan\theta & -\sec\theta \\ -\sec\theta & \tan\theta \end{bmatrix}$.

Q.17 If $X_{m\times 3} Y_{P\times 4} = Z_{2\times b}$, for three matrices X, Y, Z, find the values of m, p and b.

Q.18 Is matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$ symmetric or skew-symmetric? Give reasons.

Q.19 If $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, write (i) $R\left(\frac{\pi}{2}\right)$, (ii) R(x+y)

Q.20 For a skew-symmetric matrix $A = [a_{ij}]$, what is the nature of elements a_{ij}, if $i = j$.

Q.21 If $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, find A^{16}.

Q.22 Find x, if $[x\ 1] \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

Q.23 Find the sum of matrix $A = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$ and its additive inverse.

Q.24 Find X, if $X + \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}$

Q.25 Evaluate, $\begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix} + \begin{bmatrix} \cos^2\theta & 0 \\ -\operatorname{cosec}^2\theta & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Q.26 If A and B are symmetric matrices, show that AB is symmetric.

Q.27 If a matrix has 8 elements, what the possible orders it can have ? What if it has 5 elements?

Q.28 Evaluate the following:

$$[a, b] \begin{bmatrix} c \\ d \end{bmatrix} + [a\ b\ c\ d] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Q.29 If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^2. Hence find A^6

Q.30 Show that the element of the main diagonal of a skew-symmetric matrix are all zeros.

Q.31 Find AB, if $A = \begin{bmatrix} 0 & -4 \\ 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -7 \\ 0 & 0 \end{bmatrix}$

Q.32 If $A = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$, find values of x and y such that $A^2 - xA + yI = O$ where I is a 2×2 unit matrix and O is a 2 × 2 zero matrix.

Q.33 If $A = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 5 & 7 \end{pmatrix}$ and $A - 3B = \begin{pmatrix} 4 & 5 & -9 \\ 1 & 2 & 3 \end{pmatrix}$, find B.

Q.34 If $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then show that

$A^2 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$

Q.35 If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

prove that $(aI + bA)^3 = a^3I + 3a^2bA$.

Q.36 If $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, find the values of

p and q such that $(pI + qA)^2 = A$.

Q.37 If $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 0 & 6 \\ -2 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & 2 \\ 6 & -1 & 4 \\ 5 & 3 & -4 \end{bmatrix}$

find 2A – 3B.

Q.38 Construct a 3 × 3 matrix $[a_{ij}]$, whose elements are given by $a_{ij} = 2i - 3j$.

Q.39 If $\begin{bmatrix} x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, find x, y, z, w.

Q.40 Find matrices X and Y, if

$X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

Q.41 If $A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$;

$B = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$, then show that AB is zero

matrix, provided $(\theta - \phi)$ is an odd multiple of $\pi/2$.

Q.42 If $A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{pmatrix}$,

compute A^2B^2.

Q.43 Find the matrix X such that,

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} X + \begin{bmatrix} -1 & -8 & -10 \\ 3 & 4 & 0 \\ 10 & 20 & 10 \end{bmatrix}$$

Exercise 2

Single Correct Choice Type

Q.1 If number of elements in a matrix is 60 then how many dimensions of matrix are possible

(A) 12 (B) 6 (C) 24 (D) None of these

Q.2 Matrix A has x rows and x + 5 columns. Matrix B has y rows and 11 – y columns. Both AB and BA exist, then

(A) x = 3, y = 4 (B) x = 4, y = 3

(C) x = 3, y = 8 (D) x = 8, y = 3

Q.3 If A is square invertible matrix such that $A^2 = A$, then det.$(A^2 - I)$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Q.4 Number of distinct matrices that can be formed using all the 143 distinct elements is

(A) 4! (B) 4(143)! (C) 2(143)! (D) (143)!

Q.5 If $A^2 = A$, then $(I + A)^4$ is equal to

(A) I + A (B) I + 4A

(C) I + 15A (D) None of these

Q.6 If $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ is an orthogonal matrix, where α, β and the roots other than the common root of the equations $x^2 - px + q = 0$ & $x^2 + px - q = 0$, then

(A) $p=\pm\frac{1}{\sqrt{2}}, q=\pm\frac{1}{\sqrt{2}}$ (B) $p = 0, q = \pm\frac{1}{\sqrt{2}}$

(C) $p = \pm\frac{1}{\sqrt{2}}, q = 0$ (D) None of these

Q.7 A is a square matrix of order n and (det A) = 3. If det (λA) = 81; where λ ∈ N, then possible value of n is

(A) 3 (B) 5 (C) 2 (D) 7

Q.8 If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $f(x) = \frac{1+x}{1-x}$, then f(A)is

(A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (D) None of these

Q.9 If A is a skew symmetric matrix such that $A^TA = I$, then A^{4n-1}(n∈N) is equal to

(A) $-A^T$ (B) I (C) – I (D) A^T

Q.10 A and B are 2 × 2 matrices satisfying det A = det B and tr(A) = tr(B), further $A^2 - 3A + 14I = 0$ and $B^2 - \lambda B + \mu I = 0$, then μ is equal to

(A) 3 (B) 11 (C) –11 (D) 14

Q.11 The false statement is -

(A) The adjoint of a scalar matrix is scalar matrix.

(B) The adjoint of upper triangular matrix is lower triangular matrix.

(C) The adjoint of upper triangular matrix is upper triangular matrix.

(D) adj(adj A) = A, A is a square matrix of order 2.

Q.12 If the matrices A, B, (A + B) are non-singular, then $[A(A + B)^{-1}B]^{-1}$ is equal to

(A) A + B (B) $A^{-1} + B^{-1}$

(C) $(A + B)^{-1}$ (D) None of these

Q.13 If A is an orthogonal matrix | A | = – 1, then A^T is equal to

(A) – A (B) A

(C) – (adj A) (D) (adj A)

Q.14 If A and B are square matrices of order 3, then

(A) adj(AB) = adj A+adj B

(B) $(A + B)^{-1} = A^{-1} + B^{-1}$

(C) AB = 0 ⇒ | A | = 0 or | B | = 0

(D) AB = 0 ⇒ | A | = 0 and B = 0

Q.15 If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then | A | | adj A | is equal to

(A) a^{25} (B) a^{27}

(C) a^{81} (D) None of these

Q.16 If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then the value of $|A^TA^{-1}|$ is

(A) cos 4x (B) $\sec^2 x$

(C) – cos 4x (D) 1

Q.17 If A = $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then $19A^{-1}$ is equal to

(A) A^T (B) 2A (C) $\frac{1}{2}A$ (D) A

Q.18 If P is a two-rowed matrix satisfying $P^T = P^{-1}$, then P is

(A) $\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(C) $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ (D) None of these

Q.19 If A and B are two non-singular matrices of the same order such that $B^r = I$, for some positive integer $r > 1$, then $A^{-1}Br^{r-1} A \ A^{-1}B^{-1} A$ is equal to.

(A) 0 (B) I

(C) A^{-1} (D) None of these

Q.20 If A and B are orthogonal matrices of same order, then:

(A) A + B is also orthogonal.

(B) A – B is also orthogonal.

(C) AB is also orthogonal.

(D) AB + BA is also orthogonal.

Q.21 If C is an orthogonal matrix and A is a square matrix of same order then, trace of C^TAC is equal to

(A) Trace of C (B) Trace of AC

(C) Trace of A (D) None of these matrix

Q.22 Let A and B are idempotent matrices such that A.B = BA and A – B is non singular then | A + B | is equal to

(A) 0 (B) –1 (C) 1 (D) ±1

Q.23 If A and B are square matrices of same order and $AA^T = I$, then $(A^TBA)^{10}$ is equal to

(A) $AB^{10}A^T$ (B) $A^TB^{10}A$

(C) $A^{10}B^{10}(A^T)^{10}$ (D) $10A^TBA$

Q.24 If A = $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and

B = $\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$ then AB is equal to

(A) A^3 (B) B^2 (C) O (D) I

Q.25 If A, B, C are square matrices of same order & AB = BA, $C^2 = B$, then $(A^{-1}CA)^2$ is equal to

(A) B^2 (B) A^2 (C) C^2 (D) C

Q.26 A is a diagonal matrix of order 3, and tr(A) = 12. If all diagonal entries are positive then maximum value of det (A) is

(A) 8 (B) 16 (C) 32 (D) 64

Q.27 If A and B are two matrix such that AB = B and BA = A, then $A^2 + B^2$ is equal to

(A) 2AB (B) 2BA (C) A + B (D) AB

Q.28 A = $\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and B is column matrix such that $(A^8 + A^6 + A^4 + A^2 + I)$, B = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ where I is a unit matrix of order 2 ×2, then B is equal to

(A) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 \\ \frac{2}{11} \end{bmatrix}$ (C) $\begin{bmatrix} 0 \\ \frac{1}{|2|} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{11} \\ \frac{1}{11} \end{bmatrix}$

Q.29 If A and B are square matrices of same order such that AB = BA and $A^2 = I$, then ABA is equal to

(A) $(AB)^2$ (B) I (C) B (D) B^2

Previous Years' Questions

Q.1 The parameter, on which the value of the

determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

does not depend upon, is ***(1997)***

(A) a (B) p (C) d (D) x

Q.2 If f(x) = $\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$

then f(100) is equal to ***(1999)***

(A) 0 (B) 1 (C) 100 (D) – 100

Q.3 The number of distinct real roots of

$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval

$-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is ***(2001)***

(A) 0 (B) 2 (C) 1 (D) 3

Q.4 The number of values of k for which the system of equations ***(2004)***

(k + 1)x + 8y = 4k, kx + (k + 3)y = 3k – 1

(A) 0 (B) 1 (C) 2 (D) –1

Q.5 Given, 2x – y + 2z = 2, x – 2y + z = –4, x +y + lz = 4, then the value of λ such that the given system of equations has no solution is ***(2004)***

(A) 3 (B) 1 (C) 0 (D) –3

Q.6 If P = $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and

Q= PAP^T, then P^{T}Q^{2005}P is ***(2005)***

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2005 \\ 2005 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 2005 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q.7 If A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, 6A^{-1} = A^2 + cA + dI, then

(c, d) is ***(2005)***

(A) (–6, 11) (B) (–11, 6) (C) (11, 6) (D) (6, 11)

Q.8 Let α_1, α_2, β_1, β_2 be the roots of ax^2 + bx + c = 0 and px^2 + qx + r = 0 respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution. Then prove that $\frac{b^2}{q^2} = \frac{ac}{pr}$ ***(1987)***

Q.9 Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

where a, b, and c are respectively the pth, qth and rth terms of a harmonic progression ***(1987)***

Q.10 Suppose, f(x) is a function satisfying the following conditions: ***(1998)***

(a) f(0) = 2, f(1) =1

(b) f has a minimum value at x = $\frac{5}{2}$, and

(c) For all x,

$$f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a, b are some constants. Determine the constants a, b and the function f(x)

Q.11 Prove that for all values of θ, ***(2000)***

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Q.12 If A is an 3 × 3 non – singular matrix such that $AA' = A'A$ and $B = A^{-1}A$ then BB′ equals: ***(2014)***

(A) $I+B$ (B) I (C) B^{-1} (D) (B^{-1})

Q.13 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = I$ where I is 3 3×3 identity matrix, then the ordered pair (a, b) is equal to: ***(2015)***

(A) $(2,-1)$ (B) $(-2,1)$

(C) $(2,1)$ (D) $(-2,-1)$

Q.14 If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\,\mathrm{adj}\,A = AA^T$, then 5a + b is equal to ***(2016)***

(A) -1 (B) 5 (C) 4 (D) 13

JEE Advanced/Boards

Exercise 1

Q.1 (a) $A_{3\times3}$ is a matrix such that | A | = a, B = (adj A) such that | B | = b. Find the value of

$(ab^2 + a^2b + I)$ S Where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} +$ ...... up to ∞, and a = 3

(b) If A and B are square matrices of order 3, where | A | = –2 and | B | = I, then find (A^{-1}) adj (B^{-1}) adj$(2A^{-1})$

Q.2 Let A be the 2 × 2 matrices given by A = $[a_{ij}]$ where $a_{ij} \in \{0, 1, 2, 3, 4\}$ such that $a_{11} + a_{12} + a_{21} + a_{22} = 4$

(i) Find the number of matrices A Such that the trace of A is equal to 4.

(ii) Find the number of matrices A such that A is invertible.

(iii) Find the absolute value of the difference between maximum value and minimum value of det (A).

(iv) Find the number of matrices A such that A is either symmetric or skew-symmetric or both and det (A) is divisible by 2.

Q.3 For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^2.

Q.4 (a) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$,

Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) Find the matrix A satisfying the matrix equation $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A. \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$

Q.5 Let S be the set which contains all possible values of I, m, n, p, q, r for which

$A = \begin{bmatrix} l^2-3 & p & 0 \\ 0 & m^2-8 & q \\ r & 0 & n^2-15 \end{bmatrix}$ Be a non singular idempotent matrix. Find the absolute value of sum of the products of elements of the set S taken two at a time.

Q.6 If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that F(x). F(y) = F(x + y).

Hence, Prove that $[F(x)]^{-1} = F(-x)$.

Q.7 Let A_n and B_n be square matrices of order 3, which are defind as $A_n = [a_{ij}]$ and $B_n = [b_{ij}]$

where $a_{ij} = \frac{2i+j}{3^{2n}}$ and $b_{ij} = \frac{3i-j}{2^{2n}}$ for all i and j, $1 \le i.j \le 3$.

If $l = \lim_{n\to\infty} \text{Tr}\,(3A_1 + 3^2A_2 + 3^3A_3 + ... 3^nA_n)$ and

$m = \lim_{n\to\infty} \text{Tr}(2B_1 + 2^2B_2 + 2^3B_3 + ... 2^nB_n)$

Then find the value of (l + m).

[Note: Tr (P) denotes the trace of matrix P]

Q.8 Let A be a 3 × 3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$.

Let $A^{-1} = xA^2 + yA + zI$ then find the value of (x + y + z) where I is a unit matrix of order 3.

Q.9 Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that Cb = D.

Solve the matrix equation Ax = b.

Q.10 Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and AB ≠ 1 (where I is an identity matrix of order 3 × 3).

Find the value of Tr $(AB + (AB)^2 + (AB)^3 + ... + (AB)^{100})$ where Tr.(A) denotes the trace of matrix A.

Q.11 Let $M_n = [m_{ij}]$ denotes a square matrix of order n with entries as follows.

For $1 \le i \le n$, $m_{ii} = 10$; For $1 \le i \le n-1$, $m_{i+1,\, i} = m_{i,\, i+1} = 3$;

And all other entries in M_n are zero. Let D_n be the determinant of matrix M_n, then find the value of $(D_3 - 9D_2)$.

Q.12 Find the product of two matrices A & B,

where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the following system of linear equations

$x + y + 2z = 1$; $3x + 2y + z = 7$; $2x + y + 3z = 2$

Q.13 Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(i) Has a unique solution;

(ii) Has no solution and

(iii) Has infinitely many solutions.

Q.14 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ then solve the following matrix equations.

(a) $AX = B - 1$ (b) $(B - 1)X = IC$

(c) $CX = A$

Q.15 If A is an orthogonal matrix and B = AP where P is a non singular matrix, then show that the matrix PB^{-1} is also orthogonal.

Q.16 Let M be a 2 × 2 matrix such that $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $M^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If x_1 and x_2 $(x_1 > x_2)$ are the two values x for which $\det(M - xI) = 0$, where I is an identity matrix of order 2, then find the value of $(5x_1 + 2x_2)$.

Q.17 The set of natural numbers is divided into arrays of rows and columns in the form of matrices as $A_1 = (1)$,

$A_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $A_3 = \begin{pmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{pmatrix}$ .... So on.

Find the value of $T_r(A_{10})$.

[Note: $T_r(A)$ denotes trace of A]

Q.18 Consider $I_{n.m} = \int_0^1 \frac{x^n}{x^m - 1}dx$ and $J_{n.m} = \int_0^1 \frac{x^n}{x^m + 1}dx \;\forall\; n > m$ and $n, m \in N$.

(a) Consider a matrix $A = [a_{ij}]_{3\times3}$,

where $a_{ij} = \begin{cases} I_{6+i,3} - I_{i+3,3}, & i = j \\ 0, & i \ne j \end{cases}$. Then find trace (A^{-1}).

[Note: Trace of a square matrix is sum of the diagonal elements.]

(b) Let $A = \begin{bmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{bmatrix}$ and $B = \begin{bmatrix} I_{6,5} & 72 & I_{11,5} \\ I_{7,5} & 63 & I_{12,5} \\ I_{8,5} & 56 & I_{13,5} \end{bmatrix}$

then find the value of det (A) – det (B)

Q.19 Consider the matrices $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and

$B = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ and let P be any orthogonal matrix and

$Q = PAP^T$ and $R = P^TQ^KP$ also $S = PBP^T$ and $T = P^TS^KP$

Column I	**Column II**
(A) If we vary K from 1 to n then the first row first column elements of R will form	(p) G.P. with common ratio a
(B) If we vary K from 1 to n then the 2nd row 2nd column elements of R will form	(q) A.P. with volume difference 2
(C) If we vary K from 1 to n then the first row first column elements of T will form	(r) G.P. with common ratio b
(D) If we vary K from 3 to n then the first row 2nd column elements of T will represent the sum of	(s) A.P. with volume difference –2

Q.20 Consider a square matrix A of order 2 which has its elements as 0, 1, 2 and 4. Let N denote the number of such matrices, all elements of which are distinct.

Column I	Column II
(A) Possible non-negative value of det (A) is	(p) 2
(B) Sum of values of determinants corresponding to N matrices is	(q) 4
(C) If absolute value of (det(A)) is least, then possible value of $\|adj(adj(adj\ A))\|$	(r) – 2
(D) If det (A) is algebraically least, then possible value of det $(4A^{-1})$ is	(s) –2
	(t) 8

Exercise 2

Single Correct Choice Type

Q.1 Let A, B be two square matrices of the same dimension and let [A, B] = AB – BA, then for three 2 × 2 matrices

A, B, C, [[A, B],C] + [[B, C],A] + [[C, A],B] =

(A) 1 (B) 0

(C) ABC – CBA (D) None of these

Q.2 $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 2 & 3 & \alpha \end{vmatrix}$ & $f(x) = x^3 - 8x^2 + bx + \gamma$. If A satisfies f(x) = 0, then ordered pair (α, γ) is

(A) (2, –7) (B) (–2, 7)

(C) (2, 7) (D) (–2, –7)

Q.3 If $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is a square root of the two rowed unit matrix, then δ is equal to

(A) α (B) β

(C) γ (D) None of these

Q.4 For $A = \begin{vmatrix} 4 & 2i \\ i & 1 \end{vmatrix}$, (A – 2I) (A – 3I) is a

(A) Null-matrix (B) Hermitian matrix

(C) Unit matrix (D) None of these

Q.5 If α, β, γ are the real numbers and

$$A = \begin{pmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{pmatrix}$$ then

(A) A is skew symmetric

(B) A is invertible

(C) A is non singular

(D) | A | = 0

Q.6 The values of x for which the matrix

$$\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$$ is non-singular are

(A) R – {0}

(B) R – { – (a + b + c)}

(C) R – { 0, – (a + b + c)}

(D) None of these

Q.7 Let A is a skew symmetric matrix such $A^2 = A$, and B is a square matrix such that $B^TB = B$; | B | ≠0. If X = (A + B) (A – B), then X^TX is

(A) A – I (B) I – A

(C) A (D) None of these

Q.8 For two uni-modular complex numbers z_1 and z_2,

$$\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$$ equal to

(A) $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (D) None of these

Q.9 If $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$,

then the value of x is

(A) $\frac{a}{125}$ (B) $\frac{2a}{25}$

(C) $\frac{2a}{125}$ (D) None of these

Q.10 If A is square matrix such that $A^2 = I$, $|A| = 1$ and $B = (\text{adj } A)^{-1}$ then incorrect statement is

(A) AB = BA (B) AB = I

(C) A = B (D) B = I

Q.11 If A and B are square matrices of order 3 and adj A = B, then adj (3AB) is equal to

(A) $3 | B |^2 I_3$ (B) $9 | B | I_3$

(C) $3 | A |^2 I_3$ (D) $9 | A | I_3$

Q.12 Let A and B are square matrices of order n such that $A^T + B = O$, O is a null matrix, A = adj B, tr (A) = –1 and $A^2 = A$ then tr {adj(A^TB)} is equal to

(A) $(-1)^{n-1}$ (B) 1

(C) $(-1)^n$ (D) None of these

Q.13 If A is a non-singular matrix such that C = A + B, $| C |^2 = | A |^2 | I - (A^{-1} B)^2|$ and AB =BA, then

(A) B is null matrix (B) A is null matrix

(C) | C | = | A – B | (D) | A | = | B |

Previous Years' Questions

Q.1 Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the

form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b or c is

either ω and w^2. Then, the number of distinct matrices in the set S is ***(2011)***

(A) 2 (B) 6 (C) 4 (D) 8

Q.2 Let M and N be two 3 × 3 non-singular skew-symmetric matrices such that MN = NM. If PT denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to ***(2011)***

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

Q.3 Without expanding a determinant at any

stage, show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

= Ax + B Where A and B are determinants of order 3 not involving x. ***(1982)***

Q.4 Show that the system of equations 3x – y + 4z = 3, x + 2y – 3z = – 2, 6x + 5y + lz = – 3 has at least one solution for any real number $\lambda \neq -5$. Find the set of solutions, if $\lambda = -5$. ***(1983)***

Q.5 Consider the system of linear equations in x, y, z $(\sin 3\theta)\,x - y + z = 0$, $(\cos 2\theta)\,x + 4y + 3z = 0$ and $2x + 7y + 7z = 0$. Find the values of θ for which this system has non-trivial solution. ***(1986)***

Q.6 Let $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^8 & 3n^2-3n \end{vmatrix}$

Show that $\sum_{a=1}^{n} \Delta_a = c \in$ constant. ***(1989)***

Q.7 If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Then, find the value of $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c}$ ***(1991)***

Q.8 For a fixed positive integer n, if

$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$, then show that

$\left[\dfrac{D}{(n!)^3} - 4\right]$ is divisible by n. ***(1992)***

Q.9 Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$ $x + (\cos \alpha)y + (\sin \alpha)z = 0$ and $-x + (\sin \alpha)y - (\cos \alpha)z = 0$ has a non-trivial solution for $\lambda = 1$, find all values of α. ***(1993)***

Q.10 Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$

represents a straight line. ***(2001)***

Q.11 Let $\omega \neq 1$ be a cube root of unity and S be the set of all non–singular matrices of the form

$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each of a, b and c is either ω or ω^2.

Then the number of distinct matrices in the set S is ***(2011)***

(A) 2 (B) 6 (C) 8 (D) 4

Q.12 Let M be a 3×3 matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

Then the sum of the diagonal entries of M is ***(2011)***

(A) 5 (B) 6 (C) 9 (D) 8

Q.13 If P is 3 × 3 matrix such that $P^T = 2P + I$ where P^T is he transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **(2012)**

(A) $PX \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D) PX = – X

Q.14 Let $P = [a_{ij}]$ is 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is ***(2012)***

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

Q.15 Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $P_{ij} = \omega^{i+j}$ Then $P^2 \neq 0$, when n = ***(2013)***

(A) 57 (B) 55 (C) 58 (D) 56

Q.16 Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N_4$ then ***(2014)***

(A) determinant of (M^2+MN^2) is 0.

(B) there is a 3×3 non–zero matrix U such that (M^2+MN^2) U is the zero matrix.

(C) determinant of $(M^2+MN^2) \ge 1$.

(D) for a 3×3 matrix U, if (M^2+MN^2) U equals the zero matrix the U is the zero matrix.

Q.17 The quadratic equation p(x)=0 with real coefficients has purely imaginary roots. Then the equation p(p(x))=0 has

(A) Only purely imaginary roots.

(B) All real roots.

(C) Two real and two purely imaginary roots.

(D) Neither real nor purely imaginary roots.

Q.18 Let $z = \frac{-1+\sqrt{3}i}{2}$, where $\sqrt{-1}$,and $r, s \in \{1, 2, 3\}$.

Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of or dered pairs (r, s) for which $P^2 = -I$ is

Q.19 Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & 5 & 0 \end{bmatrix}$ where $\alpha \in R$.

Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$ where $R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$

(C) $\det(P\operatorname{adj}(Q)) = 2^9$ (D) $\det(P\operatorname{adj}(P)) = 2^{13}$

Q.20 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $p^{50} - Q = I$ then $\frac{q_{31}+q_{32}}{q_{21}}$ equals **(2016)**

(A) 52 (B) 103 (C) 201 (D) 205

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.7	Q.8	Q.17
Q.23	Q.32	Q.35
Q.38	Q.41	Q.44

Exercise 2

Q.4	Q.11	Q.14
Q.19	Q.22	Q.26

Previous Years' Questions

Q.1	Q.2	Q.6
Q.10	Q.13	

JEE Advanced/Boards

Exercise 1

Q.7	Q.10	Q.13
Q.18	Q.19	Q.20
Q.17		

Exercise 2

Q.2	Q.5	Q.8
Q.12		

Previous Years' Questions

Q.2	Q.4	Q.11
Q.12		

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $x = 2, y = 3$

Q.2 6; 6

Q.3 Order of AB is 2×3; order of BA is not defined

Q.4 (i) 3 (ii) 7 (iii) 8 (iv) 11

Q.5 $1 \times 18, 2 \times 9, 3 \times 6, 6 \times 3, 9 \times 2, 18\times 1$

Q.6 $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Q.9 Two

Q.10 $x = 1, y = -\frac{1}{2}$

Q.11 $f(A) = 3A^2 - 9S + 7I$

Q.12 AB = BA

Q.13 $\begin{bmatrix} 2 & -3 & 4 \\ 4 & -6 & 8 \\ 6 & -9 & 12 \end{bmatrix}$

Q.14 No

Q.15 $\begin{bmatrix} -1 & 1/2 \\ -2 & -1 \end{bmatrix}$

Q.16 $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Q.17 m = 2, p = 3, b = 4

Q.18 Skew-symmetric

Q.19 (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} \cos(x+y) & \sin(x+y) \\ \sin(x+y) & -\cos(x+y) \end{bmatrix}$

Q.20 Each element is zero

Q.21 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Q.22 $1 \pm \sqrt{10}$

Q.23 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Q.24 $\begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix}$

Q.25 $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Q.27 1 × 8, 2 × 4, 4 × 2, 8 × 1; 1 × 1, 5 × 1

Q.28 $[ac + bd + a^2 + b^2 + c^2 + d^2]$

Q.29 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $A^6 = A^2$

Q.31 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q.32 x = 9m, y = 14

Q.33 $B = \frac{1}{3}\begin{bmatrix} -2 & 1 & 19 \\ -5 & 8 & 11 \end{bmatrix}$

Q.36 $P \pm \frac{1}{\sqrt{3}}, q \pm \frac{1}{\sqrt{3}}$

Q.37 $\begin{bmatrix} -11 & 3 & -14 \\ -16 & 3 & 0 \\ -19 & -7 & 22 \end{bmatrix}$

Q.38 $\begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix}$

Q.39 x = 3, y = 7, z = –2, w = 14

Q.40 $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Q.42 $A^2 = A$, $B^2 = I$; $A^2B^2 = AI = A$

Q.43 $X = \begin{bmatrix} 1 & -1 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** D	**Q.4** B	**Q.5** C	**Q.6** C
Q.7 A	**Q.8** B	**Q.9** D	**Q.10** D	**Q.11** B	**Q.12** B
Q.13 C	**Q.14** C	**Q.15** D	**Q.16** D	**Q.17** D	**Q.18** B
Q.19 A	**Q.20** C	**Q.21** C	**Q.22** C	**Q.23** B	**Q.24** C
Q.25 C	**Q.26** D	**Q.27** C	**Q.28** C	**Q.29** C	

Previous Years' Questions

Q.1 B	**Q.2** A	**Q.3** C	**Q.4** B	**Q.5** B
Q.6 A	**Q.7** A	**Q.9** 0	**Q.10** $a = \frac{1}{4}, b = \frac{5}{4}$ $\quad f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$	
Q.12 B	**Q.13** D	**Q.14** B		

JEE Advanced/Boards

Exercise 1

Q.1 (a) 225(b) –8

Q.2 (i) 5 (ii) 18 (iii) 8 (iv) 5

Q.3 $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$

Q.4 (a) $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$ (b) $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

Q.5 29

Q.7 21

Q.8 1

Q.9 $x_1 = 1, x_2 = -1, x_3 = 1$

Q.10 100

Q.11 1

Q.12 x = 2, y = 1, z = – 1

Q.13 (i) $a \neq -3, b \in R$ (ii) $a = -3$ and $b \neq 1/3$ (iii) $a = -3, b = 1/3$

Q.14 (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$ (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ (c) No solution

Q.16 8`

Q.17 3355

Q.18 (a) 18 (b) 0

Q.19 $A \to q$; $B \to s$; $C \to p$; $D \to p$

Q.20 $A \to p, q, t$; $B \to s$; $C \to p, r$; $D \to r$

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** A **Q.3** A **Q.4** A **Q.5** D **Q.6** C

Q.7 B **Q.8** C **Q.9** C **Q.10** D **Q.11** B **Q.12** C

Q.13 C

Previous Years' Questions

Q.1 A

Q.2 C

Q.4 $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

Q.5 $\theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

Q6. $\sum_{a=1}^{n} \Delta_a = c$

Q.7 2

Q.8 $2n(n^2 + 4n + 5)$

Q.9 $\alpha = n\pi$ or $n\pi + \pi/4$

Q.10 0 **Q.11** A **Q.12** 9 **Q.13** D **Q.14 D** **Q.15** B, C, D

Q.16 A, B **Q.17** D **Q.18** A **Q.19** B, C **Q.20** B

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\begin{bmatrix} 2x-1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ x+y \end{bmatrix}$

$2x - 1 = 3$

$2x = 4 \Rightarrow x = 4/2 = 2$

$5 = x + y = 2 + y$

$y = 5 - 2 = 3$

$(x, y) = (2, 3)$

Sol.2 row = n

Column = m

Then total elements = mn

if $(n, m) = (2, 3) \Rightarrow nm = 2.3 = 6$

if $(n, m) = (3, 2) \Rightarrow mn = 3.2 = 6$

Sol 3: $A_{2\times 2}$ and $B_{2\times 3}$

For AB $\Rightarrow$ order will be $\Rightarrow 2 \times 3$

For BA $\Rightarrow$ row of B $\neq$ column of A

So, BA does not exist

Sol 4: $A = [a_{ij}]$, $1 \le i \le 3$, $i \le j \le 3$

$a_{ij} = i + 2j$

(i) $a_{11} = 1 + 2 = 3$ (ii) $a_{32} = 3 + 2(2) = 3 + 4 = 7$

(iii) $a_{23} = 2 + 3(2) = 8$ (iv) $a_{34} \Rightarrow$ not a element $= i \le j \le 3$ but here $4 > 3$

Sol 5: Total element = 18

Assume no of row = n

And no. of column = m

so $n \times m = 18 = 1 \times 18 = 2 \times 9 = 6 \times 3 = 3 \times 6 = 9 \times 2 \times 18 \times 1$

Sol 6: Diagonal matrix $= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

scalar matrix $= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Sol 7: Matrix $A = [a_{ij}]$ assume

$A^T = [a_{ji}]$

So $A + A^T = [a_{ij} + a_{ji}] = [b_{ij}]$ assume

Here $b_{ij} = a_{ij} = a_{ji}$

$b_{ji} = a_{ij} + a_{ji}$

Here $b_{ij} = b_{ji}$

Matrix is symmetric.

Sol 8: $A - A^T$

$\Rightarrow [a_{ij}] - [a_{ji}] = [b_{ij}]$ assume

$b_{ij} = a_{ij} - a_{ji}$

$b_{ji} = a_{ji} - a_{ij}$

$\Rightarrow b_{ij} = -b_{ij}$

This matrix is known as symmetric matrix.

Sol 9: A matrix $\rightarrow$ row n, column = m

Total element = mn

mn is prime no.

so mn could be $\rightarrow$ 2, 5, 7, 11

factor of $2 = 1 \times 2$ or 2×1

So for any prime no. of only 2 order

$\Rightarrow 1 \times n$ and $n \times 1$($n \in$ prime no.)

Sol 10: $\begin{bmatrix} x \\ 2y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x-1 \\ 2y+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$x - 1 = 0 \Rightarrow x = 1$

$2y + 4 = 3 \Rightarrow 2y = 3 - 4 = -1$

$Y = -\dfrac{1}{2}$

Sol 11: $f(x) = 3x^2 - 9x + 7$

$f(A) \Rightarrow$ if A is a matrix

$f(A) = 3A^2 - 9A + 7I$

A is a square matrix so A^2 is possible.

Sol 12: A, B and AB are symmetric matrices

$A = a_{ij}$

$B = b_{ij}$

$AB = A_{ij}\, B_{ji} = C_{ij}$

$BA = B_{ij}\,.\,A_{ji} = d_{ij}$

$$\left.\begin{array}{l}\text{but } B_{ij} = B_{ji}\\ \text{and } A_{ij} = A_{ji}\end{array}\right\}\text{ symmetric matrix's property}$$

$\therefore AB = A_{ij}\, B_{ij} = A_{ij}\,.\,B_{ij} = BA$

$AB = BA$

Sol 13: $A = \begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1}$ $B = \begin{bmatrix}2 & -2 & 4\end{bmatrix}_{1\times3}$

$$AB = \begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1}\begin{bmatrix}2 & -2 & 4\end{bmatrix}_{1\times3} = \begin{bmatrix}2 & -2 & 4\\4 & -4 & 8\\6 & -6 & 12\end{bmatrix}$$

Sol 14: $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}_{2\times2}$ and $\begin{bmatrix}1 & 2 & 0\\3 & 4 & 0\\0 & 0 & 0\end{bmatrix}_{3\times3}$

Both have different orders. So they are not same.

Sol 15: $A = \begin{bmatrix}2 & -1\\4 & 2\end{bmatrix}$, $K = -\dfrac{1}{2}$

$$KA = K\begin{bmatrix}2 & -1\\4 & 2\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}2 & -1\\4 & 2\end{bmatrix}$$

$$= \begin{bmatrix}-\dfrac{2}{2} & -\dfrac{1}{2}(-1)\\ -\dfrac{1}{2}(4) & -\dfrac{1}{2}(2)\end{bmatrix} = \begin{bmatrix}-1 & \dfrac{1}{2}\\ -2 & -1\end{bmatrix}$$

Sol 16: $\tan\theta\begin{bmatrix}\sec\theta & \tan\theta\\ \tan\theta & -\sec\theta\end{bmatrix} + \sec\theta\begin{bmatrix}-\tan\theta & -\sec\theta\\ -\sec\theta & \text{tam}\theta\end{bmatrix}$

$$= \begin{bmatrix}\tan\theta\sec\theta - \tan\theta\sec\theta & \tan^2\theta - \sec^2\theta\\ \tan^2\theta - \sec^2\theta & -\tan\theta\sec\theta + \cos\theta\sec\theta\end{bmatrix}$$

$$= \begin{bmatrix}0 & \dfrac{\sin^2\theta - 1}{\cos^2\theta}\\ \dfrac{\sin^2\theta - 1}{\cos^2\theta} & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

Sol 17: $X_{m\times3}\, Y_{p\times4} = Z_{2\times b}$

Column of x = row of y $\Rightarrow$ 3 = p and $2 \times b = (m \times 4)$

So m = 2; b = 4

Sol18: $A = \begin{bmatrix}0 & -1 & 2\\1 & 0 & -3\\-2 & 3 & 0\end{bmatrix}$

$a_{12} = -a_{21}$, $a_{13} = -a_{31}$

$a_{23} = -a_{32}$,

so A is skew symmetric.

Sol 19: $R(\theta) = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix}\cos\dfrac{\pi}{2} & \sin\dfrac{\pi}{2}\\ \sin\dfrac{\pi}{12} & -\cos\dfrac{\pi}{2}\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$R(x+y) = \begin{bmatrix}\cos(x+y) & \sin(x+y)\\ \sin(x+y) & -\cos(x+y)\end{bmatrix}$$

Sol 20: Skew symmetric $A = [a_{ij}]$

For all skew symmetric Matrix dia. l element (a_{ij}) are zero so $a_{ij} = 0$ & when i = j

Sol 21: $A = \begin{bmatrix}a & 0\\0 & 0\end{bmatrix}$

$$A^2 = \begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\0 & 0\end{bmatrix} \Rightarrow A^2 = \begin{bmatrix}a^2 & 0\\0 & 0\end{bmatrix}$$

$$A^2 \times A^2 = \begin{bmatrix}a^2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}a^2 & 0\\0 & 0\end{bmatrix} \Rightarrow A^4 = \begin{bmatrix}a^4 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow A^{16} = \begin{bmatrix}a^{16} & 0\\0 & 0\end{bmatrix}$$

Sol 22: $[X \quad 1]_{1\times2}\begin{bmatrix}1 & 0\\-2 & -3\end{bmatrix}_{2\times2}\begin{bmatrix}X\\3\end{bmatrix}_{2\times1} = 0$

$$\Rightarrow [x-2 \quad 0-3]_{1\times2}\begin{bmatrix}X\\3\end{bmatrix}_{2\times1} [x-2-3]\begin{bmatrix}X\\3\end{bmatrix} = 0$$

$[(x-2)x - 3(3)] = 0 \Rightarrow x^2 - 2x - 9 = 0$

$x = \frac{2 \pm \sqrt{2^2 - 4(-9)}}{2} = 1 \pm \sqrt{10}$

Sol 23: $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}_{2\times 2}$

Additive inverse B which is – A

So, A + B =A – A = 0

Sol 24: $x + \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}$

Assume $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1+2 & x_2-1 \\ x_3+3 & x_4-1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}$

$\Rightarrow x_1 + 2 = 2 \Rightarrow x_1 = 0$

$\Rightarrow x_2 - 1 = 4 \Rightarrow x_2 = 1 + 4 = 5$

$\Rightarrow x_3 + 3 = 5 \Rightarrow x_3 = 5 - 3 = 2$

$\Rightarrow x_4 - 1 = 0 \Rightarrow x_4 = 1$

$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix}$

Sol 25:

$\begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix} + \begin{bmatrix} \cos^2\theta & 0 \\ -\mathrm{cosec}^2\theta & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \sin^2\theta + \cos^2\theta + 0 & 1+0-1 \\ \cot^2\theta - \mathrm{cosec}^2\theta - 1 & 0+1+0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -1-1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Sol 27: Matrix has 8 element

m × n = 8 = 1 × 8 = 8 × 1 = 2 × 4 = 4 × 2

if m × n = 5 = 1 × 5 = 5 × 1 (only 2 possible order)

Sol 28: $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \times \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$[ac + bd] \times [a^2 + b^2 + c^2 + d^2]$

$\Rightarrow [a^2 + b^2 + c^2 + d^2 + ac + bd]$

Sol 29: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^6 = [A^2]^3 = [I]^3 = I$

$A^6 = I = A^2$

Sol 30: Properties of skew – symmetric matrix [a, j]

⇒ All diagonal element are zero

$\Rightarrow a_{ij} = -a_{ji}$

Sol 31: $A = \begin{bmatrix} 0 & -4 \\ 0 & -3 \end{bmatrix}$. $B = \begin{bmatrix} 5 & -7 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & -4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0.5 - 4.0 & 0(-7) \\ 0(5) & 0(-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Sol 32: $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$A^2 - XA + YI = 0$

$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 4^2 + 3.2 & 4.3 + 3.5 \\ 2.4 + 5.2 & 2.3 + 5.2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$

$A^2 - XA + YI = 0$

$\Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + \frac{1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 22 - 4x + y & 27 - 3x \\ 18 - 2x + x & 31 - 5x + y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

⇒ Compare elements

$27 - 3x = 0$

$3x = 27 \Rightarrow x = \dfrac{27}{3} = 9 \Rightarrow y = 45 - 31 = 14$

(x, y) = (9, 14)

Sol 33: $A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 5 & 7 \end{bmatrix}$

$2A - 3B = \begin{bmatrix} 4 & 5 & -9 \\ 1 & 2 & 3 \end{bmatrix}$

Assume $B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$

$\Rightarrow 2A - 3B = \begin{bmatrix} 2.1 - 3b_1 & 2.3 - 3b_2 & 2 \times 5.3b_3 \\ -4 - 3b_4 & 2.5 - 3b_5 & 2.7 - 3b_6 \end{bmatrix}$

$= \begin{bmatrix} 4 & 5 & -9 \\ 1 & 2 & 3 \end{bmatrix}$

$2 - 3b_1 = 4 \rightarrow b_1 = \dfrac{4-2}{-3} = -\dfrac{2}{3}$

$\Rightarrow 6 - 3b_2 = 1$

$\Rightarrow 3b_2 = 6 - 5 = 1$

$\Rightarrow b_2 = \dfrac{1}{3}$

Same as $b_3 = \dfrac{19}{3}$

$b_4 = -\dfrac{5}{3}, b_5 = \dfrac{8}{3}, b_6 = \dfrac{11}{3}$

So $B = \dfrac{1}{3}\begin{bmatrix} -2 & 1 & 19 \\ -5 & 8 & 11 \end{bmatrix}$

Sol 34: $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \sin\alpha\cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$

we know $\rightarrow \cos^2\alpha - \sin^2\alpha = \cos^2\alpha$

and $2\cos\alpha \sin\alpha = \sin 2\alpha$

so $A^2 = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2a \end{bmatrix}$

Sol 35: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For $(aI + bA)^3$; $aI = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

$bA = b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$aI + bA = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$(aI + bA)^3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$= \begin{bmatrix} a^2 & ab + ba \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & a^2b + 2a^2b \\ 0 & a^3 \end{bmatrix}$

$= \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$

and R. H. S. $= a^3I + 3a^2bA$

$= a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$

L. H. S. = R. H. S

Sol 36: $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$(pI + qA)^2 = A$

$pI = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, qA = q\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & q \\ -q & q \end{bmatrix}$

$pI + qA = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q \\ -q & q \end{bmatrix} = \begin{bmatrix} p & q \\ -q & p+q \end{bmatrix}$

$(pI + qA)^2 = \begin{bmatrix} p & q \\ -q & p+q \end{bmatrix}\begin{bmatrix} p & q \\ -q & p+q \end{bmatrix}$

$= \begin{bmatrix} p^2 - q^2 & pq + q(p+q) \\ -pq - q(p+q) & -q^2 + (q+q)^2 \end{bmatrix} = A \text{ (given)}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

So $p^2 - q^2 = 0 \Rightarrow p^2 = q^2 \Rightarrow p = \pm q$

$pq + qp + q^2 = 1$

$q^2 + 2qp = q^2 \pm 2q^2 = 1$

$-ve \rightarrow q^2 - 2q^2 = 1 \Rightarrow q^2 = 1$ not possible

$+ve \rightarrow q^2 + 2q^2 = 3q^2 = 1 \Rightarrow q^2 = 1/3$

So $p = q = \pm \dfrac{1}{\sqrt{3}}$

Sol 37: $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 0 & 6 \\ -2 & 1 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 5 & 1 & 2 \\ 6 & -1 & 4 \\ 5 & 3 & -4 \end{bmatrix}$

$2A - 3B = 2\begin{bmatrix} 2 & 3 & -4 \\ 1 & 0 & 6 \\ -2 & 1 & 5 \end{bmatrix} - 3\begin{bmatrix} 5 & 1 & 2 \\ 6 & -1 & 4 \\ 5 & 3 & -4 \end{bmatrix}$

$= \begin{bmatrix} -11 & 3 & -14 \\ -16 & 3 & 0 \\ -19 & -7 & 22 \end{bmatrix}$

Sol 38: $A_{3\times3} = [a_{ij}]$

$a_{ij} = 2i - 3j$

$\therefore a_{11} = 2(1) - 3(1) = -1,\ a_{12} = 2(1) - 3(2) = -4$

$a_{13} = 2(1) - 3(3) = -7,\ a_{21} = 2(2) - 3(1) = 1$

$a_{22} = 2(3) - 3(2) = -2,\ a_{23} = 2(2) - 3(3) = -5$

$a_{31} = 2(3) - 3(1) = 3,\ a_{32} = 2(3) - 3(2) = 0$

$a_{33} = 2(3) - 3(3) = -3$

So $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix}$

Sol 39: $\begin{bmatrix} x & 3x-y \\ 2x+z & 3y-w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$

Compare elements

$x = 3$

$3x - y = 3(3) - y = 9 - y = 2$

$y = 9 - 2 = 7$

$2x + z = 2(3) + 7 = 6 + 7 = 4 \Rightarrow 7 = 4 - 6 = -2$

$3y - w = 3(7) - w = 7 \Rightarrow w = 21 - 7 = 14$

$(x, y, z, w) = (3, 7, -2, 14)$

Sol 40: $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix},\ X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

sum of $X + Y,\ X - Y$

$\Rightarrow X + Y + X - Y = 2X$

$= \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5+3 & 2+6 \\ 0 & 9-1 \end{bmatrix}$

$2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = 2\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

$X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

$Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - X = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

$Y = \begin{bmatrix} 5-4 & 2-4 \\ 0 & 9-4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Sol 41: $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

$B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\theta \end{bmatrix}$

$AB = (C_{ij})$

$C_{11} = \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi$

$C_{11} = \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi)$

$C_{11} = \cos\theta\cos\phi\cos(\theta - \phi) = 0$

Similarly C_{12}, C_{21} and C_{22} will also be zero

So $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Sol 42: $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 3 \end{bmatrix},\ B = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+3-5 & -1-3+5 & 1+3-5 \\ -3-9+15 & 3+9+15 & -3-9+15 \\ -5-15+25 & 5+15-25 & -5-15+25 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = A$$

$$B^2 = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 4-3 & -12+12 & -12+12 \\ -3+3 & 4+9-12 & 3+9-12 \\ 4-4 & -4-12+16 & -3-12+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^2B^2 = A^2 I = A.\, I = A$

Sol 43: $\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} X = \begin{bmatrix} -1 & 8 & -10 \\ 3 & 4 & 0 \\ 10 & 20 & 10 \end{bmatrix}_{3\times 3}$

X 's number of row = column of $\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$

order of X = 2 × 3

assume $X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - x_4 & 2x_2 - x_5 & 2x_3 - x_6 \\ x_4 & x_5 & x_6 \\ -2x_1 + 4x_4 & -2x_2 + 4x_5 & -2x_3 + 4x_6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & -10 \\ 3 & 4 & 0 \\ 10 & 20 & 10 \end{bmatrix}$$

$x_4 = 3,\ x_5 = 4,\ x_6 = 0$

$-2x_1 + 4x_4 = 10 \Rightarrow -2x_1 + 4(3) = 10$

$2x_1 = 12 - 10 = 2 \Rightarrow x_1 = 1$

$-2x_2 + 4x_5 = -\,2x_2 + 4(4) = 20$

$-\,2x_2 + 16 = 20$

$2x_2 = 16 - 20 = -\,4 \Rightarrow x_2 = -\dfrac{4}{2} = -\,2$

$-2x_3 + 4x_6 = -\,2x_3 + 4(0) = 10 = -\,2x_3$

$x_3 = \dfrac{10}{-2} = -\,5$

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

Exercise 2

Single Correct Choice Type

Sol 1: (A) Number of elements in a matrix = 60

$60 = 2^2\, 5^1\, 3^1$

Number of order matrix can have = (2 + 1) (1 + 1) (1 + 1)

= 3 × 2 × 2 = 12

Sol 2: (C) $A_{(x)\times(x+5)}$ $\quad B_{y\times(11-y)}$

AB and BA both exist

$\Rightarrow$ for AB $\quad x + 5 = y$ ...(i)

$\Rightarrow$ for BA $\quad 11 - y = x$ ...(ii)

$\Rightarrow y = 8;\ x = 3$

Sol 3: (D) A is a square invertible matrix

$A^2 = A$

Multiply A^{-1} both sides

$A^{-1}A^2 = A^{-1}A = I$

$A = I$

So $A^2 = I$

$A^2 - I = 0$ (zero matrix)

Sol 4: (B) Total 143 elements all are different.

$143 = 1 \times 143 = 143 \times 1 = 11 \times 13 = 13 \times 11$

Total Number of order that exist = 4

Number of way to arrange 143 elements = 143!

Total not of matrix = 4×1431

Sol 5: (C) $A^2 = A$

$(I + A)^4 = (I^2 + A^2 + 2A)^2$

$= [I + A + 2A]^2 = [I + 3A]^2 \qquad (\because\ A^2 = A)$

$= I^2 + 9A^2 + 6A = I + 9A + 6A = I + 15A$

Sol 6: (C) $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Since, A is orthogonal matrix

So, $AA' = A'A = I_n$

$$AA^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^2+\beta^2 & -\alpha\beta+\alpha\beta \\ -\alpha\beta+\alpha\beta & \alpha^2+\beta^2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^2+\beta^2 & 0 \\ 0 & \alpha^2+\beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\alpha^2 + \beta^2 = 1$ ...(i)

$x^2 - px + q = 0$ & $x^2 + px - q = 0$

Sum of both equation for common roots

$x^2 - px + q + x^2 + px - q = 0$

$2x^2 = 0 \Rightarrow x = 0$

So if α is roots of $x^2 - px + q = 0$ and β is roots of $x^2 + px - q = 0$

$\Rightarrow \alpha + 0 = p$ and $\alpha(0) = q = 0$

$\beta + 0 = -p$ and $\beta(0) = -q = 0$

$\Rightarrow \alpha = p$ and $\beta = -p$

In (i) equation $a^2 + b^2 = 1$

$\Rightarrow p^2 + (-p)^2 - 1$

$\Rightarrow 2p^2 = 1$

$\Rightarrow p = \pm \dfrac{1}{\sqrt{2}}, q = 0$

Sol 7: (A) (det A) = 3

(det λA) = 81

if A's order = $n \times n$

then (det λA) = λ^n(det A) = $\lambda^x 3 = 81$

$\lambda^n = \dfrac{81}{3} = 27$

$\lambda^n = 27 = 3^n \quad \because \lambda \in N$

So, n = 3

Sol 8: (B) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $f(x) = \dfrac{1+x}{1-x}$

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = -2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{1}{\det(I-A)}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\det(I - A) = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = 0 - (-2)(-2) = -4$$

$$f(A) = \frac{(I+A)}{(I-A)} = \frac{2}{-4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0+2 & 2+0 \\ 0+2 & 2+0 \end{bmatrix} = -\frac{2}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Sol 9: (D) A is skew symmetric matrix

$\Rightarrow A^TA = I$

$\Rightarrow A^T = -A \Rightarrow (AT)^2 = (A)^2$

$\Rightarrow A^TA = -A^2 = I$

Taking square of both sides

$A^4 = I$

$\Rightarrow A^{4n} = I$ ($\because -A^2 = I$ are $A = -A^T$) \

$A^3A^{4n} = A^3I$

$A^{4n+3} = -A^2(-A)\, I$

$A^{4(n+1)-1} = (-A)I = A^T I = A^I$

$4n - 1 \in N$

$\Rightarrow A^{4n-1} = A^T$

Sol 10: (D) $|A| = |B| A_{2\times2}, B_{2\times2}$

Tr (A) = Tr(B)

$A^2 - 3A + 14I = 0$ and $B^2 - \lambda B + \mu I = 0$

if A = B, |A| = |B|

Tr |A| = Tr(B) satisfied so $A^2 - \lambda A + \mu I = 0$

$\mu = 14$ ($\because$ A's order = 2×2)

Sol 11: (B) The adjoint of upper triangular matrix is false.

$\because$ That is equal to upper triangular not lower triangular matrix.

Sol 12: (B) A, B, (A + B) are non-singular

$[A(A + B)^{-1}B]^{-1}$

$= [A^{-1}((A + B)^{-1})^{-1}B^{-1}] = (A^{-1}(A + B)\, B^{-1})$

$= [(A^{-1}A + A^{-1}B)B^{-1}) = [(I + A^{-1}B)B^{-1})$

$= [B^{-1} + A^{-1}BB^{-1}] = A^{-1} + B^{-1}$

Sol 13: (C) A is an orthogonal matrix

$|A| = -1$

$\Rightarrow AA^T = A^TA = I_n$

$\Rightarrow |A| = |A^T| = -1$

$\Rightarrow A^T = +A^{-1}$($\because$ A is an orthogonal matrix)

$\Rightarrow A^T = \dfrac{1}{\det(A)}$ (adjA) = – (adjA)

Sol 14: (C) A and B are square matrices of order 3

(A) adj(AB) = adj(A) + adj(B) is not necessary

$\Rightarrow$ option (A) is wrong.

(B) $(A + B)^{-1} \neq A^{-1} + B^{-1}$

(C)AB = 0

A, B are square matrix

So if AB = 0

$\Rightarrow |A|\,|B| = 0$

$\Rightarrow |A| = 0$ or $|B| = 0$

Sol 15: (D) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \Rightarrow |A| = a^3|I| = a^3$

$\text{adj}(A) = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$|\text{adj}\,|A|| = (a^2)^3 = a^6$

$\therefore |A|\,|\text{adj}\,(A)| = a^3\,a^6 = a^9$

Sol 16: (D) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$|A| = 1 + \tan^2 x$

$$A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & \tan x \\ \tan x & 1 \end{bmatrix} = \frac{A}{(1+\tan^2 x)}$$

$$AT = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^TA^{-1} = \frac{1}{(1+\tan^2 x)}\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan x \\ -\tan x & 1 \end{bmatrix}$$

$$= \frac{1}{(1+\tan^2 x)}\begin{bmatrix} 1+\tan^2 x & \tan x - \tan x \\ \tan x - \tan x & \tan^2 x + 1 \end{bmatrix}$$

$$= \frac{1}{(1+\tan^2 x)}\begin{bmatrix} 1+\tan^2 x & 0 \\ 0 & 1+\tan^2 x \end{bmatrix}$$

$$= \frac{1+\tan^2 x}{1+\tan^2 x}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A^T A^{-1}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Sol 17: (D) $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$\Rightarrow |A| = 2(-2) - 3(5) = -15 - 4 = -19$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \frac{-1}{-19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A$$

Sol 18: (B) $P^T = P^{-1}$

Assume $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, $P^T = \begin{bmatrix} P_1 & P_3 \\ P_2 & P_4 \end{bmatrix}$

$\Rightarrow |P| = P_1P_4 - P_2P_3$

$$P^{-1} = \frac{1}{|P|}(\text{adj P}) = \frac{1}{|P|}\begin{bmatrix} P_4 & -P_3 \\ -P_2 & P_1 \end{bmatrix}^T = \frac{1}{|P|}\begin{bmatrix} P_4 & -P_2 \\ -P_3 & P_1 \end{bmatrix}$$

$|P| = P_1P_4 - P_2P_3$

$$\begin{bmatrix} P_1 & P_3 \\ P_2 & P_4 \end{bmatrix} = \frac{1}{|P|}\begin{bmatrix} P_4 & -P_2 \\ -P_3 & P_1 \end{bmatrix}$$

$|P| = 1$

$P_2 = -P_3$

Only option (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is correct.

Sol 19: (A) $B^r = I$ r > 1

$A^{-1}B^{r-1}A - A^{-1}B^{-1}A = A^{-1}B^rB^{-1}A - A^{-1}B^{-1}A$

$= A^{-1}B^{-1}A - A^{-1}B^{-1}A = 0$

Sol 20: (C) A & B are orthogonal matrices

$\Rightarrow AA^T = A^TA = I_n$ and $BB^T = B^TB = I_n$

$AB \Rightarrow (AB)\,(AB)^T$

$\Rightarrow (AB)\,(B^TA)^T$

$\Rightarrow A(BB^T)A^t = AIA^T$

$\Rightarrow AA^T = I$

$(AB)^T(AB) = B^TA^TAB = B^TIB = B^TB = I_n$

So $(AB)^T(AB)$

$gE = AB(AB)^T = I_n$

So, AB also satisfying property of orthogonal

Sol 21: (C) C is an orthogonal matrix

$\Rightarrow CC^T = C^TC = I_n$

$Tr(C^TAC) = Tr[(C^TA)C]$

$= Tr[C(C^TA)] = Tr(CC^TA) = Tr(IA) = Tr(A)$

Sol 22: (C) A and B are idempotent matrices

so, $A^2 = A$ and $B^2 = B$

$|A|, |B| =$ or 1

$AB = BA$

$A - B$ is non-singular

$\Rightarrow |A| =$ and $|B| = 1$ or $|A| = 1$ and $|B| = 0$

Sol 23: (B) $AA^T = I$

$(A^T BA)^{10}$

$= (A^TBA)(A^TBA)(A^TBA)..........$.

$\Rightarrow A^TBIBIBI.......... BA = A^TB^{10}A$

Sol 24: (C) $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$B = \begin{bmatrix} a^2 & ab & ca \\ ab & b^2 & cb \\ ac & bc & c^2 \end{bmatrix}$

$AB = \begin{bmatrix} abc - abc & b^2c - b^2c & c^2b - bc^2 \\ -a^2bc + ab^2c & -abc + abc & -ac^2 + ac^2 \\ a^2b - a^2b^\infty & ab^2 - ab^2 & abc - abc \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Sol 25: (C) $AB = BA, C^2 = B$

$(A^{-1}CA)^2 = (A^{-1}CA)(A^{-1}CA) = A^{-1}C\,ICA = A^{-1}C^2A$

$= A^{-1}BA = A^{-1}(AB) = IB = B = C^2$

Sol 26: (D) $Tr(A) = 12$

Assume $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ ($\because$ A is diagonal matrix)

$Tr(A) = a_1 + a_2 + a_3 = 12$ and det $|A| = a_1 a_2 a_3$ for maximum of det $(A) = a_1 a_2 a_3$

$a_1 + a_2 + a_3 = 3a_1 = 3a_2 = 3a_3 = 12$

$a_1 = \frac{12}{3} = 4$

det $|A| = 4 \times 4 \times 4 = 64$

Sol 27: (C) $AB = B$

$BA = A$

$(A + B)^2 = (AB + BA)^2$

$A^2 + B^2 + AB + BA = (AB)^2 + (BA)^2 + (AB)(BA) + (BA)(AB)$

$A^2 + B^2 + AB + BA = ABAB + BABA + AB + BA$

$A^2 + B^2 = AAB + BBA = AB + BA$

$A^2 + B^2 = A + B$

Sol 28: (C) $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$, B is column matrix

$(A^8 + A^6 + A^4 + A^2 + I)B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3I$

$A^4 = A^2A^2 = 3^2I$

$A^6 = 3^3I, A^8 = 3^4I$

$A^8 + A^6 + A^4 + A^2 + I = I(1 + 3 + 3^2 + 3^3 + 3^4) = 121\,I$

$|2|\,I\,B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 0 \\ \frac{1}{|2|} \end{bmatrix}$

Sol 29: (C) $AB = BA$ and $A^2 = I$

$ABA = A(AB) = A^2B = IB = B$

Previous Years' Questions

Sol 1: (B)

$\Rightarrow \Delta = \begin{vmatrix} 1 + a^2 & a & a^2 \\ \cos(p-d)x + \cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x + \sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} 1 + a^2 & a & a^2 \\ 2\cos px \cos dx & \cos px & \cos(p+d)x \\ 2\sin px \cos dx & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_2 - 2\cos dxC_2$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$\Rightarrow \Delta = (1 + a^2 - 2a\cos dx)$ [sin (p+ d) x cos px – sin px cos (p + d) x]

$\Rightarrow \Delta = (1 + a^2 - 2a\cos dx) \sin dx$

Which is independent of p

Sol 2: (A)

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

Applying $C_3 \to C_3 - (C_1 + C_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$$

$\therefore f(x) = 0$

$\Rightarrow (100) = 0$

Sol 3: (C) Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} \sin x+2\cos x & \cos x & \cos x \\ \sin x+2\cos x & \sin x & \cos x \\ \sin x+2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (2\cos x+\sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow (2\cos x + \sin x)$

$$\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x-\cos x & 0 \\ 0 & 0 & \sin x-\cos x \end{vmatrix} = 0$$

$\Rightarrow (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$

$\Rightarrow 2\cos x + \sin x = 0$ or $\sin x - \cos x = 0$

$\Rightarrow 2\cos x = -\sin x$ or $\sin x = \cos x$

$\Rightarrow \cot x = -1/2$ gives no solution in $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

and $\sin x = \cos x \Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4$

Sol 4: (B) For infinitely many solutions, we must have

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k = 1$$

Sol 5: (B) Since, given system has no solution

$\therefore \Delta = 0$ and any one amongst D_x, D_y, D_z is non-zero.

$$\text{Let } = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1$$

Sol 6: (A) Now,

$$P^TP = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\Rightarrow P^TP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow P^TP = I \Rightarrow P^T = P^{-1}$

Since, $Q = PAP^T$

$\therefore P^TQ^{2005}P$ ... (i)

$= P^T(PAP^T)(PAP^T)$ .... 2005 times]P

$$= \underbrace{(P^TP)A(P^TP)A(P^TP)........(P^TP)A(P^TP)}_{\text{2005 times}}$$

$= IA^{2005} = A^{2005}$[from eq.(i)]

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

....

....

$$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$$\therefore P^TQ^{2005}P = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

Sol 7: (A) Every square matrix satisfied its characteristic equation

i.e. $| A - \lambda I | = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda) \{(1 - \lambda)(4 - \lambda) + 2\} = 0$

$\Rightarrow 1^3 - 61^2 + 11\lambda - 6 = 0$

$\Rightarrow A^3 - 6A^2 + 11A - 6I = O$ ... (i)

$\Rightarrow A^2 6A + 11I = 6A^{-1}$

Sol 8: Since, a_1, a_2 are the roots of $ax^2 + bx + c = 0$

$\Rightarrow a_1 + a_2 = -\dfrac{b}{a}$ and $a_1a_2 = \dfrac{c}{a}$ ... (i)

Also, b_1, b_2 are the roots of

$px^2 + qx + r = 0$

$\Rightarrow b_1 + b_2 = -\dfrac{q}{p}$ and $b_1b_2 = \dfrac{r}{p}$ ... (ii)

Given system of equations

$a_1y + a_2z = 0$

And $b_1y + b_2z = 0$, has non-trivial solution

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \qquad \Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$$

Applying componendo-dividendo

$$\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2} = \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$$

$\Rightarrow (\alpha_1 + \alpha_2)(\beta_1 - \beta_2) = (\alpha_1 - \alpha_2)(\beta_1 + \beta_2)$

$\Rightarrow (\alpha_1 + \alpha_2)^2 \{(\beta_1 - \beta_2)^2 - 4\beta_2\beta_2\}$

$= (\beta_1 + \beta_2)^2 \{(\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2\}$

From equation (i) and (ii), we get

$$\frac{b^2}{a^2}\left(\frac{q^2}{p^2} - \frac{4r}{p}\right) = \frac{q^2}{p^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$$

$$\Rightarrow \frac{b^2q^2}{a^2p^2} - \frac{4b^2r}{a^2p} = \frac{b^2q^2}{a^2p^2} - \frac{4q^2c}{ap^2}$$

$$\Rightarrow \frac{b^2r}{a} = \frac{q^2c}{p} \Rightarrow \frac{b^2}{q^2} = \frac{ac}{pr}$$

Sol 9:
$$\left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \qquad \text{... (i)}$$

$$\text{Let } \Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix},$$

[From equation (i)]

$$= abc \begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A - D) R_3 - DR_2$

$$= abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Sol 10: Given,

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 - 2R_2$, We get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} (C_2 \to C_2 - C_1)$$

$\Rightarrow f'(x) = 2ax + b$

On integrating,

we get $f(x) = ax^2 + bx + c$

Where c is an arbitrary constant

Since, f has maximum at $x = 5/2$

$\Rightarrow f'(5/2) = 0 \Rightarrow 5a + b = 0$ ... (i)

Also, $f(0) = 2 \Rightarrow c = 2$

and $f(1) = 1 \Rightarrow a + b + c = 1$ ... (ii)

On solving equation (i) and (ii) for a, b, we get $a = \frac{1}{4}$, $b = -\frac{5}{4}$

Thus, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

Sol 11:

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) + \sin\left(2\theta-\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $\sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta-\frac{2\pi}{3}\right)$

$$= 2\sin\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$= 2\sin\theta\cos\frac{2\pi}{3} = 2\sin\theta\cos\left(\pi-\frac{\pi}{3}\right)$$

$$= -2\sin\theta\cos\frac{\pi}{3} = -\sin\theta$$

and $\cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta-\frac{2\pi}{3}\right)$

$$= 2\cos\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$= 2\cos\theta\cos\left(\frac{2\pi}{3}\right) = 2\cos\theta\left(-\frac{1}{2}\right) = -\cos\theta$$

and $\sin\left(2\theta+\frac{4\pi}{3}\right) + \sin\left(2\theta-\frac{4\pi}{3}\right)$

$$= 2\sin\left(\frac{2\theta+\frac{4\pi}{3}+2\theta-\frac{4\pi}{3}}{2}\right)\cos\left(\frac{2\theta+\frac{4\pi}{3}-2\theta+\frac{4\pi}{3}}{2}\right)$$

$$= 2\sin 2q\cos\frac{4\pi}{3} = 2\sin 2q\cos\left(\pi+\frac{\pi}{3}\right)$$

$$= -2\sin 2q\cos\frac{\pi}{3} = -\sin 2\theta$$

$$\therefore \Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

= 0 (since, R_1 and R_2 are proportional)

Sol 12: (B)

$$BBT = \left(A^{-1}A^T\right)\left(A^{-1}A^{-T}\right)^T = \left(A^{-1}A^T\right)\left(A.\left(A^{-1}\right)^T\right)$$

$$= A^{-1}.\left(A^TA\right)A^T.\left(A^{-1}\right)^T = A^{-1}\left(AA^T\right)\left(A^{-1}\right)^T$$

$$= \left(A^{-1}A\right)A^T.\left(A^{-1}\right)^T = A.\left(A^{-1}\right) = \left(A^{-1}A\right)^T = I$$

Sol 13: (D) $AA^T = 9T$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9I$$

$$\Rightarrow \begin{vmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{vmatrix} \Rightarrow \begin{vmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{vmatrix}$$

Equation $a+4+2b = 0 \Rightarrow a+2b = -4$ ... (i)

$2a+2-2b = 0 \Rightarrow 2a-2b = -2$ ... (ii)

& $a^2+4+b^2 = 0 \Rightarrow a^2+b^2 = 5$ ... (iii)

Solving $a = -2, b = -1$

Sol 14: (B) $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} A\,\mathrm{adj}\,A = AA^T$$

$$\begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

$$\begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix} = \begin{bmatrix} 25a^2+b^2 & 15-2b \\ 15-2b & 13 \end{bmatrix}$$

Equate, $10a+3b = 25a^2+b^2$

and $10a+3b = 13$

and $15a-2b = 0$

$\frac{a}{2}=\frac{b}{15}=k$ (let)

Solving $a=\frac{2}{5}, b=3$

So, $5a+b=5\times\frac{2}{5}+3=5$

JEE Advanced/Boards

Exercise 1

Sol 1: (a) |A| = a, (B) = (adjA), |B| =b

$$\frac{1}{2}S=\frac{a}{b}+\frac{a^2}{b^3}+\frac{a^3}{b^5}+....$$

$$=\frac{a}{b}\left[1+\frac{a}{b^2}+\left(\frac{a}{b^2}\right)^2+....\right]$$

$$=\frac{a}{b}\left[\frac{1}{1-\frac{a}{b^2}}\right]=\frac{a}{b}\left[\frac{b^2}{b^2-a}\right]=\frac{ab}{b^2-a}$$

$$S=\frac{2ab}{b^2-a}$$

b > a, a = 3

$|B| = |A|^{n-1} = |A|^{3-1} = (3)^2 = 9 = b$

$\Rightarrow (ab^2 + a^2b + 1)\ \frac{2(3)9}{9^2-3} = (3.\ (9)^2 + 3^2.\ 9 + 1)\ \frac{2(3)9}{9^2-3}$

$= (1 + 81 + 243)\ \frac{6.9}{78} = \frac{6.9}{78} = 225$

(b) |A| = –2, |B| = 1

(A^{-1}) (adj B^{-1}) adj$(2A^{+1})$

$\Rightarrow |A^{-1}|\ |\text{adj } B^{-1}|\ 2^{3-1}\,|\text{adj } A|$

$=\frac{1}{-2}\times 1\times 2^2\times(-2)^2=\frac{4\times4}{-2}=\frac{16}{-2}=-8$

Sol 2: $A = [a_{ij}]$, $a_{ij} \in \{0, 1, 2, 3, 4\}$

$a_{11} + a_{12} + a_{21} + a_{22} = 4$

(i) $T_r(A) = a_{11} + a_{22} = 4$

and $a_{11}, a_{22} \in \{0, 1, 2, 3, 4\}$

total possibilities = 0 + 4 = 4 + 0 = 1 + 3 = 3 + 1 = 2 + 2

$\Rightarrow 5$

(ii) A is invertible $\Rightarrow$ so $|A| \neq 0$

$a_{11}\, a_{22} - a_{21}a_{21} \neq 0$

$a_{11}a_{22} \neq a_{21}\, a_{12}$

total way $\rightarrow \frac{2\times(41+3)}{{}^3C_2}=\frac{2}{3}.\ 27 = 18$

(iii) $|A|_{max} - |A|_{min}$

$\Rightarrow |A|_{max} = a_{11}a_{22} - a_{21}a_{12}$

$(a_{11}\, a_{22})_{max} = 4$ ($\because$ $a_{11} + a_{22} + a_{12} + a_{12} = 4$)

and + $(a_{21}.\ a_{12})_{max} = 3$

$a_{21}\, a_{12} = 0$ with $a_{12}\, a_{22} = 4 = (2)\ (2)$

So $|A|_{max} = 2(2) - 0 = 4$

$|A|_{min} = 0 - 2(2) = -4$

$|A|_{max} - |A|_{min} = 4 - (-4) = 8$

(iv) A is symmetric or skew symmetric or both |det A| is divisible by 2

So |det A| can be 0, 2, 4

$$A\Rightarrow\begin{bmatrix}4&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&4\end{bmatrix},\begin{bmatrix}2&0\\0&2\end{bmatrix},\begin{bmatrix}2&1\\1&0\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix}$$

Total no. $\rightarrow$ 5

Sol 3: $A=\begin{bmatrix}4&-4&5\\-2&3&-3\\3&-3&4\end{bmatrix}$

$A_{11} = 12 - 4 = 3$ A_{13} =..................... .

$A_{12} = -9 + 8 \Rightarrow -A_{21} = -15 + 16 = 1$

$A_{22} = 16 - 15 = 1, -A_{23} = -12 + 12 = 0$

$A_{31} = 12 - 15 = -3, -A_{32} = -10 + 12 = 2$

$A_{33} = 12 - 8 = 4$

$$\text{adj}A=\begin{bmatrix}3&+1&-3\\-1&1&2\\-3&0&4\end{bmatrix}$$

$|A| = 4[A_{11}] - 4(A) + 5[A_{13}]$

$= 4(3) - 4(-1) + 5(-3) = 12 + 4 - 15 = 1$

$$A^{-1}=\frac{\text{adj}A}{|A|}=\begin{bmatrix}3&+1&-3\\-1&1&2\\-3&0&4\end{bmatrix}$$

$$A^{-2}=\begin{bmatrix}3&+1&-3\\-1&1&2\\-3&0&4\end{bmatrix}\begin{bmatrix}3&+1&-3\\-1&1&2\\-3&0&4\end{bmatrix}=\begin{bmatrix}17&4&-19\\-10&0&13\\-21&-3&25\end{bmatrix}$$

Sol 4: (a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3\times3}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}_{2\times2}$

$BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3}$

Assume $P = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \end{bmatrix}_{2\times3}$

$BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$B^{-1}BPA = PA = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$|B| = 8 - 9 = -1$

$adjB = \frac{1}{-1}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

$PA = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$

$|A| = 1[4 - 3] - 1\,[2 - 2] + 1\,[6 - 8] = 1 - 2 = -1$

$adjA = \begin{bmatrix} 1 & 3-1 & 1-4 \\ 0 & -1 & 2-1 \\ 6-8 & 2-3 & +2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & +2 \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & -1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$PAA^{-1} = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} A^{-1} = P$

$P = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot A \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$

assume $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$

$|B| = 4 - 3 = 1$ $|C| = -9 - 10 = -19$

$adjB = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ adjC

$= \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix}$, $C^{-1} = \frac{1}{19}\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$

$\Rightarrow BAC = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$

$B^{-1}BAC = AC = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4-3 & 8+1 \\ -6+6 & -12-2 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 9 \\ 0 & -14 \end{bmatrix} \Rightarrow ACC^{-1}$

$= A = \begin{bmatrix} 1 & 9 \\ 0 & -14 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} - \frac{1}{19}$

$A = \frac{1}{19}\begin{bmatrix} 3+45 & 2-27 \\ -70 & 42 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

Sol 5: $A = \begin{bmatrix} \ell^2 - 3 & P & 0 \\ 0 & m^2 - 8 & q \\ r & 0 & n^2 - 15 \end{bmatrix}$

$A^2 = A$[$\because$ A is idempotent matrix]

$A^2 =$

$$\begin{bmatrix} (\ell^2-3)^2+0 & p(\ell^2-3)+p[m^2-8] & pq \\ qr & (m^2-8)^2 & q(m^2-8)+q(n^2-15) \\ r(\ell^2-3)+r(n^2+5) & rp & (n^2-15)^2 \end{bmatrix}$$

$= \begin{bmatrix} \ell^2 - 3 & p & 0 \\ 0 & m^2 - 8 & q \\ r & 0 & n^2 - 15 \end{bmatrix}$

compare elements

$\Rightarrow (1^2 - 3)^2 = \ell^2 - 3 \Rightarrow 1^2 - 3 = 0$ or 1

$\ell = \pm\sqrt{3}$ or $\pm\sqrt{4} = \pm 2$

$p[1^2 - 3 + m^2 - 8] = p \Rightarrow p = 0$ or $1^2 + m^2 - 11 = 1$

$rp = 0 \Rightarrow r = 0$ or $p = 0$

$(n^2 - 15)^2 = n^2 - 15 \Rightarrow n^2 - 15 = 1$ or 0

$q[(m^2 - 8) + n^2 - 15] = q \Rightarrow q = 0$ or $m^2 + n^2 - 23 = 0 + 1$

$(m^2 - 8)^2 = m^2 - 8 \Rightarrow m^2 - 8 =$ or 1

$m = \pm\sqrt{8}$ or $\pm\sqrt{9} = \pm 3$

if, $1_1 m_1 n$, q, q, $r \in z$

$S = \{0, \pm2, \pm3, \pm4\}$

$\Rightarrow$ Sum of products of elements = $2^2+3^2+4^2=29$

Sol 6: $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$F(x).\ F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos y\cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[F(x)]^{-1} = F(-x)$

L. H. S. $\Rightarrow |F(x)| = \cos^2(x) + \sin^2 x - 1$

$\text{adj}[F(x)] = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[F(x)]^{-1} = \dfrac{\text{adj}F(x)}{1} = \begin{bmatrix} \cos(-x) & -\sin(-x) & 0 \\ \sin(-x) & \cos(-x) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(-x)$

L. H. S. = R. H. S.

Hence proved

Sol 7: $A_n = [a_{ij}]$, $B_n = [b_{ij}]$

$a_{ij} = \dfrac{2i+j}{32n}$, $b_{ij} = \dfrac{3i-i}{2^{2n}}$

$l = \lim\limits_{n\to\infty} \text{Tr}[3A_1 + 3^2A_2 + 3^3A_3 + \ldots + 3^nAn + \ldots]$

For A_n Tr(A) = $a_{11} + a_{22} + a^{33}$

$= \dfrac{(2(1)+1)+2(2)+2+2(3)+3}{3^{2n}} = \dfrac{9+6+3}{3^{2n}} = \dfrac{18}{3^{2n}}$

$\text{Tr}(3^{nA}\ n) = \dfrac{3^n 18}{3^{2n}} = \dfrac{18}{3^n}$

$\text{Tr}\left(\sum_{n=1}^{\infty} 3^n An\right) = \sum_{i=1}^{\infty} \text{Tr}(3^n A_3) = 18\left[\dfrac{1}{3} + \dfrac{1}{3^2} + \ldots\ldots \dfrac{1}{3^\infty}\right]$

$= 18\left[\dfrac{1}{3}\left[\dfrac{1}{1-\dfrac{1}{3}}\right]\right] = \dfrac{18}{3} \times \dfrac{8}{3-1} = \dfrac{18}{2}$

$\text{Tr}(B_n) = \dfrac{3(1)-1+3(2)-2+3(3)-3}{2^{2n}} = \dfrac{2+4+6}{2^{2n}} = \dfrac{12}{2^{2n}}$

$\text{Tr}(2^n B_n) = \dfrac{12}{2^n}$

$m = \sum_{n=1}^{a} [\text{Tr}(2^n B_n)] = 12\left[\dfrac{1}{2} + \dfrac{1}{22} + \ldots\right]$

$=12\left[\dfrac{1}{2}\left[\dfrac{1}{1-\dfrac{1}{2}}\right]\right] = \dfrac{12}{2} \cdot 2 = 12$

l + m = 12 + 9 = 21

Sol 8: A is 3 × 3 matrix

$A_{11} = a_{33} = 2$, all other $a_{ij} = 1$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\begin{matrix} R_1 \Rightarrow R_1 - R_2 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_2 \Rightarrow R_2 - R_1 - R_2$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^{-1} = I = xA^2 + yA + zI$

$I = (x + y + z)I$

$(x + y + z) = 1$

Sol 9: $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$, Cb = D

$|C| = 2[2-1] + 1[1-2] + 1[2-2] = 2 - 1 = 1$

$$\text{adj}C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = |C|\, C^{-1} = C^{-1}$$

$C^{-1}Cb = C^{-1}D$

$$b = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}_{3\times3} \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}_{3\times1}$$

$$b = \begin{bmatrix} 10-9 \\ 10+13 \\ -13+18 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$|A| = 1[6 + 3] + 2[3 - 6] + 2[-2 -2] = 9 - 6 - 8 = -5$

$$\text{adj}A = \begin{bmatrix} 6+3 & -8 & 2 \\ -3 & 3-2 & 1 \\ -4 & 3 & 2-4 \end{bmatrix} = \begin{bmatrix} 9 & -8 & 2 \\ -3 & 1 & 1 \\ -4 & 3 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}A = -\frac{1}{5}\,\text{adj}A$$

$AX = b$

$$X = A^{-1}b = -\frac{1}{5}\begin{bmatrix} 9 & -8 & 2 \\ -3 & 1 & 1 \\ -4 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$X = -\frac{1}{5}\begin{bmatrix} 9-24+10 \\ -3+3+5 \\ -4+9-10 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 \\ +5 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Sol 10: $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix},\ B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$

$$AB = \begin{bmatrix} -2x+7x & 28x-28x & 14x-14x \\ 0 & 1 & 0 \\ -x+x & 14x-2-4x & 7x-2x \end{bmatrix}$$

$$= \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix}$$

$|AB| = 5x\,[5x] = 25x^2$

$$(AB)^{-1} = \frac{1}{25x^2}\begin{bmatrix} 5x & 0 & 0 \\ 0 & 25x^2 & 0 \\ 0 & -50x^2+10x & 5x \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} \frac{1}{5x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-10x+2}{5x} & \frac{1}{5x} \end{bmatrix} = AB = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix}$$

$\Rightarrow x = 1/5$

$$AB = \begin{bmatrix} \frac{-5}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & \frac{-5}{5} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & -1 \end{bmatrix}$$

$(AB)^2 = (AB)\,(AB) = (AB)\,(AB)^{-1} = I$

$\text{Tr}[AB+(AB)^2+(AB)^3+\ldots+(AB)^{100}]$

$= \text{Tr}[AB+I+AB+I+\ldots+I]$

$= \text{Tr}[50AB+50I]=50\,\text{Tr}(AB)+50\text{Tr}(I)$

$= 50[-1+1-1]+50\,[1+1+1] =-50+3(50)=100$

Sol 11: $M_n = [m_{ij}]$order $= n$

$1 \le i \le n,\ m_{ij} = 10;$

$1 \le i \le n - 1,\ m_{i+1},\ I = m_i,\ i + 1 = 3$

All other entries in Mn are zero

$$M_3 = \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 3 \\ 0 & 3 & 10 \end{bmatrix},\ |M_3| = 10[100 - 9] + 3\,[-30]$$

$= 1000 - 90 - 90 = 820$

$$M_2 = \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix} \Rightarrow |M_2| = 100 - 9 = 91$$

$D_3 - 9D_2 = 820 - 9(91) = 820 - 819 = 1$

Sol 12: $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\ \&\ B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$\begin{aligned} x+y+2z &= 1 \\ 3x+2y+z &= 7 \\ 2x+y+3z &= 2 \end{aligned} \quad AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$${}_1B^{-1} = \frac{A}{4} = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{7}{4} & \frac{1}{4} & \frac{-5}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$$

$|B| = 1[6-1] + 1[2-9] + 2[3-4]$

$= 5-7-2 = -4$

$x = \frac{|X|}{|B|}, |X| \begin{vmatrix} 1 & 1 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = [6-1] + 7[2-3]$

$+ 2[1-4] = +5-7-6 = -8$

$x = \frac{-8}{-4} = 2, Y = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 7 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$= 1[21-2] + 1[2-9] + 2[6-14] = 19-7-16 = -4$

$y = \frac{-4}{-4} = 1, Z = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 7 \\ 2 & 1 & 2 \end{vmatrix}$

$= 1[4-7] + 1[14-6] + 1[3-4] = -3+8-1 = 4$

or $\Rightarrow Bx = C$

$x = B^{-1}C$

$z = \frac{-4}{-4} = -1$

$(x, y, z) = (2, 1, -1)$

$$x = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{7}{4} & \frac{1}{4} & \frac{-5}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}_{3\times3} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}_{3\times1} = \begin{bmatrix} 1 \\ +1 \\ -1 \end{bmatrix}$$

Sol 13: $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$

$|D| = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} = 3[-8a-9] - 2[18-5a] + 1[5+16] \backslash$

$= -24a - 27 - 36 + 10a + 21 = -14a - 42$

(i) System has a unique solution $101 \neq 0$

$-140 - 42 \neq 0$

$a \neq -\frac{42}{14} = -3$

$a \neq -3$ and $b \in R$

(ii) At $a = -3$ has no solution $\Rightarrow a = -3$

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 + 2R_3$

$$\begin{bmatrix} 3 & -2 & 1 \\ 5+4 & -8+2 & 9-6 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3-2 \\ -1 \end{bmatrix} = \begin{bmatrix} b \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 9 & -6 & 3 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 1 \\ -1 \end{bmatrix}$$

Compare row 2nd and 1st

$3x - 2y + z = b$ ... (i)

$9x - 6y + 3z = 1$

$3x - 2y + z = \frac{1}{3}$ ... (ii)'

From equation (i) and (ii)

$b = \frac{1}{3}$

For no. solution $a = -3$ and $b \neq \frac{1}{3}$

(iii) Has infinitely solution

so $a = -3$ and $b = \frac{1}{3}$

so $|D| = 0$ and $|D|_x = 0$

Sol 14: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$

(a) $AX = B - I$

$X \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$|A| = 4 - 6 = -2$

$\text{adj}A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$\Rightarrow A^{-1} = -\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{|A|}\text{adj}A$

so $A^{-1}AX = X = A^{-1}(B - I)$

$$X = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 8-2 & 4+2 \\ 6+1 & -3-1 \end{bmatrix}$$

$$X = -\frac{1}{2}\begin{bmatrix} 6 & 6 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$$

(b) $(B - I)X = IC = C$

$$B - I = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$|B - I|\ Z[-1] - 1 = -3$

$$\text{adj}(B - I) = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}, (B - I)^{-1}$$

$$= \frac{\text{adj}(B-I)}{|B-I|} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$X = (B - I)^{-1}C = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} 1+2 & 2+4 \\ 1-4 & 2-8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

(c) $CX = A$

$$|C| = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 4 - 4 = 0$$

So C^{-1} does not exit $\Rightarrow$ Y has no solution

Sol 15: A is orthogonal matrix

$\Rightarrow AA' = A'A = I_n$

and $B = AP$, P is non-singular

if A is orthogonal, so A^{-1} is also orthogonal

$B = AP$

$BB^{-1} = APB^{-1}$

$I = APB^{-1}$

$A^{-1} = A^{-1}APB^{-1}$

$A^{-1} = PB^{-1}$

A^{-1} is orthogonal, so PB^{-1} is also orthogonal

Sol 16: $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$; $M^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Assume $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

so $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$a_{11} - a_{12} = -1; +a_{21} - a_{22} = 2$ ... (i)

$$M^2 = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{21}a_{12} + a_{22}^2 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{21}a_{12} + a_{22}^2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$a_{11}^2 + a_{12}a_{21} + a_{12}a_{22} = 1$ ... (ii)

$\Rightarrow a_{11}a_{21} + a_{21}a_{22} - a_{21}a_{12} + a_{22}^2 = 0$

$\Rightarrow a_{11}[a_{11} - a_{12}] + a_{12}[a_{21} + a_{22}] = 1$

$\Rightarrow a_{11}(-1) + a_{12}(2) = 1$

$\Rightarrow 2a_{12} - a_{11} = 1$

$\Rightarrow a_{12} + 1 = 1 \Rightarrow a_{12} = 0$

$\Rightarrow a_{11} = -1$

$\Rightarrow a_{21}[a_{11} - a_{12}] + a_{22}[a_{21} - a_{22}] = 0$

$\Rightarrow a_{21}[-1] + a_{22}[2] = 0$

$\Rightarrow 2a_{22} - a_{21} = 0$

$-[a_{21} - a_{22} - a_{22}] = 0$

$2 - a_{22} = 0$

$\Rightarrow a_{22} = 2$

$\Rightarrow a_{21} = 4$

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$$

$|M - XI| = 0$

$$\begin{vmatrix} -1-x & 0 \\ 4 & 2-X \end{vmatrix} = 0$$

$(1 + x)(x - 2) = 0 \Rightarrow x = -1$ or $x = 2$

$5x_1 + 2x_2 = 5(2) + 2(-1)\ 2 > -1 = 10 - 2 = 8$

Sol 17: $A_1 = 1$, $A_2 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$,

$$A_3 = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots$$

No. of element in $A_n = n^2$

For $A_n = 10^2 = 100$, (10 in each row)

S_n = sum of all element of A_n

So $S_1 = 1$, $S_2 = 2 + 3 + 4 + 5$

$S_9 = 1 + 2 + 3 + \ldots\ldots\ldots m$

Where $m = 1 + 2^2 + 3^2 + 4^2 + \ldots\ldots + 9^2$

$= \frac{(2n+1)n(n+1)}{6} = \frac{9(18+1)(9+1)}{6}$

$= \frac{3}{2} \times 10 \times 19 = 285$

So i_n $a_{10} \Rightarrow a_{11} = 285 + 1 = 286$

$a_{22} = 286 + 11$

$a_{nn} = 286 + (n - 1)11$

$tr(A) = \sum_{i=1}^{10} a_{ij} = 286 \times 10 + [11 + 11(2)$

$+ 3(11) + \ldots\ldots + 9(11)]$

$= 286^o + 11 [1 + 2 + \ldots + 9]$

$= 286^o + 11 \times \frac{9 \times 10^5}{2} = 286^o + 11 \times 45 = 3355$

Sol 18: $I_{n',m} = \int_0^1 \frac{x^n}{x^m - 1} dx \; \forall \; n, m$

$I_{n',m} = \int_0^1 \frac{x^n}{x^m + 1} dx \; \forall \; x > m, n, m \in N$

(a) $A = [a_{ij}]_{3\times3}$

$a_{ij} = \begin{cases} I_{6+i,3} & -I_{i+3,3} \quad , i = j \\ 0 & \quad , i \neq j \end{cases}$

$a_{11} = I_{6+1,3} - I_{1+3,3} = I_{7,3} - I_{4,3}$

$= \int_0^1 \frac{x_7 dx}{x^3 - 1} - \int_0^1 \frac{x^4 dx}{x^3 - 1} = \int_0^1 \frac{x^7 - x^4}{x^3 - 1} dx$

$= \int_0^1 x^4 \left(\frac{x^3 - 1}{x^3 - 1} \right) dx = \left[\frac{x^5}{5} \right]_0^1 = \frac{1}{5}$

$a_{22} = I_{8,3} - I_{5,3} = \int_0^1 x^5 dx = \left[\frac{x^6}{6} \right]_0^1 = \frac{1}{6}$

$a_{33} = I_{8^{+1},3} - I_{6,3} = \int_0^1 x^6 \left(\frac{x^3 - 1}{x^3 - 1} \right) dx = \left[\frac{x^7}{7} \right]_0^1 = \frac{1}{7}$

$A = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$, $|A| = \frac{1}{5} \cdot \frac{1}{6} \cdot \frac{1}{7} = \frac{1}{210}$

$\text{Adj } A = \begin{bmatrix} \frac{1}{42} & 0 & 0 \\ 0 & \frac{1}{35} & 0 \\ 0 & 0 & \frac{1}{30} \end{bmatrix}$, $A^{-1} = \frac{1}{|A|}$

$\text{adj}A = \frac{1}{\frac{1}{210}} \begin{bmatrix} \frac{1}{42} & & \\ & \frac{1}{35} & \\ & & \frac{1}{30} \end{bmatrix}$

$Tr(A^{-1}) = 210 \left[\frac{1}{42} + \frac{1}{35} + \frac{1}{30} \right] = 5 + 6 + 7 = 18$

(b) $A = \begin{bmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{bmatrix}$

$B = \begin{bmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{bmatrix}$

$\det(A) = -72 [J_{7,5} J_{13,5} - J_{12,5} J_{8,5}] + \ldots\ldots\ldots\ldots\ldots$

$J_{n,\alpha} J_{m,\alpha} - J_{N,\alpha} J_{M,\alpha}$

If n + m = N + N, the $\int_0^1 \left(\frac{x^{n+m}}{x^{x+1}} - \frac{x^{n+m}}{x^{\alpha+1}} \right) dx = 0$

So det (A) = 0

$|B| = 72 [I_{12'} 5I_{8,5} - I_{7,5} I_{13',5}] + \ldots\ldots\ldots$

Sum as above 12 + 8 = 7 + 13

So, |B| = 0

det(A) – det (B) = 0

Sol 19: $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

P is orthogonal matrix $\Rightarrow$ Q = PAPT,

R = PTQ^KP, S = PBPT, T = PTS^K P

Sol 20: $A \to p,q,t;\ B \to s;\ C \to p,r;\ D \to r$

$A_{2\times2} = [a_{ij}]$

Elements are 0, 1, 2, 4

(A) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$|A| = a_{11}\, a_{22} - a_{21}a_{12}$

If $|A| > 0 \Rightarrow a_{11}a_{12} > a_{21}\, a_{12}$

$\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} \Rightarrow |A| = 2$

$\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} \Rightarrow 8, \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$

(B) If |A| = m, then is also a matrix ⇒ |A| = – m

So for all matrix, have one –ve det (A) matrix so Σ det(A) = 0

(C) Least value of det(A) = 2 or – 2

$|adj(adj(adjA))| = ((\pm2)^{2-1})^{2-1})^{2-1} = \pm2$, 2 or –2

(D) det(A) is algebraically least = – 8

$4A^{-1} = \dfrac{4adjA}{|A|} = \dfrac{4}{-8}\ adjA = \left(\dfrac{1}{-2}\right)(adj\ A)$

$|4A^{-1}| = |-Z^{-1}adjA| = (-Z)^2\ |A|^{2-1}$

$= \dfrac{1}{4} \times -8 = -2$

Exercise 2

Single Correct Choice Type

Sol 1: (B) Let [A, B] = AB – BA

[[A, B], C] + [[B, C], A] + [[C, A], B]

⇒ [[A, B], C] = [AB – BA, C] = (AB – BA)C – C(AB – BA)

= ABC – BAC – CAB + CBA ...(i)

[[B, C], A] = [BC, – CB, A] = (BC – CB)A – A(BC – CB)

= BCA – CBA – ABC + ACB ...(ii)

[[C, A], B] = [CA – AC, B] = (CA – AC) B – B(CA – AC)

= CAB – A, B – BCA + BAC ...(iii)

sum of equation (i), (ii) & (iii)

[[A, B]C] + [[B, C], A] + [[C, A], B] = ABC – BAC + BAC – ABB +....... = 0

Sol 2: (A) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 2 & 3 & \alpha \end{bmatrix}$

$f(x) = x^3 - 8x^2 + bx + \gamma$

a satisfies f(x) = 0

Sol 3: (A) two rowed unit matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$F_2^2 = I_2$

So square root of $I_2 = I_2 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ (given)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\alpha = 1 = \delta,\ \gamma = \beta = 0$

Sol 4: (A) $A = \begin{bmatrix} 4 & 2i \\ i & 1 \end{bmatrix}$ (A – 2I) (A – 3I) = ?

$A - 2I = \begin{bmatrix} 4 & 2i \\ i & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 2i \\ i & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & 2i \\ i & -1 \end{bmatrix}$

$A - 3J = \begin{bmatrix} 4 & 2i \\ i & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-3 & 2i \\ i & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & 2i \\ i & -2 \end{bmatrix}$

$(A - 2J)\ (A - 3J) = \begin{bmatrix} 2 & 2i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ i & -2 \end{bmatrix}$

$= \begin{bmatrix} 2(1)+2i(i) & 4i-4i \\ i-i & 2i(i)-1(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = null matrix

Sol 5: (D) $A = \begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$

$|A = 1[1 - \cos(\beta-\gamma)\cos(\gamma-\beta)] + \cos(\alpha-\beta)\,[\cos(\beta - r)\cos(\gamma-\alpha) - \cos(\beta-\alpha)]$

$+\cos(\alpha-\gamma)[\cos(\beta-\alpha)\cos(\gamma-\beta) - \cos(\gamma-\alpha)]$

($\because$ cos(A) = cos(–A))

$= 1 - \cos^2(\beta-\gamma) + 2\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha) - \cos^2(\beta-\alpha) = \cos^2(\alpha-\gamma)$

$= 1 - \left[\cos\dfrac{(\alpha+\beta-\gamma-\alpha+\gamma-\beta)}{2}\right]^2$

$= 1 - \cos^2 0 = 1 - 1 = 0$

Sol 6: (C) A = $\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$

matrix A is non singular

$|A| \neq 0$

$$\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix} \neq 0$$

$\Rightarrow$ (x + a) [(x + b) (x + c) – bc] + b[ac – a(x + c)] + c[ab – a(x + b)] $\neq$ 0

$\Rightarrow$ (x + a) [x^2 + x(b + c)] + b [ac – ax – ac] + (c) (–ax) $\neq$ 0

$\Rightarrow x^3 + ax^2 + x^2(b + c) + ax(b + c) - abx - acx \neq 0$

$\Rightarrow x^3 + x^2(a + b + 1) \neq 0$

$\Rightarrow x^2[x + (a + b + c)] \neq 0$

$\Rightarrow x \neq 0$ and $x \neq -(a + b + c)$

Sx x = R – {0, –(a + b + c)}

Sol 7: (B) A is skew symmetric matrix

$A^2 = A$ and $B^TB = B$

$B^TB = B$

Multiply with $B^{-1} \Rightarrow (B^TB)B^{-1} = BB^{-1} = I$

$B^TI = B^T = I$

$B^T = I$. So B = I

X = (A + B) (A – B)

$X = A^2 - AB + BA - B^2 (\because B = I)$

X = A – A + A – I = A – I

$X^7 = (A - I)^T = A^T - I$

$X^7X = (A^T - 1)(A - I)$

$= AA^T - A^T - A + I$

$A^T = -A (\because$ A is skew symmetric)

$X^7X = -AA - A + A + I$

$= -A^2 + I = -A + I = I - A$

Sol 8: (C) Z, and Z_2 are uni modular complex

$$\begin{bmatrix} \bar{z}_1 - z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} = A \text{ (assume)}$$

$$= \begin{bmatrix} \bar{z}_1 z_1 + z_2\bar{z}_2 & \bar{z}_1 z_2 - z_2\bar{z}_1 \\ \bar{z}_2 z_1 - z_1\bar{z}_2 & \bar{z}_2 z_2 + z_1\bar{z}_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Sol 9: (C) $\begin{bmatrix} \frac{1}{25} & x \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$

$$= \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1} = \frac{1}{25}\begin{bmatrix} 5 & a \\ 0 & 5 \end{bmatrix}^{\frac{1}{25}} \begin{bmatrix} 5 & a \\ 0 & 5 \end{bmatrix}$$

$$= \frac{1}{625} - \begin{bmatrix} 5 & a \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & a \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/25 & x \\ 0 & 1/25 \end{bmatrix}$$

$$= \frac{1}{625}\begin{bmatrix} 25 & 5a + a5 \\ 0 & 25 \end{bmatrix} = \frac{1}{625}\begin{bmatrix} 25 & 10a \\ 0 & 25 \end{bmatrix}$$

$$x = \frac{10a}{625} = \frac{2a}{125}$$

Sol 10: (D) $A^2 = I$

$|A| = 1$, $B = (\text{adj } A)^{-1}$

$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \text{adj}(A)$

$(A^{-1})^{-1} = (\text{adj}A)^{-1}$

$A = (\text{adj } A)^{-1} = B$ given

A = B

$A^2 = I$

AA = AB = I

AB = AA = BA = I

$\Rightarrow B \neq I$ we can't say that B = I

Sol 11: (B) adj A = Border of both = 3 × 3

Adj (3AB)= 3^{3-1} adj (AB)

= 9(adj B) (adjA) = 9(adj B)B = 9|B| = I_3

$\therefore$ adj(AB) = (adjB) (adj A)

Sol 12: (C) $A^T + B = 0$

A = adjB, tr(A) = 1, $A^2 = A$

tr{adj (A^TB)}

$\Rightarrow A^T + B = 0$

$\Rightarrow A^T = -B$

$\Rightarrow$ tr[(adj B) (adj A^T)]

$\Rightarrow$ tr[A adj(–B)]

$\Rightarrow$ tr$(A(-1)^{n-1}A)$

$\Rightarrow (-1)^{n-1}$ tr$(A^2) = (-1)^{n-1}$ tr(A)

$\Rightarrow (-1)^{n-1} (-1) = (-1)^n$

Sol 13: (C) C = A + B

$|C|^2 = |A|^2 |I - (A^{-1}B)^2|$

AB = BAC = A + B

$\Rightarrow |C| = |A + B| = |A| [I + A^{-1}B]$

$|C|^2 = |A|^2 |I - A^{-1}B| |I + A^{-1}B|$ ... (ii)

Equation $\frac{(2)}{(1)} \Rightarrow \frac{|C|^2}{|C|}$

$= \frac{|A|^2 |I - A^{-1}B| |I + A^{-1}B|}{|A| |I + A^{-1}B|}$

$|C| = A^{-1} |I - A^{-1}B| = |A - B|$

$|C| = |A - B|$

Previous Years' Questions

Sol 1: (A) | A | ≠ 0, as non-singular

$\therefore \begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$

$\Rightarrow 1(1 - c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2) \neq 0$

$\Rightarrow 1 - c\omega - a\omega + ac\omega^2) \neq 0$

$\Rightarrow (1 - c\omega)(1 - a\omega) \neq 0$

$\Rightarrow a \neq \frac{1}{\omega}, c \neq \frac{1}{\omega}$

$\Rightarrow a = \omega, c = \omega$ and $b \Rightarrow \{\omega, \omega^2\}$

$\Rightarrow$ 2 solutions

Sol 2: (C) Given, $M^T = -M$, $N^T = -N$

and MN = NM ... (i)

$\therefore M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$\Rightarrow M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^TM^T$

$\Rightarrow M^2N(NN^{-1})(-M)^{-1}(N^T)^{-1}(-M)$

$\Rightarrow M^2 NI(-M^{-1})(-N^{-1})(-M)$

$\Rightarrow -M^2NM^{-1}N^{-1}M$

$\Rightarrow -M \cdot (MN)M^{-1}N^{-1} M$

$\Rightarrow -M(NM)M^{-1}N^{-1} M$

$\Rightarrow -MN(NN^{-1})N^{-1} M$

$\Rightarrow -M(NN^{-1})M$

$\Rightarrow -M^2$

Note: Here, non-singular word should not be used, since there is no non-singular 3 x 3 skew-symmetric matrix.

Sol 3: Let $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, we get

$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_1 \to R_1 + \frac{x^2}{4} R_2$

and $R_3 \to R_3 + \frac{x^2}{4} R_2$, w

$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_3 \to R_3 - 2R_1$

$= \begin{vmatrix} x+0 & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$\Rightarrow \Delta = Ax + B$

Where $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

Sol 4: The given system of equation

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

$6x + 5y + lz = -3$

Has at least one solution, if $\Delta \neq 0$

$$\therefore \Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} \neq 0$$

$\Rightarrow 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) \neq 0$

$\Rightarrow 7(\lambda + 5) \neq 0 \Rightarrow \lambda \neq -5$

For $\lambda = -5$

$\Rightarrow \Delta = 0$

$$\text{Then, } \Delta_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0$$

$$\Delta_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$$

$$\Delta_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$$

$\Delta_1 = \Delta_2 = \Delta_3 = 0$

Sol 5: The system of equations has non-trivial solution, if $\Delta = 0$

$$\Rightarrow \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1, we get

$\Rightarrow \sin 3\theta \cdot (28 - 21) - \cos 2\theta (-7 - 7) + 2(-3 - 4) = 0$

$\Rightarrow 7\sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow \sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0, \sin\theta = 1/2$

(neglecting $\sin\theta = -3/2$)

$\Rightarrow \theta = n\pi, n\pi + (-1)^n \pi/6, n \in Z$

Sol 6: Given, $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$

$$\therefore = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

= Applying $C_3 \to C_3 - 6C_1$

$$= \frac{n^3(n-1)}{12}\begin{vmatrix} 1 & 1 & 0 \\ 2n-1 & 6n & 0 \\ n-1 & 6n & 0 \end{vmatrix} = 0$$

$\Rightarrow \sum_{a=1}^{n} \Delta_a = c$, (c = 0 ie, constant)

Sol 7: Let $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = c\begin{vmatrix} a-p & q-b \\ a-p & 0 \end{vmatrix} + (r-c)\begin{vmatrix} p & b \\ a-p & q-b \end{vmatrix}$$

$= -c(a-p)(q-b) + (r-c)[p(q-b) - b(a-p)]$

$= -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p)$

Since, $\Delta = 0$

$\Rightarrow -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p) = 0$

[On dividing both side by Radding 204 th side and $-x + (\sin\alpha)y - (\cos\alpha)z = 0$ has non-

$$(a-p)(q-b)(r-c)]\ \frac{b}{b-a} + \frac{c}{r-c} + \frac{b}{q-b} + 2 = 2$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + 0 + \frac{r}{r-c} + 0 = 2$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Sol 8: Given, $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

Taking n!, (n + 1)! and (n + 2)! Common from R_1, R_2 and R_3 respectively.

$\therefore$ D = n!(n+1)!(n+2)! $\begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

D = n!(n+1)!(n+2)! $\begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$

Expanding along C_1, we get

D = (n!)(n + 1)!(n + 2)![(2n + 6) – (2n +4)]

D = (n!)(n + 1)!(n + 2)![2]

On dividing both side by $(n!)^3$

$\Rightarrow \frac{D}{(n!)^3} = \frac{(n!)(n!)(n+1)(n!)(n+1)(n+2)2}{(n!)^3}$

$\Rightarrow \frac{D}{(n!)^3} = 2(n + 1)(n + 1)(n + 2)$

$\Rightarrow \frac{D}{(n!)^3} = 2(n^3 + 4n^2 + 5n + 2) = 2n(n^2 + 4n + 5) + 4$

$\Rightarrow \frac{D}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$

Which shows that $\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n.

Sol 9: Given, $\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$

$x + (\cos\alpha)y + (\sin\alpha)z = 0$

and $-x + (\sin\alpha)y - (\cos\alpha)z = 0$ has non-trivial solution.

$\therefore \Delta = 0$

$\Rightarrow \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$

$\Rightarrow \lambda(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha(-\cos\alpha + \sin\alpha) + \cos\alpha(\sin\alpha + \cos\alpha) = 0$

$\Rightarrow -\lambda + \sin\alpha\cos\alpha + \sin\alpha\cos\alpha - \sin^2\alpha + \cos^2\alpha = 0$

$\Rightarrow \lambda = \cos 2\alpha + \sin 2a$

$\left(\because -\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}\right)$

$\therefore -\sqrt{2} \le \lambda \le \sqrt{2}$ ... (i)

Again, when $\lambda = 1$, $\cos 2\alpha + \sin 2\alpha = 1$

$\Rightarrow \frac{1}{\sqrt{2}}\cos 2\alpha + \frac{1}{\sqrt{2}}\sin 2\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos(2\alpha - \pi/4) = \cos\pi/4$

$\therefore 2\alpha - \pi/4 = 2n\pi \pm \pi/4$

$\Rightarrow 2\alpha = 2n\pi - \pi/4 + \pi/4$ or $2\alpha = 2n\pi + \pi/4 + \pi/4$

$\therefore \alpha = n\pi$ or $n\pi + \pi/4$

Sol 10: Given,

$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx+a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix} = 0$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$\Rightarrow \frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+a \\ (a^2+b^2+c^2)y & by-c-ax & b+cy \\ a^2+b^2+c^2 & b+cy & c-ax-by \end{vmatrix} = 0$

$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b+cy & c-ax-by \end{vmatrix} = 0$

$(\because a^2 + b^2 + c^2 = 1)$

Applying $C_2 \to C_2 - bC_1$

and $C_3 \to C_3 - cC_1$

$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$

$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & ax-by \end{vmatrix} = 0$

Applying $R_1 \to R_1 + yR_2 + R_3$

$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$

$\Rightarrow \frac{1}{ax}[(x^2 + y^2 + 1)\{(-c - ax)(-ax - by) - b(cy)\}] = 0$

$\Rightarrow \frac{1}{ax}[(x^2 + y^2 + 1)(acx + bcy + a^2x^2 + abxy - bcy)] = 0$

$\Rightarrow \frac{1}{ax}[(x^2 + y^2 + 1)(acx + a^2x^2 + abxy)] = 0$

$\Rightarrow \frac{1}{ax}[ax(x^2 + y^2 + 1)(c + ax + by)] = 0$

$\Rightarrow (x^2 + y^2 + 1)(ax + by + c) = 0$

$\Rightarrow ax + by + c = 0$

Which represents a straight line.

Sol 11: (A) $\Delta = 1\left(1 - c\omega - a\left(\omega - \omega^2 c\right) + b\left(0\right)\right)$

$\Delta = 1c\omega - a\omega + \omega^2 ac$

$\Delta = 1 - \omega\left(c + a\right) + \omega^2 ac$

$c = \omega \;\; a = \omega^2$ singular

$c = \omega^2 \;\; a = \omega$ singular

$c = \omega \;\; a = \omega$ non singular

$c = \omega^2 \;\; a = \omega^2$ singular

for every pair (a, c) there are two possible values of b hence 2 matrices.

Sol 12: Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow a_2 = -1, b_2 = 2, c_2 = 3$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} . \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow a_1 - a_2 = 1$$

$\Rightarrow a_1=0,\ b_1=3,\ c_1=3$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} . \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow c_1 + c_2 + c_3 = 12$$

$\Rightarrow c_3 = 12 - 5 = 7$

$\therefore$ Sum of diagonal elements $= a_1 + b_2 + c_3 = 0 + 2 + 7 = 9$

Sol 13: (D) There seems to be an ambiguity in the question since 3 × 3 skew-symmetric matrices can't be non-singular.

[**Property:** Determinant of an odd order skew-symmetric matrix is always zero]

P is a 3×3 matrix

Let $P = \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \iota & m & n \end{bmatrix}$

$$P^T = \begin{bmatrix} a & \alpha & \iota \\ b & \beta & m \\ c & \gamma & n \end{bmatrix}$$

$P^T = 2P + I$

$$\begin{bmatrix} a & \alpha & \iota \\ b & \beta & m \\ c & \gamma & n \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2\alpha & 2\beta & 2\gamma \\ 2\iota & 2m & 2n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & \alpha & \iota \\ b & \beta & m \\ c & \gamma & n \end{bmatrix} = \begin{bmatrix} 2a+1 & 2b & 2c \\ 2\alpha & 2\beta+1 & 2\gamma \\ 2\iota & 2m & 2n+1 \end{bmatrix}$$

$= 2b = \alpha, b = 2\alpha$ .It is possible when $b = \alpha, = 0$

Similarly,, $c = \iota = 0$

$m = \gamma = 0$

The matrix P is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

So , $PX = -X$

Sol 14: (D) $P = \left[a_{ij}\right]$

$Q = \left[b_{ij}\right]$

$b_{ij} = 2^{i+j}.a_{ij}$

$b_{11} = 2^2 a_{11} \;\; b_{21} = 2^3.a_{21} \;\; b_{31} = 2^4 a_{31}$

$b_{12} = 2^3 a_{12} \;\; b_{22} = 2^4.a_{22} \;\; b_{32} = 2^5 a_{32}$

$b_{13} = 2^4 a_{13} \;\; b_{23} = 2^5.a_{22} \;\; b_{33} = 2^6 a_{33}$

Given $P = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$

$$Q = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$Q = 2^2.2^3.2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Q = 2^2.2^3.2^4 \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$Q = 2^2.2^3.2^4.2^2.2^1 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Q = 2^2.2^3.2^4.2^2.2^1.2^1 = 2^{13}$$

Sol 15: (B, C, D) $P^2 = 0$ only when n is multiple of 3.

$$\text{E.g: } \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} . \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore P^2 \neq 0$ when $n = 55, 56, 58$

Sol 16: (A, B) $MN = NM$

$N^2M = N(NM) = (NM)N = (MN)N = MN^2$

$(M - N^2(M + N^2)) = M^2 + MN^2 - N^2M - N^4 = M^2 - N^4$

As $M - N^2 \neq 0 \Rightarrow |M + N^2| = 0$

$|M^2 + MN^2 = |M(M + \_N^2)|| = |M|M + N^2||$

$= 0 \Rightarrow |M + N^2| = 0$

Sol 17: (D) When roots are purely imaginary.

Then the form of equation is $x^2 + K = 0$

where K is positive no.

Let $p(x) = x^2 + K$

$p(p(x)) = (p(x))^2 + K$

$p(p(x)) = (x^2 + K)^2 + K$

$p(p(x)) = x^4 + 2Kx^2 + K \Rightarrow p(p(x)) = 0$

$x^4 + 2Kx^2 + K = 0$

All coefficients are positive and no odd degree of x are present.

Sol 18: (A) $z = \dfrac{-1 + i\sqrt{3}}{2} = \omega$

$$p = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$p^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} (-\omega)^{2r} + (\omega^{2s})^2 & \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} \\ \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} & \omega^{2r}(\omega^r + \omega) \end{bmatrix} = -I \text{ (Given)}$$

r	s	r	s
1	1	1	1
2	2	3	3

Total no. pairs = 1

Sol 19: (B, C) $\left(\dfrac{P}{K}\right).Q = I \qquad \therefore\ Q = \left(\dfrac{P}{K}\right)^{-1}$

Comparing P_{23} we get,

$$\frac{-K}{8} = \frac{-K(3\alpha + 4)}{12\alpha + 20} \Rightarrow \alpha = -1$$

Also $|P||Q| = K^3$

$$\therefore (12\alpha + 20)\frac{K^2}{2} = K^3$$

$K = 6\alpha + 10 = 4$

Sol 20: (B)

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}$$

$$\therefore\ P^n = \begin{bmatrix} 1 & 0 & 0 \\ 4n & 1 & 0 \\ 8(n^2 + n) & 4n & 1 \end{bmatrix}$$

$$\therefore\ P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 8 \times (n^2 + n) & 4n & 1 \end{bmatrix}$$

$P^{50} - Q = I$

Equation we get $200 - q_{21} = 0 \Rightarrow q^{21} = 200$

$400 \times 51 - q_{31} = 0$

$q_{31} = 400 \times 51$

$200 - q_{32} = 0 \Rightarrow q_{32} = 200$

$$\frac{q_{31} + q_{32}}{q_{21}} = \frac{400 \times 51 + 200}{200} = 103$$

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