Materials: Elasticity Theory

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University of Twente

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Elasticiteitstheorie – Elasticity Theory
module 6: 2024-202000131-1B

script & sheets, based on:
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Overview:

Index notation, matrix-vector notation, vectors and tensors
Stress, strain, (material behavior: elasticity, plasticity, viscosity)

1) Index notation and summation convention

Set of quantities: $a_1, a_2, a_3, \cdots, a_n \quad \longrightarrow$
a_i, with $i = 1, 2, 3, \cdots, n$

In 2- of 3-dim. space a_i is the i^{th} component
of the vector $\underset{\sim}{a}$, with $n = 2$ or 3

also: a_i is the i^{th} element of the (column)-vector $\{a\}$
(just a column of numbers?)

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also: a_i is the i^{th} element of the (column)vector $\{a\}$
(just a column of numbers?)

Difference:

vector as geometric, physical quantity
(e.g. distance or velocity with norm [unit] and direction)

$\underset{\sim}{a} = \vec{a}$ - or vector as matrix with one column $\{a\}$

summation convention

inner product (scalar product) of two vectors:

$$\{x\}^{\mathrm{T}}\,\{y\} = x_1\,y_1 + x_2\,y_2 + x_3\,y_3 + \;\cdots\; + x_n\,y_n = \sum_{i=1}^{n} x_i\,y_i$$

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shorter (Einstein) notation:

$$\{x\}^{\mathrm{T}}\{y\} = (\underset{\sim}{x} \cdot \underset{\sim}{y} =)\; x_i\,y_i = x_p\,y_p = x_q\,y_q$$

where i, p and q are so-called *dummy-indices*.
Those are double, in products, and have to be summed over.

So called *free indices* occur only once in product-terms.

System of equations:

$$\begin{aligned}
y_1 &= A_{11}\,x_1 &+A_{12}\,x_2 + \cdots &+A_{1n}\,x_n \\
y_2 &= A_{21}\,x_1 &+A_{22}\,x_2 + \cdots &+A_{2n}\,x_n \\
\vdots & \quad\vdots & \vdots \qquad & \qquad\vdots \\
y_m &= A_{m1}\,x_1 &+A_{m2}\,x_2 + \cdots &+A_{mn}\,x_n
\end{aligned}$$

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The i^{th} equation is:

$$y_i = A_{i1}\,x_1 + A_{i2}\,x_2 + \cdots + A_{in}\,x_n = \left(\sum_{j=1}^{n} \right) A_{ij}\,x_j$$

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shortly: $y_i = A_{ij}\,x_j \quad \left\{ \begin{array}{l} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{array} \right.$

or $\{y\} = [A]\,\{x\}$

or $\underline{\mathsf{y}} = \underline{\underline{A}} \cdot \underline{\mathsf{x}}$

Matrix $[A]$ with entries A_{ij}
(where i is the row-index, and j is the column-index)

Example: $S = \{x\}^{\mathrm{T}} \cdot \{x\} = \left(\sum_{i=1}^{n} \right) x_i \, x_i$

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Example: $S = \{x\}^{\mathrm{T}} \cdot \{x\} = \left(\sum_{i=1}^{n} \right) x_i \, x_i$

Kronecker delta: $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

Thus: $S = x_i \, x_i = \delta_{ij} \, x_i \, x_j = \{x\}^{\mathrm{T}} [I] \, \{x\}$

Also: $x_i = \delta_{ij} \, x_j = \delta_{ip} \, x_p$

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Also: $x_i = \delta_{ij} \, x_j = \delta_{ip} \, x_p$

BUT: $\delta_{ii} \neq 1$

instead, note: $\delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{nn} = 1 + 1 + \cdots + 1 = n$

Differentiation:

Consider a function: $F(x_1, x_2, \cdots, x_n)$

Partial derivative: $\frac{\partial F}{\partial x_i} = F_{,i}$

The index $, i$ means the partial derivative with respect to x_i!

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Total differential of F:

$$\mathrm{d}F = \frac{\partial F}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial F}{\partial x_2}\,\mathrm{d}x_2 + \cdots + \frac{\partial F}{\partial x_n}\,\mathrm{d}x_n = \left(\sum_{i=1}^{n}\right)\frac{\partial F}{\partial x_i}\,\mathrm{d}x_i = F_{,i}\,\mathrm{d}x_i$$

2) Scalars, vectors and Cartesian tensors

Scalar fields, functions of position, only a number with [unit].
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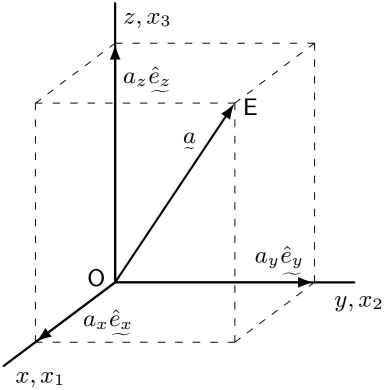
Scalar fields, functions of position, only a number with [unit].
e.g., density, temperature, ...

Vector fields, functions of position, quantity with norm [unit] and a direction, e.g., force, displacement, velocity, acceleration, ...

Cartesian coordinate system



$$\underset{\sim}{a} = a_x\, \underset{\sim}{\hat{e}_x} + a_y\, \underset{\sim}{\hat{e}_y} + a_z\, \underset{\sim}{\hat{e}_z}$$

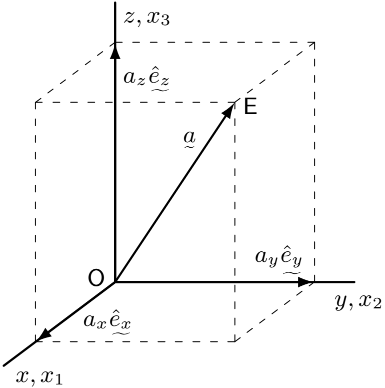
$$= \ a_1\, \underset{\sim}{\hat{e}_1} + a_2\, \underset{\sim}{\hat{e}_2} + a_3\, \underset{\sim}{\hat{e}_3}$$

$$= \sum_{i=1}^{3} a_i\, \underset{\sim}{\hat{e}_i} = a_i\, \underset{\sim}{\hat{e}_i}$$

... alternatives ...

$$= \ a_1\, \underset{\sim}{\hat{x}} + a_2\, \underset{\sim}{\hat{y}} + a_3\, \underset{\sim}{\hat{z}}$$

$$= \ a_1\, \underset{\sim}{i} + a_2\, \underset{\sim}{j} + a_3\, \underset{\sim}{k}$$



Cartesian coordinate system

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$$= \; a_1 \, \underset{\sim}{\hat{e}_1} + a_2 \, \underset{\sim}{\hat{e}_2} + a_3 \, \underset{\sim}{\hat{e}_3}$$

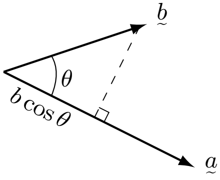
$$= \sum_{i=1}^{3} a_i \, \underset{\sim}{\hat{e}_i} = a_i \, \underset{\sim}{\hat{e}_i}$$

Norm (length) of $\underset{\sim}{a}$:

$$|\underset{\sim}{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i \, a_i}$$

inner product
(scalar product, "dot"-product):

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a\,b\,\cos\theta$$



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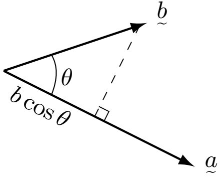

Also:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x\,b_x + a_y\,b_y + a_z\,b_z = a_1\,b_1 + a_2\,b_2 + a_3\,b_3 = a_i\,b_i$$

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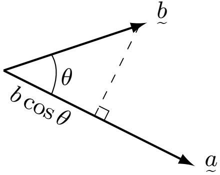


Also:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = a_i\, b_i$$

For unit-direction-vectors:

$$\underset{\sim}{\hat{e}_x} \cdot \underset{\sim}{\hat{e}_x} = \cdots = 1 \quad \text{or} \quad \underset{\sim}{\hat{e}_1} \cdot \underset{\sim}{\hat{e}_1} = \underset{\sim}{\hat{e}_2} \cdot \underset{\sim}{\hat{e}_2} = \underset{\sim}{\hat{e}_3} \cdot \underset{\sim}{\hat{e}_3} = 1$$

$$\underset{\sim}{\hat{e}_x} \cdot \underset{\sim}{\hat{e}_y} = \cdots = 0 \quad \text{or} \quad \underset{\sim}{\hat{e}_1} \cdot \underset{\sim}{\hat{e}_2} = \underset{\sim}{\hat{e}_2} \cdot \underset{\sim}{\hat{e}_3} = \underset{\sim}{\hat{e}_3} \cdot \underset{\sim}{\hat{e}_1} = 0$$

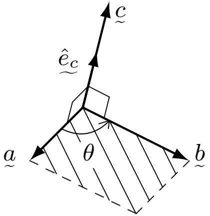
$$\Rightarrow \underset{\sim}{\hat{e}_i} \cdot \underset{\sim}{\hat{e}_j} = \delta_{ij} \quad \text{in a way that e.g.:}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i\, \underset{\sim}{\hat{e}_i} \cdot b_j\, \underset{\sim}{\hat{e}_j} = a_i\, b_j\, \underset{\sim}{\hat{e}_i} \cdot \underset{\sim}{\hat{e}_j} = a_i\, b_j\, \delta_{ij} = a_i\, b_i$$

Outer product or *vector-product*

("cross"-product)

$$\underset{\sim}{a} * \underset{\sim}{b} = (|\underset{\sim}{a}|\,|\underset{\sim}{b}|\,\sin\theta)\ \underset{\sim}{\hat{e}_c} = \underset{\sim}{c}$$



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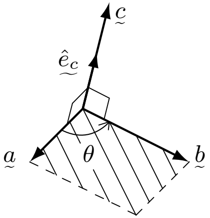


$$\underset{\sim}{a} * \underset{\sim}{b} = \underset{\sim}{a} \times \underset{\sim}{b} = (|\underset{\sim}{a}|\,|\underset{\sim}{b}|\,\sin\theta)\ \ \underset{\sim}{\hat{e}_c} = \underset{\sim}{c}$$

Also:

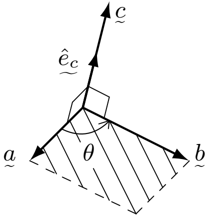
$$\underset{\sim}{a} * \underset{\sim}{b} = \det\left(\begin{bmatrix} \underset{\sim}{\hat{e}_1} & \underset{\sim}{\hat{e}_2} & \underset{\sim}{\hat{e}_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right) =$$

$$= (a_2\,b_3 - a_3\,b_2)\,\underset{\sim}{\hat{e}_1} + (a_3\,b_1 - a_1\,b_3)\,\underset{\sim}{\hat{e}_2} + (a_1\,b_2 - a_2\,b_1)\,\underset{\sim}{\hat{e}_3} =$$

$$= c_1\,\underset{\sim}{\hat{e}_1} + c_2\,\underset{\sim}{\hat{e}_2} + c_3\,\underset{\sim}{\hat{e}_3} = c_i\,\underset{\sim}{\hat{e}_i} = \underset{\sim}{c}$$

Outer product or *vector-product*

(“cross”-product)



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$$= (a_2\,b_3 - a_3\,b_2)\,\underset{\sim}{\hat{e}_1} + (a_3\,b_1 - a_1\,b_3)\,\underset{\sim}{\hat{e}_2} + (a_1\,b_2 - a_2\,b_1)\,\underset{\sim}{\hat{e}_3} =$$

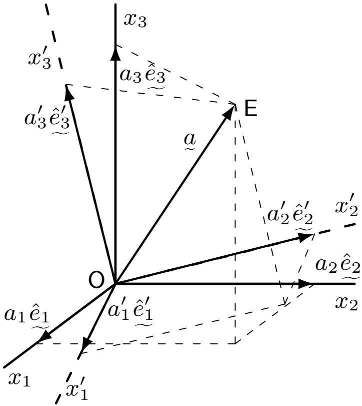
$$= \ c_1\,\underset{\sim}{\hat{e}_1} + c_2\,\underset{\sim}{\hat{e}_2} + c_3\,\underset{\sim}{\hat{e}_3} = c_i\,\underset{\sim}{\hat{e}_i} = \underset{\sim}{c}$$

permutation symbol: $c_i = \varepsilon_{ijk}\,a_j\,b_k \ \ (\underset{\sim}{c} = \varepsilon_{ijk}\,a_j\,b_k\,\underset{\sim}{\hat{e}_i})$ with:

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$; $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$; note: $\varepsilon_{ijk} = 0$

Rotation of axes: (Cartesian coordinates)
Rotations from "old" x_i-coordinates
to "new" x'_p-coordinates.
For an arbitrary vector holds:

$$
\begin{aligned}
\underset{\sim}{a} &= a_1\,\underset{\sim}{\hat{e}_1} + a_2\,\underset{\sim}{\hat{e}_2} + a_3\,\underset{\sim}{\hat{e}_3} = \\
&= a'_1\,\underset{\sim}{\hat{e}'_1} + a'_2\,\underset{\sim}{\hat{e}'_2} + a'_3\,\underset{\sim}{\hat{e}'_3}
\end{aligned}
$$



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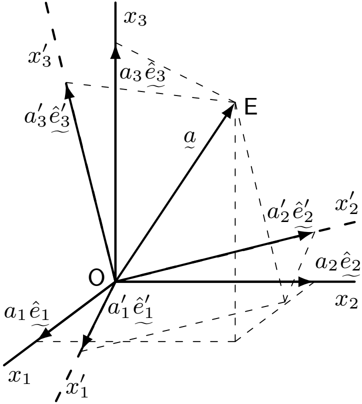
For an arbitrary vector holds:

$$\underset{\sim}{a} = a_1\, \underset{\sim}{\hat{e}_1} + a_2\, \underset{\sim}{\hat{e}_2} + a_3\, \underset{\sim}{\hat{e}_3} = \\ = a'_1\, \underset{\sim}{\hat{e}'_1} + a'_2\, \underset{\sim}{\hat{e}'_2} + a'_3\, \underset{\sim}{\hat{e}'_3}$$



In index notation: $\underset{\sim}{a} = a_i\, \underset{\sim}{\hat{e}_i} = a'_p\, \underset{\sim}{\hat{e}'_p}$

Direction-cosines:
$R_{pi} = \cos(x'_p, x_i) \quad \Rightarrow$

$$a'_1 = a_1\, \cos(x'_1, x_1) + a_2\, \cos(x'_1, x_2) + a_3\, \cos(x'_1, x_3) = \\ = a_i\, \cos(x'_1, x_i) = R_{1i}\, a_i \qquad \text{(change now } 1 \to p\text{)}$$

For $p = 1, 2, 3$:

$$\boxed{\begin{aligned} a'_p &= a_1 \cos(x'_p, x_1) + a_2 \cos(x'_p, x_2) + a_3 \cos(x'_p, x_3) = \\ &= a_i \cos(x'_p, x_i) = R_{pi}\, a_i \end{aligned}}$$

Summation over *second* index of R.

For $p = 1, 2, 3$:

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Summation over *second* index of R.
Inverse relation, from "new" to "old":

$$\boxed{\begin{aligned} a_j &= a'_1 \cos(x_j, x'_1) + a'_2 \cos(x_j, x'_2) + a'_3 \cos(x_j, x'_3) = \\ &= a'_q \cos(x_j, x'_q) = R_{qj}\, a'_q \end{aligned}}$$

Note: summation over the *first* index of R!

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Note: summation over the *first* index of R!
If this transformation rule holds, it is a vector (line with a length and direction): a so-called first-order-tensor!

Written in matrix notation $a'_p = R_{pi}\, a_i$ becomes :

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In *matrix-vector-notation* :

$$\{a'\} = [R]\{a\}$$

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The inverse transform $a_j = R_{qj}\, a'_q$ in matrix notation:

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In *matrix-vector-notation* :

$$\{a\} = [R]^{\mathrm{T}}\{a'\}$$

With $[R]^{\mathrm{T}}$ being the transposed of $[R]$. Matrix mirroring with respect to the main diagonal (top-left $\rightarrow$ bottom-right).

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Properties of the (transformation matrix) rotation matrix $[R]$.
Substitute $\{a\} = [R]^{\mathrm{T}}\{a'\}$ in $\{a'\} = [R]\{a\}$. This gives:

$$\{a'\} = [R]\{a\} = [R][R]^{\mathrm{T}}\{a'\} \Longrightarrow [R][R]^{\mathrm{T}} = [I]$$

Also: $[R][R]^{-1} = [I]$ (definition of inverted matrix), so:
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And $[R][R]^{\mathrm{T}} = [I]$:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ *Normality condition* and *orthogonality condition*

Orthogonality rotation matrix $[R]$ using index notation:

Derived before: $a'_p = R_{pi}\, a_i$ and $a_j = R_{qj}\, a'_q$

So it holds: $a'_p = R_{pi}\, a_i = R_{pi}\, R_{qi}\, a'_q$ also: $a'_p = \delta_{pq}\, a'_q$

From which follows: $\boxed{R_{pi}\, R_{qi} = \delta_{pq}}$ or $\boxed{R_{ip}\, R_{jp} = \delta_{ij}}$

Equivalent with: $[R]\,[R]^{\mathrm{T}} = [I]$

(Note: $[A]\,[B] = [C] \iff A_{ip}\, B_{pj} = C_{ij}$)

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So it holds: $a'_p = R_{pi}\, a_i = R_{pi}\, R_{qi}\, a'_q$ also: $a'_p = \delta_{pq}\, a'_q$

From which follows: $\boxed{R_{pi}\, R_{qi} = \delta_{pq}}$ or $\boxed{R_{ip}\, R_{jp} = \delta_{ij}}$

Equivalent with: $[R]\,[R]^{\mathrm{T}} = [I]$

(Note: $[A]\,[B] = [C] \iff A_{ip}\, B_{pj} = C_{ij}$)

Reverse: $a_j = R_{qj}\, a'_q = R_{qj}\, R_{qi}\, a_i$ also: $a_j = \delta_{ij}\, a_i$

From which follows: $\boxed{R_{qi}\, R_{qj} = \delta_{ij}}$ or $\boxed{R_{pi}\, R_{pj} = \delta_{ij}}$

Equivalent with: $[R]^{\mathrm{T}}\,[R] = [I]$

Second order tensor

Consider 2 vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ with components a_i and b_j

Dyadic product (per definition): $\quad \underline{S} = \underset{\sim}{a}\,\underset{\sim}{b} \quad$ with $\quad S_{ij} = a_i\,b_j$

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Rotation of axis: $S'_{ij} = a'_i\, b'_j$

Substitute transformation rule for vectors:

$$S'_{ij} = a'_i\, b'_j = R_{ip}\, a_p\, R_{jq}\, b_q = R_{ip}\, R_{jq}\, a_p\, b_q = R_{ip}\, R_{jq}\, S_{pq}$$

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Quantities that satisfy this transformation rule are called *second order tensors*. NOTE, not every second order tensor can be written as a dyadic product of two vectors.

Second order tensor: $S'_{ij} = R_{ip}\, R_{jq}\, S_{pq}$
Can be written as: $S'_{ij} = R_{ip}\, S_{pq}\, R_{jq} = R_{ip}\, S_{pq}\, R^{\mathrm{T}}_{qj}$
In matrix-vector-notation: $[S'] = [R]\, [S]\, [R]^{\mathrm{T}}$

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In matrix-vector-notation: $[S'] = [R]\,[S]\,[R]^{\mathrm{T}}$

Inverse: $S_{ij} = R_{pi}\, R_{qj}\, S'_{pq} = R^{\mathrm{T}}_{ip}\, S'_{pq}\, R_{qj}$
In matrix-vector-notation: $[S] = [R]^{\mathrm{T}}\,[S']\,[R]$

Overview tensors (⇒ means transforms to)

0. If $\Phi(x_r) \;\Rightarrow\; \Phi'(x_r)$ by $\Phi = \Phi'$

then *zero order tensor field* or *scalar field*

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then *first order tensor field* or *vector field*

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2. If $S_{ij}(x_r) \;\Rightarrow\; S'_{pq}(x_r)$ by $S'_{pq} = R_{pi}\, R_{qj}\, S_{ij}$
then *second order tensor field*

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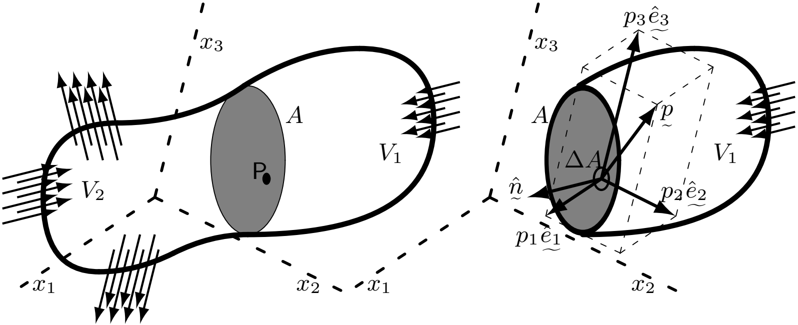
2. If $S_{ij}(x_r) \;\Rightarrow\; S'_{pq}(x_r)$ by $S'_{pq} = R_{pi}\, R_{qj}\, S_{ij}$
then *second order tensor field*
3. If $S_{ijk}(x_r) \;\Rightarrow\; S'_{pqt}(x_r)$ by $S'_{pqt} = R_{pi}\, R_{qj}\, R_{tk}\, S_{ijk}$
then *third order tensor field*
4. etc. etc. etc.

3) Stresses

3.1) Stress state in a point



3) Stresses
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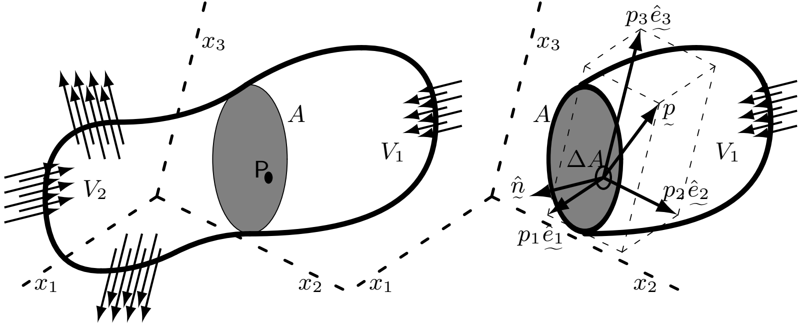

Per definition, stress vector $\underset{\sim}{p} = \lim_{\Delta A \to 0} \left(\frac{\Delta \underset{\sim}{F}}{\Delta A} \right)$

A few remarks:

- A stress vector $\underset{\sim}{p}$ has in general an arbitrary orientation

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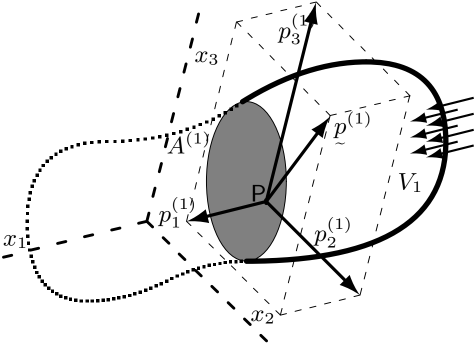
- A stress vector $\underset{\sim}{p}$ can be decomposed in a normal and a tangential component w.r.t. the orientation of ΔA
- A stress vector $\underset{\sim}{p}$ can be decomposed along a Cartesian coordinate system $\quad \underset{\sim}{p} = p_i \, \underset{\sim}{\hat{e}_i}$

Cartesian

stress- components σ_{ij}:

$$\sigma_{11} = p_1^{(1)}$$
$$\sigma_{12} = p_2^{(1)}$$
$$\sigma_{13} = p_3^{(1)}$$



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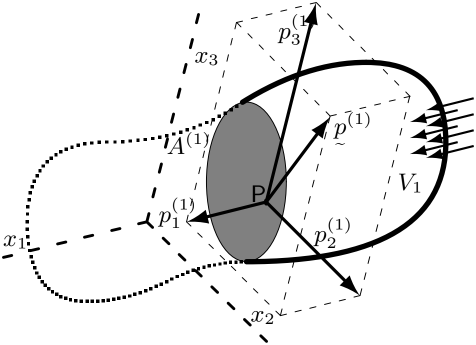


Cross-section

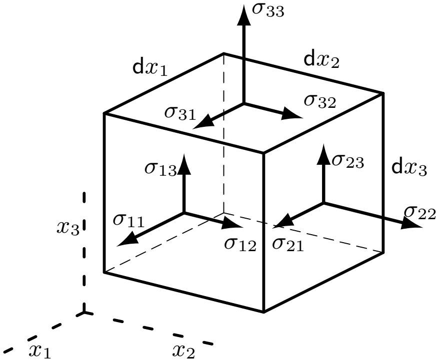
perpendicular to x_2-axis:

$$\sigma_{21} = p_1^{(2)} \qquad \sigma_{22} = p_2^{(2)} \qquad \sigma_{23} = p_3^{(2)}$$

Cross-section perpendicular to x_3-axis:

$$\sigma_{31} = p_1^{(3)} \qquad \sigma_{32} = p_2^{(3)} \qquad \sigma_{33} = p_3^{(3)}$$

Where
σ_{ij} is the stress component in the directions of x_j acting on a surface with its normal pointing in x_i-direction.



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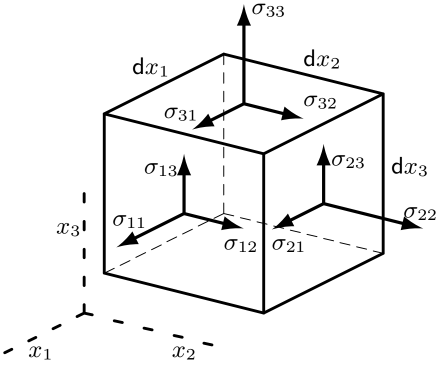
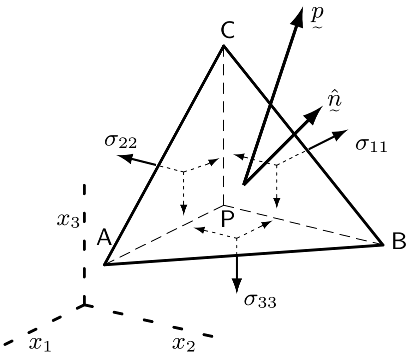


Displayed in a stress matrix:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

3.2) Stress-vector on arbitrary cross-section



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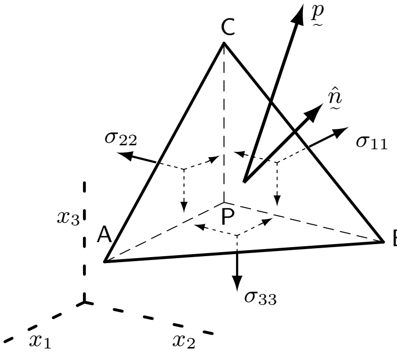
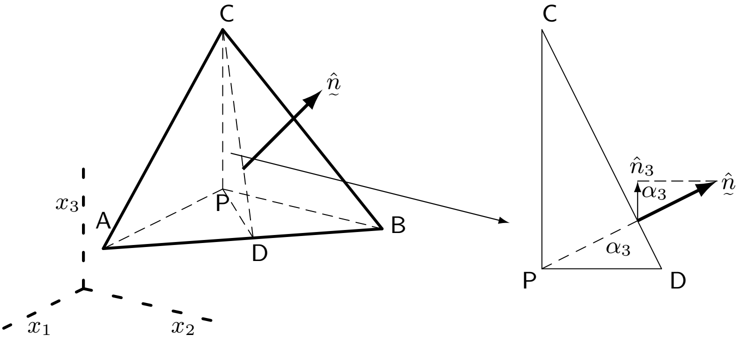
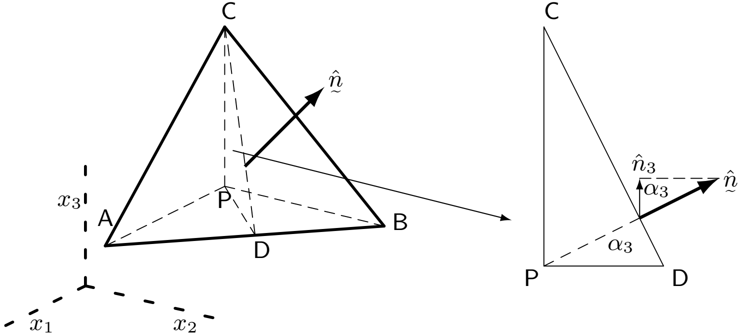


Equilibrium: $p_1 \, \Delta A_{\mathsf{ABC}} - \sigma_{11} \, \Delta A_{\mathsf{PBC}} - \sigma_{21} \, \Delta A_{\mathsf{PCA}} - \sigma_{31} \, \Delta A_{\mathsf{PAB}} = 0$

$$\Rightarrow p_1 = \sigma_{11} \, \frac{\Delta A_{\mathsf{PBC}}}{\Delta A_{\mathsf{ABC}}} + \sigma_{21} \, \frac{\Delta A_{\mathsf{PCA}}}{\Delta A_{\mathsf{ABC}}} + \sigma_{31} \, \frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}}$$



C
$\hat{\underset{\sim}{n}}$
x_3
A
P
B
D
x_1
x_2
C
$\hat{n}_3$
$\hat{\underset{\sim}{n}}$
α_3
α_3
P
D



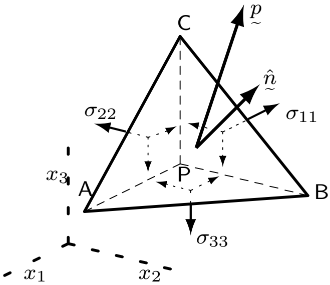
Now it holds:

$$\frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}} = \frac{\frac{1}{2}\,|\mathsf{PD}|\,|\mathsf{AB}|}{\frac{1}{2}\,|\mathsf{CD}|\,|\mathsf{AB}|} = \frac{|\mathsf{PD}|}{|\mathsf{CD}|} = \cos\alpha_3 = \frac{\hat{n}_3}{|\hat{\underset{\sim}{n}}|} = \hat{n}_3 \qquad (|\hat{\underset{\sim}{n}}| = 1)$$

And similar for the other two area ratios.

Combined this gives:

$$\frac{\Delta A_{\mathsf{PBC}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_1) = \hat{n}_1$$
$$\frac{\Delta A_{\mathsf{PCA}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_2) = \hat{n}_2$$
$$\frac{\Delta A_{\mathsf{PAB}}}{\Delta A_{\mathsf{ABC}}} = \cos(\hat{n}, x_3) = \hat{n}_3$$



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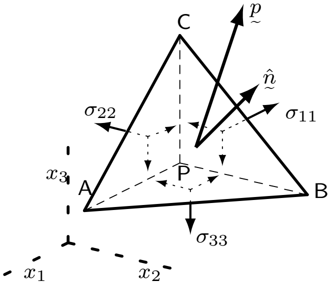

Equilibrium in x_1-direction $\quad$ (*i* = *1,2,3*):

$p_1\, \Delta A = \sigma_{11}\, \Delta A_{\mathsf{PBC}} + \sigma_{21}\, \Delta A_{\mathsf{PCA}} + \sigma_{31}\, \Delta A_{\mathsf{PAB}}$

$\Rightarrow p_1 = \sigma_{11}\, \hat{n}_1 + \sigma_{21}\, \hat{n}_2 + \sigma_{31}\, \hat{n}_3 = \sigma_{j1}\, \hat{n}_j$

Equilibrium in x_i-direction:

$\Rightarrow p_i = \sigma_{1i}\, \hat{n}_1 + \sigma_{2i}\, \hat{n}_2 + \sigma_{3i}\, \hat{n}_3 = \sigma_{ji}\, \hat{n}_j$

Cauchy's formula: $\boxed{p_i = \sigma_{ji}\,\hat{n}_j}$. The summation takes place over the index j, and index i can be either 1, 2 or 3.

$$\underset{\sim}{p} = \underset{\sim}{\hat{n}} \cdot \underline{\sigma} = \underline{\sigma}^{\mathrm{T}} \cdot \underset{\sim}{\hat{n}} \quad \text{or} \quad \{p\} = \{\hat{n}\}^{\mathrm{T}}[\sigma] = [\sigma]^{\mathrm{T}}\{\hat{n}\}$$

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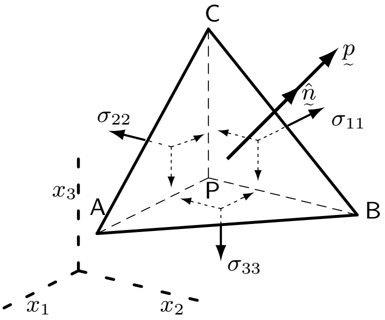
It will be shown later that the stress matrix is symmetric, so $\sigma_{ij} = \sigma_{ji}$ or $\underline{\sigma} = \underline{\sigma}^{\mathrm{T}}$ or $[\sigma] = [\sigma]^{\mathrm{T}}$. From that:

$$\boxed{p_i = \sigma_{ij}\,\hat{n}_j}$$
$$\underset{\sim}{p} = \underline{\sigma} \cdot \underset{\sim}{\hat{n}} \quad \text{or} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$
$$\{p\} = [\sigma]\{\hat{n}\}$$

3.3) Principal stresses and principal directions

Question:

Is it possible to orientate the area ABC in such a way that the direction of the stress vector is the same as the direction of the normal acting on this area.

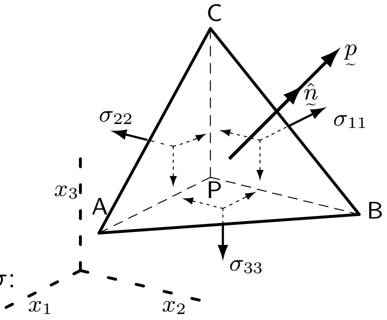


3.3) Principal stresses and principal directions

Question:

Is it possible to orientate the area ABC in such a way that the direction of the stress vector is the same as the direction of the normal acting on this area.
In that case it must hold that:
$\underset{\sim}{p} = \text{factor} \times \underset{\sim}{\hat{n}}$.

Substitute "factor" with the symbol σ:
$\underset{\sim}{p} \stackrel{?}{=} \sigma\, \underset{\sim}{\hat{n}}$ or $p_i \stackrel{?}{=} \sigma\, \hat{n}_i$



Are there multiple orientations of area ABC possible such that:

$$\underset{\sim}{p} \overset{?}{=} \sigma\,\underset{\sim}{\hat{n}} \quad \text{or} \quad p_i \overset{?}{=} \sigma\,\hat{n}_i$$

Cauchy's formula gives: $\quad p_i = \sigma_{ij}\,\hat{n}_j$

Combined: $\quad p_i = \sigma_{ij}\,\hat{n}_j \overset{?}{=} \sigma\,\hat{n}_i = \sigma\,\delta_{ij}\,\hat{n}_j$

Written as a matrix $\quad ([\sigma]\{\hat{n}\} = \sigma\,\{\hat{n}\})$:

$$\{p\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \overset{?}{=} \sigma \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$

$[\sigma]\{\hat{n}\} = \sigma\,\{\hat{n}\}$ can be written as: $\left([\sigma] - \sigma\,[I]\right)\{\hat{n}\} = \{0\}$

$$\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of algebraic equations.

$[\sigma]\{\hat{n}\} = \sigma\,\{\hat{n}\}$ can be written as: $\left([\sigma] - \sigma\,[I]\right)\{\hat{n}\} = \{0\}$

$$\begin{bmatrix} (\sigma_{11}-\sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22}-\sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33}-\sigma) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of algebraic equations.
This also yields a solution $\{\hat{n}\} \neq \{0\}$ when the determinant of the coefficient matrix is equal to zero.

$$\det\left(\begin{bmatrix} (\sigma_{11}-\sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22}-\sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33}-\sigma) \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} (\sigma_{11}-\sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22}-\sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33}-\sigma) \end{bmatrix}\right) = 0$$

Written as *characteristic equation*:

$$\boxed{\sigma^3 - I_1\,\sigma^2 + I_2\,\sigma - I_3 = 0} \qquad \text{with}$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii}$$
$$I_2 = \sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21} + \circlearrowleft = \tfrac{1}{2}\left(\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}\,\sigma_{ji}\right)$$
$$I_3 = \det([\sigma])$$

From this the *principal stresses*: $\sigma_1,\ \sigma_2$ and σ_3 are obtained for which holds that: $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Principal stresses are the *eigenvalues* of the stress matrix $[\sigma]$.
With eigenvalues comes *eigenvectors*. Here the *principal directions*.
Obtain first principal directions, related to σ_1:

$$\begin{bmatrix} (\sigma_{11}-\sigma_1) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22}-\sigma_1) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33}-\sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a *dependent* system of equations!!! e.g.:

$$\begin{bmatrix} (\sigma_{11}-\sigma_1) & \sigma_{12} \\ \sigma_{21} & (\sigma_{22}-\sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \end{bmatrix} = \begin{bmatrix} -\sigma_{13} \\ -\sigma_{23} \end{bmatrix} \hat{n}_3 \;\Rightarrow\; \begin{array}{l} \hat{n}_1 = a\ \hat{n}_3 \\ \hat{n}_2 = b\ \hat{n}_3 \end{array}$$

Found: $\hat{n}_1 = a\ \hat{n}_3$ and $\hat{n}_2 = b\ \hat{n}_3$.
Normal vector is an unit vector, so $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$
With this the first principal direction is determined.
The same could be done for the second and third principal directions
All three principal directions are perpendicular to each other!

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All three principal directions are perpendicular to each other!

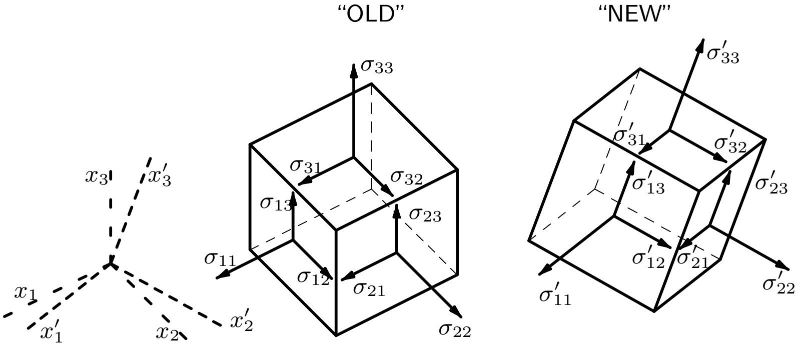
Preferably a coordinate system coinciding with the principal directions:

$$[\sigma]_{\mathsf{ps}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \Rightarrow$$

Stress invariants:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1$$
$$I_3 = \sigma_1\,\sigma_2\,\sigma_3$$

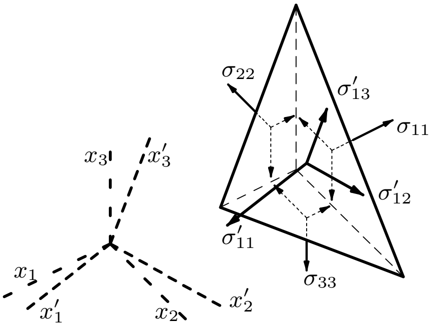
3.4) Rotation matrix



Directional cosines: $R_{pi} = \cos(x_p', x_i)$

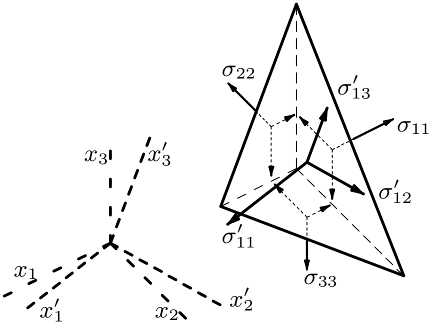
Surface
perpendicular to x_1'-axis.
Normal vector $\quad \underset{\sim}{\hat{n}}^{(1)}$

with: $\quad \hat{n}_i^{(1)} = \cos(x_1', x_i) = R_{1i}$
Cauchy's formula:
$p_j^{(1)} = \sigma_{ij}\, \hat{n}_i^{(1)} = \sigma_{ij}\, R_{1i}$



Surface
perpendicular to x'_1-axis.
Normal vector $\hat{\underset{\sim}{n}}^{(1)}$

with: $\hat{n}_i^{(1)} = \cos(x'_1, x_i) = R_{1i}$
Cauchy's formula:
$p_j^{(1)} = \sigma_{ij}\,\hat{n}_i^{(1)} = \sigma_{ij}\,R_{1i}$



After rotation:

$$\sigma'_{1q} = {p'}_q^{(1)} = R_{qj}\,p_j^{(1)} = R_{qj}\,\sigma_{ij}\,R_{1i} = R_{1i}\,R_{qj}\,\sigma_{ij}$$

Same for the x'_2- and x'_3-directions. So:

$$\sigma'_{pq} = R_{pi}\,R_{qj}\,\sigma_{ij}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During rotation of coordinate axes holds:

$$\sigma'_{pq} = R_{pi}\, R_{qj}\, \sigma_{ij} \quad \text{or} \quad [\sigma'] = [R]\,[\sigma]\,[R]^{\mathrm{T}}$$

Elaborated:

$$\begin{aligned} \sigma'_{pq} = {} & R_{p1}\, R_{q1}\, \sigma_{11} + R_{p1}\, R_{q2}\, \sigma_{12} + R_{p1}\, R_{q3}\, \sigma_{13} + \\ & + R_{p2}\, R_{q1}\, \sigma_{21} + R_{p2}\, R_{q2}\, \sigma_{22} + R_{p2}\, R_{q3}\, \sigma_{23} + \\ & + R_{p3}\, R_{q1}\, \sigma_{31} + R_{p3}\, R_{q2}\, \sigma_{32} + R_{p3}\, R_{q3}\, \sigma_{33} \end{aligned}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During transformation it holds that:

$$\sigma'_{pq} = R_{pi}\, R_{qj}\, \sigma_{ij} \quad \text{or} \quad [\sigma'] = [R]\,[\sigma]\,[R]^{\mathrm{T}}$$

Elaborated:

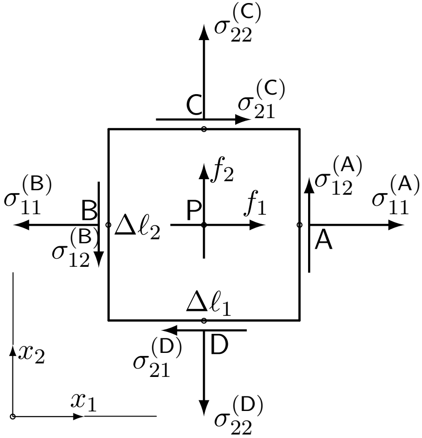
$$\begin{aligned} \sigma'_{pq} = {} & R_{p1}\, R_{q1}\, \sigma_{11} + R_{p1}\, R_{q2}\, \sigma_{12} + R_{p1}\, R_{q3}\, \sigma_{13} \; + \\ & + R_{p2}\, R_{q1}\, \sigma_{21} + R_{p2}\, R_{q2}\, \sigma_{22} + R_{p2}\, R_{q3}\, \sigma_{23} \; + \\ & + R_{p3}\, R_{q1}\, \sigma_{31} + R_{p3}\, R_{q2}\, \sigma_{32} + R_{p3}\, R_{q3}\, \sigma_{33} \end{aligned}$$

Also the inverse transformation holds ("new" to "old"):

$$\sigma_{ij} = R_{pi}\, R_{qj}\, \sigma'_{pq} \quad \text{or} \quad \sigma_{pq} = R_{ip}\, R_{jq}\, \sigma'_{ij} \quad \text{or} \quad [\sigma] = [R]^{\mathrm{T}}[\sigma']\,[R]$$

3.5) Equilibrium equations

In reader 3-dim., here 2-dim.:
Instead
of a single point now *a volume*.
Stresses are a function of x_i.



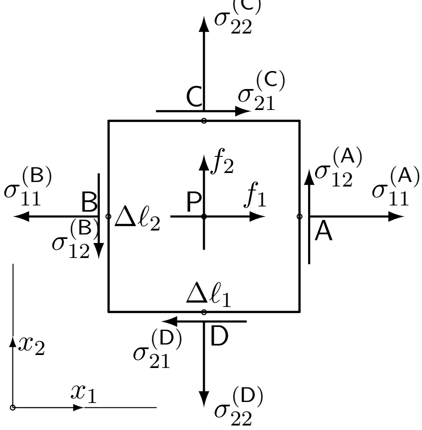
Coordinates of the points:

$$
\begin{aligned}
P:&\quad (x_1, x_2)\\
A:&\quad (x_1 + \tfrac{1}{2}\Delta\ell_1, x_2)\\
B:&\quad (x_1 - \tfrac{1}{2}\Delta\ell_1, x_2)\\
C:&\quad (x_1, x_2 + \tfrac{1}{2}\Delta\ell_2)\\
D:&\quad (x_1, x_2 - \tfrac{1}{2}\Delta\ell_2)
\end{aligned}
$$

f_1 and f_2 are volume forces or body forces, forces per unit volume.

3.5) Equilibrium equations

In reader 3-dim., here 2-dim.: Instead
of a single point now *a volume*.
Stresses are a function of x_i.



Coordinates of the points:

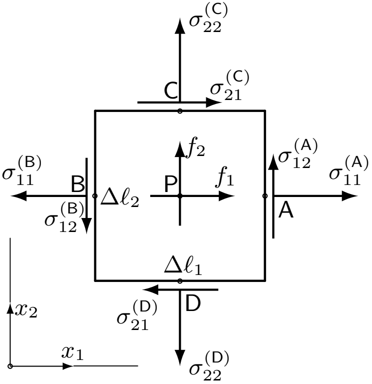
$$
\begin{array}{ll}
P: & (x_1, x_2) \\
A: & (x_1 + \frac{1}{2}\Delta\ell_1, x_2) \\
B: & (x_1 - \frac{1}{2}\Delta\ell_1, x_2) \\
C: & (x_1, x_2 + \frac{1}{2}\Delta\ell_2) \\
D: & (x_1, x_2 - \frac{1}{2}\Delta\ell_2)
\end{array}
$$

Force equilibrium in x_1-direction (volume size in x_3-direction is $\Delta\ell_3$):

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

$$(\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

$$\sigma_{11}^{(\mathsf{A})} = \sigma_{11}(x_1 + \tfrac{1}{2}\Delta\ell_1, x_2) =$$
$$= \sigma_{11}(x_1, x_2) + \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$$
$$\sigma_{11}^{(\mathsf{B})} = \sigma_{11}(x_1 - \tfrac{1}{2}\Delta\ell_1, x_2) =$$
$$= \sigma_{11}(x_1, x_2) - \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$$



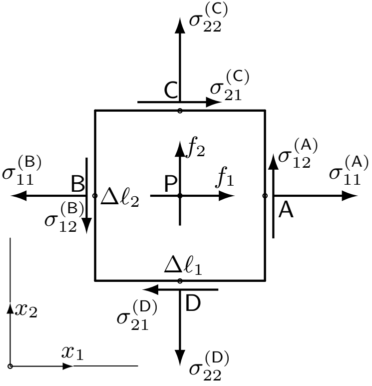
$$(\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

$$\sigma_{11}^{(\mathsf{A})} = \sigma_{11}(x_1 + \tfrac{1}{2}\Delta\ell_1, x_2) =$$
$$= \sigma_{11}(x_1, x_2) + \tfrac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$$
$$\sigma_{11}^{(\mathsf{B})} = \sigma_{11}(x_1 - \tfrac{1}{2}\Delta\ell_1, x_2) =$$
$$= \sigma_{11}(x_1, x_2) - \tfrac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1}\right) + \text{H.O.T.}$$

So (neglecting Higher Order Terms):

$$\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})} = \Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1}\right)$$

Idem: $\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})} = \Delta\ell_2 \left(\frac{\partial \sigma_{21}}{\partial x_2}\right)$



Equilibrium of forces in x_1-direction:

$$(\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})} = \Delta\ell_1\,\dfrac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})} = \Delta\ell_2\,\dfrac{\partial\sigma_{21}}{\partial x_2}$
Results in (after dividing by $\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

Shorter: $\displaystyle\sum_{j=1}^{\cdots} \frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \quad\Rightarrow\quad \sigma_{j1,j} + f_1 = 0$

Equilibrium of forces in x_1-direction:

$$(\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3 + (\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3 + f_1\,\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(\mathsf{A})} - \sigma_{11}^{(\mathsf{B})} = \Delta\ell_1 \dfrac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(\mathsf{C})} - \sigma_{21}^{(\mathsf{D})} = \Delta\ell_2 \dfrac{\partial\sigma_{21}}{\partial x_2}$

Results in (after dividing by $\Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

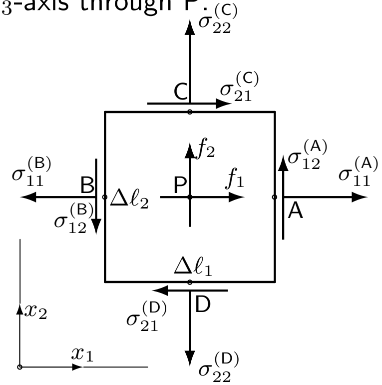
Shorter: $\displaystyle\sum_{j=1}^{\cdots} \frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \quad \Rightarrow \quad \sigma_{j1,j} + f_1 = 0$

The same for other directions, expanded to 3-dim.:

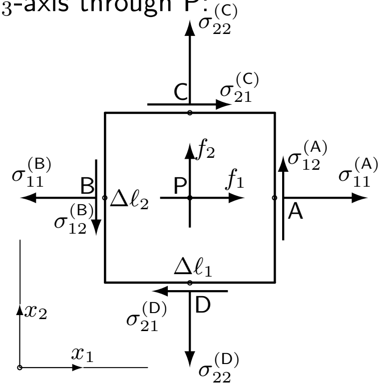
$$\boxed{\sigma_{ji,j} + f_i = 0} \qquad \left(\sum_{j=1}^{3} \frac{\partial\sigma_{ji}}{\partial x_j} + f_i = 0 \;\; \text{for } i = 1,2,3\right)$$

Equilibrium of moments around the x_3-axis through P:

$$(\sigma_{12}^{(\mathsf{A})} + \sigma_{12}^{(\mathsf{B})})\,\Delta\ell_2\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_1\ +$$
$$-\,(\sigma_{21}^{(\mathsf{C})} + \sigma_{21}^{(\mathsf{D})})\,\Delta\ell_1\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_2 = 0$$
$$\Longrightarrow$$
$$(\sigma_{12}^{(\mathsf{A})} + \sigma_{12}^{(\mathsf{B})}) - (\sigma_{21}^{(\mathsf{C})} + \sigma_{21}^{(\mathsf{D})}) = 0$$



Equilibrium of moments around the x_3-axis through P:



$$\left(\sigma_{12}^{(A)}+\sigma_{12}^{(B)}\right)\Delta\ell_2\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_1 + {}$$
$$-\left(\sigma_{21}^{(C)}+\sigma_{21}^{(D)}\right)\Delta\ell_1\,\Delta\ell_3\,\tfrac{1}{2}\Delta\ell_2 = 0$$
$$\Longrightarrow$$
$$\left(\sigma_{12}^{(A)}+\sigma_{12}^{(B)}\right)-\left(\sigma_{21}^{(C)}+\sigma_{21}^{(D)}\right)=0$$

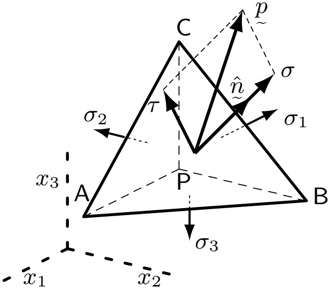
$$\sigma_{12}^{(A)} = \sigma_{12} + \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{12}}{\partial x_1}\right) + \text{H.O.T.}$$
$$\sigma_{12}^{(B)} = \sigma_{12} - \tfrac{1}{2}\Delta\ell_1\left(\frac{\partial\sigma_{12}}{\partial x_1}\right) + \text{H.O.T.}$$
$$\sigma_{21}^{(C)} = \sigma_{21} + \tfrac{1}{2}\Delta\ell_2\left(\frac{\partial\sigma_{21}}{\partial x_2}\right) + \text{H.O.T.}$$
$$\sigma_{21}^{(D)} = \sigma_{21} - \tfrac{1}{2}\Delta\ell_2\left(\frac{\partial\sigma_{21}}{\partial x_2}\right) + \text{H.O.T.}$$

$\Rightarrow\quad 2\,\sigma_{12} - 2\,\sigma_{21} = 0 \quad\Rightarrow\quad \sigma_{12} = \sigma_{21}$ in general: $\boxed{\sigma_{ij} = \sigma_{ji}}$

3.6) Maximum shear stresses, extrema

How to obtain maximum shear stress τ_{max} .

Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i-coordinate system that it will coincide with the principal directions.



3.6) Maximum shear stresses, extrema

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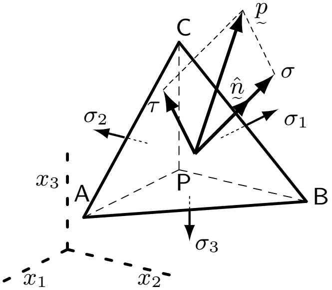
Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i-coordinate system that it will coincide with the principal directions.



The stress matrix will be:

$$[\sigma_h] = [\sigma]_{x_i} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Apply an arbitrary orientated surface, with normal vector $\underset{\sim}{\hat{n}}$.

The components of the stress vector $\underset{\sim}{p}$ to the surface (Cauchy):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \, \hat{n}_1 \\ \sigma_2 \, \hat{n}_2 \\ \sigma_3 \, \hat{n}_3 \end{bmatrix}$$

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The magnitude, σ, of the normal stress on the surface is:

$$\sigma = \{\hat{n}\}^{\mathrm{T}}\{p\} = [\ \hat{n}_1 \quad \hat{n}_2 \quad \hat{n}_3\ ] \begin{bmatrix} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{bmatrix} = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$$

The components of the stress vector $\underset{\sim}{p}$ to the surface (Cauchy):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{bmatrix}$$

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$$\sigma = \{\hat{n}\}^{\mathrm{T}}\{p\} = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{bmatrix} \begin{bmatrix} \sigma_1\,\hat{n}_1 \\ \sigma_2\,\hat{n}_2 \\ \sigma_3\,\hat{n}_3 \end{bmatrix} = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$$

The magnitude, τ, of the shear stress on the surface:

$$\tau^2 = |\underset{\sim}{p}|^2 - \sigma^2 = \sigma_1^2\,\hat{n}_1^2 + \sigma_2^2\,\hat{n}_2^2 + \sigma_3^2\,\hat{n}_3^2 - (\sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2)^2$$

Found: $\tau^2 = \sigma_1^2\,\hat{n}_1^2 + \sigma_2^2\,\hat{n}_2^2 + \sigma_3^2\,\hat{n}_3^2 - (\sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2)^2$
Eliminate the directional coefficient $\hat{n}_3$ with: $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2 \quad \Longrightarrow$

$$\begin{aligned}\tau^2 = {} & (\sigma_1^2 - \sigma_3^2)\hat{n}_1^2 + (\sigma_2^2 - \sigma_3^2)\hat{n}_2^2 + \sigma_3^2 + \\ & - \left\{(\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3\right\}^2\end{aligned}$$

Extrema can be found with: $\dfrac{\partial\tau^2}{\partial\hat{n}_1} = \dfrac{\partial\tau^2}{\partial\hat{n}_2} = 0$

Found: $\tau^2 = \sigma_1^2\,\hat{n}_1^2 + \sigma_2^2\,\hat{n}_2^2 + \sigma_3^2\,\hat{n}_3^2 - (\sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2)^2$
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Extrema can be found with: $\dfrac{\partial \tau^2}{\partial \hat{n}_1} = \dfrac{\partial \tau^2}{\partial \hat{n}_2} = 0$
The following solutions are found:

$\hat{n}_1 = 0 \quad$ with $\begin{cases} 1) & \hat{n}_2 = 0 & \hat{n_3} = \pm\,1 \\ 2) & \hat{n}_2 = \pm\,\frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp\,\frac{1}{2}\sqrt{2} \end{cases}$

The first solution is $\tau = 0$; this is trivial and not interesting

The second solution, $\hat{n}_1 = 0 \;\; ; \;\; \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} \;\; ; \;\; \hat{n}_3 = \mp \frac{1}{2}\sqrt{2}$ gives:

$$\tau^2 = (\sigma_2^2 - \sigma_3^2)\,\frac{1}{2} + \sigma_3^2 - \{(\sigma_2 - \sigma_3)\,\frac{1}{2} + \sigma_3\}^2 = \frac{1}{4}\,(\sigma_2 - \sigma_3)^2$$

An extreme case for τ is found at: $\frac{1}{2}\,|\sigma_2 - \sigma_3|$.

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An extreme case for τ is found at: $\frac{1}{2}\,|\sigma_2 - \sigma_3|$.
In a similar way the following solutions can be found:

$$\begin{array}{llll}
\hat{n}_1 = 0 & \hat{n}_2 = \pm\,\frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp\,\frac{1}{2}\sqrt{2} & \tau = \frac{1}{2}\,|\sigma_2 - \sigma_3| \\
\hat{n}_1 = \pm\,\frac{1}{2}\sqrt{2} & \hat{n}_2 = \mp\,\frac{1}{2}\sqrt{2} & \hat{n}_3 = 0 & \tau = \frac{1}{2}\,|\sigma_1 - \sigma_2| \\
\hat{n}_1 = \pm\,\frac{1}{2}\sqrt{2} & \hat{n}_2 = 0 & \hat{n}_3 = \mp\,\frac{1}{2}\sqrt{2} & \tau = \frac{1}{2}\,|\sigma_1 - \sigma_3|
\end{array}$$

The second solution, $\hat{n}_1 = 0 \;\; ; \;\; \hat{n}_2 = \pm\frac{1}{2}\sqrt{2} \;\; ; \;\; \hat{n}_3 = \mp\frac{1}{2}\sqrt{2}$ gives:

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In a similar way the following solutions can be found:

$$\begin{array}{llll}
\hat{n}_1 = 0 & \hat{n}_2 = \pm\frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp\frac{1}{2}\sqrt{2} & \tau = \frac{1}{2}\,|\sigma_2 - \sigma_3| \\
\hat{n}_1 = \pm\frac{1}{2}\sqrt{2} & \hat{n}_2 = \mp\frac{1}{2}\sqrt{2} & \hat{n}_3 = 0 & \tau = \frac{1}{2}\,|\sigma_1 - \sigma_2| \\
\hat{n}_1 = \pm\frac{1}{2}\sqrt{2} & \hat{n}_2 = 0 & \hat{n}_3 = \mp\frac{1}{2}\sqrt{2} & \tau = \frac{1}{2}\,|\sigma_1 - \sigma_3|
\end{array}$$

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then $\boxed{\tau_{\text{max}} = \frac{1}{2}\,(\sigma_1 - \sigma_3)}$

This τ_{max} originates from a 45^o rotation around the x_2-axis

45^o rotation around x_2-axis: $\quad [R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$

Then: $\quad [\sigma'] = [R]\,[\sigma_h]\,[R]^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2}(\sigma_1 + \sigma_3) & 0 & \frac{1}{2}(\sigma_1 - \sigma_3) \\ 0 & \sigma_2 & 0 \\ \frac{1}{2}(\sigma_1 - \sigma_3) & 0 & \frac{1}{2}(\sigma_1 + \sigma_3) \end{bmatrix}$

45^o rotation around x_2-axis: $\quad [R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$

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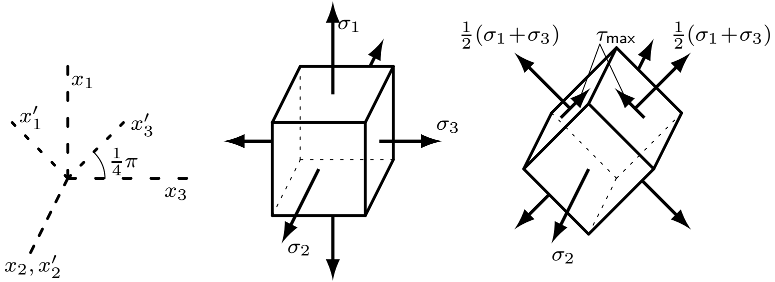

Extrema from the normal stress on a surface.
Found previously: $\sigma = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$
Elimination of $\hat{n}_3$ using $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$
$\sigma = (\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3$
Extrema when: $\dfrac{\partial\sigma}{\partial\hat{n}_1} = \dfrac{\partial\sigma}{\partial\hat{n}_2} = 0$
Resulting in: $(\sigma_1 - \sigma_3)2\hat{n}_1 = (\sigma_2 - \sigma_3)2\hat{n}_2 = 0$

Extrema from the normal stress on a surface.
Found previously: $\sigma = \sigma_1\,\hat{n}_1^2 + \sigma_2\,\hat{n}_2^2 + \sigma_3\,\hat{n}_3^2$
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$\sigma = (\sigma_1 - \sigma_3)\hat{n}_1^2 + (\sigma_2 - \sigma_3)\hat{n}_2^2 + \sigma_3$
Extrema when: $\dfrac{\partial \sigma}{\partial \hat{n}_1} = \dfrac{\partial \sigma}{\partial \hat{n}_2} = 0$
Resulting in: $(\sigma_1 - \sigma_3)2\hat{n}_1 = (\sigma_2 - \sigma_3)2\hat{n}_2 = 0$
Solution (general case): $\hat{n}_1 = \hat{n}_2 = 0 \;\; ; \;\; \hat{n}_3 = 1$
This is exactly the 3^{th} principal direction corresponding to σ_3. σ_1 and σ_2 can be found with elimination of $\hat{n}_1$ resp. $\hat{n}_2$. So, the principal stresses are the extrema of the normal stress.

$$\boxed{\text{When} \quad \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{it must hold that:} \quad \sigma_1 \geq \sigma \geq \sigma_3.}$$

3.7) Miscellaneous, including Mohr's circle(s)

Consider the following special stress state: $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$

NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a Principal stress (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

3.7) Miscellaneous, including Mohr's circle(s)

Consider the following special stress state: $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$

NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a Principal stress (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

Calculate principal stresses:

$$\det\left(\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22}-\sigma & 0 \\ 0 & 0 & \sigma_{33}-\sigma \end{bmatrix}\right) = 0$$

$$\Rightarrow \quad (\sigma_{33}-\sigma)\{(\sigma_{11}-\sigma)(\sigma_{22}-\sigma) - \sigma_{12}^2\} = 0$$

$$\Rightarrow \quad (\sigma_{33}-\sigma)\{\sigma^2 - (\sigma_{11}+\sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$$

Found: $(\sigma_{33}-\sigma)\{\sigma^2 - (\sigma_{11}+\sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$
Solutions:

$$\begin{aligned}
\sigma_{1,2} &= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \tfrac{1}{2}\sqrt{(\sigma_{11}+\sigma_{22})^2 - 4\,\sigma_{11}\sigma_{22} + 4\,\sigma_{12}^2} = \\
&= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2} \\
\sigma_3 &= \sigma_{33}
\end{aligned}$$

The order of the principal stresses is STILL random!

Found: $(\sigma_{33}-\sigma)\{\sigma^2 - (\sigma_{11}+\sigma_{22})\,\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$
Solutions:

$$\begin{aligned}
\sigma_{1,2} &= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \tfrac{1}{2}\sqrt{(\sigma_{11}+\sigma_{22})^2 - 4\,\sigma_{11}\sigma_{22} + 4\,\sigma_{12}^2} = \\
&= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2} \\
\sigma_3 &= \sigma_{33}
\end{aligned}$$

The order of the principal stresses is STILL random!
The principal directions can be found with:

$$\begin{bmatrix} \sigma_{11}-\sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22}-\sigma_i & 0 \\ 0 & 0 & \sigma_{33}-\sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1, 2, 3$$

$$\begin{bmatrix} \sigma_{11}-\sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22}-\sigma_i & 0 \\ 0 & 0 & \sigma_{33}-\sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1,2,3$$

Consecutively resulting in, for $i = 1, 2, 3$:

$$\{\hat{n}\}^{(1)} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \quad ; \quad \{\hat{n}\}^{(2)} = \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix} \quad ; \quad \{\hat{n}\}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$a_1^2 + a_2^2 = 1 \;\Rightarrow\; a_1 = \frac{\sigma_{12}}{w} \;;\; a_2 = \frac{\sigma_i - \sigma_{11}}{w} \;;\; w = \sqrt{(\sigma_i - \sigma_{11})^2 + \sigma_{12}^2}$$

What happens with a rotation of angle α around the x_3-axis?

$$\text{Rotation matrix:} \quad [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What happens with a rotation of angle α around the x_3-axis?

$$\text{Rotatiematrix:} \quad [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation (using symmetry, $\sigma_{21} = \sigma_{12}$):

$$[\sigma'] = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & 0 \\ \sigma'_{21} & \sigma'_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = [R][\sigma][R]^{\mathrm{T}} =$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

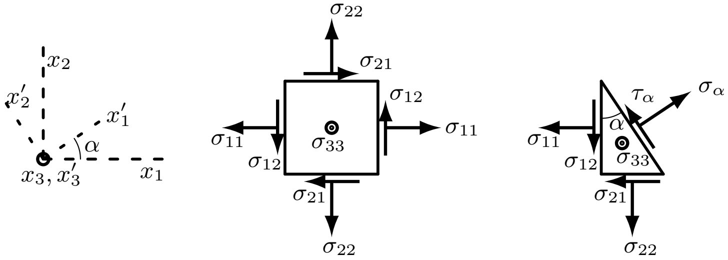
Solving matrices:

$$
\begin{aligned}
\sigma_\alpha = \sigma'_{11} &= \sigma_{11}(\cos\alpha)^2 + 2\,\sigma_{12}\sin\alpha\cos\alpha + \sigma_{22}(\sin\alpha)^2 = \\
&= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha \\
\tau_\alpha = \sigma'_{12} &= (\sigma_{22}-\sigma_{11})\sin\alpha\cos\alpha + \sigma_{12}\{(\cos\alpha)^2-(\sin\alpha)^2\} = \\
&= -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha
\end{aligned}
$$

Solving matrices:

$$
\begin{aligned}
\sigma_\alpha = \sigma'_{11} &= \sigma_{11}(\cos\alpha)^2 + 2\,\sigma_{12}\sin\alpha\cos\alpha + \sigma_{22}(\sin\alpha)^2 = \\
&= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha \\
\tau_\alpha = \sigma'_{12} &= (\sigma_{22}-\sigma_{11})\sin\alpha\cos\alpha + \sigma_{12}\{(\cos\alpha)^2-(\sin\alpha)^2\} = \\
&= -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha
\end{aligned}
$$

Visualisation:



$$
\begin{aligned}
\sigma_\alpha &= \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22}) \cos 2\alpha + \sigma_{12} \sin 2\alpha \\
\tau_\alpha &= -\tfrac{1}{2}(\sigma_{11}-\sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha
\end{aligned}
$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

$$
\left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\right\}^2 + \tau_\alpha^2 = \tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2
$$

$$\sigma_\alpha = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

$$\left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\right\}^2 + \tau_\alpha^2 = \tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2$$

Mohr's circle

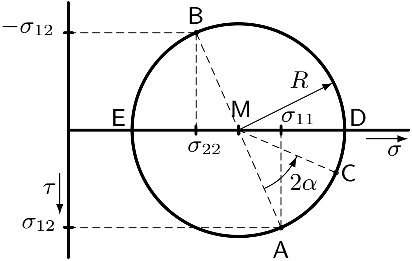
Midpoint M: $\{\tfrac{1}{2}(\sigma_{11}+\sigma_{22}), 0\}$

Radius: $R = \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$

Omit index α at σ and τ!

A: $\alpha = 0$ $(\sigma_{11}, \sigma_{12})$

B: $\alpha = \tfrac{1}{2}\pi$ $(\sigma_{22}, \sigma_{21})$



$$\sigma_\alpha = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \tfrac{1}{2}(\sigma_{11}-\sigma_{22})\cos 2\alpha + \sigma_{12}\sin 2\alpha$$
$$\tau_\alpha = -\tfrac{1}{2}(\sigma_{11}-\sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α:

$$\{\sigma_\alpha - \tfrac{1}{2}(\sigma_{11}+\sigma_{22})\}^2 + \tau_\alpha^2 = \tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2$$

Mohr's circle - ALTERNATIVE

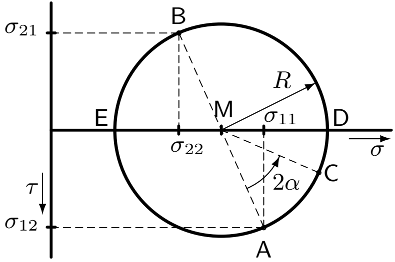
Midpoint M: $\{\tfrac{1}{2}(\sigma_{11}+\sigma_{22}), 0\}$

Radius: $R = \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$

Omit index α at σ and τ!

A: $\alpha = 0$ $(\sigma_{11}, \sigma_{12}\downarrow)$

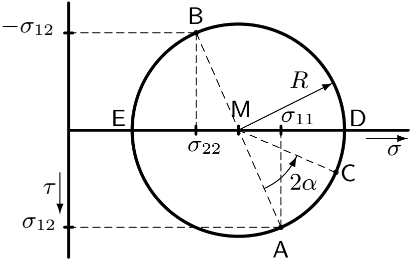
B: $\alpha = \tfrac{1}{2}\pi$ $(\sigma_{22}, \sigma_{21}\uparrow)$



C: $\alpha = ??\quad (\sigma_\alpha, \tau_\alpha)$
D: $\alpha = \alpha_1 \quad (\sigma_1, 0)$
E: $\alpha = \alpha_2 \quad (\sigma_2, 0)$
($2\alpha_1 =$ angle AMD,
$2\alpha_2 =$ angle AME)

$$\tan(2\alpha_1) = \frac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$$

$$\alpha_2 = \alpha_1 \pm \tfrac{1}{2}\pi$$



C: $\alpha = ??$ $(\sigma_\alpha, \tau_\alpha)$
D: $\alpha = \alpha_1$ $(\sigma_1, 0)$
E: $\alpha = \alpha_2$ $(\sigma_2, 0)$
$(2\alpha_1 =$ angle AMD,
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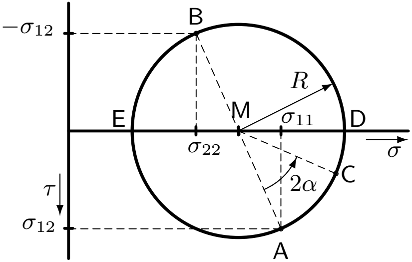
$$\tan(2\alpha_1) = \frac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$$

$$\alpha_2 = \alpha_1 \pm \tfrac{1}{2}\pi$$



(i.e. eigenvectors in real space are perpendicular!)
Eigenvectors in Mohr circle are separated by $2|\alpha_1 - \alpha_2| = \pi$ (i.e. 180°)

$$\sigma_1 = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + R = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) + \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$$

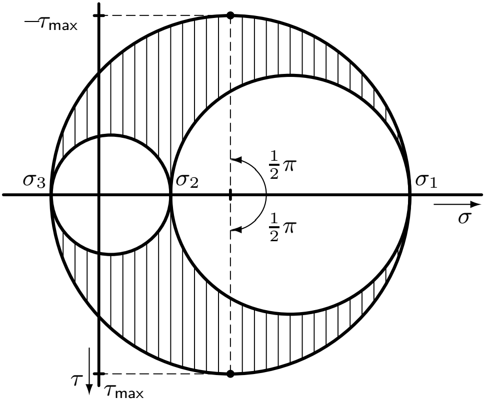
$$\sigma_2 = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) - R = \tfrac{1}{2}(\sigma_{11}+\sigma_{22}) - \sqrt{\tfrac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$$

Principal stresses are the same as found earlier!!
Memorize/practice construction procedure and the factor 2

Arbitrary, three dimensional stress state.

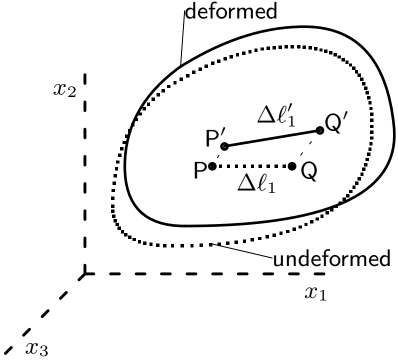
Note: $\tau_{\text{max}} = \frac{1}{2}(\sigma_1 - \sigma_3)$
Angle with first principal direction:
$\frac{1}{2}\left(\pm\,\frac{1}{2}\pi\right) = \pm\,\frac{1}{4}\pi$

Points from an
arbitrary chosen coordinate system
will end up in the striped area!



4) Deformation

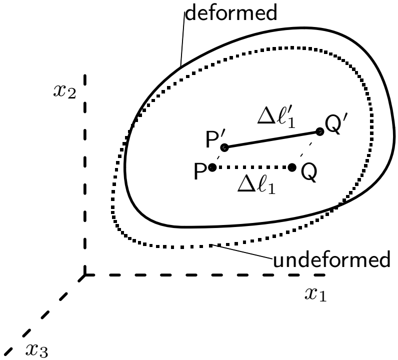
4.1) Strain



Consider a line segment PQ parallel with the x_1-axis.

4) Deformation

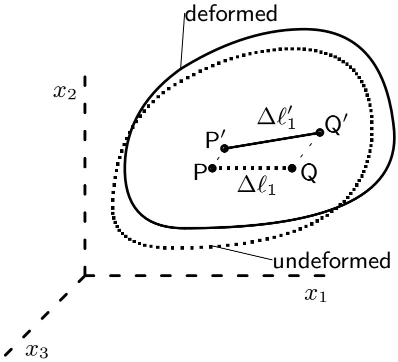
4.1) Strain



Consider a line segment PQ parallel with the x_1-axis.
The coordinates of P are x_1, x_2, x_3
Translation of P to P′
is $\underset{\sim}{u}_{\mathsf{P}}$ with components $u_i(x_1, x_2, x_3)$.

4) Deformation

4.1) Strain



Consider a line segment PQ parallel with the x_1-axis.
The
coordinates of P are x_1, x_2, x_3
Translation of P to P′ is $\underset{\sim}{u}_{\mathrm{P}}$

with components $u_i(x_1, x_2, x_3)$.
The coordinates
of Q are $x_1 + \Delta\ell_1$, x_2, x_3

Translation of Q to Q′ is $\underset{\sim}{u}_{\mathrm{Q}}$ with components $u_i(x_1+\Delta\ell_1, x_2, x_3)$.

$$\text{with: } u_i(x_1+\Delta\ell_1, x_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_1 \frac{\partial u_i}{\partial x_1} + \text{H.O.T.}$$

Coordinates of the translated points:

$$\mathrm{P}' \colon \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \qquad \mathrm{Q}' \colon \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Coordinates of the translated points:

$$\mathrm{P}'\colon \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \qquad \mathrm{Q}'\colon \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\Delta\ell_1' \quad = \quad \sqrt{\left(x_1^{(\mathrm{Q}')} - x_1^{(\mathrm{P}')}\right)^2 + \left(x_2^{(\mathrm{Q}')} - x_2^{(\mathrm{P}')}\right)^2 + \left(x_3^{(\mathrm{Q}')} - x_3^{(\mathrm{P}')}\right)^2} =$$

Coordinates of the translated points:

$$\mathrm{P}'\colon \begin{pmatrix} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{pmatrix} \qquad \mathrm{Q}'\colon \begin{pmatrix} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned} \Delta\ell_1' &= \sqrt{\left(x_1^{(\mathrm{Q}')}-x_1^{(\mathrm{P}')}\right)^2+\left(x_2^{(\mathrm{Q}')}-x_2^{(\mathrm{P}')}\right)^2+\left(x_3^{(\mathrm{Q}')}-x_3^{(\mathrm{P}')}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2+\left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2+\left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \end{aligned}$$

Coordinates of the translated points:

$$\mathrm{P}'\colon \begin{pmatrix} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{pmatrix} \qquad \mathrm{Q}'\colon \begin{pmatrix} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned}
\Delta\ell_1' &= \sqrt{\left(x_1^{(\mathrm{Q}')}-x_1^{(\mathrm{P}')}\right)^2+\left(x_2^{(\mathrm{Q}')}-x_2^{(\mathrm{P}')}\right)^2+\left(x_3^{(\mathrm{Q}')}-x_3^{(\mathrm{P}')}\right)^2} = \\
&= \sqrt{\left(\Delta\ell_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2+\left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2+\left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \\
&= \Delta\ell_1\sqrt{1+2\frac{\partial u_1}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2}
\end{aligned}$$

Coordinates of the translated points:

$$\mathrm{P}'\!: \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \qquad \mathrm{Q}'\!: \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned} \Delta\ell_1' &= \sqrt{\left(x_1^{(\mathrm{Q}')} - x_1^{(\mathrm{P}')}\right)^2 + \left(x_2^{(\mathrm{Q}')} - x_2^{(\mathrm{P}')}\right)^2 + \left(x_3^{(\mathrm{Q}')} - x_3^{(\mathrm{P}')}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1 + \tfrac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2 + \left(\tfrac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2 + \left(\tfrac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \\ &= \Delta\ell_1 \sqrt{1 + 2\tfrac{\partial u_1}{\partial x_1} + \left(\tfrac{\partial u_1}{\partial x_1}\right)^2 + \left(\tfrac{\partial u_2}{\partial x_1}\right)^2 + \left(\tfrac{\partial u_3}{\partial x_1}\right)^2} \\ &\approx \Delta\ell_1 \left(1 + \frac{\partial u_1}{\partial x_1}\right) \qquad \text{if } \frac{\partial u_i}{\partial x_1} \ll 1 \end{aligned}$$

The strain in x_1-direction is:

$$\varepsilon_{11} = \frac{\Delta\ell_1' - \Delta\ell_1}{\Delta\ell_1} \approx \frac{\Delta\ell_1\,(1 + \frac{\partial u_1}{\partial x_1} - 1)}{\Delta\ell_1} = \frac{\partial u_1}{\partial x_1}$$

The strain in x_1-direction is:

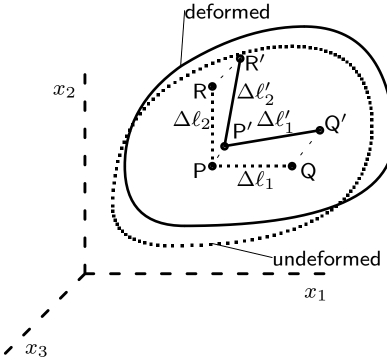
$$\varepsilon_{11} = \frac{\Delta\ell_1' - \Delta\ell_1}{\Delta\ell_1} \approx \frac{\Delta\ell_1\,(1 + \frac{\partial u_1}{\partial x_1} - 1)}{\Delta\ell_1} = \frac{\partial u_1}{\partial x_1}$$

Same for the other directions:

$$\boxed{\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}}$$

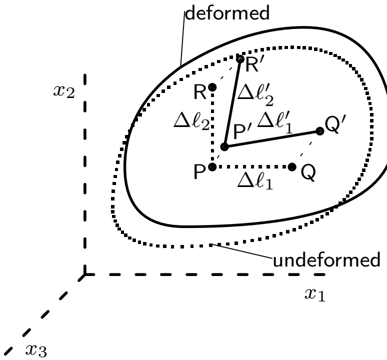
$$\text{if } \ \frac{\partial u_i}{\partial x_j} \ll 1 \ \text{ for } \left\{ \begin{array}{l} i = 1,2,3 \\ j = 1,2,3 \end{array} \right.$$

4.2) Shear strain



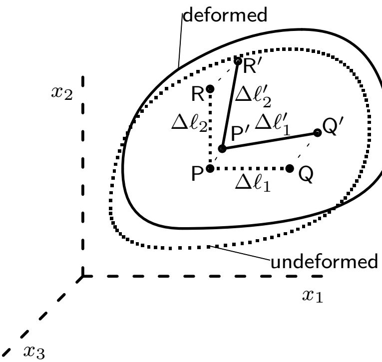
Consider line segments PQ and PR in resp. x_1- and x_2-directions.

4.2) Shear strain



Consider line segments PQ and PR in resp. x_1- and x_2-directions.
The coordinates
of point R are $x_1, x_2 + \Delta\ell_2, x_3$

4.2) Shear strain

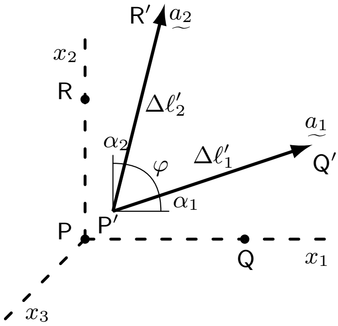


Consider line segments PQ and PR in resp. x_1- and x_2-directions. The coordinates of point R are $x_1, x_2 + \Delta\ell_2, x_3$

Besides the translations of P $\rightarrow$ P′ and Q $\rightarrow$ Q′ we also have $\underset{\sim}{u}_{\mathrm{R}}$ from R $\rightarrow$ R′, the components $u_i(x_1, x_2{+}\Delta\ell_2, x_3)$.

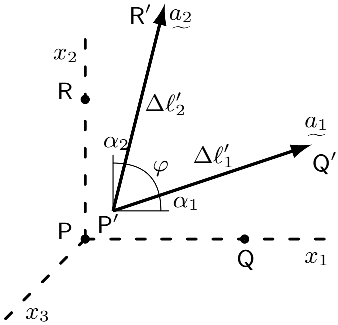
with: $u_i(x_1, x_2{+}\Delta\ell_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_2 \dfrac{\partial u_i}{\partial x_2} + \text{H.O.T.}$

For the coordinates of the points P′, Q′ and R′, see reader.



The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90^o angle.
So: $\gamma_{12} = \frac{1}{2}\pi - \varphi$

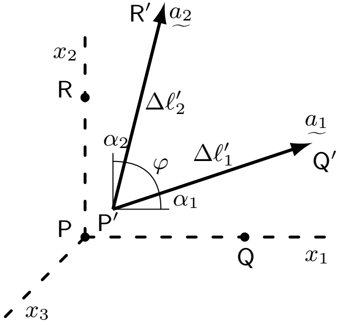
For the coordinates of the points P′, Q′ and R′, see reader.



The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90^o angle.
So: $\gamma_{12} = \frac{1}{2}\pi - \varphi$

$$
\begin{aligned}
\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 &= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\cos(\varphi) \\
&= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\cos(\tfrac{1}{2}\pi - \gamma_{12}) \\
&= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\sin(\gamma_{12})
\end{aligned}
$$

For the coordinates of the points P′, Q′ and R′, see reader.



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So: $\gamma_{12} = \frac{1}{2}\pi - \varphi$

$$
\begin{aligned}
\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 &= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\cos(\varphi) \\
&= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\cos(\tfrac{1}{2}\pi - \gamma_{12}) \\
&= |\underset{\sim}{a}_1||\underset{\sim}{a}_2|\sin(\gamma_{12})
\end{aligned}
$$

From which: $\gamma_{12} = \arcsin\left(\dfrac{\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2}{|\underset{\sim}{a}_1||\underset{\sim}{a}_2|}\right)$

After "some" calculations:

$$\boxed{\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}} \qquad \text{if} \quad \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for} \begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$$

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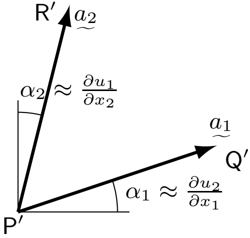
$$\boxed{\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}} \qquad \text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for } \begin{cases} i = 1,2,3 \\ j = 1,2,3 \end{cases}$$



Shear strain $\gamma_{12} = \alpha_1 + \alpha_2$

From the figure: $\tan(\alpha_1) = \dfrac{\frac{\partial u_2}{\partial x_1}\Delta\ell_1}{\Delta\ell_1 + \frac{\partial u_1}{\partial x_1}\Delta\ell_1}$

As approximation: $\alpha_1 \approx \dfrac{\partial u_2}{\partial x_1}$

And: $\alpha_2 \approx \dfrac{\partial u_1}{\partial x_2}$

In general it holds that $\alpha_1 \neq \alpha_2$!!!

Results for 3 directions:

$$\begin{aligned}
\gamma_{12} = \gamma_{21} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\
\gamma_{13} = \gamma_{31} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad \text{if} \ \ \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for} \ \left\{ \begin{array}{l} i = 1,2,3 \\ j = 1,2,3 \end{array} \right. \\
\gamma_{23} = \gamma_{32} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}
\end{aligned}$$

Results for 3 directions:

$$\gamma_{12} = \gamma_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\gamma_{13} = \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad \text{if} \quad \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for} \begin{cases} i = 1,2,3 \\ j = 1,2,3 \end{cases}$$

$$\gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

Summarised in index notation:

$$\boxed{\gamma_{ij} = \gamma_{ji} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}} \quad \text{with } i \neq j$$

4.3) Strain tensor

Strains and shear strains from the theory of *small deformations*:

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} & \qquad \gamma_{12} = \gamma_{21} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} & \qquad \gamma_{13} = \gamma_{31} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} & \qquad \gamma_{23} = \gamma_{32} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}
\end{aligned}
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\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} & \gamma_{23} = \gamma_{32} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}
\end{aligned}
$$

If $i \neq j$ is used for $\varepsilon_{ij} = \frac{1}{2}\gamma_{ij}$ then:

$$
\boxed{\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)} \qquad \text{for} \left\{ \begin{array}{l} i = 1,2,3 \\ j = 1,2,3 \end{array} \right.
$$

Notation $\left(\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right)$:

$$\underline{\varepsilon} = \varepsilon_{ij}\,\underset{\sim}{\hat{e}_i}\,\underset{\sim}{\hat{e}_j} \qquad [\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{bmatrix}$$

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Outdated notation
($x_1 \to x$, $x_2 \to y$ and $x_3 \to z$, $u_1 \to u$, $u_2 \to v$ and $u_3 \to w$):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\ (= \varepsilon_x) \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}\ (= \varepsilon_y) \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}\ (= \varepsilon_z)$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \varepsilon_{yx} = \frac{1}{2}\gamma_{yx}$$

$$\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \varepsilon_{zx} = \frac{1}{2}\gamma_{zx}$$

$$\varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \varepsilon_{zy} = \frac{1}{2}\gamma_{zy}$$

Question: Are ε_{ij} the components of a second order tensor?

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Apply rotation of coordinate axes, $x_i \rightarrow x'_p$:

$$\begin{aligned} x'_p &= x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) = \\ &= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i \end{aligned}$$

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&= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i
\end{aligned}
$$

For the deformation components holds:

$$
\begin{aligned}
\varepsilon_{ij} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x'_q}\frac{\partial x'_q}{\partial x_j} + \frac{\partial u_j}{\partial x'_p}\frac{\partial x'_p}{\partial x_i}\right) = \\
&= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi}\right)
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$$

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= \ x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i
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$$
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\varepsilon_{ij} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x'_q}\frac{\partial x'_q}{\partial x_j} + \frac{\partial u_j}{\partial x'_p}\frac{\partial x'_p}{\partial x_i}\right) \ = \\
&= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi}\right)
\end{aligned}
$$

Transformation components of displacement vector: $\quad u_i = R_{pi} u'_p \quad \Longrightarrow$

$$
\begin{aligned}
\varepsilon_{ij} &= \tfrac{1}{2}\left(R_{pi}\frac{\partial u'_p}{\partial x'_q} R_{qj} + R_{qj}\frac{\partial u'_q}{\partial x'_p} R_{pi}\right) = R_{pi} R_{qj} \tfrac{1}{2}\left(\frac{\partial u'_p}{\partial x'_q} + \frac{\partial u'_q}{\partial x'_p}\right) \ = \\
&= R_{pi} R_{qj} \varepsilon'_{pq}
\end{aligned}
$$

Summarized: (from "new" to "old" and vice versa):

$$\boxed{\varepsilon_{ij} = R_{pi}\, R_{qj}\, \varepsilon'_{pq}} \qquad \text{and} \qquad \boxed{\varepsilon'_{pq} = R_{pi}\, R_{qj}\, \varepsilon_{ij}}$$

In matrix-vector-notation:

$$\boxed{[\varepsilon] = [R]^{\mathrm{T}}[\varepsilon'][R]} \qquad \text{and} \qquad \boxed{[\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}}$$

Summarized: (from "new" to "old" and vice versa):

$$\boxed{\varepsilon_{ij} = R_{pi}\, R_{qj}\, \varepsilon'_{pq}} \qquad \text{and} \qquad \boxed{\varepsilon'_{pq} = R_{pi}\, R_{qj}\, \varepsilon_{ij}}$$

In matrix-vector-notation:

$$\boxed{[\varepsilon] = [R]^{\mathrm{T}}[\varepsilon'][R]} \qquad \text{and} \qquad \boxed{[\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}}$$

So ε_{ij} are indeed components of a second order tensor. As well as the stress tensor, this tensor is symmetric; for the deformation matrix holds: $[\varepsilon]^{\mathrm{T}} = [\varepsilon]$.

Also see the reader regarding the arbitrary oriented line segment PQ.

4.4) Principal strains and principal directions

Second order tensor, so principal directions follow from:

$$\det\left(\begin{bmatrix} \varepsilon_{11}-\varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22}-\varepsilon & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\varepsilon \end{bmatrix}\right) = \det\left([\varepsilon]-\varepsilon[I]\right) = 0$$

Resulting in the characteristic equation: $\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$

4.4) Principal strains and principal directions

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Resulting in the characteristic equation: $\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$

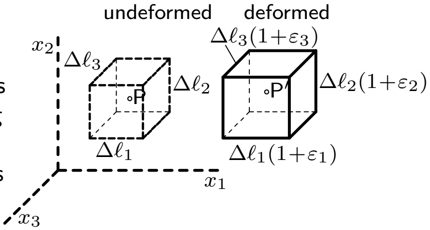
The *deformation invariants* are:

$$\begin{aligned} E_1 &= \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}=\varepsilon_{ii} \\ E_2 &= \varepsilon_{11}\,\varepsilon_{22}-\varepsilon_{12}\,\varepsilon_{21}+\circlearrowleft = \tfrac{1}{2}\left(\varepsilon_{ii}\,\varepsilon_{jj}-\varepsilon_{ij}\,\varepsilon_{ji}\right) \\ E_3 &= \det([\varepsilon]) \end{aligned}$$

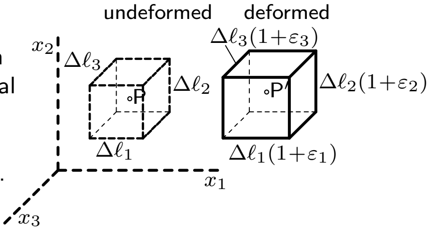
After solving we obtain the principal strains ε_1, ε_2, ε_3 and the 3 orthogonal principal directions. The invariants can be expressed in terms of principal strains.

4.5) Volumetric strain

Consider a x_i-coordinate system oriented in such a way that it coincides with the principal directions of the deformed state. A corresponding infinitesimal volume element does not have any shear strain. The surfaces remain perpendicular to each other during deformation.



4.5) Volumetric strain



Consider

a x_i-coordinate system oriented in such a way that it coincides with the principal directions of the deformed state.

A corresponding infinitesimal volume element does not have any shear strain. The surfaces remain perpendicular to each other during deformation.

Volume undeformed: $V = \Delta\ell_1\,\Delta\ell_2\,\Delta\ell_3$

Volume deformed: $V' = \Delta\ell_1(1+\varepsilon_1)\Delta\ell_2(1+\varepsilon_2)\Delta\ell_3(1+\varepsilon_3)$

Volumetric strain:

$\varepsilon_V = \frac{V'-V}{V} = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 +$ H.O.T.

For small strains we ignore H.O.T. (also including the third invariant E_3).

$\Rightarrow \quad \varepsilon_V = E_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad \text{or} \quad \boxed{\varepsilon_V = \varepsilon_{kk}} = E_1$

4.6) Engineering strain/stress vs. true strain/stress

Note that:
engineering strain $\varepsilon = \frac{\Delta L}{L_0}$ is defined relative to the original length, L_0,
while stress $\sigma = \frac{F}{A_0}$ is defined relative to the original cross-section, A_0.

Neither can be assumed constant during deformation,
so that a true strain increment becomes $\delta\varepsilon_t = \frac{\delta L}{L(t)}$,
defined relative to the momentary (at time t) size of the sample, $L(t)$,
while the true stress becomes $\sigma_t = \frac{F(t)}{A(t)}$, with momentary cross-section $A(t)$.

Integrating the true strain increment $\delta\varepsilon_t$ from 0 to ε_t,
and the right hand side from L_0 to $L(t)$,
using $\ln L(t) - \ln L(0) = \ln(L(t)/L_0)$, with $L(t) = L_0 + \Delta L$,
yields the true strain $\varepsilon_t = \ln(1 + \varepsilon)$.

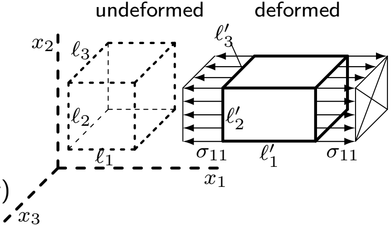
5) Material behavior
5.1) Hooke's law

Linear elastic, isotropic
material behavior.

Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$

Relation stress and strain: $\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$

E is the
Young's modulus (modulus of elasticity)



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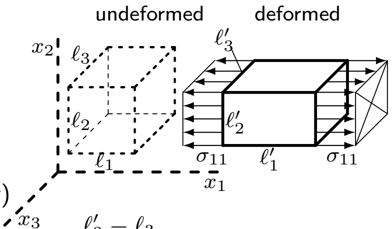

Lateral strain: $\varepsilon_{22} = \dfrac{\ell_2' - \ell_2}{\ell_2} \leq 0 \quad \text{en} \quad \varepsilon_{33} = \dfrac{\ell_3' - \ell_3}{\ell_3} \leq 0$

Relation lateral strain and stresses: $\varepsilon_{22} = \varepsilon_{33} = -\nu\dfrac{\sigma_{11}}{E}$

ν is the *poisson's ratio*

Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\frac{\sigma_{11}}{E}$$

The shear strain seems to remain equal to zero.

Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$

The shear strain seems to remain equal to zero.

Also, when we apply a single normal stress σ_{22}, again with the other stress components equal to zero:

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{33} = -\nu \frac{\sigma_{22}}{E}$$

Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\frac{\sigma_{11}}{E}$$

The shear strain seems to remain equal to zero.

Also, when we apply a single normal stress σ_{22}, again with the other stress components equal to zero:

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{33} = -\nu\frac{\sigma_{22}}{E}$$

Likewise for normal stress σ_{33}:

$$\varepsilon_{33} = \frac{\sigma_{33}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{22} = -\nu\frac{\sigma_{33}}{E}$$

Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

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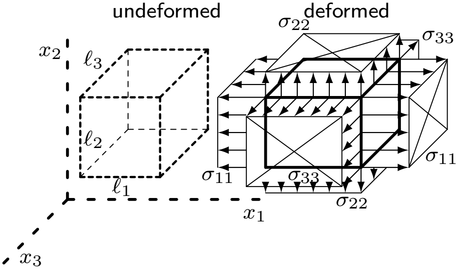
Likewise for normal stress σ_{33}:

$$\varepsilon_{33} = \frac{\sigma_{33}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{22} = -\nu\frac{\sigma_{33}}{E}$$

Also a combined stress state is possible. Superposition principle.

Result:

$$\varepsilon_{11} = \quad \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$
$$\varepsilon_{22} = -\nu\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$
$$\varepsilon_{33} = -\nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}$$

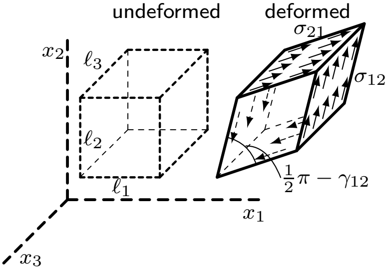


This is Hooke's law in the case that only normal stresses are present!

Shear, shear stresses and shear strain:

$$2\varepsilon_{12} = \gamma_{12} = \frac{\sigma_{12}}{G} = \frac{\sigma_{21}}{G}$$
$$2\varepsilon_{13} = \gamma_{13} = \frac{\sigma_{13}}{G} = \frac{\sigma_{31}}{G}$$
$$2\varepsilon_{23} = \gamma_{23} = \frac{\sigma_{23}}{G} = \frac{\sigma_{32}}{G}$$

The material constant G is the *shear modulus*.



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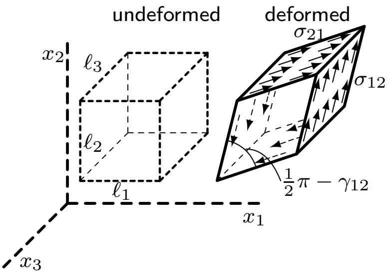


The material constant G is the *shear modulus*.

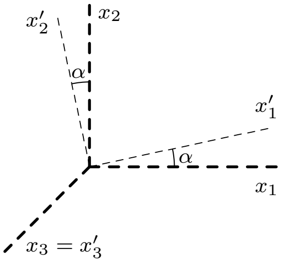
NOTE: With isotropic material behavior E, G and ν are correlated with each other. This correlation can be found via a coordinate system rotation.

Linear stress state

$$[\sigma]_{x_i} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

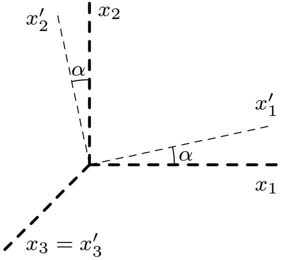
From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$



Linear stress state

$$[\sigma]_{x_i} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$

so:

$$[\varepsilon] = \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu\sigma_{11} & 0 \\ 0 & 0 & -\nu\sigma_{11} \end{bmatrix} \text{ and } [R] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve for σ'_{12} with $\sigma'_{12} = R_{1p}\, R_{q2}\, \sigma_{pq}$ or with $[\sigma'] = [R][\sigma][R]^{\mathrm{T}}$:

$$\begin{bmatrix} \diamond & \sigma'_{12} & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \diamond & -\sin\alpha & \diamond \\ \diamond & \cos\alpha & \diamond \\ \diamond & 0 & \diamond \end{bmatrix}$$

All non-relevant values for the matrix multiplication are shown as a $\diamond$.

Solve for σ'_{12} with $\sigma'_{12} = R_{1p}\, R_{q2}\, \sigma_{pq}$ or with $[\sigma'] = [R][\sigma][R]^{\mathrm{T}}$:

$$\begin{bmatrix} \diamond & \sigma'_{12} & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \diamond & -\sin\alpha & \diamond \\ \diamond & \cos\alpha & \diamond \\ \diamond & 0 & \diamond \end{bmatrix}$$

All non-relevant values for the matrix multiplication are shown as a $\diamond$. Or shorter:

$$\sigma'_{12} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} = -\sigma_{11}\cos\alpha\sin\alpha$$

The same holds for shear strain ε'_{12}. From $[\varepsilon'] = [R][\varepsilon][R]^{\mathrm{T}}$ follows:

$$\varepsilon'_{12} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \end{bmatrix} \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu\sigma_{11} & 0 \\ 0 & 0 & -\nu\sigma_{11} \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} =$$

$$= -\frac{(1+\nu)}{E}\sigma_{11}\cos\alpha\sin\alpha$$

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$$= -\frac{(1+\nu)}{E}\sigma_{11}\cos\alpha\sin\alpha$$

Due to isotropic behavior it must hold that:

$$\varepsilon'_{12} = \tfrac{1}{2}\gamma'_{12} = \tfrac{1}{2}\frac{\sigma'_{12}}{G} = -\frac{1}{2G}\sigma_{11}\cos\alpha\sin\alpha \qquad (\sigma'_{12} = -\sigma_{11}\cos\alpha\sin\alpha)$$

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When we compare both expressions, the following important equation is found:

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

NOTE: This equation only holds for *isotropic* material behavior!

Summary of Hooke's law:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) \;\; ; \;\; \gamma_{12} = 2\varepsilon_{12} = \frac{1}{G}\sigma_{12} \quad \text{(also } 12 \rightarrow 21\text{)}$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11}) \;\; ; \;\; \gamma_{13} = 2\varepsilon_{13} = \frac{1}{G}\sigma_{13} \quad \text{(also } 13 \rightarrow 31\text{)}$$

$$\varepsilon_{33} = \frac{1}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) \;\; ; \;\; \gamma_{23} = 2\varepsilon_{23} = \frac{1}{G}\sigma_{23} \quad \text{(also } 23 \rightarrow 32\text{)}$$

Hooke's law in index notation. For strains:

$$\begin{aligned}\varepsilon_{11} &= \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \\ &= \frac{(1+\nu)}{E}\left(\sigma_{11} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)\end{aligned}$$

Hooke's law in index notation. For strains:

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$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} = \frac{(1+\nu)}{E}\left(\sigma_{33} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

Hooke's law in index notation. For normal strains:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) =$$
$$= \frac{(1+\nu)}{E}\left(\sigma_{11} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} = \frac{(1+\nu)}{E}\left(\sigma_{22} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$
$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} = \frac{(1+\nu)}{E}\left(\sigma_{33} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right)$$

So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \quad \text{if} \quad i = j$$

For shear strains:

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} = \frac{(1+\nu)}{E}\sigma_{12} \qquad ; \qquad \varepsilon_{13} = \frac{\sigma_{13}}{2G} = \frac{(1+\nu)}{E}\sigma_{13}$$
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$$\varepsilon_{23} = \frac{\sigma_{23}}{2G} = \frac{(1+\nu)}{E}\sigma_{23}$$

So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} \quad \text{if} \quad i \neq j \quad \textsf{and} \quad \varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \text{ if } \quad i = j$$

For shear strains:

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So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} \ \text{ if } \ i \neq j \ \text{ and } \ \varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\right) \ \text{ if } \ i = j$$

Combined they result in Hooke's law in index notation $(\ \sigma \rightarrow \varepsilon\ )$:

$$\boxed{\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right) = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

The *inverse*, stresses expressed in deformations.

We found: $$\varepsilon_{ij} = \frac{(1+\nu)}{E}\left(\sigma_{ij} - \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij}\right)$$

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A summation of normal strains gives:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{(1+\nu)}{E}\left(\sigma_{11}+\sigma_{22}+\sigma_{33} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) =$$
$$= \frac{(1+\nu)}{E}\left(\sigma_{kk} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) = \frac{(1-2\nu)}{E}\sigma_{kk}$$

The *inverse*, stresses expressed in deformations.

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$$= \frac{(1+\nu)}{E}\left(\sigma_{kk} - 3\frac{\nu}{(1+\nu)}\sigma_{kk}\right) = \frac{(1-2\nu)}{E}\sigma_{kk}$$

Rewriting Hooke's law:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu}{(1+\nu)}\sigma_{kk}\delta_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu}{(1+\nu)}\frac{E}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

Resulting in: Hooke's law $(\ \varepsilon \rightarrow \sigma\ )$:

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right) = 2G\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right)}$$

for $i = 1, 2, 3\ ;\ j = 1, 2, 3.$

Resulting in; Hooke's law $(\ \varepsilon \to \sigma\ )$:

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right) = 2G\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right)}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.
Notation according to Lamé:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

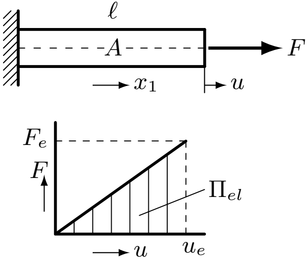
With the so called Lamé's (material) constants:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{en} \quad \mu = G = \frac{E}{2(1+\nu)}$$

5.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \, \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$



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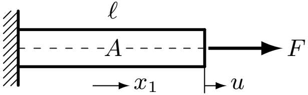
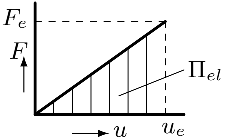


At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$



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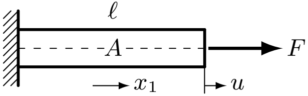

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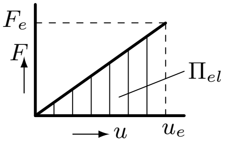

Such that:

$$\Pi_{el} = \tfrac{1}{2} A \, \sigma_{11} \, \ell \, \varepsilon_{11} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11} \, V$$

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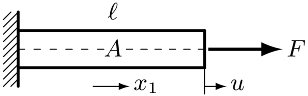
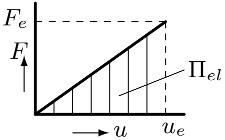


At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$



Such that:

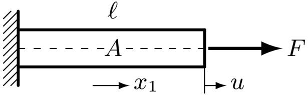
$$\Pi_{el} = \tfrac{1}{2} A \, \sigma_{11} \, \ell \, \varepsilon_{11} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11} \, V$$

Specific internal elastic energy: $\pi_{el} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11}$

5.2) Elastic energy

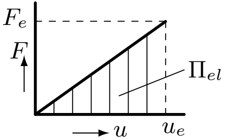
Work:

$$\Pi_{el} = \int_0^e F \; \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$



At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$



Such that:

$$\Pi_{el} = \tfrac{1}{2} A \, \sigma_{11} \, \ell \, \varepsilon_{11} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11} \, V$$

Specific internal elastic energy: $\pi_{el} = \frac{1}{2} \sigma_{11} \, \varepsilon_{11}$
Contribution to π_{el} during shearing: $\frac{1}{2} \sigma_{12} \, \gamma_{12}$

An arbitrary three dimensional stress state:

$$\pi_{el} = \frac{1}{2}\sigma_{11}\,\varepsilon_{11} + \frac{1}{2}\sigma_{22}\,\varepsilon_{22} + \frac{1}{2}\sigma_{33}\,\varepsilon_{33} + \frac{1}{2}\sigma_{12}\,\gamma_{12} + \frac{1}{2}\sigma_{13}\,\gamma_{13} + \frac{1}{2}\sigma_{23}\,\gamma_{23}$$

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Written differently (with $\gamma_{ij} = 2\,\varepsilon_{ij} = 2\,\varepsilon_{ji}$):

$$\begin{aligned}
\pi_{el} \quad &= \quad \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{12}\,\varepsilon_{12} + \tfrac{1}{2}\sigma_{13}\,\varepsilon_{13} + \\
&+ \quad \tfrac{1}{2}\sigma_{21}\,\varepsilon_{21} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{23}\,\varepsilon_{23} + \\
&+ \quad \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33}
\end{aligned}$$

An arbitrary three dimensional stress state:

$$\pi_{el} = \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} + \tfrac{1}{2}\sigma_{12}\,\gamma_{12} + \tfrac{1}{2}\sigma_{13}\,\gamma_{13} + \tfrac{1}{2}\sigma_{23}\,\gamma_{23}$$

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$$\begin{aligned} \pi_{el} &= \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{12}\,\varepsilon_{12} + \tfrac{1}{2}\sigma_{13}\,\varepsilon_{13} + \\ &+ \tfrac{1}{2}\sigma_{21}\,\varepsilon_{21} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{23}\,\varepsilon_{23} + \\ &+ \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} \end{aligned}$$

So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

An arbitrary three dimensional stress state:

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So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

Elastic energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\;\mathrm{d}V = \int_V \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\;\mathrm{d}V$$

By filling in Hooke's law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$, either the stresses or the strains can be eliminated.

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With this the specific energy π_{el} is expressed quadratic in the expression of the stresses and deformations. The specific energy π_{el} turns out to be invariant for coordinate system transformations.

By filling in Hooke's law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$, either the stresses or the strains can be eliminated.
With this the specific energy π_{el} is expressed quadratic in the expression of the stresses and deformations. See the reader. The specific energy π_{el} turns out to be invariant for coordinate system transformations.
The *specific elastic volume-change energy* is:

$$\boxed{\pi_{el_{vol}} = \frac{1}{2}\sigma_m\,\varepsilon_V} \quad \text{with}$$

$$\sigma_m = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{1}{3}\sigma_{kk} \quad \text{and} \quad \varepsilon_V = \varepsilon_{kk}$$

With Hooke's law we have seen that:

$$\varepsilon_V = \varepsilon_{kk} = \frac{(1-2\nu)}{E}\sigma_{kk} = \frac{3\,(1-2\nu)}{E}\sigma_m = \frac{\sigma_m}{C} \qquad (\sigma_m = C\,\varepsilon_V)$$

Where C is the bulk modulus.

With Hooke's law we have seen that:

$$\varepsilon_V = \varepsilon_{kk} = \frac{(1-2\nu)}{E}\sigma_{kk} = \frac{3\,(1-2\nu)}{E}\sigma_m = \frac{\sigma_m}{C} \qquad (\sigma_m = C\,\varepsilon_V)$$

Where C is the bulk modulus. Filling in $\pi_{el_{vol}}$:

$$\pi_{el_{vol}} = \tfrac{1}{2}\sigma_m\,\varepsilon_V = \tfrac{1}{2}\sigma_m\frac{3(1-2\nu)}{E}\sigma_m = \tfrac{1}{2}\frac{(1-2\nu)}{3\,E}\sigma_{hh}\sigma_{kk}$$

Turns out to be invariant for coordinate system transformations.

With Hooke's law we have seen that:

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Turns out to be invariant for coordinate system transformations.
The *specifc elastic shape-change (or distortional) energy*:

$$\begin{aligned} \pi_{el_{shape}} &= \pi_{el} - \pi_{el_{vol}} = \cdots = \\ &= \tfrac{1}{6}\,\frac{(1+\nu)}{E}\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\} \end{aligned}$$

5.3) Deviatoric stresses and strains

Deviatoric stresses: $\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

Deviatoric strains: $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

5.3) Deviatoric stresses and strains

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Deviatoric strains: $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

Also the deviatoric stresses have principal stresses and principal directions. The corresponding coefficients in the characteristic equation

$$\hat{\sigma}^3 - J_2\,\hat{\sigma} - J_3 = 0$$

i.e., the deviatoric invariants are $J_1 = 0$, $J_2 > 0$ and J_3.

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Deviatoric strains: $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

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It turns out that: Law of Hooke: $\hat{\sigma}_{ij} = 2\,G\,\hat{\varepsilon}_{ij}$

$$\pi_{el_{shape}} = \tfrac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\} = \frac{(1+\nu)}{E}J_2$$

5.3) Deviatoric stresses and strains

Deviatoric stresses: $\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

Deviatoric strains: $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

Also the deviatoric stresses have principal stresses and principal directions. The corresponding coefficients in the characteristic equation

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It turns out that: Law of Hooke: $\hat{\sigma}_{ij} = 2\,G\,\hat{\varepsilon}_{ij}$

$$\pi_{el_{shape}} = \tfrac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\} = \frac{(1+\nu)}{E}J_2$$

And: $\left(\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij} \quad ; \quad \pi_{el_{vol}} = \frac{1}{2}\sigma_m\,\varepsilon_V\right)$

$$\boxed{\pi_{el_{shape}} = \tfrac{1}{2}\frac{(1+\nu)}{E}\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \tfrac{1}{2}\frac{1}{2G}\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \tfrac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}}$$

5.4) Failure criteria

1) Tresca, bases on maximum shear stress.

$$\boxed{\sigma_{\mathsf{eq}} = 2\,\tau_{\mathsf{max}} = \max\{|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\}}$$

Or, if $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\boxed{\sigma_{\mathsf{eq}} = 2\,\tau_{\mathsf{max}} = (\sigma_1 - \sigma_3)}$$

5.4) Failure criteria

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2) Von Mises, based on the maximum specific elastic distortional energy.

$$\boxed{\begin{aligned}\sigma_{\mathsf{eq}} &= \sqrt{3\,\frac{E}{1+\nu}\pi_{el_{shape}}} = \sqrt{3J_2} = \\ &= \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}}\end{aligned}}$$

Shear stress criteria
Tresca, Coulomb, Quest, Mohr
Maximum shear stress is leading.

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Comparison 3-dim stress state with 1-dim stress state in a tensile test. $(\sigma_1 = \sigma_{\mathsf{eq}}\ ; \sigma_2 = \sigma_3 = 0)$

$$\tau_{\mathsf{max}} = \frac{1}{2}(\sigma_1 - \sigma_3) \qquad \text{(3-dim)}$$
$$\tau_{\mathsf{max}} = \frac{1}{2}\sigma_{\mathsf{eq}} \qquad \text{(1-dim)}$$

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$$\begin{array}{ll} \tau_{\text{max}} = \frac{1}{2}(\sigma_1 - \sigma_3) & \text{(3-dim)} \\ \tau_{\text{max}} = \frac{1}{2}\sigma_{\text{eq}} & \text{(1-dim)} \end{array}$$

$\sigma_{\text{eq}} \leq \bar{\sigma}$ (allowable stress)

Elastic distortional energy criteria
von Mises, Huber, Hencky
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$$\pi_{el_{shape}} = \frac{1}{6}\frac{(1+\nu)}{E}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\} \quad \text{(3-dim)}$$
$$\pi_{el_{shape}} = \frac{1}{3}\frac{(1+\nu)}{E}\sigma_{\text{eq}}^2 \quad \text{(1-dim)}$$

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Elastic distortional energy is leading.
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$$\pi_{el_{shape}} = \tfrac{1}{3}\tfrac{(1+\nu)}{E}\sigma_{\text{eq}}^2 \quad \text{(1-dim)}$$
$$\rightarrow \quad \sigma_{\text{eq}} = \sqrt{\tfrac{1}{2}\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\}}$$

$\sigma_{\text{eq}} \leq \bar{\sigma}$ (allowable stress)

Special case

Determine the equivalent stresses of a beam with circular cross section. A bending and torsional moment are acting on the beam (Hibbeler pg.195 (5.7) $T\rho/J$):

$$\sigma_{11} = \frac{M_b\,R}{I_b} \quad \text{with} \quad I_b = \tfrac{\pi}{4}\,R^4$$

$$\sigma_{12} = \frac{M_w\,R}{I_p} \quad \text{with} \quad I_p = \tfrac{\pi}{2}\,R^4 = 2\,I_b$$

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The complete stress state becomes:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses?

$$\det\left(\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix}\right) = 0 \quad \rightarrow$$

$$\det\left(\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix}\right) = 0 \quad \rightarrow$$

Solving determinant: $\quad (-\sigma)\{(\sigma_{11}-\sigma)(-\sigma)-\sigma_{12}^2\} = 0 \quad \rightarrow$

$$\sigma_1 = 0 \;\; ; \;\; \sigma^2 - \sigma_{11}\,\sigma - \sigma_{12}^2 = 0 \quad \rightarrow \quad \sigma_{2,3} = \tfrac{1}{2}\sigma_{11} \pm \tfrac{1}{2}\sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}$$

$$\det\left(\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix}\right) = 0 \quad \rightarrow$$

Solving determinant: $\quad (-\sigma)\{(\sigma_{11}-\sigma)(-\sigma)-\sigma_{12}^2\} = 0 \quad \rightarrow$

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The solutions in the right order:

$$\sigma_1 = \tfrac{1}{2}\left(\sigma_{11}+\sqrt{\sigma_{11}^2+4\,\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \tfrac{1}{2}\left(\sigma_{11}-\sqrt{\sigma_{11}^2+4\sigma_{12}^2}\right)$$

We just found:

$$\sigma_1 = \tfrac{1}{2}\left(\sigma_{11}+\sqrt{\sigma_{11}^2+4\,\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \tfrac{1}{2}\left(\sigma_{11}-\sqrt{\sigma_{11}^2+4\sigma_{12}^2}\right)$$

Tresca: $\sigma_{\mathsf{eq}} = \sigma_1 - \sigma_3 = \sqrt{\sigma_{11}^2+4\,\sigma_{12}^2}$

von Mises: $\sigma_{\mathsf{eq}} = \sqrt{\tfrac{1}{2}\{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\}} = \sqrt{\sigma_{11}^2+3\,\sigma_{12}^2}$

Note the difference!!

We just found:

$$\sigma_1 = \tfrac{1}{2}\left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \tfrac{1}{2}\left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right)$$

Tresca: $\sigma_{\mathsf{eq}} = \sigma_1 - \sigma_3 = \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2}$

von Mises: $\sigma_{\mathsf{eq}} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}} = \sqrt{\sigma_{11}^2 + 3\,\sigma_{12}^2}$

Note the difference!! Expressed in the bending and torsional moment:

Tresca: $\sigma_{\mathsf{eq}} = \frac{R}{I_b}\sqrt{M_b^2 + M_w^2}$

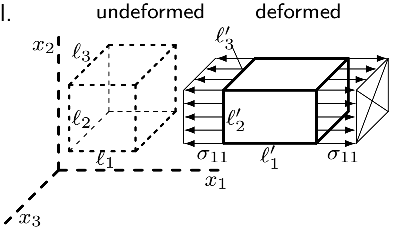
von Mises: $\sigma_{\mathsf{eq}} = \frac{R}{I_b}\sqrt{M_b^2 + \frac{3}{4}M_w^2}$

6) General material behavior
6.1) Hooke's law $\Rightarrow$ viscosity

Linear elastic, isotropic viscous material.

In the special case of 1D:

Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$



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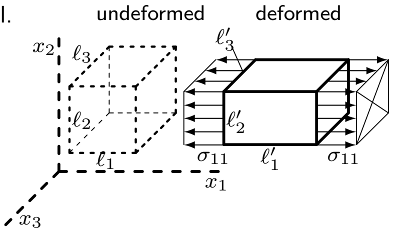
Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$

Strain-rate: $\dot{\varepsilon}_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1 \, \Delta t}$

Relation between stress and strain:

$\sigma_{11} = E\varepsilon_{11} + \eta\dot{\varepsilon}_{11}$



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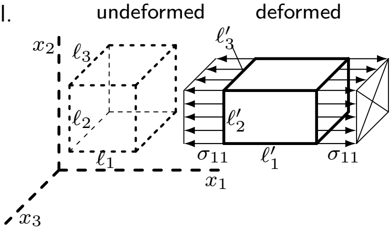


Linear elastic, isotropic viscous material.
In the special case of 1D:

Strain: $\varepsilon_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1}$

Strain-rate: $\dot{\varepsilon}_{11} = \dfrac{\ell_1' - \ell_1}{\ell_1 \, \Delta t}$

Relations between stress and strain:

$\sigma_{11} = E\varepsilon_{11} + ... + \eta\dot{\varepsilon}_{11}$

$\sigma_{12} = G\varepsilon_{12} + \eta\dot{\varepsilon}_{12}$

etc. ...

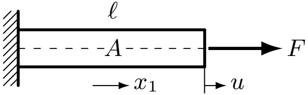
E is the *Young's modulus* (modulus of elasticity) and η is the viscosity.
Also a combined stress state is possible. Superposition principle.
Resulting in; Hooke's law + viscous stress ($\varepsilon \to \sigma$) :

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{kk}\delta_{ij}\right) + \eta\dot{\varepsilon}_{ij}}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

6.2) Elastic energy

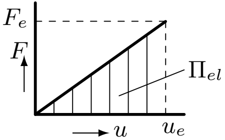
Work:



$$\Pi_{el} = \int_0^e F \, \mathrm{d}u = \tfrac{1}{2} F_e \, u_e$$

At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$



Such that:

$$\Pi_{el} = \tfrac{1}{2} A \, \sigma_{11} \, \ell \, \varepsilon_{11} = \tfrac{1}{2} \sigma_{11} \, \varepsilon_{11} \, V$$

Specific internal elastic energy: $\pi_{el} = \frac{1}{2}\sigma_{11}\,\varepsilon_{11}$
Contribution to π_{el} during shearing: $\frac{1}{2}\sigma_{12}\,\gamma_{12}$

An arbitrary three dimensional stress state:

$$\pi_{el} = \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} + \tfrac{1}{2}\sigma_{12}\,\gamma_{12} + \tfrac{1}{2}\sigma_{13}\,\gamma_{13} + \tfrac{1}{2}\sigma_{23}\,\gamma_{23}$$

Written differently (with $\gamma_{ij} = 2\,\varepsilon_{ij} = 2\,\varepsilon_{ji}$):

$$\begin{aligned} \pi_{el} \quad &= \quad \tfrac{1}{2}\sigma_{11}\,\varepsilon_{11} + \tfrac{1}{2}\sigma_{12}\,\varepsilon_{12} + \tfrac{1}{2}\sigma_{13}\,\varepsilon_{13} + \\ &+ \quad \tfrac{1}{2}\sigma_{21}\,\varepsilon_{21} + \tfrac{1}{2}\sigma_{22}\,\varepsilon_{22} + \tfrac{1}{2}\sigma_{23}\,\varepsilon_{23} + \\ &+ \quad \tfrac{1}{2}\sigma_{31}\,\varepsilon_{31} + \tfrac{1}{2}\sigma_{32}\,\varepsilon_{32} + \tfrac{1}{2}\sigma_{33}\,\varepsilon_{33} \end{aligned}$$

So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}}$$

Elastic energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\ \mathrm{d}V = \int_V \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\ \mathrm{d}V$$

Visco-elastic energy:

Hooke's and (simplest) Newtonian viscous law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$, and $\pi_{visc} = \int \sigma_{ij}^{visc}\,\mathrm{d}\varepsilon_{ij} = \eta\dot{\varepsilon}_{ij}\int\,\mathrm{d}\varepsilon_{ij}$, for constant η and ε_{ij}.

Elastic and viscous energy of a body with volume V:

$$\Pi_{el} = \int_V \pi_{el}\,\mathrm{d}V = \int_V \tfrac{1}{2}\sigma_{ij}\,\varepsilon_{ij}\,\mathrm{d}V \quad \text{and} \quad \Pi_{visc} = \int_V \pi_{visc}\,\mathrm{d}V$$

1D only: Elastic and viscous energy of a body with volume V:

$$\Pi = \Pi_{el} + \Pi_{visc} = \int_V \left(\pi_{el} + \pi_{visc}\right)\,\mathrm{d}V$$

$$= \int_V \left(\tfrac{1}{2}E\varepsilon_{11}^2 + \eta\dot{\varepsilon}_{11}\varepsilon_{11}\right)\,\mathrm{d}V$$