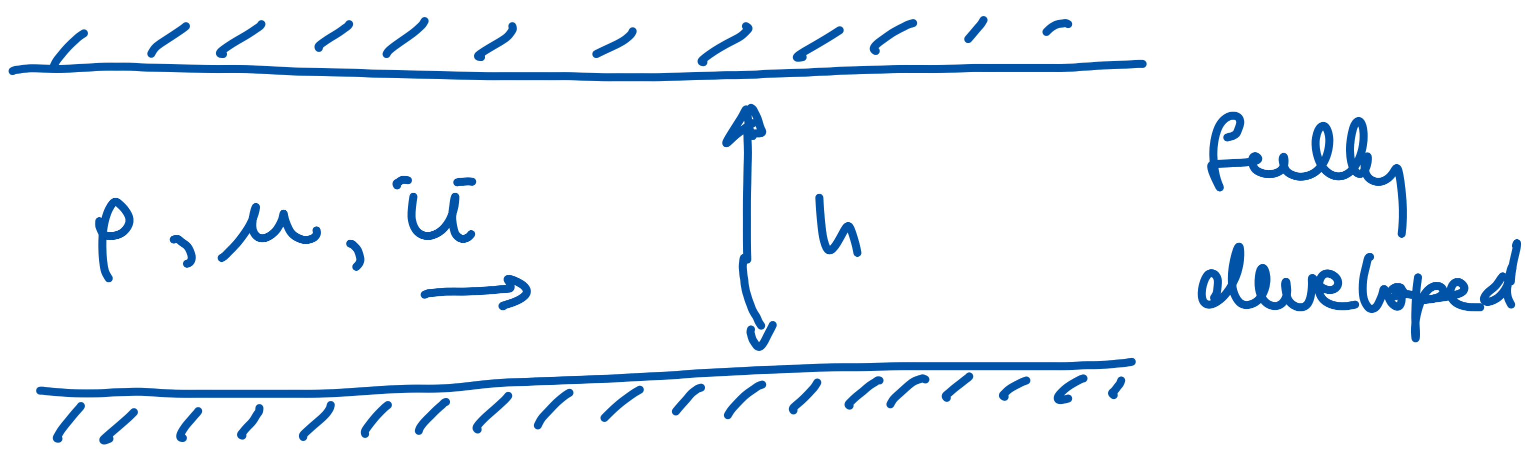
Fluid Mechanics 1

Lecture # 6: The Reynolds Number

(Dimension Analysis).

Fully Developed Flow (2D)



Question: $\frac{\partial p}{\partial x} = ?$

Assume $\frac{\partial p}{\partial x} = \text{function}(\rho, \mu, \bar{u}, h)$

$\frac{Pa}{m}$ $\frac{kg}{m^3}$ $Pa\,s$ $\frac{m}{s}$ m

$$Pa \equiv \frac{N}{m^2} \equiv kg \frac{m}{s^2} \frac{1}{m^2}$$

3

$\Rightarrow$ Physical dimensions: (kg, m, s)

$\longrightarrow$ mass, length, time

independent! You can not add or subtract!

To make the problem non-dimensional we need 3 independent parameters.

Try: $\rho, h, \bar{u}$.

are they independent?

independent means:

$$\underbrace{\left[\rho^{\alpha} h^{\beta} \bar{u}^{\gamma}\right]}_{\text{check:}} = [1] \Longleftrightarrow \alpha = \beta = \gamma = 0 \; ?$$

check: $$[\ ] = \left(\frac{kg}{m^3}\right)^{\alpha} m^{\beta} \left(\frac{m}{s}\right)^{\gamma}$$

$$= kg^{\alpha}\, m^{-3\alpha+\beta+\gamma}\, s^{-\gamma} = 1$$

$$\Rightarrow \left.\begin{aligned} \alpha &= 0 \\ -3\alpha + \beta + \gamma &= 0 \\ -\gamma &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= 0 \\ \gamma &= 0 \\ \beta &= 0 \end{aligned}$$

$\Rightarrow$ independent.

Scaling of μ: $[\mu] = [\rho^{\alpha} h^{\beta} \bar{u}^{\gamma}]$

$$[\mu] = \text{Pa s} = \text{kg} \frac{\text{m}}{\text{s}^2} \frac{1}{\text{m}^2} \cdot \text{s} = \frac{\text{kg}}{\text{m s}}$$

$$\Rightarrow \frac{\text{kg}}{\text{m s}} = \text{kg}^{\alpha}\, \text{m}^{-3\alpha + \beta + \gamma}\, \text{s}^{-\gamma}$$

$$\left.\begin{array}{lrl} \text{kg}: & 1 & = \alpha \\ \text{m}: & -1 & = -3\alpha + \beta + \gamma \\ \text{s}: & -1 & = -\gamma \end{array}\right\} \begin{array}{l} \alpha = 1 \\ \gamma = 1 \\ \beta = 1 \end{array}$$

$$\Rightarrow \tilde{\mu} \equiv \frac{\mu}{\rho h \bar{u}} \equiv \frac{1}{Re}$$

$$\Leftrightarrow \boxed{Re \equiv \frac{\rho \bar{u} h}{\mu}}$$

$$\left[\frac{\partial p}{\partial x}\right] = [\rho^{\alpha} h^{\beta} \bar{u}^{\gamma}]$$

$$\left[\frac{\partial p}{\partial x}\right] = \frac{\text{Pa}}{\text{m}} = \text{kg} \frac{\text{m}}{\text{s}^2} \frac{1}{\text{m}^2} \frac{1}{\text{m}} = \frac{\text{kg}}{\text{m}^2 \text{s}^2}$$

$$\left.\begin{aligned} kg: &\quad 1 = \alpha \\ m: &\quad -2 = -3\alpha + \beta + \gamma \\ s: &\quad -2 = -\gamma \end{aligned}\right\} \quad \begin{aligned} \alpha &= 1 \\ \gamma &= 2 \\ \beta &= -1 \end{aligned}$$

$$\Rightarrow \quad \widetilde{\frac{\partial p}{\partial x}} = \frac{\frac{\partial p}{\partial x}}{\rho\, h^{-1}\, \bar{u}^2} = \frac{\frac{\partial p}{\partial x} \cdot h}{\rho\, \bar{u}^2}$$

check dimension: (do it!)

$$\frac{\frac{Pa}{m} \cdot m}{\frac{kg}{m^3}\, \frac{m^2}{s^2}} = 1 \quad \checkmark$$

<u>How to use this?</u>

$$\Rightarrow \quad \frac{\partial p}{\partial x} = \widetilde{\frac{\partial p}{\partial x}} \cdot \frac{\rho \bar{u}^2}{h}.$$

(non-dimensional function of $\rho, \bar{u}, h, \mu$

Buckingham : $\boxed{\text{a dimensionless function only depends on dimensionless parameters.}}$

Can we make non-dimensional group(s) out of $\rho, \overline{u}, h, \mu$?

we know already: 1 possibility is the Reynolds number

$$[\rho^\alpha h^\beta \overline{u}^\gamma \mu^\delta] = 1 \quad \text{(non-dimensional)}$$

$$\Rightarrow \alpha, \beta, \gamma, \delta \, ?$$

$$\left(\frac{kg}{m^3}\right)^\alpha m^\beta \left(\frac{m}{s}\right)^\gamma \left(\frac{kg}{m\,s}\right)^\delta = 1$$

$$\left.\begin{aligned} kg &: \quad \alpha + \delta = 0 \\ m &: \quad -3\alpha + \beta + \gamma - \delta = 0 \\ s &: \quad -\gamma - \delta = 0 \end{aligned}\right\} \begin{aligned} &4 \text{ unknowns} \\ &3 \text{ equations.} \end{aligned}$$

assume δ is known:

$\Rightarrow \alpha = -\delta, \; \gamma = -\delta, \; \beta = -\delta$

$$\Rightarrow \left(\frac{\mu}{\rho \bar{U} h}\right)^{\delta} \equiv Re^{-\delta}$$

any function of $Re^{-\delta}$ can be written as another function of Re.

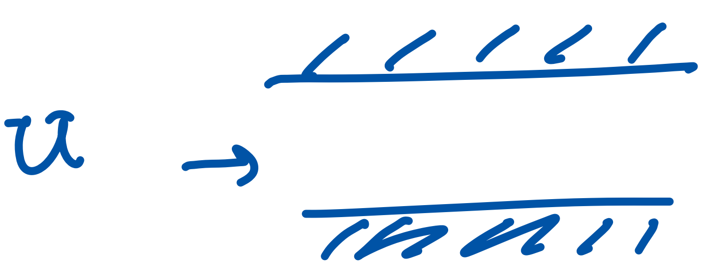
$\Rightarrow$ δ is not important.

$$\Rightarrow \boxed{\frac{\partial p}{\partial x} = \underbrace{\tilde{f}(Re)}_{\equiv \frac{\partial \tilde{p}}{\partial x}} \frac{\rho \bar{U}^2}{h}.}$$

$\Rightarrow$ Remaing problem: $\tilde{f}(Re) = ?$

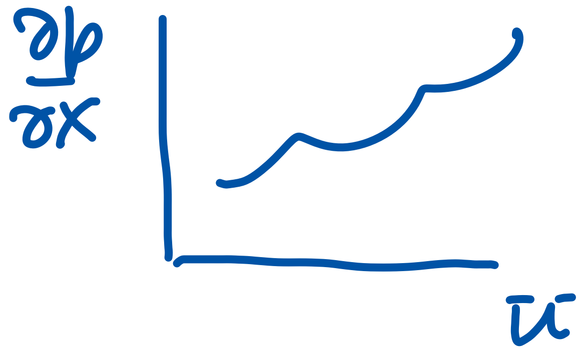
Answer: do experiments.

Build a gap, use water: ρ, μ, h are fixed constants



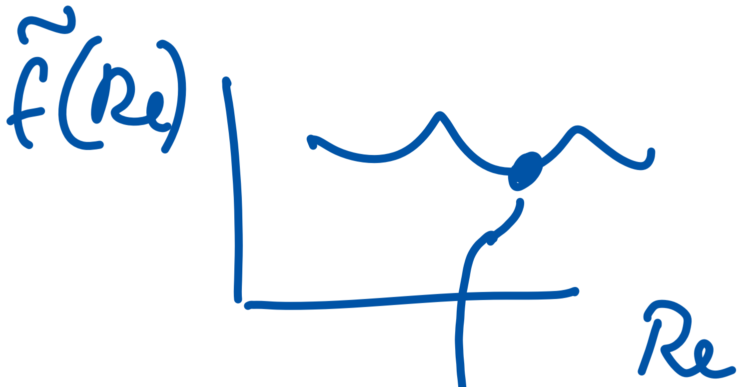
vary $\bar{U}$

Measure $\frac{\partial p}{\partial x} \equiv \frac{p_{out} - p_{in}}{L}$



$$Re \equiv \frac{\rho \bar{u} h}{\mu}$$

$$\tilde{f}(Re) \equiv \frac{\partial p}{\partial x} h / \rho \bar{u}^2$$



Usage in another flow problem: $\rho_1, \mu_1, h_1, \bar{u}_1$

$$\frac{\partial p}{\partial x} = \tilde{f}\left(\frac{\rho_1 \bar{u}_1 h_1}{\mu_1}\right) \rho_1 \bar{u}_1^2 / h_1$$

<u>Note</u>: instead of having to do $\sim 10^4$ experiments with $\sim 10^2$ different fluids

you only need to 1 experiment with water at ~ 10 different values of u

$\Rightarrow$ $\boxed{\text{Dimension analysis helps! Extremely powerful.}}$

In fluid mechanics it is convention to write

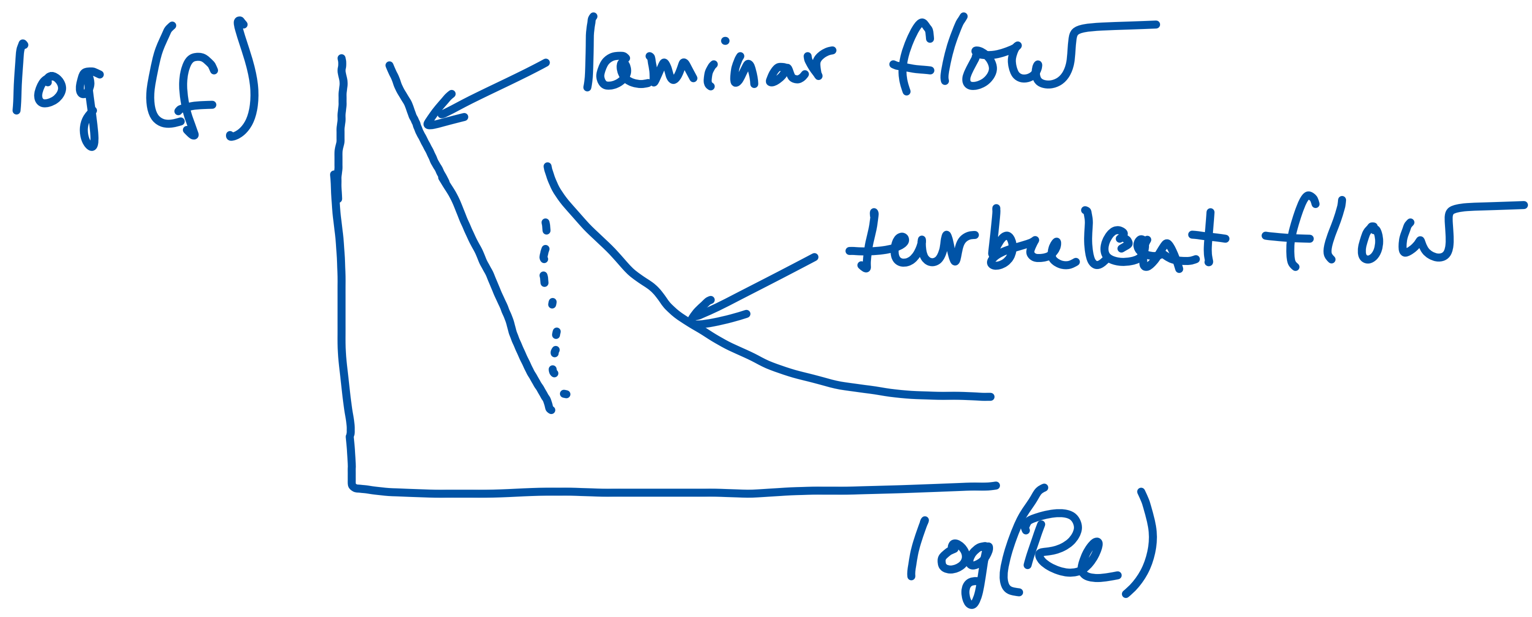
$$\frac{\partial p}{\partial x} = -f(Re)\,\frac{1}{2}\rho u^2/h$$

convenient convention.

$$\boxed{f(Re) \equiv -2\tilde{f}(Re)}$$

Darcy-Weisbach friction factor.

It appears (from the 1 experiment) that



see previous lecture.

laminar flow: $\frac{\partial p}{\partial x} = -12\,\mu \bar{u}/h^2$

theory

dimension analysis learns that

$$\frac{\partial p}{\partial x} = \underbrace{-f(Re)\cdot \tfrac{1}{2}\rho \bar{u}^2/h}_{\text{dim. analysis + experiment}}$$

$$\Rightarrow \quad -12\,\frac{\mu \bar{u}}{h^2} = -f(Re)\,\tfrac{1}{2}\,\frac{\rho \bar{u}^2}{h}$$

$\Rightarrow$ compute $f(Re)$

$$\Rightarrow \quad f(Re) = 12\,\frac{\mu \bar{u}}{h^2}\cdot 2\,\frac{h}{\rho \bar{u}^2} = 24\,\frac{\mu}{\rho \bar{u} h}$$

$$= \frac{24}{Re}$$

confirms that f depends on Re, not on $\rho, \bar{u}, h, \mu$ separately!

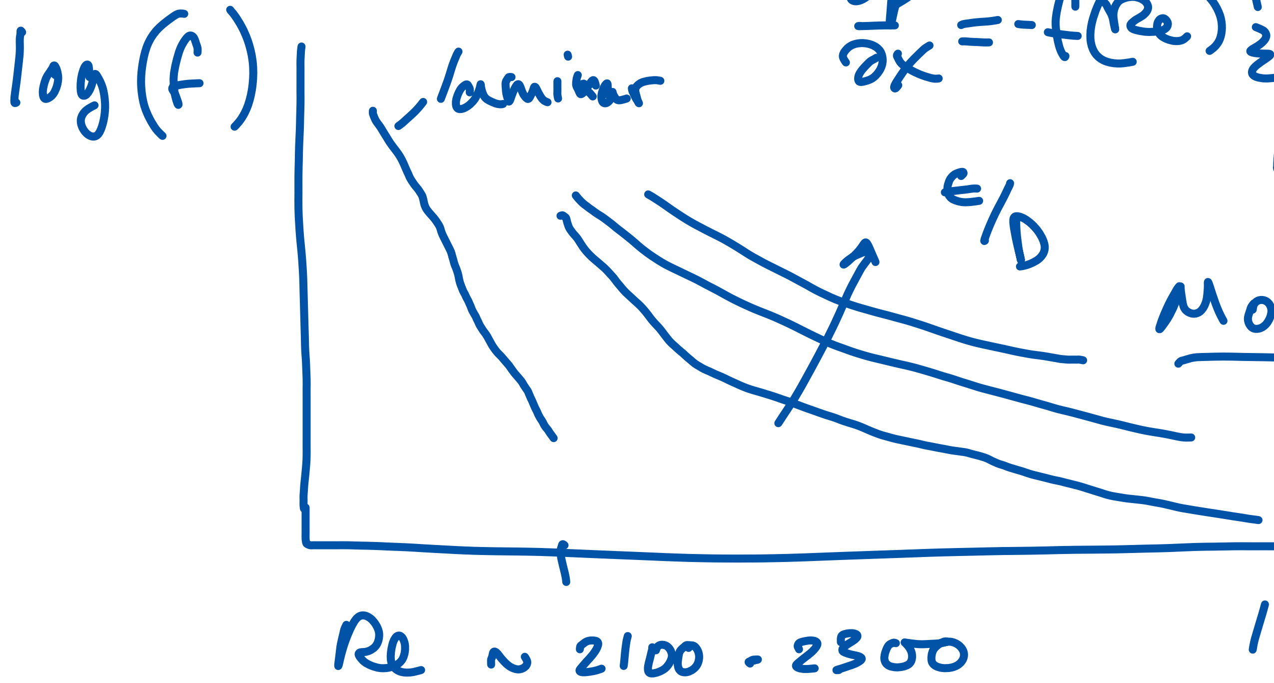
Buckingham was right!

Finally: Tube, diameter D

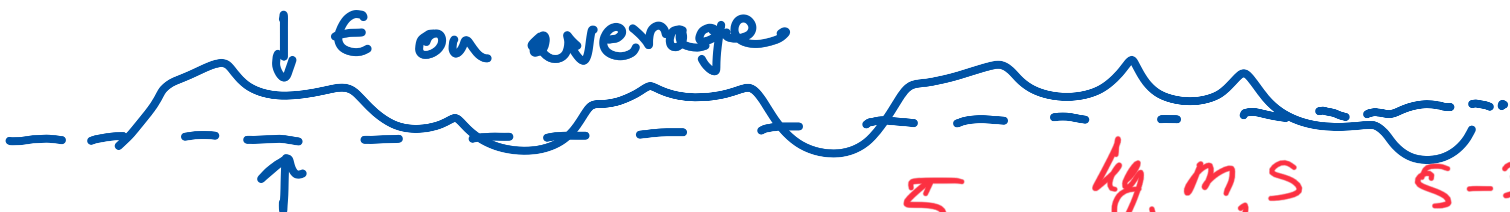
$$\frac{\partial p}{\partial x} = -f(Re)\,\tfrac{1}{2}\rho\bar{u}^2/D$$

$$Re \equiv \frac{\rho\bar{u}D}{\mu}$$

Moody chart



surface roughness: ϵ (length).



5 - kg, m, s 5-3 = 2

$$\Rightarrow \frac{\partial p}{\partial x} = F(\rho, \bar{u}, D, \mu, \epsilon). \qquad m, kg, s$$

$\Rightarrow$ two non-dimensional parameters

$$\Rightarrow \boxed{\frac{\partial p}{\partial x} = -f\left(Re, \frac{\epsilon}{D}\right)\tfrac{1}{2}\rho\bar{u}^2/D}$$