Tutorial T01 - Elasticity Basics

... based on sections 1 and 2

Exercises 1-5 are for practicing vector operations and transformations; material from sections 1,2.

Given are three vectors: $\underline{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\underline{b} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\underline{d} = \begin{bmatrix}2\\2\\1\end{bmatrix}$, in Cartesian basis.

Questions:

1. Compute the lengths (norms) of these vectors.

2. Compute the scalar (inner) products $\underline{a} \cdot \underline{b}$, $\underline{a} \cdot \underline{d}$, $\underline{b} \cdot \underline{d}$,
and compute the angles between these vector pairs.

3. Compute the outer (vector) products $\underline{a} \times \underline{b}$, $\underline{a} \times \underline{d}$, $\underline{b} \times \underline{d}$,
and from these compute the angles between these vector pairs.
Do the angles from 2. and 3. agree?

4. We define a new basis: $\underline{\hat{e}'_1} = \underline{d}/|\underline{d}|$, $\underline{\hat{e}'_2} = \underline{b}/|\underline{b}|$, and $\underline{\hat{e}'_3} = \underline{\hat{e}'_1} \times \underline{\hat{e}'_2}$.
Compute the orthogonal (transformation) rotation matrix $R_{pi} = \underline{\hat{e}'_p} \cdot \underline{\hat{e}_i}$.
Confirm its orthogonality.

5. Compute $R_{pi} a_i = a'_p$, $R_{pi} b_i = b'_p$, $R_{pi} d_i = d'_p$, and $R_{pi} \hat{e}'_{1|2|3} =$? (1|2|3 means all possibilities).

Preview/Voluntary:
... based on section 3

Exercises 6,7 cover further (tensor/matrix) operations, from linear algebra, including stress items as introduced later, in Section 3; but if you want to, you can try it now.

6.a. Give examples for tensors of rank 0, 1, and 2.

6.b. How many independent components has a symmetric stress tensor in two/three dimensions?
– and how many eigenvalues?

6.c. Under which conditions does one find no shear stress?

6.d. What mechanical/physical quantity:
is described by the trace of the stress tensor?
is the trace of the deformation tensor?
is the deviator of a tensor?

7. Given are the tensors (two-dimensional – plane-stress):
$\underline{\underline{\sigma}} = \begin{pmatrix}0.7 & 0.3\\0.3 & 1.3\end{pmatrix} p$ and $\underline{\underline{\sigma}} = \begin{pmatrix}1.0 & 0.6\\0.6 & 1.0\end{pmatrix} p$
Determine (mathematically and graphically) their eigenvalues, the traces, the deviators and the orientations of their major eigen-values with respect to the horizontal. Determine also the pressure and the maximum possible shear stress. Discuss the results – similarities and differences.

Answers T01 – Elasticity Basics

HINT: Try calculations first, before spying into the answers!

The first 5 exercises are for practicing vector operations, transformation matrix, while 6,7 cover some (tensor/matrix) operations, from linear algebra, including stress and strain. Note the matlab script `[LINK]` *that goes with these examples (not needed for the Elasticity course, practicing, exam, but maybe useful for future use).*

Given are three vectors: $\underline{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{d} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

1. Compute the lengths (norm) of these vectors.
$a = 1$
$b = 1.414213562373095 = \sqrt{2}$
$d = 3$

2. Compute the scalar (inner) products $\underline{a} \cdot \underline{b}$, $\underline{a} \cdot \underline{d}$, and $\underline{b} \cdot \underline{d}$
and compute the angles between these vector pairs.
$\underline{a} \cdot \underline{b} = 1$
$\underline{a} \cdot \underline{d} = 2$
$\underline{b} \cdot \underline{d} = 0$
$\theta_{ab} = 45$ deg.
$\theta_{ad} = 48.1897$ deg.
$\theta_{db} = 90$ deg.

3. Compute the outer (vector) products $\underline{a} \times \underline{b}$, $\underline{a} \times \underline{d}$ and $\underline{b} \times \underline{d}$
and from these compute the angles between these vector pairs.

$$\underline{a} \times \underline{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \ \underline{a} \times \underline{d} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ \underline{b} \times \underline{d} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Do the angles from 2. and 3. agree?
The angles do agree, calculated either way (not shown).

4. We define a new basis: $\underline{\hat{e}'_1} = \underline{d}/|\underline{d}|$, $\underline{\hat{e}'_2} = \underline{b}/|\underline{b}|$, and $\underline{\hat{e}'_3} = \underline{\hat{e}'_1} \times \underline{\hat{e}'_2}$. Compute the orthogonal (transformation) rotation matrix $R_{pi} = \underline{\hat{e}'_p} \cdot \underline{\hat{e}_i}$ and confirm its orthogonality.

For the Cartesian unit-vectors, use the standard basis: $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

and for the new basis, given their definitions: $\hat{e}'_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\hat{e}'_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\hat{e}'_3 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$,

so that: $\underline{\underline{R}} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & -1/3\sqrt{2} & 4/3\sqrt{2} \end{bmatrix}$, and orthogonality: $\underline{\underline{R}} \cdot \underline{\underline{R}}^T = 1$.

5. Compute $R_{pi} a_i = a'_p$, $R_{pi} b_i = b'_p$, $R_{pi} d_i = d'_p$, and $R_{pi} \hat{e}'_{1|2|3} =?$,
(where 1|2|3 means all three possibilities).
The first is not trivial, the other two are simpler.

$$\underline{a}' = \underline{\underline{R}} \cdot \hat{e}_1 = \begin{bmatrix} 2/3 \\ -1/\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}, \ \underline{b}' = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \ \underline{d}' = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},$$

$\underline{\underline{R}} \cdot \hat{e}_1 = \begin{bmatrix} 2/3 \\ -1/\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}$, $\underline{\underline{R}} \cdot \hat{e}_2 = \begin{bmatrix} 2/3 \\ 1/\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}$, $\underline{\underline{R}} \cdot \hat{e}_3 = \begin{bmatrix} 1/3 \\ 0 \\ 4/3\sqrt{2} \end{bmatrix}$, and
$\underline{\underline{R}} \cdot \hat{e}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{\underline{R}} \cdot \hat{e}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{\underline{R}} \cdot \hat{e}'_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,
i.e. the transformation matrix applied to the original Cartesian unit-vectors provides the rows of itself, and when applied to the new coordinate unit vectors, it brings back the original unit-vectors.

NOTE: for future use – the transformation behavior can be co- or contra-variant. Basis vectors behave/transform DIFFERENT from the vectors $\underline{a}$, $\underline{b}$, $\underline{d}$. Indices can, in general, be sub- or super-scripts, respectively. NOT NEEDED HERE.

Preview/Voluntary:
... based on section 3. STOP if you get stuck ...

6.a. Give examples for tensors of rank 0, 1, and 2.

R0: Scalars like density, temperature, (hydrostatic) pressure, ...
R1: Vectors like displacement, velocity, force, ...
R2: stress tensor, strain tensor, ...

6.b. How many independent components has a symmetric stress tensor in two/three dimensions? – and how many eigenvalues?

2D: three indep. components $\rightarrow$ 2 eigenvalues (and one orientation angle).
3D: six indep. components $\rightarrow$ 3 eigenvalues (and 3 orientation angles $\rightarrow$ 3 unit normals)

6.c. Under which conditions does one find no shear stress?

Shear stresses (shear strains) disappear when the stress is rotated into its eigen-system, i.e., the new coordinates are identical with the normal eigenvectors of this stress (or strain) tensor. The stress tensor then has diagonal form with eigenvalues on the diagonal and no shear stresses.

6.d. What mechanical/physical quantity:
is described by the trace of the stress tensor?
is the trace of the deformation tensor?
is the deviator of a tensor?

The trace of stress is three times the hydrostatic pressure, p, or isotropic, mean stress, $p = \sigma_m = \sigma_{\alpha\alpha}/3$. The trace of the deformation tensor (for small strains) gives the volume change. The deviator tensor $\sigma^d_{\alpha\beta} = \sigma_{\alpha\beta} - p\delta_{\alpha\beta}$ is the tensor without its isotropic, direction-independent part; this means $p^d = 0$ for the deviator stress, where the deviator relates to shear stress, and volume conservation for the deviator strain, where the deviator relates to shape change.

7. Given are the tensors (two-dimensional – plane-stress):
S1: $\underline{\underline{\sigma}} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 1.3 \end{pmatrix} p$ and **S2:** $\underline{\underline{\sigma}} = \begin{pmatrix} 1.0 & 0.6 \\ 0.6 & 1.0 \end{pmatrix} p$
Determine (mathematically and graphically) their eigenvalues, the traces, the deviators and the orientations of their major eigen-values with respect to the horizontal. Determine also the pressure and the maximum possible shear stress. Discuss the results – similarities and differences.

The purpose of this type of exercises is to work with special cases, which allows for simple solutions -– if one knows what properties to look for. The numerical solutions are given below. (Note that these 2D tensors are made 3D by adding a 1 on the 3-diagonal and zero elsewhere.)

The trace is then in both cases $3p$, where the p was a factor outside of the tensor, and $\sigma_m = tr(\sigma)/3$ is then the hydrostatic pressure p itself – simple.

The deviators are obtained, e.g., by subtracting $(tr(\sigma)/3)\delta_{ij}$ from σ_{ij}, with unit tensor components δ_{ij}. Note that the trace of the deviator is always zero – it thus contains no hydrostatic (isotropic) part.

Hint: It is advantageous for many calculations to keep the symbols and insert explicit numbers only as late as possible. Practice this!

S1:

Note that the eigenvectors below are NOT normalized (left) and normalized (right – decimal). The numbers look ugly, but one can obtain the orientation of the eigenvector corresponding to the largest eigenvalue by computing e.g. $\arctan(n1_y/n1_x)$=$3\pi/8$ (=45/2 degrees) for the angle between the horizontal and the first eigenvector n1.

The eigenvector n2 is 90 degrees from the horizontal and the eigenvector n3 is rotated - $\pi/8$, i.e. perpendicular to n1. Actually when one has a tensor like the deviator below, the orientation must be 45/2 degrees (plus or minus).

The maximal shear stress is half the difference of the largest and smallest eigenvalue, so that $\tau < 3/10^*\sqrt{2}$=0.4243.

Symbolic

```
sigma =
[  7/10,  3/10,      0]
[  3/10, 13/10,      0]
[     0,     0,      1]
V =
[          1,          0,          1]
[ 1+2^(1/2),           0, 1-2^(1/2)]
[          0,          1,          0]
D =
[ 1+3/10*2^(1/2),    0,                0]
[              0,    1,                0]
[              0,    0, 1-3/10*2^(1/2)]

sigma_D =
[ -3/10,  3/10,      0]
[  3/10,  3/10,      0]
[     0,     0,      0]
V_D =
[          1,          0,          1]
[ 1+2^(1/2),           0, 1-2^(1/2)]
[          0,          1,          0]
D_D =
[  3/10*2^(1/2),     0,                0]
[             0,     0,                0]
[             0,     0,   -3/10*2^(1/2)]
```

Decimal

```
eigenvecs =
    0.3827         0    0.9239
    0.9239         0   -0.3827
         0    1.0000         0
eigenvals =
    1.4243         0         0
         0    1.0000         0
         0         0    0.5757

eigenvecs_D =
    0.3827         0    0.9239
    0.9239         0   -0.3827
         0    1.0000         0
eigenvals_D =
    0.4243         0         0
         0         0         0
         0         0   -0.4243
```

Figure 1: Results/output from matlab script - see below.

S2:

Note that the eigenvectors below are NOT normalized (left) and normalized (right – decimal). The numbers look ugly, but one can obtain the orientation of the eigenvector corresponding to the largest eigenvalue by computing e.g. arctan($n1_y/n1_x$)=$\pi/4$ (=45 degrees) for the angle between the horizontal and the first eigenvector n1.

The eigenvector n2 is 90 degrees from the horizontal and the eigenvector n3 is rotated - $\pi/8$, i.e. perpendicular to n1. Actually when one has a tensor like the deviator below, the orientation must be 45/2 degrees (plus or minus).

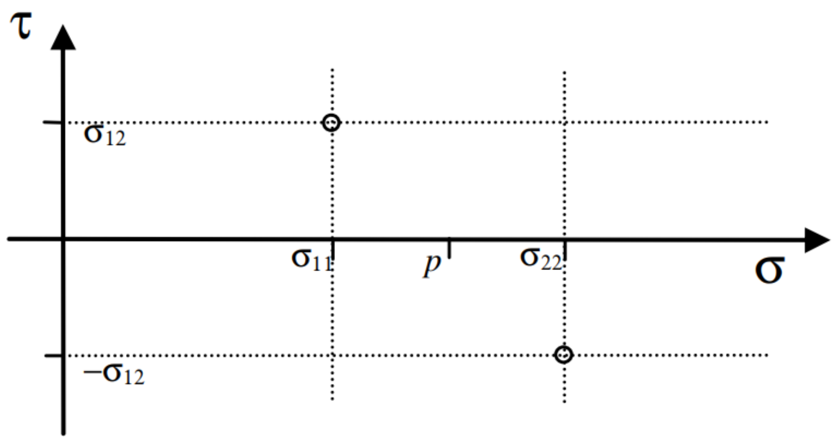
The maximal shear stress is half the difference of the largest and smallest eigenvalue so that $\tau < 6/10 = 0.6$.

```
sigma =
[    1, 6/10,    0]
[ 6/10,    1,    0]
[    0,    0,    1]
V =
[  1/2*2^(1/2),     0, -1/2*2^(1/2)]
[  1/2*2^(1/2),     0,  1/2*2^(1/2)]
[            0,     1,            0]
D =
[ 8/5,   0,   0]
[   0,   1,   0]
[   0,   0, 2/5]
sigma_D =
[   0, 3/5,   0]
[ 3/5,   0,   0]
[   0,   0,   0]
V_D =
[  1/2*2^(1/2),     0, -1/2*2^(1/2)]
[  1/2*2^(1/2),     0,  1/2*2^(1/2)]
[            0,     1,            0]
D_D =
[  3/5,     0,     0]
[    0,     0,     0]
[    0,     0,  -3/5]
```

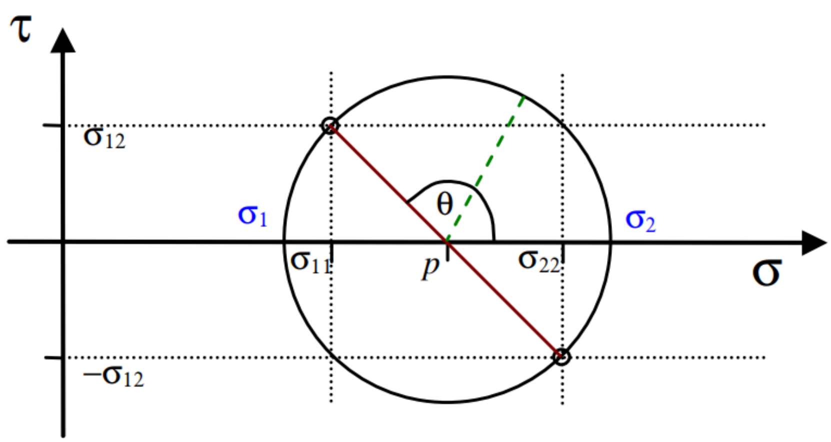
```
eigenvals =
    1.6000         0         0
         0    1.0000         0
         0         0    0.4000
eigenvecs =
    0.7071         0   -0.7071
    0.7071         0    0.7071
         0    1.0000         0

eigenvals_D =
    0.6000         0         0
         0         0         0
         0         0   -0.6000
eigenvecs_D =
    0.7071         0   -0.7071
    0.7071         0    0.7071
         0    1.0000         0
```

S1: For the graphic solution plot the elements, on the normal- and shear stress axes. The pressure p is in between. Then identify the pairs of shear and normal stresses (matter of definition and convention). Here, the 11-stress-component is paired with the positive 12-component.

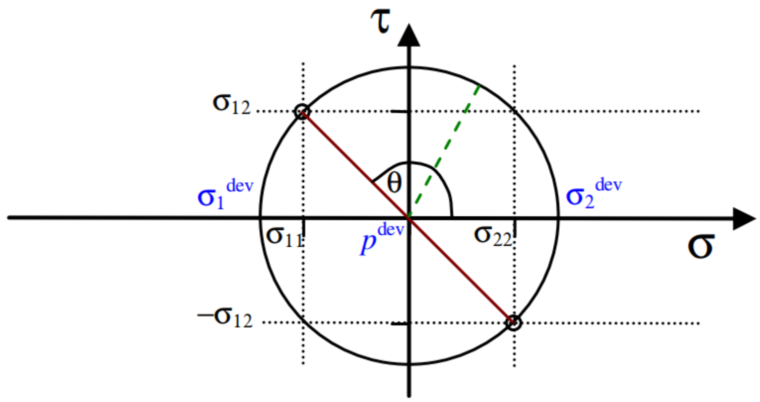


Then draw the circle through the center p and the two points (one is enough). The eigenvalues 1 and 2 (before sorting, 3 after sorting) are just the crossing points between circle and σ-axis.

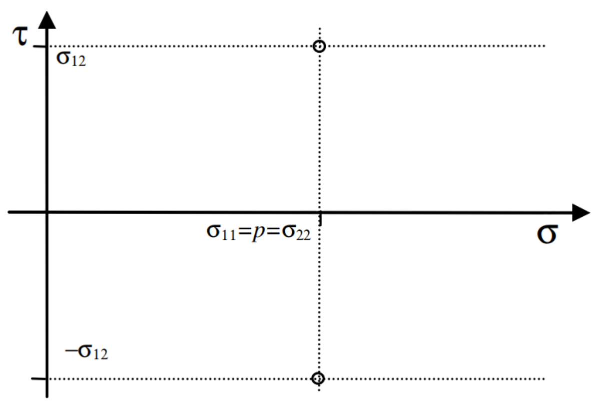


The angle θ between the σ-axis and the 11-12 point (top-left) gives $\theta = 2\phi = 6\pi/8$, so that the angle between the first eigenvector and the horizontal is $\phi = 3\pi/8$ (indicated by the dashed line).

When the deviator is considered, the vertical axis is shifted to the center of the circle. Nothing else changes concerning the circle and the graphics. The pressure p^{dev}=0 and the eigenvalues are opposite sign and equal in magnitude. The maximal shear stress corresponds to the radius of the circle in both cases (original and deviatoric stress).

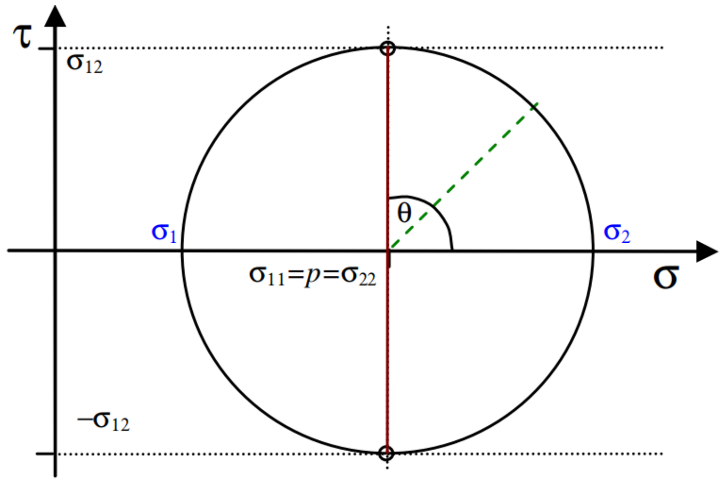


S2:
For the graphic solution plot the elements, on the normal- and shear stress axes. The pressure p is in between, i.e. here identical to the diagonal elements. Then identify the pairs of shear and normal stresses (matter of definition and convention). Here, the 11-stress-component is paired with the positive 12-component.



Then draw the circle through the center p and the two points (one is enough).

The angle θ between the σ-axis and the 11-12 point (top) gives $\theta = 2\phi = \pi/2$, so that the angle between the first eigenvector and the horizontal is $\phi = \pi/4$ (indicated by the dashed line).



Appendix: Matlab script example

```
%% ... examples for usage of MATLAB for vector and transformation operations
clc
clear all
close all
% format rat is doing symbolic as far as possible
format rat
format compact
format long
% V012 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% define colon vectors by transposing line vectors %%%%%%
veca=[1 0 0]';    % equivalent: veca=[1;0;0];
vecb=[-1 1 0]';
vecd=[2 2 1]';

%% Q1: compute lenghts of these colon vectors %%%%%%%%%%%%
a=norm(veca)
b=norm(vecb)
d=norm(vecd)

%% Q2: compute pairwise scalar products %%%%%%%%%%%%%%%%%%
adotb=veca'*vecb;
adotd=veca'*vecd;
ddotb=vecd'*vecb;
%equivalent
adotb=dot(veca,vecb)
adotd=dot(veca,vecd)
ddotb=dot(vecd,vecb)

% Q2b: comute the angle from adotb = a b cos(theta)
theta_ab=acos(adotb/a/b) /2/pi*360
```

```
theta_ad=acos(adotd/a/d) /2/pi*360
theta_db=acos(ddotb/d/b) /2/pi*360

%% Q3: compute the pairwise outer products %%%%%%%%%%%%%%%
axb=cross(veca,vecb)
axd=cross(veca,vecd)
dxb=cross(vecd,vecb)

% Q3b: compute the angle from |axb| = a b sin(theta) %%%%%%%
theta_ab=asin(norm(axb)/a/b) /2/pi*360
theta_ad=asin(norm(axd)/a/d) /2/pi*360
theta_bb=asin(norm(dxb)/d/b) /2/pi*360

%% Q4: define new basis and compute rotation matrix
format rat
e1=[1 0 0]';
e2=[0 1 0]';
e3=[0 0 1]';
e1a=vecd/d
e2a=vecb/b
e3a=cross(e1a,e2a)
% confirm that unit vectors
%norm(e1a)
%norm(e2a)
%norm(e3a)
% compute R_pi
Me =[e1  e2  e3 ]   % matrix with unit-colon vectors
Mea=[e1a e2a e3a]   % matrix with new unit-colon vectors
R_pi=Mea'*Me        % note that Mea^T * Me gives R_pi
R_ip=Me*Mea         % while     Mea * Me   gives R_ip = (R_pi)^T
R_pi*R_ip           % confirm R*R^T=1
%R_ip*R_pi           % the same ...
unittest=inv(R_pi)-R_ip  % should be approx 0 - note the prefactor

%% Q5: Rotate vectors from old basis to new basis
format long
anew=R_pi*veca
bnew=R_pi*vecb
dnew=R_pi*vecd
e1_new=R_pi*e1
e2_new=R_pi*e2
e3_new=R_pi*e3
Base_new=[e1_new e2_new e3_new]
e1a_new=R_pi*e1a
e2a_new=R_pi*e2a
e3a_new=R_pi*e3a
Basea_new=[e1a_new e2a_new e3a_new]
```