

## Lecture #9

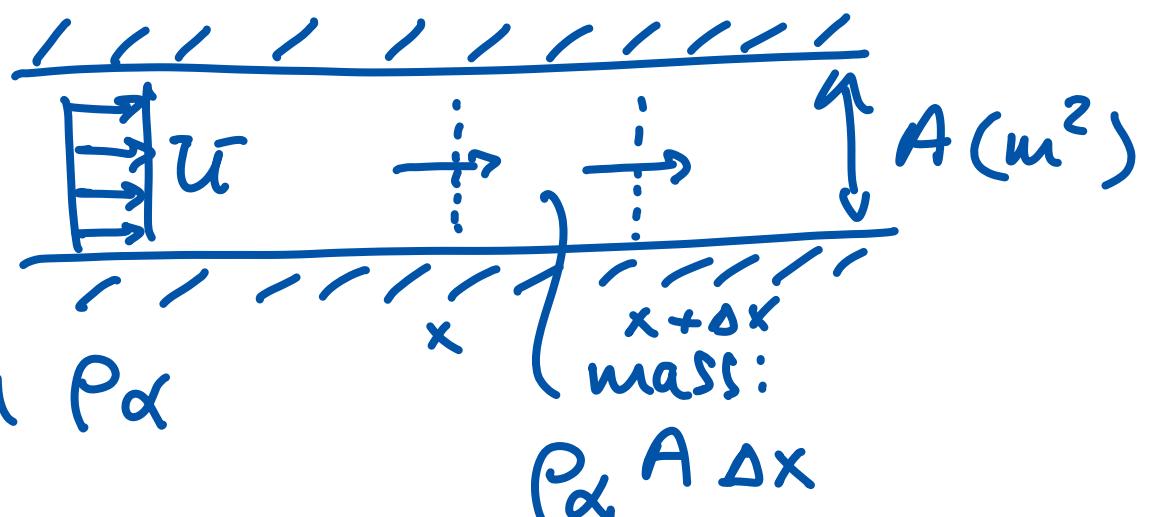
## Convection & Diffusion.



motion: convection  
spread: diffusion. } Equations?

Convection  
(1D)

inh, density  $\rho_\alpha$



$$\frac{\partial}{\partial t} \left( \rho_\alpha A / \Delta x \right) = \underbrace{(\rho_\alpha u A)}_{{\Delta x}}_x - \underbrace{(\rho_\alpha u A)}_{x + \Delta x}$$

$$\Delta x \rightarrow 0$$

$$\Rightarrow \frac{\partial \rho_\alpha}{\partial t} = - \overbrace{\frac{\partial (\rho_\alpha u)}{\partial x}}^{\text{kg/s}}$$

$$\Rightarrow \boxed{\frac{\partial \rho_\alpha}{\partial t} + u \frac{\partial \rho_\alpha}{\partial x} = 0}$$

convection equation.

$$\Rightarrow \boxed{\frac{D \rho_\alpha}{Dt} = 0}$$

$\Rightarrow \rho_\alpha$  is constant while moving with the flow

$$\Rightarrow \rho_\alpha(x, t) = f(x - ut)$$

$$\Rightarrow f(x - ut) = \text{const.} \quad \text{if } x - ut = \text{const.}$$

$$\Rightarrow x = ut$$

Question: Is  $f(x - ut)$  a solution of the derived equation?

$$\frac{\partial \rho_\alpha}{\partial t} = \frac{\partial f}{\partial t} = f'(x - ut) \cdot \frac{\partial}{\partial t}(x - ut) = -uf'$$

$$\frac{\partial \rho_\alpha}{\partial x} = \frac{\partial f}{\partial x} = f'(x - ut) \cdot \frac{\partial}{\partial x}(x - ut) = f'$$

$$\Rightarrow \frac{\partial \rho_\alpha}{\partial t} + u \frac{\partial \rho_\alpha}{\partial x} = -uf' + uf' = 0 \quad \cancel{\cancel{f}}$$

$f$ : form is ok, but details?

$t=0 \quad \rho_\alpha(x, 0) = \rho_\alpha^0(x)$

$\underbrace{\phantom{0}}$

given,  
initial condition.

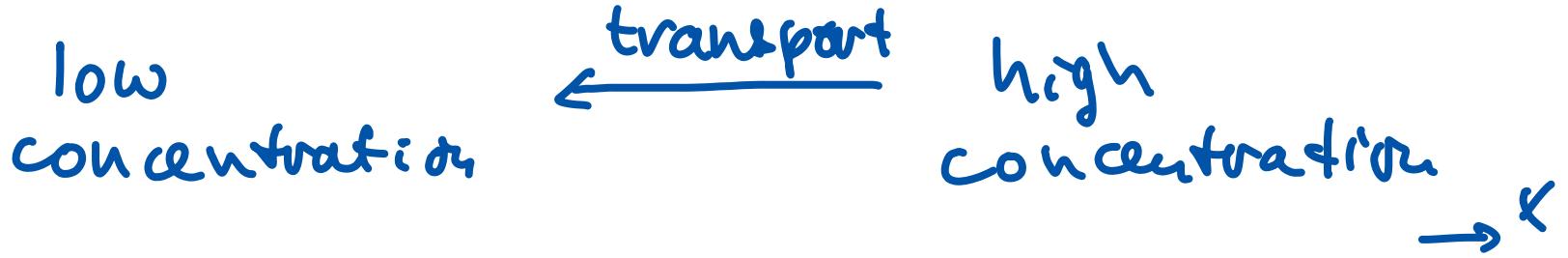
$$\Rightarrow f(x - ut_0) = \rho_\alpha^0(x)$$

$$\Rightarrow f(x) = \rho_\alpha^0(x)$$

$$f(x - ut) = \rho_\alpha^0(x - ut)$$

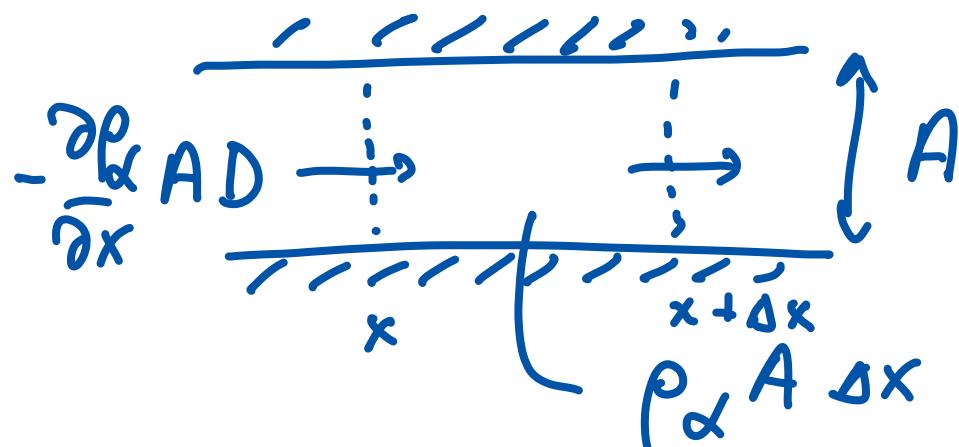
# Diffusion

Spreading  
molecules!



$$\Rightarrow \text{transport flux} \sim - \frac{\partial P_d}{\partial x}$$

Equation?



D: diffusion coefficient  
depends on the two fluids.

$$[D] = \text{m}^2/\text{s}$$

$$\frac{\partial}{\partial t} \left( \cancel{P_d A / \Delta x} \right) = \underbrace{\left( -D \frac{\partial \cancel{P_d A}}{\partial x} \right)_x - \left( -D \frac{\partial \cancel{P_d A}}{\partial x} \right)_{x+\Delta x}}_{\Delta x}$$

$$\Delta x \rightarrow 0$$

$$\Rightarrow \frac{\partial \cancel{P_d}}{\partial t} = - \frac{\partial}{\partial x} \left( -D \frac{\partial \cancel{P_d}}{\partial x} \right)$$

$$= \frac{\partial}{\partial x} \left( D \frac{\partial P_d}{\partial x} \right) = D \frac{\partial^2 P_d}{\partial x^2}$$

$$\Rightarrow \frac{\partial P_\alpha}{\partial t} = D \frac{\partial^2 P_\alpha}{\partial x^2}$$

Diffusion equation.

Does it indeed describe spreading?

Try:  $P_\alpha(x,t) = h(t) \exp\left(-\beta \frac{x^2}{t}\right)$

$$\frac{\partial P_\alpha}{\partial t} = h'(t) \cdot \exp() + h \cdot \exp() \beta \frac{x^2}{t^2}$$

$$\frac{\partial P_\alpha}{\partial x} = h \cdot \exp() \cdot -2\beta \frac{x}{t}$$

$$\frac{\partial^2 P_\alpha}{\partial x^2} = h \cdot \exp() \cdot 4\beta^2 \frac{x^2}{t^2} + h \cdot \exp() \cdot -\frac{2\beta}{t}$$

Substitute into the diffusion eq:

$$h' \cdot \cancel{\exp()} + h \cdot \cancel{\exp()} \beta \frac{x^2}{t^2} = -D \cdot h \cancel{\exp()} \frac{2\beta}{t} + D \cdot h \cancel{\exp()} 4\beta^2 \frac{x^2}{t^2}$$

$$\Rightarrow h' + h \cdot \beta \cdot \frac{x^2}{t^2} = -D h \frac{2\beta}{t} + 4D h \beta^2 \frac{x^2}{t^2}$$

$\forall x, t$ !

Choose  $\beta D = 1$   $h' = -\frac{1}{2} h/t$

$$\Rightarrow -\frac{1}{2} \frac{h}{t} + h \frac{1}{4D} \frac{x^2}{t^2} = -\frac{1}{2} \frac{h}{t} + h \frac{1}{4D} \frac{x^2}{t^2}$$

$$\Rightarrow \boxed{\beta = \frac{1}{4D}} \quad \text{but } h = ?$$

$$\frac{h'}{h} = -\frac{1}{2} \frac{1}{t} \Rightarrow \frac{d}{dt} (\ln h) = -\frac{1}{2} \frac{d}{dt} (\ln t)$$

$$\Rightarrow \ln h = -\frac{1}{2} \ln t + C = \ln t^{-1/2} + C$$

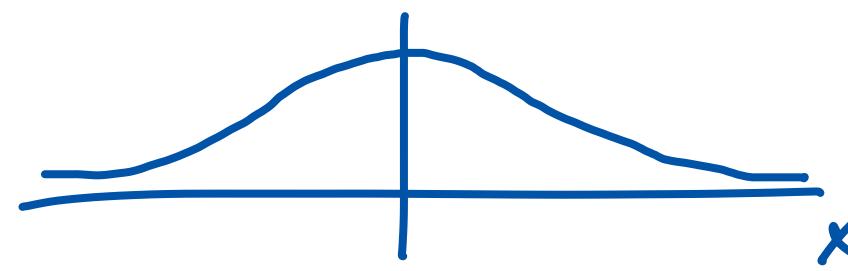
$$e^{\ln h} = e^{\ln t^{-1/2} + C} = e^C \cdot e^{\ln t^{-1/2}}$$

$$\Rightarrow h = C t^{-1/2} \Rightarrow h = \frac{\text{const}}{\sqrt{t}}$$

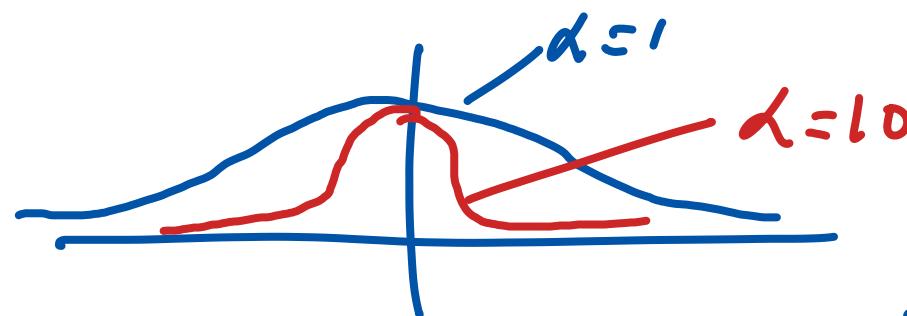
$$\Rightarrow \boxed{P_\alpha(x,t) = \frac{\text{const}}{\sqrt{t}} \cdot \exp\left(-\frac{x^2}{4Dt}\right)}$$

$$\left[ \frac{x^2}{4Dt} \right] = \frac{m^2}{m^2/s \cdot s} = 1 \quad \text{f}$$

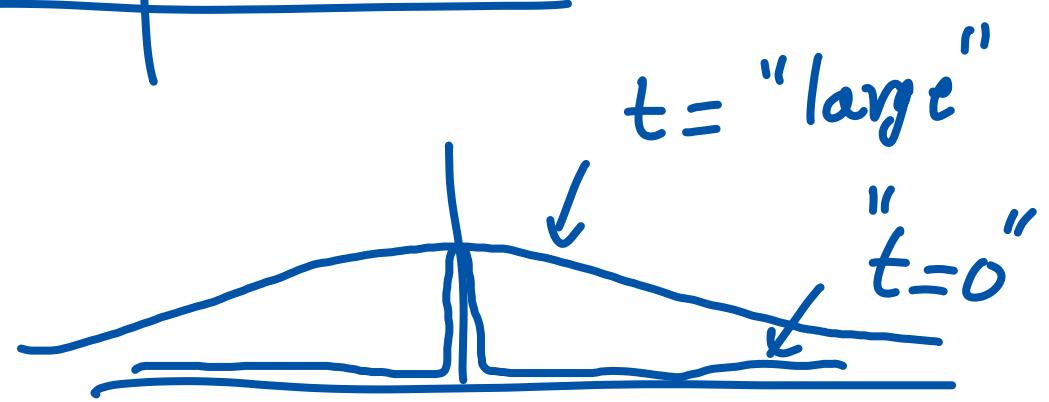
$e^{-x^2}$ :



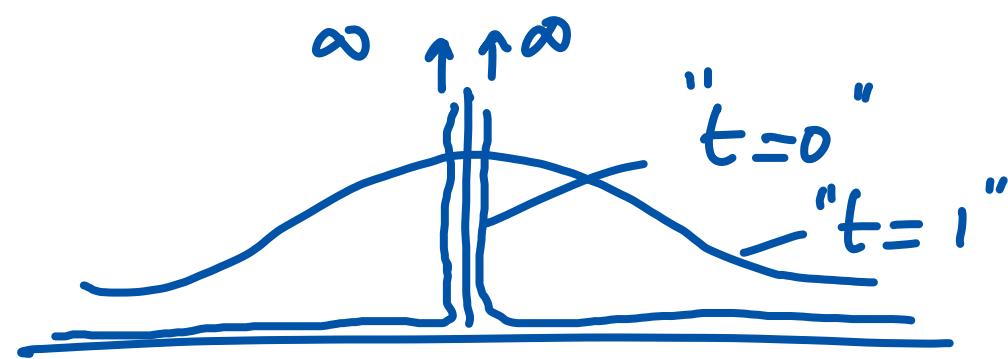
$e^{-\alpha x^2}$ :



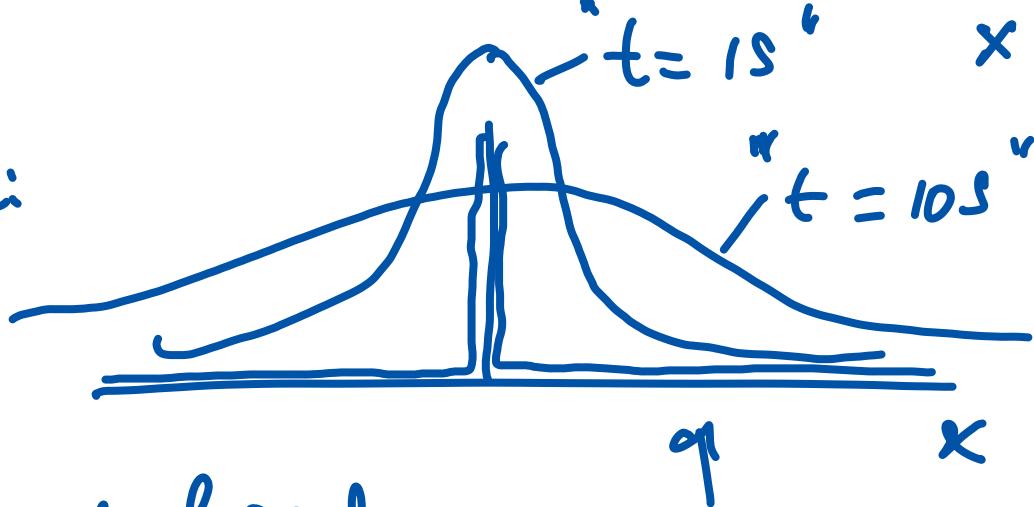
$e^{-\frac{x^2}{4Dt}}$ :  $\alpha = \frac{1}{4Dt}$



$\frac{1}{\sqrt{t}} e^{-\frac{x^2}{4Dt}}$ :



Overall picture:



Picture seems good.  
=.

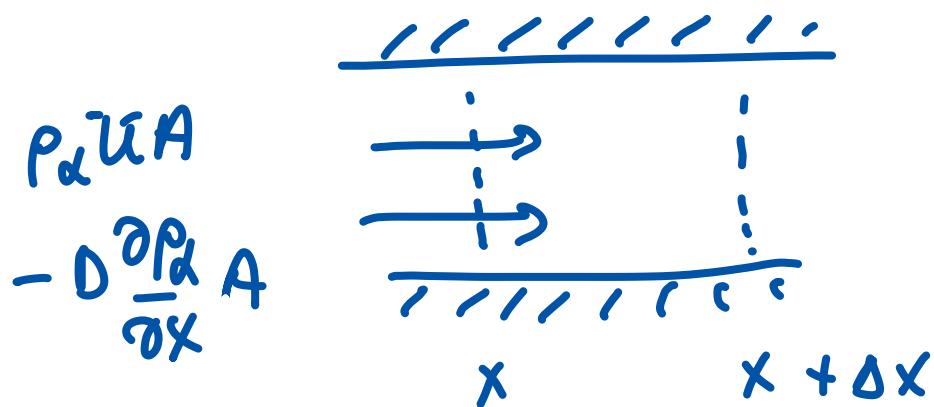
# Convection - Diffusion. (Simultaneously)

Convection:

$$\frac{\partial P_d}{\partial t} + \bar{U} \frac{\partial P_d}{\partial x} = 0$$

Diffusion:

$$\frac{\partial P_d}{\partial t} = D \frac{\partial^2 P_d}{\partial x^2}$$



Combined effect:

$$\boxed{\frac{\partial P_d}{\partial t} + \bar{U} \frac{\partial P_d}{\partial x} = D \frac{\partial^2 P_d}{\partial x^2}}$$

Convection  
Diffusion  
eq.

$$\Rightarrow \frac{DP_d}{Dt} = D \frac{\partial^2 P_d}{\partial x^2}$$

in a frame moving with  $\bar{U}$  to the right:  
pure diffusion.

:

$\Rightarrow$  in that frame:

$$P_d = \frac{\text{const}}{\sqrt{t}} \exp\left(-\frac{\xi^2}{4Dt}\right)$$

$$\xi \equiv x - Ut.$$

if  $\xi = \text{const} \Rightarrow x = Ut$

moves with the flow

$$\Rightarrow P_d(x,t) = \frac{\text{const}}{\sqrt{t}} \exp\left(-\frac{(x-Ut)^2}{4Dt}\right)$$

Solution of convection-diffusion eq.  
(can be checked by substitution).



transport and spread.