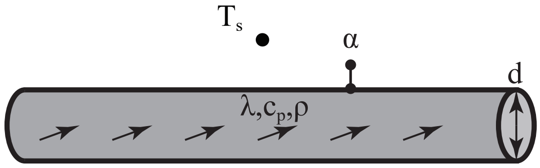
**Exercise II.13** (Cooling of a copper rod ⋆⋆):

A long copper rod is initially at a uniform temperature $T_0$. It is now exposed to an air stream at $T_\infty$ with a heat transfer coefficient $\alpha$.




**Given parameters:**

- Diameter of the copper rod: $d = 2$ cm
- Initial temperature: $T_0 = 100$ °C
- Air stream temperature: $T_\infty = 20$ °C
- Heat transfer coefficient: $\alpha = 200$ W/m$^2$K
- Thermal conductivity of copper: $\lambda = 399$ W/mK
- Specific heat capacity of copper: $c_p = 382$ J/kgK
- Density of copper: $\rho = 8930$ kg/m$^3$

**Hints:**

- Heat radiation can be neglected.
- Setup an energy balance.

**Tasks:**

a) Determine how long will it take for the copper rod to cool to a temperature of $T_1 = 25$ °C.

b) Sketch the temperature profile over the course of time.

**Solution II.13** (Cooling of a copper rod ⋆⋆):

Task a)

As mentioned earlier, the temperature profile within the solid body can be derived from the conduction equation, considering both spatial and temporal aspects.

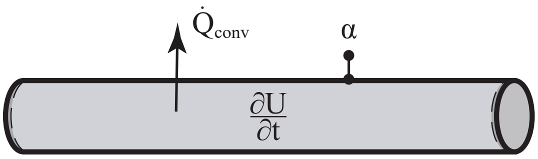
In practical scenarios, the temperature profile within the rod is never homogeneous. However, the Biot number is a useful parameter for assessing whether the temperature distribution within the rod can be approximated as homogeneous. If the Biot number allows for such an assumption, the lumped capacity model becomes a viable approach to solving the problem.

The Biot number is determined by:

$$\begin{aligned} \mathrm{Bi} &= \frac{\alpha V}{\lambda A_\mathrm{s}} = \frac{\alpha \cdot d}{4\lambda} = \\ &= \frac{200\ [\mathrm{W/m^2K}] \cdot 0.02\ [\mathrm{m}]}{4 \cdot 399\ [\mathrm{W/mK}]} = 0.025\ [-] \end{aligned} \tag{II.13.1}$$

thus Bi ≪ 1, which implies that spatial temperature differences within the copper rod are negligible.

1 **Setting up the balance:**




The copper rod is losing heat by convection and therefore the energy balance reads:

$$\frac{\partial U}{\partial t} = -\dot{Q}_\mathrm{conv} \tag{II.13.2}$$

2 **Defining the elements within the balance:**

The temporal change of internal energy can be expressed as:

$$\begin{aligned} \frac{\partial U}{\partial t} &= m c_\mathrm{p} \frac{\partial T}{\partial t} \\ &= \rho \frac{\pi d^2}{4} L c_\mathrm{p} \cdot \frac{\partial T}{\partial t} \end{aligned} \tag{II.13.3}$$

And the rate of heat loss by convection

$$\begin{aligned} \dot{Q}_\mathrm{conv} &= \alpha A_\mathrm{s} \cdot (T(t) - T_\infty) \\ &= \alpha \pi d \cdot L \cdot (T(t) - T_\infty) \end{aligned} \tag{II.13.4}$$

3 **Inserting and rearranging:**

Inserting and yields:

$$\frac{\partial T}{\partial t} = \frac{4\alpha}{d c_\mathrm{p}} \cdot (T(t) - T_\infty) \tag{II.13.5}$$

## 4 Defining the boundary and/or initial conditions:

To solve the equation, one initial condition should be given. This can be described in terms of the initial temperature of the copper rod, which is:

$$T(t = 0) = T_0 \tag{II.13.6}$$

## 5 Solving the equation:

Before solving, we first will multiply both sides with $\frac{1}{T_\infty - T_0}$ :

$$\rho \frac{\pi d^2}{4} L c_\mathrm{p} \cdot \frac{1}{T_\infty - T_0} \cdot \frac{\partial T}{\partial t} = -\alpha \pi d \cdot L \cdot \frac{T(t) - T_\infty}{T_\infty - T_0} \tag{II.13.7}$$

To solve the equation, a dimensionless temporal temperature difference will be introduced, which is:

$$\Theta^* = \frac{T(t) - T_0}{T_\infty - T_0} \tag{II.13.8}$$

Substitution of this parameter and rewriting yields:

$$\frac{1}{\Theta^* - 1} \frac{\partial \Theta^*}{\partial t} = -\frac{4\alpha}{\rho d c_\mathrm{p}} \tag{II.13.9}$$

Integration gives:

$$\begin{aligned} \ln|\Theta^* - 1| &= -\frac{4\alpha}{\rho d c_\mathrm{p}} t + A^* \\ \Rightarrow \Theta^* &= A \cdot \exp\left(-\frac{4\alpha}{\rho d c_\mathrm{p}} t\right) + 1 \end{aligned} \tag{II.13.10}$$

To find constant $A$, the initial condition that has been defined before needs to be used. To do so, first, it should be written in dimensionless form:

$$\Theta^*(t = 0) = \frac{T_0 - T_0}{T_\infty - T_0} = 0 \tag{II.13.11}$$

Substitution yields:

$$\begin{aligned} \Theta^*(t = 0) &= A \cdot \exp(0) + 1 = 0 \\ \Rightarrow A &= -1 \end{aligned} \tag{II.13.12}$$

Which yields the function of the dimensionless temporal temperature profile:

$$\Theta^*(t) = -\exp\left(-\frac{4\alpha}{\rho d c_\mathrm{p}} t\right) + 1 \tag{II.13.13}$$

At $t = t_1$, the temperature of the rod is equal to $T_1 = 25$ °C. Thus:

$$\Theta^*(t = t_1) = -\exp\left(-\frac{4\alpha}{\rho d c_\mathrm{p}} t_1\right) + 1 = \frac{T_1 - T_0}{T_\infty - T_0} \tag{II.13.14}$$

$$\Rightarrow t_1 = -\ln\left(1 - \frac{T_1 - T_0}{T_\infty - T_0}\right)\frac{\rho d c_\mathrm{p}}{4\alpha}$$

$$= -\ln\left(1 - \frac{(25-100)\ \mathrm{[°C]}}{(20-100)\ \mathrm{[°C]}}\right)\frac{8930\ [\mathrm{kg/m^3}] \cdot 0.02\ [\mathrm{m}] \cdot 382\ [\mathrm{J/kgK}]}{4 \cdot 200\ [\mathrm{W/m^2K}]} = 236\ [\mathrm{s}]$$

Conclusion

It takes almost 4 minutes for the copper rod to cool down from 100 °C to 25 °C.

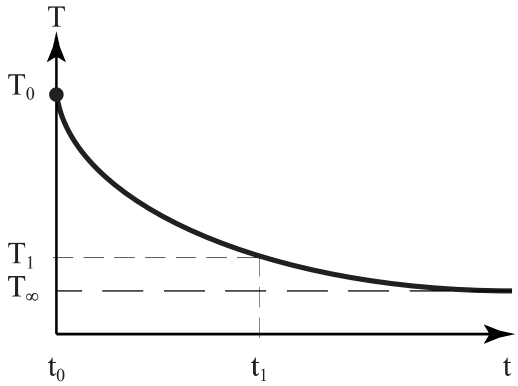
Task b)

Recall Newton's law of cooling:

$$\dot{Q} = \alpha A\left(T(t) - T_\infty\right) \tag{II.13.15}$$

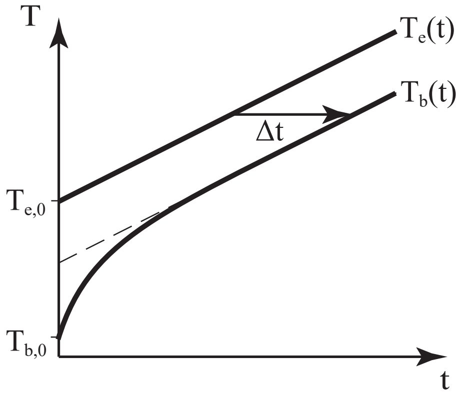
from this equation, it is evident that, as the rod cools down, the temperature difference between the body and the ambient will decrease. With this reduction in temperature difference, the rate of heat transfer will also decrease. Consequently, the temperature gradient will decrease until it matches the ambient temperature, resulting in no further cooling. At this point, a horizontal slope in the temperature profile is observed.

Conclusion



**Exercise II.14** (The temperature delay ⋆⋆):

A body with a temperature of $T_\mathrm{b}$ is located within an environment with the linearly rising temperature $T_\mathrm{e}$ and heats up accordingly to the diagram below. As $t \to \infty$, the temperature of the body follows that of the environment with a constant time delay $\Delta t$.




**Given parameters:**

- Heat transfer coefficient of the body: $\alpha$
- Surface of the body: $A$
- Mass of the body: $m$
- Heat capacity of the body: $c_\mathrm{p}$
- Temperature of the environment: $T_\mathrm{e}\,(t)$

**Hints:**

- The temperature is uniform within the body
- The environment, and its temperature, are not affected by the body.
- Heat radiation can be neglected.
- Setup an energy balance.

**Tasks:**

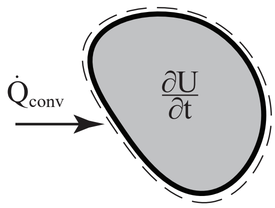
a) Determine this delay $\Delta t$.

**Solution II.14** (The temperature delay ⋆⋆):

Task a)

To determine the temperature delay $\Delta t$, an energy balance at time instance $t \rightarrow \infty$ must be set.

1 **Setting up the balance:**




The unsteady energy balance around the body at time instance $t \rightarrow \infty$ reads:

$$\frac{\partial U}{\partial t} = \dot{Q}_{\text{conv}} \tag{II.14.1}$$

2 **Defining the elements within the balance:**

Since the temperature is uniform within the body, the change of internal energy over time can be expressed as:

$$\frac{\partial U}{\partial t} = mc_{\text{p}} \frac{\partial T}{\partial t} \tag{II.14.2}$$

As our energy balance is set at $t \rightarrow \infty$, the temperature increases linearly, and thus within a time instant $\Delta t$, the temperature will have increased by the difference between the body and ambient temperature $\Delta T = (T_{\text{e}}(t \rightarrow \infty) - T_{\text{b}}(t \rightarrow \infty))$. So, we can say:

$$\frac{\partial T}{\partial t} = \frac{\Delta T}{\Delta t} \tag{II.14.3}$$

Lastly, the heat transfer rate by convection from the environment to the body can be written as,

$$\begin{aligned} \dot{Q}_{\text{conv}} &= \alpha A_{\text{s}} \left(T_{\text{e}}(t \rightarrow \infty) - T_{\text{b}}(t \rightarrow \infty)\right) \\ &= \alpha A_{\text{s}} \Delta T \end{aligned} \tag{II.14.4}$$

3 **Inserting and rearranging:**

Inserting and rewriting yields:

$$\Delta t = \frac{mc_{\text{p}}}{\alpha A_{\text{s}}} \tag{II.14.5}$$