Chapter 11

Streamline Invariants

11.1 Energy conservation differential form

By rewriting all surface integrals in the integral form of energy conservation, Eq.(10.6), into volume integrals by means of Gauss' theorem, and then using the fact that the resulting equations holds for all possible convected blobs, one obtains the differential formulation of energy conservation:

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}\left(\rho E u_j - \sigma_{ij} u_i - k \frac{\partial T}{\partial x_j}\right) = \rho g_j u_j.} \tag{11.1}$$

11.2 Isentropic flow

Friction forces always work in opposite direction of movement. Heat flows from hot spots to cold spots. Therefore, all processes in the gas during its transport are irreversible. However, when friction and heat conduction can be neglected, all processes in the gas during its transport are reversible. Therefore we expect the gas to behave isentropically, which indeed can be derived from the conservation equations as we see below.

We start with the differential form of the conservation equations of mass, momentum and energy, while we put $\mu = 0$ and $k = 0$ to reflect that there is no friction and no heat conduction:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \tag{11.2}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij} p) = \rho g_i, \tag{11.3}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(\rho E u_j + p u_j) = \rho g_j u_j. \tag{11.4}$$

Next, we rewrite these equations in terms of material derivatives:

$$\frac{D\rho}{Dt} = -\rho\frac{\partial u_j}{\partial x_j}, \tag{11.5}$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i, \tag{11.6}$$

$$\rho\frac{DE}{Dt} = -\frac{\partial p u_j}{\partial x_j} + \rho g_j u_j. \tag{11.7}$$

Using the definition of the total energy, E, we evaluate

$$\rho\frac{DE}{Dt} = \rho\frac{De}{Dt} + \rho u_j\frac{Du_j}{Dt} = \rho\frac{De}{Dt} - u_j\frac{\partial p}{\partial x_j} + \rho g_j u_j. \tag{11.8}$$

Therefore,

$$\rho\frac{De}{Dt} - u_j\frac{\partial p}{\partial x_j} = -\frac{\partial p u_j}{\partial x_j}, \tag{11.9}$$

which leads to

$$\rho\frac{De}{Dt} + p\frac{\partial u_j}{\partial x_j} = 0. \tag{11.10}$$

We can replace the velocity divergence by means of the mass conservation equation and finally obtain, after division by ρ:

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0. \tag{11.11}$$

Noting that we know from thermodynamics that an infinitesimal increase in entropy s satisfies

$$ds = de - \frac{p}{\rho^2}d\rho, \tag{11.12}$$

we recognize that

$$\boxed{\frac{Ds}{Dt} = 0}. \tag{11.13}$$

This means that, when friction and heat conduction are neglected, the entropy is constant while travelling with the flow.

In special case of a perfect gas,

$$p = (\gamma - 1)\rho e, \tag{11.14}$$

Eq.(11.11) can be evaluated to

$$\frac{D}{Dt}\left(\rho^{1-\gamma}e\right) = 0. \tag{11.15}$$

Rewriting e in terms of ρ and p again, we get

$$\boxed{\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0.} \tag{11.16}$$

This means that $p\rho^{-\gamma}$ is constant while travelling with the flow, known as Poisson's equation.

11.3 Speed of sound, Mach number

Small Perturbations The propagation of sound is one of the phenomena that is governed by the equations of continuum mechanics. To derive the equations that govern sound propagation we neglect friction and heat conduction (which is a highly accurate approximation!) and start with the conservation equations of mass (Eq.(3.18)), momentum (Eq.(4.39)), while we replace the energy conservation equation (Eq.(11.1)) by Poisson's equation (Eq.(10.16)). We consider one-dimensional small perturbations (in x-direction) with respect to a quiescent flow (zero velocity, constant pressure and density) of a perfect gas:

$$\rho = \rho_o + \rho', \quad p = p_o + p', \quad u = u', \tag{11.17}$$

where the subscript "0" indicates constant values and where the prime indicates small perturbations. When we neglect quadratic and higher order products of perturbations we end up with the following three equations:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} &= 0, \\ \frac{\partial u}{\partial t} + \rho_o^{-1} \frac{\partial p'}{\partial x} &= 0, \\ p' = a^2 \rho', \quad a \equiv \sqrt{\gamma \frac{p_o}{\rho_o}}&, \end{aligned} \tag{11.18}$$

where γ is the (constant) ratio of specific heats.

Wave Equation By combining these equations and differentiating with respect to x and t we obtain an alternative but equivalent set of three equations:

$$\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} - a^2 \frac{\partial^2 \rho'}{\partial x^2} &= 0, \\ \frac{\partial^2 p'}{\partial t^2} - a^2 \frac{\partial^2 p'}{\partial x^2} &= 0, \\ \frac{\partial^2 u'}{\partial t^2} - a^2 \frac{\partial^2 u'}{\partial x^2} &= 0. \end{aligned} \tag{11.19}$$

We see that all three perturbations, ρ', p' and u', satisfy the same equation, which is called the **wave equation**.

It is very easy to verify that any function $f(x - at)$ satisfies the wave equation. The same holds for any function $g(x + at)$. Since the wave equation is linear the sum of these arbitrary function also is a solution:

$$p'(x,t) = f(x - at) + g(x + at). \tag{11.20}$$

A solution of the form $f(x - at)$ is constant along lines of constant $x - at$ in the x, t-plane. This means that perturbations travel with speed a to the right. Similarly, a solution of the form $g(x + at)$ is constant along lines of constant $x + at$ in the x, t-plane, which means that perturbations travel with speed a to the left.

Speed of sound As is explained above, the small perturbations which we call sound travel with speed a, which therefore must be the **speed of sound**. So, for the speed of sound in a perfect gas we have

$$\boxed{a \equiv \sqrt{\gamma \frac{p_o}{\rho_o}} = \sqrt{\gamma R T_o}}. \tag{11.21}$$

Mach number We define the Mach number M as the dimensionless ratio of the gas speed and the speed of sound:

$$\boxed{M \equiv \frac{U}{a}}, \qquad U \equiv \sqrt{u_k u_k} \tag{11.22}$$

11.4 Total temperature, -pressure, and -density

Consider stationary flow and neglect friction, heat conduction and gravity. Then the differential forms of mass and energy conservation become

$$\frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0, \tag{11.23}$$

$$\frac{\partial}{\partial x_j}\left(\rho H u_j\right) = 0, \tag{11.24}$$

Using the product rule of differentiation and division by ρ, this leads to

$$u_j \frac{\partial H}{\partial x_j} = 0, \tag{11.25}$$

which, since $\frac{\partial H}{\partial t} = 0$, can be written as

$$\frac{DH}{Dt} = 0. \tag{11.26}$$

In words, in case of stationary flow without effects of viscosity, heat conduction, and gravity, the total enthalpy is constant along stream lines.

If the gas is perfect we can rewrite H as

$$H = C_v T + \frac{1}{2} u_k u_k + RT = C_p T + \frac{1}{2} u_k u_k. \tag{11.27}$$

Using the speed of sound the total enthalpy H can be written as:

$$H = C_p T \left(1 + \frac{\gamma - 1}{2} M^2\right). \tag{11.28}$$

Total temperature If $\frac{DH}{Dt} = 0$, and C_p is constant, then we also have

$$\frac{DT_t}{Dt} = 0, \quad T_t \equiv T\left(1 + \frac{\gamma - 1}{2} M^2\right), \tag{11.29}$$

where T_t is the total temperature. This means that T_t is constant along streamlines. It is noted that T_t is a possible value of the temperature, not an actual value: it is the maximum possible value of the temperature that can be reached along the streamline at hand. Obviously this maximum value can only be reached (if reached at all!) at a point where $M = 0$, in other words, where the velocity is zero. A point where the velocity is zero is called a stagnation point, and therefore the total temperature is frequently refered to as the "stagnation temperature".

Total pressure Using the perfect gas law $p = \rho RT$, and Poisson's equation $p\rho^{-\gamma} = constant$, we can rewrite Eq.(11.29) as

$$\frac{Dp_t}{Dt} = 0, \quad p_t \equiv p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}, \tag{11.30}$$

where p_t is the total pressure, often refered to as the "stagnation pressure", which apparently also is constant along stream lines.

Total density Finally, again using the perfect gas law $p = \rho RT$ and Poisson's equation $p\rho^{-\gamma} = constant$, we can also rewrite Eq.(11.29) as

$$\frac{D\rho_t}{Dt} = 0, \quad \rho_t \equiv \rho\left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}, \tag{11.31}$$

where ρ_t is the total density, often refered to as the "stagnation density", which apparently also is constant along stream lines.

11.5 Limit of small Mach number

At this point one could raise the question whether a contradiction has been introduced if one looks at Bernoulli's equation, Eq.(8.28), with $g = 0$ and multiplied by the (constant) density:

$$\frac{D}{Dt}\left[p + \frac{1}{2}\rho u_j u_j\right] = 0. \tag{11.32}$$

and at the totasl pressure equation, Eq.(11.30):

$$\frac{D}{Dt}\left[p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}\right] = 0. \tag{11.33}$$

Both equations have been derived under exactly the same assumptions (steady flow, no effects of viscosity, heat conduction, or gravity) except for the incompresibility asumption that was used in the derivation of Bernoulli's equation. Therefore, one could raise the question whether the total pressure equation reduces to Bernoulli's equation if one assumes that the density perturbations become sufficiently small.

If the density perturbations are small, Eq.(11.31) shows that the Mach number must also be small:

$$M \ll 1. \tag{11.34}$$

Using Taylor's theorem it is easily shown that for small ϵ and any α:

$$(1+\epsilon)^{\alpha} = 1 + \alpha\epsilon + \mathcal{O}\left(\epsilon^2\right), \tag{11.35}$$

so

$$\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)} = 1 + \frac{\gamma}{2} M^2 + \mathcal{O}\left(M^4\right). \tag{11.36}$$

As a result,

$$\frac{D}{Dt}\left[p\left(1 + \frac{\gamma}{2} M^2\right)\right] = \mathcal{O}\left(M^4\right). \tag{11.37}$$

With the perfect gas law $p = \rho R T$ and the equation for the speed of sound, $a = \sqrt{\gamma \frac{p}{\rho}}$, one finds

$$\frac{D}{Dt}\left[p + \frac{1}{2}\rho u_j u_j\right] = \mathcal{O}\left(M^4\right). \tag{11.38}$$

This becomes Bernoulli's equation in the limit $M \to 0$.

11.6 Example: Pitot tube

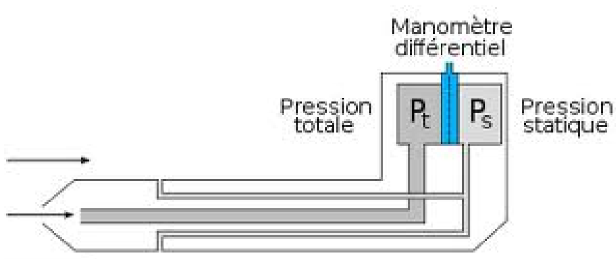

Figure 11.1: Pitot tube: basic priciple (left), and mounted under an airplane (right)

When a so-called Pitot-tube [1] is held into a flow it measures the static pressure (which is just the pressure) through a gap at the side of the probe. It also measures the total pressure through a gap at the front of the probe. The difference can be used for estimating the velocity or the Mach number.

Incompressible flow When the Mach number is small such that the flow is approximately incompressible, Bernoulli's equation gives the velocity:

$$U \equiv \sqrt{u_j u_j} = \sqrt{2\frac{p_t - p}{\rho}}. \tag{11.39}$$

Compressible flow When the Mach number is larger such that the flow is compressible, the total pressure equation gives the Mach number:

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_t}{p}\right)^{\gamma - 1} - 1\right]}. \tag{11.40}$$

11.7 Exercises

Problem 11.1. *Consider a sphere in compressible flow. Far upstream from the sphere the pressure, Mach number and temperature are known: p_∞, M_∞, and T_∞. The pressure and temperature in the stagnation point are p_o and T_o.*

(a) Express p_o in terms of p_∞, M_∞.
(b) Express p_∞ in terms of p_o, M_∞.
(c) Express T_∞ in terms of T_o, M_∞.
(d) For measured p_o, T_o, p_∞, compute the velocity U_∞ far upstream of the sphere.

Problem 11.2. *(a) Show that $\frac{1}{p}\frac{Dp}{Dt} = \frac{D}{Dt}\ln p$.*
(b) What is the meaning of $\frac{Dp}{Dt}$?

Problem 11.3. *Consider steady flow with $\mu = 0$ and $k = 0$.*

(a) Show that the mass and energy equations reduce to

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\frac{\partial}{\partial x_j}(\rho u_j H) = 0$$

(b) Show that these equations lead to $\frac{DH}{Dt} = 0$.

[1] Henri Pitot (1695 - 1771) was a French hydraulic engineer and the inventor of the Pitot tube.