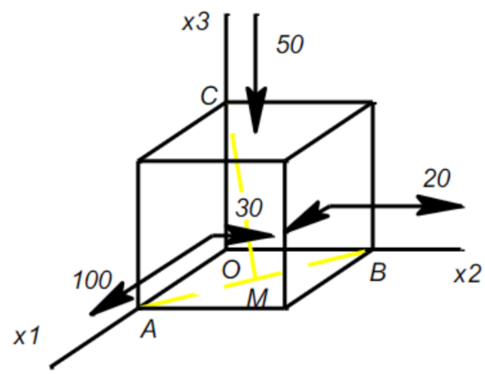
Tutorial T07 – ALL! – Elasticity and more ...

Answer the following questions as they could come up in an exam.

1 Stress basics (geometry and stress vector)

... based on section 3.1 (Exercise V1 in old material before 2022)

Given:
$E = 200\,\text{GPa}$, $\nu = 0.25$
$OA = OB = a$ and $OC = \frac{1}{2}\sqrt{2}a$



Questions:

a) Find normal stress σ_{ABC} and shear stress τ_{ABC} acting on the area ABC.
b) What are the components of the strain-tensor ε_{ij}?
c) What are the eigen-strains?
d) In this stress-state, the maximal principal stress must not be larger than: 150 MPa.
Is this stress state allowed according to the hypotheses of Tresca and von Mises?

<u>Answers:</u>

a) The stress Tensor is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix}$ MPa,
given the arrows, using symmetry, direction of arrows (sign), and non-existing (zero).

First, find the normal to the plane: *by taking the cross-product of two line vectors (in the plane).*

$$\vec{AC} \times \vec{AB} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalizing the vector using the normality condition ($\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$), one can find a:

$$a^2 \left(1^2 + 1^2 + \sqrt{2}^2\right) = 1 \Longrightarrow a = \frac{1}{2}$$

After using the cross-product, with the vectors in random order we pay close attention to the fact that the normal is facing outside the plane. With the normal you indicate which side the material is. To make the normal point away from the material, we can choose a positive.

Cauchy: Stress or traction vector: $p_i = \sigma_{ij} n_j$, so that:

$$\longrightarrow [p] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = [\sigma] \begin{bmatrix} \hat{n_1} \\ \hat{n_2} \\ \hat{n_3} \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

The normal stress on the plane ABC is: $[\sigma] = [\hat{n}]^T \ [p] = 20\,\mathrm{MPa}$

The shear stress on the plane ABC, using Pythagoras, is:

$$\tau^2 = p^2 - \sigma^2 = [p_1^2 + p_2^2 + p_3^2] - \sigma^2 = 6100 - 400 = 5700\,\mathrm{MPa}^2,$$

so that $\tau = 75.5\,\mathrm{MPa}$.

b)

Hooke's law for strain $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij})$, with $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, allows to obtain: $\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$, $\varepsilon_{12} = \frac{\sigma_{12}}{2G}$, with $G = \frac{E}{2(1+\nu)}$, and - similarly - the other components.

$$[\varepsilon] = \begin{bmatrix} 5.375 & 1.875 & 0 \\ 1.875 & 0.375 & 0 \\ 0 & 0 & -4 \end{bmatrix} 10^{-4} = \frac{1}{8} \begin{bmatrix} 43 & 15 & 0 \\ 15 & 3 & 0 \\ 0 & 0 & -32 \end{bmatrix} 10^{-4}$$

c)

Principal strains are computed, like for stress, solving:

$$\det(\varepsilon_{ij} - \varepsilon\delta_{ij}) = \begin{bmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon \end{bmatrix} = 0$$

$$\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$$

$$E_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = (14/8)\,10^{-4}$$

$$E_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2$$

$$= \left(\frac{1}{8}\,10^{-4}\right)^2 \left(43 \times 3 - 32 \times 3 - 32 \times 43 - 15^2 - 0 - 0\right) = -24.5\,10^{-8}$$

$$E_3 = \det(\varepsilon) = ... = 6\,10^{-12}$$

with solutions: $\varepsilon_I = 6\,10^{-4}$, $\varepsilon_{II} = -0.25\,10^{-4}$, $\varepsilon_{III} = -4\,10^{-4}$, sorted.

(Subscripts as in ε_{III} are used to appear different from ε_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

The third eigen-value can be read off directly from strain tensor ($\varepsilon_{III} = \varepsilon_{33}$, due to the zero values in rows and columns); the others still have to be found, from the characteristic equation (by decomposition or polynomial division), or from the invariants.

d)

To compute the allowable stress, we first need to compute (details not shown) the principal stresses: $\sigma_I = 110\,\mathrm{MPa}$, $\sigma_{II} = 10\,\mathrm{MPa}$, $\sigma_{III} = -50\,\mathrm{MPa}$, sorted.

(Subscripts as in σ_{III} are used to appear different from σ_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

According to Tresca: $\sigma_{eq}^{Tresca} = \sigma_I - \sigma_{III} = 160\,\mathrm{MPa}$.

According to von Mises: $\sigma_{eq}^{vonMises} = \sqrt{\frac{1}{2}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_I - \sigma_{III})^2 + (\sigma_{II} - \sigma_{III})^2\right]} = 140\,\mathrm{MPa}$.

Allowed stress means: $\sigma_{eq} \leq \overline{\sigma} = 150\,\mathrm{MPa}$. Thus von Mises is allowed, whereas Tresca is not.

2 Stress tensor basics

... based on sections 3.1-3.3. (Exercise V2 in old material before 2022)

Given:

- Linear elastic isotropic material with modulus $E = 2 \cdot 10^5$ N/mm^2
- The stress cube, below, in units of N/mm^2
- One principal stress is: $8\,\mathrm{N/mm^2}$

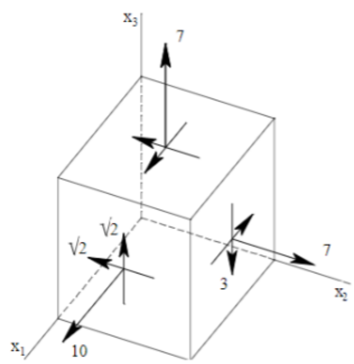


Figure 1: Stress cube → write down the stress matrix

Questions:
a) Find the other principal (eigen) stresses
b) Find the eigen-directions and plot these in a graph.
c) What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?
d) What are the equivalent stresses according to the hypotheses of Tresca and von Mises?

Answers:
a) The stress tensor from the cube is:

$$[\sigma_{ij}] = \begin{bmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{bmatrix} \text{MPa}$$

Note that the first index denotes the direction of the normal to the according surface on which this stress component works, while the second index gives the direction of the stress component.

Next, get the characteristic equation from:

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = \begin{bmatrix} 10-\sigma & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7-\sigma & -3 \\ \sqrt{2} & -3 & 7-\sigma \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

with invariants:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 24\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 176\text{MPa}^2$$

$$I_3 = \det(\sigma) = 384\text{MPa}^3$$

Here, the characteristic equation is not easily solvable; one way is to use the given eigen-value, $\sigma = 8\,N/mm^2$, and polynomial division (units dropped for simplicity, but must be added for final answer). Take the characteristic equation and divide by $(\sigma - 8)$:

$$
\begin{aligned}
&(\sigma^3 - 24\sigma^2 + 176\sigma - 384)\backslash(\sigma - 8) = \sigma^2 - 16\sigma + 48 \\
&\underline{(\sigma^3 - 8\sigma^2)} \\
&\quad -16\sigma^2 + 176\sigma - 384 \\
&\quad \underline{(-16\sigma^2 + 128\sigma)} \\
&\qquad\quad + 48\sigma - 384 \\
&\qquad\quad \underline{(+\,48\sigma - 384)} \\
&\qquad\quad \%
\end{aligned}
$$

The result is a second order polynomial, which can be solved as:
$\sigma_{1,2} = (16 \pm \sqrt{16^2 - 4 \times 48})/2 = 12$ and $4\,\text{MPa}$.

Therefore, the sorted eigen-values are: $\sigma_I = 12\,\text{MPa}$, $\sigma_{II} = 8\,\text{MPa}$, $\sigma_{III} = 4\,\text{MPa}$.
(Subscripts as in σ_{III} are used to appear different from σ_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

b) Direction of $\sigma_I = 12\text{MPa}$
There are various ways to solve for eigen-vectors, here is one example ...
Insert values, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$-2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - 5n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - 5n_3 = 0$$

The eigen-direction associated to the first, largest eigen-value:

$$\Longrightarrow \hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

Direction of $\sigma_{II} = 8$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - n_3 = 0$$

The eigen-direction associated to the second, intermediate eigen-value:

$$\Longrightarrow \hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$$

Direction of $\sigma_{III} = 4$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$6n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 + 3n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 + 3n_3 = 0$$

The eigen-direction associated to the third, smallest eigen-value:

$$\Longrightarrow \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

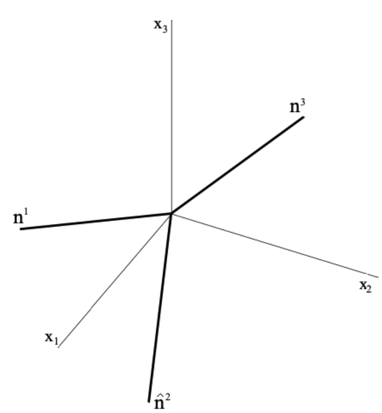


Figure 2: Sketch of the eigen-vectors (with coordinate axes) in bold.

The directions are unspecified, indicated by the plus-minus from taking a square-root; all three direction vectors are normalized (check it, if enough time in exam), $(n_i)^2 = 1$; furthermore, all three normal (eigen) vectors must be pair-wise perpendicular on each other, i.e. $n_i^{(a)} n_i^{(b)} = 0$, for all $a, b = I, II, III$ with $a \neq b$. This perpendicularity allows to obtain, alternatively, one eigen-vector by a cross-product, e.g. above $\hat{n}^{(III)} = \hat{n}^{(I)} \times \hat{n}^{(II)}$.

c)
What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?

$$\varepsilon_V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0.6 \times 10^{-4}$$

$$\varepsilon_V = \varepsilon_{kk} = \frac{1}{E}\left((1+\nu)\sigma_{kk} - \nu\sigma_{mm}\delta_{kk}\right) = \frac{1}{E}\left((1+\nu)\sigma_{kk} - 3\nu\sigma_{kk}\right) = \frac{1-2\nu}{E}\sigma_{kk}$$

The unknown Poisson ration can now be derived:

$$\rightarrow \nu = \frac{1}{2} - \frac{E\varepsilon_V}{2\sigma_{kk}} = \frac{1}{2} - \frac{200\,\mathrm{GPa}\,0.6 \times 10^{-4}}{2 \times 24\,\mathrm{MPa}} = \frac{1}{2} - \frac{20 \times 0.6}{2 \times 24} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

And, finally, the maximum deformation angle and shear strain are:

$$\varepsilon_{shear,max}/2 = \gamma_{max}/2 = \tau_{max}/2G = (\sigma_I - \sigma_{III})\,2(1+\nu)/(4E) = 8(5/2)/(800000) = (5/2) \times 10^{-5}$$

Alternative calculation can be done by computing the eigen-strains, and then from that the maximum shear strain.

d)

$$\sigma_{Tresca} = \max\{|(\sigma_I - \sigma_{II})|, |(\sigma_{II} - \sigma_{III})|, |(\sigma_{III} - \sigma_I)|\} = \max\{4, 4, 8\} = 8\,\mathrm{MPa}$$

$$\sigma_{von-Mises} = \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}} = 6.92\,\mathrm{MPa}$$

Then, Tresca is safer since it is larger and thus reaches the limit stress earlier.

3 Stress tensor basics

... based on section 3,4 (Exercise V3 in old material before 2022) **Given:**

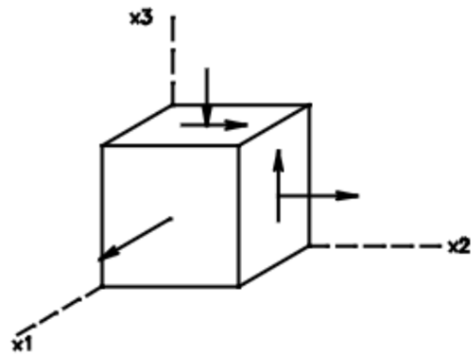


Figure 3: Stress cube, empty → fill it

The stress-state is described by the matrix: $\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix}$ N/mm^2,

with $E = 2 \cdot 10^5$ N/mm^2, and $\nu = 0.25$.

Questions:
a) Compute the principal stresses
b) Compute the eigen-directions
c) Compute the maximal shear-stress
d) Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x'_p, i.e. the coordinate system defined by the eigen-directions.
e) Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.
f) What is the strain in the direction of the normal vector from question d).

Answers:
a)
The sorted eigen-values are: $\sigma_I = 60\,\mathrm{MPa}$, $\sigma_{II} = 40\,\mathrm{MPa}$, $\sigma_{III} = -40\,\mathrm{MPa}$.
The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).

b)
Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example σ_{II}, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad n_1^2+n_2^2+n_3^2=1$$

$-20n_2 + 20\sqrt{3}n_3 = 0$

$20\sqrt{3}n_2 - 60n_3 = 0$

$n_2 = \sqrt{3}n_3$ and $n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3$ (identical due to dependency)

$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$

$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above. *The third eigenvalue calculation is similar (not shown).*

c)
The maximum shear stress is: $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50$ MPa.

d)
The maximal shear stress acts on a surface rotated by 45^o from the x_1' and x_3' directions, related to eigen-directions of σ_I and σ_{III}, respectively, see sketch.
In this coordinate system, the normalized unit vector is obtained from the $(1,0,1)$ direction, but still has to be normalized, so that: $\hat{n}^{\tau_{max}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

e) Graphic/sketch

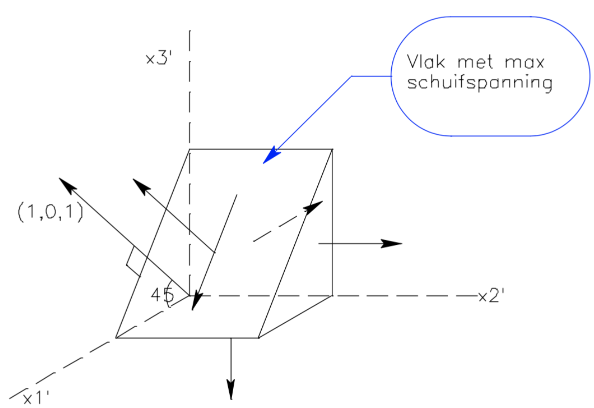


Figure 4: Sketch of the normal to the plane with maximal shear stress (in Dutch: "vlak met maximale schuifspanning"), in the coordinate system x_p' of the eigen-directions of stress, with perpendicular (sorted) intermediate stress eigen-direction x_2'.

f)
The strain in the direction of the normal vector from question d) can be obtained in various ways. Here, we compute the eigen-strains directly from the eigen-stresses using the law of Hooke: $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right)$ where in the eigen-system, all non-diagonal elements are zero (using

arabic number convention):

$$\varepsilon_1 = \frac{1}{2\,10^{-5}}\left[(5/4)\sigma_1 - (1/4)\sigma_{kk}\right] = \frac{1}{2\,10^{-5}}\left[(5/4)60 - (1/4)(60+40-40)\right] = 3\,10^{-4}$$
$$\varepsilon_2 = ... = \frac{1}{2\,10^{-5}}\left[(5/4)40 - (1/4)(60+40-40)\right] = 1.75\,10^{-4}$$
$$\varepsilon_3 = ... = \frac{1}{2\,10^{-5}}\left[(5/4)(-40) - (1/4)(60+40-40)\right] = -3.25\,10^{-4}$$

Now one can use a rotation matrix with 45 degrees about the 2-direction

$$[R] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

to compute the strain components on the surface of maximal shear stress:

$$[\varepsilon''] = [R][\varepsilon'][R]^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = ...$$

For the 11-component in the new coordinate system, pointing in the normal vector direction of d), i.e., the direction of the plane on which the maximal shear stress acts, we only need to compute:

$$\varepsilon''_{11} = (1/\sqrt{2})(1/\sqrt{2})\varepsilon_1 + (1/\sqrt{2})(1/\sqrt{2})\varepsilon_3 = (1/2)(\varepsilon_1+\varepsilon_3) = (1/2)(3-3.25)\,10^{-4} = -0.125\,10^{-4}$$

which is the normal strain (on this plane), while the maximal shear strain is simply:

$$\varepsilon_{shear,max} = \varepsilon''_{13} = (1/2)(\varepsilon_1 - \varepsilon_3) = 3.125\,10^{-4}$$

which is the shear strain (on this plane).

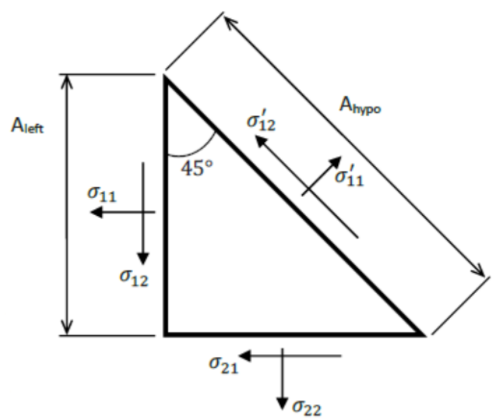
Alternatively (not shown), use the circle of Mohr for strain.

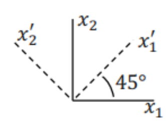
4 Stress tensor and transformation

... based on sections 3.1-3.4. (Exercise V10 in old material before 2022)

Given:

- A plane-stress state in a point P of a body with $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Given are these (mixed) stress components:
  $\sigma_{11} = 92$ MPa
  $\sigma'_{11} = 194$ MPa
  $\sigma'_{12} = -42$ MPa
  where the prime indicates the new (transformed) coordinate system.
- The material is linear elastic with $E = 2 \cdot 10^5$ MPa and $\nu = 0.25$.





Questions:
a) Give the stress tensor in the original $x_1x_2x_3$ system.
b) Give the stress tensor in the new $x'_1x'_2x'_3$ coordinate system, as obtained by a rotation of the coordinates about 45° around the x_3-axis, as sketched above.
c) Compute the eigen-stresses and the eigen-directions.
d) Give the strain tensor in the $x'_1x'_2x'_3$ coordinate system.
e) Compute the specific elastic energy in point P.

Answers:
a)
There are two ways to solve this problem. The triangle given represents all stresses on all sides, but only part of the stress components are known. By considering force equilibrium and using the respective stress components, divided by the side-lengths of the triangle (which also has a third dimension outside the plane, not shown). Assume the sides have unit-length, then the hypotenuse has, according to Pythagoras, length $\sqrt{2}$. Further assume the thickness also to be unit-length. The ratio between sides and hypotenuse is then:

$$\frac{A_l}{A_h} := \frac{A_{left}}{A_{hypo}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

With this we get:
Force balance in x_1 direction:

$$
\begin{aligned}
A_h\sigma'_{11}\cos(45^o) - A_h\sigma'_{12}\sin(45^o) - A_l\sigma_{11} - A_l\sigma_{12} = 0 \\
\Rightarrow \sigma'_{11}\cos(45^o) - \sigma'_{12}\sin(45^o) - \frac{A_l}{A_h}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow (\sigma'_{11}-\sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{11}+\sigma_{12}) = 0 \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = \sigma'_{11} - \sigma'_{12} - \sigma_{11} \\
\Rightarrow \sigma_{12} \equiv \sigma_{21} = 194 - (-42) - 92 = 144\,\mathrm{MPa}.
\end{aligned}
$$

Force balance in x_2 direction:

$$
\begin{aligned}
A_h\sigma'_{11}\sin(45^o) + A_h\sigma'_{12}\cos(45^o) - A_l\sigma_{12} - A_l\sigma_{22} = 0 \\
\Rightarrow \sigma'_{11}\sin(45^o) + \sigma'_{12}\cos(45^o) - \frac{A_l}{A_h}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow (\sigma'_{11}+\sigma'_{12})\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\sigma_{12}+\sigma_{22}) = 0 \\
\Rightarrow \sigma_{22} = \sigma'_{11} + \sigma'_{12} - \sigma_{12} \\
\Rightarrow \sigma_{22} = 194 + (-42) - 144 = 8\,\mathrm{MPa}.
\end{aligned}
$$

The stress tensor in the $x_1x_2x_3$ system is thus:

$$
\left[\sigma_{ij}\right] = \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{MPa}
$$

b)
The stress tensor in the $x'_1x'_2x'_3$ system is obtained by rotation of the original system around 45^o, as sketched, in index notation, $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$, or:

$$
\left[\sigma'\right] = \left[R\right]\left[\sigma\right]\left[R^T\right] = \begin{bmatrix} 194 & -42 & 0 \\ -42 & -94 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{MPa},
$$

using the transformation matrix:

$$
\left[R\right] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

The *alternative* way is to use the symbolic transformation rule and solve the system of equations, for each component, for the unknowns σ_{12}, σ_{22}, and σ'_{22}.

$$
\left[\sigma'\right] = \left[R\right]\left[\sigma\right]\left[R^T\right] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

in units of MPa, which allows to solve without geometry and force balance, after matrix multiplications:

$$\left[\sigma'\right] = (1/\sqrt{2}) \begin{bmatrix} 92+\sigma_{12} & \sigma_{12}+\sigma_{22} & 0 \\ -92+\sigma_{12} & -\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= (1/2) \begin{bmatrix} 92+2\sigma_{12}+\sigma_{22} & -92+\sigma_{22} & 0 \\ -92+\sigma_{22} & 92-2\sigma_{12}+\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in units of MPa, which yields from the non-diagonal 12-component $\sigma_{22} = 8\,\mathrm{MPa}$. Inserted into the 11-component, one finds $\sigma_{12} = 144\,\mathrm{MPa}$, and all inserted into the 22-component results in $\sigma'_{22} = -94\,\mathrm{MPa}$. These results are identical to the above geometry and force balance considerations.

c)
The principal stresses and eigen-directions can now be computed the usual way from

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = 0 \ ,$$

and $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$.

c.1) This stress tensor describes a plane-stress state and thus has one eigenvalue $\sigma = 0$.
The remaining characteristic equation is:

$$\sigma^2 - 100\sigma + 736 - 144^2 = 0$$

with solutions: $\sigma_{1,2} = (100 \pm \sqrt{100^2 - 4(736 - 144^2)})/2 = (100 \pm \sqrt{9.10^4})/2 = 50 \pm 150\,\mathrm{MPa}$.

The sorted eigen-values are thus: $\sigma_I = 200\,\mathrm{MPa}$, $\sigma_{II} = 0\,\mathrm{MPa}$, $\sigma_{III} = -100\,\mathrm{MPa}$.

c.2) *Insert values, solve the system of equations, and normalize the solution.*

$$\begin{bmatrix} \sigma_{11}-\sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2+n_2^2+n_3^2=1$$

$$\Longrightarrow$$

$$(92-200)n_1 + 144n_2 + 0n_3 = 0$$

$$144n_1 + (8-200)n_2 + 0n_3 = 0$$

$$\Longrightarrow$$

$$-108n_1 + 144n_2 + 0n_3 = 0$$

$$144n_1 - 192n_2 + 0n_3 = 0$$

$$\Longrightarrow$$

$$n_2 = (108/144)n_1 = (3/4)n_1$$

$$\Longrightarrow$$

$$n_1^2 + n_2^2 = (1+9/16)n_1^2 = 1$$

$$\Longrightarrow$$

$$n_1 = \sqrt{16/25} = \pm 4/5 = \pm 0.8$$

The eigen-direction associated to the first, largest eigen-value:

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0.8 \\ 0.6 \\ 0 \end{bmatrix}$$

Similarly (no details given), the eigen-direction associated to the third, smallest eigen-value:

$$\hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0.6 \\ -0.8 \\ 0 \end{bmatrix}$$

and without calculation necessary (due to structure of the matrix), for the second, intermediate:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

d)
The strain components can be computed from the linear elastic material law of Hooke, as:

$$\varepsilon'_{ij} = \frac{1}{E}\left((1+\nu)\sigma'_{ij} - \nu\sigma'_{kk}\delta_{ij}\right)$$

or (symbolic):

$$[\varepsilon'] = \frac{1}{E}\left((1+\nu)\,[\sigma'] - \nu\,\mathrm{tr}(\sigma')\,[1]\right),$$

with unit tensor [1], which results in (no details given):

$$[\varepsilon'] = \begin{bmatrix} 1087.5 & -262.5 & 0 \\ -262.5 & -712.5 & 0 \\ 0 & 0 & -125 \end{bmatrix} 10^{-6}$$

As examples: $\varepsilon'_{12} = ((1+\nu)/E)\sigma'_{12} = (5/8)10^{-5}(-42) = 262.5 \times 10^{-6}$,
and $\varepsilon'_{11} = ((1+\nu)/E)\sigma'_{11} + (\nu/E)\sigma'_{kk} = (1/2)10^{-5}((5/4)194 - (1/4)100) = 1087.5 \times 10^{-6}$,
etc.

e)
The specific elastic energy is:

$$\begin{aligned} \pi'_{el} = \pi_{el} &= \frac{1}{2}\sigma'_{ij}\varepsilon'_{ij} \\ &= \frac{1}{2}\left(\sigma'_{11}\varepsilon'_{11} + \sigma'_{12}\varepsilon'_{12} + \sigma'_{13}\varepsilon'_{13} + \sigma'_{21}\varepsilon'_{21} + \sigma'_{22}\varepsilon'_{22} + \sigma'_{23}\varepsilon'_{23} + \sigma'_{31}\varepsilon'_{31} + \sigma'_{32}\varepsilon'_{32} + \sigma'_{33}\varepsilon'_{33}\right) \\ &= \frac{1}{2}\left(\sigma'_{11}\varepsilon'_{11} + 2\sigma'_{12}\varepsilon'_{12} + 2\sigma'_{13}\varepsilon'_{13} + \sigma'_{22}\varepsilon'_{22} + 2\sigma'_{23}\varepsilon'_{23} + 2\sigma'_{33}\varepsilon'_{33}\right) \\ &= \frac{1}{2}\left(194 \cdot 1087.5 + 2 \cdot -42 \cdot -262.5 + 0 + -94 \cdot -712.5 + 0 + 0\right) \\ &= \cdots = 0.15\,\mathrm{MPa} = 0.15 \times 10^{6}\,\mathrm{J/m^3}\,. \end{aligned}$$

Note: (scalar) energy is the same in both coordinate systems, whereas the tensors are different.

5 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ($E = 2 \cdot 10^5$ MPa, $\nu = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1(\frac{x_1}{L}) - 10(\frac{x_2}{L}) + \alpha_2(\frac{x_1}{L})^2\right]$

$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8(\frac{x_1}{L}) + \beta_1(\frac{x_2}{L}) + \beta_2(\frac{x_2}{L})^2\right]$

$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10(\frac{x_1}{L}) + 7(\frac{x_2}{L}) - 8(\frac{x_1}{L})(\frac{x_2}{L})\right]$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$, and
with reference stress $\sigma_0 = 1$ MPa and reference length $L = 1$ m.

Note: Question (a) is general, symbolic, with variables x_1, x_2, x_3 and coefficients α_1, α_2, β_1, β_2; only from question (b) on, use the single, chosen point $\mathrm{P}(x_1 = 0, x_2 = 0, x_3 = 0)$.

Questions:
... based on section 3
a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the free coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium?
b) Compute the eigen-stresses in point P using linear algebra, mathematics – not the graphical circle of Mohr procedure.
Describe and name the state of stress in point P (and in all other points in the body).
c) Compute the eigen-direction of the major eigen-stress.
d) Draw the relevant circle of Mohr and confirm graphically the results of (b) and (c); explain.

Answers:
a)
Given was the plane stress-field, independent of x_3, in absence of body forces $f_i = 0$:

$$\begin{aligned}
\sigma_{11}(x_1, x_2) &= \sigma_0 \left[20 + \alpha_1 \frac{x_1}{L} - 10\frac{x_2}{L} + \alpha_2 \left(\frac{x_1}{L}\right)^2\right] \\
\sigma_{22}(x_1, x_2) &= \sigma_0 \left[10 + 8\frac{x_1}{L} + \beta_1 \frac{x_2}{L} + \beta_2 \left(\frac{x_2}{L}\right)^2\right] \\
\sigma_{12}(x_1, x_2) &= \sigma_0 \left[12 - 10\frac{x_1}{L} + 7\frac{x_2}{L} - 8\frac{x_1}{L}\frac{x_2}{L}\right]
\end{aligned}$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\begin{aligned}
\frac{d}{dx_1}\sigma_{11}(x_1, x_2) + \frac{d}{dx_2}\sigma_{12}(x_1, x_2) &= \sigma_0 \left[\frac{\alpha_1}{L} + 2\alpha_2 \frac{x_1}{L^2}\right] + \sigma_0 \left[\frac{7}{L} - 8\frac{x_1}{L^2}\right] = 0 \\
\frac{d}{dx_1}\sigma_{12}(x_1, x_2) + \frac{d}{dx_2}\sigma_{22}(x_1, x_2) &= \sigma_0 \left[\frac{-10}{L} - 8\frac{x_2}{L^2}\right] + \sigma_0 \left[\frac{\beta_1}{L} + 2\beta_2 \frac{x_2}{L^2}\right] = 0
\end{aligned}$$

From these equations, one gets the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$.

Because the field equations must be valid for all constants and points x_1, x_2, x_3, independently, one can group them accordingly: The constant terms from the first and second equations provide α_1 and β_1, respectively, while the x_1 and x_2 groups provide α_2 and β_2.

b)

The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\begin{bmatrix}\sigma_{ij}\end{bmatrix} = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0\,.$$

Knowing/recognizing that one eigen-value is zero, i.e. also $I_3 = 0$, the principal stresses can be computed from the second order polynomial as: $\sigma_I = 28\,\mathrm{MPa}$, $\sigma_{II} = 2\,\mathrm{MPa}$, $\sigma_{III} = 0\,\mathrm{MPa}$. This is a plane-stress state with all stresses on the x_3-surface equal to zero, which also has consequences for the eigen-directions ...

c)

The principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0, 0, 1)$ is directly visible from the tensor, due to the zero shear stresses in the x_3-direction.

The eigen-direction of the major stress $\sigma_I = 28\,\mathrm{MPa}$ is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that: $-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4) + 1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress, $\sigma_{II} = 2\,\mathrm{MPa}$ was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that: $18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9) + 1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

d)

Mohr's circle

Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{20+10}{2} = 15\,\mathrm{MPa}$,

and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13\,\mathrm{MPa}$.

The eigenvalues are therefore:

$\sigma_I = M + R = 28\,\mathrm{MPa}$, $\sigma_{II} = C - R = 2\,\mathrm{MPa}$.

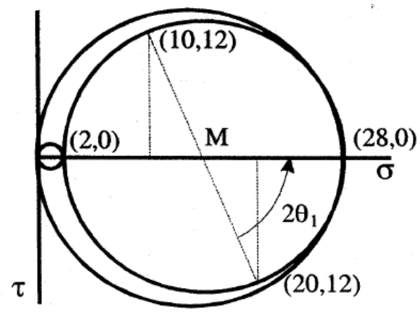


Figure 5: Sketch of a Mohr circle, focus is on the right, inner circle.

The eigen-directions are:
$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{24}{10} = 2.4 \Longrightarrow \theta_I = (1/2)\arctan(2.4) = 67.38°/2 = 33.69°$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$; and $\theta_{II} = (180° + 67.3°)/2 = 247.3°/2 = 123.7° = \arccos(-2/\sqrt{13})$.

The maximum shear stress is just the radius: $\tau^{max} = R = 13$ MPa

6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:
$E = 2 \cdot 10^{11}$ Pa, $\nu = 0.25$

Stress-state in point P: $\left[\sigma\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa

Questions:
a) Show that the principal stresses are 8, 16 and 24 MPa.
Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.
... based on sections 4,5.1
b) Compute the volumetric (isotropic) strain.
c) What is the largest angle-change (not shear-strain) in P?
... based on section 5.1
d) Which material property is implicitly used/assumed in Hooke's law?

Answers:
a)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0\,.$$

Given the eigenvalues, σ, one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*
Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 24\,\text{MPa}$, $\sigma_{II} = 16\,\text{MPa}$, and $\sigma_{III} = 8\,\text{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.
The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8\,\text{MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \to n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \to n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \to n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \to n_1 = n_2$

Inserting into line 3 yields: $n_3 = \sqrt{6}n_1$, so that: $\begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\, c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$

b)
For the volumetric (isotropic) strain, we can use the short-cut (not the full strain tensor), as: $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \left[\frac{1-2\nu}{E}\right] \sigma_{kk} = 12\ 10^{-5}$.

c)
The largest change of angle is: $\gamma_{max} = \frac{\tau_{max}}{G} = \frac{1}{2}\frac{\sigma_I - \sigma_{III}}{G} = 10^{-4}$, using $G = \frac{E}{2(1+\nu)} = (4/5)\ 10^5$ MPa, where the largest shear strain is just half of that: $\varepsilon_{max} = \gamma_{max}/2$.

d)
Isotropic (direction independent) material behavior is intrinsically assumed in the law of Hooke.

7 Displacement, strain and stress relation

... based on sections 3,4,5 (Exercise V7 in old material before 2022)

Given is the displacement field:
$u_1 = x_1 x_3$, $u_2 = -x_1 x_2$, and $u_3 = x_1^2 - x_3^2$
and material properties $E = 2\,\mathrm{GPa}$ and $\nu = 0.25$
Consider the units, drop them in calculations to save space, but give them in end-results!

Questions:

a) Compute the stress tensor (components).
b) In the point $(x_1, x_2, x_3) = (0, 0, z_0)$ compute the eigen-stresses and maximum shear stress.

Answers:
a)
To compute the components of the stress tensor,
first, the strain tensor has to be computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

resulting in the components:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = x_3 \qquad \gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 - x_2 = -x_2$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -x_1 \qquad \gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 + 0 = 0$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -2x_3 \qquad \gamma_{31} = 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2x_1 + x_1 = 3x_1$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -x_1 - x_3$$

Next, obtain σ_{kk} from Hooke's law:

$$\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\}$$

$$\varepsilon_{kk} = \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} \qquad \delta_{jj} = 1 + 1 + 1 = 3$$

$$\varepsilon_{kk} = \frac{1}{E}(1-2\nu)\sigma_{kk} \qquad \Longrightarrow \qquad \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

Substitute σ_{kk} into Hooke's law and rearrange for σ_{ij}:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

Compute the terms for the stress tensor using the previously obtained equation:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\left[x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 + x_3)\\
\sigma_{22} &= \frac{E}{1+\nu}\left[-x_1 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-x_1 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-3x_1 - x_3)\\
\sigma_{33} &= \frac{E}{1+\nu}\left[-2x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-2x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 - 5x_3)\\
\sigma_{12} &= \frac{E}{1+\nu}\left[-\frac{1}{2}x_2\right] = -\frac{4}{5}x_2\\
\sigma_{23} &= 0\\
\sigma_{31} &= \frac{E}{1+\nu}\left[\frac{3}{2}x_1\right] = \frac{12}{5}x_1
\end{aligned}$$

b)

The principal stresses in point $(x, y, z) = (0, 0, z_0)$ are identical to the diagonal components, here only (if all shear stresses are zero); and the maximal shear stress τ_{max} are:

$$\begin{aligned}
\sigma_1 = \sigma_{11} &= \frac{4}{5}(-x_1 + x_3) = \frac{4}{5}z_0 & \sigma_{12} &= -\frac{4}{5}x_2 = 0\\
\sigma_2 = \sigma_{22} &= \frac{4}{5}(-3x_1 - x_3) = -\frac{4}{5}z_0 & \sigma_{23} &= 0\\
\sigma_3 = \sigma_{33} &= \frac{4}{5}(-x_1 - 5x_3) = -4z_0 & \sigma_{31} &= \frac{12}{5}x_1 = 0
\end{aligned}$$

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 2.4 z_0 \ \mathrm{GPa\,m^{-1}}$$

for a given length z_0 with units [m].

8 Displacement, strain and stress equilibrium

... based on sections 3,4,5 (Exercise V8 in old material before 2022)

In a homogenous body made of a linear elastic, isotropic material, the displacement field is given:

$u_1 = \frac{p}{E} a \left[\frac{x_2}{a} + 2\frac{x_1 x_2}{a^2} - \frac{x_2^2}{a^2} \right]$

$u_2 = \frac{p}{E} a \left[\frac{x_1}{a} + \alpha\frac{x_1^2}{a^2} + \beta\frac{x_1 x_2}{a^2} - 2\frac{x_2^2}{a^2} \right]$

$u_3 = 0$

with coordinates x_1 and x_2, and constant coefficients p, E, a, and $\nu = 0.25$.

Questions:

In the absence of volume forces, compute the magnitude of the parameters α and β, using the information that the stress field is in mechanical equilibrium.

Answers:

A homogeneous body made from a linear elastic isotropic material ($E = 2\,\mathrm{GPa}$ and $\nu = 0.25$) is in a deformed state in mechanical equilibrium, in the absence of a volume force, with parameters reference stress p, and length a, and unknown coefficients α and β, according to the displacement field:

$$\begin{aligned}
u_1 &= \frac{pa}{E}\left[\left(\frac{x_2}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - \left(\frac{x_2}{a}\right)^2\right] \\
u_2 &= \frac{pa}{E}\left[\left(\frac{x_1}{a}\right) + \alpha\left(\frac{x_1}{a}\right)^2 + \beta\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - 2\left(\frac{x_2}{a}\right)^2\right] \\
u_3 &= 0
\end{aligned}$$

The stress components, to be computed from Hooke's law, require the strain components, computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,.$$

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{pa}{E}\left[2\left(\frac{1}{a}\right)\left(\frac{x_2}{a}\right)\right] = \frac{p}{E}\left[2\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \frac{pa}{E}\left[\beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 4\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right] = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{33} &= 0
\end{aligned}$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]$$

$$
\begin{aligned}
\gamma_{12} = 2\varepsilon_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\
&= \frac{pa}{E}\left[\left(\frac{1}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 2\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right) + \left(\frac{1}{a}\right) + 2\alpha\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) + \beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right)\right] \\
&= 2\frac{p}{E}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\gamma_{23} = 2\varepsilon_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \\
\gamma_{31} = 2\varepsilon_{31} &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0
\end{aligned}
$$

Obtain σ_{kk} with Hooke's law:

$$
\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\}
$$

$$
\begin{aligned}
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} & & \delta_{jj} = 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} & \Longrightarrow \quad & \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}
$$

Substitute σ_{kk} into Hooke's and rearrange for σ_{ij}:

$$
\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]
$$

The stress-equilibrium equations (by partial differentiation) are then obtained using the previous equations:

$$
\begin{aligned}
\sigma_{ij} &= \frac{p}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right] \\
\sigma_{11} &= \frac{p}{1+\nu}\left[\left[2\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[2(1-3\nu)\left(\frac{x_2}{a}\right)\right] + \left[\nu\beta\left(\frac{x_1}{a}\right)\right]\right] \\
\sigma_{22} &= \frac{p}{1+\nu}\left[\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta(1-\nu)\left(\frac{x_1}{a}\right)\right] + \left[(-4+6\nu)\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{33} &= \frac{p}{1+\nu}\left[0 + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta\nu\left(\frac{x_1}{a}\right) - 2\nu\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{12} &= \frac{E}{1+\nu}\left[\varepsilon_{12}\right] = \frac{p}{1+\nu}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\sigma_{23} &= 0 \\
\sigma_{31} &= 0
\end{aligned}
$$

In absence of a volume force, the three equilibrium equations for $j = 1, 2, 3$ are:

$$
\sigma_{ij,i} + 0 = (\sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3}) + 0 = 0
$$

Using partial differentiation:

$$\sigma_{11,1} = \frac{\partial}{\partial x_1}\sigma_{11} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] \qquad \sigma_{12,1} = \frac{\partial}{\partial x_1}\sigma_{12} = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]; \quad \sigma_{13,1} = 0$$

$$\sigma_{22,2} = \frac{\partial}{\partial x_2}\sigma_{22} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] \quad \sigma_{23,2} = \frac{\partial}{\partial x_2}\sigma_{23} = 0; \quad \sigma_{21,2} = \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right]$$

$$\sigma_{33,3} = \frac{\partial}{\partial x_3}\sigma_{33} = 0 \qquad \sigma_{31,3} = \frac{\partial}{\partial x_3}\sigma_{31} = 0; \quad \sigma_{32,3} = 0$$

Computing the three equations:

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0 = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right] + 0 \tag{1}$$

$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} = 0 = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] + 0 \tag{2}$$

$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0 \tag{3}$$

Inserting the value for $\nu = 0.25$:

$$(1) \implies \frac{p}{\frac{5}{4}}\left[\frac{1}{2}\beta + \left(\frac{\beta}{2}-1\right)\right]\left(\frac{1}{a}\right) = 0 \qquad \implies \qquad p\frac{4}{5}\left[\beta-1\right]\left(\frac{1}{a}\right) = 0$$

$$(2) \implies \frac{p}{\frac{5}{4}}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{\frac{5}{4}}2\left[(-\frac{5}{2})\left(\frac{1}{a}\right)\right] = 0 \qquad \implies \qquad p\frac{4}{5}\left[(\alpha+1)-5\right]\left(\frac{1}{a}\right) = 0$$

Solving for the two unknowns (α, β):

$$(1) \implies \beta = 1$$
$$(2) \implies \alpha = 4$$

This allows for mechanical equilibrium in all points in the body, because the equilibrium equations do not depend on the position. Note that either of the two equations – both have to be valid at the same time – provides one unknown.

9 Displacement, strain, stress

... based on sections 3,4,5 (Exercise V9 in old material before 2022)

Within a homogeneous body made of a linear elastic, isotropic material the displacement field:

$$u_1 = \frac{1}{3}(1-2\nu)x_1^3 - (3-2\nu)x_1x_2^2 - 3x_2 - 3x_3$$

$$u_2 = (1-2\nu)x_1^2x_2 + \frac{1}{3}(1+2\nu)x_2^3 + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

the elasticity modulus E, and the Poisson-ratio ν are given.

Questions:

... based on section 4
a) Compute the components of the strain tensor.
... based on section 5.1
b) Compute the components of the stress tensor.
... based on section 3
c) Confirm that the stress-field is in equilibrium in the absence of volume forces.

Answers:
a)
The strain tensor is computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right)$$

with the components:

$$\begin{aligned}
\varepsilon_{11} &= (1-2\nu)x_1^2 - (3-2\nu)x_2^2 \\
\varepsilon_{22} &= (1-2\nu)x_1^2 + (1+2\nu)x_2^2 \\
\varepsilon_{33} &= 0 \\
\varepsilon_{kk} &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 2(1-2\nu)\left(x_1^2 - x_2^2\right)
\end{aligned}$$

and the (symmetric) non-diagonal elements:

$$\begin{aligned}
\gamma_{12} &= 2\varepsilon_{12} = -2(3-2\nu)x_1x_2 - 3 + 2(1-2\nu)x_1x_2 + 3 = -4x_1x_2 \\
\gamma_{23} &= 2\varepsilon_{23} = -4 + 4 = 0 \\
\gamma_{13} &= 2\varepsilon_{13} = -3 + 3 = 0
\end{aligned}$$

b)
Use Hooke's law (strain-stress):

$$\begin{aligned}
\varepsilon_{ij} &= \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\} \\
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} \qquad & \delta_{jj} = 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} \qquad \Longrightarrow \qquad & \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}$$

From the stress-strain relation compute the stress tensor components:

$$\sigma_{ij} = \frac{E}{1+\nu}\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\}$$

$$
\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\{\varepsilon_{11} + \frac{\nu}{1-2\nu}\delta_{11}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{(1-2\nu)x_1^2 - (3-2\nu)x_2^2 + 2\nu\left(x_1^2 - x_2^2\right)\} = \frac{E}{1+\nu}\{x_1^2 - 3x_2^2\} \\
\sigma_{22} &= \frac{E}{1+\nu}\{\varepsilon_{22} + \frac{\nu}{1-2\nu}\delta_{22}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{(1-2\nu)x_1^2 + (1+2\nu)x_2^2 + 2\nu\left(x_1^2 - x_2^2\right)\} = \frac{E}{1+\nu}\{x_1^2 + x_2^2\} \\
\sigma_{33} &= \frac{E}{1+\nu}\{\varepsilon_{33} + \frac{\nu}{1-2\nu}\delta_{33}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{0 + 2\nu\left(x_1^2 - x_2^2\right)\} = \frac{E}{1+\nu}2\nu\{x_1^2 - x_2^2\} \\
\sigma_{12} &= \sigma_{21} = \frac{E}{1+\nu}\{\frac{1}{2}(-4x_1x_2)\} = \frac{E}{1+\nu}\{-2x_1x_2\} \\
\sigma_{23} &= \sigma_{32} = \sigma_{13} = \sigma_{31} = 0
\end{aligned}
$$

c)
Given an absence of a volume force, $f_j = 0$, the three stress-equilibrium equations $(j = 1, 2, 3)$ are:

$$\sigma_{ij,i} + f_j = (\sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3}) + 0 = 0$$

Using partial differentiation:

$$
\begin{array}{lll}
\sigma_{11,1} = \dfrac{\partial}{\partial x_1}\sigma_{11} = \dfrac{E}{1+\nu}2x_1 & \sigma_{21,2} = \dfrac{\partial}{\partial x_2}\sigma_{21} = \dfrac{E}{1+\nu}(-2x_1) & \sigma_{31,3} = \dfrac{\partial}{\partial x_3}\sigma_{31} = 0; \\
\sigma_{22,2} = \dfrac{\partial}{\partial x_2}\sigma_{22} = \dfrac{E}{1+\nu}2x_2 & \sigma_{12,1} = \dfrac{\partial}{\partial x_1}\sigma_{12} = \dfrac{E}{1+\nu}(-2x_2) & \sigma_{32,3} = \dfrac{\partial}{\partial x_3}\sigma_{32} = 0; \\
\sigma_{33,3} = \dfrac{\partial}{\partial x_3}\sigma_{33} = 0 & \sigma_{13,1} = \dfrac{\partial}{\partial x_1}\sigma_{13} = 0 & \sigma_{23,2} = \dfrac{\partial}{\partial x_2}\sigma_{23} = 0;
\end{array}
$$

and inserting the elements into the equations:

$$
\begin{aligned}
\sigma_{i1,i} &= \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0; \\
\sigma_{i2,i} &= \sigma_{22,2} + \sigma_{12,1} + \sigma_{32,3} = 0; \\
\sigma_{i3,i} &= \sigma_{33,3} + \sigma_{13,1} + \sigma_{23,2} = 0;
\end{aligned}
$$

confirms stress equilibrium.

10 Strain-stress relations

... based on sections 4,5.1 (Exercise V5 in old material before 2022)

In a Cartesian coordinate system, at point P, the strain tensor is given as:

$$[\varepsilon_{ij}] = \frac{5}{8} \begin{bmatrix} -1 & -15 & 5\sqrt{2} \\ -15 & -1 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 14 \end{bmatrix} \quad 10^{-5}$$

Questions:

a) For a material with modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$, compute the eigen-stresses and the eigen-directions.
b) Explain/argue why the eigen-directions of stress and strain are identical for a homogeneous, isotropic material.

Answers:
a)
First, the stress tensor is determined using Hooke's law $\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$.
Insert and solve for stress in the normal way ... for each component.
For example:

$\sigma_{12} = \frac{E}{1+\nu}(\epsilon_{12}) = -15\,\text{MPa}$,
$\sigma_{11} = \frac{E}{1+\nu}\left(\epsilon_{11} + \frac{\nu}{1-2\nu}\epsilon_{kk}\right) = 5\,\text{MPa}$, with $\epsilon_{kk} = (5/8)\,12\,10^{-5}$,
etc.

Another way to do this (often done for anisotropic materials and in finite element implementations – not needed for linear isotropic elasticity, to solve this rather simple problem, just for completeness) is to assemble the independent stress- and strain-tensor components in vectors and express the corresponding stiffness-matrix in moduli, using the Lamé-coeffcients (for brevity):

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad\qquad 2\mu = 2G = \frac{E}{1+\nu}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

$$[\sigma_{ij}] = \begin{bmatrix} 5 & -15 & 5\sqrt{2} \\ -15 & 5 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 20 \end{bmatrix} \text{MPa}$$

From the stress, the characteristic equation, invariants, and principal stresses can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 30\,\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = -100\,\text{MPa}^2$$

$$I_3 = \det(\sigma) = -3000\,\text{MPa}^3$$

In this case, solution by decomposition is possible, as:

$$\sigma^2(\sigma - 30) - 100(\sigma - 30) = 0$$

Therefore: $\sigma_I = 30\text{MPa}$, $\sigma_{II} = 10\text{MPa}$, $\sigma_{III} = -10\text{MPa}$.

Principal directions can be calculated as:
Direction of $\sigma_I = 30\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{II} = 10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{III} = -10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

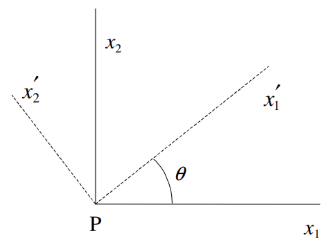
$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}$$

b)
For an isotropic material, one has the eigendirections of stress-and strain identical. Note that the term with δ_{ij} in the law of Hooke has no direction (is isotropic = direction-independent); thus the direction is carried by the terms ε_{ij} and σ_{ij} and thus the directions are equal for stress and strain.

11 Material behavior and energy

... based on sections 4,5 (Exercise V11 in old material before 2022)

At a non-loaded point P on the surface of a loaded body/construction, three normal strains are measured inside the plane parallel to the free surface, as: $\varepsilon_{11} = 750 \cdot 10^{-6}$, $\varepsilon'_{11} = 150 \cdot 10^{-6}$, and $\varepsilon_{22} = 150 \cdot 10^{-6}$. The angle between the old x_1 and new x'_1 axes is $\theta = \arctan(3/4)$, as sketched below. The material is linear elastic and isotropic with modulus of Young $E = 2.10^5$ MPa and Poisson ratio $\nu = 1/3$.



Questions:
a) Show that one strain component is $\varepsilon_{12} = -400 \cdot 10^{-6}$.
b) Why are the stress components $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$?
c) Show that the components of the stress tensor in the $x_1 - x_2 - x_3-$coordinate system are:
$\left[\sigma_{ij}\right] = \begin{bmatrix} 180 & -60 & 0 \\ -60 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa, using Hookes law. $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right)$.
d) Compute the remaining components of the strain tensor and place them in similar matrix form.
e) Compute the eigen-stresses and determine the equivalent stresses according to Tresca and von Mises. Which criterion is safer?
f) What is the specific elastic energy in point P?
Also determine the deviatoric stress tensor and the consequent specific energy related to changes of shape.
Finally determine the specific energy related to volume changes ε_V and hydrostatic stress σ_m, and compare the three values. Are the results consistent? Discuss or explain.

Related, useful formula:

$$\varepsilon'_{pq} = R_{pi}R_{qj}\varepsilon_{ij}; \quad \sigma_m = \frac{1}{3}\sigma_{kk}; \quad \sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}; \quad \varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$
$$\varepsilon_V = \varepsilon_{kk}; \quad \pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}; \quad \pi_{el_{vol}} = \frac{1}{2}\sigma_m\varepsilon_V; \quad \pi_{el_{dev}} = \frac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij}$$

Answers:
a) The rotation/transformation matrix (counter-clock-wise) around x_3 is

$$\left[R_{ij}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4} \Longrightarrow \sin\theta = \frac{3}{4}\cos\theta$$
$$\sin\theta^2 + \cos\theta^2 = 1$$

Therefore, $\sin\theta = 0.6$ and $\cos\theta = 0.8$.
The strain tensor can be rotated according to R, then $\varepsilon' = R.\varepsilon.R^T$

$$\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{12} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{13} & \varepsilon'_{23} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \varepsilon'_{11} = \cos\theta^2\varepsilon_{11} + \sin\theta^2\varepsilon_{22} + 2\cos\theta.\sin\theta\varepsilon_{12}$$

$$\Longrightarrow \varepsilon_{12} = -400.10^{-5}$$

b) In the question has been stated that there is no load applied on a surface; then, there is no load along the x_3 direction. Therefore, stress components related to this direction are zero ($\sigma_{33} = \sigma_{31} = \sigma_{32} = 0$).

c) Stress components are determined by using Hooke's law, as given. First compute σ_{kk}, then compute ε_{ij} by inserting the other stress components. The validity of the stress tensor is thus confirmed by finding agreement with the known strain values.
Several other components of stress are zero, whereas the other strain components are not necessarily zero.

d) The last component of strain must be calculated to establish the full strain tensor:

$$\varepsilon_{33} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) \Longrightarrow \varepsilon_{33} = \frac{1}{2.10^5}\left(-\frac{1}{3}.270\right) = -450.10^{-6}$$

e) Using the stress tensor, eigenvalues and invariants can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 270\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 12600\text{MPa}^2$$

$$I_3 = \det(\sigma) = 0\text{MPa}^3$$

Therefore: $\sigma_I = 210\text{MPa}$, $\sigma_{II} = 60\text{MPa}$, $\sigma_{III} = 0\text{MPa}$.
Now, the Tresca and Von-Mises criteria are investigated.

$$\sigma_{Tresca} = \text{Max}\{|(\sigma_I - \sigma_{II})|, |(\sigma_{II} - \sigma_{III})|, |(\sigma_{III} - \sigma_I)|\} = \text{Max}\{150, 60, 210\} = 210\text{MPa}$$

$$\sigma_{vonMises} = \sqrt{\frac{(\sigma_I - \sigma_{II}) + (\sigma_{II} - \sigma_{III}) + (\sigma_{III} - \sigma_I)}{2}} = 187.35\text{MPa}$$

Thus von Mises is less safe than Tresca, since the limit stress is reached at larger deformation.

f)
Elastic energy and deviatoric stress and strain

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \frac{180 + 90 + 0}{3} = 90\text{MPa}$$

$$\varepsilon_{vol} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 450.10^{-6}$$

Deviatoric stress

$$\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m.\delta_{ij} \Longrightarrow \left[\hat{\sigma}_{ij}\right] = \begin{bmatrix} 180-90 & -60 & 0 \\ -60 & 90-90 & 0 \\ 0 & 0 & -90 \end{bmatrix} = \begin{bmatrix} 90 & -60 & 0 \\ -60 & 0 & 0 \\ 0 & 0 & -90 \end{bmatrix} \text{MPa}$$

Deviatoric strain

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{vol}.\delta_{ij}}{3} \Longrightarrow \left[\hat{\varepsilon}_{ij}\right] = \begin{bmatrix} 750-150 & -400 & 0 \\ -400 & 150-150 & 0 \\ 0 & 0 & -450-150 \end{bmatrix} 10^{-6} = \begin{bmatrix} 600 & -400 & 0 \\ -400 & 0 & 0 \\ 0 & 0 & -600 \end{bmatrix} 10^{-6}$$

Specific elastic energy

$$\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13} + 2\sigma_{23}\varepsilon_{23}\right) = 98250\frac{\mathrm{J}}{\mathrm{m}^3}$$

Volumetric elastic energy

$$\pi_{vol} = \frac{1}{2}\sigma_m\varepsilon_{vol} = 20250\frac{\mathrm{J}}{\mathrm{m}^3}$$

Deviatoric elastic energy

$$\pi_{dev} = \frac{1}{2}\hat{\sigma}_{ij}\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\hat{\sigma}_{11}\hat{\varepsilon}_{11} + \hat{\sigma}_{22}\hat{\varepsilon}_{22} + \hat{\sigma}_{33}\hat{\varepsilon}_{33} + 2\hat{\sigma}_{12}\hat{\varepsilon}_{12} + 2\hat{\sigma}_{13}\hat{\varepsilon}_{13} + 2\hat{\sigma}_{23}\hat{\varepsilon}_{23}\right) = 78000\frac{\mathrm{J}}{\mathrm{m}^3}$$

And the specific elastic energy is the sum of volumetric and deviatoric elastic energy:

$$\pi_{el} = \pi_{vol} + \pi_{dev} = 20250 + 78000 = 98250\frac{\mathrm{J}}{\mathrm{m}^3}$$

12 Limits of elasticity

... based on sections 4,5 (Exercise V6 in old material before 2022)

A construction made of an elastic, isotropic material (with properties $E = 2.10^5\,\mathrm{N/mm^2}$, Poisson ratio $\nu = 0.25$, and maximally allowed stress: $160\,\mathrm{N/mm^2}$) is loaded by a force $F = 56\,\mathrm{kN}$ in a point P on the otherwise non-loaded surface; the following strains are measured:
$\varepsilon_{11} = 130.10^{-6}$, $\varepsilon_{22} = -70.10^{-6}$, $\gamma_{12} = 346,4.10^{-6}$,
where only the x_1-x_2-plane represents the surface in point P.

Questions:

a) What is the strain component ε_{33} in point P?
b) Compute the stresses in point P.
c) Up to which maximal force F can the load be increased, according to the criterion of Tresca?
d) ... and according to the criterion of von Mises?

Answers:
a)
Given is the following information:

$$F = 56\,\mathrm{kN} \qquad \text{(Load)}$$
$$E = 2 \times 10^5\,\mathrm{MPa}$$
$$\nu = 0.25$$
$$\sigma_{max} = 160\,\mathrm{MPa} \;\; \text{(Maximal allowed)}$$

In a point P on the non-loaded surface:

$$\varepsilon_{11} = 130 \times 10^{-6}$$
$$\varepsilon_{22} = -70 \times 10^{-6}$$
$$\gamma_{12} = 346.4 \times 10^{-6}$$

and the equations for Hooke's law:

$$\sigma = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

with Tresca:

$$\sigma_{eq} = \sigma_1 - \sigma_3 \leq \sigma_{max}$$

and von Mises:

$$\sigma_{eq} = \sqrt{\tfrac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]}$$

Compute ε_{33} in Point P:
Point P sits on the surface (with normal x_3), where it is not loaded, with the consequence that the stress-components on this (free) surface are zero, thus: $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$. This also means that we have a plane-stress state in the $x_1 - x_2$ plane.

The inverse of Hooke's law can now be used to compute σ_{33}:

$$\begin{aligned}
\sigma_{33} &= \frac{E}{1+\nu}\left(\varepsilon_{33} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{33}\right) \\
&= \frac{E}{1+\nu}\left(\varepsilon_{33} + \frac{\nu}{1-2\nu}\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)\right) \\
&= \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11}+\varepsilon_{22})\right) = 0
\end{aligned}$$

Rearranging to obtain the unknown ε_{33}:

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}\left(\varepsilon_{11}+\varepsilon_{22}\right) = -20 \times 10^{-6}$$

Compute σ_{11}, σ_{22}, σ_{12}:
There are three unknown stresses (σ_{11}, σ_{22}, σ_{12}), and thus we need three equations. But first, it is handy to compute the volumetric strain:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -40 \times 10^{-6}$$

By inserting the volumetric strain the three stress unknowns can be solved as:

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\left(\varepsilon_{11} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right) = 1.6 \times 10^5 (130 + \tfrac{1}{2}40)10^6 = 24\,\mathrm{MPa} \\
\sigma_{22} &= \frac{E}{1+\nu}\left(\varepsilon_{22} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\right) = 1.6 \times 10^5 (130 + \tfrac{1}{2}40)10^6 = -8\,\mathrm{MPa} \\
\sigma_{12} &= \frac{E}{(1+\nu)}\varepsilon_{12} = \frac{E}{2\,(1+\nu)}\gamma_{12} = 27.7\,\mathrm{MPa}
\end{aligned}$$

Compute F_{max} according to Tresca
The criterion of Tresca, Eq. (), requires the principal stresses in the material to be computed from the stress tensor:

$$\sigma = \begin{bmatrix} 24 & 27.7 & \sigma_{13} \\ 27.7 & -8 & \sigma_{23} \\ 0 & 0 & 0 \end{bmatrix} \mathrm{MPa}$$

In connection with moment-equilibrium, we have $\sigma_{ij} = \sigma_{ji}$ and since the question stated a plane stress-state, also $\sigma_{13} = \sigma_{23} = 0$. Applying these relations, the following system must be solved:

$$\begin{aligned}
\det([\sigma] - \sigma\,[I]) &= \det \begin{bmatrix} 24 & 27.7 & 0 \\ 27.7 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \sigma^3 + 16\sigma^2 + \frac{95929}{100}\sigma \\
&= \sigma\left(\sigma^2 + 16\sigma + \frac{95929}{100}\right) = 0
\end{aligned}$$

Solving for σ and sorting ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) yields:

$$\begin{aligned}
\sigma_1 &= 40\,\mathrm{MPa} \\
\sigma_2 &= 0\,\mathrm{MPa} \\
\sigma_3 &= -24\,\mathrm{MPa}
\end{aligned}$$

Using the limit stress of Tresca gives:

$$\sigma_{eq} = \sigma_1 - \sigma_3 = 64\,\mathrm{MPa}$$

And last, solving for the maximum force:

$$F_{max} = \frac{\sigma_{max}}{\sigma_{eq}}F = 140\,\mathrm{kN}$$

Compute F_{max} according to von Mises
Using the equivalent stress definition of von Mises gives:

$$\begin{aligned}
\sigma_{eq} &= \sqrt{\tfrac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \\
&= \sqrt{\tfrac{1}{2}\left[40^2 + 24^2 + (-64)^2\right]} = 56\,\mathrm{MPa}
\end{aligned}$$

Solving for the maximum force:

$$F_{max} = \frac{\sigma_{max}}{\sigma_{eq}}F = 160\,\mathrm{kN}$$

13 Stress and strain

... based on sections 4,5 (Exercise V13 in old material before 2022)

In a certain point P, the stress tensor: $\left[\sigma_{ij}\right] = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 20 \end{bmatrix}$ MPa
describes the stress-state in a loaded body in the x_i coordinate system. The material is linear elastic and isotropic with material parameters $E = 200\,\mathrm{GPa}$ and $\nu = 0.25$.

Questions:
a) Explain what "isotropic" material behavior means.
b) Explain what "elastic" material behavior means.
c) Explain what "linear elastic" material behavior means.
d) Compute the eigen-stresses.
e) Draw the circle of Mohr for this stress-state and compare the mathematical and graphical solution.
f) Compute the directional cosines for the minor (smallest) eigen-stress.
g) Compute the components of the strain-tensor ε_{ij} in point P.
h) Compute the volumetric strain ε_V.
i) What is the largest change of angle in point P.

<u>Answers:</u>
a)
isotropic: means that the property is direction-independent, i.e., the material behaviour is in all directions the same – this is valid for randomly structured, disordered materials, but not valid, e.g., for fibre-reinforced (anisotropic) materials, or for polymers short after deformation, since those build up anisotropy due to their entangled chains.

b)
elastic: means that the deformation (e.g. due to applied stress) is restored when the stress is removed – as mostly valid for small strains too; exception are materials like rubber that can be (nonlinearly) elastic for very large strain; beyond elasticity one can observe plastic deformations, i.e., the deformation is not restored when the applied stress is removed.

c)
linear elastic: a linear relation between stress and strain – mostly valid for small strains in all types of materials, with exception of some complex materials that might behave non-linearly already at rather small strain;

d)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = (60 - \sigma)(\sigma^2 - 70\sigma - 300) = 0\,,$$

where the calculation of the invariants, here, is not helpful; keep the decomposition above in mind to obtain one eigenvalue and a second order polynomial, that can be solved as:

$$\sigma = \frac{70 \pm \sqrt{4900 - 4 \times 600}}{2} = 35 \pm 25\,\mathrm{MPa}.$$

Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 60\,\mathrm{MPa}$, $\sigma_{II} = 60\,\mathrm{MPa}$, and $\sigma_{III} = 10\,\mathrm{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.

Not asked for, not needed, but for completeness, the invariants are:

$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 130\,\mathrm{MPa}$
$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 = 3600 + 600 + 600 = 4800\,\mathrm{MPa}^2$
$I_3 = \sigma_1\sigma_2\sigma_3 = 36000\,\mathrm{MPa}^3$

e)
The drawing is done in the standard way, not shown here, but considering the definitions of center, radius, and eigen-values – in this case for the 2,3 directions only – one has:
$M = (\sigma_{22} + \sigma_{33})/2 = 35\,\mathrm{MPa}$, and
$R^2 = (\sigma_{22} - M)^2 + \sigma_{12}^2 = 15^2 + 20^2 = 625 = 25^2$, or $R = 25\,\mathrm{MPa}$,
so that the two eigenvalues are:
$\sigma_2 = M + R = 60\,\mathrm{MPa}$, and $\sigma_3 = M - R = 10\,\mathrm{MPa}$,
ignoring the third eigen-value, $\sigma_1 = 60\,\mathrm{MPa}$,
confirming the computations done in part d).

f)
The eigen-direction of the minor eigen-stress, $\sigma_3 = 10\,\mathrm{MPa}$, is obtained by solving:

$$\begin{bmatrix} \sigma_{11}-\sigma_3 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_3 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_3 \end{bmatrix} \begin{bmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(3)} + n_2^{(3)} + n_3^{(3)} = 1$$

so that (dropping the superscript for brevity):

$$50n_1 - 0n_2 - 0n_3 = 0 \rightarrow n_1 = 0$$

$$0n_1 + 40n_2 + 20n_3 = 0 \rightarrow n_2 = -(1/2)n_3$$

$$0n_1 + 20n_2 + 10n_3 = 0 \rightarrow n_2 = -(1/2)n_3$$

$$\Longrightarrow \begin{bmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{bmatrix} = \pm c \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

where the unknown $c = 1/\sqrt{5/4} = 2/\sqrt{5}$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(3)} \\ n_2^{(3)} \\ n_3^{(3)} \end{bmatrix} = \pm \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

In order to compare this with the circle of Mohr construction from e), consider the orientation of the minor eigen value in the 2-3-plane, relative to the 2-direction: $\theta_3 = \arctan(n_3^{(3)}/n_2^{(3)} = \arctan(-2) = -63.4^o$, equivalent to the opposite direction $\theta_3 + 180^o = 116.6^o$. The Mohr circle provides $\tan 2\theta_{M2} = \frac{2\sigma_{23}}{\sigma_{22}-\sigma_{33}} = 4/3$, so that $\theta_{M2} = 26.6^o$ follows, for the larger eigenvalue direction, while $\theta_{M3} = \theta_{M2} + 90^o = 116.6^o$.

g)
From Hooke's law:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right]$$

the strain components are computed as:

$$
\begin{aligned}
[\varepsilon_{ij}] &= \frac{1+\nu}{E}\left[\begin{bmatrix}60 & 0 & 0\\ 0 & 50 & 20\\ 0 & 20 & 20\end{bmatrix}\text{ MPa} - \frac{\nu}{1+\nu}\sigma_{kk}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\\
&= \frac{5}{4\,2\,10^5\,\text{MPa}}\left[\begin{bmatrix}60 & 0 & 0\\ 0 & 50 & 20\\ 0 & 20 & 20\end{bmatrix}\text{ MPa} - \frac{1/4}{5/4}130\,\text{MPa}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\\
&= \frac{5}{8\,10^5\,\text{MPa}}\left[\begin{bmatrix}60 & 0 & 0\\ 0 & 50 & 20\\ 0 & 20 & 20\end{bmatrix}\text{ MPa} - 26\,\text{MPa}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\text{ MPa}\right]\\
&= \frac{5}{8\,10^5\,\text{MPa}}\begin{bmatrix}34 & 0 & 0\\ 0 & 24 & 20\\ 0 & 20 & -6\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}5\times 17 & 0 & 0\\ 0 & 5\times 12 & 50\\ 0 & 50 & 5\times(-3)\end{bmatrix}10^{-5}\\
&= \begin{bmatrix}85/4 & 0 & 0\\ 0 & 60/4 & 50/4\\ 0 & 50/4 & -15/4\end{bmatrix}10^{-5}\\
&= \begin{bmatrix}2.125 & 0 & 0\\ 0 & 1.5 & 1.25\\ 0 & 1.25 & -0.375\end{bmatrix}10^{-4}
\end{aligned}
$$

h)
Note that this question can be answered without need to compute the full strain tensor in g). Compute only the trace from Hooke's law:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right]$$

to get the volumetric strain:

$$\varepsilon_V = \varepsilon_{jj} = \frac{5/4}{2\,10^5\,\text{MPa}}\left[1 - \frac{1/4}{5/4}(\times 3)\right]\sigma_{kk} = \frac{5}{8\,10^5\,\text{MPa}}\frac{2}{5}130\,\text{MPa} = 32.5\,10^{-5} = 3.25\,10^{-4}$$

However, the solution can also be obtained from the trace of the full strain tensor solution in g).

i)
The largest angle-change in point P is obtained from the eigenvalues of strain as

$$\gamma_{max} = \varepsilon_1 - \varepsilon_3 = 3.125\,10^{-4}\,.$$

Note that the easiest way to compute the three eigen-values is via the law of Hooke, inserting eigenvalue components (1|2|3), i.e., using the tensor in the eigen-system, so that:

$$\varepsilon_{1|2|3} = \frac{1+\nu}{E}\left[\sigma_{1|2|3} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{1|2|3}\right] = \frac{5/4}{2\,10^5\,\text{MPa}}\left[\sigma_{1|2|3} - \frac{1}{5}130\,\text{MPa}(\times 1)\right] = \frac{5}{8\,10^5\,\text{MPa}}\left[(60|60|10) - 26\right]$$

so that: $\varepsilon_{1|2} = (5/8)\,34\,10^{-5} = 2.125\,10^{-4}$ and $\varepsilon_3 = -(5/8)\,(-16)\,10^{-5} = -1.0\,10^{-4}$

14 Materials beyond elasticity

Questions:
a) Sketch the stress-strain relation for a linear elastic material, and
b) add possible non-linear material behavior (with explanation/motivation).
c) Explain what happens for unloading of:
 c1) a linear, elastic material, or
 c2) an elastic-plastic material (for small AND for large strains).
d) Sketch the relation of shear-stress versus strain-rate, for (fluid) materials that behave:
 d1) linear,
 d2) shear-thickening,
 d3) shear-thinning, or
 d4) yield-stress fluid.

Answers:
a)
Stress-strain linear means a straight line, with slope E being the modulus, while elastic means the return path is identical (from point 2 in Fig. 6 Left), unlike after plastic deformation (from point 4 in Fig. 6 Left).

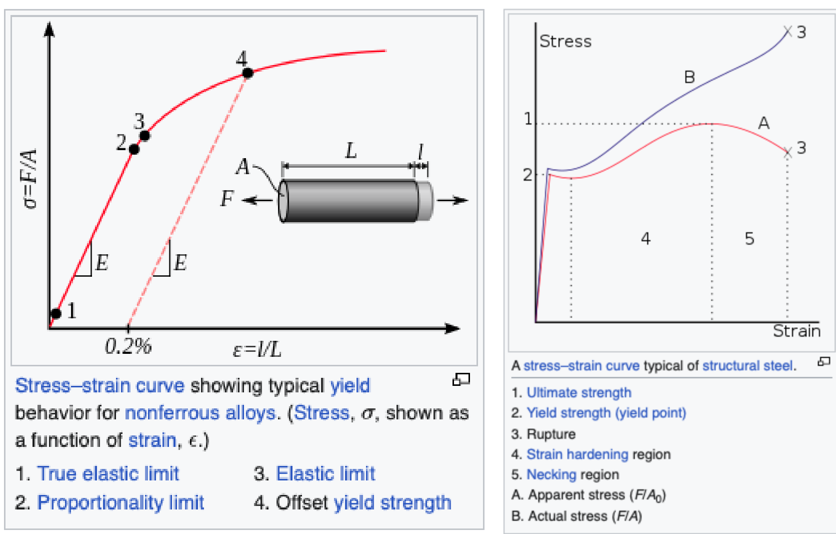
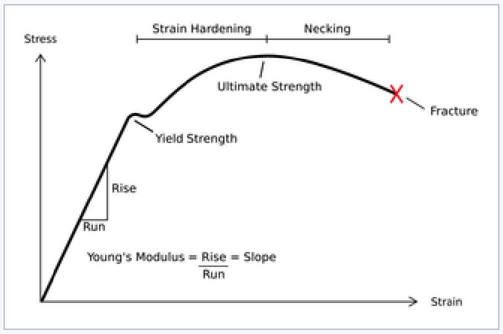


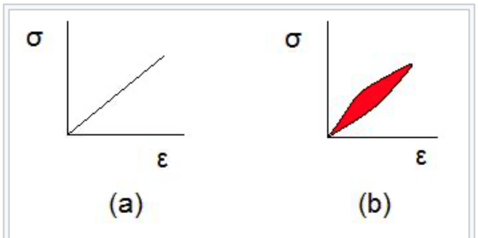
Figure 6: Plots taken from `https://en.wikipedia.org/wiki/Plasticity_(physics)`

b)
Many types of non-linear behavior are possible, see above Fig. 6, here only a few examples, linear (visco-)elastic, yield strength, strain-hardening, etc. (*note that hardening/softening for solids do not mean the same as for fluids*).
c1)
The load-unload curve is reversible in the elastic limit, and usually for small strains (see panel (a) in Fig. 8), whereas for a visco-elastic material one obtains hysteresis (red) in the stress-strain curve, which increases with the strain-rate (see panel (b) in Fig. 8), and actually its surface area (red) is the dissipated viscous energy.
c2)
for (very) small loads, the stress-strain path is typically reversible (c1) – but after a larger strain, the return (unloading) path is not indentical to the loading path. Four possible stress strain relations for plastic material are given ...



Typical stress vs. strain diagram indicating the various stages of deformation.

Figure 7: Plot taken from
`https://en.wikipedia.org/wiki/Deformation_(engineering)#Plastic_deformation`



Stress–strain curves for a purely elastic material (a) and a viscoelastic material (b). The red area is a hysteresis loop and shows the amount of energy lost (as heat) in a loading and unloading cycle. It is equal to $\oint \sigma \, d\varepsilon$, where σ is stress and ε is strain.[1]

Figure 8: Plot taken from `https://en.wikipedia.org/wiki/Viscoelasticity`

d)
Note that while in solid mechanics, mostly stress and strain are considered, for fluids we typically plot shear-stress versus strain-rate (or shear-rate), with the following material behavior: d1=Newtonian, d2=dilatant, d3=pseudo-plastic, or d4=Bingham plastic (examples).

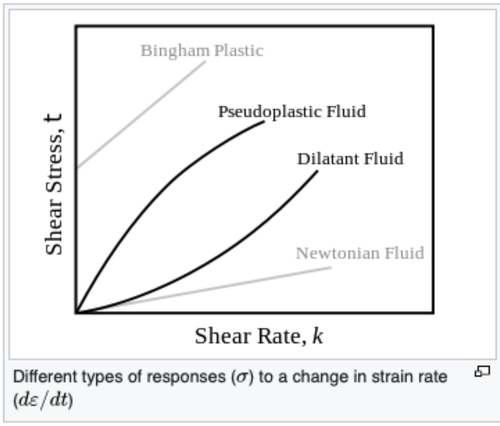


Figure 9: Plot taken from `https://en.wikipedia.org/wiki/Viscoelasticity`

15 Visco-elastic material behavior

Given is a rectangular shaped wire (length $L_0 = 0.1\,\mathrm{m}$, cross-section $A = HW$, volume $V = L_0 A = L_0 HW$) for a homogeneous, isotropic, visco-elastic, rubber-like material.

Questions:
a) What is the work necessary to quickly (or slowly) stretch the wire from stress 0 to length $3L_0$.
b) Which strain-rate is needed for making the elastic and the viscous contributions to work equally important?

Material-properties:
Kevin-Voigt viscoelastic solid (http://en.wikipedia.org/wiki/Viscoelasticity),
relation for stress = function of strain and strain-rate:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

with modulus of Young $E = 0.02\,\mathrm{MPa}$ and viscosity $\eta = 10\,\mathrm{Pa\,s}$.

Answers:
a)
Zero stress: length $L = L_0$, stretch to length $L_1 = 3L \to \varepsilon = \varepsilon_{11} = (L_1 - L_0)/L_0 = 2$.

Assume that the cross-section is not changing, i.e., H and W are constant.
Estimate the strain-rate: $\dot{\varepsilon} \approx \mathrm{const.} = (d/dt)\varepsilon = \frac{\text{length-change}}{\text{length*time}} = 2L_0/(L_0 \Delta t) = 2/\Delta t$.
The specific work is thus:
$a = \int d\varepsilon\sigma = \int d\varepsilon\,(E\varepsilon + \eta\dot{\varepsilon}) = \frac{1}{2}E\varepsilon^2\big|_0^2 + \eta\dot{\varepsilon}\varepsilon\big|_0^2 = 2\,(E + \eta\dot{\varepsilon}) = 2\left(E + \eta\frac{2}{\Delta t}\right)$,
The work in the total volume is:
$A = \int dV a = 2VE + 4V\eta/\Delta t$

b)
Estimate the speed at which viscous and elastic contributions to work equal each other:
$E \sim \eta\dot{\varepsilon}$ (after integration), so that (initially) the speed is: $v_0 = L_0\dot{\varepsilon} = 2L_0/\Delta t$,
using: $\dot{\varepsilon} = \dot{\varepsilon}_{ve} = E/\eta$ (from a), and $\Delta t = 2\eta/E$,
so that: $v_0 = L_0\dot{\varepsilon}_{ve} \to v_0 = 0.1 \times 2 \times 10^4/10 = 2 \times 10^2\,\mathrm{m/s}$,
thus a bit smaller than sound-speed.