Fluid Mechanics 1

Lecture #3 :

Continuity Equation

Integral formulation of mass conservation

see video

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho \, dV = \int_{V(t)} \frac{\partial \rho}{\partial t} \, dV + \int_{S(t)} \rho \, u_j \cdot n_j \cdot dS = 0}$$

Global equation.

We also need a detailed equation.

flow field

We will derive detailed equation from the global equation

video mathematics: gradient and divergence and Gauss Theorem!

1 Gradient $\qquad \phi(\vec{x},t) \qquad$ e.g. $p(\vec{x},t)$

$$\vec{\nabla}\phi \equiv \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix} \equiv \left(\frac{\partial \phi}{\partial x} \; \frac{\partial \phi}{\partial y} \; \frac{\partial \phi}{\partial z} \right)^T$$

(*)

Note: input is scalar (ϕ)
output is vector ($\vec{\nabla}\phi$)

2 Divergence

$\vec{u}(\vec{x},t)$

$$\vec{\nabla}\cdot\vec{u} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

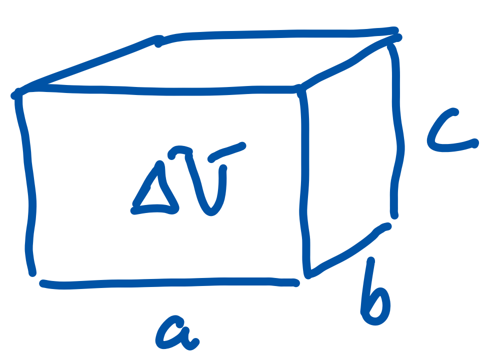
$$\equiv \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} \equiv \frac{\partial u_j}{\partial x_j}$$

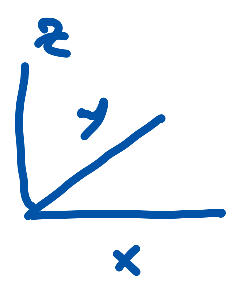
ESC

Note: input is vector $\vec{u}$
output is scalar $\vec{\nabla}\cdot\vec{u}$

What is the physical meaning of the divergence?

Consider a sufficiently small blob:





$$\frac{d}{dt}\Delta V = ?$$

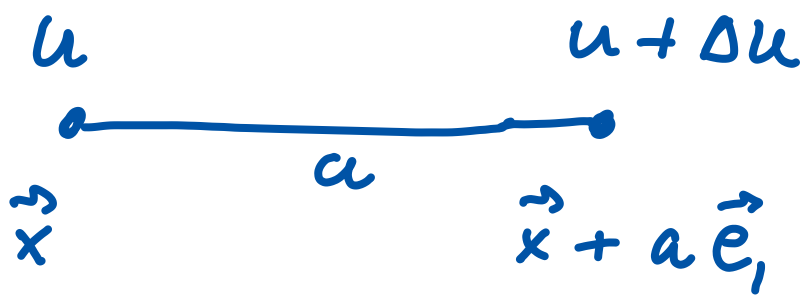
$$\frac{d}{dt}\Delta V = \frac{d}{dt}(abc)$$

chain rule

$$= \frac{da}{dt}bc + a\frac{db}{dt}c + ab\frac{dc}{dt}$$

Side "a" :

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



$$\Rightarrow \frac{da}{dt} = (u + \Delta u) - u = \Delta u \qquad \Delta u ?$$

$$\Delta u \equiv u(x + a\vec{e}_1, y, z, t) - u(x, y, z, t)$$

$$= \frac{\partial u}{\partial x}(x, y, z, t)\, a + \mathcal{O}(a^2)$$

↑ Taylor series approximation.

$$\Rightarrow \frac{da}{dt} = \frac{\partial u}{\partial x} a + \ldots$$

similar expressions for $\frac{db}{dt}$, $\frac{dc}{dt}$

$$\Rightarrow \frac{d}{dt}(\Delta V) = \left(\frac{\partial u}{\partial x} a\right) bc + a\left(\frac{\partial v}{\partial y} b\right) c + ab\left(\frac{\partial w}{\partial z} c\right) + \ldots$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) abc + \ldots$$

$$\Rightarrow \frac{d}{dt} \Delta V = (\vec{\nabla} \cdot \vec{u})\, \Delta V + \ldots$$

$\Delta V \to 0$:

$$\boxed{\frac{1}{\Delta V} \frac{d}{dt}(\Delta V) = \vec{\nabla} \cdot \vec{u}}$$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{u} \text{ is the relative change with time of a sufficiently small material blob.}}$

Question: $\frac{1}{\cancel{\Delta \bar{V}}}\frac{d}{dt}\cancel{\Delta \bar{V}} \overset{?}{=} \frac{d}{dt}$

absolutely NOT!

note $\frac{d}{dt}\Delta\bar{V} \equiv \lim_{\Delta t \to 0} \frac{\Delta\bar{V}(t+\Delta t) - \Delta\bar{V}(t)}{\Delta t}$

Similarly: $\frac{\sin x}{n} = \text{six} = 6$ (joke)

<u>But</u>: $\frac{1}{\Delta\bar{V}}\frac{d}{dt}\Delta\bar{V} \equiv \frac{d}{dt}(\ln \Delta\bar{V})$

check with chain rule:

$$\frac{d}{dt}\ln\Delta\bar{V} = \frac{d}{d\Delta\bar{V}}\ln\Delta\bar{V} \cdot \frac{d\Delta\bar{V}}{dt} = \frac{1}{\Delta\bar{V}}\frac{d\Delta\bar{V}}{dt}.$$

What does all of this mean?

Suppose $\rho = \text{const}$. $\Rightarrow$ incompressible.

$$\Rightarrow \frac{d}{dt} \Delta V = 0 \quad \Rightarrow \boxed{\vec{\nabla} \cdot \vec{u} = 0}$$

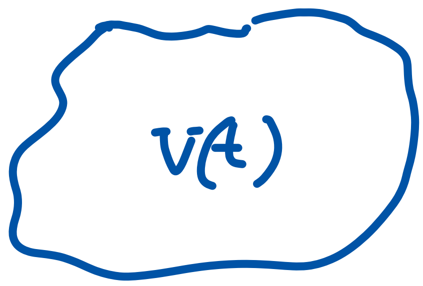
why do we use partial derivatives?

$$\cancel{\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}}$$ wrong!

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y, z, t) - u(x, y, z, t)}{\Delta x}$$

So far: sufficiently small blob ΔV

What about an arbitrary blob V?



$$\frac{dV}{dt} \overset{?}{=}$$

$$\frac{dV}{dt} = \frac{d}{dt}(V) = \frac{d}{dt}\left(\sum_j \Delta V_j\right)$$

$$= \sum_j \frac{d}{dt}(\Delta V_j) = \sum_j (\vec{\nabla}\cdot\vec{u})_j \, \Delta V_j$$

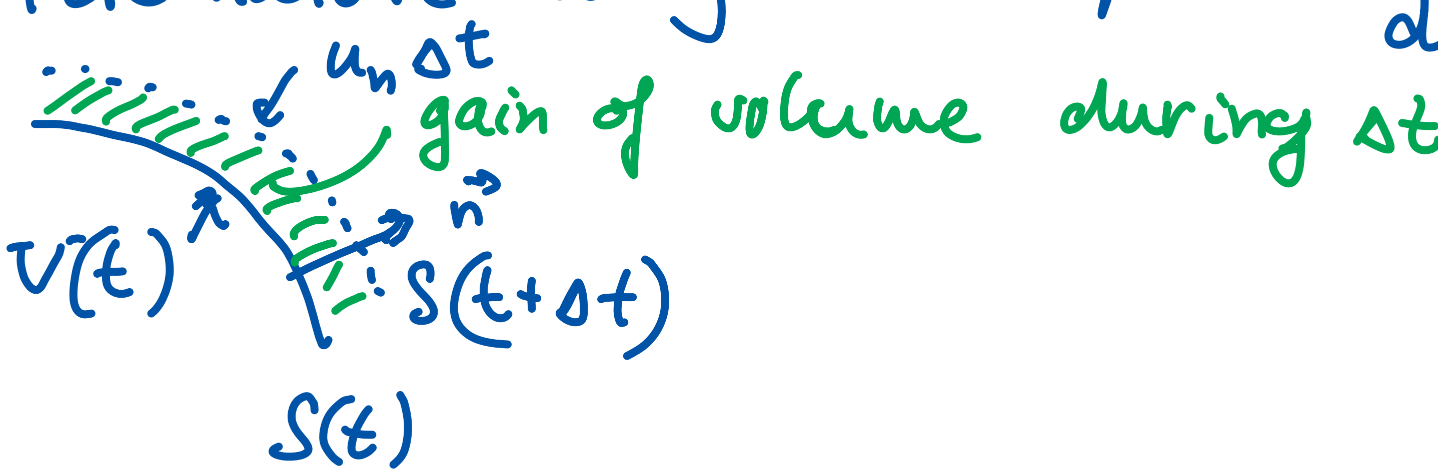
in the limit of $\Delta V_j \to 0$:

$$\boxed{\frac{dV}{dt} = \int_{V(t)} \vec{\nabla}\cdot\vec{u} \, dV}$$

dimensions

$$\frac{m^3}{s} = \frac{1}{m}\,\frac{m}{s}\cdot m^3$$

Alternative way to compute $\frac{dV}{dt}$:



$$\Rightarrow \frac{d\bar{V}}{dt} = \lim_{\Delta t \to 0} \frac{\bar{V}(t+\Delta t) - \bar{V}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\cancel{\Delta t}} \left\{ \int_{S(t)} \vec{u} \cdot \vec{n} \, \cancel{\Delta t} \, dS \right\}$$

$$\Rightarrow \boxed{\frac{d\bar{V}}{dt} = \int_{S(t)} \vec{u} \cdot \vec{n} \, dS'}$$

Equate the two expressions for $\frac{d\bar{V}}{dt}$:

$$\boxed{\int_{\bar{V}(t)} \vec{\nabla} \cdot \vec{u} \, d\bar{V} = \int_{S(t)} \vec{u} \cdot \vec{n} \, dS'}$$

Divergence Theorem of Gauss

Little guy in class room:

$$1 + 2 + 3 + \cdots + 99 + 100 =$$

$$= (1+100) + (2+99) + \cdots (50+51)$$

$$= 50 \times 101 = 5050$$

Little guy was Gauss.

Index notation: + ESC

$$\boxed{\int_{V(t)} \frac{\partial u_j}{\partial x_j}\, dV = \int_{S(t)} u_j n_j \, dS}$$

for arbitrary <u>vector field</u>

Back to mass conservation:

integral form: $\displaystyle\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{S(t)} \rho u_j n_j \, dS = 0$

$$\int_{S(t)} (\rho u_j) n_j \, dS \underset{\text{Gauss}}{=} \int_{V(t)} \frac{\partial}{\partial x_j}(\rho u_j)\, dV$$

(note: vector field is $\rho \vec{u} = \begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix}$).

$$\Rightarrow \int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{V(t)} \frac{\partial}{\partial x_j}(\rho u_j)\, dV = 0$$

$$\Rightarrow \int_{V(t)} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right\} dV = 0 \quad \underline{\underline{\forall\ V, t}}$$

$$\Longleftrightarrow \{\ \} = 0 \quad \forall\ \vec{x}, t$$

↳ equivalence!

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0}$$

Differential Formulation of Mass Conservation.

Note, via ESC:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Vector notation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$

<u>Special case:</u> $\rho =$ const (incompressible fluid)

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial x_j}(\rho u_j) = 0 \qquad \frac{\partial \rho}{\partial x_1} = 0 \text{ etc}$$

$$\rho \frac{\partial u_j}{\partial x_j} = 0 \qquad \rho \neq 0 \Rightarrow \boxed{\frac{\partial u_j}{\partial x_j} = 0}$$

<u>Other special case:</u>

Steady flow (but not $\rho =$ const).

$$\cancel{\frac{\partial \rho}{\partial t}} = 0 \Rightarrow \boxed{\frac{\partial}{\partial x_j}(\rho u_j) = 0}$$