Chapter 1

Introduction

1.1 The three fundamental conservation laws

Consider a group of N particles numbered by $i = 1, 2, ..., N$. The i-th particle has mass m_i, velocity $\boldsymbol{v}_i(t)$ and internal energy $\epsilon_i(t)$. Suppose that there are no external forces working on the particles, then, for the group of particles the following laws are observed again and again:

$$\sum_{i=0}^{N} \frac{d}{dt}(m_i) = 0, \tag{1.1}$$

$$\sum_{i=0}^{N} \frac{d}{dt}(m_i \boldsymbol{v}_i) = 0, \tag{1.2}$$

$$\sum_{i=0}^{N} \frac{d}{dt}\left(\frac{1}{2} m_i \|\boldsymbol{v}_i\|^2 + \epsilon_i\right) = 0. \tag{1.3}$$

In other words, the mass, momentum, and energy of the group of particles is conserved, and therefore we call these laws conservation laws. Internal forces between particles do not change the total momentum and energy because the loss by one particle is gained by one or more other particles. When there are external forces working on the particles, say $\boldsymbol{F}_i$ is the external force (vector) working on the i-th particle, then the equations become:

$$\sum_{i=0}^{N} \frac{d}{dt}(m_i) = 0, \tag{1.4}$$

$$\sum_{i=0}^{N} \frac{d}{dt}(m_i \boldsymbol{v}_i) = \sum_{i=0}^{N} \boldsymbol{F}_i, \tag{1.5}$$

$$\sum_{i=0}^{N} \frac{d}{dt}\left(\frac{1}{2} m_i \|\boldsymbol{v}_i\|^2 + \epsilon_i\right) = \sum_{i=0}^{N} \boldsymbol{F}_i \cdot \boldsymbol{v}_i. \tag{1.6}$$

We still call these laws conservation laws.

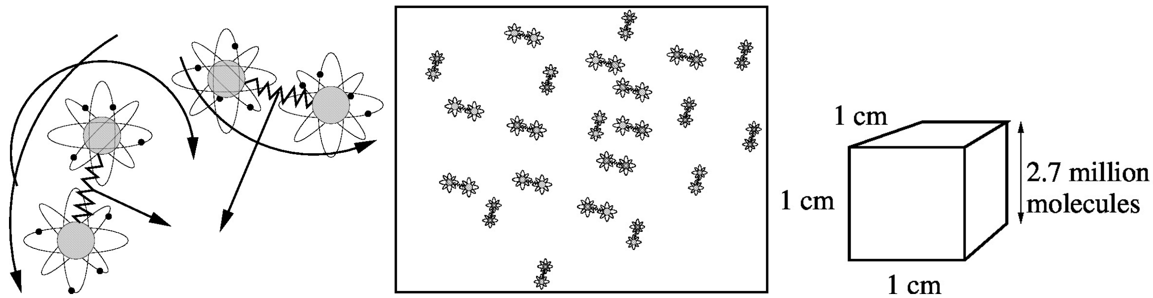

Figure 1.1: The particle model of a fluid

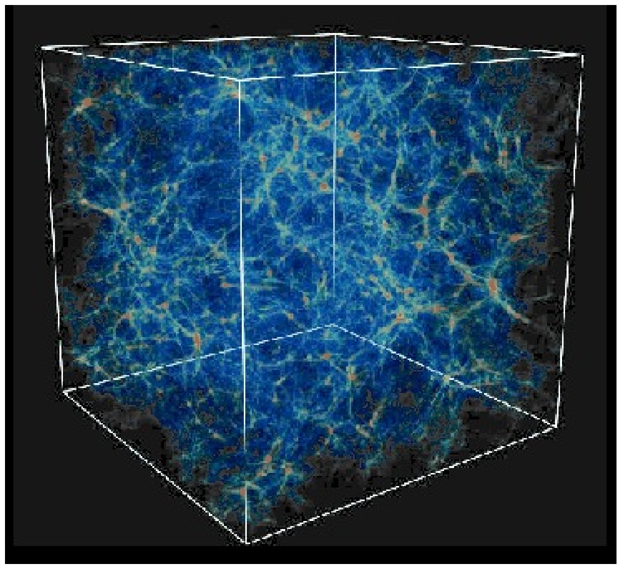

Figure 1.2: Molecular-dynamics simulation of a fluid.

1.2 Continuum model

Density At a given point in space we can construct a sphere around the point and compute the total mass of all particles within the sphere. If we then divide that mass by the volume of the sphere we have computed the amount of mass per unit volume which we will call mass density or just density. On the one hand, when we take the sphere too large, the computed density does not characterise the situation at the given point but rather the situation in a relatively large region. On the other hand, when we take the spere too small, there will be no particles within the spehere at all and the density is zero.

In order to construct the density in every point in the domain we choose the sphere sufficiently small compared to the flow domain at hand, but sufficiently large compared to the average distance between particles, assuming that this procedure still leaves a whole range of sphere sizes for which the computed density does not depend on the actual size of the sphere.

Velocity Similar to the density we can define the velocity at a given point in space by averaging the velocities of the particles over a sphere which is sufficiently small compared to the flow domain at hand, but sufficiently large compared to the average distance between particles.

Temperature The temperature at a given point in space can be computed as the average kinetic energy of particles in an appropriately sized sphere.

Pressure The pressure at a given point in space can be computed as the time-averaged force on a small area-element stuck into the flow.

1.3 Trajectories

Consider a small dust particle with time dependent position:

$$\boldsymbol{x}_p(t) = \begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix}, \tag{1.7}$$

and a surrounding fluid with velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} u(\boldsymbol{x}, t) \\ v(\boldsymbol{x}, t) \\ w(\boldsymbol{x}, t) \end{pmatrix}. \tag{1.8}$$

Note that the independent spatial coordinate $\boldsymbol{x}$ can be chosen arbirarily, whereas the spatial coordinate $\boldsymbol{x}_p(t)$ of the particle is a time-dependent variable: $\boldsymbol{x}$ and $\boldsymbol{x}_p(t)$ are different entities! When the dust particle is convected with the flow, this means that the velocity of the particle is equal to the velocity of the fluid **at the position of the particle** at all times:

$$\boxed{\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}(\boldsymbol{x}_p, t)}, \tag{1.9}$$

where we have substituted the actual particle position $\boldsymbol{x}_p(t)$ as the location where we want to know the velocity. This equation is a vector equation which in general has three components:

$$\frac{dx_p}{dt} = u(\boldsymbol{x}_p, t), \quad \frac{dy_p}{dt} = v(\boldsymbol{x}_p, t), \quad \frac{dz_p}{dt} = w(\boldsymbol{x}_p, t). \tag{1.10}$$

Example 1.1.

Consider the velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} ax + yt\cos(bz^2) \\ \exp(cxyz) \\ \sin(abx^2 z) \end{pmatrix}, \tag{1.11}$$

then the three differential equations that describe the trajectory of the particle become:

$$\begin{aligned}\frac{dx_p}{dt} &= ax_p + y_p t\cos(bz_p^2),\\ \frac{dy_p}{dt} &= \exp(cx_p y_p z_p),\\ \frac{dz_p}{dt} &= \sin(abx_p^2 z_p.\end{aligned} \tag{1.12}$$

Due to their complexity these equations can only be solved by numerical integration. There are cases however in which the velocity field is sufficiently simple and one can compute an analytical solution. For example, when the velocity field is uniform (independent of $\boldsymbol{x}$) and constant (independent of t):

$$\boldsymbol{u}(\boldsymbol{x},t) = \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{1.13}$$

the three differential equations that describe the trajectory of the particle become simply

$$\frac{dx_p}{dt} = U, \quad \frac{dy_p}{dt} = V, \quad \frac{dz_p}{dt} = W \tag{1.14}$$

and the trajectory of a particle that passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$ is a straight line

$$\boldsymbol{x}_p = \begin{pmatrix} Ut + x_o \\ Vt + y_o \\ Wt + z_o \end{pmatrix}. \tag{1.15}$$

Example 1.2.

Consider

$$\boldsymbol{u}(\boldsymbol{x},t) = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}, \tag{1.16}$$

then

$$\frac{dx_p}{dt} = ax_p, \quad \frac{dy_p}{dt} = by_p, \quad \frac{dz_p}{dt} = cz_p \tag{1.17}$$

and the trajectory of a particle that passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$ is

$$\boldsymbol{x}_p(t) = \begin{pmatrix} x_o \mathrm{e}^{at} \\ y_o \mathrm{e}^{bt} \\ z_o \mathrm{e}^{ct} \end{pmatrix}. \tag{1.18}$$

Example 1.3.

Consider

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} \frac{a}{x(t+b)} \\ cy \\ ez \end{pmatrix}, \tag{1.19}$$

then the first equation becomes

$$\frac{dx_p}{dt} = \frac{a}{x_p(t+b)}. \tag{1.20}$$

This can be rewritten as

$$x_p \frac{dx_p}{dt} = \frac{a}{t+b} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}x_p^2\right) = \frac{a}{t+b}, \tag{1.21}$$

and, when the particle passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$, integrated to

$$\frac{1}{2}x_p^2(t) = a\ln(t+b) - a\ln(b) + \frac{1}{2}x_o^2 \quad \Rightarrow \quad x_p(t) = \pm\sqrt{2a\ln(\frac{t+b)}{b}) + x_o^2}. \tag{1.22}$$

1.4 Vector-notation and index-notation

In the present lecture notes we will use two notations when it comes to vectors. Suppose we have a vector $\boldsymbol{u}$ with components u_1, u_2, and u_3, and a vector $\boldsymbol{n}$ with components n_1, n_2, and n_3. The inner product of these two vectors can be written in so-called vector notation as

$$\boldsymbol{u} \cdot \boldsymbol{n}. \tag{1.23}$$

It is noted that in vector notation we only write the vectors as a whole and not their components, and at the same time operations are written implicitly: you have to know what the dot means. In contrast, we also can write the inner product in index notation as

$$\sum_{j=1}^{3} u_j n_j. \tag{1.24}$$

This shows that in index notation we don't write vectors as a whole, but we write their components instead, and operations are written explicitly. As another example, consider the addition of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ resulting in a vector $\boldsymbol{w}$. In vector notation this becomes

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}, \tag{1.25}$$

whereas in index notation this becomes

$$w_i = u_i + v_i, \quad i = 1, 2, 3. \tag{1.26}$$

These and other examples are listed in the first two columns of Table (1.1)

Vector notation	Index notation	Index notation with ESC
$\boldsymbol{u} \cdot \boldsymbol{n}$	$\sum_{j=1}^{3} u_j n_j$	$u_j n_j$
$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$	$w_i = u_i + v_i, \quad i = 1, 2, 3$	$w_i = u_i + v_i, \quad i = 1, 2, 3$
$A\boldsymbol{x}$	$\sum_{j=1}^{3} a_{ij} x_j, \quad i = 1, 2, 3$	$a_{ij} x_j, \quad i = 1, 2, 3$
$A\boldsymbol{x} \cdot \boldsymbol{u}$	$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_j u_i$	$a_{ij} x_j u_i$
$\boldsymbol{\nabla} \cdot \boldsymbol{u}$	$\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}$	$\frac{\partial u_j}{\partial x_j}$

Table 1.1: Vector notation versus index notation

1.5 Einstein summation convention

Vector fields play an essential role in our understanding of physics and engineering. Frequently we need to compute sums of vector components, think for example of the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \Rightarrow \quad \boldsymbol{a} \cdot \boldsymbol{b} \equiv \sum_{j=1}^{3} a_j b_j, \tag{1.27}$$

or of the multiplication of a matrix A with a vector $\boldsymbol{x}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Rightarrow \quad y_i \equiv (A\boldsymbol{x})_i \equiv \sum_{j=1}^{3} a_{ij} b_j. \tag{1.28}$$

As a result, many summation symbols will enter our formulas. Einstein [(1)] encountered this problem when dealing with theories of relativity and decided to follow a convention:

Theorem 1.1 (Einstein summation convention)**.** *When a single expression carries the same index twice, summation over the index is implied. Expressions that consist of terms that are separated by either '+', '-', or '=' are not considered single.*

With this convention we write the above two examples as

$$\sum_{j=1}^{3} a_j b_j \equiv a_j b_j, \tag{1.29}$$

and

$$\sum_{j=1}^{3} a_{ij} b_j \equiv a_{ij} b_j. \tag{1.30}$$

Note that in the second example we sum over j but not over i. Other examples are

$$\sum_{k=1}^{3} a_k \frac{\partial u_k}{\partial x_i} \equiv a_k \frac{\partial u_k}{\partial x_i}, \tag{1.31}$$

[(1)]Einstein, Albert (1916); "The Foundation of the General Theory of Relativity", Annalen der Physik

$$\sum_{i=1}^{3} u_{ii} \equiv u_{ii}, \tag{1.32}$$

$$\sum_{j=1}^{3} c_j \left(a_j + b_j\right) \equiv c_j \left(a_j + b_j\right), \tag{1.33}$$

and a counter example is

$$\sum_{j=1}^{3} \left(a_j + b_j\right) \neq a_j + b_j. \tag{1.34}$$

Finaly the reader is referred to the second and third columns of Table (1.1).

1.6 Exercises

Problem 1.1. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} x \\ -y \end{pmatrix}$

(a) Draw the curves $xy = \pm 1$ *in all four quadrants of the* $x - y$ *plane.*
(b) Draw the velocity vector at several points on the curves.
(c) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.2. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.3. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.4. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.5. *Consider a little smoke particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$

(a) Draw the trajectory for $-1 \leq t \leq 1$.
(b) Compute the velocity vector.