Chapter 3

Continuity Equation

3.1 Gradient, divergence

Two important and extremely useful operators in the analysis of vector fields are the gradient and the divergence.

The gradient ($\boldsymbol{\nabla}$) operates on a scalar function, say $\phi(\boldsymbol{x})$, and the result is a vector:

$$\boldsymbol{\nabla}\phi(\boldsymbol{x}) \equiv (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z})^T. \tag{3.1}$$

In words, the gradient of a scalar function consists of a vector with its components representing the angle of inclination in the three coordinate directions.

The divergence ($\boldsymbol{\nabla}\cdot$) operates on a vector function, say $\boldsymbol{u}(\boldsymbol{x})$, and the result is a scalar:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}) \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{3.2}$$

3.2 Interpretation of the velocity field divergence

Suppose we have a tiny brick-shaped blob of fluid with edges of size a, b, and c. Then the volume of is this tiny blob is $\Delta V = abc$. When the blob is convected with the fluid in a velocity field $\boldsymbol{u} = (u, v, w)^T$ it changes in size and in shape. Its size change at the moment when we start following the blob is [1]

$$\frac{d}{dt}\Delta V = \frac{d}{dt}abc = \frac{da}{dt}bc + a\frac{db}{dt}c + ab\frac{dc}{dt}. \tag{3.3}$$

We choose the tiny blob in such a way that its edges are aligned with the three cartesian axes. The time derivative of a is

$$\frac{da}{dt} = u(x + a, y, z, t) - u(x, y, z, t). \tag{3.4}$$

[1] This is just an application of the product rule of differentiation.

In approximation [2]

$$u(x+a, y, z, t) = u(x, y, z, t) + \frac{\partial u}{\partial x} a + \mathcal{O}\left(a^2\right), \tag{3.5}$$

so

$$\frac{da}{dt} = \frac{\partial u}{\partial x} a + \mathcal{O}\left(a^2\right). \tag{3.6}$$

In a similar way we derive also that

$$\frac{db}{dt} = \frac{\partial v}{\partial x} b + \mathcal{O}\left(b^2\right), \quad \frac{dc}{dt} = \frac{\partial w}{\partial x} c + \mathcal{O}\left(c^2\right). \tag{3.7}$$

Hence, when we subtitute these expressions in the time derivative of ΔV we obtain

$$\frac{d}{dt}\Delta V = \frac{\partial u}{\partial x} abc + \frac{\partial v}{\partial x} abc + \frac{\partial w}{\partial x} abc + \text{higher order terms}, \tag{3.8}$$

or

$$\frac{d}{dt}\Delta V = \frac{\partial u_j}{\partial x_j} \Delta V + \text{higher order terms}, \tag{3.9}$$

where we have used the Einstein summation convention and $(u_1, u_2, u_3)^T \equiv (u, v, w)^T$ and similarly $(x_1, x_2, x_3)^T \equiv (x, y, z)^T$. So what we have found is

$$\frac{\partial u_j}{\partial x_j} = \frac{1}{\Delta V} \frac{d}{dt} \Delta V + \text{higher order terms}. \tag{3.10}$$

This means that, in the limit of an sufficiently small blob, the divergence of the velocity field represents the relative time derivative of its volume. From this, we see immediately that the flow is incompressible if the divergence of the velocity field is zero everywhere.

3.3 Gauss' divergence theorem

To derive the differential formulations from integral formulations we need the divergence theorem of Gauss. The theorem will not be derived mathematically here since that is beyond the scope of the present lecture notes. Instead we will make the theorem plausible based on arguments from physics. We will compute the time derivative of the volume of a large convected blob, say $V(t)$, in two different ways. Gauss' divergence theorem then follows by equating the two expressions. On the one hand we can compute the volume time derivative by subdividing the large blob into a large number N of tiny blobs and take the sum:

$$\frac{dV}{dt} \approx \sum_{i=1}^{N} \frac{d}{dt} \Delta V_i \tag{3.11}$$

[2] This is just an application of Taylor's theorem.

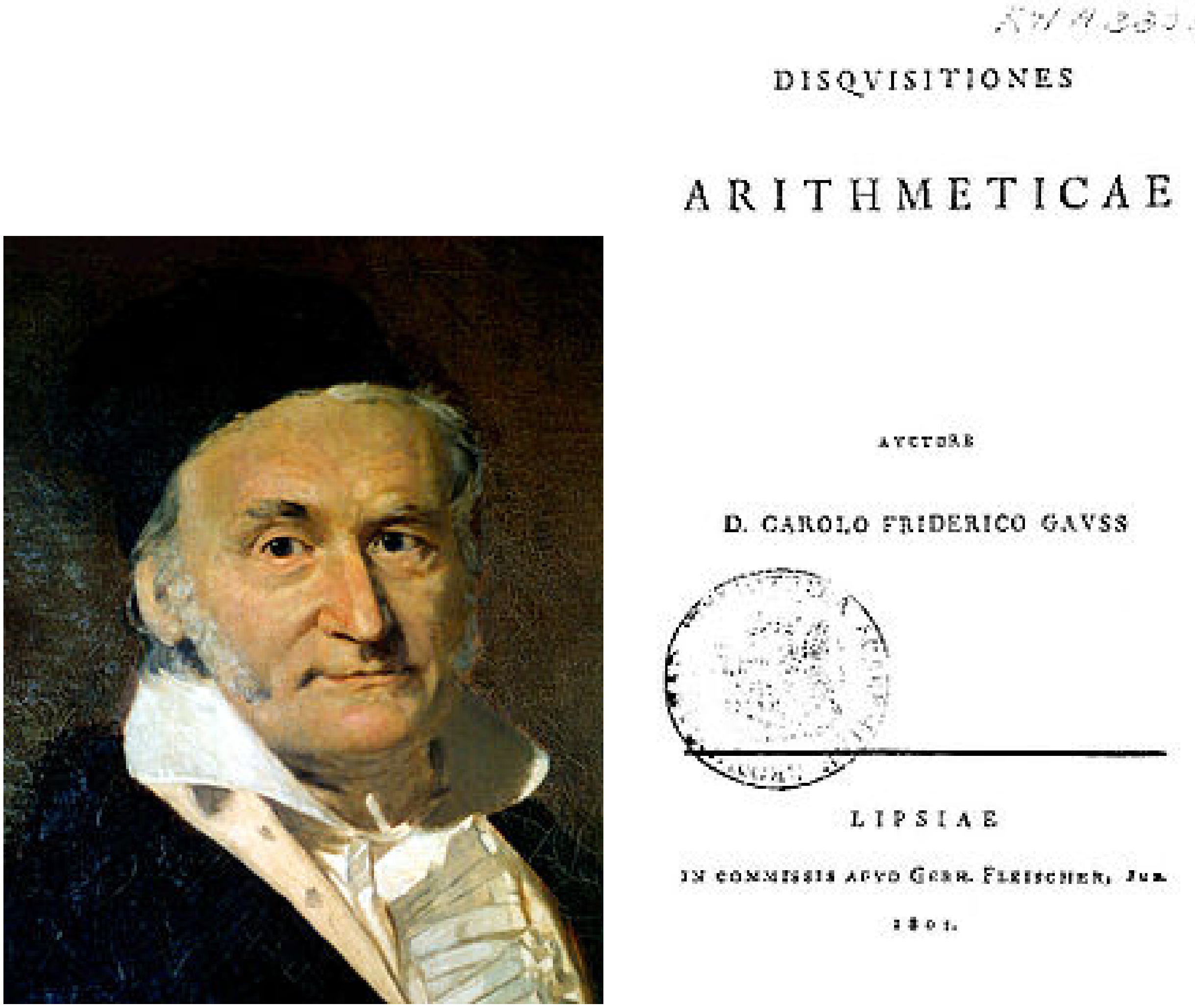

Figure 3.1: Johann Carl Friedrich Gauss (30 April 1777 23 February 1855) was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics.

where the subscript i is a counter. Using the result for the tiny blob of the previous section, Eq.(3.9), we get

$$\frac{dV}{dt} \approx \sum_{i=1}^{N} \left(\frac{\partial u_j}{\partial x_j} \right)_i \Delta V_i, \tag{3.12}$$

In the limit of $\Delta V_i \to 0$ and $N \to \infty$ this becomes an integral:

$$\frac{dV}{dt} = \int_{V(t)} \frac{\partial u_j}{\partial x_j} dV. \tag{3.13}$$

On the other hand we can compute the volume time derivative by employing the Reynolds transport theorem, Eq.(2.16):

$$\frac{dV}{dt} \equiv \frac{d}{dt}\int_{V(t)} dV = \int_{S(t)} u_j n_j dS, \tag{3.14}$$

which represents the growth rate of V at its surface.

Hence, by equating the two results for $\frac{dV}{dt}$ we obtain Gauss' divergence theorem:

$$\boxed{\int_{V(t)} \frac{\partial u_j}{\partial x_j} dV = \int_{S(t)} u_j n_j dS.} \tag{3.15}$$

Note that we have 'derived' it here for an arbitrary velocity field $\boldsymbol{u}$, so that it will hold for any vector field. (3)

3.4 Continuity equation

We derive the differential formulation of mass conservation from the integral formulation of mass conservation Eq.(2.19) by replacing the surface integral by a volume integral using Gauss' divergence theorem Eq.(3.15):

$$\int_{S(t)} \rho u_j n_j dS = \int_{V(t)} \frac{\partial \rho u_j}{\partial x_j} dV. \tag{3.16}$$

Note that here ρu_j takes the place of u_j in Eq.(3.15). With this replacement the integral form Eq.(2.19) becomes one single volume integral:

$$\int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} \right) dV = 0. \tag{3.17}$$

Since the blob V was chosen completely arbitrary, this holds for any blob, also for extremely tiny blobs. This means that

$$\boxed{\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t)}} \tag{3.18}$$

This is the differential formulation of mass conservation which is frequently called the continuity equation.

Several limiting cases are of interest. First of all, if the flow is steady, all time-derivatives ar zero, so

$$\frac{\partial \rho u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t) \tag{3.19}$$

When the flow is unsteady but incompressible ($\rho =$ constant), then $\frac{\partial \rho}{\partial t} = 0$ and we get again Eq.(3.19). But since $\rho =$ constant we can take it out of the differentiation and divide both sides by ρ (since it is larger than zero) to obtain

$$\frac{\partial u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t) \tag{3.20}$$

(3) not necessarily a velocity field, Gauss derived the theorem when working on electric fields.