Chapter 6

Reynolds number

6.1 Dimensional Analysis: Introduction by Example

Consider a pendulum of length ℓ with a weight m attached to it. The pendulum is released from an an angle θ_o and the question addressed here is: how much time does it take for the pendulum to swing back and forth? We will call the time needed the period T. Before actually solving this problem one can already say a lot about how the solution will look like. It is obvious that T is a function of the parameters mentioned and the gravitational constant g:

$$T(\ell, m, g, \theta_o). \tag{6.1}$$

The period can be made non-dimensional by using ℓ and g:

$$\tilde{T} = T / \sqrt{\frac{\ell}{g}}. \tag{6.2}$$

An extremely important notion is given by **Buckingham's theorem**:

Theorem 6.1 (Buckingham)**.** *A dimensionless function depends on dimensionless arguments only.*

This means that $\tilde{T}$ only depends on combinations of ℓ, m, g and θ_o that are non-dimensional. So the question arises whether such combinations exist. To investigate this define the group

$$\Pi \equiv \ell^{\alpha} m^{\beta} g^{\gamma} \theta_o^{\delta}, \tag{6.3}$$

and try to find values for α, β, γ and δ such that Π is non-dimensional. First determine the physical dimension of all parameters:

$$[\ell] = m, \quad [m] = kg, \quad [g] = \frac{m}{s^2}, \quad [\theta_o] = 1. \tag{6.4}$$

The dimension of θ_o is 1 because it is non-dimensional. As a result the physical dimension of Π becomes:

$$[\Pi] = m^{\alpha} kg^{\beta} \left(\frac{m}{s^2}\right)^{\gamma} 1^{\delta} = m^{\alpha+\gamma} kg^{\beta} s^{-2\gamma}. \tag{6.5}$$

To make the group Π non-dimensional, i.e. $[\Pi] = 1$, the powers of the independent dimensions m, kg and s must be zero:

$$\alpha + \gamma = 0, \quad \beta = 0, \quad -2\gamma = 0, \quad \Rightarrow \quad \alpha = \beta = \gamma = 0. \tag{6.6}$$

Hence, the only possibility to construct Π as a non-dimensional group is taking $\Pi = \theta_o^\delta$ with δ an arbitrary number. Taking $\delta = 1$ one gets:

$$\Pi = \theta_o, \tag{6.7}$$

and the expression for the period T can now be written as

$$T(\ell, m, g, \theta_o) = \tilde{T}(\theta_o)\sqrt{\frac{\ell}{g}}. \tag{6.8}$$

From linear theory for small values of θ_o it is known that $\tilde{T}(\theta_o) = 2\pi$. For larger values of θ_o there exist no closed fomulation for $\tilde{T}$.

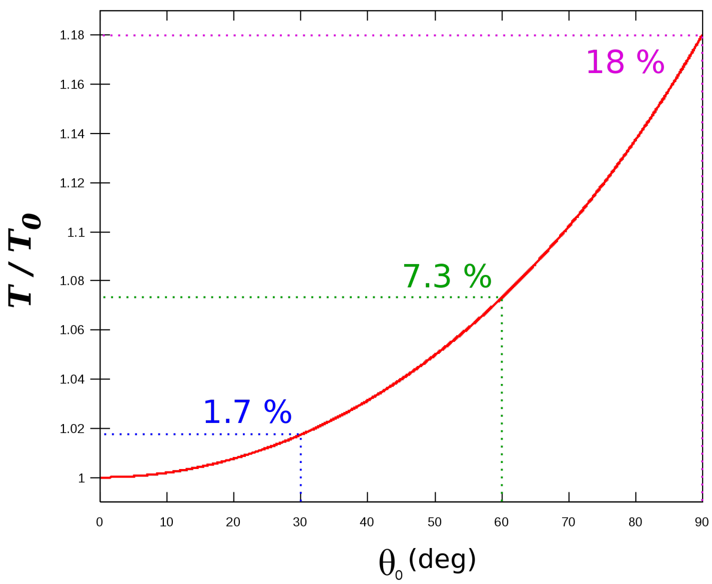

Figure 6.1: Pendulum period $\tilde{T}$ divided by 2π as a function of θ_o.

What is the benefit of this dimension analysis? Well, suppose one wants to determine the period T from experiments, then Eq.(6.1) shows that a lot of experiments are necessary with all kinds of pendula: different lengths ℓ, different weights m, different release angles θ_o and if one wants to be very accurate, different locations on earth to account for variations in the gravitational constant g. To make a high resolution data base of T would therefore be a very costly and labour intensive enterprise.

The dimension analysis however demonstrates that first of all the mass m is irrelevant for the value of T. Furthermore, it shows that one can use one pendulum at one location and only vary the release angle θ_o to make a data base of $\tilde{T}$ instead of T! The period T is then easily calculated from Eq.(6.8). This evidently means an enormous reduction of costs and effort and also makes it very easy to present the period T in reports: a simple graph of $\tilde{T}$ is sufficient.

6.2 Dimension Analysis: General Approach

The general approach to dimension analysis consists of the following steps:

(a) Identify the **dependent variable** asked for,
(b) identify M, the **number of independent parameters** that determine the variable asked for,
(c) identify N, the **number of independent physical dimensions** present,
(d) Choose N **dimensionally independent** parameters, and
(e) Scale the **dependent variable** and the remaining **independent parameters**.

In case of the pendulum these steps would result in

(a) T,
(b) ℓ, m, g, θ_o, so $M = 4$,
(c) m, kg, s, so $N = 3$,
(d) ℓ, m, g
(e) 1) $\tilde{T} = T/\sqrt{l/g}$
  2) θ_o is already non-dimensional, does not havespeher to be scaled.

which then gives Eq.(6.8). It is noted that **the number of non-dimensional parameters is equal to** $M - N$.

6.3 Dimensional analysis: Reynolds number

On the one hand, looking at the fully developed flow results of the previous chapter, we could have predicted that any calculated quantity of the flow, say Q, is a function of the variables specified:

$$Q = F(\rho, U, h, \mu), \tag{6.9}$$

simply because nothing else has been specified. The quantiy Q could be for example $\left(\frac{\partial p}{\partial x}\right)_o$. It seemes that finding an expression for the pressure gradient means looking fo a function of four variables. If such function cannot be determined analytically the only way out is doing numerical experiments. Suppose we would measure the pressure derivative for 10 different values of the four variables, that would mean a set of 10^4 tests! And not only that, we would also have to test with probably a 100 different fluids to obtain satisfactory combinations af different values of density and viscosity. Luckily there is a way out of this which reduces the four-dimensional problem to a one-dimensional problem: dimensional analysis.

In the present case, the four parameters ρ, U, h, and μ contain three fundamental physical dimensions:

$$[\rho] = \frac{M}{L^3}, \quad [U] = \frac{L}{T}, \quad [h] = L, \quad [\mu] = \frac{M}{LT}, \tag{6.10}$$

where [...] means physical dimension, M denotes mass, L denotes length, and T denotes time. This means that we can multiply powers of the parameters to construct any physical dimension, as long as it is a combination of mass, length, and time:

$$[\rho^a U^b h^c \mu^d] = M^{a+d} L^{-3a+b+c-d} T^{-b-d}. \tag{6.11}$$

Suppose we want to construct in this way a physical dimension $M^x L^y T^z$, then we have to solve the following linear system of equations:

$$a + d = x, \tag{6.12}$$

$$-3a + b + c - d = y, \tag{6.13}$$

$$-b - d = z. \tag{6.14}$$

Obviously, this system has infinitely many solutions since we have four unknowns a, b, c, and d, and only three equations. Therfore, the conclusion is that we only need three parameters to construct an arbirary physical dimension. Lets drop μ and retain ρ, U, and h, then

$$[\rho^a U^b h^c] = M^a L^{-3a+b+c} T^{-b}, \tag{6.15}$$

and the linear system becomes

$$a = \qquad x, \tag{6.16}$$

$$-3a + b + c = \qquad y, \tag{6.17}$$

$$-b = \qquad z, \tag{6.18}$$

which can be written as a matrix vector multiplication,

$$A\boldsymbol{a} = \boldsymbol{x}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{6.19}$$

From linear algebra we know that this system has a unique solution if and only if the determinant of the matrix A is not zero. (1)

The determinant of A is 1 so the system does have a unique solution which is

$$a = x, \quad b = -z, \quad c = 3x + y + z. \tag{6.20}$$

(1) If the determinant of A is not zero it means that the physical dimensions of the three parameters ρ, U, and h are dimensionally independent. In other words, $A\boldsymbol{a} = 0$ has unique solution $\boldsymbol{a} = 0$ which means that the three parameters cannot be combined into a dimensionless quantity other that taking all exponents equal to zero which is obviously irrelevant.

What can we do with this? Let's take for example $Q = \left(\frac{\partial p}{\partial x}\right)_o$ and let us try to find a combination $\rho^a U^b h^c$ such that if we divide Q by this combination we get a dimensionless quantity. The dimension of $\left(\frac{\partial p}{\partial x}\right)_o$ is

$$\left[\left(\frac{\partial p}{\partial x}\right)_o\right] = \frac{M}{L^2T^2}, \tag{6.21}$$

so

$$x = 1, \quad y = -2, \quad z = -2, \tag{6.22}$$

and, hence by Eq.(10.20) we obtain:

$$a = 1, \quad b = 2, \quad c = -1 \quad \Rightarrow \quad \rho^a U^b h^c = \frac{\rho U^2}{h}. \tag{6.23}$$

This means that when we divide $\left(\frac{\partial p}{\partial x}\right)_o$ by this combination we obtain a dimensionless quantity:

$$\left[\frac{h}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o\right] = M^0 L^0 T^0 = 1. \tag{6.24}$$

Following Buckinghams theorem, the dimensionless pressure derivative derived above depends on dimensionless quanities only. So, the question is: how many dimensionless quantities can we produce with the four parameters ρ, U, h, and μ? In other words, what are possible solutions of

$$[\rho^a U^b h^c \mu^d] = 1. \tag{6.25}$$

As mentioned before, this system has infinitely many solutions since there are four unknowns and only three equations. Therefore we treat one of the unknowns as 'known', say d, and solve for the three remaining unknowns:

$$[\rho^a U^b h^c] = [\mu^{-d}] = M^{-d} L^d T^d, \tag{6.26}$$

which leads to

$$a = -d, \quad b = -d, \quad c = -d. \tag{6.27}$$

which means that the infinitely many ways we can construct a dimensionless quantity out of the four parameters ρ, U, h, and μ is $(\rho U h/\mu)^{-d}$. The term between brackets is called the Reynolds number Re:

$$\boxed{Re \equiv \frac{\rho U h}{\mu}.} \tag{6.28}$$

So, as a result, we have derived that

$$\frac{h}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o \equiv \Phi(Re), \tag{6.29}$$

Figure 6.2: Osborne Reynolds (1842 - 1912) was a prominent innovator in the understanding of fluid dynamics. Separately, his studies of heat transfer between solids and fluids brought improvements in boiler and condenser design.

where Φ is an unknown dimensionless function and where we have absorbed the unknown value of d into the fact that Φ is unknown. Putting it slightly differently, we have revealed that the pressure derivative can be written in the following form:

$$\left(\frac{\partial p}{\partial x}\right)_o = \Phi(Re)\frac{\rho U^2}{h}. \tag{6.30}$$

The implication of this result is tremendous! By the above analysis, which is called dimension analysis, we have succeeded in reducing the four-dimensional problem of finding the pressure derivative to a one-dimensional problem of finding $\Phi(Re)$. In other words, instead of finding $\left(\frac{\partial p}{\partial x}\right)_o$ as a function of the four physical variables ρ, U, h, and μ, we have transformed the problem into finding the function Φ as a function of one variable: the Reynolds number Re. This means that we do not have to do 10,000 measurements but only 10. Or, if we require more accuracy, maybe 100. It also means that we do not have to measure with many different fluids, water will do! The same is true for the distance between the two plates, one value will do. This is due to the fact that we can vary the Reynolds number by simply varying the velocity of the flow, that's all.

It is noted that we haven't used the describing equations to perform the dimension analysis! The analysis is solely based on gathering all of the parameters that determine the quantity we are interested in and analyse their dimensions. The tricky part lies in the fact that we have to be sure that we have gathered all relevant parameters of the problem, this relies on physical insight.

For the gap flow

$$\Phi = -\frac{12}{Re}, \tag{6.31}$$

which is illustrated in Fig. (6.3)

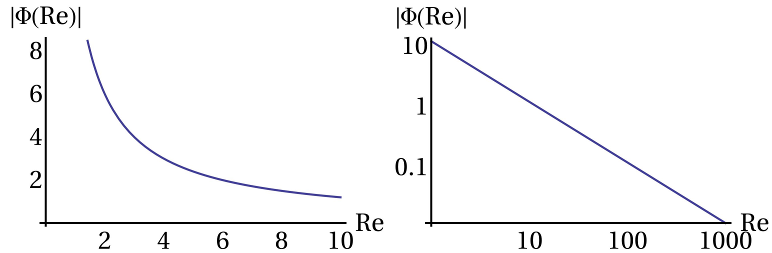

Figure 6.3: Dimensionless pressure derivative for gap flow as function of Reynolds number

For pipe flow the dimensionless pressure derivative is defined as

$$\frac{D}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o \equiv \Psi(Re_D), \qquad D \equiv 2R, \tag{6.32}$$

with

$$\boxed{Re_D \equiv \frac{\rho U D}{\mu}.} \tag{6.33}$$

This gives

$$\Psi = -\frac{32}{Re_D}, \tag{6.34}$$

which is illustrated in Fig. (6.7)

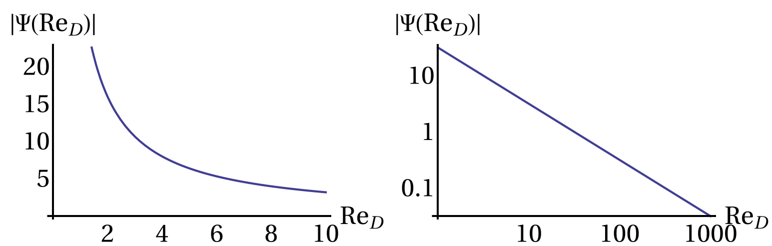

Figure 6.4: Dimensionless pressure derivative for pipe flow as function of Reynolds number

6.4 Turbulence

In the previous sections we have implicitly assumed that the flow was laminar, meaning in the case of fully developed gap flow that the flow is stationary. It is well known that when the velocity of the fluid or the gap width are increased, stationary flow does not exist anymore. Instead, the flow becomes unsteady, comprising a distribution of vortices with a large range of scales. Such flow is called turbulent. The larger vortices break up into smaller ones which again break up into even smaller ones and so on. In the end the smallest vortices vanish and their energy is released as heat. We call this energy dissipation.

The transition from laminar to turbulent flow, when looking at the dimensionless flow variables, clearly only depends on the Reynolds number: there is simply no other parameter left. The value of the Reynolds number at which transition occurs is called the critical value. Below this value, small perturbations with respect to the laminar flow solution (which are always there, imagine someone closing the door in the experiment room or a car driving by the building etc.) are damped. Above the critical value, however, perturbations are amplified and grow, leading to turbulent flow.

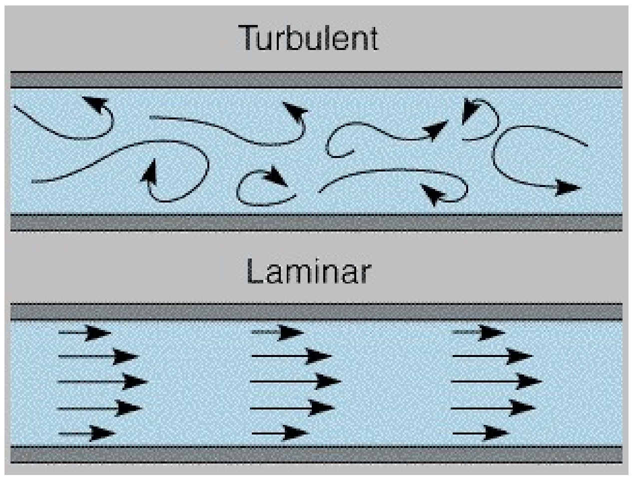

Figure 6.5: Laminar and turbulent flow regimes in a tube

6.5 Friction factor and Moody charts

In standard fluid mechanics text books it is custom to write the pressure derivative in fully developed pipe flow as

$$\left(\frac{\partial p}{\partial x}\right)_o = -f(Re_D)\,\frac{1}{2}\rho U^2/D, \tag{6.35}$$

so, in terms of the factor Ψ:

$$f(Re_D) = -2\Psi(Re_D). \tag{6.36}$$

The function $f(Re_D)$ is the called the Darcy-Weisbach friction factor. (2) Note that U is taken positive in the positive x-direction which corresponds to a negative pressure derivative. The minus sign choosen in front of the friction factor in that case ensures that f is a positive number.

Until now we have not included effects of wall roughness, we have considered perfectly smooth pipes only. Suppose now we have a pipe with a non-smooth wall characterized by

(2) The equation is named after Henry Darcy and Julius Weisbach. Henry Philibert Gaspard Darcy (1803 - 1858) was a French engineer, and Julius Ludwig Weisbach (1806 - 1871) was a German mathematician and engineer.

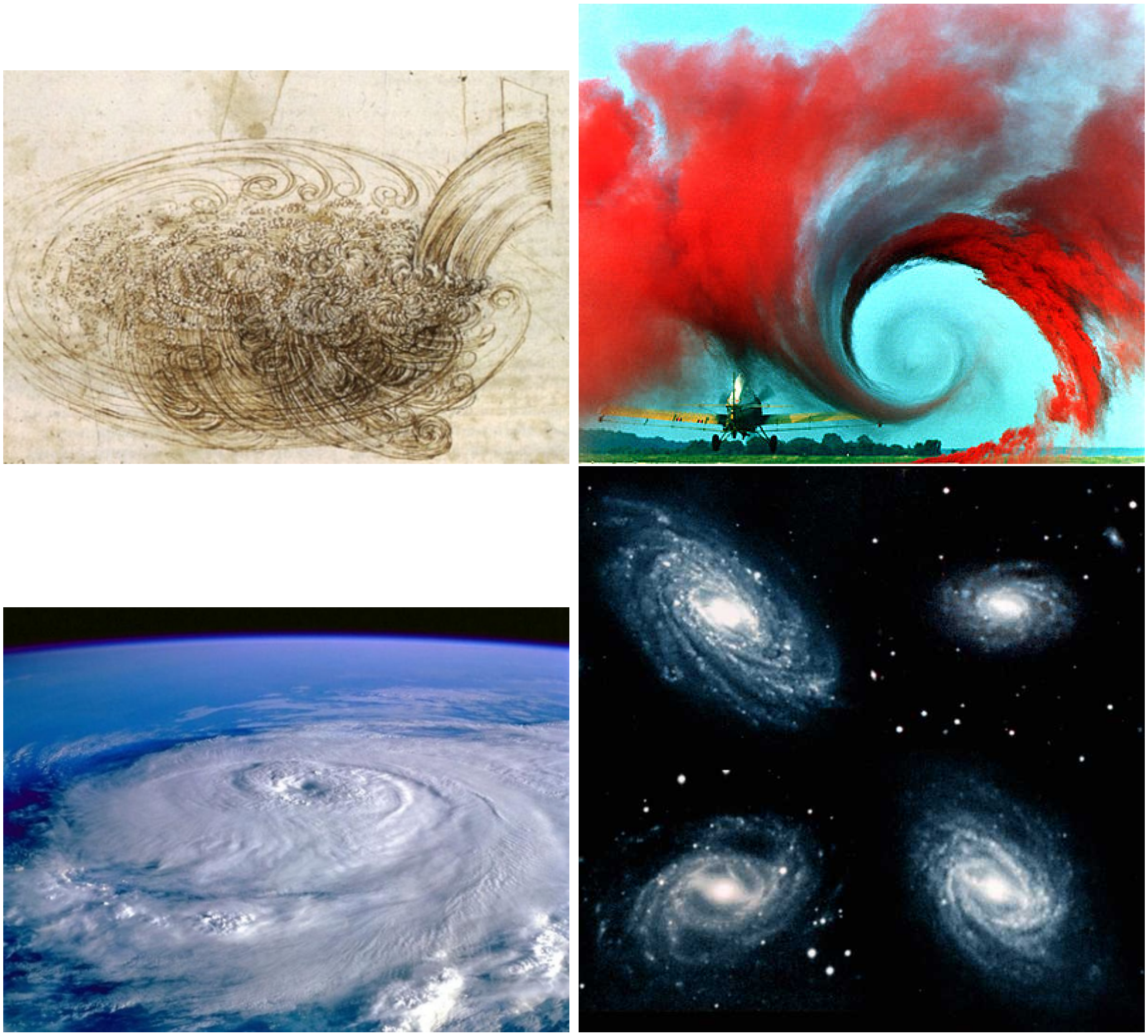

Figure 6.6: Examples of vortices on a range of scales

a mean wall roughness δ, which has the dimension of length. That means that the problem of finding the pressure derivative includes an additional parameter and dimension analysis will produce therefore an additional dimensionless number. When we scale with ρ, U, and D, the only possibility to scale δ is by dividing by D. Therefore, we define

$$\epsilon \equiv \frac{\delta}{D}. \tag{6.37}$$

as the additional dimensionless parameter. This means that the friction factor also becomes a function of ϵ:

$$\left(\frac{\partial p}{\partial x}\right)_o = -f(Re_D, \epsilon)\,\frac{1}{2}\rho U^2/D. \tag{6.38}$$

The friction factor in this form was first summarized in a single plot by Lewis Ferry Moody (1880 - 1953), and therefore the diagram is called the Moody diagram or Moody chart. [3]
For laminar flow regime the friction factor is independent of ϵ,

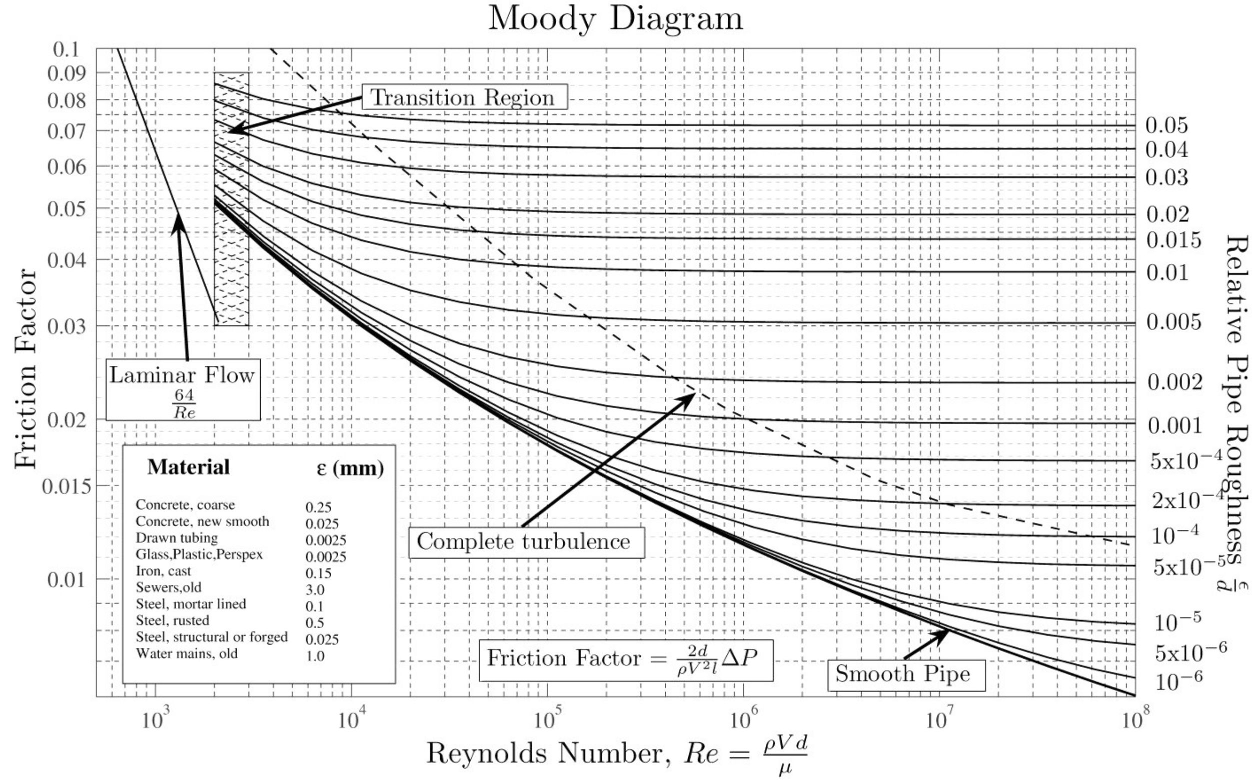

Figure 6.7: Moody chart: Darcy-Weisbach friction factor for laminar and turbulent fully developed pipe flow as function of Reynolds number Re_D and relative wall roughness ϵ.

$$f(Re_D, \epsilon) = \frac{64}{Re_D}, \quad \text{laminar flow} \tag{6.39}$$

which is just the expression found for Poiseuille flow, Eq.(5.38), whereas for the turbulent flow regime the friction factor approximately satisfies the Colebrook equation:

$$\frac{1}{\sqrt{f}} = -2\, \log_{10}\left(\frac{\epsilon}{3.7} + \frac{2.51}{Re_D\sqrt{f}}\right), \quad \text{turbulent flow.} \tag{6.40}$$

It is noted that this equation is implicit in f, in other words, it cannot be solved for f analytically.

[3] Lewis Ferry Moody (1880 - 1953), was an American engineer and professor, he was the first Professor of Hydraulics in the School of engineering at Princeton.