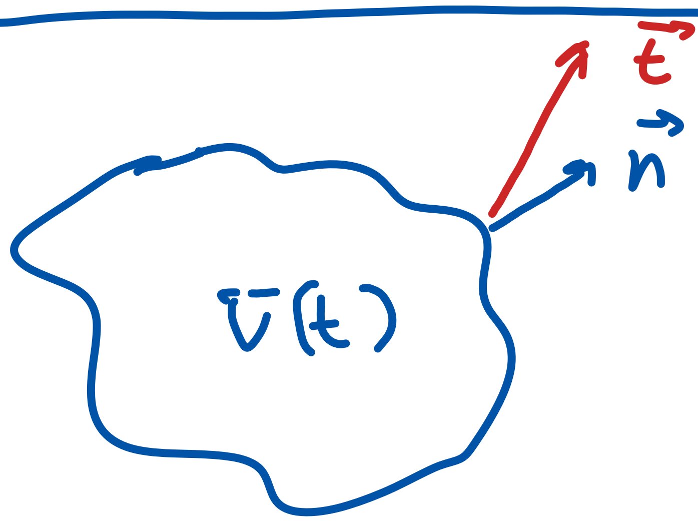
Fluid Mechanics 1

Lecture # 04

Stresses in Fluids & Momentum conservation.



stress vector

$N/m^2 \equiv Pa$

$$\vec{t} \equiv \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

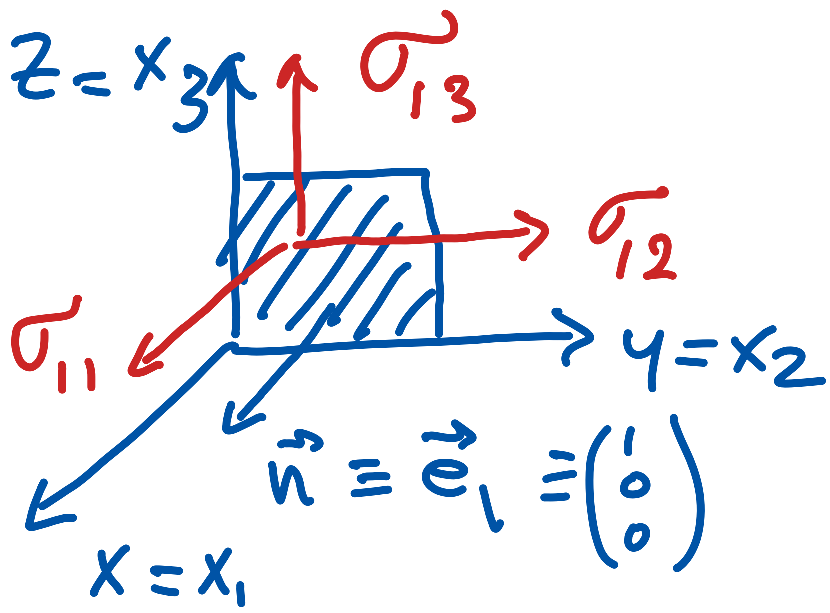
We will need the stress-tensor:

$$\overline{\overline{\sigma}} \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

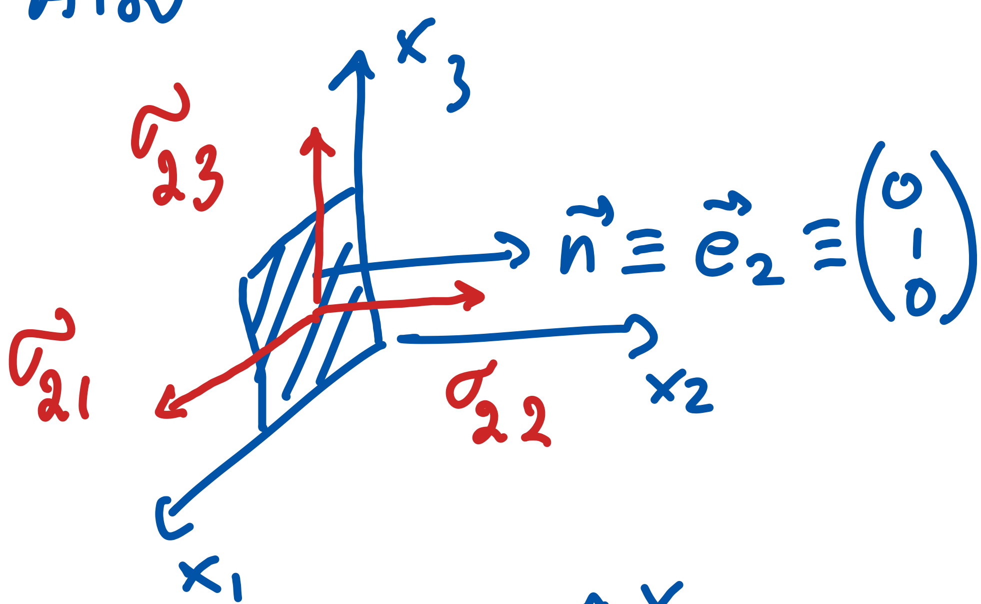
Nine components, all are functions of $\vec{x}, t$

Definition: σ_{ij}

$\equiv$ stress (Pa) on a surface with normal vector $\vec{n} \equiv \vec{e}_i$ acting in the $\vec{e}_j$ direction.



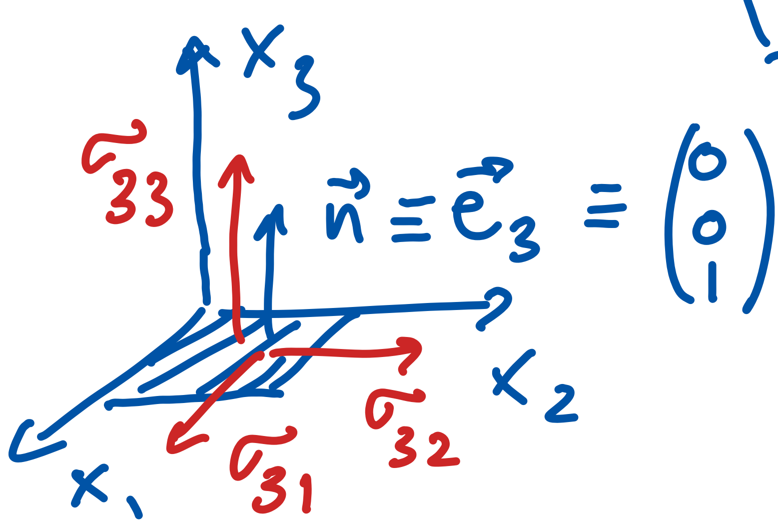
Also



Note:

if $i=j$: normal stress

if $i \neq j$: tangential stress (shear stress).

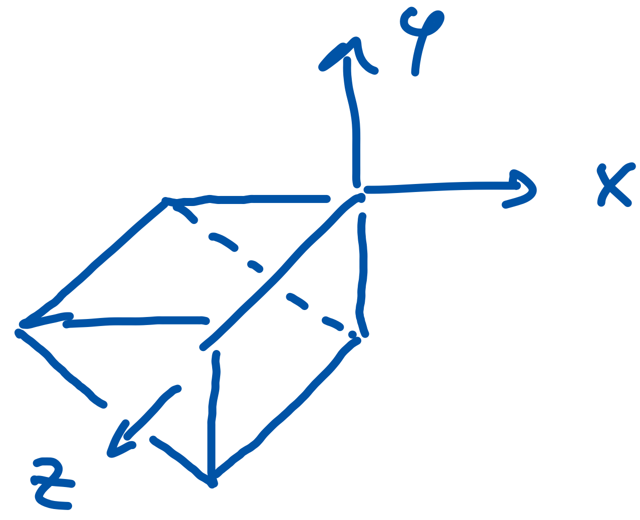


skip this part! Video?

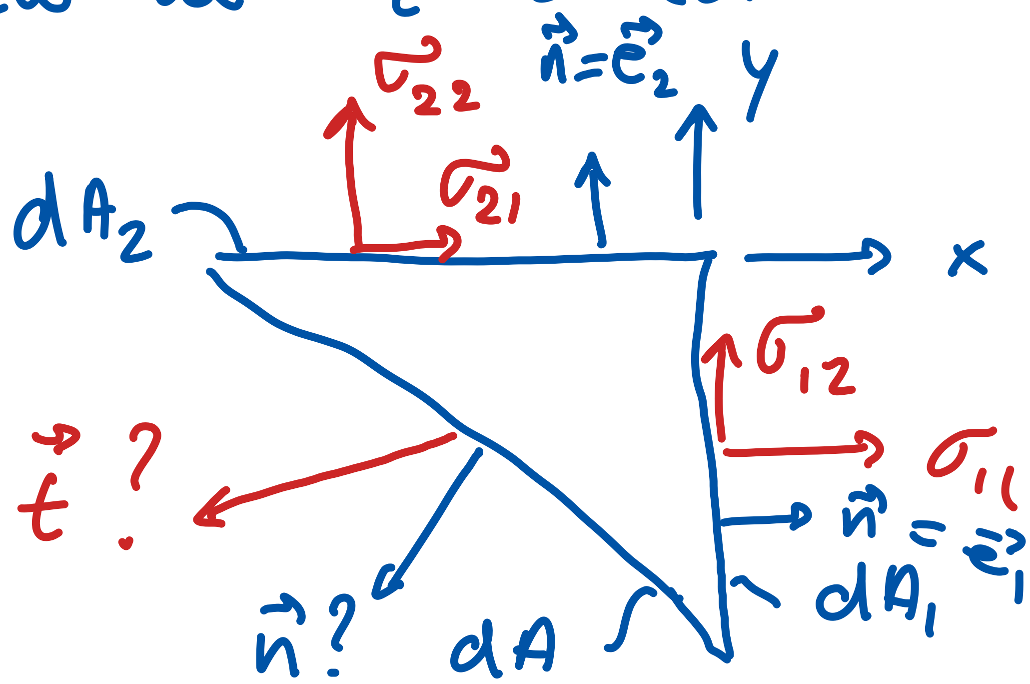
How about the stress on an arbitrary surface (normal vector $\vec{n}$).

Consider a prism*:

* not completely arbitrary, but 1st step.



View in $-z$ direction.



* assume 2D situation.

prism: sufficiently small

Apply Newton's 2nd law to determine $\vec{t}$.

$$\vec{F} = m\vec{a} \qquad 2 \text{ equations.}$$

x-dir: $\sigma_{11}\, dA_1 + \sigma_{21}\, dA_2 + t_1\, dA = m a_1$

y-dir: $\sigma_{12}\, dA_1 + \sigma_{22}\, dA_2 + t_2\, dA = m a_2$

mass $m \sim \epsilon^3$ ϵ is a size of the prism

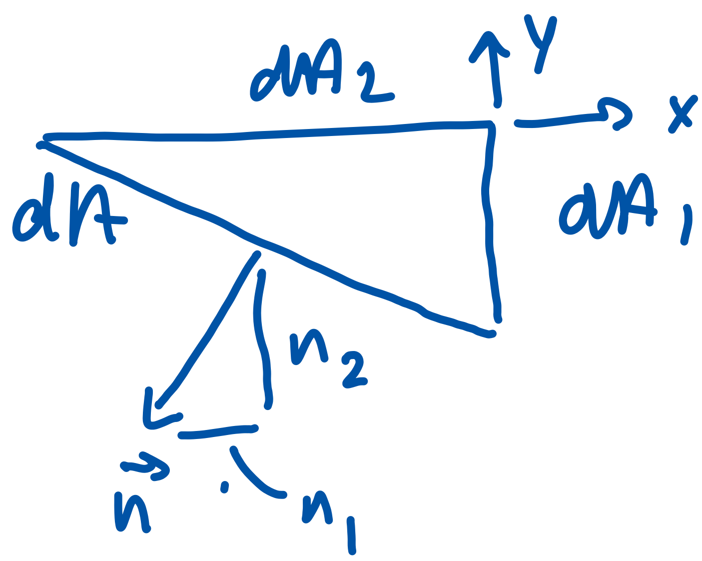
surface $\sim \epsilon^2$

let $\epsilon \downarrow 0$ $|\vec{a}| = \frac{|\vec{F}|}{m} \sim \frac{\epsilon^2}{\epsilon^3} = \frac{1}{\epsilon} \to \infty$

unacceptable so require $\vec{F} \to 0$ if $\epsilon \to 0$

$\Rightarrow \vec{a} = 0$

x-dir : $\sigma_{11} \frac{dA_1}{dA} + \sigma_{12} \frac{dA_2}{dA} + t_1 = 0$



$$\Rightarrow \boxed{\frac{n_1}{|\vec{n}|} = - \frac{dA_1}{dA}}$$

Similarly $\sigma_{21} \frac{dA_1}{dA} + \sigma_{22} \frac{dA_2}{dA} + t_2 = 0$

and $\frac{n_2}{|\vec{n}|} = - \frac{dA_2}{dA}$

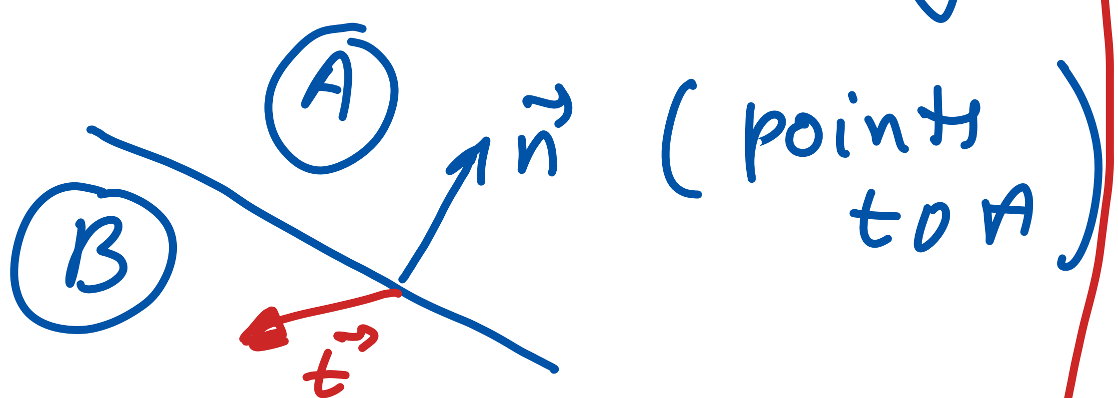
$$\Rightarrow \boxed{\begin{aligned} t_1 &= \sigma_{11} n_1 + \sigma_{21} n_2 \\ t_2 &= \sigma_{12} n_1 + \sigma_{22} n_2 \end{aligned}}$$

Vector-notation : $\boxed{\vec{t} = \underline{\underline{\sigma}}^T \vec{n}}$

Index-notation: $\boxed{t_i = \underbrace{\sigma_{ji} n_j}_{\text{sum over } j}}$

$\boxed{\text{Cauchy's relation also holds in 3D!}}$

Due to the definition of the stress tensor, we have the following:



Cauchy's relation gives the stress vector $\vec{t}$ by A on B

note, stress vector by B on A is $-\vec{t}$

(Newton's 3rd law: action = -reaction)

Remaining question: What is σ_{ij} ???

Guess that σ_{ij} depends on p

" " " " " $\frac{\partial u_i}{\partial x_j}$ $\quad i=1,2,3 \quad j=1,2,3$

Assumption: σ_{ij} linear in p and $\frac{\partial u_i}{\partial x_j}$

Take into account all kinds of symmetry conditions.

Result:

$$\sigma_{ij} = -\delta_{ij}\, p + \tau_{ij} \qquad \delta_{ij} \equiv \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\mu\, \delta_{ij} \underbrace{\frac{\partial u_k}{\partial x_k}}_{\text{Sum!}}$$

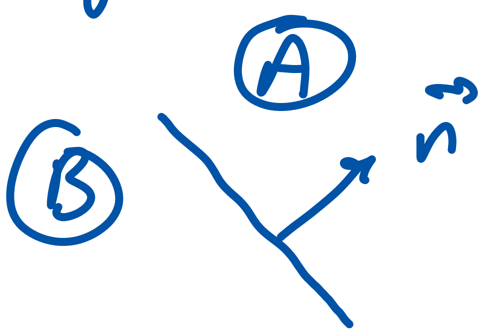
Note: if fluid is 'incompressible'
then $\frac{\partial u_k}{\partial x_k} \equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$

Note: $\frac{\partial u_i}{\partial x_j} \neq \frac{\delta u_i}{\delta x_j} \qquad \partial \neq \delta$!

example: suppose no flow: $\vec{u} = 0$

$$\Rightarrow \frac{\partial u_i}{\partial x_j} = 0 \quad \text{for all } i,j$$

$$\Rightarrow \tau_{ij} = 0 \quad \Rightarrow \sigma_{ij} = -\delta_{ij}\, p.$$



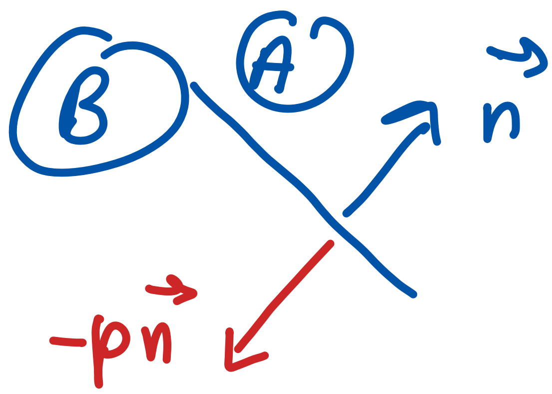
$$t_i = \sigma_{ji} n_j$$
$$= -\delta_{ji}\, p\, n_j$$

$$-\delta_{ji}\, p\, n_j = -p\left(\delta_{1i} n_1 + \delta_{2i} n_2 + \delta_{3i} n_3\right) = -p n_i$$

$$\Rightarrow t_i = -p n_i \quad \Rightarrow \boxed{\vec{t} = -p\vec{n}}$$

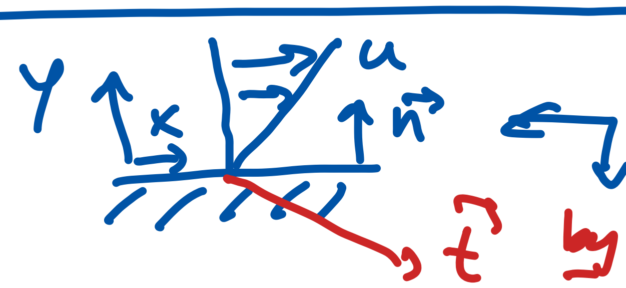
← by A on B.

$\Rightarrow$



One more example: 2D

$\vec{n} = \binom{0}{1}$



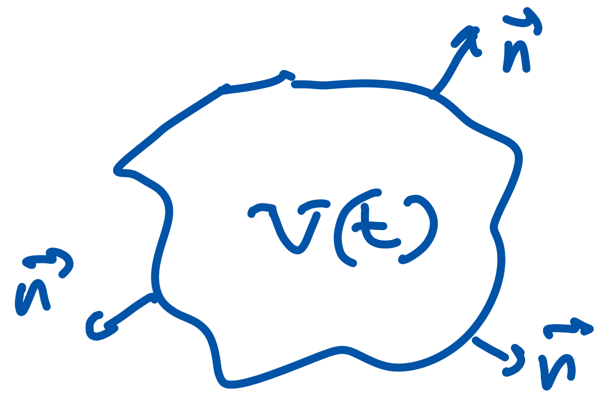
by fluid on surface

$$t_i = \sigma_{ji} n_j \quad \Rightarrow t_1 = \mu \frac{\partial u_1}{\partial x_2} = \mu \frac{\partial u}{\partial y}$$

$$t_2 = -p$$

Momentum Conservation.

$$\text{momentum} \equiv \int_{V(t)} \rho \vec{u} \, dV.$$



Newton's 2nd law: $\boxed{\dfrac{d}{dt} \int_{V(t)} \rho \vec{u} \, dV = \vec{F}}$

by environment
on blob V

Physics

Mathematics:

$$\boxed{\frac{d}{dt} \int_{V(t)} \rho \vec{u} \, dV = \int_{V(t)} \frac{\partial}{\partial t}(\rho \vec{u}) \, dV + \int_{S(t)} \rho \vec{u} \, (\vec{u} \cdot \vec{n}) \, dS}$$

Leibniz-Reynolds.

$\vec{F}$? $\quad \vec{F} = \int_{S(t)} \vec{t} \, dS + \int_{V(t)} \rho \vec{g} \, dV$

$\vec{t}$ by environment on blob surface

$t_i = \sigma_{ji} n_j \qquad \vec{n}$ outward unit normal

$$\Rightarrow \int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{S(t)} \rho u_i \underbrace{u_j n_j}_{\text{sum!}}\, dS = \int_{S(t)} \underbrace{\tau_{ji} n_j}_{\text{sum!}}\, dS + \int_{V(t)} \rho g_i \, dV$$

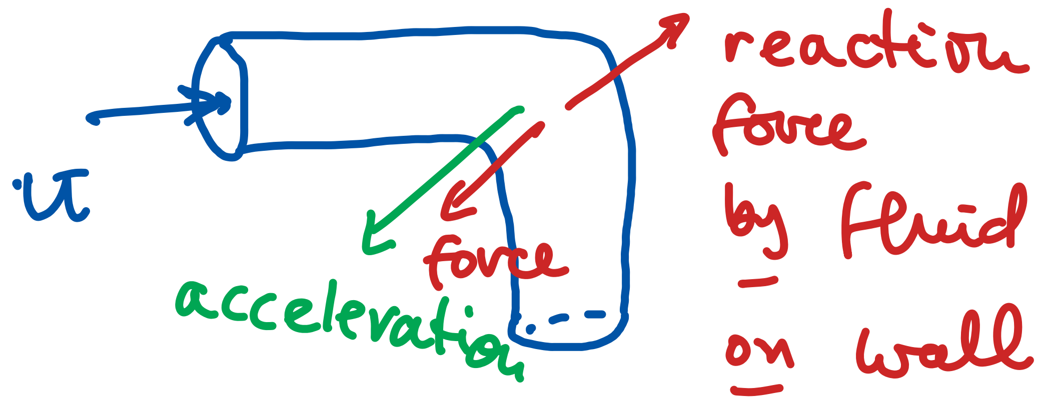
Momentum conservation, integral formulation.

note: $\downarrow \quad \vec{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$

earth

general $\vec{g}$ always works opposite with respect to increasing altitude direction.

Example :



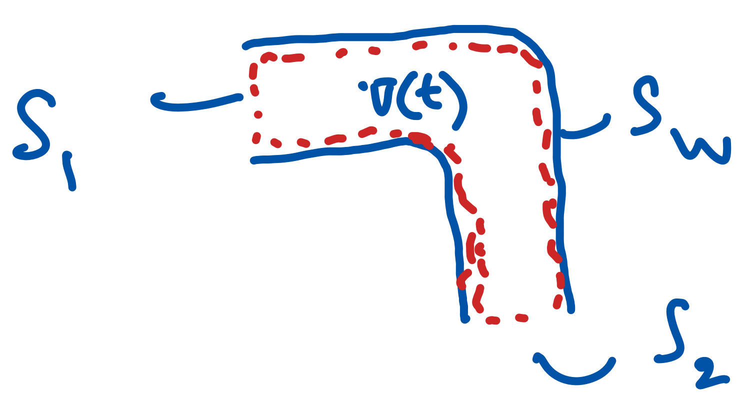
Question: how to compute this force ?

Assume: $\rho = \text{const}$ (very good for liquid).

"$g = 0$" (good if elbow is horizontal).

$\frac{\partial}{\partial t}(\ ) = 0$ (steady flow).

$$\Rightarrow \boxed{\int_{S(t)} \rho u_i \, u_j n_j \, dS = \underbrace{\int_{S(t)} \sigma_{ji} n_j \, dS}_{\text{force in } i\text{-direction by surroundings on } V(t)}}$$



$$\Rightarrow S = S_1 + S_2 + S_w$$

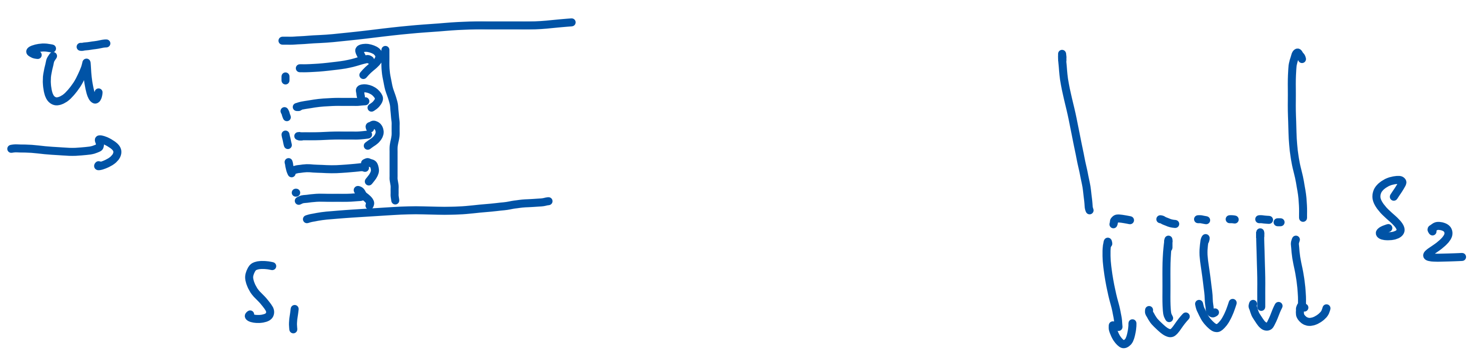
$$\int_{S_w} \sigma_{ji} n_j \, dS = \text{force in } i\text{-direction by wall on } V(t) \overset{\text{choice}}{\equiv} -F_i$$

$\Rightarrow$ F_i = force in i-direction by $V(t)$ (fluid) on wall

$$\Rightarrow \int_S \rho u_i u_j n_j \, dS = \int_{S_1} \sigma_{ji} n_j \, dS + \int_{S_2} \sigma_{ji} n_j \, dS + \underbrace{\int_{S_W} \sigma_{ji} n_j \, dS}_{= -F_i}$$

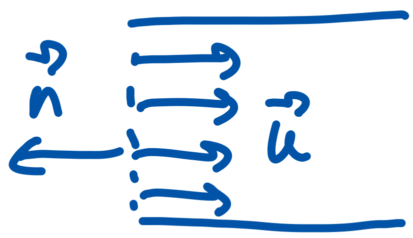
$$\Rightarrow \boxed{F_i = \int_{S_1} \sigma_{ji} n_j \, dS + \int_{S_2} \sigma_{ji} n_j \, dS - \int_S \rho u_i u_j n_j \, dS}$$

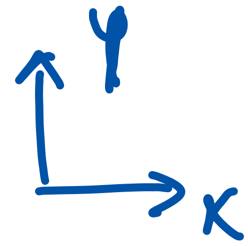
Assume in- and outflow: uniform.



$\Rightarrow$ at S_1, S_2 $\sigma_{ji} = -p\delta_{ji}$

because τ_{ji} contains $\frac{\partial u_i}{\partial x_j}$ which are zero





$$\vec{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{u} = \bar{U} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{U} \\ 0 \\ 0 \end{pmatrix}$$

note: $\bar{U}$ is the average inward velocity in x-direction.

$\Rightarrow$ at the entrance: $u_1 = \bar{U}$, $u_2 = 0$, $u_3 = 0$

also $u_j n_j = \begin{pmatrix} \bar{U} \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\bar{U} + 0 + 0 = -\bar{U}$

$$\Rightarrow \int_{S_1} \sigma_{ji} n_j \, dS = \int_{S_1} -p \delta_{ji} n_j \, dS = \int_{S_1} -p n_i \, dS$$

assume p uniform over the entrance:

$$\Rightarrow -p \int_{S_1} n_i \, dS.$$

If force in x-direction is asked: $i = 1$

$$\Rightarrow -p \int_{S_1} n_1 \, dS = -p \int_{S_1} -1 \, dS = p_{in} S_1$$

Also $\int_{S_1} \rho u_1 (u_j n_j) \, dS$

$= \rho \int_{S_1} \bar{u} \cdot -\bar{u} \, dS = -\rho \bar{u}^2 S.$

Integrals over S_2 are similar.

$\Rightarrow \quad F_1 = \underbrace{\cdots\cdots\cdots}_{\text{check dimensions!}}$