Chapter 4

Momentum Conservation

4.1 Forces and stresses in fluids

If we place an object in a flow field (for example wind or streaming water) we know that the object experiences a force. This force has a magnitude and it has a direction. Part of the force always points in the downstream direction, that part is called drag. The remaining part points in a direction perpendicular to the downstream direction, that part is called lift.

Both drag and lift are the result of stresses on the object's surface that are generated by the flow. At any point of the surface, the stress by the fluid on the object's surface is a vector, with both a magnitude and a direction. The part of the stress that points in the normal direction of the surface is called normal stress. The remaining part of the stress points in tangential direction of the surface is called tangential stress, but more often the shear stress.

Let us zoom in at a point on the surface of the object, see Fig. (4.1). The lengths of the normal and tangential components of the stress vector determine its length:

$$\|t\|^2 = t_n^2 + t_t^2. \tag{4.1}$$

Moreover, the length of the normal component, t_n, can easily be found from

$$t_n = \boldsymbol{t} \cdot \boldsymbol{n}. \tag{4.2}$$

Normal stress The question raised here is whether we can compute the stress vector when we know the flow field. Let us start with the most simple flow field that we know: no flow at all. We all know that in that case there is no shear stress. This is very different from the situation that we have when a solid exerts a force on another solid, think of standing still on a roof: you're hopefully not sliding down due to the shear stress exerted by the roof tiles on your shoes. So, if there is no flow, the only stress that is exerted on the surface is normal stress: pressure! Hence, in the case of no flow, we have

$$\boldsymbol{t} = -p\boldsymbol{n}. \tag{4.3}$$

Note that we have used a minus sign to indicate that the normal stress due to pressure is positive in the direction opposite to the outward unit normal.

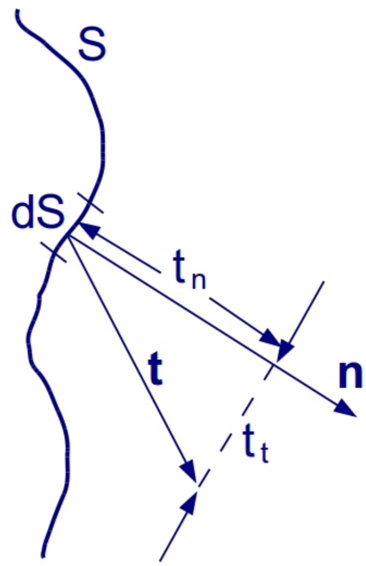

Figure 4.1: Outward unit normal vector $\boldsymbol{n}$, stress vector $\boldsymbol{t}$, normal stress t_n, and shear stress t_t at a point on the surface S of an object.

Shear stress Consider the linear shear layer flow between two plates depicted in Fig. (4.2). The velocity u in x-direction does not depend on x, but it linearly depends on y:

$$u(y) = \frac{y}{2b} U, \tag{4.4}$$

where $2b$ is the gap width between the two plates and U is the velocity of the top plate. Experimentally it has been determined that the shear stress τ experienced by both plates is

$$\tau = \mu \frac{U}{2b}, \tag{4.5}$$

where μ is a fluid-depending constant called the viscosity coefficient. Typical values are $\mu = 18 \times 10^{-6} \frac{Ns}{m^2}$ for air, $\mu = 1 \times 10^{-3} \frac{Ns}{m^2}$ for water, and $\mu = 81 \times 10^{-3} \frac{Ns}{m^2}$ for olive oil.

More generally, one can write

$$\tau = \mu \frac{\partial u}{\partial y} \tag{4.6}$$

4.2 Cauchy stress tensor

The ideas about normal stress and shear stress shown in the previous section can be extended towards a general expression for the stresses that occur in fluid flow. To this end we write the stress vector $\boldsymbol{t}$ that acts on an imaginary surface in the flow as

$$\boldsymbol{t} = S\boldsymbol{n}, \quad S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{4.7}$$

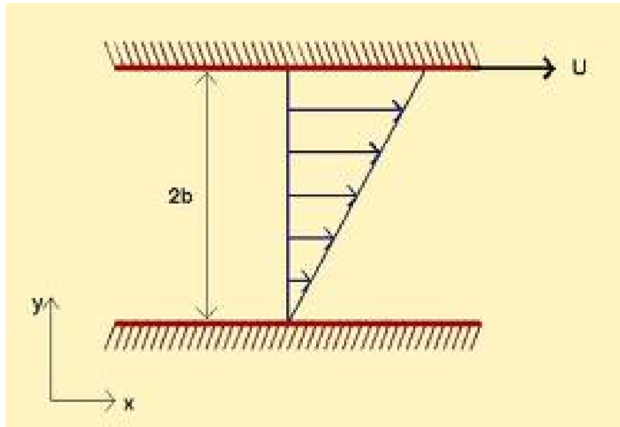

Figure 4.2: Linear shear layer between two plates, one of wich is moving.

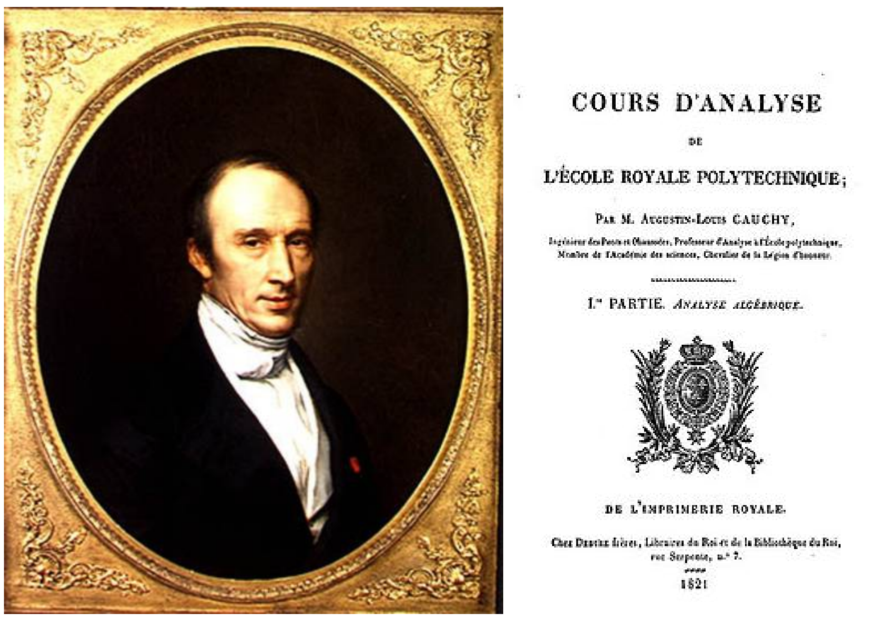

Figure 4.3: Baron Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner, rejecting the heuristic principle of the generality of algebra exploited by earlier authors. He defined continuity in terms of infinitesimals and gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. A profound mathematician, Cauchy exercised a great influence over his contemporaries and successors. His writings cover the entire range of mathematics and mathematical physics.

where $\boldsymbol{n}$ is the normal vector on the surface and the matrix S is called the Cauchy stress tensor.

The following convention is adopted: label the two adjacent sides of the surface "A" and "B". Then to calculate the stress vector caused by "A" on "B" is found by choosing the normal vector to point to "A" and to calculate $\boldsymbol{t} = S\boldsymbol{n}$. In contrast, the stress vector caused by "B" on "A" is found by choosing the normal vector to point to "B" and to calculate $\boldsymbol{t} = S\boldsymbol{n}$. Since the normal vector pointing to "A" and the normal vector pointing to "b" are each others opposite, the resulting stress vectors are also each others opposite. This observation completely agrees with Newton's third law: when an object "A" acts on another object, the other object reacts equally but in opposite direction.

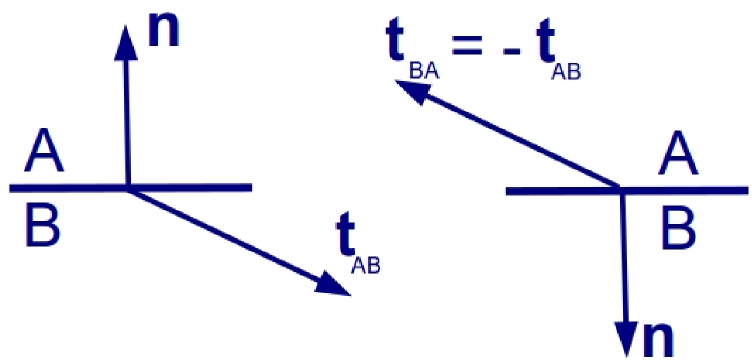

Figure 4.4: Cauchy relation convention

In index notation the stress vector is given by

$$\boxed{t_i = \sigma_{ij} n_j,} \tag{4.8}$$

which is called the Cauchy stress relation. For so-called Newtonian fluids the stress tensor is given by

$$S = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \tag{4.9}$$

or, in index notation:

$$\boxed{\sigma_{ij} = -p\delta_{ij} + \tau_{ij},} \tag{4.10}$$

with

$$\boxed{\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}.} \tag{4.11}$$

No flow or inviscid flow It is easily verified that when there is no flow, i.e. $\boldsymbol{u} = 0$, the stress relation reduces to

$$t_i = -p\delta_{ij}n_j = -pn_i, \tag{4.12}$$

which agrees with the findings above and confirms that the normal stress is pointing towards the surface and that there is no shear stress. The same reult is obtained when there is flow but the fluid is inviscid, i.e., it has a negligible viscosity ("$\mu = 0$").

Stress vector on three elemental surfaces To interpret the Cauchy stress vector let us analyse what the stress vector is on a surface aligned with the $y - z$ plane with normal vector pointing in the positive x-direction:

$$\boldsymbol{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{t} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}. \tag{4.13}$$

So, the first column of the stress tensor represents the stress vector that would act on a surface with the normal vector pointing in the positive x-direction. Similar statements can be made about the second and third columns of the stress tensor:

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{t} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}, \quad \boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{t} = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}. \tag{4.14}$$

4.3 Newton's law

Figure 4.5: Sir Isaac Newton (1642-1727) was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian. His monograph Philosophi Naturalis Principia Mathematica, published in 1687, lays the foundations for most of classical mechanics. In this work, Newton described universal gravitation and the three laws of motion, which dominated the scientific view of the physical universe for the next three centuries.

The momentum of a point particle is governed by Newton's second law. Let the mass of the particle be m, its velocity $\boldsymbol{v}(t)$, and let a force $\boldsymbol{F}(t)$ work on the particle. Then Newton's second law states that

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}(t). \tag{4.15}$$

Note that these are in fact three equations which becomes evident when we rewrite in index notation:

$$m\frac{dv_i}{dt} = F_i(t), \quad i = 1, 2, 3. \tag{4.16}$$

The question is: how can we apply this equation to a convected blob of fluid?

Assuming that the mass of the particle is constant, the equations can also be written as

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}(t) \quad \text{or} \quad \frac{d}{dt}(mv_i) = F_i(t), \quad i = 1, 2, 3. \tag{4.17}$$

Hence, the above form states that the time derivative of the momentum of a particle, $m\boldsymbol{v}$, is equal to the force acting on the particle.

4.4 Momentum conservation: integral formulation

Newton's law can be translated to a convected blob of fluid: the time derivative of the momentum of a convected blob is equal to the force acting on the blob. So then we are left with two questions: what is the momentum of the blob, and what is the force acting on the blob?

If the blob is tiny with volume ΔV, the momentum can simply be approximated by $\rho\boldsymbol{u}\Delta V$, where $\rho\boldsymbol{u}$ is the momentum density. Just like the developments with mass conservation, the momentum of the large blob can be obtained by summation of N of these tiny-blob contributions, and in the limit of $\Delta V \to 0$ and $N \to \infty$ we obtain the blob momentum as an integral:

$$\boldsymbol{M}(t) = \int_{V(t)} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t)dV. \tag{4.18}$$

The force on the blob consists of two contributions:

(a) a force $\boldsymbol{F}_S$ acting on the surface,
(b) a force $\boldsymbol{F}_V$ acting in the interior.

The force acting on the surface is caused by the surrounding fluid (or solid). On a little surface element dS we have

$$d\boldsymbol{F}_S = \boldsymbol{t}dS, \tag{4.19}$$

where $\boldsymbol{t}$ is the stress vector generated by the surroundings. The total surface force is obtained by integration:

$$\boldsymbol{F}_S = \int_{S(t)} \boldsymbol{t}dS. \tag{4.20}$$

The force acting in the interior is caused by the gravity. On a little volume element dV with mass ρdV we have

$$d\boldsymbol{F}_V = \rho\boldsymbol{g}dV, \tag{4.21}$$

where $\boldsymbol{g}$ is the gravity vector with length g, earth's gravitational constant. The total volume force is obtained by integration:

$$\boldsymbol{F}_V = \int_{V(t)} \rho \boldsymbol{g} dV. \tag{4.22}$$

Hence,

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{F} \quad \Rightarrow \quad \boxed{\frac{d}{dt}\int_{V(t)} \rho \boldsymbol{u} dV = \int_{S(t)} \boldsymbol{t} dS + \int_{V(t)} \rho \boldsymbol{g} dV} \tag{4.23}$$

Note that this momentum conservation statement is a fundamental physical statement, it cannot be proved! It has been observed again and again, it has never been observed that momentum was not conserved.

Finally, by means of the Reynolds transport theorem we obtain the integral formulation of momentum conservation(s):

$$\boxed{\boxed{\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{S(t)} \rho u_i u_j n_j dS = \int_{S(t)} t_i dS + \int_{V(t)} \rho g_i dV, \quad i = 1, 2, 3.}} \tag{4.24}$$

In summary:

(a) the first integral expresses the momentum rate of change due to the momentum density rate of change,
(b) the second integral expresses the momentum rate of change due the growth rate of the blob,
(c) the third integral expresses the momentum rate of change due the surface force, and
(d) the fourth integral expresses the momentum rate of change due the volume force.

4.5 Force by fluid on construction

The integral formulation of momentum conservation is particularly useful to calculate forces on constructions due to fluid flow. As an example consider the typical problem of a bend in a piping system as depicted in Fig. (4.7). The cross section areas at the entrance and exit are A_1 and A_2, respectively, the mean velocities at the entrance and at the exit are $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, and the pressures at the entrance and exit are p_1 and p_2, respectively. The flow is steady, the fluid mass density ρ is assumed constant, and it is assumed that viscosity and gravity effects can be neglected. The question raised here is: what is the force exerted by the fluid on the bend?

To answer this question, we construct a blob in the bend, indicated by the dashed line, and apply the integral formulation of momentum conservation (Eq.(4.24)). Since the flow is steady the first integral is zero, and since gravity effects can be neglected the last integral is also zero:

$$\int_{S(t)} \rho u_i u_j n_j dS = \int_{S(t)} t_i dS, \quad i = 1, 2, 3. \tag{4.25}$$

Figure 4.6: Examples of forces by fluids

At the instant depicted in Fig. (4.7) the surface $S(t)$ of the convected blob can be decomposed into three parts:

$$S(t) = A_1 \cup A_2 \cup A_w, \tag{4.26}$$

where A_w represents the wall of the bend. Therefore

$$\int_{A_1} \rho u_i u_j n_j dS + \int_{A_2} \rho u_i u_j n_j dS + \int_{A_w} \rho u_i u_j n_j dS = \\ \int_{A_1} t_i dS + \int_{A_2} t_i dS + \int_{A_w} t_i dS, \quad i = 1, 2, 3. \tag{4.27}$$

the first integral over A_w is zero because on the wall the normal velocity is zero: $u_j n_j$=0. The second integral over A_w represents the force in i-direction by the wall on the fluid, say

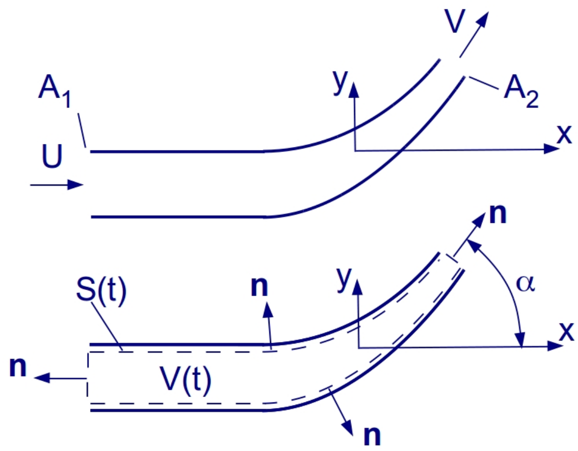

Figure 4.7: Typical problem: bend in a piping system.

F_i. Hence, the momentum equation becomes

$$\int_{A_1} \rho u_i u_j n_j dS + \int_{A_2} \rho u_i u_j n_j dS = \int_{A_1} t_i dS + \int_{A_2} t_i dS + F_i, \quad i = 1, 2, 3. \tag{4.28}$$

The force asked is the force by the fluid on the wall. We can calculate the components of this force by computing $-F_i$ with $i = 1, 2, 3$:

$$-F_i = -\int_{A_1} \rho u_i u_j n_j dS - \int_{A_2} \rho u_i u_j n_j dS + \int_{A_1} t_i dS + \int_{A_2} t_i dS, \quad i = 1, 2, 3. \tag{4.29}$$

The remaining problem is now to compute the four integrals at the right hand side.

We start by identifying the outward unit normal vectors at the entrance and exit:

$$\text{entrance: } \boldsymbol{n} = (-1, 0, 0)^T, \quad \text{exit: } \boldsymbol{n} = (\cos\alpha, \sin\alpha, 0)^T. \tag{4.30}$$

and the velocity vectors at the entrance and exit assuming uniform flow:

$$\text{entrance: } \boldsymbol{u} = (U, 0, 0)^T, \quad \text{exit: } \boldsymbol{u} = (V\cos\alpha, V\sin\alpha, 0)^T. \tag{4.31}$$

From these expressions we derive:

$$\text{entrance: } u_j n_j = -U, \quad \text{exit: } u_j n_j = V. \tag{4.32}$$

Furthermore, we use the Cauchy stress tensor to compute the stress vector $\boldsymbol{t}$ and use $\mu = 0$ since viscosity effects can be neglected:

$$t_i = \sigma_{ij} n_j = -p\delta_{ij} n_j = -p n_i. \tag{4.33}$$

With these expressions the three force components become

$$\begin{aligned}
-F_1 &= -\int_{A_1} \rho U(-U)dS - \int_{A_2} \rho V \cos\alpha (V) dS \\
&\quad + \int_{A_1} -p(-1)dS + \int_{A_2} -p\cos\alpha dS, \\
-F_2 &= -\int_{A_2} \rho V \sin\alpha (V) dS + \int_{A_2} -p\sin\alpha dS, \\
-F_3 &= 0.
\end{aligned} \tag{4.34}$$

which can be evaluated to

$$\begin{aligned}
-F_1 &= \left(\rho U^2 + p\right) A_1 - \left(\rho V^2 + p\right) \cos\alpha A_2, \\
-F_2 &= -\left(\rho V^2 + p\right) \sin\alpha A_2, \\
-F_3 &= 0.
\end{aligned} \tag{4.35}$$

Two standard checks are conducted now to verify that this answer is what we would expect and to trace any errors we may have made:

(a) are all of the terms in the answer of the right physical dimension?
(b) does the answer produce the correct limits?

The first check is easy, and it is left to the reader to show that indeed all of the terms in the answer of the physical dimension of force. The second check requires some insight to create some limiting cases for which we already now the answer. For example:

(a) due to symmetry, F_3 should be zero in all cases: ok.
(b) if $\alpha = 0$, F_2 should be zero: ok.
(c) if $\alpha = \pi$, F_2 should be zero: ok.
(d) if $\alpha = 0$, $A_1 = A_2$, and $U = V$, F_1 should be zero: ok.
(e) if $A_1 = A_2$, and $U = V$, F_1 should increase when U increases: ok.
(f) if $U = V = 0$, only pressure terms should remain: ok.
(g) when α changes sign, F_2 should change sign: ok.

4.6 Derivation of Navier-Stokes equations

We derive the differential formulation of momentum conservation from the integral formulation of momentum conservation Eq.(4.24) by replacing the surface integrals by a volume integral using Gauss' divergence theorem Eq.(3.15):

$$\int_{S(t)} \rho u_i u_j n_j dS = \int_{V(t)} \frac{\partial}{\partial x_j} \left(\rho u_i u_j\right) dV, \tag{4.36}$$

and

$$\int_{S(t)} t_i dS = \int_{S(t)} \sigma_{ij} n_j dS = \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j} dV. \tag{4.37}$$

Note that $\rho u_i u_j$ and σ_{ij} take the place of u_j in Eq.(3.15). With this replacement the integral form Eq.(4.24) becomes one single volume integral:

$$\int_{V(t)} \left(\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i \right) dV = 0, \quad i = 1, 2, 3. \tag{4.38}$$

Since the blob V was chosen completely arbitrary, this holds for any blob. This means that

$$\boxed{\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i = 0, \quad \text{for all } (\boldsymbol{x}, t)} \tag{4.39}$$

This the differential formulation of momentum conservation which is referred to as the Navier-Stokes equations.

4.7 Exercises

Problem 4.1. *By using the integral formulation of momentum conservation, show that the law of Archimedes (287 BC - 212 BC) holds: in water which is not flowing the (upward) force on a blob of water by the surrounding water is equal to the (downward) gravity force on the blob.*

Problem 4.2. *An incompressible fluid flows steadily into a T-junction of diameter D at uniform velocity U, at the opposite outlet the fluid leaves at uniform velocity V. At the lateral exit the flow leaves at unknown uniform velocity. The pressure in the T-junction is uniform: p. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.3. *An incompressible fluid flows steadily into a pipe of diameter D at uniform velocity U and pressure p_1. At the end of the pipe is a contraction of diameter d, and the fluid leaves the contraction at uniform velocity V and pressure p_2. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.4. *Incompressible water is flowing steadily through a $180°$ elbow. At the inlet the pressure is p_1 and the cross section area is A_1, at the outlet the pressure is p_2 and the cross section area is A_2. The averaged velocity at the inlet is V_1. Find the horizontal component of the force by the fluid on the elbow, neglecting viscosity and gravity.*

Problem 4.5. *An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h$ and width w. At the entrance the presure is $p1$ and the uniform velocity is U_1. At the exit the pressure is p_2 and the velocity distribution is*

$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2. \tag{4.40}$$