LECTURE

1

Algebra of Complex Numbers

BASIC CONCEPTS

1. General form of Complex Number (z)

$z = x + iy$ where, $i = \sqrt{-1}$,

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$

x = Real part of z = Re(z) and, y = – imaginary part of z = Im(z)

Note: Euler was the first mathematician to introduce the symbol i for the square root of -1.

2. Complex Number as an Ordered Pair

A complex number is defined as an ordered pair (x, y) of real numbers x and y. Thus, $z = (x, y)$, where,

abscissa = x = real part = Re(z) and

ordinate = y = imaginary part = Im(z)

3. z = purely real, if $y = 0$ i.e., imaginary part is zero = x.

4. z = purely imaginary if $x = 0$ i.e., Real part is zero = iy.

5. Integral powers of iota 'i'

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,

$i^{4n} = (i^4)^n = (1)^n = 1$,

$i^{4n+1} = (i^{4n})(i) = i$, $i^{4n+2} = -1$,

$\frac{1}{i} = -i$, $-\frac{1}{i} = i$,

Similarly, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$

Similarly i^m, for calculating integral power of i, dividing m by 4 and according to the remainder, find the value of i^m as follows

Imaginary Numbers	i^{4n-3}	i^{4n-2}	i^{4n-1}	i^{4n}	i^{4n+1}	i^{4n+2}	i^{4n+3}
Remainder	-3	-2	-1	0	1	2	3
Value	i	-1	$-i$	1	i	-1	$-i$

6. Order Relation There is no order relation between any two complex numbers i.e., if z_1 and z_2 are any two complex numbers, then, either $z_1 = z_2$ or $z_1 \neq z_2$, but not $z_1 > z_2$ or $z_1 < z_2$. i.e., $5 + 4i > 2 + 3i$ and $5 + 4i < 2 + 3i$ have no meaning.

7. Zero is only a number which is both real as well as purely imaginary.

8. Equality of Two Complex Numbers Two complex numbers are said to be equal if their corresponding real parts and imaginary parts are separately equal.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, if $z_1 = z_2 \Leftrightarrow x_1 = x_2$, $y_1 = y_2$

9. Algebra of Complex Numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then

(i) Addition:

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

(ii) Subtraction:

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$$

$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

(iii) Multiplication of Complex number:

$$z_1 = x_1 + iy_1; z_2 = x_2 + iy_2;$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(iv) Division of Complex number:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Note: Algebraic operations involving complex numbers are performed according to the same rule as in the operations involving real numbers with the convention that i^2 is replaced by -1.

10. $\sqrt{a}\sqrt{b} = \sqrt{ab}$, if at least one of a and b is nonnegative.

11. $\sqrt{a}\sqrt{b} = -\sqrt{ab}$, if both a and b are negative. e.g., $\sqrt{-2}\sqrt{-3} = -\sqrt{6}$

12. Negative of a Complex Number If $z = x + iy$, then

$-z = -x - iy$.

13. Integral powers of a Complex Number If $z = x + iy$, then

(*i*) $z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$

(ii) $z^3 = (x + iy)^3 = x^3 + i(3x^2y) - 3xy^2 - iy^3$

(iii) $z.z.z_3$ ..... k times $= z^k$

(iv) $z^0 = 1$

14. Reciprocal of a Complex Number $z = x + iy$, then reciprocal of z is denoted by

$\frac{1}{z}$ and $\frac{1}{z} = \frac{1}{x + iy}$.

It is also known as multiplicative inverse of complex number.

15. Conjugate of a Complex Numbers If $z = x + iy$, then the complex number $x - iy$ is called the complex conjugate of $\bar{z}$ and it is denoted by $\bar{z}$ which is as follows: $\bar{z} = \overline{x + iy} = x - iy$.

15.1 A complex number is purely real, if $z = \bar{z}$ $(y = 0)$ and purely imaginary, if $\bar{z} = -z$ $(x = 0)$.

Note: The sum and product of a complex number with its conjugate are both real.

16. Properties of Complex Numbers

If $z = x + iy$ then $\bar{z} = x - iy$

(i) $z + \bar{z} = 2x = 2\text{Re}(z) = 2\text{Re}(\bar{z})$

= Sum of a complex number with its conjugate.

(ii) $z - \bar{z} = 2iy = 2i\,\text{Im}(z) = -2i\,\text{Im}(\bar{z})$

(iii) $z \,.\, \bar{z} = x^2 + y^2 = |z|^2$

= Product of a complex number with its conjugate.

Note: The addition and multiplication of anytwo complex numbers are a real number, thenboth are a pair of conjugate complex numbers.

(iv) $\overline{(\bar{z})} = z$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(ix) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(*x*) $\overline{\overline{z_1} z_2} = z_1 \overline{z_2}$

(xi) $z_1 \overline{z_2} + \overline{z_1} z_2 = 2\text{Re}(\overline{z_1} z_2)$

$= 2\text{Re}(z_1 \overline{z_2})$

17. Properties of Addition and Multiplication of Complex

Numbers

(*i*) $z_1 + z_2 = z_2 + z_1$ = one complex number

(ii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iii) $z + 0 = 0 + z = z$

(iv) $z + (-z) = (-z) + z = 0$

(v) $z_1 + z_2 = z_1 + z_3 \Rightarrow z_2 = z_3$

(vi) $z_1 . z_2 = z_2 . z_1$ = one complex number

(vii) $(z_1 z_2)(z_3) = z_1 (z_2 z_3)$

(viii) $z . 1 = 1. z = z$

(ix) $z_1 z_2 = z_1 z_3 \Rightarrow z_2 = z_3$ or $z_1 = 0$

(x) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(xi) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find real values of x and y for which the complex numbers $-3 + i x^2 y$ and $x^2 + y + 4i$ are conjugate of each other.

Solution

Since $-3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugates.

Therefore, $-3 + i x^2 y = x^2 + y + 4i$

$\Rightarrow 3 + i x^2 y = x^2 + y - 4i$

$\Rightarrow -3 = x^2 + y$ (1)

and $x^2 y = -4$ (2)

$\Rightarrow -3 = x^2 - \dfrac{4}{x^2}$

[Putting $y = \dfrac{-4}{x^2}$ from (2) in (1)]

$\Rightarrow x^4 + 3x^2 - 4 = 0 \quad \Rightarrow (x^2 + 4)(x^2 - 1) = 0$

$\Rightarrow x^2 - 1 = 0$ [$\because x^2 + 4 \neq 0$ for any real x]

$\Rightarrow x = \pm 1$

From (2), $y = -4$, when $x = \pm 1$

Hence, $x = 1, y = -4$ or $x = -1, y = -4$

2. If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution

We have $x = -5 + 2\sqrt{-4}$

$\Rightarrow x + 5 = 4i \Rightarrow (x + 5)^2 = 16i^2$

$\Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2 (x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$= x^2 (0) - x(0) + 4(0) - 160$

$= -160$ [$\because x^2 + 10x + 41 = 0$]

3. If $a^2 + b^2 = 1$, then prove that $\dfrac{1 + b + ia}{1 + b + ia} = b + ia$.

Solution

$$\text{L.H.S.} = \frac{1 + b + ia}{1 + b - ia} = \left(\frac{1 + b + ia}{1 + b - ia}\right)\left(\frac{b + ia}{b + ia}\right)$$

$$= \frac{(1 + b + ia)(b + ia)}{b + ia + b^2 + iab - iab + a^2}$$

$$= \frac{(1 + b + ia)(b + ia)}{a^2 + b^2 + b + ia}$$

$$= \frac{(1 + b + ia)(b + ia)}{1 + b + ia} \quad [\text{Given } a^2 + b^2 = 1]$$

$= b + ia$ = R.H.S.

Proved

4. If $(x + iy)^3 = u + iv$, then show that $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$.

[NCERT]

Solution

Given $(x + iy)^3 = u + iv$

$\Rightarrow x^3 + i^3 y^3 + 3x (iy)(x + iy) = u + iv$

$\Rightarrow x^3 + (-i) y^3 + 3x^2 yi - 3xy^2 = u + iv$

$\Rightarrow x^3 - 3xy^2 + i(3x^2 y - y^3) = u + iv$

Equating real and imaginary parts, we get

$u = x^3 - 3xy^2$

$\Rightarrow u = x(x^2 - 3y^2)$ and $v = 3x^2 y - y^3$

$\Rightarrow v = y(3x^2 - y^2)$

$\because \dfrac{u}{x} + \dfrac{v}{y} = x^2 - 3y^2 + 3x^2 - y^2$

or $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$

5. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

[NCERT]

Solution

Given, $(x - iy)(3 + 5i) = \overline{-6 - 24i}$

$\Rightarrow x - iy = \dfrac{-6 + 24i}{3 + 5i} \times \dfrac{3 - 5i}{3 - 5i}$

$= \dfrac{-18 - 120(-1) + 72i + 30i}{9 - 25(-1)} = \dfrac{102 + 102i}{34}$

$= 3 + 3i$

$\Rightarrow x - iy = 3 + 3i$

$\Rightarrow x = 3$ and $-y = 3$

$\Rightarrow x = 3, y = -3.$

(Equating real and imaginary parts)

6. If $\left(\dfrac{1+i}{1-i}\right)^m = 1$, then find the least + ve integral value of m.

[NCERT]

Solution

Given $\left(\dfrac{1+i}{1-i}\right)^m = 1$

$\Rightarrow \left\{\dfrac{(1 \times i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)}\right\}^m = 1$

$\Rightarrow \left\{\dfrac{(1+i)^2}{1^2 - i^2}\right\}^m = 1$

$\Rightarrow \left\{\dfrac{1 + i^2 + 2i}{1 - (-1)}\right\}^m = 1$

$\Rightarrow \left(\dfrac{2i}{2}\right)^m = 1 \Rightarrow i^m = 1$

$\Rightarrow$ least positive integral value of m is 4.

7. Express each of the complex number given in the form $a + ib$.

$\left[\left(\dfrac{1}{3} + i\,\dfrac{7}{3}\right) + \left(4 + i\,\dfrac{1}{3}\right) - \left(-\dfrac{4}{3} + i\right)\right.$

Solution

Given complex number

$= \left[\left(\dfrac{1}{3} + i\,\dfrac{7}{3}\right) + \left(4 + i\,\dfrac{1}{3}\right) - \left(-\dfrac{4}{3} + i\right)\right.$

$= \left[\left(\dfrac{1}{3} + 4\right) + \left(\dfrac{7}{3} + \dfrac{1}{3}\right)i\right] - \left(-\dfrac{4}{3} + i\right)$

$= \left(\dfrac{13}{3} + \dfrac{8}{3}i\right) - \left(-\dfrac{4}{3} + i\right)$

$= \left\{\dfrac{13}{3} - \left(-\dfrac{4}{3}\right)\right\} + \left(\dfrac{8}{3} - 1\right)i = \dfrac{17}{3} + \dfrac{5}{3}i$

which is in the form $a + ib$, where $a = \dfrac{17}{3}$ and $b = \dfrac{5}{3}$.

8. Express each of the complex number givern in the form $a + ib\left(\dfrac{1}{3} + 3i\right)^3$

Solution

Given complex number

$= \left(\dfrac{1}{3} + 3i\right)^3 = \left(\dfrac{1}{3}\right)^3 + (3i)^3 + 3\left(\dfrac{1}{3}\right)(3i)\left(\dfrac{1}{3} + 3i\right)$

$= \dfrac{1}{27} + 27i^3 + 3i\left(\dfrac{1}{3} + 3i\right)$

$= \dfrac{1}{27} + 27(-i) + i + 9i^2$

$(\because i^3 = i^2 \cdot i = -i)$

$= \dfrac{1}{27} - 27i + i + 9(-1) = \left(\dfrac{1}{27} - 9\right) - 26i$

$= \left(-\dfrac{242}{27}\right) + (-26)\,i =$

which is in the form $a + ib$, where,

$a = -\dfrac{242}{27}, b = -26$

9. Express the complex number given in the form $a + ib\left(-2 - \dfrac{1}{3}i\right)^3$

Solution

Given complex number

$= \left(-2 - \dfrac{1}{3}i\right)^3 = \left\{-1\left(2 + \dfrac{1}{3}i\right)\right\}^3$

$= -\left\{2^3 + \dfrac{1}{27}i^3 + 3(2)\left(\dfrac{1}{3}i\right)\left(2 + \dfrac{1}{3}i\right)\right\}$

$= -\left\{8 + \dfrac{1}{27}(-i) + 2i\left(2 + \dfrac{1}{3}i\right)\right\}$

$= -8 + \dfrac{1}{27}i - \left(4i + \dfrac{2}{3}i^2\right)$

$= -8 + \dfrac{1}{27}i - 4i + \dfrac{2}{3}\ (\because i^2\ -1)$

$= \left(-8 + \dfrac{2}{3}\right) + \left(\dfrac{1}{27} - 4\right)i$

$= \left(-\frac{22}{3}\right) + \left(\frac{107}{27}\right)i$

which is in the form $a + ib$ where $a = -\frac{22}{3}$ and $b = -\frac{107}{27}$

10. Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

Solution

Given number

$= \left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = \{i^{16}\, i^2 + (-i)^{25}\}^3$

$\left(\because \frac{1}{i} = \frac{1}{i^2} = \frac{1}{-1}\right)$

$= \{(i^4)^4\,(-1) + (-1)(i^4)^6\, i^1\}^3$

$= (1\,(-1) - i)^3 = (-1 - i)^3 = (-1)^3\,(1 + i)^3$

$= -1\{1^3 + i^3 + 3i\,(1 + i)\}$

$= -\{1 + (-i) + 3i + 3\,(-1)\}$

$(\because i^2 = -1 \text{ and } i^3 = -i)$

$= -\{-2 + 2i\} = 2 - 2i.$

11. Prove that $\left[\frac{1+i}{1-i}\right]^n = i^n$

[MP – 1997]

Solution

$\left[\frac{1+i}{1-i}\right]^n = \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^n$

$= \left[\frac{1+i+2i}{1-i^2}\right]^n = \left[\frac{1-1+2i}{1+1}\right]^n, \quad [\because i^2 = 1]$

$= \left[\frac{2i}{2}\right] = i^n.$

12. Express each of the following in the standard form $a + ib$

$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

Solution

$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$= \frac{(6+6) + i\,(-4+9)}{(2+2) + i(4-1)} = \frac{12+5i}{4+3i}$

$= \frac{12+5i}{4+3i} \cdot \frac{4-3i}{4-3i}$

$= \frac{(48+15) + i(-36+20)}{16 - 9i^2}$

$= \frac{63}{25} - \frac{16}{25}\,i.$

13. Reduce $\left\{\frac{1}{1-4i} - \frac{2}{1+i}\right\}\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Solution

Given number

[NCERT]

$= \left\{\frac{1}{1-4i} - \frac{2}{1+i}\right\}\left(\frac{3-4i}{5+i}\right)$

$= \left\{\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\right\}\frac{3-4i}{5+i}$

$= \left(\frac{-1+9i}{1+4-3i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \frac{-3+36+27i+4i}{25+3-15i+5i}$

$= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i}$

$= \frac{33 \times 28 - 31 \times 10 + (31 \times 28 + 33 \times 10)i}{(28)^2 - (10i)^2}$

$= \frac{924 - 310 + (868 + 330)i}{784 - 100(-1)}$

$= \frac{614 + 1198i}{884} = \frac{614}{884} + \frac{1198}{884}\,i$

$= \frac{307}{442} + \frac{559}{442}\,i$

14. Find real θ such that $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is purely real.

Solution

We have $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$

$= \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{(1 - 2i\sin\theta)(1 + 2i\sin\theta)}$

$= \frac{(3 + 6i\sin\theta + 2i\sin q - 4\sin^2\theta)}{1 + 4\sin^2\theta}$

$= \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} + i\,\frac{8\sin\theta}{1 + 4\sin^2\theta}$

We are given the complex number to be real.

Therefore,

$\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0$, i.e., $\sin\theta = \theta$. Thus $\theta = n\pi$, $n \in Z$.

15. Find the value of $x^3 + 7x^2 - x + 16$, when $x = 1 + 2i$.

Solution

We have, $x = 1 + 2i$

$\Rightarrow x - 1 = 2i$

$\Rightarrow (x - 1)^2 = 4i^2$

$\Rightarrow x^2 - 2x + 1 = -4$

$\Rightarrow x^2 - 2x + 5 = 0$

Now, $x^3 + 7x^2 - x + 16$

$= x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$

$= x(0) + 9(0) + 12x - 29$ $\quad [\because x^2 - 2x + 5 = 0]$

$= 12(1 + 2i) - 29$ $\quad [\because x = 1 + 2i]$

$= -17 + 24i$

Hence, the value of the polynomial when $x = 1 + 2i$ is $-17 + 24i$.

UNSOLVED SUBJECTIVE PROBLEMS (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Evaluate the following

(i) i^{135} (ii) i^{-999}

(iii) $\left(i^{37} + \frac{1}{i^{67}}\right)$

2. Compute the following

(i) $\sqrt{-144}$ (ii) $\sqrt{-4}\sqrt{-\frac{9}{4}}$

3. Show that

(i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = -4$

(ii) $i^{107} + i^{112} + i^{117} + i^{122} = 0$

4. Add $-1 + 3i$ and $5 - 8i$

5. Subtract $7 - 3i$ from $6 + 5i$

6. Find the real values of x and y, if

(i) $(3x - 7) + 2iy = -5y + (5 + x)i$

(ii) $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

7. Express each of the following in the form $a + ib$

(i) $(3 + 4i)^2$

(ii) $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$

8. Express each of the following in the standard form $a + ib$

(i) $\frac{1}{3 - 4i}$

(ii) $\left(\frac{5 + \sqrt{2}i}{1 - \sqrt{2}i}\right)$

9. Prove that the following complex numbers are purely real:

$\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right)$

10 Find the conjugate of

(i) i^3

(ii) $\frac{2-5i}{3-2i}$

11. Find real values of x and y for which the following equalities hold. $(1 + i)y^2 + (6 + i) = (2 + i)x$

Exercise II

1. Evaluate the following:

(i) $(-\sqrt{-1})^{4n+3}$

(ii) $(-i)(3i)\left(-\frac{1}{6}i\right)^3$

2. Compute the following

$\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$

3. Show that $(1 + i)^4 \times \left(1 + \frac{1}{i}\right)^4$ for all $n \in N$.

4. Find $Z_1 + Z_2$ and $Z_1 - Z_2$ if $Z_1 = 3 + 5i$ and $Z_2 = -5 + 2i$

5. Find the real values of x and y if

(i) $(1 - i)x + (1 + i)y = 1 - 3i$

(ii) $(x + iy)(2 - 3i) = 4 + i$

6. Express each of the following in the form $a + ib$

(i) $(2 + 3i)(4 - 5i)$ (ii) $(4 - 3i)^3$

7. Express each of the following in the standard form $a + ib$

(i) $\frac{(1+i)^2}{3-i}$ (ii) $(-1 + \sqrt{3}i\ i)^{-1}$

8. Prove that the following complex numbers are purely real.

$$\left(\frac{3+2i}{2-3i}\right) + \left(\frac{3-2i}{2+3i}\right)$$

9. Find the conjugate of $(6 + 5i)^2$

ANSWERS

Exercise-I

1. (i) $-i$ (ii) i (iii) $2i$

2. (i) $12i$ (ii) -3

4. $4 - 5i$

5. $-1 + 8i$

6. (i) $x = -1, y = 2$ (ii) $x = -4, y = 6$

7. (i) $-7 + 24i$ (ii) $-7\sqrt{3}$

8. (i) $\frac{3}{25} + \frac{4}{25}i$ (ii) $(1 + 2\sqrt{2}\ i)$

10. (i) i (ii) $\frac{16}{13} + \frac{11}{13}i$

11. Ans $x = 5$ and $y = 2$ or $x = 5$ and $y = -2$

Exercise-II

1. (i) i (ii) $i/72$

2. 0

4. $Z_1 + Z_2 = -2 + 7i$, $Z_1 - Z_2 = 8 + 3i$

5. (i) $x = 2, y = -1$

(ii) $x = 5/13, y = 14/13$

6. (i) $23 + 2i$

(ii) $-44 - 117i$

7. (i) $-\frac{1}{5} + \frac{3}{5}i$ (ii) $\left(-\frac{1}{4} - \frac{\sqrt{3}}{4}i\right)$

9. $11 - 60i$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The conjugate of the complex number $\frac{(1+i)^2}{1-i}$ is

(a) $1 - i$ (b) $1 + i$

(c) $-1 + i$ (d) $-1 -$i

[Karnataka CET – 2007]

Solution

(d) $\frac{(1+i)^2}{1-i} = \frac{1+i^2+2i}{1-i} = \frac{1-1+2i}{1-i}$

$= \frac{2i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{2i(1+i)}{1-(i)^2} = \frac{2i(1+i)}{1-(-1)} = \frac{2i(1+i)}{2}$

$= i + i^2 = i - 1$

Therefore, the required conjugate is $-i - 1$

2. The real part of $(1 - \cos\theta + 2\, i \sin\theta)^{-1}$ is **[IIT – 1978, 1986]**

(a) $\frac{1}{3 + 5\cos\theta}$ (b) $\frac{1}{5 - 3\cos\theta}$

(c) $\frac{1}{3 - 5\cos\theta}$ (d) $\frac{1}{5 + \cos\theta}$

Solution

(d) $\{(1 - \cos\theta) + i.2\sin\theta\}^{-1} =$

$\left\{2\sin^2\frac{\theta}{2} + i \cdot 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2} + i2\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}$

$\frac{1}{\sin\frac{\theta}{2} + i.2\cos\frac{\theta}{2}} \times \frac{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}$

The real part is

$$\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)}$$

$$=\frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)}$$

$$=\frac{1}{5+3\cos\theta}$$

Alternative Method: Put $\theta = \pi$ to verify (d)

3. If $(a + ib)(x + iy) = (a^2 + b^2)i$, then (x, y) is equal to

(a) (a, b) (b) (b, a)

(c) $(-a, b)$ (d) $(a, -b)1$

[VIT – 2004]

Solution

(b) $(ax - by) + i(ay + bx) = (a^2 + b^2)i$

equating real and imaginary parts

$ax - by = 0 \Rightarrow ax = by$ (1)

$a^2 + b^2 = ay + bx$ (2)

$a^2 + b^2 = ay + \frac{b \times by}{2}$ from equation (1)

$\frac{a(a^2+b^2)}{a^2+b^2} = y \Rightarrow y = a$ and $x = b$

$\therefore (x, y) = (b, a)$

4. If $x = 2 + 5i$ (where $i^2 = -1$), then $x^3 - 5x^2 + 33x - 19 =$

(a) 6 (b) 8

(c) 10 (d) 12

Solution

(c) Given $x = 2 + 5i$

$\Rightarrow x - 2 = 5i$

$\Rightarrow (x - 2)^2 = 25(-1)$

$\Rightarrow x^2 - 4x + 4 + 25 = 0$

$\Rightarrow x^2 - 4x + 29 = 0$

Hence, $x^3 - 5x^2 + 33x - 10$

$= x(x^2 - 4x + 29) - x^2 + 4x - 19$

$= x(x^2 - 4x + 29) - 1(x^2 - 4x + 29) + 29 - 19$

$= x(x^2 - 4x + 29) - (x^2 - 4x + 29) + 10$

$= 0 + 10$ when $x = 2 + 5i$

$(\because x^2 - 4x + 29 = 0$ when $x = 2+5i)$

5. The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is

[AIEEE – 2008]

(a) $\frac{1}{i-1}$ (b) $\frac{-1}{i-1}$

(c) $\frac{1}{i+1}$ (d) $\frac{-1}{i+1}$

Solution

(d) $\bar{z} = \frac{1}{i-1}$ (given)

We have $z = (\bar{\bar{z}})$ giving

$$z = \frac{1}{\overline{i-1}} = \frac{1}{-i-1} = \frac{-1}{i+1}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. $2\sqrt{-9}\,\sqrt{-16}$ is equal to

(a) 24 (b) – 24

(c) 48 (d) – 48

2. If $x, y \in R$, then $x + yi$ is a non-real complex number, if

(a) $x = 0$ (b) $y = 0$

(c) $y \neq 0$ (d) $x \neq 0$

3. If n is any integer, then $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to

(a) i (b) $-i$

(c) 1 (d) 0

[WB JEE – 2009]

4. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then the least positive integral value of m is

(a) 2 (b) 4

(c) 8 (d) none of these

[IIT – 1982; MNR – 1984; UPSEAT – 2001; MPPET – 2002]

5. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then

[MPPET – 1996]

(a) $x = \frac{1}{5}$

(b) $y = \frac{3}{5}$

(c) $x + iy = \frac{1 - i}{1 - 2i}$

(d) $x - iy = \frac{1 - i}{1 + 2i}$

6. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{(x/p + y/q)}{(p^2 + q^2)}$ is equal to

(a) 2 (b) – 1

(c) 1 (d) – 2

[AIEEE – 2004]

7. Let z_1, z_2 be two complex numbers such that $z_1 + z_2$ and $z_1 z_2$ both are real, then

[RPET – 1996]

(a) $z_1 = -z_2$ (b) $z_1 = \bar{z}_2$

(c) $z_1 = -\bar{z}_2$ (d) $z_1 = z_2$

8. The real part of $\frac{1}{1 - \cos\theta + i\sin\theta}$ is equal to

(a) 1/4 (b) 1/2

(c) $\tan\theta/2$ (d) $1/1 - \cos\theta$

[Karnataka CET – 2001, 2005; MPPET – 2006]

9. Multiplicative inverse of non-zero complex number $a + ib$ $(a, b \in R)$ is

(a) $\frac{a}{a - b} - \frac{b}{a + b} \times i$

(b) $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$

(c) $-\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2} i$

(d) $\frac{a}{a + b} + \frac{b}{a + b} i$

10. If $\frac{i^4 + i^9 + i^{16}}{2 - i^8 + i^{10} + i^3} = a + ib$, then (a, b) is

[Kerala PET – 2008]

(a) (1, 2) (b) (– 1, 2)

(c) (2, 1) (d) (– 2, – 1)

11. If Z is a complex number such that $Z = -\bar{Z}$ then

(a) Z is any complex number.

(b) Real part of Z is the same as its imaginary part.

(c) Z is purely real.

(d) Z is purely imaginary.

[Karnataka CET – 2008]

12. If $z = 3 + 5i$, then $z^3 + \bar{z} + 198 =$

(a) $-3 - 5i$ (b) $-3 + 5i$

(c) $3 + 5i$ (d) $3 - 5i$

[EAMCET – 2002]

SOLUTIONS

1. (b) $2\sqrt{-9}\ \sqrt{-16} = 2\,(3i)(4i) = 24\,i^2 = -24.$

2. (c) $x + yi;\ x, y \in R$ is non real if $y \neq 0$.

3. (d) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n\,[1 + i + i^2 + i^3]$

$= i^n\,[1 + i - 1 - i] = 0.$

4. (b) $1 = \left(\frac{1 + i}{1 - i}\right)^m = \left[\frac{i(1 - i)}{(1 - i)}\right]^m = i^m$

$\Rightarrow i^m = 1 \Rightarrow m = 4.$

5. (c) Given

$\overline{(x + iy)(1 - 2i)} = 1 + i$ (1)

using formula: $\overline{(\bar{z})} = z$ we find from (1)

$(x + iy)(1 - 2i) = \overline{1 + i} = 1 - i$

$(x + iy)(1 - 2i) = 1 - i$

$x + iy = \frac{1 - i}{1 - 2i}.$

6. (d) Given $z^{1/3} = p + iq \Rightarrow z = (p + iq)^3$

$\Rightarrow x - iy = p^3 + (iq)^3 + 3piq\,(p + iq)$

$\Rightarrow x - iy = p^3 - 3pq^2 + (3p^2q - q^3)\,i$

Equating real and imaginary parts,

$x = p^3 - 3pq^2 \quad \Rightarrow \frac{x}{p} = p^2 - 3q^2$ (1)

and $-y = 3p^2q - q^3$

$\Rightarrow \dfrac{y}{q} = q^2 - 3p^2$ (2)

Add (1) and (2) to obtain

$\dfrac{x}{p} + \dfrac{y}{q} = -2(p^2 + q^2)$

7. (b) Let $z_1 = a + ib$, $z_2 = c + id$, then

$z_1 + z_2$ is real $\Rightarrow (a + c) + i(b + d)$ is real

$\Rightarrow b + d = 0 \Rightarrow d = -b$ (3)

z_1z_2 is real $\Rightarrow (ac - bd) + i\,(ad + bc)$ is real

$\Rightarrow ad + bc = 0$

$\Rightarrow a(-b) + bc = 0 \Rightarrow a = c.$

Therefore, $\Rightarrow z_1 = a + ib = c - id = \bar{z}_2$

$(\therefore\ a = c$ and $b = -d)$

8. (b) $\dfrac{1}{(1 - \cos\theta) + i\sin\theta}$

$\dfrac{\{(1 - \cos\theta) - i\sin\theta\}}{\{(1 - \cos\theta) + i\sin\theta)\}\{(1 - \cos\theta) - i\sin\theta)\}}$

$= \dfrac{(1 - \cos\theta) - i\sin\theta}{(1 - \cos\theta)^2 - i^2\sin^2\theta}$

$= \dfrac{(1 - \cos\theta) - i\sin\theta}{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}$

$= \dfrac{(1 - \cos\theta) - i\sin\theta}{2(1 - \cos\theta)}$

$= \dfrac{1}{2} - \dfrac{i\sin\theta}{2(1 - \cos\theta)}$

Real part of $\left(\dfrac{1}{(1 - \cos\theta) + i\sin\theta}\right) = \dfrac{1}{2}.$

9. (b) $(a + bi)^{-1}$

$= \dfrac{1}{a + bi} = \dfrac{a - bi}{(a + bi)(a - bi)}$

$= \dfrac{a - bi}{a^2 + b^2}.$

10. (b) $\dfrac{i^4 + i^9 + i^{16}}{2 - i^8 + i^{10} + i^3} = a$

$\dfrac{1 + i + 1}{2 - 1 + i^2 + i \times i^2} = a + ib$

$\dfrac{2 + i}{1 - 1 - i} = a + ib$

$\dfrac{(2 + i)}{(-i)} \times \dfrac{(i)}{i} = a + ib$

$2i - 1 = a + ib$

Comparing real and imaginary part $a = -1$, $b = 2$

$(a, b) = (-1, 2).$

11. (d) $z = -\bar{z} \Rightarrow x + iy = -(x - iy) = -x + iy$

i.e., $2x = 0$ or $x = 0$

i.e., $z = iy$ = purely imaginary.

12. (c) $z = 3 + 5i$

$\bar{z} = 3 - 5i$

$z^3 = (3 + 5i)^3$

$= 27 + 125i^3 + 45i\,(3 + 5i)$

$= 27 - 125\,i + 135i + 225\,i^2$

$= 27 - 225 + 10i$

$= -198 + 10i$

$z^3 + \bar{z} + 198$

$= -198 + 10i + 3 - 5i + 198$

$= 3 + 5i$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\left(\dfrac{1 + i}{1 - i}\right)^2 + \left(\dfrac{1 - i}{1 + i}\right)^2$ is equal to

(a) $2i$

(b) $-2i$

(c) -2

(d) 2

2. The conjugate of the complex number $\dfrac{2 + 5i}{4 - 3i}$ is **[MPPET – 1994]**

(a) $\dfrac{7 - 26i}{25}$ (b) $\dfrac{-7 - 26i}{25}$

(c) $\dfrac{-7 + 26i}{25}$ (d) $\dfrac{7 + 26i}{25}$

3. $\left\{\frac{2i}{1+i}\right\}^2 =$

[MNR – 1984; BIT Ranchi – 1992]

(a) 1 (b) $2i$
(c) $1 - i$ (d) $1 - 2i$

4. The imaginary part of $\frac{(1+i)^2}{2-i}$ is

(a) 1/5 (b) 3/5
(c) 4/5 (d) none of these

5. If $\frac{5(-8+6i)}{(1+i)^2} = a + ib$, then (a, b) equals

[RPET – 1986]

(a) (15, 20) (b) (20, 15)
(c) (– 15, 20) (d) none of these

6. The true statement is

[Roorkee – 1989]

(a) $1 - i < 1 + i$
(b) $2i + 1 > -2i + 1$
(c) $2i > 1$
(d) none of these

7. If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$ then value of k equals

[DCE – 2005,MPPET–2009]

(a) 2 (b) 4
(c) 6 (d) 1

8. If $x = 2 + 3i$ and $y = 2 - 3i$, then value of $x^3 + y^3$ is

(a) 92 (b) – 92
(c) 46 (d) – 46

9. Real part of $\frac{i}{3+2i}$ is

(a) 3 (b) 2/13
(c) 13/2 (d) 13

10. Conjugate of $\frac{2+3i}{-i+1}$ is

(a) $\frac{2-3i}{i+1}$ (b) $\frac{2-3i}{1-i}$
(c) $\frac{2+3i}{1+i}$ (d) $\frac{2+3i}{1+i}$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet
2. The test is of 13 minutes.
   The test consists of 13 questions.
   The maximum marks are 39.
3. Use blue / black ball point pen only for writing particulars / marking responses. Use of pencil is strictly prohibited.
4. Rough work is to be done on the space provided for this purpose on the worksheet – 1 sheet only.

1. If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{a}$ is equal to

(a) $4(a^2 + b^2)$
(b) $4(a^2 - b^2)$
(c) $4(b^2 - a^2)$
(d) none of these

[IIT – 1982, Karnataka CET – 2000]

2. The number $\frac{(1-i)^3}{(1-i^3)}$ is equal to

[Pb. CET – 1991]

(a) i (b) $-i$
(c) – 1 (d) – 2

3. The value of $-i^{51}$ is

(a) $-i$ (b) 1
(c) – 1 (d) i

4. Which of the following is not applicable for a complex number?

(a) addition (b) subtraction
(c) division (d) inequality

5. The value of

$$\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} =$$

(a) – 1 (b) – 2
(c) – 3 (d) – 4

6. The least positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$ is

(a) 2 (b) 4
(c) 1 (d) 8

7. $i^{57} + \dfrac{1}{i^{125}}$ is equal to

(a) 0 (b) $2i$
(c) $-2i$ (d) 2

8. If $2x = 3 + 5i$, then what is the value of $2x^3 + 2x^2 - 7x + 72$?

(a) 4 (b) -4
(c) 8 (d) -8

[NDA–2009]

9. If $x, y \in R$, then the complex number $x + yi$ is purely imaginary, if

(a) $x = 0, y \neq 0$
(b) $x \neq 0, y = 0$
(c) $x \neq 0, y \neq 0$
(d) $x = 0, y = 0$

10. $1 + i + i^2 + i^3$ is equal to

(a) i (b) 0
(c) $-i$ (d) 1

11. If z is a complex number such that $z \neq 0$ and $\text{Re } z = 0$, then

(a) $\text{Re}(z^2) = 0$
(b) $\text{Re}(z^2) = \text{Im}(z^2)$
(c) $\text{Im}(z^2) = 0$
(d) none of these

12. Which of the following is correct?

(a) $5 + 3i > 6 + 4i$
(b) $5 + 3i = 6 + 4i$
(c) $5 + 3i < 6 + 4i$
(d) none of these

13. The conjugate of complex number $\dfrac{2-3i}{4-i}$ is

[MPPET – 2003]

(a) $\dfrac{3i}{4}$

(b) $\dfrac{11+10i}{17}$

(c) $\dfrac{11-10i}{17}$

(d) $\dfrac{2+3i}{4i}$

ANSWER SHEET

1.	a	b	c	d
2.	a	b	c	d
3.	a	b	c	d
4.	a	b	c	d
5.	a	b	c	d
6.	a	b	c	d
7.	a	b	c	d
8.	a	b	c	d
9.	a	b	c	d
10.	a	b	c	d
11.	a	b	c	d
12.	a	b	c	d
13.	a	b	c	d

HINTS AND EXPLANATIONS

5. (b) $\dfrac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)} = i^{584-574}$

$= (i)^{10} = (i^2)^5 = (-1)^5 = -1$

8. (a) $\because\ x = \dfrac{3+5i}{2}$

$2x - 3 = 5i$

squaring we get

$2x^2 - 6x + 17 = 0$

Now

$$\begin{array}{r|l} & x + 4 \\ 2x^2 - 6x + 17 & 2x^3 + 2x^2 - 7x + 72 \\ & 2x^2 - 6x^2 + 17x \\ \hline & 8x^2 - 24x + 72 \\ & 8x^2 - 24x + 68 \\ \hline & 4 \end{array}$$

$\therefore\ 2x^3 + 2x^2 - 17x + 72$

$= (2x^2 - 6x + 12)(x + 4) + 4$

$= 4$

9. (a) $x + yi$; $x, y \in R$ is purely imaginary if $x = 0$ and $y \neq 0$.

10. (b) $1 + i + i^2 + i^3$

$= 1 + i - 1 + (-1)\, i$

$= 1 + i - 1 - i$

$= 0$

11. (c) Let $z = iy$; $y \in R, y \neq 0$

then $z^2 = (iy)^2 = -y^2 \Rightarrow \text{Im}\,(z^2) = 0$.

13. (b) $\dfrac{(2-3i)}{(4-i)} \times \dfrac{(4+i)}{(4+i)} = \dfrac{8 + 2i - 12i - 3i^2}{(4)^2 - i^2}$

$\dfrac{11 - 10i}{16 + 1} = \dfrac{11 - 10i}{17}$

Conjugate of complex number $= \dfrac{11 + 10i}{17}$

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LECTURE

2 Argand Plane Modulus and Amplitude

BASIC CONCEPTS

1. **Geometrical Representation of a Complex Number** The plane on which complex numbers are represented is known as the complex plane or Argand's plane or Gaussian plane. In this representation all real numbers lie on x-axis and imaginary numbers lie on y-axis. The x-axis is called the real axis and y-axis is known as the imaginary axis.

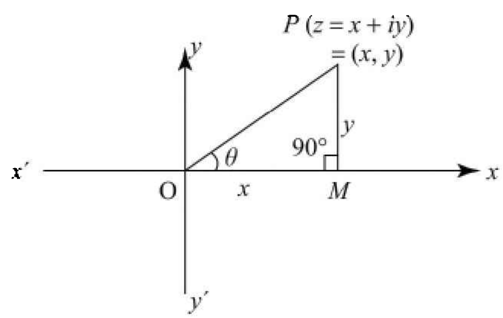
$-\bar{z} = -(\bar{z}) = -x + iy$ $= (-x, y)$	y-axis $z - x + iy = (x, y)$
Origin $-z = -x - iy$ $= (-x, y)$	Real axis $\bar{z} = x - iy$ $= (x, -y)$

The complex numbers $z = x + iy$ may be represented by a unique point in xy-plane the coordinates of which are (x, y). One-to-one correspondence is defi ned between the set of complex numbers and set of all point of Argand's plane or xy-plane.

(i) Distance between two points z_1 and z_2 = $|z_1 - z_2|$

(ii) Complex numbers are defined as vectors. The magnitude and direction of vectors are called magnitude and amplitude of complex numbers.

2. **Modulus—Amplitude Form or Polar Form of a Complex Number**



(i) $z = r(\cos\theta + i\sin\theta)$

(ii) The modulus of z is denoted by $|z|$ and $|z| = \sqrt{x^2 + y^2} = r \geq 0$

(iii) The argument or amplitude of z is denoted by arg (z) or amp (z). In the first quadrant $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ and in other quadrants, arg (z) is defined by the solution of the following two equations which are as follows

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}};\ \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

(iv) The principal value of argument of a complex number lies between $-\pi$ and π. i.e., the value of θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the principal value of the argument.

3. **Quicker Method for Finding Amplitude of a Complex Number Quadrantwise** In this method first of all quadrant of

complex number is known as follows, Here quadrantwise complex number with corresponding argument are given

$(-\bar{z}) = -(\bar{z}) = -x + iy$ amp $(-\bar{z})$ = amp $(-(\bar{z}))$ $= \pi - \theta$	$z = (x, y) = x + iy$ if amp $(z) = \theta$, then
amp $(-z) = -(\pi - \theta)$ $-z = -x - iy = (-x, -y)$	$\bar{z} = x - iy = (x, y)$ amp $(\bar{z}) = -\theta$

Note: $|z| = |-z| = |(-\bar{z})| = (\bar{z}) |=| - (\bar{z})|$

$= \sqrt{x^2 + y^2}$

Note: In this method corresponding form of a given complex number in first quadrant is obtained and argument of this complex number is obtained by the formula $\theta = \tan^{-1}\frac{y}{x}$. After this, amplitude of given complex number is obtained quadrant-wise

(v) If amp (z) i.e., θ is greater than π, then principal value of argument is equal to $\theta - 2\pi$.

(vi) If amp (z) i.e., θ is less than $-\pi$, then principal value of amplitude is $\theta + 2\pi$.

(vii) If θ is the principal value of the argument of a complex number, then its general value is denoted by $\theta + 2n\pi$, where n is any integer + ve or − ve.

(viii) Argument of the complex number 0 is not defined.

(ix) Argument of the positive real number = 0°.

(x) Argument of the negative real number = π radian = 180°

(xi) Amplitude of a positive purely imaginary number, positive imaginary part = $\pi/2$.

(xii) Amplitude of the negative purely imaginary number = $-\frac{\pi}{2}$

4. Properties of Complex Number Connectedc with Magnitudes of Complex Numbers

(i) $|z| \geq 0$

(ii) $|z_1 z_2| = |z_1| \, |z_2|$

(iii) $\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$

(iv) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

(v) $|z\bar{z}| = |z|^2$

(vi) $\frac{|z|}{|\bar{z}|} = 1$

(vii) $\left|\frac{z_1 z_2 z_3}{z_4 z_5}\right| = \frac{|z_1| \, |z_2| \, |z_3|}{|z_4| \, |z_5|}$

(viii) $|z^n| = |z|^n$

(ix) $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$

5. Properties of Complex Numbers Connected with the Amplitude of Complex Numbers

(i) If amp $(z) = \theta$, then the general value of amp (z) is $2n\pi + \theta$, $n = 0, 1, 2, \ldots\ldots$ and principle value lies between $-\pi$ and $\pi(-\pi <$ amp $(z) \leq \pi)$

(ii) amp (z) = amp$(1/z) = -$ amp(z)

(iii) amp$(-z) = -\pi +$ amp$(z) = -(\pi -$ amp$(z))$

(iv) amp$(z^n) = n$ (amp (z))

(v) amp$(z_1 z_2)$ = amp (z_1) + amp (z_2)

(vi) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) − amp (z_2)

(vii) amp $(iz) = \frac{\pi}{2}$ + amp (z)

amp$(-iz) = -\frac{\pi}{2}$ + amp (z)

(ix) amp (z) + amp $(\bar{z}) = 0$ or $2n\pi$

(x) The argument of the complex number 0 is not defined.

(xi) If k is real number,

amp $(\underline{k}) = 0$	$k > 0$
$= \pi$	$k > 0$
= not defined	$k = 0$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that the points represent the complex numbers $3 + 3i$, $-3 - 3i$, $-3\sqrt{3} + 3\sqrt{3}i$ form an equilateral triangle. Also, find the area of the triangle.

Solution

Let the complex numbers $3 + 3i$, $-3 - 3i$, and $-3\sqrt{3} + 3\sqrt{3}i$ be represented by points $A(3, 3)$, $B(-3, -3)$ and $C(-3\sqrt{3}, 3\sqrt{3})$, respectively on the Argand plane.

Then, $AB = \sqrt{(3+3)^2 + (3+3)^2}$

$= \sqrt{36\ 36} = 6\sqrt{2}$

$BC = \sqrt{(-3 + 3\sqrt{3})^2 + (-3 - 3\sqrt{3})^2}$

$= \sqrt{9 + 27 - 18\sqrt{3} + 27 + 9 + 18\sqrt{3}}$

$\Rightarrow \sqrt{72} = 6\sqrt{2}$

and $CA = \sqrt{(-3\sqrt{3} - 3)^2 + (3\sqrt{3} - 3)^2}$

$= \sqrt{27 + 9 + 18\sqrt{3} + 27 + 9 - 18\sqrt{3}}$

$\Rightarrow \sqrt{72} = 6\sqrt{2}$

Obviously, $AB = BC = CA$

Therefore, the given points form an equilateral triangle.

Therefore,

area of this triangle $= \dfrac{\sqrt{3}}{4} \times (\text{Side})^2$

$= \dfrac{\sqrt{3}}{4} \times (6\sqrt{2})^2 = 18\sqrt{3}$ square unit.

2. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$ **[NCERT]**

Solution

Given $z_1 = 2 - i$, $z_2 = 1 + i$.

$\therefore \left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

$= \dfrac{|(2-i) + (1+i) + 1|}{|2 - i - (1+i) + i|} \left(\because \left|\dfrac{z_1}{z_2}\right| = \left|\dfrac{z_1}{z_2}\right|\right)$

$= \dfrac{|4|}{|1-i|} = \dfrac{4}{\sqrt{1^2 + (-1)^2}} = \dfrac{4}{\sqrt{2}}$

$= 2\sqrt{2}$

3. If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that

$a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

[NCERT]

Solution

Given $a + ib = \dfrac{(x+i)^2}{2x^2+1}$

$\Rightarrow |a + ib| = \left|\dfrac{(x+i)^2}{2x^2+1}\right|$

$\Rightarrow |a + ib| = \left|\dfrac{(x+i)^2}{2x^2+1}\right|$

$\Rightarrow \sqrt{a^2 + b^2} = \dfrac{|x+i|^2}{\sqrt{(2x^2+1)^2 + 0^2}}$

$\Rightarrow \sqrt{a^2 + b^2} = \dfrac{\left(\sqrt{x^2 + 1^2}\right)^2}{\sqrt{(2x^2+1)^2}}$

Squaring the two sides, we get $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$

4. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

[NCERT]

(i) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\dfrac{1}{z_1\bar{z}_2}\right)$

Solution

Given $z_1 = 2 - i$, $z_2 = -2 + i$

(i) Therefore, $z_1 z_2 = (2 - i)(-2 + i)$

$= -4 - (-1) + 2i + 2i$

$= -3 + 4i$

$\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right) = \text{Re}\left(\dfrac{-3+4i}{\overline{2-i}}\right)$

$= \text{Re}\left(\dfrac{-3+4i}{2+i}\right)$

$= \text{Re}\left(\dfrac{-3+4i}{2+i} \times \dfrac{2-i}{2-i}\right)$

$= \text{Re}\left(\dfrac{-6-4(-1)+11i}{4-(-1)}\right)$

$$= \text{Re}\left(\frac{-2+11i}{5}\right)$$

$$= \text{Re}\left(-\frac{2}{5}+\frac{11}{5}i\right) = -\frac{2}{5}$$

(ii) $z_1\bar{z}_2 = (2-i)\overline{(-2+i)}$

$= (2-i)(-2-i)$

$= (-i+2)(-i-2)$

$= (-i^2) - 2^2 = -1-4 = -5$

Therefore, $I_m\left(\frac{1}{z_1\bar{z}_2}\right) = I_m\left(\frac{1}{-5}\right)$

$$= I_m\left(-\frac{1}{5}+0i\right) = 0$$

5. Find the number of non–zero integral solutions of the equation $|1-i|^x = 2^x$.

[NCERT]

Solution

Given equation is $|1-i|^x = 2^x$

$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$

$\Rightarrow (2^{1/2})^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \frac{x}{2} = x$

$\Rightarrow 2x = x \qquad \Rightarrow x = 0.$

Hence, the given equation has no nonzero integral solution.

6. If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, then show that $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$.

[NCERT]

Solution

Given $(a+ib)(c+id)(e+if)(g+ih) = A+iB$

$\Rightarrow |(a+ib)(c+id)(e+if)(g+ih)|$

$= |A+iB|$

$\Rightarrow |a+ib|\,|c+id|\,|e+if|\,|g+ih|$

$= |A+iB|$

$\Rightarrow \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}\,\sqrt{e^2+f^2}\,\sqrt{g^2+h^2}$

$= \sqrt{A^2+B^2}$

$\Rightarrow (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$

$= A^2+B^2$

Alternatively,

$(a+ib)(c+id)(e+if)(g+ih) = A+iB \quad (1)$

$\Rightarrow \overline{(a+ib)(c+id)(e+if)(g+ih)} = \overline{A+iB}$

(Taking conjugates on the two sides)

$\Rightarrow (a-ib)(c-id)(e-if)(g-ih)$

$= A-iB \quad (2)$

Multiply (1) and (2), we get

$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

($\because (a+ib)(a-ib) = a^2+b^2$ *etc.*)

Note: In this problem it should have been given that $a, b, c, d, e, f, g, h, A, B$ are real numbers.

7. For any two complex numbers z_1 and z_2, prove that $\text{Re}(z_1z_2) = \text{Re}\, z_1\, \text{Re}\, z_2 - \text{Im}\, z_1\, \text{Im}\, z_2$.

Solution

Let $z_1 = a+bi$ and $z_2 = c+di$, where a, b, c, d are real numbers, then $z_1z_2 = (a+bi)(c+di) = (ac-bd) + i(bc+ad) \Rightarrow \text{Re}(z_1z_2) = ac-bd = (\text{Re}\, z_1)(\text{Re}\, z_2) - (\text{Im}\, z_1)(\text{Im}\, z_2)$.

8. If $(a+ib)(c+id) = x+iy$, then prove that $(a-ib)(c-id) = x-iy$ and $(a^2+b^2)(c^2+d^2) = x^2+y^2$.

Solution

Given, $(a+ib)(c+id) = x+iy$

$\Rightarrow (ac-bd) + i(ad+bc) = x+iy$

Comparing the imaginary and real parts of both the sides,

$ac-bd = x. \quad (1)$

and $ad+bc = y \quad (2)$

Now, $(a-ib)(c-id) = (ac-bd) - i(ad+bc) = x-iy$ [From Equations (1) and (2)]

Proved

Again, $(a+ib)(c+id) = x+iy$ and $(a-ib)(c-id) = x-iy$

Multiplying them,

$[(a+ib)(c+id)][c-ib)(c-id)]$

$= (x+iy)(x-iy)$

$\Rightarrow [(a+ib)(a-ib)]\,[(c+id)(c-id)]$

$= (x+iy)(x-iy)$

$\Rightarrow (a^2 - i^2b^2)\,(c^2 - i^2d^2) = x^2 - i^2y^2$

$\Rightarrow (a^2 + b^2)\,(c^2 + d^2) = x^2 + y^2$

9. Find the locus of a complex variable z in the Argand plane, satisfying $|\,z - (3 - 4i)\,| = 7$.

Solution

Let $z = (x + iy)$ Then, $|\,z - (3 - 4i)| = 7$

$\Rightarrow |\,(x + iy) - (3 - 4i)|^2 = 7^2$

$\Rightarrow |\,(x - 3) + i(y + 4)|^2 = 7^2$

$\Rightarrow (x - 3)^2 + (y + 4)^2 = 7^2$

$\Rightarrow x^2 + 9 - 6x + y^2 + 16 + 8y = 49$

$\Rightarrow x^2 + y^2 - 6x + 8y - 24 = 0$

10. If $\frac{2z_1}{3z_2}$ be a purely imaginary number, then prove that

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$$

[Similar to MPPET – 1993]

Solution

Let $\frac{2z_1}{3z_2} = ai,$

$\left[\because \frac{2z_1}{3z_1} \text{ is a purely imaginary number}\right]$

$\Rightarrow \frac{z_1}{z_2} = \frac{3ai}{2} \Rightarrow \frac{z_1 - z_2}{z_1 + z_2} = \frac{3ai - 2}{3ai + 2}$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{3ai - 2}{3ai + 2}\right|$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \frac{\sqrt{9a^2 + 4}}{\sqrt{9a^2 + 4}}$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$

Proved

11. Express the following expression in the form $a + ib$

$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

Solution

Given complex number

$$= \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(3 + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

$$\frac{3^2 - (i\sqrt{5})^2}{(\sqrt{3} - \sqrt{3}) + (\sqrt{2} + \sqrt{2})i}$$

$$= \frac{9 - 5i^2}{0 + 2\sqrt{2}i} = \frac{9 - 5(-1)}{2\sqrt{2}i} \times \frac{i}{i}$$

$$= \frac{7i}{\sqrt{2}i^2} = \frac{7i}{\sqrt{2}(-1)} = 0 + \left(-\frac{7}{\sqrt{2}}\right)i$$

which is in the form $a + ib$

when $a = 0,\ b = -\frac{7}{\sqrt{2}}$.

12. Find the modulus and argument of the complex number $\frac{1 + 2i}{1 - 3i}$.

[NCERT]

Solution

Given complex number =

$$\frac{1 + 2i}{1 - 3i} = \frac{1 + 2i}{1 - 3i} \times \frac{1 + 3i}{1 + 3i}$$

$$= \frac{1 + 6(-1) + 2i + 3i}{1 - 9(-1)}$$

$$= \frac{-5 + 5i}{10} = -\frac{1}{2} + \frac{1}{2}i$$

Let $-\frac{1}{2} + \frac{1}{2}i = r(\cos\theta + i\sin\theta)$

$\Rightarrow -\frac{1}{2} = r\cos\theta$ (1)

and $\frac{1}{2} = r\sin\theta$ (2)

squaring (1) and (2) and adding

$\frac{1}{4} + \frac{1}{4} = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow r^2 = \frac{1}{2}$

$\Rightarrow r = \frac{1}{\sqrt{2}}\ (\because r \not< 0)$

substituting this value of r in (1) and (2), we get

$-\frac{1}{2} = \frac{1}{\sqrt{2}}\cos\theta$ and $\frac{1}{2} = \frac{1}{\sqrt{2}}\sin\theta$

$\Rightarrow \cos\theta = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$ and

$\sin\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

As θ lies in the second quadrant, ($\because \cos\theta < 0$ and $\sin\theta > 0$) therefore, we can write,

$\cos\theta = -\frac{1}{\sqrt{2}} = -\cos 45°$ and

$\sin\theta = \frac{1}{\sqrt{2}} = \sin 45°$

i.e $\cos\theta = (180° - 45°)\sin\theta$

$= \sin(180° - 45°)$

$\Rightarrow \theta = 135°$ or $\frac{3\pi}{4}$

Therefore, modulus of the given complex number $= \frac{1}{\sqrt{2}}$

and its amplitude $= \theta = \frac{3\pi}{4}$

13. Convert the following expression in the polar form $-1 - i$.

Solution

Let $-1 - i = r(\cos\theta + i\sin\theta)$

$\Rightarrow (-1) + (-1)i = (r\cos\theta) + i(r\sin\theta)$

$\Rightarrow r\cos\theta = -1$ (1)

and $\sin\theta = -1$ (2)

Squaring (1) and (2) and adding

$r^2(\cos\theta + \sin^2\theta) = 1 + 1$

$\Rightarrow r^2 = 2 \quad \Rightarrow r = \sqrt{2}$ $(\because r \not< 0)$

Then from (1) and (2)

$\cos\theta = -\frac{1}{r} = -\frac{1}{\sqrt{2}}$ and

$\sin\theta = -\frac{1}{r} = -\frac{1}{\sqrt{2}}$

As θ lies in the third quadrant,

$(\because \cos\theta < 0$ and also $\sin\theta < 0)$

therefore, we write

$\cos\theta = -\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4}$

$= \cos\left(\pi + \frac{\pi}{4}\right) = \cos\frac{5\pi}{4}$

and $\sin\theta = -\frac{1}{\sqrt{2}} = -\sin\frac{\pi}{4}$

$= \sin\left(\pi + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}$

$\Rightarrow \theta = \frac{5\pi}{4}$ Hence, $-1 - i$

$= \sqrt{2}\left\{\cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)\right\}$

$= \sqrt{2}\left\{\cos\left(\frac{5\pi}{4} - 2\pi\right) + \sin\left(\frac{5\pi}{4} - 2\pi\right)\right\}$

$= \sqrt{2}\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$

14. If $z = (\sqrt{2} - \sqrt{-3})$, find Re($z$), Im($z$), $\bar{z}$ and $|z|$.

Solution

$z = (\sqrt{2} - \sqrt{-3}) = \sqrt{2} - i\sqrt{3}$

$\therefore$ Re(z) $= \sqrt{2}$, Im(z) $= -\sqrt{3}$, $\bar{z} = (\sqrt{2} + i\sqrt{3})$

and $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5}$

SUBJECTIVE UNSOLVED PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If $\frac{(a+i)^2}{2a-i} = p + iq$, show that $p^2 + q^2 = \frac{(a^2+1)^2}{4a^2-1}$.

2. Prove that the sum and product of two complex numbers are real if and only if they are conjugate to each other.

3. If $(a + ib) = \sqrt{\frac{1+i}{1-i}}$, then prove that $(a^2 + b^2) = 1$.

4. Find the multiplicative inverse of the following complex numbers.

(i) $3 + 2i$ (ii) $(2 + \sqrt{3}i)^2$ (iii) $3 - 2i$

5. Separate real and imaginary parts of $\frac{4+3i}{3+i}$ and find the modulus.

6. If Z_1, Z_2 are $1 - i, -2 + 4i$, respectively, find $I_m\left(\frac{z_1 z_2}{\bar{z}_1}\right)$

7. If $\left|\frac{z-5i}{z+5i}\right| = 1$, show that z is a real number.

8. For all $Z \in C$, prove that

(i) $(\bar{\bar{z}}) = z$

(ii) $z\,\bar{z} = |z|^2$

(iii) $(z + \bar{z})$ is real

(iv) $(z - \bar{z})$ is 0 or imaginary.

9. Show that the points respresented by complex numbers $-4 + 3i$, $2 - 3i$ and $-i$ are collinear.

10. Write the following complex numbers in the polar form

(i) $-3\sqrt{2} + 3\sqrt{2}i$ (ii) 1+ı (iii) -3

Exercise II

1. If $z = 2 + 3i$, then show that $z^2 - 4z + 13 = 0$.

2. Find the modulus of $\left(\frac{1+i}{1-i} - \frac{1-i}{1+i}\right)^n$.

3. Find the multiplicative inverse of $\frac{3+4i}{3i}$.

4. If $z_1, z_2 \in C$ prove that

$\text{Im}(z_1 z_2) = \text{Re}(z_1) \,.\, \text{Im}(z_2) + \text{Im}(z_1) \,.\, \text{Re}(z_2)$.

5. For all $z_1, z_2 \in C$, prove that

(i) $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{(z - z_2)} = (\overline{z_1} - \overline{z_2})$

(iii) $\overline{(z_1 z_2)} = \overline{z_1 z_2}$.

6. Prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$, for all $z_1, z_2 \in C$.

7. Express $6(\cos 120° + i \sin 120°)$ in the form of $x + iy$.

8. Find the magnitude of $(1 + i)(1 + 2i)(1 + 3i)$.

9. Find the magnitude of (1 + i) (1 + 2i) (1 +3i).

ANSWER SHEET

Exercise I

4. (i) $\frac{3}{13} - \frac{2}{13}i$

(b) $\frac{1}{49} - \frac{4\sqrt{3}}{49}i$

(iii) $\frac{3}{13} + \frac{2}{13}i$

5. $\frac{3}{2} + \frac{1}{2}i$ and $\sqrt{\frac{5}{2}}$

6. $I_m\left(\frac{z_1 z_2}{\overline{z_1}}\right) = 2$

10. (i) $6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (iii) $3(\cos\pi + i\sin\pi)$

Exercise II

2. $|z| = 2$

3. $\frac{12}{25} + \frac{9}{25}i$

7. $Z = -3 + 3\sqrt{3}\,i$

8. 10

9. 10

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Let $\arg z < 0$ then $\arg(-z) - \arg z =$

[Orissa JEE – 2007]

(a) π (b) $\pi/2$

(c) $\pi/3$ (d) none of these

Solution

(a) $\arg z < 0$ (given), Therefore, $\arg z = -\theta < 0$

$z = |z|(\cos(-\theta) + i\sin(-\theta))$

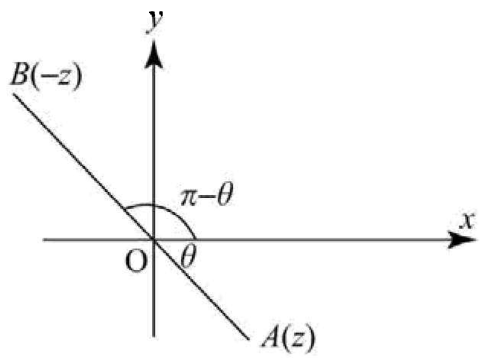
$= |z|(\cos\theta - i\sin\theta)$

$\therefore -z = |z|(-\cos\theta + i\sin\theta)$

$= |z|(\cos(\pi - \theta) + i\sin(\pi - \theta))$

$\therefore \arg(-z) = \pi - \theta = \pi + (-\theta)$

$\arg(-z) = \pi + \arg z \;\therefore\; \arg(-z) - \arg z = \pi$



OR

Clearly arg $(-z) - \arg(z) = \arg\left(\frac{-z}{z}\right)$

$= \arg(-1) \quad \Rightarrow 1\,(\cos\pi + i\sin\pi) = -1$

$= \pi$

2. The amplitude of $(1 + i)^5$ is

[Karnataka CET − 2007

(a) $3\pi/4$ (b) $-3\pi/4$

(c) $-5\pi/4$ (d) $5\pi/4$**]**

Solution

(d) $(1 + i) = (\sqrt{2})^5\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^5$

$(\sqrt{2})^5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

Therefore, Amplitude $= \frac{5\pi}{4}$

3. The locus of the point $z = x + iy$ satisfying $\left|\frac{z-2i}{z+2i}\right| = 1$ is

(a) $y = 0$ (b) $x = 0$

(c) $y = 2$ (d) $x = 2$

[EAMCET − 2007]

Solution

(a) $z = x + iy$ and $|z - 2i| = |z + 2i|$

Squaring and simplifying

$\Rightarrow x^2 + (y - 2)^2 = x^2 + (y + 2)^2$

Locus of $P(z)$ is $y = 0$

4. If $(\sqrt{8} + i)^{50} = 3^{49}(a + ib)$ then $a^2 + b^2$ is

[Kerala Engg. − 2005]

(a) 3 (b) 8

(c) 9 (d) $\sqrt{8}$

Solution

(c) $(\sqrt{8} + i)^{50} = 3^{49}(a + ib)$

Taking modulus and squaring on both the sides, we get

$(8+1)^{50} = 3^{98}(a^2+b^2) \Rightarrow 9^{50} = 3^{98}(a^2+b^2)$

$\Rightarrow 3^{100} = 3^{98}(a^2 + b^2)$

$\Rightarrow (a^2 + b^2) = 9.$

5. If $w = \left(\frac{z-i}{1+iz}\right)^n$, $n \in i$ then $|w| = 1$ for

(a) only even n (*b*) only odd n

(c) only positive n (d) all n.

Solution

(d) $|w| = 1$

$\Rightarrow \left|\left(\frac{z-i}{1+iz}\right)^n\right| = r$

$\Rightarrow \left|\frac{z-i}{i\left(z+\frac{1}{i}\right)}\right|^n = 1$

$\Rightarrow \left|\frac{z-i}{i(z-i)}\right|^n = 1 \quad \Rightarrow \left|\frac{1}{i}\right|^n = 1 \Rightarrow |-i|^n =$

which is true for all n as $|-i| = 1$.

6. If $\left|\frac{z-25}{z-1}\right| = 5$, the value of $|z|$ is

[VITEEE − 2008]

(a) 3 (b) 4 (c) 5 (d) 6

Solution

(c) Let $x + iy = z \Rightarrow (x - 25)^2 + y^2$

$= 25\,[(x - 1)^2 + y^2]$

$\Rightarrow x^2 + y^2 - 50x + 625$

$= 25\,[x^2 + y^2 - 2x + 1]$

$\Rightarrow 24x^2 + 24y^2 = 600$

$\Rightarrow x^2 + y^2 = \frac{600}{25}$

$|z|^2 = 25 \Rightarrow |z| = 5$

7. If z is a complex number such that $iz^3 + z^2 - z + i = 0$, then $|z|$ is equal to

[IIT − 1995]

(a) 2 (b) 1

(c) 1/2 (d) none of these

Solution

(b) $iz^3 + z^2 - z + i = 0$

$\Rightarrow z^3 - iz^2 + iz - i^2 = 0$ [on multiplying by $-i$]

$\Rightarrow z^2(z - i)(z - i) = 0$

$\Rightarrow (z^2 + i)(z - i) = 0$

$\Rightarrow z^2 = -i$ or $z = i$ If $\Rightarrow z^2 = -i$, then $|z| = 1$

If $z = i$, then $|z| = 1$ $\therefore$ $|z| = 1$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Amplitude of 0 is

[RPET –2000]

(a) 0 (b) π

(c) $\pi/2$ (d) not defined

2. Distance of the point representing the complex number $1 + i$ in the Argand's plane from the origin is equal to

(a) 1 (b) 2

(c) $\sqrt{2}$ (d) none of these

3. If $|z| = 4$ and Amp $z = \frac{5\pi}{6}$, then $z =$

(a) $2\sqrt{3} + 2i$ (b) $2\sqrt{3} - 2i$

(c) $-2\sqrt{3} + 2i$ (d) $-\sqrt{3} + i$

4. For any complex number z, which of the following is not true?

(a) $z.\bar{z} = |z|^2$

(b) $|z^2| = |z|^2$

(c) $|z| = \sqrt{z^2}$

(d) $z = \text{Re}(z) + i\,\text{Im}(z)$

5. If $\left|\frac{z-2}{z-4}\right| = 1$, then Re($z$) is equal to

(a) 3 (b) 0

(c) – 3 (d) none of these

6. Amp $\left\{\sin\frac{8\pi}{5} + i\left(1 + \cos\frac{8\pi}{5}\right)\right\}$ is equal to

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{10}$

(c) $\frac{4\pi}{5}$ (d) $\frac{3\pi}{10}$

7. The complex number $\frac{1+2i}{1-i}$ lies in which quadrant of the complex plane.

[MPPET – 2001]

(a) First (b) Second

(c) Third (d) Fourth

8. If z is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary then

[MPPET – 1998, 2002]

(a) $|z| = 0$ (b) $|z| = 1$

(c) $|z| > 1$ (d) $|z| < 1$

9. $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$

[MPPET–1995, 1999]

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) – 1

10. If arg $(z) = \theta$, then arg $(\bar{z}) =$

[MPPET – 1995]

(a) θ (b) $-\theta$

(c) $\pi - \theta$ (d) $\theta - \pi$

11. The argument of the complex number $-1 + i\sqrt{3}$ is

[MPPET – 1994]

(a) – 60° (b) 60° (c) 120° (d) –120°

12. If $\frac{c+i}{c-i} = a + ib$ where a, b, c are real then $a^2 + b^2 =$

(a) 1 (b) – 1 (c) c^2 (d) $-c^2$

[MPPET – 1996]

13. Which of the following is true

[MPPET – 2006]

(a) $|3 + 4i| > |5 + i|$

(b) $|6 + 7i| < |5 + 7i|$

(c) $|7 + 8i| > |6 + 8i|$

(d) $|2 + 4i| < |1 + i|$

14. If $\frac{5z_2}{7z_1}$ is purely imaginary, then the value of $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right|$ is

(a) 37/33 (b) 2 (c) 1 (d) 3

[Kerala PET – 2008]

15. The modulus of the complex number $\frac{i}{1-i}$ is

(a) $\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

16. The amplitude of the complex number $z = \sin \alpha + i(1 - \cos \alpha)$ is $\alpha \in (0, \pi)$

(a) $2 \sin \frac{\alpha}{2}$ (b) $\frac{\alpha}{2}$

(c) α (d) none of these

17. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then $(x^2 + y^2)^2 =$

[IIT–1979, RPET–1997, Karnataka CET–1999; Orissa JEE –2009]

(a) $\frac{a^2+b^2}{c^2+d^2}$ (b) $\frac{a+b}{c+d}$

(c) $\frac{c^2+d^2}{a^2+b^2}$ (d) $\left(\frac{a^2+b^2}{c^2+d^2}\right)^2$

18. The points $1 + 3i$, $5 + i$ and $3 + 2i$ in the complex plane are

(a) Vertices of a right angled triangle
(b) Collinear
(c) Vertices of an obtuse angled triangle
(d) Vertices of an equilateral triangle

[MPPET –1987]

19. The sum of amplitude of z and another complex number is π. The other complex number can be written

(a) $\bar{z}$ (b) $-\bar{z}$ (c) z (d) $-z$

[Orissa JEE –2004]

20. Let $z = \cos \theta + i\sin \theta$. Then the value of $\sum_{m=1}^{15} Im\,(z^{2m-1})$ at $\theta = 2$ is

[IIT –2009]

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3 \sin 2°}$

(c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

SOLUTIONS

1. (d) Let $0 = r(\cos \theta + i \sin \theta)$

$\therefore\ r \cos \theta = 0;\ r \sin \theta = 0$

$\Rightarrow r^2(\cos^2 \theta + \sin^2 \theta) = 0 \Rightarrow r = 0$

$\therefore\ \theta$ can have any value

$\therefore$ amp (0) can have any value.

2. (c) distance $= |1 + i| = \sqrt{1^2+1^2} = \sqrt{2}$

3. (c) $z = |z|$ cis (Arg z)

$= 4 \cos\left(\frac{5\pi}{6} + i\sin \frac{5\pi}{6}\right)$

$= 4\left[\cos\left(\pi - \frac{\pi}{6}\right) + i \sin\left(\pi - \frac{\pi}{6}\right)\right]$

$= 4\left[-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + \frac{i}{2}\right]$

$= -2\sqrt{3} + 2i$

4. (c) By definition

5. (a) Let $z = x + yi;\ x, y \in R$, then

$\left|\frac{z-2}{z-4}\right| = 1 \Rightarrow |z - 2| = |z - 4|,\ z \neq 4$

$\Rightarrow |x + yi - 2| = |x + yi - 4|,\ z \neq 4$

$\Rightarrow \sqrt{(x-2)^2+y^2} = \sqrt{(x-4)^2+y^2}$

$\Rightarrow -4x + 4 \Rightarrow -8x + 16$

$\Rightarrow 4x = 12 \Rightarrow x = 3$

6. (b) Now $\sin \frac{8\pi}{5} + i\left(1 + \cos \frac{8\pi}{5}\right)$

$= 2 \sin \frac{4\pi}{5} \cos \frac{4\pi}{5} + i\left(2\cos^2 \frac{4\pi}{5}\right)$

$= -2\cos \frac{4\pi}{5}\left\{-\sin \frac{4\pi}{5} - i \cos \frac{4\pi}{5}\right\}$

Note that $\cos \frac{4\pi}{5} < 0$

$= \left(-2\cos \frac{4\pi}{5}\right)\left\{\cos\left(\frac{3\pi}{2} - \frac{4\pi}{5}\right) + i \sin\left(\frac{3\pi}{2} - \frac{4\pi}{5}\right)\right\}$

$\left(\because -\sin \theta = \cos\left(\frac{3\pi}{2} - \theta\right)\right)$

and $-\cos \theta = \sin\left(\frac{3\pi}{2} - \theta\right)$

$= \left(-2\cos \frac{4\pi}{5}\right)\left\{\cos\left(\frac{7\pi}{10}\right) + i \sin\left(\frac{7\pi}{10}\right)\right\}$

$= r \cos \theta$, where $r = -2 \cos \frac{4\pi}{5} > 0$

and $\theta = \frac{7\pi}{10} \in (-\pi, \pi]$

Hence, amplitude of the given number is $\frac{7\pi}{10}$.

7. (b) $\frac{1+2i}{1-i} = \frac{(1+2i)\times(1+i)}{(1-i)(1+i)}$

$\frac{1+2i}{1-i} = \frac{1+2i+i+2i^2}{1+1}$

or

$\frac{1+2i}{1-i} = \frac{-1}{2} + \frac{3i}{2}$

Comparing with $x + iy$

$x \Rightarrow$ negative, $y \Rightarrow$ positive $(-, +)$ are in 2nd quadrant.

8. (b) $\frac{z-1}{z+1}$ = purely imaginary

Let $z = x + iy$ then $\frac{x+iy-1}{x+iy+1} = ai$

$\left[\frac{(x-1)+iy}{(x+1)+iy}\right]\left[\frac{(x+1)-iy}{(x+1)-iy}\right] = ai$

On comparing real parts,

$\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}$

$x^2 + y^2 - 1 = 0$ or $x^2 + y^2 = 1$ or $|z| = 1$

9. (c) $\left|\frac{(1+i)(2+i)}{3+i}\right| = \frac{|1+i|\cdot|2+i|}{|3+i|}$

$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{10}} = 1$

10. (b) Let $z = a + ib$ then arg z, $\theta = \tan^{-1}\frac{b}{a}$ and $\bar{z} = a - ib$

or arg $\bar{z}$, $\phi = -\tan^{-1}\frac{b}{a} = -\theta$

11. (c) Argument of $z = a + ib$ is $\theta = \tan^{-1}\frac{-b}{a}$

Therefore arg of $(-1+\sqrt{3})$ is $\theta = \tan^{-1}\frac{\sqrt{3}}{-1}$ $= 120°$

(since) $x < 0, y > 0$, θ is in 4th quadrant

12. (a) $a + ib = \frac{c+i}{c-i} \Rightarrow |a+ib| = \frac{|c+i|}{|c-i|}$

$\Rightarrow a^2 + b^2 = \left(\frac{c^2+1}{c^2+1}\right) \Rightarrow a^2 + b^2 = 1$

13. (c) $|7+8i| = \sqrt{7^2+8^2} = \sqrt{49+64} = \sqrt{113}$

$|6+8i| = \sqrt{6^2+8^2} = 10$

$\therefore |7+8i| > |6+8i|$

14. (c) Let $\frac{5z_2}{7z_1} = i\alpha;\ \alpha \in R,\ a \neq 0 \Rightarrow \frac{z_2}{z_1} = \frac{7\alpha i}{2}$

$\Rightarrow \left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| = \left|\frac{2+3\frac{z_2}{z_1}}{2-3\frac{z_2}{z_1}}\right| \left|\frac{2+3\left(7\frac{i\alpha}{5}\right)}{2-3\left(\frac{7i\alpha}{5}\right)}\right|$

$= \frac{|10+21\alpha)i}{|10-(21\alpha)i|} = \frac{\sqrt{10^2+(21\alpha)^2}}{\sqrt{10^2+(21\alpha)^2}} = 1$

15. (d) $\left|\frac{i}{1-i}\right| = \frac{|i|}{|1-i|} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

16. (b) $z = \sin\alpha + i(1-\cos\alpha)$

amp $(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin a}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$

$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$

17. (a) $x + iy = \left(\frac{a+ib}{c+id}\right)^{1/2} \Rightarrow |x+iy|^2$

$= \left|\frac{a+ib}{c+id}\right|$

$\Rightarrow x^2 + y^2 = \left(\frac{a^2+b^2}{c^2+d^2}\right)^{1/2}$ or $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

18. (b) Let $z_1 = 1 + 3i$, $z_2 = 5 + i$ and $z_3 = 3 + 2i$ Then the area of the triangle

$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$

Hence, z_1, z_2 and z_3 are collinear.

19. (b) We have $z = x + iy$ and let other complex number be z_2 and given that arg(z) + arg(z_2) $= \pi$ arg $z_2 = \pi -$ arg(z); arg $(z_2) = \pi +$ $\left[-\tan^{-1}\frac{y}{x}\right]$ arg $z_2 = \pi +$ [arg $(\bar{z})$]

which lies in second quadrant i.e. $-\bar{z}$.

20. (d) $X = \sin\theta + \sin 3\theta + \ldots\ldots + \sin 29\theta$

$2(\sin\theta) \times = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta$
$+ \ldots\ldots + \cos 28\theta - \cos 30\theta$

$X = \frac{1-\cos 30\theta}{2\sin\theta} = \frac{1}{4\sin 2°}$

UNSOLVED OBJECTIVE PROBLEMS: (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $z_1 = (4, 5)$ and $z_2 = (-3, 2)$ then $\frac{z_1}{z_2}$ equals **[RPET – 1996]**

(a) $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ (b) $\left(\frac{2}{13}, \frac{-23}{13}\right)$
(c) $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ (d) $\left(\frac{-2}{13}, \frac{23}{13}\right)$

2. If z is a complex number, then which of the following is not true **[MPPET – 1987]**

(a) $|z^2| = |z|^2$ (b) $|z^2| = |\bar{z}|^2$
(c) $z = \bar{z}$ (d) $\bar{z}^2 = \bar{z}^2$

3. For any complex number z, $\bar{z} = \left(\frac{1}{z}\right)$ if and only if

(a) z is a pure real number
(b) $|z| = 1$
(c) z is a pure imaginary number
(d) $z = 1$

[RPET – 1995]

4. The value of $|z - 5|$ if $z = x + iy$, is **[RPET – 1995]**

(a) $\sqrt{(x-5)^2 + y^2}$ (b) $x^2 + \sqrt{(y-5)^2}$
(c) $\sqrt{(x-y)^2 + 5^2}$ (d) $\sqrt{x^2 + (y-5)^2}$

5. $\frac{1-i}{1+i}$ is equal to **[RPET – 1984]**

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
(c) $\sin\frac{\pi}{2} + i\cos\frac{\pi}{2}$ (d) none of these

6. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is **[RPET – 1996]**

(a) 1 (b) 1/2
(c) 2 (d) $\sqrt{2}$

7. $\arg(5 - \sqrt{3}\, i) =$

(a) $\tan^{-1}\frac{5}{\sqrt{3}}$ (b) $\tan^{-1}\left(-\frac{5}{\sqrt{3}}\right)$
(c) $\tan^{-1}\frac{\sqrt{3}}{5}$ (d) $\tan^{-1}\left(-\frac{\sqrt{3}}{5}\right)$

8. If $\bar{z}$ be the conjugate of the complex number z, then which of the following relations is false **[MPPET – 1987]**

(a) $|z| = |\bar{z}|$ (b) $z \cdot \bar{z} = |\bar{z}|^2$
(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (d) $\arg z = \arg \bar{z}$

9. If z is a purely real number such that Re(z) < 0, then arg(z) is equal to

(a) π (b) $\pi/2$
(c) 0 (d) $-\pi/2$

10. If $z_1, z_2 \in C$, then amp $\left(\frac{z_1}{z_2}\right) =$

(a) amp $(z_1 \bar{z}_2)$ (b) amp $(\bar{z}_1 z_2)$
(c) amp $\left(\frac{\bar{z}_2}{\bar{z}_1}\right)$ (d) amp $\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

11. If A, B, C are represented by $3 + 4i$, $5 - 2i$, $-1 + 16i$, then A, B, C are

(a) Collinear
(b) Vertices of equilateral triangle
(c) Vertices of isosceles triangle
(d) Vertices of right angled triangle

[RPET – 1986]

12. If $z = 1 - \cos\alpha + i\sin\alpha$, then amp z =

(a) $\frac{\alpha}{2}$ (b) $-\frac{\alpha}{2}$
(c) $\frac{\pi}{2} + \frac{\alpha}{2}$ (d) $\frac{\pi}{2} - \frac{\alpha}{2}$

13. If $z = \frac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$, then arg (z) **[Roorkee – 1990, UPSEAT – 2004]**

(a) 60° (b) 120°
(c) 240° (d) 300°

14. The modulus of the complex number $\frac{(1+i)}{\left(1+\frac{1}{i}\right)}$ is

(a) $\sqrt{2}$ (b) 1
(c) $3/\sqrt{2}$ (d) none of these

15. Amplitude of $\left(\frac{1-i}{1+i}\right)$ is

(a) $-\pi/2$ (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

[RPET – 1996]

16. $(z+1)(\bar{z}+1)$ can be expressed as

(a) $z\bar{z}+1$ (b) $|z|^2+1$

(c) $|z+1|^2$ (d) $|z|^2+2$

17. The argument of the complex number $\frac{13-5i}{4-9i}$ is

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/5$ (d) $\pi/6$

[MPPET – 1997]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet
2. The test is of 13 minutes.
3. The test consists of 13 questions. The maximum marks are 39.
4. Use blue/black Ball point pen only for writing particulars / marking responses. Use of pencil is strictly prohibited.

1. The amplitude of $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$

(a) $\pi/5$ (b) $2\pi/5$

(c) $\pi/10$ (d) $\pi/15$

2. Argument of $-1-i\sqrt{3}$ is

(a) $2\pi/3$ (b) $\pi/3$

(c) $-\pi/3$ (d) $-2\pi/3$

3. $\left|\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}\right|=$

(a) $\frac{\sqrt{8}}{5}$ (b) $\frac{25}{8}$

(c) $\frac{5}{\sqrt{8}}$ (d) $\frac{8}{25}$

4. If a complex number lies in the IIIrd quadrant then its conjugate lies in quadrant number

(a) I (b) II

(c) III (d) IV

5. Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha+\beta>\pi$, then principal $\arg(z_1 z_2)$ is given by

(a) $\alpha+\beta+\pi$ (b) $\alpha+\beta-\pi$

(c) $\alpha+\beta-2\pi$ (d) $\alpha+\beta$

6. If z is a complex number, then

(a) $|z^2|>|z|^2$ (b) $|z^2|=|z|^2$

(c) $|z^2|<|z|^2$ (d) $|z^2|\geq|z|^2$

7. The complex number $i+\sqrt{3}$ in polar form can be written as

(a) $\frac{1}{\sqrt{2}}\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)$

(b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(c) $\frac{1}{2}\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)$

(d) $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

8. If $|z_1|=|z_2|$ and amp z_1 + amp $z_2=0$, then

[MPPET – 2006]

(a) $z_1=z_2$ (b) $\bar{z}_1=\bar{z}_2$

(c) $z_1+z_2=0$ (d) $|\bar{z}_1|\,|\bar{z}_2|$

9. Value of $|1-\cos\alpha+i\sin\alpha|$ is

[MPPET – 2007]

(a) $2\sin\frac{\alpha}{2}$ (b) $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

(c) $2\cos\frac{\alpha}{2}$ (b) $2\sin^2\frac{\alpha}{2}$

10. Let z be a purely imaginary number such that Im $(z)>0$. Then arg (z) is equal to:

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) $-\frac{\pi}{2}$

11. If $(\sqrt{5}+\sqrt{3}i)^{33}=2^{49}z$, then modulus of the complex number z is equal to

[Kerala PET – 2008]

(a) 1 (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) 4

12. If $\frac{z-1}{z+1}$ is purely imaginary, then

(a) $|z| = 1$ (b) $|z| = 0$

(c) $|z| < 1$ (d) $|z| > 1$

[MPPET – 1998, 2002]

13. If $(x + iy) = \sqrt{\frac{1+2i}{3+4i}}$, then $(x^2 + y^2)^2 =$

[MPPET – 2009]

(a) 5 (b) $\frac{1}{5}$

(c) 2 (d) $\frac{5}{2}$

ANSWER SHEET

1.	a	b	c	d	6.	a	b	c	d	11.	a	b	c	d
2.	a	b	c	d	7.	a	b	c	d	12.	a	b	c	d
3.	a	b	c	d	8.	a	b	c	d	13.	a	b	c	d
4.	a	b	c	d	9.	a	b	c	d					
5.	a	b	c	d	10.	a	b	c	d					

HINTS AND EXPLANATIONS

13. (b) Given, $x + iy = \sqrt{\frac{1+2i}{3+4i}} \times \frac{\sqrt{3-4i}}{\sqrt{3-4i}}$

$= \frac{1}{5}\sqrt{11+2i}$

Taking mod in both sides, and squaring

$\sqrt{x^2 + y^2} = \frac{1}{25}\sqrt{125}$

Again squaring,

Therefore, $(x^2 + y^2)^2 = \frac{125}{(25)^2} = \frac{1}{5}$

LECTURE

3 Euler's Formula

BASIC CONCEPTS

1. **Euler's Formula** $z = \cos\theta + i\sin\theta = e^{i\theta}$

$\frac{1}{z} = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

e.g. $i = 0 + i \times 1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$-i = 0 - i \times 1$

$= \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = e^{-i\pi/2}$

$\frac{1}{2} + \frac{i\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i2\pi/3}$

2. **De–Moivre's Theorem**

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\sin\theta + i\cos\theta)^n = i^n\{\cos(-n\theta) + i\sin(-n\theta)\}$

Note:

$\frac{1}{z} = \frac{1}{(\cos\theta + i\sin\theta)} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

3. **Exponential Form of Complex Number** $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$; $z = re^{i\theta}$ is exponential form.

4. **Multiplication and Division of a Complex numbers When it is in Polar Form ($z = (r(\cos\theta + i\sin\theta))$)**

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

(i) $z_1z_2 = r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$
$= r_1r_2 e^{i(\theta_1+\theta_2)}$

$r_1 = |z_1|$, $r_2 = |z_2|$, amp $(z_1) = \theta_1$, amp $(z_2) = \theta_2$.

$|z_1 z_2| = r_1r_2 = |z_1|\,|z_2|$,

$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

i.e., the magnitude of the product of two complex numbers is equal to the product of their magnitudes and the amplitude of product of two complex numbers is equal to the sum of their amplitudes.

4.1. Product of two complex numbers can be generalized for more than two complex numbers, if $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$; $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; $z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$ .... and........ $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

Then $z_1, z_2, z_3 ....z_n = r_1r_2......r_n\{\cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)\}$

Hence, $|z_1z_2z_3.........z_n| = |z_1|\,|z_2|\,|z_3|.........|z_n|$

$\arg(z_1z_2z_3...z_n) = \theta_1 + \theta_2 + \theta_3 + ... + \theta_n = \arg(z_1) + \arg(z_2) + ... + \arg(z_n)$

Note: If $z_1 = z_2 = z_3 = ... = z_n = z$, then

(i) $|z_n| = |z|^n$

(ii) $\arg(z^n) = n\arg(z)$

(iii) $\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$= \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

where $\frac{|z_1|}{|z_2|} = \frac{|r_1|}{|r_2|} = \frac{|z_1|}{|z_2|} \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 =$ $\arg(z_1) - \arg(z_2)$

The amplitude of the quotient of two complex numbers is equal to the difference of their amplitudes.

Note: If θ_1 and θ_2 are the principal values of arg z_1 and arg z_2 then $\theta_1 + \theta_2$ is not necessarily the principal value of arg $(z_1 z_2)$ nor is $\theta_1 - \theta_2$ necessarily the principal value of arg (z_1/z_2).

5. Properties of Complex Number Connected with Magnitudes of Complex Numbers

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z^2|^2 + 2\,\text{Re}\,(z_1\bar{z}_2)$ OR $= |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

(ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}\,(z_1\,\bar{z}_2)$ OR $= |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$ **(MPPET–2006)**

(iv) $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)\,\{|z_1|^2 + |z_2|^2\}$

(v) $|z_1 + z_2|^2 - |z_1 - z_2|^2 = 4\,|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

6. Properties of Complex Numbers Connected with the Amplitude of Complex Numbers

(i) If $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \left(\frac{\pi}{2}\right)$

(ii) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{z_1}{z_2}$ Purely imaginary

7. Properties of Complex Number Connected with Magnitudes of Complex Numbers

(i) Triangle's Inequality

$||z_1| - |z_2|| < |z_1 + z_2| < |z_1| + |z_2|$

The sum of any two sides is greater than the third side and the difference of any two sides is less than the third side.

(ii) If z_1 and z_2 are collinear, then

$|z_1 + z_2| = |z_1| + |z_2| = ||z_1| - |z_2||$

(iii) From inequalities (i) and (ii), we get

(iv) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Hence, maximum value of $|z_1 + z_2| = |z_1| + |z_2|$ and minimum value of $= |z_1 + z_2|$ is $||z_1| - |z_2||$

(v) $|z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)}$

$= (z_1 + z_2)\,(z_1 + z_2)$

(vi) $-|z| \le \text{Re}\,(z) \le |z|$

(vii) $-|z| \le \text{Im}\,(z) \le |z|$

(viii) $|z_1 + z_2 + z_3 + \ldots\ldots + z_n| = |z_1| + |z_2| + \ldots + |z_n|$

if and only if: amp (z_1) = amp (z_2) $= \ldots\ldots\ldots =$ amp (z_n)

i.e. $z_1, z_2, z_3, \ldots\ldots, z_n$ are collinear.

8. Square Root of a Complex Number

(i) $\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} + a\}}\right.$

$\left.\pm\, i\sqrt{\left\{\frac{1}{2}\sqrt{a^2 + b^2} - a\right\}}\right.$

where $b > 0$

$\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} + a\}}\right.$

$-i\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} - a\}}\right.$

where $b < 0$ i.e., in square root, the sign of imaginary part is same as the sign of y.

(ii) $\sqrt{a + ib} + \sqrt{a - ib} = \sqrt{2a + 2\sqrt{a^2 + b^2}}$

(iii) $\sqrt{a + ib} - \sqrt{a - ib} = i\sqrt{2\sqrt{a^2 + b^2} - 2a}$

Note: When asked $\sqrt{a + ib}$, then $\sqrt{a + ib}$ $= \pm\,(x \pm iy)$ can be verified by inspection method from the given four alternatives.

9. Logarithm of a Complex Number

$\log z = \log(a + ib) = (\log|z| + i\,\text{amp}\,(z))$ or $\left(\frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}\right)$

10. Cube Roots of Unity (i) $(1)^{1/3} = 1, \omega, \omega^2$

where $\omega = \frac{-1 + \sqrt{-3}}{2}$ or $\frac{-1 + i\sqrt{3}}{2}$

or $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ or $e^{i\frac{2\pi}{3}}$

and $\omega^2 = \frac{-1 - \sqrt{-3}}{2}$ or $\frac{-1 - i\sqrt{3}}{2}$

or $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ or $e^{i\frac{2\pi}{3}}$ (ii) $-\omega^2 = \frac{1 + i\sqrt{3}}{2}$

(iii) $i\omega = \dfrac{-i-\sqrt{3}}{2}$,

$-i\omega = \dfrac{i+\sqrt{3}}{2}$, $-i\omega^2 = \dfrac{i-\sqrt{3}}{2}$, $i\omega^2$

$= \dfrac{\sqrt{3}-i}{2}$

(iv) $-2\omega = 1 - i\sqrt{3}$, $-2\omega^2$

$= 1 + i\sqrt{3}$

(v) $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$

Note: Where $\arg(\omega) = \dfrac{2\pi}{3}$. $\arg(\omega^2) = \dfrac{4\pi}{3}$

(vi) ω and ω^2 are the roots of equation $z^2 + z + 1 = 0$.

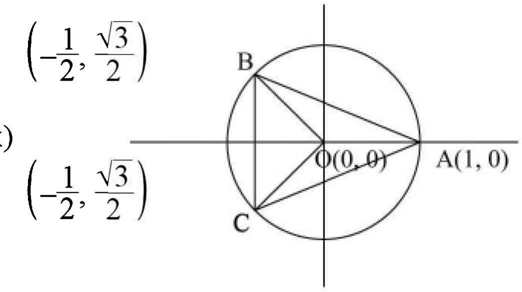
Hence, $1 + \omega + \omega^2 = 0$

i.e. sum of cube roots = 0

(vii) $(-1)^{1/3} = -1, -\omega, -\omega^2$

(viii) Each complex cube root of unity is the square of the other and $\omega . \omega^2 = \omega^3 = 1$; $\omega + \omega^2 = -1$.

(ix) Each complex cube root is the reciprocal of other complex cube root.

(x) 

If these points A, B and C which are cube roots of unity represented on a complex plane form the vertices of an equilateral triangle of area $\dfrac{3\sqrt{3}}{4}$ square unit and 1, ω, ω^2 are lying on the unit circle on a complex plane.

(xi) $1 + \omega^n + \omega^{2n}$

$$= \begin{cases} 3, \text{ when } n \text{ is multiple of } 3 \\ 0, \text{ when } n \text{ is not a multiple of } 3 \end{cases}$$

(xii) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(xiii) $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

(xiv) $a^2 + b^2 + c^2 - ab - bc - ca$
$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(xv) $a^3 + b^3 + c^3 - 3abc$
$= (a+b+c)(a + b\omega + c\omega^2)$
$(a + b\omega^2 + c\omega)$

(xvi) $1 - \omega + \omega^2 = -2\omega$

(xvii) $1 + \omega - \omega^2 = -2\omega^2$

SOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $z = \dfrac{\sqrt{3}+i}{-2}$, then z^{69} is equal to **Prove.**

Solution

$$z = \frac{\sqrt{3}+i}{-2} \Rightarrow iz = -\frac{-1 \pm \sqrt{3}i}{2} = -\omega$$

$$\Rightarrow z = \frac{-\omega}{i} = i\omega$$

$$\Rightarrow 2^{69} = i^{69}.\omega^{69} = i \ (\because \omega^{3n} = i^{4n} = 1)$$

2. Using de Moivre's theorem, find the value of $(\sqrt{3} - i)^8$.

Solution

To express $\sqrt{3} - i$ in the trigonometric form,

Let $\sqrt{3} - i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\cos\theta = \dfrac{x}{r} = \dfrac{\sqrt{3}}{2}$ and

$\sin\theta = \dfrac{y}{r} = \dfrac{-1}{2} \Rightarrow \theta = -30^\circ$

Therefore $(\sqrt{3} - i)^8 = \left[2\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)\right]^8$

$= 2^8 [\cos(-30°) + i \sin(-30°)]^8$

$= 2^8 [\cos(-240°) + i \sin(-240°)]$

("By de Moiver's theorem")

$= 2^8 [\cos(360° - 240°) + i \sin(360° - 240°)$

$= 2^8 [\cos 120° + i \sin 120°]$

$= 2^8 \left[\frac{-1}{2} + i \frac{\sqrt{3}}{2}\right]$

$= 2^7 [-1 + i\, 3]$ **Ans**

3. $\frac{(\cos\alpha + i \sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \cos(4\alpha + 5\beta) + i \sin (4\alpha + 5\beta)$ Prove.

Solution

$$\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \frac{(\cos 4\alpha + i \sin 4\alpha)}{i^5 (\cos\beta - i \sin\beta)^5}$$

$= -i (\cos 4\alpha + i \sin 4\alpha)(\cos\beta - i\sin\beta)^{-5}$

$= -i [\cos 4\alpha + i \sin 4\alpha][\cos 5\beta + i \sin 5\beta]$

$= -i [\cos(4\alpha + 5\beta) + i \sin(4\alpha + 5\beta)]$

$= \sin(4\alpha + 5\beta) - i \cos(4\alpha + 5\beta)$

4. Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$ or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$.

Solution

Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1$ and $|z_2| = r_2$ and $\bar{z}_2 = r_2(\cos\theta_2 - i \sin\theta_2)$

$\therefore z_1 \bar{z}_2 = \rho_1 (\cos\theta_1 + i \sin\theta_1) \times r_2 (\cos\theta_2 - i\sin\theta_2)$

$= r_1 r_2 [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$

$\therefore \text{Re}(z_1 \bar{z}_2) = r_1 r_2 \cos(\theta_1 - \theta_2)$

$\Rightarrow \text{Re}(z_1 z_2) = |z_1|\,|z_2| \cos(\theta_1 - z_2)$ (1)

Again, $z_1 + z_2 = r_1 (\cos\theta_1 + i\sin\theta_1) + r_2 (\cos\theta_2 + i \sin\theta_2)$

$\Rightarrow z_1 + z_2 = (r_1 \cos\theta_1 + r_2\cos\theta_2) + i(r_1 \sin\theta_1 + r_2 \sin\theta_2)$

$\therefore |z_1 + z_2| =$

$\sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2\sin\theta_2)^2}$

Squaring both the sides,

$\Rightarrow |z_1 + z_2|^2 = r_1^2 \cos^2\theta_1 + r_2^2 \cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2 \sin^2\theta_2 + 2r_1r_2 \sin\theta_1 \sin\theta_2$

$\Rightarrow |z_1 + z_2|^2 = r_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + r_2^2 (\cos^2\theta_2 + \sin^2\theta_2) + 2r_1r_2 (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$\Rightarrow |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

Or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$.

From Eqation (1).

5. Prove that: $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

Solution

We have $(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - x)$

$= \{(x + 1)^2 - i^2\}\{(x - 1)^2 - i^2\} = \{(x + 1)^2 + 1\}\{(x - 1)^2 + 1\}$

$= \{x^2 + 2x + 2\}\{x^2 + 2 - 2x\} = \{x^2 + 2 + 2x\}\{x^2 + 2 - 2x\}$

$= (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4.$

6. If $1, \omega, \omega^2$ are the cube roots of unity, then prove that

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$......

$2n$ factors $= 2^{2n}$.

[MP – 1990]

Solution

L.H.S. $= (1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$... $2n$ factors

$= (1 + \omega^2 - \omega)(1 + \omega^4 - \omega^2)(1 + \omega^8 - \omega^4)$.... $2n$ factors

$= (-\omega - \omega)(1 + \omega - \omega^2)(1 + \omega^6 . \omega^2 - \omega^4)$.. $2n$ factors

$= (-2\omega)(-2\omega^2)(-2\omega)$...... $2n$ factors,

$$\left[\because \begin{array}{c} 1 + \omega^2 = -\omega \\ 1 + \omega = -\omega^2 \\ \omega^3 = 1 \end{array}\right]$$

$= (-2)^{2\underline{n}} . (\omega^3)^n = 2^{2n} =$ R.H.S. **Proved.**

7. Find the square root of $4ab - 2(a^2 - b^2)i$.

Solution

$4ab - 2(a^2 - b^2)i = 4ab - 2(a+b)(a-b)i$
$= (a+b)^2 - (a-b)^2 - 2(a+b)(a-b)i$
$= (a+b)^2 + (a-b)^2 i^2 - 2(a+b)(a-b)i$
$= [(a+a) - (a-a)i]^2$
$\therefore \sqrt{4ab - 2(a^2 - b^2)i}$
$= \sqrt{[(a+b) - (a-b)i]^2}$
$= \pm [(a+b) - (a-b)i]$

8. Find the square root of $a^2 - 1 + 2ai$.

Solution

$a^2 - 1 + 2ai = a^2 + i^2 + 2ai = (a+i)^2$
$\therefore \sqrt{a^2 - 1 + 2ai} = \sqrt{(a+i)^2} \pm (a+i)$

9. Prove that $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... up to $2n$ factors = 1.

Solution

L.H.S.= $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... up to $2n$ factors
$= \{(1+\omega)(1+\omega^2)\}.\{(1+\omega^4)(1+\omega^8)\}$........ up to n factors
$= \{(1+\omega)(1+\omega^2)\}.\ \{(1+\omega^4)(1+\omega^8)\}$...... up to n factor
$= \{(-\omega^2)(-\omega)\}.\ \{(-\omega^2)(-\omega)\}$........ up to n factors
$= (\omega^3).(\omega^3)$............... up to n factors
$= 1.\ 1$............... up to n factors = 1
= R.H.S. **Proved.**

10. Prove that $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = x^3+y^3+z^3-3xyz$.

Solution

L.H.S. $= (x+y+z)\ [x^2 + xy(\omega+\omega^2) + xz(\omega+\omega^2)$
$+ \omega^3 y^2 + yz(\omega^2+\omega^4) + \omega^3 z^2]$
$= (x+y+z)\ [x^2 + xy(-1) + xz(-1) + y^2 + yz(\omega^2+\omega) + z^2]$
$= (x+y+z)\ [x^2 - xy - xz + y^2 - yz + z^2]$
$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$
$= x^3+y^3+z^3-3xyz$
= R.H.S. **Proved.**

11. Prove that $(x+y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2 = 6xy$

[MP PET – 2008]

Solution

L.H.S. $= (x+y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2$
$= (x^2+y^2+2xy) + (x^2\omega^2 + y^2\omega^4 + 2xy\omega^3)$
$+ (x^2\omega^4 + y^2\omega^2 + 2xy\omega^3)$
$= \xi^2(1+\omega^2+\omega^4) + y^2(1+\omega^4+\omega^2)$
$+ 2xy(1+\omega^3+\omega^3)$
$= \xi^2(1+\omega+\omega^2) + y^2(1+\omega+\omega^2) + 2xy$
$(1+1+1)$, $[\because \omega^3 = 1]$
$= x^2(0) + y^2(0) + 2xy(3) = 6xy$
= R.H.S. **Proved.**

12. Prove that

$$\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = -1$$

Solution

$$\text{L.H.S.} = \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$$

$$= \frac{\omega^2(a+b\omega+c\omega^2)}{\omega^2(c+a\omega+b\omega^2)} + \frac{\omega(a+b\omega+c\omega^2)}{\omega(b+c\omega+a\omega^2)}$$

$$= \frac{\omega^2(a+b\omega+c\omega^2)}{c\omega^2+a+b\omega} + \frac{\omega(a+b\omega+c\omega^2)}{b\omega+c\omega^2+a\omega^2}$$

$$= \frac{\omega^2(a+b\omega+c\omega^2)}{(c\omega^2+a+b\omega^4)} + \frac{\omega(a+b\omega+c\omega^2)}{(b\omega+c\omega^2+a)}$$

$= \omega^2(1) + \omega(1) \quad = \omega^2 + \omega$

$= -1\ [\because 1+\omega+\omega^2 = 0]$

13. If $i = \sqrt{-1}$, then prove that

$$4 + 5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365} = i\sqrt{3}$$

Solution

L.H.S. = 4 + 5

$$\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$$

$= 4 + 5(\omega)^{334} + 3(\omega)^{365}$,

$\left[\because \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

$= 4 + 5(\omega^3)^{111}.\ \omega + 3(\omega^3)^{121}.\ \omega^2$

$= 4 + 5(1)^{111}.\ \omega + 3\ (1)^{121}.\ \omega^2$

$= 4 + 5\ \omega + 3\ \omega^2 = 1 + 3 + 3\ \omega + 2\ \omega + 3\ \omega^2$

$= 1 + 2\ \omega + 3(1 + \omega + \omega^2)$

$= 1 + 2\ \omega + 3(0)$

$[\because 1 + \omega + \omega^2 = 0]$

$= 1 + 2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= 1 - 1 + i\sqrt{3}$

$= i\ \sqrt{3}$ **Proved.**

14. Prove that $\sqrt{15 + 8i} + \sqrt{15 - 8i} = 8$

Solution

$\sqrt{15 + 8i} + \sqrt{15 - 8i}$

$= \sqrt{15 + 2 \times 4i} + \sqrt{15 - 2 \times 4i}$

$= \sqrt{16 - 1 + 2 \times 4i} + \sqrt{16 - 1 - 2 \times 4i}$

$= \sqrt{4^2 + i^2 + 2 \times 4i} + \sqrt{4^2 + i^2 - 2 \times 4i}$

$= \sqrt{(4 + i)^2} + \sqrt{(4 - i)^2} = 4 + i + 4 - i = 8$

Proved.

15. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that $|w| = 1 \Rightarrow z$ is purely real.

Solution

We have, $|w| = \left|\frac{1 - iz}{z - i}\right| = 1$

$\Rightarrow |1 - iz| = |z - i|$

$\Rightarrow |1 - i\ (x + \text{iy})| = |\ x + iy - i|$, where $z = x + iy$

$\Rightarrow |1 + y - ix| = |x + i(y - 1)|$

$\Rightarrow \sqrt{(1 + y)^2 + (-x)^2} = \sqrt{x^2 + (y - 1)^2}$

$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2 \Rightarrow y = 0$

$\Rightarrow z = x + i^0 = x$, which is purely real.

16. Prove that one of the values of

$\sqrt{8 + 6i} - \sqrt{8 - 6i}$ is $2i$

Solution

$\sqrt{8 + 6i} - \sqrt{8 - 6i}$

$= \sqrt{8 + 2 \times 3i} - \sqrt{8 - 2 \times 3i}$

$= \sqrt{9 - 1 + 2 \times 3i} - \sqrt{9 - 1 - 2 \times 3i}$

$= \sqrt{3^2 + i^2 + 2 \times 3i} - \sqrt{3^2 + i^2 - 2 \times 3i}$

$= \sqrt{(3 + i)^2} - \sqrt{(3 - i)^2}$

$= 3 + i - 3 + i = 2i$

Proved.

17. Prove that

(i) $\sqrt{a + ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} + i\ \sqrt{(\sqrt{a^2 + b^2} + a)}\right]$

(ii) $\sqrt{a - ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} - i\ \sqrt{(\sqrt{a^2 + b^2} + a)}\right]$

(iii) $\sqrt{a + ib} + \sqrt{a - ib} = \sqrt{2(\sqrt{a^2 + b^2} + a)}$

Solution

Let $\sqrt{a + ib} = x + iy$ Then, $a + ib = (x + iy)^2$

$\Rightarrow a + ib = (x^2 - y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = a$ (1)

and $2xy = b$ (2)

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = a^2 + b^2$

$x^2 + y^2 = \sqrt{a^2 + b^2}$ (3)

Solving equation (2) and (3), we have

$x = \pm \frac{1}{2}\sqrt{[\sqrt{a^2 + b^2} + a]}$ and

$x = \pm \frac{1}{2}\sqrt{[\sqrt{a^2 + b^2} - a]}$

$\therefore\ \sqrt{a + ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} + i\sqrt{(\sqrt{a^2 + b^2} - a)}\right]$ (4)

Similarly, $\sqrt{a - ib} = \pm \frac{1}{2}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} - i\sqrt{(\sqrt{a^2 + b^2} - a)}\right]$ (5)

Adding equation (4) and (5), we get

$\sqrt{a+ib}+\sqrt{a-ib}$

$$= \pm \frac{1}{\sqrt{2}}\left[2\sqrt{(\sqrt{a^2+b^2}+a)}\right]$$

$$= \pm \sqrt{2(\sqrt{a^2+b^2}+a)}$$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Write the following complex numbers in the polar form

(i) $-1-i$ (ii) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

2. Find the modulus and argument of each of the following complex numbers

(i) $1+i\sqrt{3}$ (ii) -4

(iii) $\dfrac{1}{1+i}$ (iv) $1+i$

3. If points p represents the complex number $z = x + iy$ on the argand plane, then find the locus of the point p such that arg $(z) = 0$.

4. If z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then prove that arg $(z_1) - \arg(z_2) = \pi / 2$.

5. If arg $z < 0$, then prove that arg $(-z) - \arg z = \pi$

6. Prove that $\left(\dfrac{\sqrt{3}+i}{2}\right)^6 + \left(\dfrac{i-\sqrt{3}}{2}\right)^6 = -2$

7. Find the square roots of the following

(i) $7-24i$ (ii) $-8i$

8. If ω is one cube root of unity, then prove that $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

9. Find the cube roots of 27.

10. If ω_1 and ω_2 are complex cube roots of unity, then prove that $\omega_1^2 + \omega_2^4 = -\dfrac{1}{\omega_1\omega_2}$

Exercise ii

1. Write the following complex numbers in the polar form

(i) i (ii) $\dfrac{1+i}{1-i}$ (iii) $\dfrac{-16}{1+i\sqrt{3}}$

2. Find the modulus and argument of each of the following complex numbers

(i) $-2+2i\sqrt{3}$ (ii) $-\sqrt{3}-i$

(iii) $2\sqrt{3}-2i$

3. Express $(1-\cos\theta + i\sin\theta)$ in polar form.

4. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[\,|z_1|^2 + |z_2|^2]$

5. If z_1, z_2 are two non– zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$. Then, prove that arg $z_1 - \arg z_2 = 0$.

6. If $z = \dfrac{\sqrt{3}+i}{2}$ then prove that $Z^{69} = -i$.

7. Find the square roots of the following

(i) $5+12i$ (ii) $-15-8i$

8. If $1, \omega, \omega^2$ are the cube roots of unity, then prove that

$(1-\omega)^3 - (1+\omega^2)^3 = 0$.

9. Prove that

(i) $1+\omega^n+\omega^{2n} = 0$, if n is not a multiple of 3.

(ii) $1+\omega^n+\omega^{2n} = 3$, if n is a multiple of 3.

10. Prove that

$(1+5\omega^2+\omega^4)(1+5\omega+\omega^2)(5+\omega+\omega^2) = 64$

ANSWERS

Exercise – I

1. (i) $\sqrt{2}\left(\cos\dfrac{3\pi}{4} - i\sin\dfrac{3\pi}{4}\right)$

(ii) $2\left(\cos\dfrac{3\pi}{4} - i\sin\dfrac{3\pi}{4}\right)$

2. (i) $|z| = 2$, arg $(z) = \pi/3$

(ii) $|z| = 4$, $\arg(z) = \pi$

(iii) $|z| = 2$, $\arg(z) = -\pi/2$

(iv) $|z| = \sqrt{2}$, $\arg(z) = \frac{\pi}{4}$

3. $y = 0$ which is an equation of x − axis.

7. (i) $\pm(4 - 3i)$ (ii) $\pm 2(1 - i)$

9. $3, 3\omega, 3\omega^2$

Exercise – II

1. (i) $1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(ii) $1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(iii) $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

2. (i) $|z| = 4$, $\arg(z) = 2\pi/3$

(ii) $|z| = 2$, $\arg(z) = -5\pi/6$

(iii) $|z| = 4$, $\arg(z) = -\pi/6$

3. $2\sin\frac{\theta}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$

7. (i) $\pm(3 + 2i)$ (ii) $\pm(1 - 4i)$

SOLVED OBJECTIVE QUESTIONS: HELPING HAND

1. $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n$ is equal to

[PET (Raj.) – 1998]

(a) $\cos n\theta + i\sin n\theta$

(b) $\sin n\theta + i\cos n\theta$

(c) $\cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)$

(d) none of these

Solution

(c) Exp. =

$$\left[\frac{1 + \cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)}{1 + \cos(\pi/2 - \theta) - i\sin(\pi/2 - \theta)}\right]^n$$

$$= \cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)$$

2. If $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$, then $x^6 + \frac{1}{x^6}$ is equal to

[CET (Karnataka) – 2003]

(a) $2\cos 6\theta$ (b) $2\cos 12\theta$

(c) $2\sin 6\theta$ (d) $2\sin 12\theta$

Solution

(b) $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$

$\Rightarrow \sqrt{x} = \cos\theta + i\sin\theta$

$\Rightarrow (\sqrt{x})^{12} = x^6 = \cos 12\theta + i\sin 12\theta$

$\Rightarrow 1/x^6 = \cos 12\theta - i\sin 12\theta$

$\therefore x^6 + \frac{1}{x^6} = 2\cos 12\theta$

3. If $iz^4 + 1 = 0$, then z can take the value

[MPPET – 2006]

(a) $\frac{1+i}{\sqrt{2}}$ (b) $\cos\frac{\pi}{8}\, i\sin\frac{\pi}{8}$

(c) $\frac{1}{4i}$ (d) i

Solution

(b) Given that $iz^4 + 1 = 0$

$\Rightarrow z^4 = \frac{1}{i} = \frac{i^2}{i} = i$

Let $z^4 = \cos\frac{\pi}{2}\, i\sin\frac{\pi}{2}$

$\therefore z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/4}$

By using de Moivre's theorem, we get

$z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$

4. For any integer n, $\arg z = \frac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$ is

(a) $\pi/6$ (b) $\pi/3$

(c) $\pi/2$ (d) $2\pi/3$

Solution

$$z = \frac{2^{4n+1}}{2^{4n}} \frac{e^{i(4n+1)\pi/6}}{e^{-i4n\pi/3}} = = 2e^{i(12n+1)\pi/6}$$

$$= 2e^{2n\pi i}\, e^{i\pi/6} = 2e^{i\pi/6}$$

$\therefore$ are $z = \pi/6$

5. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then $\frac{x}{y} + \frac{y}{x}$ is equal to

[MNR – 1987]

(a) $2\cos(\theta - \phi)$ (b) $2\cos(\theta + \phi)$
(c) $2\sin(\theta - \phi)$ (d) $2\sin(\theta + \phi)$

Solution

(a) $x = \cos\theta + i\sin\theta$, $y = \cos\phi + i\sin\phi$

$\Rightarrow \frac{x}{y} = \cos(\theta - \phi) + i\sin(\theta - \phi)$

$\therefore \frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$

6. A value of n such that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^n = 1$ is

[EAMCET – 2007]

(a) 12 (b) 3 (c) 2 (d) 1

Solution

(a) $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 1 \Rightarrow \left(cis\,\frac{\pi}{6}\right) = 1$

only $12 = n$, satisfies in the given answers.

7. If $\left(\frac{1 + \cos\theta + i\sin\theta}{i + \sin\theta + i\cos\theta}\right)^4 = \cos n\theta + i\sin n\theta$, then n is equal to

[EAMCET – 1986]

(a) 1 (b) 2 (c) 3 (d) 4

Solution

(d) $D^r = i(1 + \cos 0) + \sin\theta$

$= 2i\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Therefore,

$$\text{L.H.S.} = \left[\frac{\cos(\theta/2) + i\sin(\theta/2)}{i\cos(\theta/2) + \sin(\theta/2)}\right]^4$$

$$= \frac{1}{i^4}(\cos\theta + i\sin\theta)^4$$

$$= \cos 4\theta + i\sin 4\theta$$

8. For any two complex numbers z_1, z_2 we have $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then:

(a) $\text{Re}\left(\frac{z_1}{z_2}\right) = 0$ (b) $\text{Im}\left(\frac{z_1}{z_2}\right) = 0$

(c) $\text{Re}(z_1 z_2) = 0$ (d) $\text{Im}(z_1 z_2) = 0$

Solution

(a) We have $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
$= |z_1|^2 + |z_2|^2$

where $\theta_1 = \arg(z_1)$, $\theta_2 = \arg(z_2)$

$\Rightarrow \cos(\theta_1 - \theta_2) = 0$

$\Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

$\Rightarrow \text{are}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \Rightarrow \text{Re}\left(\frac{z_1}{z_2}\right)$

$= \frac{|z_1|}{|z_2|}\cos\left(\frac{\pi}{2}\right) = 0$

Note: Also $\text{Re}\left(\frac{z_1}{z_2}\right) = 0 \Rightarrow \text{Re}(z_1\bar{z}_2) = 0$

$\Rightarrow z_1\bar{z}_2$ is purely imaginary.

9. Given $z = (1 + i\sqrt{3})^{100}$, then $\frac{\text{Re}(z)}{\text{Im}(z)}$ equals

[AMU – 2002]

(a) 2^{100} (b) 2^{50} (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Solution

(c) Let $z = (1 + i\sqrt{3})$

$r = \sqrt{3 + 1} = 2$ and $r\cos\theta = 1$, $r\sin\theta = \sqrt{3}$

$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\Rightarrow z^{100} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{100}$

$$= 2^{100}\left(\cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3}\right)$$

$$= 2^{100}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= 2^{100}\left(-\frac{1}{2}\frac{i\sqrt{3}}{2}\right)$$

$$\therefore \frac{\text{Re}(z)}{\text{Im}(z)} = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

10. If $(1 + i\sqrt{3})^9 = a + ib$, then b is equal to

[RPET – 1995]

(a) 1 (b) 256
(c) 0 (d) 93

Solution

(c) $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$= 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2e^{i\pi/3}$$

$\therefore (1 + i\sqrt{3})^9 = (2e^{i\pi/3})^9 = 2^9.e^{i(3\pi)}$

$$= 2^9 (\cos 3\pi + i \sin 3\pi) = -2^9$$

$\therefore a + ib = (1 + i\sqrt{3})^9 = -2^9; \quad \therefore b = 0$

11. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ $z = \cos\gamma + i\sin\gamma$ and $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$, then xyz is equal to

(a) i (b) 1 or −1
(c) −1 but not 1 (d) 0

[Kerala (CEE) – 2003]

Solution

(b) $\because \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

$\Rightarrow \alpha + \beta + \gamma = \pi$ or 0

$\therefore xyz = \cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)$

$= 1$ or -1.

12. If z is any complex number satisfying $|z - 1| = 1$, then which of the following is correct:

(a) $\arg(z - 1) = 2\arg z$
(b) $2\arg(z) = 2/3 \arg(z^2 - z)$
(c) $\arg(z - 1) = \arg(z + 1)$
(d) $\arg z = 2\arg(z + 1)$

Solution

(a) Therefore, $|z - 1| = 1 \therefore z - 1 = e^{i\theta} = \cos\theta + i\sin\theta$ where $\arg(z - 1) = \theta$

$\therefore z = 1 + \cos\theta + i\sin\theta$

$= 2\cos^2(\theta/2) + 2i\sin\theta/2\cos\theta/2$

$= 2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$

$= 2\cos\left(\frac{\theta}{2}\right). e^{i\pi/2}$

$\therefore \arg z = \theta/2 = 1/2 \arg(z - 1)$

Thus, $\arg(z - 1) = 2\arg z$.

13. If $|z| = 1$ and $|z| \neq \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on

(a) a line not passing through the origin
(b) $|z| = 2$ (c) z − axis
(d) y − axis

[IIT – 2007]

Solution

(d) $|z| = 1, z \neq \pm 1$

Let z be $e^{i\theta}$

$\therefore \frac{z}{1 - z^2} = \frac{e^{i\theta}}{1 - e^{2i\theta}} = \frac{e^{i\theta}}{1 - \cos 2\theta - i\sin 2\theta}$

$= \frac{e^{i\theta}}{2\sin^2\theta - 2i\sin\theta\cos\theta}$

$= \frac{e^{i\theta}}{2\sin\theta(\sin\theta - i\cos\theta)}$

$= \frac{e^{i\theta}}{2\sin\theta e^{i(\theta - \pi/2)}} = \frac{i}{2\sin\theta}$

where $\sin\theta \neq 0$ ($\because z \neq \pm 1$)

Hence, $\frac{z}{1 - z^2}$ always lies on y − axis.

14. If $\left(\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^n \in Z$, then the least positive integral value of n is:

[UPSEAT – 2002]

(a) 1 (b) 2 (c) 3 (d) 4

Solution

(c) $\left(\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^n = \left(\frac{-1 + i\sqrt{3}}{2}\right)^{2n}$

$$= \cos n\left(\frac{4\pi}{3}\right) + i\sin n\left(\frac{4\pi}{3}\right)$$

It is an integer for $n = 3$ (the least positive integral value of n)

15. The square root of i is

[NDA – 2004]

(a) $\pm\frac{1}{\sqrt{2}}(1 + i)$ (b) $\pm\frac{1}{\sqrt{2}}(1 - i)$

(c) $\pm\sqrt{2}(1 + i)$ (d) $\pm\sqrt{2}(1 - i)$

Solution

(a) $(i)^{1/2} = \pm(e^{i\pi/2})^{1/2}$

$$= \pm e^{i\pi/4} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \pm \frac{1}{\sqrt{2}}(1 + i)$$

16. If a $= \frac{1 - i\sqrt{3}}{2}$ then the correct matching of List – I from List – II is

[EAMCET – 2007]

List – I	List – II
(i) $a\bar{a}$	(A) $2\pi/3$
(ii) arg $(1/\bar{a})$	(B) $-i\sqrt{3}$
(iii) $a - \bar{a}$	(C) $2i/\sqrt{3}$
(iv) Im $(4/3a)$	(D) 1
	(E) $\pi/3$ (F) $2/\sqrt{3}$

The correct match is

	(i)	(ii)	(iii)	(iv)
(a)	D	E	C	B
(b)	D	A	B	F
(c)	F	E	B	C
(d)	D	A	B	C

Solution

(b) $a = \frac{1}{2} - \frac{\sqrt{3}}{2} i = 1 cis\left(-\frac{\pi}{3}\right)$

$\therefore\ \bar{a} = 1.cis\left(+\frac{\pi}{3}\right)$

(i) $a\bar{a} = cis\left(-\frac{\pi}{3}\right) cis\left(\frac{\pi}{3}\right) = cis0 = 1$

(ii) $Arg\left(\frac{1}{\bar{a}}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(iii) $a - \bar{a} = cis\left(-\frac{\pi}{3}\right) - cis\,\frac{\pi}{3}$

$= -2i \sin\frac{\pi}{3} = -i\sqrt{3}$

(iv) $I_m\left(\frac{4}{3a}\right) = I_m\left(\frac{4}{3}\left(\cos\frac{\pi}{3}\right)\right)$

$= \frac{4}{3} \times \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$

Note: $cis\theta = \cos\theta + i\sin\theta$

17. Find the value of the expression

$$2\left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) + 3\left(2 + \frac{1}{\omega}\right)\left(2 + \frac{1}{\omega^2}\right)$$

$$+ \ldots.. + (n + 1)\left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right)$$

[Orissa JEE – 2007]

(a) $n + \frac{n^2(n+1)^2}{4}$ (b) $n - \frac{n^2}{4(n+1)^2}$

(c) $1 - \frac{n^2}{4(n+1)^2}$ (d) $1 + \frac{n^2}{4(n+1)^2}$

Solution

(a) We have $(z + 1)(z + \omega)(z + \omega^2) = z^3 + 1$

Therefore, the given expression

$$= \sum_{r=1}^{r=n} (r + 1)(r + \omega)(r + \omega^2)$$

$$\left(\because \left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) = (1 + \omega^2)(1 + \omega)\right)$$

$$= \sum_{r=1}^{r=n} (r^3 + 1) = \sum_{r=1}^{r=n} r^3 + \sum_{r=1}^{r=n}$$

$$1 = \frac{n^2(n+1)^2}{4} + n$$

18. If 1, ω, ω^2 are the cube roots of unity then $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ is equal to

[Karnataka CET – 2007]

(a) 1 (b) 0

(c) ω^2 (d) ω

Solution

(a) If 1, ω, ω^2 are the cube roots of unity then $1 + \omega + \omega^2 = 0$ and ω 3 = 1.

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$

$= (1 + \omega)(-\omega)(1 + \omega)(-\omega)$

$= -\omega\ ^2(-\omega)(-\omega^2)(-\omega)\,\omega^3.\ \omega^3 = 1.\ 1 = 1$

19. If ω ($\neq$ 1) is a cube root of unity and $(1 + \omega)^7 = a + b\,\omega$, then A and b are respectively, the numbers

[IIT – 1995]

(a) 0, 1 (b) 1, 0

(c) 1, 1 (d) – 1, 1

Solution

(c) $(1 + \omega)^7 = a + b\,\omega$

$\Rightarrow (-\omega\ 2)^7 = a + b\,\omega$

$\Rightarrow \omega^{14} = -A - b\,\omega$

$\Rightarrow \omega^{2} . \omega\ 12 = -A - b\,\omega$

$\Rightarrow A + b\,\omega + \omega\ 2 = 0 \Rightarrow A = 1, b = 1$

$(\because 1 + \omega + \omega^{2} = 0)$

20. The maximum value of $|z|$ where z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

Solution

(b) $\left|z + \frac{2}{z}\right| = 2 \Rightarrow |z| - \frac{2}{|z|} \le 0$

$\Rightarrow |z|2 - 2|z| - 2 \le 0$

$|z| \le 2 \pm \frac{\sqrt{4+8}}{2} \le 1 \pm \sqrt{3}$

Hence maximum value of $|z|$ is $1 + \sqrt{3}$

21. If $x = a + b$, $y = a\alpha + b\beta$ and $z = \alpha\beta + b$, where α and β are the cube roots of unity then xyz is

[MP PET – 2005]

(a) $a^2 + b^2$ (b) $a^3 + b^3$

(c) a^3b^3 (d) none of these

Solution

(b) Suppose here $a = \omega$, $\alpha = \omega^2$, then

$xyz = (a + b)(a\,\omega + b\,\omega^2)(a\,\omega^2 + b\,\omega)$

$= (a + b)(a^2\,\omega^3 + ab\,\omega^2 + ba\,\omega^4 + b^2\,\omega^3)$

$= (a + b)[a^2 + ab(\omega^2 + \omega) + b^2]$

$= (a + b)(a^2 - ab + b^2) = a^3 + b^3$

22. $\frac{(-1 + i\sqrt{3})^{15}}{(1 - i)^{20}} + \frac{(-1 - i\sqrt{3})^{15}}{(1 + i)^{20}}$ is equal to

(a) -64 (b) -32

(c) -16 (d) $1/16$

Solution

(a) $2^{15}\left[\frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{15}}{(1 - i)^{20}} + \frac{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{15}}{(1 + i)^{20}}\right]$

$= 2^{15}\left[\frac{\omega^{15}}{(1 - i)^{20}} + \frac{\omega^{30}}{(1 + i)^{20}}\right]$

$= 2^{15}\left[\frac{1}{(1 - i)^{20}} + \frac{1}{(1 + i)^{20}}\right]$

$= 2^{15}\,\frac{(1 + i)^{20} + (1 - i)^{20}}{(i - 12)^{20}}$

$= \frac{2^{15}}{2^{20}}[(1 + i)^{20} + (1 - i)^{20}]$

$= \frac{1}{32}[\{(1 + i)^2\}^{10} + \{(1 - i)^{10}]$

$= \frac{1}{32}[(2i)^{10} + (2i)^{10}]$

$= \frac{1}{32}[(2^{10}\{i^{10} + i^{10}\}] = 32[-1 - 1] = -64.$

23. If $z + z^{-1} = 1$, then $z^{100} + z^{-100}$ is equal to

(a) i (b) $-i$

(c) 1 (d) -1

Solution

(d) $z + z^{-1} = 1$

$\Rightarrow z^2 - z + 1 = 0$

$\Rightarrow z = \omega$ or $-\omega^{2}$

For $z = -\omega$, $z^{100} + z^{-100} = (-\omega)^{100} + (-\omega)^{-100}$

$= \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$

For $z = -\omega^2$, $z^{100} + z^{-100} = (-\omega^2)^{100} + (-\omega^2)^{-100}$

$= \omega^{200} + \frac{1}{\omega^{200}}$

$= \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega + = -1$

24. If z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

[IIT – 1986; CET (Pb.) – 1992]

(a) real and positive

(b) real and negative

(c) purely imaginary

(d) zero or purely imaginary

Solution

(d) Let $z_1 = x + iy$ and $z_2 = p + iq$, Then

$|z_1| = |z_2| \Rightarrow x^2 + y^2 = p^2 + q^2$ (1)

Now $\dfrac{z_1 + z_2}{z_1 - z_2} = \dfrac{(x+p) + i(y+q)}{(x-p) + i(y-q)}$

$= \dfrac{2i(xq + yp)}{(x-p)^2 + (y-q)^2}$ (2)

If $xq + yp \neq 0$, then given expression is purely imaginary.

If $xq + yp = 0$, then $\frac{x}{p} = \frac{y}{-q} = \lambda$ (say) then from (1)

$p^2 + q^2 = 12(p^2 + q^2) \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = 1, -1$

For both values of λ, $z_1 \neq z_2$ but $|z_1| = |z_2|$.

$\Rightarrow z_1 = -z_2$ So in this case, the given expression is zero.

25. Solutions of the equation $z_2 + |z| = 0$ are

(a) $0, \pm 1, \pm i$ (b) $0, \pm i$

(c) $1 + i$ (d) $1 - i$

Solution

$(x + iy)^2 + \sqrt{x^2 + y^2} = 0$

$x^2 - y^2 + \sqrt{x^2 + y^2} + 2ixy = 0$

$x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ (1)

$2xy = 0$ (2)

From (2), $x = 0$ or $y = 0$. Then, from (1)

$x = 0 \Rightarrow y = 0, \pm 1$

$\therefore z = 0, \pm i$ **Ans:** *b*

26. $\tan\left[i \log \dfrac{a - ib}{a + ib}\right]$ is equal to

[DCE – 1996]

(a) $\dfrac{2ab}{a^2 + b^2}$ (b) $\dfrac{2ab}{a^2 - b^2}$

(c) $\dfrac{a^2 - b^2}{2ab}$ (d) ab

Solution

(b) $\because \log\left(\dfrac{a - ib}{a + ib}\right) = \log(a - ib) - \log(a + ib)$

$= \log r + i\tan^{-1}\left(\frac{-b}{a}\right) - \left(\log r + i\tan^{-1}\frac{b}{a}\right)$

where $r = \sqrt{a^2 + b^2}$

$= 2i\tan^{-1}(-b/a)$

$= 2i\tan^{-1} b/a$

$= -\text{i}\tan{-1}\ \dfrac{2ab}{a^2 - b^2}$

$\Rightarrow \tan\left[i\log\left(\dfrac{a - ib}{a + ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. For any complex number z, which of the following is not true?

(a) $\text{Re}(z) = \dfrac{z + \bar{z}}{2}$ (b) $\text{Im}(z)\ \dfrac{z - \bar{z}}{2i}$

(c) $|z|^2 = z\bar{z}$

(d) $-|\text{Re}(z)| \leq |z| \leq |\text{Re}(z)|$

2. If $|z - i| < |z + i|$, then

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$

(c) $\text{Im}(z) > 0$ (d) $\text{Im}(z) < 0$

3. $(\cos\theta + i\sin\theta)^2$ is equal to

(a) $\cos 2\theta + i\sin 2\theta$ (b) $\sin 2\theta + i\cos 2\theta$

(c) $\cos 2\theta - i\sin 2\theta$ (d) none of these

4. $\left(\dfrac{1}{\sqrt{3} + i}\right)^{24} =$

(a) 2^{24} (b) -2^{24}

(c) $\dfrac{1}{2^{24}}$ (d) none of these

5. $\left(\dfrac{\sqrt{3} + i}{2}\right)^{69}$ is equal to

(a) 1 (b) -1 (c) $-i$ (d) i

6. If $\left(\dfrac{1 - i}{1 + i}\right)^{100} = a + ib$, then

[MPPET – 1998]

(a) $a = 2, b = -1$ (b) $a = 1, b = 0$

(c) $a = 0, b = 1$ (d) $a = -1, b = 2$

7. If z_1 and z_2 are any two complex numbers then $|z_1 + z_2|^2 + |z_1 - z_2|^2$ is equal to

[MPPET – 1993; RPET – 1997]

(a) $2|z_1|^2|z_2|^2$ (b) $2|z_1|^2 + 2|z_2|^2$
(c) $|z_1|^2 + |z_2|^2$ (d) $2|z_1||z_2|$

8. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) 0

[IIT – 1979, 1987; EAMCET – 1986; RPET – 1997; MPPET – 2001, 2007; AIEEE – 2005]

9. If z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$, then z is equal to

(a) ω (b) $-\omega$
(c) $\overline{\omega}$ (d) $-\overline{\omega}$

[IIT – 1995; AIEEE – 2002; JEE (Orissa) – 2004]

10. $-1 + \sqrt{-3} = re^{i\theta}$, then $\theta =$

[MP PET – 1999; RPET – 1989]

(a) $\frac{2\pi}{3}$ (b) $-\frac{2\pi}{3}$
(c) $\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$

11. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\left(\frac{z_1}{z_2}\right)$ + are $\left(\frac{z_2}{z_3}\right)$ equals

(a) 0 (b) $-\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$ (d) π

12. If $z \neq 0$ is a complex number such that $\text{Arg}(z) = \pi/4$, then

(a) $\text{Im}(z^2) = 0$ (b) $\text{Re}(z^2) = 0$
(c) $\text{Re}(z) = i\text{m}(z^2)$ (d) none of these

13. The value of $(i)^i$ is

[AMU – 1998]

(a) ω (b) ω^2
(c) $e^{-\pi/2}$ (d) $2\sqrt{2}$

14. Argument of the complex number $\left(\frac{-1-3i}{2+i}\right)$ is

(a) 45° (b) 135°
(c) 225° (d) 240°

[VITEEE – 2008]

15. What is Arg (bi) where $b > 0$?

[NDA – 2008]

(a) 0 (b) $\frac{\pi}{2}$
(c) π (d) $\frac{3\pi}{2}$

16. Let c be the set of complex numbers and z_1, z_2 are in C.

(i). Argument z_1 = argument $z_2 \Rightarrow z_1 = z_2$
(ii). $|z_1| = |z_2| \Rightarrow z_1 = z_2$

Which of the statements given above is/are correct?

[NDA – 2008]

(a) 1 only (b) 2 only
(c) both 1 and 2 (d) neither 1 nor 2

17. If $y = \cos\theta + i\sin\theta$, then the value of $y + \frac{1}{y}$ is

(a) $2\cos\theta$ (b) $2\sin\theta$
(c) $2\text{cosec}\theta$ (d) $2\tan\theta$

[RPET – 1995]

18. The imaginary part of i^i is

(a) 0 (b) 1
(c) 2 (d) -1

[Karnataka CET – 2007]

19. If z_1 and z_2 are any two complex numbers, then which of the following is not true.

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$
(b) $|z_1 - z_2| \leq |z_1| + |z_2|$
(c) $|z_1 - z_2| \geq ||z_1| - |z_2||$
(d) $|z_1| - |z_2| \geq |z_1 - z_2|$

20. If $z^2 = -i$, then $z =$

(a) $\frac{1}{\sqrt{2}}(1+i)$ (b) $\frac{1}{\sqrt{2}}(1+i)$
(c) $\pm\frac{1}{\sqrt{2}}(1-i)$ (d) none of these

21. If ω is a non real cube root of unity, then $\frac{a + b\omega + c\omega^2}{a\omega + c + b\omega^2}$

(a) 1 (b) ω^2
(c) ω (d) none of these

22. $(3 + 3\omega + 5\omega^2)^3 - (2 + 4\omega + 2\omega^2)^3$ is equal to

(a) 0 (b) 3
(c) 2 (d) 1

23. If α and β are imaginary cube roots of unity, then the value of $\alpha^4 + \beta^{28} + \frac{1}{\alpha\beta}$, is

[MPPET – 1998]

(a) 1 (b) −1
(c) 0 (d) none of these

24. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2|$ is

(a) $\le |z_1| + |z_2|$ (b) $\le |z_1| - |z_2|$
(c) $< |z_1| + |z_2|$ (d) $> |z_1| + |z_2|$

[RPET – 1985; MPPET – 1987, 2004; Kerala Engg. – 2002]

25. If ω is an imaginary cube root of unity, $(1 + \omega - \omega^2)^7$ equals

(a) $128\,\omega$ (b) $-128\,\omega$
(c) $128\,\omega^2$ (d) $-128\,\omega^2$

[IIT – 1998; MPPET – 2000]

26. If z is any complex number such that $|z + 4| \le 3$, then the greatest value of $|z + 1|$ is

[AIEEE – 2007]

(a) 6 (b) 4
(c) 5 (d) 3

27. If n is a positive integer not a multiple of 3, then $1 + \omega^n + \omega^{2n} =$

[MPPET – 2004]

(a) 3 (b) 1
(c) 0 (d) none of these

28. If $1, \omega, \omega^2$ are the three cube roots of unity, then $(3 + \omega^2 + \omega^4)^6 =$

[MPPET – 1995]

(a) 64 (b) 729
(c) 2 (d) 0

29. If ω is a cube root of unity, then $(1 + \omega - \omega^2)(1 - \omega + \omega^2) =$

(a) 1 (b) 0
(c) 2 (d) 4

[MNR – 1990; MPPET – 1993, 2002]

30. One of the cube roots of unity is

[MPPET – 1994, 2003]

(a) $\frac{-1 + i\sqrt{3}}{2}$ (b) $\frac{1 + i\sqrt{3}}{2}$

(c) $\frac{1 - i\sqrt{3}}{2}$ (d) $\frac{\sqrt{3} - i}{2}$

31. If ω is the cube root of unity, then $(3 + 5\,\omega + 3\,\omega^2)^2 + (3 + 3\,\omega + 5\,\omega^2)^2 =$

[MPPET – 1999]

(a) 4 (b) 0
(c) − 4 (d) none of these

32. If ω is cube root of unity then the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) =$

(a) 0 (b) 1
(c) − 1 (d) 9

[MP PET – 2006]

33. $\left(\frac{-1 + i\sqrt{3}}{2}\right)^{20} + \left(\frac{-1 - i\sqrt{3}}{2}\right)^{20} =$

(a) $20\sqrt{3}i$ (b) 1

(c) $\frac{1}{2^{19}}$ (d) − 1

34. If $\left|Z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

[AIEEE – 2009]

(a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$
(c) 2 (d) $2 + \sqrt{2}$

35. What is the value of

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^{900} + \left(\frac{-1 + i\sqrt{3}}{2}\right)^{301} ?$$

[NDA – 2009]

(a) $\frac{-1 + i\sqrt{3}}{2}$ (b) $\frac{1 - i\sqrt{3}}{2}$

(c) $\frac{-1 - i\sqrt{3}}{2}$ (b) $\frac{1 + i\sqrt{3}}{2}$

36. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots + \left(z^6 + \frac{1}{z^6}\right)^2$ is

(a) 6 (b) 12
(c) 18 (d) 24

[MPPET – 2009, AIEEE – 2006]

37. The number of solutions of the equation $z^2 + \bar{z} = 0$ is

[MPPET – 2009]

(a) 1 (b) 2
(c) 3 (d) 4

38. If x is a positive integer, then $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})$n is equal to

[Orissa JEE – 2009]

(a) $2^{n-1} \cos \frac{n\pi}{3}$ (a) $2^n \cos \frac{n\pi}{3}$
(c) $2^{n+1} \cos \frac{n\pi}{3}$ (d) none of these

SOLUTIONS

1. (d) If $(1 - i)^n = 2^n \Rightarrow n = 0$ clearly.

2. (c) $(1 + i)^8 = ((1 + i)^2)^4 = (1 + i^2 + 2i)^4$
$= (2i)^4 = 16$
$(1 - i)^8 = ((1 - i)^2)^4 = (1 + i^2 - 2i)^4$
$= (-2i)^4 = 16$
$(1 + i)^8 + (1 - i)^8 = 32$

3. (a) $|z_1 + z_2|$

$$= \sqrt{|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)} \quad (1)$$

$|z_1 - z_2|$

$$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)} \quad (2)$$

If $|z_1 + z_2| = |z_1 - z_2| \Rightarrow \cos(\theta_1 - \theta_2) = \cos 90°$ $= 0$ i.e., $\theta_1 - \theta_2 = 90°$

or $\text{amp}(z_1) - \text{amp}(z_2) = 90°$

4. (c) $\theta = \text{amp}(x + iy) = \tan^{-1}\frac{y}{x}$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{2}}{\sqrt{3}+1}}{\frac{1}{\sqrt{3}+1}}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

5. (c) $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{69} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{69}$

$$= \left(\cos 69 \times \frac{\pi}{6} + i \sin 69 \times \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{23\pi}{2}\right) + i\sin\left(\frac{23\pi}{2}\right)$$

$$= 0 + i\left(\sin\left(10\pi + \frac{3\pi}{2}\right)\right)$$

$$= \text{i} \sin\frac{3\pi}{2} = -i$$

6. (b) $\because \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{-2i}{2} = -i$

Therefore

$$\left(\frac{(1-i)}{1+i}\right)^{100} = (-i)^{100} = i^{100} = (i^4)^{25} = 1$$

Therefore, given $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$ from 1,
$a + ib = 1$ on comparing $a = 1, b = 0$.

7. (b) Important Formula
$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1 \bar{z}_2)$
and $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1 \bar{z}_2)$

On adding, we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Alternative solution

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \quad (1)$$

$$(z_1 - z_2) = (x_1 - x_2) + i(y_1 - y_2) \quad (2)$$

$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

On adding, we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[x_1^2 + y_1^2 x_2^2 + y_2^2]$

$= 2\,\{|z_1|^2 + |z_2|^2\}$

8. (d) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1 + z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$
$= [r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)]^{1/2}$
$= [(r_1 + r_2)^2]^{1/2}$
$(\because |z_1 + z_2| = |z_1| + |z_2|)$

Therefore, $\cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$
$\Rightarrow \theta_1 = \theta_2$
$\arg(z_1) - \arg(z_2) = 0$

Alternative Solution

$|z_1 + z_2| = |z_1| + |z_2|$
z_1, z_2 are on the same line
$\therefore \arg z_1 = \arg z_2 \Rightarrow \arg z_1 - \arg z_2 = 0.$

9. (d) Let $|z| = |\omega| = r$ and Arg $\omega = \theta$

then $\omega = r \text{ cis } \theta$ and Arg $z = \pi - \theta$

Hence $z = r \text{ cis } (\pi - \theta)$
$= r\{\cos(\pi - \theta) + i \sin(\pi - \theta)\}$
$= r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta)$
$= -x + iy$
$= -\overline{\omega}$

OR

Quadrantwise complex numbers of equalmagnitude with corresponding argument

$-(z) = (-z) = -x + iy$ $amp(-(z)) = amp(-z)$ $= \pi - \theta$	$z = x + iy$ $amp(z) = \theta$
$-z = -x - iy$ $amp(z) = -(\theta - \theta)$	$z = x - iy$ $amp(z) = -\theta$

Clearly, amp (z) + amp$(-(\overline{z})) = \pi$
i.e., amp (z) + amp$(\omega) = \pi$
or $\omega = -(\overline{z})$

Note: $|z| = |\overline{z}| = |-z| = |-(\overline{z})|$

$$= |(-\overline{z})| = \sqrt{x^2 + y^2}$$

10. (a) Therefore, $e^{ix} = \cos x + i \sin x$
Therefore $-1 + \sqrt{-3} = re^{i\theta}$ may be written as $-1 + \sqrt{3}\, i = r(\cos\theta + i\sin\theta)$

Comparing real and imaginary part

$r\cos\theta = -1,\ r\sin\theta = \sqrt{3}$

or $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{\sqrt{3}}{-1} \Rightarrow \tan\theta = -\sqrt{3}$ or

$\theta = \dfrac{3\pi}{3}$

11. (a) We have $z_2 = \overline{z}_1$ and $z_4 = \overline{z}_3$, therefore, $z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

Now, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = \arg\left(\dfrac{z_1 z_2}{z_4 z_3}\right)$

$= \arg\left(\dfrac{|z_1|^2}{|z_3|^2}\right) = \arg\left(\left|\dfrac{z_1}{z_3}\right|\right)^2 = 0$

[$\because$ argument of a positive real number is zero]

12. (b) Given that $\arg(z) = \dfrac{\pi}{4}$
i.e. $\tan^{-1}\dfrac{y}{x} = \dfrac{\pi}{4}$
$\Rightarrow \dfrac{y}{x} = 1 \Rightarrow y = x$

i.e. $z = 1 + i$ (for example)
Therefore, $z^2 = 1 - 1 + 2i = 2i$ i.e., $Re(z^2)$ = real part of $z^2 = 0$

13. (c) Let $A = i^i \Rightarrow \log A = i \log i$
$\Rightarrow \log A = i\log(0 + i) = i[\log 1 + i\tan^{-1}1/0]$
$\Rightarrow \log A = i[0 + i\pi/2] = -\pi/2$
$\Rightarrow A = e^{-\pi/2}$

14. (c) $z = \dfrac{-1-3i}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{-2+i-6i-3}{5}$

$z = \dfrac{-5-5i}{5} = -1 - i$
$\arg(z)$ is given by $\sin\theta = -\dfrac{1}{\sqrt{2}}, \cos\theta = -\dfrac{1}{\sqrt{2}}$
$\Rightarrow \theta = 225°$

15. (b) $\arg(bi) = \dfrac{\pi}{2}$ ($\because b > 0$)

OR

$\arg(bi) = \tan^{-1}\left(\dfrac{b}{0}\right)$

$= \tan^{-1}(\infty)\ \dfrac{\pi}{2}$

16. (d) None of the given statements is correct.
1. their magnitudes may be different.
2. their argument may be different.

17. (a) $y = \cos\theta + i\sin\theta = e^{i\theta}, \dfrac{1}{y}\, e^{-i\theta}$

$y + \dfrac{1}{y} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$

18. (a) $i = e^{\frac{i\pi}{2}}\ \therefore\ i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}}$

19. (c) Third side is greater than or equal to the difference of two sides.

20. (c) $(1^2 + i^2 - 2i)/2 = -i$ (By verification ethod).

21. (b) $\frac{a + b\omega + c\omega^2}{a\omega + c + b\omega^2} = \frac{a\omega^3 + b\omega^4 + c\omega^2}{a\omega + c + b\omega^2}$

$= \frac{\omega^2(a\omega + b\omega^2 + c)}{a\omega + c + b\omega^2} = \omega^2.$

22. (a) $(3(1 + \omega) + 5\,\omega^2)^3 - (2(1 + \omega^2) + 4\omega)^3$

$= (-3\,\omega^2 + 5\,\omega^2)^3 - (-2\,\omega + 4\,\omega)^3$

$= (2\,\omega^2)^3 - (2\omega)^3$

$= 8 - 8 = 0.$

23. (c) Take $\alpha = \omega, \beta = \omega^2$, then $\omega^3 = 1$

$\alpha^4 + \beta^{28} + \frac{1}{\alpha\beta} + \omega^4 + \omega^{56} + \frac{1}{\omega^3}$

$= \omega^3.\ \omega + (\omega^2)^{18}\ .\ \omega^2 + 1\ \omega + \omega^2 + 1 = 0$

24. (a) By triangle identity, $|z_1 + z_2| \leq |z_1| + |z_2|$ The modulus of sum of two complex numbers is always less than or equal to the sum of modulii of two complex numbers.

25. (d) $(1 + \omega - \omega^2)^7 = (1 + \omega + \omega^2 - 2\omega^2)^7$

$= (0 - 2\,\omega^2)^7$

$= -2^7.\ \omega^{14} = -128\,\omega^2.$

26. (a) Given $|z + 4| \leq 3$ (1)

$\therefore |z + 1| = |(z + 4) + (-3)|$

$\leq |z + 4| + |(-3)|$

$\leq 3 + 3$

$\Rightarrow |z + 1| \leq 6$

$\therefore$ Maximum value of $|z + 1| = 6$.

Alternatively note that $z = -7$ satisfies (1) and for this z,

$|z + 1| = |-7 + 1| = 6.$

27. (c) If n is not the multiple of 3, then

$n = 3K + 1$ or $3K + 2$

$\therefore\ 1 + \omega^n + \omega^{2n} = 1 + \omega^{3K+1} + \omega^{2(3K+1)}$

$= 1 + \omega^{3K}\,.\ \omega + \omega^{3\times 2K}\,\omega^2$

$= 1 + \omega + \omega^2 = 0$

For $n = 3K + 2$ also $1 + \omega^n + \omega^{2n} = 0$

28. (a) $(3 + \omega^2 + \omega^4)^6 = (3 + \omega^2 + \omega)^6$

$= (3 - 1)^6$

$= 26$

$= 64.$

29. (d) $(1 + \omega - \omega^2)\,(1 - \omega + \omega^2)$

$= (-2\omega^2)\,(-2\omega) = 4\omega^3 = 4.$

30. (a) Let $x = (1)^{1/3}$ or $x^3 = 1$

Therefore, $x^3 - 1 = 0$ or $(x - 1)\,(x^2 + x + 1) = 0$

Possible roots $x = 1$ or $x = \frac{-1 \pm \sqrt{1 - 4}}{2}$

$x = 1$ or $x = \frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}$

31. (c) Formula $1 + \omega + \omega^2 = 0$

$\therefore\ (3 + 5\,\omega + 3\,\omega^2)^2$

$= (2\,\omega + 3 + 3\,\omega + 3\,\omega^2)^2 = (2\omega)^2$ (1)

and $(3 + 3\,\omega + 5\,\omega^2)$

$= (2\,\omega^2 + 3 + 3\,\omega + 3\,\omega^2)^2 = (2\,\omega^2)^2$ (2)

Adding 1 and 2

$(3 + 5\,\omega + 3\,\omega^2)^2 + (3 + 3\,\omega + 5\,\omega^2)^2$

$= 4\,\omega^2 + 4\,\omega^4 = 4(\omega^2 + \omega)\ (\because\ \omega^3 = 1)$

$= -4\ (\because\ \omega^2 + \omega = -1)$

32. (d) $(1 - \omega)\,(1 - \omega^2)\,(1 - \omega^4)\,(1 - \omega^8)$

$\{\omega^4 = \omega^3.\ \omega = \omega$ and $\omega^8 = \omega^6\,\omega^2 = \omega^2\}$

Given expression

$= (1 - \omega)\,(1 - \omega^2)\,(1 - \omega)\,(1 - \omega^2)$

$= (1 - \omega)^2\,(1 - \omega^2)^2$

$= (1 + \omega^2 - 2\,\omega)\,(1 + \omega^4 - 2\,\omega^2)$

$= (-\,\omega - 2\,\omega)\,(1 + \omega - 2\,\omega^2)$

$= (-3\,\omega)\,(-3\,\omega^2) = 9\,\omega^3 = 9.$

33. (d) As $\frac{-1 + i\sqrt{2}}{2} = \omega$ and $\frac{-1 - i\sqrt{3}}{2} = \omega^2$

$\therefore\ (\omega)^{20} + (\omega^2)^{20} = \omega^{18}.\ \omega^2 + \omega^{39}.\ \omega = \omega^2 + \omega$

$= -1.$

34. (b): $|Z| = \left|\left(Z - \frac{4}{Z}\right) + \frac{4}{Z}\right| \Rightarrow |Z|$

$= \left|Z - \frac{4}{Z} + \frac{4}{Z}\right|$

$\Rightarrow |z| \leq \left|z - \frac{4}{z}\right| + \frac{4}{|z|} \Rightarrow |z| \leq 2 + \frac{4}{|z|}$

$\Rightarrow |Z|^2 - 2\,|Z| - 4 \leq 0$

$\Rightarrow \left(|z| - (\sqrt{5} + 1)\right)\left(|z| - (1 - \sqrt{5})\right) \leq 0$

$\Rightarrow 1 - \sqrt{5} \leq |z| \leq \sqrt{5} + 1$

35. (b) $\left(\frac{-1+i\sqrt{3}}{2}\right)^{900}+\left(\frac{-1-i\sqrt{3}}{2}\right)^{301}$

$(\omega)^{900}+(\omega^2)^{301}$

$(\omega^3)^{300}+\omega^{602}$

$(1)^{300}(\omega^3)^{200}\times\omega^2$

$=1+\omega^2=-\omega=-\left(\frac{-1+i\sqrt{3}}{3}\right)$

36. (b) Given, $z^2+z+1=0$

$\Rightarrow z=\omega,\omega^2$

Take $z=\omega$,

$\therefore \left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2$

$+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^5+\left(z^6+\frac{1}{z^6}\right)^2$

$=\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega^2+\frac{1}{z^2}\right)^2+\left(\omega^3+\frac{1}{z^3}\right)^2$

$+\left(\omega^4+\frac{1}{z^4}\right)^2+\left(\omega^5+\frac{1}{z^5}\right)^5+\left(\omega^6+\frac{1}{z^6}\right)^2$

$=(\omega+\omega^{2})^2+(2+)^2+(1+1)^2$

$+(+^{2})^2+(2+)^2+(1+1)^2$

$=1+1+4+1+1+4=12$

Similarly, for $z=2$, we get the same result.

37. (d) Given, $z^2+\bar{z}=0$

Let $z=z+iy$

$\therefore (x+iy)^2+x-iy=0$

$\Rightarrow x^2-y^2+2ixy+x-iy=0$

$\Rightarrow (x^2+x-y^2)+i(2xy-y)=0$

On equating real and imaginary part, we get

$\Rightarrow x^2+x-y^2=0$ (1)

and $2xy-y=0$ $y=0$ or $x=½$

If $y=0$, then equation (1) gives $x^2+x=0$

$\Rightarrow x=0$ or -1

and if $x=\frac{1}{2}$, then equation (1) gives $\frac{1}{4}+\frac{1}{2}-y^2=0$

$y^2=\frac{3}{4}$; $y=\pm\frac{\sqrt{3}}{4}$

Hence, there are four solutions of given equation.

38. (c) $(1+i\sqrt{3})^n(1-i\sqrt{3})^n$

$=2^n\left[\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n+\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^n\right]$

$=2^n\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n\right]$

$=2^n\left[2\cos\frac{n\pi}{3}\right]=2^{n+1}\cos n\pi/3$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $(1-i)^n=2^n$, then $n=$ **[RPET – 1990]**

(a) 1 (b) 0

(c) -1 (d) none of these

2. The value of $(1+i)^8=(1-i)^8$ is:

[RPET – 2001, KCET – 2001]

(a) 16 (b) -16 (c) 32 (d) -32

3. If $|z_1+z_2|=|z_1-z_2|$, then the difference in the amplitudes of z_1 and z_2 is

[EAMCET – 1985]

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) 0

4. The amplitude of $\frac{1+\sqrt{3}i}{\sqrt{3}+1}$ is:

[Karnataka CET – 1992; Pb. CET – 2001]

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $-\frac{\pi}{6}$

5. The value of

$\frac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta)}$ is

[RPET – 2001]

(a) $\cos(\alpha+\beta-\gamma-\delta)-i\sin(\alpha+\beta-\gamma-\delta)$

(b) $\cos(\alpha+\beta-\gamma-\delta)+i\sin(\alpha+\beta-\gamma-\delta)$

(c) $\sin(\alpha+\beta-\gamma-\delta)-i\cos(\alpha+\beta-\gamma-\delta)$

(d) $\sin(\alpha+\beta-\gamma-\delta)+i\cos(\alpha+\beta-\gamma-\delta)$

6. The modulus and amplitude of $\frac{1+2i}{1-(1-i)^2}$ are:

[Karnataka CET – 2005]

(a) $\sqrt{6}$ and $\pi/6$

(b) 1 and 0

(c) 1 and $\pi/3$

(d) 1 and $\pi/4$

7. $(\sin + i\cos)^n$ is equal to

(a) $\cos n\theta + i \sin n\theta$

(b) $\sin n + i \cos n\theta$

(c) $\cos n\left(\frac{\pi}{2} - \theta\right) + i \sin n\theta$

(d) none of these

[RPET – 2001]

8. Which of the following are correct for any two complex numbers z_1 and z_2

[Roorkee – 1998]

(a) $|z_1 z_2| = |z_1|\,|z_2|$

(b) $\arg(z_1 z_2) = (\arg z_1)(\arg z_2)$

(c) $|z_1 + z_2| = |z_1| + |z_2|$

(d) $|z_1 - z_2| \geq |z_1| - |z_2|$

9. The amplitude of $\frac{1+\sqrt{3}}{\sqrt{3}-i}$ is:

[RPET – 2001]

(a) 0 (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/2$

10. If $z = \frac{-2}{1+\sqrt{3}}$ then the value of arg(z) is

[Orissa JEE – 2002]

(a) π (b) $\pi/3$ (c) 2/3 (d) $\pi/4$

11. $(1 + i)^{10}$, where $i^2 = -1$, is equal to

[AMU – 2001]

(a) $32\,i$

(b) $64 + i$

(c) $24\,i - 32$

(d) none of these

12. $\sqrt{-8-6i} =$

(a) $1 \pm 3i$

(b) $\pm(1 - 3i)$

(c) $\pm(1 + 3i)$

(d) $\pm(3 - i)$

[Roorkee – 1979, RPET – 1992]

13. The value of will

$\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} + \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2}$ be

(a) 1

(b) -1

(c) 2

(d) -2

[BIT Ranchi – 1989; Orissa JEE –2003]

14. A value of $\sqrt{i} + \sqrt{-i}$ is

[NDA – 2007]

(a) 0 (b) $\sqrt{2}$ (c) $-i$ (d) i

15. If z is a complex number, then the minimum value of $|z| + |z - 1|$ is

[Roorkee – 1992]

(a) 1

(b) 0

(c) 1/2

(d) none of these

16. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{335}$ is equal to

[IIT – 1999]

(a) $1 - i\sqrt{3}$

(b) $-1 + i\sqrt{3}$

(c) $i\sqrt{3}$

(d) $-i\sqrt{3}$

17. If $(-7 - 24\,i)^{1/2} = x - iy$, then $x^2 + y^2 =$

[RPET – 1989]

(a) 15

(b) 25

(c) -25

(d) none of these

18. The square root of $3 - 4i$ is

[RPET – 1999]

(a) $\pm(2 + i)$

(b) $\pm(2 - i)$

(c) $\pm(1 - 2i)$

(d) $\pm(1 + 2i)$

19. If $\sqrt{a + ib} = x + iy$, then possible value of $\sqrt{a - ib}$ is

[Kerala (Engg.) – 2002]

(a) $x^2 + y^2$

(b) $\sqrt{x^2 + y^2}$

(c) $x + iy$

(d) $x - iy$

20. If $a = \sqrt{2}i$ then which of the following is correct

[Roorkee – 1989]

(a) $a = 1 + i$

(b) $a = 1 - i$

(c) $a = -i$

(d) None of these

21. $(27)^{1/3} =$

[NDA – 2007]

(a) 3

(b) $3, 3i, 3i^2$

(c) $3, 3\omega, 3\omega^2$

(d) None of these

22. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{1000} =$

(a) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(b) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(d) none of these

23. If $z = \sqrt{2i} - \sqrt{-2i}$, then $|z| =$

(a) 2

(b) $\sqrt{2}$

(c) 0

(d) $2\sqrt{2}$

24. If the roots of the equation $x^3 - 1 = 0$ are 1, ω,and ω^2, then the value of $(1 - \omega)(1 - \omega^2)$ is

(a) 0 (b) 1 (c) 2 (d) 3

25. If cube root of 1 is, then the value of (3 + + $2^2)^4$ is

(a) 0 (b) 16
(c) $9\omega^2$ (d) $16\omega^2$

26. z and are two non–zero complex numbers such that $|z| = |\omega|$ and Arg z + Arg $\omega = \pi$, then z =

[AIEEE – 2002]

(a) $\overline{\omega}$ (b) $-\overline{\omega}$
(c) ω (d) $-\omega$

27. The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)^6$, where ω, ω^2 are cube roots of unity

[DCE – 2001]

(a) $128\,\omega$ (b) $-128\,\omega^2$
(c) $-128\,\omega$ (d) $128\,\omega^2$

28. The value of $(1 + i)^6 + (1 - i)^6$ is

[RPET – 2002]

(a) 0 (b) 2^7
(c) 2^6 (d) none of these

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet.
2. The test is of 17 minutes.
3. The test consists of 17 questions. The maximum marks are 51.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited

1. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer is

(a) 2 (b) 4 (c) 8 (d) 16

2. arg z + arg $\bar{z}$ $(z \neq 0)$ is

(a) 0 (b) π
(c) $\pi/2$ (d) none of these

3. If $|z_1| = |z_2|$ and arg $(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to

(a) 0
(b) purely imaginary
(c) purely real
(d) none of these

4. The polar form of the complex number $(i^{25})^3$ is

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\pi + i\sin\pi$
(c) $\cos\pi - i\sin\pi$ (d) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

5. Which one of the following is correct? If z and are complex numbers and denotes the conjugate of, then $|z +| = |z -|$ holds only if

(a) $z = 0$ or $\omega = 0$
(b) $z = 0$ and $\omega = 0$
(c) $z.\,\overline{\omega}$ is purely real
(d) $z.\overline{\omega}$ is purely imaginary

[NDA – 2008]

6. The value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^{3n} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{3n}$ is equal to

(a) 3 (b) 3/2
(c) 0 (d) 2

7. $(-1 + i\sqrt{3})^{20}$ is equal to

[RPET – 2003]

(a) $2^{20}(-1 + i\sqrt{3})^{20}$ (b) $220\,(1 - i\sqrt{3})^{20}$
(c) $2^{20}(-1 - i\sqrt{3})^{20}$ (d) none of these

8. If z_1 and z_2 are two complex number then $|z_1 + z_2|$

[MP PET – 2007]

(a) $|z_1| + |z_2|$ (b) $|z_1| - |z_2|$
(c) $< |z_1| + |z_2|$ (d) $> |z_1| + |z_2|$

9. If $2\alpha = -1 - i\sqrt{3}$ and $2\beta = -1 + i\sqrt{3}$, then $5\alpha^4 + 5\beta^4 + 7\alpha^{-1}\beta^{-1}$ is equal to

[Kerala PET – 2008]

(a) -1 (b) -2 (c) 0 (d) 2

10. What is the square root of $\frac{1}{2} - i\frac{\sqrt{3}}{2}$?

[NDA – 2008]

(a) $\pm\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ (b) $\pm\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

(c) $\pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ (d) $\pm\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

11. If cube root of 1 is ω, then the value of $(3 + \omega + 3\omega^2)^4$ is

(a) 0 (b) 16

(c) $16\,\omega$ (d) $16\omega^2$

[Karnataka CET – 2004; Pb CET – 2000; MPPET – 2001]

12. If Z_1 and Z_2 are two complex numbers, then $|z_1 - Z_2|$ is

(a) $\geq |Z_1| - |Z_2|$ (b) $\leq |Z_1| - |Z_2|$

(c) $\geq |Z_1| + |Z_2|$ (d) $\leq |Z_2| - |Z_1|$

[MPPET – 1994]

13. If α is an imaginary cube root of unity, then for $n \in N$ the value of $\alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5}$ is

[MP PET – 1996]

(a) -1 (b) 0

(c) 1 (d) 3

14. If ω is a cube root of unity, then $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 =$

(a) 32 (b) -32 (c) -16 (d) 8

[IIT – 1965; RPET – 1997; MP PET – 1997]

15. $(-8)^{1/3}$ is equal to

[MP PET – 2006]

(a) 2 (b) $1 + i\sqrt{3}$

(c) $\frac{1}{2}(1 + i\sqrt{3})$ (d) $\frac{1}{2}(1 - i\sqrt{3})$

16. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if:

(a) $z_2 = 1$ (b) $z_2 = \frac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

[MPPET– 1999; 2007; Pb.CET – 2002]

17. If z_1, z_2 C, then

[MPPET – 1995]

(a) $|z_1 + z_2| \geq |z_1| + |z_2|$

(b) $|z_1 - z_2| \geq |z_1| + |z_2|$

(c) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(d) $|z_1 + z_2| \geq ||z_1| - |z_2||$

ANSWER SHEET

1.	a	b	c	d	7.	a	b	c	d	13.	a	b	c	d
2.	a	b	c	d	8.	a	b	c	d	14.	a	b	c	d
3.	a	b	c	d	9.	a	b	c	d	15.	a	b	c	d
4.	a	b	c	d	10.	a	b	c	d	16.	a	b	c	d
5.	a	b	c	d	11.	a	b	c	d	17.	a	b	c	d
6.	a	b	c	d	12.	a	b	c	d					

HINTS AND EXPLANATIONS

11. (c) $(3 + \omega + 3\,\omega^2)^4 = (3 + 3\omega + 3\omega^2 - 2\omega)^4$

$= [3\,(1 + \omega + \omega^2) - 2\omega]^4$ [$\because$ as $1 + \omega + \omega^2 = 0$]

$= (-2\omega)^4$

$= 16\,\omega^4 = \omega$

12. (a) Third side of a triangle is greater than equal to difference of two sides.

13. (b) If is an imaginary cube root of unity then

$\alpha^3 = 1,\ 1 + \alpha + \alpha^2 = 0$

$\alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5} = \alpha^{3n}\,[\alpha^1 + \alpha^3 + \alpha^5]$

$= (\alpha^3)^n\,[\alpha + 1 + \alpha^3.\alpha^2]$

$= (1)^n\,[1 + \alpha + 2]$ ($\because \alpha^3 = 1$)

$= 1 + \alpha + \alpha^2 = 0$

14. (a) $1 + \omega + \omega^2 = 0$

$A = (1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = (-2\,\omega)^5 + (-2\,\omega^2)^5$

or, $A = -32[\omega^5 + \omega^{10}] = -32(\omega^2 + \omega) = -32\,(-1) = 32$

For $\omega^3 = 1$

15. (b) Let $x = (-8)^{1/3}$

or $x^3 = -8$ or $x^3 + 2^3 = 0$

$(x + 2)\,(x^2 - 2x + 4) = 0$

$x = -2$ or $x^2 - 2x + 4 = 0$

$$x = \frac{+2 \pm \sqrt{4 - 16}}{2}$$

$$x = \frac{+2 \pm 2\sqrt{3}i}{2} \Rightarrow x = 1 \pm \sqrt{3}i$$

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LECTURE

4 Geometry of Complex Numbers

BASIC CONCEPTS

1. *n*th Roots of Unity

(i) $(1)^{\frac{1}{n}} = \left\{1, e^{i\frac{2\pi}{n}}. \left(e^{i\frac{2\pi}{n}}\right)^2, \ldots \left(e^{i\frac{2\pi}{n}}\right)^{n-1}\right\}$

$= \{1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}\}$

(ii) $1 + \omega + \omega^2 + \ldots\ldots\ldots \omega^{n-1} = 0, \omega^n = 1.$

(iii) The nth roots of unity form a G.P. whose common ratio is $e^{i\frac{2\pi}{n}}$.

(iv) The sum of nth roots of unity = 0 and product of nth roots of unity is $(-1)^{n-1}$.

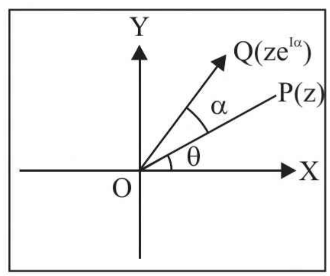
(v) nth roots of unity lie on a unit circle in Argand plane whose centre is origin. These root divide the circumference of circle into n equal parts and each part inscribed an angle $\frac{2\pi}{n}$ at the centre.

(vi) nth roots of unity form a n-sided regular polygon.

2. Complex Number as a Rotating Arrow in the Argand Plane

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ be a complex number represented by a point P in the Argand plane. Then, complex number represented by

Q is: $z = re^{i\alpha} = re^{i\theta}\, e^{i\alpha} = re^{i(\theta+\alpha)}$



(i) Multiplication by $e^{i\alpha}$ to z rotates the vector OP in anticlockwise direction through an angle α.

(ii) Similarly, multiplication by $e^{-i\alpha}$ to z rotates the vector OP in clockwise direction through an angle a.

Note $i \cos\frac{\pi}{2}\ i \sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$ Hence, angle between z and $iz = 90° = \pi/2$

3. Geometry of Complex Numbers

Let $z = x + iy$ be a complex number represented by a point in the Argand plane. Then, we say that the affix of p is z. The use of the word affix is similar to the position vector in the vectors.

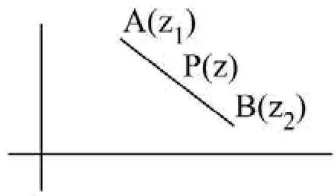
(A) Distance Between two Points Let P and Q be two points in the Argand plane having affixes z_1 and z_2 respectively.

Then $PQ = |z_2 - z_1| = |\text{affix of } Q\text{-affix of } P|$

Modulus of a complex number z represented by a plane in the Argand plane is its distance from the origin

If z_1 and z_2 are two fixed points in the argand plane; then the locus of a point z in each of the following cases

(i) $|z - z_1| + |z - z_2| = |z_1 - z_2|$, $AP + BP = AB \Rightarrow P$ lies on the line segment joining $A(z_1)$ and $B(z_2)$.



(ii) $|z - z_1| = |z - z_2| \Rightarrow AP = BP \Rightarrow P$ is equidistant from A and $B \Rightarrow P$ lies on the $\perp^r$ bisector of the line segment AB.

(iii) $|z - z_1| = k|z - z_2|, k \in r^+, k \in 1. AP = KBP$ $\Rightarrow P$ lies on a circle of a point such that the ratio of its distance from two fixed points is always constant. (Recall that circle is also defi ned as the locus.)

(iv) $|z - z_1| + |z - z_2| =$ constant $(\neq |z_1 - z_2|)$ $AP + BP =$ constant $\Rightarrow$ ellipse

(v) $|z - z_1| - |z - z_2| =$ constant $(\neq |z_1 - z_2|)$ $\Rightarrow P$ lies on a hyperbola having its foci at A and B, respectively.

(B) 1. (i) If two points P and Q have affixes z_1 and z_2, repectively in the Argand plane, then the affix of a point R dividing

PQ internally in the ratio $m : n$ is $\dfrac{mz_2\, nz_1}{m+n}$

(ii) If R is the mid point of PQ, then affix of R is $\dfrac{z_1 + z_2}{2}$

2. If z_1, z_2, z_3 are affixes of the vertices of a Δ, then affix of its centroid is $\dfrac{z_1 + z_2 + z_3}{3}$

3. If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C and D respectively.

Then $ABCD$ is a parallelogram if $z_1 + z_3 = z_2 + z_4$

4. If z_1, z_2, z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin and z is the affix of its orthocentre, then $z = z_1 + z_2 + z_3$

5. Centroid G divides line join of circumcentre and orthocentre in the ratio 1 : 2, since affix of G is $\dfrac{z_1 z_2 z_3}{3}$ and 0 is the origin.

6. (i) Equation of a circle having centre at z_0 and radius r is $|z - z_0| = R$

(ii) $z\bar{z} + az + \bar{a}z + b = 0$ where $b \in r$ represent a circle having centre at $(-a)$ and radius $\sqrt{|a|^2 - b}$

(C) 1. General Equation of a Straight Line

The general equation of a straight line is of the form $a\bar{z} + \bar{a}z + b = 0$, where a is a complex number and b is a real number.

2. Complex slope of the line segment joining two points z_1 and z_2. $\omega = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

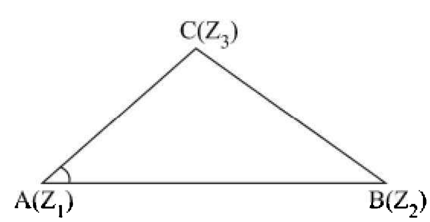
3. If ω_1 and ω_2 are the complex slope of two lines on the Argand plane

(i) If lines are $\perp^r$, if $\omega_1 + \omega_2 = 0$ (ii) parallel if $\omega_1 = \omega_2$.

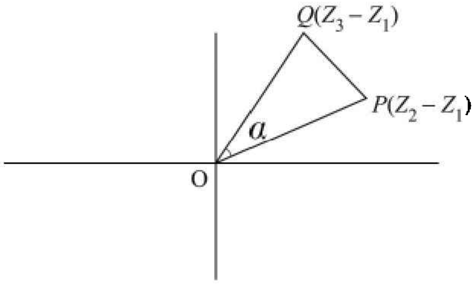
4. The slopes of the two lines are and $\dfrac{-\alpha}{\bar{\alpha}}$ and $\dfrac{-\beta}{\bar{\beta}}$ respectively. The lines will ber $\perp^r$, if $\dfrac{-\alpha}{\bar{\alpha}} + -\dfrac{\beta}{\bar{\beta}} = 0 \Rightarrow \alpha\bar{\beta} + \bar{\alpha}\beta = 0$.

5. If z_1, z_2, z_3 are the affixes of the points A, B and C in the Argand plane, then

(i) $\angle BAC = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$



(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$

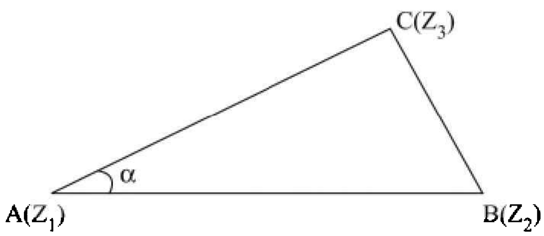


Let P and Q be two points in the Argand plane representing $z_2 - z_1$ and $z_3 - z_1$, respectively, then $\Delta OPQ = \Delta ABC$

$\therefore \angle BAC = \angle POQ = \angle XOQ - \angle XOP$

$= \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

6. Angle Between Two Lines



Angle between AC and AB:

$\angle BAC$ = amplitude of AC − amplitude of AB = amp of $(z_3 - z_1)$ − amp of $(z_2 - z_1)$

$= \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

***Note* 1**: If z_1, z_2, z_3 are collinear, then $\angle BAC = 0$

Hence, $\frac{z_3 - z_1}{z_2 - z_1}$ is purely treal because amplitude of real number = 0° i.e., $(z_3 - z_1)$ $= \arg(z_2 - z_1)$

***Note* 2**: If AC and AB are perpendicular each other then

$\alpha = \frac{\pi}{2} = \angle BAC$ Hence $\frac{z_3 - z_1}{z_2 - z_1}$ is perfectly imaginary.

***Note* 3**: If z_1, z_2, z_3 are in A.P. then they are collinear.

7. Complex numbers z_1, z_2, z_3 are vertices of an equilateral triangle iff $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or

$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_4} = 0.$

8. Some Important Results

(i) If $x + 1/x = 2\cos\theta$ or $x - 1/x = 2i\sin\theta$; then $x = \cos\theta + i\sin\theta$, $1/x = \cos\theta - i\sin\theta$ $x^n + 1/x^n = 2\cos n\theta$, $x^n - 1/x^n = 2i\sin n\theta$.

(ii) If $a = \cos\theta + i\sin\theta$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $a + b + c = 0$, then

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

$ab + bc + ca = 0$

$a^2 + b^2 + c^2 = 0$

$a^3 + b^3 + c^3 = 3abc$ etc.

(iii) $(x \pm 1)(x \pm \omega)(x \pm \omega^2) = x^3 \pm 1$

(iv) $(x \pm y)(x \pm \omega y)(x \pm \omega^2 y) = x^3 \pm y^3$

(v) $(x \pm y)(x\omega \pm y\omega^2)(x\omega^2 \pm y\omega) = x^3 \pm y^3$

(vi) $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$ $= x^3 + y^3 + z^3 - 3xyz$

(vii) $(1 + i)^2 = 2i$, $(1 - i)^2 = -2i$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $|z_1| = |z_2| = |z_3| = \ldots\ldots = |z_n| = 1$, then prove that

$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right|$

$= |z_1 + z_2 + z_3 + \ldots + z_n|$

Solution

We have $|z_1| = |z_2| = |z_3| = \ldots\ldots = |z_n| = 1$

$\Rightarrow |z_1|^2 = |z_2|^2 = |z_3|^2 = \ldots\ldots = |z_n|^2 = 1$

$\Rightarrow z_1\bar{z}_1 = 1, z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1, \ldots, z_n\bar{z}_n = 1$

$$\Rightarrow \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots\ldots+\frac{1}{zn}\right|$$
$$= |\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \ldots\ldots + \bar{z}_n|$$
$$= |\overline{z_1 + z_2 + z_3 + \ldots\ldots + z_n}|$$
$$= |z_1 + z_2 + z_3 + \ldots\ldots + z_n| \; [\because |\bar{z} = |z|\,].$$
$$\therefore \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |z_1 + z_2 + z_3 \ldots + z_n|.$$

2. If point p represents the complex number $z = x + iy$ on the Argand plane, then fi nd the locus of the point p such that $\left|\frac{z-2}{z+2}\right| = 5.$

Solution

Given $\left|\frac{z-2}{z+2}\right| = 5 \Rightarrow \left|\frac{z-2}{z+2}\right| = 5$

$\Rightarrow |z-2| = 5\,|z+2|$

$\Rightarrow |x+iy-2| = 5\,|x+iy+2|$

$\Rightarrow \sqrt{(x-2)^2+y^2} = 5\sqrt{(x+2)^2+y^2}$

$\Rightarrow (x-2)^2 + y^2 = 25\,[(x+2)^2+y^2]$

$\Rightarrow 25\,[x^2+4x+4+y^2] = x^2-4x+4+y^2$

$\Rightarrow 24x^2 + 24y^2 + 104x + 96 = 0$

$\Rightarrow 3x^2 + 3y^2 + 13x + 12 = 0$

which is an equation of a circle.

3. If point p represents the complex number $z = x + iy$ on the Argand plane, then find the locus of the point p such that $\arg(z-2-3i) = \pi/4$.

Solution

Given $\arg(z-2-3i) = \pi/4$

$\Rightarrow \arg(x+iy-2-3i) = \pi/4$

$\Rightarrow \arg(\overline{x-2} + \overline{iy-3}) = \pi/4$

$\Rightarrow \tan\frac{y-3}{x-2} = \frac{\pi}{4}$

$\Rightarrow \frac{y-3}{x-2} \tan\frac{\pi}{4} = 1$

$\Rightarrow x-2 = y-3$

$\Rightarrow x-y = -1$

4. If α and β are different complex numbers with $|\beta| = 1$, then fi nd $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ **[NCERT]**

Solution

Given $|\beta| = 1$, $\therefore \beta\bar{\beta} = |\beta|^2 = 1$, then

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta\bar{\beta}-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right|$$
$$= \frac{|\beta-\alpha|}{|\beta|\,|\overline{\beta-\alpha}|} = \frac{1}{|\beta|}\left|\frac{\beta-\alpha}{\overline{\beta-\alpha}}\right| = \frac{1}{1} = 1$$
$$\because \left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|} = 1.$$

5. If $z = 3 - 5i$, then prove that $z^3 - 10z^2 + 58z - 136 = 0$.

Solution

Given $z = 3 - 5i$

$\Rightarrow z - 3 = -5i \Rightarrow (z-3)^2 = (-5i)^2$

$\Rightarrow z^2 - 6z + 9 = 25i^2$

$\Rightarrow z^2 - 6z + 9 = -25$

$\Rightarrow z^2 - 6z + 34 = 0 \qquad (1)$

Now, $z^3 - 10z^2 + 58z - 136$

$= z^3 - 4z^2 - 6z^2 + 24z + 34z - 136$

$= z^2(z-4) - 6z(z-4) + 34(z-4)$

$= (z-4)(z^2 - 6z + 34),$

$= (z-4) \times 0$ [from Equation (1)]

$= 0$ **Proved**

6. Prove that

$$\sqrt{-1-\sqrt{-1-\sqrt{-1-\sqrt{1-1-\ldots\ldots\ldots\ldots\ldots\infty}}}}$$
$= \omega$ and ω^2.

Solution

Let $x = \sqrt{-1-\sqrt{-1-\sqrt{-1-\sqrt{1-1-\ldots\ldots\ldots\ldots\ldots\infty}}}}$

$\Rightarrow x = \sqrt{-1-x}$

Squaring both the sides, we have $x^2 = -1 - x$

$\Rightarrow x^2 + x + 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4(1)}}{2(1)} \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2} = \Rightarrow x = \omega, \omega^2.$ **Proved**

7. Find the area of the triangle whose vertices are represented by the points of complex numbers z, $z + iz$, iz.

Solution

Let $z = x + iy = (x, y)$

Then, $iz = i\,(x + iy) = -y + ix = (-y, x)$

and $z + iz = (x + iy) + i\,(x + iy)$

$= x + iy + ix - y$

$= (x - y) + i\,(x + y) = (x - y, x + y)$

Therefore, area of the triangle

$= \frac{1}{2}\,[x_1(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}\,[x(x - \overline{x + y}) - y(x + \overline{y - y}) + (x - y)\,(y - x)]$

$= \frac{1}{2}\,[x(-y) - y(x) - (x - y)^2]$

$= \frac{1}{2}\,[-xy - xy - x^2 - y^2 + 2xy]$

$= -\frac{1}{2}(x^2 + y^2) = \frac{1}{2}\,|z|^2.$

(Since, area is positive neglect negative sign)

8. If z_1, z_2 are two complex numbers and a, b are two real numbers, then prove that

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (a^2 + b^2)[\,|z_1|^2 + |z_2|^2\,]$

Solution

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1$ and $|z_2| = r_2$

$\therefore az_1 - bz_2 = ar_1\,(\cos\theta_1 + i\sin_1\theta)$

$- br_2\,(\cos\theta_2 + i\sin\theta_2)$

$= (ar_1\cos\theta_1 - br_2\cos\theta_2)$

$+ i\,(ar_1\sin\theta_1 - br_2\sin\theta_2)$

$\therefore |az_1 - bz_2|^2 = (ar_1\cos\theta_1 - br_2\cos\theta_2)^2$

$+ (ar_1\sin\theta_1 - br_2\sin\theta_2)^2$

$= a^2r_1^2\,(\cos^2\theta_1 + \sin^2\theta_1)$

$+ b^2r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$

$- 2ab\,r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |az_1 - bz_2|^2 = a^2\,r_1^2 + b_2r_2^2$

$- 2ab\,r_1r_2\cos\,(\theta_1 - \theta_2)$ (1)

Similarly,

$|bz_1 + az_2|^2$

$= b^2r_1^2 + a^2r_2^2 + 2abr_1r_2\cos(\theta_1 - \theta_2)$ (2)

Adding equations (1) and (2), we get

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)r_1^2 + (b^2 + a^2)r_2^2$

$= (a^2 + b^2)\,(r_1^2 + r_2^2)$

$= (a^2 + b^2)\,[|z_1|^2 + |z_2|^2]$

Proved

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If point p represents the complex number $z = x + iy$ on the Argand plane, then fi nd the locus of the point p such that arg $(z) = 0$.

2. Prove that the points represent the complex numbers $3 + 3i$, $-3 - 3i$, $-3\sqrt{3} + 3\sqrt{3}\,i$ form an equilateral triangle. Also, find the area of triangle.

3. If $z = (\lambda + 3) + i\sqrt{5 - \lambda^2}$, then prove that the locus of the point z is a circle.

4. Show that points represented by the complex number $1 + i$, $-2 + 3i$ and $\frac{5}{3}\,i$ are collinear.

5. Show that the vertices represented by the complex numbers $6 - i$, $7 + 3i$, $8 + 2i$ and $7 - 2i$ form a parallelogram.

6. If the points represented by the complex numbers z, $z + iz$, iz form a triangle of the area 50 square unit, then prove that $|z| = 10$.

Exercise II

1. For example values of z, solve $|z| + z = (2 + i)$.

2. If $(1 + i)\,(1 + 2i)\,(1 + 3i)\,.....\,(1 + ni) = (x + iy)$, then show that $2.5.10.......(1 + n^2) = x^2 + y^2$.

3. If z_1, z_2, z_3 are three complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then prove that $|z_1 + z_2 + z_3| = 1$.
4. Show that the points represented by complex numbers $-4 + 3i$, $2 - 3i$ and $-i$ are collinear.
5. A, B and C are three vertices of a parallelogram $ABCD$ represented by the complex.
6. Find the area of a triangle whose vertices are represented by the complex numbers $3i$, $3 + 2i$, $2 - i$.

ANSWERS

Exercise I

2. $18\sqrt{3}$ square unit.

Exercise II

1. $z = \left(\frac{3}{4} + i\right)$
5. $10 + 15i$
6. 5 units.

SOLVED OBJECTIVEQUESTIONS: HELPING HAND

1. If $\cos\alpha \cos\beta \cos\gamma = 0 = \sin\alpha + \sin\beta \sin\gamma$, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

[PET (Raj.) – 1998]

(a) 0 (b) 1
(c) 1/2 (d) 3/2

Solution

(d) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

then

$$a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 + i \times 0 = 0$$

and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = 0 - i \times 0 = 0$

$\therefore a^2 + b^2 + c^2 = (a + b + c)^2 + 2abc$

$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

$a^2 = (\cos\alpha + i\sin\alpha)^2$

$\Rightarrow \cos 2\alpha + i\sin 2\alpha$ (1)

$b^2 = (\cos\beta + i\sin\beta)^2$

$\Rightarrow \cos 2\beta + i\sin 2\beta$ (2)

$c^2 = (\cos\gamma + i\sin\gamma)^2$

$\Rightarrow \cos 2\gamma + i\sin 2\gamma$ (3)

$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i \times 0 = 0$

Therefore, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ (4)

$\sin 2\alpha \sin 2\beta \sin 2\gamma = 0$

By (4) $1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then $\sin 3\alpha + \sin 3\beta + \sin 3\gamma$ is equal to
(a) $\sin 3(\alpha + \beta + \gamma)$
(b) $3\sin 3(\alpha + \beta + \beta\gamma)$
(c) $3\sin 3(\alpha + \beta + \gamma)$
(d) $3\cos(\alpha + \beta + \gamma)$

[PET (Raj.) – 1989, 1991; Bihar (CEE) – 2000]

Solution

(b) If $a = \cos\alpha + i\sin\alpha$, $\beta = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$,

then $a + b + c =$ $(\cos\alpha + \cos\beta + \cos\gamma) + (\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0 = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \sum (\cos a + i\sin a)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \times (\cos\gamma + i\sin\gamma)$

$\Rightarrow \sum \cos 3\alpha + i\sum \sin 3\alpha$

$= 3\cos(\alpha + \beta + \gamma) + i3\sin(\alpha + \beta + \gamma)$

$\Rightarrow \sin 3\alpha + \sin 3\beta + \sin 3\gamma$

$= 3\sin(\alpha + \beta + \gamma)$.

3. $(-\sqrt{-1})^{8n+1} + (-\sqrt{-1})^{8n+3}$ $(n \in N)$ equals

[NDA – 2005]

(a) 0 (b) 1
(c) $2\sqrt{-1}$ (d) $-2\sqrt{-1}$

Solution

(a) Exp. $= (-i)^{8n+3} + (-i)^{8n+3}$

$= -i + (-i)^3 = -i + i = 0$

4. The value of is

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$$

[AIEEE – 2006]

(a) 1 (b) -1
(c) i (d) $-i$

Solution

(c) We have

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} - i\cos\frac{2k\pi}{11}\right)$$

$$\sum_{k=1}^{10} i\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$$

when

$$= i\sum_{k=1}^{10} e^{-i\frac{2k\pi}{11}} = i\sum_{k=1}^{10} a^k \text{ when } a = e^{-(2\pi i/11)}$$

$$= i\alpha\frac{(1-\alpha^{10})}{(1-\alpha)} = \mathrm{i}\,\frac{(\alpha-\alpha^{11})}{1-\alpha} = i\left(\frac{\alpha-1}{1-\alpha}\right) = -i$$

$(\therefore \alpha^{11} = e^{-i2\pi} = \cos 2\pi - i\sin 2\pi = 1)$;

$\cos\alpha + i\sin\alpha = e^{i\alpha}$.

5. Let z_1, z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further assume that the origin, z_1 and z_2 form an equilateral triangle, Then,

[AIEEE – 2003]

(a) $a^2 = 4b$ (b) $a^2 = b$
(c) $a^2 = 2b$ (d) $a^2 = 3b$

Solution

(d) $z_1 + z_2 = -a$, $z_1z_2 = b$

$\therefore 0^2 + z_1^2 + z_2^2 = z_1z_2$

(Put $z_3 = 0$ in formula $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$)

$\Rightarrow (z_1 + z_2)^2 - 2z_1z_2 = z_1z_2 \Rightarrow a^2 = 3b$

6. $\left(\left((i)^i\right)^i\right)^{i \text{.........110 times}}$ equals

[AITSE – 1999]

(a) $e^{i\pi/2}$ (b) $e^{\pi/2}$
(c) $e^{-\pi/2}$ (d) $e^{-i\pi/2}$

Solution

(d) Exp $(i)^{i^{110}} = (i)^{-1} = 1/i = -i = e^{-i\pi/2}$

7. If P, Q, R, S are represented by the complex numbers $4 + i$, $1 + 6i$, $-4 + 3i$, $-1 - 2i$ respectively, then $PQRS$ is a

[Orissa (JEE) – 2003]

(a) rectangle (b) square
(c) rhombus (d) parallelogram

Solution

(b) $|PQ| = |QR| = |RS| = |SP|$ and $\angle PQR = 90°$.

8. If $\arg.(z - a) = \frac{\pi}{4}$, where $a \in R$, then the locus of $z \in c$ is a

[MPPET – 1997]

(a) hyperbola (b) parabola
(c) ellipse (d) straight line

Solution

(d) Let $z = x + iy$ then $z - a = x + iy - a = (x - a) + iy$ given arg. $(z - a) = \frac{\pi}{4}$

Therefore, $\tan^{-1} \frac{y}{x - a} = \frac{\pi}{4}$

or $\frac{y}{x-a} = \tan\frac{\pi}{4}$ or $y = x - a$ (straight line)

9. If $|z - 4i| + |z + 4i| = 10$, then z is the locus of

[MPPET – 2006]

(a) circle (b) parabola
(c) ellipse (d) none of these

Solution

(c) $|z - 4i| + |z + 4i| = 10$

Let $z = x + iy$

$|x + iy - 4i| + |x + iy + 4i| = 10$

$|x + i(y - 4)| + |x + i(y + 4)| = 10$

$\sqrt{x^2 + (y-4)^2} + \sqrt{x^2 + (y+4)^2} = 10$

$\left\{\sqrt{x^2 + (y-4)^2}\right\}^2 = \left\{10 - \sqrt{x^2 + (y+4)^2}\right\}^2$

$x^2 + (y - 4)^2 = 100 + x^2 + (y + 4)^2 - 2 \times 10\sqrt{x^2 + (y+4)^2}$

$y^2 + 16 - 8y = 100 + y^2 + 16 + 8y - 20\sqrt{x^2 + (y+4)^2}$

$-16y - 100 = -20\sqrt{x^2 + (y+4)^2}$

$4y + 25 = 5\sqrt{x^2 (y+4)^2}$

$(4y + 25)^2 = (5\sqrt{x^2 + (y+4)^2})^2$

$16y^2 + 625 + 200y = 25(x^2 + y^2 + 16 + 8y)$

$16y^2 + 625 + 200y = 25x^2 + 25y^2 + 400 + 200y$

$25x^2 + 9y^2 = 225$

$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225} \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

which is ellipse

10. If $|z - 4| < |z - 2|$, its solution is given by

[AIEEE – 2002]

(a) Re $(z) > 0$ (b) Re $(z) < 0$
(c) Re $(z) > 3$ (d) Re $(z) > 2$

Solution

(c) $|z - 4| < |z - 2|$

or $|a - 4 + ib| < |(a - 2) + ib|$ by taking $z = a + ib$

$\Rightarrow (a - 4)^2 + b^2 < (a - 2)^2 + b^2$

$\Rightarrow -8a + 4a < -16 + 4$

$\Rightarrow 4a > 12$

$\Rightarrow a > 3$

$\Rightarrow \text{Re}(z) > 3$

11. Locus of $|z| = 1$ is

(a) $x + y = 1$ (b) $x^2 + y^2 = 1$
(c) $x^2 - y^2 = 1$ (d) $y^2 - x = 0$

[MPPET – 2007]

Solution

(b) Let $z = x + iy$ then from $|z| = 1$

$x + iy = 1$ or $x^2 + y^2 = 1$or $x^2 + y^2 = 1$

It is a circle whose radius is 1.

12. If $z = (\lambda + 3) + \sqrt{5 - \lambda^2}\, i$, then locus of z is a

[MPPET – 2006]

(a) $(x - 3)^2 + y^2 = 5$ (b) $(x - 3)^2 = 5 - y$
(c) $x - y = 8$ (d) none of these

Solution

(a) On putting $z = x + iy$

$x + iy = (\lambda + 3) + i\sqrt{5 - \lambda^2}$

On comparing real and imaginary parts

$x = \lambda + 3$ (1)

$y = \sqrt{5 - \lambda^2}$ (2)

By Equation (1) $x - 3 = \lambda$ or $\lambda^2 = (x - 3)^2$

By Equation (2) $y = \sqrt{5 - (x-3)^2}$

or $(x - 3)^2 + y^2 = 5$

13. If z is a complex number, then $|3z - 1| = 3|z - 2|$ respresents

(a) $x = 0$ (b) $x^2 + y^2 = 3x$
(c) $y = 0$ (d) $x = 7/6$

Solution

(d) Let $z = x + iy$; where x, yr, then $|3z - 1| = 3\,|z - 2|$

$\Rightarrow |3\,(x + yi) - 1| = 3\,|x + iy - 2|$

$\Rightarrow |(3x - 1) + (3y)\,i| = 3\,|x - 2 + yi|$

$\Rightarrow \sqrt{(3x-1)^2 + (3y)^2} = 3\sqrt{(x-2)^2 + y^2}$

Squaring, we get $9x^2 - 6x + 1 + 9y^2 = 9\,(x^2 - 4x + 4 + y^2)$

$\Rightarrow -6x + 1 = -36x + 36 \Rightarrow 30x = 35$

$\Rightarrow x = \frac{7}{6}$, , which means z is always at a constant distance $\frac{7}{6}$ from y–axis.

14. If z and ω are two non–zero complex numbers such that $|z\omega| = 1$and arg (z) − arg $(\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to

(a) $-i$ (b) 1 (c) -1 (d) i

[AIEEE – 2003]

Solution

(a) $|z\omega| = 1 \Rightarrow z = 1/|\omega|$, so let

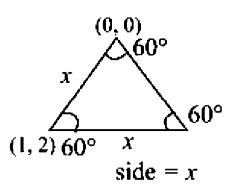
$z = (r, \theta), \omega = (1/r, \theta - \pi/2)$

$\Rightarrow \bar{z} = (r_2, -\theta) \therefore \bar{z}\omega = (1, -\pi/2) = -i$

15. The centre of a hexagon is the origin. If its one vertex is the point $(1 + 2i)$, then its perimeter is

[PET (Raj.) – 1999]

(a) $\sqrt{5}$ (b) $4\sqrt{5}$` (c) $6\sqrt{5}$ (d) $6\sqrt{2}$



Solution

(c) Side = $\sqrt{1^2 + 2^2} = \sqrt{5}$

Therefore, perimeter = $6\sqrt{5}$

16. If z_1, z_2 are two such nth roots of unity which subtend right angle at the origin, then n must be $(k \in x)$

[IIT (Screening) – 2001]

(a) $4k$ (b) $4k + 1$

(c) $4k + 2$ (d) $4k + 3$

Solution

(a) $1 = (\cos 2r\pi + i \sin 2r\pi)$

$\therefore 1^{1/n} = (\cos 2r\pi + i \sin 2r\pi\,)^{1/n}$

$= e^{\frac{i2r\pi}{n}}, r = 0, 1, 2, \ldots.(n-1)$

$r = 0, 1, 2\ldots.(n-1)$

we get $1, e^{i(2\pi/n)}, e^{i(4\pi/n)}, \ldots\ldots\ldots.e^{i.2(n-1)\pi/n}$

Let given two roots be

$z_1 = e^{i2m_1\pi/n}, z_2 = e^{im_2\pi/n}$

Since z_1, z_2 subtend right angle at the origin, so

$\left|\frac{2m_1\pi}{n} - \frac{2m_2\pi}{n}\right| = \frac{\pi}{2}$

$\Rightarrow n = 4\,|m_1 - m_2| = 4k, k \in Z$

17. A point z moves on the Argand diagram such that $|z - 3i| = 2$ then it's locus is

[MP PET – 2002]

(a) y − axis (b) a straight line

(c) a circle (d) none of these

Solution

(c) Let $z = x + iy$

$\therefore z - 3i = x + iy - 3i = x + (y - 3)i$

$\therefore |z - 3i| = 2, |z - 3i| = |x + (y - 3)i| = 2$

or $\sqrt{x^2 + (y-3)^2} = 2$

or $x^2 + (y - 3)^2 = 4$

It is the equation of a circle

Centre of circle (0, 3) Radius = 2

18. If $z = x + iy$ is a variable complex number such that arg $\frac{z-1}{z+1} = \frac{\pi}{4}$, then

[MP PET – 2004]

(a) $x^2 - y^2 - 2x = 1$ (b) $x^2 + y^2 - 2y = 1$

(c) $x^2 - 2y = 1$ (d) $y^2 + 2x = 1$

Solution

(b) $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$

Multiplying and dividing by $x + 1 - iy$

$\therefore \arg \dfrac{z-1}{z+1} = \tan^{-1} \dfrac{2y}{x^2+y^2-1}$

$\therefore \tan^{-1} \dfrac{2y}{x^2+y^2-1} = \dfrac{\pi}{4}$

or $2y = x^2 + y^2 - 1$ or $x^2 + y^2 - 2y = 1$

19. If $|z| = 2$, then the complex number $-1 + 5z$ is situated on the

(a) circle (b) straight line
(c) parabola (d) ellipse

[MP PET – 2005]

Solution

(a) Let $\omega = -1 + 5z$, then $\omega + 1 = 5z$

$\Rightarrow |\omega + 1| = = |5z| = 5\,|z| = 5.2$

$\Rightarrow |\omega + 1| = 10.$

Therefore, is a circle whose centre is -1 and radius is $\sqrt{10}$.

20. If z be a complex number, then the locus represented by $iz - 1 + z - i = 2$ is

[Roorkee (Screening) – 1999]

(a) a line
(b) a circle
(c) a pair of straight lines
(d) a coordinate axis

Solution

(d) If $z = x + iy$, then from the given relation, we have

$|i(x+iy) - 1| + |x + iy - i| = 2$

$\Rightarrow |ix - y - 1| + |x + i(y-1)| = 2$

$\Rightarrow [(y+1)^2 + x^2]^{1/2} + [x^2 + (y-1)^2]^{1/2} = 2$

$\Rightarrow (y+1)^2 + x^2 = x^2 + (y-1)^2 + 4 - 4[x^2 + (y-1)^2]^{1/2}$

$\Rightarrow (y-1)^2 = x^2 + (y-1)^2 \Rightarrow x^2 = 0$

$\Rightarrow x = 0$ which is y-axis.

21. Vertices A, B, C of an isosceles triangle ABC are represented by complex numbers z_1, z_2, z_3 respectively. If $\angle C = 90°$, then correct statement is

[PET (Raj.) – 1999; IIT, 1986;

Delhi (EEE) – 1998]

(a) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$
(b) $(z_1 - z_2)^2 = (z_1 - z_3)(z_3 - z_2)$
(c) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 z_3$
(d) none of these

Solution

(a) A, B, C are represented by z_1, z_2, z_3 respectively, so $\overline{CA} = z_1 - z_3$, $\overline{CB} = z_2 - z_3$

Also $\angle C = \pi/2$ and $CA = CB$,

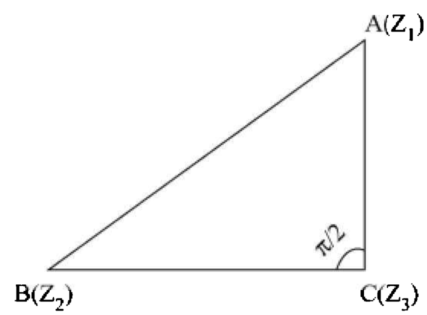
So $\overline{CB} = \overline{CA}i$ [$\because$ $CB = CA$ and CB has been given a rotation of $-\pi/2$ with respect to CA]

$\Rightarrow z_2 - z_3 = (z_1 - z_3)i$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2\, 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_2z_3 + 2z_1z_3 - 2z_3^2 - 2z_1z_2$



$\Rightarrow (z_1 - z_2)^2 = 2[(z_1z_3 - z_3^2) - (z_1z_2 - z_2z_3)]$
$= 2(z_3 - z_2)(z_1 - z_3)$

22. The centre of a regular polygon of n sides is located at the point $z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(a) $z_1\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$

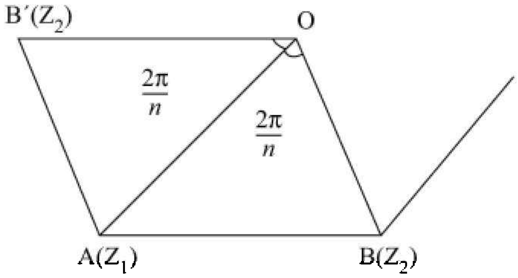
(b) $z_1\left(\cos\frac{\pi}{n} \pm i \sin\frac{\pi}{n}\right)$

(c) $z_1\left(\cos\frac{\pi}{2n} \pm i \sin\frac{\pi}{2n}\right)$

(d) none of these

Solution

(a) Let A be the vertex with affix z_1. There are two possibilities of z_2, i.e., z_2 can be obtained by rotating z_1 through $\frac{2\pi}{n}$ either in clockwise or in anticlockwise direction.



$\therefore \frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{\pm\frac{i2\pi}{n}} \Rightarrow z_2$

$= z_1 e^{\pm\frac{i2\pi}{n}}, (\because |z_2| = |z_1|)$

$\Rightarrow z_2 = z_1\left(\cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}\right)$

23. The vertices B and D of a parallelogram are $1 - 2i$ and $4 + 2i$, If the diagonals are at right angles and $AC = 2BD$, the complex number representing A is

(a) $\frac{5}{2}$ (b) $3i - \frac{3}{2}$

(c) $3i - 4$ (d) $3i + 4$

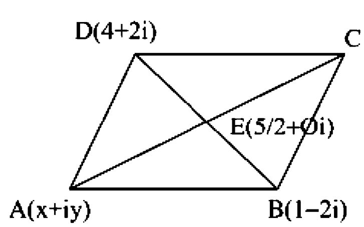
Solution

(b) We have

$|\overrightarrow{BD}| = |(4 + 2i) - (1 - 2i)| = \sqrt{9 + 16} = 5$

Let the affix of A be $z = x + iy$ The affix of the mid-point of BD is $\left(\frac{5}{2}, 0\right)$

Since the diagonals of a parallelogram bisect each other, therefore, the affix of the point of intersection of the diagonals is $\left(\frac{5}{2}, 0\right)$



We have

$|\overrightarrow{AE}| = 5\left(\because BD = \frac{1}{2}AC = AE\right)$

which is satisfied by option (b).

24. When $\frac{z+i}{z+2}$ is purely imaginary, the locus described by the point z in the Argand diagram is a

(a) circle of radius $\sqrt{\frac{5}{2}}$

(b) circle of radius $\frac{5}{4}$

(c) straight line

(d) parabola

Solution

(a) Given that Im $\left(\frac{z+i}{z+2}\right)$

Let $z = x + iy \Rightarrow \frac{x = iy + i}{x = iy + 2} = \frac{x + i(y+1)}{(x+2) + iy}$

$= \frac{[x + i(y+1)][x+2) - iy]}{[(x+2) + iy][(x+2) - iy]}$

$\left[\frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2}\right] + i\left[\frac{(y+1)(x+2) - xy}{(x+2)^2 + y^2}\right]$

If it is purely imaginary, then the real part must be equal to zero.

$= \frac{x^2 + y^2 + 2x + y}{(x+2)^2 + y^2} = 0 \Rightarrow x^2 + y^2 + 2x + y = 0$

which is a circle and its radius is given by

$\sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{1}{4} - 0} = \frac{\sqrt{5}}{2}$

Therefore, Argand diagram is circle of radius $\frac{\sqrt{5}}{2}$.

25. If the point z_1, z_2, z_3 are the vertices of an equilateral triangle in the Argand plane, then

(a) $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$

(b) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

(c) $(z_1 - z_1)^2 + (z_2 - z_3)^2 (z_3 - z_1)^2 = 0$

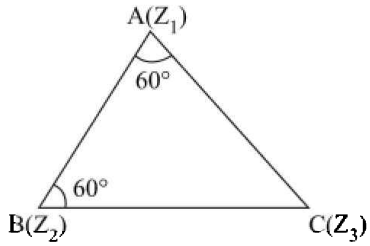
(b) $z_1^3 + z_3^3 + 3z_1z_2z_3 = 0$

Ans:*a*, *b*, *c*

Solution

Let the vertices of the ABC be represented by z_1, z_2 and z_3. By rotation in anticlockwise direction about A and B,

we get $AC = ABe^{\pi i/3}$, $BA = BCe^{\pi i/3}$



or $(z_3 - z_1) = (z_2 - z_1)e^{\pi i/3}$

and $(z_1 - z_2) = (z_3 - z_2)e^{\pi i/3}$ hence on dividing, we get

$\frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$ or $(z_3 - z_1)(z_3 - z_2)$

$= -(z_2 - z_1)^2$

or $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (1)

Above $\Rightarrow$ (b). This equation can also be written as

$\frac{1}{2}[\Sigma (z_1 - z_2)^2] = 0$

$\Rightarrow \Sigma (z_1 - z_2)^2 = 0 \Rightarrow (c)$

Again, $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$ can be written as.

$\Sigma (z_3 - z_1)(z_1 - z_2) = 0$ or $\Sigma z_1(z_3 - z_1)$

$- \Sigma z_2(z_3 - z_1) = 0$

The second sigma will be zero and the first gives $\Sigma z_1^2 = \Sigma z_1z_2$ i.e.(1) $\Rightarrow (a)$

26. The shaded region where $P(-1,0)$, $Q(-1 + \sqrt{2}, \sqrt{2})$, $R(-1 + \sqrt{2}, \sqrt{-2})$, $S(1,0)$ is represented by

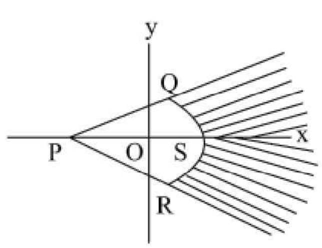
[IIT – 2005]

(a) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

(b) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$

(c) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

(d) $|z - 1| < 2, |\arg(z + 1)| > \frac{\pi}{2}$



Solution

(a) As $|PQ| = |PS| = |PR| = 2$

Therefore, shaded part represents the external part of circle having centre $(-1,0)$ and radius 2.

As we know equation of circle having centre z_0 and radius r,is

$|z - z_0| = r \therefore |z - (-1 + 0i)| > 2$

$\Rightarrow |z + 1| > 2$ (1)

Also, argument of $z + 1$ with respect to positive direction of

x – axis is $\frac{\pi}{4}$. $\therefore$ arg $(z + 1) \le \frac{\pi}{4}$

and argument of $z + 1$ in anticlockwise direction is

$\therefore -\frac{\pi}{4} \le \arg(z + 1)$ (2)

or

$|\arg(z + 1)| \le \frac{\pi}{4}$

27. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

[IIT – 2001;DCE – 2005]

(a) of area zero

(b) right–angled isosceles

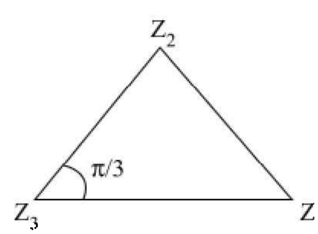
(c) equilateral

(d) obtuse–angled isosceles

Solution

(c) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2} = \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$

$= \frac{1 - i^2 3}{2(1 + i\sqrt{3})}$



$= \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$

$$\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = \frac{1 + i\sqrt{3}}{2} = \cos\frac{\pi}{3} = i\sin\frac{\pi}{3}$$

$$\Rightarrow \left|\frac{z_2 - z_3}{z_1 - z_3}\right| = \text{and } \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3}$$

Hence, the triangle is equilateral.

Passage based questions

Let A, B, C be three sets of complex numbers as defi ned here.

$A = \{z : \text{Im } z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re } (1 - i) z = \sqrt{2}\}$

28. Let x be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

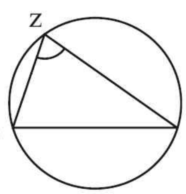
[IIT – 2008]

(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

Solution

(c) We know that $A \cap B \cap C$ contains just one point. So z is fixed, Also, z is on the circle. The points $(-1, 1)$ and $(5, 1)$ are the ends of a diameter.

z



(–1, 1) (5, 1)

Thus, $|z + 1 - i|^2 + |z - 5 - i|^2 = (\text{diameter})^2 = 6 = 36$.

Remark: This is not a problem on finding the greatest and least values of an expression which is the first impression of students after reading "lies between"

29. Let z be any point in $A \cap B \cup C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z| - |\omega| + 3$ lies between

(a) – 6 and 3 (b) – 3 and 6
(c) – 6 and 6 (d) – 3 and 9

[IIT – 2008]

Solution

(a, b, c, d)

B: $|z - 2i| = 3 \Rightarrow (x - 2)^2 + (y - 1)^2 = 3$

C: Re $(1 - i) z = \sqrt{2} \Rightarrow x + y = \sqrt{2}$

$(x - 2)^2 + (\sqrt{2} - x - 1)^2 = 3$

$x^2 - x(1 + \sqrt{2}) + 2 - 2\sqrt{2} = 0$

$x = -2, \sqrt{2} - 1;$

Corresponding $y = 2 - \sqrt{2}$, 1

Since, $y \geq 1$; $(x, y) = (\sqrt{2} - 1, 1)$

$|z| = \sqrt{(\sqrt{2} - 1)^2 + 1^2} = \sqrt{4 - 2\sqrt{2}} \cong 1.1$

$|z| - |\omega| + 3 = 1.1 = |\omega| + 3 = 4.1 - |\omega|$

Also, $|\omega - 2 - i| < |3|$

$-3 < |\omega| - |2 + i| < 3$

$\Rightarrow \sqrt{5} - 3 < |\omega| < 3 + \sqrt{5}$

since, $|\omega| \geq 0 \Rightarrow 0 < |\omega| < 3 + \sqrt{5}$

or $0 < |\omega| < 5.2$

Therfore, $|z| - |\omega| + 3 = 4.1 - (\omega)$

lies between – 1.1 to 4.1

Therefore, Ans: (a), (b), (c), (d)

Note: Though the question came in single choice, Answer given by IIT JEE had more than one option correct.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. In the Argand diagram, if O, P and Q represents, respectively the origin, the complex numbers z and $z + iz$, then the angle $\angle OPQ$ is

[MPPET – 2000]

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) $2\pi/3$

2. If $x = a$, $y = b$, $z = c\omega^2$ where ω is a complex cube root of unity, then $\frac{x}{a}\frac{y}{b}\frac{z}{c} =$

(a) 3 (b) 1
(c) 0 (d) none of these

[AMU – 1983]

3. If $z_i = \cos \frac{i\pi}{10} + i \sin \frac{i\pi}{10}$, then $z_1 z_2 z_3 z_4$ is equal to
   (a) -1 (b) 1 (c) -2 (d) 2

4. Multiplication of a complex number z by i corresponds to ($z \neq 0$)
   (a) clockwise rotation of the line joining z to the origin in Argand diagram through an angle $\pi/2$.
   (b) anticlockwise rotation of the line joining z to the origin in Argand diagram through an angle $\pi/2$.
   (c) rotation of the line joining z to origin in the Argand diagram through an angle $\pi/2$.
   (d) no rotation.

5. If $a = \text{cis}\,\alpha$, $b = \text{cis}\,\beta$, $c = \text{cis}\,\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) =$
   (a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) 0 (d) 1
   [RPET – 2001; Orissa JEE – 2007]

6. If $|z_1| = |z_2| = ... = |z_n| = 1$, then $\frac{|z_1 + z_2 + ... + z_n|}{|z_1^{-1} + z_2^{-1} + ... + z_n^{-1}|}$ is equal to
   (a) n (b) 1
   (c) $1/n$ (d) none of these

7. The solution of the equation $|z| - z = 1 + 2i$ is
   [MPPET – 1993]
   (a) $2 - \frac{3}{2}i$ (b) $\frac{3}{2} + 2i$
   (c) $\frac{3}{2} - 2i$ (d) $-2 + \frac{3}{2}i$

8. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then
   (a) $\text{Re}(z) = 0$
   (b) $\text{Im}(z) = 0$
   (c) $\text{Re}(z) > 0, \text{Im}(z) > 0$
   (d) $\text{Re}(z) > 0, \text{Im}(z) < 0$
   [MPPET – 1997]

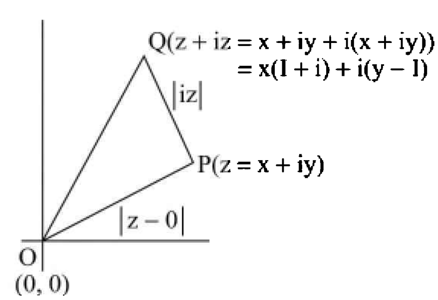
9. If $z = x + iy$, then area of the triangle whose vertices are points z, iz, $z + iz$ is
   [MP PET – 1997]
   (a) $3/2\,|z|^2$ (b) $|z|^2$
   (c) $1/2\,|z|^2$ (d) $1/4\,|z|^2$

10. The points representing the complex numbers z, for which $|z - a|^2 + |z + a|^2 = b^2$ lie on
    (a) $x + y = \frac{b - a}{2}$
    (b) $x^2 + y^2 = \frac{b^2 - a^2}{2}$
    (c) $y^2 + 2bx$
    (d) $x^2 - y^2 = 2ab$
    [MPPET – 2008]

11. Let $z = x + iy$ be a complex number where x and y are integers. Then, the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is
    [IIT – 2009]
    (a) 48 (b) 32
    (c) 40 (d) 80

SOLUTIONS

1. (c) Let $z = r(\cos\theta + i\sin\theta)$, then
   $\overline{PQ}$ = Affix of Q − Affix of P
   $= z + iz - z = iz$
   Also, $\overline{OP} = z$
   Clearly, angle between z and iz is 90°.



2. (c) Given that $x = a, y = b, z = c\omega^2$

Then, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{a}{a} + \frac{b\omega}{b} + \frac{c\omega^2}{c}$

$= 1 + \omega + \omega^2 = 0$

3. (c) $z_1 z_2 z_3 z_4 \cos\left(\frac{\pi}{10}\right) + i \sin\left(\frac{\pi}{10}\right)$

$\left(\cos\left(\frac{2\pi}{10}\right) + i \sin\left(\frac{2\pi}{10}\right)\right)$

$\left(\cos\left(\frac{3\pi}{10}\right) + i \sin\left(\frac{3\pi}{10}\right)\right)$

$\left(\cos\left(\frac{4\pi}{10}\right) + i \sin\left(\frac{4\pi}{10}\right)\right)$

$$\left\{\begin{array}{l} \cos\left(\frac{\pi}{10} + \frac{2\pi}{10} + \frac{3\pi}{10} + \frac{4\pi}{10}\right) \\ + i \sin\left(\frac{\pi}{10} + \frac{2\pi}{10} + \frac{3\pi}{10} + \frac{4\pi}{10}\right) \end{array}\right\}$$

$= \cos(\pi) + i \sin \pi = -1 + i \times 0 = -1$

4. (b) $\because i = e^{i\pi/2}$ $\therefore$ Multiplying by i, z gets shaded by $\pi/2$ in anticlockwise direction

5. (d) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$

$\Rightarrow \frac{cis\alpha}{cis\beta} + \frac{cis\beta}{cus\gamma} + \frac{cis\gamma}{cis\alpha} = 1$

$\Rightarrow cis(a - \beta) + cis(\beta - \gamma) + cis(\gamma - \alpha) = 1$

$\Rightarrow \cos(\alpha - \beta) + \cos(b - \gamma) + \cos(\gamma - \alpha) = 1$

(Equating real parts)

6. (b) $\because |z| = 1$ $z^{-1} = \bar{z}$, so

Exp. $= \frac{|z_1 + z_2 + \ldots.. + z_n|}{|\bar{z}_1 + z_2 + \ldots.. + \bar{z}_n|}$

$= \frac{|z_1 + z_2 + \ldots.. + z_n|}{|z_1 + z_2 + \ldots.. + z_n|} = 1$

$\because |z| = |\bar{z}|$

7. (c) Let $z = a + ib$ then

$|z| = |a + ib| = \sqrt{a^2 + b^2}$

$\therefore |z| - z = 1 + 2i = \frac{3}{2} - 2i$

$\Rightarrow (\sqrt{a + b^2} - a)^2 - ib = 1 + 2i$

Comparing real and imaginary parts of the both sides

$\Rightarrow \sqrt{a^2 + b^2} - a = 1, -b = 2$

$\Rightarrow a = \frac{3}{2}, b = -2$

Therfore, $z = a + ib = \frac{3}{2} - 2i$

8. (b) de-Movier's Theorem

$(\cos \theta \pm i \sin \theta)^n = \cos n\,\theta \pm i \sin n\,\theta$

$\left(\frac{\sqrt{3} + i}{2}\right)^5 = (\cos 30 + i \sin 30)^5$

$= \cos 150 + i \sin 150$ (1)

and $\left(\frac{\sqrt{3} - i}{2}\right)^5 = \cos 150 - \mathrm{i} \sin 150$ (2)

Adding $\left(\frac{\sqrt{3} + i}{2}\right)^5 + \left(\frac{\sqrt{3} - i}{2}\right)^5 = 2 \cos 150^\circ$

$= 2 - \left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$

Clearly, $(-\sqrt{3})$ is a real number. Therefore, $i_z = 0$

9. (c) Let, $z = x + iy \Rightarrow (x, y)$

$iz, i(x + iy) = -y + ix \Rightarrow (-y, x)$

$z + iz = x + iy - y + ix \Rightarrow (x - y, x + y)$

Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x - y & x + y & 1 \end{vmatrix}$

Applying, $R_3 \to R_3 - R_1 - R_2$

$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ 0 & 0 & -1 \end{vmatrix}$

$\Delta = \left|-\frac{1}{2}(x^2 + y^2)\right|$ or $\Delta = \frac{|z|^2}{2}$

10. (b) $|z - a|^2 + |z + a|^2 = b^2$

Let $z = x + iy$

$\therefore |x + iy - a|^2 + |x + iy + a|^2 = b^2$

$(x - a)^2 + y^2 + (x + a)^2 + y^2 = b^2$

$2x^2 + 2y^2 + 2a^2 = b^2$

$\therefore x^2 + y^2 = \frac{b^2}{2} - a^2$

It is a circle.

11. (a) $z\bar{z}(\bar{z}^2 + z^2) = 350$

Put $z = x + iy$

$(x^2 + y^2)(x^2 - y^2) = 175$

$(x^2 + y^2)(x^2 - y^2) = 25.7$

$x^2 + y^2 = 25$

$x^2 - y^2 = 7$

$x = \pm 4, y = \pm 3$

$x, y \in I$

Area = $8 \times 6 = 48$ sq. units.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The two numbers such that each one is square of the other are

[MPPET – 1987]

(a) ω, ω^3 (b) $-i, i$

(c) $-1, 1$ (d) ω, ω^2

2. $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$ equals

[RPET – 1996]

(a) $\sin 8\theta - i\cos 8\theta$

(b) $\cos 8\theta - i\sin 8\theta$

(c) $\sin 8\theta + i\cos 8\theta$

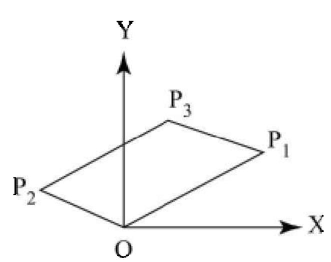
(d) $\cos 8\theta + i\sin 8\theta$

3. Let z be a complex number. Then, the angle between vectors z and $-iz$ is

(a) π (b) 0

(c) $-\pi/2$ (d) none of these

4. If the points P_1 and P_2 represent two complex numbers z_1 and z_2, then the point P_3 represents the number



(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $z_1 \times z_2$

(d) $z_1 \div z_2$

5. The point represented by the complex number $2 - i$ is rotated about origin through an angle of $\frac{\pi}{2}$ in clockwise direction. The new position of the point is

(a) $1 + 2i$

(b) $-1 - 2i$

(c) $2 + i$

(d) $-1 + 2i$

6. Let O be the origin and point p represents complex number z in a complex plane. If OP be rotated anticlockwise at an angle $\pi/2$, then the new position of p is represented by the complex number

[NDA – 2007]

(a) $z - i$ (b) $z + i$

(c) iz (d) $-iz$

7. If the amplitude of $z - 2 - 3i$ is $\frac{\pi}{4}$, then the locus of $z = x + iy$ is

[EAMCET – 2003]

(a) $x + y - 1 = 0$

(b) $x - y - 1 = 0$

(c) $x + y + 1 = 0$

(d) $x - y + 1 = 0$

8. If the areaof the triangle formed by the points z, $z + iz$ and iz on the complex plane is 18, then the value of $|z|$ is

[MPPET – 2001]

(a) 6 (b) 9

(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 5 minutes.
3. The test consists of 5 questions
   The maximum marks are 15.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The value of $\left[\dfrac{1-\cos\dfrac{\pi}{10}\ i\sin\dfrac{\pi}{10}}{1-\cos\dfrac{\pi}{10}-i\sin\dfrac{\pi}{10}}\right]^{10}$

(a) 0 (b) -1
(c) 1 (d) 2

[Karnataka CET – 2001]

2. We express

$$\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$$

in the form of $x + iy$, we get

[Karnataka CET – 2001]

(a) $\cos 49\theta - i\sin 49\theta$
(b) $\cos 23\theta - i\sin 23\theta$
(c) $\cos 49\theta + i\sin 49\theta$
(d) $\cos 21\theta + i\sin 21\theta$

3. If $z = \dfrac{7-i}{3-4i}$, then $z^{14} =$

[Kerala (Engg.) – 2005]

(a) 2^7 (b) $2^7 i$
(c) $2^{14} i$ (d) $-2^7 i$

4. If $x = 2 + 3i$, and $y = 2 - 3i$, then value of $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$ is

(a) $\dfrac{3}{23}$ (b) $-\dfrac{3}{23}$
(c) $\dfrac{5}{23}$ (d) $-\dfrac{5}{23}$

5. If 1, ω, ω^2 are cube root of unity, then the value of $(3 + 3\omega + 5\omega^2)^3 - (2 + 4\omega + 2\omega^2)^3$ is

[Pb.CET – 1998]

(a) 0 (b) 3
(c) 2 (d) 1

ANSWER SHEET

1. a b c d
2. a b c d
3. a b c d
4. a b c d
5. a b c d

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LECTURE 5

Test Your Skills

ASSERTION/REASONING

Assertion and Reasoning Type Questions

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion**

(b) **Assertion** is True, **Reason** is True and **Reason** is NOT a correct explanation for **Assertion**

(c) **Assertion** is True and **Reason** is False

(d) **Assertion** is False and **Reason** is True

1. **Assertion (A):** If $\alpha = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i$ sing and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 1$, then $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -1$

   Reason (R): $(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2) = \cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)$

2. **Assertion (A):** If the area of the triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$ is 600 square units, then $|z| = 20$

   Reason (R): Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}|z^2|$.

3. **Assertion (A):** The greatest value of the moduli of complex numbers z satisfying the equation is $\left|z - \frac{4}{z}\right| = 2$ is $\sqrt{5} + 1$

   Reason (R): For any two complex numbers z_1 and z_2, $|z_1 - z_2| \geq |z_1| - |z_2|$

4. **Assertion (A):** $7 + 4i > 5 + 3i$, where $i = \sqrt{-1}$ **Reason (R):** $7 > 5$ and $4 > 3$

5. **Assertion (A):**

   $\sqrt{(-1)}\,\sqrt{(-3)} = \sqrt{(-2)(-3)} = \sqrt{6}$

   Reason (R): If a and b both negative, then $\sqrt{a}\sqrt{b} \neq \sqrt{ab}$

6. **Assertion (A):** $\sum_{r=1}^{4n=11} i^r = i$, $i = \sqrt{-1}$

   Reason (R): Sum of the four consecutive powers of i is zero.

7. **Assertion (A):** If $\frac{5z_2}{11z_1}$ is purely imaginary, then $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = 1$

   Reason (R): $|z| = |\bar{z}|$.

8. **Assertion (A):** If $z = \sqrt{(5+12i)} + \sqrt{(12i-5)}$, then the principal values of arg (z) are $\pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$, where $i = \sqrt{-1}$.

Reason (R): If $z = a + ib$, then and for

$$\sqrt{z} = \pm\left\{\sqrt{\left(\frac{|z|+a}{2}\right)} - i\left(\frac{|z|-a}{2}\right)\right\}\ b < 0$$

9. Assertion (A): If $|z - 3 + 2i| \leq 4$, then the sum of least and greatest value of $|z|$ is 8.

Reason (R): $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$.

10. Assertion (A): The value of

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right) \text{ is } i.$$

Reason (R): It forms an A.P. series.

11. Assertion (A): For a complex number z the equation $|3z - 1| = 3|z - 2|$ represents a straight line.

Reason (R): General equation of straight line is $ax + by + c = 0$.

12. Assertion (A): If $e^{i\theta} = \cos\theta + i\sin\theta$ and the value of $e^{iA}.\ e^{iB}.\ e^{iC}$ is equal to -1.

Reason (R): $e^{i\theta} = \cos\theta + i\sin\theta$ and in any $\Delta ABC, A + B + C = 180°$.

13. Assertion (A): $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$, where z_1, z_2, z_3 and z_4 are the fourth roots of unity.

Reason (R): $(1)^{1/4} = (\cos 2r\pi + i\sin 2r\pi)^{1/4}$

14. Assertion (A): For any four complex numbers z_1, z_2, z_3 and z_4. it is given that the four points are concyclic.

Reason (R): $|z_1| = |z_2| = |z_3| = |z_4|$

15. Assertion (A): The points denoted by the complex number z lies inside the circle with radius 2 and is at the origin.

Reason (R): $|z| > 2$ represents a straight line.

16. Assertion (A): The expression $\left(\frac{2i}{1+i}\right)^n$ is a positive integer for all the values of n.

Reason (R): Here $n = 8$ is the least positive for which the above expression is a positive integer.

17. Assertion (A): we have an equation involving the complex number z is $\left|\frac{z-3i}{z+3i}\right| = 1$ which lies on the x-axis.

Reason (R): The equation of the x-axis is $y = 3$.

18. Assertion (A): The equation $|z + 1| = \sqrt{3}\,|z - 1|$ represents a circle.

Reason (R): The equation of straight line is $ax + by + c = 0$.

19. Assertion (A): The value of i^{4m+3}, when $m \in I$ is equal to $-i$.

Reason (R): $i^4 = 1$

20. Assertion (A): The roots of the equation $(x - 1)^3 + 8 = 0$ are $-1, 1 - 2\omega, 1 - 2\omega^2$.

Reason (R): $1, \omega, \omega^2$ are the cube roots of unity where $1 + \omega + \omega^2 = 0$ and $\omega^3 \neq 1$.

21. Assertion (A): If z is a complex number $(z \neq 1)$, then $\left|\frac{z}{|z|} - 1\right| < |\arg(z)|$.

Reason (R): In a unit circle, chord $AP \leq$ arc (AP).

22. Assertion (A): The least value of $|z - 3| + 5|z - 8|$, $z \in C$ is got by setting $z = \frac{8+3}{2}$

Reason (R): The least value of $|z - 3| + 5|z - 8|$ is same as that of $PA + 5\boldsymbol{PB}$, where $P = z(x, y)$ and $A = (3, 0)$, $B = (8, 0)$ and P ranges over all points in $x - y$ plane.

23. Assertion (A): If $x + \frac{1}{x} = 1$ and $p = x^{100} + \frac{1}{x^{100}}$ and q be the digit at unit place in $2^{(2n)} + 1$, $n \in N$, $n > 1$, then $p + q = 6$.

Reason (R): If $x + \frac{1}{x} = -1$, then $x^2 + \frac{1}{x^2} = -1$ and $x^3 + \frac{1}{x^3} = 2$

24. Assertion (A): Let z_1, z_2, z_3 be three points in complex plane with nonzero imaginary parts such that $z_1 + z_2 + z_3 = 0$. Then, $z_1 + z_2 + z_3$ must be vertices on an equilateral triangle.

Reason (R): If z_1, z_2, z_3 are vertices of an equilateral triangle, then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

25. Assertion (A): $ABCD$ is a parallelogram on the Argand plane. The affixes of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $8 + 5i, -7 - 5i, -5 + 5i$ respectively. Then, the affix of D is $10 + 15i$.

Reason (R): The diagonals AC and BD bisect each other.

26. **Assertion (A):** If the principal argument of a complex number z is α then principal argument of z^2 is 2α.
Reason (R): $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

27. **Assertion (A):** The modulus of the complex number $z = \frac{1-i}{3+i} + 4i$ is $\sqrt{13}$
Reason (R): Argument of z is $\tan^{-1}1\left(\frac{3}{4}\right)$

28. **Assertion (A):** If $z_1 = 3 - 4i$, $z_2 = -5 + 2i$ are two complex numbers such that $z_1 < z_2$.
Reason (R): $|z_1| < |z_2|$

29. **Assertion (A):** If $\left|\frac{zz_1 - z_2}{zz_1 + z_2}\right| = k\ (z_1, z_2 \neq 0)$, then locus of z is circle.
Reason (R): $\left|\frac{z - z_1}{z - z_2}\right| = \lambda$, represents a circle if, $\{0, 1\}$.

30. **Assertion (A):** The equation $|z - i| + |z + i| = k$, $k > 0$ can represent an ellipse, if $k > 2i$.
Reason (R): $|z - z_1| + |z - z_2| = k$, represents ellipse, if $|k| > |z_1 - z_2|$.

31. **Assertion (A):** If $1, \omega, \omega^2, \ldots\ldots\ldots \omega^{n-1}$ are the nth roots of unity, then $(2-\omega)(2-\omega^2)\ldots\ldots(2-\omega^{n-1})$ equals $2^n - 1$.
Reason (R): ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots + {}^nC_n = 2^n$.

32. **Assertion(A):** If ω is an imaginary cube root of unity, then the value of $\sin\left\{\pi + (\omega^{10} + \omega^{23})\frac{\pi}{4}\right\}$ is $\frac{1}{\sqrt{2}}$
Reason (R): $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

ASSERTION/REASONING: SOLUTIONS

1. (a) we have,
$\frac{1}{a} = \cos a - i \sin a$, $\frac{1}{b} = \cos\beta - i\sin\beta$
Now, $\frac{a}{b} = (\cos a + i \sin a)(\cos\beta - i\sin\beta)$
or, $\frac{a}{b} = \cos(a - \beta) + i\sin(a - \beta)$
Similarly, $\frac{b}{c} = \cos(\beta - \gamma) + i\sin(\beta - \gamma)$
and $\frac{c}{a} = \cos(\gamma - \alpha) + i\sin(\gamma - \alpha)$
Putting these values in $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = -1$,
We get $[\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)] + i[\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \beta)] = -1 = -1 + 0i$.
Comparing real and imaginary parts, we get, $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -1$

2. (a) Area of the triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}|z|^2$.
$\therefore \frac{3}{2}|z|^2 = 600\ |z| = 20.$

3. (a) We have, $z =$
$\left|z - \frac{4}{z}\right| \geq |z| - \left|\frac{4}{z}\right| \Rightarrow |z| - \left|\frac{4}{z}\right| \leq 2$
$\Rightarrow |z|^2 - 2|z| - 40$ or $(|z| - 1)^2 - 50 \leq 0$
$\Rightarrow (|z| - 1)^2 \leq 5$ or $|z| - 1 \leq \sqrt{5}$
$\Rightarrow |z| \leq \sqrt{5} + 1$
Hence, the greatest value of $|z|$ is $\sqrt{5} + 1$.

4. (d) Property of order i.e., $(a + ib) \lessgtr (c + id)$ is not defined. The statement $7 + 4i > 5 + 3i$ makes no sense.

5. (d) If both a and b are negative then $\sqrt{a}\sqrt{b} = -\sqrt{ab}$
$\therefore \sqrt{(-2)}\sqrt{(-3)} = -\sqrt{(-2)}\sqrt{(-3)} = -\sqrt{6}.$

6. $\sum_{r=1}^{4n+11} i^r = (i + i^2 + i^3\, i^4) + (i^5 + i^6 + \ldots\, i^8)$
$+ \ldots (i^{4n+5} + i^{4n+6} + i^{4n+7} + i^{4n+8})$
$+ i^{4n+9} + i^{4n+10}\, i^{4n+11}$
$= i - 1 - i + 0 = 1$
(Since, sum of four consecutive powers of i is zero)

7. (a) Let $\frac{5z_2}{11z_1} = i\lambda\ (\lambda \neq 0) \Rightarrow \frac{z_2}{z_1} = \frac{11i\lambda}{5}$

$$\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = \left|\frac{2 + 3\frac{z_2}{z_1}}{2 - 3\frac{z_2}{z_1}}\right| = \left|\frac{2 + \frac{33i\lambda}{5}}{2 - \frac{33i\lambda}{5}}\right|$$

$$= \left|\frac{10 + 33i\lambda}{10 - 33i\lambda}\right| = 1.$$

8. (b) Let $z = z_1 + z_2$

Since, $z_1 = \sqrt{(5 + 12i)} = \pm\left\{\begin{array}{l}\sqrt{\left(\frac{13+5}{2}\right)} \\ +i\sqrt{\left(\frac{13-5}{2}\right)}\end{array}\right\}$

$$= \pm(3 + 2i)$$

and $z_2 = \sqrt{(12i - 5)} = \sqrt{(-5 + 12i)}$

$$= \pm\left\{\sqrt{\left(\frac{13-5}{2}\right)} + i\sqrt{\left(\frac{13+5}{2}\right)}\right\}$$

$$= \pm(2 + 3i)$$

$z = \pm(3 + 2i) \pm (2 + 3i)$

$\Rightarrow z = 5 + 5i,\ 1 - i,\ -1 + i,\ -5 - 5i.$

Hence, principal values of z are

$\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}$

9. (a) Since, $|z_1 + z_2| = |z_1 - z_2|$

Therefore, $|z - 3 + 2i| = |z - (3 - 2i)| \geq ||z| - |3 - 2i||$

$= ||z| - \sqrt{13}|$

Since, $|z - 3 + 2i| \geq ||z| - \sqrt{13}|$

Since, $|z - 3 + 2i| \leq 4$

$\Rightarrow |z|\sqrt{13}| \leq |z - 3 + 2i| \leq 4$

$\Rightarrow ||z| - \sqrt{13}| \leq 4$

$-4 \leq |z| - \sqrt{13}| \leq 4$

$\Rightarrow$ or $4 - \sqrt{13} \leq |z| \leq 4 + \sqrt{13}$

$\therefore$ Greatest value of $|z| = 4 + \sqrt{13}$ and Least value of $|z| = 4 - \sqrt{13}$

$\therefore$ Sum = 8.

10. (c) $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$$= -i\sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right)$$

$$= -i\left[\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right) + \ldots\ldots + \left(\cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}\right)\right]$$

$$= -i\left[\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^2 + \ldots\ldots + \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^6\right]$$

$= -i\left[\frac{x(x^6 - 1)}{x - 1}\right]$ if $x = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$$= -i\left[\frac{x^7 - x}{x - 1}\right] = -i\left[\frac{1 - x}{x - 1}\right]$$

$$\left[\because x^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7\right]$$

$$= \cos 2\pi + i\sin 2\pi = 1$$

$$= i\left(\frac{x - 1}{x - 1}\right) = i$$

11. (b) Let $z = x + iy$, then $|3z - 1| = 3|z - 2|$

$\Rightarrow |3(x + iy) - 1| = 3|x + iy - 2|$

$\Rightarrow |(3x - 1) + 3iy| = 3|x - 2 + iy|$

$\Rightarrow (3x - 1)^2 + 9y^2 = 9[(x - 2)^2 + y^2]$

$\Rightarrow 9x^2 + 1 - 6x + 9y^2 = 9x^2 + 36 - 36x + 9y^2$

$\Rightarrow 30x = 35 \Rightarrow x = \frac{7}{6}$

i.e. a straight line parallel to y-axis.

12. (a) $e^{iA}.\ e^{iB}.\ e^{iC} = e^{i(A+B+C)} = e^{\pi i}$

[Since, $A + B + C =$ for ΔABC]

$= \cos\pi + i\sin\pi$

$= -1 + i(0) = -1.$

13. (a) $(1)^{1/4} = (\cos 2\pi r + i\sin 2\pi r)^{1/4}$

$\cos\frac{\pi r}{2} + i\sin\frac{\pi r}{2}$

where $r = 0, 1, 2, 3$

$\therefore (1)^{1/4} = 1, i, -1, -i$

$\therefore z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1 + i^2$

$= 2 - 1 - 1 = 0.$

14. (a) $|z_1|$ = the distance of the point representing z_1 from the origin. Therefore, the distances of the four points from the origin are equal. Therefore, points are concyclic.

15. (c) Since $|z| < 2$

$\therefore |z|^2 < 4$

$\Rightarrow x^2 + y^2 < 4$

16. (d) $\left(\dfrac{2i}{1+i}\right)^2 = \dfrac{4i}{(1+i)^2} = \dfrac{-4}{1+i^2+2i}$

$= \dfrac{-4}{2i} = \dfrac{-2}{i} = 2i$

$\therefore \left(\dfrac{2i}{1+i}\right)^4 = 4i^2 = -4$

$\therefore \left(\dfrac{2i}{1+i}\right)^8 = (-4)^2 = 16$

Hence, $n = 8$ is the last poitive integer

17. (c) $\left|\dfrac{z-3i}{z+3i}\right| = 1 = \Rightarrow \left|\dfrac{x+iy-3i}{x+iy+3i}\right| = 1$

$\Rightarrow \left|\dfrac{x+(y-3)i}{x+(y+3)i}\right| = 1 \Rightarrow \dfrac{x^2+(y-3)^2}{x^2+(y+3)^2} = 1$

$\Rightarrow x^2 + y^2 - 6y + 9 = x^2 + y^2 + 6y + 9$

$\Rightarrow 12y = 0$

$\Rightarrow y = 0$ which is x-axis.

Therefore, z lies on x-axis.

18. (b) $|z+1|^2 = 3\,|z-1|^2$

$\Rightarrow (x+1)^2 + y^2 = 3\,[(x-1)^2 + y^2]$

$\Rightarrow 2(x^2+y^2) - 8x + 2 = 0$

$\Rightarrow x^2 + y^2 - 4x + 1 = 0$

which is a circle.

19. (a) $i^n = i^{4m+3} = i^{4m}.\, i^3 = (i^4)^m\,(-i) = (1)^m\,(-i)$
$= -i$

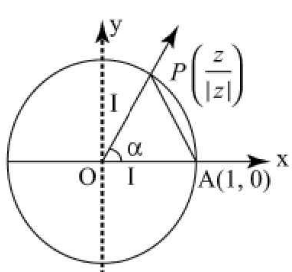
20. (a) Since $(x-1)^3 = -8 = (-2)^3$

$\therefore x - 1 = -2, -2\omega, -2\omega^2$

$\therefore x = -1, 1 - 2\omega, 1 - 2\omega^2$

21. (a) The number $\dfrac{z}{|z|}$ lies on a unit circle centred at origin

From the figure



Chord $AP = \left|\dfrac{z}{|z|} - 1\right| \le arc\ (AP)$

$\dfrac{\text{arc }(AP)}{\text{radius}} = 1 = \alpha$

$\therefore \left|\dfrac{z}{|z|} - 1\right| \le |\arg z|$

22. (b) $|z_2 z_3 + 8z_3 z_1 + 27z_1 z_2|$

$= |z_1 z_2 z_3| \left|\dfrac{1}{z_1} + \dfrac{8}{z_2} + \dfrac{27}{z_3}\right|$

$= |z_1|\,|z_2|\,|z_3| \left|\dfrac{\bar{z}}{|z_2|^2} + \dfrac{8\bar{z}_2}{|z_2|^2} + \dfrac{27\bar{z}_3}{|z_3|^2}\right|$

$= 6\,|\bar{z} + 2\bar{z}_2 + 3\bar{z}_3|$

$= 6\,|z_1 + 2z_2 + 3z_3| = 6 \times 6 = 36$

$\therefore S_1$ is true, S_2 is also true but not the correct explanation for S_1.

23. (b) $x + \dfrac{1}{x} = 1$

$\Rightarrow x^2 - x + 1 = 0$

$\Rightarrow x = -\omega, -\omega^2, \omega$ is imaginary cube root of unity.

$\therefore p\ x^{100} = + \dfrac{1}{x^{100}} = \omega + \omega^2 = -1$

For $n > 1$, $2^n = 4m$, $m \in N$

$\Rightarrow 2^{(2^n)} = 2^{4m} = 16^m$

$\Rightarrow$ unit place of $2^{(2^n)} = 6$

$\therefore\ q =$ unit place at $2^{(2^n)} + 1 = 7$

Hence, $p + q = 7 - 1 = 6$

Also, roots of $x + \dfrac{1}{x} = -1$ are ω and ω^2

$\Rightarrow x^2 + \dfrac{1}{x^2}\ = -1$ and $x^3 + \dfrac{1}{x^3} = 2$

Hence, both S_1 and S_2 are true, but S_2 is not correct explanation for S_1.

24. (d) z_1, z_2, z_3 vertices of an equilateral triangle then

$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ which

$z_1^3 + z_2^3 + z_3^3 - 3z_1 z_2 z_3 = 0$

$\therefore S_2$ is true.

But $z_1 + z_2 + z_3 = 0$ even when z_1, z_2, z_3 are collinear For example, i, $2i$ and $-3i$.

$\therefore S_1$ is false.

25. (a) Let z be the affix of D

Therefore,

$\dfrac{(8+5i) + (-5+5i)}{2} = \dfrac{(-7-5i) + z}{2}$

$z = 10 + 15i$

So, both assertion and reason are true and reason is correct explanation of assertion.

26. (d) If principal arg of z is α then argument of z^2 is $2a$. Note that it may not be principal argument

For example, Let $z = -1 + i$

$\Rightarrow \text{Arg}(z) = \frac{3\pi}{4}$

$\text{Arg}(z^2) = \frac{3\pi}{4} \times 2 = \frac{3\pi}{2}$

but principal arg $(z^2) = \frac{\pi}{4}$

So, assertion is false, reason is true.

27. (c) Converting to $a + ib$ form

$$z = \left(\frac{1-i}{3+i}\right)\left(\frac{3-i}{3-i}\right) + 4i$$

$$z = \frac{1}{5} + \frac{18}{5}i$$

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{18}{5}\right)^2}$$

$$\arg(z) = \tan^{-1}\left|\frac{\frac{18}{5}}{\frac{1}{5}}\right| = \tan^{-1}(18)$$

So, assertion is false, reason is true.

28. (d) In the set of complex number, the order relation is not defned. As such $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.

So, assertion is false, reason is true.

29. (d) If $\left|\frac{z_1 z - z_2}{z_1 z + z_2}\right| = k$

$$\Rightarrow \left|\frac{z - \frac{z_2}{z_1}}{z + \frac{z_2}{z_1}}\right| = k$$

Clearly, if $k \neq 0, 1$ then z would lie on a circle.

Case I If $k = 1$, z would be on a perpendicular bisector of line segment.

Case II If $k = 0$, $\frac{z_2}{z_1}$ and $-\frac{z_2}{z_1}$ represents a point. So, assertion is false and reason is true.

30. (d) As, we know $|z - z_1| + |z - z_2| = k$ represents an ellipse,

if $|k| > |z_1 - z_2|$

Thus, $|z - i| + |z + i| = k$ represents ellipse, if $|k| > |i + i|$ or $|k| > 2$.

So, assertion is false but reason is correct.

31. (c) Let

$x = (1)^{1/n}$

$x^n - 1 = 0$

has n roots i.e. $1, \omega, \omega^2, \ldots, \omega^{n-1}$

$(x^n - 1) = (x - 1)(x - \omega)(x - \omega^2)\ldots\ldots(x - \omega^{n-1})$

$\frac{2^n - 1}{(2-1)} = (2 - \omega)(2 - \omega^2)\ldots\ldots(2 - \omega^{n-1})$

(put $x = 2$)

i.e., $\because (2 - \omega).(2 - \omega^2)\ldots\ldots(2 - \omega^{n-1}) = 2^n - 1$

As, ${}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n = 2^n$

So, assertion is true but reason is false.

32. (a) $\left[\sin\pi + (\omega + \omega^2)\frac{\pi}{4}\right] = \sin\left[\pi - \frac{\pi}{4}\right]$

$= \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

So, both assertion and reson are true and asserition follows reason.

MENTAL PREPARATION TEST

1. If $(x + iy)^{1/3} = a + ib, x, y, a, bR$.

   Show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$

2. If $= \frac{a + ib}{c + id} = x + iy$,

   prove that $\frac{a - ib}{c - id} = x - iy$ and $\frac{a^2 + b^2}{c^2 + d^2} = x^2 + y^2$

3. If $(a + ib)(c + id) = x + iy$, then prove that $(a - ib)(c - id) = x - iy$ and $(a^2 + b^2)(c^2 + d^2) = x^2 + y^2$.

4. If $z = 3 - 5i$, then prove that $z^3 - 10z^2 + 58z - 136 = 0$.

5. Find the modulus of $\left(\frac{1+i}{1-i}, \frac{1-i}{1+i}\right)$.

6. Find the locus of a complex variable z in the argand plane, satisfying
$|z-(3-4i)|=7$.

7. Write the complex numbers $-1-i$ in the polar form.

8. If $\frac{2z_1}{3z_2}$ be a purely imaginary number, then prove that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$

9. If $i=\sqrt{-1}$, then prove that
$$4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=i\sqrt{3}$$

10. Find the square roots of the following
(i) $4ab-2(a^2-b^2)i$
(ii) $a^2-1+2ai$.

11. If $\frac{3}{2+\cos\theta+i\sin\theta}=a+ib$ them prove that $a^2+b^2=4a-3$.

12. Express the complex number $\frac{3-\sqrt{-16}}{1-\sqrt{-9}}$ in the form $a+ib$.

13. Prove that $\left(\frac{3+2i}{2-5i}\right)+\left(\frac{3-2i}{2+5i}\right)$ rational.

14. Find the values of x and y, for which the $(3x-2iy)(2+i)^2=10(1+i)$ equalities hold.

15. Prove that $x^4+4=(x+1+i)(x+1-i)(x-1+i)(x-1-i)$.

16. If $z=x+iy$ and $\omega=\frac{1-iz}{z-i}$, show that $|\omega|=1 \Rightarrow z$ is purely real.

17. If $z=-5+2\sqrt{-4}$, show that $z^2+10z+41=0$ and hence, find the value of $z^4+9z^3+35z^2-z+4$

18. If $\frac{a-ib}{a+ib}=\frac{1+i}{1-i}$, then show that $a+b=0$.

19. If $1, \omega, \omega^2$ be the cube roots of unity, prove that
$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})=49$.

20. If α and β are imaginary cube roots of unity, show that
$\alpha^4+\beta^4+\alpha^{-1}.\beta^{-1}=0$.

21. Express the numbers $\frac{1+2i}{1-3i}$ in polar form.

22. Express the $\sin 120^\circ - i\cos 120^\circ$ in polar form.

23. Find the radius and centre of the circle $|z+3+i|=5$ where z is a complex variable.

24. Show that the points representing the complex numbers
$(3+2i)$, $(2-i)$ and $-7i$ are collinear.

25. A variable complex number $z=x+iy$ is such that $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$, show that $x^2+y^2-1=0$.

TOPICWISE WARMUP TESTS

1. $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6+\left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6=$
(a) 2 (b) 1
(c) 0 (d) 4

2. The area of the triangle obtained by joining complex numbers z, iz and $z+iz$ in argand diagram is
[PET (Raj.), – 1998, 2000; MP – 1997; EAMCET – 1996; IIT – 1980; DCE – 1999; UPSEAT – 2002]
(a) $2|z|^2$ (b) $|z|^2/2$
(c) $|z|^2$ (d) none of these

3. If $\frac{z-1}{z+1}$ is purely imaginary number, then
[MP – 1998, 2002]
(a) $|z|=1$ (b) $|z|>1$
(c) $|z|<1$ (d) none of these

4. If z_1, z_2, z_3 be three complex numbers such that
$|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$ then
$|z_1+z_2+z_3|$ is equal to
[IIT (Screening) – 2000]
(a) 1 (b) less than 1
(c) greater than 3 (d) 3

5. If 1, ω, ω^2 are cube roots of unity and $a + b + c = 0$ then $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$ is equal to

(a) 0 (b) $3abc$
(c) $27abc$ (d) none of these

6. If z and ω are two nonzero complex numbers such that $|z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$, then z is equal to

[IIT – 1995; AIEEE – 2002; JEE (Orissa) – 2004]

(a) ω (b) $-\omega$
(c) $\overline{\omega}$ (d) $-\overline{\omega}$

7. If z and ω are complex numbers such that $\bar{z} + i\overline{\omega} = 0$ and $\arg(z\omega) = \pi$, then $\arg(z)$ is equal to

[AIEEE – 2004]

(a) $3\pi/4$ (b) $\pi/2$
(c) $\pi/4$ (d) $5\pi/4$

8. The polar form of

[Roorkee – 1981]

(a) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$
(b) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
(c) $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$
(d) none of these

9. If z be multiplied by $1 + i$, then in complex plane vector z will be rotated at an angle

[ICS – 2001]

(a) 90° clockwise
(b) 45° clockwise
(c) 90° anti-clockwise
(d) 45° anti-clockwise

10. If w is imaginary cube root of unity, then

$$\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} + \frac{c + a\omega + b\omega^2}{a + b\omega + c\omega^2} + \frac{b + c\omega + a\omega^2}{b + c\omega^4 + a\omega^5}$$

is equal to

[Kerala (CEE) – 2003]

(a) 1 (b) −1 (c) 0 (d) ω

11. $\frac{1 + 2\omega + 3\omega^2}{2 + 3\omega + \omega^2} + \frac{2 + 3\omega + \omega^2}{3 + \omega + 2\omega^2}$ is equal

[Orissa (JEE) – 2003]

(a) 0 (b) − 1
(c) 2ω (d) -2ω

12. If α, β are roots of the equation $x^2 + x + 1 = 0$, then $\alpha^{2001} + \beta^{2001}$ equal to

[ICS (Pre) – 2004]

(a) −2 (b) 2
(c) 0 (d) − 1

13. The modulus and amplitude of $\frac{1 + 2i}{1 - (1 - i)^2}$ are

[CET (Karnataka) – 2005]

(a) 1, 0 (b) $2, \pi$
(c) 1/2, 0 (d) $3, \pi/2$

14. If $\omega \neq 1$ be a cube root of unity and
$(1 + \omega^2)^n = (1 + \omega^4)^n$
then the least + ive value of n is

[IIT (Screening) – 2004]

(a) 2 (b) 3
(c) 4 (d) 5

15. Let z_1 and z_2 be complex numbers, then $|z_1 + z_2|^2 + |z_1 - z_2|^2$ is equal to

[MP PET – 2006]

(a) $|z_1|^2 + |z_2|^2$ (b) $2(|z_1|^2 + |z_2|^2)$
(c) $2(z_1^2 + z_2^2)$ (d) $4z_1z_2$

16. If ωis an imaginary cube root of unity, then the value of $\sin\left[(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right]$ is:

[IIT (Screening) – 1994]

(a) $\frac{-\sqrt{3}}{2}$ (b) $\frac{-1}{\sqrt{2}}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

17. What is the value of
$[(-1 + i\sqrt{3})/2]^{10} + [(-1 - i\sqrt{3})/2]^{10}$

[NDA – 2007]

(a) 1 (b) −1
(c) 2 (d) 0

18. Real part of $\frac{1}{1 + \cos\theta + i\sin\theta}$ is

[MP PET – 2006]

(a) 1/3 (b) 1/5
(c) 1/2 (d) 1/8

19. Value of $|1 - \cos\alpha + i\sin - \alpha|$is

[MP PET – 2007]

(a) $2 \sin \frac{\alpha}{2}$ (b) $2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$

(c) $2\cos \frac{\alpha}{2}$ (d) $2 \sin^2 \frac{\alpha}{2}$

20. $\left[\dfrac{1 + \cos (\pi/8) + i \sin (\pi/8)}{1 + \cos (\pi/8) - i \sin (\pi/8)}\right]^8$ is equal to

[RPET – 2001]

(a) -1 (b) 0

(c) 1 (d) 2

21. If for complex numbers z_1 and z_2, arg $(z_1/z_2) = 0$, then $|z_1 - z_2|$ is equal to

(a) $|z_1| + |z_2|$ (b) $|z_1| - |z_2|$

(c) $||z_1| - |z_2||$ (d) 0

22. If $x = \cos \theta + i \sin \theta$, then $x^4 + \dfrac{1}{x^4} =$

[MP PET – 2006]

(a) $2 \cos 4\theta$ (b) $2 i \sin 4\theta$

(c) $-2i \sin 4\theta$ (d) $-2 \cos 4\theta$

23. If $\cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha$ then $+ \sin \beta + \sin \gamma$

$\cos 3 \alpha + \cos 3\beta + \cos 3\gamma$ is equal to

[Bihar (CEE) – 2000; EAMCET – 1995]

(a) $3 \cos (\alpha + \beta + \gamma)$

(b) $\cos (3\alpha + 3\beta + 3\gamma)$

(c) $\cdots (\alpha + \beta + \gamma)$

(d) $3 \sin (\alpha + \beta + \gamma)$

24. A complex number z is such that arg $\left\{\dfrac{z-2}{z+2}\right\} = \dfrac{\pi}{2}$. The points representing this complex number will lie on

[MP PET – 2001]

(a) $\dfrac{x^2}{4} + \dfrac{y^2}{4\sqrt{3}} = 1$

(b) $y^2 = 4\sqrt{3}x$

(c) $x^2 + y^2 - 4y - 4 = 0$

(d) $x + y = 4\sqrt{3}$

25. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

[AIEEE – 2004]

(a) a circle

(b) the imaginary axis

(c) the real axis

(d) an ellipse

TOPICWISE WARMUP TESTS: SOLUTION

1. (a) L.H.S. $= \left(\dfrac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^6 + \left(\dfrac{1 - i\sqrt{3}}{1 + i\sqrt{3}}\right)^6$

$$= \left(\dfrac{\dfrac{-1 - i\sqrt{3}}{2}}{-\dfrac{-1 + i\sqrt{3}}{2}}\right)^6 + \left(\dfrac{\dfrac{-1 + i\sqrt{3}}{2}}{\dfrac{-1 - i\sqrt{3}}{2}}\right)^6$$

$$= \left(\frac{-\omega^2}{-\omega}\right)^6 + \left(\frac{-\omega}{-\omega^2}\right)^6$$

$$= \left[\because \omega = \frac{-1 + i\sqrt{3}}{2}, \omega^2 = \frac{-1 - i\sqrt{3}}{2}\right]$$

$$= \omega^6 + \frac{1}{\omega^6} = (\omega^3)^2 + \frac{1}{(\omega^3)^2} = 1 + \frac{1}{1}$$

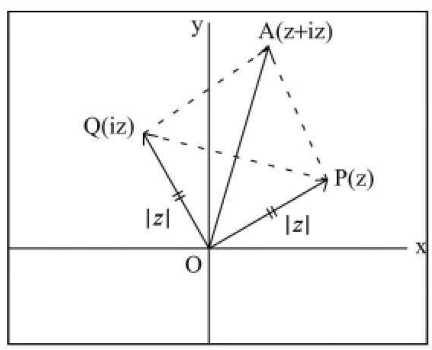
$= 1 + 1 = 2$ $=$ R.H.S. **Proved**

2. (b) Let $z = \overrightarrow{OP}$, $iz = \overrightarrow{OQ}$, $z + iz = \overrightarrow{OA}$.

Then obviously $OP \perp OQ$ and $OP = OQ$

$[\because |z| = = |iz|]$, amp (z) − amp $(iz) = -\pi/2]$

$\Rightarrow OPAQ$ is a square



Therefore, area of given $\Delta = \dfrac{1}{2}$ (area of the square)

$$= \frac{1}{2} |z|^2$$

3. (a) Let $\dfrac{z-1}{z+2} = \dfrac{i\lambda}{1}$ where ω is a real number.

$\Rightarrow \dfrac{2z}{2} = \dfrac{1 + i\lambda}{1 - i\lambda}$

$\Rightarrow |z| = \dfrac{1 + i\lambda}{1 - i\lambda}$

$= \dfrac{\sqrt{1 + 1^2}}{\sqrt{1 + 1^2}} =$

4. (a) $\because |z| = 1 \Rightarrow \dfrac{1}{2} = \bar{z}$

Hence $\dfrac{1}{z_1} = \bar{z}_1, \dfrac{1}{z_2} = \bar{z}_2, \dfrac{1}{z_3} = \bar{z}_3$

$\because \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$\Rightarrow |\overline{z_1 + z_2 + z_3}| = 1$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$ $[\because |z| = |\bar{z}|]$.

5. (c) Let $x = a + b\omega + c\omega^2$, $y = a + b\omega^2 + c\omega$.

$x^3 + y^3 = (x + y)(x\omega^2 + y\omega)(x\omega + y\omega^2)$

$= [2a + (w + \omega^2)\, b + (\omega^2 + \omega)c]$

$[(\omega^2 + \omega)\, a + 2b + (\omega + \omega^2)\, c]$

$[(\omega^2 + \omega)\, a + (\omega^2 + \omega)\, b + 2c]$

$= (2a - b - c)(-a + 2b - c)(-a - b + 2c)$

$= (3a)\ (3b)\ (3c)$

$[\because a + b + c = 0] = 27abc$.

6. (d) Let $\omega = r(\cos\theta + i\sin\theta)$, then

$z = r\{\cos(\pi - \theta) + i\sin(\pi - \theta)\}$

$[\because |z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi]$

$= r(\cos\theta\ i\sin\theta) - r(\cos\theta\ i\sin\theta) = -\bar{\omega}$

7. (a) $= \bar{z} + i\bar{\omega} = 0$

$\Rightarrow z - i\bar{\omega} \Rightarrow z = i\omega$

$\Rightarrow \arg(z) - \arg(\omega) + \pi/2$ (1)

But.

$\arg(z\omega) = \pi \Rightarrow \arg(z) + \arg(\omega) = \pi$ (2)

$(1) + (2) \Rightarrow 2\arg(z) = 3\pi/2$

$\Rightarrow \arg(z) = 3\pi/4$

8. (b) $\dfrac{1 + 7i}{(2 - i)^2} = \dfrac{1 + 7i}{3 - 4i} = \dfrac{(1 + 7i)(3 + 4i)}{25}$

$= -1 + i$

$= \sqrt{2}\,(\cos 3\pi/4 + i\sin 3\pi/4)$.

9. (d) amp $(1 + i) = 45°$,

so z will be rotated at an angle 45° in anticlockwise.

10. (c) Exp $\dfrac{1}{\omega} + \dfrac{1}{\omega^2} + 1 = \omega^2 + \omega + 1 = 0$.

11. (c) Exp $\omega + \omega = 2\omega$.

12. $x = \omega, \omega^2$.

So exp. $\omega^{2001} + \omega^{4002} = 1 + 1^2 = 2$

13. (a) $\dfrac{1 + 2i}{1} - (1 - i) = \dfrac{1 + 2i}{1 + 2i} = 1$

$\Rightarrow$ its modulus = 1,

amplitude = 0.

14. (b) Given $(1 + \omega^2)^n = (1 + \omega)^n \because \omega^4 = \omega$

or $(-\omega)^n = (-\omega^2)^n$ or $\omega^n = \omega^{2n}$

Clearly $n = 3$ is the least value of n satisfying above $\because \omega^3 = \omega^6 = 1$.

15. (b) Let z_1 and z_2 be complex numbers as follows

$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$\therefore = |z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 + y_2)^2$

$= x_1^1 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2$

$= 2\,(x_1^2 + y_1^2 + x_2^2 + y_2^2) = 2\,(|z_1|^2 + |z_2|^2)$

16. (c) Given $\sin\left[(\omega^{10} + \omega^{23})\,\pi - \dfrac{\pi}{4}\right]$

$= \sin\left[(\omega + \omega^2)\,\pi - \dfrac{\pi}{4}\right]$

$= \sin\left(-\pi - \dfrac{\pi}{4}\right) = \sin\left(\pi + \dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$.

17. (b) $(\omega)^{10} + (\omega^2)^{10} = +\,\omega^2 = -\,1$.

18. (c) $= \dfrac{1}{1 + \cos\theta + i\sin\theta}$

$$= \frac{1}{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\frac{1}{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]}$$

$$= \frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}$$

$$\frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= \text{Real part} = \frac{\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \frac{1}{2}.$$

19. (a) $|(1 - \cos\alpha) + i\sin\alpha|$

$$= \sqrt{(1-\cos\alpha)^2 + (\sin\alpha)^2}$$

$$\sqrt{\left(2\sin^2\frac{\alpha}{2}\right)^2 + \left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)^2}$$

$$= \sqrt{4\sin^2\frac{\alpha}{2} + \left(\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\right)^2}$$

$$= 2\sin\frac{\alpha}{2}.$$

20. (a) $\left[\dfrac{1+\cos(\pi/8) + i\sin(\pi/8)}{1+\cos(\pi/8) - i\sin(\pi/8)}\right]^8$

$$= \left[\frac{2\cos^2(\pi/16) + 2i\sin(\pi/16)\cos(\pi/16)}{2\cos^2(\pi/16) - 2i\sin(\pi/16)\cos(\pi/16)}\right]^8$$

$$= \frac{[\cos(\pi/16) + i\sin(\pi/16)]^8}{[\cos(\pi/16) - i\sin(\pi/16)]^8}$$

$$= \left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right]^8\left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right]^8$$

$= [\cos(\pi/16) + i\sin(\pi/16)]16$

$= \cos 16(\pi/16) + i\sin 16(\pi/16) = \cos\pi = -1.$

21. (c) We have $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ Since $\arg z_1 - \arg z_2 = 0$

$\therefore |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2$

$\Rightarrow |z_1 - z_2| = ||z_1| - |z_2||.$

22. (a) $x = \cos\theta + i\sin\theta$

$x^4 = (\cos\theta + i\sin\theta)^4$

$x^4 = (\cos 4\theta + i\sin 4\theta)$ (By Demoivers theorem)

$\frac{1}{x^4} = x^{-4} = (\cos\theta + i\sin\theta)^{-4}$

$= \cos(-4\theta) + i\sin(-4\theta)$

$= \cos 4\theta - i\sin 4\theta$

$x^4 + \frac{1}{x^4} = \cos 4\theta + i\sin 4\theta + \cos 4\theta - i\sin 4\theta$

$x^4 + \frac{1}{x^4} = 2\cos 4\theta.$

23. (a) Let $x = (1, \alpha)$, $y = (1, \beta)$, $z = (1, \gamma)$ then

$x + y + z = \Sigma\cos\alpha + i\,\Sigma\sin\alpha = 0$

$\Rightarrow x^3 + y^3 + z^2 = 3xyz$

$\Sigma(\cos\alpha + i\sin\alpha)^3 = 3[\cos(\alpha + \beta + \gamma)] + i\sin(a + \beta + \gamma)$

$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos\gamma$

$= 3\cos(\alpha + \beta + \gamma).$

24. (c) Let $z = x + iy$ $\arg\dfrac{(x+iy-2)}{x+iy-2} = \dfrac{\pi}{3}$

are $|(x-2) + iy|$ – are $|(x+2) + iy| = \frac{\pi}{3}$

Therefore, $4y = x^2 + y^2 - 4$

or $x^2 + y^2 - 4y - 4 = 0$ points representing the given complex number will lie on CIRCLE.

25. (b) $|z^2 - 1| = |z|^2 + 1$ Let $z = x + iy$

$\Rightarrow |x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$

$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = z(x^2 + (y^2 + 1)^2$

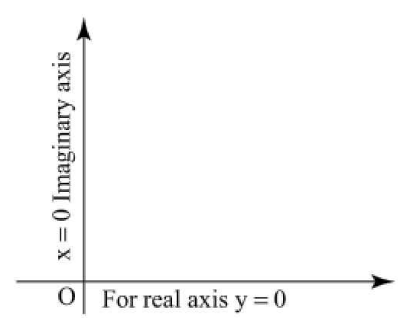
$\Rightarrow 4x^2y^2 = 4x^2(y^2 + 1)\ x = 0$

$|(x + iy)^2 - 1| = x^2 + y^2 + 1$

$\sqrt{(x^2 - y^2 - 1)^2 + (2ixy)^2} = x^2 + y^2 + 1$

Solving, we get $4x^2 = 0$

$x = 0.$



$\Rightarrow z$ lies on imaginary axis.

QUESTION BANK: SOLVE THESE TO MASTER

1. If a complex number satisfies z, $|z - 5i| \le 1$ and the argument of z is minimum then, $z =$

(a) $\frac{2}{5}\sqrt{6} + \frac{24}{5}i$ (b) $\frac{2}{5}\sqrt{6} - \frac{24}{5}i$

(c) $-\frac{2}{5}\sqrt{6} + \frac{24}{5}i$ (d) $-\frac{2}{5}\sqrt{6} - \frac{24}{5}i$

2. The conjugate of complex number $\frac{2-3i}{4-i}$ is

(a) $\frac{11+10i}{17}$ (b) $\frac{5+3i}{4-i}$

(c) $\frac{5-3i}{4-9i}$ (d) $\frac{11-10i}{17}$

3. The argument of the complex number $\frac{13-5i}{4-9i}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

4. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$ then

(a) $x = \frac{1}{5}$ (b) $y = \frac{1}{3}$

(c) $x + iy = \frac{1-i}{1+2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

5. Value of $|\,1 - \cos\alpha + i\sin\alpha|$ is:

(a) $|2\sin\alpha/2|$ (b) $\left|2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right|$

(c) $2\cos\frac{\alpha}{2}$ (d) $\left|2\sin^2\frac{\alpha}{2}\right|$

6. z and ω are two nonzero complex numbers such that $|z| = |\,\omega\,|$and Arg z + Arg ω = then z equals

(a) $\overline{\omega}$ (b) $-\overline{\omega}$ (c) ω (d) $-\omega$

7. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{p}\right)/(p^2 + q^2)$is equal to

(a) -2 (b) -1 (c) 1 (d) 2

8. The conjugate of a complex number $\frac{1}{i-1}$ is then that complex number is

(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$

(c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$

9. If $\left|\frac{z_1 - 2z_2}{2 - z_1\overline{z}_2}\right| = 1$, $|z_2| \ne 1$, then $|z_1| =$

(a) 4 (b) 2

(c) 1 (d) None of these

10. The value of $\left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8$ is equal to

(a) 4 (b) 6 (c) 8 (d) 2

11. If $\sqrt[3]{a - ib} = x - iy$, then $= \sqrt[3]{a + ib} =$

(a) $x + iy$ (b) $x - iy$

(c) $y + ix$ (d) $y - ix$

12. For any integer n, the argument of $z = \frac{(\sqrt{3}+i)^{4n+1}}{(1-i\sqrt{3})^{4n}}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

13. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right|$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

14. The real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other are

(a) 1, 4 (b) 1, -4

(c) $-1, -6$ (d) -1, 4

15. Let z be any non-zero complex number. Then, arg (z) + arg $(\overline{z})$ is equal to

(a) π (b) $-\pi$ (c) 0 (d) $\pi/2$

16. Let z be a complex number. Then the angle between vectors z and iz is:

(a) π (b) 0

(c) $\pi/2$ (d) None of these

17. If $x = 3 + i$, then $x^3 - 3x^2 - 8x + 15 =$

(a) 6 (b) 10 (c) -18 (d) -15

18. If $|z| \le 4$, then the maximum value of $|iz + 3 - 4i|$ is equal to:

(a) 2 (b) 4

(c) 3 (d) 9

19. If $z_1 = (4, 5)$ and $z_2 = (-3, 2)$, then $\frac{z_1}{z_2}$ equals

(a) $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ (b) $\left(\frac{2}{13}, \frac{-23}{12}\right)$

(c) $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ (d) $\left(\frac{-2}{13}, \frac{23}{13}\right)$

20. If $|z_1| = |z_2|$ and arg $\left(\frac{z_1}{z_2}\right) = \pi$, then $z_1 + z_2$ is equal to

(a) 0
(b) purely imaginary
(c) purely real
(d) none of these

21. $(\cos 2\theta + i \sin 2\theta) - 5 (\cos 3\theta - i \sin 3\theta)^6$ $(\sin - i \cos\theta)^3$ in the form of $A + iB$ is
(a) $(\cos 25\theta + i \sin 25\theta)$
(b) $i (\cos 25\theta + i \sin 25\theta)$
(c) $i (\cos 25\theta - i \sin 25\theta)$
(d) $(\cos 25\theta - i \sin 25\theta)$

22. The value of $\sum_{k=1}^{6} \left(\sin \frac{2\pi k}{7} - i \cos \frac{2\pi k}{7}\right)$ is
(a) -1 (b) 0 (c) $-i$ (d) i

23. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3$ $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ is equal to
(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

ANSWERS

Lecture–1: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(c)	4.	(c)	7.	(b)	10.	(a)
2.	(b)	5.	(a)	8.	(b)		
3.	(b)	6.	(d)	9.	(b)		

Lecture–1: Work Sheet: To Check Preparation Level

1.	(b)	5.	(b)	9.	(a)	13.	(b)
2.	(d)	6.	(a)	10.	(b)		
3.	(d)	7.	(a)	11.	(c)		
4.	(d)	8.	(a)	12.	(d)		

Lecture–2: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(c)	6.	(a)	11.	(a)	16.	(a)
2.	(c)	7.	(d)	12.	(d)	17.	(c)
3.	(b)	8.	(d)	13.	(a)	18.	(b)
4.	(a)	9.	(a)	14.	(c)		
5.	(b)	10.	(c)	15.	(b)		

Lecture–2: Work Sheet: To Check Preparation Level

1.	(d)	5.	(c)	9.	(a)	13.	(b)
2.	(d)	6.	(b)	10.	(b)		
3.	(d)	7.	(b)	11.	(b)		
4.	(b)	8.	(c)	12.	(a)		

Lecture–3: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(b)	8.	(d, a)	15.	(a)	22.	(c)
2.	(c)	9.	(d)	16.	(c)	23.	(a)
3.	(c)	10.	(c)	17.	(b)	24.	(d)
4.	(a)	11.	(a)	18.	(a)	25.	(c)
5.	(b)	12.	(b)	19.	(d)	26.	(b)
6.	(b)	13.	(b)	20.	(a)	27.	(c)
7.	(c)	14.	(b)	21.	(c)		

Lecture–3: Work Sheet: To Check Preparation Level

1.	(c)	5.	(a)	9.	(d)	13.	(b)
17.	(d)	2.	(a)	6.	(d)	10.	(a)
14.	(a)	3.	(a)	7.	(d)	11.	(c)
15.	(b)	4.	(d)	8.	(a)	12.	(a)
16.	(c)						

Lecture–4: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(d)	3.	(c)	5.	(b)	7.	(d)
2.	(d)	4.	(a)	6.	(c)	8.	(a)

Lecture–4: Work Sheet: To Check Preparation Level

1.	(b)	3.	(d)	5.	(a)
2.	(a)	4.	(b)		

Lecture–5: Mental Preparation Test

(5) $|z| = 2$

(6) $x^2 + y^2 - 6x + 8y - 24 = 0$

(7) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

(10) (i) $\pm[(a+b) - (a-b)i]$ (ii) $\pm(a+i)$

(12) $\frac{3}{2} + \frac{1}{2}i$

(14) $x = \frac{14}{15}, y = \frac{1}{5}$

(17) -160

(21) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(22) $\cos 30^\circ + i\sin 30^\circ$.

(23) radius = 5, centre = $(-3, -1)$.

QUESTION BANK: SOLVE THESE TO MASTER

1. (a)	2. (a)	3. (b)
4. (c)	5. (a)	6. (b)
7. (a)	8. (c)	9. (a)
10. (a)	11. (b)	12. (c)
13. (d)	14. (b)	15. (c)
16. (c)	17. (d)	18. (d)
19. (c)	20. (a)	21. (c)
22. (d)	23. (c)	