

# Chapter 5

## Fully Developed Flow

### 5.1 Fully developed flow in slender ducts

Consider the flow between two infinite walls in Fig. (5.1).

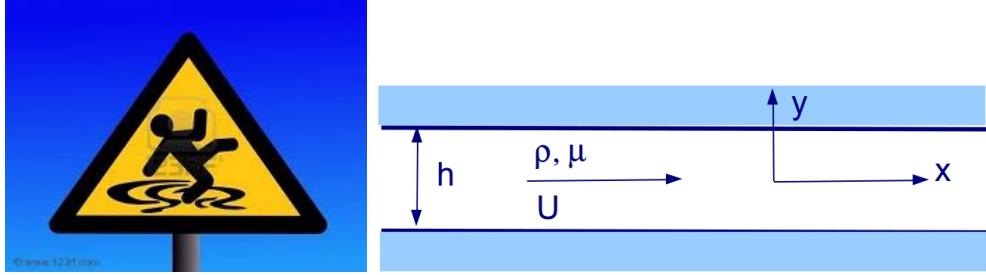


Figure 5.1: Flow between two walls: real world (left), schematic (right)

We make the following 5 assumptions:

- Two-dimensional flow:  $\frac{\partial}{\partial z}(\dots) = 0$ , and  $w=0$ ,
- Incompressible flow:  $\rho$  is a constant,
- Fully developed flow:  $\frac{\partial u}{\partial x} = 0$ ,  $\frac{\partial v}{\partial x} = 0$ ,
- Steady flow:  $\frac{\partial}{\partial t}(\dots) = 0$ , and
- No-slip at the boundaries:  $u = 0$ ,  $v = 0$  at  $y = \pm \frac{h}{2}$ .

The question is: what is the velocity field  $\mathbf{u}(x, y)$  and what is the pressure field  $p(x, y)$ ?

To determine the velocity and pressure fields we use the differential forms of the conservation equations for mass and momentum. The mass conservation equation, Eq.(3.18), becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (5.1)$$

The momentum conservation equations (the Navier-Stokes equations), Eq.(4.39), for  $i = 1$  and  $i = 2$  become

$$\frac{\partial}{\partial x} (\rho u^2 - \sigma_{11}) + \frac{\partial}{\partial y} (\rho uv - \sigma_{12}) = \rho g_1. \quad (5.2)$$

and

$$\frac{\partial}{\partial x} (\rho v u - \sigma_{21}) + \frac{\partial}{\partial y} (\rho v^2 - \sigma_{22}) = \rho g_2. \quad (5.3)$$

From the fully developed flow assumption together with mass conservation and no-slip boundary conditions we have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v(x, y) = \text{const.} \Rightarrow v(x, y) = 0. \quad (5.4)$$

Using the product rule of differentiation and the assumptions made, the velocity terms in the Navier-Stokes equations become:

$$\begin{aligned} \frac{\partial}{\partial x} (\rho u^2) &= \rho \frac{\partial}{\partial x} (u^2) = 2\rho u \frac{\partial u}{\partial x} = 0, \\ \frac{\partial}{\partial y} (\rho u v) &= 0, \\ \frac{\partial}{\partial x} (\rho v u) &= 0, \\ \frac{\partial}{\partial y} (\rho v^2) &= 0. \end{aligned} \quad (5.5)$$

Using Eq.(4.10) and the assumptions made, the four stress tensor components in the Navier-Stokes equations become:

$$\begin{aligned} \sigma_{11} &= -p + \tau_{11} = -p + 2\mu \frac{\partial u}{\partial x} = -p, \\ \sigma_{12} &= \tau_{12} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{\partial u}{\partial y}, \\ \sigma_{21} &= \tau_{21} = \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \mu \frac{\partial u}{\partial y}, \\ \sigma_{22} &= -p + \tau_{22} = -p + 2\mu \frac{\partial v}{\partial y} = -p. \end{aligned} \quad (5.6)$$

With these expressions, and noting that  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = 0$ , we obtain the so-called reduced Navier-Stokes equations:

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = \rho g_1, \quad \frac{\partial p}{\partial y} = \rho g_2.$$

(5.7)

## 5.2 Pressure gradient

The pressure gradient is

$$\nabla p \equiv \left( \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \right)^T. \quad (5.8)$$

We will show that  $\nabla p$  is constant, in other words, is independent of  $x$  and  $y$  by showing that  $\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right)$  and  $\frac{\partial}{\partial y} \left( \frac{\partial p}{\partial x} \right)$  both are zero. From the first reduced Navier-Stokes equation we find

$$\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right) = \frac{\partial}{\partial x} \left( \rho g_1 + \mu \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (5.9)$$

since  $u$  does not depend on  $x$  and all other terms are constants. From the second reduced Navier-Stokes equation we find

$$\frac{\partial}{\partial y} \left( \frac{\partial p}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial x} (\rho g_2) = 0. \quad (5.10)$$

So, indeed

$$\frac{\partial p}{\partial x} = \text{constant} \equiv \left( \frac{\partial p}{\partial x} \right)_o, \quad (5.11)$$

which means that pressure varies linearly with  $x$ . In addition, from the second reduced Navier-Stokes equation we directly find

$$\frac{\partial p}{\partial y} = \text{constant} \equiv \rho g_2, \quad (5.12)$$

which means that pressure varies linearly with  $y$ . Adding the two results we get

$$p(x, y) = p_o + \left( \frac{\partial p}{\partial x} \right)_o x + \rho g_2 y, \quad (5.13)$$

where  $p_o$  is the pressure at  $x = 0, y = 0$ . The question rising here is whether we can relate the value of  $\left( \frac{\partial p}{\partial x} \right)_o$  to the velocity, viscosity, and density of the flow.

### 5.3 Velocity profile

Since the pressure gradient is a constant, we can integrate the first reduced Navier-Stokes equation twice with respect to  $y$ :

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] y + c_1, \quad (5.14)$$

$$u(y) = \frac{1}{2\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] y^2 + c_1 y + c_2. \quad (5.15)$$

To determine the two integration constants  $c_1$  and  $c_2$  we use the two boundary conditions at  $y = \pm \frac{h}{2}$ :

$$\begin{aligned} u(-\frac{h}{2}) &= \frac{1}{2\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] \left( \frac{h}{2} \right)^2 - c_1 \frac{h}{2} + c_2 = 0, \\ u(\frac{h}{2}) &= \frac{1}{2\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] \left( \frac{h}{2} \right)^2 + c_1 \frac{h}{2} + c_2 = 0, \end{aligned}$$

which leads to

$$c_1 = 0, \quad c_2 = -\frac{1}{2\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] \left( \frac{h}{2} \right)^2. \quad (5.16)$$

Hence, we find for the velocity  $u(y)$ :

$$u(y) = \frac{1}{2\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] \left[ y^2 - \left( \frac{h}{2} \right)^2 \right]. \quad (5.17)$$

## 5.4 Pressure-velocity relation

The mean normal velocity through the gap between the two plates is defined as

$$U \equiv \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u(y) dy. \quad (5.18)$$

Since the  $u(y)$  is a known function, see Eq.(5.17), we can evaluate the integral, which represents the average velocity:

$$\frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u(y) dy = -\frac{1}{12} \frac{h^2}{\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right]. \quad (5.19)$$

Hence, we have found the following relation

$$U = -\frac{1}{12} \frac{h^2}{\mu} \left[ \left( \frac{\partial p}{\partial x} \right)_o - \rho g_1 \right] \Leftrightarrow \left( \frac{\partial p}{\partial x} \right)_o = -12 \frac{\mu U}{h^2} + \rho g_1. \quad (5.20)$$

## 5.5 Stress at boundaries

Suppose we want to calculate the stress on the lower wall. The stress actually consists of a vector  $\mathbf{t}$ , the stress vector, which satisfies Eq.(4.8). We take the normal vector  $\mathbf{n}$  pointing from the wall to the fluid since we want to know the stress caused by the fluid on the lower wall:

$$\mathbf{n} = (0, 1, 0)^T \Rightarrow t_i(x, y) = \sigma_{ij}(x, y)n_j = \sigma_{i2}(x, y) \Rightarrow \mathbf{t} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \quad (5.21)$$

Using the stress components calculated in Eq.(5.6) we see that the stress component in the  $x$ -direction is:

$$t_1(y) = \sigma_{12}(y) = \mu \frac{\partial u}{\partial y}. \quad (5.22)$$

Using the velocity distribution Eq.(5.15) and Eq.(5.20) we find:

$$t_1(-\frac{h}{2}) = 6 \frac{\mu U}{h}. \quad (5.23)$$

In a similar way we find

$$t_2(y) = \sigma_{22}(y) = -p(y), \quad t_3(y) = \sigma_{32}(y) = 0, \quad (5.24)$$

and

$$t_2\left(-\frac{h}{2}\right) = -p, \quad t_3\left(-\frac{h}{2}\right) = 0. \quad (5.25)$$

In summary, the stress vector exerted by the fluid on the lower wall is

$$\mathbf{t}\left(-\frac{h}{2}\right) = \left(6\frac{\mu U}{h}, -p\right)^T. \quad (5.26)$$

In this case, the first component,  $t_1$ , which is aligned with the wall, is called the shear stress, whereas the second component,  $t_2$ , which is oriented normal to the wall, is called the normal stress.

Note that the first component does not depend on  $x$ , but, in contrast, the second component does depend linearly on  $x$ . It is left as an exercise to the reader to show and explain that the stress vector on the upper wall is

$$\mathbf{t}\left(\frac{h}{2}\right) = \left(6\frac{\mu U}{h}, p\right)^T. \quad (5.27)$$

## 5.6 Alternative boundary conditions

Two alternative boundary conditions are treated here. First, consider a moving lower wall instead of a fixed wall, and suppose that the wall velocity is  $U_w$ . Then the no-slip boundary condition on the lower wall becomes simply

$$u(x, -\frac{h}{2}) = U_w. \quad (5.28)$$

Secondly, consider the flow of a film of water, i.e., remove the upper wall and replace it by air. The shear stress exerted by the water on the air is given by  $-\left(\mu \frac{\partial u}{\partial y}\right)_{\text{water}}$  while the shear stress exerted by the air on the water is given by  $\left(\mu \frac{\partial u}{\partial y}\right)_{\text{air}}$ . When we consider an infinitesimal volume containing both water and air, it becomes clear that both shear stresses need to be equal in absolute value to prevent an infinite acceleration of the volume:

$$\left(\mu \frac{\partial u}{\partial y}\right)_{\text{water}} = \left(\mu \frac{\partial u}{\partial y}\right)_{\text{air}}. \quad (5.29)$$

Since the viscosity coefficient of water is much larger than that of air, we have

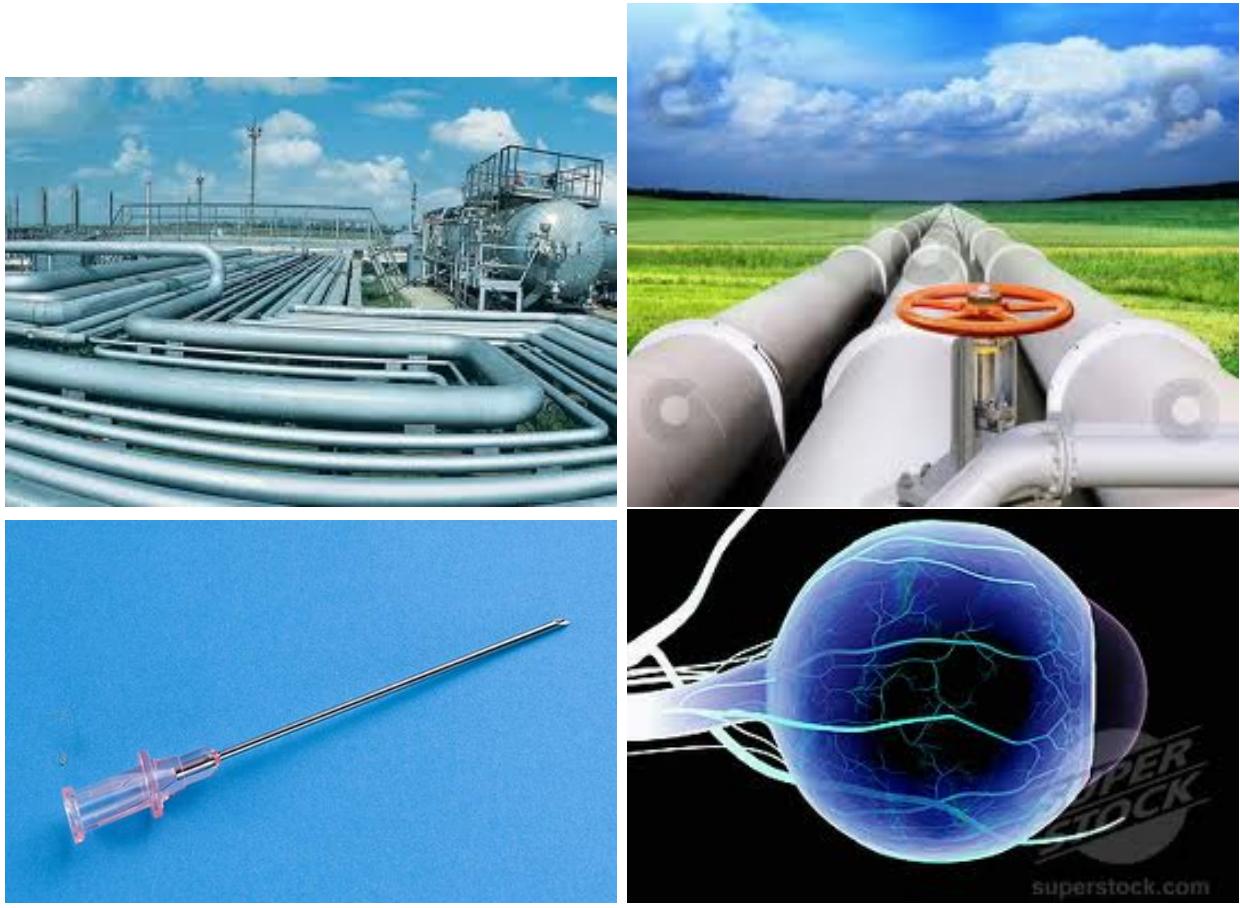
$$\mu_{\text{water}} \gg \mu_{\text{air}} \Rightarrow \left(\frac{\partial u}{\partial y}\right)_{\text{water}} \ll \left(\frac{\partial u}{\partial y}\right)_{\text{air}}. \quad (5.30)$$

Therefore, when the velocity derivative in the air is moderate, we can approximate

$$\left(\frac{\partial u}{\partial y}\right)_{\text{water}} \approx 0. \quad (5.31)$$

This equation can be used as the boundary condition at the upper boundary.

## 5.7 Fully developed flow in slender pipes



*Figure 5.2: Flow in slender pipes*

We can extend the developed flow analysis to pipes; think of refineries, engine cooling, hydraulics, injection needles, lung tubes, blood vessels, etc, etc. We start by transforming the partial differential equations from cartesian coordinates  $(x, y, z)$  to cylinder coordinates  $(r, \theta, z)$ :

$$x = x, \quad y = r \cos(\theta), \quad z = r \sin(\theta). \quad (5.32)$$

The velocities in the directions of  $x$ ,  $r$ , and  $\theta$  will be denoted by  $u$ ,  $u_r$ , and  $u_\theta$ , respectively (see Fig. (5.3)):

$$u = u, \quad (5.33)$$

$$u_r = v \cos(\theta) + w \sin(\theta), \quad (5.34)$$

$$u_\theta = -v \sin(\theta) + w \cos(\theta). \quad (5.35)$$

We will make the following assumptions:

- (a) Axi-symmetric flow:  $\frac{\partial}{\partial \theta}(\dots) = 0$ ,  $u_\theta = 0$ ,

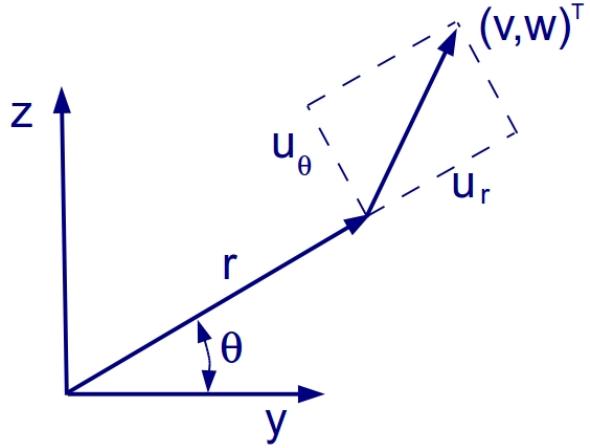


Figure 5.3: Transformation to cylindrical coordinates

- (b) Incompressible flow:  $\rho$  is a constant,
- (c) Fully developed flow:  $\frac{\partial u}{\partial x} = 0$ ,  $\frac{\partial u_\theta}{\partial x} = 0$ ,
- (d) Steady flow:  $\frac{\partial}{\partial t}(\dots) = 0$ ,
- (e) No-slip at the boundary:  $\mathbf{u} = 0$  at  $r = R$ , and
- (f) Zero gravity:  $g = 0$ .

In this case the reduced Navier-Stokes equations become:

$$\frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad (5.36)$$

$$\frac{\partial p}{\partial r} = 0. \quad (5.37)$$

It can be shown that  $(\frac{\partial p}{\partial x})$  is constant, say  $(\frac{\partial p}{\partial r})_o$ , and with the no-slip boundary condition the solution becomes

$$u(r) = -\frac{1}{4\mu} \left( \frac{\partial p}{\partial r} \right)_o (R^2 - r^2).$$

(5.38)

This solution is referred to as Hagen-Poiseuille flow.

## 5.8 Exercises

**Problem 5.1.** *Incompressible viscous oil flows steadily between stationary parallel plates. The flow is laminar and fully developed. The total gap width between the plates is  $h$ . The oil viscosity is  $\mu$  and the pressure drop over a distance  $L$  is  $\Delta p$ .*

- (a) Derive an expression for the shear stress on the upper plate.
- (b) Derive an expression for the volume flow rate through the channel over a width  $w$ .