E1: Examples + Results/Details

Exercise sheet E1 in English is for practicing vector operations, transformation matrix, while E2 is covering all the other tensor operations including stress, strain and materials.

Note the **matlab** script that goes with this examples (not needed for elasticity, practicing, exam, but maybe useful for future use and reference).

Given are three vectors: $\underline{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\underline{d} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

1. Compute the lengths of these vectors
   a = 1
   b = 1.414213562373095=sqrt(2)
   d = 3
2. Compute the scalar products $\underline{a}\cdot\underline{b}$, $\underline{a}\cdot\underline{d}$, and $\underline{b}\cdot\underline{d}$
   and compute the angles between these vector pairs.
   a.b = 1
   a.d = 2
   d.b = 0
   θ_ab = 45 deg.
   θ_ad = 48.189 deg.
   θ_db = 90 deg.
3. Compute the outer (vector, x or *) products $\underline{a}*\underline{b}$, $\underline{a}*\underline{d}$, and $\underline{b}*\underline{d}$
   and from these compute the angles between these vector pairs.
   $\underline{a}\times\underline{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\underline{a}\times\underline{d} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, and $\underline{d}\times\underline{b} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$.
   Do the angles from 2. and 3. agree? *The angles do agree both ways (not shown).*
4. We define the new basis: $\underline{\hat{e}}'_1 = \underline{d}/|\underline{d}|$, $\underline{\hat{e}}'_2 = \underline{b}/|\underline{b}|$, and $\underline{\hat{e}}'_3 = \underline{\hat{e}}'_1 * \underline{\hat{e}}'_2$.
   Compute the orthogonal (transformation) rotation matrix $R_{pi} = \underline{\hat{e}}'_p \cdot \underline{\hat{e}}_i$ and confirm its orthogonality …
   $$\underline{\underline{R}} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix}$$ and $\underline{\underline{R}} \cdot \underline{\underline{R}}^T = \underline{\underline{1}}$

5. Compute $R_{pi}a_i = a'_p$, $R_{pi}b_i = b'_p$, $R_{pi}d_i = d'_p$, and $R_{pi}\underline{\hat{e}_{1|2|3}}'_i = ?$ (1|2|3 means all).
   The first is not trivial: but the other two are simpler – length unchanged:

$$\underline{a}' = \begin{bmatrix} 2/3 \\ 1/\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}, \quad \underline{b}' = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \text{ and } \quad \underline{d}' = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

NOTE for future use – the transformation behavior can be co- or contra-variant.
Basis vector behave/transform DIFFERENT from the vectors a, b, d.
Indices can in the general case be sub- or super-scripts, respectively. NOT HERE.

The following exercises use knowledge of the future lectures, but if you want to, you can try already. Otherwise, jump to the next lectures and exercises.

E2: Examples stress, strain and materials

Given is the stress tensor from Script(NL) Voorbeeld 1 pg.53: $\underline{\underline{\sigma}} = \begin{bmatrix} 10 & 0 & 20 \\ 0 & -10 & 10 \\ 20 & 10 & 0 \end{bmatrix}$.

6 Compute the eigenvalues and eigenvectors from the stress tensor. Build the rotation matrix from these eigenvectors (used as new basis) and the original basis – and rotate stress into this new system.

EV1	EV2	EV3
0.4320	0.4913	-0.7563
0.5932	-0.7864	-0.1720
-0.6793	-0.3744	-0.6312

EVals = -21.4510, -5.2398, 26.6908

R_pi =

0.4320	0.5932	-0.6793
0.4913	-0.7864	-0.3744
-0.7563	-0.1720	-0.6312

σ’ =

-21.4510	-0.0000	-0.0000
-0.0000	-5.2398	0.0000
-0.0000	0.0000	26.6908

with R_pi=matrix(EV)^T and σ’_pq = R_pi σ’_ij R_jq

7 Assuming E=10GPa and ν=0.3 compute the strain assuming isotropic Hooke

ε = 1.0e-8 *

0.1300	0	0.2600
0	-0.1300	0.1300
0.2600	0.1300	0

Vε =

0.4320	0.4913	-0.7563
0.5932	-0.7864	-0.1720
-0.6793	-0.3744	-0.6312

Dε = 1.0e-8 *

-0.2789	0	0
0	-0.0681	0
0	0	0.3470

8 Confirm that the strain eigenvectors are parallel to the stress eigenvectors

Stress:

EV1	EV2	EV3
0.4320	0.4913	-0.7563
0.5932	-0.7864	-0.1720
-0.6793	-0.3744	-0.6312

Vε =

0.4320	0.4913	-0.7563
0.5932	-0.7864	-0.1720
-0.6793	-0.3744	-0.6312

NOTE: only true for isotropic material behavior!

9a Compute isotropic (volumetric) contributions of stress and strain
… their eigenvalues and eigenvectors
sigmaV = epsilonV = 0*I (in this case – not always)

9b Compute the deviatoric contributions of stress and strain
… their eigenvalues and eigenvectors
sigmaD =

10	0	20
0	-10	10
20	10	0

epsilonD = 1.0e-8 *

0.1300	0	0.2600
0	-0.1300	0.1300
0.2600	0.1300	0

Eigenvectors and –values are the same as for the stress.

9c Compute the invariants of the tensors, volumetric tensors and deviatoric tensors

I1 = 0

I2 = -600
I3 = 3000
J1 = 0
J2 = -600
J3 = 3000

10 Compute the equivalent stresses according to Mises and Tresca.
Which of the two criteria is “safer”? Discuss.

tau_Mises = 24.4949
tau_Tresca = 48.1418

The larger eq.stress is safer since it reaches the limit stress earlier (for smaller strain).

ASSIGNMENT – TENSORS – Solutions

1.a	Give examples for tensors of rank 0, 1, and 2.	2
	R0: Scalars like density, temperature, (hydrostatic) pressure, … *R1: Vectors like displacement, velocity, force, …* *R2: stress tensor, strain tensor, …*	
1.b	How many independent components has a symmetric stress tensor in two/three dimensions ? – and how many eigenvalues ?	2
	2D: three indep. components -> 2 eigenvalues (and one orientation angle) *3D: six indep. components -> 3 eigenvalues (and 3 orientation angles -> 3 unit normals)*	
1.c	Under which conditions does one find no shear stress?	2
	Shear stresses (shear strains) disappear when the stress is rotated into its eigen-system, i.e., the new coordinates are identical with the normal eigenvectors of this stress (or strain) tensor. The stress tensor then has diagonal form with eigenvalues on the diag.	
1.d	What physical quantity is described by the trace of the stress tensor ? What is the trace of the deformation tensor ? What is the meaning of the deviator ? What is the meaning of symmetric and anti-symmetric tensors – give examples.	3
	The trace of stress is three times the hydrostatic pressure. *The trace of the deformation tensor (for small strains) gives the volume change.* *Non-symmetric matrices are for example the rotation matrices R_ip, which are composed of a symmetric and a anti-symmetric (=fully non-symmetric) part. Anti-symmtric tensors are e.g. the e_ijk which is used for the outer cross-product of vectors. One anti-symmetric tensors is a special case of rotation matrices for phi=90.* *Note that eigenvalue computation of non-symmetric tensors gives complex results.*	
2.	Given are the tensors (two-dimensional – plane-stress): $\underline{\underline{\sigma}} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 1.3 \end{pmatrix} p$ and $\underline{\underline{\sigma}} = \begin{pmatrix} 1.0 & 0.6 \\ 0.6 & 1.0 \end{pmatrix} p$ Determine (mathematically and graphically) their eigenvalues, the traces, the deviators and the orientations of their major eigen-values with respect to the horizontal. Determine also the pressure and the maximum possible shear stress. Discuss the results – similarities and differences.	5
	The purpose of this exercise is to work with special cases which allow for simple solutions – if one knows what properties to look for. The numerical solutions are given below. (Note that these 2D tensors are made 3D by adding a 1 on the 3-diagonal and zero elsewhere.) *The trace is then in both cases 3*p, where the p was a factor outside of the tensor. Tr(sigma)/3 is then the hydrostatic pressure p – simple.* *The deviators are obtained by subtracting (tr(σ)/3) I_ij from σ, with unit tensor I. Note that the trace of the deviator is always zero – it thus contains no hydrostatic (isotropic) part.* *Note that it is advantageous for many calculations to keep the symbols and insert explicit numbers only as late as possible. Practice this!*	

S1	*Note that the eigenvectors below are NOT normalized (left) and normalized (right – decimal). The numbers look ugly, but one can obtain the orientation of the eigenvector corresponding to the largest eigenvalue by computing e.g. arctan(n1_y/n1_x)=3π/8 (=45/2 degrees) for the angle between the horizontal and the first eigenvector n1.* *The eigenvector n2 is 90 degrees from the horizontal and the eigenvector n3 is rotated -π/8, i.e. perpendicular to n1. Actually when one has a tensor like the deviator below, the orientation must be 45/2 degrees (plus or minus).* *The maximal shear stress is half the difference of the largest and smallest eigenvalue so that* τ *< 3/10*sqrt(2)=0.4243.*	

Symbolic

```
sigma =
[  7/10,  3/10,     0]
[  3/10, 13/10,     0]
[     0,     0,     1]
V =
[          1,          0,          1]
[ 1+2^(1/2),           0, 1-2^(1/2)]
[          0,          1,          0]
D =
[ 1+3/10*2^(1/2),   0,               0]
[               0,  1,               0]
[               0,  0, 1-3/10*2^(1/2)]

sigma_D =
[ -3/10,  3/10,     0]
[  3/10,  3/10,     0]
[     0,     0,     0]
V_D =
[          1,          0,          1]
[ 1+2^(1/2),           0, 1-2^(1/2)]
[          0,          1,          0]
D_D =
[  3/10*2^(1/2),   0,               0]
[            0,    0,               0]
[            0,    0,  -3/10*2^(1/2)]
```

Decimal

```
eigenvecs =
    0.3827         0    0.9239
    0.9239         0   -0.3827
         0    1.0000         0
eigenvals =
    1.4243         0         0
         0    1.0000         0
         0         0    0.5757

eigenvecs_D =
    0.3827         0    0.9239
    0.9239         0   -0.3827
         0    1.0000         0
eigenvals_D =
    0.4243         0         0
         0         0         0
         0         0   -0.4243
```

S2

Note that the eigenvectors below are NOT normalized (left) and normalized (right – decimal). The numbers look ugly, but one can obtain the orientation of the eigenvector corresponding to the largest eigenvalue by computing e.g. arctan(n1_y/n1_x)=$\pi/4$ (=45 degrees) for the angle between the horizontal and the first eigenvector n1.

The eigenvector n2 is 90 degrees from the horizontal and the eigenvector n3 is rotated -$\pi/4$, i.e. perpendicular to n1. Actually when one has a tensor like the deviator below, the orientation must be 45 degrees (plus or minus).

The maximal shear stress is half the difference of the largest and smallest eigenvalue so that $\tau < 6/10=0.6$.

Symbolic

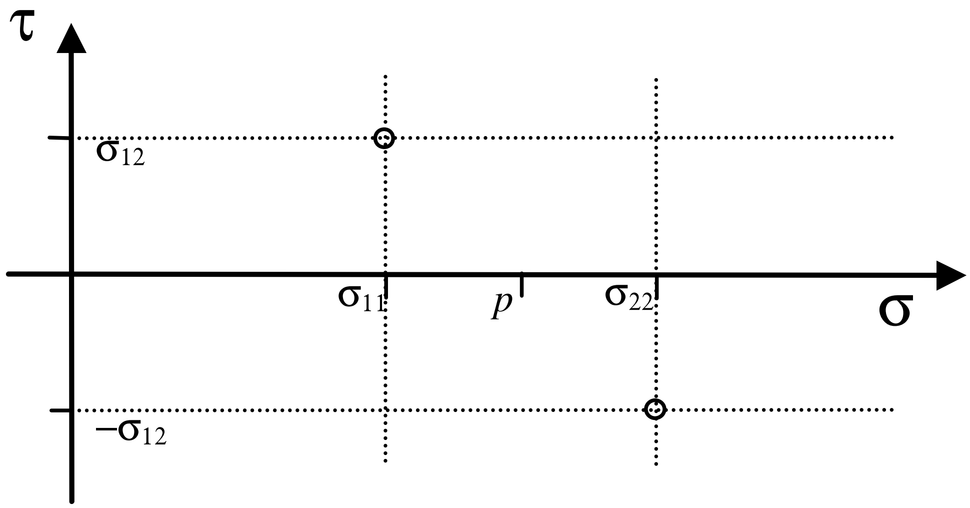
```
sigma =
[    1, 6/10,    0]
[ 6/10,    1,    0]
[    0,    0,    1]
V =
[  1/2*2^(1/2),     0, -1/2*2^(1/2)]
[  1/2*2^(1/2),     0,  1/2*2^(1/2)]
[        0,         1,            0]
D =
[ 8/5,   0,   0]
[   0,   1,   0]
[   0,   0, 2/5]
sigma_D =
[   0, 3/5,   0]
[ 3/5,   0,   0]
[   0,   0,   0]
V_D =
[  1/2*2^(1/2),     0, -1/2*2^(1/2)]
[  1/2*2^(1/2),     0,  1/2*2^(1/2)]
[        0,         1,            0]
D_D =
[  3/5,    0,    0]
[    0,    0,    0]
[    0,    0, -3/5]
```

Decimal

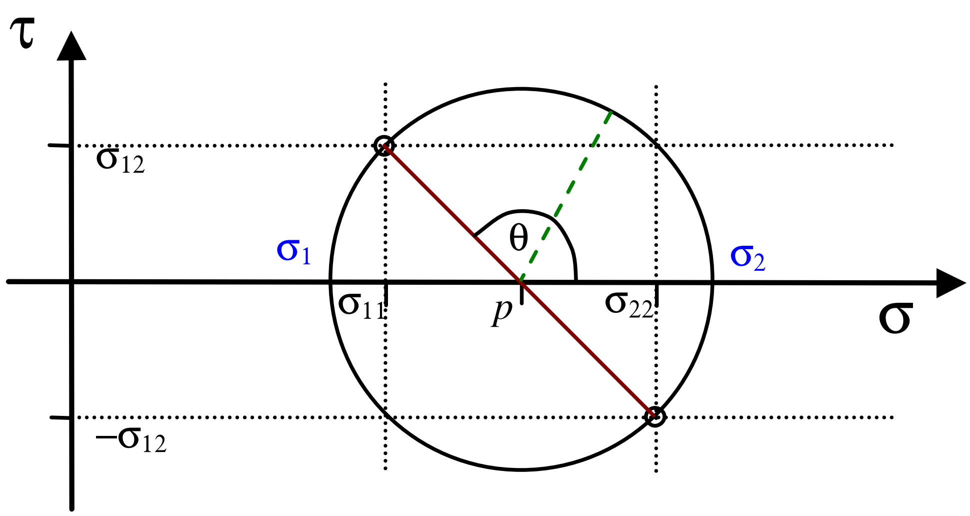
```
eigenvals =
    1.6000        0         0
         0   1.0000         0
         0        0    0.4000
eigenvecs =
    0.7071        0   -0.7071
    0.7071        0    0.7071
         0   1.0000         0
eigenvals_D =
    0.6000        0         0
         0        0         0
         0        0   -0.6000
eigenvecs_D =
    0.7071        0   -0.7071
    0.7071        0    0.7071
         0   1.0000         0
```

S1:

For the graphic solution plot the elements, on the normal- and shear stress axes. The pressure p is in between. Then identify the pairs of shear and normal stresses (matter of definition and convention). Here, the 11-stress-component is paired with the positive 12-component.

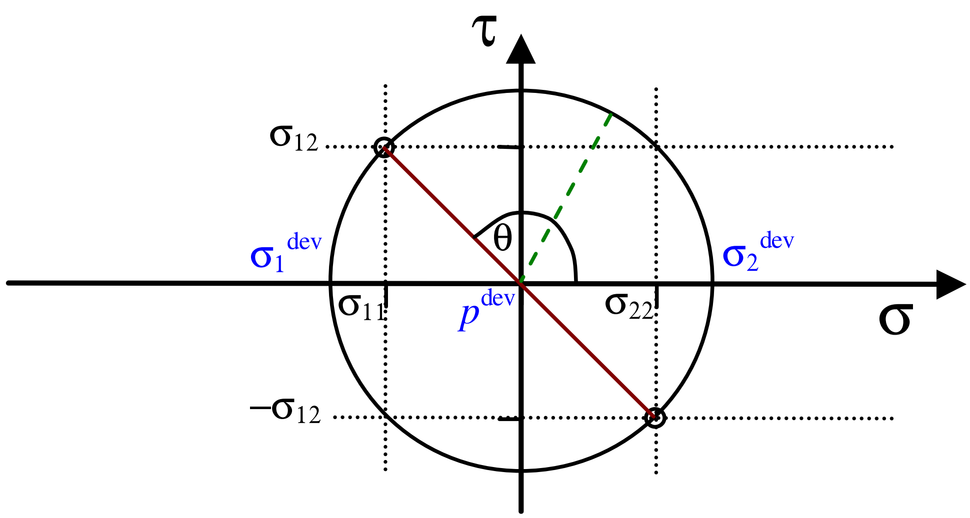


Then draw the circle through the center p and the two points (one is enough). The eigenvalues 1 and 2 (before sorting, 3 after sorting) are just the crossing points between circle and σ-axis.



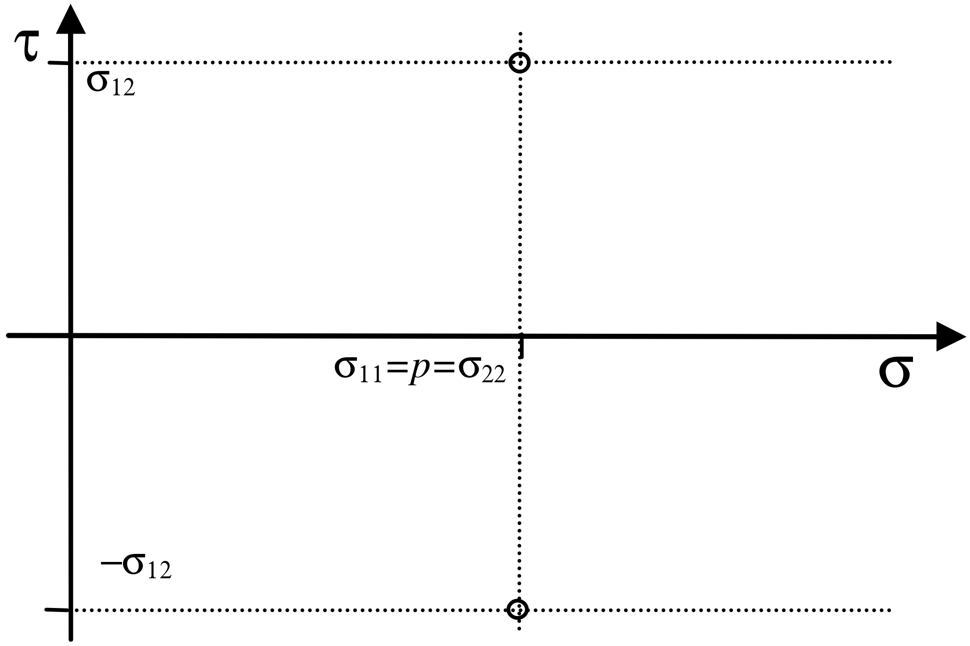
The angle θ between the σ-axis and the 11-12 point (top-left) gives $\theta=2\phi=6\pi/8$, so that the angle between the first eigenvector and the horizontal is $\phi=3\pi/8$ (indicated by the dashed line).

When the deviator is considered, the vertical axis is shifted to the center of the circle. Nothing else changes concerning the circle and the graphics. The pressure p^{dev}=0 and the eigenvalues are opposite sign and equal in magnitude. The maximal shear stress corresponds to the radius of the circle in both cases (original and deviatoric stress).



S2:

For the graphic solution plot the elements, on the normal- and shear stress axes. The pressure p is in between, i.e. here identical to the diagonal elements. Then identify the pairs of shear and normal stresses (matter of definition and convention). Here, the 11-stress-component is paired with the positive 12-component.



Then draw the circle through the center p and the two points (one is enough).

The angle θ between the σ-axis and the 11-12 point (top) gives $\theta=2\phi=\pi/2$, so that the angle between the first eigenvector and the horizontal is $\phi=\pi/4$ (indicated by the dashed line).