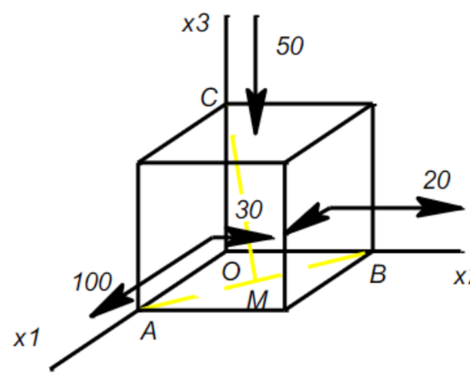
Tutorial T04 – Elasticity – stress and strain

Answer the following questions as they could come up in an exam.

1 Stress basics (geometry and stress vector)

... based on section 3.1 (Exercise V1 in old material before 2022)

Given:
$E = 200\,\text{GPa}$, $\nu = 0.25$
$OA = OB = a$ and $OC = \frac{1}{2}\sqrt{2}a$



Questions:

a) Find normal stress σ_{ABC} and shear stress τ_{ABC} acting on the area ABC.
b) What are the components of the strain-tensor ε_{ij}?
c) What are the eigen-strains?
d) In this stress-state, the maximal principal stress must not be larger than: 150 MPa.
Is this stress state allowed according to the hypotheses of Tresca and von Mises?

<u>**Answers:**</u>

a) The stress Tensor is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix}$ MPa,
given the arrows, using symmetry, direction of arrows (sign), and non-existing (zero).

First, find the normal to the plane: *this can be done by taking the cross-product of two line vectors (that describe a plane).*

$$\vec{AC} \times \vec{AB} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalizing the vector using the normality condition ($\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$), one can find a:

$$a^2 \left(1^2 + 1^2 + \sqrt{2}^2\right) = 1 \Longrightarrow a = \frac{1}{2}$$

After using the cross-product, with the vectors in random order we pay close attention to the fact that the normal is facing outside the plane. With the normal you indicate which side the material is. To make the normal point away from the material, we can choose a positive.

Cauchy: Stress or traction vector: $p_i = \sigma_{ij} n_j$, so that:

$$\longrightarrow [p] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = [\sigma] \begin{bmatrix} \hat{n_1} \\ \hat{n_2} \\ \hat{n_3} \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

The normal stress on the plane ABC is: $[\sigma] = [\hat{n}]^T \; [p] = 20\,\text{MPa}$

The shear stress on the plane ABC, using Pythagoras, is:

$$\tau^2 = p^2 - \sigma^2 = [p_1^2 + p_2^2 + p_3^2] - \sigma^2 = 6100 - 400 = 5700\,\text{MPa}^2,$$

so that $\tau = 75.5\,\text{MPa}$.

b)
Hooke's law for strain $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij})$, with $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, allows to obtain: $\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$, $\varepsilon_{12} = \frac{\sigma_{12}}{2G}$, with $G = \frac{E}{2(1+\nu)}$, and - similarly - the other components.

$$[\varepsilon] = \begin{bmatrix} 5.375 & 1.875 & 0 \\ 1.875 & 0.375 & 0 \\ 0 & 0 & -4 \end{bmatrix} 10^{-4} = \frac{1}{8} \begin{bmatrix} 43 & 15 & 0 \\ 15 & 3 & 0 \\ 0 & 0 & -32 \end{bmatrix} 10^{-4}$$

c)
Principal strains are computed, like for stress, solving:

$$\det(\varepsilon_{ij} - \varepsilon\delta_{ij}) = \begin{bmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon \end{bmatrix} = 0$$

$$\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$$

$$E_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = (14/8)\,10^{-4}$$

$$E_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2$$

$$= \left(\frac{1}{8}\,10^{-4}\right)^2 \left(43 \times 3 - 32 \times 3 - 32 \times 43 - 15^2 - 0 - 0\right) = -24.5\,10^{-8}$$

$$E_3 = \det(\varepsilon) = ... = 6\,10^{-12}$$

with solutions: $\varepsilon_I = 6\,10^{-4}$, $\varepsilon_{II} = -0.25\,10^{-4}$, $\varepsilon_{III} = -4\,10^{-4}$, sorted.

The third eigen-value can be read off directly from strain tensor ($\varepsilon_{III} = \varepsilon_{33}$, due to the zero values in rows and columns); the others still have to be found, from the characteristic equation (by decomposition or polynomial division), or from the invariants.

d)
To compute the allowable stress, we first need to compute (details not shown) the principal stresses: $\sigma_I = 110\,\text{MPa}$, $\sigma_{II} = 10\,\text{MPa}$, $\sigma_{III} = -50\,\text{MPa}$, sorted.

According to Tresca: $\sigma_{eq}^{Tresca} = \sigma_I - \sigma_{III} = 160\,\text{MPa}$.
According to von Mises: $\sigma_{eq}^{vonMises} = \sqrt{\frac{1}{2}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_I - \sigma_{III})^2 + (\sigma_{II} - \sigma_{III})^2\right]} = 140\,\text{MPa}$.
Allowed stress means: $\sigma_{eq} \leq \overline{\sigma} = 150\,\text{MPa}$. Thus von Mises is allowed, whereas Tresca is not.

2 Stress tensor basics

... based on sections 3.1-3.3. (Exercise V2 in old material before 2022)

Given:

- Linear elastic isotropic material with modulus $E = 2 \cdot 10^5$ N/mm^2
- The stress cube, below, in units of N/mm^2
- One principal stress is: $8\,\text{N/mm}^2$

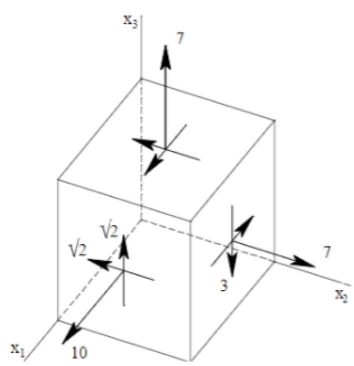


Figure 1: Stress cube → write down the stress matrix

Questions:
a) Find the other principal (eigen) stresses
b) Find the eigen-directions and plot these in a graph.
c) What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?
d) What are the equivalent stresses according to the hypotheses of Tresca and von Mises?

Answers:
a) The stress tensor from the cube is:

$$[\sigma_{ij}] = \begin{bmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{bmatrix} \text{MPa}$$

Note that the first index denotes the direction of the normal to the according surface on which this stress component works, while the second index gives the direction of the stress component.

Next, get the characteristic equation from:

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = \begin{bmatrix} 10-\sigma & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7-\sigma & -3 \\ \sqrt{2} & -3 & 7-\sigma \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

with invariants:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 24\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 176\text{MPa}^2$$

$$I_3 = \det(\sigma) = 384\text{MPa}^3$$

Here, the characteristic equation is not easily solvable; one way is to use the given eigen-value, $\sigma = 8\,N/mm^2$, and polynomial division (units dropped for simplicity, but must be added for final answer). Take the characteristic equation and divide by $(\sigma - 8)$:

$$
\begin{aligned}
&(\sigma^3 - 24\sigma^2 + 176\sigma - 384)\backslash(\sigma - 8) = \sigma^2 - 16\sigma + 48 \\
&\underline{(\sigma^3 - 8\sigma^2)} \\
&\quad -16\sigma^2 + 176\sigma - 384 \\
&\quad \underline{(-16\sigma^2 + 128\sigma)} \\
&\qquad\quad + 48\sigma - 384 \\
&\qquad\quad \underline{(+48\sigma - 384)} \\
&\qquad\quad \%
\end{aligned}
$$

The result is a second order polynomial, which can be solved as:
$\sigma_{1,2} = (16 \pm \sqrt{16^2 - 4 \times 48})/2 = 12$ and $4\,\text{MPa}$.

Therefore, the sorted eigen-values are: $\sigma_I = 12\,\text{MPa}$, $\sigma_{II} = 8\,\text{MPa}$, $\sigma_{III} = 4\,\text{MPa}$.

b) Direction of $\sigma_I = 12\text{MPa}$
There are various ways to solve for eigen-vectors, here is one example ...
Insert values, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$-2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - 5n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - 5n_3 = 0$$

The eigen-direction associated to the first, largest eigen-value:

$$\Longrightarrow \hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

Direction of $\sigma_{II} = 8$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - n_3 = 0$$

The eigen-direction associated to the second, intermediate eigen-value:

$$\Longrightarrow \hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$$

Direction of $\sigma_{III} = 4$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$6n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 + 3n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 + 3n_3 = 0$$

The eigen-direction associated to the third, smallest eigen-value:

$$\Longrightarrow \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

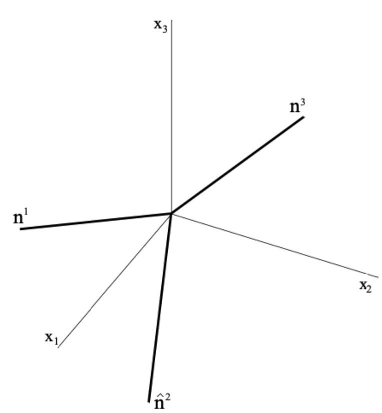


Figure 2: Sketch of the eigen-vectors (with coordinates and vectors)

The directions are unspecified, indicated by the plus-minus from taking a square-root; all three direction vectors are normalized (check it, if enough time in exam), $(n_i)^2 = 1$*; furthermore, all three normal (eigen) vectors must be pair-wise perpendicular on each other, i.e.* $n_i^{(a)} n_i^{(b)} = 0$*, for all* $a, b = I, II, III$ *with* $a \neq b$*. This perpendicularity allows to obtain, alternatively, one eigen-vector by a cross-product, e.g. above* $\hat{n}^{(III)} = \hat{n}^{(I)} \times \hat{n}^{(II)}$*.*

c)

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0.6 \times 10^{-4}$$

Since the material is isotropic:

$$\varepsilon_v = 3\varepsilon_{xx} = 0.6 \times 10^{-4}$$

$$\varepsilon_{xx} = 0.2 \times 10^{-4}$$

$$\sigma_{xx} = E\ \varepsilon_{xx} = 2 \times 10^5\ 0.2 \times 10^{-4} = 4\,\mathrm{MPa}$$

d)

$$\sigma_{Tresca} = \max\{|(\sigma_I - \sigma_{II})|, |(\sigma_{II} - \sigma_{III})|, |(\sigma_{III} - \sigma_I)|\} = \max\{4, 4, 8\} = 8\,\mathrm{MPa}$$

$$\sigma_{von-Mises} = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}} = 6.92\,\mathrm{MPa}$$

Then, Tresca is safer since it is larger and thus reaches the limit stress earlier.

3 Stress tensor basics

... based on sections 3.1-3.3. (Exercise V3 in old material before 2022)

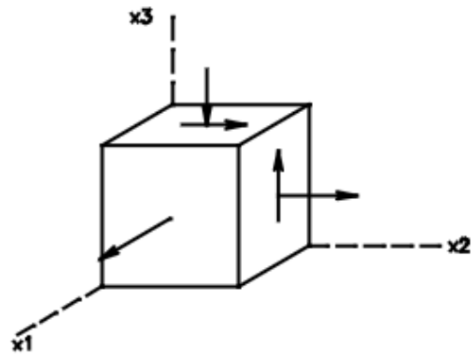


Figure 3: Stress cube, empty → fill it

Given:

The stress-state is described by the matrix: $\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix}$ N/mm^2,

with $E = 2 \cdot 10^5$ N/mm^2, and $\nu = 0.25$.

Questions:
a) Compute the principal stresses
b) Compute the eigen-directions
c) Compute the maximal shear-stress
d) Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x'_p, i.e. the coordinate system defined by the eigen-directions.
e) Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.
f) What is the strain in the direction of the normal vector from question d).

Answers:
a)
The sorted eigen-values are: $\sigma_I = 60\,\text{MPa}$, $\sigma_{II} = 40\,\text{MPa}$, $\sigma_{III} = -40\,\text{MPa}$.
The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).

b)
Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example σ_{II}, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad n_1^2+n_2^2+n_3^2=1$$

$$-20n_2 + 20\sqrt{3}n_3 = 0$$

$$20\sqrt{3}n_2 - 60n_3 = 0$$

$n_2 = \sqrt{3}n_3$ and $n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3$ (identical due to dependency)

$$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$$

$$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above. *The third eigenvalue calculation is similar (not shown).*

c)
The maximum shear stress is: $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50$ MPa.

d)
The maximal shear stress acts on a surface rotated by 45^o from the x'_1 and x'_3 directions, related to eigen-directions of σ_I and σ_{III}, respectively, see sketch.

In this coordinate system, the normalized unit vector is obtained from the $(1,0,1)$ direction, but has to be normalized, so that: $\hat{n}^{\tau_{max}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

e) Graphic/sketch

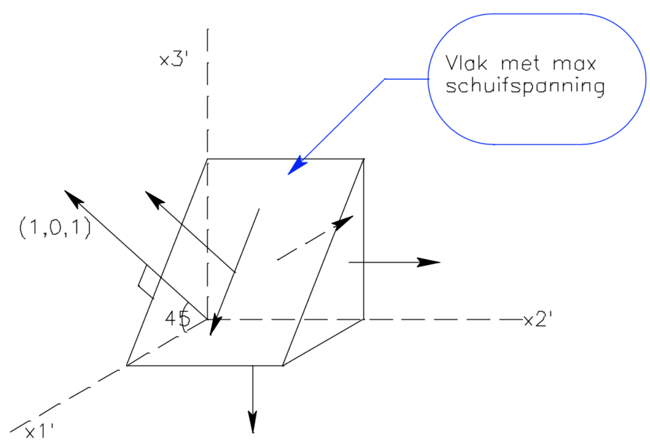


Figure 4: Sketch of the normal to the plane with maximal shear stress, in the coordinate system x'_p of the eigen-directions of stress, with perpendicular (sorted) intermediate stress eigen-direction x'_2.

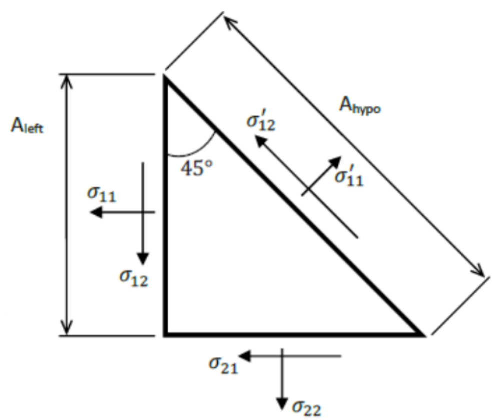
f)
The strain in the direction of the normal vector from question d) can be obtained in various ways. Here, we compute the eigen-strains directly from the eigen-stresses using the law of Hooke: $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right)$ where in the eigen-system, all non-diagonal elements are zero.

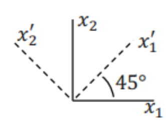
4 Stress tensor and transformation

... based on sections 3.1-3.4. (Exercise V10 in old material before 2022)

Given:

- A plane-stress state in a point P of a body with $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Given are these (mixed) stress components:
  $\sigma_{11} = 92$ MPa
  $\sigma'_{11} = 194$ MPa
  $\sigma'_{12} = -42$ MPa
  where the prime indicates the new (transformed) coordinate system.
- The material is linear elastic with $E = 2 \cdot 10^5$ MPa and $\nu = 0.25$.





Questions:
a) Give the stress tensor in the original $x_1x_2x_3$ system.
b) Give the stress tensor in the new $x'_1x'_2x'_3$ coordinate system, as obtained by a rotation of the coordinates about 45° around the x_3-axis, as sketched above.
c) Compute the eigen-stresses and the eigen-directions.

<u>**Answers:**</u>
a)
There are two ways to solve this problem. The triangle given represents all stresses on all sides, but only part of the stress components are known. By considering force equilibrium and using the respective stress components, divided by the side-lengths of the triangle (which also has a third dimension outside the plane, not shown). Assume the sides have unit-length, then the hypotenuse has, according to Pythagoras, length $\sqrt{2}$. Further assume the thickness also to be unit-length. The ratio between sides and hypotenuse is then:

$$\frac{A_l}{A_h} := \frac{A_{left}}{A_{hypo}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

.
With this we get:
Force balance in x_1 direction:

$$\begin{gathered}
A_h\sigma'_{11}\cos(45^o) - A_h\sigma'_{12}\sin(45^o) - A_l\sigma_{11} - A_l\sigma_{12} = 0 \\
\Rightarrow \sigma'_{11}\cos(45^o) - \sigma'_{12}\sin(45^o) - \frac{A_l}{A_h}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow (\sigma'_{11} - \sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = \sigma'_{11} - \sigma'_{12} - \sigma_{11} \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = 194 - (-42) - 92 = 144\,\text{MPa}.
\end{gathered}$$

Force balance in x_2 direction:

$$\begin{gathered}
A_h\sigma'_{11}\sin(45^o) + A_h\sigma'_{12}\cos(45^o) - A_l\sigma_{12} - A_l\sigma_{22} = 0 \\
\Rightarrow \sigma'_{11}\sin(45^o) + \sigma'_{12}\cos(45^o) - \frac{A_l}{A_h}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow (\sigma'_{11} + \sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow \sigma_{22} = \sigma'_{11} + \sigma'_{12} - \sigma_{12} \\
\Rightarrow \sigma_{22} = 194 + (-42) - 144 = 8\,\text{MPa}.
\end{gathered}$$

The stress tensor in the $x_1x_2x_3$ system is thus:

$$[\sigma_{ij}] = \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ \text{MPa}$$

b)
The stress tensor in the $x'_1x'_2x'_3$ system is obtained by rotation of the original system around 45^o, as sketched, in index notation, $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$, or:

$$[\sigma'] = [R]\,[\sigma]\,[R^T] = \begin{bmatrix} 194 & -42 & 0 \\ -42 & -94 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ \text{MPa},$$

using the transformation matrix:

$$[R] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The *alternative* way is to use the symbolic transformation rule and solve the system of equations, for each component, for the unknowns σ_{12}, σ_{22}, and σ'_{22}.

$$[\sigma'] = [R]\,[\sigma]\,[R^T] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 92 & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which allows to solve without geometry and force balance, after matrix multiplications:

$$[\sigma'] = (1/\sqrt{2}) \begin{bmatrix} 92+\sigma_{12} & \sigma_{12}+\sigma_{22} & 0 \\ -92+\sigma_{12} & -\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(1/2) \begin{bmatrix} 92+2\sigma_{12}+\sigma_{22} & -92+\sigma_{22} & 0 \\ -92+\sigma_{22} & 92-2\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which yields from the non-diagonal 12-component $\sigma_{22} = 8\,\mathrm{MPa}$. Inserted into the 11-component, one finds $\sigma_{12} = 144\,\mathrm{MPa}$, and all inserted into the 22-component results in $\sigma'_{22} = -94\,\mathrm{MPa}$. These results are identical to the above geometry and force balance considerations.

c)

The principal stresses and eigen-directions can now be computed the usual way from

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = 0 \ ,$$

and $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$.

c.1) This stress tensor describes a plane-stress state and thus has one eigenvalue $\sigma = 0$.
The remaining characteristic equation is:

$$\sigma^2 - 100\sigma + 736 - 144^2 = 0$$

with solutions: $\sigma_{1,2} = (100 \pm \sqrt{100^2 - 4(736 - 144^2)})/2 = (100 \pm \sqrt{9.10^4})/2 = 50 \pm 150\,\mathrm{MPa}$.

The sorted eigen-values are thus: $\sigma_I = 200\,\mathrm{MPa}$, $\sigma_{II} = 0\,\mathrm{MPa}$, $\sigma_{III} = -100\,\mathrm{MPa}$.

c.2) *Insert values, solve the system of equations, and normalize the solution.*

$$\begin{bmatrix} \sigma_{11}-\sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$\Longrightarrow$

$$(92 - 200)n_1 + 144n_2 + 0n_3 = 0$$

$$144n_1 + (8 - 200)n_2 + 0n_3 = 0$$

$\Longrightarrow$

$$-108n_1 + 144n_2 + 0n_3 = 0$$

$$144n_1 - 192n_2 + 0n_3 = 0$$

$\Longrightarrow$

$$n_2 = (108/144)n_1 = (3/4)n_1$$

$\Longrightarrow$

$$n_1^2 + n_2^2 = (1 + 9/16)n_1^2 = 1$$

$\Longrightarrow$

$$n_1 = \sqrt{16/25} = \pm 4/5 = \pm 0.8$$

The eigen-direction associated to the first, largest eigen-value:

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0.8 \\ 0.6 \\ 0 \end{bmatrix}$$

Similarly (no details given), the eigen-direction associated to the third, smallest eigen-value:

$$\hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0.6 \\ -0.8 \\ 0 \end{bmatrix}$$

and without calculation necessary (due to structure of the matrix), for the second, intermediate:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ($E = 2 \cdot 10^5$ MPa, $\nu = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1(\frac{x_1}{L}) - 10(\frac{x_2}{L}) + \alpha_2(\frac{x_1}{L})^2\right]$

$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8(\frac{x_1}{L}) + \beta_1(\frac{x_2}{L}) + \beta_2(\frac{x_2}{L})^2\right]$

$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10(\frac{x_1}{L}) + 7(\frac{x_2}{L}) - 8(\frac{x_1}{L})(\frac{x_2}{L})\right]$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$, and
with reference stress $\sigma_0 = 1\,\text{MPa}$ and reference length $L = 1\,\text{m}$.

Note: Question (a) is general, symbolic, with variables x_1, x_2, x_3 and coefficients α_1, α_2, β_1, β_2; only from question (b) on, use the single, chosen point $\mathrm{P}(x_1 = 0, x_2 = 0, x_3 = 0)$.

Questions:
... based on section 3
a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the free coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium?
b) Compute the eigen-stresses in point P using linear algebra, mathematics -– not circle of Mohr. Describe and name the state of stress in point P (and in all other points in the body).
c) Compute the eigen-direction of the major eigen-stress.
d) Draw the relevant circle of Mohr and confirm graphically the results of (b) and (c); explain.

Answers:
a)
Given was the plane stress-field, independent of x_3, in absence of body forces $f_i = 0$:

$$\begin{aligned}
\sigma_{11}(x_1, x_2) &= \sigma_0 \left[20 + \alpha_1 \frac{x_1}{L} - 10\frac{x_2}{L} + \alpha_2 \left(\frac{x_1}{L}\right)^2\right] \\
\sigma_{22}(x_1, x_2) &= \sigma_0 \left[10 + 8\frac{x_1}{L} + \beta_1 \frac{x_2}{L} + \beta_2 \left(\frac{x_2}{L}\right)^2\right] \\
\sigma_{12}(x_1, x_2) &= \sigma_0 \left[12 - 10\frac{x_1}{L} + 7\frac{x_2}{L} - 8\frac{x_1}{L}\frac{x_2}{L}\right]
\end{aligned}$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\begin{aligned}
\frac{d}{dx_1}\sigma_{11}(x_1, x_2) + \frac{d}{dx_2}\sigma_{12}(x_1, x_2) &= \sigma_0 \left[\frac{\alpha_1}{L} + 2\alpha_2 \frac{x_1}{L^2}\right] + \sigma_0 \left[\frac{7}{L} - 8\frac{x_1}{L^2}\right] = 0 \\
\frac{d}{dx_1}\sigma_{12}(x_1, x_2) + \frac{d}{dx_2}\sigma_{22}(x_1, x_2) &= \sigma_0 \left[\frac{-10}{L} - 8\frac{x_2}{L^2}\right] + \sigma_0 \left[\frac{\beta_1}{L} + 2\beta_2 \frac{x_2}{L^2}\right] = 0
\end{aligned}$$

From these equations, one gets the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$.

Because the field equations must be valid for all constants and points x_1, x_2, x_3, independently, one can group them accordingly: The constant terms from the first and second equations provide α_1 and β_1, respectively, while the x_1 and x_2 groups provide α_2 and β_2.

b)

The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\begin{bmatrix}\sigma_{ij}\end{bmatrix} = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0\,.$$

Knowing/recognizing that one eigen-value is zero, i.e. also $I_3 = 0$, the principal stresses can be computed from the second order polynomial as: $\sigma_I = 28\,\text{MPa}$, $\sigma_{II} = 2\,\text{MPa}$, $\sigma_{III} = 0\,\text{MPa}$. This is a plane-stress state with all stresses on the x_3-surface equal to zero, which also has consequences for the eigen-directions ...

c)
The principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$ is directly visible from the tensor, due to the zero shear stresses in the x_3-direction.

The eigen-direction of the major stress $\sigma_I = 28\,\text{MPa}$ is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that: $-8n_1^{(I)} + 12n_2^{(I)} = 0 \to n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \to n_2^{(I)} = 2/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress, $\sigma_{II} = 2\,\text{MPa}$ was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that: $18n_1^{(II)} + 12n_2^{(II)} = 0 \to n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9) + 1]n_2^{(II)} = 1 \to n_2^{(II)} = 3/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

d)
Mohr's circle
Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{20+10}{2} = 15\,\text{MPa}$,
and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13\,\text{MPa}$.

The eigenvalues are therefore:
$\sigma_I = M + R = 28\,\text{MPa}$, $\sigma_{II} = C - R = 2\,\text{MPa}$.

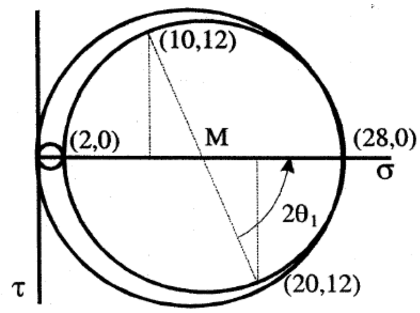


Figure 5: Sketch of a Mohr circle, focus is on the right, inner circle.

The eigen-directions are:
$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{24}{10} = 2.4 \implies \theta_I = (1/2)\arctan(2.4) = 67.38°/2 = 33.69°$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$; and $\theta_{II} = (180° + 67.3°)/2 = 247.3°/2 = 123.7° = \arccos(-2/\sqrt{13})$.

The maximum shear stress is just the radius: $\tau^{max} = R = 13$ MPa

6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:
$E = 2 \cdot 10^{11}$ Pa, $\nu = 0.25$

Stress-state in point P: $\left[\sigma\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa

Questions:
a) Show that the principal stresses are 8, 16 and 24 MPa.
Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.
... based on sections 4,5
b) Compute the volumetric (isotropic) strain.
c) What is the largest angle-change (not shear-strain) in P?
... based on section 5
d) Which material property is implicitly used/assumed in Hookes law?

Answers:
a)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0\,.$$

Given the eigenvalues, σ, one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*
Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 24\,\text{MPa}$, $\sigma_{II} = 16\,\text{MPa}$, and $\sigma_{III} = 8\,\text{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.
The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8\,\text{MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$

Inserting into line 3 yields: $n_3 = \sqrt{6}n_1$, so that: $\begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\, c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$

b)
For the volumetric (isotropic) strain, we can use the short-cut (not the full strain tensor), as: $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \left[\frac{1-2\nu}{E}\right] \sigma_{kk} = 12\ 10^{-5}$.

c)
The largest change of angle is: $\gamma_{max} = \frac{\tau_{max}}{G} = \frac{1}{2}\frac{\sigma_I - \sigma_{III}}{G} = 10^{-4}$, using $G = \frac{E}{2(1+\nu)} = (4/5)\ 10^5$ MPa, where the largest shear strain is just half of that: $\varepsilon_{max} = \gamma_{max}/2$.

d)
Isotropic (direction independent) material behavior is intrinsically assumed in the law of Hooke.

7 Displacement, strain and stress relation

... based on sections 3,4,5 (Exercise V7 in old material before 2022)

Given is the displacement field:
$u_1 = x_1 x_3$, $u_2 = -x_1 x_2$, and $u_3 = x_1^2 - x_3^2$
and material properties $E = 2\,\text{GPa}$ and $\nu = 0.25$
Consider the units, drop them in calculations to save space, but give them in end-results!

Questions:

a) Compute the stress tensor (components).
b) In the point $(x_1, x_2, x_3) = (0, 0, z_0)$ compute the eigen-stresses and maximum shear stress.

Answers:
a)
To compute the components of the stress tensor,
first, the strain tensor has to be computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

resulting in the components:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = x_3 \qquad \gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 - x_2 = -x_2$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -x_1 \qquad \gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 + 0 = 0$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -2x_3 \qquad \gamma_{31} = 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2x_1 + x_1 = 3x_1$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -x_1 - x_3$$

Next, obtain σ_{kk} from Hooke's law:

$$\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} + \nu\delta_{ij}\sigma_{kk}\}$$

$$\varepsilon_{kk} = \frac{1}{E}\{(1+\nu)\sigma_{kk} + \nu\delta_{jj}\sigma_{kk}\} \qquad \delta_{jj} = 1 + 1 + 1 = 3$$

$$\varepsilon_{kk} = \frac{1}{E}(1-2\nu)\sigma_{kk} \qquad \Longrightarrow \qquad \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Substitute σ_{kk} into Hooke's law and rearrange for σ_{ij}:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

Compute the terms for the stress tensor using the previously obtained equation:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\left[x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 + x_3) \\
\sigma_{22} &= \frac{E}{1+\nu}\left[-x_1 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-x_1 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-3x_1 - x_3) \\
\sigma_{33} &= \frac{E}{1+\nu}\left[-2x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-2x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 - 5x_3) \\
\sigma_{12} &= \frac{E}{1+\nu}\left[-\frac{1}{2}x_2\right] = -\frac{4}{5}x_2 \\
\sigma_{23} &= 0 \\
\sigma_{31} &= \frac{E}{1+\nu}\left[\frac{3}{2}x_1\right] = \frac{12}{5}x_1
\end{aligned}$$

b)
The principal stresses in point $(x, y, z) = (0, 0, z_0)$ are identical to the diagonal components, here only (if all shear stresses are zero); and the maximal shear stress τ_{max} are:

$$\begin{aligned}
\sigma_1 = \sigma_{11} &= \frac{4}{5}(-x_1 + x_3) = \frac{4}{5}z_0 & \sigma_{12} &= -\frac{4}{5}x_2 = 0 \\
\sigma_2 = \sigma_{22} &= \frac{4}{5}(-3x_1 - x_3) = -\frac{4}{5}z_0 & \sigma_{23} &= 0 \\
\sigma_3 = \sigma_{33} &= \frac{4}{5}(-x_1 - 5x_3) = -4z_0 & \sigma_{31} &= \frac{12}{5}x_1 = 0
\end{aligned}$$

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 2.4z_0 \ \mathrm{GPa\,m^{-1}}$$

for a given length z_0 with units [m].

8 Displacement, strain and stress equilibrium

... based on sections 3,4,5 (Exercise V8 in old material before 2022)

In a homogenous body made of a linear elastic, isotropic material, the displacement field is given:

$u_1 = \frac{p}{E} a \left[\frac{x_2}{a} + 2\frac{x_1 x_2}{a^2} - \frac{x_2^2}{a^2} \right]$

$u_2 = \frac{p}{E} a \left[\frac{x_1}{a} + \alpha\frac{x_1^2}{a^2} + \beta\frac{x_1 x_2}{a^2} - 2\frac{x_2^2}{a^2} \right]$

$u_3 = 0$

with coordinates x_1 and x_2, and constant coefficients p, E, a, and $\nu = 0.25$.

Questions:

In the absence of volume forces, compute the magnitude of the parameters α and β, using the information that the stress field is in mechanical equilibrium.

Answers:
a)
A homogeneous body made from a linear elastic isotropic material ($E = 2\,\mathrm{GPa}$ and $\nu = 0.25$) is in a deformed state in mechanical equilibrium, in the absence of a volume force, with parameters reference stress p, and length a, and unknown coefficients α and β, according to the displacement field:

$$
\begin{aligned}
u_1 &= \frac{pa}{E}\left[\left(\frac{x_2}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - \left(\frac{x_2}{a}\right)^2\right] \\
u_2 &= \frac{pa}{E}\left[\left(\frac{x_1}{a}\right) + \alpha\left(\frac{x_1}{a}\right)^2 + \beta\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - 2\left(\frac{x_2}{a}\right)^2\right] \\
u_3 &= 0
\end{aligned}
$$

The stress components, to be computed from Hooke's law, require the strain components, computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,.$$

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{pa}{E}\left[2\left(\frac{1}{a}\right)\left(\frac{x_2}{a}\right)\right] = \frac{p}{E}\left[2\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \frac{pa}{E}\left[\beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 4\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right] = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{33} &= 0
\end{aligned}
$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]$$

$$
\begin{aligned}
\gamma_{12} = 2\varepsilon_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\
&= \frac{pa}{E}\left[\left(\frac{1}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 2\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right) + \left(\frac{1}{a}\right) + 2\alpha\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) + \beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right)\right] \\
&= 2\frac{p}{E}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\gamma_{23} = 2\varepsilon_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \\
\gamma_{31} = 2\varepsilon_{31} &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0
\end{aligned}
$$

Obtain σ_{kk} with Hooke's law:

$$
\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} + \nu\delta_{ij}\sigma_{kk}\}
$$

$$
\begin{aligned}
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} + \nu\delta_{jj}\sigma_{kk}\} & & \delta_{jj} = 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} & \Longrightarrow \quad & \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}
$$

Substitute σ_{kk} into Hooke's and rearrange for σ_{ij}:

$$
\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]
$$

The stress-equilibrium equations (by partial differentiation) are then obtained using the previous equations:

$$
\begin{aligned}
\sigma_{ij} &= \frac{p}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right] \\
\sigma_{11} &= \frac{p}{1+\nu}\left[\left[2\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[2(1-3\nu)\left(\frac{x_2}{a}\right)\right] + \left[\nu\beta\left(\frac{x_1}{a}\right)\right]\right] \\
\sigma_{22} &= \frac{p}{1+\nu}\left[\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta(1-\nu)\left(\frac{x_1}{a}\right)\right] + \left[(-4+6\nu)\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{33} &= \frac{p}{1+\nu}\left[0 + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta\nu\left(\frac{x_1}{a}\right) - 2\nu\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{12} &= \frac{E}{1+\nu}\left[\varepsilon_{12}\right] = \frac{p}{1+\nu}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\sigma_{23} &= 0 \\
\sigma_{31} &= 0
\end{aligned}
$$

In absence of a volume force, the three equilibrium equations for $j = 1, 2, 3$ are:

$$
\sigma_{ij,i} + 0 = (\sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3}) + 0 = 0
$$

Using partial differentiation:

$$\sigma_{11,1} = \frac{\partial}{\partial x_1}\sigma_{11} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] \qquad \sigma_{12,1} = \frac{\partial}{\partial x_1}\sigma_{12} = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]; \quad \sigma_{13,1} = 0$$
$$\sigma_{22,2} = \frac{\partial}{\partial x_2}\sigma_{22} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] \quad \sigma_{23,2} = \frac{\partial}{\partial x_2}\sigma_{23} = 0; \quad \sigma_{21,2} = \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right]$$
$$\sigma_{33,3} = \frac{\partial}{\partial x_3}\sigma_{33} = 0 \qquad \sigma_{31,3} = \frac{\partial}{\partial x_3}\sigma_{31} = 0; \quad \sigma_{32,3} = 0$$

Computing the three equations:

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0 = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right] + 0 \tag{1}$$
$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} = 0 = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] + 0 \tag{2}$$
$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0 \tag{3}$$

Inserting the value for $\nu = 0.25$:

$$(1) \implies \frac{p}{\frac{5}{4}}\left[\frac{1}{2}\beta + \left(\frac{\beta}{2}-1\right)\right]\left(\frac{1}{a}\right) = 0 \qquad \implies \qquad p\frac{4}{5}\left[\beta - 1\right]\left(\frac{1}{a}\right) = 0$$
$$(2) \implies \frac{p}{\frac{5}{4}}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{\frac{5}{4}}\cdot 2\left[(-\frac{5}{2})\left(\frac{1}{a}\right)\right] = 0 \qquad \implies \qquad p\frac{4}{5}\left[(\alpha+1) - 5\right]\left(\frac{1}{a}\right) = 0$$

Solving for the two unknowns (α, β):

$$(1) \implies \beta = 1$$
$$(2) \implies \alpha = 4$$

This allows for mechanical equilibrium in all points in the body, because the equilibrium equations do not depend on the position.

9 Displacement, strain, stress

... based on sections 3,4,5 (Exercise V9 in old material before 2022)

Within a homogeneous body made of a linear elastic, isotropic material the displacement field:

$$u_1 = \frac{1}{3}(1-2\nu)x_1^3 - (3-2\nu)x_1x_2^2 - 3x_2 - 3x_3$$

$$u_2 = (1-2\nu)x_1^2x_2 + \frac{1}{3}(1+2\nu)x_2^3 + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

the elasticity modulus E, and the Poisson-ratio ν are given.

Questions:

a) Compute the components of the strain tensor.

... based on section 5.1
b) Compute the components of the stress tensor.

... based on section 3,5.1
c) Confirm that the stress-field is in equilibrium in the absence of volume forces.

Answers:
a)
...

10 Strain-stress relations

... based on sections 4,5 (Exercise V5 in old material before 2022)

In a Cartesian coordinate system, at point P, the strain tensor is given as:

$$[\varepsilon_{ij}] = \frac{5}{8}\begin{bmatrix} -1 & -15 & 5\sqrt{2} \\ -15 & -1 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 14 \end{bmatrix} \; 10^{-5}$$

Questions:

a) For a material with modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$, compute the eigen-stresses and the eigen-directions.
b) Explain/argue why the eigen-directions of stress and strain are identical for a homogeneous, isotropic material.

Answers:
a)
First, the stress tensor is determined using Hooke's law $\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$.
Insert and solve for stress in the normal way ... for each component.
For example:

$\sigma_{12} = \frac{E}{1+\nu}(\epsilon_{12}) = -15\,\text{MPa}$,
$\sigma_{11} = \frac{E}{1+\nu}\left(\epsilon_{11} + \frac{\nu}{1-2\nu}\epsilon_{kk}\right) = 5\,\text{MPa}$, with $\epsilon_{kk} = (5/8)\,12\,10^{-5}$,
etc.

Another way to do this (often done for anisotropic materials and in finite element implementations – not needed for linear isotropic elasticity, to solve this rather simple problem, just for completeness) is to assemble the independent stress- and strain-tensor components in vectors and express the corresponding stiffness-matrix in moduli, using the Lamé-coeffcients (for brevity):

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad\qquad 2\mu = 2G = \frac{E}{1+\nu}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

$$[\sigma_{ij}] = \begin{bmatrix} 5 & -15 & 5\sqrt{2} \\ -15 & 5 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 20 \end{bmatrix} \text{MPa}$$

From the stress, the characteristic equation, invariants, and principal stresses can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 30\,\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = -100\,\text{MPa}^2$$

$$I_3 = \det(\sigma) = -3000\,\text{MPa}^3$$

In this case, solution by decomposition is possible, as:

$$\sigma^2\,(\sigma - 30) - 100\,(\sigma - 30) = 0$$

Therefore: $\sigma_I = 30\text{MPa}$, $\sigma_{II} = 10\text{MPa}$, $\sigma_{III} = -10\text{MPa}$.

Principal directions can be calculated as:
Direction of $\sigma_I = 30\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{II} = 10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{III} = -10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}$$

b)
For an isotropic material, one has the eigendirections of stress-and strain identical. Note that the term with δ_{ij} in the law of Hooke has no direction (is isotropic = direction-independent); thus the direction is carried by the terms ε_{ij} and σ_{ij} and thus the directions are equal for stress and strain.