Chapter 7

Flat Plate Boundary Layer

7.1 Boundary layer equations

We consider two dimensional, incompressible, steady-state flow over a half-infinite plate, see Fig. (7.1). The Navier-Stokes equations, Eq.(4.39) in this case become:

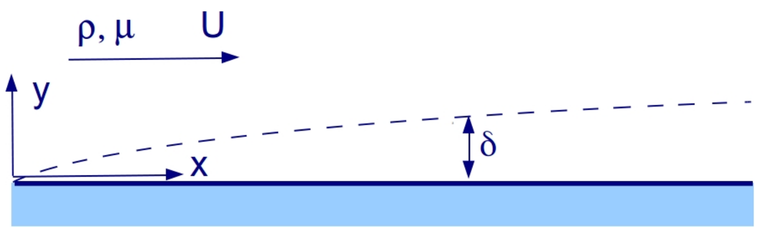

Figure 7.1: Two dimensional, incompressible, steady-state flow over a half-infinite plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{7.3}$$

The viscous flow over a flat plate satifies the following boundary conditions:

$$u(x,0) = v(x,0) = 0, \quad u(x,\infty) = U. \tag{7.4}$$

We will make two powerful assumptions based on the observation that streamlines in the boundarylayer are almost parallel with the plate:

(a) u varies much slower with x than with y, so $\|\frac{\partial^2 u}{\partial x^2}\| \ll \|\frac{\partial^2 u}{\partial y^2}\|$ in Eq.(7.2), and

(b) pressure is almost constant across the boundarylayer, so $\frac{\partial p}{\partial y} \approx 0$.

The second assumption and the fact that p is independent of x at $y = \infty$, immediately leads to the conclusion that

$$p \text{ is constant for all } x, y. \tag{7.5}$$

Furthermore, we non-dimensionalize these equations by using the only parameters we have available: U, ρ and μ. The independent variables become

$$\tilde{x} \equiv \frac{\rho U x}{\mu}, \quad \tilde{y} \equiv \frac{\rho U y}{\mu}, \tag{7.6}$$

and the dependent variables become:

$$\tilde{u} \equiv \frac{u}{U}, \quad \tilde{v} \equiv \frac{v}{U}, \quad \tilde{p} \equiv \frac{p}{\rho U^2}. \tag{7.7}$$

As a result we obtain dimensionless approximate equations first derived by Prandtl which

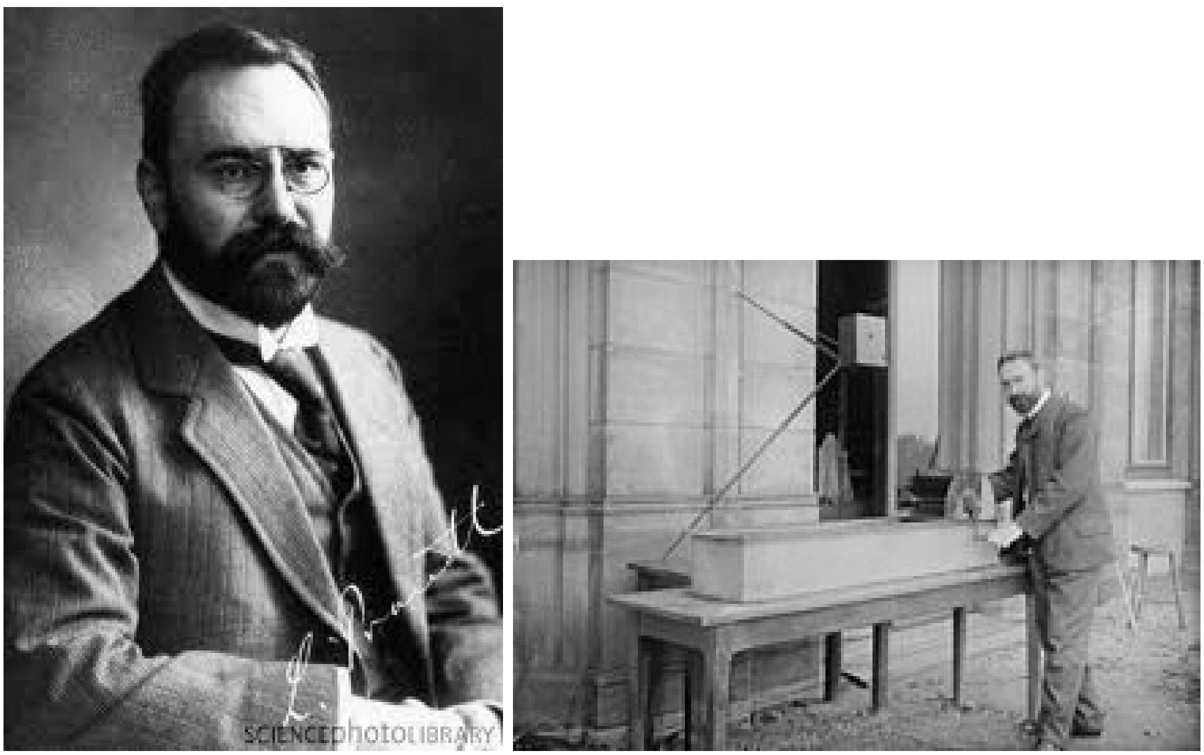

Figure 7.2: Ludwig Prandtl (1875 - 1953) was a German scientist. He was a pioneer in the development of rigorous systematic mathematical analyses which he used for underlaying the science of aerodynamics, which have come to form the basis of the applied science of aeronautical engineering. In the 1920s he developed the mathematical basis for the fundamental principles of subsonic aerodynamics in particular; and in general up to and including transonic velocities. His studies identified the boundary layer, thin-airfoils, and lifting-line theories. The Prandtl number was named after him.

are called the boundary layer equations for a flat plate:

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, \\
u\frac{\partial \tilde{u}}{\partial \tilde{x}} + v\frac{\partial \tilde{u}}{\partial \tilde{y}} &= \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}.
\end{aligned} \tag{7.8}$$

and dimensionless boundary conditions:

$$\tilde{u}(\tilde{x}, 0) = \tilde{v}(\tilde{x}, 0) = 0, \quad \tilde{u}(\tilde{x}, \infty) = 1. \tag{7.9}$$

7.2 Laminar flow: Blasius solution

Blasius[(1)] solved the boundary layer equations Eq.(7.8) with boundary conditions Eq.(7.9) by using a stream function that measures the volume flow rate between $\tilde{y} = 0$ and $\tilde{y} = \tilde{y}$:

$$\tilde{\Psi}(\tilde{x}, \tilde{y}) = \int_0^{\tilde{y}} \tilde{u}(\tilde{x}, \tilde{s}) d\tilde{s}, \tag{7.10}$$

which leads to

$$\tilde{u} = \frac{\partial \tilde{\Psi}}{\partial \tilde{y}} \equiv \tilde{\Psi}_{\tilde{y}}, \quad \tilde{v} = -\frac{\partial \tilde{\Psi}}{\partial \tilde{x}} \equiv -\tilde{\Psi}_{\tilde{x}}. \tag{7.11}$$

As a result, the first equation of Eq.(7.8) is satisfied automatically, and the second equation of Eq.(7.8) becomes an partial differential equation for $\tilde{\Psi}$ only:

$$\tilde{\Psi}_{\tilde{y}}\tilde{\Psi}_{\tilde{x}\tilde{y}} - \tilde{\Psi}_{\tilde{x}}\tilde{\Psi}_{\tilde{y}\tilde{y}} = \tilde{\Psi}_{\tilde{y}\tilde{y}\tilde{y}}, \tag{7.12}$$

By writing the unknown function $\tilde{\Psi}$ of two variables as another unknown function f of one variable

$$\tilde{\Psi}(\tilde{x}, \tilde{y}) = \sqrt{\tilde{x}} f\left(\frac{\tilde{y}}{\sqrt{\tilde{x}}}\right), \tag{7.13}$$

the partial differential equation for $\tilde{\Psi}$, Eq.(7.12), becomes an ordinary differential equation for f called the Blasius equation:

$$\boxed{ff'' + 2f''' = 0,} \tag{7.14}$$

and

$$\tilde{u}(\tilde{x}, \tilde{y}) = f'\left(\frac{\tilde{y}}{\sqrt{\tilde{x}}}\right). \tag{7.15}$$

Evidently, finding such transformation is not straight forward at all. Furthermore, the boundary conditions become

$$\boxed{f(0) = f'(0) = 0, \quad f'(\infty) = 1.} \tag{7.16}$$

(1) Paul Richard Heinrich Blasius (1883 - 1970) was a German fluid dynamics engineer.

It appears that

$$\tilde{u}(\tilde{x}, \tilde{y}) = 0.99 \text{ when } \frac{\tilde{y}}{\sqrt{\tilde{x}}} \approx 5, \tag{7.17}$$

so the dimensionless boundary layer thickness, defined as $\tilde{\delta} \equiv \frac{U\delta}{\nu}$ and $\tilde{u}(\tilde{x}, \tilde{\delta}) \equiv 0.99$, becomes

$$\tilde{\delta}(\tilde{x}) \approx 5\sqrt{\tilde{x}}, \tag{7.18}$$

and the dimensional boundary layer thickness becomes

$$\delta(x) \approx 5\sqrt{\frac{x\nu}{U}}. \tag{7.19}$$

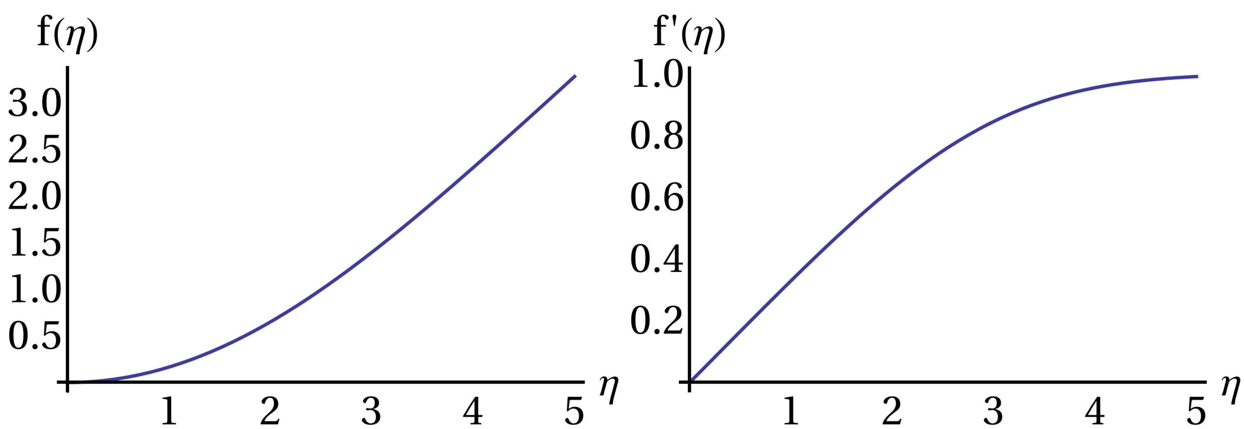

Figure 7.3: Solution of Blasius' equation

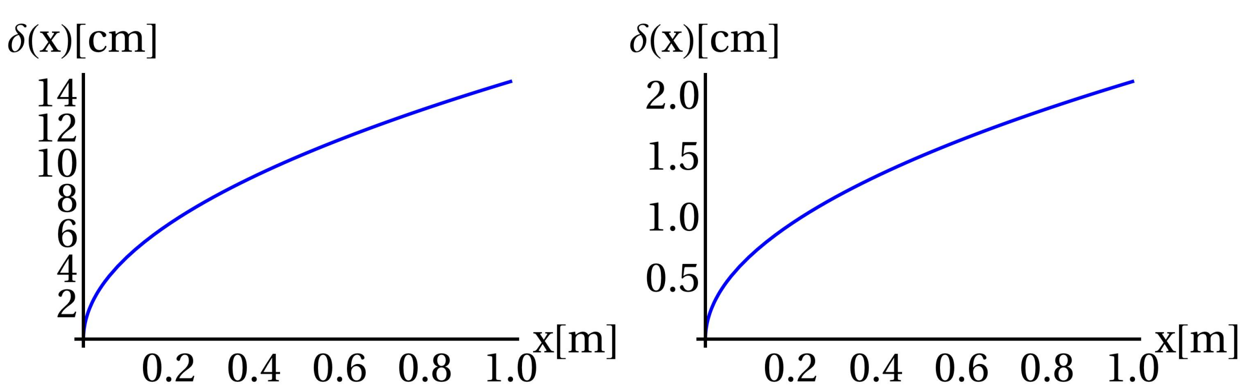

Figure 7.4: Flat plate boundary layer thickness based on Blasius' solution for water (left, $\nu = 894 \times 10^{-6}\, Pa\, s$) and air (right, $\nu = 18.6 \times 10^{-6}\, Pa\, s$) with $U = 1\, m/s$, confirming that $\delta \sim \sqrt{\nu}$. [Note: the solution for air does not hold beyond a few centimeters in downstream direction because turbulence sets in and the boundary layer would be much thicker.]

7.3 Shear stress at the wall

The stress-vector $\boldsymbol{t}$ acting on the wall is given by Cauchy's relation Eq.(4.8):

$$t_i = \sigma_{ij} n_j, \tag{7.20}$$

where in this case the normal vector is $\boldsymbol{n} = (0,1)^T$ (pointing to the acting medium!).

The shear stress at the wall is the first component of the stress vector at $\tilde{y} = y = 0$:

$$\tau_w \equiv t_1 = \sigma_{1j} n_j = \sigma_{12} = \mu \left(\frac{\partial u}{\partial y} \right)_o, \tag{7.21}$$

where the subscript o denotes $\tilde{y} = y = 0$. The Blasius solution is given in terms of dimensionless functions, therefore to actually calculate the wall shear stress we have to rewrite:

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_o = \mu \left(\frac{\partial U \tilde{u}}{\partial y} \right)_o = \mu U \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o \frac{\partial \tilde{y}}{\partial y} = \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o \rho U^2. \tag{7.22}$$

This is frequently written as

$$\tau_w = C_f \frac{1}{2} \rho U^2, \quad C_f \equiv 2 \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o. \tag{7.23}$$

The dimensionless quantity C_f is called the friction coefficient which is a little bit similar to the friction factor used in fully developed flows.

From Eq.(7.15) we see that

$$\frac{\partial \tilde{u}}{\partial \tilde{y}} = f'' \left(\frac{\tilde{y}}{\sqrt{\tilde{x}}} \right) \frac{1}{\sqrt{\tilde{x}}} \Rightarrow \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o = f''(0) \frac{1}{\sqrt{\tilde{x}}}. \tag{7.24}$$

The numerical solution of the Blasius equation shows that $f''(0) \approx 0.332$, so

$$C_f = \frac{0.664}{\sqrt{\tilde{x}}}. \tag{7.25}$$

7.4 Exercises

Problem 7.1. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u}$ *by using the definition of the streamfunction* $\tilde{\Psi}$, *and the definition of its partial derivative:*

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u} \equiv \lim_{\Delta \tilde{y} \to 0} \frac{\tilde{\Psi}(\tilde{x}, \tilde{y} + \Delta \tilde{y}) - \tilde{\Psi}(\tilde{x}, \tilde{y})}{\Delta \tilde{y}}.$$

Problem 7.2. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{x}} = -\tilde{v}$ *by using mass conservation and the definition of the streamfunction* $\tilde{\Psi}$