Task 1 Shaping Machine (Whitworth Mechanism)

Solution

For dimensioning the foundation of a shaping machine, the inertial forces must be determined. The masses of the sliders are negligible. The shaft rotates with constant rpm n_1.

a) The components of translational accelerations of the centres of gravity S_1, S_2, and S_3 in x- and y- direction should be determined for the positions $\varphi = 0$ and $\varphi = \pi$ as well as for both dead positions.

Generally, the following rule for differentiation over time, from which φ depends, holds:

$$\xi = \xi(\varphi) \tag{1.1}$$

$$\dot{\xi} = \frac{d\xi}{dt} = \frac{d\xi}{d\varphi} \cdot \frac{d\varphi}{dt} = \dot{\varphi} \cdot \frac{d\xi}{d\varphi} = \dot{\varphi} \cdot \xi' \tag{1.2}$$

$$\begin{aligned}\ddot{\xi} = \frac{d\dot{\xi}}{dt} = \frac{d(\dot{\varphi} \cdot \xi')}{dt} = \ddot{\varphi} \cdot \xi' + \dot{\varphi} \cdot \frac{d\xi'}{dt} = \ddot{\varphi} \cdot \xi' + \dot{\varphi} \cdot \frac{d\xi'}{d\varphi} \cdot \frac{d\varphi}{dt} \\ = \ddot{\varphi} \cdot \xi' + \dot{\varphi}^2 \cdot \xi''\end{aligned} \tag{1.3}$$

For all three components the translational accelerations in their centers of gravity need to be derived in x- and y-direction:

1. Crank (Component 1):

From the geometric relations it follows:

$$x_{S1} = l_{S1} \cdot sin(\varphi) \tag{1.4}$$

$$y_{S1} = l_0 - l_{S1} \cdot cos(\varphi) \tag{1.5}$$

Taking into account the aforementioned rule of differentiation this gives:

$$\dot{x}_{S1} = l_{S1} \cdot \omega \cdot cos(\varphi) \tag{1.6}$$

$$\dot{y}_{S1} = l_{S1} \cdot \omega \cdot sin(\varphi) \tag{1.7}$$

$$\ddot{x}_{S1} = \ddot{\varphi} \cdot l_{S1} \cdot cos(\varphi) - \omega^2 \cdot l_{S1} \cdot sin(\varphi) \tag{1.8}$$

$$\ddot{y}_{S1} = \ddot{\varphi} \cdot l_{S1} \cdot sin(\varphi) + \omega^2 \cdot l_{S1} \cdot cos(\varphi) \tag{1.9}$$

The crank rotates at constant speed. Thus, the angular acceleration is zero:

$$\ddot{x}_{S1} = -\omega^2 \cdot l_{S1} \cdot sin(\varphi) \tag{1.10}$$

$$\ddot{y}_{S1} = \omega^2 \cdot l_{S1} \cdot cos(\varphi) \tag{1.11}$$

2. Rocker (Component 2):

From the geometric relations it follows:

$$x_{S2} = l_{S2} \cdot sin(\psi) \tag{1.12}$$

$$y_{S2} = l_{S2} \cdot cos(\psi) \tag{1.13}$$

Applying the chain rule gives ($\psi = \psi(t)$):

$$\dot{x}_{S2} = l_{S2} \cdot \dot{\psi} \cdot cos(\psi) \tag{1.14}$$

$$\dot{y}_{S2} = -l_{S2} \cdot \dot{\psi} \cdot sin(\psi) \tag{1.15}$$

$$\ddot{x}_{S2} = \ddot{\psi} \cdot l_{S2} \cdot cos(\psi) - \dot{\psi}^2 \cdot l_{S2} \cdot sin(\psi) \tag{1.16}$$

$$\ddot{y}_{S2} = -\ddot{\psi} \cdot l_{S2} \cdot sin(\psi) - \dot{\psi}^2 \cdot l_{S2} \cdot cos(\psi) \tag{1.17}$$

ψ depends on φ:

$$\psi = \psi(\varphi) \tag{1.18}$$

$$\dot{\psi} = \frac{d\psi}{dt} = \frac{d\psi}{d\varphi} \cdot \frac{d\varphi}{dt} = \dot{\varphi} \cdot \psi' \tag{1.19}$$

$$\ddot{\psi} = \frac{d\dot{\psi}}{dt} = \frac{d(\dot{\varphi} \cdot \psi')}{d\varphi} \cdot \dot{\varphi} = \frac{d\dot{\varphi}}{d\varphi} \cdot \psi' \cdot \dot{\varphi} + \dot{\varphi}^2 \cdot \frac{d\psi'}{d\varphi} \tag{1.20}$$

Since $\omega = \dot{\varphi} = const.$, it follows:

$$\frac{d\dot{\varphi}}{d\varphi} = 0 \tag{1.21}$$

Thus:

$$\ddot{\psi} = \dot{\varphi}^2 \cdot \psi'' \tag{1.22}$$

From (1.16) and (1.17) it follows:

$$\begin{aligned} \ddot{x}_{S2} &= \dot{\varphi}^2 \cdot \psi'' \cdot l_{S2} \cdot cos(\psi) - l_{S2} \cdot \dot{\varphi}^2 \cdot \psi'^2 sin(\psi) \\ &= -l_{S2} \cdot \dot{\varphi}^2 \cdot \left(\psi'^2 \cdot sin(\psi) - \psi'' \cdot cos(\psi)\right) \end{aligned} \tag{1.23}$$

$$\begin{aligned} \ddot{y}_{S2} &= -\dot{\varphi}^2 \cdot \psi'' \cdot l_{S2} \cdot sin(\psi) - l_{S2} \cdot \dot{\varphi}^2 \cdot \psi'^2 cos(\psi) \\ &= -l_{S2} \cdot \dot{\varphi}^2 \cdot \left(\psi'^2 \cdot cos(\psi) + \psi'' \cdot sin(\psi)\right) \end{aligned} \tag{1.24}$$

A relation between φ and ψ needs to be found. Therefore, the position of A is derived:

$$x_A = l_1 \cdot sin(\varphi) \tag{1.25}$$

$$y_A = l_0 - l_1 \cdot cos(\varphi) \tag{1.26}$$

The distance from B_0 to A is:

$$\overline{B_0A} = \sqrt{x_A^2 + y_A^2} = \sqrt{\left(l_1 \cdot sin(\varphi)\right)^2 + \left(l_0 - l_1 \cdot cos(\varphi)\right)^2} \\ = \sqrt{l_1^2 + l_0^2 - 2 \cdot l_0 \cdot l_1 \cdot cos(\varphi)} \tag{1.27}$$

With:

$$\lambda = \frac{l_1}{l_0} \tag{1.28}$$

it follows:

$$\overline{B_0A} = l_0 \cdot \sqrt{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2} \tag{1.29}$$

From the trigonometry one can obtain:

$$tan(\psi) = \frac{x_A}{y_A} = \frac{\lambda \cdot sin(\varphi)}{1 - \lambda \cdot cos(\varphi)} \tag{1.30}$$

$$sin(\psi) = \frac{x_A}{\overline{B_0A}} = \frac{\lambda \cdot sin(\varphi)}{\sqrt{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2}} \tag{1.31}$$

$$cos(\psi) = \frac{y_A}{\overline{B_0A}} = \frac{1 - \lambda \cdot cos(\varphi)}{\sqrt{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2}} \tag{1.32}$$

Differentiation of (1.30) over φ gives:

$$\frac{dtan(\psi)}{d\varphi} = \frac{dtan(\psi)}{d\psi} \cdot \frac{d\psi}{d\varphi} = \frac{1}{cos^2(\psi)} \cdot \psi' \tag{1.33}$$

$$\Leftrightarrow \psi' = cos^2(\psi) \cdot \frac{dtan(\psi)}{d\varphi} = cos^2(\psi) \cdot \frac{d}{d\varphi}\left(\frac{\lambda \cdot sin(\varphi)}{1 - \lambda \cdot cos(\varphi)}\right) \tag{1.34}$$

With (1.32) it follows:

$$\psi' = \frac{\left(1 - \lambda \cdot cos(\varphi)\right)^2}{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2} \cdot \frac{\lambda \cdot (cos(\varphi) - \lambda)}{\left(1 - \lambda \cdot cos(\varphi)\right)^2} \\ = \frac{\lambda \cdot (cos(\varphi) - \lambda)}{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2} \tag{1.35}$$

Differentiation:

$$\psi'' = \frac{d\psi'}{d\varphi} = \frac{d}{d\varphi}\left(\frac{\lambda \cdot (cos(\varphi) - \lambda)}{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2}\right) = \frac{\lambda \cdot sin(\varphi) \cdot (\lambda^2 - 1)}{(1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2)^2} \tag{1.36}$$

3. Slider (Component 3):

From the geometric relations it follows:

$$x_{S3} = x_C - c_x = h \cdot tan(\psi) - c_x \tag{1.37}$$

$$y_{S3} = h + c_y \tag{1.38}$$

Differentiation:

$$\dot{x}_{S3} = \frac{h \cdot \lambda \cdot \omega \cdot (cos(\varphi) - \lambda)}{\left(1 - \lambda \cdot cos(\varphi)\right)^2} \tag{1.39}$$

$$\dot{y}_{S3} = 0 \tag{1.40}$$

$$\ddot{x}_{S3} = \frac{h \cdot \lambda \cdot \omega^2 \cdot sin(\varphi) \cdot (2 \cdot \lambda^2 - \lambda \cdot cos(\varphi) - 1)}{\left(1 - \lambda \cdot cos(\varphi)\right)^3} \tag{1.41}$$

$$\ddot{y}_{S3} = 0 \tag{1.42}$$

The two dead center positions of the mechanism occur, when the crank is perpendicular to the rocker. This means that the slider changes the direction of motion. In this reversal point, a change of φ does not result in a change of ψ. Thus:

$$\frac{d\psi}{d\varphi} = \psi' = 0 \tag{1.43}$$

$$\Leftrightarrow \frac{\lambda \cdot (cos(\varphi) - \lambda)}{1 - 2 \cdot \lambda \cdot cos(\varphi) + \lambda^2} = 0 \tag{1.44}$$

$$\Leftrightarrow cos(\varphi) = \lambda = \frac{l_1}{l_0} = \frac{150\,mm}{400\,mm} = 0{,}375 \tag{1.45}$$

$$\varphi_1 = 68^0 \text{ und } \varphi_2 = 292^0$$

Numerical values:

$$\omega = \dot{\varphi} = \frac{2 \cdot \pi \cdot n}{60} = 10{,}472\ \frac{1}{s}$$

ϕ	[°]	0	180	67,975	292,024
$\ddot{x}_{S1}$	[m/s^2]	0	0	-4,07	4,07
$\ddot{y}_{S1}$	[m/s^2]	4,39	-4,39	1,64	1,64
$\ddot{x}_{S2}$	[m/s^2]	0	0	-6,17	6,17
$\ddot{y}_{S2}$	[m/s^2]	-5,92	-1,22	2,49	2,49
$\ddot{x}_{S3}$	[m/s^2]	0	0	-33,5	33,5

b) The components of corresponding inertial forces should be determined for these positions

From the principle of linear momentum it follows:

$$\vec{F}_{Si} = -\frac{d}{dt}(m_i \vec{v}_{Si}) = -m_i \vec{a}_{Si} \tag{1.46}$$

$$F_{Si,x} = -m_i \ddot{x}_{Si} \tag{1.47}$$

$$F_{Si,y} = -m_i \ddot{y}_{Si} \tag{1.48}$$

For the entire system it follows:

$$F_x = -m_1 \ddot{x}_{S1} - m_2 \ddot{x}_{S2} - m_3 \ddot{x}_{S3} \tag{1.49}$$

$$F_y = -m_1 \ddot{y}_{S1} - m_2 \ddot{y}_{S2} - m_3 \ddot{y}_{S3} \tag{1.50}$$

$$\vec{F} = -m_1 \vec{a}_{S1} - m_2 \vec{a}_{S2} - m_3 \vec{a}_{S3} \tag{1.51}$$

Numerical values:

ϕ	[°]	0	180	67,975	292,024
F_{x_1}	[N]	0	0	61,0	-61,0
F_{y_1}	[N]	-65,8	65,8	-24,6	24,6
F_{x_2}	[N]	0	0	185,0	-184,0
F_{y_2}	[N]	177,7	36,7	-74,8	-74,8
F_{x_3}	[N]	0	0	1677	-1677
$\Sigma F_x=F_{x_1}+F_{x_2}+F_{x_3}$	[N]	0	0	1923	-1923
$\Sigma F_y=F_{y_1}+F_{y_2}$	[N]	111,9	102,5	-99,4	-99,4