Chapter 8

Equations of Euler and Bernoulli

8.1 Total derivative

The surface temperature T on the globe depends on the location, say x and y, and the time-instant, say t. Consider the following three scenarios:

(a) increase t by Δt while keeping x and y fixed,
(b) increase x by Δx while keeping t and y fixed, and
(c) increase y by Δy while keeping t and x fixed.

The three increments in temperature corresponding to these scenarios, in case of small increments, become

(a) $\Delta T \approx \frac{\partial T}{\partial t}\Delta t$,
(b) $\Delta T \approx \frac{\partial T}{\partial x}\Delta x$, and
(c) $\Delta T \approx \frac{\partial T}{\partial y}\Delta y$,

which follow from applying Taylor's theorem.

As a fourth scenario, suppose we travel over the surface of the globe. This means that our x and y coordinates are functions of time, say $x_p(t)$ and $y_p(t)$. Then by increasing time, we also increase our x and y coordinates: $\Delta x \approx \frac{dx_p}{dt}\Delta t$ and $\Delta y \approx \frac{dx_p}{dt}\Delta t$. As a consequence, the temperature increment becomes a sum of the contributions of the three scenarios:

$$\Delta T \approx \frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\frac{dx_p}{dt}\Delta t + \frac{\partial T}{\partial y}\frac{dy_p}{dt}\Delta t. \tag{8.1}$$

In the limit of $\Delta t \to 0$ we get

$$\frac{d}{dt}T\left(x_p(t), y_p(t), t\right) \equiv \lim_{\Delta t \to 0}\frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p}{dt} + \frac{\partial T}{\partial y}\frac{dy_p}{dt}, \tag{8.2}$$

which is called the total-derivative of T (with respect to t). We write d's in stead of ∂'s since $T\left(x_p(t), y_p(t), t\right)$ is a function of t only. A straight-forward generalization towards three dimensions is

$$\frac{d}{dt}T\left(x_p(t), y_p(t), z_p(t), t\right) \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p}{dt} + \frac{\partial T}{\partial y}\frac{dy_p}{dt} + \frac{\partial T}{\partial z}\frac{dz_p}{dt}, \tag{8.3}$$

which may represent the temperature in the atmosphere examined by an airplane with time-dependent position $(x_p(t), y_p(t), z_p(t))^T$. Note that $\frac{dx_p}{dt}$, $\frac{dy_p}{dt}$, and $\frac{dz_p}{dt}$ denote the velocity components in x, y, and z-direction, respectively.

8.2 Material derivative

In fluid dynamics it is often convenient to compute the time derivative of a quantity at a position that is convected by the flow. So, this is just the total derivative introduced in the previous section but with the time-derivatives of the position taken equal to the velocity of the fluid <u>at that position</u>:

$$\frac{dx_p}{dt} = u\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.4}$$

$$\frac{dy_p}{dt} = v\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.5}$$

$$\frac{dz_p}{dt} = w\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.6}$$

Since this special derivative appears very frequently, a special symbol is introduced. Leaving out the arguments for readability, one writes

$$\boxed{\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z},} \tag{8.7}$$

which is called the <u>material derivative</u> of T, refering to the fact that the position at which the time derivative is taken is 'attached' to the material (fluid). Finally it is noted that the material derivative can be written in index notation as

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j}. \tag{8.8}$$

Steady flow When the flow is steady,

$$\frac{DT}{Dt} = u_j\frac{\partial T}{\partial x_j}. \tag{8.9}$$

Trajectory invariants Suppose that the material derivative of T is zero, then this means that that T does not vary along trajectories:

$$\frac{DT}{Dt} = 0 \Leftrightarrow T \text{ is constant along trajectories} \tag{8.10}$$

Streamline invariants Suppose that the material derivative of T is zero, and the flow is steady, then this means that that T does not vary along streamlines:

$$\frac{\partial T}{\partial t} = 0, \quad \frac{DT}{Dt} = 0 \Leftrightarrow T \text{ is constant along streamlines} \tag{8.11}$$

8.3 Euler's equation

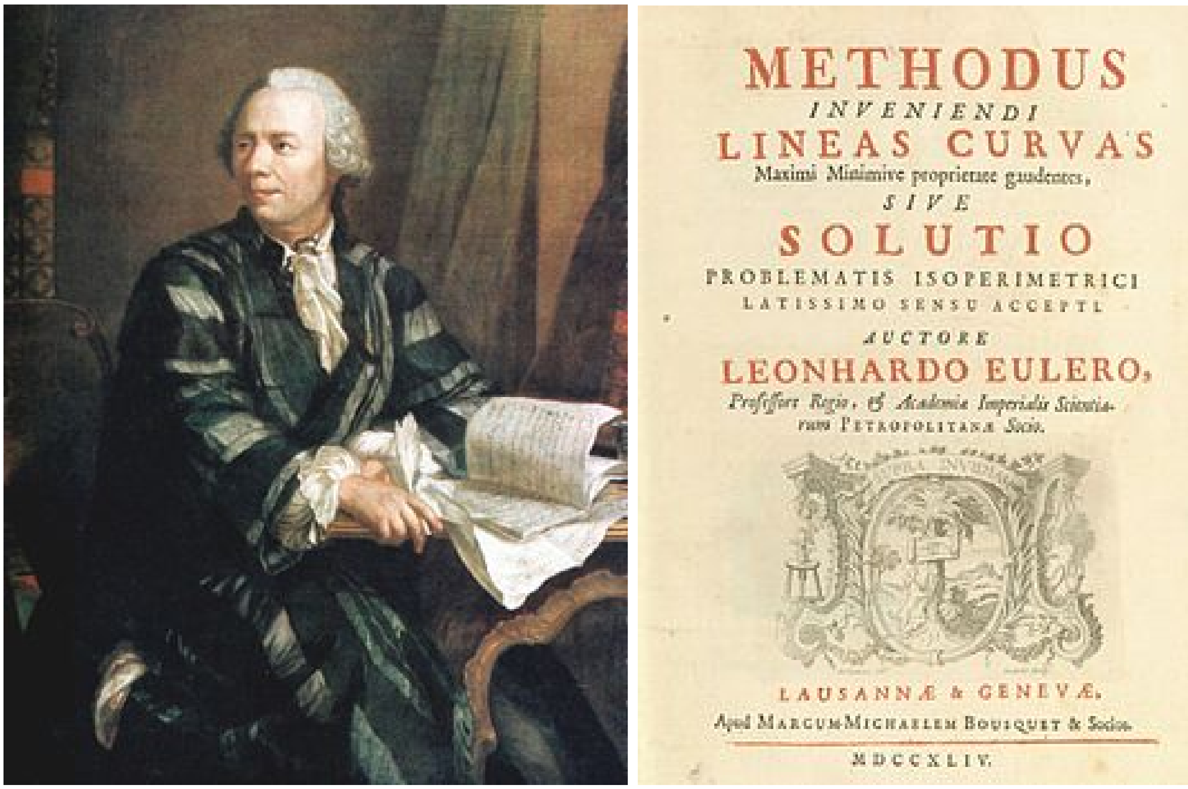

Figure 8.1: Leonhard Euler (1707-1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy.

When the flow is inviscid, i.e., $\mu = 0$, then the stress tensor Eq.(4.10) becomes

$$\sigma_{ij} = -p\delta_{ij}, \tag{8.12}$$

and the Navier-Stokes equations (momentum conservation) given by Eq.(4.39)) reduce to:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p\delta_{ij}}{\partial x_j} - \rho g_i = 0. \tag{8.13}$$

With the properties of the delta function and the summation convention one finds

$$\frac{\partial p\delta_{ij}}{\partial x_j} = \frac{\partial p}{\partial x_i}. \tag{8.14}$$

Furthermore, by means of the product rule of differentiation:

$$\frac{\partial}{\partial t}\left(\rho u_i\right) = \frac{\partial \rho}{\partial t}u_i + \rho\frac{\partial u_i}{\partial t}, \tag{8.15}$$

and

$$\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = \frac{\partial \rho u_j}{\partial x_j}u_i + \frac{\partial u_i}{\partial x_j}\rho u_j. \tag{8.16}$$

In view of the continuity equation (mass conservation) given by Eq.(3.18), the sum of the first terms at the right hand sides of these two equations is zero:

$$\frac{\partial \rho}{\partial t}u_i + \frac{\partial \rho u_j}{\partial x_j}u_i = \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right)u_i = 0, \tag{8.17}$$

So, after division by ρ, Eq.(8.13) can be written as

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i. \tag{8.18}$$

The left hand side appears to be the material derivative (introduced in the previous section) of u_i, so finally we obtain Euler's equation:

$$\boxed{\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i.} \tag{8.19}$$

Note that $\frac{Du_i}{Dt}$ represents the time-derivative of u_i while traveling with the flow, i.e., it represents the acceleration of the flow in i-direction. This shows that Euler's equation is closely related to Newton's second law as is expected since both equations express conservation of momentum.

8.4 Bernoulli's equation

One of the most popular equations in fluid mechanics is Bernoulli's equation, because of both its power and its simplicity. Basically, it is an equation for the material derivative of the kinetic energy of the fluid.

To derive Bernoulli's equation we start by making a number of assumptions:

(a) the flow is steady ($\frac{\partial \ldots}{\partial t} = 0$),
(b) the flow is incompressible (ρ=constant), and
(c) the flow is inviscid ($\mu = 0$).

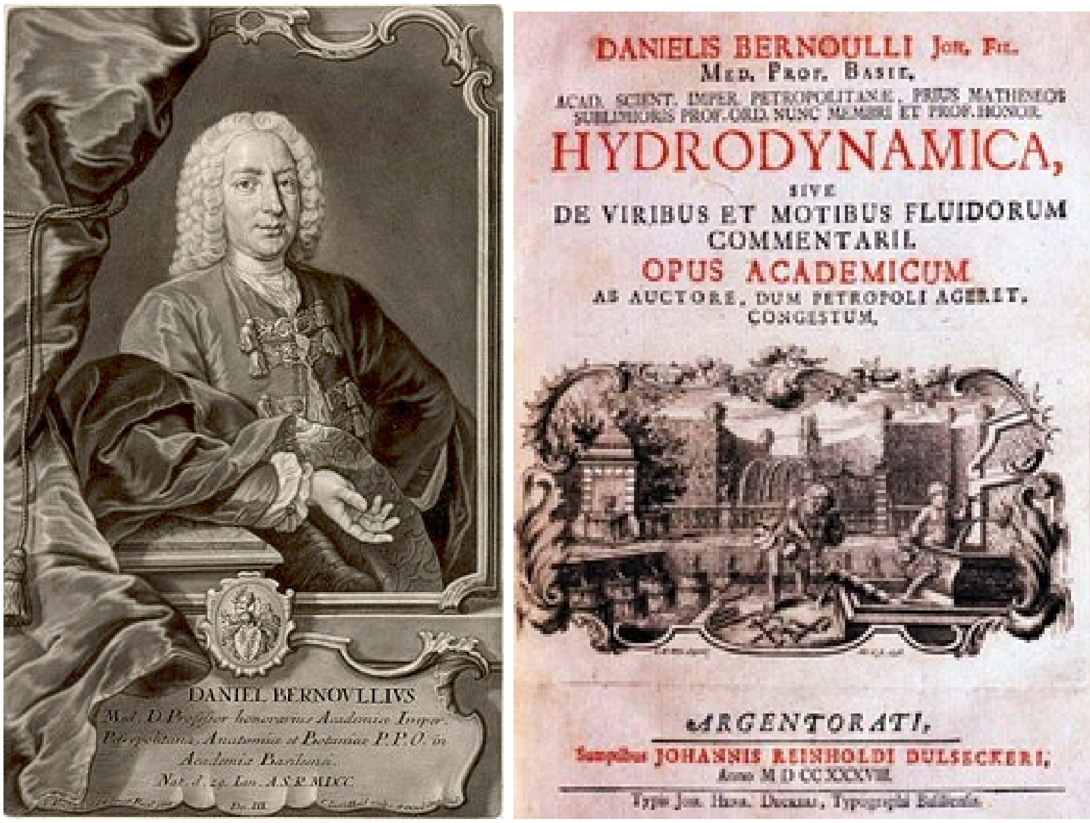

Figure 8.2: Daniel Bernoulli (1700 - 1782) was a Dutch-Swiss mathematician and was one of the many prominent mathematicians in the Bernoulli family. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics. Bernoulli's work is still studied at length by many schools of science throughout the world.

Next, we use Euler's equation Eq.(8.19), which is allowed since the flow is inviscid, and rewrite each of the terms. In view of the steady flow assumption may rewrite the first term as

$$\frac{Du_i}{Dt} = u_j \frac{\partial u_i}{\partial x_j}. \tag{8.20}$$

In view of incompressibility we may rewrite the second term as

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right). \tag{8.21}$$

Finally we rwerite the third term as:

$$g_i = -\frac{\partial g\zeta}{\partial x_i}, \tag{8.22}$$

where ζ is the height above the earth surface. This makes sense, since if ζ does not depend on x_i, then $g_i = 0$, and, on the other hand, when ζ depends on x_i only, we have $g_i = \pm g$. All intermediate situations are also covered.

Now we multiply the result by u_i:

$$u_i \left[u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i} \left(\frac{p}{\rho} \right) + \frac{\partial g\zeta}{\partial x_i} \right] = 0. \tag{8.23}$$

The first term can be rewritten as

$$u_i u_j \frac{\partial u_i}{\partial x_j} = u_j u_i \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right), \tag{8.24}$$

and in view of the summation convention the indices can be interchanged

$$u_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right) \equiv u_i \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_j u_j \right), \tag{8.25}$$

So,

$$u_i \left[\frac{\partial}{\partial x_i} \left(\frac{1}{2} u_j u_j \right) + \frac{\partial}{\partial x_i} \left(\frac{p}{\rho} \right) + \frac{\partial g\zeta}{\partial x_i} \right] = 0, \tag{8.26}$$

or

$$u_i \frac{\partial}{\partial x_i} \left[\frac{1}{2} u_j u_j + \frac{p}{\rho} + g\zeta \right] = 0. \tag{8.27}$$

Finally, in view of the flow being steady, we recognize the material derivative:

$$\boxed{\frac{D}{Dt} \left[\frac{p}{\rho} + \frac{1}{2} u_j u_j + g\zeta \right] = 0.} \tag{8.28}$$

This is Bernoulli's equation which expresses that the term between brackets is constant along streamlines.

8.5 Exercises

Problem 8.1. *Let* $T(x, y) = T_0 + ax + by$

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y$.

Problem 8.2. *Let* $T(x, y) = T_0 + ax + by + cxy$.

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y$.
(c) Under what condition do the answers of (a) and (b) coincide?

Problem 8.3. *Let* $T(x, y) = T_0 + ax + by$ *and* $x_p(t) = x_o + ut$, $y_p(t) = y_o + vt$.

(a) Compute $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.