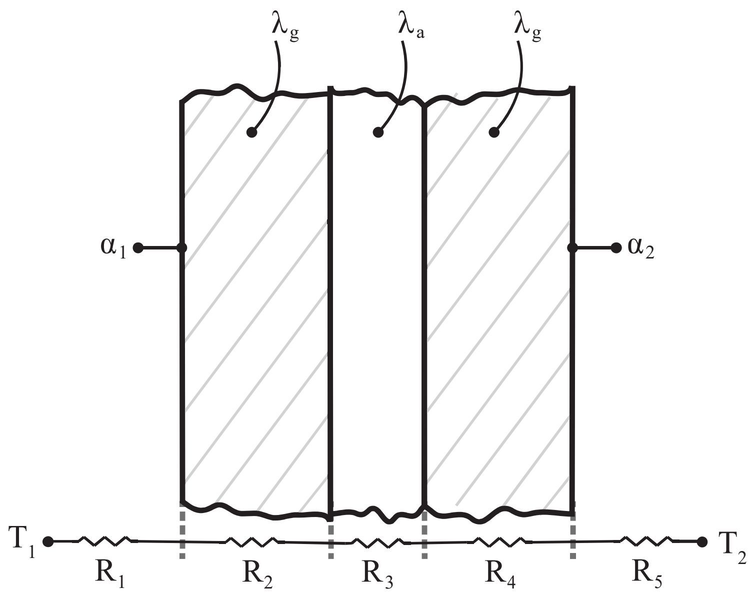
2.4 Window insulation ★

a) *Determine the rate of heat transfer $\dot{Q}$.*

The given problem can be solved by setting up the energy balance of the system, which is analog to sketching the thermal resistance network for a multi-layer wall problem. Afterwards, the thermal resistances can be determined and so the rate of heat loss in both situations.

1) Setting up an energy balance/sketching the system:



It is important before starting the calculations that it is clear what the thermal resistance network looks like. In the given situation, we are dealing with 5 resistors that are connected in series. With this given we can determine each individual thermal resistance and eventually the total thermal resistance of the system.

2) Defining the thermal resistances:

Area:

$$A = 1.2 \text{ [m]} \cdot 2 \text{ [m]} = 2.4 \text{ [m}^2\text{]} \tag{2.29}$$

Thermal resistances:

$$R_1 = \frac{1}{\alpha_1 \cdot A} = \frac{1}{10 \text{ [W/m}^2\text{K]} \cdot 2.4 \text{ [m}^2\text{]}} = 0.04167 \text{ [K/W]} \tag{2.30}$$

$$R_2 = R_4 = \frac{\delta_g}{\lambda_g \cdot A} = \frac{0.003\ [\text{m}]}{0.78\ [\text{W/mK}] \cdot 2.4\ [\text{m}^2]} = 0.00160\ [\text{K/W}] \tag{2.31}$$

$$R_3 = \frac{\delta_a}{\lambda_a \cdot A} = \frac{0.015\ [\text{m}]}{0.026\ [\text{W/mK}] \cdot 2.4\ [\text{m}^2]} = 0.2404\ [\text{K/W}] \tag{2.32}$$

$$R_5 = \frac{1}{\alpha_2 \cdot A} = \frac{1}{25\ [\text{W/m}^2\text{K}] \cdot 2.4\ [\text{m}^2]} = 0.01667\ [\text{K/W}] \tag{2.33}$$

Total resistance:

$$R_{\text{total}} = R_1 + R_2 + R_3 + R_4 + R_5 = 0.30192\ [\text{K/W}] \tag{2.34}$$

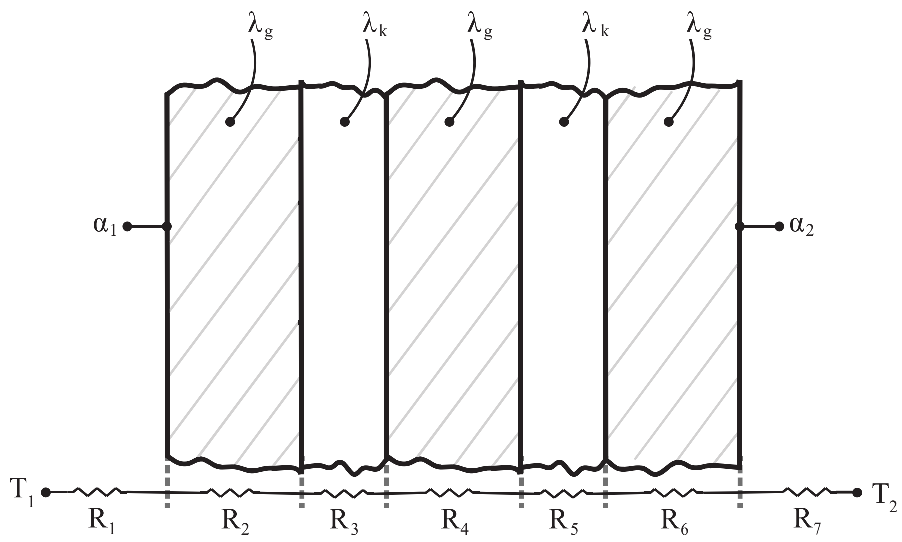
3) Inserting and rearranging:

Rate of heat transfer:

$$\boxed{\rightarrow \dot{Q} = \frac{T_1 - T_2}{R_{\text{total}}} = \frac{22\ [\text{K}] - -7\ [\text{K}]}{0.30192\ [\text{K/W}]} = 96.05\ [\text{W}]} \tag{2.35}$$

b) *Determine the rate of heat transfer* $\dot{Q}$.

1) Setting up an energy balance/sketching the system:



In the new situation we have 7 thermal resistances that are connected in series. With this given we can determine each individual thermal resistance, eventually the total thermal resistance and thus the rate of heat transfer.

2) Defining the thermal resistances:

Thermal resistances:

$$R_1 = \frac{1}{\alpha_\mathrm{g} \cdot A} = \frac{1}{10\ [\mathrm{W/m^2K}] \cdot 2.4\ [\mathrm{m^2}]} = 0.04167\ [\mathrm{K/W}] \tag{2.36}$$

$$R_2 = R_4 = R_6 = \frac{\delta_\mathrm{g}}{\lambda_\mathrm{g} \cdot A} = \frac{0.003\ [\mathrm{m}]}{0.78\ [\mathrm{W/mK}] \cdot 2.4\ [\mathrm{m^2}]} = 0.00160\ [\mathrm{K/W}] \tag{2.37}$$

$$R_3 = R_5 = \frac{\delta_\mathrm{k}}{\lambda_\mathrm{k} \cdot A} = \frac{0.008\ [\mathrm{m}]}{0.00949\ [\mathrm{W/mK}] \cdot 2.4\ [\mathrm{m^2}]} = 0.3512\ [\mathrm{K/W}] \tag{2.38}$$

$$R_7 = \frac{1}{\alpha_2 \cdot A} = \frac{1}{25\ [\mathrm{W/m^2K}] \cdot 2.4\ [\mathrm{m^2}]} = 0.01667\ [\mathrm{K/W}] \tag{2.39}$$

Total resistance:

$$R_\mathrm{total} = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 = 0.76563\ [\mathrm{K/W}] \tag{2.40}$$

3) Inserting and rearranging:

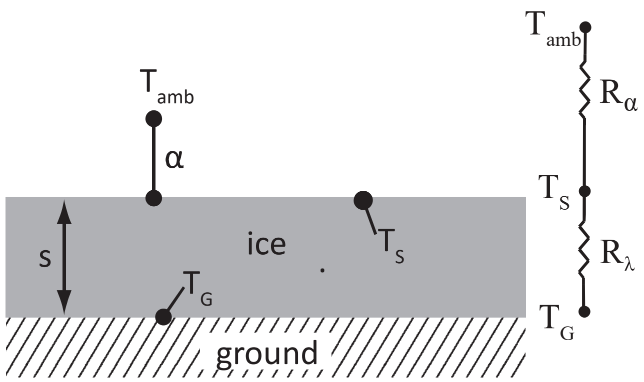
Rate of heat transfer:

$$\boxed{\rightarrow \dot{Q} = \frac{T_1 - T_2}{R_\mathrm{total}} = \frac{22\ [\mathrm{K}] - -7\ [\mathrm{K}]}{0.76563\ [\mathrm{K/W}]} = 37.88\ [\mathrm{W}]} \tag{2.41}$$

2.5 Ice layer ★★

a) *Determine the thickness s of the ice layer.*

1) Setting up an energy balance/sketching the system:



It is important to have an idea of what the thermal resistance network in the given problem looks like. In the described situation we are dealing with 2 resistances that are connected in series. One resistance describes the conductivity of heat through the ice and the other one convective heat transfer at the surface of the ice.

2) Defining the fluxes and thermal resistances:

Conductive heat flux ($T_{\mathrm{s}} \rightarrow T_{\mathrm{G}}$):

$$\dot{Q} = \frac{T_{\mathrm{s}} - T_{\mathrm{G}}}{\frac{s}{\lambda \cdot A}} \tag{2.42}$$

Convective heat flux ($T_{\mathrm{amb}} \rightarrow T_{\mathrm{s}}$):

$$\dot{Q} = \frac{T_{\mathrm{amb}} - T_{\mathrm{s}}}{\frac{1}{\alpha \cdot A}} \tag{2.43}$$

3) Inserting and rearranging:

As we are dealing with one-dimensional steady-state heat transfer without sources/sinks, the fluxes are equal to each other:

$$\frac{T_{\mathrm{s}} - T_{\mathrm{G}}}{\frac{s}{\lambda \cdot A}} = \frac{T_{\mathrm{amb}} - T_{\mathrm{s}}}{\frac{1}{\alpha \cdot A}} \tag{2.44}$$

Rewriting yields:

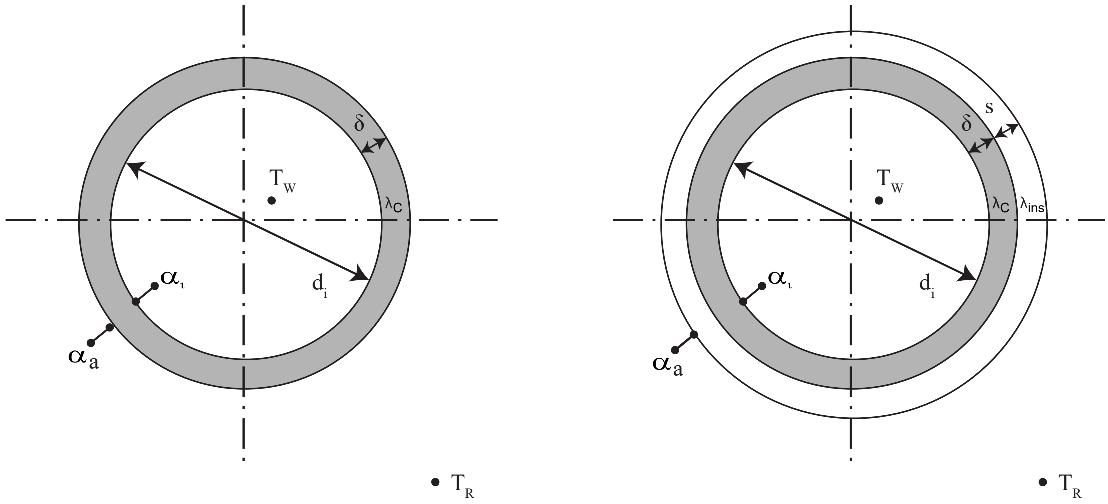
$$\boxed{\rightarrow s = \frac{\lambda}{\alpha} \cdot \frac{T_\mathrm{s} - T_\mathrm{G}}{T_\mathrm{amb} - T_\mathrm{s}} = \frac{2.2\ [\mathrm{W/mK}]}{10\ [\mathrm{W/m^2K}]} \cdot \frac{(-3+10)\ [\mathrm{K}]}{(5+3)\ [\mathrm{K}]} = 0.19\ [\mathrm{m}]} \tag{2.45}$$

2.6 Warm-water pipe ★ ★ ★

a. *Determine the heat transfered per unit pipe length for $\dot{q}'$.*

1) Setting up an energy balance/sketching the system:

System without and with insulation:



Resistance theorem:

$$\dot{Q} = \frac{\Delta T}{R_{\text{total}}} \tag{2.46}$$

Where the temperature difference is:

$$\Delta T = T_{\text{W}} - T_{\text{R}} \tag{2.47}$$

Calculations for the system without insulation:

2) Defining the thermal resistances:

It is important to have an idea of what the thermal resistance network in the given problem looks like. In the described situation we are dealing with 3 resistances that are connected in series. One resistance describes convective heat transfer inside the pipe, another one the conductivity of heat through the pipe and the other one convective heat transfer at the surface of the pipe.

With this we can determine the rate of heat transferred per unite pipe length.

The total thermal resistance:

$$R_{\text{total,wo,ins}} = R_{\text{conv,i}} + R_{\text{C}} + R_{\text{conv,a,wo}} \tag{2.48}$$

Expression of the thermal resistance due to convection on the inside, with L as the unknown length of the pipe, but this term will cancel out later in the calculation.:

$$R_{\text{conv,i}} = \frac{1}{\alpha_{\text{i}} \cdot \pi \cdot d_{\text{i}} \cdot L} \tag{2.49}$$

$$R_{\text{conv,i}} = \frac{1}{2300\ [\text{W/m}^2\text{K}] \cdot \pi \cdot 0.006\ [\text{m}] \cdot L} = \frac{0.0231}{L}\ [\text{K/W}] \tag{2.50}$$

Thermal resistance copper layer:

$$R_{\text{C}} = \frac{\ln\left[\left(d_i + 2\delta\right)/d_i\right]}{2 \cdot \pi \cdot L \cdot \lambda_{\text{C}}} \tag{2.51}$$

$$R_{\text{C}} = \frac{\ln\left[\left(0.006\ [\text{m}] + 2 \cdot 0.001\ [\text{m}]\right)/0.006\ [\text{m}]\right]}{2 \cdot \pi \cdot L \cdot 372\ [\text{W/mK}]} = \frac{0.0001}{L}\ [\text{K/W}] \tag{2.52}$$

Thermal resistance convection on the outside:

$$R_{\text{conv,a,wo}} = \frac{1}{\alpha_{\text{a}} \cdot \pi \cdot \left(d_{\text{i}} + 2\delta\right) \cdot L} \tag{2.53}$$

$$R_{\text{conv,a,wo}} = \frac{1}{6\ [\text{W/m}^2\text{K}] \cdot \pi \cdot \left(0.006\ [\text{m}] + 2 \cdot 0.001\ [\text{m}]\right) \cdot L} = \frac{6.6315}{L}\ [\text{K/W}] \tag{2.54}$$

3) Inserting and rearranging:

Filling in:

$$\dot{q}' = \frac{\dot{Q}}{L} = \frac{T_{\text{W}} - T_{\text{R}}}{R_{\text{total,wo,ins}} \cdot L} \tag{2.55}$$

$$\boxed{\rightarrow \dot{q}' = \frac{60\ [\text{K}]}{\left(0.0231 + 0.0001 + 6.6315\right)\ \left[\frac{\text{K}\cdot\text{m}}{\text{W}}\right]} = 9\ [\text{W/m}]} \tag{2.56}$$

Calculations for the system with insulation:

It is important to have an idea of what the thermal resistance network in the given problem looks like. In the described situation we are dealing with 4 resistances that are connected in series. One resistance describes convective heat transfer inside the pipe, the two in the middle describe the conductivity of heat through the pipe and insulation layer and the last one convective heat transfer at the surface of the pipe.

With this we can determine the rate of heat transferred per unite pipe length.

2) Defining the thermal resistances:

The total thermal resistance:

$$R_{\text{total,w,ins}} = R_{\text{conv,i}} + R_{\text{C}} + R_{\text{ins}} + R_{\text{conv,a,w}} \tag{2.57}$$

Thermal resistance insulation layer:

$$R_{\text{ins}} = \frac{\ln\left[\left(d_i + 2\delta + 2s\right) / \left(d_i + 2\delta\right)\right]}{2 \cdot \pi \cdot L \cdot \lambda_{\text{ins}}} \tag{2.58}$$

$$R_{\text{ins}} = \frac{\ln\left[(0.006\ [\text{m}] + 2 \cdot 0.001\ [\text{m}] + 2 \cdot 0.004\ [\text{m}]) / (0.006\ [\text{m}] + 2 \cdot 0.001\ [\text{m}])\right]}{2 \cdot \pi \cdot L \cdot 0.042\ [\text{W/mK}]} = \frac{2.6266}{L}\ [\text{K/W}] \tag{2.59}$$

Thermal resistance convection on the outside:

$$R_{\text{conv,a,w}} = \frac{1}{\alpha_{\text{a}} \cdot \pi \cdot \left(d_{\text{i}} + 2\delta + 2s\right) \cdot L} \tag{2.60}$$

$$R_{\text{conv,a,w}} = \frac{1}{6\ [\text{W/m}^2\text{K}] \cdot \pi \cdot (0.006\ [\text{m}] + 2 \cdot 0.001\ [\text{m}] + 2 \cdot 0.004\ [\text{m}]) \cdot L} = \frac{3.3157}{L}\ [\text{K/W}] \tag{2.61}$$

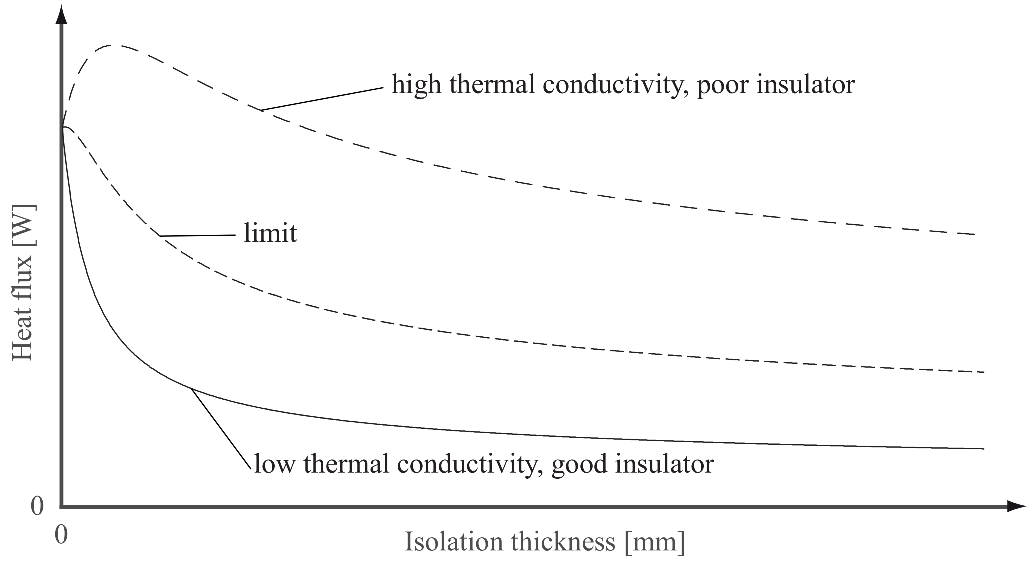
3) Inserting and rearranging:

Filling in:

$$\dot{q}' = \frac{\dot{Q}}{L} = \frac{T_{\text{W}} - T_{\text{R}}}{R_{\text{total,w,ins}} \cdot L} \tag{2.62}$$

$$\boxed{\rightarrow \dot{q}' = \frac{60\ [\text{K}]}{(0.0231 + 0.0001 + 2.6266 + 3.3157)\ \left[\frac{\text{K}\cdot\text{m}}{\text{W}}\right]} = 10\ [\text{W/m}]} \tag{2.63}$$

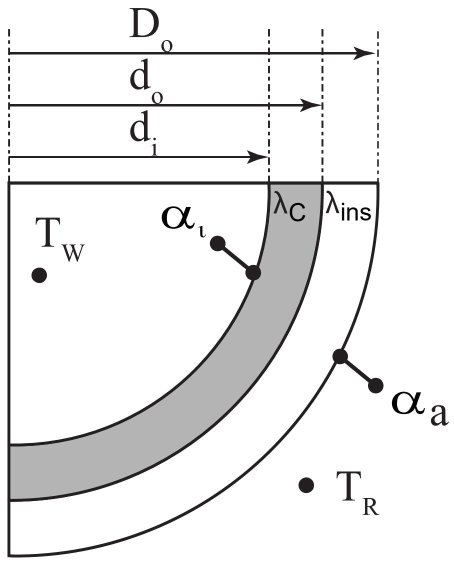
b. *Qualitatively sketch the heat emission profile $\dot{q}'$ as a function of the insulation thickness for different thermal conductivities of the insulation material. Explain the underlying physical principles.*



The line describing the behavior of a poor insulator shows that up to a critical thickness, the heat flux is increasing instead of decreasing. This is due to the fact that the effect of the outer surface area increasing, and therefore the resistance for transferring by convection decreasing, is stronger than the effect of adding additional insulation and therefore increasing the resistance for the conduction of heat.

This phenomena does not occur in the case where a good insulator is used, as the effect of increasing the thermal resistance for heat transfer by conduction outweighs the effect decreasing the resistance for heat transfer by convection on the outside.

c. *Determine the necessary thermal conductivity λ_{ins} for the insulating material to obtain a general reduction in heat loss.*



Insulation only has a positive effect when the rate of heat transfer $\dot{q}'$ decreases. In some scenario's adding an insulation layer will result in an increase of the rate of heat transfer, instead of an decrease. This is due to the phenomena that the thermal resistance for convection on the outside decreases more rapidly, than the addition of thermal resistance due to insulation. The rate of heat transfer $\dot{q}'$ starts decreasing from a certain point where the insulation layer thickness has reached a critical value. The critical value, where the rate of heat transfer $\dot{q}'$ is maximum, is when R_{total} is minimum. After this point, the rate of heat transfer will decrease.

Differentiating R_{total} with respect to the outer diameter D_{o} and equalling it to zero helps finding the critical condition where R_{total} is minimal:

$$\frac{d}{dD_{\text{o}}}\left[R_{\text{conv,i}} + R_{\text{C}} + R_{\text{ins}} + R_{\text{conv,a,w}}\right] = 0 \tag{2.64}$$

$$\frac{d}{dD_{\text{o}}}\left[R_{\text{ins}} + R_{\text{conv,a,w}}\right] = 0 \tag{2.65}$$

Inserting and simplifying:

$$\frac{d}{dD_{\text{o}}}\left[\frac{\ln\left[D_{\text{o}}/d_{\text{o}}\right]}{2\cdot\lambda_{\text{ins}}} + \frac{1}{\alpha_{\text{a}}\cdot D_{\text{o}}}\right] = 0 \tag{2.66}$$

$$\frac{1}{2\cdot\lambda_{\text{iso}}}\cdot\frac{1}{D_{\text{o}}} - \frac{1}{\alpha_{\text{a}} D_{\text{o}}^2} = 0 \tag{2.67}$$

$$D_{\mathrm{o,max}} = \frac{2 \cdot \lambda_{\mathrm{iso}}}{\alpha_{\mathrm{a}}} \tag{2.68}$$

Equation 2.68 tells us that maximum heat transfer would occur if $D_{\mathrm{o}} = \frac{2 \cdot \lambda_{\mathrm{iso}}}{\alpha_{\mathrm{a}}}$. Rearranging this equation and inserting the condition $D_{\mathrm{o}} = d_{\mathrm{o}}$ and $\lambda^{*}_{\mathrm{iso}}$ tells us that for a certain thermal conductivity, the effect of adding insulation will be the same as if the thickness was equal the diameter without insulation:

$$\frac{2 \cdot \lambda^{*}_{\mathrm{iso}}}{\alpha_{\mathrm{a}}} = d_{\mathrm{o}} \tag{2.69}$$

Equation 2.69 tells us that for a specific value of $\lambda^{*}_{\mathrm{iso}}$, isolation has no effect (in other words, the rate of heat transfer with an additional insulation layer of thickness D_o, is equal to the rate of heat transfer without an additional insulation layer).
With this given, we can rearrange and insert the numeric values for the known parameters.

$$\lambda^{*}_{\mathrm{iso}} = \alpha_{\mathrm{a}} \cdot \frac{d_{\mathrm{o}}}{2} = 6\ [\mathrm{W/m^2K}] \cdot \frac{0.008\ [\mathrm{m}]}{2} = 0.024\ [\mathrm{W/mK}] \tag{2.70}$$

$$\boxed{\rightarrow \lambda^{*}_{\mathrm{iso}} \geq 0.024\ [\mathrm{W/mK}]} \tag{2.71}$$