Tutorial T02 – Elasticity – Stress basics

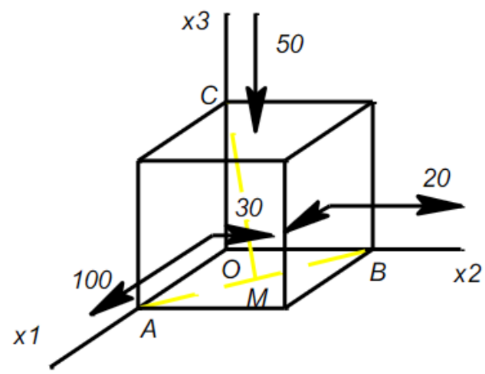
November 25, 2024

Answer the following questions as they could come up in an exam.

1 Stress basics (geometry and stress vector)

... based on section 3.1 (Exercise V1 in old material before 2022)

Given:
$E = 200\,\text{GPa}$, $\nu = 0.25$
$OA = OB = a$ and $OC = \frac{1}{2}\sqrt{2}a$



Questions:

a) Find normal stress σ_{ABC} and shear stress τ_{ABC} acting on the area ABC.

Answers:

a) The stress Tensor is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix}$ MPa,
given the arrows, using symmetry, direction of arrows (sign), and non-existing (zero).

First, find the normal to the plane: *by taking the cross-product of two line vectors (in the plane).*

$$\vec{AC} \times \vec{AB} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalizing the vector using the normality condition ($\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$), one can find b:

$$b^2 \left(1^2 + 1^2 + \sqrt{2}^2\right) = 1 \Longrightarrow b = \frac{1}{2}$$

After using the cross-product, with the vectors in random order we pay close attention to the fact that the normal is facing outside the plane. With the normal you indicate which side the material is. To make the normal point away from the material, we can choose a positive.

Cauchy: Stress or traction vector: $p_i = \sigma_{ij} n_j$, so that:

$$\longrightarrow \left[p\right] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \left[\sigma\right] \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

The normal stress on the plane ABC is: $\sigma = \left[\hat{n}\right]^T \cdot \left[p\right] = 20\,\mathrm{MPa}$
The shear stress on the plane ABC, using Pythagoras, is:

$$\tau^2 = p^2 - \sigma^2 = \left[p_1^2 + p_2^2 + p_3^2\right] - \sigma^2 = 6100 - 400 = 5700\,\mathrm{MPa}^2,$$

so that $\tau = 75.5\,\mathrm{MPa}$.

2 Stress tensor basics

... based on sections 3.1-3.3. (Exercise V2 in old material before 2022)

Given:

- Linear elastic isotropic material with modulus $E = 2 \cdot 10^5$ N/mm^2
- The stress cube, below, in units of N/mm^2
- One principal stress is: $8\,\text{N/mm}^2$

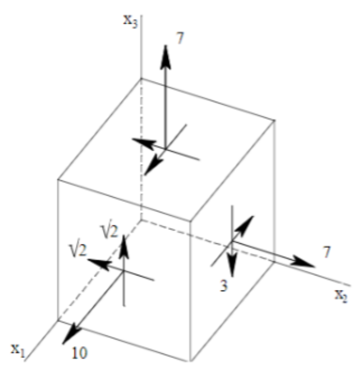


Figure 1: Stress cube → write down the stress matrix

Questions:
a) Find the other principal (eigen) stresses
b) Find the eigen-directions and plot these in a graph.

Answers:
a) The stress tensor from the cube is:

$$[\sigma_{ij}] = \begin{bmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{bmatrix} \text{ MPa}$$

Note that the first index denotes the direction of the normal to the according surface on which this stress component works, while the second index gives the direction of the stress component.

Next, get the characteristic equation from:

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = \begin{bmatrix} 10-\sigma & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7-\sigma & -3 \\ \sqrt{2} & -3 & 7-\sigma \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

with invariants:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 24\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 176\text{MPa}^2$$

$$I_3 = \det(\sigma) = 384\text{MPa}^3$$

Here, the characteristic equation is not easily solvable; one way is to use the given eigen-value, $\sigma = 8\,N/mm^2$, and polynomial division (units dropped for simplicity, but must be added for final answer). Take the characteristic equation and divide by $(\sigma - 8)$:

$$(\sigma^3 - 24\sigma^2 + 176\sigma - 384)\backslash(\sigma - 8) = \sigma^2 - 16\sigma + 48$$

$$\underline{(\sigma^3 - 8\sigma^2)}$$

$$-16\sigma^2 + 176\sigma - 384$$

$$\underline{(-16\sigma^2 + 128\sigma)}$$

$$+\ 48\sigma - 384$$

$$\underline{(+\ 48\sigma - 384)}$$

$$\%$$

The result is a second order polynomial, which can be solved as:
$\sigma_{1,2} = (16 \pm \sqrt{16^2 - 4 \times 48})/2 = 12$ and $4\,\text{MPa}$.

Therefore, the sorted eigen-values are: $\sigma_I = 12\,\text{MPa}$, $\sigma_{II} = 8\,\text{MPa}$, $\sigma_{III} = 4\,\text{MPa}$.
(Subscripts as in σ_{III} are used to appear different from σ_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

b) Direction of $\sigma_I = 12\text{MPa}$
There are various ways to solve for eigen-vectors, here is one example ...
Insert values, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$-2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - 5n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - 5n_3 = 0$$

The eigen-direction associated to the first, largest eigen-value:

$$\Longrightarrow \hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

Direction of $\sigma_{II} = 8\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - n_3 = 0$$

The eigen-direction associated to the second, intermediate eigen-value:

$$\Longrightarrow \hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$$

Direction of $\sigma_{III} = 4\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$6n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 + 3n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 + 3n_3 = 0$$

The eigen-direction associated to the third, smallest eigen-value:

$$\Longrightarrow \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

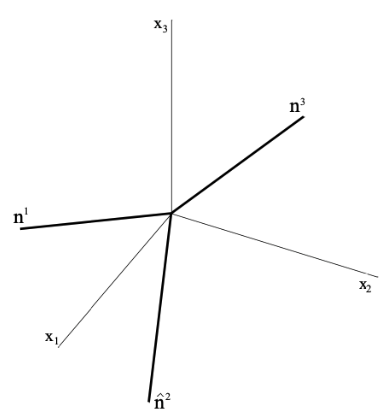


Figure 2: Sketch of the eigen-vectors (with coordinate axes) in bold.

The directions are unspecified, indicated by the plus-minus from taking a square-root; all three direction vectors are normalized (check it, if enough time in exam), $(n_i)^2 = 1$*; furthermore, all three normal (eigen) vectors must be pair-wise perpendicular on each other, i.e.* $n_i^{(a)} n_i^{(b)} = 0$*, for all* $a, b = I, II, III$ *with* $a \neq b$*. This perpendicularity allows to obtain, alternatively, one eigen-vector by a cross-product, e.g. above* $\hat{n}^{(III)} = \hat{n}^{(I)} \times \hat{n}^{(II)}$.

3 Stress tensor basics

... based on section 3,4 (Exercise V3 in old material before 2022)

Given:

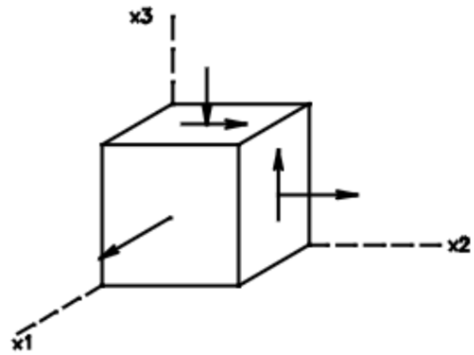


Figure 3: Stress cube, empty → fill it

The stress-state is described by the matrix: $\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix}$ N/mm^2,
with $E = 2 \cdot 10^5$ N/mm^2, and $\nu = 0.25$.

Questions:
a) Compute the principal stresses
b) Compute the eigen-directions
c) Compute the maximal shear-stress

Answers:
a)
The sorted eigen-values are: $\sigma_I = 60\,\text{MPa}$, $\sigma_{II} = 40\,\text{MPa}$, $\sigma_{III} = -40\,\text{MPa}$.
The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).

b)
Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example σ_{II}, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad n_1^2+n_2^2+n_3^2=1$$

$-20n_2 + 20\sqrt{3}n_3 = 0$

$20\sqrt{3}n_2 - 60n_3 = 0$

$n_2 = \sqrt{3}n_3$ and $n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3$ (identical due to dependency)

$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$

$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above.
The third eigenvalue calculation is similar (not shown).

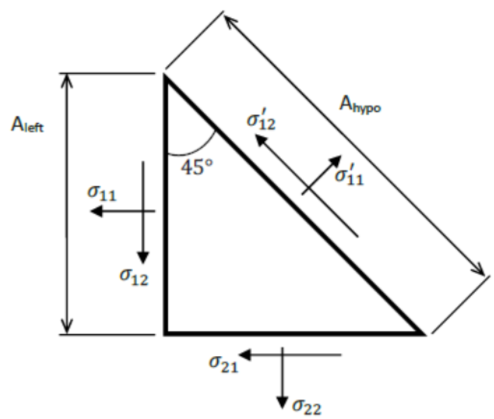
c)
The maximum shear stress is: $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50$ MPa.

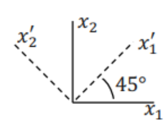
4 Stress tensor and transformation

... based on sections 3.1-3.4. (Exercise V10 in old material before 2022)

Given:

- A plane-stress state in a point P of a body with $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Given are these (mixed) stress components:
  $\sigma_{11} = 92$ MPa
  $\sigma'_{11} = 194$ MPa
  $\sigma'_{12} = -42$ MPa
  where the prime indicates the new (transformed) coordinate system.
- The material is linear elastic with $E = 2 \cdot 10^5$ MPa and $\nu = 0.25$.





Questions:
a) Give the stress tensor in the original $x_1x_2x_3$ system.
b) Give the stress tensor in the new $x'_1x'_2x'_3$ coordinate system, as obtained by a rotation of the coordinates about 45° around the x_3-axis, as sketched above.
c) Compute the eigen-stresses and the eigen-directions.

<u>**Answers:**</u>
a)
There are two ways to solve this problem. The triangle given represents all stresses on all sides, but only part of the stress components are known. By considering force equilibrium and using the respective stress components, divided by the side-lengths of the triangle (which also has a third dimension outside the plane, not shown). Assume the sides have unit-length, then the hypotenuse has, according to Pythagoras, length $\sqrt{2}$. Further assume the thickness also to be unit-length. The ratio between sides and hypotenuse is then:

$$\frac{A_l}{A_h} := \frac{A_{left}}{A_{hypo}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

With this we get:
Force balance in x_1 direction:

$$\begin{aligned}
A_h\sigma'_{11}\cos(45^o) - A_h\sigma'_{12}\sin(45^o) - A_l\sigma_{11} - A_l\sigma_{12} = 0 \\
\Rightarrow \sigma'_{11}\cos(45^o) - \sigma'_{12}\sin(45^o) - \frac{A_l}{A_h}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow (\sigma'_{11} - \sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = \sigma'_{11} - \sigma'_{12} - \sigma_{11} \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = 194 - (-42) - 92 = 144\,\text{MPa}.
\end{aligned}$$

Force balance in x_2 direction:

$$\begin{aligned}
A_h\sigma'_{11}\sin(45^o) + A_h\sigma'_{12}\cos(45^o) - A_l\sigma_{12} - A_l\sigma_{22} = 0 \\
\Rightarrow \sigma'_{11}\sin(45^o) + \sigma'_{12}\cos(45^o) - \frac{A_l}{A_h}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow (\sigma'_{11} + \sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow \sigma_{22} = \sigma'_{11} + \sigma'_{12} - \sigma_{12} \\
\Rightarrow \sigma_{22} = 194 + (-42) - 144 = 8\,\text{MPa}.
\end{aligned}$$

The stress tensor in the $x_1x_2x_3$ system is thus:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

b)
The stress tensor in the $x'_1x'_2x'_3$ system is obtained by rotation of the original system around 45^o, as sketched, in index notation, $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$, or:

$$\left[\sigma'\right] = \left[R\right]\left[\sigma\right]\left[R^T\right] = \begin{bmatrix} 194 & -42 & 0 \\ -42 & -94 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa},$$

using the transformation matrix:

$$\left[R\right] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The *alternative* way is to use the symbolic transformation rule and solve the system of equations, for each component, for the unknowns σ_{12}, σ_{22}, and σ'_{22}.

$$\left[\sigma'\right] = \left[R\right]\left[\sigma\right]\left[R^T\right] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in units of MPa, which allows to solve without geometry and force balance, after matrix multiplications:

$$\begin{bmatrix}\sigma'\end{bmatrix} = (1/\sqrt{2}) \begin{bmatrix} 92+\sigma_{12} & \sigma_{12}+\sigma_{22} & 0 \\ -92+\sigma_{12} & -\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= (1/2) \begin{bmatrix} 92+2\sigma_{12}+\sigma_{22} & -92+\sigma_{22} & 0 \\ -92+\sigma_{22} & 92-2\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in units of MPa, which yields from the non-diagonal 12-component $\sigma_{22} = 8\,\mathrm{MPa}$. Inserted into the 11-component, one finds $\sigma_{12} = 144\,\mathrm{MPa}$, and all inserted into the 22-component results in $\sigma'_{22} = -94\,\mathrm{MPa}$. These results are identical to the above geometry and force balance considerations.

c)
The principal stresses and eigen-directions can now be computed the usual way from

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = 0 \ ,$$

and $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$.

c.1) This stress tensor describes a plane-stress state and thus has one eigenvalue $\sigma = 0$.
The remaining characteristic equation is:

$$\sigma^2 - 100\sigma + 736 - 144^2 = 0$$

with solutions: $\sigma_{1,2} = (100 \pm \sqrt{100^2 - 4(736 - 144^2)})/2 = (100 \pm \sqrt{9.10^4})/2 = 50 \pm 150\,\mathrm{MPa}$.

The sorted eigen-values are thus: $\sigma_I = 200\,\mathrm{MPa}$, $\sigma_{II} = 0\,\mathrm{MPa}$, $\sigma_{III} = -100\,\mathrm{MPa}$.

c.2) *Insert values, solve the system of equations, and normalize the solution.*

$$\begin{bmatrix} \sigma_{11}-\sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2+n_2^2+n_3^2=1$$

$$\Longrightarrow$$

$$(92-200)n_1+144n_2+0n_3=0$$

$$144n_1+(8-200)n_2+0n_3=0$$

$$\Longrightarrow$$

$$-108n_1+144n_2+0n_3=0$$

$$144n_1-192n_2+0n_3=0$$

$$\Longrightarrow$$

$$n_2=(108/144)n_1=(3/4)n_1$$

$$\Longrightarrow$$

$$n_1^2+n_2^2=(1+9/16)n_1^2=1$$

$$\Longrightarrow$$

$$n_1=\sqrt{16/25}=\pm 4/5=\pm 0.8$$

The eigen-direction associated to the first, largest eigen-value:

$$\hat{n}^{(I)}=\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}=\pm\begin{bmatrix} 0.8 \\ 0.6 \\ 0 \end{bmatrix}$$

Similarly (no details given), the eigen-direction associated to the third, smallest eigen-value:

$$\hat{n}^{(III)}=\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}=\pm\begin{bmatrix} 0.6 \\ -0.8 \\ 0 \end{bmatrix}$$

and without calculation necessary (due to structure of the matrix), for the second, intermediate:

$$\hat{n}^{(II)}=\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}=\pm\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$