

Tutorial T03 – Elasticity Stress

Answer the following questions as they could come up in an exam.

Exercises 1,2,3,4 will be continued in tutorial T04

5 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ($E = 2 \cdot 10^5$ MPa, $\nu = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1 \left(\frac{x_1}{L} \right) - 10 \left(\frac{x_2}{L} \right) + \alpha_2 \left(\frac{x_1}{L} \right)^2 \right]$$

$$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8 \left(\frac{x_1}{L} \right) + \beta_1 \left(\frac{x_2}{L} \right) + \beta_2 \left(\frac{x_2}{L} \right)^2 \right]$$

$$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10 \left(\frac{x_1}{L} \right) + 7 \left(\frac{x_2}{L} \right) - 8 \left(\frac{x_1}{L} \right) \left(\frac{x_2}{L} \right) \right]$$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$, and
with reference stress $\sigma_0 = 1$ MPa and reference length $L = 1$ m.

Note: Question (a) is general, symbolic, with variables x_1 , x_2 , x_3 and coefficients α_1 , α_2 , β_1 , β_2 ;
only from question (b) on, use the single, chosen point P($x_1 = 0, x_2 = 0, x_3 = 0$).

Questions:

... based on section 3

- Does the stress field agree with the stress-equilibrium equations in absence of volume-forces?
Which relations have to be valid for the free coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium?
- Compute the eigen-stresses in point P using linear algebra, mathematics -- not circle of Mohr.
Describe and name the state of stress in point P (and in all other points in the body).
- Compute the eigen-direction of the major eigen-stress.
- Draw the relevant circle of Mohr and confirm graphically the results of (b) and (c); explain.

Answers:

a)

Given was the plane stress-field, independent of x_3 , in absence of body forces $f_i = 0$:

$$\begin{aligned}\sigma_{11}(x_1, x_2) &= \sigma_0 \left[20 + \alpha_1 \frac{x_1}{L} - 10 \frac{x_2}{L} + \alpha_2 \left(\frac{x_1}{L} \right)^2 \right] \\ \sigma_{22}(x_1, x_2) &= \sigma_0 \left[10 + 8 \frac{x_1}{L} + \beta_1 \frac{x_2}{L} + \beta_2 \left(\frac{x_2}{L} \right)^2 \right] \\ \sigma_{12}(x_1, x_2) &= \sigma_0 \left[12 - 10 \frac{x_1}{L} + 7 \frac{x_2}{L} - 8 \frac{x_1}{L} \frac{x_2}{L} \right]\end{aligned}$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\begin{aligned}\frac{d}{dx_1} \sigma_{11}(x_1, x_2) + \frac{d}{dx_2} \sigma_{12}(x_1, x_2) &= \sigma_0 \left[\frac{\alpha_1}{L} + 2\alpha_2 \frac{x_1}{L^2} \right] + \sigma_0 \left[\frac{7}{L} - 8 \frac{x_1}{L^2} \right] = 0 \\ \frac{d}{dx_1} \sigma_{12}(x_1, x_2) + \frac{d}{dx_2} \sigma_{22}(x_1, x_2) &= \sigma_0 \left[\frac{-10}{L} - 8 \frac{x_2}{L^2} \right] + \sigma_0 \left[\frac{\beta_1}{L} + 2\beta_2 \frac{x_2}{L^2} \right] = 0\end{aligned}$$

From these equations, one gets the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$.

Because the field equations must be valid for all constants and points x_1, x_2, x_3 , independently, one can group them accordingly: The constant terms from the first and second equations provide α_1 and β_1 , respectively, while the x_1 and x_2 groups provide α_2 and β_2 .

b)

The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $[\sigma_{ij}] = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0.$$

Knowing/recognizing that one eigen-value is zero, i.e. also $I_3 = 0$, the principal stresses can be computed from the second order polynomial as: $\sigma_I = 28$ MPa, $\sigma_{II} = 2$ MPa, $\sigma_{III} = 0$ MPa. This is a plane-stress state with all stresses on the x_3 -surface equal to zero, which also has consequences for the eigen-directions ...

c)

The principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$ is directly visible from the tensor, due to the zero shear stresses in the x_3 -direction.

The eigen-direction of the major stress $\sigma_I = 28$ MPa is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that: $-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Rightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress, $\sigma_{II} = 2$ MPa was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that: $18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9) + 1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Rightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

d) Mohr's circle

Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{20+10}{2} = 15$ MPa,

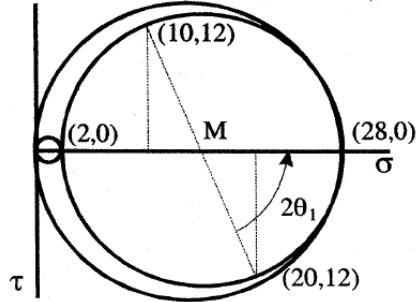


Figure 1: Sketch of a Mohr circle, focus is on the right, inner circle.

and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13 \text{ MPa.}$

The eigenvalues are therefore:

$$\sigma_I = M + R = 28 \text{ MPa}, \sigma_{II} = C - R = 2 \text{ MPa.}$$

The eigen-directions are:

$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{24}{10} = 2.4 \implies \theta_I = (1/2) \arctan(2.4) = 67.38^\circ / 2 = 33.69^\circ$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$; and $\theta_{II} = (180^\circ + 67.3^\circ)/2 = 247.3^\circ / 2 = 123.7^\circ = \arccos(-2/\sqrt{13})$.

The maximum shear stress is just the radius: $\tau^{max} = R = 13 \text{ MPa}$

6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:

$$E = 2 \cdot 10^{11} \text{ Pa}, \nu = 0.25$$

Stress-state in point P: $[\sigma] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} \text{ MPa}$

Questions:

a) Show that the principal stresses are 8, 16 and 24 MPa.

Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.

Answers:

a)

From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0.$$

Given the eigenvalues, σ , one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*

Sorting the eigen-values is convention and part of the answer:

$$\sigma_I = 24 \text{ MPa}, \sigma_{II} = 16 \text{ MPa}, \text{ and } \sigma_{III} = 8 \text{ MPa}.$$

it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.

The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8 \text{ MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$

$$\text{Inserting into line 3 yields: } n_3 = \sqrt{6}n_1, \text{ so that: } \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\implies \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$