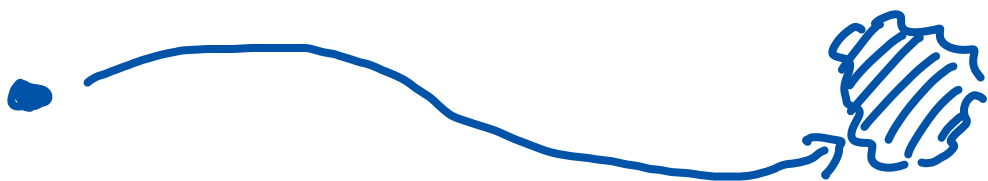


Lecture #9

Convection & Diffusion.

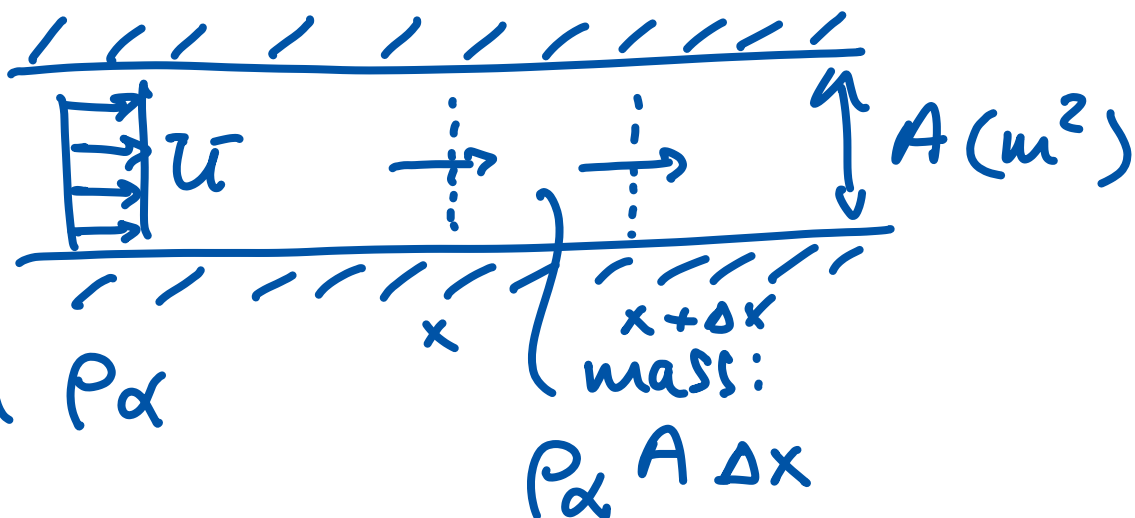


motion: convection
spread: diffusion. } Equation?

Convection

(1D)

inh, density ρ_α



$$\frac{\partial}{\partial t}(\cancel{\rho_\alpha A \Delta x}) = \underbrace{(\cancel{\rho_\alpha u A})_x}_{\text{kg/s}} - \underbrace{(\cancel{\rho_\alpha u A})_{x+\Delta x}}_{\Delta x}$$

$$\Delta x \rightarrow 0$$

$$\Rightarrow \frac{\partial \rho_\alpha}{\partial t} = - \frac{\partial}{\partial x}(\rho_\alpha u)$$

$$\Rightarrow \boxed{\frac{\partial \rho_\alpha}{\partial t} + u \frac{\partial \rho_\alpha}{\partial x} = 0} \Rightarrow \boxed{\frac{D \rho_\alpha}{D t} = 0}$$

convection equation.

$\Rightarrow \rho_\alpha$ is constant while moving
with the flow

$$\Rightarrow \rho_\alpha(x, t) \stackrel{?}{=} f(x - \bar{u}t)$$

$$\Rightarrow f(x - \bar{u}t) = \text{const if } x - \bar{u}t = \text{const.}$$

$$\Rightarrow x = \bar{u}t$$

Question: is $f(x - \bar{u}t)$ a solution
of the derived equation?

$$\frac{\partial \rho_\alpha}{\partial t} = \frac{\partial f}{\partial t} = f'(x - \bar{u}t) \cdot \frac{\partial}{\partial t}(x - \bar{u}t) = -\bar{u}f'$$

$$\frac{\partial \rho_\alpha}{\partial x} = \frac{\partial f}{\partial x} = f'(x - \bar{u}t) \cdot \frac{\partial}{\partial x}(x - \bar{u}t) = f'$$

$$\Rightarrow \frac{\partial \rho_\alpha}{\partial t} + \bar{u} \frac{\partial \rho_\alpha}{\partial x} = -\bar{u}f' + \bar{u}f' = \underline{\underline{0}} \quad f$$

f: form is OK, but details?

$$t=0 \quad \rho_{\alpha}(x, 0) = \underbrace{\rho_{\alpha}^0(x)}_{\text{given, initial condition.}}$$

$$\Rightarrow f(x - u \cdot 0) = \rho_{\alpha}^0(x)$$

$$\Rightarrow f(x) = \rho_{\alpha}^0(x)$$

$$\boxed{f(x - ut) = \rho_{\alpha}^0(x - ut)}$$

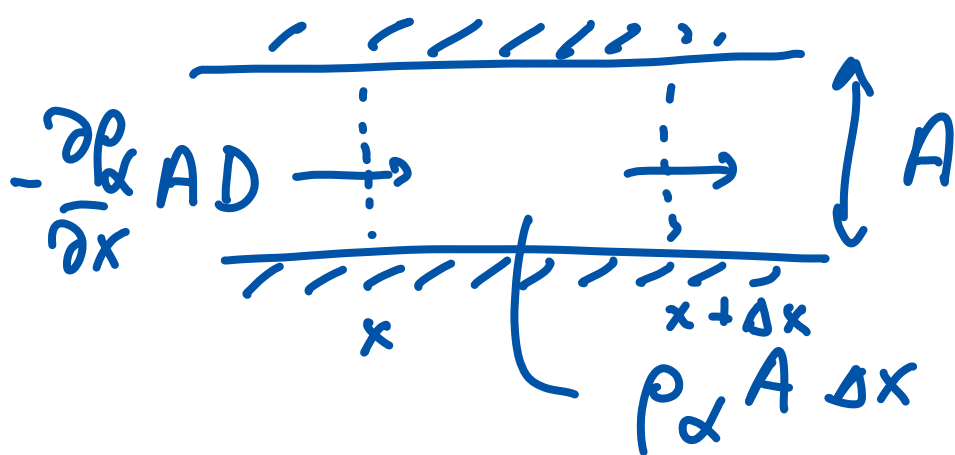
Diffusion

Spreading
molecules!

low concentration $\xleftarrow{\text{transport}}$ high concentration $\rightarrow x$

$$\Rightarrow \text{transport flux} \sim -\frac{\partial \rho_\alpha}{\partial x}$$

Equation?



D : diffusion coefficient depends on the two fluids.
 $[D] = \text{m}^2/\text{s}$

$$\frac{\partial}{\partial t} (\rho_\alpha A \Delta x) = \underbrace{\left(-D \frac{\partial \rho_\alpha}{\partial x} A \right)_x - \left(-D \frac{\partial \rho_\alpha}{\partial x} A \right)_{x+\Delta x}}_{\Delta x}$$

$$\Delta x \rightarrow 0 \Rightarrow \frac{\partial \rho_\alpha}{\partial t} = -\frac{\partial}{\partial x} \left(-D \frac{\partial \rho_\alpha}{\partial x} \right) = \frac{\partial}{\partial x} \left(D \frac{\partial \rho_\alpha}{\partial x} \right) = D \frac{\partial^2 \rho_\alpha}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial \rho_\alpha}{\partial t} = D \frac{\partial^2 \rho_\alpha}{\partial x^2}}$$

Diffusion equation.

Does it indeed describe spreading?

Try: $\rho_\alpha(x, t) = h(t) \exp\left(-\beta \frac{x^2}{t}\right)$

$$\frac{\partial \rho_\alpha}{\partial t} = h'(t) \cdot \exp(\cdot) + h \cdot \exp(\cdot) \beta \frac{x^2}{t^2}$$

$$\frac{\partial \rho_\alpha}{\partial x} = h \cdot \exp(\cdot) \cdot -2\beta \frac{x}{t}$$

$$\frac{\partial^2 \rho_\alpha}{\partial x^2} = h \cdot \exp(\cdot) \cdot 4\beta^2 \frac{x^2}{t^2} + h \cdot \exp(\cdot) \cdot -\frac{2\beta}{t}$$

Substitute into the diffusion eq:

$$\cancel{h' \cdot \exp(\cdot)} + \cancel{h \cdot \exp(\cdot)} \beta \frac{x^2}{t^2} =$$

$$- D \cdot \cancel{h \exp(\cdot)} \frac{2\beta}{t} + D \cdot \cancel{h \exp(\cdot)} 4\beta^2 \frac{x^2}{t^2}$$

$$\Rightarrow h' + h \cdot \beta \cdot \frac{x^2}{t^2} = -Dh \frac{2\beta}{t} + 4Dh \beta^2 \frac{x^2}{t^2}$$

$\forall x, t!$

Choose $4\beta D = 1$ $h' = -\frac{1}{2} h/t$

$$\Rightarrow -\frac{1}{2} \frac{h}{t} + h \frac{1}{4D} \frac{x^2}{t^2} = -\frac{1}{2} \frac{h}{t} + h \frac{1}{4D} \frac{x^2}{t^2}$$

$$\Rightarrow \boxed{\beta = \frac{1}{4D}}$$

but $h = ?$

$$\frac{h'}{h} = -\frac{1}{2} \frac{1}{t} \Rightarrow \frac{d}{dt}(\ln h) = -\frac{1}{2} \frac{d}{dt}(\ln t)$$

$$\Rightarrow \ln h = -\frac{1}{2} \ln t + C = \ln t^{-1/2} + C$$

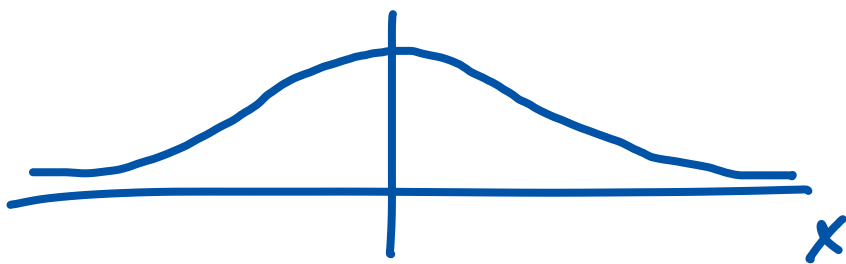
$$e^{\ln h} = e^{\ln t^{-1/2} + C} = e^C \cdot e^{\ln t^{-1/2}}$$

$$\Rightarrow h = C t^{-1/2} \Rightarrow h = \frac{\text{const}}{\sqrt{t}}$$

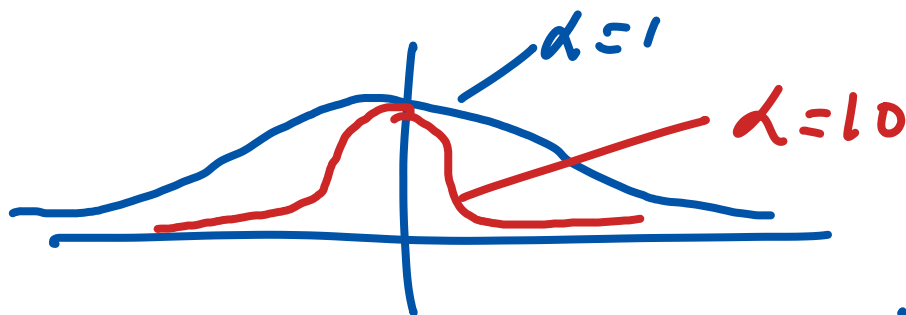
$$\Rightarrow \boxed{p_\alpha(x, t) = \frac{\text{const}}{\sqrt{t}} \cdot \exp\left(-\frac{x^2}{4Dt}\right)}$$

$$\left[\frac{x^2}{4Dt} \right] = \frac{m^2}{m^2/s \cdot s} = 1 \quad \int$$

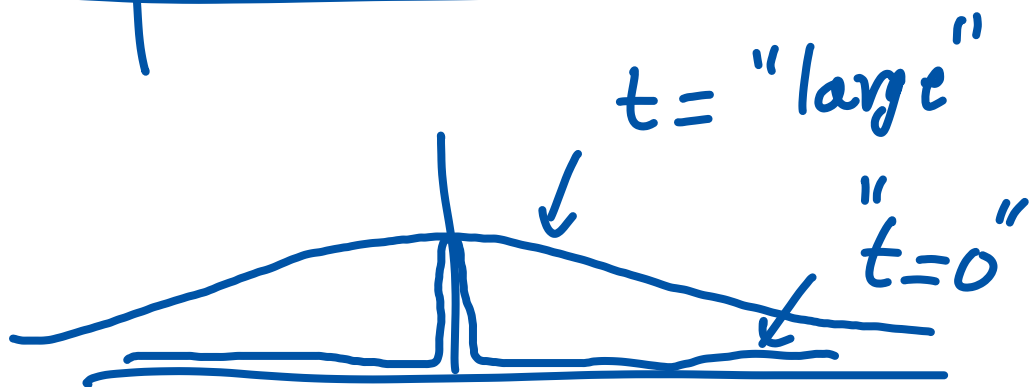
$$e^{-x^2}:$$



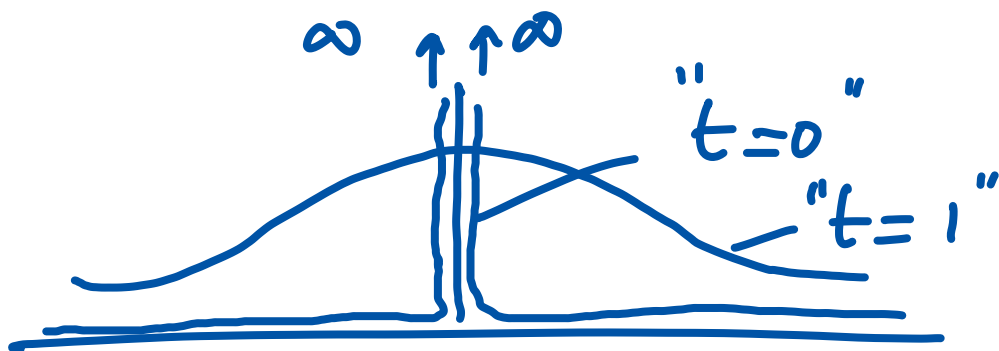
$$e^{-dx^2}:$$



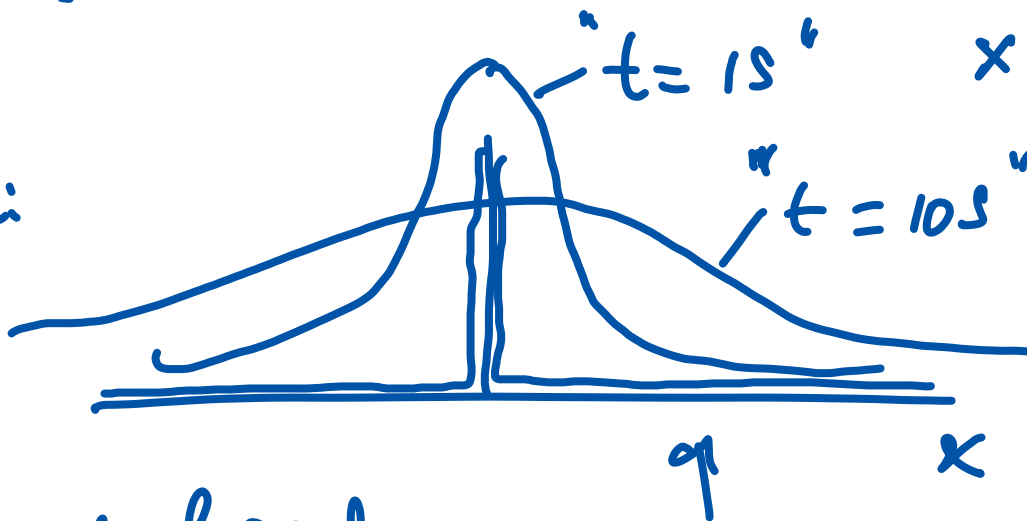
$$e^{-\frac{x^2}{4Dt}}: d = \frac{1}{4Dt}$$



$$\frac{1}{\sqrt{t}} e^{-\frac{x^2}{4Dt}}:$$



Overall picture:



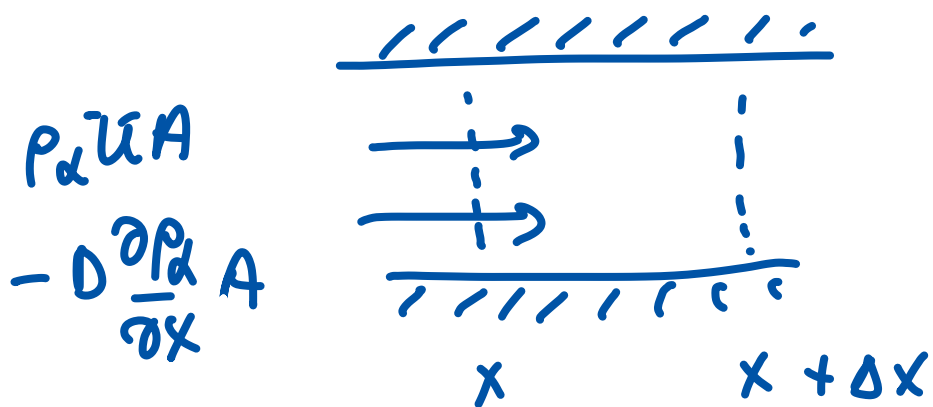
Picture seems good.

Convection - Diffusion.

(Simultaneously)

Convection: $\frac{\partial \rho_d}{\partial t} + \bar{u} \frac{\partial \rho_d}{\partial x} = 0$

Diffusion: $\frac{\partial \rho_d}{\partial t} = D \frac{\partial^2 \rho_d}{\partial x^2}$



Combined effect:

$$\boxed{\frac{\partial \rho_d}{\partial t} + \bar{u} \frac{\partial \rho_d}{\partial x} = D \frac{\partial^2 \rho_d}{\partial x^2}}$$

Convection
Diffusion
eq.

$$\Rightarrow \frac{D \rho_d}{D t} = D \frac{\partial^2 \rho_d}{\partial x^2}$$

in a frame moving with \bar{u} to the right:
pure diffusion.

:

\Rightarrow in that frame:

$$P_x = \frac{\text{const}}{\sqrt{t}} \exp\left(-\frac{\xi^2}{4Dt}\right)$$

$$\xi \equiv x - \bar{u}t.$$

$$\text{if } \xi = \text{const} \Rightarrow x = \bar{u}t$$

moves with the flow

$$\Rightarrow P_x(x,t) = \frac{\text{const}}{\sqrt{t}} \exp\left(-\frac{(x - \bar{u}t)^2}{4Dt}\right)$$

Solution of convection-diffusion eq.
(can be checked by substitution).



transport and spread.