

# Tutorial T03 – Elasticity – Stress

December 10, 2024

*Answer the following questions as they could come up in an exam.*

Exercises 1,2,4 will be continued in tutorial T04

## 3 Stress tensor basics

*... based on section 3,4 (Exercise V3 in old material before 2022)*

**Given:**

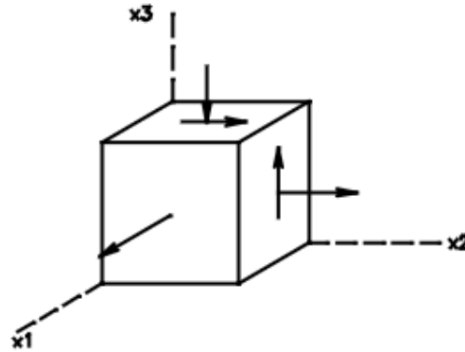


Figure 1: Stress cube, empty → fill it

The stress-state is described by the matrix: 
$$\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix} \text{ N/mm}^2,$$

with  $E = 2 \cdot 10^5 \text{ N/mm}^2$ , and  $\nu = 0.25$ .

### Questions:

- Compute the principal stresses
- Compute the eigen-directions
- Compute the maximal shear-stress
- Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in  $x'_p$ , i.e. the coordinate system defined by the eigen-directions.
- Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.

### Answers:

a)

The sorted eigen-values are:  $\sigma_I = 60 \text{ MPa}$ ,  $\sigma_{II} = 40 \text{ MPa}$ ,  $\sigma_{III} = -40 \text{ MPa}$ .

*The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).*

b)

Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from  $(\sigma_{ij} - \sigma \delta_{ij})n_j = 0$ , with normalization  $n_j^2 = 1$ :

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example  $\sigma_{II}$ , solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^2 + n_2^2 + n_3^2 = 1$$

$$-20n_2 + 20\sqrt{3}n_3 = 0$$

$$20\sqrt{3}n_2 - 60n_3 = 0$$

$$n_2 = \sqrt{3}n_3 \text{ and } n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3 \text{ (identical due to dependency)}$$

$$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$$

$$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above.  
The third eigenvalue calculation is similar (not shown).

c)

The maximum shear stress is:  $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50 \text{ MPa}$ .

d)

The maximal shear stress acts on a surface rotated by  $45^\circ$  from the  $x'_1$  and  $x'_3$  directions, related to eigen-directions of  $\sigma_I$  and  $\sigma_{III}$ , respectively, see sketch.

In this coordinate system, the normalized unit vector is obtained from the  $(1, 0, 1)$  direction,

$$\text{but still has to be normalized, so that: } \hat{n}^{\tau_{max}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This vector also can be obtained from:  $\hat{n}^{(I)'} + \hat{n}^{(III)'}$ , in the *same* coordinate system, with coordinate unit vectors:  $x'_1 = \hat{n}^{(I)'} = (1, 0, 0)$  and  $x'_3 = \hat{n}^{(III)'} = (0, 0, 1)$ , and normalization.  
*Note:* The solution is different in the original coordinate system, where the normal unit vectors (without prime) from answer b) are to be used.

e) Graphic/sketch

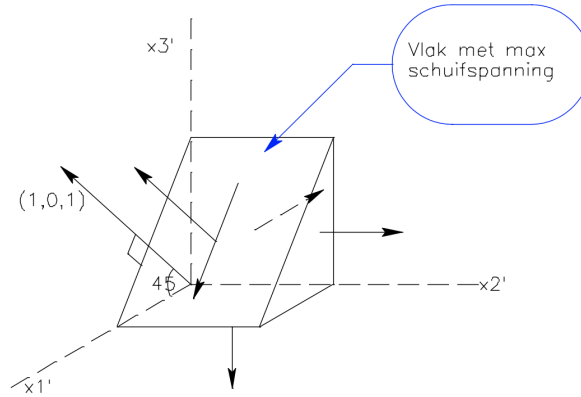


Figure 2: Sketch of the normal to the plane with maximal shear stress (in Dutch: “vlak met maximale schuifspanning”), in the coordinate system  $x'_p$  of the eigen-directions of stress, with perpendicular (sorted) intermediate stress eigen-direction  $x'_2$ .

## 5 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ( $E = 2 \cdot 10^5$  MPa,  $\nu = 0.25$ ) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian  $x_1 - x_2 - x_3$  coordinate system as:

$$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[ 20 + \alpha_1 \left( \frac{x_1}{L} \right) - 10 \left( \frac{x_2}{L} \right) + \alpha_2 \left( \frac{x_1}{L} \right)^2 \right]$$

$$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[ 10 + 8 \left( \frac{x_1}{L} \right) + \beta_1 \left( \frac{x_2}{L} \right) + \beta_2 \left( \frac{x_2}{L} \right)^2 \right]$$

$$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[ 12 - 10 \left( \frac{x_1}{L} \right) + 7 \left( \frac{x_2}{L} \right) - 8 \left( \frac{x_1}{L} \right) \left( \frac{x_2}{L} \right) \right]$$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$ , and  
with reference stress  $\sigma_0 = 1$  MPa and reference length  $L = 1$  m.

*Note: Question (a) is general, symbolic, with variables  $x_1, x_2, x_3$  and coefficients  $\alpha_1, \alpha_2, \beta_1, \beta_2$ ; only from question (b) on, use the single, chosen point  $P(x_1 = 0, x_2 = 0, x_3 = 0)$ .*

### Questions:

... based on section 3

a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the free coefficients  $\alpha_1, \alpha_2, \beta_1, \beta_2$  due to stress equilibrium?

b) Compute the eigen-stresses in point P using linear algebra, mathematics – not the graphical circle of Mohr procedure.

Describe and name the state of stress in point P (and in all other points in the body).

c) Compute the eigen-direction of the major eigen-stress.

d) Draw the relevant circle of Mohr and confirm graphically the results of (b) and (c); explain.

### Answers:

a)

Given was the plane stress-field, independent of  $x_3$ , in absence of body forces  $f_i = 0$ :

$$\begin{aligned}\sigma_{11}(x_1, x_2) &= \sigma_0 \left[ 20 + \alpha_1 \frac{x_1}{L} - 10 \frac{x_2}{L} + \alpha_2 \left( \frac{x_1}{L} \right)^2 \right] \\ \sigma_{22}(x_1, x_2) &= \sigma_0 \left[ 10 + 8 \frac{x_1}{L} + \beta_1 \frac{x_2}{L} + \beta_2 \left( \frac{x_2}{L} \right)^2 \right] \\ \sigma_{12}(x_1, x_2) &= \sigma_0 \left[ 12 - 10 \frac{x_1}{L} + 7 \frac{x_2}{L} - 8 \frac{x_1}{L} \frac{x_2}{L} \right]\end{aligned}$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\begin{aligned}\frac{d}{dx_1} \sigma_{11}(x_1, x_2) + \frac{d}{dx_2} \sigma_{12}(x_1, x_2) &= \sigma_0 \left[ \frac{\alpha_1}{L} + 2\alpha_2 \frac{x_1}{L^2} \right] + \sigma_0 \left[ \frac{7}{L} - 8 \frac{x_1}{L^2} \right] = 0 \\ \frac{d}{dx_1} \sigma_{12}(x_1, x_2) + \frac{d}{dx_2} \sigma_{22}(x_1, x_2) &= \sigma_0 \left[ \frac{-10}{L} - 8 \frac{x_2}{L^2} \right] + \sigma_0 \left[ \frac{\beta_1}{L} + 2\beta_2 \frac{x_2}{L^2} \right] = 0\end{aligned}$$

From these equations, one gets the coefficients that solve them:  $\alpha_1 = -7$ ,  $\alpha_2 = 4$ ,  $\beta_1 = 10$ ,  $\beta_2 = 4$ .

*Because the field equations must be valid for all constants and points  $x_1, x_2, x_3$ , independently, one can group them accordingly: The constant terms from the first and second equations provide  $\alpha_1$  and  $\beta_1$ , respectively, while the  $x_1$  and  $x_2$  groups provide  $\alpha_2$  and  $\beta_2$ .*

b)

The stress Tensor in point  $P = (x_1 = 0, x_2 = 0, x_3 = 0)$  is:  $[\sigma_{ij}] = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0.$$

Knowing/recognizing that one eigen-value is zero, i.e. also  $I_3 = 0$ , the principal stresses can be computed from the second order polynomial as:  $\sigma_I = 28$  MPa,  $\sigma_{II} = 2$  MPa,  $\sigma_{III} = 0$  MPa. This is a plane-stress state with all stresses on the  $x_3$ -surface equal to zero, which also has consequences for the eigen-directions ...

c)

The principal directions can be calculated the usual way, where  $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$  is directly visible from the tensor, due to the zero shear stresses in the  $x_3$ -direction.

The eigen-direction of the major stress  $\sigma_I = 28$  MPa is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that:  $-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$  and thus:  $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Rightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress,  $\sigma_{II} = 2$  MPa was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that:  $18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$  and thus:  $[(4/9) + 1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Rightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

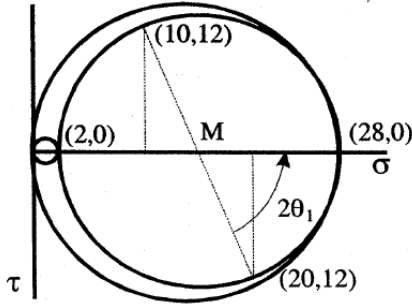


Figure 3: Sketch of a Mohr circle, focus is on the right, inner circle.

d)

Mohr's circle

Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is:  $M = \sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{20+10}{2} = 15$  MPa,

and its radius is:  $R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13$  MPa.

The eigenvalues are therefore:

$\sigma_I = M + R = 28$  MPa,  $\sigma_{II} = M - R = 2$  MPa.

The eigen-directions are:

$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{24}{10} = 2.4 \Rightarrow \theta_I = (1/2) \arctan(2.4) = 67.38^\circ/2 = 33.69^\circ$ , which corresponds to the orientation of the first eigenvector relative to the horizontal  $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$ ; and  $\theta_{II} = (180^\circ + 67.3^\circ)/2 = 247.3^\circ/2 = 123.7^\circ = \arccos(-2/\sqrt{13})$ .

The maximum shear stress is just the radius:  $\tau^{max} = R = 13$  MPa

## 6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

**Given:**

$$E = 2 \cdot 10^{11} \text{ Pa}, \nu = 0.25$$

$$\text{Stress-state in point P: } [\sigma] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} \text{ MPa}$$

**Questions:**

a) Show that the principal stresses are 8, 16 and 24 MPa.

Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.

**Answers:**

a)

From  $\det(\sigma_{ij} - \sigma \delta_{ij}) = 0$ , the characteristic equation follows as:

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = \sigma^3 - 48 \sigma^2 + 704 \sigma - 3072 = 0.$$

Given the eigenvalues,  $\sigma$ , one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*

Sorting the eigen-values is convention and part of the answer:

$$\sigma_I = 24 \text{ MPa}, \sigma_{II} = 16 \text{ MPa}, \text{ and } \sigma_{III} = 8 \text{ MPa}.$$

it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.

The eigen-direction of the minor eigen-stress,  $\sigma_{III} = 8 \text{ MPa}$  is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

$$\text{Subtracting line 2 from 1 yields: } n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$$

$$\text{Inserting into line 3 yields: } n_3 = \sqrt{6}n_1, \text{ so that: } \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$$

where the unknown  $c = 1/\sqrt{8} = \sqrt{2}/4$  is obtained from normalization, resulting in:

$$\Rightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$