Section III

Convection solutions

Exercise III.1: (Walking man ⋆)

A man has a body surface area of A and a skin temperature of T_s, with an average surface temperature of the clothed person of T_c. The convection heat transfer coefficient α for a clothed man walking in the air with temperature T_A is expressed as:

$$\alpha = C \cdot \sqrt{V},$$

for $0.5 < V < 5$ m/s, and where $C = 8.2\ \frac{\text{J}}{\text{m}^{2.5}\text{s}^{0.5}\text{K}}$, and V is the relative velocity of the man with respect to the air.

Given parameters:

- Surface area of the body: $A = 1.8\ \text{m}^2$
- Thermal conductivity of the skin: $\lambda_s = 0.25$ W/mK
- Thermal conductivity of clothes: $\lambda_s = 0.03$ W/mK
- Thermal conductivity of the air: $\lambda_a = 0.026$ W/mK
- Skin temperature of the man: $T_s = 33$ °C
- Surface temperature of the clothed man: $T_c = 30$ °C
- Air temperature: $T_A = 15$ °C

Hints:

- Assume steady-state operating conditions.
- Assume the heat transfer coefficient to be constant over the entire surface.

Tasks:

a) Determine the rate of heat loss from the man by convection while walking in still air at a speed of 1 m/s.

b) Determine the rate of heat loss from the man walking in the air when walking in the same direction of the wind with a velocity of 1.5 m/s, while the wind is blowing at a velocity of 2 m/s.

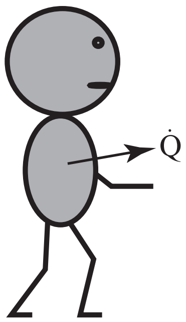
c) Determine the rate of heat loss and the relative velocity from the man while walking in still air with a Nusselt number of Nu = 510, and a characteristic length of $L = 1$ m.

Solution III.1: (Walking man ⋆)

Task a)

1 **Setting up the balance:**

A person walking in the air loses heat by convection due to the difference in temperature between their body and the surrounding air. Convection is the process of heat transfer between a surface and a fluid in motion. As the person walks, air moves around them, creating a flow of air near their skin.



The rate at which heat dissipates via convection follows Newton's law of cooling, which stipulates that the rate of heat transfer is proportional to the temperature difference between the surface and its surrounding environment:

$$\dot{Q} = \alpha A_s \left(T_c - T_A\right). \tag{III.1.1}$$

2 **Defining the elements within the balance:**

The relative velocity in the given problem is 1 $\frac{\mathrm{m}}{\mathrm{s}}$, and thus the heat transfer coefficient yields to be:

$$\begin{aligned} \alpha &= C \cdot \sqrt{V} \\ &= 8.2 \left(\frac{\mathrm{J}}{\mathrm{m}^{2.5}\mathrm{s}^{0.5}\mathrm{K}}\right) \cdot \sqrt{1} \left(\frac{\mathrm{m}^{0.5}}{\mathrm{s}^{0.5}}\right) = 8.2 \left(\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}\right). \end{aligned} \tag{III.1.2}$$

3 **Inserting and rearranging:**

$$\begin{aligned} \dot{Q} &= \alpha A_s (T_c - T_A) \\ &= 8.2 \left(\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}\right) \cdot 1.8 \ \left(\mathrm{m}^2\right) \cdot (30 - 15) \ (^\circ\mathrm{C}) = 221 \ (\mathrm{W}). \end{aligned} \tag{III.1.3}$$

Conclusion

The man loses 221 W of heat by convection while walking in still air at a velocity of 1 $\frac{\mathrm{m}}{\mathrm{s}}$.

Task b)

1 **Setting up the balance:**

The rate of heat loss due to convection is described by Newton's law of cooling:

$$\dot{Q} = \alpha A_s \left(T_c - T_A\right). \tag{III.1.4}$$

2 Defining the elements within the balance:

The relative velocity of the man to the air in the given problem is 0.5 $\frac{\text{m}}{\text{s}}$ and the heat transfer coefficient results from:

$$\begin{aligned} \alpha &= C \cdot \sqrt{V} \qquad \text{(III.1.5)} \\ &= 8.2 \left(\frac{\text{J}}{\text{m}^{2.5}\text{s}^{0.5}\text{K}}\right) \cdot \sqrt{0.5} \left(\frac{\text{m}^{0.5}}{\text{s}^{0.5}}\right) = 8.2 \left(\frac{\text{W}}{\text{m}^2\text{K}}\right). \end{aligned}$$

3 Inserting and rearranging:

Inserting all numbers:

$$\begin{aligned} \dot{Q} &= \alpha A_s (T_c - T_A) \qquad \text{(III.1.6)} \\ &= 5.8 \left(\frac{\text{W}}{\text{m}^2\text{K}}\right) \cdot 1.8 \ \left(\text{m}^2\right) \cdot (30 - 15) \ (^{\circ}\text{C}) = 157 \ (\text{W}). \end{aligned}$$

Conclusion

The man loses 157 W of heat by convection when walking in the same direction of the wind with a velocity of 1.5 $\frac{\text{m}}{\text{s}}$, while the wind is blowing at a velocity of 2 $\frac{\text{m}}{\text{s}}$.

Task c)

1 Setting up the balance:

The rate of heat loss due to convection is described as:

$$\dot{Q} = \alpha A_s \left(T_c - T_A\right). \tag{III.1.7}$$

2 Defining the elements within the balance:

The heat transfer coefficient is found by rewriting the definition of the Nusselt number:

$$\begin{aligned} \alpha &= \frac{\text{Nu}\lambda_f}{L} \qquad \text{(III.1.8)} \\ &= \frac{510 \ (-) \cdot 0.026 \ \left(\text{W/m}^2\text{K}\right)}{1 \ (\text{m})} = 13.3 \ \left(\text{W/m}^2\text{K}\right). \end{aligned}$$

Rewriting the given equation of the heat transfer coefficient, the velocity is found:

$$V = \left(\frac{\alpha}{8.2}\right)^2 \tag{III.1.9}$$

$$\left(\frac{13.3 \ \left(\frac{\text{J}}{\text{m}^2\text{sK}}\right)}{8.2 \ \left(\frac{\text{J}}{\text{m}^{2.5}\text{s}^{0.5}\text{K}}\right)}\right)^2 = 2.6 \left(\frac{\text{m}}{\text{s}}\right). \tag{III.1.10}$$

3 Inserting and rearranging:

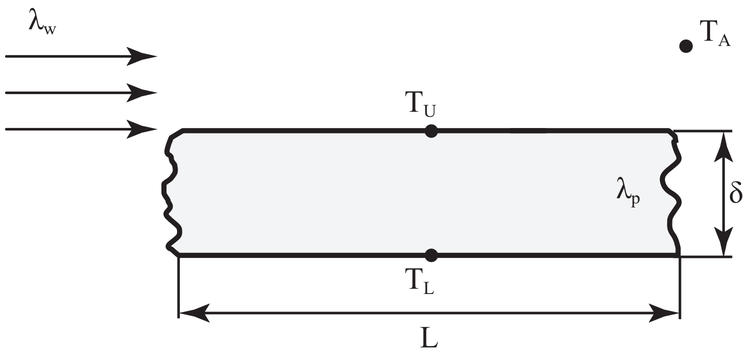
$$\begin{aligned}\dot{Q} &= \alpha A_s (T_c - T_A) \\ &= 13.3 \left(\frac{W}{m^2K}\right) \cdot 1.8 \left(m^2\right) \cdot (30 - 15) \ (°C) = 358 \ (W).\end{aligned} \tag{III.1.11}$$

Conclusion

The man loses 358 W of heat by convection when walking at a velocity of 2.6 $\frac{m}{s}$ with a Nusselt number of Nu = 510, and a characteristic length of $L = 1$ m.

Exercise III.2: (Thick solid plate ⋆)

The top surface of a thick solid plate is cooled by water flowing. The upper and lower surfaces of the solid plate are maintained at constant temperatures T_U and T_L respectively.



Given parameters:

- Thickness of the plate: δ
- Length of the plate: L
- Thermal conductivity of the plate: λ_p
- Thermal conductivity of the water: λ_w
- Upper surface temperature of the plate: T_U
- Lower surface temperature of the plate: T_L
- Ambient temperature: T_A

Hints:

- Assume steady-state operating conditions.
- Assume the heat transfer coefficient to be constant over the entire surface.
- $T_L > T_U$.

Tasks:

a) Determine the Nusselt number in terms of the given variables, using the length L of the plat as the characteristic length.

b) Determine the temperature gradient inside the water at the interface in terms of the given variables.

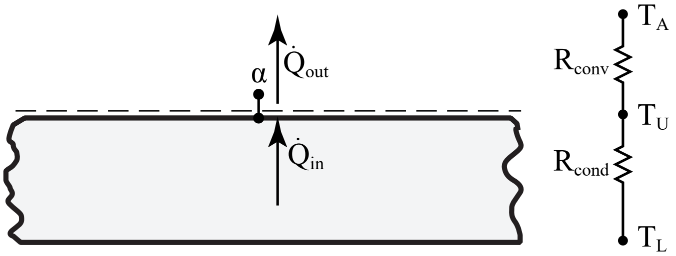
Solution III.2: (Thick solid plate ⋆)

Task a)

The given situation is identified as a multi-layer wall problem without any generation of heat. The problem is solved by establishing the energy balance within the system. This is achieved through the delineation of the thermal resistance network. Subsequently, the thermal resistances are determined.

1 Setting up the balance:

The thermal resistance network comprises a conductive and a convective resistance arranged in series.



The energy balance at the interface between both layers reads:

$$0 = \dot{Q}_{in} - \dot{Q}_{out}. \tag{III.2.1}$$

2 Defining the elements within the balance:

The heat transfer rate through conduction is elucidated using the conductive thermal resistance and the corresponding temperature difference within the solid layer:

$$\dot{Q}_{in} = \frac{T_L - T_U}{R_{cond}}. \tag{III.2.2}$$

Similarly, the rate of heat transfer through convection can be determined by employing the convective thermal resistance and the corresponding temperature difference within the water:

$$\dot{Q}_{out} = \frac{T_L - T_U}{R_{conv}}, \tag{III.2.3}$$

where the thermal resistances are defined as:

$$R_{cond} = \frac{\delta}{A\lambda_p}, \tag{III.2.4}$$

and:

$$R_{conv} = \frac{1}{A\alpha}. \tag{III.2.5}$$

3 Inserting and rearranging:

Inserting and rewriting yields:

$$\alpha = \frac{\lambda_p}{\delta} \cdot \frac{T_L - T_U}{T_U - T_A}. \tag{III.2.6}$$

Using the definition of the Nusselt number:

$$\begin{aligned}\mathrm{Nu} &= \frac{\alpha L}{\lambda_\mathrm{w}} \\ &= \frac{\lambda_\mathrm{p} L}{\lambda_\mathrm{w}\delta}\cdot\frac{T_\mathrm{L}-T_\mathrm{U}}{T_\mathrm{U}-T_\mathrm{A}}.\end{aligned} \tag{III.2.7}$$

Task b)

1 Setting up the balance:

Taking the same energy balance at the interface:

$$0 = \dot{Q}_\mathrm{in} - \dot{Q}_\mathrm{out}. \tag{III.2.8}$$

2 Defining the elements within the balance:

The rate of heat transfer by conduction in a plane wall is described by the use of Fourier's law:

$$\begin{aligned}\dot{Q}_\mathrm{in} &= \left(-\lambda_\mathrm{p} A \frac{\partial T_\mathrm{p}}{\partial x}\right)_\mathrm{int} \\ &= -\lambda_\mathrm{p} A \frac{T_\mathrm{L}-T_\mathrm{U}}{\delta}.\end{aligned} \tag{III.2.9}$$

At the interface, the fluid remains stagnant due to the no-slip condition. Consequently, the rate of convection is expressed accordingly:

$$\dot{Q}_\mathrm{out} = \left(-\lambda_\mathrm{w} A \frac{\partial T_\mathrm{w}}{\partial x}\right)_\mathrm{int}. \tag{III.2.10}$$

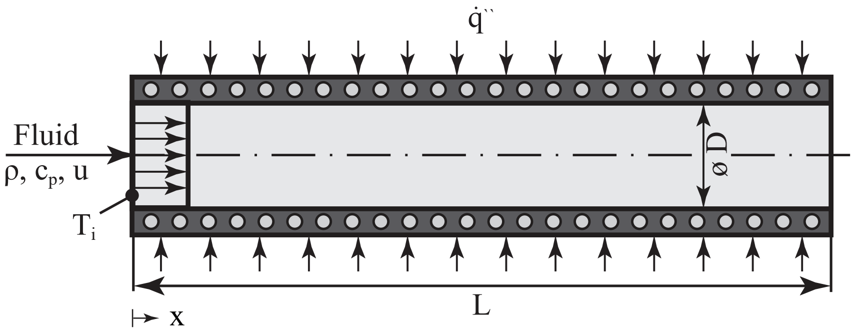
Conclusion

3 Inserting and rearranging:

$$\left(\frac{\partial T_\mathrm{w}}{\partial x}\right)_\mathrm{int} = \frac{\lambda_\mathrm{p}}{\lambda_\mathrm{w}}\frac{T_\mathrm{L}-T_\mathrm{U}}{\delta}. \tag{III.2.11}$$

Exercise III.3: (Pipe flow ⋆⋆)

A fluid flows through a long cylindrical tube. A constant heat flux density $\dot{q}''$ is imposed on the fluid.



Given parameters:

- Diameter of the pipe: D
- Length of the plate: L
- Heat flux density: $\dot{q}''$
- Density of the fluid: ρ
- Specific heat capacity of the fluid: c_p
- Average velocity of the fluid: u
- Fluid inlet temperature: T_i

Hints:

- Assume one-dimensional heat transfer in the axial direction.
- Assume steady-state operating conditions.
- Conduction in the fluid is negligible.

Tasks:

a) Determine the temperature profile of the fluid.

b) Determine the temperature of the fluid at 75% of the pipe length.

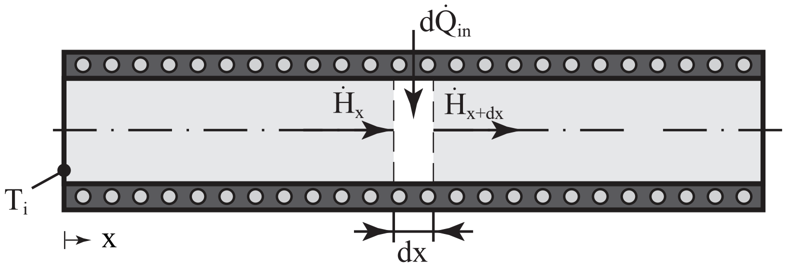
Solution III.3: (Pipe flow ★★)

Task a)

The temperature profile is derived from the equation of energy conservation for a specific system. To derive this equation, the energy balance for an infinitesimal element within the relevant domain should be derived and solved.

1 **Setting up the balance:**

The presented problem is characterized as a scenario where a medium flows through a heated pipe. As stated, heat conduction is negligible. Thus solely the incoming heat flux and enthalpy transport must be considered.



The one-dimensional energy balance for an infinitesimal element within the pipe reads:

$$0 = \underbrace{\dot{H}_{\mathrm{x}} - \dot{H}_{\mathrm{x+dx}}}_{\text{Net rate of advection}} + \underbrace{d\dot{Q}_{\mathrm{in}}}_{\text{External heating}}. \tag{III.3.1}$$

2 **Defining the elements within the balance:**

The rate of heat transport entering due to the motion of the fluid is stated as:

$$\begin{aligned} \dot{H}_{\mathrm{x}} &= \dot{m} c_{\mathrm{p}} T(x) \\ &= \rho u \frac{\pi D^2}{4} c_{\mathrm{p}} T(x). \end{aligned} \tag{III.3.2}$$

and the rate of heat transfer transport leaving due to the motion of the fluid is approximated by the use of the Taylor series expansion:

$$\begin{aligned} \dot{H}_{\mathrm{x+dx}} &= \dot{H}_{\mathrm{x}} + \frac{\partial \dot{H}_{\mathrm{x}}}{\partial x} \cdot dx \\ &= \rho u \frac{\pi D^2}{4} c_{\mathrm{p}} T(x) + \frac{\partial}{\partial x}\left(\rho u \frac{\pi D^2}{4} c_{\mathrm{p}} T(x)\right) \cdot dx. \end{aligned} \tag{III.3.3}$$

The incoming rate of heat transfer from the uniform heat flux is written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{in}} &= \dot{q}'' \cdot dA_{\mathrm{s}} \\ &= \dot{q}'' \cdot \pi D \cdot dx. \end{aligned} \tag{III.3.4}$$

3 **Inserting and rearranging:**

$$\frac{\partial T}{\partial x} = \frac{4\dot{q}''}{\rho u c_{\mathrm{p}} D} \tag{III.3.5}$$

4 **Defining the boundary and/or initial conditions:**

To solve the differential equation, a single boundary condition is required since the temperature has been differentiated once with respect to position.

The inlet temperature is specified, which yields the required boundary condition:

$$T(x = 0) = T_{\mathrm{in}}. \tag{III.3.6}$$

5 **Solving the equation:**

The differential equation must be integrated once with respect to x:

$$T(x) = \frac{4\dot{q}''}{\rho u c_{\mathrm{p}} D} \cdot x + C_1. \tag{III.3.7}$$

Using the defined boundary condition:

$$T(x = 0) = \frac{4\dot{q}''}{\rho u c_{\mathrm{p}} D} \cdot 0 + C_1 \tag{III.3.8}$$

$$\Rightarrow C_1 = T_{\mathrm{in}}.$$

Conclusion

Which results in the fluid's temperature profile:

$$T(x) = \frac{4\dot{q}''}{\rho u c_{\mathrm{p}} D} \cdot x + T_{\mathrm{in}}. \tag{III.3.9}$$

Task b)

To determine the temperature at 75% of the length of the pipe, the derived temperature profile can be used.

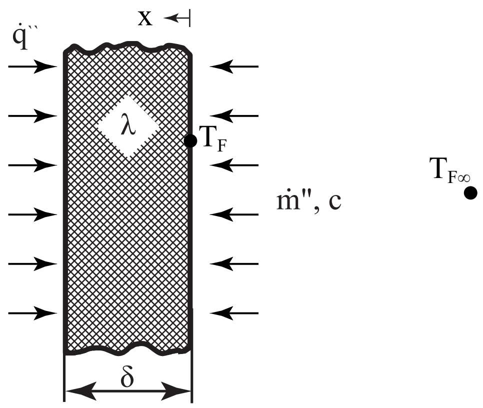
Conclusion

This gives:

$$T\left(x = \frac{3}{4}L\right) = \frac{3\dot{q}''L}{\rho u c_{\mathrm{p}} D} + T_{\mathrm{in}}. \tag{III.3.10}$$

Exercise III.4: (Porous wall ⋆⋆⋆)

The surface of a porous wall, impermeable to radiation, absorbs a radiative heat flux. For cooling purposes, a coolant is circulated through the wall with an inlet temperature is T_F.



Given parameters:

- Imposed radiative heat flux: $\dot{q}'' = 150 \cdot 10^3$ W/m^2
- Wall thickness: $\delta = 50$ mm
- Wall Thermal conductivity: $\lambda = 8$ W/mK
- Coolant specific heat capacity: $c = 1000$ J/kgK
- Coolant inlet temperature: $T_F = -15$ °C
- Coolant area specific mass flux: $\dot{m}'' = 0.6$ kg/m$^2 \cdot$ s

Hints:

- Within the wall, conduction of the imposed radiative heat flux is negligible.
- The local fluid and wall temperatures can be assumed to be identical.

Tasks:

a) Determine the temperature profile $T(x)$ for the porous wall.

b) Determine the maximum temperature T_{max} reached within the wall.

c) Determine the heat flux $\dot{q}''_F$ per unit area, which is transmitted into the fluid at $x = 0$.

d) Which temperature $T_{F,\infty}$ does the fluid reach far away from the wall?

e) Sketch the temperature profiles for two different mass fluxes and mark each curve.

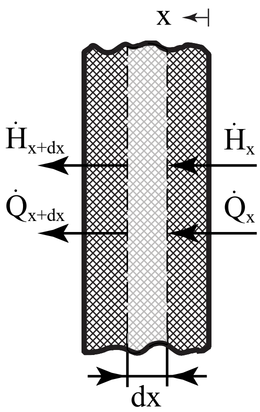
Solution III.4: (Porous wall ⋆ ⋆ ⋆)

Task a)

The temperature profile can be derived from the equation of energy conservation for a specific system. To derive this equation, the energy balance for an infinitesimal element within the relevant domain must be derived and solved.

1 **Setting up the balance:**

Heat is transferred by means of diffusion and fluid motion. Therefore the rate of diffusion and advection need to be incorporated within the steady-state balance.



The energy balance for the steady-state infinitesimal element is written as:

$$0 = \underbrace{\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}}_{\text{Net rate of diffusion}} + \underbrace{\dot{H}_{\mathrm{x}} - \dot{H}_{\mathrm{x+dx}}}_{\text{Net rate of advection}}. \tag{III.4.1}$$

Note that one-dimensional heat transfer assumes that energy transfer happens in the positive x-direction by convention. The direction is accounted for by the definition of the fluxes and the boundary conditions.

2 **Defining the elements within the balance:**

Diffusive heat transport is expressed by Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A \frac{\partial T}{\partial x}. \tag{III.4.2}$$

Rate of energy transport due to the fluid motion yields from:

$$\begin{aligned} \dot{H}_{\mathrm{x}} &= \dot{m}\, c\, T(x) \\ &= \dot{m}''\, A\, c\, T(x). \end{aligned} \tag{III.4.3}$$

Where the outgoing diffusive heat flow and enthalpy flow for an infinitesimal element is approximated by use of Taylor series expansion:

$$\dot{Q}_{\mathrm{x+dx}} = \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \cdot dx \\ = -\lambda A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A \frac{\partial T}{\partial x}\right) \cdot dx, \tag{III.4.4}$$

and:

$$\dot{H}_{\mathrm{x+dx}} = \dot{H}_{\mathrm{x}} + \frac{\partial \dot{H}_{\mathrm{x}}}{\partial x} \cdot dx \\ = \dot{m}''\, A\, c\, T(x) + \frac{\partial}{\partial x}\left(\dot{m}''\, A\, c\, T(x)\right) \cdot dx. \tag{III.4.5}$$

3 Inserting and rearranging:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\dot{m}''\, c}{\lambda} \cdot \frac{\partial T}{\partial x}. \tag{III.4.6}$$

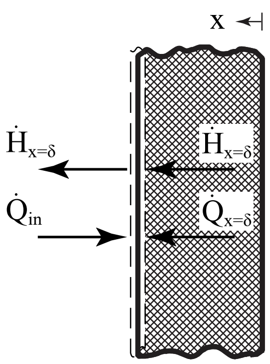
4 Defining the boundary and/or initial conditions:

Because the differential equation is of 2$^{\mathrm{nd}}$ order, two boundary conditions are required.

At $x = 0$ the temperature is already specified, which gives the first boundary condition:

$$T(x = 0) = T_{\mathrm{F}}. \tag{III.4.7}$$

The other boundary condition is obtained by setting up an energy balance at $x = \delta$



At this location, the following energy balance is observed:

$$\dot{Q}_{\mathrm{x}=\delta} + \dot{Q}_{\mathrm{in}} + \underbrace{\cancel{\dot{H}_{x=\delta} - \dot{H}_{x=\delta}}}_{0} = 0, \tag{III.4.8}$$

where:

$$\dot{Q}_{\mathrm{in}} = \dot{q}'' A,$$

and:

$$\dot{Q}_{\mathrm{x}=\delta} = -\lambda A \left.\frac{\partial T}{\partial x}\right|_{x=\delta}.$$

Inserting and rewriting yields the second boundary condition::

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta} = \frac{\dot{q}''}{\lambda}. \tag{III.4.9}$$

5 Solving the equation:

As a linear differential equation is being dealt with, this equation can be attempted to be solved by "guessing" what the solution temperature profile will look like.

$$T(x) = \exp(sx). \tag{III.4.10}$$

Differentiating once with respect to x:

$$\frac{\partial T}{\partial x} = s\exp(sx). \tag{III.4.11}$$

Differentiating twice with respect to x:

$$\frac{\partial^2 T}{\partial x^2} = s^2\exp(sx). \tag{III.4.12}$$

Substitution of the 1$^{\text{st}}$ and 2$^{\text{nd}}$ derivative into the energy balance gives:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\dot{m}''\, c}{\lambda}\cdot\frac{\partial T}{\partial x}, \tag{III.4.13}$$

and:

$$s^2\exp(sx) - \frac{\dot{m}''\, c}{\lambda}\cdot s\exp(sx) = 0. \tag{III.4.14}$$

Rewriting:

$$\exp(sx)\left(s^2 - \frac{\dot{m}''\, c}{\lambda}\right) = 0. \tag{III.4.15}$$

The most straightforward solution is $\exp(sx) = 0$. However, a problem-specific solution is required and therefore $\exp(sx) = 0$ is not satisfactory. Therefore the solution follows from:

$$s^2 - \frac{\dot{m}''\, c}{\lambda}\cdot s = 0. \tag{III.4.16}$$

This gives:

$$s_1 = 0 \qquad \text{and} \qquad s_2 = \frac{\dot{m}''\, c}{\lambda}. \tag{III.4.17}$$

Therefore the general solution is:

$$T(x) = c_1\exp(s_1 x) + c_2\exp(s_2 x). \tag{III.4.18}$$

Which is rewritten to:

$$T(x) = c_1 + c_2\exp\left(\frac{\dot{m}''\, c}{\lambda}x\right). \tag{III.4.19}$$

Now having found a general solution, the integration constants c_1 and c_2 must be determined by use of the boundary conditions.

$T(x=0) = T_F$ gives that:

$$T(x=0) = c_1 + c_2 \exp(0) = T_F \qquad \text{(III.4.20)}$$
$$\Rightarrow c_1 = T_F - c_2.$$

$\frac{\partial T}{\partial x}\Big|_{x=\delta} = \frac{\dot{q}''}{\lambda}$ results:

$$\frac{\partial T}{\partial x}\Big|_{x=\delta} = c_2 \frac{\dot{m}''\, c}{\lambda} \cdot \exp\left(\frac{\dot{m}''\, c}{\lambda} \cdot \delta\right) = \frac{\dot{q}''}{\lambda} \qquad \text{(III.4.21)}$$
$$\Rightarrow c_2 = \frac{\dot{q}''}{\dot{m}''\, c} \exp\left(-\frac{\dot{m}''\, c}{\lambda} \cdot \delta\right).$$

Conclusion

Substitution of c_1 and c_2 into the expression of $T(x)$ gives:

$$T(x) = T_F - \frac{\dot{q}''}{\dot{m}''\, c} \cdot \exp\left(-\frac{\dot{m}''\, c}{\lambda} \cdot \delta\right) \cdot \left[1 - \exp\left(\frac{\dot{m}''\, c}{\lambda} \cdot x\right)\right]. \qquad \text{(III.4.22)}$$

Task b)

Determining the position of the maximum temperature is achieved by either taking the derivative of the temperature profile, identifying the critical points (where the derivative equals zero), or by logical reasoning.

Considering that the fluid is heated when entering the porous wall, the point at which the fluid exits the wall corresponds to the maximum temperature, denoted as $x = \delta$. Thus:

$$T_{\max} = T_F + \frac{\dot{q}_s''}{\dot{m}''\, c} \cdot \left[1 - \exp\left(-\frac{\dot{m}''\, c}{\lambda} \cdot \delta\right)\right] \qquad \text{(III.4.23)}$$
$$= -15\ (^\circ\mathrm{C}) + \frac{150 \cdot 10^3\ \left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)}{0.6\ \left(\frac{\mathrm{kg}}{\mathrm{m}^2\cdot\mathrm{s}}\right) \cdot 1000\ \left(\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{K}}\right)} \cdot \left[1 - \exp\left(\frac{0.6\ \left(\frac{\mathrm{kg}}{\mathrm{m}^2\cdot\mathrm{s}}\right) \cdot 1000\ \left(\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{K}}\right)}{8\ \left(\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}\right)} \cdot 0.05\ (\mathrm{m})\right)\right] = 229\ (^\circ\mathrm{C})\,.$$

Conclusion

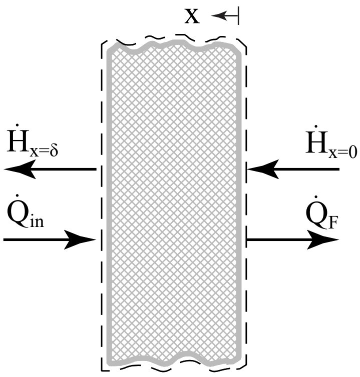
The maximum temperature reached within the wall is thus 229 °C.

Task c)

The heat flux $\dot{q}_F''$ at $x = 0$ is determined by setting up a global energy balance spanning from $x = 0$ to $x = \delta$.

1 Setting up the balance:

In this situation, the fluid is moving in the positive x-direction, resulting in advective transport occurring in the same direction as the fluid's motion. As the fluid traverses, it undergoes heating, causing diffusive heat transfer to operate in the opposite direction to the fluid's travel.



The following equation describes the local energy balance at exactly $x = 0$:

$$0 = \underbrace{-\dot{H}_{x=0} - \dot{H}_{x=0}}_{\substack{\text{Net rate of}\\ \text{advection}}} + \underbrace{\dot{Q}_{\text{in}}}_{\substack{\text{Radiative}\\ \text{heating}}} - \underbrace{\dot{Q}_{\text{F}}}_{\substack{\text{Diffusion}\\ \text{into fluid at } x=0}}. \tag{III.4.24}$$

2 Defining the elements within the balance:

The rate of heat transferred into the fluid at $x = 0$ is expressed in terms of the cross-sectional area and the heat flux $\dot{q}''_{\text{F}}$ per unit area:

$$\dot{Q}_{\text{F}} = \dot{q}''_{\text{F}} A \tag{III.4.25}$$

The rate of heat transferred into the fluid at $x = \delta$ is expressed in terms of the cross-sectional area and the heat flux $\dot{q}''$ per unit area:

$$\dot{Q}_{\text{in}} = \dot{q}'' A. \tag{III.4.26}$$

The energy transferred due to the motion of the fluid at $x = 0$ and $x = \delta$ is written as:

$$\begin{aligned} \dot{H}_{\text{x=0}} &= \dot{m}''\, A\, c\, T\,(x = 0) \\ &= \dot{m}''\, A\, c\, T_{\text{F}}, \end{aligned} \tag{III.4.27}$$

and:

$$\dot{H}_{\text{x=}\delta} = \dot{m}''\, A\, c\, T\,(x = \delta)\,. \tag{III.4.28}$$

3 Inserting and rearranging:

$$\begin{aligned} \dot{q}''_{\text{F}} &= \dot{q}'' - \dot{m}''\, c\, \left(T\,(x = \delta) - T_{\text{F}}\right) \\ &= 150 \cdot 10^3 \left(\frac{\text{W}}{\text{m}^2}\right) - 0.6 \left(\frac{\text{kg}}{\text{m}^2\text{s}}\right) 1000 \left(\frac{\text{J}}{\text{kg} \cdot \text{K}}\right) (229 - 15)\ (\text{K}) = 3.5 \left(\frac{\text{kW}}{\text{m}^2}\right). \end{aligned} \tag{III.4.29}$$

Conclusion

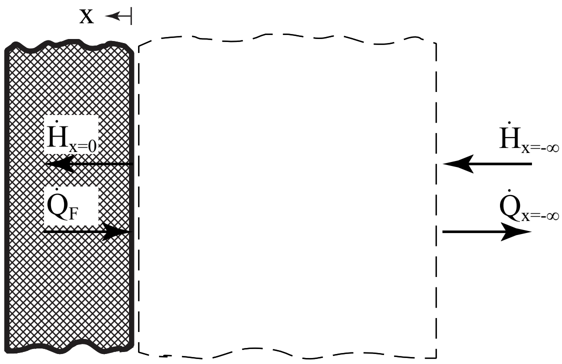
The heat flux per unit area transmitted into the fluid at $x = 0$ is thus 3.5 $\frac{\text{kW}}{\text{m}^2}$.

Task d)

$T_{\text{F},\infty}$ is determined by setting up an energy balance. A global energy balance is taken for $-\infty \leq x \leq 0$.

1 Setting up the balance:

The fluid's positive x-direction movement induces advective transport parallel to its flow. Simultaneously, heating of the fluid causes diffusive heat transfer to act in the opposite direction to its motion.



The energy balance for the given domain is written as:

$$0 = \underbrace{\dot{Q}_{\mathrm{F}} - \dot{Q}_{\mathrm{x}=-\infty}}_{\text{Net rate of diffusion}} + \underbrace{\dot{H}_{x=-\infty} - \dot{H}_{x=0}}_{\text{Net rate of advection}}. \tag{III.4.30}$$

2 Defining the elements within the balance:

The rate of heat transfer towards the fluid is described as:

$$\dot{Q}_{\mathrm{F}} = \dot{q}''_{\mathrm{F}} A. \tag{III.4.31}$$

The energy transferred due to the motion of the fluid at $x = 0$ and $x = -\infty$ is written as:

$$\begin{aligned} \dot{H}_{\mathrm{x}=0} &= \dot{m}''\; A\, c\, T\,(x = 0) \\ &= \dot{m}''\; A\, c\, T_{\mathrm{F}}, \end{aligned} \tag{III.4.32}$$

and:

$$\dot{H}_{\mathrm{x}=-\infty} = \dot{m}''\; A\; c\; T_{\mathrm{F},\infty}. \tag{III.4.33}$$

Lastly, sufficiently far away, the temperature gradient $\frac{\partial T_{\mathrm{F}}}{\partial x}\big|_{x=-\infty} = 0$, and therefore:

$$\dot{Q}_{x=-\infty} = -\lambda_{\mathrm{F}} A \left.\frac{\partial T_{\mathrm{F}}}{\partial x}\right|_{x=-\infty} = 0. \tag{III.4.34}$$

3 Inserting and rearranging:

$$\begin{aligned} T_{\mathrm{F},\infty} &= -\frac{\dot{q}''_{\mathrm{F}}}{\dot{m}''\; c} + T_{\mathrm{F}} \\ &= -\frac{3.5 \cdot 10^{3}\ \left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)}{0.6\ \left(\frac{\mathrm{kg}}{\mathrm{m}^2 \cdot \mathrm{s}}\right) \cdot 1000\ \left(\frac{\mathrm{J}}{\mathrm{kg} \cdot \mathrm{K}}\right)} - 15\ \ (^{\circ}\mathrm{C}) = -21\ \ (^{\circ}\mathrm{C}). \end{aligned} \tag{III.4.35}$$

Conclusion

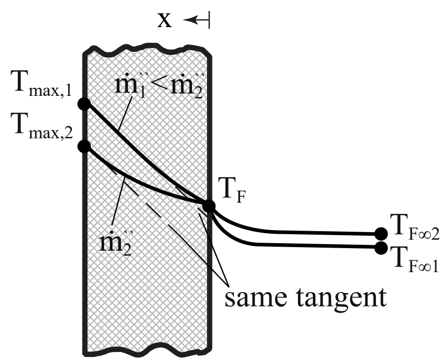
The temperature from the fluid far away from the wall is thus −15 °C.

Task e)

The derived equation for the temperature profile inside the wall yields that the maximum temperature is higher for a smaller mass flux. But at $x = 0$, the temperature is fixed to T_F. Therefore, inside the wall, the profile is steeper for a smaller mass flux.

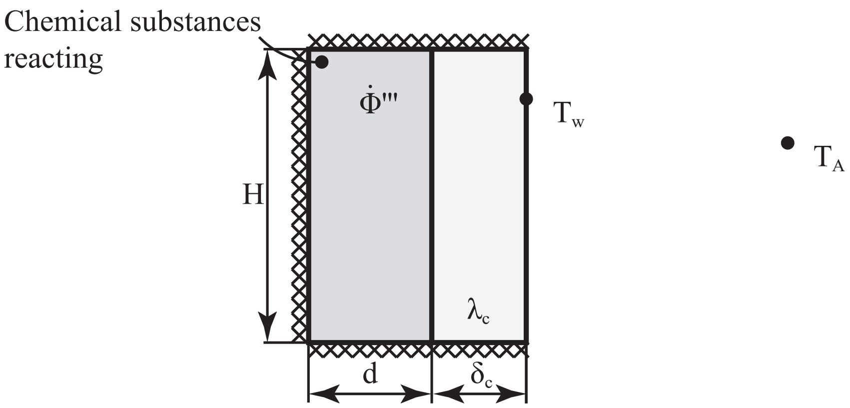
The expression found for $T_{\mathrm{F},\infty}$, tells that this becomes higher in the case that the mass flux is smaller. Still, in both cases, the profile will have a zero-slope gradient for $x \to -\infty$.

Conclusion



Exercise III.5: (Substance container ⋆)

Imagine you are involved in the design of a chemical substance container. These containers house substances that generate heat during chemical reactions. The top and back are adiabatically insulated. During this reaction heat is dissipated to the surrounding air.



Given parameters:

- Height of the container: $H = 80$ cm
- Depth of the container: $d = 50$ cm
- Wall thickness of the container: $\delta_c = 10$ cm
- Thermal conductivity of the wall: $\lambda_c = 0.3$ W/mK
- Thermal conductivity of the air: $\lambda = 0.025$ W/mK
- Prandtl number of the air: $\mathrm{Pr} = 0.72$
- Kinematic viscosity of the air: $\nu = 1.5 \cdot 10^{-5}$ m^2/s
- Outside temperature of the wall: $T_w = 30$ °C
- Temperature of the ambient air: $T_A = 20$ °C

Hints:

- Assume one-dimensional heat transfer.
- Assume steady-state operating conditions.

Tasks:

a) Determine the interface temperature between the chemical substances and their container.

b) Determine the heat generated by the substances per unit volume $\dot{\Phi}'''$.

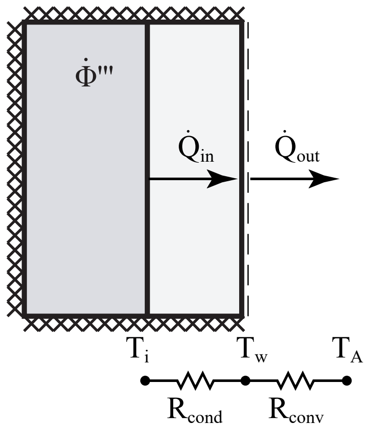
Solution III.5: (Substance container ⋆)

Task a)

The interface temperature between the chemical substances and the container wall is determined by setting up an energy balance at the interface between the container and the ambient.

1 Setting up the balance:

The problem can be seen as a multi-layer wall problem where the first layer generates heat, subsequently conducted into the second layer. At the interface between these layers and the ambient surroundings, heat is transferred by convection. In this particular problem, only the conductive and convective resistances hold significance. Hence, the network is depicted for only two layers.



The energy balance reads:

$$0 = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}. \tag{III.5.1}$$

2 Defining the elements within the balance:

The conductive flux is written by use of the thermal resistance theorem:

$$\dot{Q}_{\text{in}} = \frac{T_{\text{i}} - T_{\text{w}}}{R_{\text{cond}}}. \tag{III.5.2}$$

The same goes for the convective flux:

$$\dot{Q}_{\text{out}} = \frac{T_{\text{w}} - T_{\text{A}}}{R_{\text{conv}}}, \tag{III.5.3}$$

where the thermal resistances are defined as:

$$R_{\text{cond}} = \frac{\delta_{\text{c}}}{\lambda_{\text{c}} A}, \tag{III.5.4}$$

and:

$$R_{\text{conv}} = \frac{1}{\overline{\alpha} A}. \tag{III.5.5}$$

Nevertheless, the average heat transfer coefficient is not given. Given the absence of forced flow within the container, natural convection is the prevalent mode. Estimating the heat transfer coefficient involves employing a Nusselt correlation applicable to natural flow along a vertical plate. To initiate this process, the Grashof must be determined. The property to evaluate the fluid properties at results from the average temperature of the boundary layer:

$$T_{prop} = \frac{T_w - T_A}{2} = \frac{(30-20)\ (^\circ C)}{2} = 25\ (^\circ C). \tag{III.5.6}$$

Assuming air to act as an ideal gas, the volume expansion coefficient is determined from:

$$\beta = \frac{1}{T_{prop}} = \frac{1}{(25+273)\ (K)} = 3.4 \cdot 10^{-3}\ \left(K^{-1}\right). \tag{III.5.7}$$

Besides, the height of the container is the characteristic length to assess all relevant parameters:

$$L = H = 0.8\ (m). \tag{III.5.8}$$

The Grashof number is found from:

$$\mathrm{Gr}_L = \frac{\beta g (T_w - T_A) L^3}{\nu^2} = \frac{3.4 \cdot 10^{-3}\ \left(K^{-1}\right) \cdot 9.81\ \left(\frac{m}{s^2}\right) (30-20)\ (K) \cdot 0.8^3\ \left(m^3\right)}{(1.5 \cdot 10^{-5})^2\ \left(\frac{m^4}{s^2}\right)} = 7.5 \cdot 10^8\ (-). \tag{III.5.9}$$

Having found the value of the Grashof number, HTC.17 can be used to determine the average Nusselt number.

HTC **Average Nusselt number for natural laminar flow along a vertical plate with isothermal surface:**

$$\overline{\mathrm{Nu}}_L = C\left(\mathrm{Gr}_L \mathrm{Pr}\right)^{\frac{1}{4}}, \tag{HTC.17}$$

for $\mathrm{Gr}_L \cdot \mathrm{Pr} < 4 \cdot 10^9$, and:

Pr	0.003	0.01	0.03	0.72	1	2	10	100	1000	∞
C	0.182	0.242	0.305	0.516	0.535	0.568	0.620	0.653	0.665	0.670

Where for $\mathrm{Pr} = 0.72$, the constant $C = 0.516$:

$$\overline{\mathrm{Nu}}_L = C\left(\mathrm{Gr}_L\ \mathrm{Pr}\right)^{1/4} = 0.516\left(\left(7.5 \cdot 10^8\right) \cdot 0.72\right)^{1/4} = 78.6\ (-). \tag{III.5.10}$$

Rewriting the definition of the Nusselt number gives the average heat transfer coefficient:

$$\overline{\alpha} = \frac{\overline{\mathrm{Nu}}_L \lambda}{L} = \frac{78.6\ (-) \cdot 0.025\ (W/mK)}{0.8\ (m)} = 2.5\ \left(\frac{W}{m^2K}\right). \tag{III.5.11}$$

3 **Inserting and rearranging:**

$$T_\mathrm{i} = T_\mathrm{w} + \frac{\alpha \delta_\mathrm{c}}{\lambda_\mathrm{c}} \left(T_\mathrm{w} - T_\mathrm{A}\right) \tag{III.5.12}$$

$$30\ (^\circ\mathrm{C}) + \frac{2.5\ \left(\mathrm{W/m^2K}\right) \cdot 0.1\ (\mathrm{m})}{0.3\ (\mathrm{W/mK})} (30 - 20)\ \ (^\circ\mathrm{C}) = 38.2\ (^\circ\mathrm{C}).$$

Conclusion

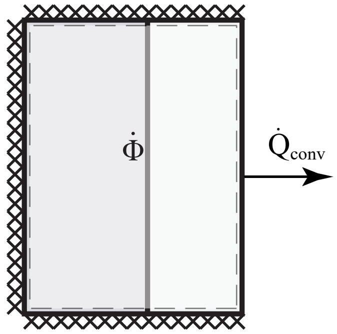
The interface temperature between the chemical substances and their container is thus 38.2 °C.

Task b)

To determine the heat generated by the substances per unit volume, a global energy balance around the container is set.

1 Setting up the balance:

A balance around the entire container can be set. Within the container, a constant rate of heat is generated. At the same time, this heat is being lost by convective losses to the environment.



The steady-state energy balance reads:

$$0 = \underbrace{\dot{\Phi}}_{\text{Internal heat generation}} - \underbrace{\dot{Q}_{\text{conv}}}_{\text{Convective losses}}, \tag{III.5.13}$$

which states that all heat generated by the substances is dissipated through convection.

2 Defining the elements within the balance:

The heat generated is written as:

$$\begin{aligned} \dot{\Phi} &= \dot{\Phi}''' \, V \\ & \dot{\Phi}''' \, H \, d \, w, \end{aligned} \tag{III.5.14}$$

where w represents the width of the container.

The heat dissipated by convection is stated as:

$$\begin{aligned} \dot{Q}_{\text{conv}} &= \overline{\alpha} A \left(T_{\text{w}} - T_{\text{A}}\right) \\ &= \overline{\alpha} \, H \, w \, \left(T_{\text{w}} - T_{\text{A}}\right). \end{aligned} \tag{III.5.15}$$

3 Inserting and rearranging:

$$\begin{aligned} \dot{\Phi}''' &= \frac{\overline{\alpha}}{d} \cdot \left(T_{\text{w}} - T_{\text{A}}\right) \\ &= \frac{2.5 \, \left(\frac{\text{W}}{\text{m}^2\text{K}}\right)}{0.5 \, (\text{m})} \cdot (30 - 20) \, \left(^{\circ}\text{C}\right) = 49.1 \left(\frac{\text{W}}{\text{m}^3}\right). \end{aligned} \tag{III.5.16}$$

Conclusion

The rate of heat generation by the substances per unit volume is thus 49.1 $\frac{\mathrm{W}}{\mathrm{m}^3}$.