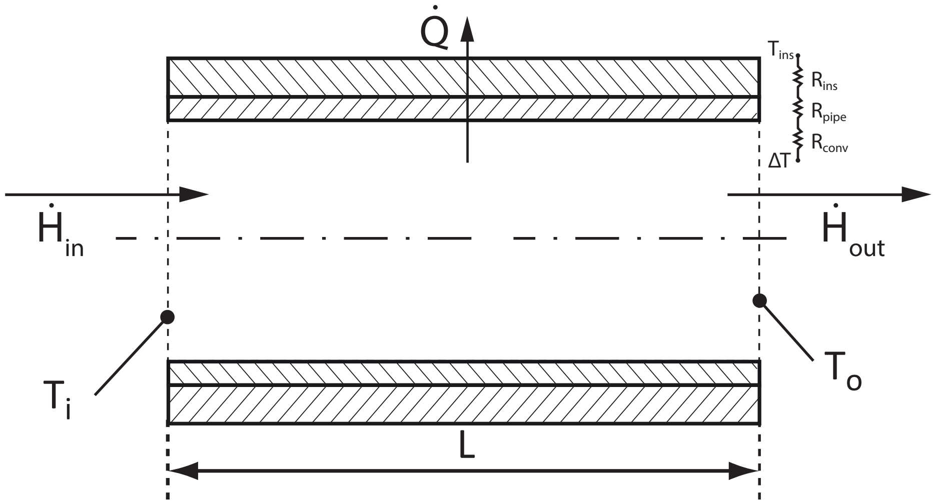
2.9 Insulated pipe ★★

a) *Find an expression for the exit temperature T_2 in terms of given parameters.*

The starting point of finding the expression for the outlet temperature T_2 is a global energy balance around the whole pipe.

1) Setting up an energy balance:

The energy balance for the pipe flow can be described as:



We have the following energy balance:

$$\cancel{\frac{\partial U}{\partial t}}^{0} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.145}$$

$$\boxed{\rightarrow 0 = \dot{H}_{\text{in}} - \dot{H}_{\text{out}} - \dot{Q}} \tag{2.146}$$

2) Defining the fluxes:

The rate of ingoing enthalpy flow can be described as:

$$\boxed{\rightarrow \dot{H}_{\text{in}} = \dot{m} c_{\text{p}} T_1} \tag{2.147}$$

The outgoing enthalpy flow can be described as:

$$\boxed{\rightarrow \dot{H}_{\text{out}} = \dot{m} c_{\text{p}} T_2} \tag{2.148}$$

Lastly, the rate of heat transfer leaving the system can be described by use determining an overall heat transfer coefficient k.

$$\dot{Q} = k \cdot A_{\mathrm{s,fluid}} \Delta T \tag{2.149}$$

Where k is the overall heat transfer coefficient [W/m^{2}K] and $A_{\mathrm{s,fluid}}$ the contact area between the fluid and the pipe wall.

The average temperature difference in the case of a constant wall temperature can be expressed as:

$$\Delta T = \frac{T_2 - T_1}{\ln\left(\frac{T_{\mathrm{ins}} - T_1}{T_{\mathrm{ins}} - T_2}\right) = \frac{}{\dot{m} c_{\mathrm{p}}}} \tag{2.150}$$

The overall heat transfer coefficient k can be determined by use of thermal resistances:

$$\frac{1}{R_{\mathrm{tot}}} = \frac{1}{R_{\mathrm{conv}}} + \frac{1}{R_{\mathrm{pipe}}} + \frac{1}{R_{\mathrm{ins}}} \tag{2.151}$$

$$k \pi d_{\mathrm{i}} L = \alpha \pi d_{\mathrm{i}} L + \frac{2\pi L \lambda_{\mathrm{p}}}{\ln\left(\frac{d_{\mathrm{o}}}{d_{\mathrm{i}}}\right)} + \frac{2\pi L \lambda_{\mathrm{ins}}}{\ln\left(\frac{d_{\mathrm{ins}}}{d_{\mathrm{o}}}\right)} \tag{2.152}$$

Rewriting:

$$\boxed{\rightarrow k = \alpha + \frac{2}{d_{\mathrm{i}}} \left(\frac{\lambda_{\mathrm{p}}}{\ln\left(\frac{d_{\mathrm{o}}}{d_{\mathrm{i}}}\right)} + \frac{\lambda_{\mathrm{ins}}}{\ln\left(\frac{d_{\mathrm{ins}}}{d_{\mathrm{o}}}\right)} \right)} \tag{2.153}$$

Substitution of the average temperature difference, the overall heat transfer coefficient and $A_{\mathrm{s,fluid}} = \pi d_i L$ gives us:

$$\boxed{\rightarrow \dot{Q} = \left[\alpha + \frac{2}{d_{\mathrm{i}}} \left(\frac{\lambda_{\mathrm{p}}}{\ln\left(\frac{d_{\mathrm{o}}}{d_{\mathrm{i}}}\right)} + \frac{\lambda_{\mathrm{ins}}}{\ln\left(\frac{d_{\mathrm{ins}}}{d_{\mathrm{o}}}\right)} \right) \right] \cdot \pi d_i L \cdot \frac{T_2 - T_1}{\ln\left(\frac{T_{\mathrm{ins}} - T_1}{T_{\mathrm{ins}} - T_2}\right)}} \tag{2.154}$$

3) Inserting and rearranging:

Inserting the found fluxes into the energy balance yields:

$$0 = \dot{H}_{\mathrm{in}} - \dot{H}_{\mathrm{out}} - \dot{Q} \tag{2.155}$$

$$0 = \dot{m} c_{\mathrm{p}} \left(T_1 - T_2\right) - k \cdot \pi d_i L \cdot \frac{T_2 - T_1}{\ln\left(\frac{T_{\mathrm{ins}} - T_1}{T_{\mathrm{ins}} - T_2}\right)} \tag{2.156}$$

Rewriting yields:

$$\ln\left(\frac{T_{\mathrm{ins}} - T_1}{T_{\mathrm{ins}} - T_2}\right) = \frac{k \cdot \pi d_i L}{\dot{m} c_{\mathrm{p}}} \tag{2.157}$$

$$\boxed{\rightarrow T_2 = T_{\mathrm{ins}} + \left(T_1 - T_{\mathrm{ins}}\right) \exp\left(- \frac{k \cdot \pi d_i L}{\dot{m} c_{\mathrm{p}}} \right)} \tag{2.158}$$

2.10 Heating of a pipe ★ ★ ★

a) *Determine the mean convective heat transfer coefficient $\overline{\alpha}$ for both cases.*

1) Setting up the definition of the heat transfer coefficient:

The average heat transfer coefficient for pipe flow can be defined as:

$$\boxed{\rightarrow \overline{\alpha} = \frac{\lambda \cdot \overline{\mathrm{Nu}}_{\mathrm{D}}}{D}} \tag{2.159}$$

2) Defining the required parameters:

The thermal conductivity λ and diamete D are specified within the known parameters. Therefore only the Nusselt number needs to be determined. Due to the given information for case 1 that the Reynolds number is below the critcal value of 2300 there is a laminar flow over an isothermal surface. In addition to that the flow is thermally and hydrodynamically developed which leads to a constant Nusselt number of:

$$\boxed{\rightarrow \overline{\mathrm{Nu}}_{\mathrm{D,1}} = 3.66 \cancelto{1}{\left(\frac{\eta}{\eta_{\mathrm{w}}}\right)^{0.14}} = 3.66} \tag{2.160}$$

If instead of the wall temperature, the heat flow at the wall remains constant, then the heat transfer coefficients have values increased by 20%. Therefore:

$$\boxed{\rightarrow \overline{\mathrm{Nu}}_{\mathrm{D,2}} = 1.2 \cdot \overline{\mathrm{Nu}}_{\mathrm{D,1}} = 4.392} \tag{2.161}$$

3) Inserting and rearranging:

$$\boxed{\rightarrow \overline{\alpha}_1 = \frac{3.66 \cdot \lambda}{D}} \tag{2.162}$$

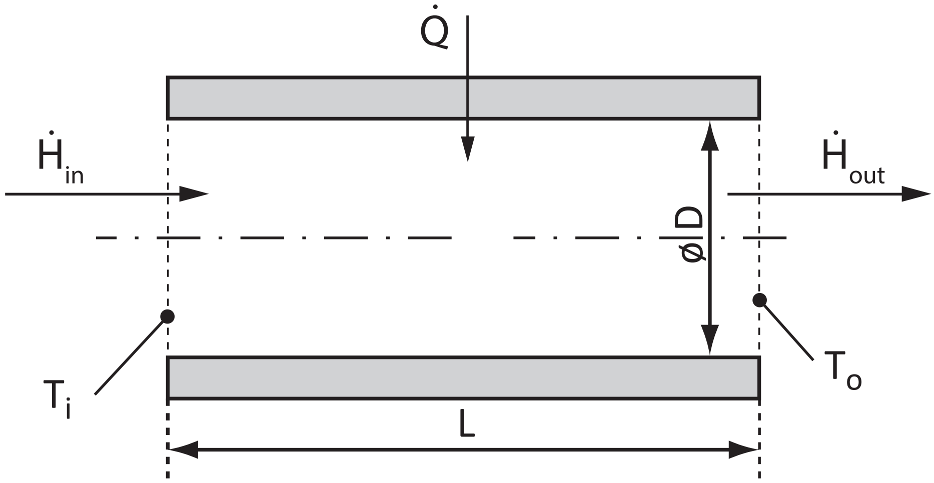
$$\boxed{\rightarrow \overline{\alpha}_2 = \frac{4.392 \cdot \lambda}{D}} \tag{2.163}$$

b) *Give the respective mean temperature difference ΔT_m between the inner wall of the tube and the fluid.*

The mean temperature difference is used in the expression for pipe flow subjected to convection:

$$\dot{Q} = \overline{\alpha} \cdot A_\mathrm{s} \cdot \Delta T \tag{2.164}$$

1) Setting up an energy balance:



For both cases we have the following energy balance:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_\mathrm{in} - \sum \dot{Q}_\mathrm{out} \tag{2.165}$$

$$\boxed{\rightarrow 0 = \dot{Q} + \dot{H}_\mathrm{in} - \dot{H}_\mathrm{out}} \tag{2.166}$$

2) Defining the fluxes:

The rate of ingoing enthalpy flow can be described as:

$$\boxed{\rightarrow \dot{H}_\mathrm{in} = \dot{m} c_\mathrm{p} T_\mathrm{in}} \tag{2.167}$$

The outgoing enthalpy flux can be described as:

$$\boxed{\rightarrow \dot{H}_\mathrm{out} = \dot{m} c_\mathrm{p} T_\mathrm{out}} \tag{2.168}$$

Lastly, the rate of heat transfer entering the system differs in both cases.

For case 1 this can be described by Newton's law of cooling:

$$\boxed{\rightarrow \dot{Q}_1 = \overline{\alpha_1} \cdot \pi \cdot D \cdot L \cdot \Delta T} \tag{2.169}$$

For case 2 it can be described by Newton's law of cooling and the constantly imposed heat flux:

$$\boxed{\rightarrow \dot{Q}_2 = \overline{\alpha_2} \cdot \pi \cdot D \cdot L \cdot \Delta T = \dot{q}'' \cdot \pi \cdot D \cdot L} \tag{2.170}$$

3) Inserting and rearranging:

Inserting the found values for both cases yields in the following energy balances.

Case 1:

$$0 = \overline{\alpha_1} \cdot \pi \cdot D \cdot L \cdot \Delta T_1 + \dot{m}c_\mathrm{p}T_\mathrm{in} - \dot{m}c_\mathrm{p}T_\mathrm{out} \tag{2.171}$$

Case 2:

$$0 = \overline{\alpha_2} \cdot \pi \cdot D \cdot L \cdot \Delta T_2 + \dot{m}c_\mathrm{p}T_\mathrm{in} - \dot{m}c_\mathrm{p}T_\mathrm{out} \tag{2.172}$$

Rewriting yields:

Case 1:

$$\boxed{\rightarrow \Delta T_1 = \frac{\dot{m}c_\mathrm{p}}{\overline{\alpha_1} \cdot \pi \cdot D \cdot L}\left(T_\mathrm{out} - T_\mathrm{in}\right)} \tag{2.173}$$

Case 2:

$$\boxed{\rightarrow \Delta T_2 = \frac{\dot{m}c_\mathrm{p}}{\overline{\alpha_2} \cdot \pi \cdot D \cdot L}\left(T_\mathrm{out} - T_\mathrm{in}\right)} \tag{2.174}$$

Alternative formulations of the logarithmic temperature differences of both cases could also be found.

Case 1:

$$\boxed{\rightarrow \Delta T_1 = \frac{T_\mathrm{o} - T_\mathrm{i}}{\ln\left(\frac{T_\mathrm{w} - T_\mathrm{i}}{T_\mathrm{w} - T_\mathrm{o}}\right)}} \tag{2.175}$$

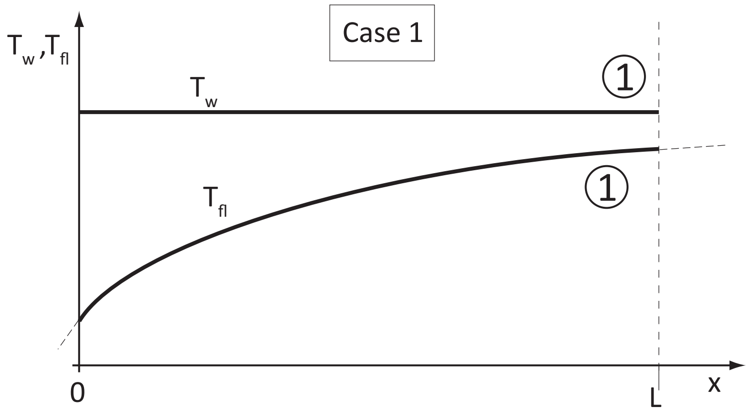
This could be found from deriving the temperature profile inside the pipe and expressing and writing the term $\frac{\dot{m}c_\mathrm{p}}{\overline{\alpha_1} \cdot \pi \cdot D \cdot L}$ in terms of the known temperatures.

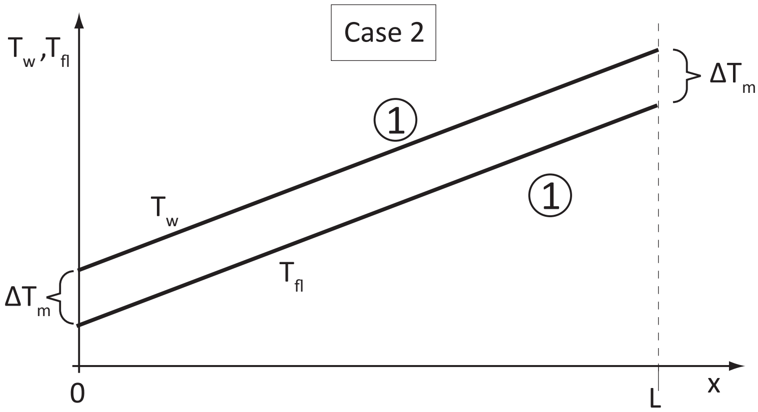
Case 2:

$$\boxed{\rightarrow \Delta T_2 = \frac{\dot{q}''}{\overline{\alpha}_2}} \tag{2.176}$$

This expression could be found by rewriting $\dot{Q}_2 = \overline{\alpha_2}\pi D L \Delta T = \dot{q}''\pi D L$

c) *Draw qualitatively for both cases the profile of the wall temperature T_w and the mean fluid temperature T_{fl}*





For case 1 we are dealing with a constant wall temperature. Therefore the fluid temperature will keep on increasing, and depending on the length of the pipe it will converge to the wall temperature.

For case 2 we are dealing with a constant heat flux. The wall temperature is increasing with the same rate as the fluid temperature. Therefore we have two temperature profiles with the same slope, where the difference is the mean temperature difference.

2.11 Flow through a grid ★ ★ ★

a) *Determine the volumetric heat release $\dot{\Phi}'''$ created by the electrically heated grid.*

1) Setting up the definition of a volumetric heat source:

A volumetric heat source can be defined as:

$$\boxed{\rightarrow \dot{\Phi}''' = \frac{\dot{\Phi}}{V}} \tag{2.177}$$

2) Defining the required parameters:

The generated heat can be described in terms of the average heat flux on the surface of the grid $\dot{q}''$ and the heat transfer area of the grid A_{G}:

$$\boxed{\rightarrow \dot{\Phi} = \dot{q}'' \cdot A_{\mathrm{G}}} \tag{2.178}$$

Furthermore, the volume of Region II can be described as:

$$\boxed{\rightarrow V = \frac{1}{4}\pi D^2 L} \tag{2.179}$$

3) Inserting and rearranging:

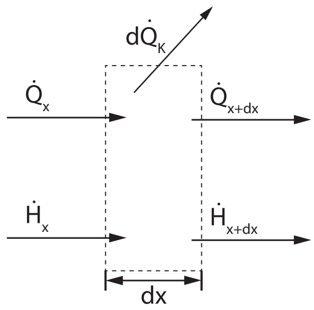
Inserting yields:

$$\boxed{\rightarrow \dot{\Phi}''' = \frac{4 \cdot \dot{q}'' \cdot A_{\mathrm{G}}}{\pi D^2 L}} \tag{2.180}$$

b) *Derive the differential equations for the temperature profile of the water in the pipe in regions I and II.*

1) Setting up an energy balance:

By an infinitesimal energy balance for region I, the following equation can be obtained:

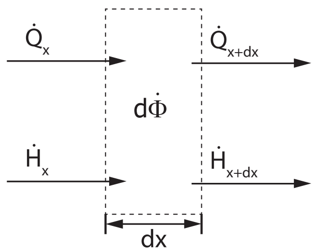


$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.181}$$

Resulting in:

$$\boxed{\rightarrow 0 = \dot{Q}_x + \dot{H}_x - \dot{Q}_{x+\mathrm{d}x} - \dot{H}_{x+\mathrm{d}x} - \mathrm{d}\dot{Q}_{\mathrm{K}}} \tag{2.182}$$

By an infinitesimal energy balance for region II, the following equation can be obtained:



$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.183}$$

Resulting in:

$$\boxed{\rightarrow 0 = \dot{Q}_x + \dot{H}_x + d\dot{\Phi} - \dot{Q}_{x+\mathrm{d}x} - \dot{H}_{x+\mathrm{d}x}} \tag{2.184}$$

2) Defining the fluxes:

The ingoing conductive heat flux can be described by use of Fourier's Law, which is:

$$\boxed{\rightarrow \dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}}\frac{\partial T}{\partial x} = -\frac{1}{4}\lambda\pi D^2\frac{\partial T}{\partial x}} \tag{2.185}$$

The outgoing conductive heat flux for an infinitesimal element can be approximated by use of the Taylor series expansion:

$$\dot{Q}_{\mathrm{x+dx}} = \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x}dx \tag{2.186}$$

$$\boxed{\rightarrow \dot{Q}_{\mathrm{x+dx}} = -\frac{1}{4}\lambda\pi D^2\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\frac{1}{4}\lambda\pi D^2\frac{\partial T}{\partial x}\right)\cdot dx} \tag{2.187}$$

The rate of ingoing enthalpy flow can be described as:

$$\boxed{\rightarrow \dot{H}_{\mathrm{x}} = \dot{m}c_{\mathrm{p}}T} \tag{2.188}$$

Similarly as for the outgoing conductive heat flux, the outgoing enthalpy flux can be approximated by use of the Taylor series expansion, which yields:

$$\boxed{\rightarrow \dot{H}_{\mathrm{x+dx}} = \dot{m}c_{\mathrm{p}}T + \frac{\partial}{\partial x}\left(\dot{m}c_{\mathrm{p}}T\right)\cdot dx} \tag{2.189}$$

For region I, where heat is lost to the environment, the flux can be described by use of the overall heat transfer coefficient k:

$$\boxed{\rightarrow d\dot{Q}_{\mathrm{K}} = kdA_{\mathrm{s}}\left(T_{\mathrm{I}}(x) - T_{\mathrm{e}}\right) = k\cdot\pi\cdot D\cdot dx\left(T_{\mathrm{I}}(x) - T_{\mathrm{e}}\right)} \tag{2.190}$$

For region II, where heat is generated, the source for the infinitesimal element can be described as:

$$\boxed{\rightarrow d\dot{\Phi} = \dot{\Phi}'''\cdot dV = \frac{1}{4}\cdot\dot{\Phi}'''\cdot\pi\cdot D^2\cdot dx} \tag{2.191}$$

3) Inserting and rearranging:

Inserting the found expressions in the energy balance of the infinitesimal element in region I yields:

$$0 = \dot{Q}_x + \dot{H}_x - \dot{Q}_{x+\mathrm{d}x} - \dot{H}_{x+\mathrm{d}x} - \mathrm{d}\dot{Q}_{\mathrm{K}} \tag{2.192}$$

$$0 = -\frac{\partial}{\partial x}\left(-\frac{1}{4}\lambda\pi D^2\frac{\partial T_{\mathrm{I}}}{\partial x}\right)\cdot dx - \frac{\partial}{\partial x}\left(\dot{m}c_{\mathrm{p}}T_{\mathrm{I}}\right)\cdot dx - kdA_{\mathrm{s}}\left(T_{\mathrm{I}}(x) - T_{\mathrm{e}}\right) \tag{2.193}$$

Rewriting yields:

$$\boxed{\rightarrow 0 = \lambda\frac{\partial^2 T_{\mathrm{I}}}{\partial x^2} - \frac{4\dot{m}c_{\mathrm{p}}}{\pi D^2}\frac{\partial T_{\mathrm{I}}}{\partial x} - \frac{4k}{D}\left(T_{\mathrm{I}}(x) - T_{\mathrm{e}}\right)} \tag{2.194}$$

Inserting the found expressions in the energy balance of the infinitesimal element in region II yields:

$$0 = \dot{Q}_x + \dot{H}_x + d\dot{\Phi} - \dot{Q}_{x+\mathrm{d}x} - \dot{H}_{x+\mathrm{d}x} \tag{2.195}$$

$$0 = -\frac{\partial}{\partial x}\left(-\frac{1}{4}\lambda\pi D^2\frac{\partial T_{\mathrm{II}}}{\partial x}\right)\cdot dx - \frac{\partial}{\partial x}\left(\dot{m}c_{\mathrm{p}}T_{\mathrm{II}}\right)\cdot dx + \frac{1}{4}\cdot\dot{\Phi}'''\cdot\pi\cdot D^2\cdot dx \tag{2.196}$$

Rewriting yields:

$$\boxed{\rightarrow 0 = \lambda\frac{\partial^2 T_{\mathrm{II}}}{\partial x^2} - \frac{4\dot{m}c_{\mathrm{p}}}{\pi D^2}\frac{\partial T_{\mathrm{II}}}{\partial x} + \dot{\Phi}'''.} \tag{2.197}$$

c) *Provide all the coupling or boundary conditions required for the solution of the problem (regions I and II).*

4) Defining the boundary/coupling conditions:

Regions I and II both are a 2nd order differential equation and therefore require four boundary/coupling conditions in total.

The water flowing entering region I is equal to T_{e} and is not changing yet at that point. Therefore one of the following two boundary conditions can be taken for Region I:

$$\boxed{\rightarrow \lim_{x\to-\infty} T_{\mathrm{I}}(x) = T_{\mathrm{e}}} \tag{2.198}$$

or

$$\boxed{\rightarrow \lim_{x\to-\infty} \frac{\mathrm{d}T_{\mathrm{I}}}{\mathrm{d}x} = 0} \tag{2.199}$$

The remaining three can be picked from the latter four.

We know that the temperature and the rate of heat transfer at the interface are equal to each other.

$$\boxed{\rightarrow T_{\mathrm{I}}(x=0) = T_{\mathrm{II}}(x=0)} \tag{2.200}$$

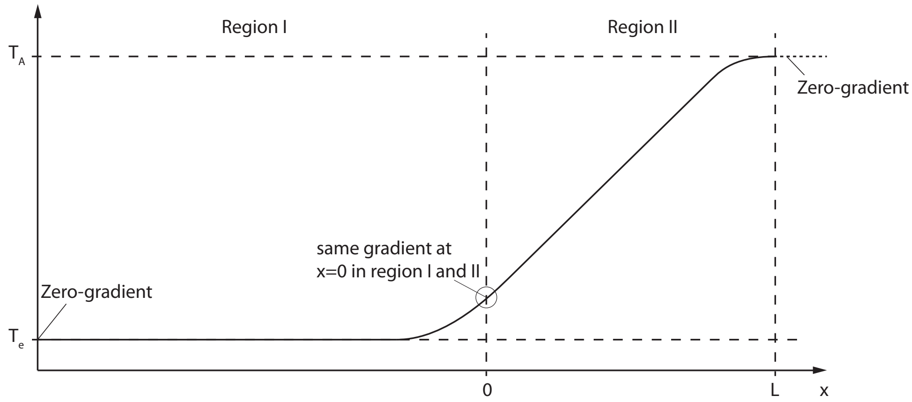
$$\boxed{\rightarrow \left.\frac{\mathrm{d}T_{\mathrm{I}}}{\mathrm{d}x}\right|_{x=0} = \left.\frac{\mathrm{d}T_{\mathrm{II}}}{\mathrm{d}x}\right|_{x=0}} \tag{2.201}$$

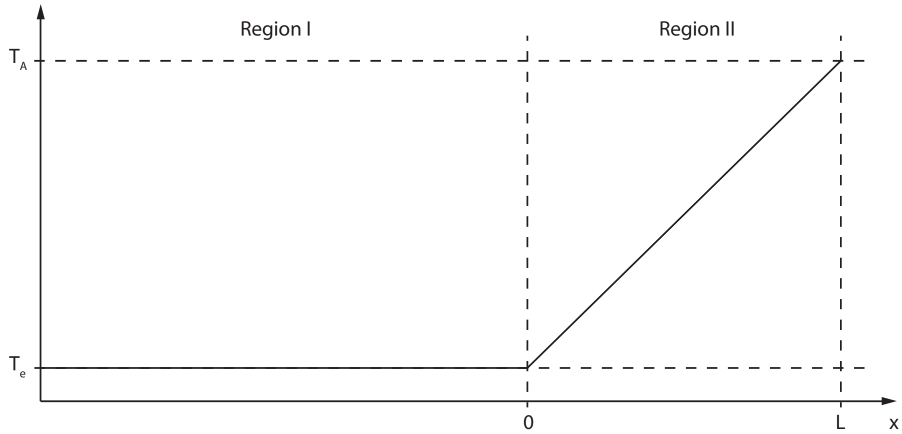
Furthermore it is known that at $x = L$ the temperature in region II equals T_{A} and no heat is transferred anymore at that position. So therefore:

$$\boxed{\rightarrow T_{\mathrm{II}}(x=L) = T_{\mathrm{A}}} \tag{2.202}$$

$$\boxed{\rightarrow \left.\frac{\mathrm{d}T_{\mathrm{II}}}{\mathrm{d}x}\right|_{x=L} = 0} \tag{2.203}$$

d) *Sketch the temperature profiles of the water in the pipe with and without consideration of the diffusive heat transport.*





In the first drawing diffusion (=conduction of heat) is present. Which means that due to the temperature difference in Region I and region II, some heat is transferred back into region I, which causes the increase of heat in Region I already. Similarly, as the temperature is constant in Region II, at $x = L$ the temperature profile should already approximate the zero-gradient slope.

In the second drawing no diffusion occurs and therefore the temperature only increases linearly due to the grid.