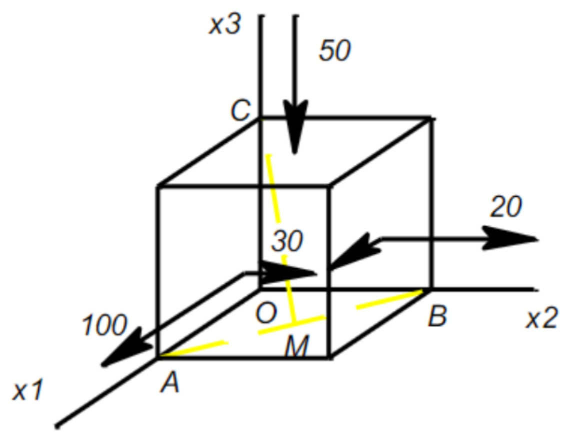
1 Exercise V-1a

Given: $E = 200GPa$, $\upsilon = 0.25$
$OA = OB = a$ and $OC = \frac{1}{2}\sqrt{2} \cdot a$
In this stress-state, the maximal principal stress must not be larger than: 150 MPa.
Questions:
a) Find σ_{ABC} and τ_{ABC}



2 Exercise V.4

Given: $E = 2 \cdot 10^{11} Pa$, $\upsilon = 0.25$
Stress-state in point P: $\sigma = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa
Questions:
A) Show that the principal stresses are 8, 16 and 24 MPa. Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.
B) Compute the volumetric (isotropic) strain.
C) What is the largest angle-change (not shear-strain) in P?
D) Which material property is implicitly used in Hookes law?

3 Exercise V.12abc

In a linear elastic ($E = 2 \cdot 10^5 MPa$, $\upsilon = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$$\sigma_{11}(x_1, x_2, x_3) = \sigma_0[20 + \alpha_1(\tfrac{x_1}{L}) - 10(\tfrac{x_2}{L}) + \alpha_2(\tfrac{x_1}{L})^2]$$

$$\sigma_{22}(x_1, x_2, x_3) = \sigma_0[10 + 8(\tfrac{x_1}{L}) + \beta_1(\tfrac{x_2}{L}) + \beta_2(\tfrac{x_2}{L})^2]$$

$$\sigma_{12}(x_1, x_2, x_3) = \sigma_0[12 - 10(\tfrac{x_1}{L}) + 7(\tfrac{x_2}{L}) - 8(\tfrac{x_1}{L})(\tfrac{x_2}{L})]$$

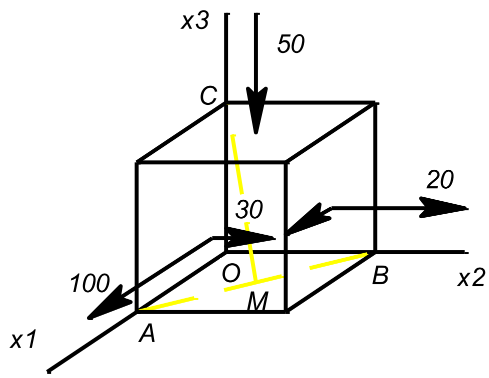
$$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$$

with reference stress $\sigma_0 = 1MPa$ and reference length $L = 1m$. Note that all stresses are independent on x_3 and that the calculation in question (a) below is general with variables x_1, x_2, and x_3; from question (b) on, use the point $P(x_1 = 0, x_2 = 0, x_3 = 0)$.

Questions:
a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the four free parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium.
b) Compute the eigen-stresses in point P using linear algebra mathematics – not the circle of Mohr.
Describe and name the state of stress in point P (and in all other points in the body).
c) Compute the eigen-direction of the major eigen-stress.

Exercise V-1

Problem



Given:

$E = 200$ GPa & $\nu = 0.25$

$OA = OB = a \quad \& \quad OC = \frac{1}{2}\sqrt{2} \cdot a$

In this stress-state, the maximal principal stress must not be larger than: 150 MPa.

Questions:

a) σ_{ABC} & τ_{ABC},

Solutions

a)

Stress tensor, from the sketch: $\sigma = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} MPa$

Compute the normal to the surface ABC: for example by using the cross-product of two vectors inside this plane.

$$A\vec{C} \times A\vec{B} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalisation: $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$: $\qquad \alpha^2\left(1^2 + 1^2 + \left(\sqrt{2}\right)^2\right) = 1 \quad \Rightarrow \quad \alpha = \frac{1}{2}$

Comment

After using the cross-product, with vectors in arbitrary order, one must check/confirm that the normal points out of the plane, away from the cube-backside point O, where the material still exists. The normal should point away from the material. Then choosing α (in this case) positive, one gets the normal in the right direction; the other solution to normalisation is not valid here.

The stress-vector on surface ABC: $p_i = \sigma_{ji} \cdot n_j$

$$\rightarrow \qquad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = [\sigma] \cdot \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

Normal-stress: $\sigma = p^{(n)}$ on ABC: $\{\sigma\} = \{\hat{n}\}^T \cdot \{p\} = 20MPa$

Shear-stress τ on ABC (Pythagoras):

$$\tau^2 = |\{p\}|^2 - \sigma^2 = \left[p_1^2 + p_2^2 + p_3^2\right] - \sigma^2 = 6100 - 400 = 5700MPa \quad \Rightarrow \quad \tau = 75{,}5Mpa$$

Exercise V-4

Problem

Given:

$E = 2.10^{11} GPa \quad \& \quad \nu = 0.25$

Stress-state in point P: $\sigma = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} MPa$

Questions:

A) Show that the principal stresses are 8, 16 and 24 MPa. Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.

B) Compute the volumetric (isotropic) strain.

C) What is the largest angle-change (not shear-strain) in P?

D) Which material property is implicitly used in Hookes law?

Solutions

A)

$$\det([\sigma] - \sigma[I]) = \det \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} = 0$$

→ characteristic equation: $\sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0 = 0$

From this follow the principal stresses (one can insert them and show that the characteristic equation gets zero for everyone; or one can factorize the equation; or one computes the invariants from the eigen-values and identifies them with the euqation):

$$\sigma_1 = 24MPa$$
$$\sigma_2 = 16MPa$$
$$\sigma_3 = 8MPa$$

For the smallest principal stress, compute the eigen-direction:

$$\begin{bmatrix} \sigma_{11} - 8 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - 8 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - 8 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1^3 \\ \hat{n}_2^3 \\ \hat{n}_3^3 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this equation system, for example, yields:

$$\begin{Bmatrix} \hat{n}_1^3 \\ \hat{n}_2^3 \\ \hat{n}_3^3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \sqrt{6} \end{Bmatrix}$$

which still must be normalized →

$$\left(\hat{n}_1^3\right)^2+\left(\hat{n}_1^3\right)^2+\left(\sqrt{6}\cdot\hat{n}_1^3\right)^2=1 \quad \rightarrow \quad \left(\hat{n}_1^3\right)^2=\frac{1}{8} \quad \rightarrow \quad \hat{n}_1^3=\frac{\sqrt{2}}{4}$$

which gives the eigen-direction: $$\left\{\begin{array}{c}\hat{n}_1^3\\ \hat{n}_2^3\\ \hat{n}_3^3\end{array}\right\}=\left\{\begin{array}{c}\frac{1}{4}\sqrt{2}\\ \frac{1}{4}\sqrt{2}\\ \frac{1}{2}\sqrt{3}\end{array}\right\}$$

which actually are the directional cosines (three entries R_{3i}).

B)

For the volumetric strain we get: $\varepsilon_V=\varepsilon_1+\varepsilon_2+\varepsilon_3=\left[\frac{1-2\nu}{E}\right]\cdot\sigma_{kk}=12.10^{-5}$

C)

The largest change of angle is: $\gamma_{\max}=\frac{\tau_{\max}}{G}=\frac{\frac{1}{2}\left(\sigma_1-\sigma_3\right)}{G}=1.10^{-4}$,

where the largest shear strain is just half of that.

D)

Isotropy is intrinsic to using the law of Hooke.

Solutions of V-12

a) Given was the stress-field in absence of body forces $f_i = 0$:

$$\sigma_{11}(x_1, x_2) = \sigma_0 \left[20 + \alpha_1.\frac{x_1}{L} - 10.\frac{x_2}{L} + \alpha_2.\left(\frac{x_1}{L}\right)^2\right]$$
$$\sigma_{22}(x_1, x_2) = \sigma_0 \left[10 + 8.\frac{x_1}{L} + \beta_1.\frac{x_2}{L} + \beta_2.\left(\frac{x_2}{L}\right)^2\right]$$
$$\sigma_{12}(x_1, x_2) = \sigma_0 \left[12 - 10.\frac{x_1}{L} + 7.\frac{x_2}{L} - 8.\frac{x_1}{L}.\frac{x_2}{L}\right]$$

Using the stress-equilibrium equations, i.e. derivatives with displacement-directions with respect to the coordinate system, one obtains:

$$\frac{d}{dx_1}\sigma_{11}(x_1, x_2) + \frac{d}{dx_2}\sigma_{12}(x_1, x_2) = \sigma_0 \left[\frac{\alpha_1}{L} + 2\alpha_2.\frac{x_1}{L^2}\right] + \sigma_0 \left[\frac{7}{L} - 8.\frac{x_1}{L^2}\right] = 0$$
$$\frac{d}{dx_1}\sigma_{12}(x_1, x_2) + \frac{d}{dx_2}\sigma_{22}(x_1, x_2) = \sigma_0 \left[\frac{-10}{L} - 8.\frac{x_2}{L^2}\right] + \sigma_0 \left[\frac{\beta_1}{L} + 2\beta_2.\frac{x_2}{L^2}\right] = 0$$

From these equations, one obtains the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$

b) The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\left[\sigma_{ij}\right]$ = $\begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

Using the stress tensor, the characteristic equation can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

and from this, knowing that one eigen-value is zero, i.e. $I_3 = 0$, the principal stresses can be computed from a second order polynomial as: $\sigma_I = 28$MPa, $\sigma_{II} = 2$MPa, $\sigma_{III} = 0$MPa. This is a plane-stress state with all stresses on the x_3-surface being equal to zero.

c) And the principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$ is directly visible from the tensor, due to zero- shear stresses in the x_3-direciton, while the others require to insert:

Direction of the major stress $\sigma_I = 28$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and normalization: $n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$

$-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

Direction of the middle (was not asked, for completeness) $\sigma_{II} = 2$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and normalization: $n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$

$18n_1^{(II)} + 12n_2^{(II)} = 0 \to n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9) + 1]n_2^{(II)} = 1 \to n_2^{(II)} = 3/\sqrt{13}$

$$\implies \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$