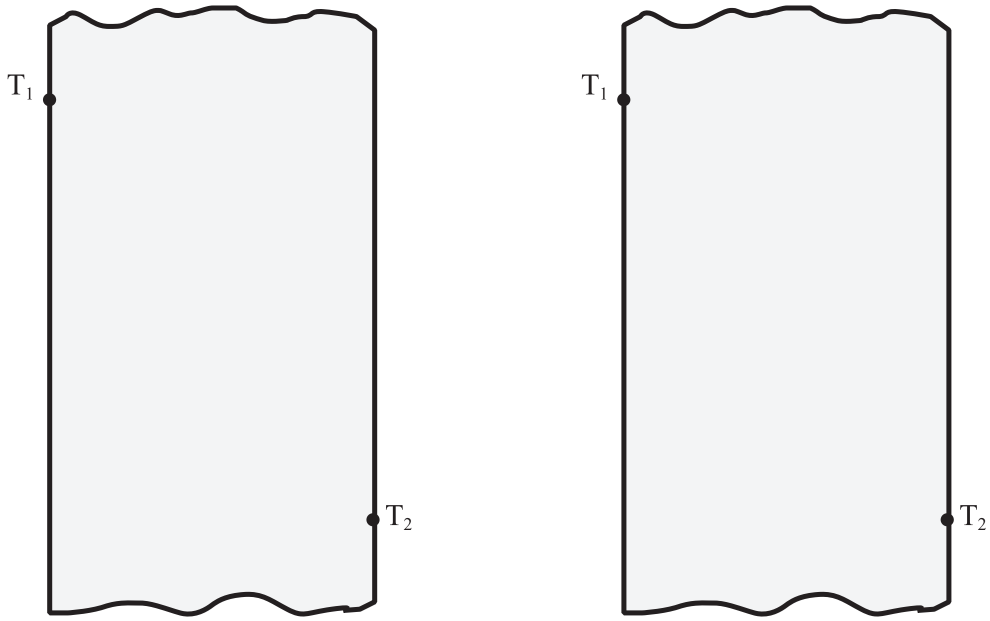
SECTION II

Conduction solutions

Exercise II.1 (Temperature profiles in planar walls ⋆)**:**

Both sides of a planar wall are heated to a constant temperature of T_1 and T_2, where $T_1 > T_2$.



λ = constant $\qquad\qquad$ $\lambda = \lambda_0 \cdot (1+\gamma \cdot (T - T_0))$ with $\gamma > 0$

Tasks:
Qualitatively sketch the temperature profile for steady-state conditions, if

a) the conductivity remains constant

b) the conductivity is a function of the temperature following the equation:

$$\lambda = \lambda_0(1 + \gamma(T - T_0)) \text{ with } \gamma > 0$$
$$(\lambda_0 = \text{thermal conductivity at reference temperature } T_0)$$

Solution II.1 (Temperature profiles in planar walls ⋆):

Task a)

From Fourier's law:

$$\dot{Q} = \lambda A \frac{\partial T}{\partial x} \tag{II.1.1}$$

it becomes evident that in the case of a plane wall with a uniform cross-sectional area A and constant thermal conductivity λ, the temperature gradient $\frac{\partial T}{\partial x}$ should remain constant. Consequently, a steady linear slope should be drawn.

Task b)

Similarly, we are once again considering a plane wall, implying a constant cross-sectional area A. As we move from left to right, it follows that the temperature decreases within the plane wall.

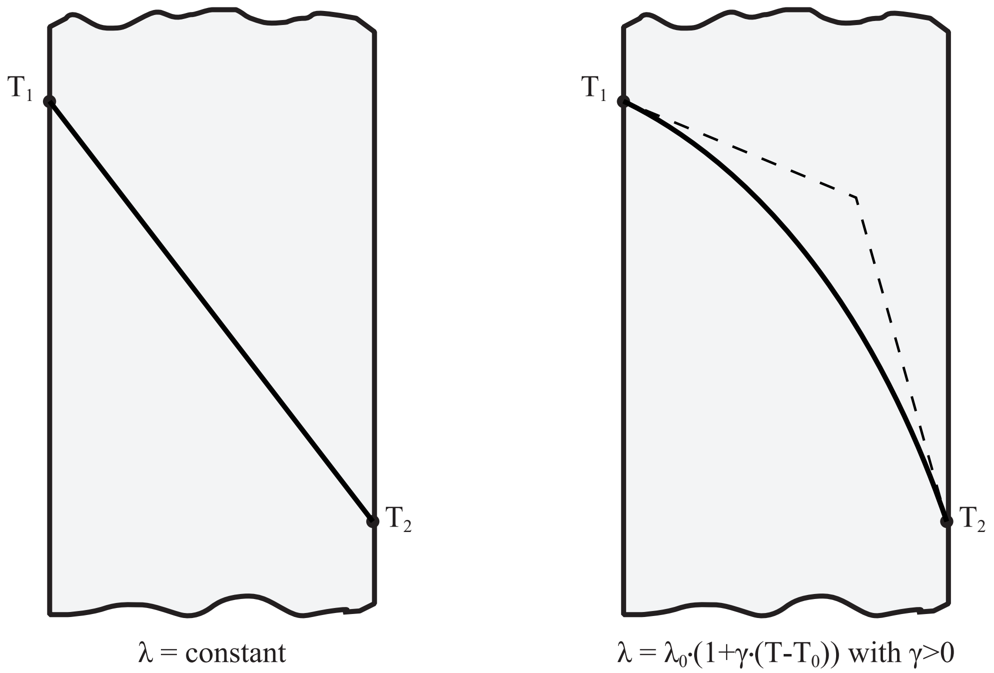
If we recall the equation for thermal conductivity λ:

$$\lambda = \lambda_0(1 + \gamma(T - T_0)) \text{ with } \gamma > 0 \tag{II.1.2}$$

it becomes apparent that as the temperature decreases, so does the thermal conductivity.

Hence, as we move from left to right, the temperature gradient should increase $\frac{\partial T}{\partial x}$, resulting in a steeper profile due to the diminishing thermal conductivity.

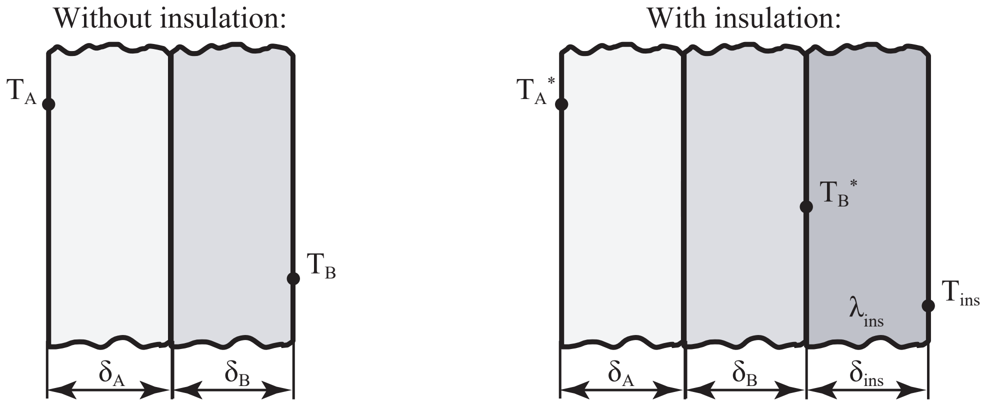
Conclusion



λ = constant $\quad\quad$ $\lambda = \lambda_0 \cdot (1+\gamma \cdot (T-T_0))$ with $\gamma>0$

Exercise II.2 (Onion layer principle ★★):

A solar panel manufacturer makes use of heat processing applications that include preheating, curing, heat treating, and finishing. The manufacturer has an old and new type of industrial oven. The newer one has an additional insulation layer.



Given parameters:

Old oven:

- Surface temperature of layer A: $T_A = 260$ °C
- Surface temperature of layer B: $T_B = 32$ °C
- Thickness of layer A: $\delta_A = 125$ mm
- Thickness of layer B: $\delta_B = 200$ mm

New oven:

- Surface temperature of layer A: $T_A^* = 305$ °C
- Surface temperature of layer B: $T_B^* = 219$ °C
- Surface temperature of insulation layer: $T_{ins} = 27$ °C
- Thickness of layer A: $\delta_A = 125$ mm
- Thickness of layer B: $\delta_B = 200$ mm
- Thickness of insulation layer: $\delta_{ins} = 25$ mm
- Thermal conductivity of insulation layer: $\lambda_{ins} = 0.075$ W/mK

Hint:

- Assume steady-state conditions.

Tasks:

a) Determine the heat flux per unit area $\dot{q}''$ for the situations without and with the insulating layer.

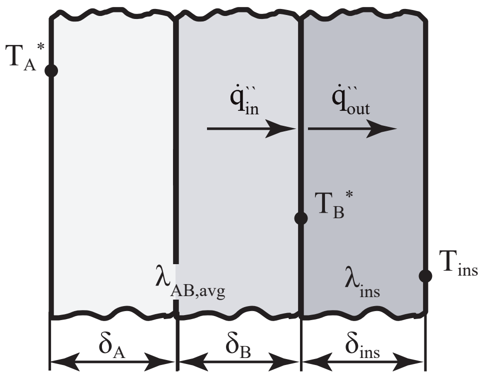
Solution II.2 (Onion layer principle ★★):

Task a)

The heat flux can be determined by setting up an energy balance at an interface within both systems. Since the thermal conductivity of layers A and B are unknown, their average value needs to be determined. This can be done, by determining the heat flux for the insulated case first and rewriting this expression.

System with insulation:

1 **Setting up the balance:**



The energy balance at the interface between layer B and the insulation layer reads:

$$0 = \dot{q}_{in}'' - \dot{q}_{out}'' \tag{II.2.1}$$

2 **Defining the elements within the balance:**

The outgoing heat flux can be easily determined by the use of Fourier's law for a plane wall:

$$\begin{aligned} \dot{q}_{out}'' &= -\lambda_{ins} \cdot \frac{T_{ins} - T_B^*}{\delta_{ins}} \\ &= -0.075\ [\mathrm{W/mK}] \cdot \frac{(27 - 219)\ [\mathrm{K}]}{0.025\ [\mathrm{m}]} = 576\ \left[\mathrm{W/m^2}\right] \end{aligned} \tag{II.2.2}$$

We can express the incoming heat flux using Fourier's law with the average thermal conductivity of layers A and B since the heat flux through each layer remains constant for a plane wall:

$$\dot{q}_{in}'' = -\lambda_{AB,avg} \cdot \frac{T_B^* - T_A^*}{\delta_A + \delta_B} \tag{II.2.3}$$

3 **Inserting and rearranging:**

Inserting the terms and rewriting the energy balance yields in the average thermal conductivity

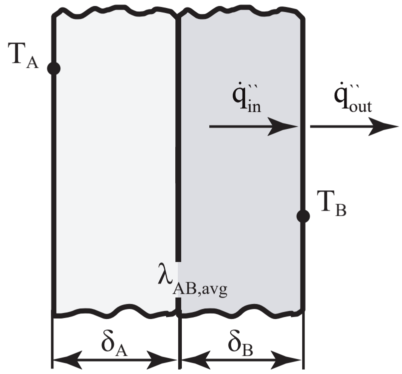
$$\begin{aligned} \lambda_{AB,avg} &= \dot{q}_{out}'' \cdot \frac{(\delta_A + \delta_B)}{T_A^* - T_B^*} \\ &= 576\ [\mathrm{W/m^2}] \cdot \frac{(0.125 + 0.200)\ [\mathrm{m}]}{(305 - 219)\ [\mathrm{K}]} = 2.18\ [\mathrm{W/mK}] \end{aligned} \tag{II.2.4}$$

Conclusion

The heat flux per unit area for the system with insulation is 576 $\mathrm{W/m^2}$. Additionally, the combined average thermal conductivity of layers A and B has been calculated to be 2.18 W/mK, as this parameter is necessary to determine the heat flux for the system without any insulation.

System without insulation:

1 Setting up the balance:



The energy balance at the interface between layer B and the ambient reads:

$$0 = \dot{q}''_{\text{in}} - \dot{q}''_{\text{out}} \tag{II.2.5}$$

2 Defining the elements within the balance:

We can express the incoming heat flux using Fourier's law with the average thermal conductivity of layers A and B since the heat flux through each layer remains constant for a plane wall:

$$\begin{aligned} \dot{q}''_{\text{in}} &= -\lambda_{\text{AB,avg}} \cdot \frac{T_{\text{B}} - T_{\text{A}}}{(\delta_{\text{A}} + \delta_{\text{B}})} \\ &= -2.18\ [\text{W/mK}] \cdot \frac{(32 - 260)\ [\text{K}]}{(0.125 - 0.200)\ [\text{m}]} = 1527\ \left[\text{W/m}^2\right] \end{aligned} \tag{II.2.6}$$

3 Inserting and rearranging:

Inserting and rearranging the energy balance yields:

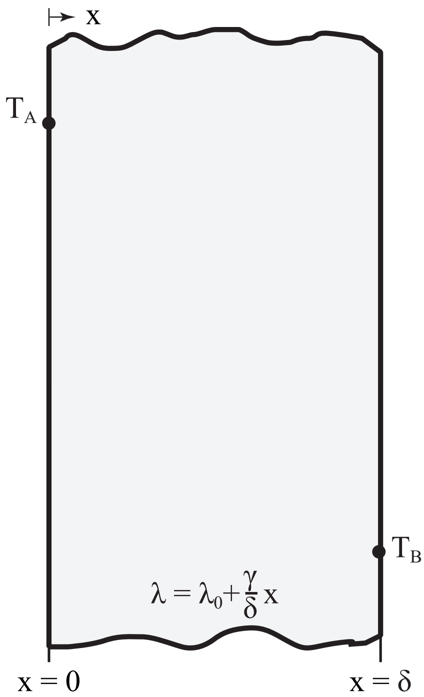
$$\dot{q}''_{\text{in}} = 1527\ \left[\text{W/m}^2\right] \tag{II.2.7}$$

Conclusion

The heat flux per unit area for the system without insulation is 1527 $\mathrm{W/m^2}$.

Exercise II.3 (Heat conduction equation ⋆ ⋆ ⋆)**:**

Both sides of a planar wall are heated to a constant temperature of T_A and T_B, respectively; where $T_A > T_B$.



Given parameters:

- Thermal conductivity as a function of the position in the wall:

$$\lambda(x) = \lambda_0 + \frac{\gamma}{\delta} \cdot x$$

Hints:

- Assume one-dimensional steady-state heat transfer in x-direction.

Tasks:

a) Derive the function of the temperature profile inside the plane wall.

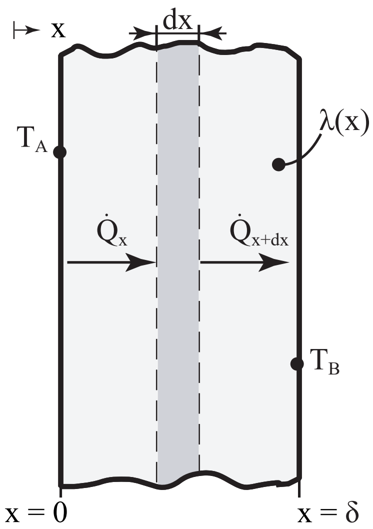
b) Sketch the temperature profile inside the plane wall in the x-direction.

Solution II.3 (Heat conduction equation ⋆⋆⋆):

Task a)

The heat conduction equation for a system is derived based on the energy balance of an infinitesimal element.

1 **Setting up the balance:**



The one-dimensional steady-state energy balance for an infinitesimal element with no heat sources/sinks can be described as:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} \tag{II.3.1}$$

2 **Defining the elements within the balance:**

The ingoing flux can be described by use of Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda(x) A \frac{\partial T}{\partial x} \tag{II.3.2}$$

For an infinitesimal element, the outgoing conductive heat flux can be approximated by use of the Taylor series expansion, which results in:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \cdot dx \\ &= -\lambda(x) A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda(x) A \frac{\partial T}{\partial x}\right) \cdot dx \end{aligned} \tag{II.3.3}$$

3 **Inserting and rearranging:**

Substituting the definitions of $\dot{Q}_{\mathrm{x}}$, $\dot{Q}_{\mathrm{x+dx}}$, and $\lambda(x)$ into the energy balance and doing some rewriting yields:

$$0 = \frac{\partial}{\partial x}\left(\left[\lambda_0 + \frac{\gamma}{\delta} x\right] \cdot \frac{\partial T}{\partial x}\right) \tag{II.3.4}$$

4 Defining the boundary and/or initial conditions:

To solve the given differential equation, two boundary conditions will be required since the temperature has been differentiated twice to x.

The following two boundary conditions are known from the given figure:

$$T(x=0) = T_{\mathrm{A}} \tag{II.3.5}$$

$$T(x=\delta) = T_{\mathrm{B}} \tag{II.3.6}$$

5 Solving the equation:

The function of the temperature profile can be derived by solving the heat conduction equation.

First-time integration of the heat conduction equation yields:

$$\left[\lambda_0 + \frac{\gamma}{\delta}x\right] \cdot \frac{\partial T}{\partial x} = C_1 \tag{II.3.7}$$

Rewriting gives:

$$\frac{\partial T}{\partial x} = \frac{C_1}{\lambda_0 + \frac{\gamma}{\delta}x} \tag{II.3.8}$$

Second time integrating:

$$T(x) = \frac{C_1 \delta}{\gamma} \ln\left|\lambda_0 + \frac{\gamma}{\delta}x\right| + C_2 \tag{II.3.9}$$

From the first boundary condition, it can be found that:

$$T(x=0) = \frac{C_1 \delta}{\gamma} \ln|\lambda_0| + C_2 = T_{\mathrm{A}} \tag{II.3.10}$$

$$\Rightarrow C_2 = T_{\mathrm{A}} - \frac{C_1 \delta}{\gamma} \ln|\lambda_0|$$

From the second boundary condition, it can be found that:

$$T(x=\delta) = \frac{C_1 \delta}{\gamma} \ln\left|\lambda_0 + \frac{\gamma}{\delta} \cdot \delta\right| + C_2 = T_{\mathrm{B}} \tag{II.3.11}$$

$$\Rightarrow C_1 = \frac{\gamma\,(T_{\mathrm{B}} - T_{\mathrm{A}})}{\delta \ln\left|1 + \frac{\gamma}{\lambda_0}\right|}$$

Conclusion

Substitution of C_1 and C_2 into the temperature profile containing both constants, stated in equation II.3.9 and doing some rewriting:

$$T(x) = \frac{(T_{\mathrm{B}} - T_{\mathrm{A}})}{\ln\left|1 + \frac{\gamma}{\lambda_0}\right|} \cdot \left(\ln\left|\lambda_0 + \frac{\gamma}{\delta}x\right| - \ln|\lambda_0|\right) + T_{\mathrm{A}} \tag{II.3.12}$$

Task b)

The function of the thermal conductivity function $\lambda(x)$ indicates an increase in thermal conductivity as we progress in the positive x-direction. According to Fourier's law:

$$\dot{Q} = -\lambda(x) \cdot A \cdot \frac{\partial T}{\partial x} \tag{II.3.13}$$

it is apparent that the temperature gradient $\frac{\partial T}{\partial x}$ should decrease in the x-direction. This phenomenon arises due to the constancy of the rate of heat transfer $\dot{Q}$ and the cross-sectional area A for a plane wall under steady-state conditions.

Conclusion