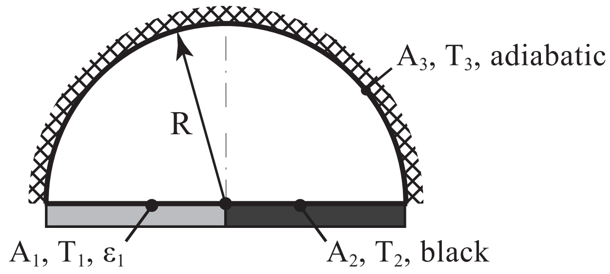
Exercise IV.5 (Cupola ⋆⋆⋆):

Both semi-circular slabs A_1 and A_2 of the geometric configuration depicted below are conditioned to maintain a constant temperature of and, respectively. Surface A_2 can be considered a black body, and the hemispherical surface A_3 above the slabs is adiabatic.



Given parameters:

- Temperature of slab 1: $T_1 = 150\,°\mathrm{C}$
- Temperature of slab 2: $T_2 = 20\,°\mathrm{C}$
- Emissivity of slab 1: $\varepsilon_1 = 0.6$
- Radius of the dome: $R = 3$ m

Hints:

- Surfaces A_1 and A_3 are grey bodies and emit diffuse radiation.
- The hemispherical volume is filled with a vacuum.
- All surfaces are opaque.

Tasks:

a) Compute the amount of heat transferred through radiation between the surfaces A_1 and A_2 (= net radiative flux to surface A_2).

b) Which temperature T_3 is obtained for surface A_3?

Solution IV.5 (Cupola ⋆ ⋆ ⋆)**:**

Task a)

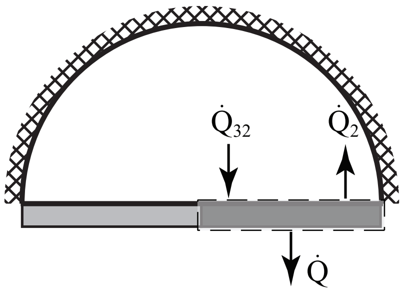
Slab 2 is maintained at a constant temperature of 20 °C. In contrast, slab 1 is at a higher temperature. Consequently, slab 2 must be cooled to sustain this lower temperature. According to thermodynamics, it is understood that an equilibrium situation would eventually occur over time, resulting in both slabs reaching the same temperature.

Therefore the amount of additional cooling $\dot{Q}$ is exactly equal to the net rate of heat transferred through radiation from slab 1 to slab 2. The amount of heat transferred $\dot{Q}$ to slab 2, can be determined from an outer or inner energy balance around body 2.

Outer energy balance:

1 **Setting up the balance:**

Partially, the surface brightness of surface A_3 is radiated on slab 2. Besides, slab 2 radiates its surface brightness towards surface 3. To maintain surface 2 at its lower temperature, it will be externally cooled at a rate of $\dot{Q}$.

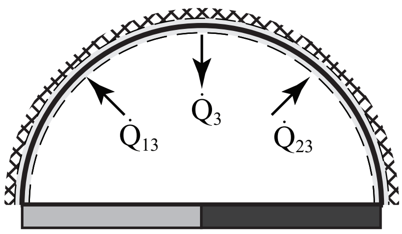


The outer energy balance around slab 2 can be written as:

$$0 = \dot{Q}_{32} - \dot{Q}_2 - \dot{Q} \tag{IV.5.1}$$

The surface brightness of body 3 is intricate and requires establishing an outer energy balance around body 3. It cannot be determined directly using the equation for a black body radiator, as the temperature of body 3 is unknown.

Surfaces A_1, A_2 and A_3 partially radiate their surface brightness on surface A_3 and therefore the sum of these fluxes equals the incoming rate of heat transfer.



$$0 = \dot{Q}_{13} + \dot{Q}_{23}\dot{Q}_{33} - \dot{Q}_3, \tag{IV.5.2}$$

where the respective fluxes are defined as: $\dot{Q}_{13} = \Phi_{13}\dot{Q}_1$, $\dot{Q}_{23} = \Phi_{23}\dot{Q}_2$, and $\dot{Q}_{33} = \Phi_{33}\dot{Q}_3$.

Rewriting the outer energy balance around surface 3 yields:

$$\dot{Q}_3 = \frac{\dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23}}{1 - \Phi_{33}} \tag{IV.5.3}$$

2 Defining the elements within the balance:

The surface brightness for the slab equals the surface brightness of a black body radiator:

$$\begin{aligned} \dot{Q}_2 &= \sigma A_2 T_2^4 \\ &= 5.67 \cdot 10^{-8}\ [\mathrm{W/m^2K^4}] \cdot \frac{\pi \cdot 3^2}{2}\ [\mathrm{m^2}] \cdot 293^4\ [\mathrm{K^4}] = 5{,}920\ [\mathrm{W}] \end{aligned} \tag{IV.5.4}$$

The part of the surface brightness transferred from surface 3 to surface 2 can be written as:

$$\dot{Q}_{32} = \Phi_{32}\dot{Q}_3 \tag{IV.5.5}$$

To determine view factor Φ_{32}, view factor Φ_{23} is needed. Surface 2 only sees surface 3, and therefore:

$$\Phi_{23} = 1 \tag{IV.5.6}$$

Using the reciprocity rule

$$\begin{aligned} \Phi_{32} A_3 &= \Phi_{23} A_2 \\ \Rightarrow \Phi_{32} &= \Phi_{23}\frac{\frac{1}{2}\pi R^2}{\frac{4}{2}\pi R^2} = \frac{1}{4} \end{aligned} \tag{IV.5.7}$$

Once the view factor has been determined, the surface brightness $\dot{Q}_3$ needs to be assessed.

To obtain the surface brightness of body 3, the respective view factors and the surface brightness of slab 1 need to be determined.

From the sketch it can be seen that body 1 can only see body 3, and therefore:

$$\Phi_{13} = 1 \tag{IV.5.8}$$

From symmetry it can be seen that Φ_{32} and Φ_{31} are identical:

$$\Phi_{31} = \frac{1}{4} \tag{IV.5.9}$$

View factor Φ_{33} can be determined from the summation rule:

$$\begin{aligned} \Phi_{31} + \Phi_{32} + \Phi_{33} &= 1 \\ \Rightarrow \Phi_{33} &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \end{aligned} \tag{IV.5.10}$$

The surface brightness of surface 1 can be expressed as:

$$\dot{Q}_1 = \dot{Q}_{\epsilon,1} + \dot{Q}_{\rho,1} + \dot{Q}_{\tau,1} \tag{IV.5.11}$$

Where the emissive term can be written as:

$$\dot{Q}_{\epsilon,1} = \epsilon_1 \sigma A_1 T_1^4, \tag{IV.5.12}$$

the reflective term yields from the partial surface brightness of body 3 being reflected:

$$\dot{Q}_{\rho,1} = \rho_1 \Phi_{31} \dot{Q}_3, \tag{IV.5.13}$$

and since all bodies are opaque no radiation is transmitted:

$$\dot{Q}_{\tau,1} = \rho_1 \Phi_{31} \dot{Q}_3. \tag{IV.5.14}$$

Note that slab 1 is a grey body and thus Kirchoff's law yields that

$$\epsilon_1 = \alpha_1 \tag{IV.5.15}$$

Moreover, considering that all surfaces are opaque, and utilizing the relationship between absorptivity, reflectivity, and transmissivity, it results:

$$\rho_1 = 1 - \alpha_1 \tag{IV.5.16}$$

Inserting the expressions of the surface brightness into equation (IV.7.12) yields the surface brightness of body 3 and rearranging yields:

$$\begin{aligned}\dot{Q}_3 &= \frac{\epsilon_1 \sigma A_1 T_1^4 + \dot{Q}_2}{1 - \Phi_{33} - (1 - \epsilon_1)\,\Phi_{31}} \\ &= \frac{0.6\ [-] \cdot 5.67 \cdot 10^{-8}\ \left[\mathrm{W/m^2K^4}\right] \cdot \frac{1}{2}\pi \cdot 3^2\ [\mathrm{m^2}] \cdot 423^4\ [\mathrm{K^4}] + 5{,}920\ [\mathrm{W}]}{\left(1 - \frac{1}{2} - (1 - 0.6) \cdot \frac{1}{4}\right)\ [-]} = 53{,}348\ [\mathrm{W}]\end{aligned} \tag{IV.5.17}$$

Conclusion

3 Inserting and rearranging:

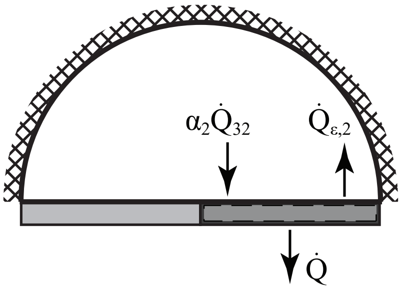
Substituting the determined surface brightness values into the outer energy balance around body 2 and rearranging determines the net heat transferred through radiation between surfaces 1 and 2:

$$\begin{aligned}\dot{Q} &= \Phi_{32} \dot{Q}_3 - \dot{Q}_2 \\ &= \frac{1}{4}\ [-] \cdot 53{,}348\ [\mathrm{W}] - 5{,}920\ [\mathrm{W}] = 7{,}417\ [\mathrm{W}]\end{aligned} \tag{IV.5.18}$$

Inner energy balance:

1 Setting up the balance:

Partially, the surface brightness of surface A_3 is radiated on slab 2 and absorved. Besides, slab 2 emits radiation towards surface 3. To maintain surface 2 at its lower temperature, it will be externally cooled at a rate of $\dot{Q}$.

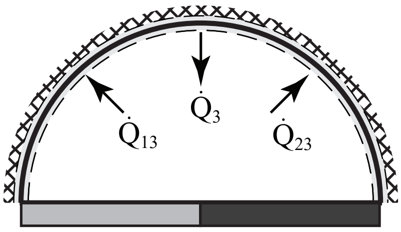


The inner energy balance around slab 2 can be written as:

$$0 = \alpha_2 \dot{Q}_{32} - \dot{Q}_{\epsilon,2} - \dot{Q} \tag{IV.5.19}$$

The surface brightness of body 3 is intricate and requires establishing an outer energy balance around body 3. It cannot be determined directly using the equation for a black body radiator, as the temperature of body 3 is unknown.

Surfaces A_1, A_2 and A_3 partially radiate their surface brightness on surface A_3 and therefore the sum of these fluxes equals the incoming rate of heat transfer.



$$0 = \dot{Q}_{13} + \dot{Q}_{23}\dot{Q}_{33} - \dot{Q}_3, \tag{IV.5.20}$$

where the respective fluxes are defined as: $\dot{Q}_{13} = \Phi_{13}\dot{Q}_1$, $\dot{Q}_{23} = \Phi_{23}\dot{Q}_2$, and $\dot{Q}_{33} = \Phi_{33}\dot{Q}_3$.

Rewriting the inner energy balance around surface 3 yields:

$$\dot{Q}_3 = \frac{\dot{Q}_1\Phi_{13} + \dot{Q}_2\Phi_{23}}{1 - \Phi_{33}} \tag{IV.5.21}$$

2 Defining the elements within the balance:

The emitted radiation of slab 2 equals the surface brightness of a black body radiator:

$$\begin{aligned} \dot{Q}_{\epsilon,2} &= \sigma A_2 T_2^4 \\ &= 5.67 \cdot 10^{-8}\ [\mathrm{W/m^2K^4}] \cdot \frac{\pi \cdot 3^2}{2}\ [\mathrm{m^2}] \cdot 293^4\ [\mathrm{K^4}] = 5{,}920\ [\mathrm{W}] \end{aligned} \tag{IV.5.22}$$

The part of the surface brightness transferred from surface 3 to surface 2 can be written as:

$$\alpha_2 \dot{Q}_{32} = \Phi_{32} \dot{Q}_3, \tag{IV.5.23}$$

where $\alpha_2 = 1$ since slab 2 is a black body.

To determine view factor Φ_{32}, view factor Φ_{23} is needed. Surface 2 only sees surface 3, and therefore:

$$\Phi_{23} = 1 \tag{IV.5.24}$$

Using the reciprocity rule

$$\Phi_{32} A_3 = \Phi_{23} A_2 \tag{IV.5.25}$$
$$\Rightarrow \Phi_{32} = \Phi_{23} \frac{\frac{1}{2}\pi R^2}{\frac{4}{2}\pi R^2} = \frac{1}{4}$$

Once the view factor has been determined, the surface brightness $\dot{Q}_3$ needs to be assessed.

To obtain the surface brightness of body 3, the respective view factors and the surface brightness of slab 1 need to be determined.

From the sketch it can be seen that body 1 can only see body 3, and therefore:

$$\Phi_{13} = 1 \tag{IV.5.26}$$

From symmetry it can be seen that Φ_{32} and Φ_{31} are identical:

$$\Phi_{31} = \frac{1}{4} \tag{IV.5.27}$$

View factor Φ_{33} can be determined from the summation rule:

$$\Phi_{31} + \Phi_{32} + \Phi_{33} = 1 \tag{IV.5.28}$$
$$\Rightarrow \Phi_{33} = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

The surface brightness of surface 1 can be expressed as:

$$\dot{Q}_1 = \dot{Q}_{\epsilon,1} + \dot{Q}_{\rho,1} + \dot{Q}_{\tau,1} \tag{IV.5.29}$$

Where the emissive term can be written as:

$$\dot{Q}_{\epsilon,1} = \epsilon_1 \sigma A_1 T_1^4, \tag{IV.5.30}$$

the reflective term yields from the partial surface brightness of body 3 being reflected:

$$\dot{Q}_{\rho,1} = \rho_1 \Phi_{31} \dot{Q}_3, \tag{IV.5.31}$$

and since all bodies are opaque no radiation is transmitted:

$$\dot{Q}_{\tau,1} = \rho_1 \Phi_{31} \dot{Q}_3. \tag{IV.5.32}$$

Note that slab 1 is a grey body and thus Kirchoff's law yields that

$$\epsilon_1 = \alpha_1 \tag{IV.5.33}$$

Moreover, considering that all surfaces are opaque, and utilizing the relationship between absorptivity, reflectivity, and transmissivity, it results:

$$\rho_1 = 1 - \alpha_1 \tag{IV.5.34}$$

Inserting the expressions of the surface brightness into equation (IV.7.12) yields the surface brightness of body 3 and rearranging yields:

$$\begin{aligned}\dot{Q}_3 &= \frac{\epsilon_1 \sigma A_1 T_1^4 + \dot{Q}_2}{1 - \Phi_{33} - (1-\epsilon_1)\,\Phi_{31}} \\ &= \frac{0.6\ [-] \cdot 5.67 \cdot 10^{-8}\ \left[\mathrm{W/m^2K^4}\right] \cdot \frac{1}{2}\pi \cdot 3^2\ \left[\mathrm{m^2}\right] \cdot 423^4\ \left[\mathrm{K^4}\right] + 5{,}920\ [\mathrm{W}]}{\left(1 - \frac{1}{2} - (1-0.6)\cdot\frac{1}{4}\right)\ [-]} = 53{,}348\ [\mathrm{W}]\end{aligned} \tag{IV.5.35}$$

Conclusion

3 Inserting and rearranging:

Substituting the determined surface brightness values into the outer energy balance around body 2 and rearranging determines the net heat transferred through radiation between surfaces 1 and 2:

$$\begin{aligned}\dot{Q} &= \Phi_{32}\dot{Q}_3 - \dot{Q}_2 \\ &= \frac{1}{4}\ [-] \cdot 53{,}348\ [\mathrm{W}] - 5{,}920\ [\mathrm{W}] = 7{,}417\ [\mathrm{W}]\end{aligned} \tag{IV.5.36}$$

Task b)

In such a scenario, where the body doesn't transmit radiation and is adiabatic, the surface brightness essentially becomes equal to the emission of a black body radiator. This is because the body, being adiabatic, absorbs all incident radiation and emits radiation at maximum efficiency across all wavelengths, similar to a black body radiator. Therefore the surface brightness of body 3 can be written as a black body radiator:

$$\dot{Q}_3 = \sigma A_3 T_3^4, \tag{IV.5.37}$$

which yielded to have a value of 53,348 W in the previous task.

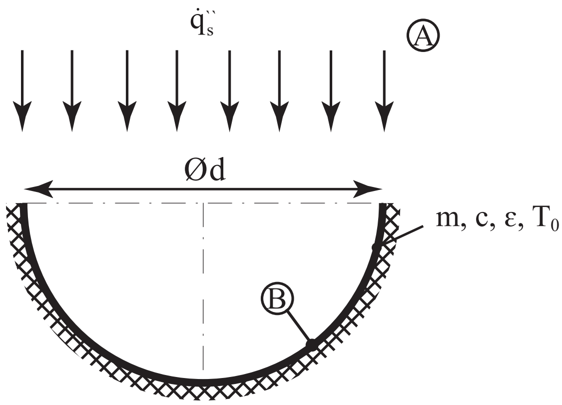
Conclusion

Rewriting yields the temperature of the body 3:

$$\begin{aligned}T_3 &= \sqrt[4]{\frac{\dot{Q}_3}{\sigma A_3}} \\ &= \sqrt[4]{\frac{53{,}348\ [\mathrm{W}]}{5.67 \cdot 10^{-8}\ \left[\mathrm{W/m^2K^4}\right] \cdot 2\pi \cdot 3^2\ \left[\mathrm{m^2}\right]}} = 359\ [\mathrm{K}]\end{aligned} \tag{IV.5.38}$$

Exercise IV.6 (Pokè bowl ⋆ ⋆ ⋆):

An empty bowl, that is used for serving the typical traditional Hawaiian dish called pokè bowl, has the homogeneous temperature T_0 and is adiabatically insulated at its convex side. At the time t_0, the bowl is suddenly exposed to parallel radiation from the sun.



Given parameters:

- Mass of the bowl: m
- Specific heat capacity of the bowl: c
- Emissivity of the bowl: $\epsilon \approx 0.5$
- Starting temperature of the bowl: T_0
- Diameter of the bowl: d
- Heat flux of the solar radiation on the ground: $\dot{q}''_{\mathrm{S}}$
- View factor of the bowl to the ambient: Φ_{BA}
- View factor of the bowl to itself: Φ_{BB}

Hints:

- The bowl radiates grey and diffuse and has a homogeneous temperature at any time.
- Influences from the ambient or the atmosphere can be neglected.
- The sun is a black body.

Tasks:

a) Determine the surface brightness of the bowl $\dot{Q}_{\mathrm{B}}$.

   Hint: In a nonsteady state, the surface brightness of a grey, adiabatic body is not the same as the surface brightness of a black body.

b) Derive the differential equation for the temperature as a function of time and the necessary initial condition to solve this differential equation.

c) Determine the steady-state final temperature T_{S} of the bowl.

d) Draw the temperature as a function of time qualitatively.

Solution IV.6 (Pokè bowl ⋆ ⋆ ⋆):

Task a)

The surface brightness encompasses both the emission and reflection of solar radiation, including reflection from the bowl itself.

1 **Setting up the balance:**

The surface brightness of the bowl can be expressed as:

$$\dot{Q}_{\mathrm{B}} = \dot{Q}_{\epsilon,\mathrm{B}} + \dot{Q}_{\rho,\mathrm{B}} + \dot{Q}_{\tau,\mathrm{B}} \tag{IV.6.1}$$

2 **Defining the elements within the balance:**

Where the emission of the bowl can be described as a grey body radiator:

$$\begin{aligned} \dot{Q}_{\epsilon,\mathrm{B}} &= \epsilon\sigma A_{\mathrm{B}} T_{\mathrm{B}}(t)^4 \\ &= \epsilon\sigma \frac{\pi d^2}{2} T_{\mathrm{B}}(t)^4 \end{aligned} \tag{IV.6.2}$$

In this scenario, transmission is negligible as the convex surface of the bowl is adiabatic:

$$\dot{Q}_{\tau,\mathrm{B}} = 0 \tag{IV.6.3}$$

The reflected radiation from the bowl results from the reflected incident radiation from the sun and the bowl itself:

$$\dot{Q}_{\rho,\mathrm{B}} = \rho\left(\dot{Q}_{\mathrm{BB}} + \dot{Q}_{\mathrm{SB}}\right), \tag{IV.6.4}$$

where $\dot{Q}_{\mathrm{BB}}$ states the surface brightness from the bowl that falls on the surface itself, and $\dot{Q}_{\mathrm{SB}}$ describes the radiation from the sun that falls on the bowl.

For a grey body that is not transmitting any radiation ($\tau = 0$) the reflection term yields to be:

$$\begin{aligned} \epsilon + \rho + \tau &= 1 \\ \Rightarrow \rho &= 1 - \epsilon \end{aligned} \tag{IV.6.5}$$

The part of the surface brightness that falls on the bowl itself is written as:

$$\dot{Q}_{\mathrm{BB}} = \Phi_{\mathrm{BB}} \dot{Q}_{\mathrm{B}} \tag{IV.6.6}$$

The radiation transferred from the sun yields the product of the heat flux density and the incident area:

$$\begin{aligned} \dot{Q}_{\mathrm{SB}} &= \dot{q}''_{\mathrm{S}} A_{\mathrm{incident}} \\ &= \dot{q}''_{\mathrm{S}} \frac{\pi d^2}{4}, \end{aligned} \tag{IV.6.7}$$

where the incident area is equal to that of a circle.

Conclusion

③ **Inserting and rearranging:**

Inserting and rewriting yields:

$$\dot{Q}_{\mathrm{B}} = \frac{\epsilon\sigma\frac{\pi d^2}{2}T_{\mathrm{B}}(t)^4 + (1-\epsilon)\,\dot{q}_{\mathrm{S}}''\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_{\mathrm{BB}}} \tag{IV.6.8}$$

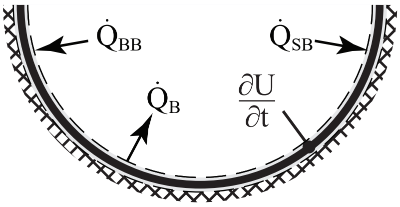
Task b)

To derive the differential equation for the temperature as a function of time, an outer or inner energy balance around the bowl should be established.

Outer energy balance:

① **Setting up the balance:**

The bowl receives radiation from the sun, as well as partially its surface brightness which it radiates.



The outer energy balance reads:

$$\frac{\partial U}{\partial t} = \dot{Q}_{\mathrm{BB}} + \dot{Q}_{\mathrm{SB}} - \dot{Q}_{\mathrm{B}} \tag{IV.6.9}$$

② **Defining the elements within the balance:**

The temporal change in the inner energy of the bowl is described in terms of the temperature, mass, and specific heat capacity:

$$\frac{\partial U}{\partial t} = mc\frac{\partial T_{\mathrm{B}}}{\partial t} \tag{IV.6.10}$$

Furthermore, the terms $\dot{Q}_{\mathrm{BB}}$, $\dot{Q}_{\mathrm{SB}}$, and $\dot{Q}_{\mathrm{B}}$ had been defined in the previous task.

Conclusion

③ **Inserting and rearranging:**

Inserting the definition of the fluxes and internal energy yields:

$$mc\frac{\partial T_{\mathrm{B}}}{\partial t} = \frac{\epsilon\sigma\frac{\pi d^2}{2}T_{\mathrm{B}}(t)^4 + (1-\epsilon)\,\dot{q}_{\mathrm{S}}''\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_{\mathrm{BB}}}\left(\Phi_{\mathrm{BB}}-1\right) + \dot{q}_{\mathrm{s}}''\frac{\pi d^2}{4} \tag{IV.6.11}$$

④ **Defining the boundary and/or initial conditions:**

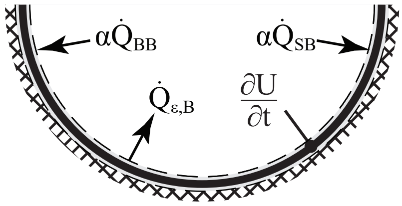
To solve the differential equation, one initial condition is required, which reads as follows:

$$T(t=t_0) = T_0 \tag{IV.6.12}$$

Inner energy balance:

1 **Setting up the balance:**

The bowl receives radiation from the sun, as well as partially its surface brightness which it radiates.



The inner energy balance reads:

$$\frac{\partial U}{\partial t} = \alpha \dot{Q}_{\mathrm{BB}} + \alpha \dot{Q}_{\mathrm{SB}} - \dot{Q}_{\epsilon,\mathrm{B}} \tag{IV.6.13}$$

2 **Defining the elements within the balance:**

The temporal change in the inner energy of the bowl is described in terms of the temperature, mass, and specific heat capacity:

$$\frac{\partial U}{\partial t} = mc\frac{\partial T_{\mathrm{B}}}{\partial t} \tag{IV.6.14}$$

Furthermore, the terms $\dot{Q}_{\mathrm{BB}}$, $\dot{Q}_{\mathrm{SB}}$, and $\dot{Q}_{\epsilon,\mathrm{B}}$ had been defined in the previous task.

Since the bowl acts as a grey body, using Kirchoff's law it yields that:

$$\alpha = \epsilon \tag{IV.6.15}$$

Conclusion

3 **Inserting and rearranging:**

Inserting the definition of the fluxes and internal energy yields:

$$mc\frac{\partial T_{\mathrm{B}}}{\partial t} = \frac{\epsilon^2 \sigma \frac{\pi d^2}{2} T_{\mathrm{B}}(t)^4 + \left(\epsilon - \epsilon^2\right) \dot{q}_{\mathrm{S}}'' \frac{\pi d^2}{4}}{1 - (1-\epsilon)\,\Phi_{\mathrm{BB}}} \Phi_{\mathrm{BB}} + \epsilon \left(\dot{q}_{\mathrm{s}}'' \frac{\pi d^2}{4} - \sigma \frac{\pi d^2}{2} T_{\mathrm{B}}(t)^4 \right) \tag{IV.6.16}$$

4 **Defining the boundary and/or initial conditions:**

To solve the differential equation, one initial condition is required, which reads as follows:

$$T(t = t_0) = T_0 \tag{IV.6.17}$$

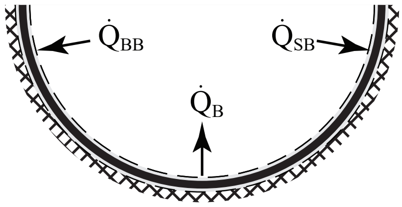
Task c)

After some time the system will reach equilibrium conditions and so the internal energy will not change anymore.

To determine the equilibrium temperature, an energy balance should be derived that describes the equilibrium state. This energy balance is similar to the energy balance derived in question b), but without the transient term describing the change in internal energy over time.

1 **Setting up the balance:**

The bowl receives radiation from the sun, as well as partially its surface brightness which it radiates.



Therefore, the steady-state outer energy balance equals:

$$0 = \dot{Q}_{\mathrm{BB}} + \dot{Q}_{\mathrm{SB}} - \dot{Q}_{\mathrm{B}} \tag{IV.6.18}$$

2 **Defining the elements within the balance:**

The surface brightness of the bowl now can be written as: Inserting and rewriting yields:

$$\dot{Q}_{\mathrm{B}} = \frac{\epsilon\sigma\frac{\pi d^2}{2}T_{\mathrm{S}}^4 + (1-\epsilon)\,\dot{q}_{\mathrm{S}}''\frac{\pi d^2}{4}}{1-(1-\epsilon)\,\Phi_{\mathrm{BB}}} \tag{IV.6.19}$$

The fluxes $\dot{Q}_{\mathrm{BB}}$ and $\dot{Q}_{\mathrm{SB}}$ have already been defined in the previous tasks.

Conclusion

3 **Inserting and rearranging:**

$$T_{\mathrm{S}} = \sqrt[4]{\frac{\dot{q}_{\mathrm{S}}''}{2\sigma}\left[\frac{1}{(1-\Phi_{\mathrm{BB}})}\right]} \tag{IV.6.20}$$

Alternative solution:

A different result could have been obtained by simplifying the fact that in the stationary case, the surface brightness of the bowl is equal to that of a black body since the body is gray and its back is adiabatic. This would yield:

2 **Defining the elements within the balance:**

$$\dot{Q}_{\mathrm{B}} = \sigma A_{\mathrm{B}} T_{\mathrm{S}} \tag{IV.6.21}$$

3 Inserting and rearranging:

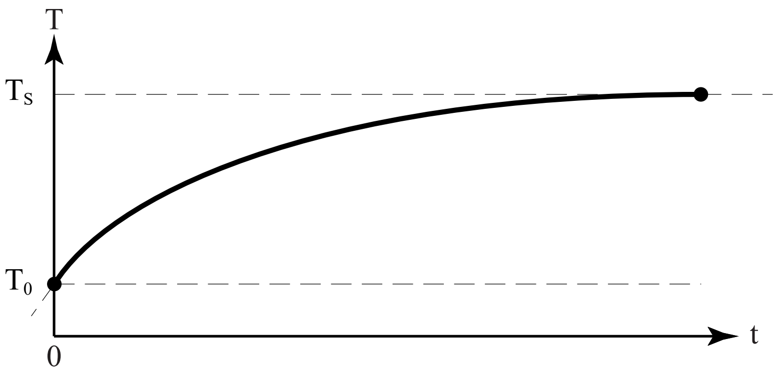
This approach would have yielded the identical expression:

$$T_S = \sqrt[4]{\frac{\dot{q}''_S}{2\sigma}\left[\frac{1}{(1-\Phi_{BB})}\right]} \tag{IV.6.22}$$

Task d)

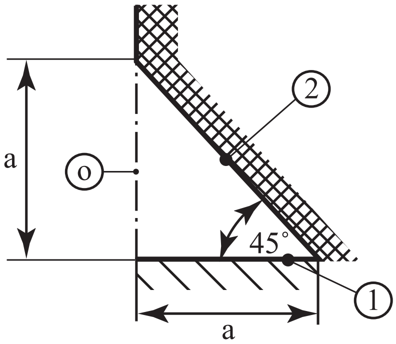
As temperature rises, the gradient must gradually decrease, ultimately approaching zero as time tends towards infinity. This phenomenon can be understood through physical principles: the absorbed heat flux from the sun remains constant. Initially, with low temperatures, the emitted heat flux is minimal, resulting in a significant temperature rise (high net heat gain). Over time, this temperature increase diminishes as the absorbed and emitted heat fluxes converge, reaching equilibrium when both fluxes balance out in a steady state.

Conclusion



Exercise IV.7 (Radiation within a wedge-shaped opening ⋆⋆⋆):

Consider an infinitely long opening with a wedge-shaped cross-section as shown in the figure below.



Given parameters:

- Temperature of surface 1: $T_1 = 1000\,\mathrm{K}$
- Temperature of space surrounding: $T_o = 0\,\mathrm{K}$
- Emissivity of surface 1: $\varepsilon_1 = 1$
- Width: $a = 30$ cm

Hints:

- Surface 2 is a grey body and adiabatically insulated at the back.
- The space surrounding the opening can be considered to be a black body.
- Influences due to convection shall be disregarded.

Tasks:

a) Determine all view factors.

b) Determine the energy through the opening $\dot{q}'_{o,\mathrm{loss}}$ for a unit length of the opening.

c) Determine the temperature T_2 of surface 2.

Solution IV.7 (Radiation within a wedge-shaped opening ⋆⋆⋆):

Task a)

When determining the view factors, recall the general rules that apply to view factors. First the summation rule:

$$\sum_{j=1}^{n} \Phi_{ij} = \Phi_{i1} + \Phi_{i2} + \Phi_{i3} + \ldots + \Phi_{in} = 1,$$

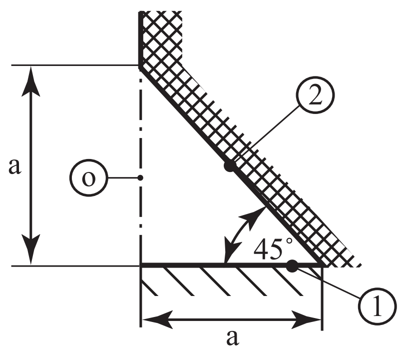
second, the reciprocity rule:

$$A_i \Phi_{ij} = A_j \Phi_{ji},$$

and third the symmetry rule:

$$\Phi_{ij} = \Phi_{ik},$$

which applies if two or more surfaces display symmetry about a third surface, they will have identical view factors from that surface.



Surfaces 1, 2 and o cannot see themselves and therefore it can be stated that:

$$\Rightarrow \mathbf{\Phi_{11} = 0} \tag{IV.7.1}$$

$$\Rightarrow \mathbf{\Phi_{22} = 0} \tag{IV.7.2}$$

$$\Rightarrow \mathbf{\Phi_{oo} = 0} \tag{IV.7.3}$$

Furthermore, from symmetry, it yields that Φ_{2o} and Φ_{21} should be the same and thus:

$$\Rightarrow \mathbf{\Phi_{2o} = \frac{1}{2}} \tag{IV.7.4}$$

$$\Rightarrow \mathbf{\Phi_{21} = \frac{1}{2}} \tag{IV.7.5}$$

The reciprocity rule ca be used to determine Φ_{o2}:

$$A_o \Phi_{o2} = A_2 \cdot \Phi_{2o} \tag{IV.7.6}$$

$$\Rightarrow \mathbf{\Phi_{o2} = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{2}},$$

and Φ_{12}:

$$A_1 \cdot \Phi_{12} = A_2 \cdot \Phi_{21} \tag{IV.7.7}$$

$$\Rightarrow \boldsymbol{\Phi_{12} = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{2}}.$$

The summation rule can be used to determine Φ_{1o}

$$\Phi_{11} + \Phi_{12} + \Phi_{1o} = 1 \tag{IV.7.8}$$

$$\Rightarrow \boldsymbol{\Phi_{1o} = 1 - \frac{\sqrt{2}}{2}},$$

and Φ_{o1}:

$$\Phi_{oo} + \Phi_{o1} + \Phi_{o2} = 1 \tag{IV.7.9}$$

$$\Rightarrow \boldsymbol{\Phi_{o1} = 1 - \frac{\sqrt{2}}{2}}.$$

Conclusion

The view factors of surface 1 can thus be written as:

$$\Phi_{11} = 0 \qquad \Phi_{12} = \frac{\sqrt{2}}{2} \qquad \Phi_{1o} = 1 - \frac{\sqrt{2}}{2},$$

and for surface 2 as:

$$\Phi_{21} = \frac{1}{2} \qquad \Phi_{22} = 0 \qquad \Phi_{2o} = \frac{1}{2},$$

and for surface o:

$$\Phi_{o1} = 1 - \frac{\sqrt{2}}{2} \qquad \Phi_{o2} = \frac{\sqrt{2}}{2} \qquad \Phi_{oo} = 0.$$

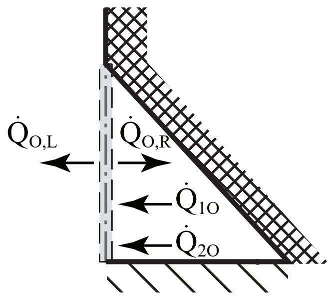
Task b)

To determine the energy loss through the opening, the first step is setting up an energy balance around the opening, where it is considered to be a surface that transmits all incident radiation $\tau_o = 1$ and therefore does not reflect, nor emit due to its temperature being 0 K.

Outer energy balance:

1 **Setting up the balance:**

For the outer energy balance around the opening, it is emitting its surface brightness on the left- and right sides of the opening. Besides, partially the surface brightness of surfaces 1 and 1 is radiated on surface o.



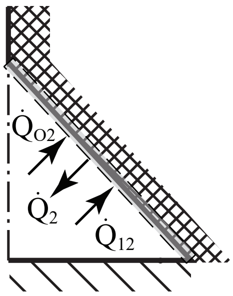
The outer energy balance reads:

$$0 = \dot{Q}_{1o} + \dot{Q}_{2o} - \dot{Q}_{o,L} - \dot{Q}_{o,R}, \tag{IV.7.10}$$

where the surface brightness $\dot{Q}_{o,L}$ radiated by the left surface equals the energy loss from the opening.

The surface brightness of body 2 is intricate and requires establishing an outer energy balance around body 2. It cannot be determined directly, as the temperature of the body is unknown.

Surfaces 1 and o partially radiate their surface brightness on surface 2, whereas surface 2 emits its surface brightness.



$$0 = \dot{Q}_{12} + \dot{Q}_{o2} - \dot{Q}_2, \tag{IV.7.11}$$

where the respective fluxes are defined as: $\dot{Q}_{12} = \Phi_{12}\dot{Q}_1$, and $\dot{Q}_{o2} = \Phi_{o2}\dot{Q}_{O,R}$.

Rewriting the inner energy balance around surface 2 yields:

$$\dot{Q}_2 = \Phi_{12}\dot{Q}_1 + \Phi_{o2}\dot{Q}_{O,R} \tag{IV.7.12}$$

2 Defining the elements within the balance:

The surface brightness from body 1 radiated through the opening can be written as:

$$\dot{Q}_{1o} = \Phi_{1o}\dot{Q}_1, \tag{IV.7.13}$$

where the surface brightness of body 1 can be written to be that of a black body radiator ($\epsilon_1 = 1$):

$$\begin{aligned}\dot{Q}_1 &= \sigma A_1 T_1^4 \\ &= \sigma\,(a \times L)\,T_1^4,\end{aligned} \tag{IV.7.14}$$

where L describes the length of the wedge.

As the opening transmits all its radiation, the surface brightness for the right side can be written as:

$$\dot{Q}_{o,R} = \dot{Q}_{\epsilon,o,R} + \dot{Q}_{\rho,o,R} + \dot{Q}_{\tau,o,R} \tag{IV.7.15}$$

But since the opening cannot emit (because $T_o = 0$ K), nor reflect (because $\rho = 0$) it can be written that:

$$\dot{Q}_{\epsilon,o,R} = 0, \tag{IV.7.16}$$

and:

$$\dot{Q}_{\rho,o,R} = 0. \tag{IV.7.17}$$

Furthermore, it does transmit receive any radiation from the ambient, and therefore no radiation is transmitted through the right side:

$$\dot{Q}_{\tau,o,R} = 0 \tag{IV.7.18}$$

The surface brightness from body 2 radiated through the opening can be written as:

$$\dot{Q}_{2o} = \Phi_{2o}\dot{Q}_2, \tag{IV.7.19}$$

where the surface brightness of body 2 yields from the outer energy balance set around it:

$$\dot{Q}_2 = \Phi_{12}\sigma\,(a \times L)\,T_1^4 \tag{IV.7.20}$$

3 Inserting and rearranging:

Plugging the definitions of the surface brightnesses into the outer energy balance around the opening results in:

$$\dot{Q}_{o,L} = (\Phi_{1o} + \Phi_{2o}\Phi_{12})\,\sigma\,(a \times L)\,T_1^4 \tag{IV.7.21}$$

Dividing it by the respective length L yields the energy through the opening for a unit length:

$$\begin{aligned}\dot{q}'_{\text{loss}} &= \frac{\dot{Q}_{o,L}}{L} \\ &= (\Phi_{1o} + \Phi_{2o}\Phi_{12})\,\sigma a T_1^4 \\ &= \left(1 - \frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} +\right)[-]\cdot 5.67\cdot 10^{-8}\ [\mathrm{W/m^2K^4}]\cdot 0.3\ [\mathrm{m}]\cdot 1000^4\ [\mathrm{K^4}] = 11\ [\mathrm{kW/m}]\end{aligned} \tag{IV.7.22}$$

Conclusion

The energy through the opening for a unit length is thus 11 kW/m.

Task c)

In the previous task the surface brightness of surface 2 was already found:

$$\begin{aligned}\dot{Q}_2 &= \Phi_{12}\sigma A_1 T_1^4 \\ &= \Phi_{12}\sigma \left(a \times L\right) T_1^4\end{aligned} \tag{IV.7.23}$$

1 **Setting up the balance:**

Besides, it can be expressed in terms of the emitted, reflected, and transmitted radiation.

$$\dot{Q}_2 = \dot{Q}_{\epsilon,2} + \dot{Q}_{\rho,2} + \dot{Q}_{\tau,2} \tag{IV.7.24}$$

2 **Defining the elements within the balance:**

Where the emitted term can be written as:

$$\begin{aligned}\dot{Q}_{\epsilon,2} &= \epsilon_2 \sigma A_2 T_2^4 \\ &= \epsilon_2 \sigma \left(a\sqrt{2} \times L\right) T_2^4,\end{aligned} \tag{IV.7.25}$$

the reflection term as:

$$\begin{aligned}\dot{Q}_{\rho,2} &= \rho_2 \dot{Q}_{12} \\ &= \left(1-\epsilon_2\right) \Phi_{12}\sigma \left(a \times L\right) T_1^4,\end{aligned} \tag{IV.7.26}$$

and since the back of surface 2 is adiabatic, no radiation can be transmitted, thus:

$$\dot{Q}_{\tau,2} = 0. \tag{IV.7.27}$$

3 **Inserting and rearranging:**

By equating the expression obtained from the previous task with the newly derived one and rearranging, we obtain:

$$\begin{aligned}T_2 &= \sqrt[4]{\frac{\Phi_{12} T_1^4}{\sqrt{2}}} \\ &= \sqrt[4]{\frac{1000^4\ [\mathrm{K}^4]}{2\ [-]}} = 841\ [\mathrm{K}]\end{aligned} \tag{IV.7.28}$$

Conclusion

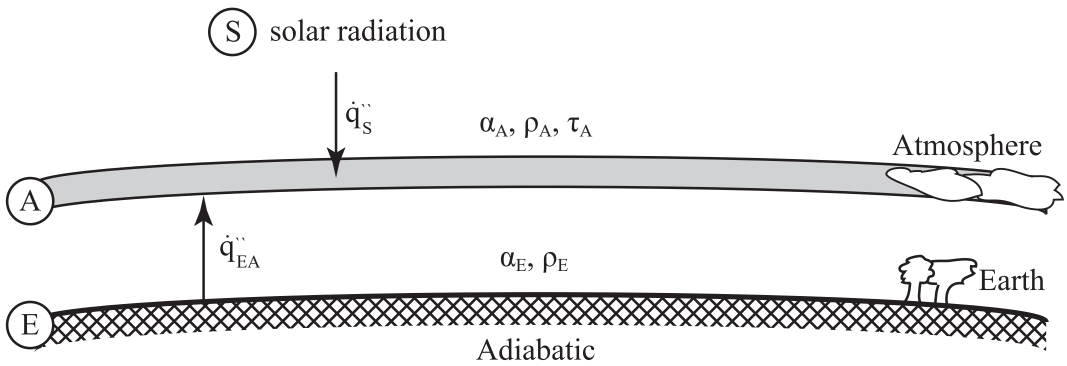
The temperature of surface 2 is thus 841 K.

Exercise IV.8 (Earth's atmosphere ⋆⋆⋆)**:**

The climate on Earth is influenced by the atmosphere to a great extent. To describe heat transfer between Earth and space, it is assumed that the atmosphere surrounds the Earth as a thin, distinct layer.

When balancing radiative heat flows, long-wave and short-wave radiation must be distinguished (indices LW and SW). Earth and atmosphere (indices E and A) have specific absorption, reflection, and transmission coefficients (α, ρ, τ) for long-wave and short-wave radiation each. The spectrum of solar radiation ($\dot{q}''_{\mathrm{S}}$) is assumed to be in the short-wave range only, whereas emission from earth and atmosphere is in the long-wave range only.

Additionally to the radiative heat fluxes, a net heat flux $\dot{q}''_{\mathrm{EA}}$ is carried from the earth into the atmosphere, which leads back to convective heat transfer and vaporization.



Given parameters:

- Short-wave solar radiation: $\dot{q}''_{\mathrm{S,SW}} = 341\ \mathrm{W/m^2}$
- Long-wave solar radiation: $\dot{q}''_{\mathrm{S,LW}} = 0\ \mathrm{W/m^2}$
- Convection and vaporization: $\dot{q}''_{\mathrm{EA}} = 101\ \mathrm{W/m^2}$

	Short-wave	**Long-wave**
Atmosphere	$\rho_{\mathrm{A,SW}} = 0.23$	$\rho_{\mathrm{A,LW}} = 0.34$
	$\tau_{\mathrm{A,SW}} = 0.54$	$\tau_{\mathrm{A,LW}} = 0.10$
	$\alpha_{\mathrm{A,SW}} = 0.23$	$\alpha_{\mathrm{A,LW}} = 0.56$
	emission negligible	emission
Earth	$\rho_{\mathrm{E,SW}} = 0.16$	$\alpha_{\mathrm{E,LW}} = 1.00$
	$\alpha_{\mathrm{E,SW}} = 0.84$	
	emission negligible	emission
Solar radiation	emission	emission negligible

Hints:

- Curvature is negligible, i.e. earth and atmosphere have the same surface area and the atmosphere does not radiate onto itself.
- The atmosphere emits equally in both directions.
- The given heat fluxes are averaged across the entire earth and over multiple years. Do not distinguish between the light and dark hemispheres.

- Assume steady state.

Tasks:

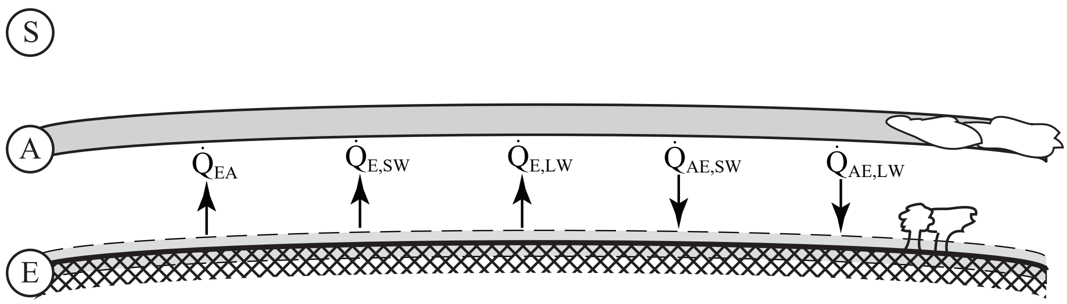
a) Determine the flux of short-wave radiation which hits onto the earth's surface $\dot{q}''_{\mathrm{SW\ to\ E}}$.

b) Give all energy balances and surface brightnesses necessary to determine the temperature at the earth's surface. You may assume that the spectrum of black body radiation is completely within the long-wave range for that temperature.

Solution IV.8 (Earth's atmosphere ⋆⋆⋆):

Task a)

The earth only receives radiation from the atmosphere. Therefore its outer energy balance can be drawn as follows:



From this, it yields that the only short wave radiation which hits the earth's surface is the surface brightness yielding from the bottom of the atmosphere.

1 **Setting up the balance:**

The short-wave surface brightness of the bottom of the atmosphere can be expressed as:

$$\dot{Q}_{A,bot,SW} = \dot{Q}_{\epsilon,A,bot,SW} + \dot{Q}_{\rho,A,bot,SW} + \dot{Q}_{\tau,A,bot,SW} \tag{IV.8.1}$$

2 **Defining the elements within the balance:**

Given that the short-wave emission is negligible, it can be written that:

$$\dot{Q}_{\epsilon,A,bot,SW} = 0 \tag{IV.8.2}$$

Some of the incident solar radiation is transmitted through the bottom of the atmosphere, and therefore the shortwave transmitted radiation of the atmosphere can be defined as:

$$\begin{aligned}\dot{Q}_{\tau,A,bot,SW} &= \tau_{A,SW}\dot{Q}_{SA,SW} \\ &= \tau_{A,SW}A_A\dot{q}''_{S,SW}\end{aligned} \tag{IV.8.3}$$

Furthermore, some of the radiation of the shortwave surface brightness of the earth is reflected at the earth by the inside of the atmosphere. Therefore, the reflected radiation on the inside of the atmosphere can be expressed as:

$$\begin{aligned}\dot{Q}_{\rho,A,bot,SW} &= \rho_{A,SW}\dot{Q}_{EA,SW} \\ &= \rho_{A,SW}\dot{Q}_{E,SW},\end{aligned} \tag{IV.8.4}$$

where $\Phi_{EA} = 1$ because the earth and atmosphere only see each other and not themselves. Heat transport through convection and vaporization is not considered when determining reflected radiation because convection and vaporization primarily involve the transfer of latent heat and sensible heat, rather than the reflection of electromagnetic radiation.

The issue with the expression is that it uses the unknown short-wave surface brightness of the earth. Therefore this expression has to be defined as well:

$$\dot{Q}_{E,SW} = \dot{Q}_{\epsilon,E,SW} + \dot{Q}_{\rho,E,SW} + \dot{Q}_{\tau,E,SW} \tag{IV.8.5}$$

Since emission for the short wavelength of the earth is negligible, the emitted term can be written as:

$$\dot{Q}_{\epsilon,\mathrm{E,SW}} = 0, \tag{IV.8.6}$$

similarly, due to the back of the earth being adiabatic, no radiation can be transmitted:

$$\dot{Q}_{\tau,\mathrm{E,SW}} = 0. \tag{IV.8.7}$$

Lastly, the reflected radiation by the earth yields from the reflected radiation received from the atmosphere:

$$\begin{aligned} \dot{Q}_{\rho,\mathrm{E,SW}} &= \rho_{\mathrm{E,SW}}\dot{Q}_{\mathrm{AE,SW}} \\ &= \rho_{\mathrm{E,SW}}\dot{Q}_{\mathrm{A,bot,SW}}, \end{aligned} \tag{IV.8.8}$$

where $\Phi_{\mathrm{AE}} = 1$ because the earth and atmosphere only see each other and not themselves.

3 **Inserting and rearranging:**

Inserting and rearranging yields:

$$\begin{aligned} \dot{Q}_{\mathrm{A,bot,SW}} &= \frac{\tau_{\mathrm{A,SW}} A_{\mathrm{A}} \dot{q}''_{\mathrm{S,SW}}}{1-\rho_{\mathrm{A,SW}}\rho_{\mathrm{E,SW}}} \\ \Rightarrow \dot{q}''_{\mathrm{A,bot,SW}} &= \frac{\tau_{\mathrm{A,SW}} \dot{q}''_{\mathrm{S,SW}}}{1-\rho_{\mathrm{A,SW}}\rho_{\mathrm{E,SW}}} \\ &= \frac{0.54\,[-]\cdot 341\ [\mathrm{W/m^2}]}{(1-0.23\cdot 0.16)\,[-]} = 191\ [\mathrm{W/m^2}] \end{aligned} \tag{IV.8.9}$$

Conclusion

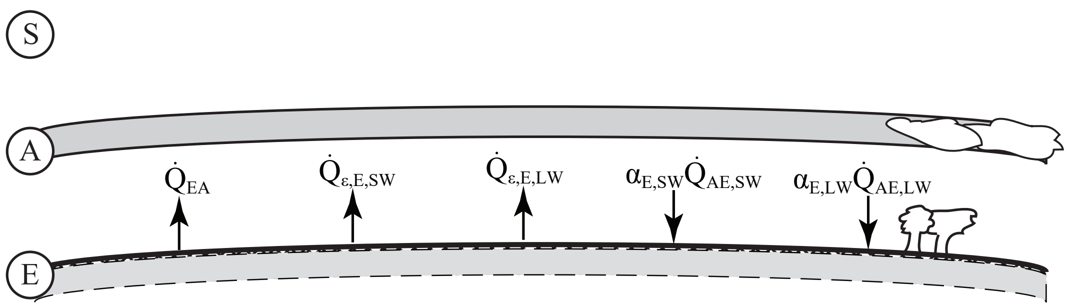
The short-wave radiation that hits on earth's surface is thus 191 $\mathrm{W/m^2}$.

Task b)

The Earth's temperature can be determined through an inner energy balance radiation process around the planet. This balance encompasses the short- and long-wave emissions emitted by the Earth itself, as well as the short- and long-wave surface brightness of the atmosphere's interior radiated toward the Earth. Additionally, heat is transferred as a result of convection and vaporization.

1 **Setting up the balance:**

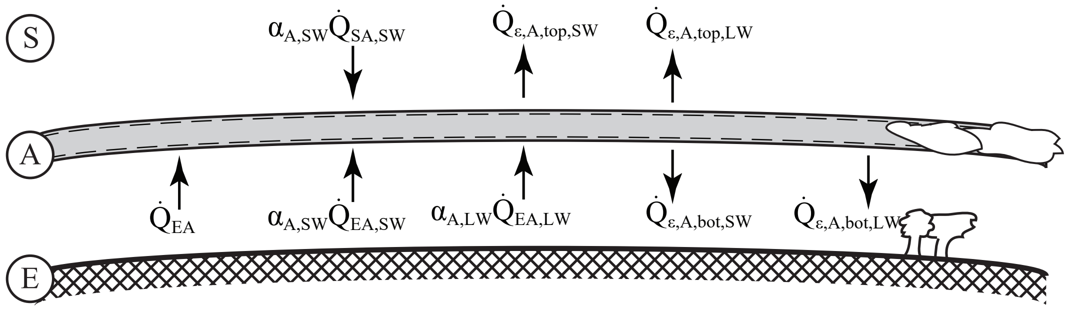
The inner energy balance comprises terms that account for heat loss due to convection and evaporation, as well as emissions in both long and short wavelengths. Additionally, the Earth absorbs short and long-wavelength radiation received from the atmosphere.



Inner energy balance around the Earth reads:

$$0 = \alpha_{E,SW}\dot{Q}_{AE,SW} + \alpha_{E,LW}\dot{Q}_{AE,LW} - \dot{Q}_{\epsilon,E,SW} - \dot{Q}_{\epsilon,E,LW} - \dot{Q}_{EA} \tag{IV.8.10}$$

The long-wave emission of the bottom of the atmosphere is intricate and requires establishing an outer energy balance around the body 2. It cannot be determined directly, as the temperature of the atmosphere is unknown.



$$0 = \dot{Q}_{SA} + \dot{Q}_{EA} + \dot{Q}_{EA,SW} + \dot{Q}_{EA,LW} - \dot{Q}_{A,top,SW} - \dot{Q}_{A,top,LW} - \dot{Q}_{A,bot,SW} - \dot{Q}_{A,bot,LW}, \tag{IV.8.11}$$

where the respective fluxes are defined as: $\dot{Q}_{EA,SW} = \dot{Q}_{A,SW}$, $\dot{Q}_{EA,LW} = \dot{Q}_{A,LW}$, $\dot{Q}_{\epsilon,A,bot,SW} = 0$, $\dot{Q}_{\epsilon,A,top,SW} = 0$, $\dot{Q}_{\epsilon,A,bot,LW} = \dot{Q}_{\epsilon,A,top,LW}$.

Rewriting the inner balance around yields:

$$\dot{Q}_{\epsilon,A,bot,LW} = \frac{1}{2}\left(\dot{Q}_{SA} + \dot{Q}_{EA} + \dot{Q}_{E,SW} + \dot{Q}_{E,LW}\right) \tag{IV.8.12}$$

2 Defining the elements within the balance:

The heat emitted due to convection and vaporization $\dot{q}''_{EA}$ is written as:

$$\dot{Q}_{EA} = \dot{q}''_{EA} A_E \tag{IV.8.13}$$

The Earth's emission at long wavelengths can be described as that of a black body radiator, because $\alpha_{E,LW} = \epsilon_{E,LW} = 1$.

$$\dot{Q}_{\epsilon,E,LW} = \sigma A_E T_E^4 \tag{IV.8.14}$$

Similarly to the short-wave surface brightness of the inside of the atmosphere, as in question a), the long-wave surface brightness of the inside of the atmosphere can be expressed in terms of emission, reflection, and transmission.

$$\dot{Q}_{A,LW} = \dot{Q}_{\epsilon,A,LW} + \dot{Q}_{\rho,A,LW} + \dot{Q}_{\tau,A,LW} \tag{IV.8.15}$$

The transmission term is zero because only solar radiation, which consists of short-wave radiation, can be transmitted toward the Earth. However, as long-wave radiation surface brightness focuses on emissions in the long wavelength spectrum, it does not include short-wave radiation.

$$\dot{Q}_{\tau,A,LW} = 0 \tag{IV.8.16}$$

The emission term has been defined previously by setting up the inner energy balance around the atmosphere. To determine this parameter, the radiation received by the atmosphere from the sun can be written as:

$$\dot{Q}_{\mathrm{SA}} = \dot{q}''_{\mathrm{S,SW}} A_{\mathrm{A}}, \tag{IV.8.17}$$

the short-wave surface brightness of the earth can be written as:

$$\dot{Q}_{\mathrm{E,SW}} = \dot{Q}_{\epsilon,\mathrm{E,SW}} + \dot{Q}_{\rho,\mathrm{E,SW}} + \dot{Q}_{\tau,\mathrm{E,SW}} \tag{IV.8.18}$$

where the emission and transmission is negligible:

$$\dot{Q}_{\epsilon,\mathrm{E,SW}} = 0, \tag{IV.8.19}$$

$$\dot{Q}_{\tau,\mathrm{E,SW}} = 0, \tag{IV.8.20}$$

and the reflection yields from the reflection of the long-wave surface brightness of the atmosphere:

$$\dot{Q}_{\rho,\mathrm{E,SW}} = \rho_{\mathrm{E,SW}} \dot{Q}_{\mathrm{AE,SW}} \tag{IV.8.21}$$

Lastly, the long-wave surface brightness of the earth is expressed as that of a black-body radiator:

$$\dot{Q}_{\mathrm{E,LW}} = \sigma A_{\mathrm{E}} T_{\mathrm{E}}^4 \tag{IV.8.22}$$

Furthermore the short-wave surface brightness of the atmosphere to the earth $\dot{Q}_{\mathrm{AE,SW}}$ and the short-wave emission term of the earth $\dot{Q}_{\epsilon,\mathrm{E,SW}}$ were already determined in the previous task.

Conclusion

3 **Inserting and rearranging:**

Therefore we have found just as many equations as undetermined parameters, which makes the problem solvable.