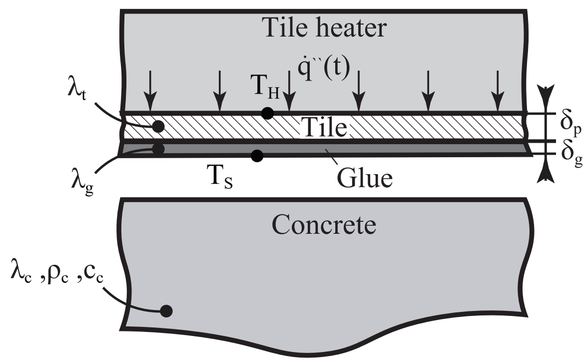
Exercise II.15: (Tile setting ⋆⋆)

A tile setter employs a modern technique for tile installation, involving preheating the tile and glue before affixing them to the concrete. The tile and glue are heated until they reach a steady-state condition, achieving a uniform heating temperature T_H and a constant heat flux $\dot{q}_0''$. Once these conditions are met, the tile setter places the heated tile and glue on the concrete, maintaining a constant temperature T_S throughout the process. After reaching a critical temperature T_crit at a distance δ_crit within the concrete, the heater is removed. Initially, the concrete used to be at a homogeneous temperature T_0



Given parameters:

- Steady-state heat flux: $\dot{q}_0''$ = 7.5 kW/m^2
- Thickness of the pile: δ_p = 10 mm
- Thickness of the glue: δ_g = 2 mm
- Conductivity of the pile: λ_p = 1.0 W/mK
- Conductivity of the glue: λ_g = 0.35 W/mK
- Conductivity of the concrete: λ_c = 2.3 W/mK
- Heat capacity of the concrete: c_c = 1,000 J/kgK
- Density of the concrete: ρ_c = 2,400 kg/m^3
- Initial temperature of the concrete: T_0 = 20 °C
- Heating temperature of the tile heater: T_H = 200 °C
- Critical temperature: T_crit = 35 °C
- Critical distance: δ_crit = 10 mm

Hints:

- Heat will never penetrate entirely through the concrete.

Tasks:

a) Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the concrete. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.

b) Determine the time t_{crit} at which the heater can be removed.

c) Illustrate the concrete's temperature profile that depicts both temporal and spatial variations.

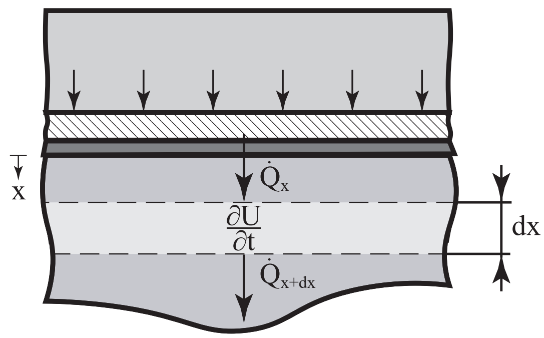
Solution II.15: (Tile setting ⋆⋆)

Task a)

As observed earlier, the temperature profile is obtained from the conduction equation within an element.

1 Setting up the balance:

In this problem, the concrete undergoes continuous heating from the tile heater. As a result, the concrete gradually increases in temperature over time. With a temperature gradient established within the concrete body, heat diffusion becomes prevalent. Consequently, both spatial and temporal temperature differences are observed within the concrete material.



The energy balance for an infinitesimal element within the concrete reads:

$$\underbrace{\frac{\partial U}{\partial t}}_{\substack{\text{Temporal change}\\\text{of inner energy}}} = \underbrace{\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}}_{\substack{\text{Net rate of}\\\text{diffusion}}}. \tag{II.15.1}$$

2 Defining the elements within the balance:

Temporal change of internal energy:

$$\begin{aligned}\frac{\partial U}{\partial t} &= dm \cdot c_{\mathrm{c}} \frac{\partial T}{\partial t} \\ &= \rho_{\mathrm{c}} c_{\mathrm{c}} A dx \frac{\partial T}{\partial t}.\end{aligned} \tag{II.15.2}$$

The ingoing conductive flux is defined by the use of Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda_{\mathrm{c}} A \frac{\partial T}{\partial x}, \tag{II.15.3}$$

and the outgoing flux is approximated by the use of the Taylor series expansion:

$$\begin{aligned}\dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \\ &= \frac{\partial}{\partial x}\left(-\lambda_{\mathrm{c}} A \frac{\partial T}{\partial x}\right) \cdot dx.\end{aligned} \tag{II.15.4}$$

Conclusion

3 Inserting and rearranging:

$$\frac{\partial T}{\partial t} = a_{\mathrm{c}} \frac{\partial^2 T}{\partial x^2}, \tag{II.15.5}$$

where the thermal diffusivity is defined as $a = \frac{\lambda}{\rho c_{\mathrm{p}}}$.

4 Defining the boundary and/or initial conditions:

For solving the differential equation, one initial condition and two boundary conditions are necessitated. This is due to the fact that the temperature T has undergone differentiation once concerning time (t) and twice concerning space (x).

Initially, the temperature within the concrete has a homogeneous temperature:

$$\left.\begin{array}{r} t = 0 \\ 0 < x < \infty \end{array}\right\} T = T_0. \tag{II.15.6}$$

As mentioned, heat will never penetrate entirely through the concrete. This implies that at all times, far away from the surface, the temperature will still be equal to the initial temperature. Therefore, the concrete can be observed as a semi-infinite body and the temperature for $x \to \infty$ will remain to be T_0 at all times:

$$\left.\begin{array}{r} t > 0 \\ x \to \infty \end{array}\right\} T = T_0. \tag{II.15.7}$$

When $t > 0$ the temperature at $x = 0$ is remained at a constant temperature T_{S}:

$$\left.\begin{array}{r} t > 0 \\ x = 0 \end{array}\right\} T = T_{\mathrm{S}}. \tag{II.15.8}$$

However, to determine this temperature T_{S}, an energy balance around the tile and the glue needs to be set:

$$\begin{aligned} 0 &= \dot{Q}_{\mathrm{in}} - \dot{Q}_{\mathrm{out}} \\ 0 &= \dot{q}_0'' A - \frac{T_{\mathrm{H}} - T_{\mathrm{S}}}{\frac{\delta_{\mathrm{p}}}{\lambda_{\mathrm{p}} A} + \frac{\delta_{\mathrm{g}}}{\lambda_{\mathrm{g}} A}} \\ \Rightarrow T_{\mathrm{S}} &= T_{\mathrm{H}} - \dot{q}_0'' \left(\frac{\delta_{\mathrm{p}}}{\lambda_{\mathrm{p}}} + \frac{\delta_{\mathrm{g}}}{\lambda_{\mathrm{g}}} \right) \\ &= 200\ (^{\circ}\mathrm{C}) - 7{,}500\ \left(\mathrm{W/m^2}\right) \left(\frac{0.01\ (\mathrm{m})}{1.0\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} + \frac{0.002\ (\mathrm{m})}{0.35\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} \right) = 82\ (^{\circ}\mathrm{C}). \end{aligned} \tag{II.15.9}$$

From the differential equation, the boundary conditions, and the initial condition, it is found that the problem is of the type semi-infinite body with a constant surface temperature.

Task b)

5 **Solving the equation:**

The solution to the given differential equation is written as:

$$\theta^* = 1 - \text{erf}\left(\frac{x}{\sqrt{4at}}\right), \tag{II.15.10}$$

where:

$$\theta^* = \frac{T(x,t) - T_0}{T_S - T_0}. \tag{II.15.11}$$

To determine, the critical time, the equation must be rewritten:

$$\frac{T_{crit} - T_0}{T_S - T_0} = 1 - \text{erf}\left(\frac{\delta_{crit}}{\sqrt{\frac{4\lambda_c}{\rho_c c_c} t_{crit}}}\right) \tag{II.15.12}$$

$$\Rightarrow t_{crit} = \frac{\rho_c c_c}{4\lambda_c} \cdot \left[\frac{\delta_{crit}}{\text{erf}^{-1}\left(1 - \frac{T_{crit}-T_0}{T_S-T_0}\right)}\right]^2$$

$$= \frac{2{,}400\ \left(\frac{kg}{m^3}\right) \cdot 1{,}000\ \left(\frac{J}{kgK}\right)}{4 \cdot 2.3\ \left(\frac{W}{mK}\right)} \cdot \left(\frac{0.01\ (m)}{\text{erf}^{-1}\left(1 - \frac{(35-20)\ (°C)}{(82-20)\ (°C)}\right)}\right)^2 = 38\ (s).$$

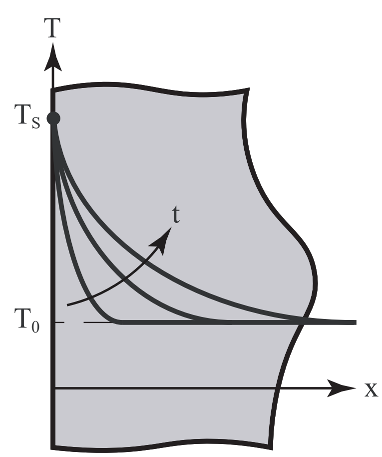
Conclusion

The heater must be removed after 38 s.

Task c)

Initially, the temperature within the concrete is uniformly at temperature T_0. As soon as $t > 0$, the surface temperature is held constant at T_S. The gradient of the temperature profile diminishes in the direction of penetration until the initial temperature is reached, where the profile has a horizontal slope. Additionally, with time, the temperature within the body rises.

Conclusion



Exercise II.16: (Heating and quenching of a sphere ⋆⋆⋆)

A sphere, initially at a homogeneous temperature of T_0, is put into an oven. The oven temperature remains constant at a homogeneous temperature of T_o.

Given parameters:

- Initial temperature of the sphere: $T_0 = 25$ °C
- Intermediate temperature of the sphere: $T_h = 150$ °C
- Oven temperature: $T_o = 200$ °C
- Quenching temperature: $T_q = 30$ °C
- Heat transfer coefficient: $\alpha = 110$ W/m^{2}K
- Radius of the sphere: $r_1 = 1.5$ cm
- Thermal conductivity of the sphere: $\lambda = 1.52$ W/mK
- Density the sphere: $\rho = 1.45 \cdot 10^3$ kg/m^3
- Specific heat capacity the sphere: $c_p = 0.88$ kJ/kg $\cdot$ K

Hints:

- Heat radiation can be neglected.
- It always remains that Fo > 0.2.

Tasks:

a) Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the sphere. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.

b) Determine the temperature of the center of the sphere after 3 minutes.

After some time the sphere has a hot homogeneous temperature T_h and is being quenched. During this process, the quenching temperature is constant at T_q. Further, in time, the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

c) Determine the time instant when the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

d) Determine the amount of heat dissipated at this time instant.

Solution II.16: (Heating and quenching of a sphere ⋆ ⋆ ⋆)

Task a)

The temperature profile within the solid body is derived from the conduction equation, considering both spatial and temporal aspects.

In practical scenarios, the temperature profile within the sphere is never homogeneous. However, the Biot number is a useful parameter for assessing whether the temperature distribution within the sphere is approximated as homogeneous. If the Biot number allows for such an assumption, the lumped capacity model becomes a viable approach to solving the problem.

The characteristic length of the Biot number is written as:

$$\begin{aligned} L &= \frac{V}{A} \\ &= \frac{\frac{4\pi r_1^3}{3}}{4\pi r_1^2} \\ &= \frac{0.015\ (\mathrm{m})}{3} = 0.005\ (\mathrm{m}). \end{aligned} \tag{II.16.1}$$

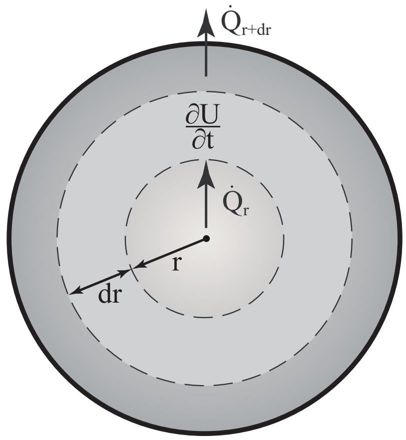
Substitution into the definition of the Biot number:

$$\begin{aligned} \mathrm{Bi} &= \frac{\alpha L}{\lambda} \\ &= \frac{\alpha r_1}{3\lambda} \\ &= \frac{110\ \left(\frac{\mathrm{W}}{\mathrm{m^2 K}}\right) \cdot 0.015\ (\mathrm{m})}{3 \cdot 1.52\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} = 0.36\ (-). \end{aligned} \tag{II.16.2}$$

Given that the Biot number is not significantly smaller than 1, both spatial and temporal variations in the temperature of the sphere must be considered to ensure a high level of accuracy in the results.

1 **Setting up the balance:**

In this scenario, the phenomenon is viewed as a transient problem wherein a hot sphere, initially at elevated temperature, is introduced into a colder environment, prompting immediate cooling. With the interface between the sphere and the ambient environment cooler than the sphere's center, a temperature gradient emerges within the sphere, initiating heat diffusion. Consequently, both temporal and spatial temperature variations are observed.



Due to symmetry, the problem is simplified to one-dimensional in the radial direction. As the temperature is not uniform within the body, an infinitesimal element within the sphere is considered as the control volume to derive the necessary differential equation. The energy balance for this element is given by:

$$\underbrace{\frac{\partial U}{\partial t}}_{\substack{\text{Temporal change}\\ \text{of inner energy}}} = \underbrace{\dot{Q}_\mathrm{r} - \dot{Q}_\mathrm{r+dr}}_{\substack{\text{Net rate of}\\ \text{diffusion}}}. \tag{II.16.3}$$

2 Defining the elements within the balance:

The temporal change of internal energy within the infinitesimal element is:

$$\begin{aligned}\frac{\partial U}{\partial t} &= dm \cdot c_\mathrm{p} \frac{\partial T}{\partial t} \\ &= \rho \cdot dV \cdot c_\mathrm{p} \cdot \frac{\partial T}{\partial t},\end{aligned} \tag{II.16.4}$$

where the volume of the infinitesimal element yields from:

$$\begin{aligned}dV &= \frac{4\pi}{3}\left[(r+dr)^3 - r^3\right] \\ &= \frac{4\pi}{3}\left[3r^2 \cdot dr + 3r \cdot dr^2 + dr^3\right] \\ &\approx 4\pi r^2 \cdot dr.\end{aligned} \tag{II.16.5}$$

Note that $3r \cdot dr^2 \ll 3r^2 \cdot dr$ and $dr^3 \ll 3r^2 \cdot dr$ and are thus neglected.

The incoming rate of heat conduction is:

$$\begin{aligned}\dot{Q}_\mathrm{r} &= -\lambda A(r) \frac{\partial T}{\partial r} \\ &= -4\lambda \pi r^2 \frac{\partial T}{\partial r}.\end{aligned} \tag{II.16.6}$$

For an infinitesimal element, the outgoing rate of heat conduction is approximated by the use of the Taylor series expansion:

$$\dot{Q}_{\mathrm{r+dr}} = \dot{Q}_{\mathrm{r}} + \frac{\partial \dot{Q}_{\mathrm{r}}}{\partial r} \cdot dr = -4\lambda\pi r^2 \frac{\partial T}{\partial r} + \frac{\partial}{\partial r}\left(-4\lambda\pi r^2 \frac{\partial T}{\partial r}\right) \cdot dr. \tag{II.16.7}$$

Conclusion

3 **Inserting and rearranging:**

$$\frac{\partial T}{\partial t} = \frac{a}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right), \tag{II.16.8}$$

where the thermal diffusivity is defined as $a = \frac{\lambda}{\rho c_{\mathrm{p}}}$.

4 **Defining the boundary and/or initial conditions:**

To solve the obtained differential equation, two boundary conditions are needed, as the temperature has been differentiated twice with respect to r, and one initial condition is necessary because the temperature has been differentiated once to t.

Initially, the entire temperature of the sphere is to be at T_0, therefore the initial condition is written as:

$$T(t = 0) = T_0. \tag{II.16.9}$$

Furthermore, since the temperature profile of the sphere is symmetrical, implying a horizontal slope at the center, the first boundary condition is expressed as:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0. \tag{II.16.10}$$

Lastly, the second boundary condition is found from an energy balance at the surface of the sphere, which states that $\dot{Q} = -\lambda A \left.\frac{\partial T}{\partial r}\right|_{r=r_1} = \alpha A \left(T(r = r_1) - T_{\mathrm{O}}\right)$. Rewriting gives:

$$\left.\frac{\partial T}{\partial r}\right|_{r=r_1} = -\frac{\alpha}{\lambda}\left(T(r = r_1) - T_{\mathrm{O}}\right).$$

The differential equation, initial condition, and boundary conditions align with the temperature profiles described by the Heisler diagrams. Hence, there is no necessity to solve this equation, and the Heisler diagrams are directly applied without undergoing the mathematical derivation.

Task b)

5 **Solving the equation:**

To determine the temperature at the center of the sphere after 3 minutes the chart on the next page is used which describes the temperature of the center in a sphere. The inverse of the Biot number and the Fourier number at that specific time instant need to be calculated.

Note that the Biot number defined in the Heisler diagrams is slightly different from how this number was defined before, and is a factor 3 larger in the case of spheres. This factor has been accounted for within the chart.

The inverse of the Biot number for the Heisler diagrams is found from:

$$\frac{1}{\mathrm{Bi}} = \frac{\lambda}{\alpha r_1} \tag{II.16.11}$$
$$= \frac{1.52\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)}{110\ \left(\frac{\mathrm{W}}{\mathrm{m^2K}}\right)\cdot 0.015\ (\mathrm{m})} = 0.92\ (-)\,.$$

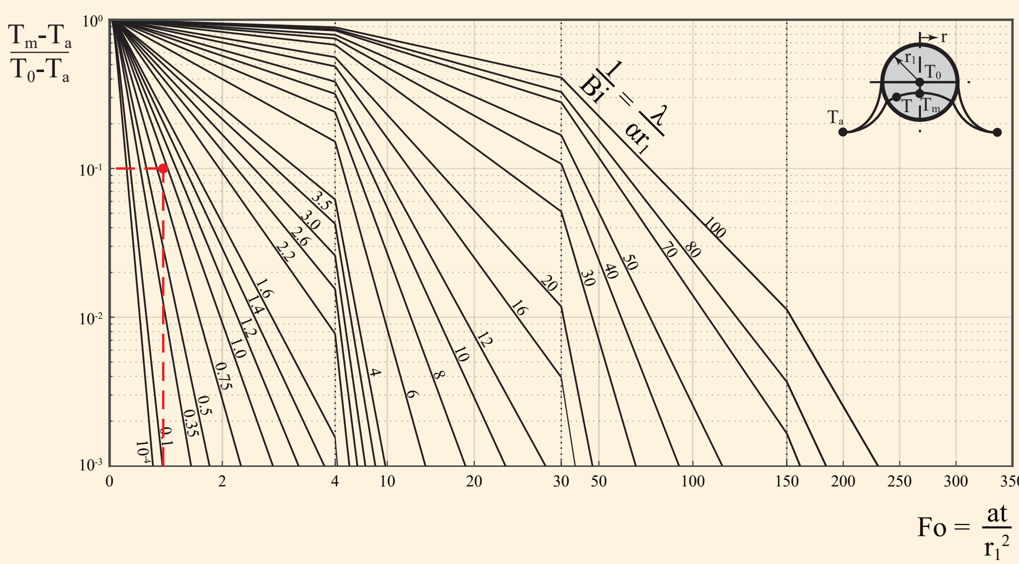
The Fourier number results from:

$$\mathrm{Fo} = \frac{a t_1}{{r_1}^2} = \frac{\lambda t_1}{\rho c_\mathrm{p} {r_1}^2} \tag{II.16.12}$$
$$= \frac{1.52\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)\cdot 180\ (\mathrm{s})}{1450\ \left(\frac{\mathrm{kg}}{\mathrm{m^3}}\right)\cdot 880\ \left(\frac{\mathrm{J}}{\mathrm{kgK}}\right)\ 0.015^2\ (\mathrm{m^2})} = 0.95\ (-)\,.$$

Using the Heisler diagram for the temperature in the center of a sphere.

Fundamental EQ

Temperature in the centre of a sphere with radius r_1:



Reading the chart yields:

$$\frac{T_\mathrm{m} - T_\mathrm{a}}{T_0 - T_\mathrm{a}} \approx 10^{-1}\ (-)\,. \tag{II.16.13}$$

Note that in the given case, the ambient temperature T_a is the oven temperature T_O. Substitution of this and rewriting:

$$T_\mathrm{m} = 10^{-1}\cdot(T_0 - T_\mathrm{O}) + T_\mathrm{O} \tag{II.16.14}$$
$$= 10^{-1}\ (-)\cdot(25 - 200)\ (^\circ\mathrm{C}) + 200\ (^\circ\mathrm{C}) = 182.5\ (^\circ\mathrm{C})\,.$$

Conclusion

The temperature of the center of the sphere after 3 minutes equals 183 °C.

Task c)

Now the sphere is being quenched, the ambient temperature and the convection coefficient have been changed, and therefore also the Biot number. The chart on the next page is used to determine the inverse of the Biot number when knowing the center and the surface temperature of the sphere.

To do so, the dimensionless temperature difference needs to be determined, where the ambient temperature T_a is equal to the quenching temperature T_q:

$$\begin{aligned} \frac{T - T_\mathrm{a}}{T_\mathrm{m} - T_\mathrm{a}} &= \frac{T - T_\mathrm{q}}{T_\mathrm{m} - T_\mathrm{q}} \\ &= \frac{(44.4 - 30)\ (^\circ\mathrm{C})}{(54 - 30)\ (^\circ\mathrm{C})} = 0.6\ (-)\,, \end{aligned} \tag{II.16.15}$$

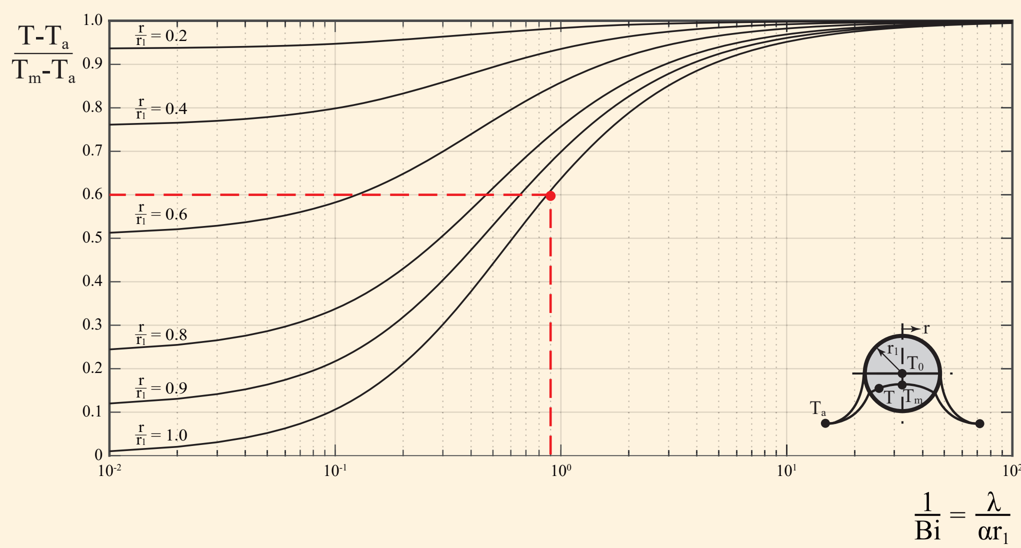
and the ratio between radii:

$$\begin{aligned} \frac{r}{r_1} &= \frac{r_1}{r_1} \\ &= 1\ (-)\,. \end{aligned} \tag{II.16.16}$$

With this, the following diagram can be read:

Fundamental EQ

Temperature distribution in a sphere with radius r_1:



which is for Fo > 0.2.

Which gives the inverse of the Biot number:

$$\frac{1}{\mathrm{Bi}} \approx 0.9\ (-)\,. \tag{II.16.17}$$

Now the inverse of the Biot number is known, the chart describing the center temperature of the sphere over time is used to determine the Fourier number at the specific time instant.

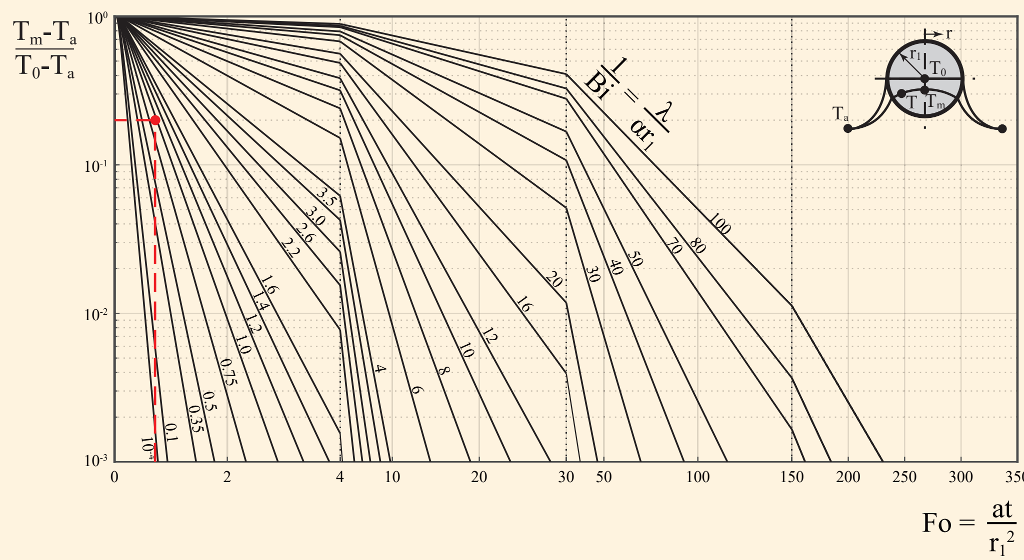
To do so, besides the Biot number, the dimensionless temperature difference is needed as well. In this case, the ambient temperature T_a is equal to the quenching temperature T_q.

$$\frac{T_\mathrm{m} - T_\mathrm{a}}{T_0 - T_\mathrm{a}} = \frac{T_\mathrm{m} - T_\mathrm{q}}{T_0 - T_\mathrm{q}} \tag{II.16.18}$$

$$= \frac{(54 - 30)\ (^\circ\mathrm{C})}{(150 - 30)\ (^\circ\mathrm{C})} = 0.2\ (-)\,.$$

Fundamental EQ

Temperature in the centre of a sphere with radius r_1:



Reading the Heisler diagram as used in the previous task yields the Fourier number at this specific time instant t_2.

$$\mathrm{Fo} \approx 0.7\ (-)\,. \tag{II.16.19}$$

Lastly, rewriting the definition of the Fourier number leads to:

$$\mathrm{Fo} = \frac{a t_2}{{r_1}^2} = \frac{\lambda t_2}{\rho c_\mathrm{p} {r_1}^2} \tag{II.16.20}$$

$$\Rightarrow t_2 = \frac{\mathrm{Fo}\rho c_\mathrm{p} {r_1}^2}{\lambda}$$

$$= \frac{0.7\ (-) \cdot 1450\ \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right) \cdot 880\ \left(\frac{\mathrm{J}}{\mathrm{kgK}}\right) \cdot 0.015^2\ \left(\mathrm{m}^2\right)}{1.52\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} = 132\ (\mathrm{s}).$$

Conclusion

After 2.2 minutes the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

Task d)

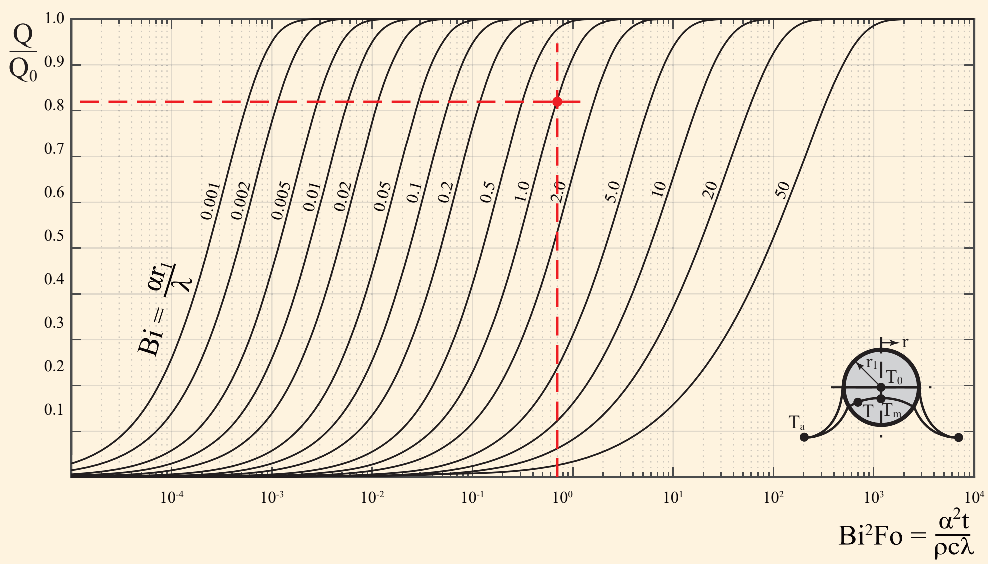
To determine the amount of heat dissipated after 2.2 min, the chart below is used. To do so, first, the product of the Biot number squared and the Fourier number needs to be determined.

This yields from:

$$\mathrm{Bi}^2 \cdot \mathrm{Fo} = 1^2 \ (-) \cdot 0.7 \ (-) = \ 0.7 \ (-) \,. \tag{II.16.21}$$

Fundamental EQ

Heat loss of a sphere:



where $Q = mc\,(T(t) - T_\mathrm{a})$ and $Q_0 = mc\,(T_0 - T_\mathrm{a})$.

Reading the Heisler diagram describing the heat loss by a sphere gives:

$$\frac{Q}{Q_o} \approx 0.82 \ (-) \,. \tag{II.16.22}$$

Rewriting and substituting the definition of the initial heat within the sphere results:

$$\begin{aligned} Q &= 0.82 Q_0 \\ &= 0.82 \rho \frac{4\pi {r_1}^3}{3} c_\mathrm{p} \cdot (T_0 - T_\mathrm{q}) \\ &= 0.82 \ (-) \cdot 1.45 \cdot 10^3 \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right) \cdot \frac{4\pi \cdot 0.015^3 \ (\mathrm{m}^2)}{3} \cdot 880 \left(\frac{\mathrm{J}}{\mathrm{kgK}}\right) \cdot (150 - 30) \ (\mathrm{K}) = 1.8 \ (\mathrm{kJ}) \,. \end{aligned} \tag{II.16.23}$$

Conclusion

Thus 1.8 kJ of heat has been dissipated after 2.2 min.