Chapter 4
Convection

We discussed the energy transport by heat conduction in chapter 3. Molecular properties, described by the thermal conductivity of the material play a key role in heat transport.

In some of the energy transport examples discussed previously, we had to formulate the boundary conditions for heat transfer from the body to the flowing fluid along the body. Without detailed analysis of the physical relationships, we assumed that

$$\frac{\dot{Q}_\mathrm{W}}{A} \equiv \dot{q}''_\mathrm{W} = \alpha(T_\mathrm{W} - T_\mathrm{a}) \tag{4.1}$$

whereat the heat transfer coefficient α was assumed to be known. In this section, we will discuss the basic principles of this transport mechanism.

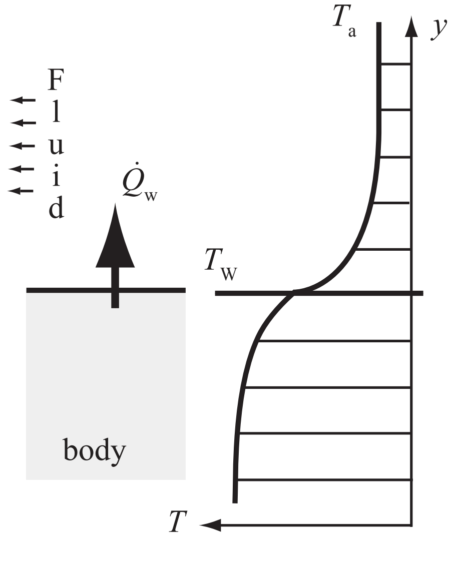

Figure 4.1: Heat transfer at a surface

The major difference between the transport mechanisms of heat conduction and heat convection can be demonstrated by a simple experiment, where a hot, horizontal wall is let to cool down.

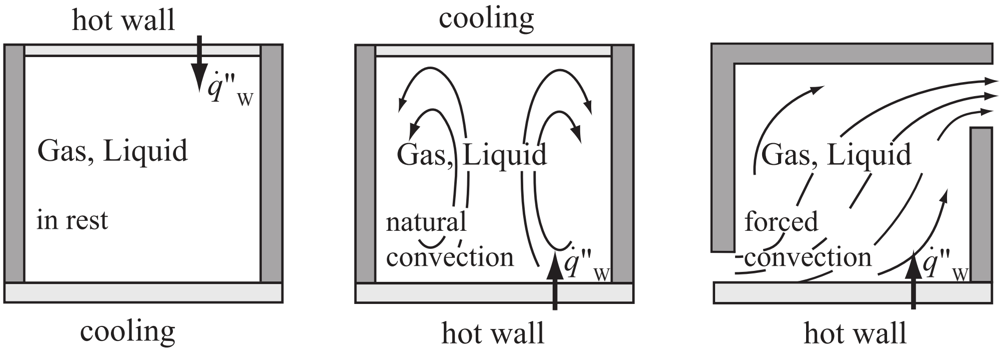

Figure 4.2: Cooling of a hot surface by conduction and convection

In the first experiment, the wall is cooled from below by a cold fluid, liquid or gas. For this arrangement we have a stagnant fluid, and the heat is transported solely by conduction to the area of lower heat. If the hot wall is placed at the bottom, the fluid will become unstable, and the warmer areas (in the lower part) will rise up due to their lower density and hence flow in the fluid will be initiated. This so-called *natural convetion* amplifies the heat transfer. In the third case shown (rightmost diagram), the hot wall is cooled by a medium, which is pumped through the system. As was the case of natural convection, so again, in this case of *forced convection*, it is not only the thermal conductivity of the fluid that determines the characteristics of the heat transfer. An important, additional parameter that dictates the heat transfer is obviously the velocity of the fluid, which depends on the density differences as well as the electrical power of the pump.

Only in areas directly adjacent to the wall, where because of the boundary layer condition which states that the velocity drops to zero (non-slip condition), the Fourier equation for the heat transfer in the fluid still applies

$$\dot{q}''_{\mathrm{W}} = -\left(\lambda_{\mathrm{fluid}} \frac{dT_{\mathrm{fluid}}}{dy}\right)_{\mathrm{W}} \qquad \text{(regarding the fluid)} \qquad (4.2)$$

as well as for the heat transfer in the body

$$\dot{q}''_{\mathrm{W}} = -\left(\lambda_{\mathrm{body}} \frac{dT_{\mathrm{body}}}{dy}\right)_{\mathrm{W}} \qquad \text{(regarding the solid body)} \qquad (4.3)$$

The heat transfer coefficient defined by equation (4.1) can be determined from the temperature gradient at the fluid side of the wall by comparison with equation (4.2).

$$\alpha = \frac{-\left(\lambda_{\text{fluid}} \frac{dT_{\text{fluid}}}{dy}\right)_{\text{W}}}{T_{\text{W}} - T_{\text{a}}} \tag{4.4}$$

The temperature field in the fluid and hence the temperature gradient at the wall surface is described by the equation of energy conservation (1st Law of thermodynamics). In this section, we will have to expand the energy balance from chapter 3 by including the enthalpy flowing in and out of the volume element. To describe these flows, the velocities, in addition to the temperatures, have to be known. Thus, in addition to the energy equation, the momentum equation and the continuity equation will have to be formulated and solved. The necessary basic principles have been discussed in introductory lectures on fluid mechanics and will only be briefly presented here.

4.1 Conservation laws for laminar, steady state, two-dimensional flow

The general solution of a three-dimensional flow, for which the flow field and the temperature field are interdependent can only be determined by determining the three velocity components u, v and w, the temperature T and pressure p, as well as the material properties of the fluid, the density ρ , the dynamic viscosity η and the thermal conductivity λ. Since these parameters are not independent of each other, a simultaneous solution of the conservation equations of mass, momentum and energy is required. To avoid too much writing, we will focus on the two-dimensional, steady state case.

The flow can be either laminar or turbulent, i.e. the streamlines are either parallel or superimposed to the mean, steady state fluctuating flow. These three-dimensional fluctuating velocities and their impact on the momentum and heat exchange will be discussed in section 4.4.

The following derivations are limited to laminar flow.

4.1.1 Equation of continuity

For steady state flow, the difference between mass flowing in and out must vanish.

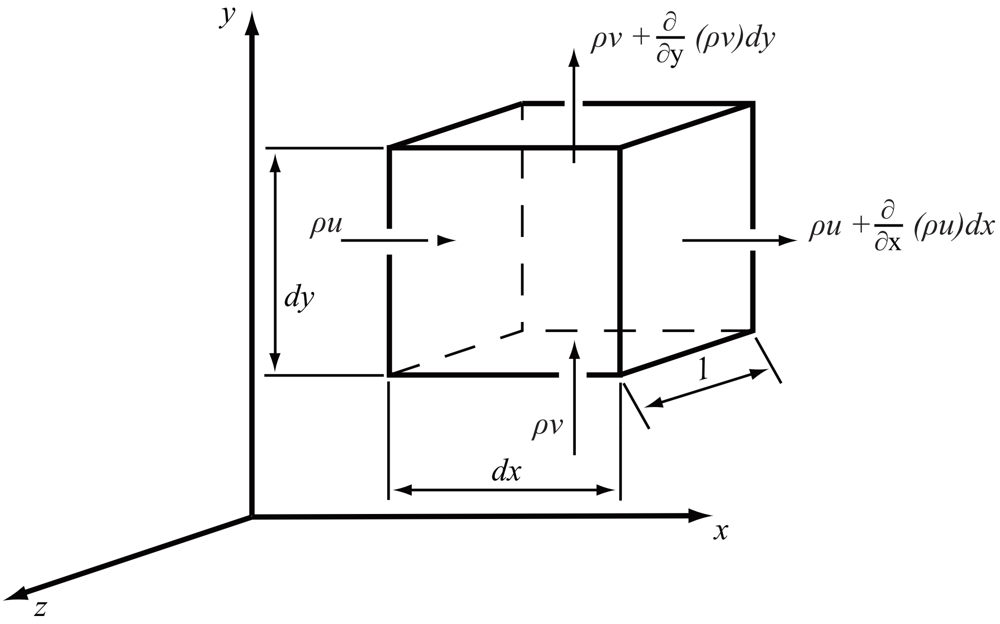

Figure 4.3: Mass balance at the control volume

Mass flowing in

$$\begin{array}{c} \rho\ u\ dy\ l \\ \rho\ v\ dx\ l \end{array}$$

Mass flowing out

$$\begin{array}{c} \rho\ u\ dy\ l + \frac{\partial}{\partial x}\rho\ u\ dx\ dy\ l \\ \rho\ v\ dx\ l + \frac{\partial}{\partial y}\rho\ v\ dy\ dx\ l \end{array}$$

Difference

$$\frac{\partial}{\partial x}\rho\ u\ dx\ dy\ l + \frac{\partial}{\partial y}\rho\ v\ dx\ dy\ l$$

This yields the equation of continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad \text{(equation of continuity)} \tag{4.5}$$

or vectorically

$$div\left(\rho\,\vec{w}\right) = 0$$

For incompressible fluids ($\rho =$ constant), this equation simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.6}$$

4.1.2 Equations of momentum

The momentum equation states that the difference between the momentum in and out flow of the volume element is equal to the external forces acting on the volume element, whereat intertial forces and surface forces can be relevant.

4.1.2.1 Equation of momentum in x-direction

The difference between momentum flowing in and out is

$$\left(\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y}\right) dxdy\; l = \left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + u\left(\underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}}_{=0 \text{ (eq.4.5)}}\right)\right) dxdy\; l \tag{4.7}$$

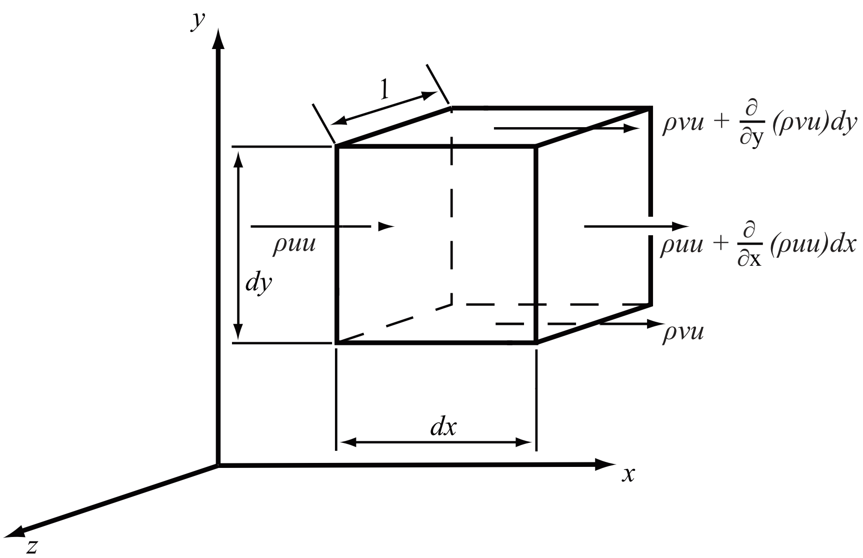

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Figure 4.4: x-momentum balance at the control volume

The surface forces, i.e. pressure, shear and normal forces can be expressed by p_{xy}, where the first index indicates the orientation of the surface at which the force is applied (e.g. here perpendicular to the x-direction) and the second index shows the direction of the force (e.g. here in the y-direction).

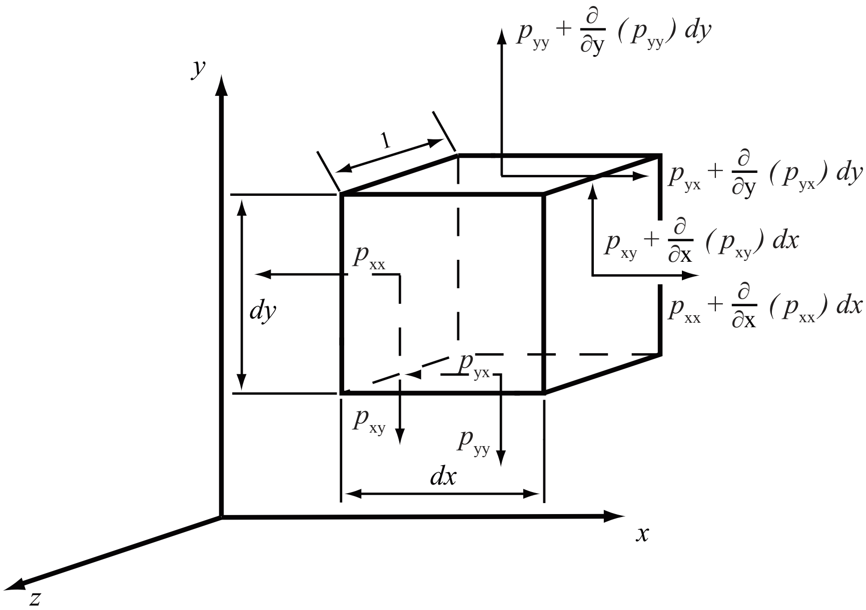

Figure 4.5: Forces at the control volume

In the equation of momentum for the x-direction, only the forces acting in x-direction are considered.

Hence, the balance of surface forces applied at the volume element is

$$\frac{\partial p_{\text{xx}}}{\partial x} dxdy\; l + \frac{\partial p_{\text{yx}}}{\partial y} dy\; dx\; l$$

with the normal stress p_{xx} and the shear stress p_{yx},related to the pressure u and v and the dynamic viscosity η [kg/ms] given by Navier (1827) and Stokes (1845), see Schlichting u. Gersten (2006).

$$p_{\text{xx}} = -p + \sigma_{\text{x}} = -p - \frac{2}{3}\eta \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\eta \frac{\partial u}{\partial x}$$

and

$$p_{\text{yx}} = \tau_{\text{yx}} = \tau_{\text{xy}} = \eta \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

This yields

$$\left[-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x}-\frac{2}{3}\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\eta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)\right] dxdy\ l$$

At the volume element, inertial forces are also applied, of which we will discuss only the force of gravity

$$\rho\ g_{\mathrm{x}} dxdy\ l$$

Summarizing the momentum, surface and inertial forces, we get the momentum equation, the so-called Navier-Stokes equation

$$\begin{aligned}\rho u\frac{\partial u}{\partial x}&+\rho v\frac{\partial u}{\partial y}\\ &=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x}-\frac{2}{3}\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\eta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\rho\ g_{\mathrm{x}}\end{aligned}\tag{4.8}$$

(equation of momentum, x-direction)

4.1.2.2 Equation of momentum in y-direction

$$\begin{aligned}\rho u\frac{\partial v}{\partial x}&+\rho v\frac{\partial v}{\partial y}\\ &=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}-\frac{2}{3}\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial x}\left(\eta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\rho\ g_{\mathrm{y}}\end{aligned}\tag{4.9}$$

(equation of momentum, y-direction)

Assuming constant material properties (density, viscosity), the equations of momentum are further simplified to

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho\ g_{\mathrm{x}}\tag{4.10}$$

(equation of momentum, x-direction, (ρ,η constant))

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho\, g_{\mathrm{y}} \tag{4.11}$$

(equation of momentum, y-direction, (ρ,η constant))

or as vector equation

$$\rho\left(\vec{w}\times\text{grad}\,\right)\vec{w}=-\text{grad}\,p+\eta\nabla^2\,\vec{w}+\rho\,\vec{g} \tag{4.12}$$

(equation of momentum, (ρ,η constant))

4.1.3 Equation of energy conservation

The fist law of thermodynamics

$$d\dot{Q}+dP=d\dot{U}+d\dot{E}_a \tag{4.13}$$

applied to a volume element of a flowing medium states that any heat or work added to the system will result in change of inner, potential or kinetic energy.

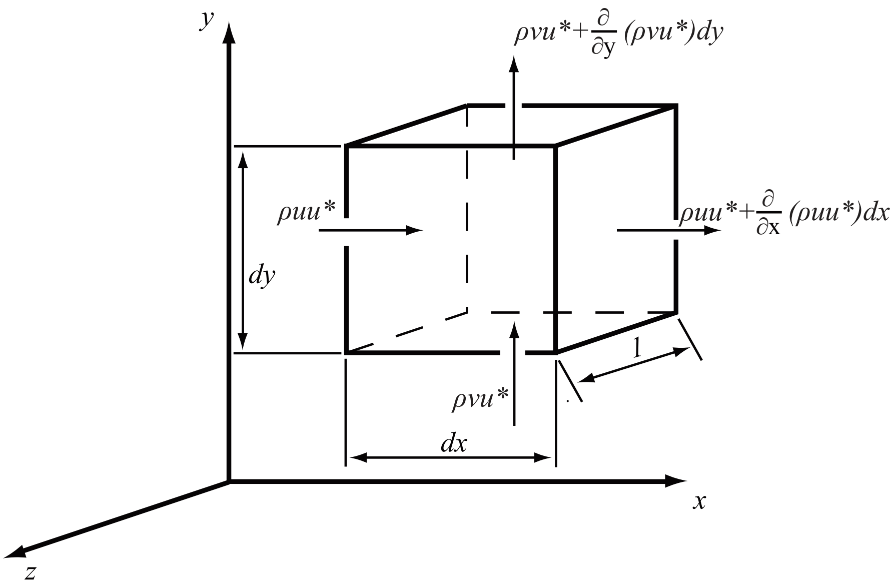

Figure 4.6: Energy flows at the control volume

The kinetic energy is to be considered for the net energy only for flow velocities in the order of the speed of sound, and thus will be neglected here. The difference of inner energy flowing in and out is

$$\frac{\partial}{\partial x}\left(\rho u u^*\right) dxdyl + \frac{\partial}{\partial y}\left(\rho v u^*\right) dxdyl$$

where the specific inner energy is named u^* $[\frac{\mathrm{J}}{\mathrm{kg}}]$. If we introduce the specific enthalpy $h \equiv u^* + \frac{p}{\rho}$ instead of the specific inner energy we get

$$\left(\frac{\partial}{\partial x}\left(\rho u h\right) + \frac{\partial}{\partial y}\left(\rho v h\right) - \frac{\partial}{\partial x}\left(up\right) - \frac{\partial}{\partial y}\left(vp\right)\right) dxdy\, l$$
$$= \left(\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} + h\left(\underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}}_{=0\ (\text{eq.4.5})}\right) - \frac{\partial}{\partial x}\left(up\right) - \frac{\partial}{\partial y}\left(vp\right)\right) dxdy\, l$$

Resulting from Fourier's equation, the total *added heat* is

$$\left(\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)\right) dxdy\, l.$$

The volume element receives work from the surface (friction) forces as well as from the inertial forces.

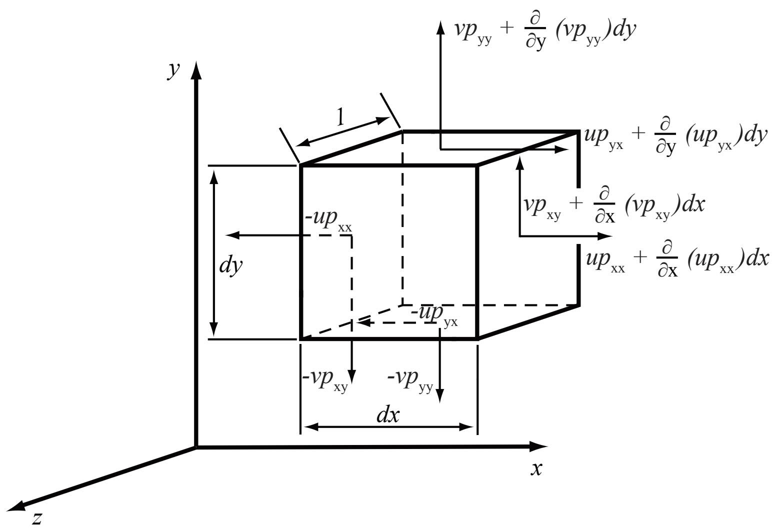

Figure 4.7: Contribution of the forces to the energy balance of the volume element

$$\left(\frac{\partial}{\partial x}\left(u\sigma_{\mathrm{x}}\right) - \frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(u\tau_{\mathrm{xy}}) + \rho u g_{\mathrm{x}} \right.$$
$$\left. + \frac{\partial}{\partial y}(v\sigma_{\mathrm{y}}) - \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial x}(v\tau_{\mathrm{xy}}) + \rho v g_{\mathrm{y}}\right) dxdy\ l$$

Hence, the energy equation is

$$\begin{aligned}\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} =& \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) \\ &+ \frac{\partial}{\partial x}\left(u\sigma_{\mathrm{x}} + v\tau_{\mathrm{xy}}\right) + \frac{\partial}{\partial y}\left(u\tau_{\mathrm{xy}} + v\sigma_{\mathrm{y}}\right) + \rho u g_{\mathrm{x}} + \rho v g_{\mathrm{y}}\end{aligned} \tag{4.14}$$

The stress contributions of the normal, shear and inertial forces will be omitted since their values are, in most cases, negligibly small and do not contribute significantly to the net energy balance.

The specific enthalpy may be substituted by the temperatures for systems without phase changes, no chemical reactions and, if the fluid can be considered an ideal gas ($dh = c_p\ dT$) or an incompressible liquid ($dh = c\ dT$)

Hence,

$$\rho u c_p\frac{\partial T}{\partial x} + \rho v c_p\frac{\partial T}{\partial y} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) \tag{4.15}$$

Comparing this equation with the Fourier equation for heat conduction, equation 3.2, for steady state problems without heat sources, it can be shown that under the assumptions above the energy transport by heat conduction is extended by convective energy transport through enthalpy flow. For unsteady state processes, the left-hand side of equation 4.15 is to be supplemented by a term characteristic for the energy storage $\rho c_p\frac{\partial T}{\partial t}$.

If, in addition, the thermal conductivity is independent of location, following from equation (4.15)

$$\rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y} = \frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4.16}$$

or

$$\rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y} = \frac{\eta}{\mathrm{Pr}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4.17}$$

with the Prandtl number

$$\mathrm{Pr} \equiv \frac{\eta}{\lambda/c_p} \tag{4.18}$$

or in vector form

$$\rho \vec{w} \times \text{grad}\, T = \frac{\eta}{\text{Pr}} \nabla^2 T \tag{4.19}$$

The temperature field of a flowing medium and hence the temperature gradient at its boundary, from which the heat transfer can be calculated, equation (4.2), is sufficiently described by the energy equation, (4.17), momentum equation, (4.10), the equation of continuity, (4.6), and appropriate estimates of the material properties as well as the boundary conditions of the system itself.

This system of interdependent partial differential equations can only be solved numerically using methods of finite differences or finite volumes, which nowadays can be obtained commercially and which will be discussed in depth at other lectures. Such computational fluid dynamic (CFD) methods reach the ultimate limits of modern high performance computers in cases of geometries or boundary conditions with a high degree of complexity, so that often it is more cost-efficient to carry out experiments to obtain heat transfer coefficients.

A major group of technologically important flows, *boundary layer flows,* uses a simplified form of the derived conservation equations. This leads to parabolic instead of elliptical partial differential equations, which in turn are much easier to handle numerically, and in some cases can be transformed into ordinary differential equations.

These solution methods will be briefly discussed further on, a detailed investigation of these problems can be found for example in Bayley u. a. (1972).

4.2 Forced convection - boundary layer equations for laminar, steady state flow

For flow along solid walls the influence of the viscous forces is limited to an area in the vicinity of the wall, called *velocity boundary layer* , with thickness δ_u The viscosity is of no influence outside of this boundary layer the flow behaves as a potential flow. The same applies for the area, where heat conduction dominates. The *thermal boundary layer* with thickness δ_T outlines the area of heat conduction from the area of undisturbed flow.

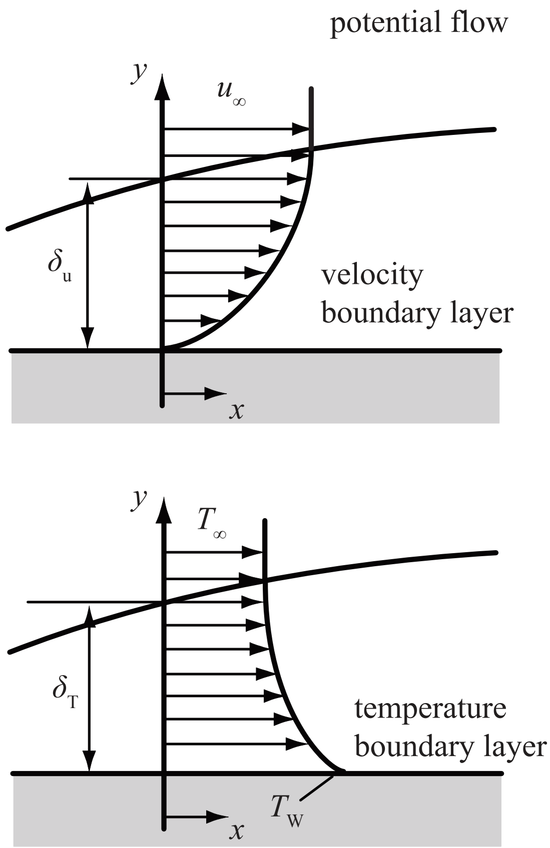

Figure 4.8: Velocity and thermal boundary layers of a surface in a stream flow

Prandtl (1904) derived the equations valid for boundary layers from the conservation laws through appropriate assumptions for the individual terms. These assumptions, which can also be found at Schlichting and Gersten (2006), state that for flow of the boundary layer type, i.e. $\frac{\delta}{L} \ll 1$, the following assumptions are valid

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2} \tag{4.20}$$

Thus, the parabolic differential equations for boundary layers with constant material properties are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.21}$$

(equation of continuity)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} + \rho g_{\mathrm{x}} \tag{4.22}$$

(momentum equation, x-direction)

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\mathrm{Pr}} \frac{\partial^2 T}{\partial y^2} \tag{4.23}$$

(equation of energy conservation)

In general, the thicknesses of the velocity and thermal boundary layers are not equal. Yet, as a consequence of the approximations for the boundary layers, with $\delta_u \approx \eta^{\frac{1}{2}}$ yields:

$$\frac{\delta_u}{\delta_T} \approx \mathrm{Pr}^{\frac{1}{2}} \tag{4.24}$$

The Prandtl number for gases is approximately 1, so that the velocity and thermal boundary layer almost coincide in thickness.

4.2.1 Exact solutions for the equations of boundary layers

Transformation of coordinates is possible for a flat plate ($dp/dx = 0$), without inertial forces ($\rho g_{\mathrm{x}} = 0$) which leads to an ordinary differential equation with an exact solution, see Schlichting u. Gersten (2006).

The solution of the momentum equation, the velocity profile in the boundary layer, was firstly published by Blasius (1908) and is shown in 4.9 as a function of the dimensionless wall distance. The dimensionless wall shear stress, the friction coefficient $\frac{c_{\mathrm{f}}}{2}$, derives from the velocity gradients at the wall.

$$\frac{c_{\mathrm{f}}}{2} \equiv \frac{\tau_{\mathrm{W}}}{\rho u_\infty^2} = \frac{\left(\eta \frac{du}{dy}\right)_{\mathrm{W}}}{\rho u_\infty^2} \tag{4.25}$$

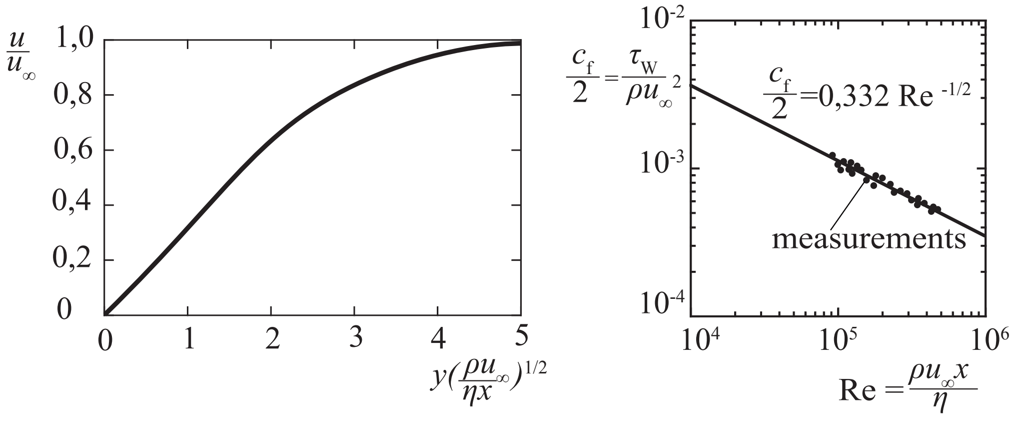

Figure 4.9: Laminar flow over a flat plate - velocity profile and friction coefficient, according to Blasius (1908)

The corresponding equation of energy conservation (equation 4.23) for the flat plate with constant wall temperature was solved by Pohlhausen (1921) and leads to the temperature profiles, which are additionally dependent on the Prandtl number. The temperature gradients at the wall and hence the local heat transfer coefficient α derive from these temperature profiles, see equation (4.4), or its mean value $\overline{\alpha}$:

$$\alpha = \frac{-\left(\lambda \frac{dT}{dy}\right)_{\mathrm{W}}}{(T_{\mathrm{W}} - T_\infty)} \quad \text{and} \quad \overline{\alpha} = \frac{1}{L}\int_0^L \alpha(x)\, dx \tag{4.26}$$

The dimensionless temperature profile and the dimensionless heat transfer coefficient, the Nusselt number

$$\mathrm{Nu} \equiv \frac{\alpha\, x}{\lambda} \quad \text{and} \quad \overline{\mathrm{Nu}} \equiv \frac{\overline{\alpha} L}{\lambda} \tag{4.27}$$

are shown in Fig. 4.10.

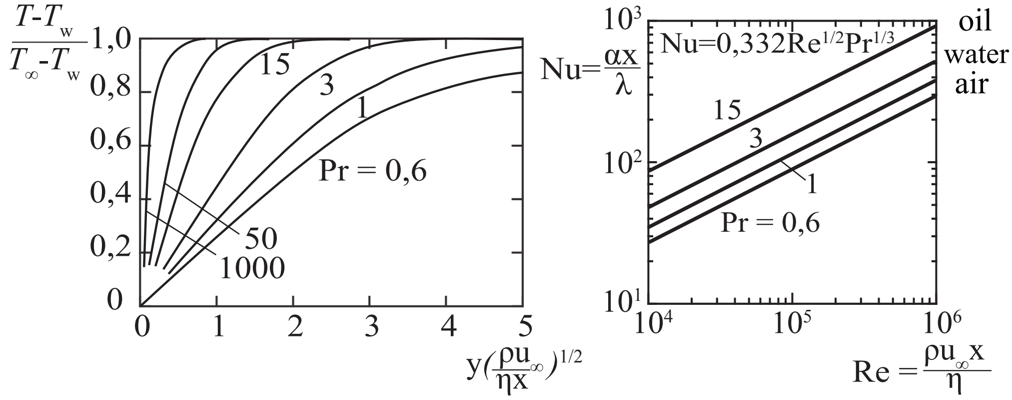

Figure 4.10: Laminar flow over a flat plate - temperature profile and Nusselt number according to Pohlhausen (1921)

For Pr = 1 the momentum equation, equation (4.22), and the energy, equation (4.23), are identical if the pressure gradients and inertial forces are neglected. Hence, the dimensionless velocity and temperature profile must also be identical and thus

$$\left(\frac{d\left(\frac{u}{u_\infty}\right)}{dy}\right)_{\mathrm{W}} = \left(\frac{d\left(\frac{T-T_W}{T_\infty-T_W}\right)}{dy}\right)_{\mathrm{W}}$$

From equation (4.4) and (4.27) follows

$$\mathrm{Nu} = \frac{x}{\lambda}\left(\lambda\frac{d\left(\frac{T-T_{\mathrm{W}}}{T_\infty-T_{\mathrm{W}}}\right)}{dy}\right)_{\mathrm{W}}$$

and with equation (4.25)

$$\frac{c_{\mathrm{f}}}{2} = \frac{\eta}{\rho u_\infty}\left(\frac{d\frac{u}{u_\infty}}{dy}\right)_{\mathrm{W}}, \text{(for Pr = 1)} \tag{4.28}$$

With the equality of the dimensionless temperature and velocity profiles, the ratio between the Nusselt number, the friction coefficient and Reynolds number, $\mathrm{Re} = \frac{\rho u_\infty \mathrm{x}}{\eta}$ is:

$$\mathrm{Nu} = \frac{c_{\mathrm{f}}}{2}\mathrm{Re}, \text{(for Pr = 1)} \tag{4.29}$$

Taking the apprriximate influence of the Prantl number into account, the enhanced relation is:

$$\mathrm{Nu} = \frac{c_{\mathrm{f}}}{2}\mathrm{Re}\ \mathrm{Pr}^{\frac{1}{3}} \tag{4.30}$$

4.2.2 A simple approximation for the boundary layer equations

In the previous section the results of the exact solutions of the laminar, steady state boundary layer equations for plane flows were shown. The necessary mathematical methods are so complex that they cannot be included in the scope of this introductory course.

However, a very simple approximation for the boundary layer equations, which yields results comparable to those of the exact solutions and above all, shows the physical principles, will be discussed next.

We examine again the boundary layer flow over a flat plate. A control volume, bound by the planes $\overline{11}$, $\overline{12}$, $\overline{22}$ and the wall will be used to formulate the conservation equations. Plane $\overline{12}$ is located within the potential flow, outside of the boundary layer.

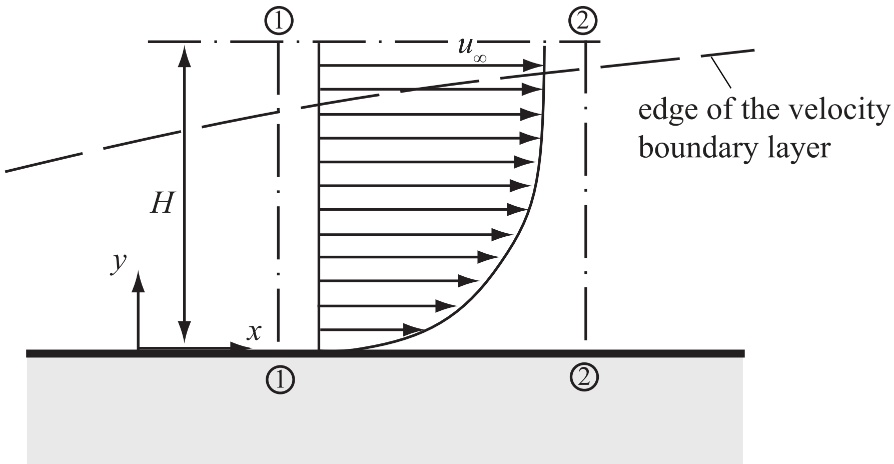

Figure 4.11: Integral momentum balance of a boundary layer flow

For the control volume described above, follows the integral *equation of continuity*

$$\frac{d}{dx}\int_0^H \rho\, u\, dy\, dx\, l + \rho v_\infty dx\, l = 0 \tag{4.31}$$

(equation of continuity)

and the integral momentum equation in x-direction

$$\frac{d}{dx}\int_0^H \rho\, u^2\, dy\, dx\, l + \rho u_\infty v_\infty dx\, l = -\tau_{\mathrm{W}} dx\, l$$

At the upper boundary plane, which is located in the potential flow, no shear forces are present. Normal forces in the planes $\overline{11}$ and $\overline{22}$ are to be neglected following the boundary layer approximation. At the wall, only the shear stress $\tau_{\mathrm{yx}} = \left(\eta \frac{du}{dy}\right)_{\mathrm{W}}$ should be considered. Considering the equation of continuity, equation (4.31), yields:

$$\frac{d}{dx}\int_0^H \rho\, u^2\, dy - \frac{d}{dx}\int_0^H \rho\, u u_\infty\, dy = -\left(\eta \frac{du}{dy}\right)_{\mathrm{W}}$$

or

$$\frac{d}{dx}\int_0^H \rho u\, (u_\infty - u)\, dy = \left(\eta \frac{du}{dy}\right)_{\mathrm{W}} \tag{4.32}$$

(momentum equation, x-direction)

The integration is possible if the velocity profile in the boundary layer is known.

If, a *linear velocity profile* is assumed, as a crude approximation

$$\frac{u}{u_\infty} = \frac{y}{\delta_{\mathrm{u}}} \tag{4.33}$$

and density is constant within the boundary layer, equation (4.32) yields

$$\rho \mathrm{u}_\infty \frac{d}{dx}\int_0^{\delta_{\mathrm{u}}} \frac{y}{\delta_{\mathrm{u}}}\left(1 - \frac{y}{\delta_{\mathrm{u}}}\right) dy = \frac{\eta}{\delta_{\mathrm{u}}}$$

Eventually a relationship for the thickness of the velocity boundary layer is found by integration

$$\delta_{\mathrm{u}} = \sqrt{\frac{12\eta}{\rho u_\infty} x} \tag{4.34}$$

or dimensionless

$$\frac{\delta_\mathrm{u}}{x} = \sqrt{\frac{12\eta}{\rho \mathrm{u}_\infty \mathrm{x}}} = \sqrt{\frac{12}{\mathrm{Re}_\mathrm{x}}} \tag{4.35}$$

with the Reynolds number, $\mathrm{Re} \equiv \frac{\rho u_\infty x}{\eta}$, based on the length x.

If the thickness of the boundary layer is known, the *shear stress at the wall* can be determined

$$\tau_\mathrm{W} = \left(\eta \frac{du}{dy}\right)_\mathrm{W} = \eta \frac{u_\infty}{\delta_\mathrm{u}} = \frac{1}{\sqrt{12}\sqrt{\mathrm{Re}}} \rho\, u_\infty^2 \tag{4.36}$$

and in its dimensionless form, the friction coefficient $\frac{c_\mathrm{f}}{2}$

$$\frac{c_\mathrm{f}}{2} = \frac{\tau_\mathrm{W}}{\rho u_\infty^2} = \frac{1}{\sqrt{12}\sqrt{\mathrm{Re}}} = 0{,}289\ \mathrm{Re}^{-\frac{1}{2}} \tag{4.37}$$

This relationship shows the same dependence of the friction coefficient on the Reynolds number as the previously given exact solution of the momentum equation

$$\frac{c_\mathrm{f}}{2} = 0{,}332\ \mathrm{Re}^{-\frac{1}{2}} \tag{4.38}$$

merely the constant is 15% too low.

The *integral energy equation* can be derived using a similar control volume that includes the thermal boundary layer.

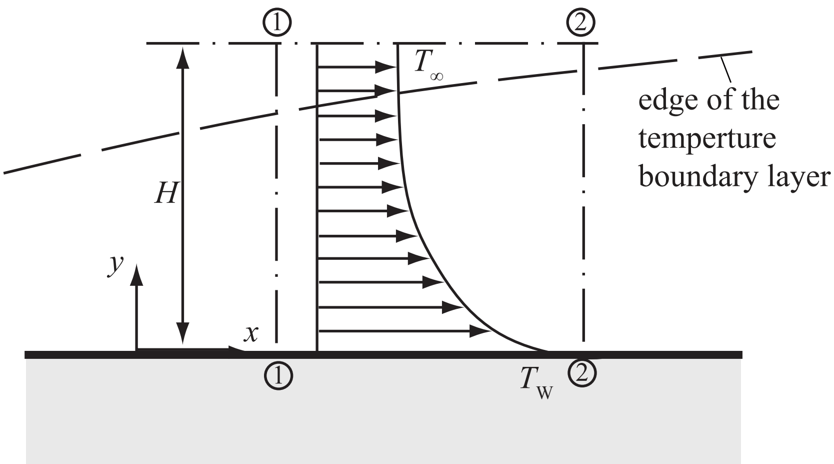

Figure 4.12: Integral energy balance at a boundary layer

Neglecting the dissipated heat of the forces acting on the volume and the heat conduction in direction of the flow, which is negligible compared to the convective energy transport in this direction, results in

$$\frac{d}{dx}\int_0^H \rho\, u\, c_p(T-T_0)dx\; dy\; l+\rho v_\infty c_p(T_\infty-T_0)dx\; l=-\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}} dx\; l$$

where T_0 is the reference temperature used to determine the enthalpy reference value.

Implementing the equation of continuity, equation (4.31), again results in

$$\frac{d}{dx}\int_0^H \rho\, u\, c_p(T-T_0)dy-\frac{d}{dx}\int_0^H \rho\, u\, c_p(T_\infty-T_0)dy=-\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}}$$

or

$$\frac{d}{dx}\int_0^H \rho\, u\, c_p(T-T_\infty)dy=-\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}} \qquad \text{(energy equation)} \tag{4.39}$$

Assuming constant properties (ρ,λ,c_p) the energy equation can be integrated assuming a linear temperature profile in the boundary layer

$$\frac{T-T_{\mathrm{W}}}{T_\infty-T_{\mathrm{W}}}\equiv\frac{\theta}{\theta_\infty}=\frac{y}{\delta_{\mathrm{T}}} \tag{4.40}$$

Thus,

$$\rho c_p\frac{d}{dx}\int_0^H u_\infty\frac{y}{\delta_{\mathrm{u}}}\left(\frac{y}{\delta_T}\theta_\infty-\theta_\infty\right)dy=-\lambda\frac{\theta_\infty}{\delta_T}$$

Assuming that the thermal boundary layer is as thick or thinner than the velocity boundary layer, yields $H=\delta_T$. After integration:

$$\delta_{\mathrm{T}} d\left(\frac{\delta_{\mathrm{T}}^2}{\delta_{\mathrm{u}}}\right)=\frac{6\lambda}{\rho c_p u_\infty}dx \tag{4.41}$$

In case the ratio of the two boundary layer thicknesses remains constant:

$$\left(\frac{\delta_{\mathrm{T}}}{\delta_{\mathrm{u}}}\right)^3\delta_{\mathrm{u}}\; d\delta_{\mathrm{u}}=\frac{6\lambda}{\rho c_p u_\infty}dx \tag{4.42}$$

Applying equation (4.34) leads to a relationship for the thickness of the thermal boundary layer

$$\frac{\delta_{\mathrm{T}}}{\delta_{\mathrm{u}}}=\left(\frac{\lambda}{\eta c_p}\right)^{\frac{1}{3}}=\frac{1}{\mathrm{Pr}^{\frac{1}{3}}} \tag{4.43}$$

Gases have Prandtl numbers $\mathrm{Pr} \approx 1$, so the assumption $\delta_u = \delta_T$ was legitimate. If the thickness δ_T of the thermal boundary layer is known, the heat flux at the wall can be calculated

$$\dot{q}''_{\mathrm{W}} = -\lambda \left(\frac{dT}{dy}\right)_{\mathrm{W}} = -\lambda \frac{\theta_\infty}{\delta_T} = -\frac{\lambda \theta_\infty \mathrm{Pr}^{\frac{1}{3}}}{\delta_{\mathrm{u}}} \tag{4.44}$$

applying equation (4.35) for the thickness of the velocity boundary layer

$$\dot{q}''_{\mathrm{W}} = \frac{\lambda}{x} 0{,}289\ \mathrm{Re}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}} \left(T_{\mathrm{W}} - T_\infty\right) \tag{4.45}$$

Comparing these relationships with the often used empirical assumption for convective heat transfer

$$\dot{q}''_{\mathrm{W}} = \alpha \left(T_{\mathrm{W}} - T_\infty\right)$$

leads to the following equation for the heat transfer coefficient α

$$\alpha = \frac{\lambda}{x} 0{,}289\ \mathrm{Re}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}} \tag{4.46}$$

or in dimensionless form using the Nusselt number, $\mathrm{Nu} \equiv \frac{\alpha x}{\lambda}$,

$$\mathrm{Nu} = 0{,}289\ \mathrm{Re}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}}. \tag{4.47}$$

As comparison to the exact solution according to Pohlhausen (1921) shows,

$$\mathrm{Nu} = 0{,}332\ \mathrm{Re}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}} \tag{4.48}$$

the heat transfer coefficients, calculated in this manner, have the same function profile with only minor deviation from the absolute value, in spite of the simplified assumptions. With only little additional effort, integrating these equations using polinominals of 3rd order to describe the profiles, the solution's conformity to the exact solution can be considerably improved.

4.3 Natural convection - boundary layer equations for laminar, steady state flow

In the previous section we discussed the heat transfer between a surface and a fluid, where the flow was initiated e.g. by a ventilator. Inertial forces were discarded in the energy balance because of their small contribution.

In other cases the flow along a surface is not forced, but induced by a mass force, usually gravity. This *natural convection* has velocity and thermal boundary layers as shown in the figure below. These profiles result from the solutions of the boundary layer equations, the equation of continuity, equation (4.21), the momentum equation in x-direction, equation (4.22), and the equation of energy conservation, equation (4.23).

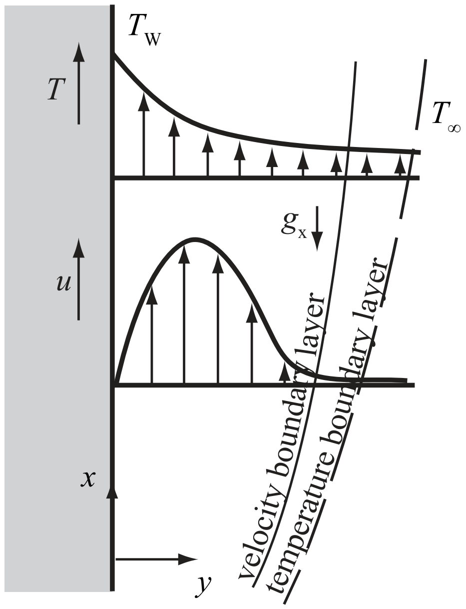

Figure 4.13: Natural convection at a vertical plate

In the momentum equation

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} + \rho g_x \tag{4.49}$$

the sign of the term that describes the impact of gravity force has been changed, since gravity acts against the positive x-direction in this example.

The pressure gradient $\frac{dp}{dx}$ was zero for forced convection over a flat plate. For problems of natural convection, the pressure change over the height dx equals the weight of the liquid head over a unit area.

$$\frac{dp}{dx} = -\rho_\infty g \tag{4.50}$$

Hence, the momentum equation can be rewritten

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g\left(\rho_\infty - \rho\right) + \eta \frac{\partial^2 u}{\partial y^2} \tag{4.51}$$

Usually, the difference in density in the buoyancy term is written in terms of the volumetric expansion coefficient β for homogeneous mediums

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \frac{\rho_\infty - \rho}{\rho\left(T - T_\infty\right)}$$

hence,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g\rho\beta\left(T - T_\infty\right) + \eta \frac{\partial^2 u}{\partial y^2} \tag{4.52}$$

(momentum equation, x-direction)

The volumetric expansion coefficient is listed for many fluids, and for ideal gases it is valid that

$$\beta = \frac{1}{T}$$

Assuming constant properties is justified for many cases, except for the buoyancy term, although it is exactly the density deviations that causes the fluid to flow.

In order to determine the velocity and temperature fields and hence friction and heat transfer at a vertical surface, the following conservation equations must be solved:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.53}$$

(equation of continuity, ρ constant)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g\rho\beta\left(T - T_\infty\right) + \eta \frac{\partial^2 u}{\partial y^2} \tag{4.54}$$

(momentum equation, x-direction, ρ,η constant)

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\mathrm{Pr}} \frac{\partial^2 T}{\partial y^2} \tag{4.55}$$

(energy equation, λ constant)

The solution renders more difficult than the one of the boundary layer equations for forced convection, since knowledge of the temperature field is necessary in order to solve the momentum equation.

Yet, for the simple geometry of the vertical flat plate with constant surface temperature, an exact solution by transformation of the partial differential equations into ordinary differential equations is possible again.

These ordinary differential equations have been solved by Ostrach (1953). The resulting velocity and temperature profiles are shown in a dimensionless form in the following diagrams.

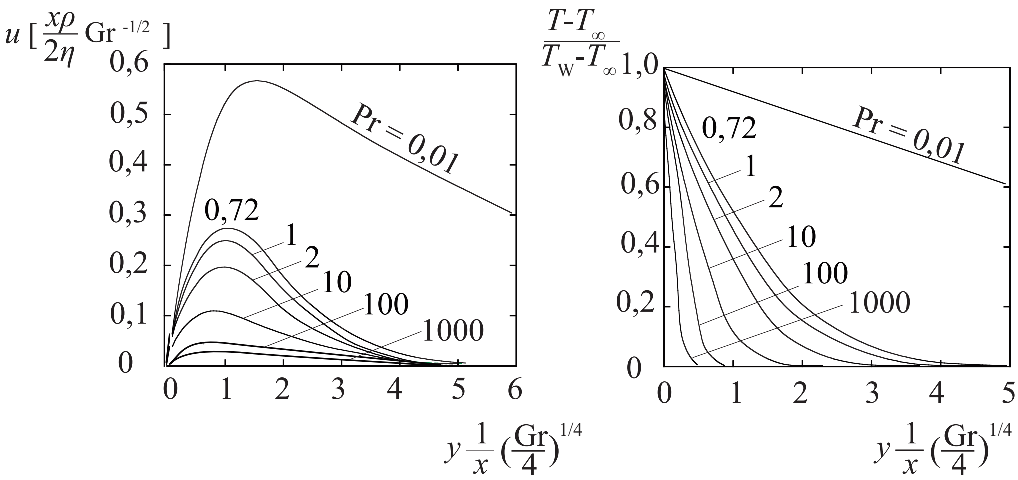

Figure 4.14: Velocity and temperature profiles at a vertical wall for natural convection according to Ostrach (1953)

Here, the new dimensionless number, the *Grashof number*

$$\mathrm{Gr} \equiv \frac{\beta g \rho^2 \left(T_{\mathrm{W}} - T_\infty\right) x^3}{\eta^2} \tag{4.56}$$

is the ratio of the buoyancy forces to the viscosity forces.

With the gradient of the temperature profile at the wall, the heat flux can be calculated according to the Fourier equation as well as the heat transfer coefficient using equation (4.25). Local mean Nusselt numbers can be written in the form

$$\overline{\mathrm{Nu}} = C\,(\mathrm{Gr}\;\mathrm{Pr})^{\frac{1}{4}} \tag{4.57}$$

where $\overline{\mathrm{Nu}}$ is the mean value of the Nusselt number over the height of the plate and C is a coefficient that depends on the Prandtl number, and is given in the following table

Pr	0,003	0,01	0,03	0,72	1	2	10	100	1000	∞
C	0,182	0,242	0,305	0,516	0,535	0,568	0,620	0,653	0,665	0,670

Table 4.1: Values for the coefficient C of equation (4.57) according to Ostrach (1953)

A comparison between the theoretical results and measured values is shown in the following diagrams. In Fig. (4.15) the calculated values of the velocity and temperature fields at a vertical, heated plate in air are compared to experimental values. In (4.16) the mean Nusselt numbers are compared with measured values according to Whitaker (1974).

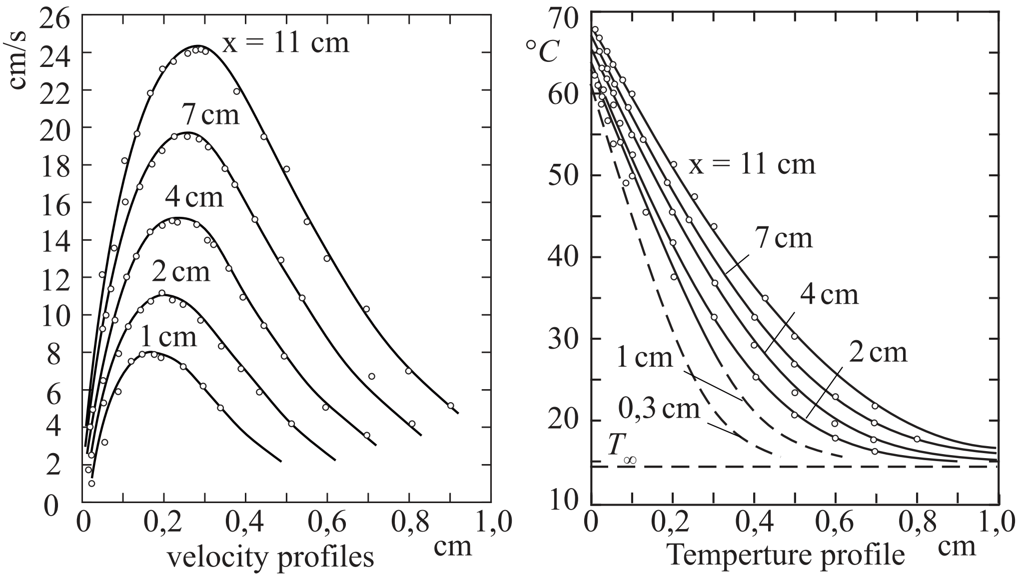

Figure 4.15: Calculated and measured velocity and temperature profiles at a vertical plate for natural convection in air, from Whitaker (1974)

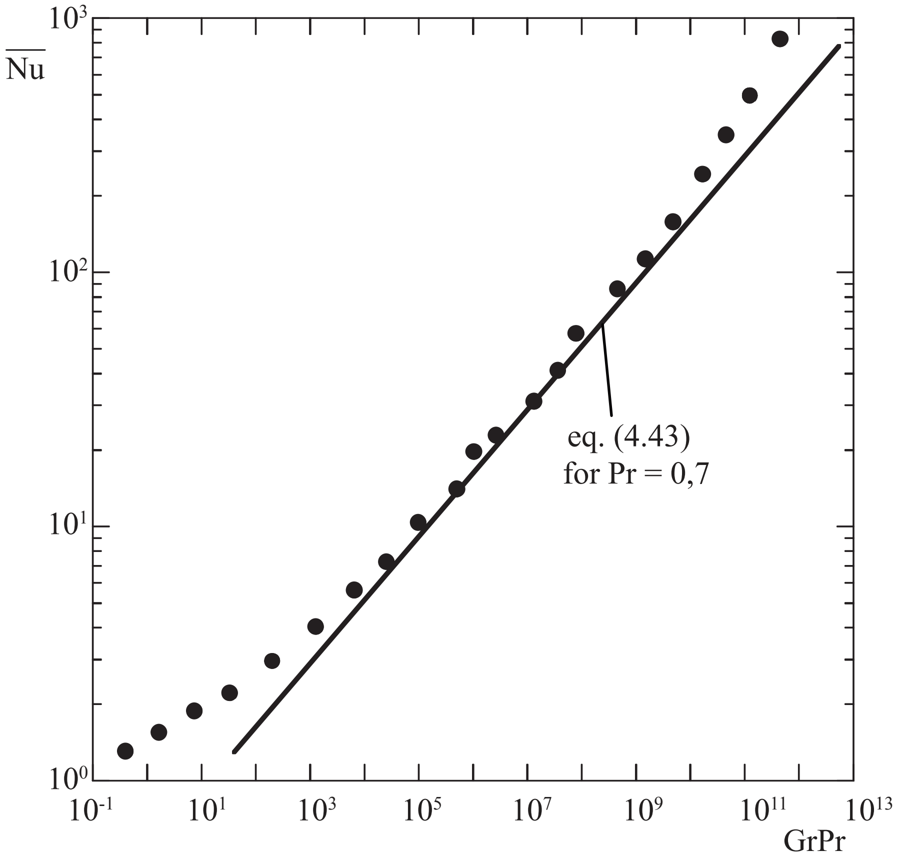

Figure 4.16: Calculated and measured mean values of the Nusselt number of a vertical plate for natural convection in air, from Whitaker (1974)

Comparing these profiles in air shows a very good accordance and proves the validity of the simplified assumptions used during the derivation of the conservation laws. For the local mean values of the Nusselt number, the theoretical values conform to the experimental values only in the middle range of (GrPr)-values. For GrPr $< 10^3$ these boundary layer approximations are obviously no longer valid. In the range of GrPr $> 10^9$ heat transfer is intensified by the shift from laminar to turbulent flow.

These two effects, which are also observed during forced convection, make the theoretical solution of the conservation equations significantly more difficult. If the boundary layer assumptions are dropped, instead of the parabolic differential equations elliptical differential equations have to be solved and numerical methods or eventually experiments will be nescessary. In addition, when the flow shifts from laminar to turbulent, models have to be developed that include the impact of turbulence to the momentum and heat exchange.

The characteristics of turbulent flow will be discussed in the following section 4.4.

4.4 Heat transfer in turbulent flows

The previous sections described the heat transfer from a wall to a laminar flowing fluid. It was shown that adjacent to the wall, the dominating energy transport mechanisms is heat conduction, whereas in the layers further away, enthalpy transport by the flow dominates.

In many problems from practice though, the flow becomes turbulent, depending on a characteristic Reynolds number, so that additional exchange mechanisms have to be considered. In many cases on average the flow shows steady state characteristics yet. Measuring the instantaneous values of velocity or temperature by e.g. a hot wire anemometer, or a very thin thermocouple results in profiles as shown in the following diagrams.

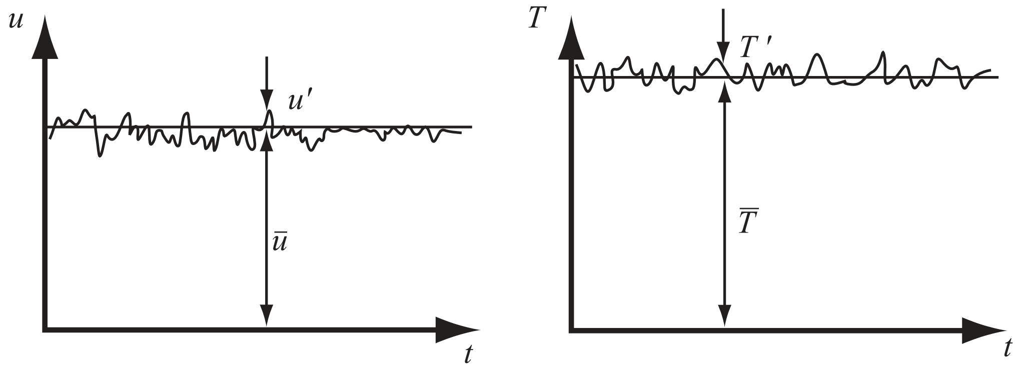

Figure 4.17: Velocity and temperature as a function of time for an - on the average - steady state, turbulent flow

For the instantaneous value of the velocity u, and the mean value $\overline{u}$ with deviation $u^{'}$,the following relationships are valid

$$u = \overline{u} + u^{'}$$

and respectively,

$$v = \overline{v} + v^{'} \;\; ; \;\; T = \overline{T} + T^{'} \;\; \text{and} \;\; p = \overline{p} + p^{'}$$

Putting these instantaneous values in the equations of continuity, momentum and energy, and then calculate the mean values it follows, e.g. for flow with boundary layer character and assuming constant properties, from equations (4.21) to (4.23) a new set of conservation equations

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \qquad \text{(equation of continuity)} \qquad (4.58)$$

$$\rho\overline{u}\frac{\partial\overline{u}}{\partial x} + \rho\overline{v}\frac{\partial\overline{u}}{\partial y} = -\frac{d\overline{p}}{dx} + \eta\frac{\partial^2\overline{u}}{\partial y^2} - \rho\frac{\partial\overline{u'v'}}{\partial y} \qquad \text{(momentum equation)} \quad (4.59)$$

$$\rho\overline{u}c_p\frac{\partial\overline{T}}{\partial x} + \rho\overline{v}c_p\frac{\partial\overline{T}}{\partial y} = \lambda\frac{\partial^2\overline{T}}{\partial y^2} - \rho c_p\frac{\partial\overline{v'T'}}{\partial y} \qquad \text{(energy equation)} \quad (4.60)$$

The full derivation of these equations can be found for example in Schlichting u. Gersten (2006). The new equations valid for steady state turbulent flows differ from those for laminar flows by a newly introduced term on the right-hand side of the momentum and energy equation, which describes the additional transport of momentum and energy by the eddy movements of the flow.

Usually these additional terms are labeled *"turbulent shear stress"* or *"turbulent heat conduction"* , respectively

$$\tau_\mathrm{t} = -\rho\overline{u'v'} \;\text{ and }\; \dot{q}_\mathrm{t}'' = \rho c_p\overline{v'T'}.$$

If, in analogy to molecular transport of heat and momentum, a corresponding turbulent viscosity η_t and turbulent heat conductivity λ_t is defined:

$$\tau_\mathrm{t} \equiv \eta_\mathrm{t}\frac{\partial\overline{u}}{\partial y} \;\text{ and }\; \dot{q}_\mathrm{t}'' \equiv -\lambda_\mathrm{t}\frac{\partial\overline{T}}{\partial y}.$$

then the molecular transport properties and the influence of the turbulence can be summarised in the *effective transport properties* , e.g. $\eta_\mathrm{eff} = \eta + \eta_t$. This yields for example for a turbulent boundary layer flow to the following conservation equations

$$\frac{\partial\overline{u}}{\partial x} + \frac{\partial\overline{v}}{\partial y} = 0 \qquad \text{(equation of continuity)} \quad (4.61)$$

$$\rho\overline{u}\frac{\partial\overline{u}}{\partial x} + \rho\overline{v}\frac{\partial\overline{u}}{\partial y} = -\frac{d\overline{p}}{dx} + \frac{\partial}{\partial y}\left(\eta_\mathrm{eff}\frac{\partial\overline{u}}{\partial y}\right) \qquad \text{(momentum equation)} \quad (4.62)$$

$$\rho\overline{u}c_p\frac{\partial\overline{T}}{\partial x} + \rho\overline{v}c_p\frac{\partial\overline{T}}{\partial y} = \frac{\partial}{\partial y}\left(\lambda_\mathrm{eff}\frac{\partial\overline{T}}{\partial y}\right) \qquad \text{(energy equation)} \quad (4.63)$$

Even though the equations are formally very similar to those for laminar boundary flows, it should not be forgotten that the turbulent properties describe the impact of turbulence and hence depend not only on the location in the flow, but also on the geometry and history of the flow. The turbulent transport properties are not material properties as is the case for their molecular counterparts. References for semi-empirical approches used to describe them can be found under the keyword "turbulence models", e.g. Jischa (1982).

These special properties render the exact solution of the conservation equations, in the way presented for laminar flow, impossible. Therefore complex numerical methods, like the method of finite differences or finite elements, are used. The way in which these methods are applied will be the topic of specialised courses on the subject. The application of advanced commercially available program codes for solving problems of energy engineering and combustion technology will be presented in the lecture "Design of Burners and Furnaces".

4.5 Application of dimensional analysis for heat transfer

Although with the development in computer technology, major advances have been made in the numerical calculation of heat transfer, it is still necessary for most technical cases to use empirical heat transfer laws for design purposes.

These laws are a result of measurements, which are shown graphically or in a form of empirical formulas.

As will be shown in this section, it is not always necessary to repeat experiments if either fluid, geometry, or the flow conditions are changed.

The analytical solutions for heat transfer for forced and natural convection, dealt with previously, have shown that the results can be represented in an appropriate form by a few characteristic numbers

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\mathrm{Pr}) \text{ for forced convection} \tag{4.64}$$

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Gr},\mathrm{Pr}) \text{ for natural convection} \tag{4.65}$$

The solutions rendered not only the characteristic numbers that describe the process, but also their functional relationship.

For cases in which a solution of the conservation laws is not known, it is often possible to reduce the number of parameters that influence the heat transfer by the method of the *dimensional analysis* . The functional relationships of these coefficients are obtained by specific measurements.

To determine the characteristic numbers, the conservation equations are rewritten in a dimensionless form. In order to reduce complexitiy, the two-dimensional, laminar boundary layer equations with constant properties in the form of equation (4.53) to (4.55), with an additional pressure gradient $\left(\frac{dp}{dx}\right)_{\text{kin}}$ in the momentum equation will be regarded. It should be noted that while deriving the momentum equation (4.54) for the vertical flat plate only the pressure gradient, describing the fluid head $\left(\frac{dp}{dx}\right)_{\text{pot}}$ was effective, and the pressure gradient caused by the acceleration of the fluid $\left(\frac{dp}{dx}\right)_{\text{kin}}$, as for example in a tube flow, was omitted. Hence the more general case is discussed here

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(equation of continuity)} \quad (4.66)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\left(\frac{dp}{dx}\right)_{\text{kin}} + \eta \frac{\partial^2 u}{\partial y^2} - g_{\text{x}} \rho \beta \left(T - T_\infty\right) \quad \text{(momentum equation)} \quad (4.67)$$

$$\rho\ u\ c_p \frac{\partial T}{\partial x} + \rho\ v\ c_p \frac{\partial T}{\partial y} = \lambda \frac{\partial^2 T}{\partial y^2} \qquad \text{(energy equation)} \quad (4.68)$$

All variables in these equations shall be made dimensionless by appropriate characteristic parameters, arising from the specific case. Hence, the coordinates x and y are referenced to a characteristic length L, which can be the diameter D for a sphere or tube, or the length of a plate for flow over a plate. The velocity components u and v are divided by the characteristic velocity u_∞ ,for flow in tubes the mean velocity, and for flow over a plate the free stream velocity. The pressure is always rated over the double of the dynamic pressure and the temperature over a characteristic temperature difference, e.g. the temperature difference between the surface and the fluid at a greater distance from the body.

Labeling the dimensionless variables with *

$$\text{x}^* \equiv \frac{\text{x}}{L}; \quad \text{y}^* \equiv \frac{\text{y}}{L}; \quad u^* \equiv \frac{u}{u_\infty}; \quad v^* \equiv \frac{v}{u_\infty}; \quad p^* \equiv \frac{p}{\rho u_\infty^2}; \quad \theta^* \equiv \frac{T - T_\infty}{T_{\text{W}} - T_\infty}$$

yields the dimensionless conservation equations

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad \text{(equation of continuity)} \quad (4.69)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\left(\frac{dp^*}{dx^*}\right)_{\text{kin}} + \left(\frac{\eta}{\rho u_\infty L}\right) \frac{\partial^2 u^*}{\partial y^{*2}} + \left(\frac{g_x \beta L (T_{\text{W}} - T_\infty)}{u_\infty^2}\right) \theta^* \quad (4.70)$$

(momentum equation)

$$u^*\frac{\partial\theta^*}{\partial x^*}+v^*\frac{\partial\theta^*}{\partial y^*}=\left(\frac{\lambda}{\rho c_p u_\infty L}\right)\frac{\partial^2\theta^*}{\partial y^{*2}} \tag{4.71}$$

(energy equation)

In addition to the dimensionless variables, these equations contain dimensionless quantities in the brackets. A closer look at these expressions, reveals that these are the dimensionless numbers from the previous sections.

The momentum equation, equation (4.70),contains two of these terms.

The expression for the friction term is the reciprocal of *the Reynolds number*

$$\mathrm{Re}\equiv\frac{\rho u_\infty L}{\eta}$$

The expression for the buoyancy term is called *Archimedes number*

$$\mathrm{Ar}\equiv\frac{g_\mathrm{x}\beta L(T_\mathrm{W}-T_\infty)}{u_\infty^2}$$

which in the literature is often expressed as a quotient of *the Grashof number* and the *Reynolds number* squared

$$\mathrm{Ar}=\left(\frac{g_\mathrm{x}\rho^2\beta L^3(T_\mathrm{W}-T_\infty)}{\eta^2}\right)\left(\frac{\eta}{\rho u_\infty L}\right)^2=\frac{\mathrm{Gr}}{\mathrm{Re}^2}$$

The dimensionless energy equation defines the *Péclet number*

$$\mathrm{Pe}\equiv\frac{\rho c_p u_\infty L}{\lambda}=\frac{u_\infty L}{a}$$

which in turn, can be rewritten as a product of the *Reynolds* and *Prandtl number*

$$\mathrm{Pe}=\left(\frac{\rho u_\infty L}{\eta}\right)\left(\frac{\eta c_p}{\lambda}\right)=\mathrm{Re\ Pr}$$

If, for different applications, e.g. different fluids, different pipe diameters the coefficients or a combination of the same, given in the expressions in the brackets, are equal, then so are the solutions of the differential equations, i.e. the dimensionless fields of velocity, temperature and pressure. Yet, the boundary conditions must be equivalent, too, for the above statement to be valid. Hence, the velocity and temperature fields can be expressed as a function of these parameters

$$\vec{w}=\vec{w}\ (\mathrm{x}^*,\mathrm{y}^*,\mathrm{Re},\mathrm{Gr},\mathrm{Pr}) \tag{4.72}$$

and

$$\theta^* = \theta^* \left(\mathrm{x}^*,\mathrm{y}^*,\mathrm{Re},\mathrm{Gr},\mathrm{Pr}\right) \tag{4.73}$$

Velocity and temperature fields of the fluid are less interesting than the heat that is transferred between the fluid and the body and which can be derived from the gradient of the temperature profile in the fluid at the surface of the body.

$$\dot{q}'' = -\left(\lambda \frac{\partial T}{\partial y}\right)_{\mathrm{W}}$$

A comparison with the equation that defined the heat transfer coefficient α

$$\dot{q}'' \equiv \alpha \left(T_{\mathrm{W}} - T_{\infty}\right)$$

yields

$$\alpha = \frac{-\left(\lambda \frac{dT}{dy}\right)_{\mathrm{W}}}{\left(T_{\mathrm{W}} - T_{\infty}\right)}$$

or written in a dimensionless form, the *Nusselt number* :

$$\mathrm{Nu} \equiv \frac{\alpha L}{\lambda} = -\left(\frac{d\theta^*}{dy^*}\right)_{\mathrm{W}}$$

The Nusselt number, already introduced in previous sections as a dimensionless heat transfer coefficient, is thus equal to the dimensionless temperature gradient and hence, can be expressed as the temperature field

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\mathrm{Gr},\mathrm{Pr}) \tag{4.74}$$

This shows that heat transfer for problems with similar geometries and boundary conditions can formally be expressed by 4 dimensionless numbers.

This statement is also valid for turbulent flow, as long as the levels of turbulence are comparable.

For many practical purposes, the relationship (4.74), can be further simplified.

For *forced convection* the inertial and friction forces are greater than the buoyant forces resulting from the changes in temperature. Neglecting the buoyant term in the equation of motion leads to

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\, \mathrm{Pr}) \tag{4.75}$$

For *natural convection* the velocity field is caused by temperature differences, and a velocity u_∞ induced by external sources is not present. The inertial forces may be neglected, hence

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Gr}, \mathrm{Pr}) \tag{4.76}$$

The dimensional analysis cannot state the form of the functional relationship between the Nu number and the Re, Gr, Pr numbers. This relationship can either be found from the analytical solutions, if these are expressed in characteristic numbers, or from experiments. Often, it is aimed at representing the experimental results as a power law, i.e. in the form

$$\mathrm{Nu} = C\ \mathrm{Re}^{\mathrm{m}}\mathrm{Pr}^{\mathrm{n}}\mathrm{Gr}^{\mathrm{p}}$$

Important examples of such power laws will be shown in the next chapter.