



# HEATQUIZ

## Heat transfer: Convection

### Lecture notes

December 2021

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Conservation laws for laminar, steady state, two-dimensional flow</b>	<b>4</b>
2.1	Equation of continuity . . . . .	5
2.2	Equations of momentum . . . . .	7
2.2.1	Equation of momentum in x-direction . . . . .	7
2.2.2	Equation of momentum in y-direction . . . . .	9
2.3	Equation of energy conservation . . . . .	10
<b>3</b>	<b>Forced convection - boundary layer equations for laminar, steady state flow</b>	<b>14</b>
3.1	Exact solutions for the equations of boundary layers . . . . .	16
3.2	An approximation for the boundary layer equations . . . . .	19
<b>4</b>	<b>Natural convection - boundary layer equations for laminar, steady state flow</b>	<b>24</b>
<b>5</b>	<b>Heat transfer in turbulent flows</b>	<b>30</b>
<b>6</b>	<b>Application of dimensional analysis for heat transfer</b>	<b>34</b>
<b>7</b>	<b>Heat transfer correlations</b>	<b>39</b>
7.1	Introduction to heat transfer correlations . . . . .	39
7.2	External forced convection . . . . .	42
7.3	Internal forced convection . . . . .	53
7.4	External natural convection . . . . .	57
7.5	Internal natural convection . . . . .	62

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# Chapter 1

## Introduction

We discussed the energy transport by heat conduction in the previous learning path. Molecular properties, described by the thermal conductivity of the material play a key role in heat transport.

In some of the energy transport examples discussed previously, we had to formulate the boundary conditions for heat transfer from the body to the flowing fluid along the body. Without detailed analysis of the physical relationships, we assumed that

$$\frac{\dot{Q}_W}{A} \equiv \dot{q}_W'' = \alpha(T_W - T_a) \quad (1.1)$$

whereat the heat transfer coefficient  $\alpha$  was assumed to be known. In this section, we will discuss the basic principles of this transport mechanism.

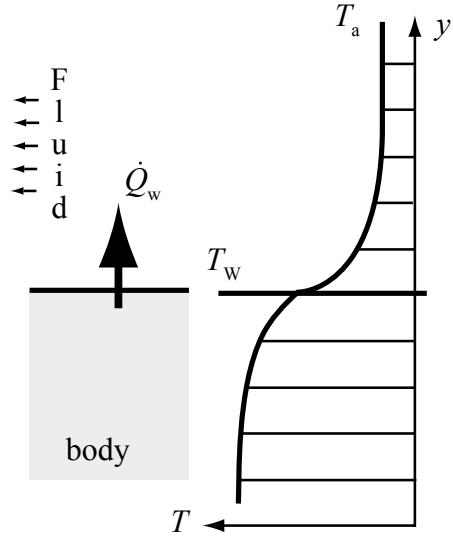


Figure 1.1: Heat transfer at a surface.

The major difference between the transport mechanisms of heat conduction and heat convection can be demonstrated by a simple experiment, where a hot, horizontal wall is let to cool down.

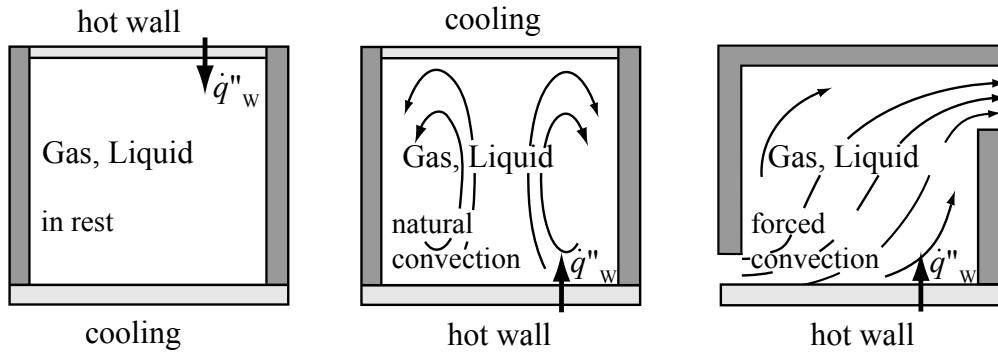


Figure 1.2: Cooling of a hot surface by conduction and convection.

In the first experiment, the wall is cooled from below by a cold fluid, liquid or gas. For this arrangement we have a stagnant fluid, and the heat is transported solely by conduction to the area of lower heat. If the hot wall is placed at the bottom, the fluid will become unstable, and the warmer areas (in the lower part) will rise up due to their lower density and hence flow in the fluid will be initiated. This so-called *natural convection* amplifies the heat transfer. In the third case shown (rightmost diagram), the hot wall is cooled by a medium, which is pumped through the system. As was

the case of natural convection, so again, in this case of *forced convection*, it is not only the thermal conductivity of the fluid that determines the characteristics of the heat transfer. An important, additional parameter that dictates the heat transfer is obviously the velocity of the fluid, which depends on the density differences as well as the electrical power of the pump.

Only in areas directly adjacent to the wall, where because of the boundary layer condition which states that the velocity drops to zero (non-slip condition), the Fourier equation for the heat transfer in the fluid still applies

$$\dot{q}_W'' = - \left( \lambda_{\text{fluid}} \frac{dT_{\text{fluid}}}{dy} \right)_W \quad (\text{regarding the fluid}) \quad (1.2)$$

as well as for the heat transfer in the body

$$\dot{q}_W'' = - \left( \lambda_{\text{body}} \frac{dT_{\text{body}}}{dy} \right)_W \quad (\text{regarding the solid body}) \quad (1.3)$$

The heat transfer coefficient defined by equation (1.1) can be determined from the temperature gradient at the fluid side of the wall by comparison with equation (1.2).

$$\alpha = \frac{- \left( \lambda_{\text{fluid}} \frac{dT_{\text{fluid}}}{dy} \right)_W}{T_W - T_a} \quad (1.4)$$

The temperature field in the fluid and hence the temperature gradient at the wall surface is described by the equation of energy conservation (1st Law of thermodynamics). In this section, we will have to expand the energy balance from chapter 3 by including the enthalpy flowing in and out of the volume element. To describe these flows, the velocities, in addition to the temperatures, have to be known. Thus, in addition to the energy equation, the momentum equation and the continuity equation will have to be formulated and solved. The necessary basic principles have been discussed in introductory lectures on fluid mechanics and will only be briefly presented here.

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# **Chapter 2**

## **Conservation laws for laminar, steady state, two-dimensional flow**

The general solution of a three-dimensional flow, for which the flow field and the temperature field are interdependent can only be determined by determining the three velocity components  $u$ ,  $v$  and  $w$ , the temperature  $T$  and pressure  $p$ , as well as the material properties of the fluid, the density  $\rho$ , the dynamic viscosity  $\eta$  and the thermal conductivity  $\lambda$ . Since these parameters are not independent of each other, a simultaneous solution of the conservation equations of mass, momentum and energy is required. To avoid too much writing, we will focus on the two-dimensional, steady state case.

The flow can be either laminar or turbulent, i.e. the streamlines are either parallel or superimposed to the mean, steady state fluctuating flow. These three-dimensional fluctuating velocities and their impact on the momentum and heat exchange will be discussed in section 5.

The following derivations are limited to laminar flow.

## 2.1 Equation of continuity

For steady state flow, the difference between mass flowing in and out must vanish.

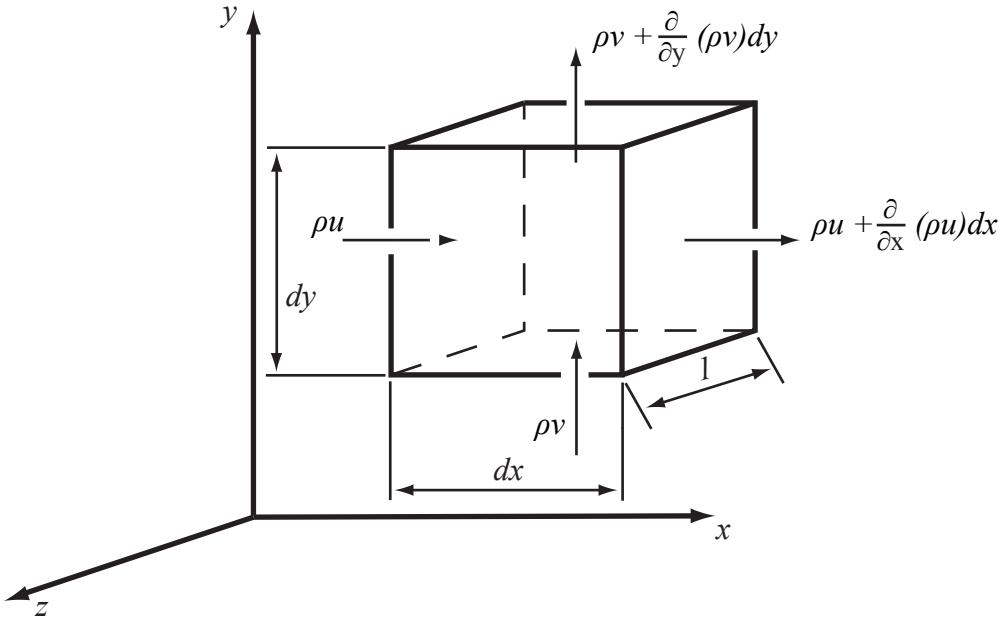


Figure 2.1: Mass balance at the control volume.

The **mass balance** accounting for the ingoing and outgoing mass flows yields:

$$0 = \dot{m}_{x,in} - \dot{m}_{x,out} + \dot{m}_{y,in} - \dot{m}_{y,out}$$

with the ingoing mass flows defined at the location  $x$  as

$$\dot{m}_{x,in} = \dot{m}_x(x) = \rho u dy l$$

$$\dot{m}_{y,in} = \dot{m}_y(y) = \rho v dx l$$

and the outgoing mass flows defined at the location  $x + dy$  by using the Taylor-series approximation

$$\dot{m}_{x,out} = \dot{m}_x(x + dx) = \rho u dy l + \frac{\partial}{\partial x} \rho u dx dy l$$

$$\dot{m}_{y,out} = \dot{m}_y(y + dx) = \rho v dx l + \frac{\partial}{\partial y} \rho v dy dx l$$

Substituting all the terms in the mass balance by the specified mass flows yields

$$0 = \frac{\partial}{\partial x} \rho u dx dy l + \frac{\partial}{\partial y} \rho v dx dy l$$

Now, all dimensions of the differential element cancel out, yielding the equation of continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad (\text{equation of continuity}) \quad (2.1)$$

or written in vectorical form

$$\operatorname{div}(\rho \vec{w}) = 0$$

For incompressible fluids ( $\rho = \text{constant}$ ), this equation simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2)$$

## 2.2 Equations of momentum

The momentum equation states that the difference between the momentum in and out flow of the volume element is equal to the external forces acting on the volume element, whereat inertial forces and surface forces can be relevant.

### 2.2.1 Equation of momentum in x-direction

The difference between momentum flowing in and out is

$$\left( \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} \right) dx dy l = \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + u \left( \underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}}_{=0 \text{ (eq.2.1)}} \right) \right) dx dy l \quad (2.3)$$

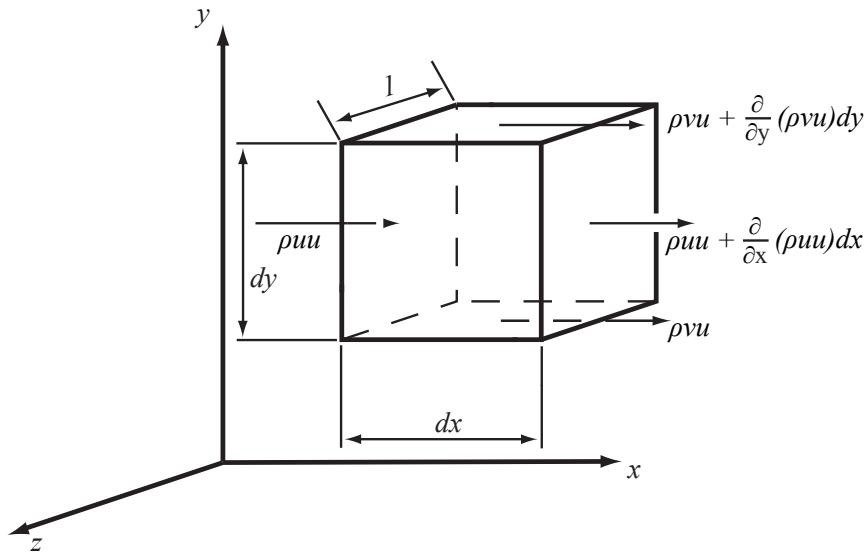


Figure 2.2: x-momentum balance at the control volume.

The surface forces, i.e. pressure, shear and normal forces can be expressed by  $p_{xy}$ , where the first index indicates the orientation of the surface at which the force is applied (e.g. here perpendicular to the x-direction) and the second index shows the direction of the force (e.g. here in the y-direction).

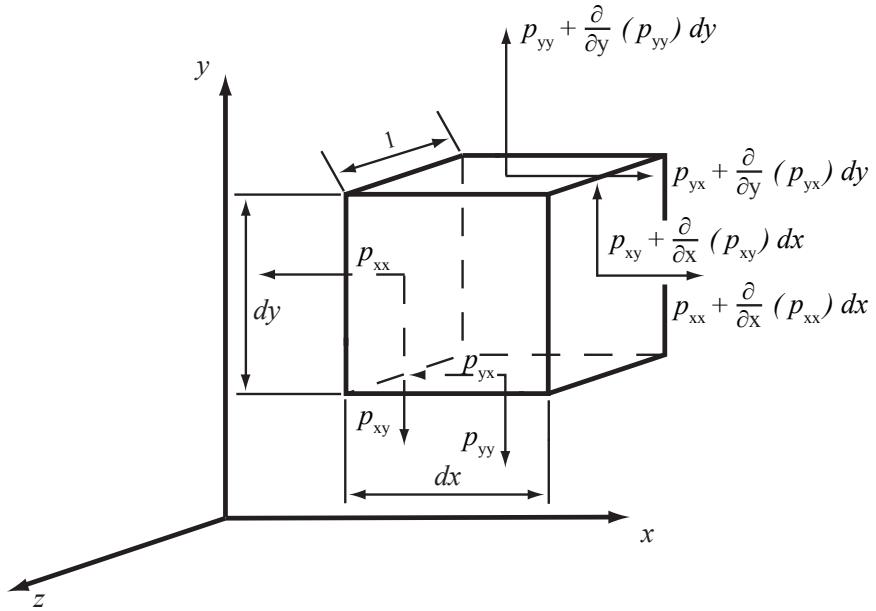


Figure 2.3: Forces at the control volume.

In the equation of momentum for the x-direction, only the forces acting in x-direction are considered.

Hence, the balance of surface forces applied at the volume element is

$$\frac{\partial p_{xx}}{\partial x} dx dy l + \frac{\partial p_{yx}}{\partial y} dy dx l$$

with the normal stress  $p_{xx}$  and the shear stress  $p_{yx}$ , related to the pressure  $u$  and  $v$  and the dynamic viscosity  $\eta$  [kg/ms] given by Navier (1827) and Stokes (1845), see Schlichting u. Gersten (2006).

$$p_{xx} = -p + \sigma_x = -p - \frac{2}{3}\eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\eta \frac{\partial u}{\partial x}$$

and

$$p_{yx} = \tau_{yx} = \tau_{xy} = \eta \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

This yields

$$\left[ -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( 2\eta \frac{\partial u}{\partial x} - \frac{2}{3}\eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left( \eta \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \right] dx dy l$$

At the volume element, inertial forces are also applied, of which we will discuss only the force of gravity

$$\rho g_x dx dy l$$

Summarizing the momentum, surface and inertial forces, we get the momentum equation, the so-called Navier-Stokes equation

$$\begin{aligned} & \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} \\ &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( 2\eta \frac{\partial u}{\partial x} - \frac{2}{3}\eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left( \eta \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \rho g_x \end{aligned} \quad (2.4)$$

(equation of momentum, x-direction)

### 2.2.2 Equation of momentum in y-direction

$$\begin{aligned} & \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} \\ &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left( 2\eta \frac{\partial v}{\partial y} - \frac{2}{3}\eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial x} \left( \eta \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \rho g_y \end{aligned} \quad (2.5)$$

(equation of momentum, y-direction)

Assuming constant material properties (density, viscosity), the equations of momentum are further simplified to

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x \quad (2.6)$$

(equation of momentum, x-direction, ( $\rho, \eta$  constant))

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \eta \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y \quad (2.7)$$

(equation of momentum, y-direction,  $(\rho, \eta$  constant))

or as vector equation

$$\rho (\vec{w} \times \text{grad } ) \vec{w} = -\text{grad } p + \eta \nabla^2 \vec{w} + \rho \vec{g} \quad (2.8)$$

(equation of momentum,  $(\rho, \eta$  constant))

## 2.3 Equation of energy conservation

The first law of thermodynamics

$$d\dot{Q} + dP = d\dot{U} + d\dot{E}_a \quad (2.9)$$

applied to a volume element of a flowing medium states that any heat or work added to the system will result in change of inner, potential or kinetic energy.

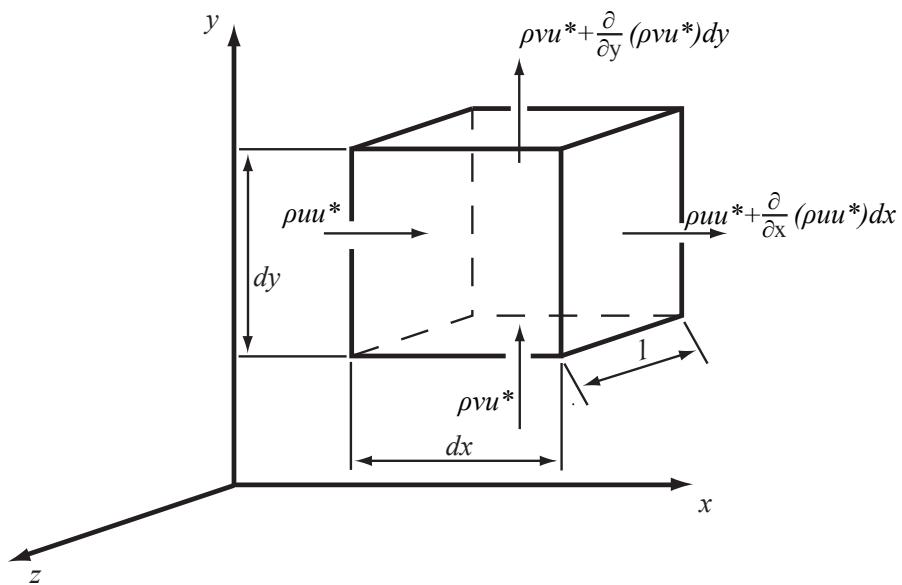


Figure 2.4: Energy flows at the control volume.

The kinetic energy is to be considered for the net energy only for flow velocities in

the order of the speed of sound, and thus will be neglected here. The difference of inner energy flowing in and out is

$$\frac{\partial}{\partial x} (\rho u u^*) dx dy l + \frac{\partial}{\partial y} (\rho v u^*) dx dy l$$

where the specific inner energy is named  $u^* [\frac{J}{kg}]$ . If we introduce the specific enthalpy  $h \equiv u^* + \frac{p}{\rho}$  instead of the specific inner energy we get

$$\begin{aligned} & \left( \frac{\partial}{\partial x} (\rho u h) + \frac{\partial}{\partial y} (\rho v h) - \frac{\partial}{\partial x} (u p) - \frac{\partial}{\partial y} (v p) \right) dx dy l \\ &= \left( \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} + h \left( \underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}}_{=0 \text{ (eq.2.1)}} \right) - \frac{\partial}{\partial x} (u p) - \frac{\partial}{\partial y} (v p) \right) dx dy l \end{aligned}$$

Resulting from Fourier's equation, the total *added heat* is

$$\left( \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) \right) dx dy l.$$

The volume element receives work from the surface (friction) forces as well as from the inertial forces.

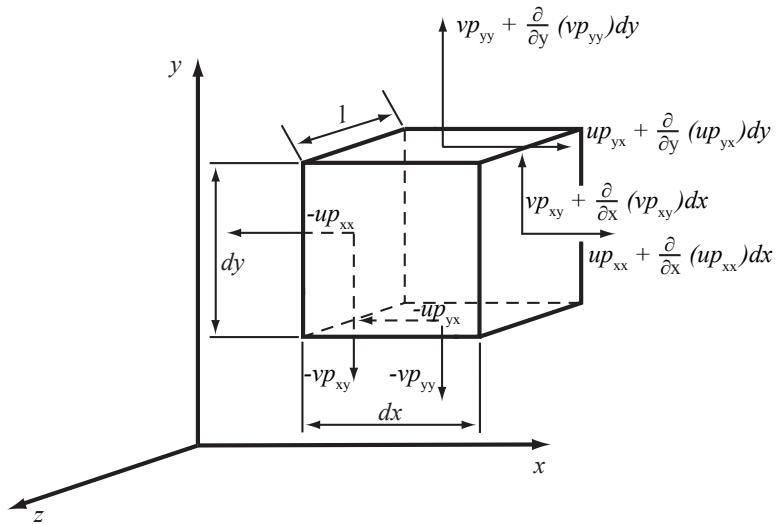


Figure 2.5: Contribution of the forces to the energy balance of the volume element.

$$\left( \frac{\partial}{\partial x} (u\sigma_x) - \frac{\partial}{\partial x} (up) + \frac{\partial}{\partial y} (u\tau_{xy}) + \rho ug_x + \frac{\partial}{\partial y} (v\sigma_y) - \frac{\partial}{\partial y} (vp) + \frac{\partial}{\partial x} (v\tau_{xy}) + \rho vg_y \right) dx dy l$$

Hence, the energy equation is

$$\begin{aligned} \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} &= \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) \\ &\quad + \frac{\partial}{\partial x} (u\sigma_x + v\tau_{xy}) + \frac{\partial}{\partial y} (u\tau_{xy} + v\sigma_y) + \rho ug_x + \rho vg_y \end{aligned} \quad (2.10)$$

The stress contributions of the normal, shear and inertial forces will be omitted since their values are, in most cases, negligibly small and do not contribute significantly to the net energy balance.

The specific enthalpy may be substituted by the temperatures for systems without phase changes, no chemical reactions and, if the fluid can be considered an ideal gas ( $dh = c_p dT$ ) or an incompressible liquid ( $dh = c dT$ )

Hence,

$$\rho u c_p \frac{\partial T}{\partial x} + \rho v c_p \frac{\partial T}{\partial y} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) \quad (2.11)$$

Comparing this equation with the Fourier equation for heat conduction, equation ??, for steady state problems without heat sources, it can be shown that under the assumptions above the energy transport by heat conduction is extended by convective energy transport through enthalpy flow. For unsteady state processes, the left-hand side of equation 2.11 is to be supplemented by a term characteristic for the energy storage  $\rho c_p \frac{\partial T}{\partial t}$ .

If, in addition, the thermal conductivity is independent of location, following from equation (2.11)

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\lambda}{c_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (2.12)$$

or

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\text{Pr}} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (2.13)$$

with the Prandtl number

$$\text{Pr} \equiv \frac{\eta}{\lambda/c_p} \quad (2.14)$$

or in vector form

$$\rho \vec{w} \times \text{grad } T = \frac{\eta}{\text{Pr}} \nabla^2 T \quad (2.15)$$

The temperature field of a flowing medium and hence the temperature gradient at its boundary, from which the heat transfer can be calculated, equation (1.2), is sufficiently described by the energy equation, (2.13), momentum equation, (2.6), the equation of continuity, (2.2), and appropriate estimates of the material properties as well as the boundary conditions of the system itself.

This system of interdependent partial differential equations can only be solved numerically using methods of finite differences or finite volumes, which nowadays can be obtained commercially and which will be discussed in depth at other lectures. Such computational fluid dynamic (CFD) methods reach the ultimate limits of modern high performance computers in cases of geometries or boundary conditions with a high degree of complexity, so that often it is more cost-efficient to carry out experiments to obtain heat transfer coefficients.

A major group of technologically important flows, *boundary layer flows*, uses a simplified form of the derived conservation equations. This leads to parabolic instead of elliptical partial differential equations, which in turn are much easier to handle numerically, and in some cases can be transformed into ordinary differential equations.

These solution methods will be briefly discussed further on, a detailed investigation of these problems can be found for example in Bayley u. a. (1972).

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# Chapter 3

## Forced convection - boundary layer equations for laminar, steady state flow

For flow along solid walls the influence of the viscous forces is limited to an area in the vicinity of the wall, called *velocity boundary layer*, with thickness  $\delta_u$ . The viscosity is of no influence outside of this boundary layer the flow behaves as a potential flow. The same applies for the area, where heat conduction dominates. The *thermal boundary layer* with thickness  $\delta_T$  outlines the area of heat conduction from the area of undisturbed flow.

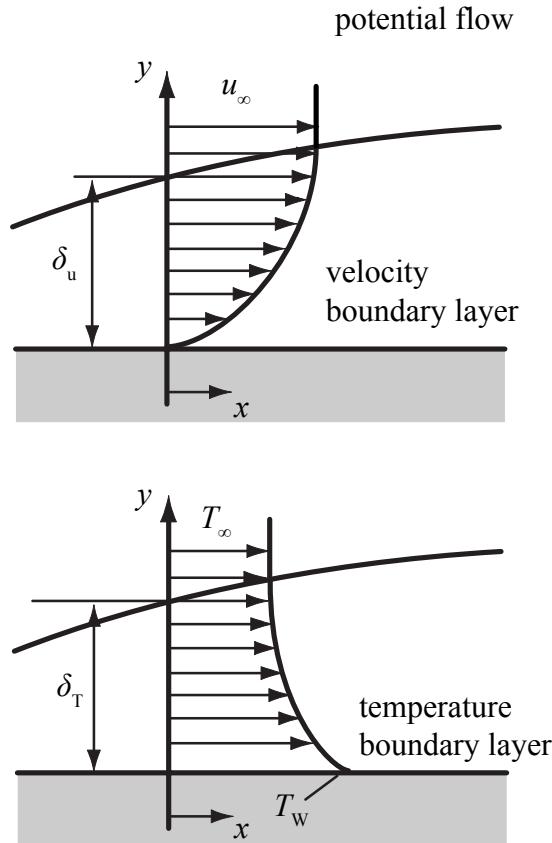


Figure 3.1: Velocity and thermal boundary layers of a surface in a stream flow.

Prandtl (1904) derived the equations valid for boundary layers from the conservation laws through appropriate assumptions for the individual terms. These assumptions, which can also be found at Schlichting and Gersten (2006), state that for flow of the boundary layer type, i.e.  $\frac{\delta}{L} \ll 1$ , the following assumptions are valid

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2} \quad (3.1)$$

Thus, the parabolic differential equations for boundary layers with constant material properties are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.2)$$

(equation of continuity)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} + \rho g_x \quad (3.3)$$

(momentum equation, x-direction)

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\text{Pr}} \frac{\partial^2 T}{\partial y^2} \quad (3.4)$$

(equation of energy conservation)

In general, the thicknesses of the velocity and thermal boundary layers are not equal. Yet, as a consequence of the approximations for the boundary layers, with  $\delta_u \approx \eta^{\frac{1}{2}}$  yields:

$$\frac{\delta_u}{\delta_T} \approx \text{Pr}^{\frac{1}{2}} \quad (3.5)$$

The Prandtl number for gases is approximately 1, so that the velocity and thermal boundary layer almost coincide in thickness.

### 3.1 Exact solutions for the equations of boundary layers

Transformation of coordinates is possible for a flat plate ( $dp/dx = 0$ ), without inertial forces ( $\rho g_x = 0$ ) which leads to an ordinary differential equation with an exact solution, see Schlichting u. Gersten (2006).

The solution of the momentum equation, the velocity profile in the boundary layer, was firstly published by Blasius (1908) and is shown in 3.2 as a function of the dimensionless wall distance. The dimensionless wall shear stress, the friction coefficient  $\frac{c_f}{2}$ , derives from the velocity gradients at the wall.

$$\frac{c_f}{2} \equiv \frac{\tau_w}{\rho u_\infty^2} = \frac{\left(\eta \frac{du}{dy}\right)_w}{\rho u_\infty^2} \quad (3.6)$$

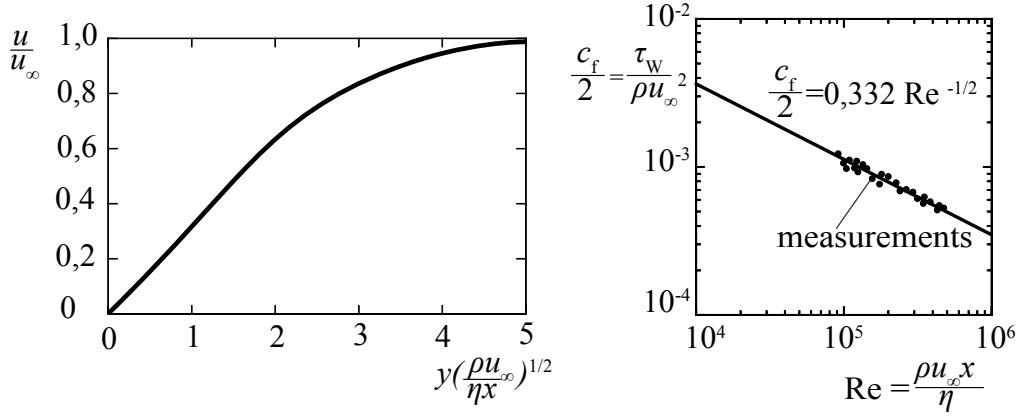


Figure 3.2: Laminar flow over a flat plate - velocity profile and friction coefficient, according to Blasius (1908).

The corresponding equation of energy conservation (equation 3.4) for the flat plate with constant wall temperature was solved by Pohlhausen (1921) and leads to the temperature profiles, which are additionally dependent on the Prandtl number. The temperature gradients at the wall and hence the local heat transfer coefficient  $\alpha$  derive from these temperature profiles, see equation (1.4), or its mean value  $\bar{\alpha}$ :

$$\alpha = \frac{-\left(\lambda \frac{dT}{dy}\right)_W}{(T_W - T_\infty)} \text{ and } \bar{\alpha} = \frac{1}{L} \int_0^L \alpha(x) dx \quad (3.7)$$

The dimensionless temperature profile and the dimensionless heat transfer coefficient, the Nusselt number

$$Nu \equiv \frac{\alpha x}{\lambda} \text{ and } \overline{Nu} \equiv \frac{\bar{\alpha} L}{\lambda} \quad (3.8)$$

are shown in Fig. 3.3.

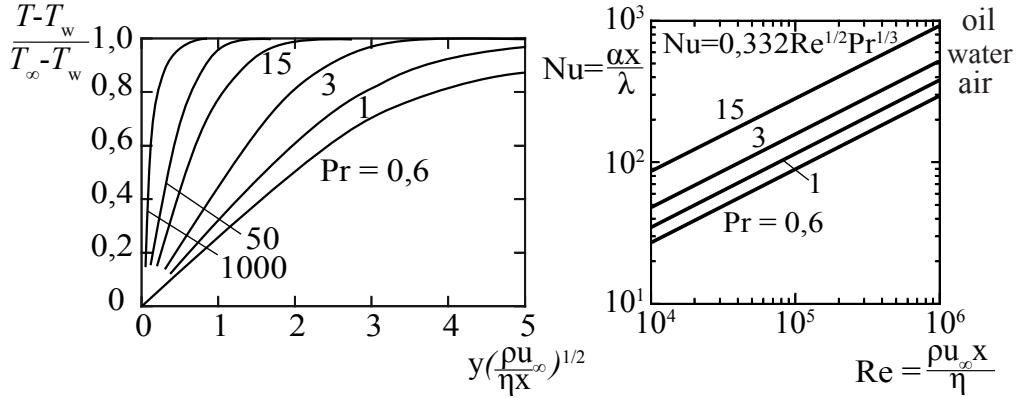


Figure 3.3: Laminar flow over a flat plate - temperature profile and Nusselt number according to Pohlhausen (1921).

For  $\text{Pr} = 1$  the momentum equation, equation (3.3), and the energy, equation (3.4), are identical if the pressure gradients and inertial forces are neglected. Hence, the dimensionless velocity and temperature profile must also be identical and thus

$$\left( \frac{d \left( \frac{u}{u_\infty} \right)}{dy} \right)_W = \left( \frac{d \left( \frac{T-T_W}{T_\infty - T_W} \right)}{dy} \right)_W$$

From equation (1.4) and (3.8) follows

$$\text{Nu} = \frac{x}{\lambda} \left( \lambda \frac{d \left( \frac{T-T_W}{T_\infty - T_W} \right)}{dy} \right)_W$$

and with equation (3.6)

$$\frac{c_f}{2} = \frac{\eta}{\rho u_\infty} \left( \frac{d \frac{u}{u_\infty}}{dy} \right)_W, \quad (\text{for } \text{Pr} = 1) \quad (3.9)$$

With the equality of the dimensionless temperature and velocity profiles, the ratio between the Nusselt number, the friction coefficient and Reynolds number,  $\text{Re} = \frac{\rho u_\infty x}{\eta}$  is:

$$\text{Nu} = \frac{c_f}{2} \text{Re}, \quad (\text{for } \text{Pr} = 1) \quad (3.10)$$

Taking the approximate influence of the Prantl number into account, the enhanced relation is:

$$\text{Nu} = \frac{c_f}{2} \text{Re} \text{Pr}^{\frac{1}{3}} \quad (3.11)$$

### 3.2 An approximation for the boundary layer equations

In the previous section the results of the exact solutions of the laminar, steady state boundary layer equations for plane flows were shown. The necessary mathematical methods are so complex that they cannot be included in the scope of this introductory course.

However, a very simple approximation for the boundary layer equations, which yields results comparable to those of the exact solutions and above all, shows the physical principles, will be discussed next.

We examine again the boundary layer flow over a flat plate. A control volume, bound by the planes  $\overline{11}$ ,  $\overline{12}$ ,  $\overline{22}$  and the wall will be used to formulate the conservation equations. Plane  $\overline{12}$  is located within the potential flow, outside of the boundary layer.

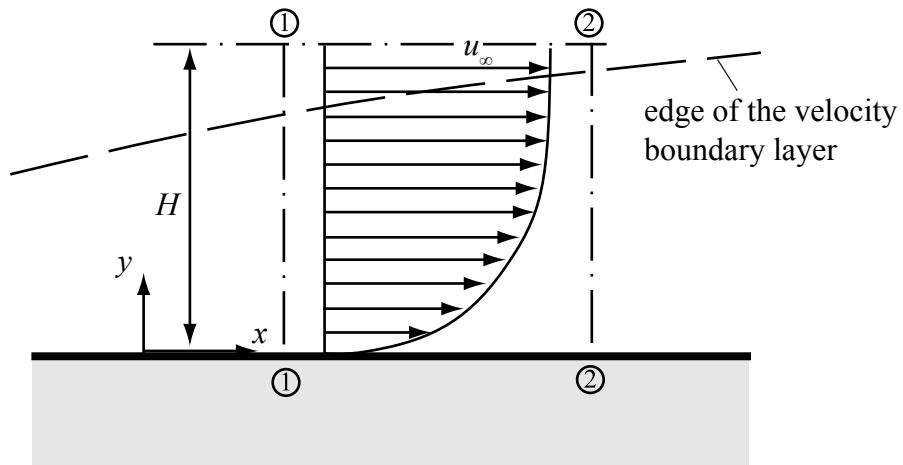


Figure 3.4: Integral momentum balance of a boundary layer flow.

For the control volume described above, follows the integral *equation of continuity*

$$\frac{d}{dx} \int_0^H \rho u dy dx l + \rho v_\infty dx l = 0 \quad (3.12)$$

(equation of continuity)

and the integral momentum equation in x-direction

$$\frac{d}{dx} \int_0^H \rho u^2 dy dx l + \rho u_\infty v_\infty dx l = -\tau_w dx l$$

At the upper boundary plane, which is located in the potential flow, no shear forces are present. Normal forces in the planes  $\overline{11}$  and  $\overline{22}$  are to be neglected following the boundary layer approximation. At the wall, only the shear stress  $\tau_{yx} = (\eta \frac{du}{dy})_w$  should be considered. Considering the equation of continuity, equation (3.12), yields:

$$\frac{d}{dx} \int_0^H \rho u^2 dy - \frac{d}{dx} \int_0^H \rho u u_\infty dy = - \left( \eta \frac{du}{dy} \right)_w$$

or

$$\frac{d}{dx} \int_0^H \rho u (u_\infty - u) dy = \left( \eta \frac{du}{dy} \right)_w \quad (3.13)$$

(momentum equation, x-direction)

The integration is possible if the velocity profile in the boundary layer is known.

If, a *linear velocity profile* is assumed, as a crude approximation

$$\frac{u}{u_\infty} = \frac{y}{\delta_u} \quad (3.14)$$

and density is constant within the boundary layer, equation (3.13) yields

$$\rho u_\infty \frac{d}{dx} \int_0^{\delta_u} \frac{y}{\delta_u} \left( 1 - \frac{y}{\delta_u} \right) dy = \frac{\eta}{\delta_u}$$

Eventually a relationship for the thickness of the velocity boundary layer is found by integration

$$\delta_u = \sqrt{\frac{12\eta}{\rho u_\infty} x} \quad (3.15)$$

or dimensionless

$$\frac{\delta_u}{x} = \sqrt{\frac{12\eta}{\rho u_\infty x}} = \sqrt{\frac{12}{Re_x}} \quad (3.16)$$

with the Reynolds number,  $Re \equiv \frac{\rho u_\infty x}{\eta}$ , based on the length  $x$ .

If the thickness of the boundary layer is known, the *shear stress at the wall* can be determined

$$\tau_w = \left( \eta \frac{du}{dy} \right)_w = \eta \frac{u_\infty}{\delta_u} = \frac{1}{\sqrt{12}\sqrt{Re}} \rho u_\infty^2 \quad (3.17)$$

and in its dimensionless form, the friction coefficient  $\frac{c_f}{2}$

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_\infty^2} = \frac{1}{\sqrt{12}\sqrt{Re}} = 0.289 Re^{-\frac{1}{2}} \quad (3.18)$$

This relationship shows the same dependence of the friction coefficient on the Reynolds number as the previously given exact solution of the momentum equation

$$\frac{c_f}{2} = 0.332 Re^{-\frac{1}{2}} \quad (3.19)$$

merely the constant is 15% too low.

The *integral energy equation* can be derived using a similar control volume that includes the thermal boundary layer.

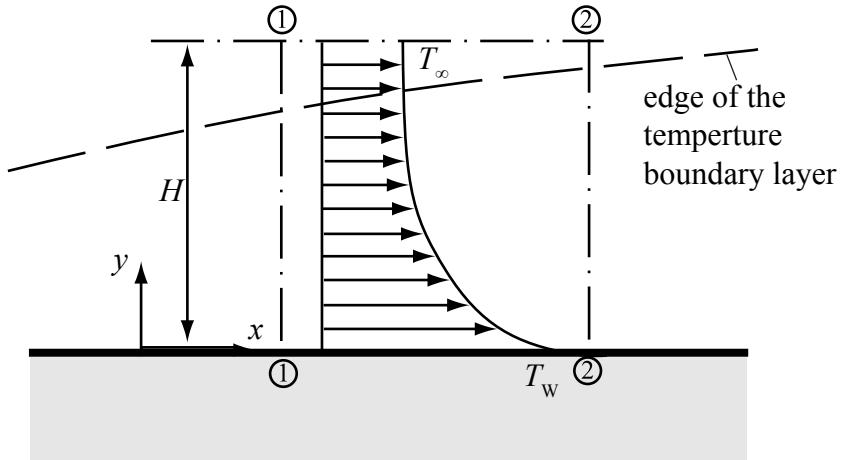


Figure 3.5: Integral energy balance at a boundary layer.

Neglecting the dissipated heat of the forces acting on the volume and the heat conduction in direction of the flow, which is negligible compared to the convective energy transport in this direction, results in

$$\frac{d}{dx} \int_0^H \rho u c_p (T - T_0) dy = - \left( \lambda \frac{dT}{dy} \right)_W dx$$

where  $T_0$  is the reference temperature used to determine the enthalpy reference value.

Implementing the equation of continuity, equation (3.12), again results in

$$\frac{d}{dx} \int_0^H \rho u c_p (T - T_0) dy = - \left( \lambda \frac{dT}{dy} \right)_W$$

or

$$\frac{d}{dx} \int_0^H \rho u c_p (T - T_\infty) dy = - \left( \lambda \frac{dT}{dy} \right)_W \quad (\text{energy equation}) \quad (3.20)$$

Assuming constant properties ( $\rho, \lambda, c_p$ ) the energy equation can be integrated assuming a linear temperature profile in the boundary layer

$$\frac{T - T_W}{T_\infty - T_W} \equiv \frac{\theta}{\theta_\infty} = \frac{y}{\delta_T} \quad (3.21)$$

Thus,

$$\rho c_p \frac{d}{dx} \int_0^H u_\infty \frac{y}{\delta_u} \left( \frac{y}{\delta_T} \theta_\infty - \theta_\infty \right) dy = - \lambda \frac{\theta_\infty}{\delta_T}$$

Assuming that the thermal boundary layer is as thick or thinner than the velocity boundary layer, yields  $H = \delta_T$ . After integration:

$$\delta_T d \left( \frac{\delta_T^2}{\delta_u} \right) = \frac{6\lambda}{\rho c_p u_\infty} dx \quad (3.22)$$

In case the ratio of the two boundary layer thicknesses remains constant:

$$\left( \frac{\delta_T}{\delta_u} \right)^3 \delta_u d \delta_u = \frac{6\lambda}{\rho c_p u_\infty} dx \quad (3.23)$$

Applying equation (3.15) leads to a relationship for the thickness of the thermal boundary layer

$$\frac{\delta_T}{\delta_u} = \left( \frac{\lambda}{\eta c_p} \right)^{\frac{1}{3}} = \frac{1}{Pr^{\frac{1}{3}}} \quad (3.24)$$

Gases have Prandtl numbers  $\text{Pr} \approx 1$ , so the assumption  $\delta_u = \delta_T$  was legitimate. If the thickness  $\delta_T$  of the thermal boundary layer is known, the heat flux at the wall can be calculated

$$\dot{q}_W'' = -\lambda \left( \frac{dT}{dy} \right)_W = -\lambda \frac{\theta_\infty}{\delta_T} = -\frac{\lambda \theta_\infty \text{Pr}^{\frac{1}{3}}}{\delta_u} \quad (3.25)$$

applying equation (3.16) for the thickness of the velocity boundary layer

$$\dot{q}_W'' = \frac{\lambda}{x} 0.289 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} (T_W - T_\infty) \quad (3.26)$$

Comparing these relationships with the often used empirical assumption for convective heat transfer

$$\dot{q}_W'' = \alpha (T_W - T_\infty)$$

leads to the following equation for the heat transfer coefficient  $\alpha$

$$\alpha = \frac{\lambda}{x} 0.289 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \quad (3.27)$$

or in dimensionless form using the Nusselt number,  $\text{Nu} \equiv \frac{\alpha x}{\lambda}$ ,

$$\text{Nu} = 0.289 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}}. \quad (3.28)$$

As comparison to the exact solution according to Pohlhausen (1921) shows,

$$\text{Nu} = 0.332 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \quad (3.29)$$

the heat transfer coefficients, calculated in this manner, have the same function profile with only minor deviation from the absolute value, in spite of the simplified assumptions. With only little additional effort, integrating these equations using polynomials of 3rd order to describe the profiles, the solution's conformity to the exact solution can be considerably improved.

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# Chapter 4

## Natural convection - boundary layer equations for laminar, steady state flow

In the previous section we discussed the heat transfer between a surface and a fluid, where the flow was initiated e.g. by a ventilator. Inertial forces were discarded in the energy balance because of their small contribution.

In other cases the flow along a surface is not forced, but induced by a mass force, usually gravity. This *natural convection* has velocity and thermal boundary layers as shown in the figure below. These profiles result from the solutions of the boundary layer equations, the equation of continuity, equation (3.2), the momentum equation in x-direction, equation (3.3), and the equation of energy conservation, equation (3.4).

In the momentum equation

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} + \rho g_x \quad (4.1)$$

the sign of the term that describes the impact of gravity force has been changed, since gravity acts against the positive x-direction in this example.

The pressure gradient  $\frac{dp}{dx}$  was zero for forced convection over a flat plate. For problems of natural convection, the pressure change over the height  $dx$  equals the weight of the liquid head over a unit area.

$$\frac{dp}{dx} = -\rho_\infty g \quad (4.2)$$

Hence, the momentum equation can be rewritten

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g (\rho_\infty - \rho) + \eta \frac{\partial^2 u}{\partial y^2} \quad (4.3)$$

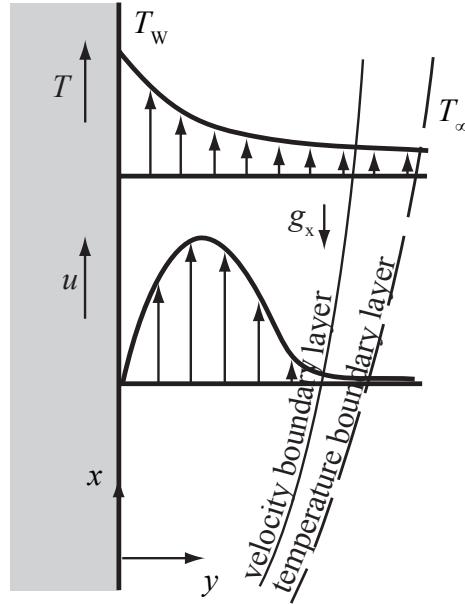


Figure 4.1: Natural convection at a vertical plate.

Usually, the difference in density in the buoyancy term is written in terms of the volumetric expansion coefficient  $\beta$  for homogeneous mediums

$$\beta \equiv \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p = \frac{\rho_\infty - \rho}{\rho(T - T_\infty)}$$

hence,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g \rho \beta (T - T_\infty) + \eta \frac{\partial^2 u}{\partial y^2} \quad (4.4)$$

(momentum equation, x-direction)

The volumetric expansion coefficient is listed for many fluids, and for ideal gases it is valid that

$$\beta = \frac{1}{T}$$

Assuming constant properties is justified for many cases, except for the buoyancy term, although it is exactly the density deviations that causes the fluid to flow.

In order to determine the velocity and temperature fields and hence friction and heat transfer at a vertical surface, the following conservation equations must be solved:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.5)$$

(equation of continuity,  $\rho$  constant)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g \rho \beta (T - T_{\infty}) + \eta \frac{\partial^2 u}{\partial y^2} \quad (4.6)$$

(momentum equation, x-direction,  $\rho, \eta$  constant)

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\text{Pr}} \frac{\partial^2 T}{\partial y^2} \quad (4.7)$$

(energy equation,  $\lambda$  constant)

The solution renders more difficult than the one of the boundary layer equations for forced convection, since knowledge of the temperature field is necessary in order to solve the momentum equation.

Yet, for the simple geometry of the vertical flat plate with constant surface temperature, an exact solution by transformation of the partial differential equations into ordinary differential equations is possible again.

These ordinary differential equations have been solved by Ostrach (1953). The resulting velocity and temperature profiles are shown in a dimensionless form in the following diagrams.

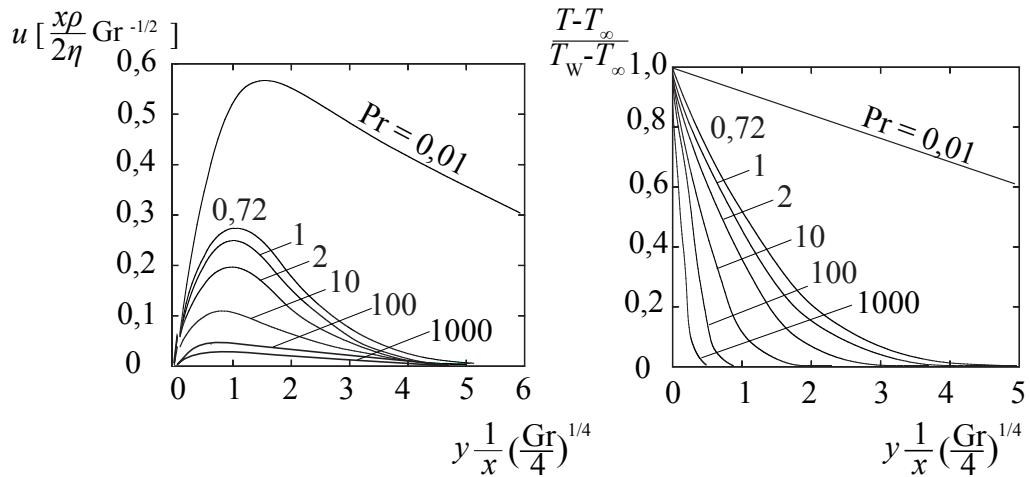


Figure 4.2: Velocity and temperature profiles at a vertical wall for natural convection according to Ostrach (1953).

Here, the new dimensionless number, the *Grashof number*

$$\text{Gr} \equiv \frac{\beta g \rho^2 (T_w - T_\infty) x^3}{\eta^2} \quad (4.8)$$

is the ratio of the buoyancy forces to the viscosity forces.

With the gradient of the temperature profile at the wall, the heat flux can be calculated according to the Fourier equation as well as the heat transfer coefficient using equation (3.6). Local mean Nusselt numbers can be written in the form

$$\overline{\text{Nu}} = C (\text{Gr} \text{ Pr})^{\frac{1}{4}} \quad (4.9)$$

where  $\overline{\text{Nu}}$  is the mean value of the Nusselt number over the height of the plate and  $C$  is a coefficient that depends on the Prandtl number, and is given in the following table

Pr	0,003	0,01	0,03	0,72	1	2	10	100	1000	$\infty$
C	0,182	0,242	0,305	0,516	0,535	0,568	0,620	0,653	0,665	0,670

Table 4.1: Values for the coefficient  $C$  of equation (4.9) according to Ostrach (1953).

A comparison between the theoretical results and measured values is shown in the following diagrams. In Fig. (4.3) the calculated values of the velocity and temperature fields at a vertical, heated plate in air are compared to experimental values. In (4.4) the mean Nusselt numbers are compared with measured values according to Whitaker (1974).

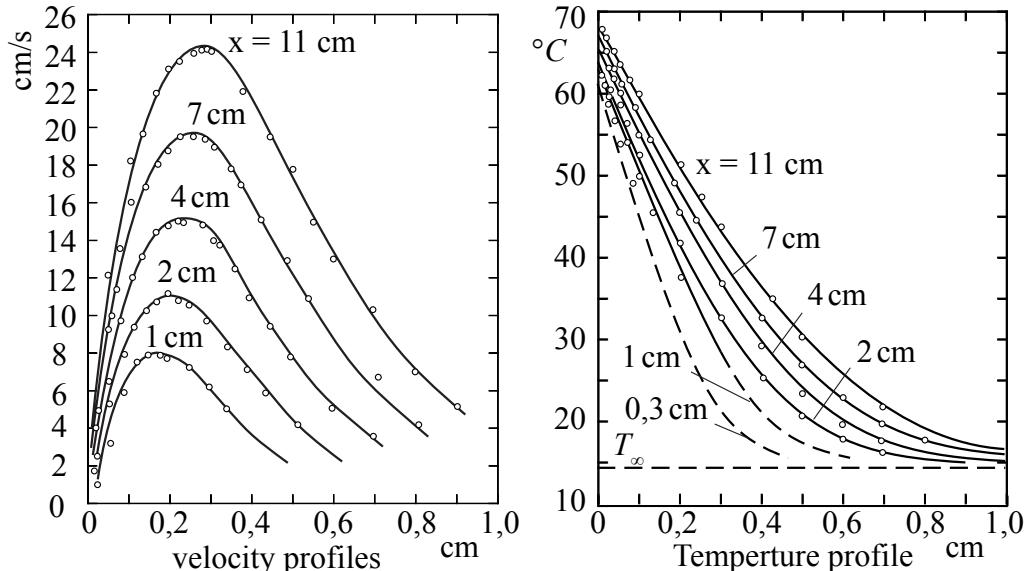


Figure 4.3: Calculated and measured velocity and temperature profiles at a vertical plate for natural convection in air, from Whitaker (1974).

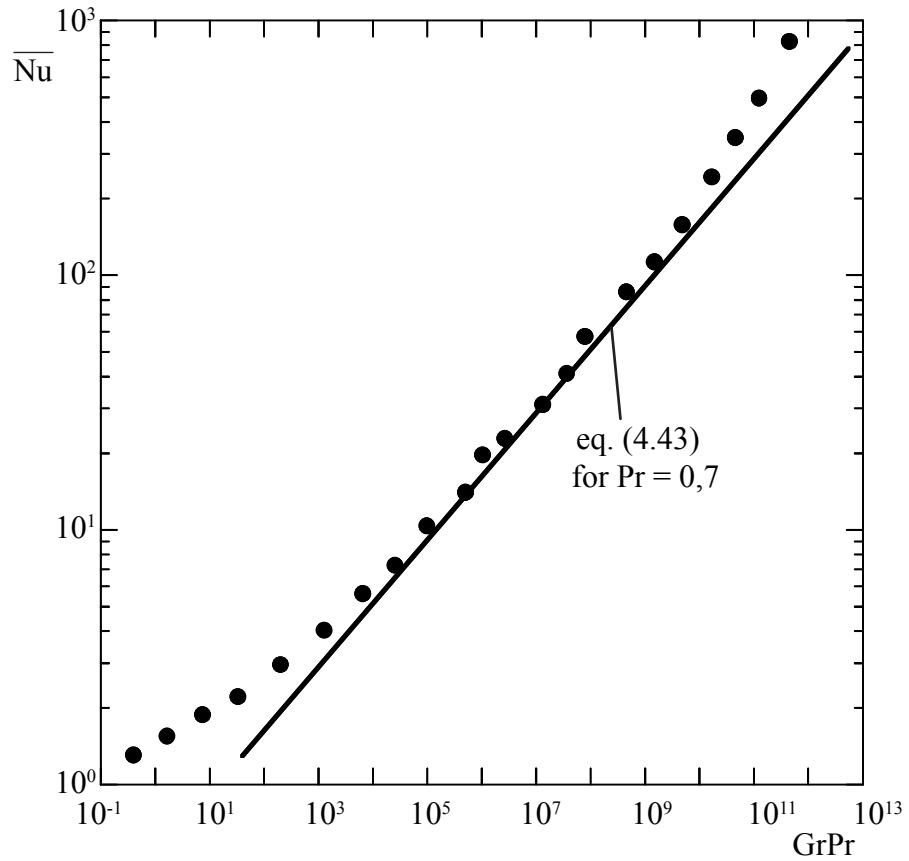


Figure 4.4: Calculated and measured mean values of the Nusselt number of a vertical plate for natural convection in air, from Whitaker (1974).

Comparing these profiles in air shows a very good accordance and proves the validity of the simplified assumptions used during the derivation of the conservation laws. For the local mean values of the Nusselt number, the theoretical values conform to the experimental values only in the middle range of  $(GrPr)$ -values. For  $GrPr < 10^3$  these boundary layer approximations are obviously no longer valid. In the range of  $GrPr > 10^9$  heat transfer is intensified by the shift from laminar to turbulent flow.

These two effects, which are also observed during forced convection, make the theoretical solution of the conservation equations significantly more difficult. If the boundary layer assumptions are dropped, instead of the parabolic differential equations elliptical differential equations have to be solved and numerical methods or eventually experiments will be necessary. In addition, when the flow shifts from laminar to turbulent, models have to be developed that include the impact of turbulence to the momentum and heat exchange.

The characteristics of turbulent flow will be discussed in the following section 5.

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# Chapter 5

## Heat transfer in turbulent flows

The previous sections described the heat transfer from a wall to a laminar flowing fluid. It was shown that adjacent to the wall, the dominating energy transport mechanisms is heat conduction, whereas in the layers further away, enthalpy transport by the flow dominates.

In many problems from practice though, the flow becomes turbulent, depending on a characteristic Reynolds number, so that additional exchange mechanisms have to be considered. In many cases on average the flow shows steady state characteristics yet. Measuring the instantaneous values of velocity or temperature by e.g. a hot wire anemometer, or a very thin thermocouple results in profiles as shown in the following diagrams.

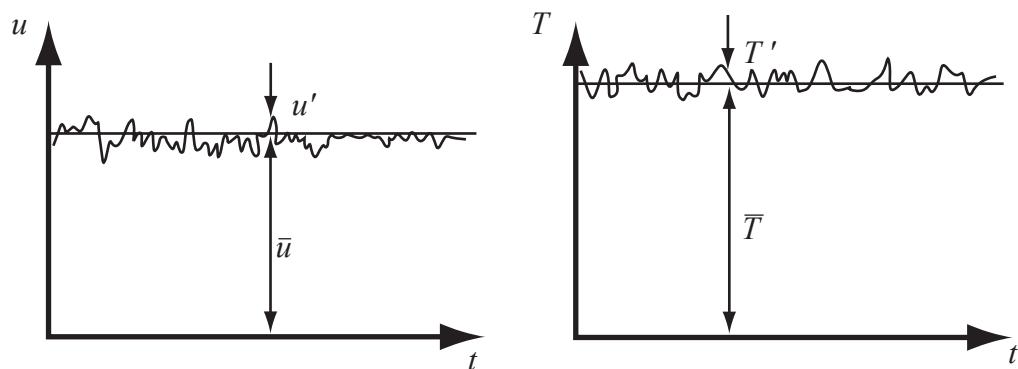


Figure 5.1: Velocity and temperature as a function of time for an - on the average - steady state, turbulent flow.

For the instantaneous value of the velocity  $u$ , and the mean value  $\bar{u}$  with deviation  $u'$ , the following relationships are valid

$$u = \bar{u} + u'$$

and respectively,

$$v = \bar{v} + v' ; \quad T = \bar{T} + T' \quad \text{and} \quad p = \bar{p} + p'$$

Putting these instantaneous values in the equations of continuity, momentum and energy, and then calculate the mean values it follows, e.g. for flow with boundary layer character and assuming constant properties, from equations (3.2) to (3.4) a new set of conservation equations

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (\text{equation of continuity}) \quad (5.1)$$

$$\rho \bar{u} \frac{\partial \bar{u}}{\partial x} + \rho \bar{v} \frac{\partial \bar{u}}{\partial y} = - \frac{d \bar{p}}{dx} + \eta \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial \bar{u}' v'}{\partial y} \quad (\text{momentum equation}) \quad (5.2)$$

$$\rho \bar{u} c_p \frac{\partial \bar{T}}{\partial x} + \rho \bar{v} c_p \frac{\partial \bar{T}}{\partial y} = \lambda \frac{\partial^2 \bar{T}}{\partial y^2} - \rho c_p \frac{\partial \bar{v}' T'}{\partial y} \quad (\text{energy equation}) \quad (5.3)$$

The full derivation of these equations can be found for example in Schlichting u. Gersten (2006). The new equations valid for steady state turbulent flows differ from those for laminar flows by a newly introduced term on the right-hand side of the momentum and energy equation, which describes the additional transport of momentum and energy by the eddy movements of the flow.

Usually these additional terms are labeled "*turbulent shear stress*" or "*turbulent heat conduction*", respectively

$$\tau_t = -\rho \bar{u}' v' \quad \text{and} \quad \dot{q}_t'' = \rho c_p \bar{v}' T'.$$

If, in analogy to molecular transport of heat and momentum, a corresponding turbulent viscosity  $\eta_t$  and turbulent heat conductivity  $\lambda_t$  is defined:

$$\tau_t \equiv \eta_t \frac{\partial \bar{u}}{\partial y} \quad \text{and} \quad \dot{q}_t'' \equiv -\lambda_t \frac{\partial \bar{T}}{\partial y}.$$

then the molecular transport properties and the influence of the turbulence can be summarised in the *effective transport properties*, e.g.  $\eta_{\text{eff}} = \eta + \eta_t$ . This yields for example for a turbulent boundary layer flow to the following conservation equations

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (\text{equation of continuity}) \quad (5.4)$$

$$\rho \bar{u} \frac{\partial \bar{u}}{\partial x} + \rho \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{d \bar{p}}{dx} + \frac{\partial}{\partial y} \left( \eta_{\text{eff}} \frac{\partial \bar{u}}{\partial y} \right) \quad (\text{momentum equation}) \quad (5.5)$$

$$\rho \bar{u} c_p \frac{\partial \bar{T}}{\partial x} + \rho \bar{v} c_p \frac{\partial \bar{T}}{\partial y} = \frac{\partial}{\partial y} \left( \lambda_{\text{eff}} \frac{\partial \bar{T}}{\partial y} \right) \quad (\text{energy equation}) \quad (5.6)$$

Even though the equations are formally very similar to those for laminar boundary flows, it should not be forgotten that the turbulent properties describe the impact of turbulence and hence depend not only on the location in the flow, but also on the geometry and history of the flow. The turbulent transport properties are not material properties as is the case for their molecular counterparts. References for semi-empirical approaches used to describe them can be found under the keyword “turbulence models”, e.g. Jischka (1982).

These special properties render the exact solution of the conservation equations, in the way presented for laminar flow, impossible. Therefore complex numerical methods, like the method of finite differences or finite elements, are used. The way in which these methods are applied will be the topic of specialised courses on the subject. The application of advanced commercially available program codes for solving problems of energy engineering and combustion technology will be presented in the lecture “Design of Burners and Furnaces”.

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# Chapter 6

## Application of dimensional analysis for heat transfer

Although with the development in computer technology, major advances have been made in the numerical calculation of heat transfer, it is still necessary for most technical cases to use empirical heat transfer laws for design purposes.

These laws are a result of measurements, which are shown graphically or in a form of empirical formulas.

As will be shown in this section, it is not always necessary to repeat experiments if either fluid, geometry, or the flow conditions are changed.

The analytical solutions for heat transfer for forced and natural convection, dealt with previously, have shown that the results can be represented in an appropriate form by a few characteristic numbers

$$\text{Nu} = \text{Nu}(\text{Re}, \text{Pr}) \text{ for forced convection} \quad (6.1)$$

$$\text{Nu} = \text{Nu}(\text{Gr}, \text{Pr}) \text{ for natural convection} \quad (6.2)$$

The solutions rendered not only the characteristic numbers that describe the process, but also their functional relationship.

For cases in which a solution of the conservation laws is not known, it is often possible to reduce the number of parameters that influence the heat transfer by the method of the *dimensional analysis*. The functional relationships of these coefficients are obtained by specific measurements.

To determine the characteristic numbers, the conservation equations are rewritten in a dimensionless form. In order to reduce complexity, the two-dimensional, laminar boundary layer equations with constant properties in the form of equation (4.5) to (4.7), with an additional pressure gradient  $\left(\frac{dp}{dx}\right)_{\text{kin}}$  in the momentum equation will be regarded. It should be noted that while deriving the momentum equation (4.6) for the vertical flat plate only the pressure gradient, describing the fluid head

$\left(\frac{dp}{dx}\right)_{\text{pot}}$  was effective, and the pressure gradient caused by the acceleration of the fluid  $\left(\frac{dp}{dx}\right)_{\text{kin}}$ , as for example in a tube flow, was omitted. Hence the more general case is discussed here

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{equation of continuity}) \quad (6.3)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \left( \frac{dp}{dx} \right)_{\text{kin}} + \eta \frac{\partial^2 u}{\partial y^2} - g_x \rho \beta (T - T_\infty) \quad (\text{momentum equation}) \quad (6.4)$$

$$\rho u c_p \frac{\partial T}{\partial x} + \rho v c_p \frac{\partial T}{\partial y} = \lambda \frac{\partial^2 T}{\partial y^2} \quad (\text{energy equation}) \quad (6.5)$$

All variables in these equations shall be made dimensionless by appropriate characteristic parameters, arising from the specific case. Hence, the coordinates  $x$  and  $y$  are referenced to a characteristic length  $L$ , which can be the diameter  $D$  for a sphere or tube, or the length of a plate for flow over a plate. The velocity components  $u$  and  $v$  are divided by the characteristic velocity  $u_\infty$ , for flow in tubes the mean velocity, and for flow over a plate the free stream velocity. The pressure is always rated over the double of the dynamic pressure and the temperature over a characteristic temperature difference, e.g. the temperature difference between the surface and the fluid at a greater distance from the body.

Labeling the dimensionless variables with \*

$$x^* \equiv \frac{x}{L}; \quad y^* \equiv \frac{y}{L}; \quad u^* \equiv \frac{u}{u_\infty}; \quad v^* \equiv \frac{v}{u_\infty}; \quad p^* \equiv \frac{p}{\rho u_\infty^2}; \quad \theta^* \equiv \frac{T - T_\infty}{T_w - T_\infty}$$

yields the dimensionless conservation equations

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (\text{equation of continuity}) \quad (6.6)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = - \left( \frac{dp^*}{dx^*} \right)_{\text{kin}} + \left( \frac{\eta}{\rho u_\infty L} \right) \frac{\partial^2 u^*}{\partial y^{*2}} + \left( \frac{g_x \beta L (T_w - T_\infty)}{u_\infty^2} \right) \theta^* \quad (6.7)$$

(momentum equation)

$$u^* \frac{\partial \theta^*}{\partial x^*} + v^* \frac{\partial \theta^*}{\partial y^*} = \left( \frac{\lambda}{\rho c_p u_\infty L} \right) \frac{\partial^2 \theta^*}{\partial y^{*2}} \quad (6.8)$$

(energy equation)

In addition to the dimensionless variables, these equations contain dimensionless quantities in the brackets. A closer look at these expressions, reveals that these are the dimensionless numbers from the previous sections.

The momentum equation, equation (6.7), contains two of these terms.

The expression for the friction term is the reciprocal of *the Reynolds number*

$$\text{Re} \equiv \frac{\rho u_\infty L}{\eta}$$

The expression for the buoyancy term is called *Archimedes number*

$$\text{Ar} \equiv \frac{g_x \beta L (T_w - T_\infty)}{u_\infty^2}$$

which in the literature is often expressed as a quotient of *the Grashof number* and the *Reynolds number* squared

$$\text{Ar} = \left( \frac{g_x \rho^2 \beta L^3 (T_w - T_\infty)}{\eta^2} \right) \left( \frac{\eta}{\rho u_\infty L} \right)^2 = \frac{\text{Gr}}{\text{Re}^2}$$

The dimensionless energy equation defines the *Péclet number*

$$\text{Pe} \equiv \frac{\rho c_p u_\infty L}{\lambda} = \frac{u_\infty L}{a}$$

which in turn, can be rewritten as a product of the *Reynolds* and *Prandtl number*

$$\text{Pe} = \left( \frac{\rho u_\infty L}{\eta} \right) \left( \frac{\eta c_p}{\lambda} \right) = \text{Re} \text{Pr}$$

If, for different applications, e.g. different fluids, different pipe diameters the coefficients or a combination of the same, given in the expressions in the brackets, are equal, then so are the solutions of the differential equations, i.e. the dimensionless fields of velocity, temperature and pressure. Yet, the boundary conditions must be equivalent, too, for the above statement to be valid. Hence, the velocity and temperature fields can be expressed as a function of these parameters

$$\vec{w} = \vec{w} (x^*, y^*, \text{Re}, \text{Gr}, \text{Pr}) \quad (6.9)$$

and

$$\theta^* = \theta^* (x^*, y^*, \text{Re}, \text{Gr}, \text{Pr}) \quad (6.10)$$

Velocity and temperature fields of the fluid are less interesting than the heat that is transferred between the fluid and the body and which can be derived from the gradient of the temperature profile in the fluid at the surface of the body.

$$\dot{q}'' = - \left( \lambda \frac{\partial T}{\partial y} \right)_W$$

A comparison with the equation that defined the heat transfer coefficient  $\alpha$

$$\dot{q}'' \equiv \alpha (T_W - T_\infty)$$

yields

$$\alpha = \frac{- \left( \lambda \frac{dT}{dy} \right)_W}{(T_W - T_\infty)}$$

or written in a dimensionless form, the *Nusselt number* :

$$Nu \equiv \frac{\alpha L}{\lambda} = - \left( \frac{d\theta^*}{dy^*} \right)_W$$

The Nusselt number, already introduced in previous sections as a dimensionless heat transfer coefficient, is thus equal to the dimensionless temperature gradient and hence, can be expressed as the temperature field

$$Nu = Nu(Re, Gr, Pr) \quad (6.11)$$

This shows that heat transfer for problems with similar geometries and boundary conditions can formally be expressed by 4 dimensionless numbers.

This statement is also valid for turbulent flow, as long as the levels of turbulence are comparable.

For many practical purposes, the relationship (6.11), can be further simplified.

For *forced convection* the inertial and friction forces are greater than the buoyant forces resulting from the changes in temperature. Neglecting the buoyant term in the equation of motion leads to

$$Nu = Nu(Re, Pr) \quad (6.12)$$

For *natural convection* the velocity field is caused by temperature differences, and a velocity  $u_\infty$  induced by external sources is not present. The inertial forces may be neglected, hence

$$Nu = Nu(Gr, Pr) \quad (6.13)$$

The dimensional analysis cannot state the form of the functional relationship between the Nu number and the Re, Gr, Pr numbers. This relationship can either be found from the analytical solutions, if these are expressed in characteristic numbers, or from experiments. Often, it is aimed at representing the experimental results as a power law, i.e. in the form

$$\text{Nu} = C \text{Re}^m \text{Pr}^n \text{Gr}^p$$

Important examples of such power laws will be shown in the next chapter.

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# Chapter 7

## Heat transfer correlations

### 7.1 Introduction to heat transfer correlations

The heat flux from a wall with the surface area  $A$  and temperature  $T_W$  to a fluid with the temperature  $T_f$  is:

$$\dot{Q}_W = A\alpha (T_W - T_f) \quad (7.1)$$

To determine the heat transfer coefficient  $\alpha$ , Nusselt number correlations for a number of important applications are given below.

These correlations have the general form

$$\begin{aligned} \text{Nu} &= \text{Nu}(\text{Re}, \text{Pr}) \text{ for forced convection} \\ \text{Nu} &= \text{Nu}(\text{Gr}, \text{Pr}) \text{ for natural convection} \end{aligned}$$

With the definitions for Nusselt number

$$\text{Nu} \equiv \frac{\alpha L}{\lambda} \quad (7.2)$$

Reynolds number

$$\text{Re} \equiv \frac{\rho u_\infty L}{\eta} \quad (7.3)$$

Grashof number

$$\text{Gr} \equiv \frac{\rho^2 g \beta (T_W - T_f) L^3}{\eta^2} \quad (7.4)$$

Prandtl number

$$\text{Pr} \equiv \frac{\eta c_p}{\lambda} \quad (7.5)$$

The applicability of the given correlations should be checked for each case according to the following criteria:

The *geometry* of the given problem must be analogous to the geometry of the experiment from which the heat transfer law is estimated.

The flow profile must be checked by means of the critical Reynolds and Grashof numbers in the range describing the shift from laminar to turbulent flow. A law valid for laminar flow cannot be applied for turbulent flow.

The *thermal boundary conditions* of the investigated case should be compared to those under which the applied heat transfer law was derived. Laws that are valid for example for flow in tubes with constant wall temperature can be used only with restrictions for cases with a constant heat flux rate on the wall. For cases, where high temperature changes occur in the system, care should be taken to use the same reference temperature at which the properties were determined. In most of the heat transfer laws the properties are to be determined at a mean temperature. In special cases, where the variable properties are to be described, the correlations are extended to include additional terms, either those of temperature ratios or viscosity ratios.

An additional problem is the assumption for the fluid temperature  $T_{\text{fl}}$  for the *local heat flux* of an enclosed flow. In this case, normally a caloric mean value is used to determine the temperature:

$$T_{\text{fl}} = \frac{\int_{A_c} \rho u c_p T \, dA_c}{\int_{A_c} \rho u c_p \, dA_c} \quad (7.6)$$

Many heat transfer laws are a result of integral measurement on a system and thus yield only the *averaged heat transfer coefficients*  $\bar{\alpha}$  of a heat transporting medium. The same applies for the *average heat flux*

$$\dot{Q}_W = \bar{\alpha} A (T_W - T_{\text{fl}})_m = \bar{\alpha} A \Delta T_m \quad (7.7)$$

where both the wall temperature as well as the fluid temperature may depend on the location. It can be shown that for many practical purposes, for which the heat transfer coefficient is not dependent on the temperature, the so-called *logarithmic temperature difference*, made up of the temperature differences at the entrance and exit of the heat exchanger, may be used

$$(T_W - T_{\text{fl}})_m = \Delta T_{\ln} \equiv \frac{\Delta T_{\text{inlet}} - \Delta T_{\text{outlet}}}{\ln \left( \frac{\Delta T_{\text{inlet}}}{\Delta T_{\text{outlet}}} \right)}$$

This will be discussed in detail in the lecture “Heat Exchangers and Steam Generators”.

Eventually it must be verified, that the field of characteristic numbers in which the law was determined matches the one of the problem under investigation.

This is especially important for laws that have been determined experimentally, which describe the functional relationships only approximately.

Next, a selection of important heat transfer laws for forced and natural convection are presented. A detailed summary of the heat transfer laws can be found, for example in VDI Wärmeatlas, 1997.

## 7.2 External forced convection

The properties used in the characteristic numbers are to be determined, if not stated otherwise, at

$$T_{\text{prop.}} = \frac{T_w + T_\infty}{2}$$

**Flat plate - laminar boundary layer flow**  $\text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$

Note: The heat transfer laws for laminar boundary layers are ‘exact’ solutions of

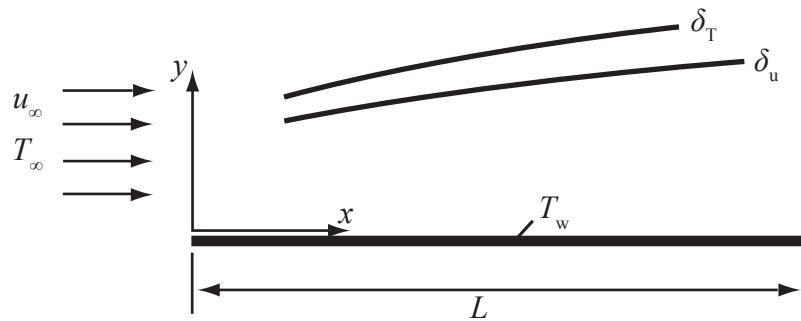


Figure 7.1: Laminar boundary layer of a plate with isothermal surface

the conservation laws.

Simultaneous hydrodynamic and thermal inflow, isothermal surface

- Nusselt law for the local heat transfer:

$$\text{Nu}_x = 0.332 \text{Re}_x^{1/2} \text{Pr}^{1/3} \quad (\text{HTC.1})$$

for  $0.6 < \text{Pr} < 10$   
and  $\text{Re}_x < \text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$

- Nusselt law for the mean heat transfer:

$$\overline{\text{Nu}_L} = \frac{\bar{\alpha}L}{\lambda} = \frac{1}{L} \int_0^L \alpha(x) dx = 2 \text{ Nu}_{x=L}$$

$$\overline{\text{Nu}_L} = 0.664 \text{ Re}_L^{1/2} \text{ Pr}^{1/3} \quad (\text{HTC.2})$$

for  $0,6 < \text{Pr} < 10$   
and  $\text{Re}_L < \text{Re}_{L,\text{crit}} \approx 2 \cdot 10^5$

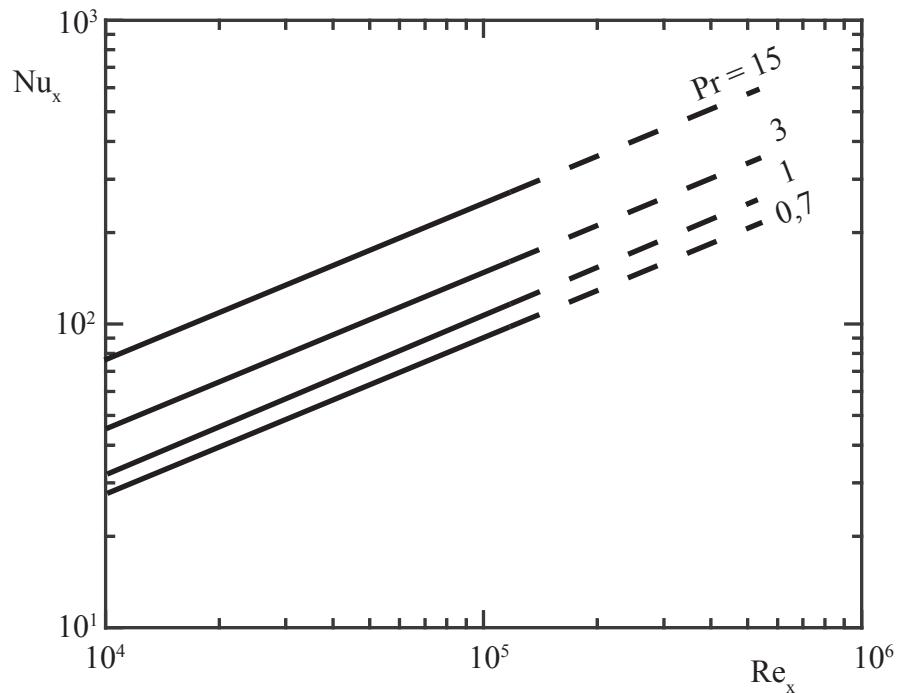


Figure 7.2: Local Nusselt number for the laminar boundary layer of a plate

Hydrodynamical inflow, heated or cooled from position  $x = x_0$ , isothermal surface

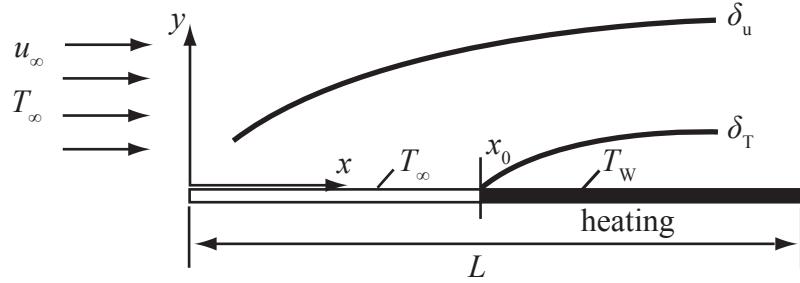


Figure 7.3: Laminar boundary layer of a plate, heated from  $x = x_0$  with isothermal surface

- Nusselt law for the local heat transfer:

$$Nu_x = 0.332 \text{ Re}_x^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \left[ 1 - \left( \frac{x_0}{x} \right)^{\frac{3}{4}} \right]^{-\frac{1}{3}} \quad (\text{HTC.3})$$

for  $0.6 < \text{Pr} < 10$   
and  $\text{Re}_x < \text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$

- Nusselt law for the mean heat transfer:

$$Nu_L = \frac{L}{L - x_0} \frac{1}{\lambda} \int_{x_0}^L \alpha(x) dx$$

$$Nu_L = 0.664 \text{ Re}_L^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \frac{\left[ 1 - \left( \frac{x_0}{L} \right)^{\frac{3}{4}} \right]^{\frac{2}{3}}}{\left[ 1 - \frac{x_0}{L} \right]} \quad (\text{HTC.4})$$

for  $0.6 < \text{Pr} < 10$   
and  $\text{Re}_x < \text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$

**Flat plate - turbulent boundary layer flow**  $\text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$ 

Note: The shift from laminar to turbulent flow depends on the degree of turbulence of the undisturbed flow, the surface roughness, etc, see below. For flow over flat surfaces the critical Reynolds number, i.e. the beginning of the eddy area is expected at 1 to  $2 \cdot 10^5$ . In extremely low turbulence wind tunnels, critical Reynolds numbers in the order of  $\text{Re}_{x,\text{crit}} \approx 1 \cdot 10^6$  can be reached.

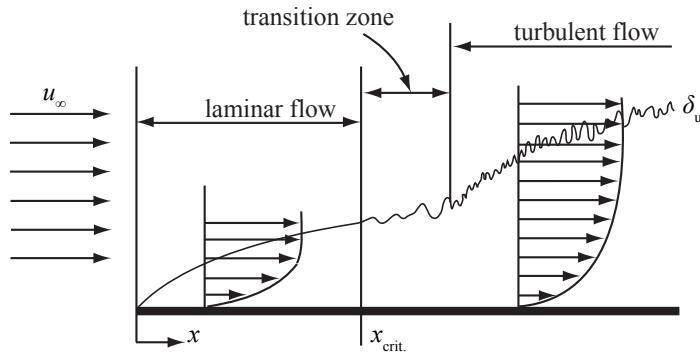


Figure 7.4: Laminar / turbulent boundary layer of a plate

The given relationships are derived from experimental data.

Isothermal surface, simultaneous hydrodynamic and thermal inflow

- Nusselt correlation for local heat transfer

$$\text{Nu}_x = 0.0296 \text{ Re}_x^{0.8} \text{ Pr}^{0.43} \quad \text{for } 5 \cdot 10^5 < \text{Re}_x < 10^7 \quad (\text{HTC.5})$$

- Nusselt law for mean heat transfer

$$\overline{\text{Nu}_L} = \frac{\overline{\alpha}L}{\lambda} = \frac{1}{\lambda} \left( \int_0^{x_{\text{crit}}} \alpha(x)_{\text{eq. (HTC.1)}} dx + \int_{x_{\text{crit}}}^L \alpha(x)_{\text{eq. (HTC.5)}} dx \right) \quad (7.8)$$

Note : The average heat transfer of a plate with length L can be calculated by integrating equations (HTC.1) and HTC.5. Hence, the result depends on the actual critical Reynolds number.

- Nusselt law for mean heat transfer:

$$\overline{\text{Nu}_L} \approx 0.036 \text{ Pr}^{0.43} \left( \text{Re}_L^{0.8} - 9400 \right) \quad (\text{HTC.6})$$

$$\text{for } \text{Re}_{L,\text{crit}} \approx 2 \cdot 10^5$$

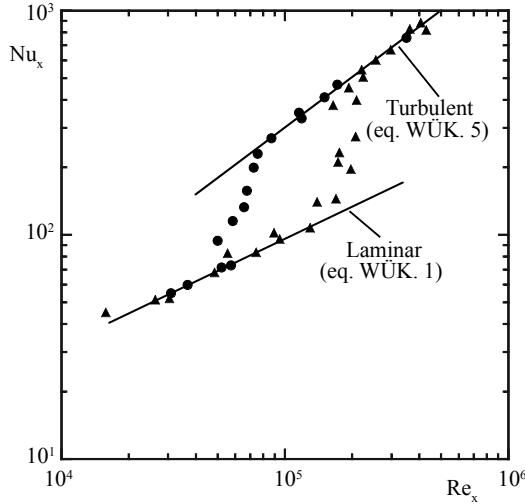


Figure 7.5: Local Nusselt number for a flat plate in air flow, Level of turbulence  $T_a = 0.7$  to  $1.6\%$ ; Level of turbulence  $T_a = 2.4$  to  $3.8\%$  according to Kestin u. a. (1961)

### Cylinders in a flow parallel to their longitudinal axis

As long as the diameter of the body is much greater than the thickness of the boundary layer, cylinders in longitudinal flow can be regarded as flat plates.

### Cylinders in a flow perpendicular to their longitudinal axis

- Local heat transfer

The local Nusselt number depicted in figure 7.7 shows a strong influence of the circumference angle, which is a result of the transition from laminar to turbulent flow and of the detachment of the flow.

- Mean heat transfer:

Nusselt law for the mean heat transfer

$$\overline{\text{Nu}_d} = C \text{Re}_d^m \text{Pr}^{0.4} \quad (\text{HTC.7})$$

With the constants C and m relevant for the given ranges of Reynolds numbers.

For all Reynolds numbers, the following simple approximation is valid according to Whitaker (1976):

$$\overline{\text{Nu}_d} = \left[ 0.40 \text{Re}_d^{\frac{1}{2}} + 0.06 \text{Re}_d^{\frac{2}{3}} \right] \text{Pr}^{0.4} \left( \frac{\eta_{\infty}}{\eta_w} \right)^{\frac{1}{4}} \quad (\text{HTC.8})$$

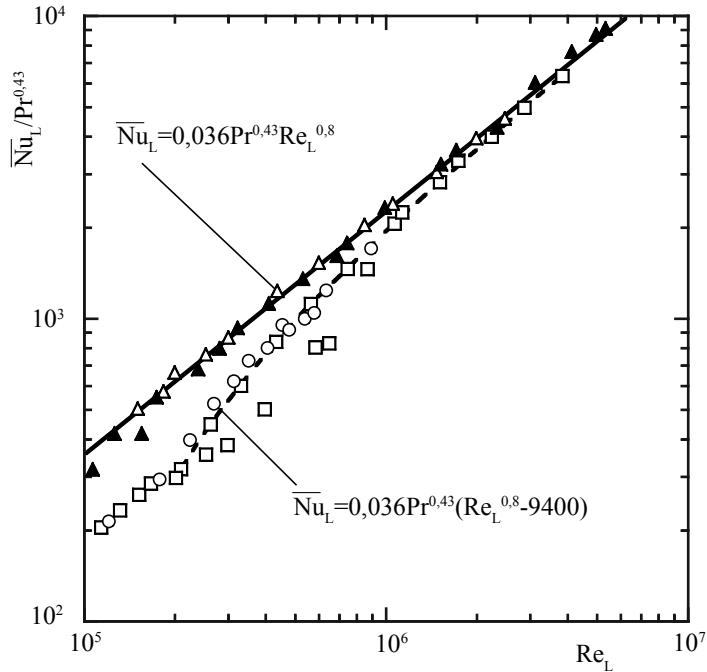


Figure 7.6: Mean Nusselt number for the turbulent boundary layer of a plate  
 (— Air (heating) with low level of turbulence,  $\Delta$  Air (heating) with 5% level of turbulence, Air, water, oil (heating and cooling) with unknown level of turbulence, Whitaker (1976))

Note: the properties used for the characteristic numbers in (HTC.8) are to be determined at free-flow temperature. The last factor, the ratio between the viscosity at free-flow temperature and the viscosity at the temperature of the wall, approximates the dependence of the temperature on the properties.

- Mean heat transfer for non circular cylinders

$$\overline{Nu}_d = C \text{ } Re_d^m \text{ } Pr^{0.4} \quad (\text{HTC.9})$$

Constants of equation (HTC.9) according to Jakob (1949)

Table 7.1: Constants C and m for corresponding Reynolds numbers

$Re_d$	$C$	$m$
0.4 - 4	0.989	0.330
4 - 40	0.911	0.385
40 - 4000	0.683	0.466
4000 - 40000	0.193	0.618
40000 - 400000	0.0266	0.805

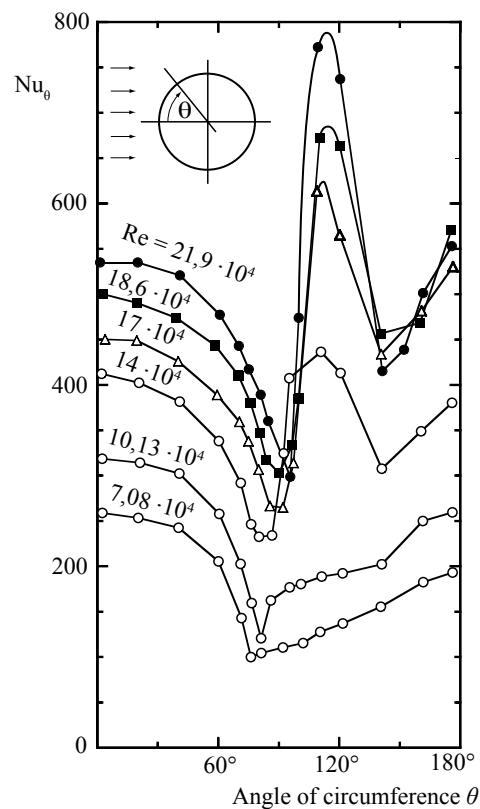


Figure 7.7: Local Nusselt number for cylinders in a flow perpendicular to their longitudinal axis, according to Holman (1976)

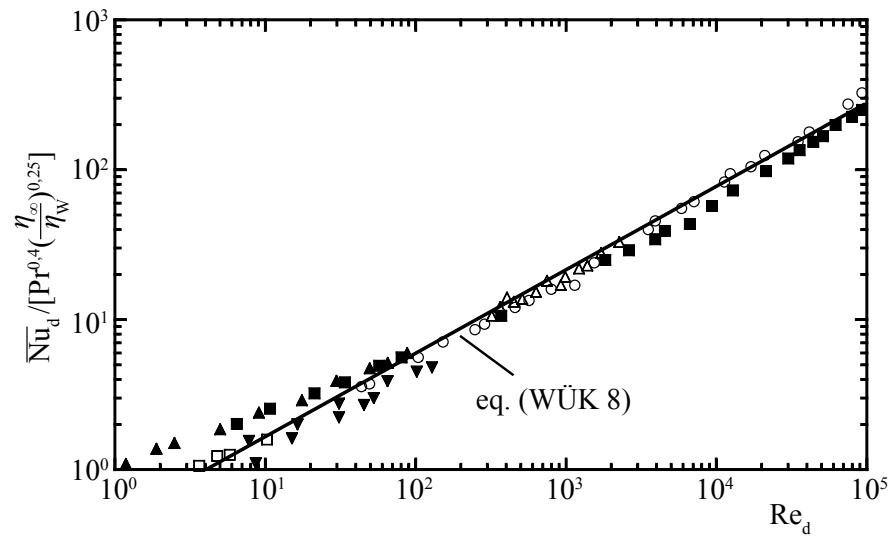


Figure 7.8: Mean Nusselt number for circular cylinders in a flow perpendicular to their longitudinal axis ( $\mp$  Water, glycol;  $\Delta$  nitrogen;  $\circ$  water;  $\blacktriangle$  air;  $\blacktriangledown$  water, paraffin, oil), Whitaker (1976)

Table 7.2: Constants C and m of equation (HTC.9)

Geometry	$Re_d$	$C$	$m$
	$5 \cdot 10^3 - 10^5$	0.246	0.588
	$5 \cdot 10^3 - 10^5$	0.102	0.675
	$5 \cdot 10^3 - 1.94 \cdot 10^4$	0.160	0.638
	$1.95 \cdot 10^4 - 10^5$	0.0385	0.782
	$5 \cdot 10^3 - 10^5$	0.153	0.638
	$5 \cdot 10^3 - 10^5$	0.228	0.731

### Bundles of smooth tubes in an orthogonal flow

Note: Since the temperature of the fluid in the tube bundle changes from row to row, a representative mean temperature difference must be used calculating the heat transfer.

$$\dot{Q}_W = \bar{\alpha} A (T_W - T_{fl})_m = \bar{\alpha} A \Delta T_m \quad (7.9)$$

For many practical purposes, the logarithmic temperature difference is a good approximation

$$\Delta T_m = \Delta T_{ln} \equiv \frac{\Delta T_{inlet} - \Delta T_{outlet}}{\ln \frac{\Delta T_{inlet}}{\Delta T_{outlet}}}$$

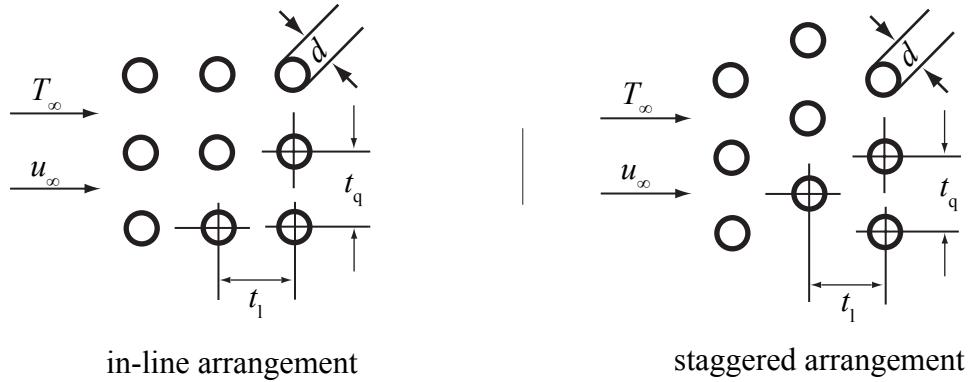


Figure 7.9: Arrangements of tubes for flow at right angles to tube bundles

In most industrial cases, where bundles of smooth tubes are used, the following equation for the heat exchange at the outer side of the tubes is valid according to Witte (1974)

$$\overline{Nu}_d = 0.287 \text{ Re}_d^{0.6} \text{Pr}^{0.36} \cdot f_e \quad (\text{HTC.10})$$

The Reynolds number is calculated using the free-flow velocity. The properties are to be determined at a mean temperature of the fluid inlet and outlet temperatures.

The tube arrangement factor  $f_e$  depends on the relative longitudinal distance/diameter ratio  $t_l/d$ , the relative transverse distance/diameter ratio  $t_q/d$  of the tubes, as well as the tube arrangement. This factor is shown in the following diagrams.

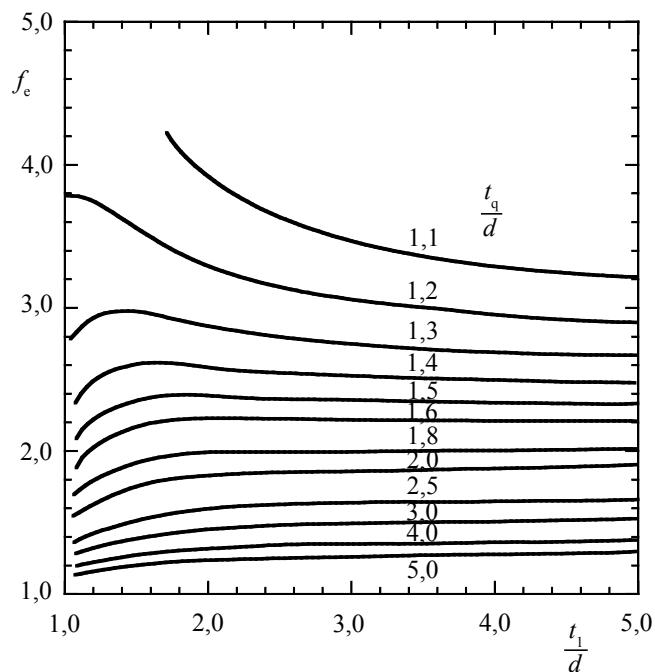


Figure 7.10: Arrangement factor, in-line arrangement, Witte (1974)

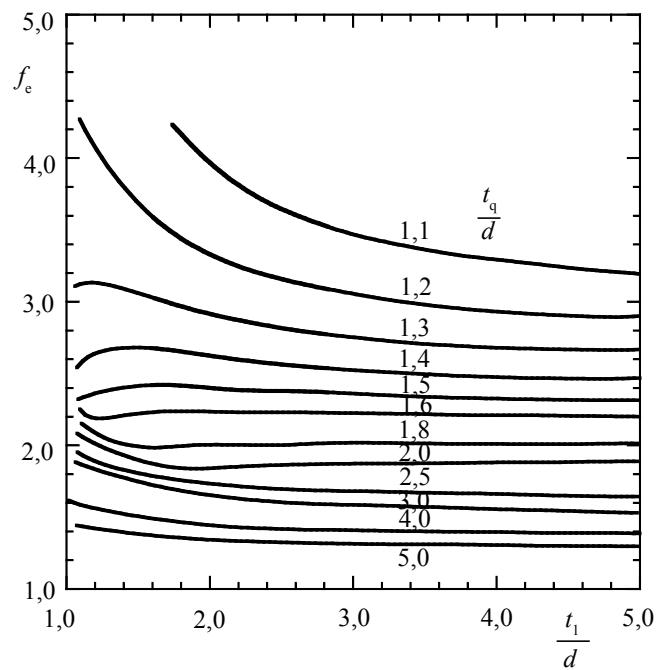


Figure 7.11: Arrangement factor, staggered arrangement, Witte (1974)

### Flow around spheres

In analogy to the example with the cylinders in flow perpendicular to their longitudinal axis, according to Whitaker (1976) the experimental results can be summarised in the following equation:

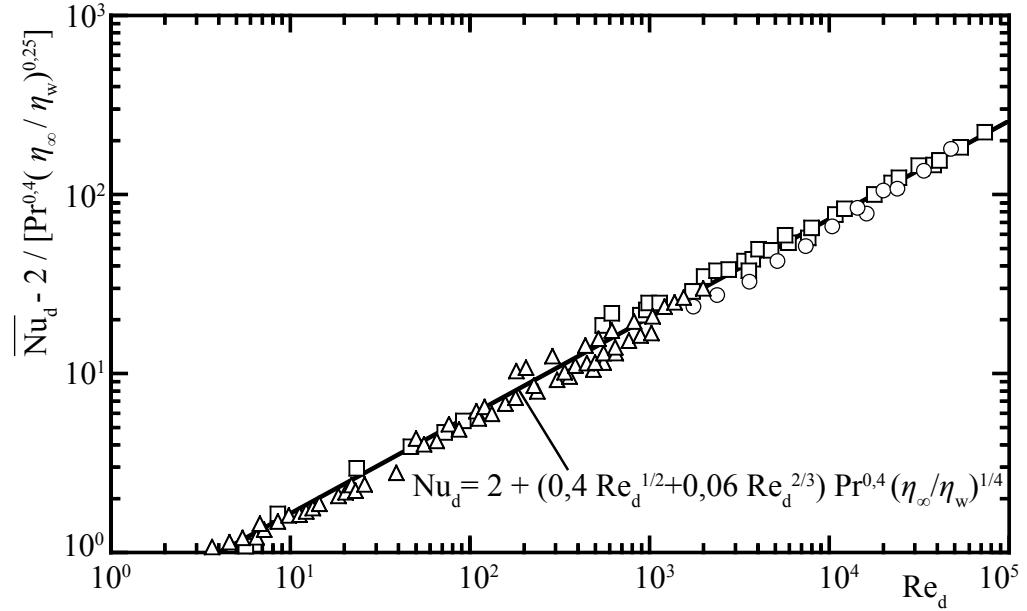


Figure 7.12: Mean Nusselt number for flow around spheres ( $\mp$  water,  $\circ$  air, according to Whitaker (1976))

dinal axis, according to Whitaker (1976) the experimental results can be summarised in the following equation:

$$\overline{Nu}_d = 2 + \left(0.4 Re_d^{\frac{1}{2}} + 0.06 Re_d^{\frac{2}{3}}\right) Pr^{0.4} \left(\frac{\eta_{\infty}}{\eta_w}\right)^{\frac{1}{4}} \quad (\text{HTC.11})$$

$$\begin{aligned} \text{for } & 0.7 < Pr < 380 \\ \text{and } & 3.5 < Re_d < 8 \cdot 10^4 \end{aligned}$$

In this case too, the properties used for the characteristic numbers have to be determined at free-flow temperature.

### 7.3 Internal forced convection

For flow in tubes, channels or nozzles, the heat transfer coefficient is defined by

$$\dot{Q}_W = \bar{\alpha} A \Delta T_m$$

where  $\Delta T_m$  is a representative temperature difference between the temperature at the wall  $T_W$  and the energetically averaged mean temperature of the fluid  $T_{fl}$  (see introductory remarks). For greater changes of the fluid temperature along the flow path the logarithmic temperature difference should be used.

The properties used for the characteristic numbers are to be determined, if not stated otherwise, at a mean fluid temperature over the tube length

$$T_{mat} = \frac{T_{fl,outlet} + T_{fl,inlet}}{2}$$

The Reynolds number  $Re_d$  is determined using the mean fluid velocity and the diameter.

The relationships given for turbulent flow of tubes can also be used for channels with non circular cross section, if the diameter is replaced by the so-called hydraulic mean diameter  $d_h$ :

$$d_h = 4 \frac{\text{cross-section area}}{\text{wetted perimeter}} = 4 \frac{A_c}{U}$$

**Laminar flow in tubes**  $Re_{d,crit} \approx 2300$

Isothermal surface, fully developed flow at the start of the heated (cooled) section of a tube

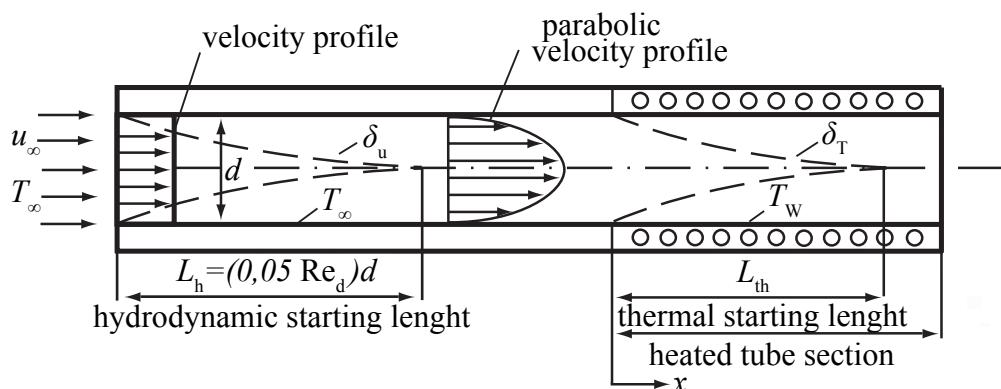


Figure 7.13: Heated tube with fully developed flow

There are theoretical solutions for the local and mean heat transfer in the form of Taylor series, shown in the following diagram. It can be seen from the diagram that

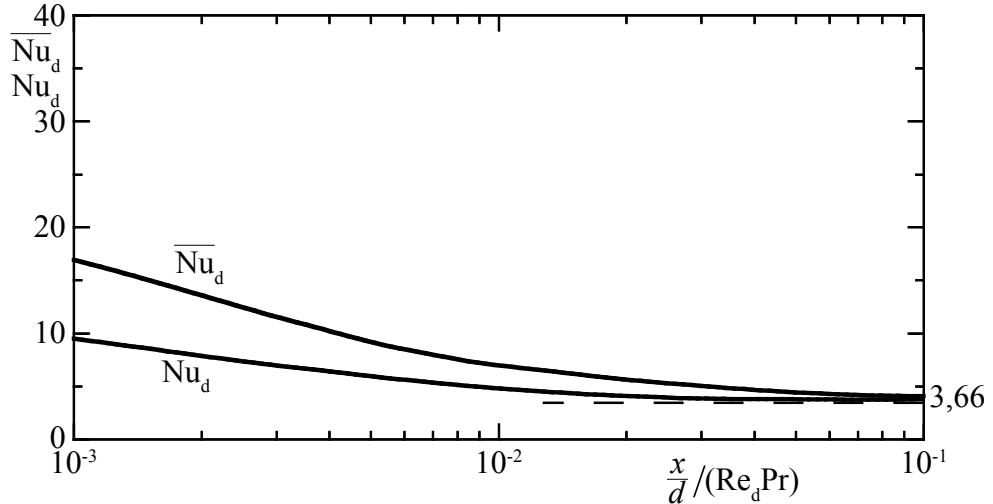


Figure 7.14: Local and mean Nusselt number for laminar fully developed tube flow, Rohsenow u. Harnett (1973)

after a thermal inflow of

$$\frac{L_{\text{th}}}{d} \approx 0.05 \text{ Re}_d \text{Pr}$$

is reached, the Nusselt number has a constant final value of  $\text{Nu}_{d,\infty} = 4.36 \left( \frac{\eta}{\eta_w} \right)^{0.14}$ .

The results for the mean Nusselt number  $\overline{\text{Nu}}_d$  can be approximated by

$$\overline{\text{Nu}}_d = \left( 3.66 + \frac{0.19 \left( \text{Re}_d \text{Pr} \frac{d}{L} \right)^{0.8}}{1 + 0.117 \left( \text{Re}_d \text{Pr} \frac{d}{L} \right)^{0.467}} \right) \left( \frac{\eta}{\eta_w} \right)^{0.14} \quad (\text{HTC.12})$$

If instead of the wall temperature, the heat flux at the wall remains constant, the heat transfer coefficients increase by about 20%. The constant final value for long tubes is  $\text{Nu}_{d,\infty} = 4.36 \left( \frac{\eta}{\eta_w} \right)^{0.14}$  in this case.

Isothermal surface, simultaneous hydrodynamic and thermal inflow

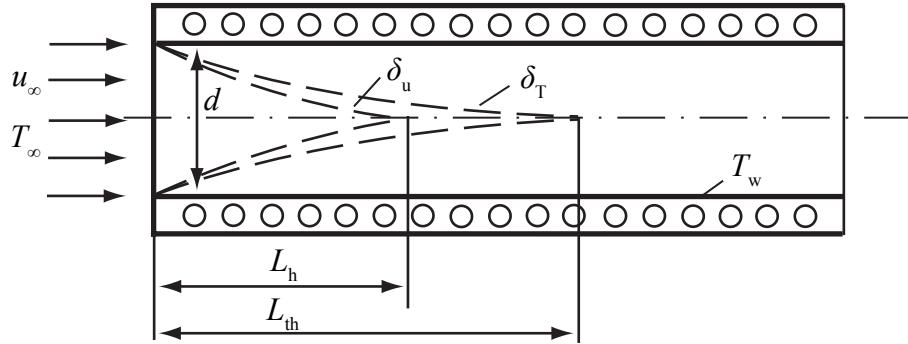


Figure 7.15: Heated tube with simultaneous hydrodynamic and thermal inflow

This problem can be solved theoretically as well. The results are additionally dependent on the Prandtl number of the fluid. The mean Nusselt number is shown in Fig. (7.16).

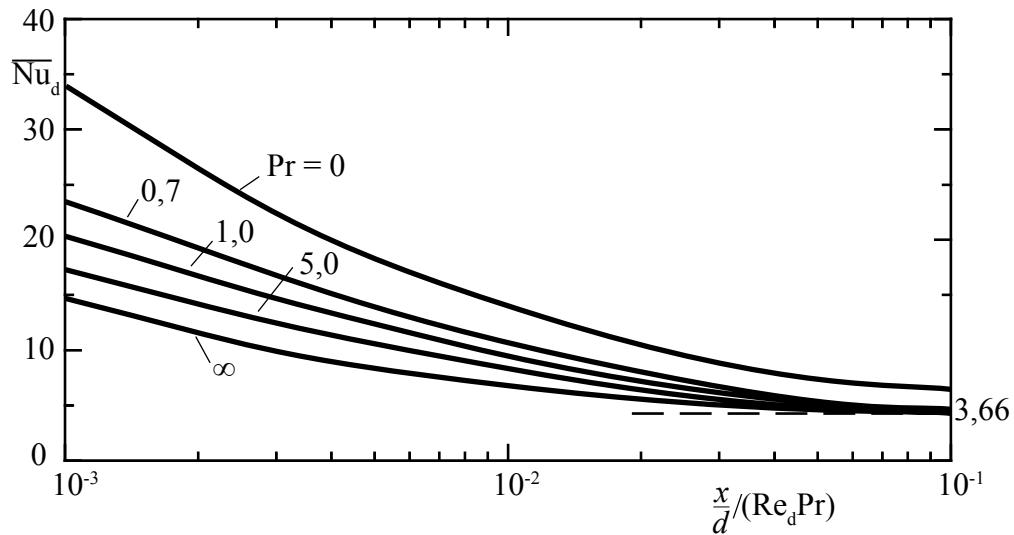


Figure 7.16: Mean Nusselt number for laminar flow in tubes and simultaneous hydrodynamic and thermal inflow

The solution can be approximated by

$$\overline{Nu}_d = \left( 3.66 + \frac{0.0677 \left( Re_d Pr \frac{d}{L} \right)^{1.33}}{1 + 0.1 Pr \left( Re_d \frac{d}{L} \right)^{0.83}} \right) \left( \frac{\eta}{\eta_w} \right)^{0.14} \quad (\text{HTC.13})$$

**Turbulent flow in tubes,  $Re_{d,crit} \approx 2300$**

Isothermal surface, simultaneous hydrodynamic and thermal inflow

An empirical relationship for the mean Nusselt number for turbulent pipe flows heated from the beginning based on a large number of experiments can be found in VDI Wärmeatlas, 1997.

$$\overline{Nu_d} = 0.0235 \left( Re_d^{0.8} - 230 \right) \left( 1.8 Pr^{0.3} - 0.8 \right) \left( 1 + \left( \frac{d}{L} \right)^{\frac{2}{3}} \right) \left( \frac{\eta}{\eta_w} \right)^{0.14} \quad (\text{HTC.14})$$

$$\begin{aligned} \text{for } & 0.6 < Pr < 500 \quad \frac{L}{d} > 1 \\ \text{and } & Re_d > 2300 \end{aligned}$$

which also approximates part of the thermal inflow by the term  $\left( 1 + \left( \frac{d}{L} \right)^{\frac{2}{3}} \right)$ . The thermal inflow ends at a length of 10 to 40 times the tube diameter, for turbulent flow, depending on the value of the Reynolds number.

Heat transfer coefficients that have been measured in conditions of constant heat flux deviate by 5% from the values obtained at constant wall temperature, with exception of flow of liquid metals with very low Prandtl number.

In many cases the following relationship, instead of equation (HTC.14) is a sufficient approximation:

$$\overline{Nu_d} = 0.027 Re_d^{0.8} Pr^{\frac{1}{3}} \left( \frac{\eta}{\eta_w} \right)^{0.14} \quad (\text{HTC.15})$$

$$\text{for } 3000 < Re_d > 10^5 \quad \frac{L}{d} > 40$$

## 7.4 External natural convection

**Vertical plate - laminar boundary layer flow,**  $\text{Gr}_{x,\text{crit}} \cdot \text{Pr} \approx 4 \cdot 10^9$

Note: All properties, if not stated otherwise, are to be determined at a mean boundary layer temperature, excluding the isobaric expansion coefficient  $\beta$ , which for ideal gases is determined by  $\beta = \frac{1}{T_\infty}$ .

### Isothermal surface

- Local heat transfer

As already mentioned in section 4 a theoretical relationship for the Nusselt number can be derived from an approximation of the conservation equations.

$$\text{Nu}_x = 0.508 \left( \frac{\text{Pr}}{0.952 + \text{Pr}} \right)^{\frac{1}{4}} (\text{Gr}_x \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.16})$$

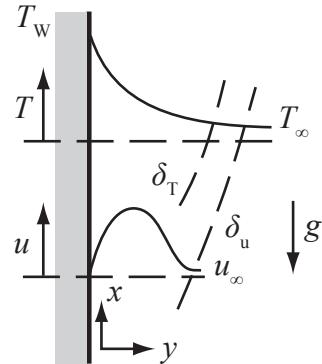


Figure 7.17: Heated plate, natural convection

- Mean heat transfer ( $\text{Gr}_{L,\text{crit}} \cdot \text{Pr} \approx 4 \cdot 10^9$ )

$$\overline{\text{Nu}_L} = \frac{\overline{\alpha} L}{\lambda} = \frac{1}{\lambda} \int_0^L \alpha(x) dx \quad (7.10)$$

$$\overline{\text{Nu}_L} = C (\text{Gr}_L \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.17})$$

for  $\text{GrPr} < \text{Gr}_{L,\text{crit}} \cdot \text{Pr}$

with the constant  $C$ , dependent on the Prandtl number

Note: This exact solution deviates insignificantly from the solution obtained by integration of equation (HTC.16).

Table 7.3: Constant C of equation (HTC.17)

Pr	0.003	0.01	0.03	0.72	1	2	10	100	1000	$\infty$
C	0.182	0.242	0.305	0.516	0.535	0.568	0.620	0.653	0.665	0.670

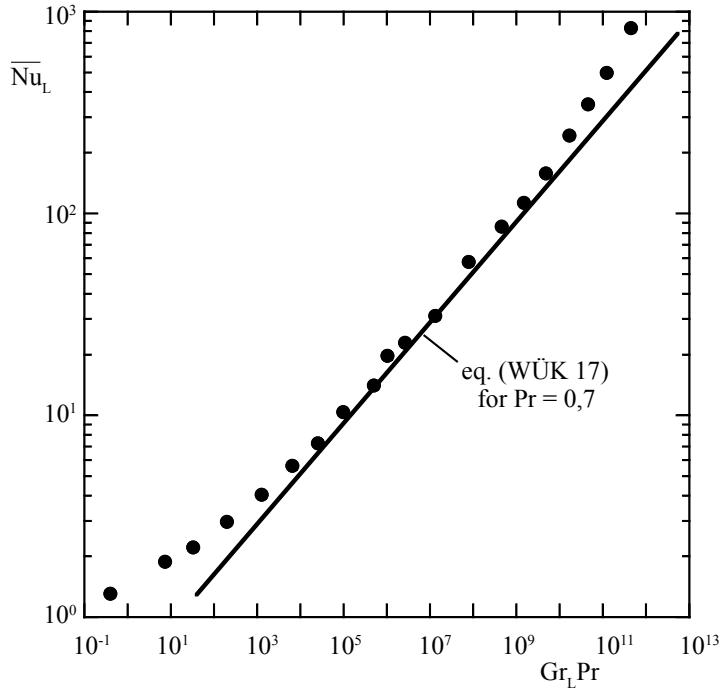


Figure 7.18: Mean Nusselt number for natural convection at a vertical plate, • air, according to Whitaker (1976)

#### Constant heat flux

Note: Since the heat flux is given as a boundary condition, for simplicity, a modified Grashof number  $\text{Gr}_x^*$  shall be defined  $\text{Gr}_x^* \equiv \text{Gr}_x \text{Nu}_x = \frac{\rho^2 g \beta q_w'' x^4}{\lambda \eta^2}$ .

- local heat transfer

Derived from experiments the following equation is valid for local heat transfer, according to Holman (1976)

$$\text{Nu}_x = 0.60 (\text{Gr}_x^* \text{Pr})^{\frac{1}{5}} \quad (\text{HTC.18})$$

$$\text{for } 10^5 < \text{Gr}_x^* < 10^{11}$$

**Vertical plate, turbulent boundary layer flow,  $\text{Gr}_{x,\text{crit}} \text{Pr} \approx 4 \cdot 10^9$**

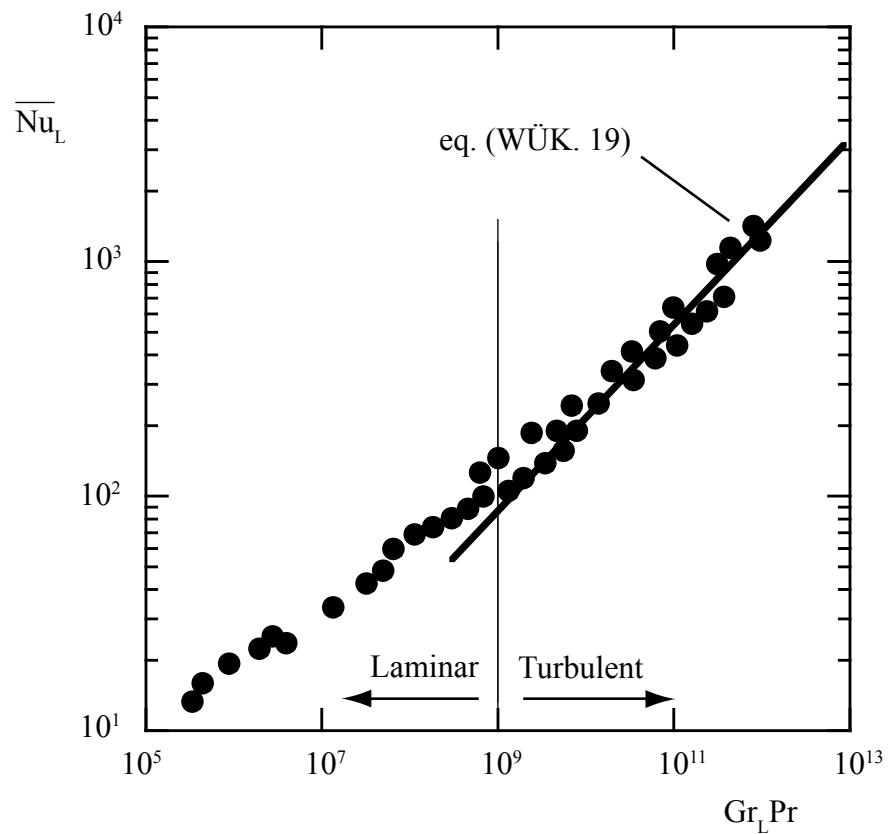


Figure 7.19: Mean Nusselt number for turbulent, natural convection at a vertical plate, • air, according to Whitaker (1976)

- Mean heat transfer

$$\bar{Nu}_L = 0.13 (Gr_L Pr)^{\frac{1}{3}} \quad (\text{HTC.19})$$

$$\text{for } 10^9 < (Gr_L Pr) < 10^{12}$$

Note: For turbulent boundary layer flow the exponent  $\frac{1}{3}$  of the Grashof number makes the heat transfer coefficient independent of the height of the plate.

**Vertical cylinder - laminar and turbulent boundary layer flow**

As long as the diameter of the cylinder is significantly higher than the developed boundary layer thickness, the relationships valid for the vertical plate can be applied to this case. The limiting factor is approximately  $\frac{d}{L} > 35 \cdot Gr_L^{-\frac{1}{4}}$ . The characteristic geometrical length is the length of the cylinder.

**Horizontal cylinder - laminar and turbulent boundary layer flow, isothermal surface**

Note: the characteristic length  $L$  is in this case the cylinder diameter  $d$ .

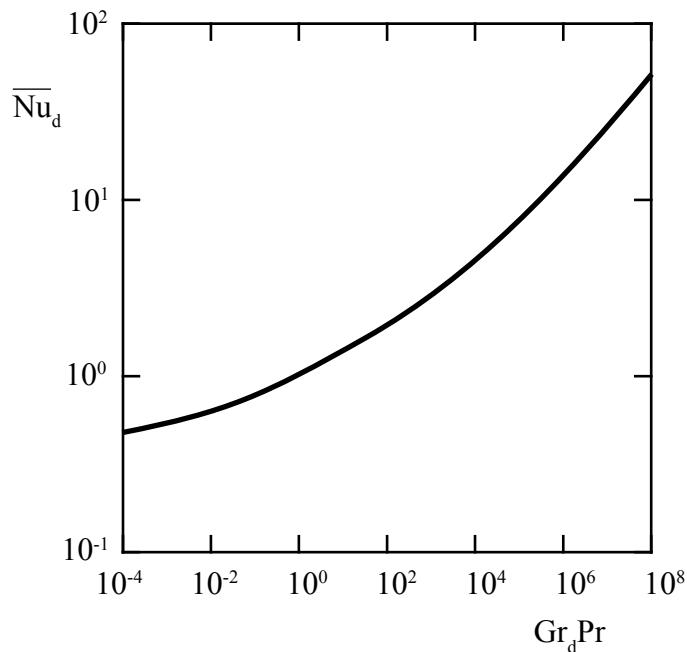


Figure 7.20: Mean Nusselt number for natural convection at a horizontal cylinder,  
McAdams (1954)

Using the approximations introduced by Bayley u. a. (1972)

- for the **laminar** range  $10^4 < Gr_d \cdot Pr < 10^9$

$$\overline{Nu}_d = 0.53 (Gr_d \cdot Pr)^{\frac{1}{4}} \quad (\text{HTC.20})$$

- for the **turbulent** range  $10^9 < Gr_d \cdot Pr < 10^{12}$

$$\overline{Nu}_d = 0.13 (Gr_d \cdot Pr)^{\frac{1}{3}} \quad (\text{HTC.21})$$

### **Horizontal plates - laminar and turbulent boundary layer flow, Holman (1976)**

Note: the relationships presented below are derived from measurements on square plates. They are approximately valid for rectangular and circular surfaces, as long as one mean side length or ( $0.9d$ ), respectively is taken as the characteristic length.

#### Isothermal surface, heated upper plate side or cooled lower side

- **laminar** range  $2 \cdot 10^4 < \text{Gr}_L \text{Pr} < 8 \cdot 10^6$

$$\overline{\text{Nu}_L} = 0.54 (\text{Gr}_L \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.22a})$$

- **turbulent** range  $8 \cdot 10^6 < \text{Gr}_L \text{Pr} < 10^{11}$

$$\overline{\text{Nu}_L} = 0.15 (\text{Gr}_L \text{Pr})^{\frac{1}{3}} \quad (\text{HTC.23a})$$

#### Isothermal surface, cooled upper plate side or heated lower side

- **laminar** range  $10^5 < \text{Gr}_L \text{Pr} < 10^{11}$

$$\overline{\text{Nu}_L} = 0.27 (\text{Gr}_L \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.24a})$$

#### Constant heat flux, heated upper plate side or cooled lower side

- **laminar** range  $\text{Gr}_L \text{Pr} < 2 \cdot 10^8$

$$\overline{\text{Nu}_L} = 0.13 (\text{Gr}_L \text{Pr})^{\frac{1}{3}} \quad (\text{HTC.22b})$$

- **turbulent** range  $2 \cdot 10^8 < \text{Gr}_L \text{Pr} < 10^{11}$

$$\overline{\text{Nu}_L} = 0.16 (\text{Gr}_L \text{Pr})^{\frac{1}{3}} \quad (\text{HTC.23b})$$

#### Constant heat flux, cooled upper plate side or heated lower side

- **laminar** range  $10^6 < \text{Gr}_L \text{Pr} < 10^{11}$

$$\overline{\text{Nu}_L} = 0.58 (\text{Gr}_L \text{Pr})^{\frac{1}{5}} \quad (\text{HTC.24b})$$

## 7.5 Internal natural convection

Note: The characteristic length  $L$  is the distance  $s$  between the heated and cooled surface, the effective temperatures are the heated and cooled wall temperatures.

### Fluid layers between isothermal, vertical walls with the height/distance ratio

$3.1 < \frac{H}{s} < 42.2$ , Bayley u. a. (1972)

For  $\text{Gr}_s < 2 \cdot 10^3$  (heat conduction only)  $\overline{\text{Nu}}_s = 1$

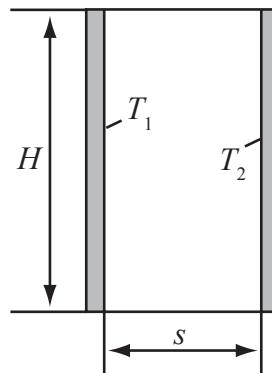


Figure 7.21: Vertical, enclosed fluid layers

For the **laminar** range  $2 \cdot 10^3 < \text{Gr}_s < 2 \cdot 10^4$

$$\overline{\text{Nu}}_s = 0.20 \left( \frac{H}{s} \right)^{-\frac{1}{9}} (\text{Gr}_s \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.25})$$

For the **turbulent** range  $2 \cdot 10^5 < \text{Gr}_s < 10^7$

$$\overline{\text{Nu}}_s = 0.071 \left( \frac{H}{s} \right)^{-\frac{1}{9}} (\text{Gr}_s \text{Pr})^{\frac{1}{3}} \quad (\text{HTC.26})$$

**Fluid layers between isothermal, horizontal walls, lower wall is heated, Holman (1976)**

Note: If heated from above, stable layers are formed. The heat is transferred by

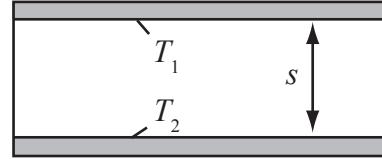


Figure 7.22: Horizontal, enclosed fluid layers

conduction.

For  $\text{Gr}_s < 2 \cdot 10^3$  (**heat conduction only**)  $\overline{\text{Nu}}_s = 1$

For the **laminar** range  $10^4 < \text{Gr}_s < 3.2 \cdot 10^5$

$$\overline{\text{Nu}}_s = 0.21 (\text{Gr}_s \text{Pr})^{\frac{1}{4}} \quad (\text{HTC.27})$$

For the **turbulent** range  $3.2 \cdot 10^5 < \text{Gr}_s < 10^7$

$$\overline{\text{Nu}}_s = 0.075 (\text{Gr}_s \text{Pr})^{\frac{1}{3}} \quad (\text{HTC.28})$$

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