

# Chapter 9

## Convection and Diffusion

### 9.1 Mass Conservation

Consider a vein or a long tube with cross-sectional area  $A(x)$  with  $x$  the coordinate along the tube. Imagine the steady flow of a fluid of mass density  $\rho$  and cross-sectional averaged velocity  $u$  in the positive  $x$ -direction, then the mass flow rate (kg/s) through the tube cross section is the same at every position along the tube:

$$\frac{d}{dx}(\rho u A) = 0 \quad \Rightarrow \quad \rho u A = \text{constant}, \quad (9.1)$$

If  $\rho$  is constant this leads to  $uA = \text{constant}$  and if also the cross section is constant this finally leads to  $u = \text{constant}$ .

If a small amount of a substance is added to the fluid it will be transported by the fluid and will also be spread over the fluid. The transportation is called **convection** and the spreading is called **diffusion**. Both mechanisms play an essential role in bio-fluid mechanics. The mass density of the added substance will be denoted by  $\rho_\alpha$  and its concentration by  $c_\alpha$  which can be computed from

$$c_\alpha \equiv \rho_\alpha / \rho \quad \Rightarrow \quad \rho_\alpha = \rho c_\alpha. \quad (9.2)$$

### 9.2 Convection

Imagine the flow of the fluid with the added substance through a tube with constant cross section  $A_o$ , one could think of a medical drug added to blood flowing through an idealized vein. Then the accumulation of the added drug in a small section of the tube of length  $\Delta x$  is described by the time derivative of its mass density which, if there is no spreading (i.e. no diffusion) depends on the inflow and outflow rates of the drug:

$$\frac{\partial \rho_\alpha}{\partial t} \Delta x A_o = (\rho_\alpha u A_o)_x - (\rho_\alpha u A_o)_{x+\Delta x} \quad (9.3)$$

By dividing by  $A_o\Delta x$ , noting that  $u = U$  is a constant and by taking the limit  $\Delta x \rightarrow 0$  one obtains

$$\frac{\partial \rho_\alpha}{\partial t} = -U \frac{\partial \rho_\alpha}{\partial x}, \quad (9.4)$$

which can further be simplified by using Eq.(9.2) and  $\rho = \text{constant}$  which finally leads to the **convection equation**:

$$\boxed{\frac{\partial c_\alpha}{\partial t} + U \frac{\partial c_\alpha}{\partial x} = 0.} \quad (9.5)$$

The solution to the convection equation is easily guessed based on the observation that, in absense of spreading (i.e. diffusion), a given concentration profile will be just transported with velocity  $u$  without deformation. Hence it is natural to try a solution of the form  $g(x - Ut)$ . Substitution into the convection equation Eq.(9.5) gives:

$$g'(x - Ut) \cdot (-U) + U g'(x - Ut) = 0, \quad (9.6)$$

which shows that  $g(x - Ut)$  is indeed a solution for any function  $c$ . If the initial condition is  $c_\alpha(x, 0) = c_\alpha^o(x)$  then  $g(x) = c_\alpha^o(x)$  and so  $g(x - Ut) = c_\alpha^o(x - Ut)$  so the solution becomes

$$\boxed{c_\alpha(x, t) = c_\alpha^o(x - Ut)}. \quad (9.7)$$

This expression indeed reflects the intuitive idea that the concentration profile is transported to the right with speed  $U$  without deformation.

### 9.3 Diffusion

Imagine the same tube with a quiescent fluid with a locally added substance. From experience one knows that after a while the substance will have spreaded due to Brownian motion of the molecules.

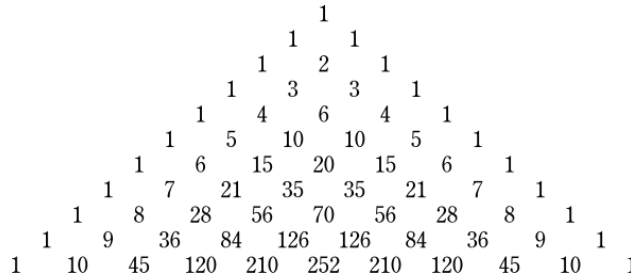


Figure 9.1: Newton's binomium as a model for diffusion

At any position  $x$  the flux  $f$  of substance is proportional to **minus** the concentration derivative:

$$f = -D \frac{\partial \rho_\alpha}{\partial x}, \quad (9.8)$$

where  $D$  is the diffusion coefficient which is assumed to be constant. This means that if concentration increases in positive  $x$ -direction then the actual flux is in the negative  $x$  direction.

Again the accumulation of the added drug in a small section of the tube of length  $\Delta x$  is described by the time derivative of its mass density which, if there is no fluid motion (i.e. no convection) depends on the inflow and outflow rates of the drug by means of Brownian motion:

$$\frac{\partial \rho_\alpha}{\partial t} \Delta x A_o = \left( -D \frac{\partial \rho_\alpha}{\partial x} A_o \right)_x - \left( -D \frac{\partial \rho_\alpha}{\partial x} A_o \right)_{x+\Delta x} \quad (9.9)$$

By dividing by  $A_o \Delta x$ , noting that  $D$  is a constant and by taking the limit  $\Delta x \rightarrow 0$  one obtains

$$\frac{\partial \rho_\alpha}{\partial t} = D \frac{\partial}{\partial x} \left( \frac{\partial \rho_\alpha}{\partial x} \right), \quad (9.10)$$

which can further be simplified by using Eq.(9.2) and  $\rho = \text{constant}$  which finally leads to the **diffusion equation**.

$$\boxed{\frac{\partial c_\alpha}{\partial t} = D \frac{\partial^2 c_\alpha}{\partial x^2}}. \quad (9.11)$$

The general solution of the diffusion equation can be derived by the method of separation of variables which will not be undertaken here. Instead a fundamental solution will be derived which is of interest when initially the added substance is confined to a compact area, think of an instantaneous injection at a point  $\tilde{x}$ . The trial solution has the form

$$c_\alpha(x, t) = h(t) \exp \left( -\beta \frac{(x - \tilde{x})^2}{t} \right), \quad (9.12)$$

where the function  $h(t)$  and the constant  $\beta$  are determined by solution of the trial solution into the diffusion equation:

$$h(t) = \frac{c}{\sqrt{t}}, \quad \beta = \frac{1}{4D}, \quad (9.13)$$

with  $c$  a constant. If the initial condition is  $c_\alpha(x, 0) = c_\alpha^o(x)$  then the general solution can be written as an integral over the fundamental solutions in Eq.(9.12):

$$\boxed{c_\alpha(x, t) = \int_{-\infty}^{\infty} \frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi Dt}} \exp \left( -\frac{(x - \tilde{x})^2}{4Dt} \right) d\tilde{x}}. \quad (9.14)$$

This solution is an integration over Gaussian distributions with standard deviation  $\sqrt{2Dt}$ . It is interesting that in the limit of small  $t$  the standard deviation goes to zero and the solution becomes an integration over Dirac delta functions which recovers the initial condition:

$$\lim_{t \rightarrow 0} c_\alpha(x, t) = \int_{-\infty}^{\infty} c_\alpha^o(\tilde{x}) \delta(x - \tilde{x}) d\tilde{x} = c_\alpha^o(x). \quad (9.15)$$

## 9.4 Convection & diffusion

In this section simultaneous convection and diffusion is considered which is described by the so-called **convection-diffusion equation**:

$$\boxed{\frac{\partial c_\alpha}{\partial t} + U \frac{\partial c_\alpha}{\partial x} = D \frac{\partial^2 c_\alpha}{\partial x^2}}. \quad (9.16)$$

To produce a solution describing an injected substance which then is both convected and diffused the above equation is first transformed to a reference frame **moving with the flow**. The idea behind it is that when one moves with the flow the velocity is zero and only the spreading mechanism (diffusion) should be present. So, after transformation one expects a diffusion equation in the new reference frame. The transformation applied is

$$c_\alpha(x, t) = \bar{c}_\alpha(\xi, \tau), \quad \xi \equiv x - Ut, \quad \tau = t. \quad (9.17)$$

Note that when  $\xi$  is constant  $x - Ut$  is constant which means that  $x$  moves with time at speed  $U$ , exactly what is intended. The derivatives transform as

$$\frac{\partial c_\alpha}{\partial t} = \frac{\partial \bar{c}_\alpha}{\partial \tau} - U \frac{\partial \bar{c}_\alpha}{\partial \xi}, \quad \frac{\partial c_\alpha}{\partial x} = \frac{\partial \bar{c}_\alpha}{\partial \xi}, \quad \frac{\partial^2 c_\alpha}{\partial x^2} = \frac{\partial^2 \bar{c}_\alpha}{\partial \xi^2}, \quad (9.18)$$

which after substitution into the convection-diffusion equation Eq.(9.16) indeed gives

$$\frac{\partial \bar{c}_\alpha}{\partial \tau} = D \frac{\partial^2 \bar{c}_\alpha}{\partial \xi^2}. \quad (9.19)$$

The solution due to a local injection of substance at  $(x, t) = (0, 0)$  in the convection-diffusion problem corresponds to local injection at  $(\xi, \tau) = (0, 0)$  in the diffusion problem which gives the solution derived in the previous section:

$$\bar{c}_\alpha(\xi, \tau) = \int_{-\infty}^{\infty} \frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi D\tau}} \exp\left(-\frac{(\xi - \tilde{x})^2}{4D\tau}\right) d\tilde{x}. \quad (9.20)$$

The corresponding solution to the convection-diffusion problem is now directly available by backward transformation:

$$\boxed{c_\alpha(x, t) = \int_{-\infty}^{\infty} \frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x - Ut - \tilde{x})^2}{4Dt}\right) d\tilde{x}}. \quad (9.21)$$

which is an integral over spreading Gaussian distributions which is convected with speed  $U$ .

## 9.5 Péclet number

When a convection diffusion problem is considered, the initial condition typically exists of a given concentration profile with say a maximum value of  $c_\alpha^{max}$  on an interval of length  $L$ .

The other two parameters involved are the convection speed  $U$  and the diffusion coefficient  $D$ . Dimension analysis shows that there exist a dimensionless number, the **Péclet number**, which expresses the relative importance of convection compared to diffusion:

$$Pe \equiv \frac{UL}{D}. \quad (9.22)$$

As an example, consider the flow of oxygen in the human lung. The trachea is the entrance of the lung and one expects that convection is much more important than diffusion so  $Pe \gg 1$ . On the other hand, in the alveolar region where the lung tubes end in the alveoli the flow is nearly stagnant and one expects diffusion to be much more important than convection so  $Pe \ll 1$ .

## 9.6 Exercises

**Problem 9.1.** Consider the following convection-diffusion equation:

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2},$$

where  $\phi(x, t)$  is some property, and the units of  $x$ ,  $t$ , and  $u$  are  $m$ ,  $s$ , and  $m/s$ , respectively. Compute the unit of  $\alpha$ , and check the answers for the cases  $\alpha = \frac{k}{\rho C_v}$ , and  $\alpha = \frac{\mu}{\rho}$ .

**Problem 9.2.** One-dimensional sound waves are described by the wave equation

$$\frac{\partial^2 p}{\partial t^2} - a^2 \frac{\partial^2 p}{\partial x^2} = 0$$

, where  $p$  is the pressure disturbance and  $a$  the speed of sound. Show that  $f(x - at)$  and  $g(x + at)$  are solutions of the wave equation, with  $f$  and  $g$  arbitrary functions.

**Problem 9.3.** Show, by substitution, that  $T(x, t) (a \cos(\lambda x) + b \sin(\lambda x)) \exp(-\alpha \lambda^2 t)$  is a solution of the diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}.$$

**Problem 9.4.** Let  $T(x, t) = \bar{T}(\xi(x, t), t)$ ,  $\xi(x, t) = x - Ut$ .

(a) Express  $\frac{\partial T}{\partial t}$ ,  $\frac{\partial T}{\partial x}$ , and  $\frac{\partial^2 T}{\partial x^2}$  in terms of derivatives of  $\bar{T}$  with respect to  $\xi$  and  $t$ .

(b) Show that the convection-diffusion equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}$$

transforms into a diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \alpha \frac{\partial^2 \bar{T}}{\partial \xi^2}.$$