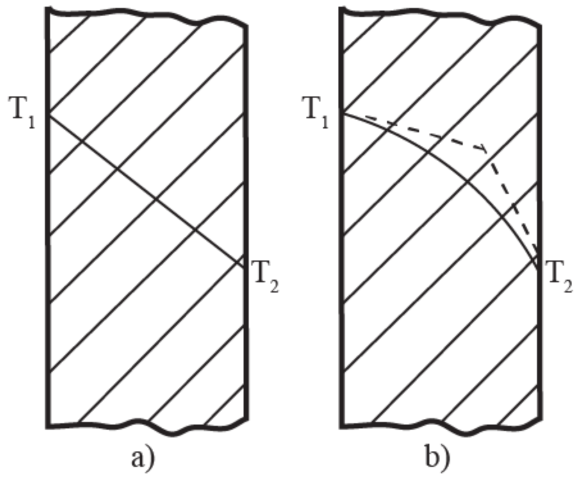
1.1 Temperature profiles in planar walls ⋆

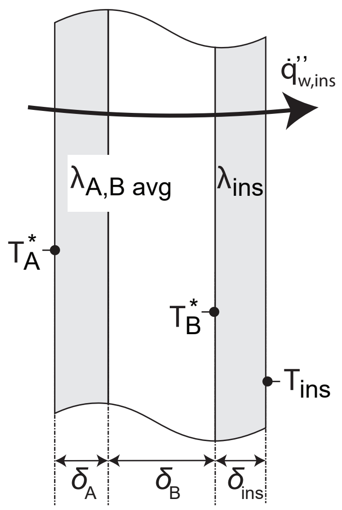


1.2 Onion principle ★★

a) *Determine the difference in transmitted heat flux per unit area $\dot{q}''$ with and without the insulating layer; assume steady-state conditions.*

System with insulation:

1) Setting up an energy balance/sketching the system:



The warmd by us of Fourier's law:

$$\boxed{\dot{q}'' = -\lambda \frac{\partial T}{\partial x}} \tag{1.1}$$

2) Defining the fluxes:

Fourier's law:

$$\boxed{\dot{q}''_{\mathrm{w,ins}} = -\lambda_{\mathrm{ins}} \cdot \frac{T_{\mathrm{ins}} - T_{\mathrm{B}}^*}{\delta_{\mathrm{ins}}} = -0.075\ [\mathrm{W/mK}] \cdot \frac{(27 - 219)\ \ [\mathrm{K}]}{0.025\ [\mathrm{m}]} = 576\ [\mathrm{W/m^2}]} \tag{1.2}$$

3) Inserting and rearranging:

We could write Fourier's law as in equation 1.2 also in terms of the properties of layer A and B, because the heat flux through each layer remains constant for a plane wall:

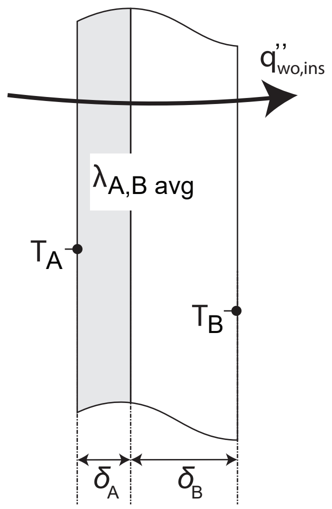
$$\dot{q}''_{\mathrm{w,ins}} = -\lambda_{\mathrm{A,B\ \ avg}} \cdot \frac{T_{\mathrm{B}}^* - T_{\mathrm{A}}^*}{\delta_{\mathrm{A}} + \delta_{\mathrm{B}}} \tag{1.3}$$

Rewriting yields in the average heat conductivity of layer A and B:

$$\lambda_{\mathrm{A,B\ avg}} = \dot{q}''_{\mathrm{w,ins}} \cdot \frac{(\delta_{\mathrm{A}} + \delta_{\mathrm{B}})}{T^*_{\mathrm{A}} - T^*_{\mathrm{B}}} = 576\ [\mathrm{W/m^2}] \cdot \frac{(0.125 + 0.200)\ [\mathrm{m}]}{(305 - 219)\ [\mathrm{K}]} = 2.1767\ [\mathrm{W/mK}] \quad (1.4)$$

<u>System without insulation:</u>

1) Setting up an energy balance/sketching the system:



The rate of heat transfer can be described by us of Fourier's law:

$$\boxed{\dot{q}'' = -\lambda \frac{\partial T}{\partial x}} \quad (1.5)$$

2) Defining the fluxes:

Fourier's law:

$$\dot{q}''_{\mathrm{wo,ins}} = -\lambda_{\mathrm{A,B\ avg}} \cdot \frac{T_{\mathrm{B}} - T_{\mathrm{A}}}{(\delta_{\mathrm{A}} + \delta_{\mathrm{B}})} \quad (1.6)$$

$$\boxed{\dot{q}''_{\mathrm{wo,ins}} = -2.1767\ [\mathrm{W/mK}] \cdot \frac{(32 - 260)\ [\mathrm{K}]}{(0.125 - 0.200)\ [\mathrm{m}]} = 1527\ [\mathrm{W/m^2}]} \quad (1.7)$$

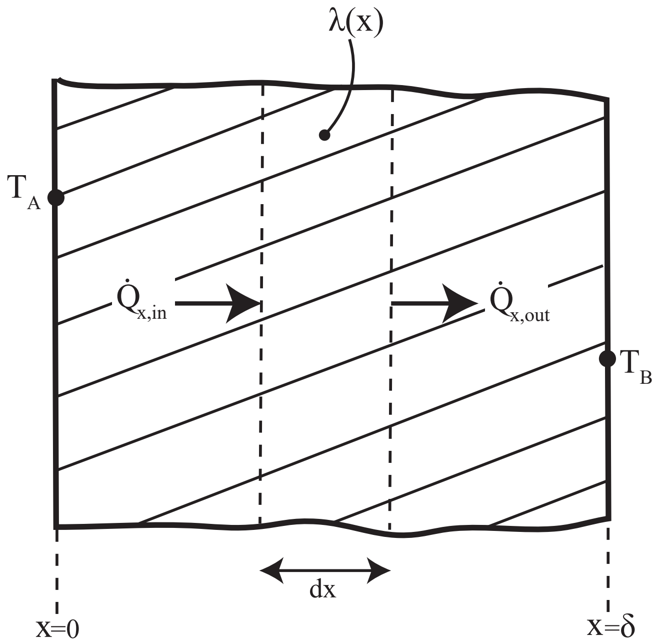
3) Inserting and rearranging:

The difference in the rate of heat transfer:

$$\boxed{\rightarrow \Delta\dot{q}'' = \dot{q}''_{\mathrm{wo,ins}} - \dot{q}''_{\mathrm{w,ins}} = 1527 - 576\ [\mathrm{W/m^2}] = 951\ [\mathrm{W/m^2}]} \quad (1.8)$$

1.3 Heat Conduction Equation ⋆ ⋆ ⋆

a) *Setup an energy balance and derive the heat conduction equation.*



1) Setting up the energy balance

The heat conduction equation for a system is derived based on the energy balance of an infinitesimal element.

The one-dimensional steady-state energy balance for an infinitesimal element with no heat sources/sinks can be described as:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{1.9}$$

$$\boxed{\rightarrow 0 = \dot{Q}_{\text{x,in}} - \dot{Q}_{\text{x,out}}} \tag{1.10}$$

2) Defining the fluxes:

The ingoing flux can be described by use of Fourier's law:

$$\boxed{\rightarrow \dot{Q}_{\text{x,in}} = -\lambda(x) \cdot A_{\text{c}} \cdot \frac{dT}{dx}} \tag{1.11}$$

For an infinitesimal element, the outgoing conductive heat flux can be approximated by use of the Taylor series expansion, which results in:

$$\boxed{\rightarrow \dot{Q}_{\text{x,out}} = \dot{Q}_{\text{x,in}} + \frac{d\dot{Q}_{\text{x,in}}}{dx} \cdot dx = -\lambda(x) \cdot A_{\text{c}} \cdot \frac{dT}{dx} + \frac{d}{dx}\left(-\lambda(x) \cdot A_{\text{c}} \cdot \frac{dT}{dx}\right) \cdot dx} \tag{1.12}$$

3) Inserting and rewriting:

Substituting the definitions of $\dot{Q}_{\text{x,in}}$ and $\dot{Q}_{\text{x,out}}$ into the energy balance results in:

$$0 = \dot{Q}_{\text{x,in}} - \dot{Q}_{\text{x,out}} \tag{1.13}$$

$$0 = \frac{d}{dx}\left(\lambda(x) \cdot A_{\text{c}} \cdot \frac{dT}{dx}\right) dx \tag{1.14}$$

Dividing both sides by the constants A_{c} and dx and substitution of $\lambda(x) = \lambda_0 + \frac{\gamma}{\delta} \cdot x$ yields:

$$\boxed{\rightarrow 0 = \frac{d}{dx}\left(\left[\lambda_0 + \frac{\gamma}{\delta} \cdot x\right] \cdot \frac{dT}{dx}\right)} \tag{1.15}$$

b) Derive the function of the temperature profile inside the planar wall with use of the heat conduction equation obtained in question a).

4) Defining the boundary conditions:

The following two boundary conditions are known:

$$\boxed{\rightarrow T(x = 0) = T_{\text{A}}} \tag{1.16}$$

$$\boxed{\rightarrow T(x = \delta) = T_{\text{B}}} \tag{1.17}$$

5) Solving the equation:

The function of the temperature profile can be derived by use of the heat conduction equation. It should be integrated twice, and the integration constants will result from plugging in the boundary conditions into the integrated function.

$$0 = \frac{d}{dx}\left(\left[\lambda_0 + \frac{\gamma}{\delta} \cdot x\right] \cdot \frac{dT}{dx}\right) \tag{1.18}$$

First integration of the heat conduction equation yields:

$$\left[\lambda_0 + \frac{\gamma}{\delta} \cdot x\right] \cdot \frac{dT}{dx} = C_1 \tag{1.19}$$

For simplicity we make the following substitutions:

$$a = \frac{\gamma}{\delta}$$

$$b = \lambda_0$$

And therefore the heat conduction equation can be described as:

$$[a \cdot x + b] \cdot \frac{dT}{dx} = C_1 \tag{1.20}$$

Rewriting:

$$\frac{dT}{dx} = \frac{C_1}{a \cdot x + b} \tag{1.21}$$

Second time integrating:

$$T(x) = \frac{C_1}{a} \ln|a \cdot x + b| + C_2 \tag{1.22}$$

From $T(x = 0) = T_\mathrm{A}$ it can be found that:

$$T(0) = \frac{C_1}{a} \ln|a \cdot 0 + b| + C_2 = T_\mathrm{A} \tag{1.23}$$

$$C_2 = T_\mathrm{A} - \frac{C_1}{a} \ln|b| \tag{1.24}$$

And from $T(x = \delta) = T_\mathrm{B}$ it can be found that

$$T(\delta) = \frac{C_1}{a} \ln|a \cdot \delta + b| + T_\mathrm{A} - \frac{C_1}{a} \ln|b| = T_\mathrm{B} \tag{1.25}$$

$$\frac{C_1}{a} \ln\left|\frac{a \cdot \delta + b}{b}\right| + T_\mathrm{A} = T_\mathrm{B} \tag{1.26}$$

$$C_1 = \frac{a \cdot (T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{a \cdot \delta + b}{b}\right|} \tag{1.27}$$

Substitution of $a = \frac{\gamma}{\delta}$ and $b = \lambda_0$ into the definitions of C_1 and C_2:

$$\rightarrow C_1 = \frac{a \cdot (T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{a \cdot \delta + b}{b}\right|} = \frac{\gamma \cdot (T_\mathrm{B} - T_\mathrm{A})}{\delta \ln\left|\frac{\gamma}{\lambda_0} + 1\right|} \tag{1.28}$$

$$\rightarrow C_2 = T_\mathrm{A} - \frac{1}{a} \ln|b| \cdot \frac{a \cdot (T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{a \cdot \delta + b}{b}\right|} = -\frac{(T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{\gamma}{\lambda_0} + 1\right|} \cdot \ln|\lambda_0| + T_\mathrm{A} \tag{1.29}$$

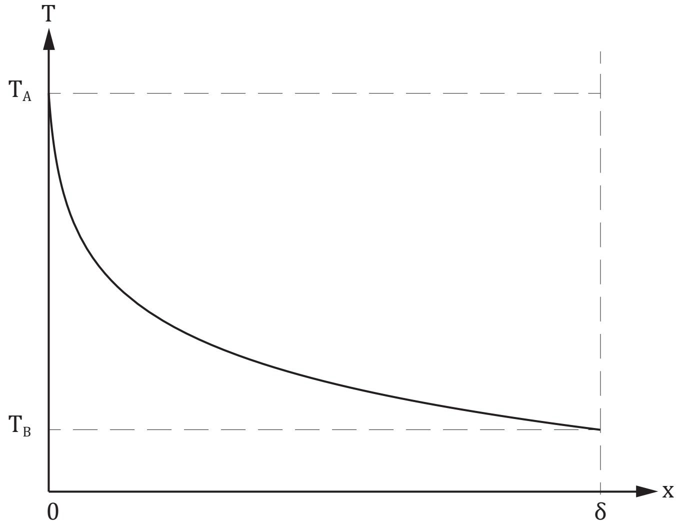
Substitution of C_1 and C_2 and a and b into equation 1.30:

$$T(x) = \frac{C_1 \delta}{\gamma} \ln\left|\frac{\gamma}{\delta} \cdot x + \lambda_0\right| + C_2 \tag{1.30}$$

$$T(x) = \frac{(T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{\gamma}{\lambda_0} + 1\right|} \cdot \ln\left|\frac{\gamma}{\delta} \cdot x + \lambda_0\right| - \frac{(T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{\gamma}{\lambda_0} + 1\right|} \cdot \ln|\lambda_0| + T_\mathrm{A} \tag{1.31}$$

$$\boxed{\rightarrow T(x) = \frac{(T_\mathrm{B} - T_\mathrm{A})}{\ln\left|\frac{\gamma}{\lambda_0} + 1\right|} \cdot \left(\ln\left|\frac{\gamma}{\delta} \cdot x + \lambda_0\right| - \ln|\lambda_0|\right) + T_\mathrm{A}} \tag{1.32}$$

c) Make a sketch of the temperature profile inside the plane wall in x-direction.



The function of the thermal conductivity $\lambda(x)$ tells that the thermal conductivity increases as we move further in positive x-direction. From Fourier's law ($\dot{Q} = -\lambda(x) \cdot A_c \cdot \frac{dT}{dx}$), it can be seen that the temperature gradient $\frac{dT}{dx}$ therefore should decrease in x-direction. This is because the rate of heat transfer $\dot{Q}$ and the cross-sectional area A_c remain constant for a plane wall under steady-state conditions.