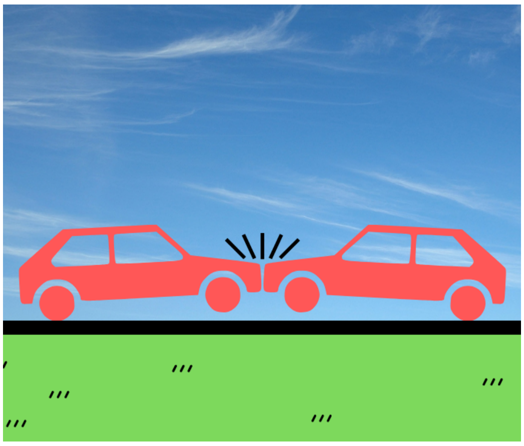
When Angular Momentum Holds



Moments and angular momenta can are defined around a reference point R. When does the following equation for an induced change in angular momentum for a single rigid body hold?

$$\mathbf{H_{R1}} + \int_{t_1}^{t_2} \Sigma\mathbf{M_R}\mathrm{d}t = \mathbf{H_{R2}}$$

Solution:

This is possibly the trickiest principle to fully understand in Dynamics I. *Knowing how to use this principle in problems is enough.* The entire proof, although given below, is not required to know. Essentially, the shown equation for angular momentum always holds with respect to any virtual fixed reference point R and will be equivalent to the equation for angular momentum with respect to any other. However, that does not mean the angular momentum to that reference point can be easily written as $I_\mathrm{R}\omega$. That only holds for the center of mass and for fixed points of a body/system.

Proof:

The before and after-state of the system is given with 1,2 after the index respectively. Also no change of mass occurs. We start with defining angular momentum is its most general form for an arbitrary reference point R in both instance 1 and instance 2.

$$\mathbf{H}_{\mathrm{R1}} \equiv \sum_i m_i \mathbf{v}_{i1} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} \qquad \mathbf{H}_{\mathrm{R2}} \equiv \sum_i m_i \mathbf{v}_{i2} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}}$$

Note that this definition simplifies to $I_{\mathrm{R}}\omega$ when taking the center of gravity or a fixed point of a body/system. Combining these and rewriting gives:

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \sum_i m_i \mathbf{v}_{i2} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} - \sum_i m_i \mathbf{v}_{i2} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}}$$

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \sum_i m_i \mathbf{v}_{i2} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} - m_i \mathbf{v}_{i2} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}}$$

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \sum_i (\mathbf{G}_{i2} - \mathbf{G}_{i1}) \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} \tag{1}$$

The rigid body has been segmented into smaller masses (in the limit) with mass m_i at P_i. The resultant force acting on mass m_i is described with $\mathbf{F}_i$; This includes external forces and reaction forces between the connected segments. We know from linear momentum that the following holds:

$$\mathbf{G}_{i2} - \mathbf{G}_{i1} = \int_{t_1}^{t_2} \mathbf{F}_i \mathrm{d}t \tag{2}$$

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \sum_i \int_{t_1}^{t_2} \mathbf{F}_i \mathrm{d}t \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}}$$

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \sum_i \int_{t_1}^{t_2} \mathbf{F}_i \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} \mathrm{d}t$$

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \int_{t_1}^{t_2} \sum_i \mathbf{F}_i \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} \mathrm{d}t$$

The integrand exists of contributions of external forces on each mass. The internal force contributions between masses will cancel out via Newton's third law, giving:

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \int_{t_1}^{t_2} \sum_i \mathbf{F}_{\mathrm{ext},i} \times \mathbf{r}_{\mathrm{P}_i/\mathrm{R}} \mathrm{d}t$$

This integrand is the total external moment $\Sigma\mathbf{M}_{\mathrm{R}}$ on the system with respect to R.

$$\mathbf{H}_{\mathrm{R}2} - \mathbf{H}_{\mathrm{R}1} = \int_{t_1}^{t_2} \Sigma\mathbf{M}_{\mathrm{R}} \mathrm{d}t$$

This shows that when the equation of linear momentum holds, the equation of angular momentum holds for with respect to any reference point R (we have not put any restrictions or assumptions on R in the proof).