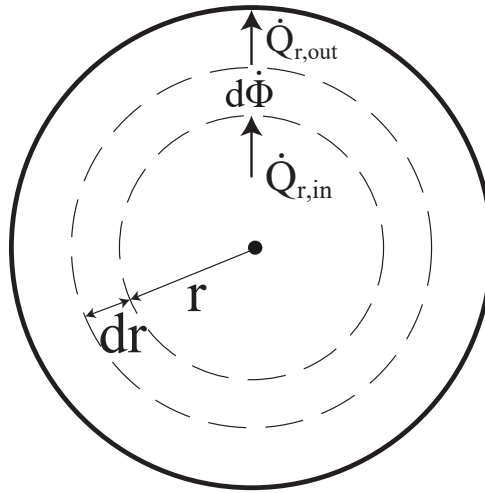## 2.10 Resistance wire ★

a) *Derive the heat conduction equation by setting up an energy balance.*

1) Setting up the energy balance:

Energy balance of infinitesimal element within the wire:

$$\cancelto{0}{\frac{\partial U}{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.175}$$

$$\boxed{\to 0 = \dot{Q}_{\text{r,in}} - \dot{Q}_{\text{r,out}} + d\dot{\Phi}} \tag{2.176}$$

2) Defining the fluxes:

The ingoing conductive heat flux can be described by use of Fourier's law:

$$\dot{Q}_{\text{r,in}} = -\lambda \cdot A(r) \cdot \frac{\partial T}{\partial r} \tag{2.177}$$

$$\boxed{\to \dot{Q}_{\text{r,in}} = -\lambda \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{\partial T}{\partial r}} \tag{2.178}$$

The outgoing heat flux can be approximated for an infinitesimal element by the Taylor series expansion:

$$\dot{Q}_{\text{r,out}} = \dot{Q}_{\text{r,in}} + \frac{\partial \dot{Q}_{\text{r,in}}}{\partial r} \cdot dr \tag{2.179}$$

$$\boxed{\to \dot{Q}_{\text{r,out}} = -\lambda \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{\partial T}{\partial r} + \frac{\partial}{\partial r}\left(-\lambda \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{\partial T}{\partial r}\right) \cdot dr} \tag{2.180}$$

Heat source for the infinitesimal element:

$$d\dot{\Phi} = \dot{\Phi}''' \cdot dV \tag{2.181}$$

Where dV is:

$$dV = \pi \cdot L \cdot \left((r+dr)^2 - \pi \cdot r^2\right) = \pi \cdot L \cdot \left(2 \cdot r \cdot dr + dr^2\right) \approx \pi \cdot L \cdot \left(2 \cdot r \cdot dr\right) \tag{2.182}$$

Note that $dr^2 << 2 \cdot r \cdot dr$ and therefore can be neglected:

$$dV \approx \pi \cdot L \cdot \left(2 \cdot r \cdot dr\right) = 2 \cdot \pi \cdot r \cdot L \cdot dr \tag{2.183}$$

$$\boxed{\rightarrow d\dot{\Phi} = \dot{\Phi}''' \cdot 2 \cdot \pi \cdot r \cdot L \cdot dr} \tag{2.184}$$

3) Inserting and rearranging:

Substitution of the definition of the heat fluxes into the energy balance :

$$0 = \dot{Q}_{r,\text{in}} - \dot{Q}_{r,\text{out}} + d\dot{\Phi} \tag{2.185}$$

$$0 = \frac{\partial}{\partial r}\left(\lambda \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{\partial T}{\partial r}\right) \cdot dr + \dot{\Phi}''' \cdot 2 \cdot \pi \cdot r \cdot L \cdot dr \tag{2.186}$$

Rewriting of yields:

$$\boxed{\rightarrow 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda \cdot r \cdot \frac{\partial T}{\partial r}\right) + \dot{\Phi}'''} \tag{2.187}$$

b) *Determine the temperature at $r = 3.5$mm.*

In order to determine the temperature at $r = 3.5$mm, the function of the temperature profile should be determined, by solving the heat conduction equation.

4) Defining the boundary conditions:

Boundary conditions:

$$\boxed{\rightarrow \frac{\partial T}{\partial r}(r = 0) = 0} \tag{2.188}$$

$$\boxed{\rightarrow T(r = r_0) = T_s} \tag{2.189}$$

5) Solving the equation:

1-dimensional heat transportation equation with source cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right) + \dot{\Phi}''' = 0 \tag{2.190}$$

Integration:

$$\frac{\partial T}{\partial r} = -\frac{\dot{\Phi}'''}{2\lambda}r + C_1 \tag{2.191}$$

$$T(r) \;=\; -\frac{\dot{\Phi}'''}{4\lambda}r^2 \;+\; C_1 r + C_2 \tag{2.192}$$

Using the boundary conditions results in:

$$\rightarrow C_1 \;=\; 0 \tag{2.193}$$

$$\rightarrow C_2 \;=\; T_{\mathrm{s}} + \frac{\dot{\Phi}'''}{4\lambda}r_0^2 \tag{2.194}$$

Giving the temperature equation:

$$\boxed{\rightarrow T(r) \;=\; -\frac{\dot{\Phi}'''}{4\lambda}r^2 \;+\; T_s + \frac{\dot{\Phi}'''}{4\lambda}r_0^2} \tag{2.195}$$

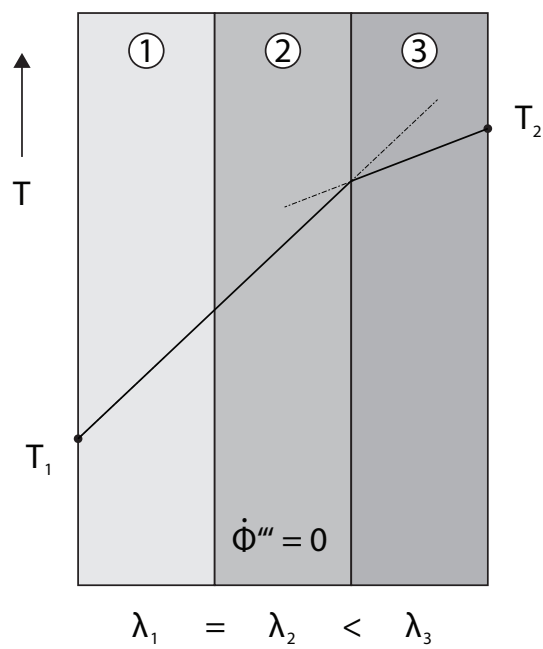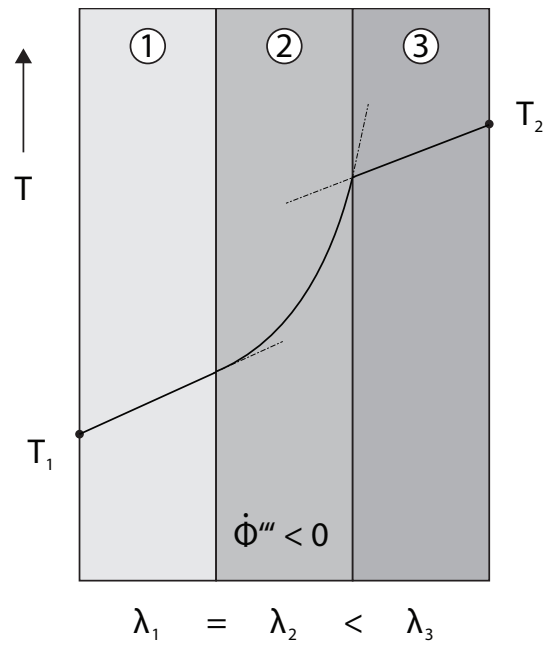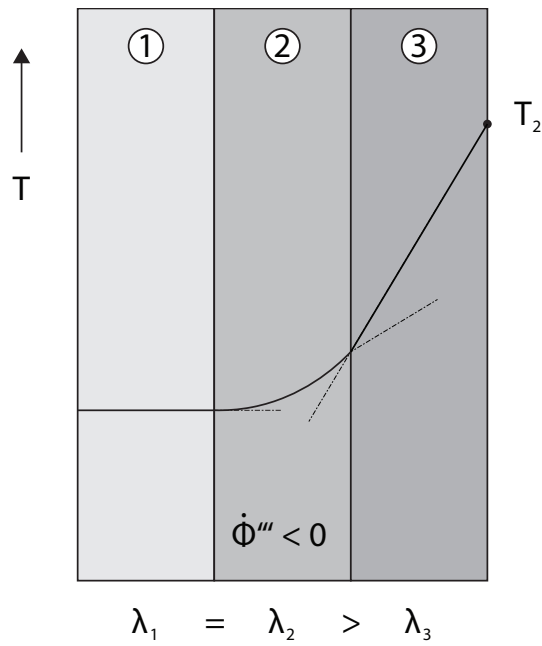## 6). Determining the temperature at the specified location:

And thus:

$$T(r = 0.0035 \text{ m}) \;=\; -\frac{5 \cdot 10^7 \ [\mathrm{Wm}^{-3}]}{4 \cdot 6 \ [\mathrm{Wm}^{-1}\mathrm{K}^{-1}]} \cdot 0.0035^2 \ [\mathrm{m}^2] + 453.15 \ [\mathrm{K}] \;+\; \frac{5 \cdot 10^7 \ [\mathrm{Wm}^{-3}]}{4 \cdot 66 \ [\mathrm{Wm}^{-1}\mathrm{K}^{-1}]} \cdot 0.005^2 \ [\mathrm{m}^2] \tag{2.196}$$

$$\boxed{\rightarrow T(r = 0.0035 \text{ m}) \;=\; 478.71 \ [\mathrm{K}]} \tag{2.197}$$

2.11 Draw temperature profiles   ★★

$$\dot{\Phi}''' < 0$$

$$\lambda_1 \quad = \quad \lambda_2 \quad > \quad \lambda_3$$

$$\dot{\Phi}''' < 0$$

$$\lambda_1 \quad = \quad \lambda_2 \quad < \quad \lambda_3$$

$$\dot{\Phi}''' = 0$$

$$\lambda_1 \quad = \quad \lambda_2 \quad < \quad \lambda_3$$

2.12 Copper rod   ★ ★ ★

a) *Derive the equation for the temperature profile in the rod by setting up an energy balance.*

As the thermal boundary conditions differ at the right (convective heat transfer) and left (adiabatic insulation $\Rightarrow \dot{Q} = 0$) side of the rod, both halves of the rod need to be regarded separately from each other. The coupling of both individual systems is ensured by the boundary conditions at $x = 0$. (For the left end x=-L)
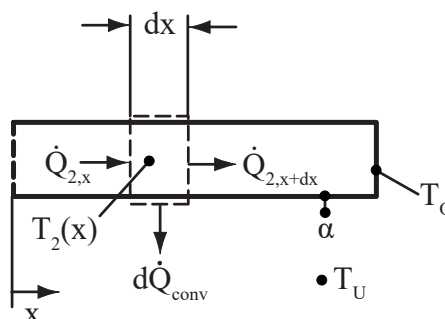
1) Setting up the energy balance:
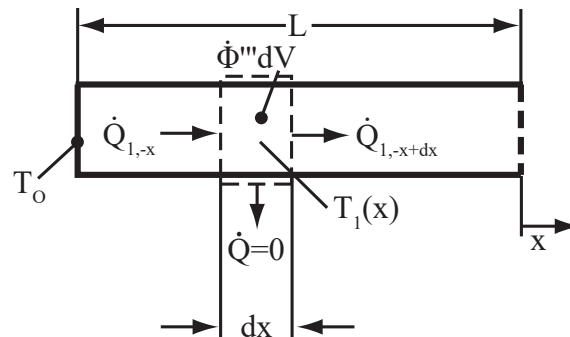
Energy balance for the <u>left-side</u> of the rod:

$$\cancel{\frac{\partial U}{\partial t}}^{0} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.198}$$

$$\boxed{\rightarrow 0 = \dot{Q}_{1,\text{x}} - \dot{Q}_{1,\text{x+dx}} + \dot{\Phi}''' \cdot dV} \tag{2.199}$$

Energy balance for the right-side of the rod:

$$\frac{\cancel{\partial U}^{\,0}}{\cancel{\partial t}} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}} \tag{2.200}$$

$$\boxed{\rightarrow 0 = \dot{Q}_{2,\text{x}} - \dot{Q}_{2,\text{x+dx}} + \dot{\Phi}''' \cdot dV} \tag{2.201}$$

2) Defining the fluxes:

Left-side of the rod:

$$\boxed{\rightarrow \dot{Q}_{1,-\text{x}} = -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\partial T_1}{\partial x}} \tag{2.202}$$

$$\boxed{\rightarrow \dot{Q}_{1,-\text{x+dx}} = \dot{Q}_{1,-\text{x}} + \frac{\partial \dot{Q}_{1,-\text{x}}}{\partial x} dx} \tag{2.203}$$

$$\boxed{\rightarrow \dot{\Phi}''' dV = \dot{\Phi}''' \frac{\pi d^2}{4} dx} \tag{2.204}$$

Right-side of the rod:

$$\boxed{\rightarrow \dot{Q}_{2,\text{x}} = -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\partial T_2}{\partial x}} \tag{2.205}$$

$$\boxed{\rightarrow \dot{Q}_{2,\text{x+dx}} = \dot{Q}_{2,\text{x}} + \frac{\partial \dot{Q}_{2,\text{x}}}{\partial x} dx} \tag{2.206}$$

$$\boxed{\rightarrow d\dot{Q}_{\text{conv}} = \alpha \cdot \pi d \cdot dx \left(T_2 - T_{\text{U}}\right)} \tag{2.207}$$

3) Inserting and rearranging:

Substituting these definitions into the left-side energy balance and rearranging yields in:

$$0 = \dot{Q}_{1,\text{x}} - \dot{Q}_{1,\text{x+dx}} + \dot{\Phi}''' \cdot dV \tag{2.208}$$

$$\boxed{\rightarrow \frac{\partial^2 T_1}{\partial x^2} = -\frac{\dot{\Phi}'''}{\lambda}} \tag{2.209}$$

Substituting these definitions into the right-side energy balance and rearranging yields in:

$$0 = \dot{Q}_{2,\text{x}} - \dot{Q}_{2,\text{x+dx}} + \dot{\Phi}''' \cdot dV \tag{2.210}$$

$$\boxed{\rightarrow 0 = \frac{\partial^2 T_2}{\partial x^2} - \frac{4 \cdot \alpha}{\lambda \cdot d} \left(T_2 - T_{\text{A}}\right)} \tag{2.211}$$

4) Defining the boundary and coupling conditions:

In order to determine the temperature profiles inside the system, we need in total 4 boundary and/or coupling conditions.

The temperature at the left and right-side of the system are known:

$$\boxed{\rightarrow T_1(x = -L) = T_{\mathrm{O}}} \tag{2.212}$$

$$\boxed{\rightarrow T_2(x = L) = T_{\mathrm{O}}} \tag{2.213}$$

The temperature at the intersection for the left and right side should be equal to each other:

$$\boxed{\rightarrow T_1(x = 0) = T_2(x = 0)} \tag{2.214}$$

The heat flux flowing from the rod's left half at $x = 0$ must be equal to the flow into the rod's right half.

$$\boxed{\rightarrow \left(\frac{\partial T_1}{\partial x}\right)_{x=0} = \left(\frac{\partial T_2}{\partial x}\right)_{x=0}} \tag{2.215}$$

5) Solving the equation:

Integrating the differential equation for the <u>left-side</u> twice yields:

$$\frac{\partial^2 T_1}{\partial x^2} = -\frac{\dot\Phi'''}{\lambda} \tag{2.216}$$

$$\frac{\partial T_1}{\partial x} = -\frac{\dot\Phi'''}{\lambda} \cdot x + c_1 \tag{2.217}$$

$$\boxed{\rightarrow T_1 = -\frac{\dot\Phi'''}{2 \cdot \lambda} \cdot x^2 + c_1 \cdot x + c_2} \tag{2.218}$$

The differential equation for the <u>right-side</u> is a linear second order differential equation with constant coefficients. Therefore, this differential equation can be solved with the same approach as for a fin system. Where the first step is introducing the homogenized temperature and the fin parameter ($\Theta_2 = T_2(x) - T_{\mathrm{A}}$, $\frac{\partial \Theta_2}{\partial x} = \frac{\partial T_2}{\partial x}$ and $m = \sqrt{\frac{4 \cdot \alpha}{\lambda \cdot d}}$)

$$0 = \frac{\partial^2 T_2}{\partial x^2} - \frac{4 \cdot \alpha}{\lambda \cdot d}(T_2 - T_{\mathrm{A}}) \tag{2.219}$$

$$0 = \frac{\partial^2 \Theta_2}{\partial x^2} - m \cdot \Theta_2 \tag{2.220}$$

which has its standard solution:

$$\Theta_2 = c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx) \tag{2.221}$$

And therefore:

$$\boxed{\rightarrow T_2 = T_{\mathrm{A}} + c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx)} \tag{2.222}$$

Using the boundary and coupling conditions to find constants c_1, c_2, c_3 and c_4 results in:

$$\rightarrow c_1 = \frac{1}{L} \cdot \frac{(T_O - T_A) - \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right) \cdot \cosh(m \cdot L)}{\frac{\sinh(m \cdot L)}{m \cdot L} + \cosh(m \cdot L)} \tag{2.223}$$

$$\rightarrow c_2 = T_A + \frac{(T_O - T_A) + \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right) \cdot \frac{\sinh(m \cdot L)}{m \cdot L}}{\frac{\sinh(m \cdot L)}{m \cdot L} + \cosh(m \cdot L)} \tag{2.224}$$

$$\rightarrow c_3 = \frac{c_1}{m} \tag{2.225}$$

$$\rightarrow c_4 = c_2 - T_A \tag{2.226}$$

And with the substitution of these constants in both temperature functions, the temperature profile in the entire rod is defined for a given heat source flux density $\dot{\Phi}'''$.

b) *Determine the value of $\dot{\Phi}'''$ so that the temperature in the center of the rod equals the temperature T_O at its ends.*

In order to find $\dot{\Phi}'''$, we should find the expressions for $T_1(x = 0)$ and $T_2(x = 0)$. So:

Through the condition $T_1(x = 0) = T_2(x = 0) = T_O$, boundary condition 2 is further constrained. So:

$$T_1(x = 0) = T_2(x = 0) = c_2 = T_O \tag{2.227}$$

With the relation for the temperature profile the stipulation for the source flux density $\dot{\Phi}'''$ is obtained.

$$c_2 = T_O = T_A + \frac{T_O - T_A + \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right) \cdot \frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)} \tag{2.228}$$

This expression can be rewritten to:

$$\dot{\Phi}''' = \frac{2 \cdot \lambda \cdot m}{L} \cdot (T_O - T_A) \cdot \frac{\cosh(mL) - 1}{\sinh(mL)} \tag{2.229}$$

And further simplification with the addition theorems for hyperbolic functions:

$$\rightarrow \dot{\Phi}''' = \frac{2 \cdot \lambda \cdot m}{L} \cdot (T_O - T_A) \cdot \tanh\left(\frac{mL}{2}\right) \tag{2.230}$$

c) *Calculate $\dot{\Phi}'''$ using the following data:* $L = 1\,\text{m}$; $d = 5.2\,\text{mm}$; $T_O = 120\,°\text{C}$; $T_A = 100\,°\text{C}$; $\alpha = 6\,\text{W}/\text{m}^2\,\text{K}$; $\lambda = 372\,\text{W}/\text{m}\,\text{K}$ *for the conditions postulated in b).*

$$m = \sqrt{\frac{4 \cdot \alpha}{\lambda \cdot d}} = 3.52\ [1/\text{m}] \tag{2.231}$$

$$\dot{\Phi}''' = \frac{2 \cdot \lambda \cdot m}{L} \cdot (T_O - T_A) \cdot \tanh\left(\frac{mL}{2}\right) \tag{2.232}$$

$$\dot{\Phi}''' = \frac{2 \cdot 372\ [\text{W/mK}] \cdot 3.52\ [1/\text{m}]}{1\ [1]} \cdot (120\ [°\text{C}] - 100\ [°\text{C}]) \cdot \tanh\left(\frac{3.52\ [1/\text{m}] \cdot 1\ [\text{m}]}{2}\right) \tag{2.233}$$

$$\boxed{\rightarrow \dot{\Phi}''' = 49.4\ [\text{kW/m}^3]} \tag{2.234}$$

Determine the extremes of the temperature distribution for the given values. Give their position and values, additionally, sketch the temperature profile.

For reasons of symmetry the location of the maximum temperature is

$$x_{\max} = -\frac{L}{2} \tag{2.235}$$

Similiarly the location of the minimum temperature is

$$x_{\min} = \frac{L}{2}. \tag{2.236}$$

Maximum temperature:

$$\boxed{\rightarrow T_{\max} = T_1\left(x = -\frac{L}{2}\right) = T_O + \frac{\dot{\Phi}''' \cdot L^2}{8\lambda} = 136.6\ [°\text{C}]} \tag{2.237}$$

Minimum temperature

$$\boxed{\rightarrow T_{\min} = T_2\left(x = \frac{L}{2}\right) = T_U + \frac{T_O - T_A}{\cosh\left(\frac{mL}{2}\right)} = 106.7\ [°\text{C}]} \tag{2.238}$$

Which results in the following temperature profile: