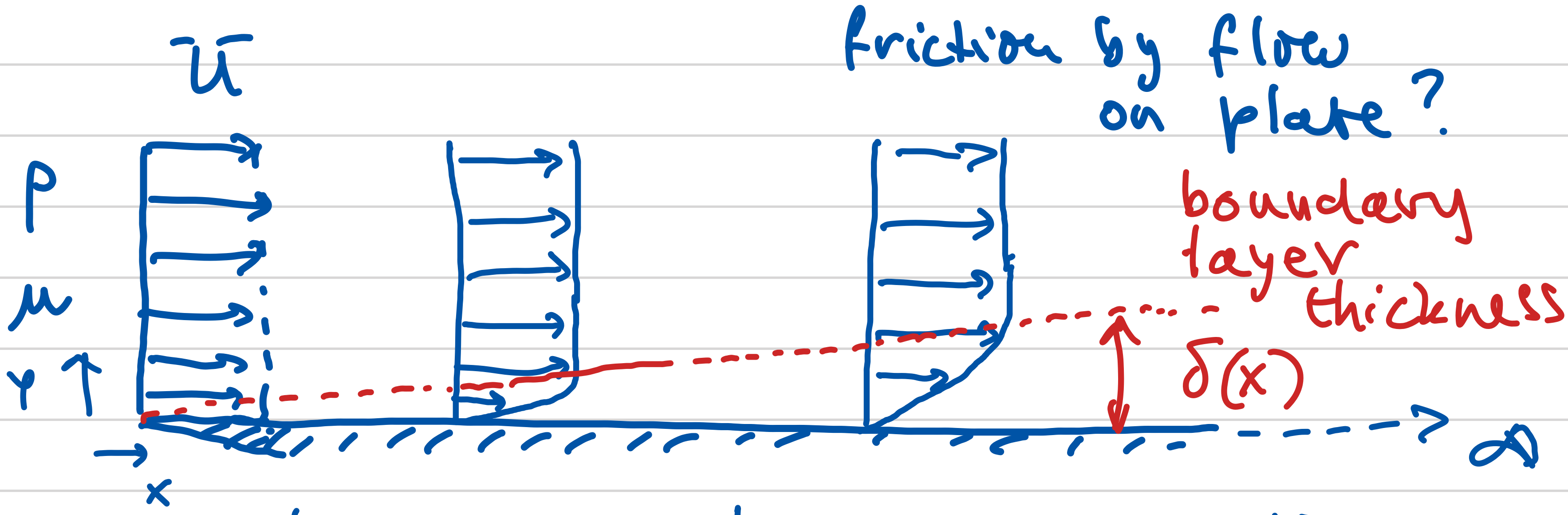
Fluid Mechanics 1

Lect #7 Flat Plate Boundary layer



at $y=0$: $u=0$! no-slip condition.

Question : $\delta(x)$? $\longrightarrow$ $\tau(x)$?

Assumption : incompressible $\rho = \text{const.}$
gravity neglected : "$g=0$"
2D
steady
μ = constant.

Mass conservation.

$$\cancel{\frac{\partial \rho}{\partial t}}_{\text{steady}} + \frac{\partial}{\partial x_j}(\overbrace{\rho}^{\rho = \text{const}} u_j) = 0$$

$$\rho \frac{\partial u_j}{\partial x_j} = 0 \qquad \rho \neq 0 \Rightarrow \frac{\partial u_j}{\partial x_j} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{=0 \text{ 2D}} = 0 \qquad \Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

Momentum conservation:

$$\cancel{\frac{\partial \rho u_i}{\partial t}} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j - \sigma_{ji}\right) = \cancelto{0}{g}$$

$$\Rightarrow \rho \frac{\partial}{\partial x_j}\left(u_i u_j\right) - \frac{\partial}{\partial x_j}\left\{-p\,\delta_{ji} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + 0\right\} = 0$$

$$\frac{\partial}{\partial x_j}\left(u_i u_j\right) = \frac{\partial u_i}{\partial x_j} u_j + u_i \cancelto{\rho = \text{const}}{\frac{\partial u_j}{\partial x_j}}$$

$$\frac{\partial}{\partial x_j}\left(p\,\delta_{ij}\right) = \frac{\partial p}{\partial x_i}$$

$$\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \leftarrow \text{sum!}$$

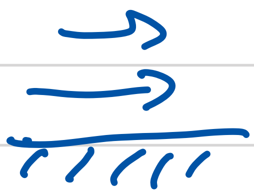
$$= \mu\left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_j}{\partial x_i}\right) = \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) = \mu \frac{\partial}{\partial x_i}\left(\cancelto{=0}{\frac{\partial u_j}{\partial x_j}}\right)$$

mass: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$i=1$: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

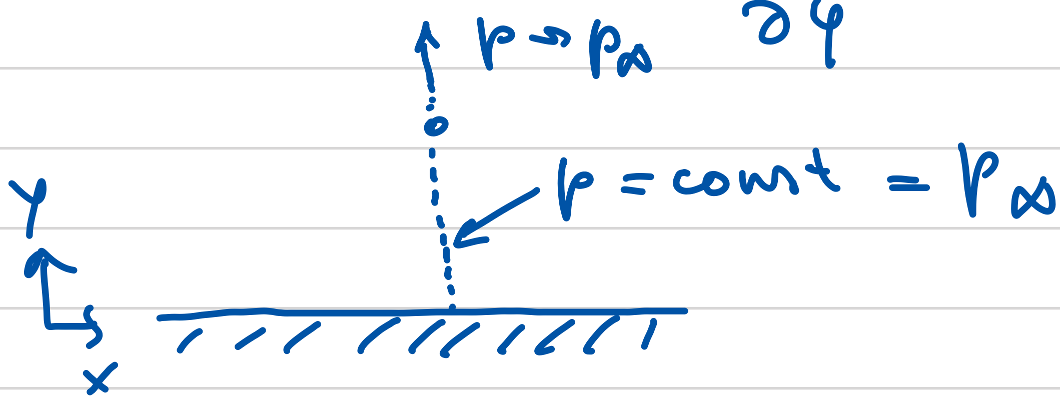
$i=2$: $$0 \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0$$



streamlines ~ straight.

3 eq's, 3 unknowns: $u(x,y), v(x,y), p(x,y)$.

Let's start with $\frac{\partial p}{\partial y} = 0$



p is uniform:

$$\boxed{p(x,y) = p_\infty}$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0$$

Final assumption by Prandtl $$\left|\frac{\partial^2 u}{\partial x^2}\right| \ll \left|\frac{\partial^2 u}{\partial y^2}\right|$$

Final step: make non-dimensional.

parameter: ρ, μ, U dimensions: kg, m, s

$\Rightarrow$ zero non-dimensional number.

$$\tilde{x} \equiv \frac{\rho \bar{U} x}{\mu} \qquad \tilde{y} \equiv \frac{\rho \bar{U} y}{\mu} \qquad \text{independent var.}$$

$$\tilde{u} \equiv u/\bar{U} \qquad \tilde{v} \equiv v/\bar{U} \qquad \text{dependent var.}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial v}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial y} = 0$$

$$\Rightarrow \frac{\partial \tilde{u} \bar{U}}{\partial \tilde{x}} \underbrace{\frac{\partial \tilde{x}}{\partial x}}_{\frac{\rho \bar{U}}{\mu}} + \cdots = 0$$

$$\Rightarrow \frac{\rho \bar{U}^2}{\mu} \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) = 0$$

$$\Rightarrow \boxed{\begin{array}{ccc} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 & & \tilde{u} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \dfrac{\partial \tilde{u}}{\partial \tilde{y}} = \dfrac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \\ \tilde{u}(\tilde{x}, 0) = 0 & \tilde{v}(\tilde{x}, 0) = 0 & \tilde{u}(\tilde{x}, \infty) = 1 \end{array}}$$

Boundary layer Equations

Prandtl 1904

Compare: 1903 Wright brothers
1st motorized flight

1905 Einstein:
special relativity.

1^{st} step: define $\tilde{\psi}(\tilde{x},\tilde{y})$ ("psi") such that:

$$\tilde{u} = \frac{\partial \tilde{\psi}}{\partial \tilde{y}} \qquad -\tilde{v} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}$$

$$\Rightarrow \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = \frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right) + \frac{\partial}{\partial \tilde{y}}\left(-\frac{\partial \tilde{\psi}}{\partial \tilde{x}}\right)$$

$$= \frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}\partial \tilde{y}} - \frac{\partial \tilde{\psi}}{\partial \tilde{y}\partial \tilde{x}} = 0$$

mass conservation automatically satisfied.

instead of unknowns $\tilde{u}$ and $\tilde{v}$ we only have unknown $\tilde{\psi}$

stream function.

<u>momentum eq:</u>

$$\boxed{\tilde{\psi}_{\tilde{y}}\,\tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}}\,\tilde{\psi}_{\tilde{y}\tilde{y}} = \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}}$$

$$\tilde{\psi}_{\tilde{x}} \equiv \frac{\partial \tilde{\psi}}{\partial \tilde{x}}$$

Partial Diff. Eq

3rd order, <u>non-linear</u>

Blasius (PhD-student of Prandtl).

<u>try</u> $\tilde{\psi}(\tilde{x},\tilde{y}) = \sqrt{\tilde{x}}\, f\left(\frac{\tilde{y}}{\sqrt{\tilde{x}}}\right)$

$$\frac{\tilde{y}}{\sqrt{\tilde{x}}} \equiv \eta$$

$\Rightarrow$ momentum equation becomes:

$$\boxed{f f'' + 2 f''' = 0} \qquad f(\eta) \qquad \eta = \frac{\tilde{y}}{\sqrt{\tilde{x}}}$$

Ordinary Diff. Eq
non-linear 3rd order

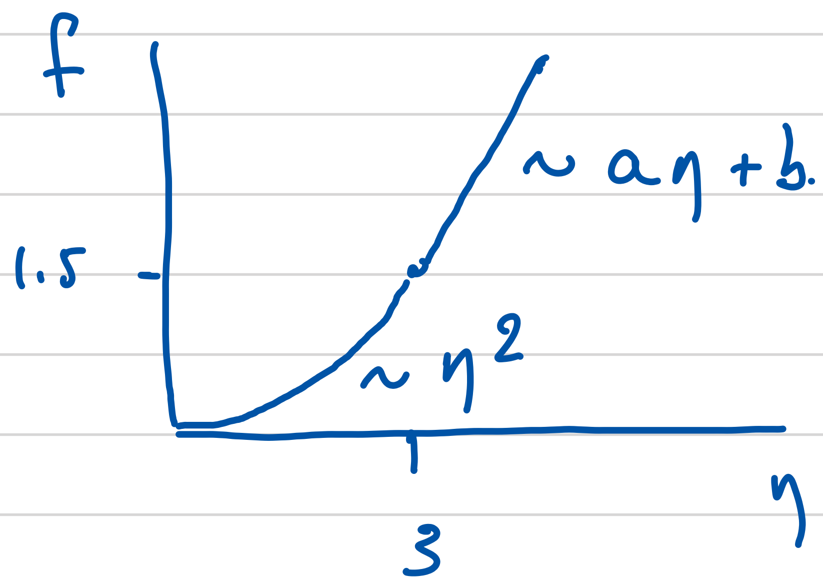
$$f'(\eta) \equiv \frac{df}{d\eta} \text{ etc.}$$

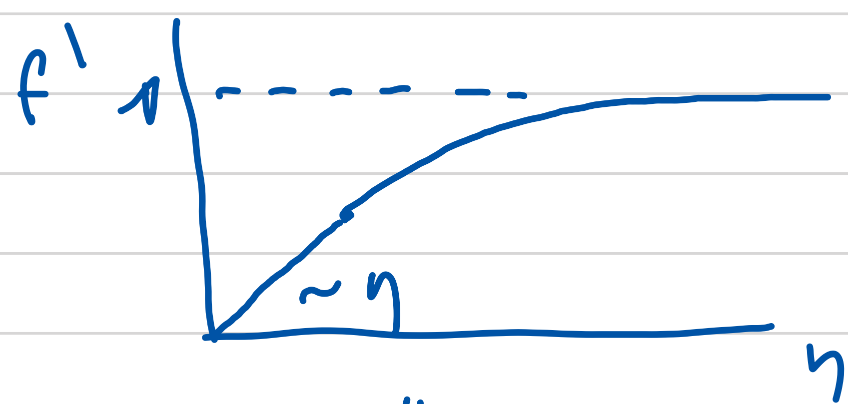
Blasius Equation

Solution: by computer.

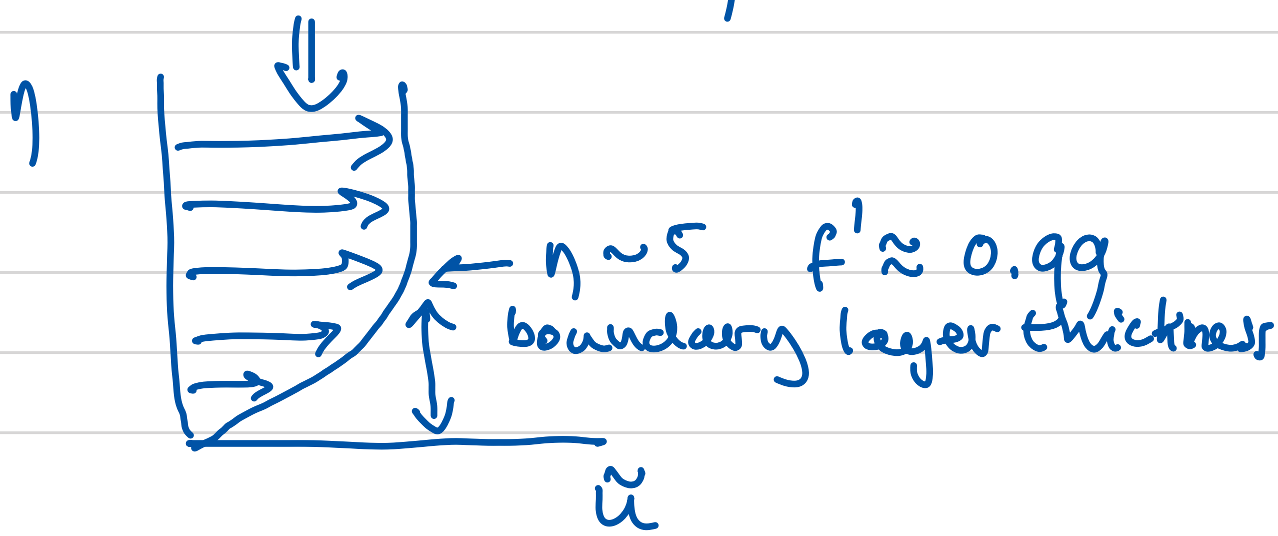
3 b.c.'s: $f'(0) = 0 \qquad f(0) = 0 \qquad f'(\infty) = 1$

Solution:



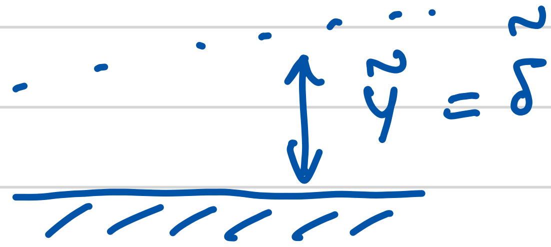


note: $\tilde{u} = f'$

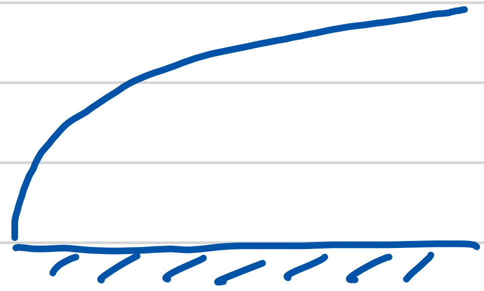


$$\eta = 5 \Rightarrow \frac{\tilde{y}}{\sqrt{\tilde{x}}} = 5$$

$$\Rightarrow \tilde{y} = 5 \cdot \sqrt{\tilde{x}}$$



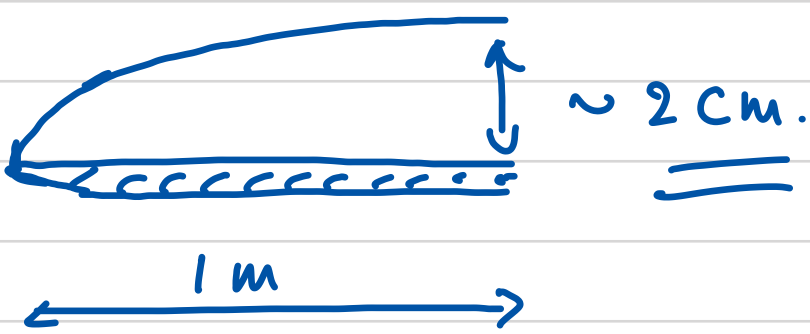
$$\Rightarrow \tilde{\delta}(\tilde{x}) = 5 \cdot \sqrt{\tilde{x}}$$



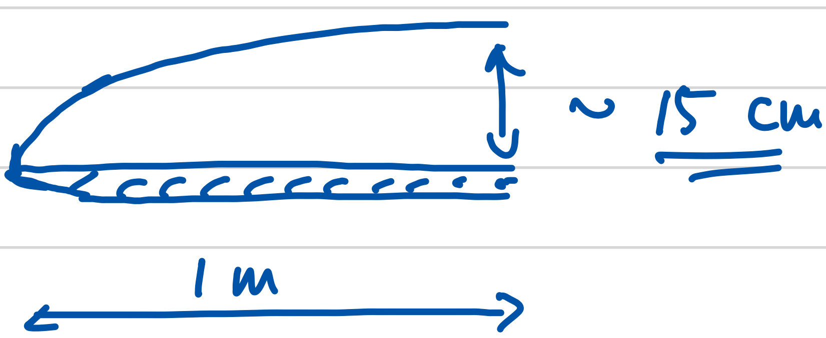
$$\Rightarrow \boxed{\delta(x) = 5 \cdot \sqrt{\frac{x\nu}{U}}}$$

$\nu = \mu / \rho$ kinematic viscosity

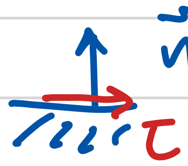
Example 1: air, $U = 1\ m/s$, $\nu = 18 \cdot 10^{-6}\ m^2/s$



Example 2: water, $U = 1\ m/s$, $\nu = 9 \cdot 10^{-4}\ m^2/s$



Finally: shear stress on the wall.



$\vec{n} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \tau = \mu \dfrac{\partial u}{\partial y}$

by fluid on wall

Blasius solution gives $f''(0) = 0.332$

$$\Rightarrow \boxed{\tau_w = \frac{0.332}{\sqrt{\frac{\bar{U} x}{\nu}}} \cdot \frac{1}{2} \rho \bar{U}^2}$$

Friction coefficient:

$$\boxed{c_f \equiv \frac{\tau_w}{\frac{1}{2}\rho \bar{U}^2} = \frac{0.332}{\sqrt{\frac{\bar{U} x}{\nu}}}} \sim \frac{1}{\sqrt{x}}$$

local infinite shear stress but $\int_0^L \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_0^L$ = finite.