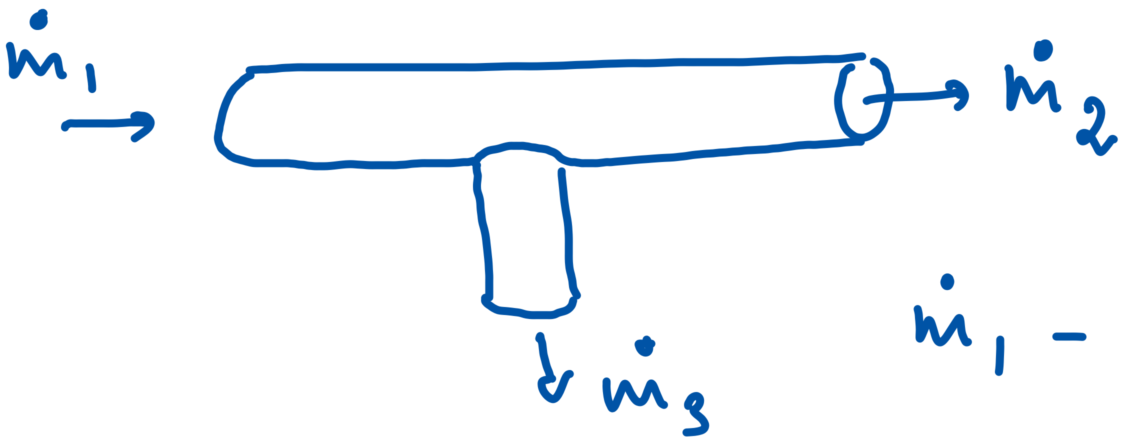
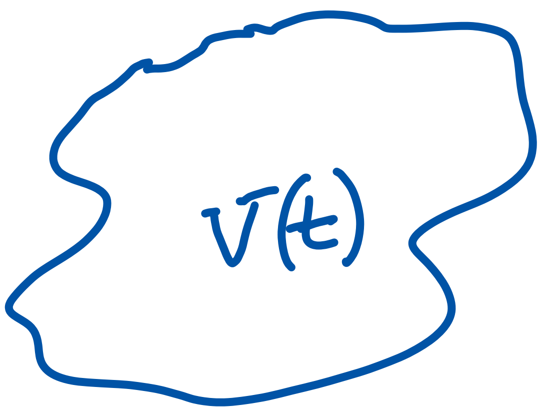
Fluid Mechanics 1
Lecture #2:
Mass Conservation.



steady state

$$\dot{m}_1 - \dot{m}_2 - \dot{m}_3 = 0$$



blob of fluid

Mass inside blob:

$$\boxed{Mass(t) \equiv \int_{V(t)} \rho \, dV}$$

$\rho(\vec{x},t)$: mass density $\left(\frac{kg}{m^3}\right)$

Physics: $Mass(t) = \text{constant}$

$$\Rightarrow \frac{d}{dt} Mass(t) = 0 \quad \Rightarrow \boxed{\boxed{\frac{d}{dt} \int_{V(t)} \rho \, dV = 0}}$$

Mathematical ToolBox

How to convert this into a useful equation?

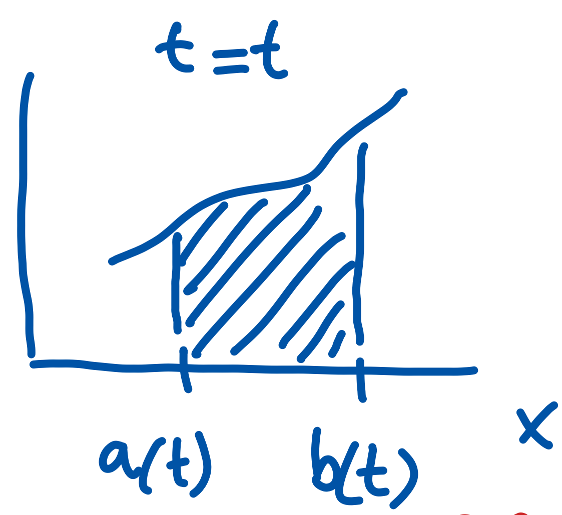
Special case: if $\mathcal{V}$ is not moving

$$\Rightarrow \quad \underbrace{\frac{d}{dt}\int_{\mathcal{V}(t)} \rho \, d\mathcal{V}}_{\text{ordinary derivative}} = \underbrace{\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V}}_{\text{partial derivative.}}$$

(*) Demonstration → minivideo Leibniz Reynolds

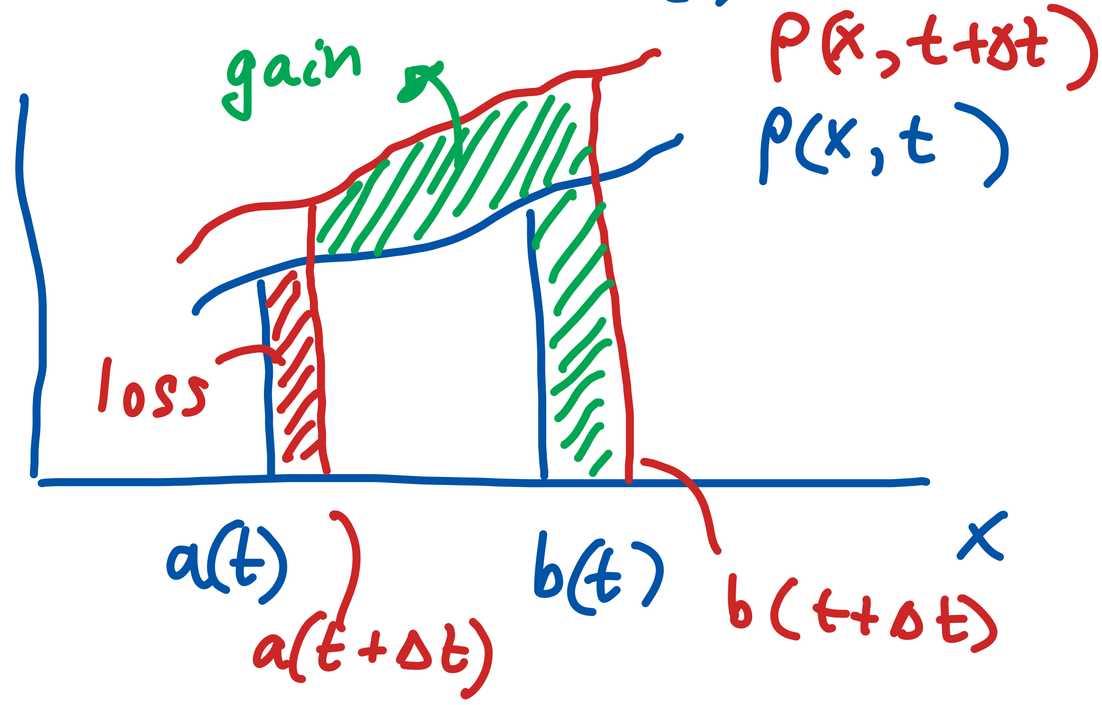
How to take the motion of $\mathcal{V}$ into account?

1D example: $\rho(x,t)$



$$M(t) \equiv \int_{a(t)}^{b(t)} \rho(x,t)\, dx$$

$$\frac{dM}{dt} = ?$$



$$a(t+\Delta t) = a(t) + \underbrace{\frac{da}{dt}}\Delta t + \cdots$$

$$\frac{dM}{dt} \equiv \lim_{\Delta t \to 0} \frac{M(t+\Delta t) - M(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\cancel{\Delta t}} \Big\{ \overset{\text{loss}}{-}\overset{\text{height}}{\rho(a(t),t)} \cdot \overbrace{\frac{da}{dt} \cdot \cancel{\Delta t}}^{\text{width}} + \rho(b(t),t) \cdot \frac{db}{dt} \cdot \cancel{\Delta t} + \int_{a(t)}^{b(t)} \frac{\partial \rho}{\partial t} \cancel{\Delta t}\, dx + \frac{\overbrace{\text{error}}^{\text{behaves like } \Delta t^2}}{\Delta t} \Big\}$$

$$\Rightarrow \boxed{\frac{dM}{dt} = -\rho(a(t),t)\frac{da}{dt} + \rho(b(t),t)\frac{db}{dt} + \int_{a(t)}^{b(t)} \frac{\partial \rho}{\partial t}\, dx + 0}$$

Leibniz' rule.

Check dimensions.

$M : \frac{kg}{m^3}\, m = \frac{kg}{m^2}$

$\Rightarrow \frac{dM}{dt} : \frac{kg}{m^2 s}$ $\qquad \rho \frac{da}{dt} : \frac{kg}{m^3} \frac{m}{s}$ ✓

$\int_a^b \frac{\partial \rho}{\partial t} dx : \frac{kg}{m^3} \frac{1}{s}\, m$ ✓

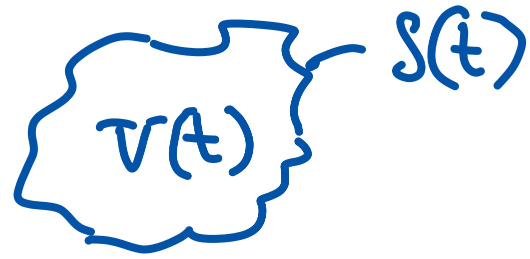
Remark: $a(t+\Delta t) = a(t) + \frac{da}{dt}(t)\Delta t + O(\Delta t^2)$

Taylor series.

order symbol of Landau.

How does this translate to

$$\frac{d}{dt}\int_{V(t)} \rho \, dV \quad ?$$



answer:

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho(\vec{x},t)\,dV = \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \int_{S(t)} \rho\, u_n \, dS}$$

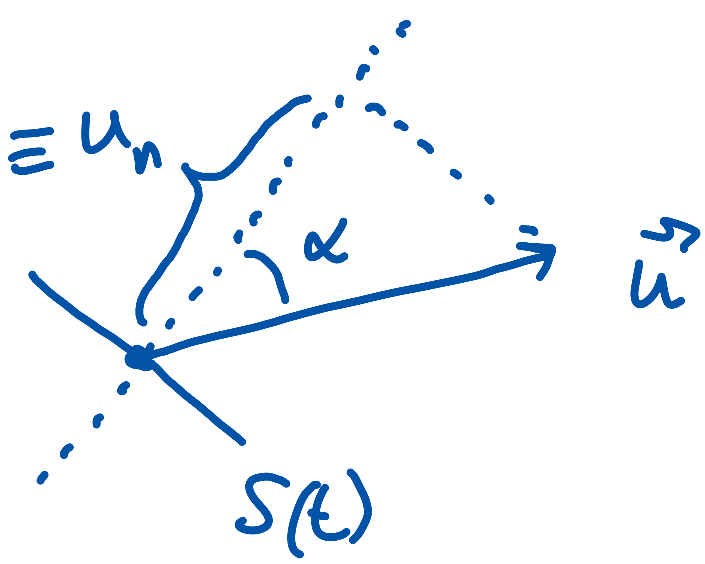
Reynolds-Leibniz transport theorem.

u_n = outward normal velocity

Mathematical Toolbox u_n and inner product

mini videos

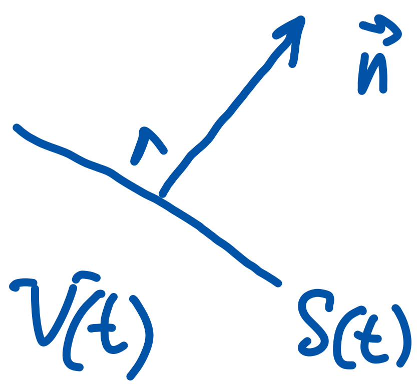
How to compute u_n?



u_n is scalar, not a vector.

$$u_n = |\vec{u}| \cos\alpha$$

To compute $\cos\alpha$, we need the outward unit normal vector.



$\vec{n}$:
- outward
- $|\vec{n}| = 1$
- perpendicular to S (normal).

$\vec{n}$: non-dimensional

We also need the <u>inner-product</u>:

$\vec{a}, \vec{b} \in \mathbb{R}^3 \qquad \vec{a}\cdot\vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\vec{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \vec{b} \equiv \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \equiv \sum_{j=1}^{3} a_j b_j$$

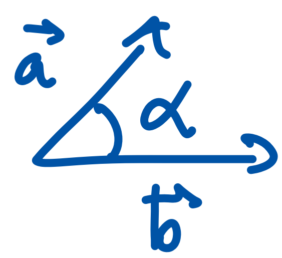
$$\equiv a_j b_j$$

separate ESC video.

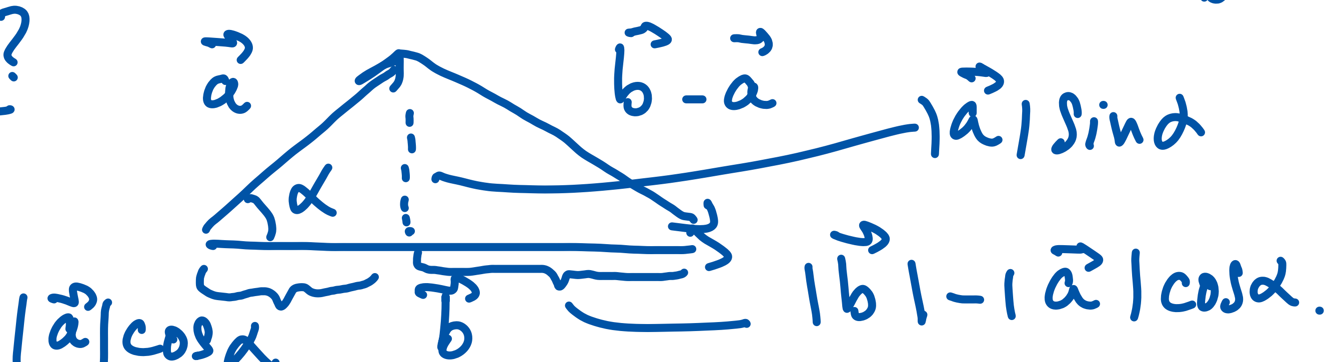
$$\boxed{\text{Einstein Summation convention on repeated indices.}}$$ (ESC)

<u>Property</u> of the inner-product:

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}| \cos\alpha$$



<u>Proof?</u>



idea: compute $|\vec{b}-\vec{a}|$ in two ways.

1 method:

Pythagoras: $$|\vec{b}-\vec{a}|^2 = |\vec{a}|^2 \sin^2\alpha + (|\vec{b}| - |\vec{a}|\cos\alpha)^2$$

$|\vec{a}| \equiv a \quad |\vec{b}| \equiv b$

$$\Rightarrow |\vec{b}-\vec{a}|^2 = a^2\sin^2\alpha + b^2 - 2ab\cos\alpha + a^2\cos^2\alpha$$

$$= \underline{a^2 + b^2 - 2ab\cos\alpha}.$$

method 2: $$|\vec{b}-\vec{a}|^2 \equiv (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$$

$$= \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{a}$$

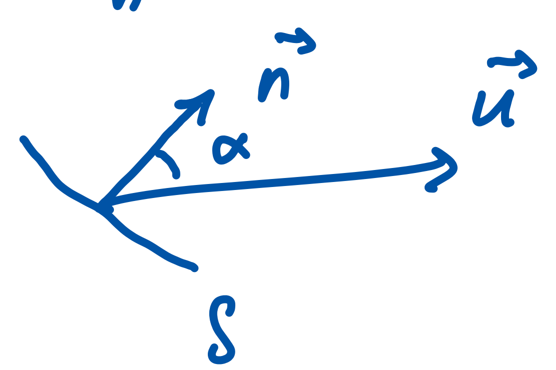
$$= |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b} + |\vec{a}|^2$$

$$= \underline{a^2 + b^2 - 2\,\vec{a}\cdot\vec{b}}$$

$$\Rightarrow \vec{a}\cdot\vec{b} = ab\cos\alpha. \qquad \square$$

Now return to computing $u_n = |\vec{u}|\cos\alpha$

Consider $\vec{u}\cdot\vec{n}$



$\vec{u}\cdot\vec{n} = |\vec{u}||\vec{n}|\cos\alpha \qquad |\vec{n}| \equiv 1$

$$\Rightarrow \vec{u}\cdot\vec{n} = |\vec{u}|\cos\alpha = u_n$$

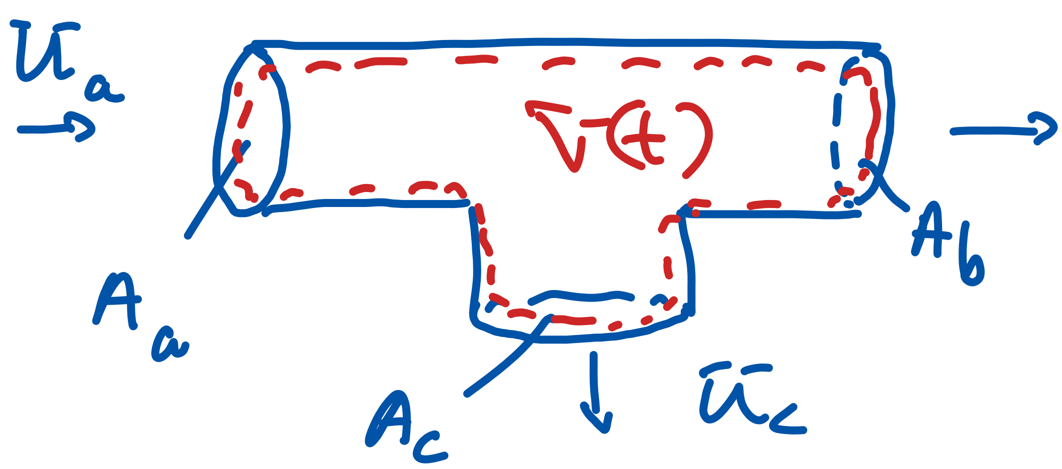
$$\Rightarrow \boxed{u_n = \vec{u}\cdot\vec{n}} \equiv u_j n_j. \qquad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\Rightarrow \boxed{\frac{d}{dt}\int_{V(t)} \rho \, dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho \, u_j n_j \, dS = 0}$$

mathematics

physics.

Application example



$\bar{U}_a, \bar{U}_b$ given.

A_a, A_b, A_c given.

$\bar{U}_c$?

Given: $\rho = \text{constant}$

Given: steady. $\frac{\partial}{\partial t}(\,) = 0$

steady, also $\rho = \text{const.}$

use: $$\int_{V(t)} \cancel{\frac{\partial \rho}{\partial t}} dV + \int_{\boxed{S(t)}} \rho \, u_j n_j \, dS = 0$$

$\rho = \text{const.}$

$$\Rightarrow \rho \int_{S(t)} u_j n_j \, dS = 0 \qquad \rho \neq 0 \Rightarrow \boxed{\int_{S(t)} u_j n_j \, dS = 0}$$

Solid steady wall

$$\Rightarrow u_j n_j = 0$$

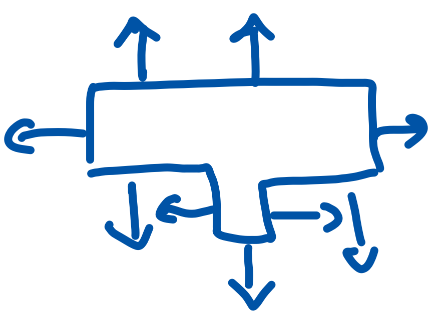
wall.

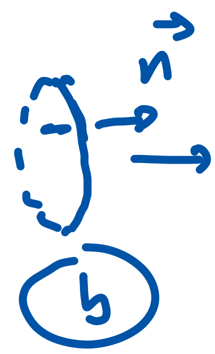
$$S(t) = A_a + A_b + A_c + A_w$$

$$\Rightarrow \int_{A_a} \cdots + \int_{A_b} \cdots + \int_{A_c} \cdots + \cancel{\int_{A_w} \cdots} = 0$$

$=0$ $u_j n_j = 0$

$$\Rightarrow \int_{A_a} u_j n_j \, dS + \int_{A_b} u_j n_j \, dS + \int_{A_c} u_j n_j \, dS = 0$$

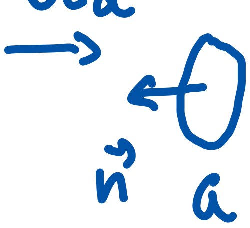




$\bar{U}_b \equiv$ average outward normal velocity.

$$\frac{1}{A_b} \int_{A_b} u_j n_j \, dS \equiv \bar{U}_b$$

$\Rightarrow$ 2nd integral is $A_b \bar{U}_b$



$\bar{U}_a$ is average inward normal velocity !

$$-\frac{1}{A_a} \int_{A_a} u_j n_j \, dS \equiv \bar{U}_a$$

$\Rightarrow$ 1st integral is $-A_a \bar{U}_a$

3rd integral is $A_c \bar{u}_c$

$$\Rightarrow \quad -A_a \bar{u}_a + A_b \bar{u}_b + A_c \bar{u}_c = 0$$

$$\Rightarrow \boxed{\bar{u}_c = \frac{A_a}{A_c} \bar{u}_a - \frac{A_b}{A_c} \bar{u}_b}$$

check dimensions: $\frac{A_a}{A_c}$: 1 ✓

other checks: suppose $A_b = 0$ ⓑ is closed

$$\Rightarrow \bar{u}_c = \frac{A_a}{A_c} \bar{u}_a$$

if $A_a = A_c \Rightarrow \bar{u}_c = \bar{u}_a$ ✓.

→ - Leibniz - Reynolds
- normal velocity and inner product
- Einstein summation convention