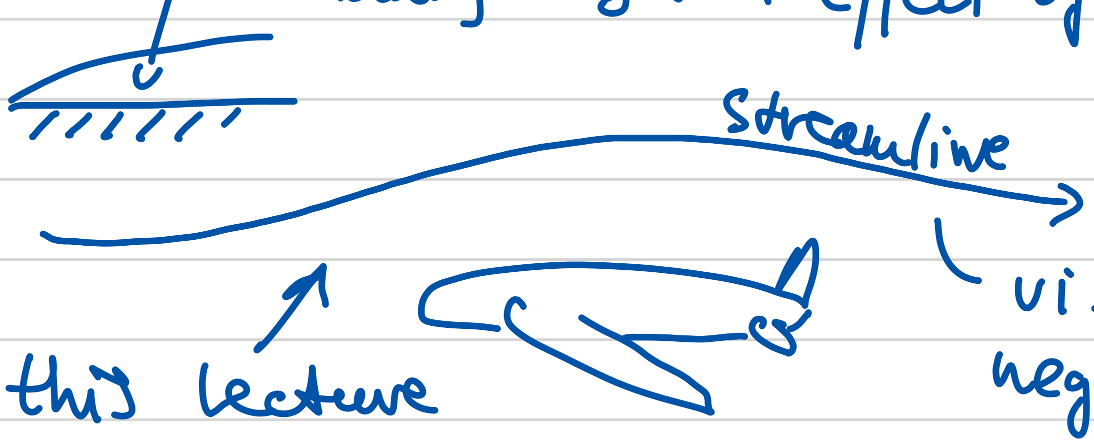
Fluid Mechanics I

Lecture #8   Euler & Bernoulli eqs.

Lect #7   boundary layer : effect of viscosity

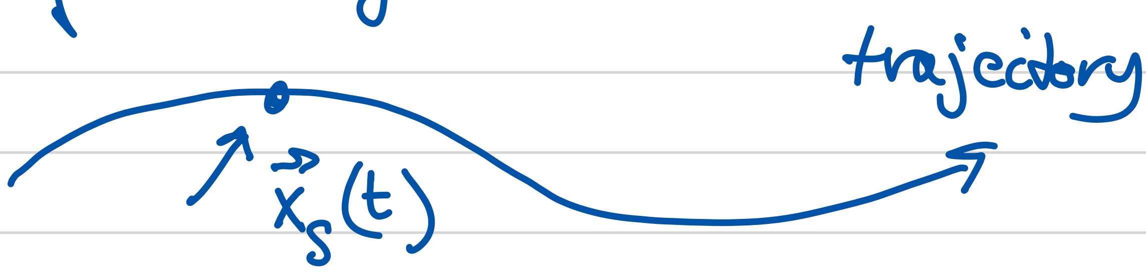


viscosity negligible?

this lecture
"$\mu = 0$"

First: Material Derivative.

Imagine micro-sensor for pressure following the fluid:



$$\frac{d}{dt}\vec{x}_s(t) = \vec{u}\left(\vec{x}_s(t), t\right)$$

Pressure measured by sensor:

$$p\left(\vec{x}_s(t), t\right)$$

measured
✓ pressure changes with time due to

- pressure field changes with time
- sensor position changes with time

$$\frac{d}{dt}\, p(\vec{x}_s(t), t) \overset{\text{chain rule}}{=}$$

$$= \underbrace{\frac{\partial p}{\partial x}\frac{dx_s}{dt}}_{\equiv u(\vec{x}_s, t)} + \underbrace{\frac{\partial p}{\partial y}\frac{dy_s}{dt}}_{\equiv v(\vec{x}_s(t), t)} + \underbrace{\frac{\partial p}{\partial z}\frac{dz_s}{dt}}_{\equiv w(\vec{x}_s(t), t)} + \frac{\partial p}{\partial t}\cancel{\frac{dt}{dt}}^{=1}$$

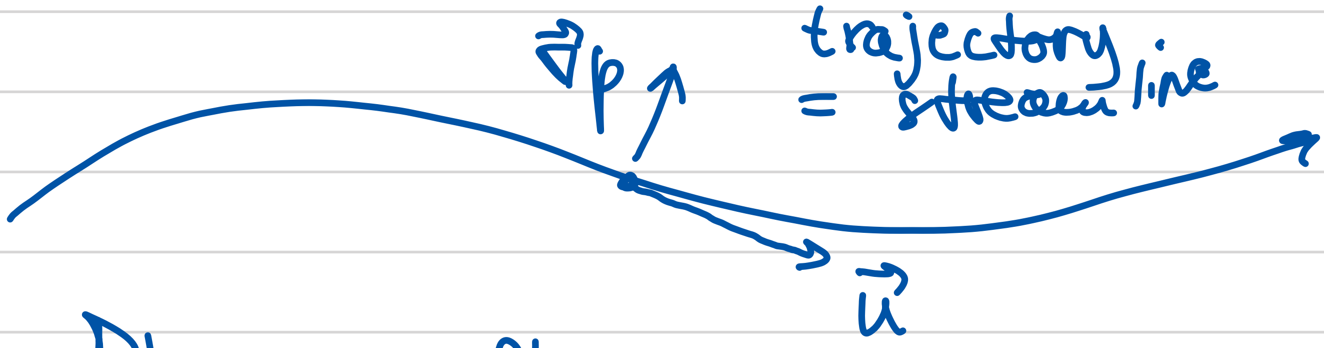
$$\Rightarrow \underbrace{\frac{d}{dt}\, p(\vec{x}_s(t), t)}_{\equiv \frac{Dp}{Dt}} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

$\equiv \frac{Dp}{Dt}$: material derivative*.

$$\Leftrightarrow \quad \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \underbrace{u_j \frac{\partial p}{\partial x_j}}_{\text{sum!}}$$

* time-derivative while traveling with the fluid.

Example: Steady flow



$$\Rightarrow \frac{Dp}{Dt} = u_j \frac{\partial p}{\partial x_j}$$

Suppose $\frac{Dp}{Dt} = 0 \Rightarrow p$ is constant along streamline

c.a.s.

$\Rightarrow \vec{\nabla} p$ does not have a component in the direction of the streamline $\Rightarrow \vec{\nabla} p \perp \vec{u} \Leftrightarrow \vec{\nabla} p \cdot \vec{u} = 0$

Euler's equation.

Momentum equation with $\mu = 0$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij} - \cancel{\mu(\ )}\right) = \rho g_i$$

$$\frac{\partial}{\partial x_j}(p\delta_{ij}) = \frac{\partial}{\partial x_1}(p\delta_{i1}) + \frac{\partial}{\partial x_2}(p\delta_{i2}) + \frac{\partial}{\partial x_3}(p\delta_{i3})$$

$$= \frac{\partial p}{\partial x_i}$$

$$\Rightarrow \frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \underbrace{-\frac{\partial p}{\partial x_i} + \rho g_i}$$

$$\underbrace{\frac{\partial \rho}{\partial t} u_i} + \rho \frac{\partial u_i}{\partial t} + \underbrace{\frac{\partial}{\partial x_j}(\rho u_j) u_i} + \rho u_j \frac{\partial u_i}{\partial x_j} = \cdots$$

$$: \underbrace{\cancel{\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right\}}}_{= 0 \text{ mass cons.}} u_i$$

$$\Rightarrow \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i$$

$$\Rightarrow \boxed{\frac{D u_i}{D t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i}$$

Euler's eq.

"$\mu = 0$"

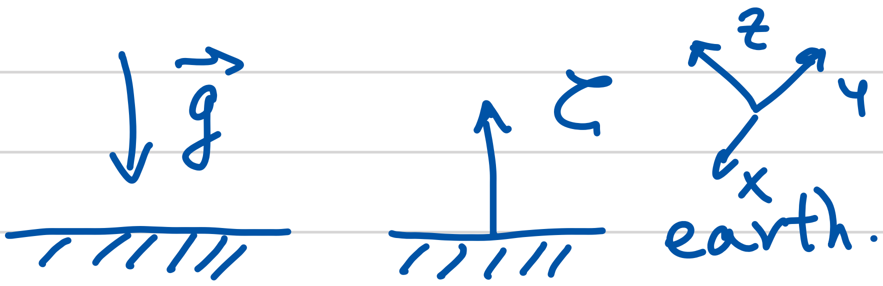
Note $\frac{D u_i}{D t}$ = time derivative of u_i while traveling with fluid

$\Rightarrow$ acceleration of fluid

$\rightarrow$ Newton's 2nd law.

Vector notion: $\boxed{\frac{D \vec{u}}{D t} = -\frac{1}{\rho}\vec{\nabla} p + \vec{g}}$

Gravity:

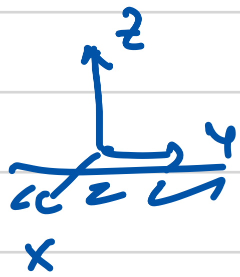


Define ζ as the altitude.

$$\sim \vec{\nabla}\zeta \quad \Rightarrow \quad \vec{g} \sim -\vec{\nabla}\zeta$$

$$|\vec{\nabla}\zeta| = 1 \quad \Rightarrow \quad \boxed{\vec{g} = -g\,\vec{\nabla}\zeta}$$

Test:



$$\Rightarrow \quad \zeta = z \qquad \vec{\nabla}\zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{g} = -g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \checkmark.$$

index notation: $\boxed{g_i = -g \dfrac{\partial \zeta}{\partial x_i}}$

Back to Euler's equation:

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\frac{\partial \zeta}{\partial x_i}.$$

Assume:
- steady flow
- incompressible.

$$\Rightarrow \quad u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right) - \frac{\partial}{\partial x_i}\left(g\zeta\right)$$

Take the inner product of this equation with the velocity vector.

$$\Rightarrow \quad \underbrace{u_i u_j \frac{\partial u_i}{\partial x_j}}_{\text{double sum}} = - \underbrace{u_i \frac{\partial}{\partial x_i}}_{\text{sum}}\left(\frac{p}{\rho}\right) - \underbrace{u_i \frac{\partial}{\partial x_i}}_{\text{sum}}(g\zeta)$$

$$u_i u_j \frac{\partial u_i}{\partial x_j} = u_j u_i \frac{\partial u_i}{\partial x_j} = u_j \cdot \frac{\partial}{\partial x_j}\left(\frac{1}{2} u_i u_i\right)$$

check: $\underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{2} u_i u_i\right)}_{\text{sum}} = \frac{1}{2}\left\{\underbrace{\frac{\partial u_i}{\partial x_j} u_i}_{\text{sum}} + \underbrace{u_i \frac{\partial u_i}{\partial x_j}}_{\text{sum}}\right\} = \underbrace{u_i \frac{\partial u_i}{\partial x_j}}_{\text{sum}}$

Rewrite $\underbrace{u_j \frac{\partial}{\partial x_j}\left(\frac{1}{2} u_i u_i\right)}_{\text{double sum}} \equiv \underbrace{u_i \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j u_j\right)}_{\text{double sum.}}$

Put everything together:

$$u_i \frac{\partial}{\partial x_i}\left\{ \frac{1}{2} u_j u_j + \frac{p}{\rho} + g\zeta \right\} = 0$$

$$\Rightarrow \vec{u} \cdot \vec{\nabla}\left\{ \frac{p}{\rho} + \frac{1}{2}\vec{u}\cdot\vec{u} + g\zeta \right\} = 0$$

$$\Rightarrow \vec{u} \perp \vec{\nabla}\{\ \}$$

$$\Rightarrow \{\ \} = \text{c.a.s.} \ !$$

$$\Rightarrow \frac{p}{\rho} + \frac{1}{2}\bar{u}^2 + g\zeta = c.a.s. \qquad \bar{u} = |\vec{u}|$$

$\rho = const \Rightarrow$

$$\boxed{p + \frac{1}{2}\rho\bar{u}^2 + \rho g\zeta = c.a.s.}$$

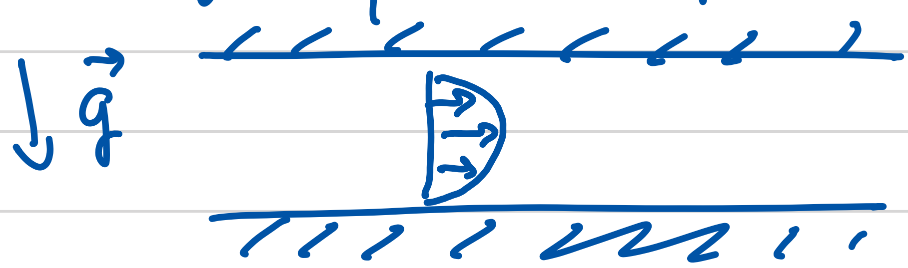
Bernoulli's equation.

if − $\mu = 0$ (viscosity neglected)

− steady

− incompressible

Note: fully developed flow: lect 5?



steady ✓

incompressible ✓

viscous ✗

Bernoulli? ✗

$$\underbrace{p}_{\neq c.a.s.} + \underbrace{\frac{1}{2}\rho\bar{u}^2}_{=c.a.s.} + \underbrace{\rho g\zeta}_{=c.a.s.} \neq c.a.s.$$

($\neq$ c.a.s. see ch 5 $\frac{dp}{dx} < 0$

Example Application:

assume: $\mu = 0$, $\rho = \text{const}$, $\frac{\partial}{\partial t}(\,) = 0$

What is the pressure in the stagnation point?
$\hookrightarrow p_0$

Bernoulli's equation is allowed to be used:

$$\left\{ p + \tfrac{1}{2}\rho|\vec{u}|^2 + \rho g \zeta \right\} = \text{c.a.s.}$$

so here $\{\ \}_\infty = \{\ \}_0$

$$\Rightarrow p_\infty + \tfrac{1}{2}\rho\bar{u}^2 + \cancel{\rho g \zeta} = p_0 + 0 + \cancel{\rho g \zeta}$$

$$\Rightarrow \boxed{p_0 = p_\infty + \tfrac{1}{2}\rho\bar{u}^2} \equiv \text{total pressure}$$

How to use this in practice?

$\bar{u}, p$

$u \approx 0$

airplane

pressure sensor in stagnation point $\Rightarrow p_0$ is known.

suppose you also know p_∞ from METEO

Bernoulli: $$p_0 = p_\infty + \frac{1}{2}\rho \bar{u}^2$$

$$\Rightarrow \bar{u} = \sqrt{2\,\frac{p_0 - p_\infty}{\rho}}$$

(with respect to the atmosphere