Elasticity Theory - Formula sheet

Stress	**Strain**
Rotation/Transformation: $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$, where $R_{pi} = \cos(x'_p, x_i)$	Rotation/Transformation: $\epsilon'_{pq} = R_{pi}R_{qj}\epsilon_{ij}$, where $R_{pi} = \cos(x'_p, x_i)$
Principal stresses & directions: $(\sigma_{ij} - \sigma\delta_{ij})\hat{n}_j = 0$ with $\hat{n}_j\hat{n}_j = 1$	Principal strains & directions: $(\epsilon_{ij} - \epsilon\delta_{ij})\hat{n}_j = 0$ with $\hat{n}_j\hat{n}_j = 1$
Deviatoric stress: $\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$	Deviatoric strain: $\hat{\epsilon}_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}$
Hydrostatic (isotropic) stress: $\sigma_m = \frac{1}{3}\sigma_{kk}$	Volumetric (isotropic) strain: $\epsilon_v = \epsilon_{kk}$
Constitutive relation: $\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$ $(Hooke's\ law)$	Constitutive relation: $\epsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right)$ $(Hooke's\ law)$
Shear modulus: $G = \frac{E}{2(1+\nu)}$	
Traction vector: $p_i = \sigma_{ji}\hat{n}_j$	Displacement and strain relation : $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
Force balance, symmetry: $\sigma_{ji,j} + f_i = 0$, $\sigma_{ij} = \sigma_{ji}$	

Principal stresses(σ): Characteristic equation: $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma_m = tr(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{23}^2 = \frac{1}{2}\left[(\sigma_{kk}^2 - \sigma_{ij}\sigma_{ij})\right] = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{13}\sigma_{23} = \det(\boldsymbol{\sigma}) = \sigma_1\sigma_2\sigma_3$$

Strain relation: $\sigma \to \epsilon$ and $I \to E$

Deviatoric stress: $\sigma \to \hat{\sigma}$ and $\hat{\sigma}^3 - J_2\hat{\sigma} - J_3 = 0$, with $J_1 = 0$, $J_2 = I_1^2/3 - I_2 > 0$, and $J_3 = \det(\hat{\sigma})$

Failure criteria (with $\sigma_1 \geq \sigma_2 \geq \sigma_3$)

Maximum stress:

Normal stress: $\to$ principal stresses.

max. shear stress: $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

Tresca:

$\sigma_{eq} = \sigma_1 - \sigma_3 \leq \sigma_{toel}$

von Mises:

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}}$$

Energy	Visco-elastic
Elastic energy (or work): $\Pi_{el} = \int_V dV \pi_{el}$	
Specific elastic energy: $\pi_{el} = \int_\varepsilon d\varepsilon_{ij}\ \sigma_{ij}(\boldsymbol{\varepsilon})$	$\pi_{visc} = \int_\varepsilon d\varepsilon_{ij}\ \sigma_{ij}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) = \int_{\Delta t} dt\ \dot{\varepsilon}_{ij}\sigma_{ij}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$
Linear stress-strain (integrated): $\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ and: $\pi_{el} = \pi_{el_{vol}} + \pi_{el_{ged}} = \frac{1}{2}\sigma_m\varepsilon_V + \frac{1+\nu}{E}J_2$ $= \frac{1}{2}\sigma_m\varepsilon_V + \frac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}$	for $\sigma_{ij}^v = \eta\dot{\varepsilon}_{ij} = const.$: $\pi_{visc} = \sigma_{ij}^v\dot{\varepsilon}_{ij}\Delta t$