Elasticity Theory Exercises

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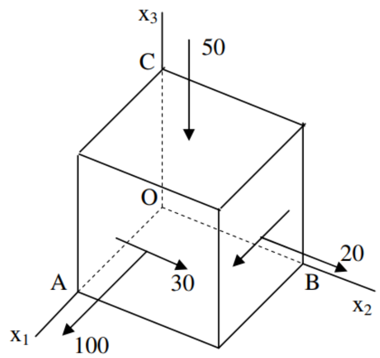
The following Exercises V1-V13 are based on old exams from the last 20 years and allow you to practice all types of calculations taught during the course Elasticity Theory and will be tested during the exam.

See the Canvas-Overview to see which of the exercises can be done in association with the Lectures.

Exercise V-1



On the surfaces of a block of linear elastic, isotropic material with modulus $E = 2.10^{11}$ Pa and Poisson ratio $\nu = 0.25$ act the sketched stresses (with units MPa). The geometry is $\mathrm{OA} = \mathrm{OB} = a$ and $\mathrm{OC} = \sqrt{2}a/2$ with base length a. In a linear stress state, the only non-zero eigen-stress should not exceed 150 MPa.

Questions:

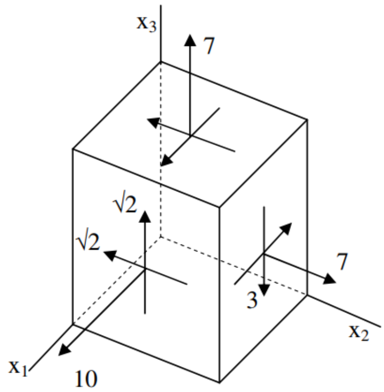
a. What are the normal and shear stresses on the surface ABC?

b. What are the components of the strain-tensor ε_{ij}?

c. What are the eigen-strains?

d. Is this stress state allowed according to the hypotheses of Tresca and von Mises?

Exercise V-2



On the surfaces of a cube of linear elastic, isotropic material (modulus $E = 2.10^5$ N/mm^2) the sketched stresses (in units N/mm^2) are measured. One of the eigen-stresses is known as 8 N/mm^2.

Questions:

a. What are the other eigen-stresses?

b. What are the eigen-directions? and plot these in a graph.

c. What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?

d. What are the equivalent stresses according to the hypotheses of Tresca and von Mises?

Exercise V- 3

The stress state in a point inside a volume of linear elastic, isotropic material (with modulus $E = 2\,10^5$ N/mm^2 and Poisson ratio $\nu = 0.25$) is described by the stress tensor:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix} \text{ MPa}$$

with respect to the (Cartesian) coordinate system x_i.

Questions:

a. What are the eigen-stresses?

b. What are the eigen-directions that also form a new coordinate system x'_p?

c. What is the maximal shear stress?

d. Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x'_p.

e. Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.

f. What is the strain in the direction of the normal vector from question d.

Exercise V-4

At a point P in a linear elastic, isotropic (modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$) body the stress-tensor is given by:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} \text{ MPa}$$

Questions:

a. Show that the eigen-stresses are 8, 16, and 24 MPa.

Determine the direction-cosinus (transformation) matrix entries for the smallest eigen-value.

b. Compute the volumetric (isotropic) strain.

c. What is the largest change of angle γ in point P?

d. Which material property is implicitly used in Hookes law?

Exercise V-5

In a Cartesian coordinate system, at point P, the strain tensor is given as:

$$\left[\varepsilon_{ij}\right] = \tfrac{5}{8} \begin{bmatrix} -1 & -15 & 5\sqrt{2} \\ -15 & -1 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 14 \end{bmatrix} \cdot 10^{-5}$$

Questions:

a. For a material with modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$, compute the eigen-stresses and the eigen-directions.

b. Explain/argue why the eigen-directions of stress and strain are identical for a homogeneous, isotropic material.

Exercise V-6

A construction made of an elastic, isotropic material (with properties $E = 2.10^5$ N/mm^2, Poisson ratio $\nu = 0.25$, and maximally allowed stress: 160 N/mm^2) is loaded by a force $F = 56$ kN. In a point P on the non-loaded surface, the following strains are measured:

$\varepsilon_{11} = 130.10^{-6}$, $\varepsilon_{22} = -70.10^{-6}$, $\gamma_{12} = 346,4.10^{-6}$,

where the $x_1 - x_2-$plane represents the surface/plane in point P.

Questions:

a. What is the strain component ε_{33} in point P?

b. Compute the stresses in point P.

c. What is the maximal value up to which the force F can be increased according to the criterion of Tresca?

d. ... and according to the criterion of von Mises?

Exercise V-7

Given is the displacement-field:
$u_1 = x_1 x_3$, $u_2 = -x_1 x_2$ and $u_3 = x_1^2 - x_3^2$
and material properties $E = 2$ (discuss the units, but drop them in calculatons to save space) and $\nu = 0.25$.
Questions:
a. Compute the stress tensor (components).
b. In the point $(x_1, x_2, x_3) - (0, 0, z_0)$ compute the eigen-stresses and maximal shear stress.

Exercise V-8

In a homogeneous body that is made of a linear elastic, isotropic material, the displacement field is given as:

$$u_1 = \frac{p}{E} a \left[\frac{x_2}{a} + 2\frac{x_1 x_2}{a^2} - \frac{x_2^2}{a^2} \right]$$

$$u_2 = \frac{p}{E} a \left[\frac{x_1}{a} + \alpha \frac{x_1^2}{a^2} + \beta \frac{x_1 x_2}{a^2} - 2\frac{x_2^2}{a^2} \right]$$

$$u_3 = 0$$

with coordinates x_1 and x_2, and variables p, E, a, with $\nu = 0.25$.
Questions:
In absence of volume forces, compute the magnitude of the parameters α and β using the information that the stress field is in mechanical equilibrium.

Exercise V-9

Within a homogeneous body made of a linear elastic, isotropic material the displacement field:

$$u_1 = \frac{1}{3}(1 - 2\nu)x_1^3 - (3 - 2\nu)x_1 x_2^2 - 3x_2 - 3x_3$$

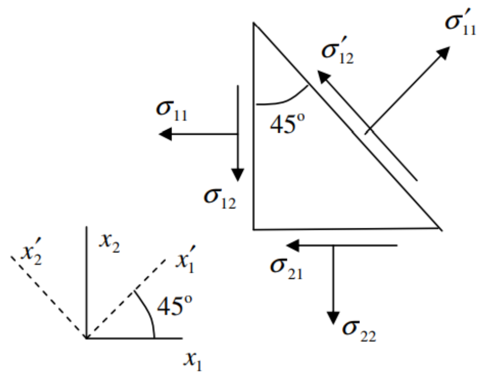
$$u_2 = (1 - 2\nu)x_1^2 x_2 + \frac{1}{3}(1 + 2\nu)x_2^3 + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

the elasticity modulus E, and the Poisson-ratio ν are given.
Questions:
a. Compute the components of the strain tensor.
b. Compute the components of the stress tensor.
c. Confirm that the stress-field is conform with the stress-equilibrium conditions in absence of volume forces.

Exercise V-10

In point P in a linear elastic ($E = 2.10^5$ MPa and $\nu = 0.25$) body under load, we have a plane-stress state with: $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The known (measured) stress components are $\sigma_{11} = 92$, $\sigma'_{11} = 194$ and $\sigma'_{12} = -42$ MPa, where the primes denote quantities in the new coordinate system.



Questions:

a. Give the stress tensor in the original $x_1 - x_2 - x_3$—coordinate system.

b. Give the stress tensor in the $x'_1 - x'_2 - x_3$—coordinate system, as obtained by a rotation of the coordinates about 45^o around the x_3-axis, as sketched above.

c. Compute the eigen-stresses and the eigen-directions.

d. Give the strain tensor in the $x'_1 x'_2 x_3$ coordinate system.

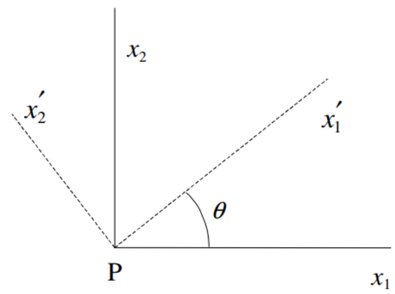
e. Compute the specific elastic energy in point P.

Related, useful formulas:

$$
\begin{aligned}
\sigma'_{pq} &= R_{pi} R_{qj} \sigma_{ij} \\
R_{ij} &= \cos\left(x'_i, x_j\right) \\
\varepsilon_{ij} &= \frac{1}{E}\big((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\big) \\
\pi_{el} &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij}
\end{aligned}
$$

Exercise V-11

At a non-loaded point P on the surface of a loaded body/construction, three normal strains are measured inside the plane parallel to the free surface, as: $\varepsilon_{11} = 750 \cdot 10^{-6}$, $\varepsilon'_{11} = 150 \cdot 10^{-6}$, and $\varepsilon_{22} = 150 \cdot 10^{-6}$. The angle between the old x_1 and new x'_1 axes is $\theta = \arctan(3/4)$, as sketched below. The material is linear elastic and isotropic with modulus of Young $E = 2.10^5$ MPa and Poisson ratio $\nu = 1/3$.



Questions:

a. Show that one strain component is $\varepsilon_{12} = -400 \cdot 10^{-6}$.

b. Why are the stress components $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$?

c. Show that the components of the stress tensor in the $x_1 - x_2 - x_3-$coordinate system are:

$\left[\sigma_{ij}\right] = \begin{bmatrix} 180 & -60 & 0 \\ -60 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa, using Hookes law. $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right)$.

d. Compute the remaining components of the strain tensor and place them in similar matrix form.

e. Compute the eigen-stresses and determine the equivalent stresses according to Tresca and von Mises. Which criterion is safer?

f. What is the specific elastic energy in point P?

Also determine the deviatoric stress tensor and the consequent specific energy related to changes of shape. Finally determine the specific energy related to volume changes ε_V and hydrostatic stress σ_m, and compare the three values. Are the results consistent? Discuss or explain.

Related, useful formulas:

$$\varepsilon'_{pq} = R_{pi}R_{qj}\varepsilon_{ij}; \quad \sigma_m = \frac{1}{3}\sigma_{kk}; \quad \sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}; \quad \varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$\varepsilon_V = \varepsilon_{kk}; \quad \pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}; \quad \pi_{el_{vol}} = \frac{1}{2}\sigma_m\varepsilon_V; \quad \pi_{el_{dev}} = \frac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}$$

Exercise V-12

In a linear elastic ($E = 2.10^5$ MPa and $\nu = 1/4$) body under load, the strain-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3-$coordinate system as:

$$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1 \left(\frac{x_1}{L}\right) - 10 \left(\frac{x_2}{L}\right) + \alpha_2 \left(\frac{x_1}{L}\right)^2 \right]$$

$$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8 \left(\frac{x_1}{L}\right) + \beta_1 \left(\frac{x_2}{L}\right) + \beta_2 \left(\frac{x_2}{L}\right)^2 \right]$$

$$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10 \left(\frac{x_1}{L}\right) + 7 \left(\frac{x_2}{L}\right) - 8 \left(\frac{x_1}{L}\right) \left(\frac{x_2}{L}\right) \right]$$

$$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$$

with reference stress $\sigma_0 = 1$ MPa and reference length $L = 1$ m. Note that all stresses are independent on x_3 and that the calculation in question (a) below is general with variables x_1, x_2, and x_3; from question (b) on, use the point P ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$).

Questions:

a. Does the displacement field agree with the stress-equilibrium equations in absence of volume-forces?
Which relations have to be valid for the four free parameters α_1, α_2, β_1 and β_2 due to stress equilibrium.

b. Compute the eigen-stresses in point P using linear algebra mathematics – not the circle of Mohr.
Describe and name the state of stress in point P (and in all other points in the body).

c. Compute the eigen-direction of the major eigen-stress.

d. Draw the relevent circle of Mohr and confirm graphically the results of (b) and (c); explain.

e. Compute the equivalent stress according to Tresca.
What is the origin of the limit-stress hypothesis of Tresca?

f. Compute the equivalent stress according to von Mises.
What is the origin of the limit-stress hypothesis of von Mises?

g. Compute the specific elastic energy π_{el} in point P.

Related, useful formulas:

$$\varepsilon_{ij} = \frac{1}{E}\big((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\big); \quad \pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}; \quad \sigma_{ij,j} + f_i = 0 \ ,$$

and, von Mises: $\sigma_{eq} = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}}$

Exercise V-13

In a certain point P, the stress tensor: $\left[\sigma_{ij}\right] = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{bmatrix}$ MPa

describes the stress-state in a loaded body in the x_i coordinate system. The material is linear elastic and isotropic with material parameters $E = 200$ GPa and $\nu = 0.25$.

Questions:

a. Explain what "isotropic" material behavior means.

b. Explain what "elastic" material behavior means.

c. Explain what "linear elastic" material behavior means.

d. Compute the eigen-stresses.

e. Draw the circle of Mohr for this stress-state and compare the mathematical and graphical solution.

f. Compute the directional cosines for the minor (smallest) eigen-stress.

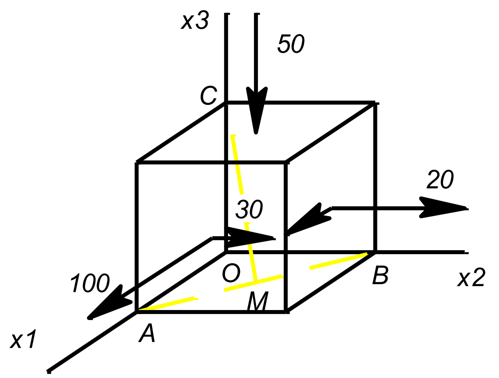
g. Compute the components of the strain-tensor ε_{ij} in point P.

h. Compute the volumetric strain ε_V.

i. What is the largest change of angle in point P.

Exercise V-1

Problem



Given:
$E = 200$ GPa & $\nu = 0.25$

$OA = OB = a$ & $OC = \frac{1}{2}\sqrt{2} \cdot a$

In this stress-state, the maximal principal stress must not be larger than: 150 MPa.

Questions:

a) σ_{ABC} & τ_{ABC},

b) components of the strain tensor ε_{ij}

c) Principal strains

d) Maxima according to:
(1) Tresca and (2) von Mises?

Solutions

a)

Stress tensor, from the sketch: $\sigma = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} MPa$

Compute the normal to the surface ABC: for example by using the cross-product of two vectors inside this plane.

$$\vec{AC} \times \vec{AB} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalisation: $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$: $\quad \alpha^2\left(1^2 + 1^2 + \left(\sqrt{2}\right)^2\right) = 1 \quad \Rightarrow \quad \alpha = \frac{1}{2}$

Comment

After using the cross-product, with vectors in arbitrary order, one must check/confirm that the normal points out of the plane, away from the cube-backside point O, where the material still exists. The normal should point away from the material. Then choosing α (in this case) positive, one gets the normal in the right direction; the other solution to normalisation is not valid here.

The stress-vector on surface ABC: $p_i = \sigma_{ji} \cdot n_j$

$$\rightarrow \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = [\sigma] \cdot \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

Normal-stress: $\sigma = p^{(n)}$ on ABC: $\{\sigma\} = \{\hat{n}\}^T \cdot \{p\} = 20MPa$

Shear-stress τ on ABC (Pythagoras):

$$\tau^2 = \left|\{p\}\right|^2 - \sigma^2 = \left[p_1^2 + p_2^2 + p_3^2\right] - \sigma^2 = 6100 - 400 = 5700MPa \quad \Rightarrow \quad \tau = 75{,}5Mpa$$

b)

$$\varepsilon_{ij} = \frac{1}{E} \cdot \left\{(1+\nu) \cdot \sigma_{ij} - \nu \cdot \sigma_{kk} \cdot \delta_{ij}\right\}$$

with: $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$

Computing now each term, we get the strain-tensor, with:

e.g.: $$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu \cdot \frac{\sigma_{22}}{E} - \nu \cdot \frac{\sigma_{33}}{E}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}$$

Using the shear modulus $G = \dfrac{E}{2 \cdot (1+\nu)}$ gives:

$$[\varepsilon] = \begin{bmatrix} 5.375 & 1.875 & 0 \\ 1.875 & 0.375 & 0 \\ 0 & 0 & -4 \end{bmatrix} \cdot 10^{-4} = \frac{10^{-4}}{8} \cdot \begin{bmatrix} 43 & 15 & 0 \\ 15 & 3 & 0 \\ 0 & 0 & -32 \end{bmatrix}$$

c)

Computing the principal strains, just as done for stress:

$$\det([\varepsilon] - \varepsilon[I]) = \det \begin{bmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon \end{bmatrix} = 0$$

The characteristic equation is:

$$\left[-32 \cdot \frac{10^{-4}}{8} - \varepsilon\right] \cdot \left(\varepsilon^2 - 46 \cdot \frac{10^{-4}}{8} \varepsilon - 96 \cdot \left(\frac{10^{-4}}{8}\right)^2\right) = 0$$

with solutions (sorted from large to small):

$$\varepsilon_1 = 6.10^{-4}$$

$$\varepsilon_2 = -0.25.10^{-4}$$

$$\varepsilon_3 = -4.10^{-4}$$

d)
In order to compute the allowable stress according to Tresca and von Mises, we first need to compute the principal stresses.

$$\det([\sigma]-\sigma[I]) = \det\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} = 0$$

gives the characteristic equation: → $[-50-\sigma]\cdot(\sigma^2 - 120\cdot\sigma + 1100) = 0$
From which we get the sorted principal stresses:

$$\sigma_1 = 110MPa$$
$$\sigma_2 = 10MPa$$
$$\sigma_3 = -50MPa$$

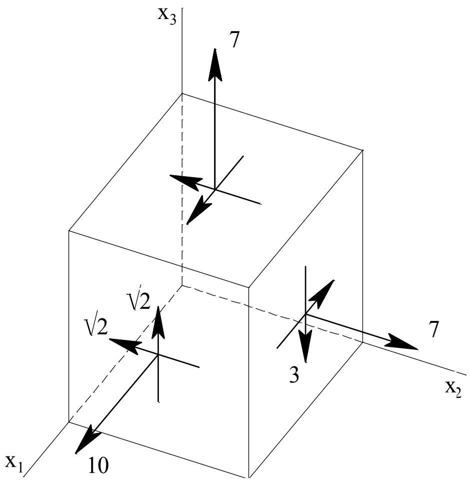
Stress-hypothesis according to Tresca: $\sigma_{eq,tresca} = (\sigma_1 - \sigma_3) = 160MPa$
Stress-hypothesis according to von Mises:

$$\sigma_{eq,mises} = [\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]]^{1/2} = 140MPa$$

Toelaatbaar belasten betekent: $\sigma_{eq} \le \bar{\sigma}$
Given a maximally allowed stress of 15 MPa (error in the old version of the exercise), both stresses are inacceptable by far.

Given 150 MPa (this was the supposed correct value to be used) for the maximal allowed stress, we find that according to von Mises the stresses are acceptable, but not acceptable according to Tresca.

Exercise V-2



Given:
Linear elastic isotropic material
$E=2\cdot 10^5$ N/mm²
One principal stress is given: 8 N/mm²

Questions:
a) the other principal (eigen) stresses
b) the eigen-directions and a sketch
c) the maximal shear-strain for a given volumetric strain $\varepsilon_V = 0{,}6{\cdot}10^{-4}$
d) the equivalent stresses according to Tresca and von Mises

Solutions:

a) The principal stresses can be computed as the eigen-values of the stress-matrix/tensor. This means one has to find the solutions to: $\det([\sigma]-\sigma[I])=0$

$$\det([\sigma]-\sigma[I])=\det\begin{bmatrix}\sigma_{11}-\sigma & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma\end{bmatrix}=\det\begin{bmatrix}10-\sigma & -\sqrt{2} & \sqrt{2}\\ -\sqrt{2} & 7-\sigma & -3\\ \sqrt{2} & -3 & 7-\sigma\end{bmatrix}=0$$

Note that the first index denotes the direction of the normal to the accoding surface on which this stress component is working, while the second index gives the direction in which the stress component works.

The determinant yields the characteristic equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0, \quad \text{with invariants:} \quad I_1 = \sigma_{11}+\sigma_{22}+\sigma_{33} = 24\ N/mm^2$$

$$I_2 = \sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}\sigma_{21}-\sigma_{23}\sigma_{32}-\sigma_{31}\sigma_{13} = 176\left(N/mm^2\right)^2$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33}-\sigma_{11}\sigma_{23}\sigma_{32}-\sigma_{12}\sigma_{21}\sigma_{33}+\sigma_{12}\sigma_{23}\sigma_{31}+\sigma_{13}\sigma_{21}\sigma_{32}-\sigma_{13}\sigma_{31}\sigma_{22} = 384\left(N/mm^2\right)^3$$

so that:

$$\sigma^3 - 24\sigma^2 + 176\sigma - 384 = 0$$

This equation has three solutions, but cannot be solved in general; however, one is known alredy, 8 *N/mm²*, thus after decomposition, the equation will look like:

$$(\sigma-8)(a\sigma^2+b\sigma+c)=0$$

with unknown *a*, *b* and *c*. To get these coefficients, divide the whole characteristic equation by: $\sigma-8$ using polynomial division:

$$\sigma-8/\sigma^3-24\sigma^2+176\sigma-384\backslash\sigma^2-16\sigma+48$$

$$\begin{array}{r}\underline{\sigma^3-8\sigma^2\ -}\\ -16\sigma^2+176\sigma\\ \underline{-16\sigma^2+128\sigma\ -}\\ 48\sigma-384\\ \underline{48\sigma-384\ -}\\ 0\end{array}$$

The equation decomposed into its three factors is now:

$$(\sigma-8)(\sigma^2-16\sigma+48)=(\sigma-8)(\sigma-4)(\sigma-12)=0,$$

from which we get the principal stresses:

$$\sigma_1 = 12\ MPa$$
$$\sigma_2 = 8\ MPa \qquad\qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$$
$$\sigma_3 = 4\ MPa$$

b) The eigen-directions are obtained by inserting one eigen-stress into $([\sigma]-\sigma[I])$, and multiplying the matrix with a unit-direction vector with three unknowns to solve:

Hoofdrichting 1:

$$\begin{bmatrix} -2 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -5 & -3 \\ \sqrt{2} & -3 & -5 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From the rows of the matrix, one gets the following three relations:

$$-2\hat{n}_1-\sqrt{2}\hat{n}_2+\sqrt{2}\hat{n}_3=0 \qquad [1]$$
$$-\sqrt{2}\hat{n}_1-5\hat{n}_2-3\hat{n}_3=0 \qquad [2]$$
$$\sqrt{2}\hat{n}_1-3\hat{n}_2-5\hat{n}_3=0 \qquad [3]$$

From [3] follows:

$$\hat{n}_1=\tfrac{1}{2}\sqrt{2}(3\hat{n}_2+5\hat{n}_3) \qquad [4]$$

Inserted in [1] gives:

$$\hat{n}_2=-\hat{n}_3 \qquad [5]$$

[5] back in [4] gives $\hat{n}_1$ expressed in terms of $\hat{n}_3$:

$$\hat{n}_1=\sqrt{2}\hat{n}_3$$

Using the normalisation condition: $\hat{n}_1{}^2+\hat{n}_2{}^2+\hat{n}_3{}^2=1$ ne gets the value of $\hat{n}_3$

$$\left(\sqrt{2}^2+1+1\right)\hat{n}_3{}^2=1 \qquad \Leftrightarrow \qquad \hat{n}_3=\pm\tfrac{1}{2}$$

The first eigen-direction (associated to the first, major principal stress) is thus:

$$\{\hat{n}_i\}^1=\pm\frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ -1 \\ 1 \end{Bmatrix}$$

Hoofdrichting 2:

$$\begin{bmatrix} 2 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -1 & -3 \\ \sqrt{2} & -3 & -1 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Similar to above, after inerting the second eigen-stress:

$$2\hat{n}_1-\sqrt{2}\hat{n}_2+\sqrt{2}\hat{n}_3=0 \qquad [6]$$
$$-\sqrt{2}\hat{n}_1-\hat{n}_2-3\hat{n}_3=0 \qquad [7]$$
$$\sqrt{2}\hat{n}_1-3\hat{n}_2-\hat{n}_3=0 \qquad [8]$$

From [8] follows

$$\hat{n}_1 = \tfrac{1}{2}\sqrt{2}\left(3\hat{n}_2 + \hat{n}_3\right) \qquad [9]$$

Inserting into [6] gives

$$\hat{n}_2 = -\hat{n}_3 \qquad [10]$$

[10] back in [9] gives $\hat{n}_1$ expressed in terms of $\hat{n}_3$:

$$\hat{n}_1 = -\sqrt{2}\hat{n}_3$$

Again, using the normalization to get $\hat{n}_3$ results in:

$$\left(\left(-\sqrt{2}\right)^2 + \left(-1\right)^2 + 1^2\right)\hat{n}_3{}^2 = 1 \qquad \Leftrightarrow \qquad \hat{n}_3 = \pm\tfrac{1}{2}$$

The second eigen-direction is thus:

$$\{\hat{n}_i\}^2 = \pm\frac{1}{2}\begin{Bmatrix} -\sqrt{2} \\ -1 \\ 1 \end{Bmatrix} = \pm\frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ 1 \\ -1 \end{Bmatrix}$$

Hoofdrichting 3:

$$\begin{bmatrix} 6 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 3 & -3 \\ \sqrt{2} & -3 & 3 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

As above, after inserting the third eigen-stress:

$$6\hat{n}_1 - \sqrt{2}\hat{n}_2 + \sqrt{2}\hat{n}_3 = 0 \qquad [11]$$
$$-\sqrt{2}\hat{n}_1 + 3\hat{n}_2 - 3\hat{n}_3 = 0 \qquad [12]$$
$$\sqrt{2}\hat{n}_1 - 3\hat{n}_2 + 3\hat{n}_3 = 0 \qquad [13]$$

Uit [13] volgt dat

$$\hat{n}_1 = \tfrac{1}{2}\sqrt{2}\left(3\hat{n}_2 - 3\hat{n}_3\right) \qquad [14]$$

Inserted in [11] yields

$$\hat{n}_2 = \hat{n}_3 \qquad [15]$$

[15] back in [14] gives

$$\hat{n}_1 = 0$$

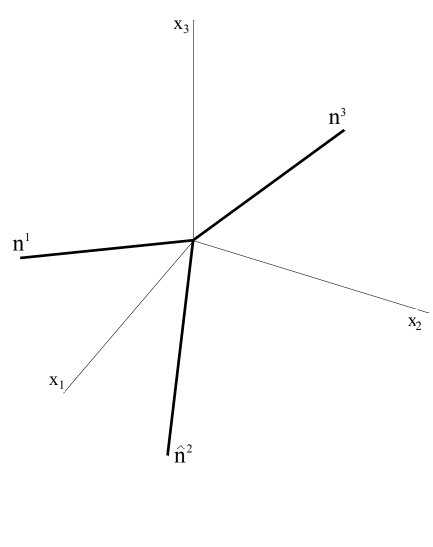
And the normalisation gives for $\hat{n}_3$

$$\left(0^2 + 1^2 + 1^2\right)\hat{n}_3{}^2 = 1 \qquad \Leftrightarrow \qquad \hat{n}_3 = \pm\sqrt{\tfrac{1}{2}} = \pm\tfrac{1}{2}\sqrt{2}$$

The third eigen-direction is thus:

$$\{\hat{n}_i\}^3 = \pm\frac{1}{2}\sqrt{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

Plotting the three direction-vectors:



c) The maximal shear strain and the maximal deformation angle are

$$2\varepsilon_{max} = \gamma_{max} = \frac{\tau_{max}}{G} = \frac{\sigma_1 - \sigma_3}{2}\frac{2(1+\nu)}{E}$$

With: τ_{max} the maximal shear-stress

ν the Poisson ratio

Here, unusually, all quantities are known, except for the Poisson ratio.

$$\varepsilon_V = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{1}{E}\left[(1+\nu)\sigma_{kk} - \nu \quad \sigma_{kk}\delta_{kk}\right] = \frac{1}{E}\left[(1+\nu)\sigma_{kk} - \nu \quad \sigma_{kk} \cdot 3\right] =$$

$$\frac{1-2\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}I_1 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

Rearranging this as an expression for ν provides its value:

$$\nu = \tfrac{1}{2} - E\frac{\varepsilon_V}{2(\sigma_1 + \sigma_2 + \sigma_3)} = \tfrac{1}{4}$$

Given all necessary parameters allows to compute the maximal angle and shear strain:

$\gamma_{max} = 0{,}5 \cdot 10^{-4}$ and $\varepsilon_{max} = 0{,}25 \ 10^{-4}$

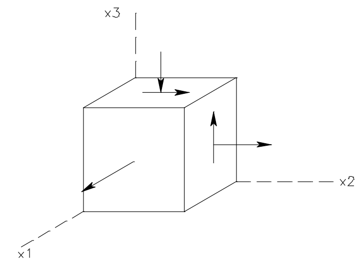
d) The equivalent stress according to Tresca:

$$\sigma_{eq,Tresca} = \sigma_1 - \sigma_3 = 8 \ MPa.$$

The equivalent stress according to von Mises:

$$\sigma_{eq,VonMises} = \sqrt{\tfrac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]} = 6{,}93 \ MPa.$$

Exercise V.3



Given:

a stress-state:

$$\sigma_{ij} = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix} MPa$$

$E = 2 \cdot 10^5$ MPa $\qquad \nu = 0.25$

Solutions

a) Compute the principal stresses

From the stress matrix, one can see directly that one of the eigenvalues must be equal to σ_{11}. The shear-stresses on the '1-surface' are namely all zero ($\sigma_{12} = \sigma_{13} = 0$). Also plot the stresses in the stress-cube picture. From the linear algebra procedure follows:

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \Rightarrow$$

$$\det\begin{pmatrix} 60-\sigma & 0 & 0 \\ 0 & 20-\sigma & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-\sigma \end{pmatrix} = (60-\sigma)\left\{(20-\sigma)(-20-\sigma) - \left(20\sqrt{3}\right)^2\right\} =$$

$$(60-\sigma)\left\{-400+\sigma^2-1200\right\} = (60-\sigma)\left\{\sigma^2-1600\right\} = 0$$

$$\Rightarrow \sigma_1 = 60 \text{ MPa}, \ \sigma_2 = 40 \text{ MPa}, \ \sigma_3 = -40 \text{ MPa}$$

b) Compute the eigen-directions

$$\left(\sigma_{ij} - \sigma\delta_{ij}\right)\hat{n}_j = 0 \ \& \ \hat{n}_j\hat{n}_j = 1$$

1st principal stress inserted:

$$\begin{bmatrix} 60-60 & 0 & 0 \\ 0 & 20-60 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-60 \end{bmatrix}\left\{\hat{n}_j^1\right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -80 \end{bmatrix}\left\{\hat{n}_j^1\right\} = 0 \Rightarrow \left\{\hat{n}_j^1\right\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

(compare to the sketch of the stress-cube.)

2nd principal stress inserted:

$$\begin{bmatrix} 60-40 & 0 & 0 \\ 0 & 20-40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-40 \end{bmatrix}\left\{\hat{n}_j^2\right\} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & -20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -60 \end{bmatrix}\left\{\hat{n}_j^2\right\} = 0$$

$$20 \cdot n_1^2 = 0 \Rightarrow n_1^2 = 0$$

$$-20 \cdot n_2^2 + 20\sqrt{3} \cdot n_3^2 = 0 \Rightarrow n_3^2 = \frac{20 \cdot n_2^2}{20\sqrt{3}} = \frac{n_2^2}{\sqrt{3}}$$

$$20\sqrt{3} \cdot n_2^2 - 60 \cdot n_3^2 = 0 \Rightarrow n_3^2 = \frac{20\sqrt{3} \cdot n_2^2}{60} = \frac{\sqrt{3} \cdot n_2^2}{3} = \frac{n_2^2}{\sqrt{3}}$$ (Equal to the above, due to dependence

$$\hat{n}_j\hat{n}_j = 1 \Rightarrow 0^2 + \left(n_2^2\right)^2 + \left(\frac{n_2^2}{\sqrt{3}}\right)^2 = 1 \Rightarrow \left(n_2^2\right)^2 = \frac{1}{1\frac{1}{3}} = \frac{3}{4} \Rightarrow n_2^2 = \frac{\sqrt{3}}{2} \Rightarrow n_3^2 = \frac{1}{2} \Rightarrow \left\{\hat{n}_j^2\right\} = \frac{1}{2}\begin{Bmatrix} 0 \\ \sqrt{3} \\ 1 \end{Bmatrix}$$

3rd stress eigenvalue inserted:

$$\begin{bmatrix} 60+40 & 0 & 0 \\ 0 & 20+40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20+40 \end{bmatrix} \left\{ \hat{n}_j^3 \right\} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 60 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & 20 \end{bmatrix} \left\{ \hat{n}_j^3 \right\} = 0$$

$$100 \cdot n_1^3 = 0 \Rightarrow n_1^3 = 0$$

$$60 \cdot n_2^3 + 20\sqrt{3} \cdot n_3^3 = 0 \Rightarrow n_3^3 = -\frac{60 \cdot n_2^3}{20\sqrt{3}} = -\sqrt{3} n_2^3$$

Equal to the above, due to dependence

$$\hat{n}_j \hat{n}_j = 1 \Rightarrow 0^2 + \left(n_2^3\right)^2 + \left(-\sqrt{3} n_2^3\right)^2 = 1 \Rightarrow \left(n_2^3\right)^2 = \frac{1}{4} \Rightarrow n_2^3 = \frac{1}{2} \Rightarrow n_3^2 = -\frac{\sqrt{3}}{2} \Rightarrow \left\{ \hat{n}_j^2 \right\} = \frac{1}{2} \begin{Bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{Bmatrix}$$

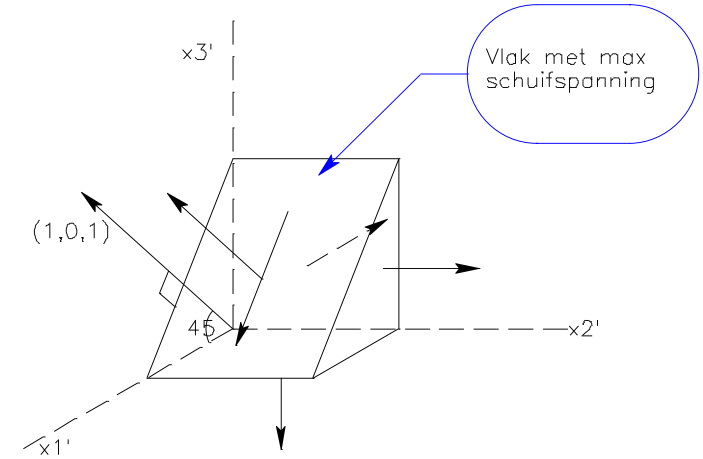
c) The maximal shear-stress

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{1}{2}\left(60 - -40\right) = 50 \text{ MPa}$$

d) and e) the surface on which the maximal shear stress acts?

<table>
<tr>
<td>The max. shear stress acts on a surface rotated by 45º from x'_1 and x'_3 which are respectively the directions of σ_1 en σ_3
(see sketch)

The normal-direction of this surface is (1,0,1) , but this still has to be normalised, i.e. divide by the length:

Thus: $\hat{n}_i^{\tau\max} = \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$</td>
<td>
</td>
</tr>
</table>

f) The strain in this direction

Compute from the eigen-stresses (principal stresses) directly the eigen-strains (principal strains)

$$\varepsilon_{ij} = \tfrac{1}{E}\left\{(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right\}$$

$$\Rightarrow$$

$$\varepsilon_1 = \tfrac{1}{2\cdot 10^5}\left\{(1+0.25)\sigma_1 - 0.25(\sigma_1+\sigma_2+\sigma_3)\right\} = \tfrac{1}{2\cdot 10^5}\left\{(1+0.25)60 - 0.25(60+40-40)\right\} = 3\cdot 10^{-4}$$

$$\varepsilon_2 = \tfrac{1}{2\cdot 10^5}\left\{(1+0.25)\sigma_2 - 0.25(60+40-40)\right\} = 1.75\cdot 10^{-4}$$

$$\varepsilon_3 = \tfrac{1}{2\cdot 10^5}\left\{(1+0.25)\sigma_3 - 0.25(60+40-40)\right\} = -3.25\cdot 10^{-4}$$

Now one can use the rotation matrix for the above rotation about 45 degrees to compute the strain components in this surface.

The x' coordinate-sytem is rotated such that the x_1''-axis (after rotation) points in the direction of the normal computed in d , namely: $\hat{n}_i^{\tau\max}$. What is then ε''_{11} in this new coordinate-system?

$$[R] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$[\varepsilon''] = [R][\varepsilon'][R]^T$$

$$\begin{bmatrix} \varepsilon''_{11} & \varepsilon''_{12} & \varepsilon''_{13} \\ \varepsilon''_{21} & \varepsilon''_{22} & \varepsilon''_{23} \\ \varepsilon''_{31} & \varepsilon''_{32} & \varepsilon''_{33} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}\,\varepsilon_1 & 0 & -1/\sqrt{2}\,\varepsilon_1 \\ 0 & \varepsilon_2 & 0 \\ 1/\sqrt{2}\,\varepsilon_3 & 0 & 1/\sqrt{2}\,\varepsilon_3 \end{bmatrix}$$

$$\varepsilon''_{11} = 1/\sqrt{2}\;1/\sqrt{2}\,\varepsilon_1 + 1/\sqrt{2}\;1/\sqrt{2}\,\varepsilon_3 = \frac{1}{2}(\varepsilon_1+\varepsilon_3) = \frac{1}{2}(3-3.25)\cdot 10^{-4} = -12.5\cdot 10^{-6}$$

Exercise V-4

Problem

Given:

$E = 2.10^{11} GPa \quad \& \quad \nu = 0.25$

Stress-state in point P: $\sigma = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix} MPa$

Questions:

A) Show that the principal stresses are 8, 16 and 24 MPa. Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.

B) Compute the volumetric (isotropic) strain.

C) What is the largest angle-change (not shear-strain) in P?

D) Which material property is implicitly used in Hookes law?

Solutions

A)

$$\det([\sigma]-\sigma[I]) = \det\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} = 0$$

→ characteristic equation: $\sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0 = 0$

From this follow the principal stresses (one can insert them and show that the characteristic equation gets zero for everyone; or one can factorize the equation; or one computes the invariants from the eigen-values and identifies them with the euqation):

$\sigma_1 = 24MPa$

$\sigma_2 = 16MPa$

$\sigma_3 = 8MPa$

For the smallest principal stress, compute the eigen-direction:

$$\begin{bmatrix} \sigma_{11}-8 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-8 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-8 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1^3 \\ \hat{n}_2^3 \\ \hat{n}_3^3 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this equation system, for example, yields:

$$\begin{Bmatrix} \hat{n}_1^3 \\ \hat{n}_2^3 \\ \hat{n}_3^3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \sqrt{6} \end{Bmatrix}$$

which still must be normalized →

$$\left(\hat{n}_1^3\right)^2 + \left(\hat{n}_1^3\right)^2 + \left(\sqrt{6}\cdot\hat{n}_1^3\right)^2 = 1 \qquad \rightarrow \qquad \left(\hat{n}_1^3\right)^2 = \frac{1}{8} \qquad \rightarrow \qquad \hat{n}_1^3 = \frac{\sqrt{2}}{4}$$

which gives the eigen-direction: $\begin{Bmatrix} \hat{n}_1^3 \\ \hat{n}_2^3 \\ \hat{n}_3^3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} \\ \frac{1}{2}\sqrt{3} \end{Bmatrix}$

which actually are the directional cosines (three entries R_{3i}).

B)

For the volumetric strain we get: $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \left[\frac{1-2\nu}{E}\right]\cdot\sigma_{kk} = 12.10^{-5}$

C)

The largest change of angle is: $\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{\frac{1}{2}(\sigma_1 - \sigma_3)}{G} = 1.10^{-4}$,

where the largest shear strain is just half of that.

D)

Isotropy is intrinsic to using the law of Hooke.

Solutions of V-5

a)

First, the stress tensor is determined using Hooke's law ($\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$). Insert and solve – thats the normal way.

One other way to do this (often done for anisotropic materials and in finite element implementations – not needed, and highly redundant – here we show it as an example only, but it is not needed to solve this rather simple problem) is to assemble the independent stress- and strain-tensor components in vectors and express the corresponding stiffness-matrix in moduli, using the Lame-coeffcients (for brevity):

$$\lambda = \frac{E * \nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{1+\nu}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{31}\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix}2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{31}\\ 2\varepsilon_{23}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{ij}\end{bmatrix} = \begin{bmatrix}5 & -15 & 5\sqrt{2}\\ -15 & 5 & -5\sqrt{2}\\ 5\sqrt{2} & -5\sqrt{2} & 20\end{bmatrix} MPa$$

Using the stress tensor, the characteristic equation and principal stresses can be computed

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 30MPa$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = -100MPa^2$$

$$I_3 = \det(\sigma) = -3000MPa^3$$

Therefore: $\sigma_I = 30$MPa, $\sigma_{II} = 10$MPa, $\sigma_{III} = -10$MPa.

Principal direction can be calculated as:

- Direction of $\sigma_I = 30$MPa

$$\begin{bmatrix}\sigma_{11}-\sigma_I & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22}-\sigma_I & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma I\end{bmatrix}\begin{bmatrix}n_1^{(I)}\\ n_2^{(I)}\\ n_3^{(I)}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{and} \quad n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

$$\Longrightarrow \begin{bmatrix}n_1^{(I)}\\ n_2^{(I)}\\ n_3^{(I)}\end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix}1\\-1\\ \sqrt{2}\end{bmatrix}$$

- Direction of $\sigma_{II} = 10$MPa

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma II \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

- Direction of $\sigma_{III} = -10$MPa

$$\begin{bmatrix} \sigma_{11}-\sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

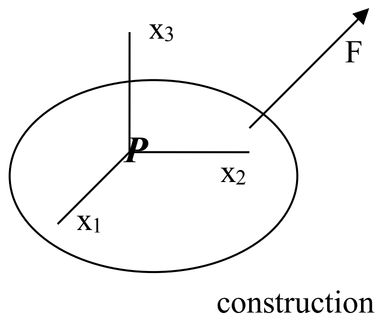
$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}$$

b)

Only for an isotropic material, one has the eigendirections of stress-and strain identical. Note that the term with δ_{ij} in the law of Hooke has no direction (is isotropic = direction-independent); thus the direction is carried by the terms ε_{ij} and σ_{ij} and thus the directions are equal for stress and strain.

Exercise V-6

Problem



Normal in P // x_3-axes (parallel)

Given:
Load F = 56 kN
E = $2*10^5$ MPa
$\upsilon = 0{,}25$
maximal allowed:
σ_{max} = 160 MPa

in a point P on the non-loaded surface:

$\varepsilon_{11} = 130 \cdot 10^{-6}$

$\varepsilon_{22} = -70 \cdot 10^{-6}$

$\gamma_{12} = 346{,}4 \cdot 10^{-6}$

Questions:

a) ε_{33} in point P
b) σ_{11}, σ_{22}, σ_{12}
c) F_{max} according to Tresca
d) F_{max} according to von Mises

Equations to be used:

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) \text{ (inverse Hooke)}$$

$$\sigma_{eq} = \sigma_1 - \sigma_3 \le \sigma_{toel} \text{ (Tresca); } (\sigma_1 \ge \sigma_2 \ge \sigma_3)$$

$$\text{(v.Mises) } \sigma_{eq} = \sqrt{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}}$$

Solutions

a)

Point P sits on the surface (with normal x_3) where it is not loaded, with consequence that the stress-components on this (free) surface are zero, thus: $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$. This means also that we have a plane-stress state in the x_1-x_2 plane.

The inverse law of Hooke can now be used to compute the unknown 33-strain:

$$\sigma_{33} = \frac{E}{1+\nu}\left(\varepsilon_{33} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{33}\right) = \frac{E}{1+\nu}\left(\varepsilon_{33} + \frac{\nu}{1-2\nu}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)\right) =$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11} + \varepsilon_{22})\right) = 0$$

Insertion of the given components yields:

$$\Rightarrow \quad \varepsilon_{33} = -\frac{\nu}{1-\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right) = -20 \cdot 10^{-6}$$

b)

There are three unknown stresses (σ_{11}, σ_{22}, σ_{12}), and thus we need three equations:

As usual it is handy to compute the volumetric strain: $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 40 \cdot 10^{-6}$

Therefore:

$$\sigma_{11} = \frac{E}{1+\nu}\left(\varepsilon_{11} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right) = \tfrac{8}{5}\cdot 10^5\left(130 + \tfrac{1}{2}\cdot 40\right)\cdot 10^{-6} = 24\,\text{MPa}$$

$$\sigma_{22} = \frac{E}{1+\nu}\left(\varepsilon_{22} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right) = \tfrac{8}{5}\cdot 10^5\left(-70 + \tfrac{1}{2}\cdot 40\right)\cdot 10^{-6} = -8\,\text{MPa}$$

$$\sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12} = \frac{E}{2(1+\nu)}\gamma_{12} = 27{,}7\ \text{MPa}$$

c)
The criterion of Tresca requires the principal stresses in the material; to be computed from the stress-tensor:

$$\sigma = \begin{bmatrix} 24 & 27{,}7 & \sigma_{13} \\ 27{,}7 & -8 & \sigma_{23} \\ 0 & 0 & 0 \end{bmatrix}\text{MPa}$$

In connection with moment-equilibrium we have: $\sigma_{21} = \sigma_{12}$; $\sigma_{13} = \sigma_{31}$; $\sigma_{23} = \sigma_{32}$ and thus also $\sigma_{13} = \sigma_{23} = 0$ (because a plane stress-state was given in the instructions).

The following system must be solved:

$$\det([\sigma] - \sigma[I]) = \det\begin{bmatrix} 24-\sigma & 27{,}7 & 0 \\ 27{,}7 & -8-\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix} = 0$$

$$\Rightarrow \quad \det([\sigma] - \sigma[I]) = \sigma^3 + 16\sigma^2 + \frac{95929}{100}\sigma = \sigma\left(\sigma^2 + 16\sigma + \frac{95929}{100}\right) = 0$$

Thus: $\sigma_1 = 40$ Mpa
$\sigma_2 = 0$ $\qquad (\sigma_1 \geq \sigma_2 \geq \sigma_3)$
$\sigma_3 = -24$ Mpa

Using the limit stress of Tresca gives:

$$\sigma_{eq} = \sigma_1 - \sigma_3 = 64\ \text{MPa} \quad \text{and thus:} \quad F_{max} = \frac{\sigma_{max}}{\sigma_{eq}}\cdot F = 140\,\text{kN}$$

d)
Using the equivalent stress definition of van Von Mises gives:

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left((\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right)} = 56\,\text{MPa}$$

and thus: $F_{max} = \dfrac{\sigma_{max}}{\sigma_{eq}}\cdot F = 160\ \text{kN}$

Exercise V-7

Given:
A deformed homogeneous body made of an isotropic linear elastic materiaal (E=2GPa, en $\upsilon = 0.25$), has the displacement field:

$$u_1 = x_1 x_3$$
$$u_2 = -x_1 x_2$$
$$u_3 = x_1^2 - x_3^2$$

Questions:
o) Not asked for, but necessary first: compute the components of the strain tensor.
a) Compute the components of the stress tensor?
b) The principal stresses in the point $(x,y,z)=(0,0,z_0)$, and the maximal shear stress.

Uitwerking
o) The strain tensor is computed from the displacement field: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = x_3$$
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -x_1$$
$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -2x_3$$
$$\varepsilon_{kk} = -x_1 - x_3$$
$$\gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 - x_2 = -x_2$$
$$\gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 + 0 = 0$$
$$\gamma_{31} = 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2x_1 + x_1 = 3x_1$$

a) Hooke: $\varepsilon_{ij} = \frac{1}{E}\left\{(1+\upsilon)\sigma_{ij} - \upsilon\delta_{ij}\sigma_{kk}\right\}$

$$\varepsilon_{kk} = \frac{1}{E}\left\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\right\} \quad \delta_{jj} = 1+1+1 = 3$$
$$\varepsilon_{kk} = \frac{1}{E}(1-2\nu)\sigma_{kk} \;\Rightarrow\; \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Note: strain should be dimensionless and thus should have a pre-factor with unit [1/m].

substitute and invert:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \;\Rightarrow\; \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

b) the components of stress (in GPa): $\sigma_{ij} = \dfrac{E}{1+\nu}\left[\varepsilon_{ij} + \dfrac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$

$$\sigma_{11} = \frac{E}{1+\nu}\left[x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{E}{1+\nu}\left[\frac{1}{1-2\nu}(-x_1\nu + x_3(1-3\nu))\right]$$

$$= \frac{8}{5}\left[2(-0.25x_1 + 0.25x_3)\right] = \frac{4}{5}\left[(-x_1 + x_3)\right]$$

$$\sigma_{22} = \frac{E}{1+\nu}\left[-x_1 + \frac{\upsilon}{1-2\upsilon}(-x_1 - x_3)\right] = \frac{E}{1+\nu}\left[\frac{1}{1-2\nu}(-x_3\nu - x_1(1-2\nu+\nu))\right]$$

$$= \frac{8}{5}\left[2(-0.25x_3 - 0.75x_1)\right] = \frac{4}{5}\left[(-3x_1 - x_3)\right]$$

$$\sigma_{33} = \frac{E}{1+\nu}\left[-2x_3 + \frac{\upsilon}{1-2\nu}(-x_1 - x_3)\right] = \frac{E}{1+\nu}\left[\frac{1}{1-2\nu}(-x_1\nu - x_3(2-4\nu+\nu))\right]$$

$$= \frac{8}{5}\left[2(-0.25x_1 - 1.25x_3)\right] = \frac{4}{5}\left[(-x_1 - 5x_3)\right]$$

$$\sigma_{12} = \frac{E}{1+\nu}\left[-x_2/2\right] = \frac{4}{5}(-x_2)$$

$$\sigma_{23} = 0$$

$$\sigma_{31} = \frac{E}{1+\nu}\left[3x_1/2\right] = \frac{12}{5}(x_1)$$

in the point => insert not earlier! only now: x_1=0; x_2=0; x_3=z_0

$$\sigma_{11} = \frac{4}{5}\left[(-x_1 + x_3)\right] = \frac{4}{5}z_0$$

$$\sigma_{22} = \frac{4}{5}\left[(-3x_1 - x_3)\right] = -\frac{4}{5}z_0$$

$$\sigma_{33} = \frac{4}{5}\left[(-x_1 - 5x_3)\right] = -4z_0$$

$$\sigma_{12} = \frac{4}{5}(-x_2) = 0$$

$$\sigma_{23} = 0$$

$$\sigma_{31} = \frac{12}{5}(x_1) = 0$$

$$\tau_{\max} = \tfrac{1}{2}\left(\sigma_1 - \sigma_3\right) = 2.4z_0$$

with units [GPa/m].

Exercise V-8

Given:
A homogeneous body made from a linear elastic isotropic material (E=2GPa, $\upsilon = 0.25$) is in a deformed state in mechanical equilibrium, in absence of a volume force, with parameters reference stress p, and length a, and unknown coefficients α and β, according to the displacement field:

$$u_1 = \frac{pa}{E}\left[\left(\frac{x_2}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - \left(\frac{x_2}{a}\right)^2\right]$$

$$u_2 = \frac{pa}{E}\left[\left(\frac{x_1}{a}\right) + \alpha\left(\frac{x_1}{a}\right)^2 + \beta\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - 2\left(\frac{x_2}{a}\right)^2\right]$$

$$u_3 = 0$$

Questions:
o) The strain field is not asked for – but you will have to compute it.
a) The stress components, via the Hooke law from the strain components.
b) The stress-equilibrium equations (by partial differentiation)

Solutions

o) The strain field: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{pa}{E}\left[2\left(\frac{1}{a}\right)\left(\frac{x_2}{a}\right)\right] = \frac{p}{E}\left[2\left(\frac{x_2}{a}\right)\right]$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{pa}{E}\left[\beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 4\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right] = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right]$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

$$\varepsilon_{kk} = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]$$

$$\gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{pa}{E}\left[\left(\frac{1}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 2\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right) + \left(\frac{1}{a}\right) + 2\alpha\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) + \beta\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right]$$

$$\Rightarrow \varepsilon_{12} = \frac{p}{E}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + (\tfrac{\beta}{2} - 1)\left(\frac{x_2}{a}\right)\right]$$

$$\gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0$$

$$\gamma_{31} = 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0$$

a) Thus stress is derived as:

$$\varepsilon_{kk} = \frac{1}{E}\left\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\right\} \qquad \delta_{jj} = 1+1+1 = 3$$

$$\varepsilon_{kk} = \frac{1}{E}(1-2\nu)\sigma_{kk} \;\Rightarrow\; \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Substitution gives:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \Rightarrow \quad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

b) The stress components: $\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$

$$\sigma_{11} = \frac{E}{1+\nu}\left[\varepsilon_{11} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right] = \frac{p}{1+\nu}\left[\frac{1-2\nu}{1-2\nu}\left[2\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right]$$

$$= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[2(1-3\nu)\left(\frac{x_2}{a}\right)\right] + \left[\nu\beta\left(\frac{x_1}{a}\right)\right]\right]$$

$$\sigma_{22} = \frac{E}{1+\nu}\left[\varepsilon_{22} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right] = \frac{p}{1+\nu}\left[\frac{1-2\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right]$$

$$= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[(1-\nu)\beta\left(\frac{x_1}{a}\right)\right] + \left[(-4+6\nu)\left(\frac{x_2}{a}\right)\right]\right]$$

$$\sigma_{33} = \frac{E}{1+\nu}\left[\varepsilon_{33} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right] = \frac{p}{1+\nu}\left[0 + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right]$$

$$= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta\nu\left(\frac{x_1}{a}\right) - 2\nu\left(\frac{x_2}{a}\right)\right]\right]$$

$$\sigma_{12} = \frac{E}{1+\nu}\left[\varepsilon_{12}\right] = \frac{p}{1+\nu}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + (\tfrac{\beta}{2}-1)\left(\frac{x_2}{a}\right)\right]$$

$$\sigma_{23} = 0$$

$$\sigma_{31} = 0$$

b) Equilibrium equations (volume-force = 0);

THhe 3 equations for j=1,2,3: $\sigma_{ij,i} + 0 = 0 (= \sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3})$

Using partial differentiation:

$$\sigma_{11,1} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\nu\beta\left(\frac{1}{a}\right)\right]\right]$$

$$\sigma_{22,2} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[(-4+6\nu)\left(\frac{1}{a}\right)\right]\right]$$

$$\sigma_{33,3} = 0$$

$$\sigma_{12,1} = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]; \sigma_{13,1} = 0$$

$$\sigma_{23,2} = 0; \sigma_{21,2} = \frac{p}{1+\nu}\left[(\tfrac{\beta}{2}-1)\left(\frac{1}{a}\right)\right]$$

$$\sigma_{31,3} = 0; \sigma_{32,3} = 0$$

Gives the equation(s) (3 times):

$$(1)\ \sigma_{11,1}+\sigma_{21,2}+\sigma_{31,3}=0=\frac{p}{1+v}\frac{1}{1-2v}\left[v\beta\left(\frac{1}{a}\right)\right]+\frac{p}{1+v}\left[(\tfrac{\beta}{2}-1)\left(\frac{1}{a}\right)\right]+0$$

$$(2)\ \sigma_{12,1}+\sigma_{22,2}+\sigma_{32,3}=0=\frac{p}{1+v}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]+\frac{p}{1+v}\frac{1}{1-2v}\left[\left[(-4+6v)\left(\frac{1}{a}\right)\right]\right]+0$$

$$(3)\ \sigma_{13,1}+\sigma_{23,2}+\sigma_{33,3}=0$$

v=0.25 insertion yields:

$$(1) \Rightarrow \frac{p}{5/4}\left[0.5\beta+(\tfrac{\beta}{2}-1)\right]\left(\frac{1}{a}\right)=0 \Rightarrow p\frac{4}{5}\left[\beta-1\right]\left(\frac{1}{a}\right)=0$$

$$(2) \Rightarrow \frac{p}{5/4}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]+\frac{p}{5/4}2\left[\left[(-2.5)\left(\frac{1}{a}\right)\right]\right]=0 \Rightarrow p\frac{4}{5}\left[(\alpha+1)-5\right]\left(\frac{1}{a}\right)=0$$

so that:

$(1) \Rightarrow \beta=1$

$(2) \Rightarrow \alpha=4$

allow for mechanical equilibrium in all points in the body.

Exercise V-9

Given:

A loaded homogeneous linear elastic, isotropic material has the displacement field:

$$u_1 = \frac{1}{3}(1-2\nu)x_1^{\ 3} - (3-2\nu)x_1x_2^{\ 2} - 3x_2 - 3x_3$$

$$u_2 = (1-2\nu)x_1x_2^{\ 2} + \frac{1}{3}(1+2\nu)x_2^{\ 3} + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

(where the units of the single terms are wrong! There should be factors with the respectively right units to compensate, but we dropped those to reduce the writing)

Questions:

a) The components of the strain-field.
b) The components of the stress-field.
c) Show that the stress field comlies to the stress-balance equation in absence of a volume-force.

Solutions:

a) The strain-stress relation: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = (1-2\nu)x_1^{\ 2} - (3-2\nu)x_2^{\ 2}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = (1-2\nu)x_1^{\ 2} + (1+2\nu)x_2^{\ 2}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

$$\varepsilon_{kk} = 2(1-2\nu)x_1^{\ 2} + (-3+2\nu+1+2\nu)x_2^{\ 2} = 2(1-2\nu)(x_1^{\ 2} - x_2^{\ 2})$$

$$\gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -(3-2\nu)2x_1x_2 - 3 + 2(1-2\nu)x_1x_2 + 3 = 2x_1x_2(-3+2\nu+1-2\nu) = -4x_1x_2$$

$$\gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = -4 + 4 = 0$$

$$\gamma_{31} = 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 3 - 3 = 0$$

Hooke: $\varepsilon_{ij} = \frac{1}{E}\left\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right\}$

$$\varepsilon_{kk} = \frac{1}{E}\left\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\right\} \qquad \delta_{jj} = 1+1+1 = 3$$

$$\varepsilon_{kk} = \frac{1}{E}(1-2\nu)\sigma_{kk} \;\; => \;\; \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Substitute:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}$$

$$=> \quad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

b) The stress-field: $\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$

$$\sigma_{11} = \frac{E}{1+\nu}\left[(1-2\nu)x_1^{\,2} - (3-2\nu)x_2^{\,2} + \frac{\nu}{1-2\nu}2(1-2\nu)(x_1^{\,2} - x_2^{\,2})\right] = \frac{E}{1+\nu}(x_1^{\,2} \text{-} \, 3x_2^{\,2})$$

$$\sigma_{22} = \frac{E}{1+\nu}\left[(1-2\upsilon)x_1^{\,2} + (1+2\nu)x_2^{\,2} + \frac{\nu}{1-2\upsilon}2(1-2\nu)(x_1^{\,2} - x_2^{\,2})\right] = \frac{E}{1+\nu}(x_1^{\,2} + x_2^{\,2})$$

$$\sigma_{33} = \frac{E}{1+\nu}\left[0 + \frac{\upsilon}{1-2\nu}2(1-2\nu)(x_1^{\,2} - x_2^{\,2})\right] = \frac{E}{1+\nu}2\nu(x_1^{\,2} - x_2^{\,2})$$

$$\sigma_{12} = \frac{E}{1+\nu}\left[\frac{1}{2}(-4x_1x_2)\right] = \frac{E}{1+\nu}(-2x_1x_2) \;\; ; \;\; \sigma_{23} = \sigma_{31} = 0$$

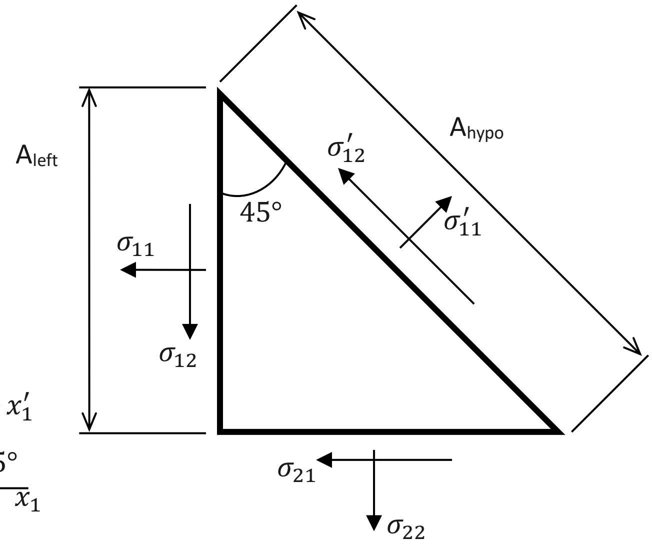
c) Stress equilibrium equations: $\sigma_{ji,j} + F_i = 0$ with $F_i = 0$

$$i = 1 \;\; ; \;\; \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} = \frac{E}{1+\nu}(2x_1 - 2x_1 + 0) = 0$$

$$i = 2 \;\; ; \;\; \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} = \frac{E}{1+\nu}(-2x_2 + 2x_2 + 0) = 0$$

$$i = 3 \;\; ; \;\; \frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} = \frac{E}{1+\nu}(0 + 0 + 0) = 0$$

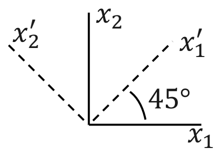
Exercise V-10



Problem:

Given:

- In a point P of a body we have a plane-stress state with: $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Given are these (mixed) stress components:
  $\sigma_{11} = 92$ MPa
  $\sigma'_{11} = 194$ MPa
  $\sigma'_{12} = -42$ MPa
- The material is linear elastic with:
  $E = 2 \cdot 10^5$ MPa
  $\nu = 0.25$



Questions:

- Give the stress tensor in the original $x_1x_2x_3$-system.
- Give the stress tensor in the new $x'_1x'_2x_3$ coordinate system, as obtained by a rotation of the coordinates about 45° around the x_3-axis, as sketched above.
- Compute the eigen-stresses and the eigen-directions.
- Give the strain tensor in the $x'_1x'_2x_3$ coordinate system.
- Compute the specific elastic energy in point P.

Solutions:

- There are two ways to solve this problem. The triangle above represents all stresses on all sides, but only part of the stress components are given.
  By considering force equilibrium and using the respective stress components divided by the side-lengths of the triangle (which also has a third dimension outside the plane, not shown). Assume the sides have unit-length, then the hypotenuse has according to Phytagoras length $\sqrt{2}$. Further assume the thickness also to be unit-length. The ratio between sides and hypothenuse is then:

$$\frac{A_{left}}{A_{hypo}} = \frac{A_l}{A_h} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

  With this we can get:.
  - Force-balance in x_1-direction:

$$A_h\sigma'_{11}\cos(45°) - A_h\sigma'_{12}\sin(45°) - A_l\sigma_{11} - A_l\sigma_{21} =$$
$$\sigma'_{11}\cos(45°) - \sigma'_{12}\sin(45°) - \frac{A_l}{A_h}\sigma_{11} - \frac{A_l}{A_h}\sigma_{21} =$$
$$\frac{1}{2}\sqrt{2}\sigma'_{11} - \frac{1}{2}\sqrt{2}\sigma'_{12} - \frac{1}{2}\sqrt{2}\sigma_{11} - \frac{1}{2}\sqrt{2}\sigma_{21} = 0$$

    So that: $\sigma_{21} \equiv \sigma_{12} = \sigma'_{11} - \sigma'_{12} - \sigma_{11} = 194 - -42 - 92 = 144$ MPa
  - Force-balance in x_2-direction:

$$A_h\sigma'_{11}\sin(45°) + A_h\sigma'_{12}\cos(45°) - A_l\sigma_{12} - A_l\sigma_{22} =$$
$$\sigma'_{11}\sin(45°) + \sigma'_{12}\cos(45°) - \frac{A_l}{A_h}\sigma_{12} - \frac{A_l}{A_h}\sigma_{22} =$$
$$\frac{1}{2}\sqrt{2}\sigma'_{11} + \frac{1}{2}\sqrt{2}\sigma'_{12} - \frac{1}{2}\sqrt{2}\sigma_{12} - \frac{1}{2}\sqrt{2}\sigma_{22} = 0$$

    so that: $\sigma_{22} = \sigma'_{11} + \sigma'_{12} - \sigma_{12} = 194 + -42 - 144 = 8$ MPa

The stress tensor in the $x_1x_2x_3$-system is thus:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

- The stress tensor in the $x_1'x_2'x_3$-system is obtained by rotating it from the $x_1x_2x_3$-system by $45°$. For this we have to build the rotationmatrix $R_{pi} = \cos\,(x_p, x_i)$. The stress tensor in the new system is then: $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$ or in matrix-vector notation: $[\sigma'] = [R][\sigma][R]^T$:

$$[\sigma'] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 194 & -42 & 0 \\ -42 & -94 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

The alternative for calculation uses the unknowns in this transformation relation:

$$[\sigma'] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

and, after spelling them out allows to solve without the geometry and force-balance:

$$[\sigma'] = \frac{1}{2}\sqrt{2} \begin{bmatrix} 92 + \sigma_{12} & \sigma_{12} + \sigma_{22} & 0 \\ -92 + \sigma_{12} & -\sigma_{12} + \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

and second step:

$$[\sigma'] = \frac{1}{2} \begin{bmatrix} 92 + 2\sigma_{12} + \sigma_{22} & -92 + \sigma_{22} & 0 \\ -92 + \sigma_{22} & 92 - 2\sigma_{12} + \sigma_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

which yields from the non-diagonal: $\sigma_{22} = 8$, inserted in the 11-component: $\sigma_{12} = 144$, and inserted into 22: $\sigma'_{22} = -94$, as above. Thus the tensor transformation equation, given only the rotation angle, ignoring the geometry/area considerations, leads to the same result.

- The principal stresses and eigen-directions can now be computed the usual way with: $(\sigma_{ij} - \sigma\delta_{ij})\hat{n}_j = 0$ and $\hat{n}_j\ \hat{n}_j = 1$ or in vector-matrix notation: $([\sigma] - \sigma[I])\{\hat{n}\} = \{0\}$ and $\|\hat{n}\| = 1$ for the directions; while the principal stresses follow from $([\sigma] - \sigma[I]) = 0$, for which the determinant has to vanish:

$$det([\sigma] - \sigma[I]) = \begin{vmatrix} 92 - \sigma & 144 & 0 \\ 144 & 8 - \sigma & 0 \\ 0 & 0 & -\sigma \end{vmatrix} = [(92 - \sigma) \cdot (8 - \sigma) - 144^2] \cdot (-\sigma) = 0$$

The second term gives one eigenvalue, while first term is a polynomial of second order:

$$\sigma^2 - 100\sigma + 736 - 144^2 = 0$$

which has the solutions:

$$\sigma^{(1)} = \frac{100 + \sqrt{100^2 - 4 \cdot (736 - 144^2)}}{2} = 200 \text{ MPa}$$

$$\sigma^{(3)} = \frac{100 - \sqrt{100^2 - 4 \cdot (736 - 144^2)}}{2} = -100 \text{ MPa}$$

$$\sigma^{(3)} = 0 \text{ MPa}$$

The zero eigenvalue can already be deduced from the information that we have a plane-stess state.

The eigen-direction for the first eigen-value can be computed as:

$$(92-200)n_1^{(1)} + 144 \cdot n_2^{(1)} = 0 \quad \rightarrow \quad -3 \cdot n_1^{(1)} + 4 \cdot n_2^{(1)} = 0$$
$$144 \cdot n_1^{(1)} + (8-200)n_2^{(1)} = 0 \quad \rightarrow \quad 3 \cdot n_1^{(1)} - 4 \cdot n_2^{(1)} = 0$$
$$-200 \cdot n_3^{(1)} = 0 \quad \rightarrow \quad n_3^{(1)} = 0$$

After normalization, $\left|n^{(1)}\right| = 1$, this results in:

$$n^{(1)} = \frac{n^{(1)}}{\left|n^{(1)}\right|} = \begin{Bmatrix} 0.8 \\ 0.6 \\ 0 \end{Bmatrix}$$

Similar calculations – not details given here – yield:

$n^{(3)} = \begin{Bmatrix} 0.6 \\ -0.8 \\ 0 \end{Bmatrix}$ and (without calculation, due to plane-stress in 3-direction) $n^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

- The strain components can be calculated using the linear elastic material law:

$$\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\} \quad \text{of} \quad \varepsilon' = \frac{1}{E}\{(1+\nu)[\sigma'] - \nu[I]tr[\sigma']\}$$

which result in: $\varepsilon' = \begin{bmatrix} 1087.5 & -262.5 & 0 \\ -262.5 & -712.5 & 0 \\ 0 & 0 & -125 \end{bmatrix} \cdot 10^{-6}$

- The specific elastic energy is then:

$$\pi_{el} = \frac{1}{2}\left(\sigma'_{11}\varepsilon'_{1,1} + 2\sigma'_{12}\varepsilon'_{1,2} + \sigma'_{22}\varepsilon'_{2,2} + 0 + 0 + 0\right) = 0.15 \text{ MPa}$$

Solutions of V-11

a) The rotation/transformation matrix (counter-clock-wise) around x_3 is

$$\left[R_{ij}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4} \Longrightarrow \sin\theta = \frac{3}{4}\cos\theta$$

$$\sin\theta^2 + \cos\theta^2 = 1$$

Therefore, $\sin\theta = 0.6$ and $\cos\theta = 0.8$.
Strain tensor can be rotated according to R, then $\varepsilon' = R.\varepsilon.R^T$

$$\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{12} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{13} & \varepsilon'_{23} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \varepsilon'_{11} = \cos\theta^2\varepsilon_{11} + \sin\theta^2\varepsilon_{22} + 2\cos\theta.\sin\theta\varepsilon_{12}$$

$$\Longrightarrow \varepsilon_{12} = -400.10^{-5}$$

b) In the question has been stated that there is no load applied on a surface; then, there is no load along the x_3 direction. Therefore, stress components related to this direction are zero ($\sigma_{33} = \sigma_{31} = \sigma_{32} = 0$).

c) Stress components are determined by using Hooke's law, as given. First compute σ_{kk}, then compute ε_{ij} by inserting the other stress components. The validity of the stress tensor is thus confirmed by finding agreement with the known strain values.
Several other components of stress are zero, whereas the other strain components are not necessarily zero.

d) The last component of strain must be calculated to establish the full strain tensor:

$$\varepsilon_{33} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) \Longrightarrow \varepsilon_{33} = \frac{1}{2.10^5}\left(-\frac{1}{3}.270\right) = -450.10^{-6}$$

e) Using the stress tensor, eigenvalues and invariants can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 270\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 12600\text{MPa}^2$$

$$I_3 = \det(\sigma) = 0\text{MPa}^3$$

Therefore: $\sigma_I = 210$MPa, $\sigma_{II} = 60$MPa, $\sigma_{III} = 0$MPa.
Now, the Tresca and Von-Mises criteria are investigated.

$$\sigma_{Tresca} = \text{Max}\{|(\sigma_I - \sigma_{II})|, |(\sigma_{II} - \sigma_{III})|, |(\sigma_{III} - \sigma_I)|\} = \text{Max}\{150, 60, 210\} = 210\text{MPa}$$

$$\sigma_{Von-Mises} = \sqrt{\frac{(\sigma_I - \sigma_{II}) + (\sigma_{II} - \sigma_{III}) + (\sigma_{III} - \sigma_I)}{2}} = 187.35\text{MPa}$$

Thus Von-Mises is less safe than Tresca, since it the limit stress is reached at larger deformation.
f) Elastic energy and deviatoric stress and strain

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \frac{180 + 90 + 0}{3} = 90\text{MPa}$$

$$\varepsilon_{vol} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 450.10^{-6}$$

Deviatoric stress

$$\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m.\delta_{ij} \Longrightarrow \left[\hat{\sigma}_{ij}\right] = \begin{bmatrix} 180 - 90 & -60 & 0 \\ -60 & 90 - 90 & 0 \\ 0 & 0 & -90 \end{bmatrix} = \begin{bmatrix} 90 & -60 & 0 \\ -60 & 0 & 0 \\ 0 & 0 & -90 \end{bmatrix} \text{MPa}$$

Deviatoric strain

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{vol}.\delta_{ij}}{3} \Longrightarrow \left[\hat{\varepsilon}_{ij}\right] = \begin{bmatrix} 750 - 150 & -400 & 0 \\ -400 & 150 - 150 & 0 \\ 0 & 0 & -450 - 150 \end{bmatrix} .10^{-6} = \begin{bmatrix} 600 & -400 & 0 \\ -400 & 0 & 0 \\ 0 & 0 & -600 \end{bmatrix} .10^{-6}$$

Specific elastic energy

$$\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13} + 2\sigma_{23}\varepsilon_{23}\right) = 98250\frac{\text{J}}{\text{m}^3}$$

Volumetric elastic energy

$$\pi_{vol} = \frac{1}{2}\sigma_m\varepsilon_{vol} = 20250\frac{\text{J}}{\text{m}^3}$$

Deviatoric elastic energy

$$\pi_{dev} = \frac{1}{2}\hat{\sigma_{ij}}\hat{\varepsilon_{ij}} = \frac{1}{2}\left(\hat{\sigma_{11}}\hat{\varepsilon_{11}} + \hat{\sigma_{22}}\hat{\varepsilon_{22}} + \hat{\sigma_{33}}\hat{\varepsilon_{33}} + 2\hat{\sigma_{12}}\hat{\varepsilon_{12}} + 2\hat{\sigma_{13}}\hat{\varepsilon_{13}} + 2\hat{\sigma_{23}}\hat{\varepsilon_{23}}\right) = 78000\frac{\text{J}}{\text{m}^3}$$

And the specific elastic energy is the sum of volumetric and deviatoric elastic energy:

$$\pi_{el} = \pi_{vol} + \pi_{dev} = 20250 + 78000 = 98250\frac{\text{J}}{\text{m}^3}$$

Solutions of V-12

a) Given was the stress-field in absence of body forces $f_i = 0$:

$$\sigma_{11}(x_1, x_2) = \sigma_0 \left[20 + \alpha_1.\frac{x_1}{L} - 10.\frac{x_2}{L} + \alpha_2.\left(\frac{x_1}{L}\right)^2\right]$$
$$\sigma_{22}(x_1, x_2) = \sigma_0 \left[10 + 8.\frac{x_1}{L} + \beta_1.\frac{x_2}{L} + \beta_2.\left(\frac{x_2}{L}\right)^2\right]$$
$$\sigma_{12}(x_1, x_2) = \sigma_0 \left[12 - 10.\frac{x_1}{L} + 7.\frac{x_2}{L} - 8.\frac{x_1}{L}.\frac{x_2}{L}\right]$$

Using the stress-equilibrium equations, i.e. derivatives with displacement-directions with respect to the coordinate system, one obtains:

$$\frac{d}{dx_1}\sigma_{11}(x_1, x_2) + \frac{d}{dx_2}\sigma_{12}(x_1, x_2) = \sigma_0 \left[\frac{\alpha_1}{L} + 2\alpha_2.\frac{x_1}{L^2}\right] + \sigma_0 \left[\frac{7}{L} - 8.\frac{x_1}{L^2}\right] = 0$$
$$\frac{d}{dx_1}\sigma_{12}(x_1, x_2) + \frac{d}{dx_2}\sigma_{22}(x_1, x_2) = \sigma_0 \left[\frac{-10}{L} - 8.\frac{x_2}{L^2}\right] + \sigma_0 \left[\frac{\beta_1}{L} + 2\beta_2.\frac{x_2}{L^2}\right] = 0$$

From these equations, one obtains the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$

b) The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\left[\sigma_{ij}\right]$ = $\begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

Using the stress tensor, the characteristic equation can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

and from this, knowing that one eigen-value is zero, i.e. $I_3 = 0$, the principal stresses can be computed from a second order polynomial as: $\sigma_I = 28$MPa, $\sigma_{II} = 2$MPa, $\sigma_{III} = 0$MPa. This is a plane-stress state with all stresses on the x_3-surface being equal to zero.

c) And the principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$ is directly visible from the tensor, due to zero- shear stresses in the x_3-direciton, while the others require to insert:

Direction of the major stress $\sigma_I = 28$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and normalization: ${n_1^{(I)}}^2 + {n_2^{(I)}}^2 + {n_3^{(I)}}^2 = 1$

$-8n_1^{(I)} + 12n_2^{(I)} = 0$ $\quad n_1^{(I)} = -(3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

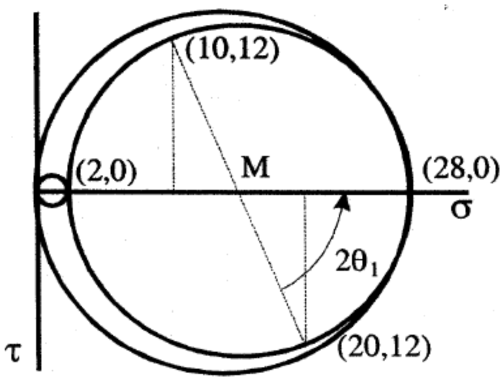
$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

Direction of the middle (was not asked, for completeness) $\sigma_{II} = 2$MPa

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and normalization: $n_1^{(II)^2} + n_2^{(II)^2} + n_3^{(II)^2} = 1$

$18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9)+1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$



d) Mohr's circle
(consider only the two non-zero eigenvalues that characterise the plane-stress state in point P):
The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{20+10}{2} = 15$ MPa,
and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13$ MPa.
The eigenvalues are therefore:
$\sigma_I = M + R = 28$ MPa, $\sigma_{II} = C - R = 2$MPa.
The eigen-directions are:
$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{24}{10} = 2.4 \Longrightarrow \theta_I = (1/2)\arctan(2.4) = 67.38°/2 = 33.69°$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$;
and $\theta_{II} = (180° + 67.3°)/2 = 247.3°/2 = 123.7° = \arccos(-2/\sqrt{13})$.

The max. shear stress is just the radius: $\tau^{max} = R = 13$ MPa

e) Failure criteria according to the (double) maximal shear stress:
$\tau_{max} = \frac{1}{2} \mid \sigma_{max} - \sigma_{min} \mid = 14$ MPa, $\sigma_{eq}^{Tresca} = 2\tau_{max} = 28$ MPa

f) Failure criteria according to the shape-change (distortion) energy:
$\sigma_{eq}^{von-Mises} = \sqrt{\frac{1}{2}[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2]} \approx 27$ MPa

$\sigma_{eq}^{von-Mises} < \sigma_{tresca}$ thus the Tresca criterion is safer, since limit is reached earlier.

g) $\varepsilon_{ij} = \frac{1}{E}.[(1+\nu).\sigma_{ij} - \nu\sigma_{kk}.\delta_{ij})$, with $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$
from the given stress, using $E = 2.10^5$ MPa and $\nu = 1/4$, one obtains:
$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = 20/E - (10/4)/E = (35/2)E^{-1}$,
$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{33}}{E} = 10/E - (20/4)/E = 5E^{-1}$,
$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} = -(10/4)/E - (20/4)/E = -(15/2)E^{-1}$,

$\varepsilon_{12} = \frac{\sigma_{12}}{2G} = 6/G = 15E^{-1}$ (with $G = \frac{E}{2(1+\nu)} = 2E/5$) ,
and $\varepsilon_{13} = \varepsilon_{23} = 0$. Note that $\varepsilon_{33} \neq 0$, even though $\sigma_{33} = 0$.

$$\left[\varepsilon\right] = \begin{bmatrix} 35/2 & 15 & 0 \\ 15 & 10/2 & 0 \\ 0 & 0 & -15/2 \end{bmatrix} \text{MPa} E^{-1} = \begin{bmatrix} 35 & 30 & 0 \\ 30 & 10 & 0 \\ 0 & 0 & -15 \end{bmatrix} \frac{10^{-5}}{4} = \begin{bmatrix} 87.5 & 75 & 0 \\ 75 & 25 & 0 \\ 0 & 0 & -37.5 \end{bmatrix} 10^{-6}\,.$$

Elastic energy:

$$\pi_{el} = 0.5 \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\varepsilon_{ij} = 1.9\,10^{-3}\,\text{MPa} \left(= \frac{\text{Energy}}{\text{Volume}}\right)$$

in detail (diamonds are not needed):

$$\begin{aligned}
\pi_{el} &= \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 87.5 & 75 & 0 \\ 75 & 25 & 0 \\ 0 & 0 & -37.5 \end{bmatrix} \frac{10^{-6}}{2}\,\text{MPa} \\
&= \text{tr} \begin{bmatrix} 20 * 87.5 + 12 * 75 & \diamond & 0 \\ \diamond & 12 * 75 + 10 * 25 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{10^{-6}}{2}\,\text{MPa} \\
&= (1750 + 900 + 900 + 250)\frac{10^{-6}}{2}\,\text{MPa} = 1.9\,10^{-3}\,\text{MPa}
\end{aligned}$$

Solutions V-13

a) **isotropic**: means that the property is direction-independent, e.g. the material behaviour is in all directions the same – this is valid for randomly structured, disordered materials, but not valid e.g. for fibre-reinforced (anisotropic) materials or for polymers short after deformation, since those build-up anisotropy due to their interconnected chains.

b) **elastic**: means that the deformation (e.g. due to applied stress) is restored when the stress is removed – mostly valid for small strains too, with exception of materials like rubber that can be (nonlinear) elastic for very larges strain; beyond elasticity one can observe plastic deformations, i.e., the deformation is not restored when the applied stress is removed;

c) **linear**: a linear relation between stress and strain – mostly valid for small strains in all types of materials, with exception of some complex materials that might behave non-linear already at rather small strain;

d) **Compute the principal stresses in point P, where the invariants are not explicitly asked for here,** I_1, I_2, I_3 (hint: solve the characteristic equation $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$).

$$|\sigma_{ij} - \sigma\delta_{ij}| = 0$$

$$det\begin{pmatrix} 60-\sigma & 0 & 0 \\ 0 & 50-\sigma & 20 \\ 0 & 20 & 20-\sigma \end{pmatrix} = (60-\sigma)\{(50-\sigma)(20-\sigma) - 400\}$$

$$= (60-\sigma)\{\sigma^2 - 70\sigma - 300\}$$

$$\rightarrow \sigma = \frac{70 \pm \sqrt{4900 - 4 * 600}}{2} = 35 \pm 25 \rightarrow \sigma_1 = 60, \sigma_2 = 60;\ \sigma_1 = 10, \text{all in units MPa}$$

Not asked for:

$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 130$ MPa

$I_1 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 = 3600 + 600 + 600 = 4800$ MPa^2

$I_1 = \sigma_1\sigma_2\sigma_3 = 36000$ MPa^3

e) **Draw the circle of Mohr** (asked for, not shown here) **and compare the results with d)**

The centre of the circle in the 2-3-plane is the mean of the eigenvalues M=(50+20)/2=35 MPa. The radius of the circle is (Pythagoras) $R^2 = (50-M)^2+20^2 = 225+400 = 625 = 25^2$ => R = 25 MPa. So that the two eigenvalues are: M+R=60 and M-R=10 MPa.
(This can always be done analytically, but if the circle is asked for, better draw it.)

f) **Determine the eigen-direction of the smallest eigenvalue**

$|\sigma_{ij} - \sigma\delta_{ij}|n_j = 0$ and normalization: $n_j n_j = 1$

$$\begin{pmatrix} 60-10 & 0 & 0 \\ 0 & 50-10 & 20 \\ 0 & 20 & 20-10 \end{pmatrix}\begin{pmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{pmatrix} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{pmatrix}\begin{pmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{pmatrix} \rightarrow 40n_2^{(3)} + 20n_3^{(3)} = 0$$

$$\rightarrow n_2^{(3)} = -\tfrac{1}{2}n_3^{(3)} \text{ and norm: } 1 = \left(\tfrac{1}{2}n_3^{(3)}\right)^2 + \left(n_3^{(3)}\right)^2 \rightarrow n_3^{(3)} = \tfrac{2}{\sqrt{5}} \rightarrow \begin{pmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{pmatrix} = \pm\begin{pmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

To compare the direction of this unit-vector with the circle of Mohr construction, we compare its orientation in the 2-3-plane relative to the 2-direction: $\theta_3 = \arctan\left(n_3^{(3)}/n_2^{(3)}\right) = \arctan(-2) = -63.4^{\circ}$, equivalent to the opposite direction $\theta_3 + \pi = 116.6^{\circ}$, while the circle provides: $\tan 2\theta_{M2} = \frac{2\sigma_{yz}}{\sigma_{yy}-\sigma_{zz}} = \frac{4}{3} \rightarrow \theta_{M2} = 26.6^{o}$, which is perpendicular, because it relates to the direction of the larger eigen-value, θ_2. We are interested in the orientation in relation to the smaller eigenvalue, thus $\theta_3 = \theta_{M2} + \pi = 116^{\circ}$.

g) Compute the components of the strain tensor in point P

$$
\begin{aligned}
[\varepsilon_{ij}] &= \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right] \\
&= \frac{1+\nu}{E}\left[\begin{pmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{pmatrix} - \frac{\nu}{1+\nu}\begin{pmatrix} 130 & 0 & 0 \\ 0 & 130 & 0 \\ 0 & 0 & 130 \end{pmatrix}\right] \text{MPa} \\
&= \frac{5/4}{2.10^5}\left[\begin{pmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{pmatrix} - \frac{1/4}{5/4}\begin{pmatrix} 130 & 0 & 0 \\ 0 & 130 & 0 \\ 0 & 0 & 130 \end{pmatrix}\right] \\
&= \frac{5}{8.10^5}\left[\begin{pmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 130 & 0 & 0 \\ 0 & 130 & 0 \\ 0 & 0 & 130 \end{pmatrix}\right] \\
&= \frac{1}{8.10^5}\left[\begin{pmatrix} 300 & 0 & 0 \\ 0 & 250 & 100 \\ 0 & 100 & 100 \end{pmatrix} - \begin{pmatrix} 130 & 0 & 0 \\ 0 & 130 & 0 \\ 0 & 0 & 130 \end{pmatrix}\right] \\
&= \frac{1}{8.10^5}\left[\begin{pmatrix} 170 & 0 & 0 \\ 0 & 120 & 100 \\ 0 & 100 & -30 \end{pmatrix}\right] = \frac{1}{4.10^5}\left[\begin{pmatrix} 85 & 0 & 0 \\ 0 & 60 & 50 \\ 0 & 50 & -15 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 21.25 & 0 & 0 \\ 0 & 15 & 12.5 \\ 0 & 12.5 & -3.75 \end{pmatrix}\right] 10^{-5} \sim \left[\begin{pmatrix} 2.13 & 0 & 0 \\ 0 & 1.50 & 1.25 \\ 0 & 1.25 & -0.38 \end{pmatrix}\right] 10^{-4}
\end{aligned}
$$

h) Compute the volumetric strain in point P

Note that this question can be answered much faster, in case no time for the full strain in g)

$$
\begin{aligned}
[\varepsilon_{kk}] &= \frac{1+\nu}{E}\left[\sigma_{kk} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{mm}\right] \\
&= \frac{1+\nu}{E}\left[1 - \frac{3\nu}{1+\nu}\right]\sigma_{kk} = \frac{1+\nu}{E}\left[\frac{1-2\nu}{1+\nu}\right]\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{kk} \\
&= \frac{1/2}{E}130\text{MPa} = \frac{1}{4.10^5\text{MPa}}130\text{MPa} = 32.5\ 10^{-5} = 3.25\ 10^{-4}
\end{aligned}
$$

But the solution can also be taken from the trace of the full strain tensor in g)

i) What is the largest change of angle in point P?

Inserting the eigenvalues (1;2;3) into the law of Hooke, gives: $\gamma_{max} = \varepsilon_1 - \varepsilon_3 = 3.125\ 10^{-4}$

$$
[\varepsilon_{1;2;3}] = \frac{1+\nu}{E}\left[\sigma_{1;2;3} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{1;2;3}\right] = \frac{5}{8.10^5}\left[\sigma_{1;2;3} - \frac{1}{5}130\delta_{1;2;3}\right] = \frac{5}{8}[60;10-26]10^{-5}
$$

$\varepsilon_1 = \varepsilon_2 = \frac{5}{8}[34]10^{-5} = [170/8]10^{-5} = [2.125]10^{-4}$; $\varepsilon_3 = \frac{5}{8}[-16]10^{-5} = [-1.0]10^{-4}$