Elasticity Theory Exercises

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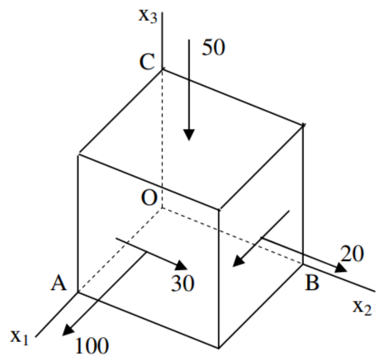
The following Exercises V1-V13 are based on old exams from the last 20 years and allow you to practice all types of calculations taught during the course Elasticity Theory and will be tested during the exam.

See the Canvas-Overview to see which of the exercises can be done in association with the Lectures.

Exercise V-1



On the surfaces of a block of linear elastic, isotropic material with modulus $E = 2.10^{11}$ Pa and Poisson ratio $\nu = 0.25$ act the sketched stresses (with units MPa). The geometry is $\mathrm{OA} = \mathrm{OB} = a$ and $\mathrm{OC} = \sqrt{2}a/2$ with base length a. In a linear stress state, the only non-zero eigen-stress should not exceed 150 MPa.

Questions:

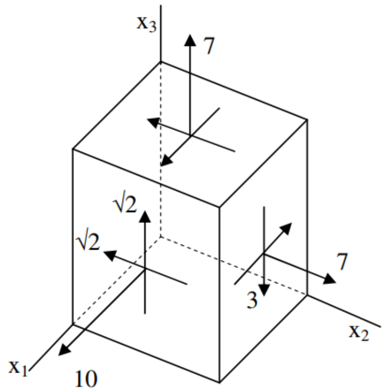
a. What are the normal and shear stresses on the surface ABC?

b. What are the components of the strain-tensor ε_{ij}?

c. What are the eigen-strains?

d. Is this stress state allowed according to the hypotheses of Tresca and von Mises?

Exercise V-2



On the surfaces of a cube of linear elastic, isotropic material (modulus $E = 2.10^5$ N/mm^2) the sketched stresses (in units N/mm^2) are measured. One of the eigen-stresses is known as 8 N/mm^2.

Questions:

a. What are the other eigen-stresses?

b. What are the eigen-directions? and plot these in a graph.

c. What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?

d. What are the equivalent stresses according to the hypotheses of Tresca and von Mises?

Exercise V- 3

The stress state in a point inside a volume of linear elastic, isotropic material (with modulus $E = 2\,10^5$ N/mm^2 and Poisson ratio $\nu = 0.25$) is described by the stress tensor:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix} \text{ MPa}$$

with respect to the (Cartesian) coordinate system x_i.

Questions:

a. What are the eigen-stresses?

b. What are the eigen-directions that also form a new coordinate system x_p'?

c. What is the maximal shear stress?

d. Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x_p'.

e. Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.

f. What is the strain in the direction of the normal vector from question d.

Exercise V-4

At a point P in a linear elastic, isotropic (modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$) body the stress-tensor is given by:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} \text{ MPa}$$

Questions:

a. Show that the eigen-stresses are 8, 16, and 24 MPa.

Determine the direction-cosinus (transformation) matrix entries for the smallest eigen-value.

b. Compute the volumetric (isotropic) strain.

c. What is the largest change of angle γ in point P?

d. Which material property is implicitly used in Hookes law?

Exercise V-5

In a Cartesian coordinate system, at point P, the strain tensor is given as:

$$\left[\varepsilon_{ij}\right] = \frac{5}{8} \begin{bmatrix} -1 & -15 & 5\sqrt{2} \\ -15 & -1 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 14 \end{bmatrix} \cdot 10^{-5}$$

Questions:

a. For a material with modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$, compute the eigen-stresses and the eigen-directions.

b. Explain/argue why the eigen-directions of stress and strain are identical for a homogeneous, isotropic material.

Exercise V-6

A construction made of an elastic, isotropic material (with properties $E = 2.10^5$ N/mm^2, Poisson ratio $\nu = 0.25$, and maximally allowed stress: 160 N/mm^2) is loaded by a force $F = 56$ kN. In a point P on the non-loaded surface, the following strains are measured:

$\varepsilon_{11} = 130.10^{-6}$, $\varepsilon_{22} = -70.10^{-6}$, $\gamma_{12} = 346,4.10^{-6}$,

where the $x_1 - x_2-$plane represents the surface/plane in point P.

Questions:

a. What is the strain component ε_{33} in point P?

b. Compute the stresses in point P.

c. What is the maximal value up to which the force F can be increased according to the criterion of Tresca?

d. ... and according to the criterion of von Mises?

Exercise V-7

Given is the displacement-field:
$u_1 = x_1 x_3$, $u_2 = -x_1 x_2$ and $u_3 = x_1^2 - x_3^2$
and material properties $E = 2$ (discuss the units, but drop them in calculatons to save space) and $\nu = 0.25$.
Questions:
a. Compute the stress tensor (components).
b. In the point $(x_1, x_2, x_3) - (0, 0, z_0)$ compute the eigen-stresses and maximal shear stress.

Exercise V-8

In a homogeneous body that is made of a linear elastic, isotropic material, the displacement field is given as:

$$u_1 = \frac{p}{E} a \left[\frac{x_2}{a} + 2\frac{x_1 x_2}{a^2} - \frac{x_2^2}{a^2} \right]$$

$$u_2 = \frac{p}{E} a \left[\frac{x_1}{a} + \alpha \frac{x_1^2}{a^2} + \beta \frac{x_1 x_2}{a^2} - 2\frac{x_2^2}{a^2} \right]$$

$$u_3 = 0$$

with coordinates x_1 and x_2, and variables p, E, a, with $\nu = 0.25$.
Questions:
In absence of volume forces, compute the magnitude of the parameters α and β using the information that the stress field is in mechanical equilibrium.

Exercise V-9

Within a homogeneous body made of a linear elastic, isotropic material the displacement field:

$$u_1 = \frac{1}{3}(1 - 2\nu)x_1^3 - (3 - 2\nu)x_1 x_2^2 - 3x_2 - 3x_3$$

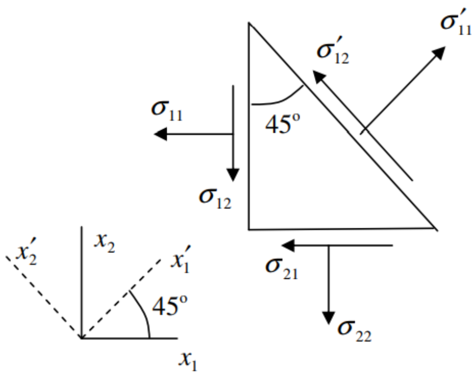
$$u_2 = (1 - 2\nu)x_1^2 x_2 + \frac{1}{3}(1 + 2\nu)x_2^3 + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

the elasticity modulus E, and the Poisson-ratio ν are given.
Questions:
a. Compute the components of the strain tensor.
b. Compute the components of the stress tensor.
c. Confirm that the stress-field is conform with the stress-equilibrium conditions in absence of volume forces.

Exercise V-10

In point P in a linear elastic ($E = 2.10^5$ MPa and $\nu = 0.25$) body under load, we have a plane-stress state with: $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The known (measured) stress components are $\sigma_{11} = 92$, $\sigma'_{11} = 194$ and $\sigma'_{12} = -42$ MPa, where the primes denote quantities in the new coordinate system.



Questions:

a. Give the stress tensor in the original $x_1 - x_2 - x_3-$coordinate system.

b. Give the stress tensor in the $x'_1 - x'_2 - x_3-$coordinate system, as obtained by a rotation of the coordinates about 45^o around the x_3-axis, as sketched above.

c. Compute the eigen-stresses and the eigen-directions.

d. Give the strain tensor in the $x'_1 x'_2 x_3$ coordinate system.

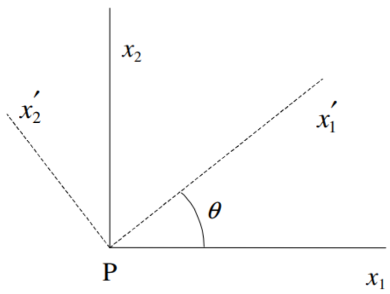
e. Compute the specific elastic energy in point P.

Related, useful formulas:

$$\begin{aligned}
\sigma'_{pq} &= R_{pi} R_{qj} \sigma_{ij} \\
R_{ij} &= \cos\left(x'_i, x_j\right) \\
\varepsilon_{ij} &= \frac{1}{E}\big((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\big) \\
\pi_{el} &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij}
\end{aligned}$$

Exercise V-11

At a non-loaded point P on the surface of a loaded body/construction, three normal strains are measured inside the plane parallel to the free surface, as: $\varepsilon_{11} = 750 \cdot 10^{-6}$, $\varepsilon'_{11} = 150 \cdot 10^{-6}$, and $\varepsilon_{22} = 150 \cdot 10^{-6}$. The angle between the old x_1 and new x'_1 axes is $\theta = \arctan(3/4)$, as sketched below. The material is linear elastic and isotropic with modulus of Young $E = 2.10^5$ MPa and Poisson ratio $\nu = 1/3$.



Questions:

a. Show that one strain component is $\varepsilon_{12} = -400 \cdot 10^{-6}$.

b. Why are the stress components $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$?

c. Show that the components of the stress tensor in the $x_1 - x_2 - x_3-$coordinate system are:

$\left[\sigma_{ij}\right] = \begin{bmatrix} 180 & -60 & 0 \\ -60 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa, using Hookes law. $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right)$.

d. Compute the remaining components of the strain tensor and place them in similar matrix form.

e. Compute the eigen-stresses and determine the equivalent stresses according to Tresca and von Mises. Which criterion is safer?

f. What is the specific elastic energy in point P?

Also determine the deviatoric stress tensor and the consequent specific energy related to changes of shape. Finally determine the specific energy related to volume changes ε_V and hydrostatic stress σ_m, and compare the three values. Are the results consistent? Discuss or explain.

Related, useful formulas:

$$\varepsilon'_{pq} = R_{pi}R_{qj}\varepsilon_{ij}; \quad \sigma_m = \frac{1}{3}\sigma_{kk}; \quad \sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}; \quad \varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$\varepsilon_V = \varepsilon_{kk}; \quad \pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}; \quad \pi_{el_{vol}} = \frac{1}{2}\sigma_m\varepsilon_V; \quad \pi_{el_{dev}} = \frac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}$$

Exercise V-12

In a linear elastic ($E = 2.10^5\,$MPa and $\nu = 1/4$) body under load, the strain-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3-$coordinate system as:

$$\sigma_{11}(x_1,x_2,x_3) = \sigma_0\left[20 + \alpha_1\left(\frac{x_1}{L}\right) - 10\left(\frac{x_2}{L}\right) + \alpha_2\left(\frac{x_1}{L}\right)^2\right]$$

$$\sigma_{22}(x_1,x_2,x_3) = \sigma_0\left[10 + 8\left(\frac{x_1}{L}\right) + \beta_1\left(\frac{x_2}{L}\right) + \beta_2\left(\frac{x_2}{L}\right)^2\right]$$

$$\sigma_{12}(x_1,x_2,x_3) = \sigma_0\left[12 - 10\left(\frac{x_1}{L}\right) + 7\left(\frac{x_2}{L}\right) - 8\left(\frac{x_1}{L}\right)\left(\frac{x_2}{L}\right)\right]$$

$$\sigma_{13}(x_1,x_2,x_3) = \sigma_{23}(x_1,x_2,x_3) = \sigma_{33}(x_1,x_2,x_3) = 0$$

with reference stress $\sigma_0 = 1\,$MPa and reference length $L = 1\,$m. Note that all stresses are independent on x_3 and that the calculation in question (a) below is general with variables x_1, x_2, and x_3; from question (b) on, use the point P ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$).

Questions:

a. Does the displacement field agree with the stress-equilibrium equations in absence of volume-forces?
Which relations have to be valid for the four free parameters α_1, α_2, β_1 and β_2 due to stress equilibrium.

b. Compute the eigen-stresses in point P using linear algebra mathematics – not the circle of Mohr.
Describe and name the state of stress in point P (and in all other points in the body).

c. Compute the eigen-direction of the major eigen-stress.

d. Draw the relevent circle of Mohr and confirm graphically the results of (b) and (c); explain.

e. Compute the equivalent stress according to Tresca.
What is the origin of the limit-stress hypothesis of Tresca?

f. Compute the equivalent stress according to von Mises.
What is the origin of the limit-stress hypothesis of von Mises?

g. Compute the specific elastic energy π_{el} in point P.

Related, useful formulas:

$$\varepsilon_{ij} = \frac{1}{E}\big((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\big); \quad \pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}; \quad \sigma_{ij,j} + f_i = 0\ ,$$

and, von Mises: $\sigma_{eq} = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}}$

Exercise V-13

In a certain point P, the stress tensor: $\left[\sigma_{ij}\right] = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{bmatrix}$ MPa
describes the stress-state in a loaded body in the x_i coordinate system. The material is linear elastic and isotropic with material parameters $E = 200$ GPa and $\nu = 0.25$.

Questions:

a. Explain what "isotropic" material behavior means.

b. Explain what "elastic" material behavior means.

c. Explain what "linear elastic" material behavior means.

d. Compute the eigen-stresses.

e. Draw the circle of Mohr for this stress-state and compare the mathematical and graphical solution.

f. Compute the directional cosines for the minor (smallest) eigen-stress.

g. Compute the components of the strain-tensor ε_{ij} in point P.

h. Compute the volumetric strain ε_V.

i. What is the largest change of angle in point P.

V.14

a) Sketch de stress-strain relation for a linear, elastic material, and
b) add possible non-linear material behavior (with explanation/motivation).
c) Explain what happens for unloading of c1) a linear, elastic material, of c2) a elastic-plastic material (for small AND for large strains).
d) Sketch the relation of shear-stress versus strain- or deformation-rate, for d1) a lineair, d2) a shear-thickening, d3) a shear-thinning, or d4) a yield-stress-fluid.

V.15

Given is a wire (length *L*=0.1m, cross-section *HW,* volume *V=LHW*) for a homogeneous, elastic, isotropic, rubber-like material. What is the work necessary to quickly (or very slowly) stretch the wirefrom stress 0 to length 3*L*. Which strain-rate is needed for making the elastic and the viscous contribution equaly important?

Material-properties:
Kevin-Voigt viscoelastic solid (http://en.wikipedia.org/wiki/Viscoelasticity):

relation for stress = function of strain and strain-rate:
$\sigma = E\varepsilon + \eta\dot{\varepsilon}$ with modulus of Young *E*=0.02 MPa and viscosity η = 10 Pa s.