1.6 Exercises

Problem 1.1. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} x \\ -y \end{pmatrix}$

(a) Draw the curves $xy = \pm 1$ *in all four quadrants of the* $x - y$ *plane.*
(b) Draw the velocity vector at several points on the curves.
(c) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.2. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.3. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.4. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.5. *Consider a little smoke particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$

(a) Draw the trajectory for $-1 \leq t \leq 1$.
(b) Compute the velocity vector.

Problem 1.6. *Consider a little dust particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

(a) Draw the trajectory for $0 \leq t \leq 2\pi$.
(b) Compute the velocity vector $\boldsymbol{u}$
(c) Construct a formula for the velocity vector as a function of $x(t)$ *and* $y(t)$.
(d) Draw the velocity vector at several points on the trajectory.
(e) Compute the acceleration vector.
(f) Construct a formula for the acceleration vector as a function of $x(t)$ *and* $y(t)$.
(g) Draw the acceleration vector at several points on the trajectory.

Problem 1.7. *Using the index summation convention of Einstein, write in full:*

(a) $\frac{1}{2}u_i u_i$,
(b) $\frac{1}{2}u_i u_j$,
(c) $\frac{1}{2}u_j u_j$,

(d) $\frac{\partial u_k}{\partial x_k}$,
(e) $u_j \frac{\partial u_i}{\partial x_j}$,
(f) $u_i + u_j$,
(g) $\delta_{ij} u_i$, *where* $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
(h) $\delta_{ij} u_j$,
(i) $a_{ij} u_i u_j$.

Problem 1.8. *Rewrite the following vector-notation expressions in index notation:*

(a) $\boldsymbol{x} \cdot \boldsymbol{y}$,
(b) $\boldsymbol{\nabla} \cdot \boldsymbol{u}$,
(c) the i*-th component of the vector* $A\boldsymbol{x}$*, where* A *is a* 3×3 *matrix,*
(d) the i, j*-th component of the matrix* AB*, where* A *and* B *are a* 3×3 *matrices,*

Problem 1.9. *The velocity at the plane defined by the normal* $\boldsymbol{n} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *is* $\boldsymbol{u} = \begin{pmatrix} 15 \\ 34 \end{pmatrix} \frac{m}{s}$. *Calculate the normal and tangential velocities.*

Problem 1.10. *Given the Eulerian field*

$$\boldsymbol{u}(x, y, z, t) = 3t\boldsymbol{e}_1 + xz\boldsymbol{e}_2 + ty^2\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ *and* $\boldsymbol{e}_3$ *are the unit vectors along the coordinate axis, determine the flow acceleration.*

Problem 1.11. *A two-dimensional velocity field is described by*

$$u = \frac{x}{a + bt}, \quad v = \frac{y}{a + 2bt}.$$

Calculate the trajectories that pass by (x_o, y_o) *at* $t = 0$.

Problem 1.12. *Using polar coordinates, the velocity field in a tornado can be approximated as*

$$\boldsymbol{u} = -\frac{a}{r}\boldsymbol{e}_r + \frac{b}{r}\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_r$ *and* $\boldsymbol{e}_\theta$ *are the unit vectors in the directions* r *and* θ. *Show that the trajectories satisfy the so-called logarithmic spiral equation:*

$$r(\theta) = C \exp\left(-\frac{a}{b}\theta\right).$$

Problem 1.13. *A two-dimensional velocity field is given by*

$$u = 5ax(t + t_o), \quad v = 5ay(t - t_o).$$

Find the trajectories that pass x_o, y_o *at time* $t = 0$.

Problem 1.14. *The ideal flow around a corner placed at the origin is given by*

$$u = ax, \quad v = -ay,$$

with $a > 0$ a constant. Determine the trajectories and draw the trajectory that passes the point (x_o, y_o) at time $t = 0$ and indicate the flow direction. Calculate the material derivative of the velocity vector.

Problem 1.15. *The velocity field in a vortex like the one present in a cyclone, is given by:*

$$u = -\frac{Ky}{x^2 + y^2}, \quad v = \frac{Kx}{x^2 + y^2},$$

with $K > 0$. Determine the trajectories and draw a few of them.

Alternatively suppose that we know that the flow is incompressible, in other words, the mass density is a known constant. In that case the volume integral in Eq.(2.29) is again zero since $\frac{\partial \rho}{\partial t}$ is zero. The result is

$$\dot{m} = -\rho \int_{S_1(t)} u_j n_j dS. \tag{2.31}$$

2.7 Exercises

Problem 2.1. *Compute the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ if*

(a) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (1, 0, 0)^T$.
(b) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (0, 1, 0)^T$.
(c) $\boldsymbol{a} = (a_1, a_2, a_3)^T$, $\boldsymbol{b} = (b_1, b_2, b_3)^T$.
(d) $\boldsymbol{a} = (x, y^2, x)^T$, $\boldsymbol{b} = (y, y, z)^T$.
(e) $\boldsymbol{a} = (u, v, w)^T$, $\boldsymbol{b} = (n_1, n_2, n_3)^T$.

Problem 2.2. *Compute the inner product $\boldsymbol{u} \cdot \boldsymbol{n}$ if*

(a) $\boldsymbol{u} = U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.
(b) $\boldsymbol{u} = -U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.

Problem 2.3. *A tube has cross-sectional area A_a at the entrance and cross-sectional area A_b at the exit, and the fluid flowing through the tube is incompressible.*

(a) If the volume flow rate at the exit is Q, compute the average normal velocity at the exit.
(b) If the volume flow rate at the exit is Q, compute the average normal velocity at the entrance.

Problem 2.4. *A channel with rectangular cross section has sides b and h at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u, v, w)^T$, *with* $u = U(1 - z/h)$, $v = \ln yz$, $w = yz^2$.
(b) $(u, v, w)^T$, *with* $u = U(1 - y/b)$, $v = \sin z$, $w = \cos y$.
(c) $(u, v, w)^T$, *with* $u = U(1 - y/b)(1 - z/h)$, $v = 0$, $w = yz$.

Problem 2.5. *A tube with circular cross section has radius R at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u(r), 0, 0)^T$*, with* $u(r) = U(1 - r/R)$*, compute the average normal velocity.*
(b) $(u(r), 0, 0)^T$*, with* $u(r) = U(1 - (r/R)^2)$*, compute the average normal velocity.*

Problem 2.6. *Show how Eq.(2.19) reduces in the following two cases:*

(a) steady flow
(b) Incompressible flow

$$\boxed{\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho u_j n_j dS = 0 \quad \text{for all } (V(t), t)} \tag{2.19}$$

Problem 2.7. *Consider steady, incompressible flow through the device shown. Given:* U_1*,* A_1*,* U_2*,* A_2*,* A_3*. Derive an expression for the volume flow rate through port 3.*

Problem 2.8. *Incompressible oil flows steadily in a thin layer down an inclined plane with width* w*. The velocity profile is*

$$u = \frac{\rho g \sin\theta}{\mu}\left[hy - \frac{1}{2}y^2\right]. \tag{2.32}$$

Derive formulas for the volume flow rate and mass flow rate in terms of ρ*,* μ*,* g*,* θ*, and* h*.*

Problem 2.9. *Incompressible water flows steadily through a pipe of length* L *and radius* R*. Derive an expression for the uniform inlet velocity,* U*, if the velocity distribution across the outlet is given by*

$$u = V\left[1 - \frac{r^2}{R^2}\right]. \tag{2.33}$$

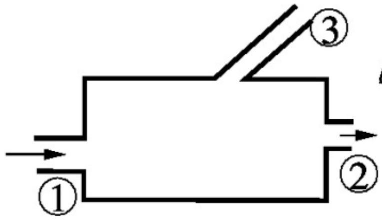
Problem 2.10. *A bend with rectangular cross section and width* w *has a linear velocity profile at port 1. The flow is uniform at ports 2 and 3. The fluid is incompressible and the flow is steady. Derive an expression for the uniform velocity at port 3.*

Problem 2.11. *Water enters a two-dimensional square channel of constant width* w*, and constant height,* h*, with uniform velocity,* U*. The channel makes a* $90°$ *bend that distorts the flow to produce the linear velocity profile shown at the exit, with* $V_{max} = 2V_{min}$*. The flow is steady and the fluid is incompressible. Derive an expression for* V_{min}*.*

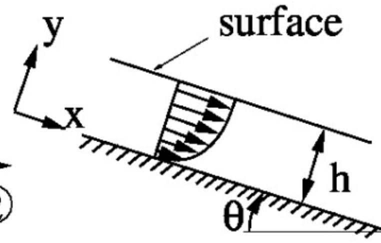
Problem 2.12. *Incompressible water flows steadily past a porous plate of width* w *and length* L*. Constant suction is applied along the plate with normal velocity* V *(towards the plate). The velocity profile at the outflow plane is:*

$$\frac{u}{U} = 3\left[\frac{y}{\delta}\right] - 2\left[\frac{y}{\delta}\right]^{3/2}. \tag{2.34}$$

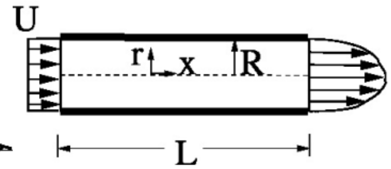
Derive a formula for the mass flow rate through the boundary at the top of the domain $(y = \delta)$*.*



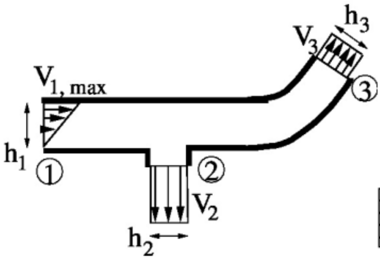
(a) Problem 2.7



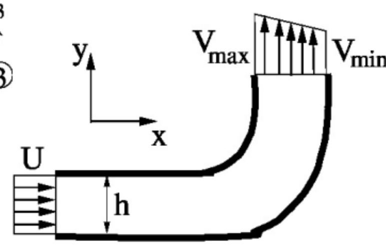
(b) Problem 2.8



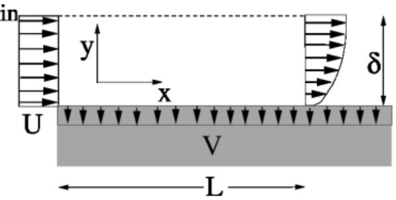
(c) Problem 2.9



(a) Problem 2.10



(b) Problem 2.11



(c) Problem 2.12