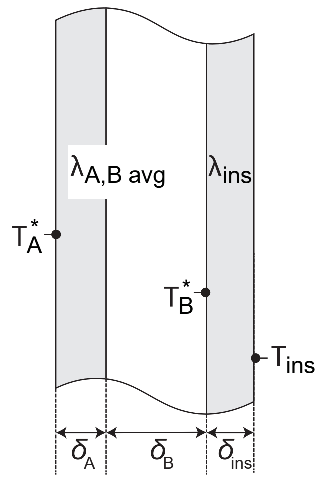
## 2.1 Onion layer principle

a. Determine the transmitted heat flux per unit area $\dot{q}''$.

Problem type:
Steady-state one-dimensional heat transfer through a multi-layer pipe wall.

System with insulation:



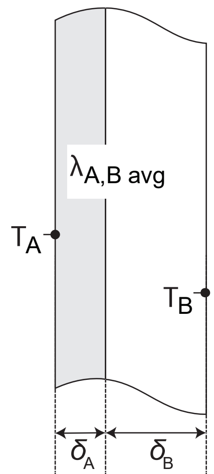

Fourier's law:

$$\dot{q}''_{\mathrm{w,ins}} = -\lambda_{\mathrm{ins}} \cdot \frac{T_{\mathrm{ins}} - T_{\mathrm{B}}^*}{\delta_{\mathrm{ins}}} = -0.075\ [\mathrm{W/mK}] \cdot \frac{(27 - 219)\ \ [\mathrm{K}]}{0.025\ [\mathrm{m}]} = 576\ [\mathrm{W/m^2}] \quad (2.1)$$

Average heat conductivity of layer A and B (resulting from Fourier's law):

$$\lambda_{\mathrm{A,B\ \ avg}} = \dot{q}''_{\mathrm{w,ins}} \cdot \frac{(\delta_{\mathrm{A}} + \delta_{\mathrm{B}})}{T_{\mathrm{A}}^* - T_{\mathrm{B}}^*} = 576\ [\mathrm{W/m^2}] \cdot \frac{(0.125 + 0.200)\ \ [\mathrm{m}]}{(305 - 219)\ \ [\mathrm{K}]} = 2.1767\ [\mathrm{W/mK}] \quad (2.2)$$

System without insulation:



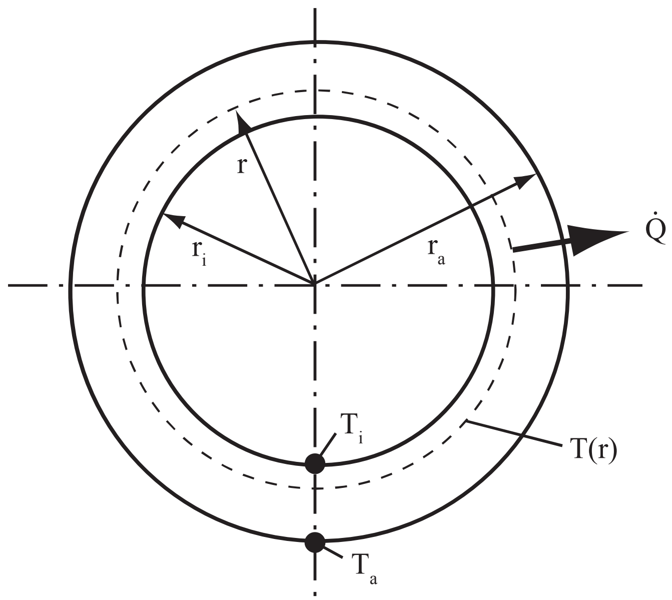

Fourier's law:

$$\dot{q}''_{\mathrm{wo,ins}} = -\lambda_{\mathrm{A,B\ avg}} \cdot \frac{T_{\mathrm{B}} - T_{\mathrm{A}}}{(\delta_{\mathrm{A}} + \delta_{\mathrm{B}})} = -2.1767\,[\mathrm{W/mK}] \cdot \frac{(32 - 260)\ \ [\mathrm{K}]}{(0.125 - 0.200)\ \ [\mathrm{m}]} = 1527\,[\mathrm{W/m^2}] \tag{2.3}$$

## 2.2 Wooden cylinder

1. Problem type:
   Steady-state, one-dimensional heat conduction through a curved surface without heat sources.

2. System boundaries and general approach:




As the temperature at the pipe's surface are constant the isothermal in the pipe's cross section are circular, therefore the temperature $T$ is only dependent on the radius $r$.
For a constant heat flux and no heat sources the heat flux in radial direction is constant.

$$\dot{Q}_\mathrm{r} = \dot{Q} = \text{const.} \tag{2.4}$$

According to Fourier's law the heat flux through the jacket surface of a cylinder of radius $r$ is equal to:

$$\begin{aligned}\dot{Q} &= -\lambda \cdot A \cdot \frac{\mathrm{d}T}{\mathrm{d}r} \\ &= -\lambda \cdot 2\pi r \cdot L \cdot \frac{\mathrm{d}T}{\mathrm{d}r} \\ &= -2\pi L \cdot \lambda_0 \left(1 + \gamma \left(T - T_0\right)\right) \cdot r \cdot \frac{\mathrm{d}T}{\mathrm{d}r}\end{aligned} \tag{2.5}$$

Because of $\dot{Q} = \text{const}$ the ODE can be solved immediately after separating the variables.

3. Boundary conditions:

$$\begin{aligned} T(r = r_\mathrm{i}) &= T_\mathrm{i} \\ T(r = r_\mathrm{o}) &= T_\mathrm{o} \end{aligned} \tag{2.6}$$

4. Solution

a) Calculation of the heat flux throug the pipe for a variable thermal conductivity.

Separation of variables:

$$-\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \cdot \frac{\mathrm{d}r}{r} = \left(1 + \gamma\left(T - T_0\right)\right) \cdot \mathrm{d}T \tag{2.7}$$

Applying the boundary conditions yields:

$$\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \int_{r=r_\mathrm{i}}^{r=r_\mathrm{o}} \frac{\mathrm{d}r}{r} = \int_{T=T_\mathrm{i}}^{T=T_\mathrm{o}} \left(1 + \gamma\left(T - T_0\right)\right) \, \mathrm{d}T \tag{2.8}$$

After integration the heat flux reads:

$$\begin{aligned} \dot{Q} &= -\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{o} - T_\mathrm{i} + \frac{\gamma}{2}\left(\left(T_\mathrm{o} - T_0\right)^2 - \left(T_\mathrm{i} - T_0\right)^2\right)\right) \\ \dot{Q} &= +\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{i} - T_\mathrm{o} + \frac{\gamma}{2}\left(T_\mathrm{i}^2 - T_\mathrm{o}^2 - 2T_0 \cdot \left(T_\mathrm{i} - T_\mathrm{o}\right)\right)\right) \end{aligned} \tag{2.9}$$

Heat flux for constant thermal conductivity:

$$\dot{Q} = \frac{2\pi L \cdot \lambda_\mathrm{m}}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{i} - T_\mathrm{o}\right) \tag{2.10}$$

Through reformulation of eq. 2.9 one obtains

$$\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{i} - T_\mathrm{o}\right) \cdot \left(1 + \gamma\left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) = \frac{2\pi L \cdot \lambda_\mathrm{m}}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \left(T_\mathrm{i} - T_\mathrm{o}\right) \tag{2.11}$$

Thus follows

$$\lambda_\mathrm{m} = \lambda_0 \cdot \left(1 + \gamma \left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right)$$
$$= \lambda_0 \cdot (1 + \gamma\,(T_\mathrm{m} - T_0)) \tag{2.12}$$
and
$$T_\mathrm{m} = \frac{1}{2}\,(T_\mathrm{i} + T_\mathrm{o}) \tag{2.13}$$

5. Result
For a thermal conductivity varying linearly with the temperature the relation for constant thermal conductivities can be utilised as long as the arithmetic mean values of the temperatures at the respective boundaries are used.

b) Temperature profile

Integrating eq. 2.7 within the boundaries

$$\begin{aligned} r &= r_\mathrm{i} \qquad & T &= T_\mathrm{i} \\ r &= r \qquad & T &= T \end{aligned} \tag{2.14}$$

yields

$$-\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \cdot \ln\left(\frac{r}{r_\mathrm{i}}\right) = T - T_\mathrm{i} + \frac{\gamma}{2}\left(T^2 - T_\mathrm{i}^2 - 2T_0 \cdot (T - T_\mathrm{i})\right) \tag{2.15}$$

Employing the relation for the heat flux laid out in eq. 2.11, 2.12 and 2.13

$$\dot{Q} = \frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(1 + \gamma \cdot \left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) \cdot (T_\mathrm{i} - T_\mathrm{o}) \tag{2.16}$$

one then obtains

$$T^2 + 2\,(T - T_\mathrm{i}) \cdot \left(\frac{1}{\gamma} - T_0\right) - T_\mathrm{i}^2 + \frac{2}{\gamma} \cdot (T_\mathrm{i} - T_\mathrm{o}) \cdot \ldots$$
$$\ldots \cdot \left(1 + \gamma \left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) \cdot \frac{\ln\left(\frac{r}{r_\mathrm{i}}\right)}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} = 0$$

$\rightarrow$ second order polynomial eq. for $T$.