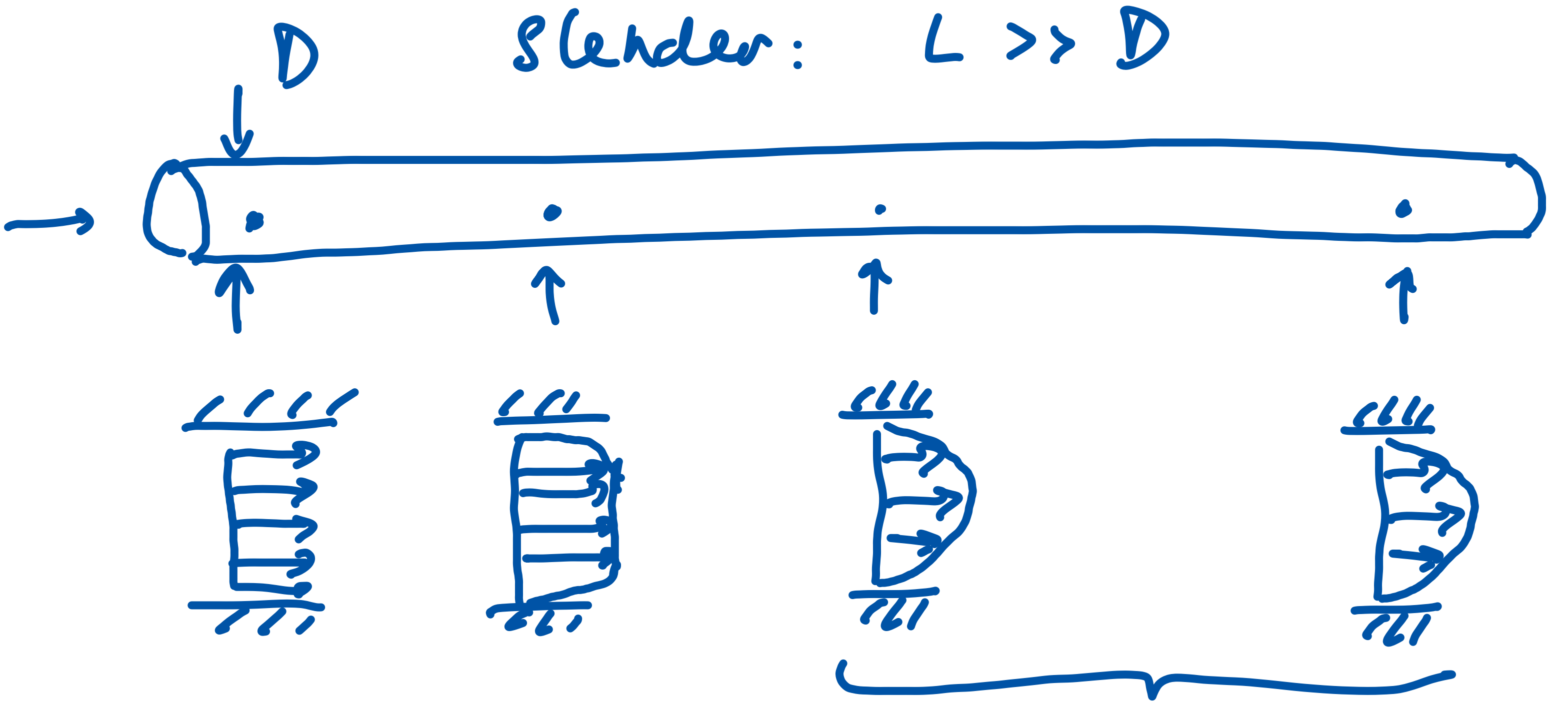
(*) add Drag and heat transfer of sphere!

Fluid Mechanics 1

Lecture #5:

Navier-Stokes Equation & Fully Developed Flow

slender: $L \gg D$



velocity profile development. $\longrightarrow$

fully developed.

→ Can we compute the fully developed profile?

→ Can we establish the relation between $\bar{U}$ and $\frac{dp}{dx}$?

$\bar{U}$: average velocity

$\frac{dp}{dx}$: pressure derivative

We need a differential form of momentum conservation.

$$\int_{S(t)} \rho u_i u_j n_j \, dS = \int_{S(t)} (\rho u_i u_j) \underbrace{n_j}_{\text{sum!}} dS$$

inner product

Gauss.

$$= \int_{V(t)} \frac{\partial}{\partial x_j}(\rho u_i u_j) \, dV$$

Likewise

$$\int_{S(t)} \underbrace{\sigma_{ji} n_j}_{\text{sum!}} \, dS \overset{\text{Gauss}}{=} \int_{V(t)} \frac{\partial}{\partial x_j} \sigma_{ji} \, dV$$

$$\Rightarrow \int_{V(t)} \{ \dots \} dV = 0 \quad \forall V, t$$

See previous lecture for mom. cons. in integral form.

$$\Leftrightarrow \{ \ \} = 0 \quad \forall \vec{x}, t.$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \rho u_i + \frac{\partial}{\partial x_j}(\rho u_i u_j - \sigma_{ji}) = \rho g_i} \quad i = 1, 2, 3$$

Navier – Stokes equation.

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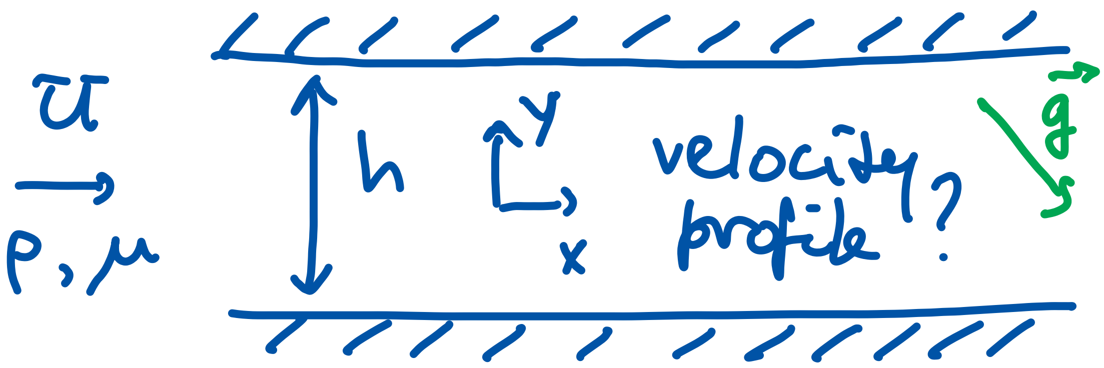
Application example: fully developed flow:



assume:
- incompressible
- steady
- 2D
- fully developed $\frac{\partial u}{\partial x} = 0$ $\frac{\partial v}{\partial x} = 0$

Apply Navier-Stokes equation

$$\cancel{\frac{\partial \rho u_i}{\partial t}} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j - \sigma_{ji}\right) = \rho g_i$$

steady

$$i=1 \quad \frac{\partial}{\partial x_1}\left(\rho u_1 u_1 - \sigma_{11}\right) + \frac{\partial}{\partial x_2}\left(\rho u_1 u_2 - \sigma_{21}\right) = \rho g_1$$

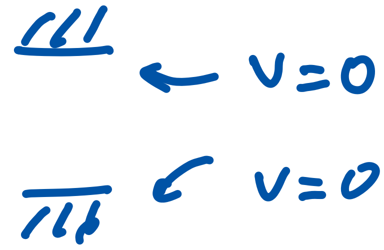
$$\Rightarrow \frac{\partial}{\partial x}\left(\rho u^2 - \sigma_{11}\right) + \frac{\partial}{\partial y}\left(\rho u v - \sigma_{21}\right) = \rho g_1$$

$\rho = \text{const}$ and $\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u^2}{\partial x} = 0$

$$\Rightarrow -\frac{\partial \sigma_{11}}{\partial x} + \cdots = \rho g_1$$

mass conservation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
$\rho = \text{const}$

$$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 0$$



$$\Rightarrow \boxed{v \equiv 0}$$

$$\Rightarrow \boxed{-\frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{21}}{\partial y} = \rho g_1}$$

Navier-Stokes in x-dir, fully developed flow

$$\sigma_{ji} = -p\delta_{ji} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3}\delta_{ij}\mu \cancel{\frac{\partial u_k}{\partial x_k}}$$

incompressible.

$$\Rightarrow \sigma_{11} = -p\underbrace{\delta_{11}}_{=1} + \mu\left(\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0}\right) = -p.$$

fully dev.

$$\Rightarrow \sigma_{21} = -p\underbrace{\cancel{\delta_{21}}}_{=0} + \mu\left(\underbrace{\cancel{\frac{\partial v}{\partial x}}}_{=0} + \frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y}.$$

fully developed

$$\Rightarrow \frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \rho g_1$$

Assume $\mu = \text{const}$

$$\Rightarrow \boxed{\frac{\partial p}{\partial x} - \mu\frac{\partial^2 u}{\partial y^2} = \rho g_1}$$

Navier-Stokes in $i=2$ direction (y)

$$\frac{\partial}{\partial x}\Big(\underbrace{\cancel{\rho v u}}_{\text{fully dev.}} - \tau_{12}\Big) + \frac{\partial}{\partial y}\Big(\underbrace{\cancel{\rho v^2}}_{v \equiv 0} - \tau_{22}\Big) = \rho g_2$$

$$\Rightarrow \quad -\frac{\partial \tau_{12}}{\partial x} - \frac{\partial \tau_{22}}{\partial y} = \rho g_2$$

Symmetry

$$\tau_{12} = \tau_{21} = \mu \frac{\partial u}{\partial y}$$

$$\tau_{22} = -p \underbrace{\cancel{\delta_{22}}}_{=1} + \mu\Big(\underbrace{\cancel{\frac{\partial v}{\partial y}}}_{0} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v \equiv 0}\Big) = -p$$

$$\Rightarrow \quad \underbrace{-\frac{\partial}{\partial x}\Big(\mu \frac{\partial u}{\partial y}\Big)}_{} + \frac{\partial p}{\partial y} = \rho g_2 .$$

$$= -\mu \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) \equiv -\mu \frac{\partial}{\partial y}\Big(\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{=0 \text{ fully dev.}}\Big) = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial p}{\partial y} = \rho g_2} \qquad (y\text{-dir})$$

$$\boxed{\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = \rho g_1} \qquad (x\text{-dir}).$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{\mu}\left(\rho g_1 - \frac{\partial p}{\partial x}\right)$$

$$= \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right)$$

$$\int \frac{\partial^2 u}{\partial y^2}\, dy = \frac{\partial u}{\partial y} \qquad \int \frac{\partial u}{\partial y}\, dy = u$$

$$\frac{\partial u}{\partial y} = \underbrace{\frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right)} y + c_1$$

is correct if $\quad$ = independent of y

μ, ρ, g_1 are constants ✓

but what about $\frac{\partial p}{\partial x}$?

(in principle function of x, y)

Note $\quad \frac{\partial p}{\partial y} = \rho g_2$

constants

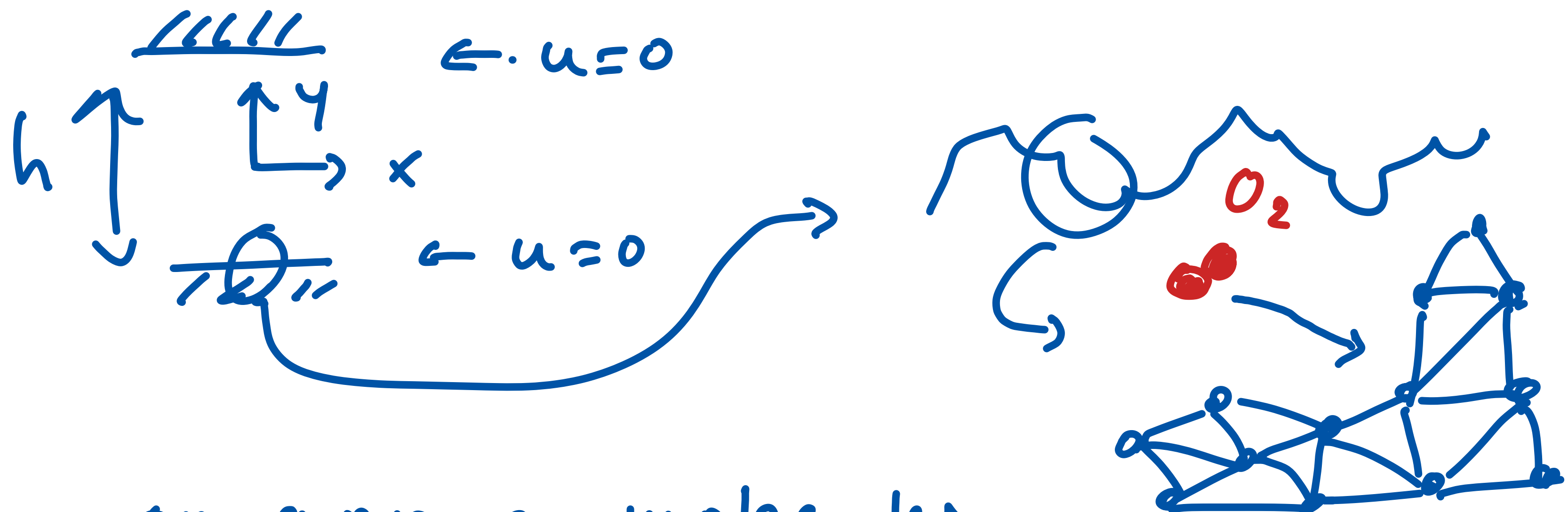
$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}\left(\rho g_2\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = 0$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right) y + c_1$$

$$\Rightarrow \boxed{u(y) = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right)\frac{1}{2}y^2 + c_1 y + c_2}$$

Problem left is: c_1? c_2?



on average molecules have zero velocity in x, y - direction

$$u\left(-\frac{h}{2}\right) = \frac{1}{\mu}(\quad)\,\frac{1}{2}\left(-\frac{h}{2}\right)^2 + c_1\left(-\frac{h}{2}\right) + c_2 = 0$$

$$u\left(\frac{h}{2}\right) = \frac{1}{\mu}(\quad)\,\frac{1}{2}\left(\frac{h}{2}\right)^2 + c_1\left(\frac{h}{2}\right) + c_2 = 0$$

2 eq's, 2 unknowns: c_1, c_2

add: $$\frac{1}{\mu}(\quad)\left(\frac{h}{2}\right)^2 + 0 + 2c_2 = 0$$

$$\Rightarrow c_2 = -\frac{1}{2\mu}(\quad)\left(\frac{h}{2}\right)^2$$

Subtract: $0 \quad - c_1 h \quad + 0 = 0$

$$\Rightarrow c_1 = 0 .$$

$$\Rightarrow \boxed{u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right)\left(y^2 - \left(\frac{h}{2}\right)^2\right)}$$

→ dimensions? "$\tau = \mu \frac{\partial u}{\partial y}$" $Pa = [\mu] \frac{1}{s}$

$$\frac{m}{s} = \frac{1}{Pa \cdot s}\left(\frac{Pa}{m} - \frac{kg}{m^3}\frac{m}{s^2}\right) m^2$$

$$\frac{1}{Pa\, s}\frac{Pa}{m} m^2 = \frac{m}{s} ✓ \qquad \frac{kg}{m^3}\frac{m}{s^2} m^2 = kg \frac{m}{s^2}\frac{1}{m^2} \cdot m$$

$$= Pa \cdot m$$

$$\frac{1}{Pa\, s} \cdot Pa \cdot m = \frac{m}{s} ✓ .$$

→ boundary conditions?

$$y = -\frac{h}{2} \Rightarrow u = 0 ✓$$

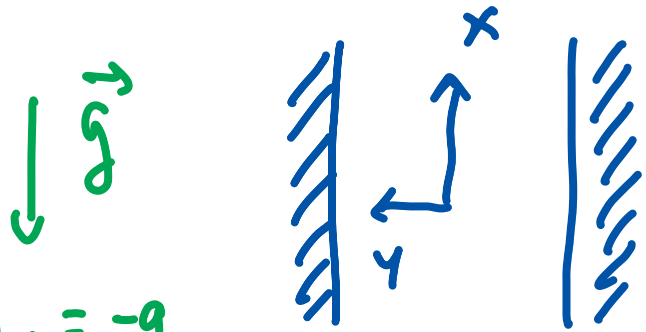
$$y = \frac{h}{2} \Rightarrow u = 0 ✓ .$$

→ sign(s)? if $p_{in} > p_{out}$

u must be > 0.

$(y^2 - (\frac{h}{2})^2) \leq 0$ if $g_1 = 0$, $\frac{\partial p}{\partial x} < 0 \Rightarrow u > 0$ ✓.

Note: if $\frac{\partial p}{\partial x} = \rho g_1 \Rightarrow u = 0$



$g_1 = -g$

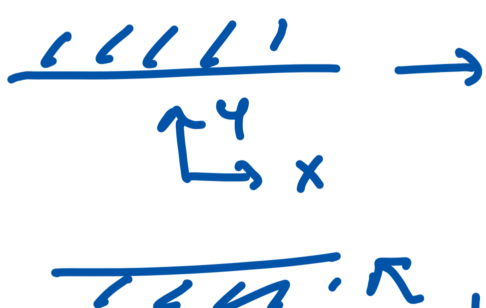
$\Rightarrow \frac{\partial p}{\partial x} = -\rho g$

$\Rightarrow p = -\rho g x + c$

$\Rightarrow$ p is decreasing with x

hydrostatic case: $\boxed{u = 0}$

More general problem:

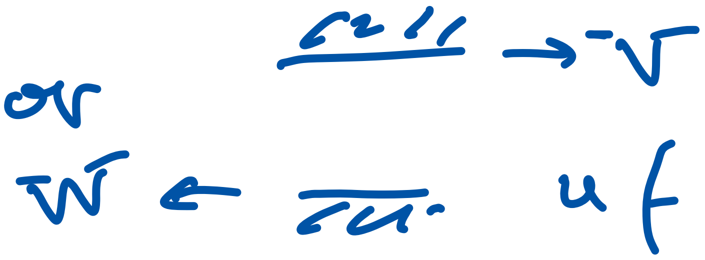


$\Rightarrow u\left(\frac{h}{2}\right) = V$

no-slip condition.

$u\left(-\frac{h}{2}\right) = 0$

or



$u\left(-\frac{h}{2}\right) = -W$

What is the relation between

$\frac{\partial p}{\partial x} - \rho g_1$ and the average velocity $\bar{u}$?

$$\bar{u} \equiv \frac{1}{h} \int_{-h/2}^{h/2} u(y)\,dy$$

$$\frac{1}{m} \cdot \frac{m}{s} \cdot m = \frac{m}{s} \checkmark$$

$$\Rightarrow \boxed{\bar{u} = -\frac{1}{12}\frac{h^2}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_1\right)}$$

$$\Leftrightarrow \quad \frac{\partial p}{\partial x} = -12\frac{\mu \bar{u}}{h^2} + \rho g_1$$

$$\frac{Pa}{m} = \frac{Pa \cdot s \ \frac{m}{s}}{m^2} + \frac{Pa}{m}$$

$$\checkmark \qquad \checkmark$$