2.13 Cooling of a copper rod ★

a) *Determine the time it takes for the copper rod to cool to an average temperature of 25 °C:*

The could possibly be solved by use of the lumped capacity model in the case of Bi << 1.

Therefore checking whether the lumped capacity model can be used:

$$\mathrm{Bi} = \frac{\alpha \cdot V}{\lambda \cdot A_{\mathrm{s}}} = \frac{\alpha \cdot d}{4 \cdot 399\ [\mathrm{W/mK}]} = 0.025 \tag{2.239}$$

$$\boxed{\rightarrow \mathrm{Bi} << 1} \tag{2.240}$$

The lumped capacity model can be found in the slides, or it can be derived by use of an energy balance (steps 1-5).

1) Setting up an energy balance:

Energy balance around the copper rod:

$$\frac{\partial U}{\partial t} = \cancelto{0}{\sum \dot{Q}_{\mathrm{in}}} - \sum \dot{Q}_{\mathrm{out}} \tag{2.241}$$

$$\boxed{\rightarrow \frac{\partial U}{\partial t} = -\dot{Q}_{\mathrm{conv}}} \tag{2.242}$$

2) Defining the fluxes:

Where the change of internal energy over the course of time can be expressed as:

$$\boxed{\rightarrow \frac{\partial U}{\partial t} = V \cdot \rho \cdot c_{\mathrm{p}} \cdot \frac{\partial T}{\partial t}} \tag{2.243}$$

And the rate of heat loss by convection

$$\boxed{\rightarrow \dot{Q}_{\mathrm{conv}} = \alpha \cdot A_{\mathrm{s}} \cdot (T(t) - T_{\infty})} \tag{2.244}$$

3) Inserting and rearranging:

Inserting yields:

$$\frac{\partial U}{\partial t} = -\dot{Q}_{\mathrm{conv}} \tag{2.245}$$

$$\boxed{\rightarrow V \cdot \rho \cdot c_{\mathrm{p}} \cdot \frac{\partial T}{\partial t} = -\alpha \cdot A_{\mathrm{s}} \cdot (T(t) - T_{\infty})} \tag{2.246}$$

4) Defining the initial condition:

In order to solve the equation, an initial condition should be given. This can be described in terms of the initial temperature of the copper rod, which is:

$$\boxed{\rightarrow T(t=0) = T_0} \tag{2.247}$$

5) Solving the equation:

In order to solve the equation, an homogenized temperature should be introduced, which is:

$$\Theta^* = \frac{T(t) - T_0}{T_\infty - T_0} \tag{2.248}$$

And:

$$\frac{\partial \Theta^*}{\partial t} = \frac{\partial}{\partial t}\left(\frac{T(t) - T_0}{T_\infty - T_0}\right) = \frac{1}{T_\infty - T_0}\frac{\partial T}{\partial t} \tag{2.249}$$

And therefore the energy balance can be written to:

$$V \cdot \rho \cdot c_\mathrm{p} \cdot \frac{\partial T}{\partial t} = -\alpha \cdot A_\mathrm{s} \cdot (T(t) - T_\infty) \tag{2.250}$$

$$V \cdot \rho \cdot c_\mathrm{p} \cdot \frac{1}{T_\infty - T_0} \cdot \frac{\partial T}{\partial t} = -\alpha \cdot A_\mathrm{s} \cdot \frac{T(t) - T_\infty}{T_\infty - T_0} \tag{2.251}$$

$$\frac{\partial \Theta^*}{\partial t} = -\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \cdot (\Theta^* - 1) \tag{2.252}$$

Rewriting yields:

$$\frac{1}{\Theta^* - 1}\frac{\partial \Theta^*}{\partial t} = -\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \tag{2.253}$$

Integration:

$$\ln|\Theta^* - 1| = -\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \cdot t + c_1 \tag{2.254}$$

$$\Theta^* = C_1 \cdot \exp\left(-\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \cdot t\right) + 1 \tag{2.255}$$

At $T(t=0) = T_0$, thus:

$$\Theta^*(t=0) = \frac{T_0 - T_0}{T_\infty - T_0} = 0 \tag{2.256}$$

And therefore:

$$C_1 = -1 \tag{2.257}$$

So:

$$\boxed{\rightarrow \Theta^*(t) = -\exp\left(-\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \cdot t\right) + 1} \tag{2.258}$$

6) Determining the time where the temperature of the rod is equal to 25 °C:

At $t = t_1$, the temperature of the rod is equal to 25 °C. Thus:

$$\Theta^*(t = t_1) = -\exp\left(-\frac{\alpha \cdot A_\mathrm{s}}{V \cdot \rho \cdot c_\mathrm{p}} \cdot t_1\right) + 1 = \frac{T(t = t_1) - T_0}{T_\infty - T_0} = 0.9375 \tag{2.259}$$

Rewriting yields:

$$t_1 = -\ln(0.0625)\,\frac{V \cdot \rho \cdot c_\mathrm{p}}{\alpha \cdot A_\mathrm{s}} \tag{2.260}$$

$$t_1 = -\ln(0.0625)\,\frac{d \cdot \rho \cdot c_\mathrm{p}}{4 \cdot \alpha} = -\ln(0.0625)\,\frac{0.02\ [\mathrm{m}] \cdot 8930\ [\mathrm{kg/m^3}] \cdot 382\ [\mathrm{J/kgK}]}{4 \cdot 200\ [\mathrm{W/m^2K}]} \tag{2.261}$$

$$\boxed{\rightarrow t_1 = 236.45\ [\mathrm{s}]} \tag{2.262}$$

2.14 The temperature delay ★★

a) *Determine Δt:*

1) Setting up an energy balance:

Energy balance:

$$\frac{\partial U}{\partial t} = \sum \dot{Q}_{\text{in}} - \cancelto{0}{\sum \dot{Q}_{\text{out}}} \tag{2.263}$$

$$\boxed{\rightarrow \frac{dU}{dt} = \dot{Q}_{\text{conv}}} \tag{2.264}$$

2) Defining the fluxes:

Where:

$$\boxed{\rightarrow \frac{dU}{dt} = m \cdot c_{\text{p}} \cdot \frac{dT}{dt}} \tag{2.265}$$

$$\boxed{\rightarrow \dot{Q}_{\text{conv}} = \alpha \cdot A_{\text{s}} \cdot \Delta T} \tag{2.266}$$

3) Inserting and rearranging:

Inserting yields:

$$\frac{dU}{dt} = \dot{Q}_{\text{conv}} \tag{2.267}$$

$$\boxed{\rightarrow m \cdot c_{\text{p}} \cdot \frac{dT}{dt} = \alpha \cdot A_{\text{s}} \cdot \Delta T} \tag{2.268}$$

For $t \rightarrow \infty$ the temperature increases linearly and therefore we can say: $\frac{dT}{dt} = \frac{\Delta T}{\Delta t}$
Substitution of $\frac{dT}{dt}$ results in:

$$m \cdot c_{\text{p}} \cdot \frac{\Delta T}{\Delta t} = \alpha \cdot A_{\text{s}} \cdot \Delta T \tag{2.269}$$

And for that reason:

$$\boxed{\rightarrow \Delta t = \frac{m \cdot c_{\text{p}}}{\alpha \cdot A_{\text{s}}}} \tag{2.270}$$