Materials: Elasticity Theory

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University of Twente

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Elasticiteitstheorie – Elasticity Theory
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script & sheets, based on:
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Overview:

Index notation, matrix-vector notation, vectors and tensors
Stress, strain, (material behavior: elasticity, plasticity, viscosity)

1) Index notation and summation convention

Set of quantities: $\quad a_1, a_2, a_3, \cdots, a_n \longrightarrow$
$\qquad\qquad a_i$, with $i = 1, 2, 3, \cdots, n$

In 2- of 3-dim. space a_i is the ith component
of the vector $\underset{\sim}{a}$, with $n = 2$ or 3

also: a_i is the ith element of the (column)-vector $\{a\}$

$\qquad\qquad\qquad\qquad$ (just a column of numbers?)

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Difference:

vector as geometric, physical quantity
(e.g. distance or velocity with norm [unit] and direction)

$\underset{\sim}{a} = \vec{a}$ - or vector as matrix with one column $\{a\}$

summation convention
inner product (scalar product) of two vectors:

$$\{x\}^{\mathrm{T}}\{y\} = x_1 \, y_1 + x_2 \, y_2 + x_3 \, y_3 + \cdots + x_n \, y_n = \sum_{i=1}^{n} x_i \, y_i$$

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shorter (Einstein) notation:

$$\{x\}^{\mathrm{T}}\{y\} = (\underline{x}\cdot\underline{y}=)\; x_i\,y_i = x_p\,y_p = x_q\,y_q$$

where i, p and q are so-called *dummy-indices*.
Those are double, in products, and have to be summed over.

So called *free indices* occur only once in product-terms.

System of equations:

$$
\begin{aligned}
y_1 &= A_{11}\, x_1 &+ A_{12}\, x_2 + \cdots &+ A_{1n}\, x_n \\
y_2 &= A_{21}\, x_1 &+ A_{22}\, x_2 + \cdots &+ A_{2n}\, x_n \\
&\;\;\vdots &\vdots \qquad \vdots &\qquad\quad \vdots \\
y_m &= A_{m1}\, x_1 &+ A_{m2}\, x_2 + \cdots &+ A_{mn}\, x_n
\end{aligned}
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\end{aligned}
$$

The ith equation is:

$$
y_i = A_{i1}\,x_1 + A_{i2}\,x_2 + \cdots + A_{in}\,x_n = \left(\sum_{j=1}^{n}\right) A_{ij}\,x_j
$$

System of equations:

$$y_1 = A_{11} x_1 + A_{12} x_2 + \cdots + A_{1n} x_n$$
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$$\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots$$
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$$y_i = A_{i1} x_1 + A_{i2} x_2 + \cdots + A_{in} x_n = \left(\sum_{j=1}^{n} \right) A_{ij} x_j$$

shortly: $\quad y_i = A_{ij} x_j \quad \begin{cases} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{cases}$

or $\quad \{y\} = [A] \{x\}$

or $\quad \underline{y} = \underline{\underline{A}} \cdot \underline{x}$

Matrix $[A]$ with entries A_{ij}
(where i is the row-index, and j is the column-index)

Example: $S = \{x\}^{\mathrm{T}} \cdot \{x\} = \left(\sum_{i=1}^{n} \right) x_i \, x_i$

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Example: $S = \{x\}^{\mathrm{T}} \cdot \{x\} = \left(\displaystyle\sum_{i=1}^{n}\right) x_i\, x_i$

Kronecker delta: $\quad \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

Thus: $\quad S = x_i\, x_i = \delta_{ij}\, x_i\, x_j = \{x\}^{\mathrm{T}}[I]\,\{x\}$

Also: $\quad x_i = \delta_{ij}\, x_j = \delta_{ip}\, x_p$

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Also: $x_i = \delta_{ij}\, x_j = \delta_{ip}\, x_p$

BUT: $\delta_{ii} \neq 1$

instead, note: $\delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{nn} = 1 + 1 + \cdots + 1 = n$

Differentiation:

Consider a function: $F(x_1, x_2, \cdots, x_n)$

Partial derivative: $\frac{\partial F}{\partial x_i} = F_{,i}$

The index $,i$ means the partial derivative with respect to x_i!

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Total differential of F:

$$\mathrm{d}F = \frac{\partial F}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial F}{\partial x_2}\,\mathrm{d}x_2 + \cdots + \frac{\partial F}{\partial x_n}\,\mathrm{d}x_n = \left(\sum_{i=1}^{n}\right)\frac{\partial F}{\partial x_i}\,\mathrm{d}x_i = F_{,i}\,\mathrm{d}x_i$$

2) Scalars, vectors and Cartesian tensors

Scalar fields, functions of position, only a number with [unit].
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Scalar fields, functions of position, only a number with [unit].
e.g., density, temperature, ...

Vector fields, functions of position, quantity with norm [unit]
and a direction, e.g., force, displacement, velocity, acceleration, ...

Cartesian coordinate system

$$\underset{\sim}{a} = a_x \, \hat{\underset{\sim}{e}}_x + a_y \, \hat{\underset{\sim}{e}}_y + a_z \, \hat{\underset{\sim}{e}}_z$$

$$= a_1 \, \hat{\underset{\sim}{e}}_1 + a_2 \, \hat{\underset{\sim}{e}}_2 + a_3 \, \hat{\underset{\sim}{e}}_3$$

$$= \sum_{i=1}^{3} a_i \, \hat{\underset{\sim}{e}}_i = a_i \, \hat{\underset{\sim}{e}}_i$$

... alternatives ...

$$= a_1 \, \hat{\underset{\sim}{x}} + a_2 \, \hat{\underset{\sim}{y}} + a_3 \, \hat{\underset{\sim}{z}}$$

$$= a_1 \, \underset{\sim}{i} + a_2 \, \underset{\sim}{j} + a_3 \, \underset{\sim}{k}$$

$$\underset{\sim}{a} = a_x \,\hat{\underset{\sim}{e}}_x + a_y \,\hat{\underset{\sim}{e}}_y + a_z \,\hat{\underset{\sim}{e}}_z$$

$$= a_1 \,\hat{\underset{\sim}{e}}_1 + a_2 \,\hat{\underset{\sim}{e}}_2 + a_3 \,\hat{\underset{\sim}{e}}_3$$

$$= \sum_{i=1}^{3} a_i \,\hat{\underset{\sim}{e}}_i = a_i \,\hat{\underset{\sim}{e}}_i$$

Norm (length) of $\underset{\sim}{a}$:

$$|\underset{\sim}{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i \, a_i}$$

inner product
(scalar product, "dot"-product):

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a\, b\, \cos \theta$$

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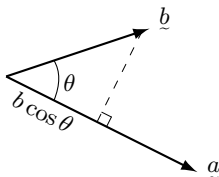
Also:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = a_i\, b_i$$

inner product

(scalar product, "dot"-product):

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a\, b\, \cos\theta$$

Also:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = a_i\, b_i$$

For unit-direction-vectors:

$$\hat{\underset{\sim}{e}}_x \cdot \hat{\underset{\sim}{e}}_x = \cdots = 1 \qquad \text{or} \qquad \hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_1 = \hat{\underset{\sim}{e}}_2 \cdot \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{e}}_3 \cdot \hat{\underset{\sim}{e}}_3 = 1$$

$$\hat{\underset{\sim}{e}}_x \cdot \hat{\underset{\sim}{e}}_y = \cdots = 0 \qquad \text{or} \qquad \hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{e}}_2 \cdot \hat{\underset{\sim}{e}}_3 = \hat{\underset{\sim}{e}}_3 \cdot \hat{\underset{\sim}{e}}_1 = 0$$

$$\Rightarrow \hat{\underset{\sim}{e}}_i \cdot \hat{\underset{\sim}{e}}_j = \delta_{ij} \qquad \text{in a way that e.g.:}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i\, \hat{\underset{\sim}{e}}_i \cdot b_j\, \hat{\underset{\sim}{e}}_j = a_i\, b_j\, \hat{\underset{\sim}{e}}_i \cdot \hat{\underset{\sim}{e}}_j = a_i\, b_j\, \delta_{ij} = a_i\, b_i$$

Outer product or *vector-product*

("cross"-product)

$$\underset{\sim}{a} * \underset{\sim}{b} = (|\underset{\sim}{a}|\,|\underset{\sim}{b}|\,\sin\theta)\ \hat{\underset{\sim}{e}}_c = \underset{\sim}{c}$$

Outer product or *vector-product*

("cross"-product)

$$\underset{\sim}{a} * \underset{\sim}{b} = \underset{\sim}{a} \times \underset{\sim}{b} = (|\underset{\sim}{a}| \, |\underset{\sim}{b}| \, \sin\theta) \; \hat{\underset{\sim}{e}}_c = \underset{\sim}{c}$$

Also:

$$\underset{\sim}{a} * \underset{\sim}{b} = \det\left(\begin{bmatrix} \hat{\underset{\sim}{e}}_1 & \hat{\underset{\sim}{e}}_2 & \hat{\underset{\sim}{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right) =$$

$$= (a_2 \, b_3 - a_3 \, b_2) \, \hat{\underset{\sim}{e}}_1 + (a_3 \, b_1 - a_1 \, b_3) \, \hat{\underset{\sim}{e}}_2 + (a_1 \, b_2 - a_2 \, b_1) \, \hat{\underset{\sim}{e}}_3 =$$

$$= c_1 \, \hat{\underset{\sim}{e}}_1 + c_2 \, \hat{\underset{\sim}{e}}_2 + c_3 \, \hat{\underset{\sim}{e}}_3 = c_i \, \hat{\underset{\sim}{e}}_i = \underset{\sim}{c}$$

Outer product or *vector-product*

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$$\underset{\sim}{a} * \underset{\sim}{b} = \underset{\sim}{a} \times \underset{\sim}{b} = (|\underset{\sim}{a}|\,|\underset{\sim}{b}|\,\sin\theta)\ \hat{e}_c = \underset{\sim}{c}$$

Also:

$$\underset{\sim}{a} * \underset{\sim}{b} = \det\left(\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right) =$$
$$= (a_2\,b_3 - a_3\,b_2)\,\hat{e}_1 + (a_3\,b_1 - a_1\,b_3)\,\hat{e}_2 + (a_1\,b_2 - a_2\,b_1)\,\hat{e}_3 =$$
$$= c_1\,\hat{e}_1 + c_2\,\hat{e}_2 + c_3\,\hat{e}_3 = c_i\,\hat{e}_i = \underset{\sim}{c}$$

permutation symbol: $\quad c_i = \varepsilon_{ijk}\,a_j\,b_k \quad (\underset{\sim}{c} = \varepsilon_{ijk}\,a_j\,b_k\,\hat{e}_i) \quad$ with:
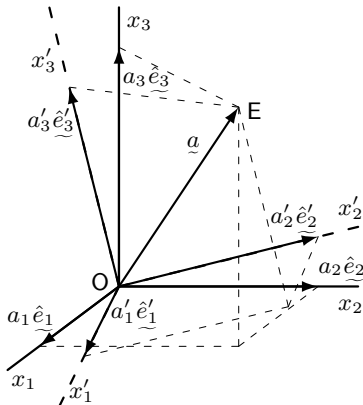
$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1\ ;\ \varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1\ ;\ \text{note:}\ \varepsilon_{ijk} = 0$$

Rotation of axes: (Cartesian coordinates)
Rotations from "old" x_i-coordinates
to "new" x'_p-coordinates.
For an arbitrary vector holds:

$$\underset{\sim}{a} = a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3 =$$
$$= a'_1\,\underset{\sim}{\hat{e}'_1} + a'_2\,\underset{\sim}{\hat{e}'_2} + a'_3\,\underset{\sim}{\hat{e}'_3}$$

Rotation of axes: (Cartesian coordinates)
Rotations from "old" x_i-coordinates
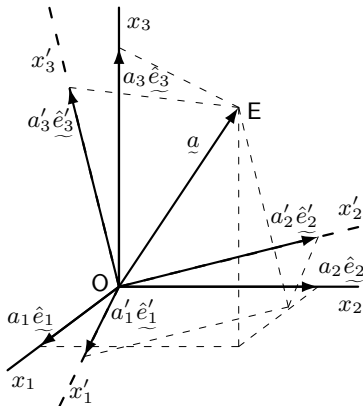to "new" x'_p-coordinates.
For an arbitrary vector holds:

$$\underset{\sim}{a} = a_1 \, \underset{\sim}{\hat{e}_1} + a_2 \, \underset{\sim}{\hat{e}_2} + a_3 \, \underset{\sim}{\hat{e}_3} =$$
$$= a'_1 \, \underset{\sim}{\hat{e}'_1} + a'_2 \, \underset{\sim}{\hat{e}'_2} + a'_3 \, \underset{\sim}{\hat{e}'_3}$$

In index notation: $\quad \underset{\sim}{a} = a_i \, \underset{\sim}{\hat{e}_i} = a'_p \, \underset{\sim}{\hat{e}'_p}$

Direction-cosines:

$R_{pi} = \cos(x'_p, x_i) \quad \Rightarrow$

$a'_1 = a_1 \, \cos(x'_1, x_1) + a_2 \, \cos(x'_1, x_2) + a_3 \, \cos(x'_1, x_3) =$
$\quad = a_i \, \cos(x'_1, x_i) = R_{1i} \, a_i \qquad$ (change now $1 \rightarrow p$)

For $p = 1, 2, 3$:

$$a'_p = a_1 \cos(x'_p, x_1) + a_2 \cos(x'_p, x_2) + a_3 \cos(x'_p, x_3) =$$
$$= a_i \cos(x'_p, x_i) = R_{pi}\, a_i$$

Summation over *second* index of R.

For $p = 1, 2, 3$:

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$$= a_i \cos(x'_p, x_i) = R_{pi}\, a_i$$

Summation over *second* index of R.
Inverse relation, from "new" to "old":

$$a_j = a'_1 \cos(x_j, x'_1) + a'_2 \cos(x_j, x'_2) + a'_3 \cos(x_j, x'_3) =$$
$$= a'_q \cos(x_j, x'_q) = R_{qj}\, a'_q$$

Note: summation over the *first* index of R!

For $p = 1, 2, 3$:

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$$= a'_q \cos(x_j, x'_q) = R_{qj} \, a'_q$$

Note: summation over the *first* index of R!
If this transformation rule holds, it is a vector (line with a length and direction): a so-called first-order-tensor!

Written in matrix notation $a'_p = R_{pi} a_i$ becomes :

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In *matrix-vector-notation* :
$$\{a'\} = [R]\{a\}$$

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The inverse transform $a_j = R_{qj}\, a'_q$ in matrix notation:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \quad \begin{array}{l} \text{In } matrix\text{-}vector\text{-}notation: \\ \{a\} = [R]^{\mathrm{T}}\{a'\} \end{array}$$

With $[R]^{\mathrm{T}}$ being the transposed of $[R]$. Matrix mirroring with respect to the main diagonal (top-left $\rightarrow$ bottom-right).

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In $matrix\text{-}vector\text{-}notation$:
$$\underline{a}' = \underline{\underline{R}} \cdot \underline{a}$$

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$$\underline{a} = \underline{\underline{R}}^{\mathrm{T}} \cdot \underline{a}'$$

With $\underline{\underline{R}}^{\mathrm{T}}$ being the transposed of $[R]$. Matrix mirroring with respect to the main diagonal (top-left $\rightarrow$ bottom-right).

Properties of the (transformation matrix) rotation matrix $[R]$.
Substitute $\{a\} = [R]^{\mathrm{T}}\{a'\}$ in $\{a'\} = [R]\{a\}$. This gives:
$$\{a'\} = [R]\{a\} = [R][R]^{\mathrm{T}}\{a'\} \Longrightarrow [R][R]^{\mathrm{T}} = [I]$$

Also: $[R][R]^{-1} = [I]$ (definition of inverted matrix), so:
$[R]^{\mathrm{T}} = [R]^{-1}$ Which is called an *orthogonal* matrix.

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And $[R][R]^{\mathrm{T}} = [I]$:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ *Normality condition* and *orthogonality condition*

Orthogonality rotation matrix $[R]$ using index notation:

Derived before: $\quad a'_p = R_{pi}\, a_i \quad$ and $\quad a_j = R_{qj}\, a'_q$

So it holds: $\quad a'_p = R_{pi}\, a_i = R_{pi}\, R_{qi}\, a'_q \quad$ also: $\quad a'_p = \delta_{pq}\, a'_q$

From which follows: $\quad \boxed{R_{pi}\, R_{qi} = \delta_{pq}} \quad$ or $\quad \boxed{R_{ip}\, R_{jp} = \delta_{ij}}$

Equivalent with: $\quad [R]\,[R]^{\mathrm{T}} = [I]$

(Note: $\quad [A]\,[B] = [C] \quad \Longleftrightarrow \quad A_{ip}\, B_{pj} = C_{ij}$)

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Equivalent with: $\quad [R]\,[R]^{\mathrm{T}} = [I]$

(Note: $[A]\,[B] = [C] \iff A_{ip}\, B_{pj} = C_{ij}$)

Reverse: $\quad a_j = R_{qj}\, a'_q = R_{qj}\, R_{qi}\, a_i \quad$ also: $\quad a_j = \delta_{ij}\, a_i$

From which follows: $\quad \boxed{R_{qi}\, R_{qj} = \delta_{ij}} \quad$ or $\quad \boxed{R_{pi}\, R_{pj} = \delta_{ij}}$

Equivalent with: $\quad [R]^{\mathrm{T}}\,[R] = [I]$

Second order tensor

Consider 2 vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ with components a_i and b_j

Dyadic product (per definition): $\quad \underline{S} = \underset{\sim}{a}\,\underset{\sim}{b} \quad$ with $\quad S_{ij} = a_i\,b_j$

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Rotation of axis: $\quad S'_{ij} = a'_i\,b'_j$

Substitute transformation rule for vectors:

$S'_{ij} = a'_i\,b'_j = R_{ip}\,a_p\,R_{jq}\,b_q = R_{ip}\,R_{jq}\,a_p\,b_q = R_{ip}\,R_{jq}\,S_{pq}$

Second order tensor

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Quantities that satisfy this transformation rule are called *second order tensors*. NOTE, not every second order tensor can be written as a dyadic product of two vectors.

Second order tensor: $\quad S'_{ij} = R_{ip}\,R_{jq}\,S_{pq}$

Can be written as: $\quad S'_{ij} = R_{ip}\,S_{pq}\,R_{jq} = R_{ip}\,S_{pq}\,R^{\mathrm{T}}_{qj}$

In matrix-vector-notation: $\quad [S'] = [R]\,[S]\,[R]^{\mathrm{T}}$

Second order tensor: $\quad S'_{ij} = R_{ip}\,R_{jq}\,S_{pq}$
Can be written as: $\quad S'_{ij} = R_{ip}\,S_{pq}\,R_{jq} = R_{ip}\,S_{pq}\,R^{\mathrm{T}}_{qj}$
In matrix-vector-notation: $\quad [S'] = [R]\,[S]\,[R]^{\mathrm{T}}$

Inverse: $\quad S_{ij} = R_{pi}\,R_{qj}\,S'_{pq} = R^{\mathrm{T}}_{ip}\,S'_{pq}\,R_{qj}$
In matrix-vector-notation: $\quad [S] = [R]^{\mathrm{T}}\,[S']\,[R]$

<u>Overview tensors</u>   ($\Rightarrow$ means transforms to)
0. If $\Phi(x_r) \Rightarrow \Phi'(x_r)$ by $\Phi = \Phi'$
then *zero order tensor field* or *scalar field*

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then *second order tensor field*

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then *second order tensor field*

3. If $S_{ijk}(x_r) \; \Rightarrow \; S'_{pqt}(x_r)$ by $S'_{pqt} = R_{pi}\, R_{qj}\, R_{tk}\, S_{ijk}$
then *third order tensor field*

4. etc. etc. etc.

3) Stresses
3.1) Stress state in a point

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Per definition, stress vector $\underset{\sim}{p} = \lim\limits_{\Delta A \to 0} \left(\dfrac{\Delta \underset{\sim}{F}}{\Delta A} \right)$

A few remarks:

▶ A stress vector $\underset{\sim}{p}$ has in general an arbitrary orientation

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▶ A stress vector $\underset{\sim}{p}$ has in general an arbitrary orientation

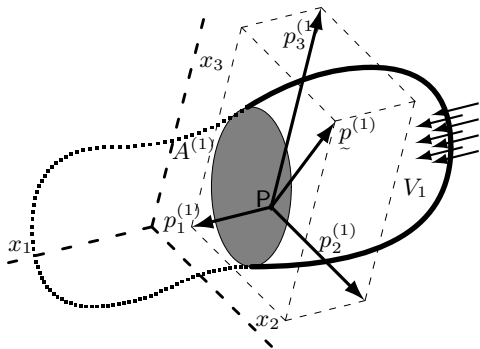▶ The magnitude and direction of $\underset{\sim}{p}$ is depending on the location of P and the orientation of ΔA

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- A stress vector $\underset{\sim}{p}$ can be decomposed in a normal and a tangential component w.r.t. the orientation of ΔA
- A stress vector $\underset{\sim}{p}$ can be decomposed along a Cartesian coordinate system $\quad \underset{\sim}{p} = p_i \, \underset{\sim}{\hat{e}}_i$

Cartesian
stress- components σ_{ij}:

$$\sigma_{11} = p_1^{(1)}$$
$$\sigma_{12} = p_2^{(1)}$$
$$\sigma_{13} = p_3^{(1)}$$

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Cross-section
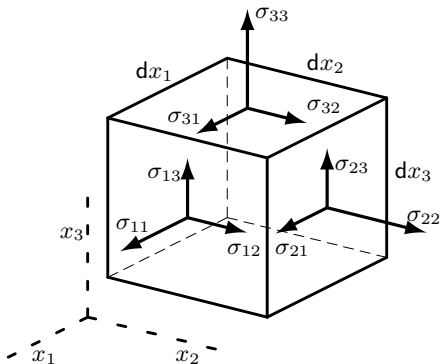
perpendicular to x_2-axis:

$$\sigma_{21} = p_1^{(2)} \qquad \sigma_{22} = p_2^{(2)} \qquad \sigma_{23} = p_3^{(2)}$$

Cross-section perpendicular to x_3-axis:

$$\sigma_{31} = p_1^{(3)} \qquad \sigma_{32} = p_2^{(3)} \qquad \sigma_{33} = p_3^{(3)}$$

Where
σ_{ij} is the stress component
in the directions of x_j acting
on a surface with its normal
pointing in x_i-direction.

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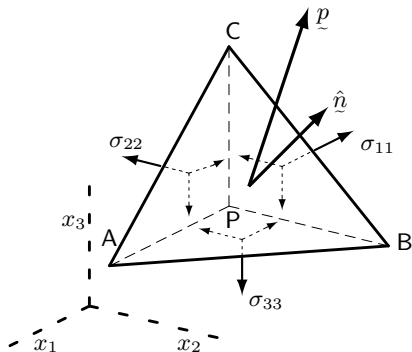

Displayed in a stress matrix:

$$[\sigma] = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{array} \right] = \left[\begin{array}{ccc} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{array} \right]$$
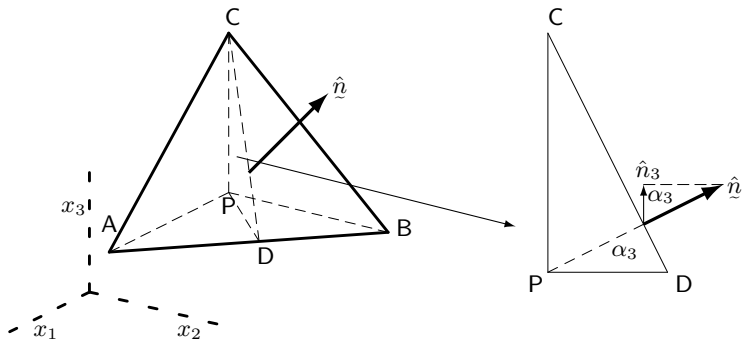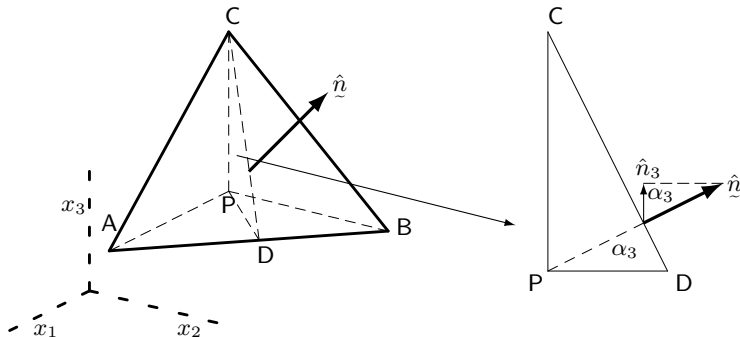
3.2) Stress-vector on arbitrary cross-section

3.2) Stress-vector on arbitrary cross-section

Equilibrium: $p_1 \, \Delta A_{\text{ABC}} - \sigma_{11} \, \Delta A_{\text{PBC}} - \sigma_{21} \, \Delta A_{\text{PCA}} - \sigma_{31} \, \Delta A_{\text{PAB}} = 0$

$\Rightarrow p_1 = \sigma_{11} \dfrac{\Delta A_{\text{PBC}}}{\Delta A_{\text{ABC}}} + \sigma_{21} \dfrac{\Delta A_{\text{PCA}}}{\Delta A_{\text{ABC}}} + \sigma_{31} \dfrac{\Delta A_{\text{PAB}}}{\Delta A_{\text{ABC}}}$

Now it holds:
$$\frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}} = \frac{\frac{1}{2}\,|\mathsf{PD}|\,|\mathsf{AB}|}{\frac{1}{2}\,|\mathsf{CD}|\,|\mathsf{AB}|} = \frac{|\mathsf{PD}|}{|\mathsf{CD}|} = \cos\alpha_3 = \frac{\hat{n}_3}{|\hat{\underset{\sim}{n}}|} = \hat{n}_3 \quad (|\hat{\underset{\sim}{n}}| = 1)$$
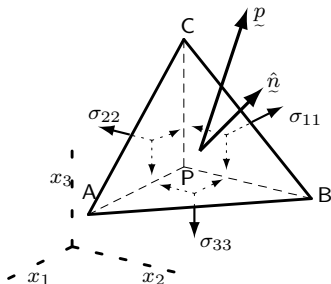
And similar for the other two area ratios.

Combined this gives:

$$\frac{\Delta A_{\mathsf{PBC}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_1) = \hat{n}_1$$

$$\frac{\Delta A_{\mathsf{PCA}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_2) = \hat{n}_2$$

$$\frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_3) = \hat{n}_3$$

Combined this gives:

$$\frac{\Delta A_{\mathsf{PBC}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_1) = \hat{n}_1$$

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$$\frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_3) = \hat{n}_3$$

Equilibrium in x_1-direction $(i = 1,2,3)$:

$p_1 \, \Delta A = \sigma_{11} \, \Delta A_{\mathsf{PBC}} + \sigma_{21} \, \Delta A_{\mathsf{PCA}} + \sigma_{31} \, \Delta A_{\mathsf{PAB}}$

$\Rightarrow p_1 = \sigma_{11} \, \hat{n}_1 + \sigma_{21} \, \hat{n}_2 + \sigma_{31} \, \hat{n}_3 = \sigma_{j1} \, \hat{n}_j$

Equilibrium in x_i-direction:

$\Rightarrow p_i = \sigma_{1i} \, \hat{n}_1 + \sigma_{2i} \, \hat{n}_2 + \sigma_{3i} \, \hat{n}_3 = \sigma_{ji} \, \hat{n}_j$

Cauchy's formula: $\boxed{p_i = \sigma_{ji}\,\hat{n}_j}$.   The summation takes place over the index j,   and index i can be either 1, 2 or 3.

$$\underset{\sim}{p} = \underset{\sim}{\hat{n}} \cdot \underline{\sigma} = \underline{\sigma}^{\mathrm{T}} \cdot \underset{\sim}{\hat{n}} \quad \text{or} \quad \{p\} = \{\hat{n}\}^{\mathrm{T}}[\sigma] = [\sigma]^{\mathrm{T}}\{\hat{n}\}$$

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It will be shown later that the stress matrix is symmetric, so
$\sigma_{ij} = \sigma_{ji}$ or $\underline{\sigma} = \underline{\sigma}^{\mathrm{T}}$ or $[\sigma] = [\sigma]^{\mathrm{T}}$. From that:

$$\boxed{p_i = \sigma_{ij}\,\hat{n}_j}$$
$$\underset{\sim}{p} = \underline{\sigma} \cdot \hat{\underset{\sim}{n}} \quad \text{or} \quad \left[\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array}\right] = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array}\right] \left[\begin{array}{c} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{array}\right]$$
$$\{p\} = [\sigma]\{\hat{n}\}$$

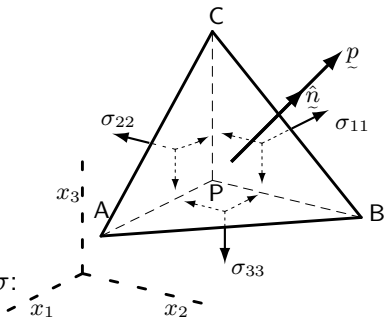### 3.3) Principal stresses and principal directions
Question:
Is it possible to orientate
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Is it possible to orientate
the area ABC in such a way
that the direction of the stress
vector is the same as the direction
of the normal acting on this area.
In that case it must hold that:

$\underset{\sim}{p} = $ factor $\times \hat{\underset{\sim}{n}}$.

Substitute "factor" with the symbol σ:

$\underset{\sim}{p} \overset{?}{=} \sigma \hat{\underset{\sim}{n}}$ or $p_i \overset{?}{=} \sigma \hat{n}_i$

Are there multiple orientations of area ABC possible such that:

$$\underset{\sim}{p} \overset{?}{=} \sigma \, \hat{\underset{\sim}{n}} \quad \text{or} \quad p_i \overset{?}{=} \sigma \, \hat{n}_i$$

Cauchy's formula gives: $\quad p_i = \sigma_{ij} \, \hat{n}_j$

Combined: $\quad p_i = \sigma_{ij} \, \hat{n}_j \overset{?}{=} \sigma \, \hat{n}_i = \sigma \, \delta_{ij} \, \hat{n}_j$

Written as a matrix $\quad ([\sigma]\{\hat{n}\} = \sigma \, \{\hat{n}\})$:

$$\{p\} = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] \left[\begin{array}{c} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{array} \right] \overset{?}{=} \sigma \left[\begin{array}{c} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{array} \right]$$

$[\sigma]\{\hat{n}\} = \sigma\{\hat{n}\}$ can be written as: $([\sigma] - \sigma[I])\{\hat{n}\} = \{0\}$

$$\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of algebraic equations.

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This is a homogeneous system of algebraic equations.
This also yields a solution $\{\hat{n}\} \neq \{0\}$ when the determinant of the coefficient matrix is equal to zero.

$$\det\left(\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix}\right) = 0$$

$$\det \left(\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix} \right) = 0$$

Written as *characteristic equation*:

$$\boxed{\sigma^3 - I_1\,\sigma^2 + I_2\,\sigma - I_3 = 0} \qquad \text{with}$$

$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii}$

$I_2 = \sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21} + \circlearrowleft = \frac{1}{2}\,(\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}\,\sigma_{ji})$

$I_3 = \det([\sigma])$

From this the *principal stresses*: σ_1, σ_2 and σ_3 are obtained for which holds that: $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Principal stresses are the *eigenvalues* of the stress matrix $[\sigma]$.
With eigenvalues comes *eigenvectors*. Here the *principal directions*.
Obtain first principal directions, related to σ_1:

$$\begin{bmatrix} (\sigma_{11} - \sigma_1) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma_1) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a *dependent* system of equations!!! e.g.:

$$\begin{bmatrix} (\sigma_{11} - \sigma_1) & \sigma_{12} \\ \sigma_{21} & (\sigma_{22} - \sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \end{bmatrix} = \begin{bmatrix} -\sigma_{13} \\ -\sigma_{23} \end{bmatrix} \hat{n}_3 \implies \begin{array}{l} \hat{n}_1 = a \ \hat{n}_3 \\ \hat{n}_2 = b \ \hat{n}_3 \end{array}$$

Found: $\hat{n}_1 = a \, \hat{n}_3$ and $\hat{n}_2 = b \, \hat{n}_3$.

Normal vector is an unit vector, so $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$
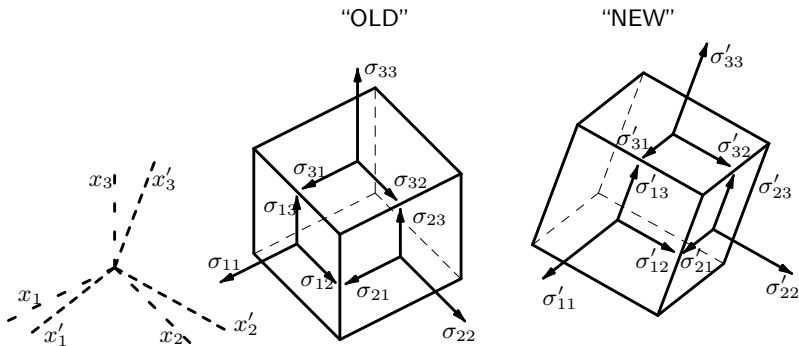
With this the first principal direction is determined.

The same could be done for the second and third principal directions

All three principal directions are perpendicular to each other!

Found: $\hat{n}_1 = a\,\hat{n}_3$ and $\hat{n}_2 = b\,\hat{n}_3$.

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All three principal directions are perpendicular to each other!

Preferably a coordinate system coinciding with the principal directions:

$$[\sigma]_{\mathsf{ps}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \Rightarrow \quad$$

Stress invariants:
$I_1 = \sigma_1 + \sigma_2 + \sigma_3$
$I_2 = \sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1$
$I_3 = \sigma_1\,\sigma_2\,\sigma_3$

3.4) Rotation matrix

"OLD" "NEW"

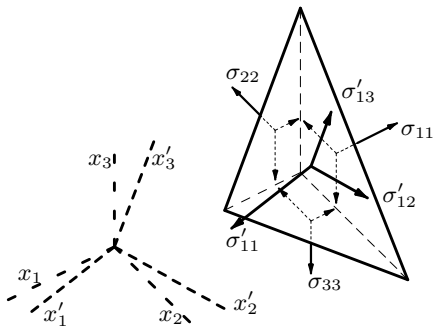Directional cosines: $\quad R_{pi} = \cos(x'_p, x_i)$

Surface
perpendicular to x'_1-axis.
Normal vector $\hat{\underset{\sim}{n}}^{(1)}$

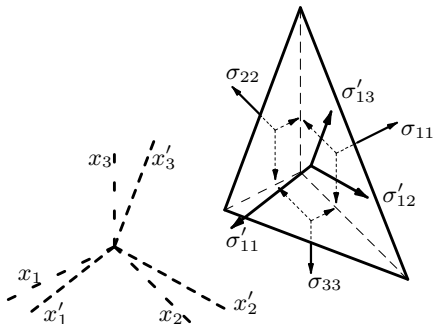with: $\hat{n}_i^{(1)} = \cos(x'_1, x_i) = R_{1i}$
Cauchy's formula:
$p_j^{(1)} = \sigma_{ij}\,\hat{n}_i^{(1)} = \sigma_{ij}\,R_{1i}$

Surface
perpendicular to x'_1-axis.
Normal vector $\hat{\underset{\sim}{n}}^{(1)}$

with: $\hat{n}_i^{(1)} = \cos(x'_1, x_i) = R_{1i}$
Cauchy's formula:
$p_j^{(1)} = \sigma_{ij}\,\hat{n}_i^{(1)} = \sigma_{ij}\,R_{1i}$

After rotation:

$$\sigma'_{1q} = p'^{(1)}_q = R_{qj}\,p_j^{(1)} = R_{qj}\,\sigma_{ij}\,R_{1i} = R_{1i}\,R_{qj}\,\sigma_{ij}$$

Same for the x'_2- and x'_3-directions. So:

$$\sigma'_{pq} = R_{pi}\,R_{qj}\,\sigma_{ij}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During rotation of coordinate axes holds:

$$\sigma'_{pq} = R_{pi}\, R_{qj}\, \sigma_{ij} \quad \text{or} \quad [\sigma'] = [R]\, [\sigma]\, [R]^{\mathrm{T}}$$

Elaborated:

$$\sigma'_{pq} = R_{p1}\, R_{q1}\, \sigma_{11} + R_{p1}\, R_{q2}\, \sigma_{12} + R_{p1}\, R_{q3}\, \sigma_{13} +$$
$$+ R_{p2}\, R_{q1}\, \sigma_{21} + R_{p2}\, R_{q2}\, \sigma_{22} + R_{p2}\, R_{q3}\, \sigma_{23} +$$
$$+ R_{p3}\, R_{q1}\, \sigma_{31} + R_{p3}\, R_{q2}\, \sigma_{32} + R_{p3}\, R_{q3}\, \sigma_{33}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During transformation it holds that:

$$\sigma'_{pq} = R_{pi}\,R_{qj}\,\sigma_{ij} \quad \text{or} \quad [\sigma'] = [R]\,[\sigma]\,[R]^{\mathrm{T}}$$

Elaborated:

$$\begin{aligned}
\sigma'_{pq} = R_{p1}\,R_{q1}\,\sigma_{11} + R_{p1}\,R_{q2}\,\sigma_{12} + R_{p1}\,R_{q3}\,\sigma_{13} + \\
+ R_{p2}\,R_{q1}\,\sigma_{21} + R_{p2}\,R_{q2}\,\sigma_{22} + R_{p2}\,R_{q3}\,\sigma_{23} + \\
+ R_{p3}\,R_{q1}\,\sigma_{31} + R_{p3}\,R_{q2}\,\sigma_{32} + R_{p3}\,R_{q3}\,\sigma_{33}
\end{aligned}$$

Also the inverse transformation holds ("new" to "old"):

$$\sigma_{ij} = R_{pi}\,R_{qj}\,\sigma'_{pq} \ \text{ or } \ \sigma_{pq} = R_{ip}\,R_{jq}\,\sigma'_{ij} \ \text{ or } \ [\sigma] = [R]^{\mathrm{T}}[\sigma']\,[R]$$

3.5) Equilibrium equations

In reader 3-dim., here 2-dim.:
Instead
of a single point now *a volume*.
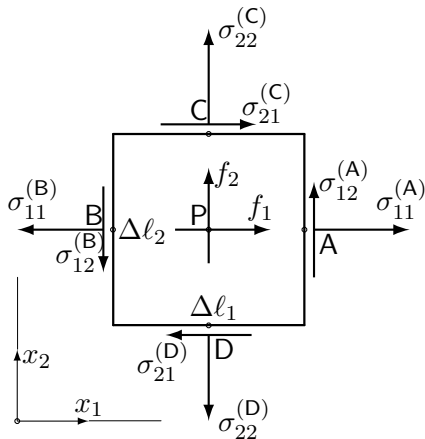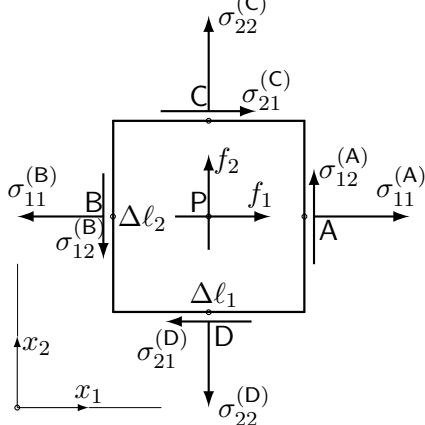Stresses are a function of x_i.

Coordinates of the points:

$P: \quad (x_1, x_2)$

$A: \quad (x_1 + \frac{1}{2}\Delta\ell_1, x_2)$

$B: \quad (x_1 - \frac{1}{2}\Delta\ell_1, x_2)$

$C: \quad (x_1, x_2 + \frac{1}{2}\Delta\ell_2)$

$D: \quad (x_1, x_2 - \frac{1}{2}\Delta\ell_2)$

f_1 and f_2 are volume forces or body forces, forces per unit volume.

3.5) Equilibrium equations

In reader 3-dim., here 2-dim.:
Instead
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Stresses are a function of x_i.

Coordinates of the points:

P : (x_1, x_2)
A : $(x_1 + \frac{1}{2}\Delta\ell_1, x_2)$
B : $(x_1 - \frac{1}{2}\Delta\ell_1, x_2)$
C : $(x_1, x_2 + \frac{1}{2}\Delta\ell_2)$
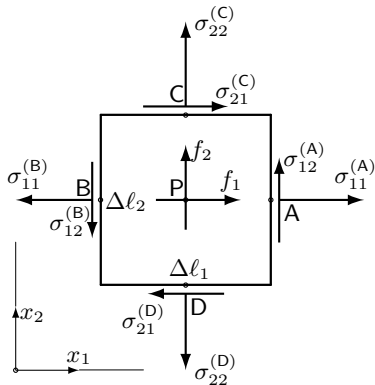D : $(x_1, x_2 - \frac{1}{2}\Delta\ell_2)$

Force equilibrium in x_1-direction (volume size in x_3-direction is $\Delta\ell_3$):

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \, \Delta\ell_2 \, \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \, \Delta\ell_1 \, \Delta\ell_3 + f_1 \, \Delta\ell_1 \, \Delta\ell_2 \, \Delta\ell_3 = 0$$

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)})\, \Delta\ell_2\, \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)})\, \Delta\ell_1\, \Delta\ell_3 + f_1\, \Delta\ell_1\, \Delta\ell_2\, \Delta\ell_3 = 0$$
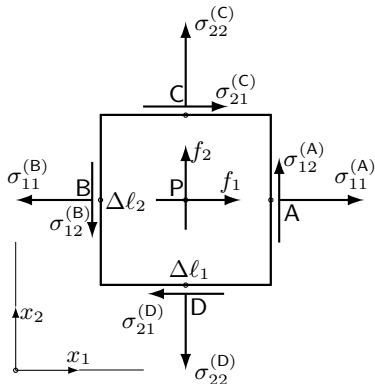
$\sigma_{11}^{(A)} = \sigma_{11}(x_1 + \tfrac{1}{2}\Delta\ell_1, x_2) =$
$= \sigma_{11}(x_1, x_2) + \tfrac{1}{2}\Delta\ell_1 \left(\dfrac{\partial\sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$

$\sigma_{11}^{(B)} = \sigma_{11}(x_1 - \tfrac{1}{2}\Delta\ell_1, x_2) =$
$= \sigma_{11}(x_1, x_2) - \tfrac{1}{2}\Delta\ell_1 \left(\dfrac{\partial\sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$

$$(\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

$\sigma_{11}^{(\mathsf{A})} = \sigma_{11}(x_1 + \frac{1}{2}\Delta\ell_1, x_2) =$
$= \sigma_{11}(x_1, x_2) + \frac{1}{2}\Delta\ell_1\left(\dfrac{\partial\sigma_{11}}{\partial x_1}\right) + \mathsf{H.O.T.}$

$\sigma_{11}^{(\mathsf{B})} = \sigma_{11}(x_1 - \frac{1}{2}\Delta\ell_1, x_2) =$
$= \sigma_{11}(x_1, x_2) - \frac{1}{2}\Delta\ell_1\left(\dfrac{\partial\sigma_{11}}{\partial x_1}\right) + \mathsf{H.O.T.}$

So (neglecting Higher Order Terms):
$$\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})} = \Delta\ell_1\left(\dfrac{\partial\sigma_{11}}{\partial x_1}\right)$$
Idem: $\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})} = \Delta\ell_2\left(\dfrac{\partial\sigma_{21}}{\partial x_2}\right)$

Equilibrium of forces in x_1-direction:

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(A)} - \sigma_{11}^{(B)} = \Delta\ell_1\,\dfrac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(C)} - \sigma_{21}^{(D)} = \Delta\ell_2\,\dfrac{\partial\sigma_{21}}{\partial x_2}$

Results in (after dividing by $\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

Shorter: $\displaystyle\sum_{j=1}^{\cdots}\frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \quad\Rightarrow\quad \sigma_{j1,j} + f_1 = 0$

Equilibrium of forces in x_1-direction:

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)})\, \Delta\ell_2\, \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)})\, \Delta\ell_1\, \Delta\ell_3 + f_1\, \Delta\ell_1\, \Delta\ell_2\, \Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(A)} - \sigma_{11}^{(B)} = \Delta\ell_1\, \dfrac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(C)} - \sigma_{21}^{(D)} = \Delta\ell_2\, \dfrac{\partial\sigma_{21}}{\partial x_2}$

Results in (after dividing by $\Delta\ell_1\, \Delta\ell_2\, \Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

Shorter: $\displaystyle\sum_{j=1}^{\cdots} \frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \;\Rightarrow\; \sigma_{j1,j} + f_1 = 0$

The same for other directions, expanded to 3-dim.:

$$\boxed{\sigma_{ji,j} + f_i = 0} \qquad \left(\sum_{j=1}^{3} \frac{\partial\sigma_{ji}}{\partial x_j} + f_i = 0 \;\text{ for } i = 1, 2, 3\right)$$
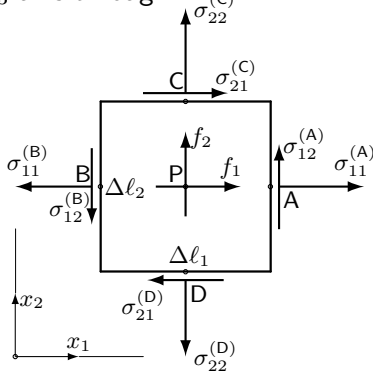
Equilibrium of moments around the x_3-axis through P:

$(\sigma_{12}^{(A)} + \sigma_{12}^{(B)})\, \Delta\ell_2\, \Delta\ell_3\, \frac{1}{2}\Delta\ell_1 +$
$-\, (\sigma_{21}^{(C)} + \sigma_{21}^{(D)})\, \Delta\ell_1\, \Delta\ell_3\, \frac{1}{2}\Delta\ell_2 = 0$
$\Longrightarrow$
$(\sigma_{12}^{(A)} + \sigma_{12}^{(B)}) - (\sigma_{21}^{(C)} + \sigma_{21}^{(D)}) = 0$

Equilibrium of moments around the x_3-axis through P:

$$\left(\sigma_{12}^{(A)} + \sigma_{12}^{(B)}\right)\Delta\ell_2\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_1 +$$
$$-\left(\sigma_{21}^{(C)} + \sigma_{21}^{(D)}\right)\Delta\ell_1\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_2 = 0$$
$$\Longrightarrow$$
$$\left(\sigma_{12}^{(A)} + \sigma_{12}^{(B)}\right) - \left(\sigma_{21}^{(C)} + \sigma_{21}^{(D)}\right) = 0$$

$$\sigma_{12}^{(A)} = \sigma_{12} + \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{12}}{\partial x_1}\right) + \text{H.O.T.}$$

$$\sigma_{12}^{(B)} = \sigma_{12} - \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{12}}{\partial x_1}\right) + \text{H.O.T.}$$

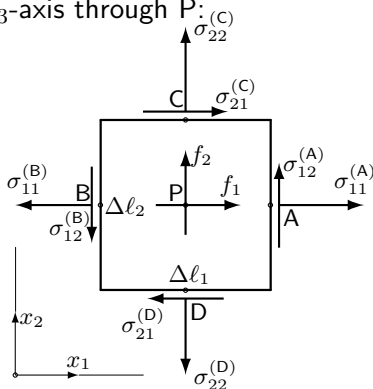$$\sigma_{21}^{(C)} = \sigma_{21} + \tfrac{1}{2}\Delta\ell_2\left(\frac{\partial\sigma_{21}}{\partial x_2}\right) + \text{H.O.T.}$$

$$\sigma_{21}^{(D)} = \sigma_{21} - \tfrac{1}{2}\Delta\ell_2\left(\frac{\partial\sigma_{21}}{\partial x_2}\right) + \text{H.O.T.}$$

$$\Rightarrow\quad 2\,\sigma_{12} - 2\,\sigma_{21} = 0 \quad\Rightarrow\quad \sigma_{12} = \sigma_{21} \qquad \text{in general:} \qquad \boxed{\sigma_{ij} = \sigma_{ji}}$$
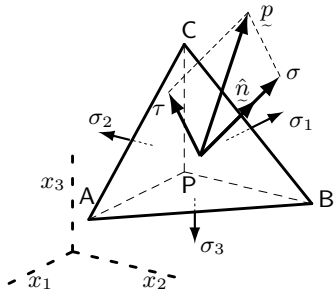
3.6) Maximum shear stresses, extrema

How to obtain maximum shear stress τ_{max} .

Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i-coordinate system that it will coincide with the principal directions.
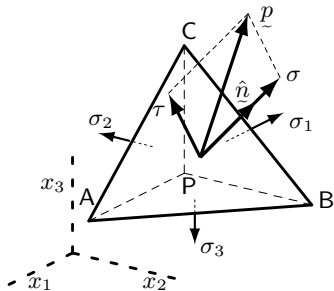
3.6) Maximum shear stresses, extrema

How to obtain maximum shear stress $\tau_{\max}$.

Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i-coordinate system that it will coincide with the principal directions.

The stress matrix will be:

$$[\sigma_h] = [\sigma]_{x_i} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Apply an arbitrary orientated surface, with normal vector $\hat{\underset{\sim}{n}}$.

The components of the stress vector $\underset{\sim}{p}$ to the surface (Cauchy):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \left[\begin{array}{ccc} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{array}\right] \left[\begin{array}{c} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{array}\right] = \left[\begin{array}{c} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{array}\right]$$

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The magnitude, σ, of the normal stress on the surface is:

$$\sigma = \{\hat{n}\}^{\mathrm{T}}\{p\} = [\begin{array}{ccc} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{array}] \left[\begin{array}{c} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{array}\right] = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$$

The components of the stress vector $\underset{\sim}{p}$ to the surface (Cauchy):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \left[\begin{array}{ccc} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{array}\right] \left[\begin{array}{c} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{array}\right] = \left[\begin{array}{c} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{array}\right]$$

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$$\sigma = \{\hat{n}\}^{\mathrm{T}}\{p\} = \left[\begin{array}{ccc} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{array}\right] \left[\begin{array}{c} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{array}\right] = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$$

The magnitude, τ, of the shear stress on the surface:

$$\tau^2 = |\underset{\sim}{p}|^2 - \sigma^2 = \sigma_1^2\,\hat{n}_1^2 + \sigma_2^2\,\hat{n}_2^2 + \sigma_3^2\,\hat{n}_3^2 - (\sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2)^2$$

Found: $\tau^2 = \sigma_1^2 \, \hat{n}_1^2 + \sigma_2^2 \, \hat{n}_2^2 + \sigma_3^2 \, \hat{n}_3^2 - (\sigma_1 \, \hat{n}_1^2 + \sigma_2 \, \hat{n}_2^2 + \sigma_3 \, \hat{n}_3^2)^2$

Eliminate the directional coefficient $\hat{n}_3$ with: $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$ $\implies$

$$\tau^2 = (\sigma_1^2 - \sigma_3^2)\hat{n}_1^2 + (\sigma_2^2 - \sigma_3^2)\hat{n}_2^2 + \sigma_3^2 +$$
$$- \left\{ (\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3 \right\}^2$$

Extrema can be found with: $\dfrac{\partial \tau^2}{\partial \hat{n}_1} = \dfrac{\partial \tau^2}{\partial \hat{n}_2} = 0$

Found: $\tau^2 = \sigma_1^2\,\hat{n}_1^2 + \sigma_2^2\,\hat{n}_2^2 + \sigma_3^2\,\hat{n}_3^2 - (\sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2)^2$

Eliminate the directional coefficient $\hat{n}_3$ with: $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2 \quad \Longrightarrow$

$$\tau^2 = (\sigma_1^2 - \sigma_3^2)\hat{n}_1^2 + (\sigma_2^2 - \sigma_3^2)\hat{n}_2^2 + \sigma_3^2 +$$
$$- \left\{(\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3\right\}^2$$

Extrema can be found with: $\dfrac{\partial \tau^2}{\partial \hat{n}_1} = \dfrac{\partial \tau^2}{\partial \hat{n}_2} = 0$

The following solutions are found:

$\hat{n}_1 = 0 \quad$ with $\begin{cases} 1) & \hat{n}_2 = 0 & \hat{n}_3 = \pm 1 \\ 2) & \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} \end{cases} \quad \circlearrowleft$

The first solution is $\tau = 0$; this is trivial and not interesting

The second solution, $\hat{n}_1 = 0$; $\hat{n}_2 = \pm\frac{1}{2}\sqrt{2}$; $\hat{n}_3 = \mp\frac{1}{2}\sqrt{2}$ gives:

$$\tau^2 = (\sigma_2^2 - \sigma_3^2)\tfrac{1}{2} + \sigma_3^2 - \{(\sigma_2 - \sigma_3)\tfrac{1}{2} + \sigma_3\}^2 = \tfrac{1}{4}(\sigma_2 - \sigma_3)^2$$

An extreme case for τ is found at: $\frac{1}{2}|\sigma_2 - \sigma_3|$.

The second solution, $\quad \hat{n}_1 = 0 \; ; \; \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} \; ; \; \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} \quad$ gives:

$$\tau^2 = (\sigma_2^2 - \sigma_3^2) \tfrac{1}{2} + \sigma_3^2 - \left\{ (\sigma_2 - \sigma_3) \tfrac{1}{2} + \sigma_3 \right\}^2 = \tfrac{1}{4} (\sigma_2 - \sigma_3)^2$$

An extreme case for τ is found at: $\quad \frac{1}{2} |\sigma_2 - \sigma_3|$.
In a similar way the following solutions can be found:

$$
\begin{array}{llll}
\hat{n}_1 = 0 & \hat{n}_2 = \pm \tfrac{1}{2}\sqrt{2} & \hat{n}_3 = \mp \tfrac{1}{2}\sqrt{2} & \tau = \tfrac{1}{2} |\sigma_2 - \sigma_3| \\
\hat{n}_1 = \pm \tfrac{1}{2}\sqrt{2} & \hat{n}_2 = \mp \tfrac{1}{2}\sqrt{2} & \hat{n}_3 = 0 & \tau = \tfrac{1}{2} |\sigma_1 - \sigma_2| \\
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\end{array}
$$

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\hat{n}_1 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_2 = 0 & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} & \tau = \frac{1}{2}|\sigma_1 - \sigma_3|
\end{array}
$$

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then $\boxed{\tau_{\mathsf{max}} = \frac{1}{2}(\sigma_1 - \sigma_3)}$
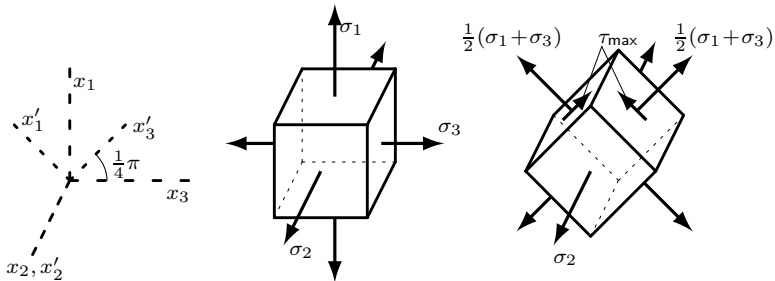
This τ_{max} originates from a 45° rotation around the x_2-axis

45° rotation around x_2-axis: $\quad [R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$

Then: $\quad [\sigma'] = [R]\,[\sigma_h]\,[R]^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2}(\sigma_1 + \sigma_3) & 0 & \frac{1}{2}(\sigma_1 - \sigma_3) \\ 0 & \sigma_2 & 0 \\ \frac{1}{2}(\sigma_1 - \sigma_3) & 0 & \frac{1}{2}(\sigma_1 + \sigma_3) \end{bmatrix}$

45° rotation around x_2-axis: $\quad [R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$

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Extrema from the normal stress on a surface.

Found previously: $\quad \sigma = \sigma_1\, \hat{n}_1^2 + \sigma_2\, \hat{n}_2^2 + \sigma_3\, \hat{n}_3^2$

Elimination of $\hat{n}_3$ $\quad$ using $\quad \hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$

$\sigma = (\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3$

Extrema when: $\quad \dfrac{\partial \sigma}{\partial \hat{n}_1} = \dfrac{\partial \sigma}{\partial \hat{n}_2} = 0$

Resulting in: $\quad (\sigma_1 - \sigma_3)2\hat{n}_1 = (\sigma_2 - \sigma_3)2\hat{n}_2 = 0$

Extrema from the normal stress on a surface.

Found previously: $\quad \sigma = \sigma_1 \, \hat{n}_1^2 + \sigma_2 \, \hat{n}_2^2 + \sigma_3 \, \hat{n}_3^2$

Elimination of $\hat{n}_3$ using $\quad \hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$

$\sigma = (\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3$

Extrema when: $\quad \dfrac{\partial \sigma}{\partial \hat{n}_1} = \dfrac{\partial \sigma}{\partial \hat{n}_2} = 0$

Resulting in: $\quad (\sigma_1 - \sigma_3)2\hat{n}_1 = (\sigma_2 - \sigma_3)2\hat{n}_2 = 0$

Solution (general case): $\quad \hat{n}_1 = \hat{n}_2 = 0 \ ; \ \hat{n}_3 = 1$

This is exactly the 3th principal direction corresponding to σ_3. σ_1 and σ_2 can be found with elimination of $\hat{n}_1$ resp. $\hat{n}_2$. So, the principal stresses are the extrema of the normal stress.

When $\quad \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad$ it must hold that: $\quad \sigma_1 \geq \sigma \geq \sigma_3$.

3.7) Miscellaneous, including Mohr's circle(s)

Consider the following special stress state:
$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a Principal stress (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

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NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a Principal stress (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

Calculate principal stresses:

$$\det \left(\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma & 0 \\ 0 & 0 & \sigma_{33} - \sigma \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad (\sigma_{33} - \sigma)\{(\sigma_{11} - \sigma)(\sigma_{22} - \sigma) - \sigma_{12}^2\} = 0$$

$$\Rightarrow \quad (\sigma_{33} - \sigma)\{\sigma^2 - (\sigma_{11} + \sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$$

Found: $(\sigma_{33}-\sigma)\{\sigma^2 - (\sigma_{11}+\sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$
Solutions:

$$\sigma_{1,2} = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \tfrac{1}{2}\sqrt{(\sigma_{11}+\sigma_{22})^2 - 4\,\sigma_{11}\sigma_{22} + 4\,\sigma_{12}^2} =$$

$$= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$$

$$\sigma_3 = \sigma_{33}$$

The order of the principal stresses is STILL random!

Found: $\quad (\sigma_{33}-\sigma)\{\sigma^2 - (\sigma_{11}+\sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$

Solutions:

$$\begin{aligned}
\sigma_{1,2} &= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \tfrac{1}{2}\sqrt{(\sigma_{11}+\sigma_{22})^2 - 4\,\sigma_{11}\sigma_{22} + 4\,\sigma_{12}^2} = \\
&= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2} \\
\sigma_3 &= \sigma_{33}
\end{aligned}$$

The order of the principal stresses is STILL random!

The principal directions can be found with:

$$\begin{bmatrix} \sigma_{11} - \sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma_i & 0 \\ 0 & 0 & \sigma_{33} - \sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1, 2, 3$$

$$\begin{bmatrix} \sigma_{11} - \sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma_i & 0 \\ 0 & 0 & \sigma_{33} - \sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1, 2, 3$$

Consecutively resulting in, for $i = 1, 2, 3$:

$$\{\hat{n}\}^{(1)} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \quad ; \quad \{\hat{n}\}^{(2)} = \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix} \quad ; \quad \{\hat{n}\}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$a_1^2 + a_2^2 = 1 \ \Rightarrow \ a_1 = \frac{\sigma_{12}}{w} \ ; \ a_2 = \frac{\sigma_i - \sigma_{11}}{w} \ ; \ w = \sqrt{(\sigma_i - \sigma_{11})^2 + \sigma_{12}^2}$$

What happens with a rotation of angle α around the x_3-axis?

$$\text{Rotation matrix:} \quad [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What happens with a rotation of angle α around the x_3-axis?

$$\text{Rotatiematrix:} \quad [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation (using symmetry, $\sigma_{21} = \sigma_{12}$):

$$[\sigma'] = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & 0 \\ \sigma'_{21} & \sigma'_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = [R][\sigma][R]^{\mathrm{T}} =$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving matrices:

$$\sigma_\alpha = \sigma'_{11} = \sigma_{11}(\cos\alpha)^2 + 2\,\sigma_{12}\sin\alpha\cos\alpha + \sigma_{22}(\sin\alpha)^2 =$$
$$= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = \sigma'_{12} = (\sigma_{22}-\sigma_{11})\sin\alpha\cos\alpha + \sigma_{12}\{(\cos\alpha)^2-(\sin\alpha)^2\} =$$
$$= -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

Solving matrices:

$$\sigma_\alpha = \sigma'_{11} = \sigma_{11}(\cos\alpha)^2 + 2\,\sigma_{12}\sin\alpha\cos\alpha + \sigma_{22}(\sin\alpha)^2 =$$
$$= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = \sigma'_{12} = (\sigma_{22}-\sigma_{11})\sin\alpha\cos\alpha + \sigma_{12}\{(\cos\alpha)^2-(\sin\alpha)^2\} =$$
$$= -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

Visualisation:

$$\sigma_\alpha = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

$$\left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\right\}^2 + \tau_\alpha^2 = \tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2$$
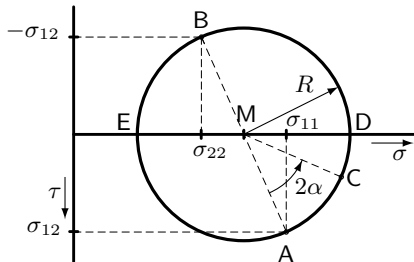
$$\sigma_\alpha = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

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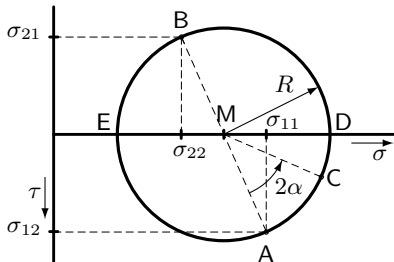
Mohr's circle

Midpoint M: $\{\tfrac{1}{2}(\sigma_{11}+\sigma_{22}), 0\}$

Radius: $R = \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$

Omit index α at σ and τ!

A: $\alpha = 0$ $(\sigma_{11}, \sigma_{12})$

B: $\alpha = \tfrac{1}{2}\pi$ $(\sigma_{22}, \sigma_{21})$

$$\sigma_\alpha = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

$$\{\sigma_\alpha - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\}^2 + \tau_\alpha^2 = \tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2$$

Mohr's circle - ALTERNATIVE
Midpoint M: $\{\tfrac{1}{2}(\sigma_{11}+\sigma_{22}), 0\}$
Radius: $R = \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$
Omit index α at σ and τ!
A: $\alpha = 0$ $(\sigma_{11}, \sigma_{12}\downarrow)$
B: $\alpha = \tfrac{1}{2}\pi$ $(\sigma_{22}, \sigma_{21}\uparrow)$

C: $\alpha = ??$ $(\sigma_\alpha, \tau_\alpha)$
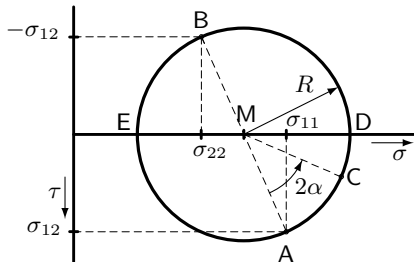D: $\alpha = \alpha_1$ $(\sigma_1, 0)$
E: $\alpha = \alpha_2$ $(\sigma_2, 0)$
$(2\alpha_1 = $ angle AMD,
$2\alpha_2 = $ angle AME$)$
$\tan(2\alpha_1) = \dfrac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$
$\alpha_2 = \alpha_1 \pm \frac{1}{2}\pi$

C: $\alpha =$?? $(\sigma_\alpha, \tau_\alpha)$
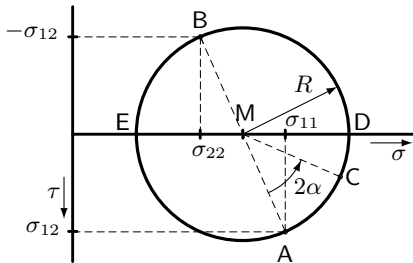D: $\alpha = \alpha_1$ $(\sigma_1, 0)$
E: $\alpha = \alpha_2$ $(\sigma_2, 0)$
$(2\alpha_1 =$ angle AMD,
$2\alpha_2 =$ angle AME)
$\tan(2\alpha_1) = \dfrac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$
$\alpha_2 = \alpha_1 \pm \frac{1}{2}\pi$

(i.e. eigenvectors in real space are perpendicular!)
Eigenvectors in Mohr circle are separated by $2|\alpha_1 - \alpha_2| = \pi$ (i.e. $180°$)

$\sigma_1 = \frac{1}{2}(\sigma_{11}+\sigma_{22}) + R = \frac{1}{2}(\sigma_{11}+\sigma_{22}) + \sqrt{\frac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$

$\sigma_2 = \frac{1}{2}(\sigma_{11}+\sigma_{22}) - R = \frac{1}{2}(\sigma_{11}+\sigma_{22}) - \sqrt{\frac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$

Principal stresses are the same as found earlier!!
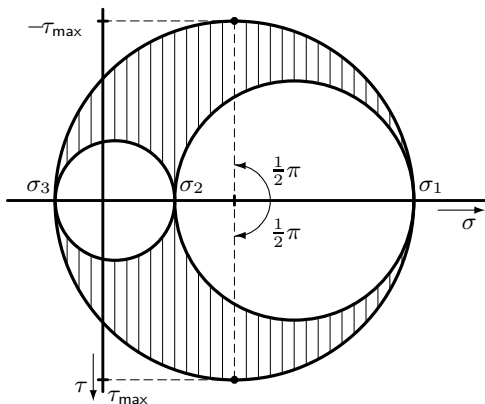Memorize/practice construction procedure and the factor 2

Arbitrary, three dimensional stress state.

Note: $\tau_{\text{max}} = \frac{1}{2}(\sigma_1 - \sigma_3)$

Angle with first principal direction:
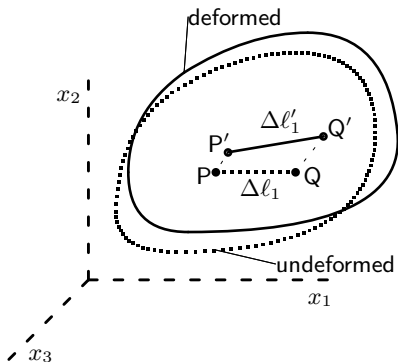$\frac{1}{2}\left(\pm\frac{1}{2}\pi\right) = \pm\frac{1}{4}\pi$

Points from an
arbitrary chosen coordinate system
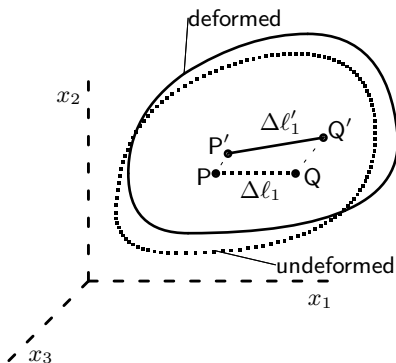will end up in the striped area!

4) Deformation
4.1) Strain

Consider a line segment PQ parallel with the x_1-axis.

4) Deformation
4.1) Strain

deformed

undeformed
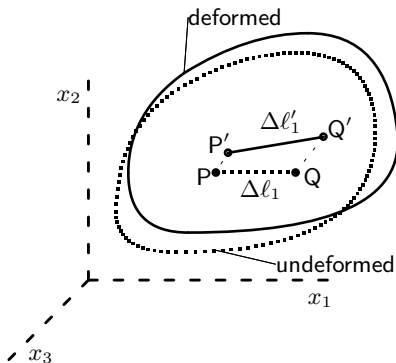
Consider a line segment PQ parallel with the x_1-axis.
The coordinates of P are x_1, x_2, x_3
Translation of P to P$'$
is $\underline{u}_P$ with components $u_i(x_1, x_2, x_3)$.

4) Deformation
4.1) Strain

Consider a line segment PQ parallel with the x_1-axis. The coordinates of P are x_1, x_2, x_3

Translation of P to P′ is $\underset{\sim}{u_P}$ with components $u_i(x_1, x_2, x_3)$.

The coordinates of Q are $x_1 + \Delta\ell_1$, x_2, x_3

Translation of Q to Q′ is $\underset{\sim}{u_Q}$ with components $u_i(x_1 + \Delta\ell_1, x_2, x_3)$.

with: $u_i(x_1 + \Delta\ell_1, x_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_1 \dfrac{\partial u_i}{\partial x_1} + \text{H.O.T.}$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \qquad Q': \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Coordinates of the translated points:

$$\mathrm{P}': \begin{pmatrix} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{pmatrix} \qquad \mathrm{Q}': \begin{pmatrix} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\Delta\ell_1' \;=\; \sqrt{\left(x_1^{(\mathrm{Q}')} - x_1^{(\mathrm{P}')}\right)^2 + \left(x_2^{(\mathrm{Q}')} - x_2^{(\mathrm{P}')}\right)^2 + \left(x_3^{(\mathrm{Q}')} - x_3^{(\mathrm{P}')}\right)^2} \;=\;$$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \qquad Q': \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned} \Delta\ell'_1 &= \sqrt{\left(x_1^{(Q')} - x_1^{(P')}\right)^2 + \left(x_2^{(Q')} - x_2^{(P')}\right)^2 + \left(x_3^{(Q')} - x_3^{(P')}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \end{aligned}$$

Coordinates of the translated points:

$$
\mathrm{P}': \left(\begin{array}{c} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{array}\right) \qquad \mathrm{Q}': \left(\begin{array}{c} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{array}\right)
$$

Length of line segment PQ in the deformed shape:

$$
\begin{aligned}
\Delta\ell_1' &= \sqrt{\left(x_1^{(\mathrm{Q}')} - x_1^{(\mathrm{P}')}\right)^2 + \left(x_2^{(\mathrm{Q}')} - x_2^{(\mathrm{P}')}\right)^2 + \left(x_3^{(\mathrm{Q}')} - x_3^{(\mathrm{P}')}\right)^2} = \\
&= \sqrt{\left(\Delta\ell_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \\
&= \Delta\ell_1 \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}
\end{aligned}
$$

Coordinates of the translated points:

$$\mathrm{P'}: \left(\begin{array}{c} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{array} \right) \qquad \mathrm{Q'}: \left(\begin{array}{c} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{array} \right)$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned} \Delta\ell_1' &= \sqrt{\left(x_1^{(\mathrm{Q'})} - x_1^{(\mathrm{P'})}\right)^2 + \left(x_2^{(\mathrm{Q'})} - x_2^{(\mathrm{P'})}\right)^2 + \left(x_3^{(\mathrm{Q'})} - x_3^{(\mathrm{P'})}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \\ &= \Delta\ell_1 \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} \\ &\approx \Delta\ell_1 \left(1 + \frac{\partial u_1}{\partial x_1}\right) \qquad \text{if } \frac{\partial u_i}{\partial x_1} \ll 1 \end{aligned}$$
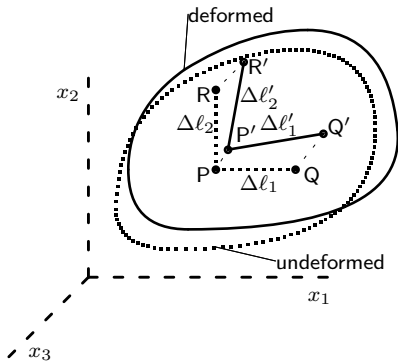
The strain in x_1-direction is:

$$\varepsilon_{11} = \frac{\Delta \ell_1' - \Delta \ell_1}{\Delta \ell_1} \approx \frac{\Delta \ell_1 \left(1 + \frac{\partial u_1}{\partial x_1} - 1\right)}{\Delta \ell_1} = \frac{\partial u_1}{\partial x_1}$$

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Same for the other directions:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$
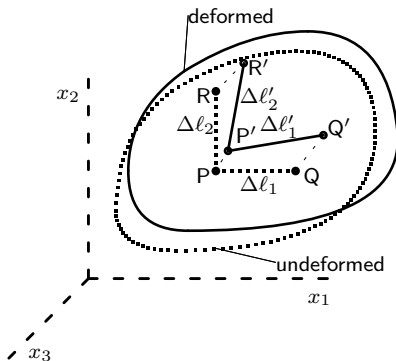
$$\text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \text{ for } \left\{ \begin{array}{l} i = 1, 2, 3 \\ j = 1, 2, 3 \end{array} \right.$$

4.2) Shear strain

Consider line segments PQ and PR in resp. x_1- and x_2-directions.

4.2) Shear strain

deformed

undeformed
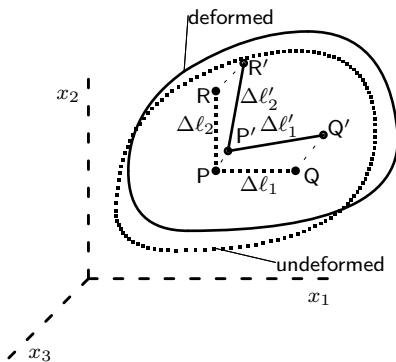
Consider line segments PQ and PR in resp. x_1- and x_2-directions.
The coordinates
of point R are $x_1, x_2 + \Delta\ell_2, x_3$

4.2) Shear strain

deformed

undeformed

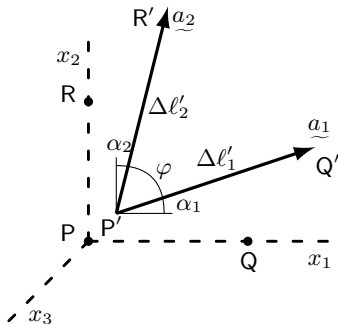Consider line segments PQ and PR in resp. x_1- and x_2-directions. The coordinates of point R are $x_1, x_2 + \Delta\ell_2, x_3$ Besides the translations of P $\to$ P' and Q $\to$ Q' we also have $\underline{u}_R$ from R $\to$ R', the components $u_i(x_1, x_2 + \Delta\ell_2, x_3)$.

with: $u_i(x_1, x_2 + \Delta\ell_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_2 \dfrac{\partial u_i}{\partial x_2} + \text{H.O.T.}$
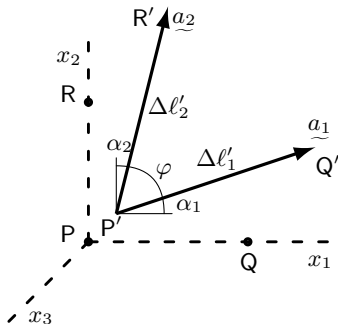
For the coordinates of the points P′, Q′ and R′, see reader.

The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90° angle. So: $\quad \gamma_{12} = \frac{1}{2}\pi - \varphi$

For the coordinates of the points P', Q' and R', see reader.

The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90° angle. So: $\quad \gamma_{12} = \frac{1}{2}\pi - \varphi$

$$
\begin{aligned}
\underset{\sim}{a_1} \cdot \underset{\sim}{a_2} &= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\cos(\varphi) \\
&= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\cos(\tfrac{1}{2}\pi - \gamma_{12}) \\
&= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\sin(\gamma_{12})
\end{aligned}
$$

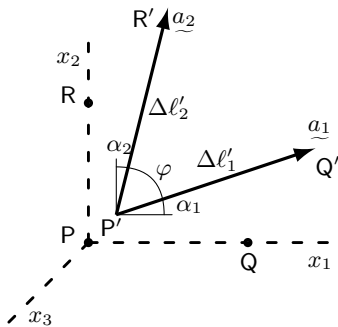For the coordinates of the points P′, Q′ and R′, see reader.

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$$
\begin{aligned}
\underset{\sim}{a_1} \cdot \underset{\sim}{a_2} &= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\cos(\varphi) \\
&= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\cos(\tfrac{1}{2}\pi - \gamma_{12}) \\
&= |\underset{\sim}{a_1}||\underset{\sim}{a_2}|\sin(\gamma_{12})
\end{aligned}
$$

From which: $\quad \gamma_{12} = \arcsin\left(\dfrac{\underset{\sim}{a_1} \cdot \underset{\sim}{a_2}}{|\underset{\sim}{a_1}||\underset{\sim}{a_2}|}\right)$

After "some" calculations:

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \qquad \text{if} \quad \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for} \quad \left\{ \begin{array}{l} i = 1, 2, 3 \\ j = 1, 2, 3 \end{array} \right.$$
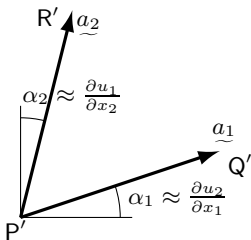
After "some" calculations:

$$\boxed{\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}}$$

if $\dfrac{\partial u_i}{\partial x_j} \ll 1$ for $\begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$

Shear strain $\quad \gamma_{12} = \alpha_1 + \alpha_2$

From the figure: $\quad \tan(\alpha_1) = \dfrac{\frac{\partial u_2}{\partial x_1} \Delta \ell_1}{\Delta \ell_1 + \frac{\partial u_1}{\partial x_1} \Delta \ell_1}$

As approximation: $\quad \alpha_1 \approx \dfrac{\partial u_2}{\partial x_1}$

And: $\alpha_2 \approx \dfrac{\partial u_1}{\partial x_2}$

In general it holds that $\alpha_1 \neq \alpha_2$!!!

Results for 3 directions:

$$\gamma_{12} = \gamma_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\gamma_{13} = \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad \text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for } \left\{ \begin{array}{l} i = 1, 2, 3 \\ j = 1, 2, 3 \end{array} \right.$$

$$\gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

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$$\gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

Summarised in index notation:

$$\boxed{\gamma_{ij} = \gamma_{ji} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}} \quad \text{with } i \neq j$$

4.3) Strain tensor

Strains and shear strains from the theory of *small deformations*:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \gamma_{12} = \gamma_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \gamma_{13} = \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \qquad \gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

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$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \qquad \gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

If $i \neq j$ is used for $\varepsilon_{ij} = \frac{1}{2}\,\gamma_{ij}$ then:

$$\boxed{\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)} \qquad \text{for } \left\{ \begin{array}{l} i = 1, 2, 3 \\ j = 1, 2, 3 \end{array} \right.$$

Notation $\left(\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right)$:

$$\underset{\sim}{\varepsilon} = \varepsilon_{ij}\,\underset{\sim}{\hat{e}_i}\,\underset{\sim}{\hat{e}_j} \qquad [\varepsilon] = \left[\begin{array}{ccc} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{array}\right] = \left[\begin{array}{ccc} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{array}\right]$$

Notation $\left(\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right)$:

$$\underline{\underline{\varepsilon}} = \varepsilon_{ij}\,\hat{e}_i\,\hat{e}_j \qquad [\varepsilon] = \left[\begin{array}{ccc} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{array}\right] = \left[\begin{array}{ccc} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{array}\right]$$

Outdated notation
$(x_1 \to x,\ \ x_2 \to y \text{ and } x_3 \to z,\ \ u_1 \to u,\ \ u_2 \to v \text{ and } u_3 \to w)$:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\ (= \varepsilon_x) \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}\ (= \varepsilon_y) \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}\ (= \varepsilon_z)$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \varepsilon_{yx} = \frac{1}{2}\gamma_{yx}$$

$$\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \varepsilon_{zx} = \frac{1}{2}\gamma_{zx}$$

$$\varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \varepsilon_{zy} = \frac{1}{2}\gamma_{zy}$$

Question: Are ε_{ij} the components of a second order tensor?

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Apply rotation of coordinate axes, $x_i \rightarrow x'_p$:

$$x'_p = x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) =$$
$$= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i$$

Question: Are ε_{ij} the components of a second order tensor?
Apply rotation of coordinate axes, $x_i \to x'_p$:

$$x'_p = x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) =$$
$$= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i$$

For the deformation components holds:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} \frac{\partial x'_q}{\partial x_j} + \frac{\partial u_j}{\partial x'_p} \frac{\partial x'_p}{\partial x_i} \right) =$$
$$= \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi} \right)$$

Question: Are ε_{ij} the components of a second order tensor?
Apply rotation of coordinate axes, $x_i \rightarrow x'_p$:

$$x'_p = x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) =$$
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$$= \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi} \right)$$

Transformation components of displacement vector: $u_i = R_{pi} u'_p \implies$

$$\varepsilon_{ij} = \frac{1}{2} \left(R_{pi} \frac{\partial u'_p}{\partial x'_q} R_{qj} + R_{qj} \frac{\partial u'_q}{\partial x'_p} R_{pi} \right) = R_{pi} R_{qj} \frac{1}{2} \left(\frac{\partial u'_p}{\partial x'_q} + \frac{\partial u'_q}{\partial x'_p} \right) =$$
$$= R_{pi} R_{qj} \varepsilon'_{pq}$$

Summarized: (from "new" to "old" and vice versa):

$$\varepsilon_{ij} = R_{pi}\, R_{qj}\, \varepsilon'_{pq} \qquad \text{and} \qquad \varepsilon'_{pq} = R_{pi}\, R_{qj}\, \varepsilon_{ij}$$

In matrix-vector-notation:

$$[\varepsilon] = [R]^{\mathrm{T}}[\varepsilon'][R] \qquad \text{and} \qquad [\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}$$

Summarized: (from "new" to "old" and vice versa):

$$\varepsilon_{ij} = R_{pi} R_{qj} \varepsilon'_{pq} \qquad \text{and} \qquad \varepsilon'_{pq} = R_{pi} R_{qj} \varepsilon_{ij}$$

In matrix-vector-notation:

$$[\varepsilon] = [R]^{\mathrm{T}}[\varepsilon'][R] \qquad \text{and} \qquad [\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}$$

So ε_{ij} are indeed components of a second order tensor. As well as the stress tensor, this tensor is symmetric; for the deformation matrix holds: $[\varepsilon]^{\mathrm{T}} = [\varepsilon]$.

Also see the reader regarding the arbitrary oriented line segment PQ.

4.4) Principal strains and principal directions

Second order tensor, so principal directions follow from:

$$\det\left(\left[\begin{array}{ccc} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon \end{array}\right]\right) = \det\left([\varepsilon] - \varepsilon[I]\right) = 0$$

Resulting in the characteristic equation: $\quad \varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$

4.4) Principal strains and principal directions

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Resulting in the characteristic equation: $\quad \varepsilon^3 - E_1 \varepsilon^2 + E_2 \varepsilon - E_3 = 0$

The *deformation invariants* are:

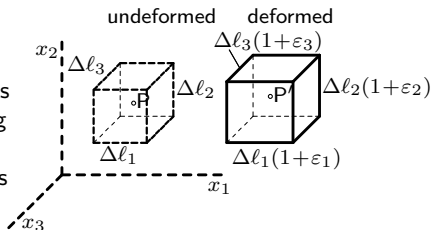$$\begin{aligned} E_1 &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii} \\ E_2 &= \varepsilon_{11}\,\varepsilon_{22} - \varepsilon_{12}\,\varepsilon_{21} + \circlearrowleft = \tfrac{1}{2}\left(\varepsilon_{ii}\,\varepsilon_{jj} - \varepsilon_{ij}\,\varepsilon_{ji} \right) \\ E_3 &= \det([\varepsilon]) \end{aligned}$$

After solving we obtain the principal strains ε_1, ε_2, ε_3 and the 3 orthogonal principal directions. The invariants can be expressed in terms of principal strains.

4.5) Volumetric strain

Consider a x_i-coordinate system oriented in such a way that it coincides with the principal directions of the deformed state. A corresponding infinitesimal volume element does not have any shear strain. The surfaces remain perpendicular to each other during deformation.
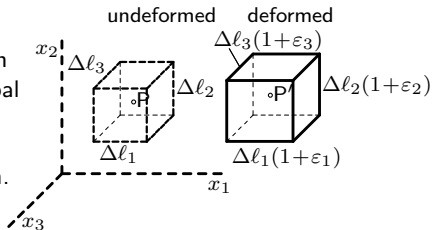
4.5) Volumetric strain

Consider
a x_i-coordinate system oriented in such
a way that it coincides with the principal
directions of the deformed state.
A corresponding infinitesimal volume
element does not have any shear strain.
The surfaces remain perpendicular to
each other during deformation.

Volume undeformed: $\quad V = \Delta\ell_1 \, \Delta\ell_2 \, \Delta\ell_3$
Volume deformed: $\quad V' = \Delta\ell_1(1 + \varepsilon_1)\Delta\ell_2(1 + \varepsilon_2)\Delta\ell_3(1 + \varepsilon_3)$
Volumetric strain:
$\varepsilon_V = \frac{V'-V}{V} = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \mathsf{H.O.T.}$
For small strains we ignore H.O.T. (also including the third invariant E_3).
$\Rightarrow \quad \varepsilon_V = E_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad$ or $\quad \boxed{\varepsilon_V = \varepsilon_{kk}} = E_1$

4.6) Engineering strain/stress vs. true strain/stress

Note that:
engineering strain $\varepsilon = \frac{\Delta L}{L_0}$ is defined relative to the original length, L_0,
while stress $\sigma = \frac{F}{A_0}$ is defined relative to the original cross-section, A_0.

Neither can be assumed constant during deformation,
so that a true strain increment becomes $\delta\varepsilon_t = \frac{\delta L}{L(t)}$,
defined relative to the momentary (at time t) size of the sample, $L(t)$,
while the true stress becomes $\sigma_t = \frac{F(t)}{A(t)}$, with momentary cross-section $A(t)$.

Integrating the true strain increment $\delta\varepsilon_t$ from 0 to ε_t,
and the right hand side from L_0 to $L(t)$,
using $\ln L(t) - \ln L(0) = \ln(L(t)/L_0)$, with $L(t) = L_0 + \Delta L$,
yields the true strain $\varepsilon_t = \ln(1 + \varepsilon)$.
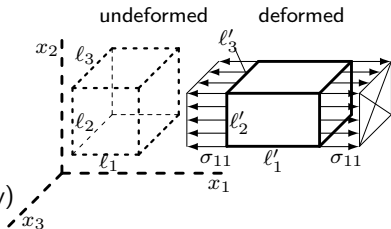
5) Material behavior
5.1) Hooke's law

Linear elastic, isotropic
material behavior.

Strain: $\varepsilon_{11} = \dfrac{\ell'_1 - \ell_1}{\ell_1}$

Relation stress and strain: $\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$

E is the
Young's modulus (modulus of elasticity)

5) Material behavior
5.1) Hooke's law

Linear elastic, isotropic
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Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$

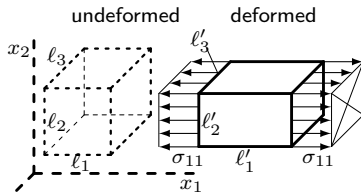Relation stress and strain: $\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$

E is the
Young's modulus (modulus of elasticity)

Lateral strain: $\varepsilon_{22} = \dfrac{\ell_2' - \ell_2}{\ell_2} \leq 0$ and $\varepsilon_{33} = \dfrac{\ell_3' - \ell_3}{\ell_3} \leq 0$

Relation lateral strain and stresses: $\varepsilon_{22} = \varepsilon_{33} = -\nu \dfrac{\sigma_{11}}{E}$

ν is the *Poisson's ratio*

Isotropic material behavior!!!
So, with a single normal stress σ_{11} (other stress components are zero!):
$\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu\dfrac{\sigma_{11}}{E}$
The shear strain seems to remain equal to zero.

So, with a single normal stress σ_{11} (other stress components are zero!):
$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\frac{\sigma_{11}}{E}$$
The shear strain seems to remain equal to zero.
Also, when we apply a single normal stress σ_{22}, again with the other stress components equal to zero:
$$\varepsilon_{22} = \frac{\sigma_{22}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{33} = -\nu\frac{\sigma_{22}}{E}$$

Isotropic material behavior!!!

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Likewise for normal stress σ_{33}:

$\varepsilon_{33} = \dfrac{\sigma_{33}}{E}$ and $\varepsilon_{11} = \varepsilon_{22} = -\nu\dfrac{\sigma_{33}}{E}$

Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

$\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu\dfrac{\sigma_{11}}{E}$

The shear strain seems to remain equal to zero.

Also, when we apply a single normal stress σ_{22}, again with the other stress components equal to zero:

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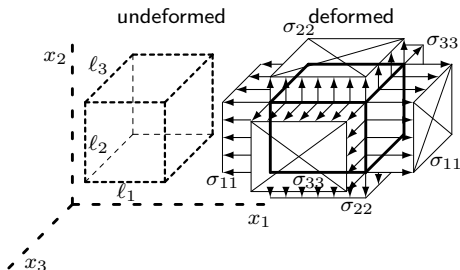Likewise for normal stress σ_{33}:

$\varepsilon_{33} = \dfrac{\sigma_{33}}{E}$ and $\varepsilon_{11} = \varepsilon_{22} = -\nu\dfrac{\sigma_{33}}{E}$

Also a combined stress state is possible. Superposition principle.

Result:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$

$$\varepsilon_{22} = -\nu\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$

$$\varepsilon_{33} = -\nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}$$

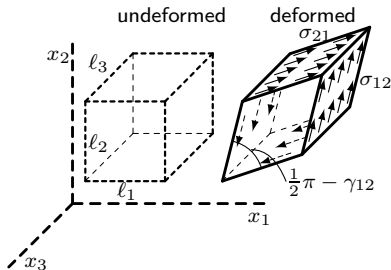This is Hooke's law in the case that only normal stresses are present!

Shear, shear stresses and shear strain:

$$2\varepsilon_{12} = \gamma_{12} = \frac{\sigma_{12}}{G} = \frac{\sigma_{21}}{G}$$
$$2\varepsilon_{13} = \gamma_{13} = \frac{\sigma_{13}}{G} = \frac{\sigma_{31}}{G}$$
$$2\varepsilon_{23} = \gamma_{23} = \frac{\sigma_{23}}{G} = \frac{\sigma_{32}}{G}$$

The material
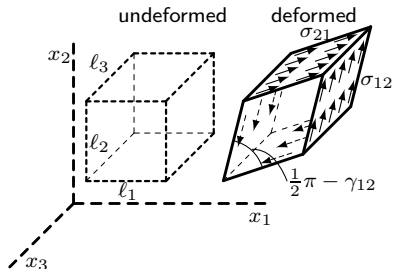constant G is the *shear modulus*.

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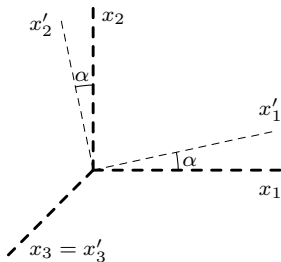constant G is the *shear modulus*.

NOTE: With isotropic material behavior E, G and ν are correlated with each other. This correlation can be found via a coordinate system rotation.

Linear stress state

$$[\sigma]_{x_i} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$
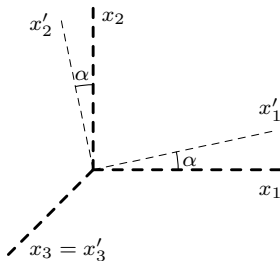
Linear stress state

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From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\frac{\sigma_{11}}{E}$$

so:

$$[\varepsilon] = \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu\sigma_{11} & 0 \\ 0 & 0 & -\nu\sigma_{11} \end{bmatrix} \quad \text{and} \quad [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve for σ'_{12} with $\sigma'_{12} = R_{1p} R_{q2} \sigma_{pq}$ or with $[\sigma'] = [R][\sigma][R]^{\mathrm{T}}$:

$$
\begin{bmatrix} \diamond & \sigma'_{12} & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \diamond & -\sin\alpha & \diamond \\ \diamond & \cos\alpha & \diamond \\ \diamond & 0 & \diamond \end{bmatrix}
$$

All non-relevant values for the matrix multiplication are shown as a $\diamond$.

Solve for σ'_{12} with $\sigma'_{12} = R_{1p}\, R_{q2}\, \sigma_{pq}$ or with $[\sigma'] = [R][\sigma][R]^{\mathrm{T}}$:

$$\begin{bmatrix} \diamond & \sigma'_{12} & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \diamond & -\sin\alpha & \diamond \\ \diamond & \cos\alpha & \diamond \\ \diamond & 0 & \diamond \end{bmatrix}$$

All non-relevant values for the matrix multiplication are shown as a $\diamond$. Or shorter:

$$\sigma'_{12} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} = -\sigma_{11}\cos\alpha\sin\alpha$$

The same holds for shear strain ε'_{12}. From $[\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}$ follows:

$$\varepsilon'_{12} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \end{bmatrix} \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu\sigma_{11} & 0 \\ 0 & 0 & -\nu\sigma_{11} \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} =$$

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$$= -\frac{(1+\nu)}{E}\sigma_{11}\cos\alpha\sin\alpha$$

Due to isotropic behavior it must hold that:

$$\varepsilon'_{12} = \tfrac{1}{2}\gamma'_{12} = \tfrac{1}{2}\frac{\sigma'_{12}}{G} = -\frac{1}{2G}\sigma_{11}\cos\alpha\sin\alpha \qquad (\sigma'_{12} = -\sigma_{11}\cos\alpha\sin\alpha)$$

The same holds for shear strain ε'_{12}. From $[\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}$ follows:

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When we compare both expressions, the following important equation is found:

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

NOTE: This equation only holds for *isotropic* material behavior!

Summary of Hooke's law:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) \;\; ; \;\; \gamma_{12} = 2\varepsilon_{12} = \frac{1}{G}\sigma_{12} \;\; \text{(also } 12 \to 21)$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11}) \;\; ; \;\; \gamma_{13} = 2\varepsilon_{13} = \frac{1}{G}\sigma_{13} \;\; \text{(also } 13 \to 31)$$

$$\varepsilon_{33} = \frac{1}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) \;\; ; \;\; \gamma_{23} = 2\varepsilon_{23} = \frac{1}{G}\sigma_{23} \;\; \text{(also } 23 \to 32)$$

Hooke's law in index notation. For strains:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) =$$
$$= \frac{(1+\nu)}{E}\left(\sigma_{11} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

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Hooke's law in index notation. For normal strains:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) =$$
$$= \frac{(1+\nu)}{E}\left(\sigma_{11} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} = \frac{(1+\nu)}{E}\left(\sigma_{22} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$
$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} = \frac{(1+\nu)}{E}\left(\sigma_{33} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \quad \text{if } i = j$$

For shear strains:

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} = \frac{(1+\nu)}{E}\sigma_{12} \quad ; \quad \varepsilon_{13} = \frac{\sigma_{13}}{2G} = \frac{(1+\nu)}{E}\sigma_{13}$$
$$\varepsilon_{23} = \frac{\sigma_{23}}{2G} = \frac{(1+\nu)}{E}\sigma_{23}$$

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So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} \ \text{ if } \ i \neq j \ \text{ and } \ \varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \ \text{ if } \ i = j$$

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So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} \text{ if } i \neq j \text{ and } \varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \text{ if } i = j$$

Combined they result in <u>Hooke's law in index notation</u> $(\sigma \to \varepsilon)$:

$$\boxed{\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right) = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

The *inverse*, stresses expressed in deformations.

We found:
$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)$$

The *inverse*, stresses expressed in deformations.

We found: $\varepsilon_{ij} = \dfrac{(1+\nu)}{E}\left(\sigma_{ij} - \dfrac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)$

A summation of normal strains gives:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{(1+\nu)}{E}\left(\sigma_{11}+\sigma_{22}+\sigma_{33} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) =$$

$$= \frac{(1+\nu)}{E}\left(\sigma_{kk} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) = \frac{(1-2\nu)}{E}\sigma_{kk}$$

The *inverse*, stresses expressed in deformations.

We found: $\varepsilon_{ij} = \dfrac{(1+\nu)}{E}\left(\sigma_{ij} - \dfrac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)$

A summation of normal strains gives:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{(1+\nu)}{E}\left(\sigma_{11}+\sigma_{22}+\sigma_{33} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) =$$

$$= \frac{(1+\nu)}{E}\left(\sigma_{kk} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) = \frac{(1-2\nu)}{E}\sigma_{kk}$$

Rewriting Hooke's law:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu}{(1+\nu)}\frac{E}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

Resulting in: <u>Hooke's law</u>　　　　$(\varepsilon \to \sigma)$:

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk}\delta_{ij} \right) = 2G \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk}\delta_{ij} \right)$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

Resulting in; <u>Hooke's law</u>   ($\varepsilon \to \sigma$) :

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right) = 2G\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right)$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.
Notation according to Lamé:

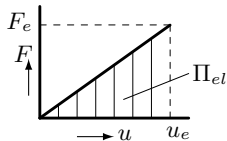$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$
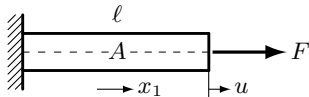
With the so called Lamé's (material) constants:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = G = \frac{E}{2(1+\nu)}$$

5.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \, \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$

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$$\sigma_{11} = \frac{F_e}{A} \quad \text{and} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$

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Such that:

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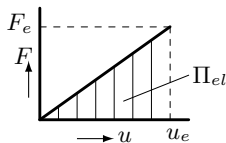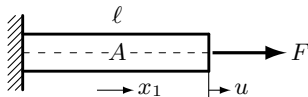
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Specific internal elastic energy: $\quad \pi_{el} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11}$
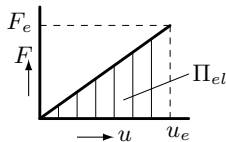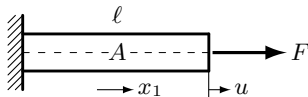
5.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \ \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$

At the end:

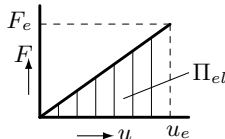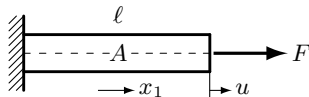$$\sigma_{11} = \frac{F_e}{A} \quad \text{and} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$

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Specific internal elastic energy: $\quad \pi_{el} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11}$
Contribution to π_{el} during shearing: $\quad \tfrac{1}{2} \sigma_{12} \, \gamma_{12}$

An arbitrary three dimensional stress state:

$$\pi_{el} = \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} + \tfrac{1}{2}\sigma_{12}\,\gamma_{12} + \tfrac{1}{2}\sigma_{13}\,\gamma_{13} + \tfrac{1}{2}\sigma_{23}\,\gamma_{23}$$

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Written differently (with $\gamma_{ij} = 2\,\varepsilon_{ij} = 2\,\varepsilon_{ji}$):

$$\begin{aligned}
\pi_{el} &= \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{12}\,\varepsilon_{12} + \tfrac{1}{2}\sigma_{13}\,\varepsilon_{13} + \\
&+ \tfrac{1}{2}\sigma_{21}\,\varepsilon_{21} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{23}\,\varepsilon_{23} + \\
&+ \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33}
\end{aligned}$$

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Written differently (with $\gamma_{ij} = 2\,\varepsilon_{ij} = 2\,\varepsilon_{ji}$):

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So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

An arbitrary three dimensional stress state:

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&+ \quad \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33}
\end{aligned}
$$

So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

Elastic energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\ \mathrm{d}V = \int_V \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\ \mathrm{d}V$$

By filling in Hooke's law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$, either the stresses or the strains can be eliminated.

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The *specific elastic volume-change energy* is:

$$\boxed{\pi_{el_{vol}} = \tfrac{1}{2}\sigma_m\,\varepsilon_V}\quad \text{with}$$
$$\sigma_m = \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \tfrac{1}{3}\sigma_{kk}\quad \text{and}\quad \varepsilon_V = \varepsilon_{kk}$$

With Hooke's law we have seen that:

$$\varepsilon_V = \varepsilon_{kk} = \frac{(1-2\nu)}{E}\sigma_{kk} = \frac{3(1-2\nu)}{E}\sigma_m = \frac{\sigma_m}{C} \qquad (\sigma_m = C\,\varepsilon_V)$$

Where C is the bulk modulus.

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Where C is the bulk modulus. Filling in $\pi_{el_{vol}}$:

$$\pi_{el_{vol}} = \tfrac{1}{2}\sigma_m\,\varepsilon_V = \tfrac{1}{2}\sigma_m\frac{3(1-2\nu)}{E}\sigma_m = \tfrac{1}{2}\frac{(1-2\nu)}{3\,E}\sigma_{hh}\sigma_{kk}$$

Turns out to be invariant for coordinate system transformations.

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Turns out to be invariant for coordinate system transformations.
The *specifc elastic shape-change (or distortional) energy*:

$$\pi_{el_{shape}} = \pi_{el} - \pi_{el_{vol}} = \cdots =$$
$$= \tfrac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}$$

5.3) Deviatoric stresses and strains

Deviatoric stresses: $\quad \hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

Deviatoric strains: $\quad \hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

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Deviatoric strains: $\qquad \hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

Also the deviatoric stresses have principal stresses and principal directions. The corresponding coefficients in the characteristic equation

$$\hat{\sigma}^3 - J_2\,\hat{\sigma} - J_3 = 0$$

i.e., the deviatoric invariants are $J_1 = 0$, $J_2 > 0$ and J_3.

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It turns out that: $\quad$ Law of Hooke: $\quad \hat{\sigma}_{ij} = 2\,G\,\hat{\varepsilon}_{ij}$

$$\pi_{el_{shape}} = \frac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\} = \frac{(1+\nu)}{E}J_2$$

5.3) Deviatoric stresses and strains

Deviatoric stresses: $\quad\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m\delta_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

Deviatoric strains: $\quad\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V\delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

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And: $\quad (\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij} \;\; ; \;\; \pi_{el_{vol}} = \tfrac{1}{2}\sigma_m\,\varepsilon_V)$

$$\boxed{\pi_{el_{shape}} = \tfrac{1}{2}\frac{(1+\nu)}{E}\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \tfrac{1}{2}\frac{1}{2G}\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \tfrac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}}$$

5.4)   Failure criteria

1) Tresca, bases on maximum shear stress.

$$\sigma_{\mathsf{eq}} = 2\,\tau_{\mathsf{max}} = \max\{|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\}$$

Or, if $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

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5.4) Failure criteria

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Or, if $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\sigma_{\mathsf{eq}} = 2\,\tau_{\mathsf{max}} = (\sigma_1 - \sigma_3)$$

2) Von Mises, based on the maximum specific elastic distortional energy.

$$\sigma_{\mathsf{eq}} = \sqrt{3\,\frac{E}{1+\nu}\pi_{el_{shape}}} = \sqrt{3J_2} =$$
$$= \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}}$$

Shear stress criteria
Tresca, Coulomb, Quest, Mohr
Maximum shear stress is leading.

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Comparison 3-dim stress state with 1-dim stress state in a tensile test. $(\sigma_1 = \sigma_{eq} \; ; \sigma_2 = \sigma_3 = 0)$

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) \quad \text{(3-dim)}$$
$$\tau_{max} = \tfrac{1}{2}\sigma_{eq} \quad\quad\; \text{(1-dim)}$$

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$\sigma_{\text{eq}} \leq \bar{\sigma}$ (allowable stress)

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von Mises, Huber, Hencky
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$$\pi_{el_{shape}} = \frac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\} \quad \text{(3-dim)}$$
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Comparison 3-dim stress state with 1-dim stress state in a tensile
test. $\quad (\sigma_1 = \sigma_{\text{eq}} \; ; \sigma_2 = \sigma_3 = 0)$

$\pi_{el_{shape}} = \frac{1}{6} \frac{(1+\nu)}{E} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \}$ $\quad$ (3-dim)

$\pi_{el_{shape}} = \frac{1}{3} \frac{(1+\nu)}{E} \sigma_{\text{eq}}^2$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (1-dim)

$\rightarrow \quad \sigma_{\text{eq}} = \sqrt{\frac{1}{2} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \}}$

$\sigma_{\text{eq}} \le \bar{\sigma}$ $\quad$ (allowable stress)

Special case
Determine the equivalent stresses of a beam with circular cross section. A bending and torsional moment are acting on the beam (Hibbeler pg.195 (5.7) $T\rho/J$):

$$\sigma_{11} = \frac{M_b \, R}{I_b} \quad \text{with} \quad I_b = \tfrac{\pi}{4} \, R^4$$
$$\sigma_{12} = \frac{M_t \, R}{I_t} \quad \text{with} \quad I_t = \tfrac{\pi}{2} \, R^4 = 2 \, I_b$$

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$$\sigma_{12} = \frac{M_t R}{I_t} \quad \text{with} \quad I_t = \frac{\pi}{2} R^4 = 2 I_b$$

The complete stress state becomes:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses?

$$\det \left(\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix} \right) = 0 \quad \rightarrow$$

$$\det \left(\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix} \right) = 0 \quad \rightarrow$$

Solving determinant: $\quad (-\sigma)\{(\sigma_{11} - \sigma)(-\sigma) - \sigma_{12}^2\} = 0 \quad \rightarrow$

$$\sigma_1 = 0 \;\; ; \;\; \sigma^2 - \sigma_{11}\,\sigma - \sigma_{12}^2 = 0 \quad \rightarrow \quad \sigma_{2,3} = \tfrac{1}{2}\sigma_{11} \pm \tfrac{1}{2}\sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}$$

$$\det\left(\left[\begin{array}{ccc} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{array}\right]\right) = 0 \quad \rightarrow$$

Solving determinant: $\quad (-\sigma)\{(\sigma_{11} - \sigma)(-\sigma) - \sigma_{12}^2\} = 0 \quad \rightarrow$

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The solutions in the right order:

$$\sigma_1 = \frac{1}{2}\left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \frac{1}{2}\left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right)$$

We just found:

$$\sigma_1 = \tfrac{1}{2}\left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \tfrac{1}{2}\left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right)$$

Tresca: $\quad \sigma_{\mathsf{eq}} = \sigma_1 - \sigma_3 = \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2}$

von Mises: $\quad \sigma_{\mathsf{eq}} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}} =$
$\quad\quad\quad\quad\quad = \sqrt{\sigma_{11}^2 + 3\,\sigma_{12}^2}$

Note the difference!!

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Note the difference!! $\quad$ Expressed in the bending and torsional moment:

$$\text{Tresca:} \quad \sigma_{\text{eq}} = \tfrac{R}{I_b}\sqrt{M_b^2 + M_t^2}$$

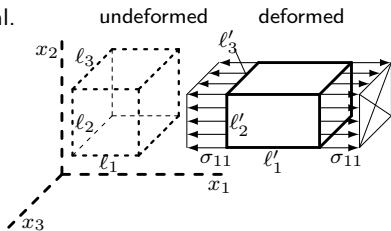$$\text{von Mises:} \quad \sigma_{\text{eq}} = \tfrac{R}{I_b}\sqrt{M_b^2 + \tfrac{3}{4}M_t^2}$$

6) General material behavior
6.1) Hooke's law $\Rightarrow$ viscosity

Linear elastic, isotropic viscous material.
In the special case of 1D:

Strain: $\varepsilon_{11} = \dfrac{\ell'_1 - \ell_1}{\ell_1}$

6) General material behavior
6.1) Hooke's law $\Rightarrow$ viscosity

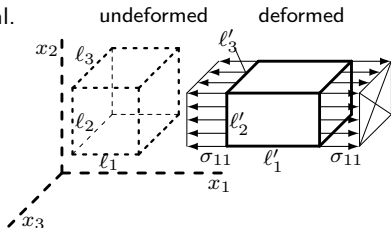Linear elastic, isotropic viscous material.
In the special case of 1D:

Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$

Strain-rate: $\dot{\varepsilon}_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1 \, \Delta t}$

Relation between stress and strain:
$\sigma_{11} = E\varepsilon_{11} + \eta\dot{\varepsilon}_{11}$

6) General material behavior
6.1) Hooke's law $\Rightarrow$ viscosity

Linear elastic, isotropic viscous material.
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Strain-rate: $\dot{\varepsilon}_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1 \, \Delta t}$

Relations between stress and strain:

$\sigma_{11} = E\varepsilon_{11} + ... + \eta\dot{\varepsilon}_{11}$

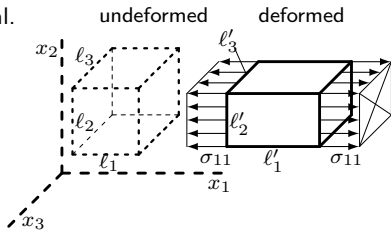$\sigma_{12} = G\varepsilon_{12} + \eta\dot{\varepsilon}_{12}$

etc. ...

E is the *Young's modulus* (modulus of elasticity) and η is the viscosity.
Also a combined stress state is possible. Superposition principle.
Resulting in; Hooke's law + viscous stress   ($\varepsilon \to \sigma$) :

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk}\delta_{ij} \right) + \eta\dot{\varepsilon}_{ij}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

6.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \, \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$

At the end:
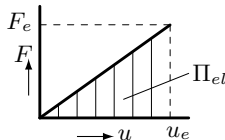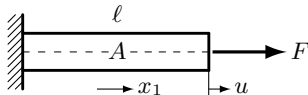
$$\sigma_{11} = \frac{F_e}{A} \quad \text{and} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$

Such that:

$$\Pi_{el} = \tfrac{1}{2} A \, \sigma_{11} \, \ell \, \varepsilon_{11} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11} \, V$$

Specific internal elastic energy: $\quad \pi_{el} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11}$
Contribution to π_{el} during shearing: $\quad \tfrac{1}{2} \sigma_{12} \, \gamma_{12}$

An arbitrary three dimensional stress state:

$$\pi_{el} = \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} + \tfrac{1}{2}\sigma_{12}\,\gamma_{12} + \tfrac{1}{2}\sigma_{13}\,\gamma_{13} + \tfrac{1}{2}\sigma_{23}\,\gamma_{23}$$

Written differently (with $\gamma_{ij} = 2\,\varepsilon_{ij} = 2\,\varepsilon_{ji}$):

$$
\begin{aligned}
\pi_{el} \quad = \quad & \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{12}\,\varepsilon_{12} + \tfrac{1}{2}\sigma_{13}\,\varepsilon_{13} + \\
+ \quad & \tfrac{1}{2}\sigma_{21}\,\varepsilon_{21} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{23}\,\varepsilon_{23} + \\
+ \quad & \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33}
\end{aligned}
$$

So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

Elastic energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\ \mathrm{d}V = \int_V \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\ \mathrm{d}V$$

Visco-elastic energy:

Hooke's and (simplest) Newtonian viscous law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$, and
$\pi_{visc} = \int \sigma_{ij}^{visc}\,\mathrm{d}\varepsilon_{ij} = \eta\dot{\varepsilon}_{ij}\int \mathrm{d}\varepsilon_{ij} = \eta\dot{\varepsilon}_{ij}\,[\varepsilon_{ij} - 0]$,
$\qquad\qquad$ for constant η and ε_{ij} independent of $\dot{\varepsilon}_{ij}$.

Elastic and viscous energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\,\mathrm{d}V = \int_V \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\,\mathrm{d}V \;\;\text{and}\;\; \Pi_{visc} = \int_V \pi_{visc}\,\mathrm{d}V$$

1D only: Elastic and viscous energy of a body with volume V:

$$\Pi = \Pi_{el} + \Pi_{visc} = \int_V \left(\pi_{el} + \pi_{visc}\right)\,\mathrm{d}V$$

$$= \int_V \left(\tfrac{1}{2}E\varepsilon_{11}^2 + \eta\dot{\varepsilon}_{11}\varepsilon_{11}\right)\,\mathrm{d}V$$