Chapter 2

Mass Conservation

2.1 Mass conservation: integral formulation

Suppose we want to compute the mass M of a fluid contained in a volume V. If the mass density ρ is independent of position $\boldsymbol{x}$, the answer is simply

$$M = \rho V. \tag{2.1}$$

When the mass density varies with position $\boldsymbol{x}$ the answer has to be modified. We do this by dividing V in a number of N smaller volumes, ΔV_i, with i serving as a counter. If the volumes are sufficiently small we can take the density approximately constant within each volume, say ρ_i, and the answer becomes

$$M \approx \sum_{i=1}^{N} \rho_i \Delta V_i. \tag{2.2}$$

In the limit of infinitely small volumes and $N \to \infty$ we get [(1)]:

$$M = \lim_{N\to\infty} \sum_{i=1}^{N} \rho_i \Delta V_i \equiv \int_V \rho(\boldsymbol{x}) dV. \tag{2.3}$$

When the mass density is not only a function of position $\boldsymbol{x}$ but also function of time we simply get [(2)]:

$$M(t) = \int_V \rho(\boldsymbol{x}, t) dV. \tag{2.4}$$

We will now take a very special volume: think of a fluid in motion, for example the earth atmosphere on a windy day, and imagine that we are able to stop time for a moment. We could imagine colorizing a blob of fluid. When we let time run again we see the blob moving

[(1)] The mass M is not a function of $\boldsymbol{x}$! We have integrated over $\boldsymbol{x}$ so the answer does not contain $\boldsymbol{x}$ anymore. Compare this to $\int_a^b f(x)dx$ which only depends on a and b and not on x.

[(2)] The mass M is a function of t since ρ is a function of t and the integration is only over $\boldsymbol{x}$

around: it deforms and changes in size. We will call such volume of fluid a 'convected blob'. Our formula for the mass contained in the blob now becomes

$$M(t) = \int_{V(t)} \rho(\boldsymbol{x}, t) dV, \tag{2.5}$$

where $V(t)$ indicates that the volume of the blob is time-dependent: not only in size but also at least in shape. Evidently, each point of the blob travels with the local fluid velocity $\boldsymbol{u}$ which depends on both position $\boldsymbol{x}$ and time t. This means that also the surface of the blob travels with the local fluid velocity which, in turn, means that no fluid is leaving or entering the blob. Since we know that mass cannot be created nor destroyed (at non-relativistic speeds) we conclude that the mass contained in the blob is constant, in other words it is conserved:

$$\frac{dM}{dt} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} \int_{V(t)} \rho(\boldsymbol{x}, t) dV = 0} \tag{2.6}$$

Note that this mass conservation statement is a fundamental physical statement, it cannot be proved! It has been observed again and again, it has never been observed that mass was not conserved.

The challenge of the present chapter is to transform this fundamental physical statement into a practicle tool that can be used by scientists and engineers to make computations involving moving fluids. Of course we need mathematics to do so: mathematics has been invented, at least for a large part, to solve problems from physics and engineering.

2.2 Normal velocity, inner product

To develop a useful and practical tool to do computations for moving fluids we need to develop an expression for the velocity with which the surface of a convected blob travels in the direction normal to the surface. In Fig. (2.1) the blob is depicted, including the velocity vector $\boldsymbol{u}$ at a particular point at the surface of the blob. At the same location the so-called outer unit normal vector [3] $\boldsymbol{n}$ is depicted. Note that this vector has three properties that can be derived from its name:

(a) $\boldsymbol{n}$ is of unit length: $\|\boldsymbol{n}\| = 1$,
(b) $\boldsymbol{n}$ points outward with respect to the blob,
(c) $\boldsymbol{n}$ is oriented normal to the surface of the blob.

The normal velocity is defined as the component (part) of the velocity vector $\boldsymbol{u}$ that points in the same direction as the outward unit normal $\boldsymbol{n}$. We will call the length of this component u_n [4]. Hence, if the angle between the two vectors is ϕ, then

$$u_n = \|\boldsymbol{u}\| \cos \phi. \tag{2.7}$$

[3] The outward unit normal vector is often called 'outward unit normal'.
[4] Note that u_n is a scalar (number), not a vector

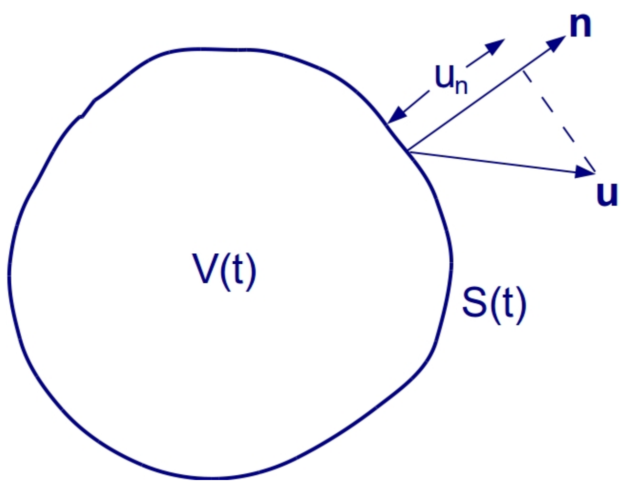

Figure 2.1: Outward unit normal vector $\boldsymbol{n}$*, velocity vector* $\boldsymbol{u}$*, and normal velocity* u_n *at a point on the surface* $S(t)$ *of a convected blob* $V(t)$*.*

The problem of course is that we don't know the value of ϕ. Luckily there is a way out by using the so-called inner product. The inner product of two vectors, say $\boldsymbol{a}$ and $\boldsymbol{b}$, is written as $\boldsymbol{a} \cdot \boldsymbol{b}$ and defined as [5]:

$$\boldsymbol{a} = (a_1, a_2, a_3)^T, \quad \boldsymbol{b} = (b_1, b_2, b_3)^T, \quad \boxed{\boldsymbol{a} \cdot \boldsymbol{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3} \tag{2.8}$$

How does this help us with finding the normal velocity? Well, the inner product defined above satisfies an extremely helpful property

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\phi \quad \Rightarrow \quad \cos\phi = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \|\boldsymbol{b}\|}, \tag{2.9}$$

where ϕ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. So, taking $\boldsymbol{a} = \boldsymbol{u}$ and $\boldsymbol{b} = \boldsymbol{n}$ we find an elegant expression for the normal velocity [6]:

$$\cos\phi = \frac{\boldsymbol{u} \cdot \boldsymbol{n}}{\|\boldsymbol{u}\| \|\boldsymbol{n}\|} \quad \Rightarrow \quad \boxed{u_n = \|\boldsymbol{u}\| \cos\phi = \boldsymbol{u} \cdot \boldsymbol{n}.} \tag{2.10}$$

2.3 Leibniz-Reynolds transport theorem

Leibniz' rule Suppose we have a time-dependent function $f(x,t)$ and we integrate it over a time-dependent interval $[a(t), b(t)]$, then the result (the area under the function between time-dependent boundaries $a(t)$ abnd $b(t)$) is a function of a, b, and t:

$$\int_{a(t)}^{b(t)} f(x,t)\, dx \equiv F(a(t), b(t), t). \tag{2.11}$$

When time advances the function F will change, see Fig. (2.3) Three effects can be observed:

[5] Note that the inner product has two vectors as input, whereas the result is a scalar
[6] Note that $\|\boldsymbol{n}\| = 1$

Figure 2.2: Gottfried Wilhelm Leibniz (left) (1646 - 1716) was a German philosopher and mathematician. He wrote in different languages, primarily in Latin, French, and German. Leibniz occupies a prominent place in the history of mathematics and the history of philosophy. He developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published.

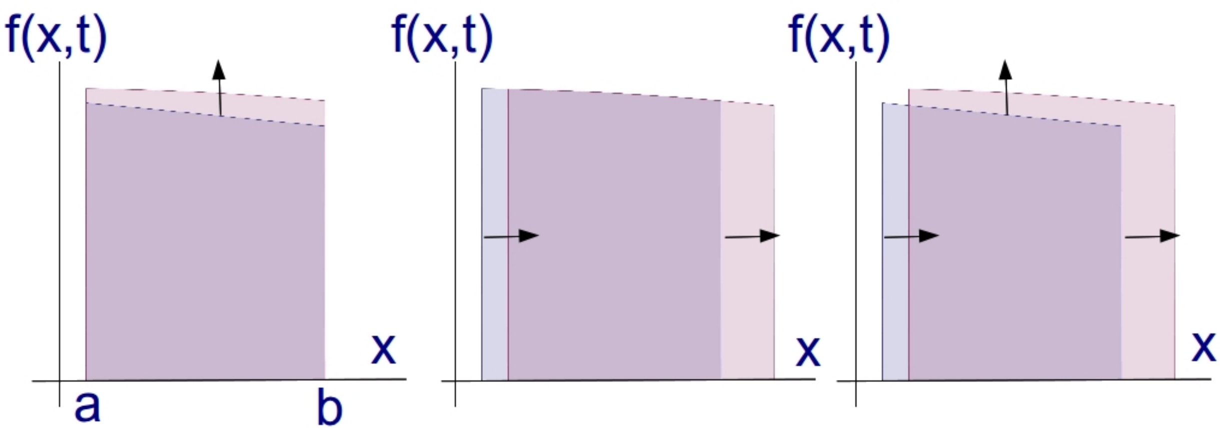

Figure 2.3: Integral variation with time: time-dependent integrand (left), time-dependent boundaries (middle), time-dependent integrand and boundaries (right).

(a) the effect of time-variation of the function $f(x,t)$ (Fig. (2.3).(left)),
(b) the effect of time-variation of the boundaty $a(t)$ (Fig. (2.3).(middle)), and
(c) the effect of time-variation of the boundaty $b(t)$ (Fig. (2.3).(middle)).

The combined effect is illustrated in (Fig. (2.3).(right).

If the time increment is small, say Δt, the time increments of f, a, and b can be estimated by means of a Tayor series expansion:

(a) $\Delta f = \frac{\partial f}{\partial t}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$,
(b) $\Delta a = \frac{da}{dt}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$, and
(c) $\Delta b = \frac{db}{dt}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$.

and with these expressions the three increments of F can be estimated:

(a) the effect of time-variation of the function $f(x,t)$: $\int_{a(t)}^{b(t)} \Delta f \, dx$,
(b) the effect of time-variation of the boundaty $a(t)$: $-f(a)\Delta a$, and
(c) the effect of time-variation of the boundaty $b(t)$: $f(b)\Delta b$.

Note that we have neglected the "intersections" of these increments in the left-top and right-top corner of the area but that is allowed for smal values of Δt since these are $\mathcal{O}\left((\Delta t)^2\right)$. With these expressions and

$$\frac{\partial F}{\partial t} \equiv \lim_{\Delta t \to 0} \frac{F(a(t+\Delta t), b(t+\Delta t), t+\Delta t) - F(a(t), b(t), t)}{\Delta t} \tag{2.12}$$

we arrive at

$$\boxed{\frac{\partial F}{\partial t} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b)\frac{db}{dt} - f(a)\frac{da}{dt}.} \tag{2.13}$$

This relation may be referred to as Leibniz' rule

(Leibniz-)Reynolds transport theorem The Reynolds transport theorem explains how we can compute the time derivative of an integral of a function, say $\rho(\boldsymbol{x}, t)$, that depends on time, over a convected blob $V(t)$. Let $S(t)$ denote the surface of $V(t)$, then the Reynolds transport theorem states that

$$\frac{d}{dt}\int_{V(t)} \rho(\boldsymbol{x},t) dV = \int_{V(t)} \frac{\partial \rho}{\partial t}(\boldsymbol{x},t) dV + \int_{S(t)} \rho(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t) dS. \tag{2.14}$$

We will frequently leave out the arguments $\boldsymbol{x}$ and t for reasons of readability [7]:

$$\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho \boldsymbol{u}\cdot\boldsymbol{n} dS. \tag{2.15}$$

Furthermore, we will use the Einstein summation convention:

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho u_j n_j dS} \tag{2.16}$$

[7] Which does not mean that ρ, $\boldsymbol{u}$ and $\boldsymbol{n}$ are constants!

How can we understand this theorem?

Well, on the one hand, if the blob does not move at all (one could think of a vessel containing the blob of fluid) the answer would clearly be

$$\frac{dV}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV, \tag{2.17}$$

since $\boldsymbol{x}$ and t are independent variables which alows us to interchange integration over $\boldsymbol{x}$ and differentiation with respect to t.

On the other hand, if $V(t)$ moves but ρ is constant, then the answer would be

$$\frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_{V(t)} \rho dV = \int_{S(t)} \rho u_j n_j dS. \tag{2.18}$$

This makes sense since the only way for the integral to increase in time if ρ is constant is to expand due to the motion of the surface. Evidently, if the surface moves in a direction parallel to the surface itself, this does not contribute to expansion. It is therefore clear that it is the normal velocity of the surface that determines the expansion of V. If the surface locally moves in outward normal direction the expansion contribution per unit time is $\boldsymbol{u} \cdot \boldsymbol{n} dS = u_j n_j dS$. The contribution to the increase per unit time of the volume integral over ρ is thus $\rho \boldsymbol{u} \cdot \boldsymbol{n} dS = \rho u_j n_j dS$. Finally, the overall increase per unit time of the volume integral over ρ is obtained by integrating $\rho u_j n_j$ over the complete surface $S(t)$.

So, in conclusion, we can understand both situations: on the one hand a non-moving blob but time-dependent density, and on the other hand a moving blob but time-independent density. Based on mathematical argumenst one can prove that in the general case of a moving blob ănd a time-dependent density the answer is the sum of the two answers found for the restricted cases. Such prove is left out here because it is out of the scope of these lecture notes.

2.4 Integral formulation of mass conservation

We are now ready to take the fundamental physical statement of mass conservation, Eq.(2.6), and combine it with the Reynolds transport theorem, Eq.(2.16). As a result, we obtain the integral formulation of mass conservation:

$$\boxed{\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho u_j n_j dS = 0 \quad \text{for all } \ (V(t), t)} \tag{2.19}$$

In summary:

(a) the first integral expresses the mass rate of change due to the density rate of change, and

(b) the second integral expresses the mass rate of change due the growth rate of the blob.

2.5 Flow rates and averaged velocity

We now want to describe the volume flow rate $\dot{v}$ through (part of) a surface S, think for example of the entrance or exit of a pump. First we define a normal vector on the surface which determines in which direction the volume flow rate is considered positive. Note that for a non-closed surface we are allowed to choose the direction of the normal vector. In case of the pump: when we define the normal vector **inward** the volume flow rate is considered **positive** when its directed **inward**. If we define the normal vector **outward** the volume flow rate is considered **positive** when its directed **outward**.

The volume flow rate through a small surface element dS is:

$$d\dot{v} = u_n dS = \boldsymbol{u} \cdot \boldsymbol{n} \, dS, \tag{2.20}$$

where u_n is the normal velocity on S. It is evident that we should use the normal velocity instead of the velocity itself since the tangential component of the velocity vector does not contribute to the volme flow rate. If we had wanted to calculate the volume flow rate from right to left we should have taken $u_n \equiv -u_j n_j$, or we should have chosen the unit normal vector in opposite direction. The **volume flow rate** through S now simply follows by integration:

$$\dot{v} = \int_S \boldsymbol{u} \cdot \boldsymbol{n} \, dS. \tag{2.21}$$

It is easy to see that the dimension of this expression is volume per unit time since the normal vector is dimensionless and the dimension of dS is length to the power two.

From this expression we can directly calculate the **averaged normal velocity** U over S:

$$U \equiv \frac{1}{S} \int_S \boldsymbol{u} \cdot \boldsymbol{n} \, dS, \quad S \equiv \int_S dS. \tag{2.22}$$

Similarly, one can calculate the **mass flow rate** $\dot{m}$ through the surface S, starting with the contribution of a surface element dS:

$$d\dot{m} = \rho u_n dS, \tag{2.23}$$

and then integrating over the surface:

$$\dot{m} = \int_S \rho u_j n_j dS, \tag{2.24}$$

where it again easily verified that the dimension of this expression is mass per unit time.

Finally, at solid walls the velocity vector is parallel to the wall. This means that the velocity component normal to the wall is zero, or

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ at solid walls.} \tag{2.25}$$

This observation directly leads to a daily-life experience: the volume and mass flow rates through solid walls are identically zero.

2.6 Example problem

We now return to the problem raised in the beginning of this chapter, see Fig. (2.4). For a given velocity field at the tube entrance we have to calculate the mass flow rate at the tube exit. We will do this by using the derived integral formulation of mass conservation given by Eq.(2.19). To do so we define a fluid blob in the tube that is alligned with the tube wall and with the entance and exit surfaces, see Fig. (2.4). Next we split the surface integral in

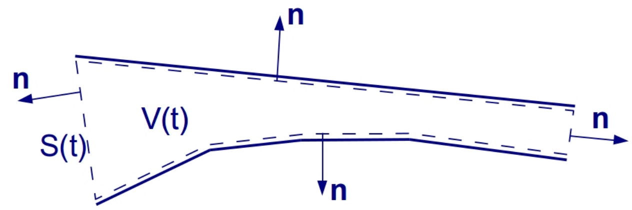

Figure 2.4: Fluid blob in tube with outward unit normals

three parts: the entrance S_1, the wall S_w, and the exit S_2:

$$\int_{S(t)} \rho u_j n_j dS = \int_{S_1(t)} \rho u_j n_j dS + \int_{S_w(t)} \rho u_j n_j dS + \int_{S_2(t)} \rho u_j n_j dS. \tag{2.26}$$

The second of these integrals is zero since integration is over a solid wall where always $u_j n_j = 0$. The third of these integrals is the mass flow rate at the exit:

$$\dot{m} = \int_{S_2(t)} \rho u_j n_j dS. \tag{2.27}$$

Therefore, substitution of these expressions into the integral formulation of mass conservation Eq.(2.19) gives

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S_1(t)} \rho u_j n_j dS + \dot{m} = 0. \tag{2.28}$$

So finally we obtain

$$\dot{m} = -\int_{V(t)} \frac{\partial \rho}{\partial t} dV - \int_{S_1(t)} \rho u_j n_j dS. \tag{2.29}$$

We see that we cannot evaluate this expression further since we need information about $\frac{\partial \rho}{\partial t}$ in the interior of $V(t)$. Suppose now that we know that the flow is steady, in other words, the flow is independent of time at all times. In that case the volume integral is zero since $\frac{\partial \rho}{\partial t}$ is zero and the result is

$$\dot{m} = -\int_{S_1(t)} \rho u_j n_j dS. \tag{2.30}$$

This makes sense because the remaining integral represents the outgoing mass flow rate at the entrance with a minus sign which is just the incoming mass flow rate at the entrance. The remaining integral still cannot be evaluated since no information is given about the mass density at the entrance.

Alternatively suppose that we know that the flow is incompressible, in other words, the mass density is a known constant. In that case the volume integral in Eq.(2.29) is again zero since $\frac{\partial \rho}{\partial t}$ is zero. The result is

$$\dot{m} = -\rho \int_{S_1(t)} u_j n_j dS. \tag{2.31}$$

2.7 Exercises

Problem 2.1. *Compute the inner product* $\boldsymbol{a} \cdot \boldsymbol{b}$ *if*

(a) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (1, 0, 0)^T$.
(b) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (0, 1, 0)^T$.
(c) $\boldsymbol{a} = (a_1, a_2, a_3)^T$, $\boldsymbol{b} = (b_1, b_2, b_3)^T$.
(d) $\boldsymbol{a} = (x, y^2, x)^T$, $\boldsymbol{b} = (y, y, z)^T$.
(e) $\boldsymbol{a} = (u, v, w)^T$, $\boldsymbol{b} = (n_1, n_2, n_3)^T$.

Problem 2.2. *Compute the inner product* $\boldsymbol{u} \cdot \boldsymbol{n}$ *if*

(a) $\boldsymbol{u} = U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.
(b) $\boldsymbol{u} = -U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.

Problem 2.3. *A tube has cross-sectional area A_a at the entrance and cross-sectional area A_b at the exit, and the fluid flowing through the tube is incompressible.*

(a) If the volume flow rate at the exit is Q, compute the average normal velocity at the exit.
(b) If the volume flow rate at the exit is Q, compute the average normal velocity at the entrance.

Problem 2.4. *A channel with rectangular cross section has sides b and h at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u, v, w)^T$, *with* $u = U(1 - z/h)$, $v = \ln yz$, $w = yz^2$.
(b) $(u, v, w)^T$, *with* $u = U(1 - y/b)$, $v = \sin z$, $w = \cos y$.
(c) $(u, v, w)^T$, *with* $u = U(1 - y/b)(1 - z/h)$, $v = 0$, $w = yz$.

Problem 2.5. *A tube with circular cross section has radius R at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u(r), 0, 0)^T$, *with* $u(r) = U(1 - r/R)$, *compute the average normal velocity.*
(b) $(u(r), 0, 0)^T$, *with* $u(r) = U(1 - (r/R)^2)$, *compute the average normal velocity.*

Problem 2.6. *Show how Eq.(2.19) reduces in the following two cases:*