Tutorial T03 – Elasticity Stress

Answer the following questions as they could come up in an exam.

1 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ($E = 2 \cdot 10^5$ MPa, $\nu = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1(\frac{x_1}{L}) - 10(\frac{x_2}{L}) + \alpha_2(\frac{x_1}{L})^2\right]$

$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8(\frac{x_1}{L}) + \beta_1(\frac{x_2}{L}) + \beta_2(\frac{x_2}{L})^2\right]$

$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10(\frac{x_1}{L}) + 7(\frac{x_2}{L}) - 8(\frac{x_1}{L})(\frac{x_2}{L})\right]$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$, and
with reference stress $\sigma_0 = 1\,\text{MPa}$ and reference length $L = 1\,\text{m}$.

Note: Question (a) is general, symbolic, with variables x_1, x_2, x_3 and coefficients α_1, α_2, β_1, β_2; only from question (b) on, use the single, chosen point P$(x_1 = 0, x_2 = 0, x_3 = 0)$.

Questions:
... based on section 3
a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the free coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium?
b) Compute the eigen-stresses in point P using linear algebra, mathematics -– not circle of Mohr. Describe and name the state of stress in point P (and in all other points in the body).
c) Compute the eigen-direction of the major eigen-stress.
d) Draw the relevent circle of Mohr and confirm graphically the results of (b) and (c); explain.
... based on section 5
e) Compute the equivalent stress according to Tresca.
What is the origin of the limit-stress hypothesis of Tresca?
f) Compute the equivalent stress according to von Mises.
What is the origin of the limit-stress hypothesis of von Mises?
g) Compute the specific elastic energy π_{el} in point P.

Answers:
a) Given was the plane stress-field, independent of x_3, in absence of body forces $f_i = 0$:

$$\sigma_{11}(x_1, x_2) = \sigma_0 \left[20 + \alpha_1 \frac{x_1}{L} - 10\frac{x_2}{L} + \alpha_2 \left(\frac{x_1}{L}\right)^2\right]$$
$$\sigma_{22}(x_1, x_2) = \sigma_0 \left[10 + 8\frac{x_1}{L} + \beta_1 \frac{x_2}{L} + \beta_2 \left(\frac{x_2}{L}\right)^2\right]$$
$$\sigma_{12}(x_1, x_2) = \sigma_0 \left[12 - 10\frac{x_1}{L} + 7\frac{x_2}{L} - 8\frac{x_1}{L}\frac{x_2}{L}\right]$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\frac{d}{dx_1}\sigma_{11}(x_1,x_2) + \frac{d}{dx_2}\sigma_{12}(x_1,x_2) = \sigma_0\left[\frac{\alpha_1}{L} + 2\alpha_2\frac{x_1}{L^2}\right] + \sigma_0\left[\frac{7}{L} - 8\frac{x_1}{L^2}\right] = 0$$
$$\frac{d}{dx_1}\sigma_{12}(x_1,x_2) + \frac{d}{dx_2}\sigma_{22}(x_1,x_2) = \sigma_0\left[\frac{-10}{L} - 8\frac{x_2}{L^2}\right] + \sigma_0\left[\frac{\beta_1}{L} + 2\beta_2\frac{x_2}{L^2}\right] = 0$$

From these equations, one gets the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$.

Because the field equations must be valid for all constants and points x_1, x_2, x_3, independently, one can group them accordingly: The constant terms from the first and second equations provide α_1 and β_1, respectively, while the x_1 and x_2 groups provide α_2 and β_2.

b)

The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0\,.$$

Knowing/recognizing that one eigen-value is zero, i.e. also $I_3 = 0$, the principal stresses can be computed from the second order polynomial as: $\sigma_I = 28\,\text{MPa}$, $\sigma_{II} = 2\,\text{MPa}$, $\sigma_{III} = 0\,\text{MPa}$. This is a plane-stress state with all stresses on the x_3-surface equal to zero, which also has consequences for the eigen-directions ...

c)
The principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0,0,1)$ is directly visible from the tensor, due to the zero shear stresses in the x_3-direction.

The eigen-direction of the major stress $\sigma_I = 28\,\text{MPa}$ is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that: $-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress, $\sigma_{II} = 2\,\text{MPa}$ was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that: $18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9) + 1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

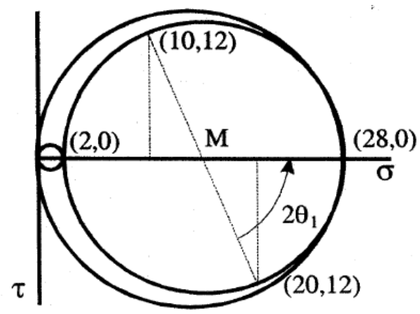

Figure 1: Sketch of a Mohr circle, focus is on the right, inner circle.

d) Mohr's circle

Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{20+10}{2} = 15\,\text{MPa}$,
and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13\,\text{MPa}$.

The eigenvalues are therefore:
$\sigma_I = M + R = 28\,\text{MPa}$, $\sigma_{II} = C - R = 2\,\text{MPa}$.

The eigen-directions are:
$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{24}{10} = 2.4 \Longrightarrow \theta_I = (1/2)\arctan(2.4) = 67.38°/2 = 33.69°$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$; and $\theta_{II} = (180° + 67.3°)/2 = 247.3°/2 = 123.7° = \arccos(-2/\sqrt{13})$.

The max. shear stress is just the radius: $\tau^{max} = R = 13$ MPa

e) Failure criteria according to the (double) maximal shear stress:
$\tau_{max} = \frac{1}{2} \mid \sigma_{max} - \sigma_{min} \mid = 14$ MPa, $\sigma_{eq}^{Tresca} = 2\tau_{max} = 28$ MPa

f) Failure criteria according to the shape-change (distortion) energy:
$\sigma_{eq}^{von-Mises} = \sqrt{\frac{1}{2}[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2]} \approx 27$ MPa

$\sigma_{eq}^{von-Mises} < \sigma_{eq}^{Tresca}$, thus the Tresca criterion is safer, since the limit stress is reached earlier.

g) Hooke's law for strain as function of stress: $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}]$,
with $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, using $E = 2.10^5$ MPa and $\nu = 1/4$, one obtains:
$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E} = 20/E - (10/4)/E = (35/2)\,\text{MPa}\ E^{-1}$,
$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{33}}{E} = 10/E - (20/4)/E = 5\,\text{MPa}\ E^{-1}$,
$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} = -(10/4)/E - (20/4)/E = -(15/2)\,\text{MPa}\ E^{-1}$,
$\varepsilon_{12} = \frac{\sigma_{12}}{2G} = 6/G = 15\,\text{MPa}\ E^{-1}$ (with $G = \frac{E}{2(1+\nu)} = 2E/5$) ,
and $\varepsilon_{13} = \varepsilon_{23} = 0$. *Note that* $\varepsilon_{33} \neq 0$, *even though* $\sigma_{33} = 0$.

$$[\varepsilon] = \begin{bmatrix} 35/2 & 15 & 0 \\ 15 & 10/2 & 0 \\ 0 & 0 & -15/2 \end{bmatrix} \text{MPa}\ E^{-1} = \begin{bmatrix} 35 & 30 & 0 \\ 30 & 10 & 0 \\ 0 & 0 & -15 \end{bmatrix} \frac{10^{-5}}{4} = \begin{bmatrix} 87.5 & 75 & 0 \\ 75 & 25 & 0 \\ 0 & 0 & -37.5 \end{bmatrix} 10^{-6}\,.$$

Elastic energy:

$$\pi_{el} = 0.5 \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \varepsilon_{ij} = \sigma_{ij} \varepsilon_{ij} = 1.9\,10^{-3}\,\mathrm{MPa} \left(= \frac{\text{Energy}}{\text{Volume}} \right)$$

in detail (diamonds entries not needed):

$$\begin{aligned}
\pi_{el} &= \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 87.5 & 75 & 0 \\ 75 & 25 & 0 \\ 0 & 0 & -37.5 \end{bmatrix} \frac{10^{-6}}{2}\,\mathrm{MPa} \\
&= \mathrm{tr} \begin{bmatrix} 20 * 87.5 + 12 * 75 & \diamond & 0 \\ \diamond & 12 * 75 + 10 * 25 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{10^{-6}}{2}\,\mathrm{MPa} \\
&= (1750 + 900 + 900 + 250) \frac{10^{-6}}{2}\,\mathrm{MPa} = 1.9\,10^{-3}\,\mathrm{MPa}
\end{aligned}$$

Note that stress and energy density have the same units.

2 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:
$E = 2 \cdot 10^{11}$ Pa, $\nu = 0.25$

Stress-state in point P: $\left[\sigma\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa

Questions:
a) Show that the principal stresses are 8, 16 and 24 MPa.
Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.
... based on sections 4,5
b) Compute the volumetric (isotropic) strain.
c) What is the largest angle-change (not shear-strain) in P?
... based on section 5
d) Which material property is implicitly used/assumed in Hookes law?

Answers:
a)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0\,.$$

Given the eigenvalues, σ, one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation, watch the signs in the definitions.
Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 24\,\text{MPa}$, $\sigma_{II} = 16\,\text{MPa}$, and $\sigma_{III} = 8\,\text{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.
The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8\,\text{MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$

Inserting into line 3 yields: $n_3 = \sqrt{6}n_1$, so that: $\begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$

b)
For the volumetric (isotropic) strain, we can use the short-cut (not the full strain tensor), as: $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \left[\frac{1-2\nu}{E}\right] \sigma_{kk} = 12\ 10^{-5}$.

c)
The largest change of angle is: $\gamma_{max} = \frac{\tau_{max}}{G} = \frac{1}{2}\frac{\sigma_I - \sigma_{III}}{G} = 10^{-4}$, using $G = \frac{E}{2(1+\nu)} = (4/5)\ 10^5$ MPa, where the largest shear strain is just half of that: $\varepsilon_{max} = \gamma_{max}/2$.

d)
Isotropic (direction independent) material behavior is intrinsically assumed in the law of Hooke.