2.6 Cupola ★ ★ ★

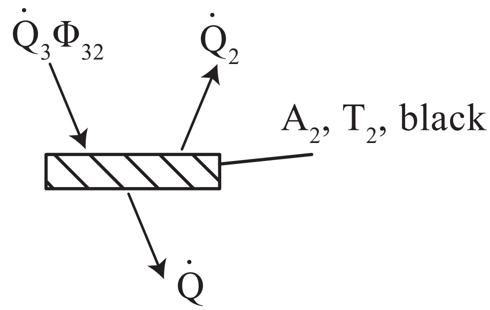
a) *Compute the amount of heat transferred through radiation between the surfaces A_1 and A_2.*

To compute the amount of heat transferred $\dot{Q}$ through surface A_2, an outer energy balance around surface 2 should be established.

Partially, the surface brightness of surface A_3 is radiated on surface A_2. Besides, surface A_2 radiates its surface brightness towards the environment.

The difference between these fluxes equals the rate of heat transferred through surface A_2.

1) Setting up an energy balance:



Resulting in the outer energy balance around surface A_2:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.57}$$

$$0 = \dot{Q}_3 \Phi_{32} - \dot{Q}_2 - \dot{Q} \tag{2.58}$$

Rewriting yields:

$$\boxed{\rightarrow \dot{Q} = \dot{Q}_3 \Phi_{32} - \dot{Q}_2} \tag{2.59}$$

2) Defining the fluxes and view factors:

It can already been seen from the figure that Φ_{11}, Φ_{12}, Φ_{21} and Φ_{22} are zero.

$$\boxed{\rightarrow \Phi_{11} = \Phi_{12} = \Phi_{21} = \Phi_{22}} = 0 \tag{2.60}$$

With this given, Φ_{13} and Φ_{23} can be determined using the sum rule:

$$\cancelto{0}{\Phi_{11}} + \cancelto{0}{\Phi_{12}} + \Phi_{13} = 1 \tag{2.61}$$

$$\boxed{\rightarrow \Phi_{13} = 1} \tag{2.62}$$

$$\cancelto{0}{\Phi_{21}} + \cancelto{0}{\Phi_{22}} + \Phi_{23} = 1 \tag{2.63}$$

$$\boxed{\rightarrow \Phi_{23} = 1} \tag{2.64}$$

Using the reciprocity rule we can determine Φ_{31}:

$$\Phi_{31} A_3 = \Phi_{13} A_1 \tag{2.65}$$

Rewriting: and inserting:

$$\boxed{\rightarrow \Phi_{31} = \Phi_{13} \frac{A_1}{A_3} = \Phi_{13} \frac{\frac{R^2\pi}{2}}{\frac{4^2}{2}} = \frac{1}{4}} \tag{2.66}$$

Furthermore, due to symmetry we know that $\Phi_{31} = \Phi_{32}$:

$$\boxed{\rightarrow \Phi_{32} = \frac{1}{4}} \tag{2.67}$$

And again using the sum rule, Φ_{33} can be determined:

$$\Phi_{31} + \Phi_{32} + \Phi_{33} = 1 \tag{2.68}$$

$$\boxed{\rightarrow \Phi_{33} = 1 - \Phi_{31} - \Phi_{32} = \frac{1}{2}} \tag{2.69}$$

After having determined the view factors, the surface brightnesses $\dot{Q}_2$ and $\dot{Q}_3$ should be determined.

The surface brightness for surface A_2 equals the surface brightness of a black body radiator:

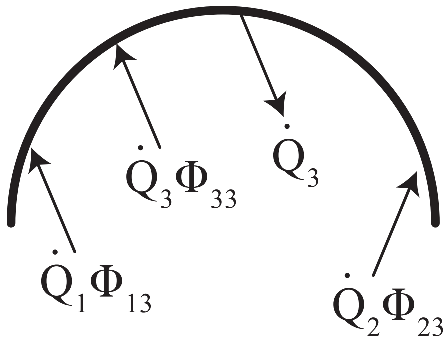
$$\dot{Q}_2 = \sigma \cdot \frac{\pi \cdot R^2}{2} \cdot T_2^4 = 5.67 \cdot 10^{-8}\ [\mathrm{W/m^2K^4}] \cdot \frac{\pi \cdot 3^2}{2}\ [\mathrm{m^2}] \cdot 293.15\ [\mathrm{K^4}] \tag{2.70}$$

$$\boxed{\rightarrow \dot{Q}_2 = 5,920\ [\mathrm{W}]} \tag{2.71}$$

The surface brightness of surface A_3 is more complex and can be obtained by setting up an outer energy balance around surface A_3.

It can't be determined using the equation for a black body radiator, as as its temperature is unknown.

Surfaces A_1, A_2 and A_3 partially radiate their surface brightness on surface A_3 and therefore the sum of these fluxes equals the incoming rate of heat transfer.



$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.72}$$

$$0 = \dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23} + \dot{Q}_3\Phi_{33} - \dot{Q}_3 \tag{2.73}$$

Rewriting yields:

$$\dot{Q}_3 = \frac{\dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23}}{1 - \Phi_{33}} \tag{2.74}$$

To obtain the surface brightness of surface A_3, the surface brightness of surface A_1 needs to be determined.

Note that surface A_1 is a grey body and thus $\epsilon_1 = \alpha_1 = 0.6$ and that $\rho_1 = 1\alpha_1 - \cancelto{0}{\tau_1} = 0.4$. The surface brightness of surface A1 can be expressed as:

$$\dot{Q}_1 = \dot{Q}_{\epsilon,1} + \dot{Q}_{\rho,1} + \cancelto{0}{\dot{Q}_{\tau,1}} \tag{2.75}$$

$$\dot{Q}_1 = \epsilon_1 \sigma A_1 T_1^4 + \rho_1 \dot{Q}_3 \Phi_{31} \tag{2.76}$$

Inserting Equation 2.76 into 2.74 yields:

$$\dot{Q}_3 = \frac{\left(\epsilon_1 \sigma A_1 T_1^4 + \rho_1 \dot{Q}_3 \Phi_{31}\right)\cancelto{1}{\Phi_{13}} + \dot{Q}_2\cancelto{1}{\Phi_{23}}}{1 - \Phi_{33}} \tag{2.77}$$

Rearranging $\dot{Q}_3$ to the left hand side:

$$\dot{Q}_3 = \frac{\epsilon_1 \sigma A_1 T_1^4 + \dot{Q}_2}{1 - \Phi_{33} - \rho_1 \Phi_{31}} \tag{2.78}$$

Inserting numerical values yields:

$$\dot{Q}_3 = \frac{0.6 \cdot 5.67 \cdot 10^{-8}\ [\mathrm{W/m^2K^4}] \cdot \frac{\pi \cdot 3^2}{2}\ [\mathrm{m^2}] \cdot 423.15^4\ [\mathrm{K^4}] + 5,920\ [\mathrm{W}]}{1 - \frac{1}{2} - 0.4 \cdot \frac{1}{4}} \tag{2.79}$$

$$\boxed{\rightarrow \dot{Q}_3 = 53,348\ [\mathrm{W}]} \tag{2.80}$$

3) Inserting and rearranging: Substituting the values for the found surface brightnesses yields:

$$\dot{Q} = \dot{Q}_3 \Phi_{32} - \dot{Q}_2 \tag{2.81}$$

$$\dot{Q} = 53,348\ [\mathrm{W}] \cdot \frac{1}{4} - 5,920\ [\mathrm{W}] \tag{2.82}$$

$$\boxed{\rightarrow \dot{Q} = 7,417\ [\mathrm{W}]} \tag{2.83}$$

b) *Which temperature T_3 is obtained for surface A_3?*

Due to the fact that body 3 is grey and adiabatic it acts like a black body. Therefore we can equal the previously determined surface brightness to the surface brightness of a black body radiator.

1) Setting up the definition of the surface brightness:

$$\boxed{\rightarrow \dot{Q}_3 = \sigma A_3 T_3^4} \tag{2.84}$$

2) Setting up the definition of the fluxes:

The surface brightness of surface A_3 has already been obtained in question a):

$$\boxed{\rightarrow \dot{Q}_3 = 53,348\ [\mathrm{W}]} \tag{2.85}$$

3) Inserting and rearranging:

Therefore the temperature can be expressed as:

$$T_3 = \sqrt[4]{\frac{\dot{Q}_3}{\sigma A_3}} \tag{2.86}$$

$$T_3 = \sqrt[4]{\frac{53,348\ [\mathrm{W}]}{5.67 \cdot 10^{-8}\ [\mathrm{W/m^2K^4}] \cdot 2\pi \cdot 3^2\ [\mathrm{m^2}]}} \tag{2.87}$$

$$\boxed{\rightarrow T_3 = 359\ [\mathrm{K}]} \tag{2.88}$$

2.7 Poké bowl ★ ★ ★

a) *Determine the surface brightness of the bowl* $\dot{Q}_{\mathrm{H}}$.

The surface brightness contains not only the emission but also the reflection of the sun radiation and the reflection of the surface brightness facing itself. And usually the transmitted radiation, but in the given situation this equals zero due to the fact that the convex surface of the bowl is adiabatic. When defining the term for the solar radiation it has to be taken into account that the cross sectional area of the bowl AB is representative, since it is perpendicular to the solar radiation.

1) Setting up the surface brightness:

Therefore the surface brightness of the bowl can be expressed as:

$$\dot{Q}_{\mathrm{B}} = \dot{Q}_{\epsilon,\mathrm{B}} + \dot{Q}_{\rho,\mathrm{B}} + \cancelto{0}{\dot{Q}_{\tau,\mathrm{B}}} \tag{2.89}$$

$$\boxed{\rightarrow \dot{Q}_{\mathrm{B}} = \dot{Q}_{\epsilon,\mathrm{B}} + \dot{Q}_{\rho,\mathrm{B}}} \tag{2.90}$$

2) Defining the fluxes:

Where the emission of the bowl can be described as a grey body radiator:

$$\dot{Q}_{\epsilon,\mathrm{B}} = \epsilon\sigma A_{\mathrm{S,B}} T_{\mathrm{B}}(t)^4 \tag{2.91}$$

$$\boxed{\rightarrow \dot{Q}_{\epsilon,\mathrm{B}} = \frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{B}}(t)^4} \tag{2.92}$$

Note that the surface area of the bowl is equal to half or the surface area of a sphere with diameter d:

$$A_{\mathrm{S,B}} = \frac{1}{2}A_{\mathrm{S,sphere}} = \frac{\pi}{2}d^2 \tag{2.93}$$

The reflected radiation from the bowl results from the reflected incident radiation from the sun and the bowl itself:

$$\dot{Q}_{\rho,\mathrm{B}} = \rho\left(\Phi_{\mathrm{BB}}\dot{Q}_{\mathrm{B}} + \dot{q}''_{\mathrm{S}} A_{\mathrm{incident}}\right) \tag{2.94}$$

$$\boxed{\rightarrow \dot{Q}_{\rho,\mathrm{B}} = (1-\epsilon)\left(\Phi_{\mathrm{BB}}\dot{Q}_{\mathrm{B}} + \frac{\pi}{4}\pi d^2 \dot{q}''_{\mathrm{S}}\right)} \tag{2.95}$$

Note that the incident surface area of the bowl is equal to the surface area of a circle with diameter d:

$$A_{\mathrm{incident}} = A_{\mathrm{S,circle}} = \frac{\pi}{4}d^2 \tag{2.96}$$

And for a grey body that is not transmitting any radiation:

$$\epsilon + \rho + \cancelto{0}{\tau} = 1 \leftrightarrow \rho = 1 - \epsilon \tag{2.97}$$

3) Inserting and rearranging:

Inserting the definition of the fluxes into the description of the surface brightness yields:

$$\dot{Q}_{\mathrm{B}} = \dot{Q}_{\epsilon,\mathrm{B}} + \dot{Q}_{\rho,\mathrm{B}} \tag{2.98}$$

$$\dot{Q}_{\mathrm{B}} = \frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{B}}(t)^4 + (1-\epsilon)\left(\Phi_{\mathrm{BB}}\dot{Q}_{\mathrm{B}} + \frac{\pi}{4}\pi d^2 \dot{q}''_{\mathrm{S}}\right) \tag{2.99}$$

Rewriting $\dot{Q}_{\mathrm{B}}$ to the left side of the equation yields in the surface brightness:

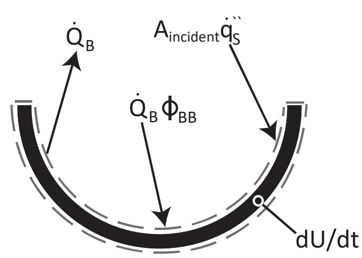
$$\boxed{\rightarrow \dot{Q}_{\mathrm{B}} = \frac{\frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{B}}(t)^4 + (1-\epsilon)\left(\frac{\pi}{4}\pi d^2 \dot{q}''_{\mathrm{S}}\right)}{\left(1 + (\epsilon - 1)\,\Phi_{\mathrm{BB}}\right)}} \tag{2.100}$$

b) *Derive the differential equation for the temperature as function of time and the necessary starting condition to solve this differential equation.*

To derive the differential equation for the temperature as function of time, a energy balance around the bowl should be established.

The bowl receives radiation from the sun, as well partially its own surface brightness which it radiates.

1) Setting up an energy balance:



$$\frac{\partial U}{\partial t} = \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}} \tag{2.101}$$

$$\frac{\partial U}{\partial t} = \dot{Q}_{\mathrm{B}}\Phi_{\mathrm{BB}} + A_{\mathrm{incident}}\dot{q}''_{\mathrm{s}} - \dot{Q}_{\mathrm{B}} \tag{2.102}$$

$$\boxed{\rightarrow \frac{\partial U}{\partial t} = \dot{Q}_{\mathrm{B}} \left(\Phi_{\mathrm{BB}} - 1\right) + A_{\mathrm{incident}} \dot{q}_{\mathrm{s}}''} \tag{2.103}$$

2) Defining the fluxes and internal energy:

The change in internal energy of the bowl over the course of time can be described in terms of the temperature, mass and specific heat capacity:

$$\boxed{\rightarrow \frac{\partial U}{\partial t} = mc \frac{\partial T_{\mathrm{B}}}{\partial t}} \tag{2.104}$$

The surface brightness of the bowl was already obtained in question a).

$$\boxed{\rightarrow \dot{Q}_{\mathrm{B}} = \frac{\frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{B}}(t)^4 + (1-\epsilon)\left(\frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''\right)}{\left(1 + (\epsilon - 1)\,\Phi_{\mathrm{BB}}\right)}} \tag{2.105}$$

As well as the solar radiation and the incident area:

$$\boxed{\rightarrow A_{\mathrm{incident}} \dot{q}_{\mathrm{S}}'' = \frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''}$$

3) Inserting and rearranging:

Inserting the definition of the fluxes and internal energy yields:

$$\frac{\partial U}{\partial t} = \dot{Q}_{\mathrm{B}} \left(\Phi_{\mathrm{BB}} - 1\right) + A_{\mathrm{incident}} \dot{q}_{\mathrm{s}}'' \tag{2.106}$$

$$\boxed{\rightarrow mc \frac{\partial T_{\mathrm{B}}}{\partial t} = \left(\Phi_{\mathrm{BB}} - 1\right) \frac{\frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{B}}^4 + (1-\epsilon)\left(\frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''\right)}{\left(1 + (\epsilon - 1)\,\Phi_{\mathrm{BB}}\right)} + \frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''} \tag{2.107}$$

To solve the differential equation, one initial condition is required, which reads as follows:

$$\boxed{\rightarrow T(t = t_0) = T_0} \tag{2.108}$$

c) *Determine the steady-state final temperature T_{S} of the bowl.*

After some time the system will reach equilibrium conditions and so the internal energy will not change anymore.

To determine the equilibrium temperature, an energy balance should be derived that describes the equilibrium state. This energy balance is similar to the energy balance derived in question b), but without the transient term describing the change in internal energy over the course of time.

1) Setting up an energy balance:

Therefore, the steady-state energy balance equals:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}} \tag{2.109}$$

$$\boxed{\rightarrow 0 = \dot{Q}_{\mathrm{B}} \left(\Phi_{\mathrm{BB}} - 1\right) + A_{\mathrm{incident}} \dot{q}_{\mathrm{s}}''} \tag{2.110}$$

2) Defining the fluxes:

The fluxes have already been defined in question b), but instead of the temperature of the bowl being a function of time $T_{\mathrm{B}}(t)$ it remains a stationary temperature T_{S}. Inserting yields:

$$\boxed{\rightarrow 0 = \left(\Phi_{\mathrm{BB}} - 1\right) \frac{\frac{\pi}{2}\epsilon\sigma d^2 T_{\mathrm{S}}^4 + (1-\epsilon)\left(\frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''\right)}{\left(1 + (\epsilon - 1)\,\Phi_{\mathrm{BB}}\right)} + \frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''} \tag{2.111}$$

3) Inserting and rearranging:

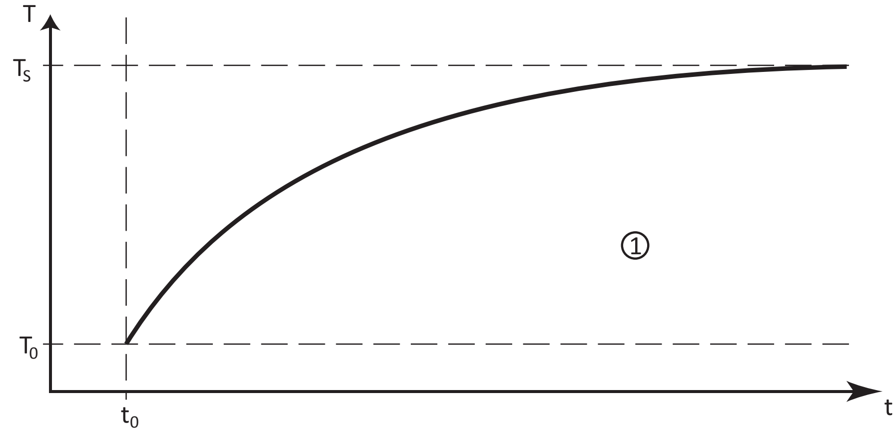
$$T_{\mathrm{S}}^4 = \left(\frac{-\left(\frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''\right)\left(1 + (\epsilon - 1)\,\Phi_{\mathrm{BB}}\right)}{\left(\Phi_{\mathrm{BB}} - 1\right)} - (1-\epsilon)\left(\frac{\pi}{4}\pi d^2 \dot{q}_{\mathrm{S}}''\right)\right) \cdot \frac{1}{\frac{\pi}{2}\epsilon\sigma d^2} \tag{2.112}$$

$$\boxed{\rightarrow T_{\mathrm{S}} = \sqrt[4]{\left(\frac{\left(A_{\mathrm{incident}} \dot{q}_{\mathrm{S}}''\right)\left(-\rho\Phi_{\mathrm{BB}} - 1\right)}{\left(\Phi_{\mathrm{BB}} - 1\right)} - \rho\left(A_{\mathrm{incident}} \dot{q}_{\mathrm{S}}''\right)\right) \cdot \frac{1}{A_{\mathrm{S,B}}\epsilon\sigma}}} \tag{2.113}$$

<u>**An alternative solution**</u> for the found temperature could have been obtained by simplifying the fact that in the stationary case, the surface brightness of the bowl is equal to that of a black body with the same temperature. $\dot{Q}_{\mathrm{B}} = \sigma A_{\mathrm{S,B}} T_{\mathrm{S}}$, which would yield:

$$\boxed{\rightarrow T_{\mathrm{S}} = \sqrt[4]{\frac{A_{\mathrm{incident}} \dot{q}_{\mathrm{s}}''}{\left(1 - \Phi_{\mathrm{BB}}\right)\sigma A_{\mathrm{S,B}}}}} \tag{2.114}$$

d) *Draw the temperature as a function of time qualitatively in the given diagram.*



As the temperature increases the gradient has to steadily decrease, thus the gradient is aiming towards zero for $t \to \infty$. But the curve progression can also be derived from physical consideration: The absorbed heat flux coming from the sun is always the same. In the beginning the emission is low (due to low temperature), which leads to a big increase in temperature (high net heat gain). The increase in temperature decays as the absorbed heat flux and the emitted heat flux converge until eventually both are equal and in steady state.

2.8 Radiation within a wedge-shaped opening ★ ★ ★

a) *Determine the view factors* $\Phi_{1,2}$,$\Phi_{2,1}$,$\Phi_{1,o}$,$\Phi_{2,o}$.

It can already be seen that $\Phi_{1,1}$=$\Phi_{2,2}$=$\Phi_{o,o}$=0 and due to symmetry that $\Phi_{2,o} = \Phi_{2,1} = \frac{1}{2}$.

$$\boxed{\rightarrow \Phi_{2,o} = \Phi_{2,1} = \frac{1}{2}} \tag{2.115}$$

As we know $\Phi_{2,o}$ and $\Phi_{2,1}$, we can use the reciprocity rule to determine $\Phi_{o,2}$ and $\Phi_{1,2}$:

$$a \cdot \Phi_{1,2} = a\sqrt{2} \cdot \Phi_{2,1} \tag{2.116}$$

$$\boxed{\rightarrow \Phi_{1,2} = \frac{1}{2} \cdot \sqrt{2}} \tag{2.117}$$

$$a \cdot \Phi_{o,2} = a\sqrt{2} \cdot \Phi_{2,o} \tag{2.118}$$

$$\rightarrow \Phi_{o,2} = \frac{1}{2}\sqrt{2} \tag{2.119}$$

With this given, we can use the sum rule to determine $\Phi_{1,o}$ and $\Phi_{o,1}$:

$$\Phi_{1,2} + \Phi_{1,o} = 1 \tag{2.120}$$

$$\boxed{\rightarrow \Phi_{1,o} = 1 - \frac{1}{2} \cdot \sqrt{2}} \tag{2.121}$$

$$\Phi_{o,1} + \Phi_{o,2} = 1 \tag{2.122}$$

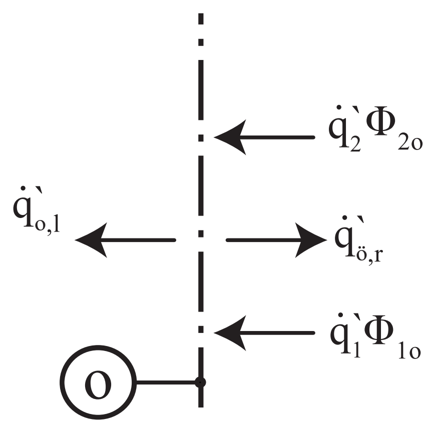
$$\rightarrow \Phi_{o,1} = 1 - \frac{1}{2}\sqrt{2} \tag{2.123}$$

b) *Determine the energy loss through the opening $\dot{q}'_{\mathrm{o,loss}}$ for a unit length of the opening.*

To determine the energy loss through the opening, the first step is setting up an energy balance around the opening, where we consider it to be a surface that transmits all incident radiation $\tau_{\mathrm{o}} = 1$.

1) Setting up an energy balance:

For the outer energy balance around opening, it is emitting its own surface brightness on the left- and right-side of the opening. Besides, partially the surface brightness of surfaces (1) and (2) is emitted on the opening surface.



Energy balance:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{q}'_{\mathrm{in}} - \sum \dot{q}'_{\mathrm{out}} \tag{2.124}$$

$$0 = \Phi_{2\mathrm{o}}\dot{q}'_2 + \Phi_{1\mathrm{o}}\dot{q}'_1 - \dot{q}'_{\mathrm{o},l} - \dot{q}'_{\mathrm{o},r} \tag{2.125}$$

The energy loss is described by the surface brightness that is emitted towards the environment and leaves the system. It can be described as:

$$\boxed{\rightarrow \dot{q}'_{\mathrm{o,loss}} = \dot{q}'_{\mathrm{o},l} = \Phi_{2\mathrm{o}}\dot{q}'_2 + \Phi_{1\mathrm{o}}\dot{q}'_1 - \dot{q}'_{\mathrm{o},r}} \tag{2.126}$$

2) Definition of the fluxes:

As the opening transmits all its radiation, the surface brightness for the left- and right-side of the opening can be described as:

$$\dot{q}'_{\mathrm{o},l} = \cancelto{0}{\dot{q}'_{\epsilon,\mathrm{o},l}} + \cancelto{0}{\dot{q}'_{\rho,\mathrm{o},l}} + \dot{q}'_{\tau,\mathrm{o},l} = \cancelto{1}{\tau} \cdot \left(\Phi_{2\mathrm{o}}\dot{q}'_2 + \Phi_{1\mathrm{o}}\dot{q}'_1\right) \tag{2.127}$$

$$\boxed{\rightarrow \dot{q}'_{\text{o},l} = \tau \cdot (\Phi_{2\text{o}}\dot{q}'_2 + \Phi_{1\text{o}}\dot{q}'_1)} \tag{2.128}$$

$$\dot{q}'_{\text{o},r} = \cancelto{0}{\dot{q}'_{\epsilon,\text{o},r}} + \cancelto{0}{\dot{q}'_{\rho,\text{o},r}} + \cancelto{0}{\dot{q}'_{\tau,\text{o},r}} = 0 \tag{2.129}$$

$$\boxed{\rightarrow \dot{q}'_{\text{o},r} = 0} \tag{2.130}$$

Furthermore, as for surface (1) $\epsilon = 1$, it acts as a black body radiator and it's surface brightness can be described as:

$$\dot{q}'_1 = \dot{q}'_{\epsilon,1} + \cancelto{0}{\dot{q}'_{\rho,1}} + \cancelto{0}{\dot{q}'_{\tau,1}} \tag{2.131}$$

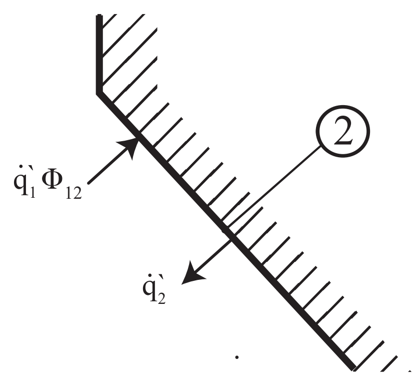
$$\boxed{\rightarrow \dot{q}'_1 = \sigma a T_1^4} \tag{2.132}$$

The surface brightness of surface (2) equals that of a black body radiator due to the fact that that its back is adiabatic. But due to the fact that it's temperature is unknown it can't be expressed directly in the terms of $\dot{q}'_2 = \sigma\sqrt{2}aT_2^4$.

The surface brightness of surface (2) should be determined by use of an energy balance around surface (2)

Surface (2) emits its own surface brightness, and besides partially the surface brightness of surface (2) is emitted on surface (2)

Setting up the outer energy balance around surface 2 results:



$$\frac{\cancelto{0}{\partial U}}{\partial t} = \sum \dot{q}'_{\text{in}} - \sum \dot{q}'_{\text{out}} \tag{2.133}$$

$$0 = \dot{q}'_2 - \Phi_{12}\dot{q}'_1 \tag{2.134}$$

Rearranging and inserting results in the expression for the surface brightness of surface (2):

$$\boxed{\rightarrow \dot{q}_2' = \Phi_{12}\dot{q}_1' = \Phi_{12}\sigma a T_1^4} \tag{2.135}$$

3) Inserting and rearranging

Plugging the definitions of the surface brightnesses into the outer energy balance around the opening results in:

$$\dot{q}_{\text{o,loss}}' = \dot{q}_{\text{o},l}' = \Phi_{2\text{o}}\dot{q}_2' + \Phi_{1\text{o}}\dot{q}_1' - \dot{q}_{\text{o},r}' \tag{2.136}$$

Inserting the found terms and rearranging:

$$\dot{q}_{\text{o,loss}}' = \dot{q}_{\text{o},l}' = \sigma a T_1^4 \left(\Phi_{2\text{o}}\Phi_{12} + \Phi_{1\text{o}}\right) \tag{2.137}$$

$$\dot{q}_{\text{o,loss}}' = 5.67 \cdot 10^{-8}\ [\text{W/m}^2\text{K}^4] \cdot 0.3\ [\text{m}] \cdot 1000^4\ [\text{K}^4] \left(\frac{1}{2} \cdot \frac{1}{2}\sqrt{2} + 1 - \frac{1}{2}\sqrt{2}\right) = 11\ [\text{kW/m}] \tag{2.138}$$

$$\boxed{\rightarrow \dot{q}_{\text{o,loss}}' = 11\ [\text{kW/m}]} \tag{2.139}$$

c) *Determine the temperature T_2 of surface (2)*

To determine the temperature of surface (2), its surface brightness should be rewritten.

In question *b)* the surface brightness of surface (2) was already found ($\dot{q}_2' = \Phi_{12}\dot{q}_1'$). Besides, it can be expressed in terms of the emitted, reflected and transmitted radiation.

1) Setting up the definition of the surface brightness:

$$\dot{q}_2' = \dot{q}_{\epsilon,2}' + \dot{q}_{\rho,2}' + \cancelto{0}{\dot{q}_{\tau,2}'} = \Phi_{12}\dot{q}_1' \tag{2.140}$$

$$\boxed{\rightarrow \dot{q}_2' = \dot{q}_{\epsilon,2}' + \dot{q}_{\rho,2}' + = \Phi_{12}\dot{q}_1'} \tag{2.141}$$

2) Definition of the fluxes:

Surface (2) does not transmit any radiation, as its back is adiabatic. Furthermore, the emitted radiation can be described in terms of a grey body radiator and the reflected radiation results from the radiation that is reflected from surface (1):

$$\boxed{\rightarrow \dot{q}_{\epsilon,2}' = \epsilon_2\sigma\left(\frac{a}{\cos\left(45^{\text{o}}\right)}\right)T_2^4 = \epsilon_2\sigma\sqrt{2}aT_2^4} \tag{2.142}$$

$$\boxed{\rightarrow \dot{q}'_{\rho,2} = \rho_2 \Phi_{12} \dot{q}'_1 = (1 - \epsilon_2)\, \Phi_{12} \dot{q}'_1} \tag{2.143}$$

3) Inserting and rearranging:

Therefore:

$$\dot{q}'_2 = \epsilon_2 \sigma \sqrt{2} a T_2^4 + (1 - \epsilon_2)\, \Phi_{12} \dot{q}'_1 = \Phi_{12} \dot{q}'_1 \tag{2.144}$$

Rearranging:

$$T_2^4 = \frac{\Phi_{12} \dot{q}'_1}{\sigma \sqrt{2} a} \tag{2.145}$$

Inserting the definition of $\dot{q}'_1 = \sigma a T_1^4$ and $\Phi_{12} = \frac{1}{2}\sqrt{2}$

$$T_2^4 = \frac{T_1^4}{2} \tag{2.146}$$

$$\boxed{\rightarrow T_2 = \sqrt[4]{\frac{1000^4\ [\mathrm{K}^4]}{2}} = 840.9\ [\mathrm{K}]} \tag{2.147}$$

2.9 Earth's atmosphere ★ ★ ★

a) *Determine the flux of short-wave radiation which hits onto the earth's surface* $\dot{q}''_{SW\ to\ E}$.

Since all surfaces are equal, it is allowed to use area-specific energy balances and surface brightnesses. The short-wave radiation hitting the earth is the short-wave surface brightness of the atmosphere. This is composed of the transmitted sun light and the reflected short-wave radiation from earth. The atmosphere itself does not radiate on this wavelength band.

1) Setting up the surface brightness:

Therefore the short-wave surface brightness of the atmosphere at the inside $\dot{q}''_{\mathrm{A,SW}}$ can be expressed as:

$$\dot{q}''_{\mathrm{A,SW}} = \cancelto{0}{\dot{q}''_{\epsilon,\mathrm{A,SW}}} + \dot{q}''_{\tau,\mathrm{A,SW}} + \dot{q}''_{\rho,\mathrm{A,SW}} \tag{2.148}$$

$$\boxed{\rightarrow \dot{q}''_{\mathrm{A,SW}} = \dot{q}''_{\tau,\mathrm{A,SW}} + \dot{q}''_{\rho,\mathrm{A,SW}}} \tag{2.149}$$

2) Defining the fluxes:

Some of the incident solar radiation is transmitted by the atmosphere, and therefore the shortwave transmitted radiation of the atmosphere can be defined as:

$$\boxed{\rightarrow \dot{q}''_{\tau,\mathrm{A,SW}} = \dot{q}''_{\mathrm{S}}\, \tau_{\mathrm{A,SW}}} \tag{2.150}$$

Furthermore, the some of the radiation of the shortwave surface brightness of the earth $\dot{q}''_{\mathrm{E,SW}}$ is reflected back at the earth by the inside of the atmosphere. Therefore, the reflected radiation on the inside of the atmosphere can be expressed as:

$$\dot{q}''_{\rho,\mathrm{A,SW}} = \dot{q}''_{\mathrm{E,SW}}\, \rho_{\mathrm{A,SW}} \tag{2.151}$$

The issue with this expression is that it gives us an unknown short-wave surface brightness of the earth $\dot{q}''_{\mathrm{E,SW}}$, but as the short-wave surface brightness of the earth only consists of reflected short-wave surface brightness of the atmosphere, it can be expressed as:

$$\dot{q}''_{\mathrm{E,SW}} = \cancelto{0}{\dot{q}''_{\epsilon,\mathrm{E,SW}}} + \cancelto{0}{\dot{q}''_{\tau,\mathrm{E,SW}}} + \dot{q}''_{\rho,\mathrm{E,SW}} = \dot{q}''_{\mathrm{A,SW}}\, \rho_{\mathrm{E,SW}} \tag{2.152}$$

With this known, the short-wave reflection of the atmosphere can be expressed as:

$$\boxed{\rightarrow \dot{q}''_{\rho,\mathrm{A,SW}} = \dot{q}''_{\mathrm{A,SW}}\, \rho_{\mathrm{E,SW}}\, \rho_{\mathrm{A,SW}}} \tag{2.153}$$

3) Inserting and rewriting:

Now with the short-wave reflected and transmitted radiation by the atmosphere towards the earth known and plugging in, the flux of short-wave radiation which hits onto the earth can be expressed ad:

$$\dot{q}''_{\mathrm{A,SW}} = \cancelto{0}{\dot{q}''_{\epsilon,\mathrm{A,SW}}} + \dot{q}''_{\tau,\mathrm{A,SW}} + \dot{q}''_{\rho,\mathrm{A,SW}} \tag{2.154}$$

Plugging in the found expressions:

$$\dot{q}''_{\mathrm{A,SW}} = \dot{q}''_{\mathrm{S}}\,\tau_{\mathrm{A,SW}} + \dot{q}''_{\mathrm{A,SW}}\,\rho_{\mathrm{E,SW}}\,\rho_{\mathrm{A,SW}} \tag{2.155}$$

Rewriting yields:

$$\dot{q}''_{\mathrm{A,SW}} = \frac{\tau_{\mathrm{A,SW}}}{1-\rho_{\mathrm{E,SW}}\rho_{\mathrm{A,SW}}}\,\dot{q}''_{\mathrm{S}} \tag{2.156}$$

$$\boxed{\rightarrow \dot{q}''_{\mathrm{A,SW}} = \frac{0.54}{1-0.16\cdot 0.23}\cdot 341\ [\mathrm{W/m^2}] = 191.2\ [\mathrm{W/m^2}]} \tag{2.157}$$

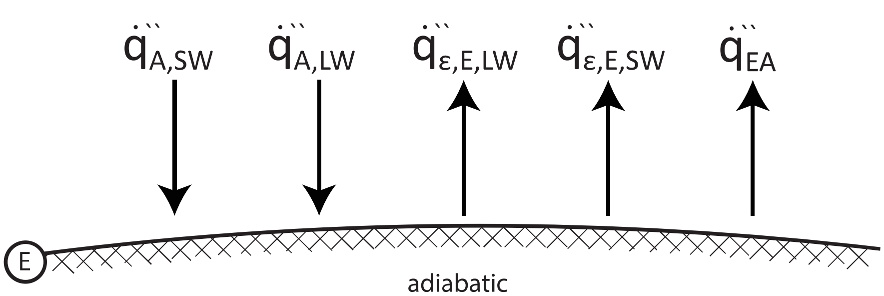
b) *Give all energy balances and surface brightnesses necessary to determine the temperature at the earth's surface. You may assume that the spectrum of black body radiation is completely within the long-wave range for that temperature.*

The earth's temperature can be derived from an inner energy balance radiation around the earth.

This energy balance consists out of short- and long-wave emission that is emitted by the earth itsself, short- and long-wave surface brightness of the inside of the atmosphere that is radiated towards the earth and lastly heat that is emitted due to convection and vaporization.

1) Setting up an energy balance:

Outer energy balance around the atmosphere:



Inner energy balance around the earth:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \alpha_{\mathrm{E}} \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{emitted}} \tag{2.158}$$

Note that the absorption is wavelength dependent, which results in:

$$\boxed{\rightarrow 0 = \alpha_{\mathrm{E,LW}} \dot{q}''_{\mathrm{A,LW}} + \alpha_{\mathrm{E,SW}} \dot{q}''_{\mathrm{A,SW}} - \dot{q}''_{\epsilon,\mathrm{E,LW}} - \dot{q}''_{\epsilon,\mathrm{E,SW}} - \dot{q}''_{\mathrm{EA}}} \tag{2.159}$$

2) Definition of the fluxes:

The heat emitted due to convection and vaporization $\dot{q}''_{\mathrm{EA}}$ is given in the problem:

$$\boxed{\rightarrow \dot{q}''_{\mathrm{EA}} = 101\ [\mathrm{W/m^2}]} \tag{2.160}$$

Furthermore was the short-wave surface brightness of the atmosphere $\dot{q}''_{\mathrm{A,SW}}$ already determined in question a).

$$\boxed{\rightarrow \dot{q}''_{\mathrm{A,SW}} = \frac{\tau_{\mathrm{A,SW}}}{1 - \rho_{\mathrm{E,SW}} \rho_{\mathrm{A,SW}}} \dot{q}''_{\mathrm{S}} = \frac{0.54}{1 - 0.16 \cdot 0.23} \cdot 341\ [\mathrm{W/m^2}] = 191.2\ [\mathrm{W/m^2}]} \tag{2.161}$$

Emission of the earth at short wavelength is as stated negligible, and therefore:

$$\boxed{\rightarrow \dot{q}''_{\epsilon,\mathrm{E,SW}} = 0}$$

Emission of the earth at short wavelength can be described as a grey body radiator ($\alpha_{\mathrm{E,LW}} = \epsilon \mathrm{E,LW} = 1$):

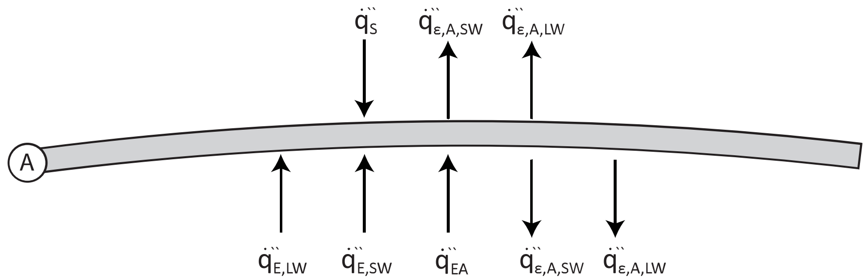
$$\boxed{\rightarrow \dot{q}''_{\epsilon,\mathrm{E,LW}} = \sigma T_{\mathrm{E}}^4} \tag{2.162}$$

Similarly to the short-wave surface brightness of the inside of the atmosphere, as in question a), the long-wave surface brightness of the inside of the atmosphere can be expressed in terms of emission, reflection and transmission. For the transmission term it equals zero, as only solar radiation can be transmitted towards the earth, but this is short-wave radiation and is therefore not included into the long-wave radiation surface brightness:

$$\dot{q}''_{\mathrm{A,LW}} = \dot{q}''_{\epsilon,\mathrm{A,LW}} + \cancelto{0}{\dot{q}''_{\tau,\mathrm{A,LW}}} + \dot{q}''_{\rho,\mathrm{A,LW}} \tag{2.163}$$

Furthermore, the long-wave emissivity of the inside atmosphere could be expressed as $\dot{q}''_{\epsilon,\mathrm{A,LW}} = \epsilon_{\mathrm{A,LW}} \sigma T_{\mathrm{A}}$. But the temperature of the atmosphere is unknown. Another option of expressing the emissivity of the atmosphere is by setting up an inner energy balance around the atmosphere.

This energy balance consists the short wave emission and long wave emission (where it is given in the exercise that short wave emission is negligible) that is emitted by the atmosphere. Furthermore, it receives heat from solar radiation and the earth due to convection and vaporization. Lastly, the earth emits its short- and long-wave surface brightness towards the atmosphere.



$$\alpha_\mathrm{A} \sum \dot{Q}_\mathrm{in} - \sum \dot{Q}_\mathrm{emitted} = \cancelto{0}{\frac{\partial U}{\partial t}} \tag{2.164}$$

Note that the absorption is wavelength dependent, which results in:

$$\dot{q}''_\mathrm{EA} + \alpha_\mathrm{A,LW}\dot{q}''_\mathrm{E,LW} + \alpha_\mathrm{A,SW}\dot{q}''_\mathrm{E,SW} + \alpha_\mathrm{A,SW}\dot{q}''_\mathrm{S} - 2\dot{q}''_{\epsilon,\mathrm{A,LW}} - 2\cancelto{0}{\dot{q}''_{\epsilon,\mathrm{A,SW}}} = 0 \tag{2.165}$$

Furthermore, it might be assumed that the long-wave surface brightness of the earth equals that of a black body radiator, due to the fact that its back is adiabatic and the spectrum of black body radiation is completely within the long-wave range for that temperature ($\dot{q}''_\mathrm{E,LW} = \dot{q}''_{\epsilon,\mathrm{E,LW}}$).
And rewriting yields

$$\dot{q}''_{\epsilon,\mathrm{A,LW}} = \frac{1}{2}\dot{q}''_\mathrm{EA} + \frac{1}{2}\alpha_\mathrm{A,LW}\dot{q}''_{\epsilon,\mathrm{E,LW}} + \frac{1}{2}\alpha_\mathrm{A,SW}\dot{q}''_\mathrm{E,SW} + \frac{1}{2}\alpha_\mathrm{A,SW}\dot{q}''_\mathrm{S} \tag{2.166}$$

The long-wave reflection of the inside atmosphere consists of the long-wave surface brightness of the earth that is reflected, and note that the long-wave surface brightness of the earth equals the long-wave emission of the earth:

$$\dot{q}''_{\rho,\mathrm{A,LW}} = \rho_\mathrm{A,LW}\dot{q}''_\mathrm{E,LW} = \rho_\mathrm{A,LW}\dot{q}''_{\epsilon,\mathrm{E,LW}} = \rho_\mathrm{A,LW}\sigma T_\mathrm{E}^4 \tag{2.167}$$

Therefore the long-wave radiation surface brightness of the atmosphere can be expressed as:

$$\dot{q}''_\mathrm{A,LW} = \dot{q}''_{\epsilon,\mathrm{A,LW}} + \cancelto{0}{\dot{q}''_{\tau,\mathrm{A,LW}}} + \dot{q}''_{\rho,\mathrm{A,LW}} \tag{2.168}$$

$$\boxed{\rightarrow \dot{q}''_\mathrm{A,LW} = \frac{1}{2}\left(\dot{q}''_\mathrm{EA} + \alpha_\mathrm{A,LW}\dot{q}''_{\epsilon,\mathrm{E,LW}} + \alpha_\mathrm{A,SW}\dot{q}''_\mathrm{E,SW} + \alpha_\mathrm{A,SW}\dot{q}''_\mathrm{S}\right) + \rho_\mathrm{A,LW}\dot{q}''_{\epsilon,\mathrm{E,LW}}} \tag{2.169}$$

3) Evaluation:

We have 6 unknowns, and 6 equations (see the boxed equations) and therefore the system is solvable.
The unknowns are:

$$\dot{q}''_{\mathrm{A,LW}} \qquad \dot{q}''_{\mathrm{A,SW}} \qquad \dot{q}''_{\epsilon,\mathrm{E,LW}} \qquad \dot{q}''_{\epsilon,\mathrm{E,SW}} \qquad \dot{q}''_{\mathrm{EA}} \qquad \dot{q}''_{\mathrm{S}}$$