Solution IV.5: (Cupola ⋆⋆⋆)

Task a)

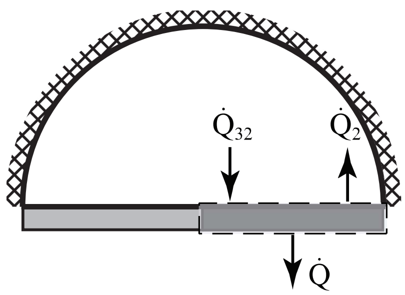
Slab 2 is maintained at a constant temperature of 20 °C. In contrast, slab 1 is at a higher temperature. Consequently, slab 2 must be cooled to sustain this lower temperature. According to thermodynamics, an equilibrium situation must eventually occur over time, resulting in both slabs reaching the same temperature.

Therefore the amount of additional cooling $\dot{Q}$ is exactly equal to the net rate of heat transferred through radiation from slab 1 to slab 2. The amount of heat transferred $\dot{Q}$ to slab 2, can be determined from an outer or inner energy balance around slab 2.

Outer energy balance:

1 Setting up the balance:

Partially, the surface brightness of surface A_3 is radiated on slab 2. Besides, slab 2 radiates its surface brightness towards dome 3. To maintain slab 2 at a lower temperature, the slab must be cooled at a rate of $\dot{Q}$.

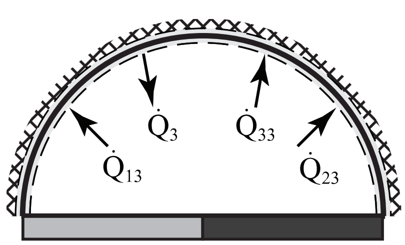


The outer energy balance around slab 2 can be written as:

$$0 = \underbrace{\dot{Q}_{32}}_{\substack{\text{S.B. dome 3}\\ \text{on slab 2}}} - \underbrace{\dot{Q}_{2}}_{\substack{\text{S.B.}\\ \text{slab 2}}} - \underbrace{\dot{Q}}_{\substack{\text{External}\\ \text{cooling}}}. \tag{IV.5.1}$$

The surface brightness of dome 3 is intricate and requires establishing an outer energy balance around dome 3. The surface brightness of dome 3 cannot be determined directly using the equation for a black body radiator, as the temperature of dome 3 is unknown.

Surfaces A_1, A_2 and A_3 partially radiate their surface brightness on surface A_3 and therefore the sum of these fluxes equals the incoming rate of heat transfer.



$$0 = \underbrace{\dot{Q}_{13}}_{\substack{\text{S.B. slab 1}\\ \text{on dome 3}}} + \underbrace{\dot{Q}_{23}}_{\substack{\text{S.B. slab 2}\\ \text{on dome 3}}} + \underbrace{\dot{Q}_{33}}_{\substack{\text{S.B. dome 3}\\ \text{on dome 3}}} - \underbrace{\dot{Q}_{3}}_{\substack{\text{S.B.}\\ \text{dome 3}}}, \tag{IV.5.2}$$

where the respective terms are defined as: $\dot{Q}_{13} = \Phi_{13}\dot{Q}_1$, $\dot{Q}_{23} = \Phi_{23}\dot{Q}_2$, and $\dot{Q}_{33} = \Phi_{33}\dot{Q}_3$.

Rewriting the outer energy balance around dome 3:

$$\dot{Q}_3 = \frac{\dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23}}{1 - \Phi_{33}}. \tag{IV.5.3}$$

2 Defining the elements within the balance:

The surface brightness for the slab equals the surface brightness of a black body radiator:

$$\begin{aligned}\dot{Q}_2 &= \sigma A_2 T_2^4 \\ &= 5.67\cdot 10^{-8}\left(\frac{\mathrm{W}}{\mathrm{m^2K^4}}\right)\cdot\frac{\pi\cdot 3^2}{2}\left(\mathrm{m^2}\right)\cdot 293^4\left(\mathrm{K^4}\right) = 5{,}920\ (\mathrm{W}).\end{aligned} \tag{IV.5.4}$$

The part of the surface brightness transferred from dome 3 to slab 2 can be written as:

$$\dot{Q}_{32} = \Phi_{32}\dot{Q}_3. \tag{IV.5.5}$$

To determine view factor Φ_{32}, view factor Φ_{23} is needed. slab 2 only sees dome 3, and therefore:

$$\Phi_{23} = 1\ (-). \tag{IV.5.6}$$

Using the reciprocity rule

$$\begin{aligned}\Phi_{32}A_3 &= \Phi_{23}A_2 \\ \Rightarrow \Phi_{32} &= \Phi_{23}\frac{\frac{1}{2}\pi R^2}{\frac{4}{2}\pi R^2} = \frac{1}{4}\ (-).\end{aligned} \tag{IV.5.7}$$

Once the view factor has been determined, the surface brightness $\dot{Q}_3$ needs to be assessed.

To obtain the surface brightness of dome 3, the respective view factors and the surface brightness of slab 1 need to be determined.

The sketch shows that slab 1 can only see dome 3, and therefore:

$$\Phi_{13} = 1\ (-). \tag{IV.5.8}$$

Slabs 1 and 2 display symmetry about dome 3, thus Φ_{32} and Φ_{31} are identical:

$$\Phi_{31} = \frac{1}{4}\ (-). \tag{IV.5.9}$$

View factor Φ_{33} can be determined from the summation rule:

$$\begin{aligned}\Phi_{31} + \Phi_{32} + \Phi_{33} &= 1 \\ \Rightarrow \Phi_{33} &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}\ (-).\end{aligned} \tag{IV.5.10}$$

The surface brightness of slab 1 can be expressed as:

$$\dot{Q}_1 = \dot{Q}_{\epsilon,1} + \dot{Q}_{\rho,1} + \dot{Q}_{\tau,1}, \tag{IV.5.11}$$

where the emissive term can be written as:

$$\dot{Q}_{\epsilon,1} = \epsilon_1\sigma A_1 T_1^4, \tag{IV.5.12}$$

the reflective term is determined using the partial surface brightness of dome 3 being reflected:

$$\dot{Q}_{\rho,1} = \rho_1 \Phi_{31} \dot{Q}_3, \tag{IV.5.13}$$

and since all bodies are opaque no radiation is transmitted:

$$\dot{Q}_{\tau,1} = \rho_1 \Phi_{31} \dot{Q}_3. \tag{IV.5.14}$$

Note that slab 1 is a grey body, for grey bodies Kirchoff's law states:

$$\epsilon_1 = \alpha_1. \tag{IV.5.15}$$

Moreover, considering that all surfaces are opaque, and utilizing the relationship between absorptivity, reflectivity, and transmissivity:

$$\rho_1 = 1 - \alpha_1. \tag{IV.5.16}$$

Inserting the expressions of the surface brightness into equation (IV.7.12) and rearranging gives the surface brightness of dome 3:

$$\begin{aligned}\dot{Q}_3 &= \frac{\epsilon_1 \sigma A_1 T_1^4 + \dot{Q}_2}{1 - \Phi_{33} - (1 - \epsilon_1)\, \Phi_{31}} \\ &= \frac{0.6\ (-) \cdot 5.67 \cdot 10^{-8}\ \left(\frac{\mathrm{W}}{\mathrm{m^2 K^4}}\right) \cdot \frac{1}{2}\pi \cdot 3^2\ \left(\mathrm{m^2}\right) \cdot 423^4\ \left(\mathrm{K^4}\right) + 5{,}920\ (\mathrm{W})}{\left(1 - \frac{1}{2} - (1 - 0.6) \cdot \frac{1}{4}\right)\ (-)} = 53{,}348\ (\mathrm{W}).\end{aligned} \tag{IV.5.17}$$

Conclusion

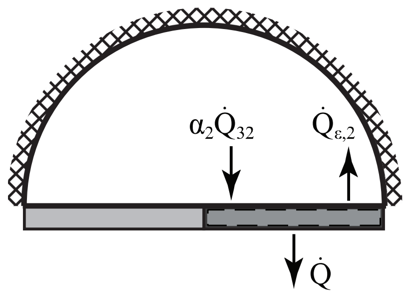
3 **Inserting and rearranging:**

$$\begin{aligned}\dot{Q} &= \Phi_{32} \dot{Q}_3 - \dot{Q}_2 \\ &= \frac{1}{4}\ (-) \cdot 53{,}348\ (\mathrm{W}) - 5{,}920\ (\mathrm{W}) = 7{,}417\ (\mathrm{W}).\end{aligned} \tag{IV.5.18}$$

Inner energy balance:

① **Setting up the balance:**

Partially, the surface brightness of surface A_3 is radiated on slab 2 and absorbed. Besides, slab 2 emits radiation towards dome 3. To maintain slab 2 at a lower temperature, the slab must be cooled at a rate of $\dot{Q}$.

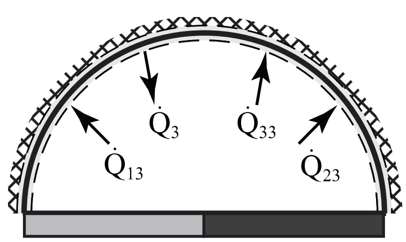


The inner energy balance around slab 2 can be written as:

$$0 = \underbrace{\alpha_2\dot{Q}_{32}}_{\substack{\text{S.B. dome 3}\\ \text{absorbed by slab 2}}} - \underbrace{\dot{Q}_{\epsilon,2}}_{\substack{\text{Emission}\\ \text{slab 2}}} - \underbrace{\dot{Q}}_{\substack{\text{External}\\ \text{cooling}}} . \tag{IV.5.19}$$

The surface brightness of dome 3 is intricate and requires establishing an outer energy balance around dome 3. The surface brightness cannot be determined directly using the equation for a black body radiator, as the temperature of dome 3 is unknown.

Surfaces A_1, A_2 and A_3 partially radiate their surface brightness on surface A_3 and therefore the sum of these fluxes equals the incoming rate of heat transfer.



$$0 = \underbrace{\dot{Q}_{13}}_{\substack{\text{S.B. slab 1}\\ \text{on dome 3}}} + \underbrace{\dot{Q}_{23}}_{\substack{\text{S.B. slab 2}\\ \text{on dome 3}}} + \underbrace{\dot{Q}_{33}}_{\substack{\text{S.B. dome 3}\\ \text{on dome 3}}} - \underbrace{\dot{Q}_{3},}_{\substack{\text{S.B.}\\ \text{dome 3}}} \tag{IV.5.20}$$

where the respective fluxes are defined as: $\dot{Q}_{13} = \Phi_{13}\dot{Q}_1$, $\dot{Q}_{23} = \Phi_{23}\dot{Q}_2$, and $\dot{Q}_{33} = \Phi_{33}\dot{Q}_3$.

Rewriting the inner energy balance around dome 3:

$$\dot{Q}_3 = \frac{\dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23}}{1 - \Phi_{33}}. \tag{IV.5.21}$$

② **Defining the elements within the balance:**

The emitted radiation of slab 2 equals the surface brightness of a black body radiator:

$$\begin{aligned}\dot{Q}_{\epsilon,2} &= \sigma A_2 T_2^4 \qquad \text{(IV.5.22)}\\ &= 5.67\cdot 10^{-8}\left(\frac{\mathrm{W}}{\mathrm{m^2K^4}}\right)\cdot\frac{\pi\cdot 3^2}{2}\left(\mathrm{m^2}\right)\cdot 293^4\left(\mathrm{K^4}\right) = 5{,}920\ (\mathrm{W}).\end{aligned}$$

The part of the surface brightness transferred from dome 3 to slab 2 can be written as:

$$\alpha_2\dot{Q}_{32} = \Phi_{32}\dot{Q}_3, \qquad \text{(IV.5.23)}$$

where $\alpha_2 = 1$ since slab 2 is a black body.

To determine view factor Φ_{32}, view factor Φ_{23} is needed. slab 2 only sees dome 3, and therefore:

$$\Phi_{23} = 1\ \left(-\right). \qquad \text{(IV.5.24)}$$

Using the reciprocity rule

$$\begin{aligned}\Phi_{32}A_3 &= \Phi_{23}A_2 \qquad \text{(IV.5.25)}\\ \Rightarrow \Phi_{32} &= \Phi_{23}\frac{\frac{1}{2}\pi R^2}{\frac{4}{2}\pi R^2} = \frac{1}{4}\ \left(-\right).\end{aligned}$$

Once the view factor has been determined, the surface brightness $\dot{Q}_3$ needs to be assessed.

To obtain the surface brightness of dome 3, the respective view factors and the surface brightness of slab 1 need to be determined.

The sketch shows that slab 1 can only see dome 3, and therefore:

$$\Phi_{13} = 1\ \left(-\right). \qquad \text{(IV.5.26)}$$

Slabs 1 and 2 display symmetry about dome 3, thus Φ_{32} and Φ_{31} are identical:

$$\Phi_{31} = \frac{1}{4}\ \left(-\right). \qquad \text{(IV.5.27)}$$

View factor Φ_{33} can be determined from the summation rule:

$$\begin{aligned}\Phi_{31} + \Phi_{32} + \Phi_{33} &= 1 \qquad \text{(IV.5.28)}\\ \Rightarrow \Phi_{33} &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}\ \left(-\right).\end{aligned}$$

The surface brightness of slab 1 can be expressed as:

$$\dot{Q}_1 = \dot{Q}_{\epsilon,1} + \dot{Q}_{\rho,1} + \dot{Q}_{\tau,1}\ \left(-\right), \qquad \text{(IV.5.29)}$$

where the emissive term can be written as:

$$\dot{Q}_{\epsilon,1} = \epsilon_1\sigma A_1 T_1^4, \qquad \text{(IV.5.30)}$$

the reflective term is determined using the partial surface brightness of dome 3 being reflected:

$$\dot{Q}_{\rho,1} = \rho_1\Phi_{31}\dot{Q}_3, \qquad \text{(IV.5.31)}$$

and since all bodies are opaque no radiation is transmitted:

$$\dot{Q}_{\tau,1} = \rho_1 \Phi_{31} \dot{Q}_3. \tag{IV.5.32}$$

Note that slab 1 is a grey body, for grey bodies Kirchoff's law states:

$$\epsilon_1 = \alpha_1. \tag{IV.5.33}$$

Moreover, considering that all surfaces are opaque, and utilizing the relationship between absorptivity, reflectivity, and transmissivity:

$$\rho_1 = 1 - \alpha_1. \tag{IV.5.34}$$

Inserting the expressions of the surface brightness into equation (IV.7.12) and rearranging gives the surface brightness of dome 3:

$$\begin{aligned}\dot{Q}_3 &= \frac{\epsilon_1 \sigma A_1 T_1^4 + \dot{Q}_2}{1 - \Phi_{33} - (1 - \epsilon_1)\,\Phi_{31}} \\ &= \frac{0.6\ (-) \cdot 5.67 \cdot 10^{-8}\ \left(\frac{\mathrm{W}}{\mathrm{m^2K^4}}\right) \cdot \frac{1}{2}\pi \cdot 3^2\ \left(\mathrm{m^2}\right) \cdot 423^4\ \left(\mathrm{K^4}\right) + 5{,}920\ (\mathrm{W})}{\left(1 - \frac{1}{2} - (1 - 0.6) \cdot \frac{1}{4}\right)\ (-)} = 53{,}348\ (\mathrm{W}).\end{aligned} \tag{IV.5.35}$$

Conclusion

3 **Inserting and rearranging:**

$$\begin{aligned}\dot{Q} &= \Phi_{32} \dot{Q}_3 - \dot{Q}_2 \\ &= \frac{1}{4}\ (-) \cdot 53{,}348\ (\mathrm{W}) - 5{,}920\ (\mathrm{W}) = 7{,}417\ (\mathrm{W}).\end{aligned} \tag{IV.5.36}$$

Task b)

In such a scenario, where the body doesn't transmit radiation and is adiabatic, the surface brightness essentially becomes equal to the emission of a black body radiator. This is because the body, being adiabatic, absorbs all incident radiation and emits radiation at maximum efficiency across all wavelengths, similar to a black body radiator. Therefore the surface brightness of dome 3 can be written as a black body radiator:

$$\dot{Q}_3 = \sigma A_3 T_3^4, \tag{IV.5.37}$$

which was found to have a value of 53,348 W.

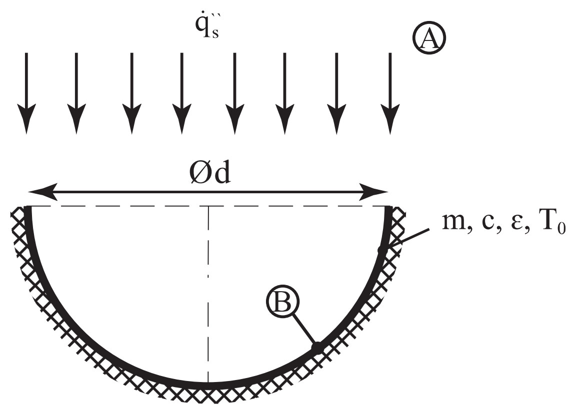
Conclusion

Rewriting gives the temperature of the dome 3:

$$\begin{aligned}T_3 &= \sqrt[4]{\frac{\dot{Q}_3}{\sigma A_3}} \\ &= \sqrt[4]{\frac{53{,}348\ (\mathrm{W})}{5.67 \cdot 10^{-8}\ \left(\frac{\mathrm{W}}{\mathrm{m^2K^4}}\right) \cdot 2\pi \cdot 3^2\ (\mathrm{m^2})}} = 359\ (\mathrm{K}).\end{aligned} \tag{IV.5.38}$$

Exercise IV.6: (Pokè bowl ⋆ ⋆ ⋆)

An empty bowl, that is used for serving the typical traditional Hawaiian dish called pokè bowl, has the homogeneous temperature T_0 and is adiabatically insulated at its convex side. At the time t_0, the bowl is suddenly exposed to parallel radiation from the sun.



Given parameters:

- Mass of the bowl: m
- Specific heat capacity of the bowl: c
- Emissivity of the bowl: $\epsilon \approx 0.5$
- Starting temperature of the bowl: T_0
- Diameter of the bowl: d
- Heat flux of the solar radiation on the ground: $\dot{q}''_{\mathrm{S}}$
- View factor of the bowl to the ambient: Φ_{BA}
- View factor of the bowl to itself: Φ_{BB}

Hints:

- The bowl radiates grey and diffuse and has a homogeneous temperature at any time.
- Influences from the ambient or the atmosphere can be neglected.
- The sun is a black body.

Tasks:

a) Determine the surface brightness of the bowl $\dot{Q}_{\mathrm{B}}$.

Hint: In a nonsteady state, the surface brightness of a grey, adiabatic body is not the same as the surface brightness of a black body.

b) Derive the differential equation for the temperature as a function of time and the necessary initial condition to solve this differential equation.

c) Determine the steady-state final temperature T_{S} of the bowl.

d) Draw the temperature as a function of time qualitatively.

Solution IV.6: (Pokè bowl ⋆ ⋆ ⋆)

Task a)

The surface brightness encompasses both the emission and reflection of solar radiation, including reflection from the bowl itself.

1 **Setting up the balance:**

The surface brightness of the bowl is expressed as:

$$\dot{Q}_{\mathrm{B}} = \dot{Q}_{\epsilon,\mathrm{B}} + \dot{Q}_{\rho,\mathrm{B}} + \dot{Q}_{\tau,\mathrm{B}}. \tag{IV.6.1}$$

2 **Defining the elements within the balance:**

Where the emission of the bowl is described as a grey body radiator:

$$\begin{aligned}\dot{Q}_{\epsilon,\mathrm{B}} &= \epsilon\sigma A_{\mathrm{B}} T_{\mathrm{B}}(t)^4 \\ &= \epsilon\sigma \frac{\pi d^2}{2} T_{\mathrm{B}}(t)^4.\end{aligned} \tag{IV.6.2}$$

In this scenario, transmission is negligible as the convex surface of the bowl is adiabatic:

$$\dot{Q}_{\tau,\mathrm{B}} = 0. \tag{IV.6.3}$$

The reflected radiation from the bowl results from the reflected incident radiation from the sun and the bowl self:

$$\dot{Q}_{\rho,\mathrm{B}} = \rho\left(\dot{Q}_{\mathrm{BB}} + \dot{Q}_{\mathrm{SB}}\right), \tag{IV.6.4}$$

where $\dot{Q}_{\mathrm{BB}}$ states the surface brightness from the bowl that falls on the surface itself, and $\dot{Q}_{\mathrm{SB}}$ describes the radiation from the sun that falls on the bowl.

For a grey body that is not transmitting any radiation ($\tau = 0$) the reflection term is found to be:

$$\begin{aligned}&\epsilon + \rho + \tau = 1 \\ &\Rightarrow \rho = 1 - \epsilon.\end{aligned} \tag{IV.6.5}$$

The part of the surface brightness that falls on the bowl is written as:

$$\dot{Q}_{\mathrm{BB}} = \Phi_{\mathrm{BB}} \dot{Q}_{\mathrm{B}}. \tag{IV.6.6}$$

The radiation transferred from the sun yields the product of the heat flux density and the incident area:

$$\begin{aligned}\dot{Q}_{\mathrm{SB}} &= \dot{q}''_{\mathrm{S}} A_{\mathrm{incident}} \\ &= \dot{q}''_{\mathrm{S}} \frac{\pi d^2}{4},\end{aligned} \tag{IV.6.7}$$

where the incident area is equal to that of a circle.

Conclusion

3 Inserting and rearranging:

$$\dot{Q}_\mathrm{B} = \frac{\epsilon\sigma\frac{\pi d^2}{2}T_\mathrm{B}(t)^4 + (1-\epsilon)\,\dot{q}''_\mathrm{S}\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_\mathrm{BB}}. \tag{IV.6.8}$$

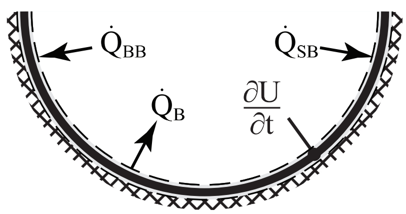
Task b)

To derive the differential equation for the temperature as a function of time, an outer or inner energy balance around the bowl must be established.

Outer energy balance:

1 Setting up the balance:

The bowl receives radiation from the sun, as well as partially the surface brightness which the bowl radiates.



The outer energy balance reads:

$$\underbrace{\frac{\partial U}{\partial t}}_{\substack{\text{Temporal change}\\ \text{of inner energy}}} = \underbrace{\dot{Q}_\mathrm{BB}}_{\substack{\text{S.B. bowl}\\ \text{on bowl}}} + \underbrace{\dot{Q}_\mathrm{SB}}_{\substack{\text{S.B. sun}\\ \text{on bowl}}} - \underbrace{\dot{Q}_\mathrm{B}}_{\substack{\text{S.B.}\\ \text{bowl}}}. \tag{IV.6.9}$$

2 Defining the elements within the balance:

The temporal change in the inner energy of the bowl is described in terms of the temperature, mass, and specific heat capacity:

$$\frac{\partial U}{\partial t} = mc\frac{\partial T_\mathrm{B}}{\partial t}. \tag{IV.6.10}$$

Furthermore, the terms $\dot{Q}_\mathrm{BB}$, $\dot{Q}_\mathrm{SB}$, and $\dot{Q}_\mathrm{B}$ had been defined in the previous task.

Conclusion

3 Inserting and rearranging:

$$mc\frac{\partial T_\mathrm{B}}{\partial t} = \frac{\epsilon\sigma\frac{\pi d^2}{2}T_\mathrm{B}(t)^4 + (1-\epsilon)\,\dot{q}''_\mathrm{S}\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_\mathrm{BB}}\left(\Phi_\mathrm{BB}-1\right) + \dot{q}''_\mathrm{s}\frac{\pi d^2}{4}. \tag{IV.6.11}$$

4 Defining the boundary and/or initial conditions:

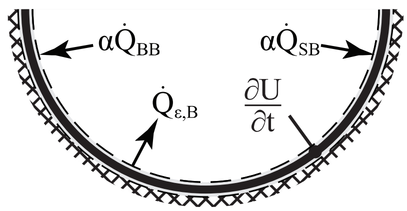
To solve the differential equation, one initial condition is required, which reads as follows:

$$T(t=t_0) = T_0. \tag{IV.6.12}$$

Inner energy balance:

1 Setting up the balance:

The bowl receives radiation from the sun, as well as partially the surface brightness which the bowl radiates.



The inner energy balance reads:

$$\underbrace{\frac{\partial U}{\partial t}}_{\substack{\text{Temporal change of}\\\text{inner energy}}} = \underbrace{\alpha\dot{Q}_{\mathrm{BB}}}_{\substack{\text{S.B. bowl}\\\text{absorbed by bowl}}} + \underbrace{\alpha\dot{Q}_{\mathrm{SB}}}_{\substack{\text{S.B. sun}\\\text{absorbed by bowl}}} - \underbrace{\dot{Q}_{\epsilon,\mathrm{B}}}_{\substack{\text{Emission}\\\text{bowl}}}. \tag{IV.6.13}$$

2 Defining the elements within the balance:

The temporal change in the inner energy of the bowl is described in terms of the temperature, mass, and specific heat capacity:

$$\frac{\partial U}{\partial t} = mc\frac{\partial T_{\mathrm{B}}}{\partial t}. \tag{IV.6.14}$$

Furthermore, the terms $\dot{Q}_{\mathrm{BB}}$, $\dot{Q}_{\mathrm{SB}}$, and $\dot{Q}_{\epsilon,\mathrm{B}}$ had been defined in the previous task.

The bowl acts as a grey body, for these bodies Kirchoff's law states:

$$\alpha = \epsilon. \tag{IV.6.15}$$

Conclusion

3 Inserting and rearranging:

$$mc\frac{\partial T_{\mathrm{B}}}{\partial t} = \frac{\epsilon^2\sigma\frac{\pi d^2}{2}T_{\mathrm{B}}(t)^4 + \left(\epsilon - \epsilon^2\right)\dot{q}''_{\mathrm{S}}\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_{\mathrm{BB}}}\Phi_{\mathrm{BB}} + \epsilon\left(\dot{q}''_{\mathrm{s}}\frac{\pi d^2}{4} - \sigma\frac{\pi d^2}{2}T_{\mathrm{B}}(t)^4\right). \tag{IV.6.16}$$

4 Defining the boundary and/or initial conditions:

To solve the differential equation, one initial condition is required, which reads as follows:

$$T(t=t_0) = T_0. \tag{IV.6.17}$$

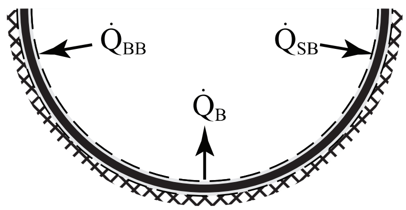
Task c)

After some time the system reaches equilibrium conditions and so the internal energy does not change anymore.

To determine the equilibrium temperature, an energy balance must be derived that describes the equilibrium state. This energy balance is similar to the energy balance derived in question b), but without the transient term describing the change in internal energy over time.

1 **Setting up the balance:**

The bowl receives radiation from the sun, as well as partially the surface brightness which the bowl radiates.



Therefore, the steady-state outer energy balance equals:

$$0 = \underbrace{\dot{Q}_{\mathrm{BB}}}_{\substack{\text{S.B. bowl}\\ \text{on bowl}}} + \underbrace{\dot{Q}_{\mathrm{SB}}}_{\substack{\text{S.B. sun}\\ \text{on bowl}}} - \underbrace{\dot{Q}_{\mathrm{B}}.}_{\substack{\text{S.B.}\\ \text{bowl}}} \tag{IV.6.18}$$

2 **Defining the elements within the balance:**

The surface brightness of the bowl now is written as: Inserting and rewriting:

$$\dot{Q}_{\mathrm{B}} = \frac{\epsilon\sigma \frac{\pi d^2}{2} T_{\mathrm{S}}^4 + (1-\epsilon)\,\dot{q}_{\mathrm{S}}'' \frac{\pi d^2}{4}}{1 - (1-\epsilon)\,\Phi_{\mathrm{BB}}}. \tag{IV.6.19}$$

The fluxes $\dot{Q}_{\mathrm{BB}}$ and $\dot{Q}_{\mathrm{SB}}$ have already been defined in the previous tasks.

Conclusion

3 **Inserting and rearranging:**

$$T_{\mathrm{S}} = \sqrt[4]{\frac{\dot{q}_{\mathrm{S}}''}{2\sigma}\left[\frac{1}{(1-\Phi_{\mathrm{BB}})}\right]}. \tag{IV.6.20}$$

Alternative solution:

A different result could have been obtained by simplifying the fact that in the stationary case, the surface brightness of the bowl is equal to that of a black body since the body is gray and the bowl's back is adiabatic. This results:

2 **Defining the elements within the balance:**

$$\dot{Q}_{\mathrm{B}} = \sigma A_{\mathrm{B}} T_{\mathrm{S}}. \tag{IV.6.21}$$

3 Inserting and rearranging:

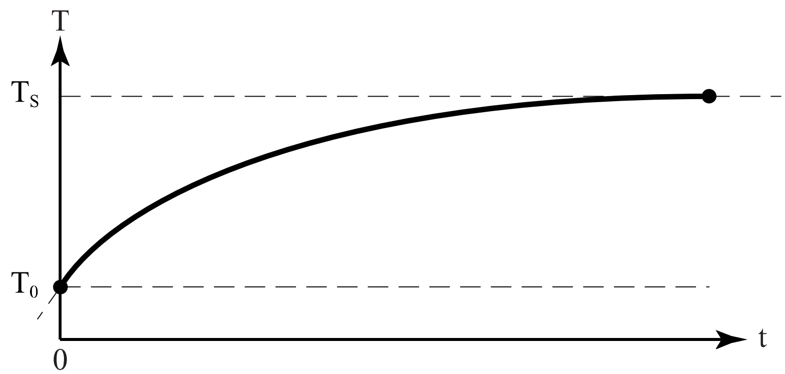
$$T_S = \sqrt[4]{\frac{\dot{q}''_S}{2\sigma}\left[\frac{1}{(1-\Phi_{BB})}\right]}, \tag{IV.6.22}$$

which is identical to the found solution before.

Task d)

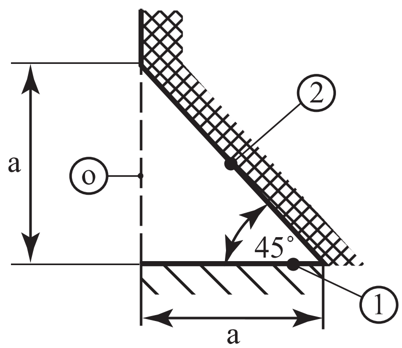
As temperature rises, the gradient must gradually decrease, ultimately approaching zero as time tends towards infinity. This phenomenon is understood through physical principles: the absorbed heat flux from the sun remains constant. Initially, with low temperatures, the emitted heat flux is minimal, resulting in a significant temperature rise (high net heat gain). Over time, this temperature increase diminishes as the absorbed and emitted heat fluxes converge, reaching equilibrium when both fluxes balance out in a steady state.

Conclusion



Exercise IV.7: (Radiation within a wedge-shaped opening ⋆⋆⋆)

Consider an infinitely long opening with a wedge-shaped cross-section as shown in the figure below.



Given parameters:

- Temperature of surface 1: $T_1 = 1000\,\mathrm{K}$
- Temperature of space surrounding: $T_o = 0\,\mathrm{K}$
- Emissivity of surface 1: $\varepsilon_1 = 1$
- Width: $a = 30$ cm

Hints:

- Surface 2 is a grey body and adiabatically insulated at the back.
- The space surrounding the opening can be considered to be a black body.
- Influences due to convection shall be disregarded.

Tasks:

a) Determine all view factors.

b) Determine the energy through the opening $\dot{q}'_{\mathrm{o,loss}}$ for a unit length of the opening.

c) Determine the temperature T_2 of surface 2.

Solution IV.7: (Radiation within a wedge-shaped opening ⋆ ⋆ ⋆)

Task a)

When determining the view factors, recall the general rules that apply to view factors. First the summation rule:

$$\sum_{j=1}^{n} \Phi_{ij} = \Phi_{i1} + \Phi_{i2} + \Phi_{i3} + ... + \Phi_{in} = 1,$$

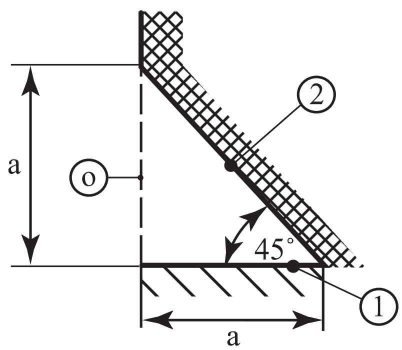
second, the reciprocity rule:

$$A_i \Phi_{ij} = A_j \Phi_{ji},$$

and third the symmetry rule:

$$\Phi_{ij} = \Phi_{ik},$$

which applies if two or more surfaces display symmetry about a third surface, they will have identical view factors from that surface.



Surfaces 1, 2 and o cannot see themselves and therefore:

$$\Phi_{11} = 0 \ \ (-)\,, \tag{IV.7.1}$$

$$\Phi_{22} = 0 \ \ (-)\,, \tag{IV.7.2}$$

and:

$$\Phi_{oo} = 0 \ \ (-)\,. \tag{IV.7.3}$$

Surfaces o and 1 display symmetry around surface 2, thus Φ_{2o} and Φ_{21} must be the same:

$$\Phi_{2o} = \frac{1}{2} \ \ (-)\,, \tag{IV.7.4}$$

and:

$$\Phi_{21} = \frac{1}{2} \ \ (-)\,. \tag{IV.7.5}$$

The reciprocity rule is used to determine Φ_{o2}:

$$A_o \Phi_{o2} = A_2 \cdot \Phi_{2o} \tag{IV.7.6}$$

$$\Phi_{o2} = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{2} \ \ (-)\,,$$

and Φ_{12}:

$$A_1 \cdot \Phi_{12} = A_2 \cdot \Phi_{21} \tag{IV.7.7}$$
$$\Phi_{12} = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{2} \;\; (-)\,.$$

Using the summation rule to determine $\Phi_{1\mathrm{o}}$

$$\Phi_{11} + \Phi_{12} + \Phi_{1\mathrm{o}} = 1 \tag{IV.7.8}$$
$$\Phi_{1\mathrm{o}} = 1 - \frac{\sqrt{2}}{2} \;\; (-)\,,$$

and $\Phi_{\mathrm{o}1}$:

$$\Phi_{\mathrm{oo}} + \Phi_{\mathrm{o}1} + \Phi_{\mathrm{o}2} = 1 \tag{IV.7.9}$$
$$\Rightarrow \boldsymbol{\Phi_{\mathrm{o}1} = 1 - \frac{\sqrt{2}}{2}} \;\; (-)\,.$$

Conclusion

Al view factors are written as:

$$\Phi_{11} = 0, \qquad \Phi_{12} = \frac{\sqrt{2}}{2}, \qquad \Phi_{1\mathrm{o}} = 1 - \frac{\sqrt{2}}{2}.$$
$$\Phi_{21} = \frac{1}{2}, \qquad \Phi_{22} = 0, \qquad \Phi_{2\mathrm{o}} = \frac{1}{2}.$$
$$\Phi_{\mathrm{o}1} = 1 - \frac{\sqrt{2}}{2}, \qquad \Phi_{\mathrm{o}2} = \frac{\sqrt{2}}{2}, \qquad \Phi_{\mathrm{oo}} = 0.$$

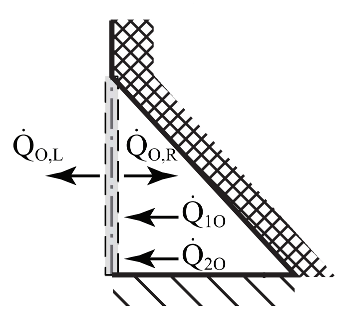
Task b)

To determine the energy loss through the opening, the first step is setting up an energy balance around the opening. The simplification is made that the opening is considered to be a surface that transmits all incident radiation $\tau_{\mathrm{o}} = 1$ and therefore does not reflect, nor emit due to the temperature being 0 K.

Outer energy balance:

1 **Setting up the balance:**

The opening is radiating its surface brightness on the left- and right sides of the opening. Besides, partially the surface brightness of surfaces 1 and 2 is radiated towards the opening.



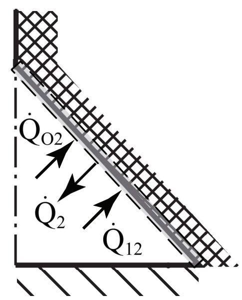
The outer energy balance reads:

$$0 = \underbrace{\dot{Q}_{1\mathrm{o}}}_{\substack{\text{S.B. surface 1}\\ \text{on opening}}} + \underbrace{\dot{Q}_{2\mathrm{o}}}_{\substack{\text{S.B. surface 2}\\ \text{on opening}}} - \underbrace{\dot{Q}_{\mathrm{o,L}}}_{\substack{\text{S.B. opening}\\ \text{left-side}}} - \underbrace{\dot{Q}_{\mathrm{o,R}}}_{\substack{\text{S.B. opening}\\ \text{right-side}}}, \tag{IV.7.10}$$

where the surface brightness $\dot{Q}_{\mathrm{o,L}}$ radiated by the left surface equals the energy loss from the opening.

The surface brightness of body 2 is intricate and requires establishing an outer energy balance around body 2. The expression of this surface brightness cannot be determined directly, because the temperature of the body is unknown.

Surfaces 1 and o partially radiate their surface brightness on surface 2, whereas surface 2 emits its surface brightness.



The outer energy balance around surface 2 reads:

$$0 = \underbrace{\dot{Q}_{12}}_{\substack{\text{S.B. surface 1}\\ \text{on surface 2}}} + \underbrace{\dot{Q}_{\mathrm{o}2}}_{\substack{\text{S.B. opening}\\ \text{on surface 2}}} - \underbrace{\dot{Q}_{2}}_{\substack{\text{S.B.}\\ \text{surface 2}}}, \tag{IV.7.11}$$

where the respective fluxes are defined as: $\dot{Q}_{12} = \Phi_{12}\dot{Q}_1$, and $\dot{Q}_{\mathrm{o}2} = \Phi_{\mathrm{o}2}\dot{Q}_{\mathrm{O,R}}$.

Rewriting the inner energy balance around surface 2 gives:

$$\dot{Q}_2 = \Phi_{12}\dot{Q}_1 + \Phi_{\mathrm{o}2}\dot{Q}_{\mathrm{O,R}}. \tag{IV.7.12}$$

2 **Defining the elements within the balance:**

The surface brightness from body 1 radiated through the opening is written as:

$$\dot{Q}_{1\mathrm{o}} = \Phi_{1\mathrm{o}}\dot{Q}_1, \tag{IV.7.13}$$

where the surface brightness of body 1 is written to be that of a black body radiator ($\epsilon_1 = 1$):

$$\begin{aligned}\dot{Q}_1 &= \sigma A_1 T_1^4 \\ &= \sigma\left(a \times L\right) T_1^4,\end{aligned} \tag{IV.7.14}$$

where L describes the length of the wedge.

As the opening transmits all radiation, the surface brightness for the right side is described by:

$$\dot{Q}_{\mathrm{o,R}} = \dot{Q}_{\epsilon,\mathrm{o,R}} + \dot{Q}_{\rho,\mathrm{o,R}} + \dot{Q}_{\tau,\mathrm{o,R}}. \tag{IV.7.15}$$

Since the opening cannot emit ($T_o = 0$ K), the emission term is written as:

$$\dot{Q}_{\epsilon,o,R} = 0. \tag{IV.7.16}$$

Nor can the opening reflect reflect (because $\rho = 0$), and thus the reflection term is:

$$\dot{Q}_{\rho,o,R} = 0. \tag{IV.7.17}$$

Furthermore, the opening does not receive any radiation from the ambient, and therefore no radiation is transmitted through the right side:

$$\dot{Q}_{\tau,o,R} = 0. \tag{IV.7.18}$$

The surface brightness from body 2 radiated through the opening is stated as:

$$\dot{Q}_{2o} = \Phi_{2o}\dot{Q}_2, \tag{IV.7.19}$$

where the surface brightness of body 2 yields from the outer energy balance set around this body:

$$\dot{Q}_2 = \Phi_{12}\sigma\left(a \times L\right)T_1^4. \tag{IV.7.20}$$

3 **Inserting and rearranging:**

$$\dot{Q}_{o,L} = \left(\Phi_{1o} + \Phi_{2o}\Phi_{12}\right)\sigma\left(a \times L\right)T_1^4. \tag{IV.7.21}$$

Dividing by the respective unit length L results in the energy through the opening for a unit length:

$$\begin{aligned} \dot{q}'_{\text{loss}} &= \frac{\dot{Q}_{o,L}}{L} \\ &= \left(\Phi_{1o} + \Phi_{2o}\Phi_{12}\right)\sigma a T_1^4 \\ &= \left(1 - \frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} +\right)(-)\cdot 5.67\cdot 10^{-8}\left(\frac{\text{W}}{\text{m}^2\text{K}^4}\right)\cdot 0.3\ (\text{m})\cdot 1000^4\ \left(\text{K}^4\right) = 11\left(\frac{\text{kW}}{\text{m}}\right). \end{aligned} \tag{IV.7.22}$$

Conclusion

The energy through the opening for a unit length is thus 11 $\frac{\text{kW}}{\text{m}}$.

Task c)

In the previous task, the surface brightness of surface 2 was already determined:

$$\begin{aligned} \dot{Q}_2 &= \Phi_{12}\sigma A_1 T_1^4 \\ &= \Phi_{12}\sigma\left(a \times L\right)T_1^4. \end{aligned} \tag{IV.7.23}$$

1 **Setting up the balance:**

Alternatively, the surface brightness can be written in terms of emission, reflection, and transmission.

$$\dot{Q}_2 = \dot{Q}_{\epsilon,2} + \dot{Q}_{\rho,2} + \dot{Q}_{\tau,2}. \tag{IV.7.24}$$

2 **Defining the elements within the balance:**

The emission term must be written as:

$$\begin{aligned} \dot{Q}_{\epsilon,2} &= \epsilon_2\sigma A_2 T_2^4 \\ &= \epsilon_2\sigma\left(a\sqrt{2} \times L\right)T_2^4, \end{aligned} \tag{IV.7.25}$$

the reflection term as:

$$\dot{Q}_{\rho,2} = \rho_2 \dot{Q}_{12} \\ = (1-\epsilon_2)\,\Phi_{12}\sigma\,(a \times L)\,T_1^4, \tag{IV.7.26}$$

and since the back of surface 2 is adiabatic, no radiation is transmitted:

$$\dot{Q}_{\tau,2} = 0. \tag{IV.7.27}$$

3 **Inserting and rearranging:**

Equating the expression obtained from the previous task with the newly derived one:

$$T_2 = \sqrt[4]{\frac{\Phi_{12}T_1^4}{\sqrt{2}}} \\ = \sqrt[4]{\frac{1000^4\ (\mathrm{K}^4)}{2}} = 841\ (\mathrm{K}). \tag{IV.7.28}$$

Conclusion

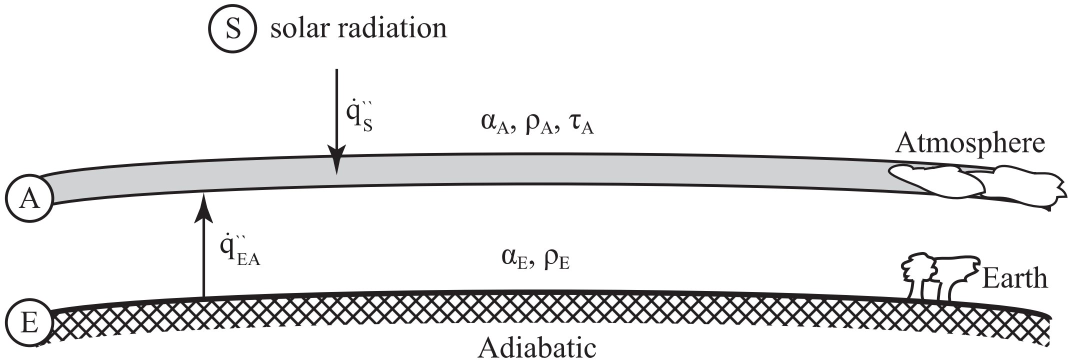
The temperature of surface 2 is thus 841 K.

Exercise IV.8: (Earth's atmosphere ⋆ ⋆ ⋆)

The climate on Earth is influenced by the atmosphere to a great extent. To describe heat transfer between Earth and space, it is assumed that the atmosphere surrounds the Earth as a thin, distinct layer.

When balancing radiative heat flows, long-wave and short-wave radiation must be distinguished (indices LW and SW). Earth and atmosphere (indices E and A) have specific absorption, reflection, and transmission coefficients (α, ρ, τ) for long-wave and short-wave radiation each. The spectrum of solar radiation ($\dot{q}''_{\mathrm{S}}$) is assumed to be in the short-wave range only, whereas emission from earth and atmosphere is in the long-wave range only.

Additionally to the radiative heat fluxes, a net heat flux $\dot{q}''_{\mathrm{EA}}$ is carried from the earth into the atmosphere, which leads back to convective heat transfer and vaporization.



Given parameters:

- Short-wave solar radiation: $\dot{q}''_{\mathrm{S,SW}} = 341\ \mathrm{W/m^2}$
- Long-wave solar radiation: $\dot{q}''_{\mathrm{S,LW}} = 0\ \mathrm{W/m^2}$
- Convection and vaporization: $\dot{q}''_{\mathrm{EA}} = 101\ \mathrm{W/m^2}$

	Short-wave	**Long-wave**
Atmosphere	$\rho_{\mathrm{A,SW}} = 0.23$	$\rho_{\mathrm{A,LW}} = 0.34$
	$\tau_{\mathrm{A,SW}} = 0.54$	$\tau_{\mathrm{A,LW}} = 0.10$
	$\alpha_{\mathrm{A,SW}} = 0.23$	$\alpha_{\mathrm{A,LW}} = 0.56$
	emission negligible	emission
Earth	$\rho_{\mathrm{E,SW}} = 0.16$	$\alpha_{\mathrm{E,LW}} = 1.00$
	$\alpha_{\mathrm{E,SW}} = 0.84$	
	emission negligible	emission
Solar radiation	emission	emission negligible

Hints:

- Curvature is negligible, i.e. earth and atmosphere have the same surface area and the atmosphere does not radiate onto itself.
- The atmosphere emits equally in both directions.
- The given heat fluxes are averaged across the entire earth and over multiple years. Do not distinguish between the light and dark hemispheres.

- Assume steady state.

Tasks:

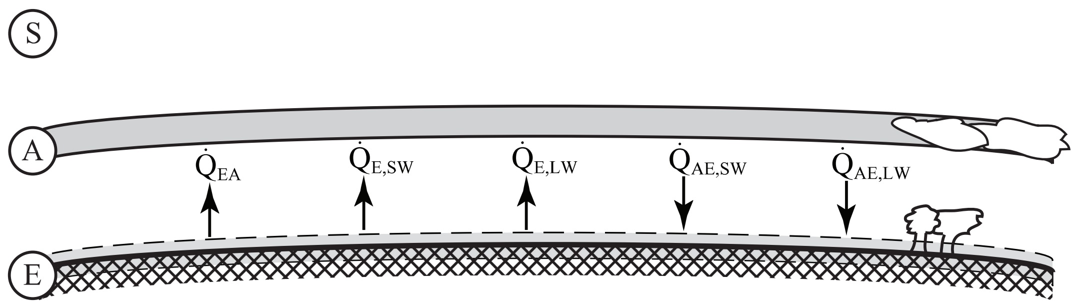
a) Determine the flux of short-wave radiation which hits onto the earth's surface $\dot{q}''_{\mathrm{SW\ to\ E}}$.

b) Give all energy balances and surface brightnesses necessary to determine the temperature at the earth's surface. You may assume that the spectrum of black body radiation is completely within the long-wave range for that temperature.

Solution IV.8: (Earth's atmosphere ⋆ ⋆ ⋆)

Task a)

The Earth only receives radiation from the atmosphere. This radiation must be distinguished based on the short- and long-wavelength ranges. Furthermore, the Earth radiates short- and longe-wavelength radiation and transports heat due to convection and vaporization.



This shows that the only short-wave radiation that hits the Earth's surface is the surface brightness hitting the bottom of the atmosphere.

1 **Setting up the balance:**

The short-wave surface brightness of the bottom of the atmosphere is stated as:

$$\dot{Q}_{\mathrm{A,bot,SW}} = \dot{Q}_{\epsilon,\mathrm{A,bot,SW}} + \dot{Q}_{\rho,\mathrm{A,bot,SW}} + \dot{Q}_{\tau,\mathrm{A,bot,SW}}. \tag{IV.8.1}$$

2 **Defining the elements within the balance:**

Given that the short-wave emission is negligible:

$$\dot{Q}_{\epsilon,\mathrm{A,bot,SW}} = 0. \tag{IV.8.2}$$

Some of the incident solar radiation is transmitted through the bottom of the atmosphere, and therefore the shortwave transmitted radiation of the atmosphere is defined as:

$$\begin{aligned} \dot{Q}_{\tau,\mathrm{A,bot,SW}} &= \tau_{\mathrm{A,SW}} \dot{Q}_{\mathrm{SA,SW}} \\ &= \tau_{\mathrm{A,SW}} A_{\mathrm{A}} \dot{q}''_{\mathrm{S,SW}}. \end{aligned} \tag{IV.8.3}$$

Furthermore, some of the radiation of the shortwave surface brightness of the Earth is reflected at the Earth by the inside of the atmosphere. Therefore, the reflected radiation on the inside of the atmosphere is expressed as:

$$\begin{aligned} \dot{Q}_{\rho,\mathrm{A,bot,SW}} &= \rho_{\mathrm{A,SW}} \dot{Q}_{\mathrm{EA,SW}} \\ &= \rho_{\mathrm{A,SW}} \dot{Q}_{\mathrm{E,SW}}, \end{aligned} \tag{IV.8.4}$$

where $\Phi_{\mathrm{EA}} = 1$ because the Earth and atmosphere only see each other and not themselves. Heat transport through convection and vaporization is not considered when determining reflected radiation because convection and vaporization primarily involve the transfer of latent heat and sensible heat, rather than the reflection of electromagnetic radiation.

The issue with the expression is that the unknown short-wave surface brightness of the Earth is used. Therefore this expression has to be defined as well:

$$\dot{Q}_{\mathrm{E,SW}} = \dot{Q}_{\epsilon,\mathrm{E,SW}} + \dot{Q}_{\rho,\mathrm{E,SW}} + \dot{Q}_{\tau,\mathrm{E,SW}}. \tag{IV.8.5}$$

The emission for the short wavelength of the Earth is negligible:

$$\dot{Q}_{\epsilon,\mathrm{E,SW}} = 0. \tag{IV.8.6}$$

Similarly, due to the back of the Earth being adiabatic, no radiation is transmitted:

$$\dot{Q}_{\tau,\mathrm{E,SW}} = 0. \tag{IV.8.7}$$

Lastly, the reflected radiation by the Earth yields from the reflected radiation received from the atmosphere:

$$\begin{aligned}\dot{Q}_{\rho,\mathrm{E,SW}} &= \rho_{\mathrm{E,SW}}\dot{Q}_{\mathrm{AE,SW}} \\ &= \rho_{\mathrm{E,SW}}\dot{Q}_{\mathrm{A,bot,SW}},\end{aligned} \tag{IV.8.8}$$

where $\Phi_{\mathrm{AE}} = 1$ because the Earth and atmosphere only see each other and not themselves.

3 **Inserting and rearranging:**

$$\begin{aligned}\dot{Q}_{\mathrm{A,bot,SW}} &= \frac{\tau_{\mathrm{A,SW}} A_{\mathrm{A}} \dot{q}''_{\mathrm{S,SW}}}{1-\rho_{\mathrm{A,SW}}\rho_{\mathrm{E,SW}}} \\ \Rightarrow \dot{q}''_{\mathrm{A,bot,SW}} &= \frac{\tau_{\mathrm{A,SW}} \dot{q}''_{\mathrm{S,SW}}}{1-\rho_{\mathrm{A,SW}}\rho_{\mathrm{E,SW}}} \\ &= \frac{0.54\,[-]\cdot 341\,\left[\frac{\mathrm{W}}{\mathrm{m^2}}\right]}{(1-0.23\cdot 0.16)\,[-]} = 191\left(\frac{\mathrm{W}}{\mathrm{m^2}}\right).\end{aligned} \tag{IV.8.9}$$

Conclusion

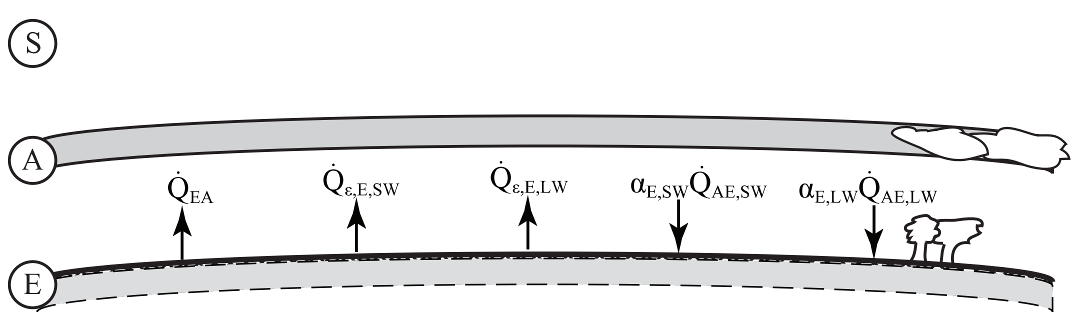
The short-wave radiation that hits Earth's surface is thus 191 $\frac{\mathrm{W}}{\mathrm{m^2}}$.

Task b)

The Earth's temperature is determined through an inner energy balance radiation process around the planet. This balance encompasses the short- and long-wave emissions emitted by the Earth itself, as well as the short- and long-wave surface brightness of the atmosphere's interior radiated toward the Earth. Additionally, heat is transferred as a result of convection and vaporization.

1 **Setting up the balance:**

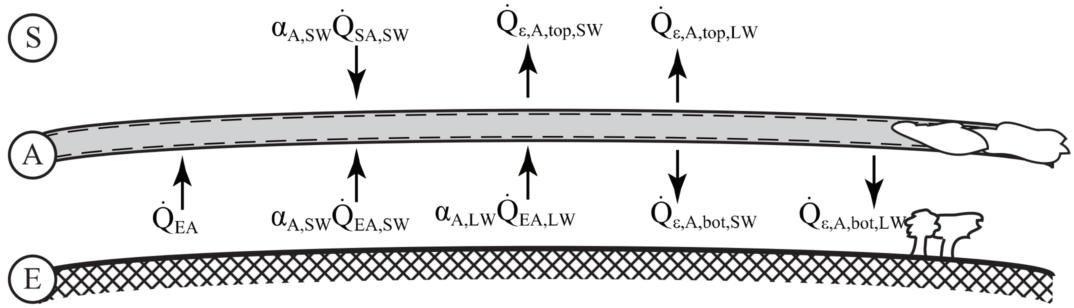
The inner energy balance comprises terms that account for heat loss due to convection and evaporation, as well as emissions in both long and short wavelengths. Additionally, the Earth absorbs short and long-wavelength radiation received from the atmosphere.



The inner energy balance around the Earth reads:

$$0 = \underbrace{\alpha_{\text{E,SW}}\dot{Q}_{\text{AE,SW}}}_{\substack{\text{SW S.B. atmosphere}\\ \text{absorbed by Earth}}} + \underbrace{\alpha_{\text{E,LW}}\dot{Q}_{\text{AE,LW}}}_{\substack{\text{LW S.B. atmosphere}\\ \text{absorbed by Earth}}} - \underbrace{\dot{Q}_{\epsilon,\text{E,SW}}}_{\substack{\text{SW emission}\\ \text{by Earth}}} - \underbrace{\dot{Q}_{\epsilon,\text{E,LW}}}_{\substack{\text{LW emission}\\ \text{by Earth}}} - \underbrace{\dot{Q}_{\text{EA}}}_{\substack{\text{vaporization}\\ \text{and convection}}}. \tag{IV.8.10}$$

The long-wave emission of the bottom of the atmosphere is intricate and requires establishing an outer energy balance around the atmosphere. The long-wave surface brightness of the atmosphere cannot be determined directly, as the temperature of the atmosphere is unknown.



The inner energy balance around the atmosphere reads:

$$\begin{aligned}0 = &\underbrace{\dot{Q}_{\text{SA}}}_{\substack{\text{S.B. solar}\\ \text{to Earth}}} + \underbrace{\dot{Q}_{\text{EA}}}_{\substack{\text{vaporization}\\ \text{and convection}}} + \underbrace{\dot{Q}_{\text{EA,SW}}}_{\substack{\text{SW S.B. Earth}\\ \text{to atmosphere}}} + \underbrace{\dot{Q}_{\text{EA,LW}}}_{\substack{\text{LW S.B. Earth}\\ \text{to atmosphere}}}\\ &- \underbrace{\dot{Q}_{\text{A,top,SW}}}_{\substack{\text{SW S.B.}\\ \text{top atmosphere}}} - \underbrace{\dot{Q}_{\text{A,top,LW}}}_{\substack{\text{LW S.B.}\\ \text{top atmosphere}}} - \underbrace{\dot{Q}_{\text{A,bot,SW}}}_{\substack{\text{SW S.B.}\\ \text{bottom atmosphere}}} - \underbrace{\dot{Q}_{\text{A,bot,LW}}}_{\substack{\text{LW S.B.}\\ \text{bottom atmosphere}}},\end{aligned} \tag{IV.8.11}$$

where the respective fluxes are defined as: $\dot{Q}_{\text{EA,SW}} = \dot{Q}_{\text{A,SW}}$, $\dot{Q}_{\text{EA,LW}} = \dot{Q}_{\text{A,LW}}$, $\dot{Q}_{\epsilon,\text{A,bot,SW}} = 0$, $\dot{Q}_{\epsilon,\text{A,top,SW}} = 0$, $\dot{Q}_{\epsilon,\text{A,bot,LW}} = \dot{Q}_{\epsilon,\text{A,top,LW}}$.

Rewriting the balance:

$$\dot{Q}_{\epsilon,\text{A,bot,LW}} = \frac{1}{2}\left(\dot{Q}_{\text{SA}} + \dot{Q}_{\text{EA}} + \dot{Q}_{\text{E,SW}} + \dot{Q}_{\text{E,LW}}\right). \tag{IV.8.12}$$

2 **Defining the elements within the balance:**

The heat transported due to convection and vaporization $\dot{q}''_{\text{EA}}$ is written as:

$$\dot{Q}_{\text{EA}} = \dot{q}''_{\text{EA}} A_{\text{E}}. \tag{IV.8.13}$$

The Earth's emission at long wavelengths is described as that of a black body radiator, because $\alpha_{\text{E,LW}} = \epsilon_{\text{E,LW}} = 1$.

$$\dot{Q}_{\epsilon,\text{E,LW}} = \sigma A_{\text{E}} T_{\text{E}}^4. \tag{IV.8.14}$$

Similarly to the short-wave surface brightness of the inside of the atmosphere, as in question a), the long-wave surface brightness of the inside of the atmosphere is expressed in terms of emission, reflection, and transmission.

$$\dot{Q}_{\text{A,LW}} = \dot{Q}_{\epsilon,\text{A,LW}} + \dot{Q}_{\rho,\text{A,LW}} + \dot{Q}_{\tau,\text{A,LW}}. \tag{IV.8.15}$$

The transmission term is zero because only solar radiation, which consists of short-wave radiation, is transmitted toward the Earth. However, as long-wave radiation surface brightness focuses on emissions in the long wavelength spectrum, this does not include short-wave radiation.

$$\dot{Q}_{\tau,\mathrm{A,LW}} = 0. \tag{IV.8.16}$$

The emission term has been defined previously by setting up the inner energy balance around the atmosphere. To determine this parameter, the radiation received by the atmosphere from the sun is written as:

$$\dot{Q}_{\mathrm{SA}} = \dot{q}''_{\mathrm{S,SW}} A_{\mathrm{A}}, \tag{IV.8.17}$$

The short-wave surface brightness of the Earth is expressed as:

$$\dot{Q}_{\mathrm{E,SW}} = \dot{Q}_{\epsilon,\mathrm{E,SW}} + \dot{Q}_{\rho,\mathrm{E,SW}} + \dot{Q}_{\tau,\mathrm{E,SW}}, \tag{IV.8.18}$$

where the emission is negligible:

$$\dot{Q}_{\epsilon,\mathrm{E,SW}} = 0, \tag{IV.8.19}$$

as well as the transmission term:

$$\dot{Q}_{\tau,\mathrm{E,SW}} = 0, \tag{IV.8.20}$$

and the reflection is found from the reflection of the long-wave surface brightness of the atmosphere:

$$\dot{Q}_{\rho,\mathrm{E,SW}} = \rho_{\mathrm{E,SW}} \dot{Q}_{\mathrm{AE,SW}}. \tag{IV.8.21}$$

Lastly, the long-wave surface brightness of the Earth is expressed as that of a black-body radiator:

$$\dot{Q}_{\mathrm{E,LW}} = \sigma A_{\mathrm{E}} T_{\mathrm{E}}^4. \tag{IV.8.22}$$

Furthermore the short-wave surface brightness of the atmosphere to the Earth $\dot{Q}_{\mathrm{AE,SW}}$ and the short-wave emission term of the Earth $\dot{Q}_{\epsilon,\mathrm{E,SW}}$ were already determined in the previous task.

Conclusion

3 **Inserting and rearranging:**

Therefore exactly as many equations as undetermined parameters are found, which makes the problem solvable.