Lecture #11. Streamline Invariants.

Topics:
- energy conservation, diff. form
- Poisson's relation
- Speed-of-sound
- Streamline invariants
- Bernoulli revisited.

$$\text{Energy}(t) \equiv \int_{V(t)} \rho E \, dV \qquad E \equiv e + \tfrac{1}{2} u_k u_k$$

conservation: $$\int_{V(t)} \frac{\partial}{\partial t}(\rho E) \, dV + \int_S \{\cdots\}_j n_j \, dS = \int_V \rho g_i u_i \, dV$$

Gauss: $$\int_S \{\ \}_j n_j \, dS = \int_V \frac{\partial}{\partial x_j} \{\ \}_j \, dV$$

Result: $$\int_{V(t)} [\cdots\cdots] \, dV = 0 \quad \forall V(t) \ \forall t.$$

$$\Longleftrightarrow [\cdots] = 0 \quad \forall \vec{x}, t.$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j} \left\{ \rho E u_j - \sigma_{ij} u_i - k \frac{\partial T}{\partial x_j} \right\} = \rho g_j u_j}$$

Energy conservation, differential form.

momentum: $\rho \frac{Du_i}{Dt} \overset{\text{if } \mu=0}{=} -\frac{\partial p}{\partial x_i}$ Euler's equation.

mass: $\frac{D\rho}{Dt} = -\rho \frac{\partial u_j}{\partial x_j}$

$g=0$

$k=0$

energy: $\rho \frac{DE}{Dt} = -\frac{\partial}{\partial x_j}(p u_j)$

Combine all three equations.

$$\Rightarrow \boxed{\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0}$$

$\mu=0$ $k=0$

g drops out anyway

From thermodynamics: $d\bar{U} + p\,d\bar{V} = 0$ reversible process.

Here $\bar{U}$: internal energy

$\bar{V}$: volume,

$$\Rightarrow de + p\,d\left(\frac{1}{\rho}\right) = 0 \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = 0$$

$$\frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{D\rho}{Dt} \Rightarrow \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0$$

$$d\bar{U} + p\,d\bar{V} \underset{\text{reversible}}{\equiv} T\,dS$$

entropy.

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0 \Rightarrow \boxed{\frac{Ds}{Dt} = 0}$$

$$\Rightarrow \boxed{s = \text{c.a.s.}} \qquad \mu = 0 \quad k = 0$$

Poisson's relation.

Assume ideal gas: $p = \rho R T$ $\quad e = c_v T$

$c_v = \text{const}$. $\quad c_p = \text{const}$ $\quad c_p - c_v = R$ $\quad c_p/c_v = \gamma$

$$\Rightarrow p = (\gamma - 1)\rho e.$$

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{De}{Dt} - (\gamma-1)\frac{e}{\rho}\frac{D\rho}{Dt} = 0$$

$$\frac{1}{e}\frac{De}{Dt} - (\gamma-1)\frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

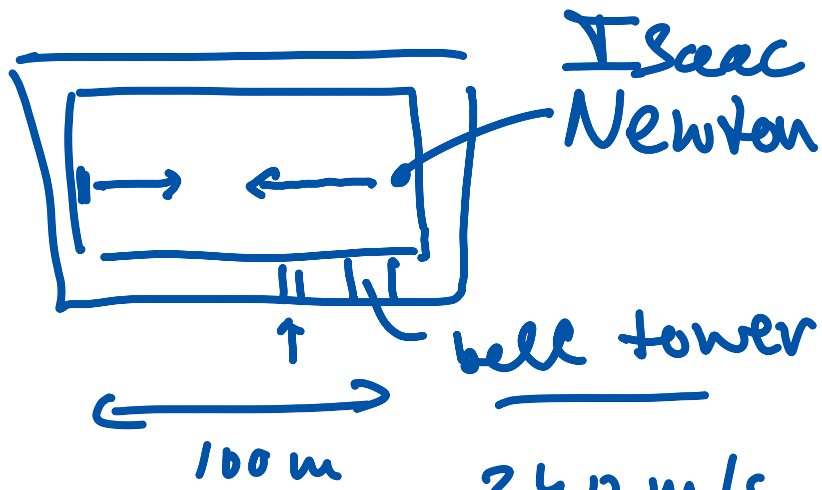
$$\Rightarrow \frac{D}{Dt}\left\{\ln e - (\gamma-1)\ln\rho\right\} = 0$$

$$\Rightarrow \frac{D}{Dt}\ln\left(e\rho^{-(\gamma-1)}\right) = 0 \qquad \rho e = \frac{p}{\gamma-1}$$

$$\Rightarrow \frac{D}{Dt}\left(p\rho^{-\gamma}\right) = 0 \Rightarrow \boxed{p\rho^{-\gamma} = \text{c.a.s.}}$$

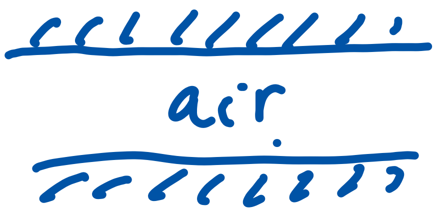
Poissons relation.

Speed of sound?



340 m/s

1666



1D

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\rho(x,t) = \rho_0 + \underbrace{\rho'(x,t)}_{?}$$

$$u(x,t) = \underbrace{\cancel{u_0}}_{=0} + \underbrace{u'(x,t)}_{?}$$

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + \frac{\partial}{\partial x}\left((\rho_0 + \rho')u'\right) = 0$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_0 u') + \underbrace{\frac{\partial}{\partial x}\cancel{(\rho' u')}}_{\text{small}} = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0}$$ approximation.

Similar from momentum conservation:
($\mu = 0$, $g = 0$)

$$\boxed{\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0}$$ approximation.

Energy: $p\rho^{-\gamma} = \text{c.a.s.}$

$t=0 \quad p = p_0 \,,\; \rho = \rho_0 \quad \forall x$

$\Rightarrow \quad p\rho^{-\gamma} = \text{const} = p_0 \rho_0^{-\gamma}$

$$\Rightarrow \frac{\partial}{\partial x}\left(p\rho^{-\gamma}\right) = \frac{\partial p}{\partial x}\rho^{-\gamma} + -\gamma p \rho^{-\gamma-1}\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \frac{\partial p}{\partial x} - \gamma \frac{p}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \frac{\partial p'}{\partial x} - \gamma \frac{p_0 + p'}{\rho_0 + \rho'}\frac{\partial \rho'}{\partial x} = 0$$

$$\Rightarrow \frac{\partial p'}{\partial x} = \gamma \frac{p_0}{\rho_0}\frac{\partial \rho'}{\partial x} \Rightarrow \boxed{\begin{array}{l} \dfrac{\partial p'}{\partial x} = c_0^2 \dfrac{\partial \rho'}{\partial x} \\ c_0^2 \equiv \gamma \dfrac{p_0}{\rho_0} \end{array}}$$

$$\boxed{\begin{array}{l} \Rightarrow \dfrac{\partial \rho'}{\partial t} + \rho_0 \dfrac{\partial u'}{\partial x} = 0 \\ \rho_0 \dfrac{\partial u'}{\partial t} + c_0^2 \dfrac{\partial \rho'}{\partial x} = 0 \end{array}} \quad \begin{array}{l} \dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial x} \end{array}$$

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$$\frac{\partial^2 \rho'}{\partial t^2} + \cancel{\rho_0 \frac{\partial^2 u'}{\partial x \partial t}} - \cancel{\rho_0 \frac{\partial^2 u'}{\partial x \partial t}} - c_0^2 \frac{\partial^2 \rho'}{\partial x^2} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 p'}{\partial t^2} - c_0^2 \frac{\partial^2 p'}{\partial x^2} = 0}$$

Try $p'(x,t) = f(x - c_0 t)$

$$\Rightarrow \frac{\partial p'}{\partial t} = f' \cdot \frac{\partial}{\partial t}(x - c_0 t) = -c_0 f'$$

$$\Rightarrow \frac{\partial^2 p'}{\partial t^2} = -c_0 f'' \cdot -c_0 = c_0^2 f''$$

Similarly: $\frac{\partial^2 p'}{\partial x^2} = f''$

$$\Rightarrow c_0^2 f'' - c_0^2 \cdot f'' = 0$$

$\Rightarrow f(x - c_0 t)$ is a solution!

if $x - c_0 t = \text{const} \Rightarrow f$ is const

$x(t) = c_0 t \Rightarrow f$ is const.

$\Rightarrow$ wave, traveling with speed c_0 to the right.

$$c_0^2 = \gamma \frac{p_0}{\rho_0} \Rightarrow \{c_0^2\} = \frac{N/m^2}{kg/m^3} = \frac{\cancel{kg} \frac{m}{s^2} \frac{1}{m^2}}{\cancel{kg} \frac{1}{m^3}}$$

$$= \frac{m^2}{s^2}$$

c_0 is speed of sound!

$$p = \rho R T \Rightarrow \gamma \frac{p_0}{\rho_0} = \gamma R T_0$$

$$\Rightarrow \boxed{c_0 = \sqrt{\gamma R T_0}}$$

$T_0 \sim$ kinetic energy of molecules

$\Rightarrow \sqrt{T_0} \sim$ speed of molecules
thermal

intuition: sound wave is propagated by molecular collision.

Streamline Invariants.

$\mu = 0 \quad k = 0 \quad q = 0 \quad$ steady.

mass : $\frac{\partial}{\partial x_j}(\rho u_j) = 0$

energy : $\frac{\partial}{\partial x_j}(\rho u_j H) = 0 \qquad H \equiv E + \frac{p}{\rho}$

$$\Rightarrow \underbrace{\frac{\partial}{\partial x_j}(\rho u_j)}_{=0} H + \rho u_j \frac{\partial H}{\partial x_j} = 0 \quad \longleftarrow \text{sum!}$$

$$\Rightarrow \rho u_j \frac{\partial H}{\partial x_j} = 0 \qquad \rho \neq 0 \quad \Rightarrow u_j \frac{\partial H}{\partial x_j} = 0$$

Sum!

steady $\Rightarrow \frac{\partial H}{\partial t} = 0 \Rightarrow \frac{\partial H}{\partial t} + u_j \frac{\partial H}{\partial x_j} = 0$

$\Rightarrow \frac{DH}{Dt} = 0 \Rightarrow \boxed{H = \text{c.a.s.}}$

streamline invariant

Assume ideal gas: $p = \rho R T$ $e = C_v T$

$$H \equiv E + \frac{p}{\rho} = e + \frac{1}{2} u_k u_k + \frac{p}{\rho}$$

$$= C_v T + \frac{1}{2} u_k u_k + RT = C_p T + \frac{1}{2} u_k u_k.$$

$$= C_p T \left\{ 1 + \frac{1}{2} \frac{u_k u_k}{C_p T} \right\}$$

$$= C_p T \left\{ 1 + \frac{1}{2} \frac{\gamma R \cancel{T}}{C_p \cancel{T}} \frac{u_k u_k}{\gamma R T} \right\}$$

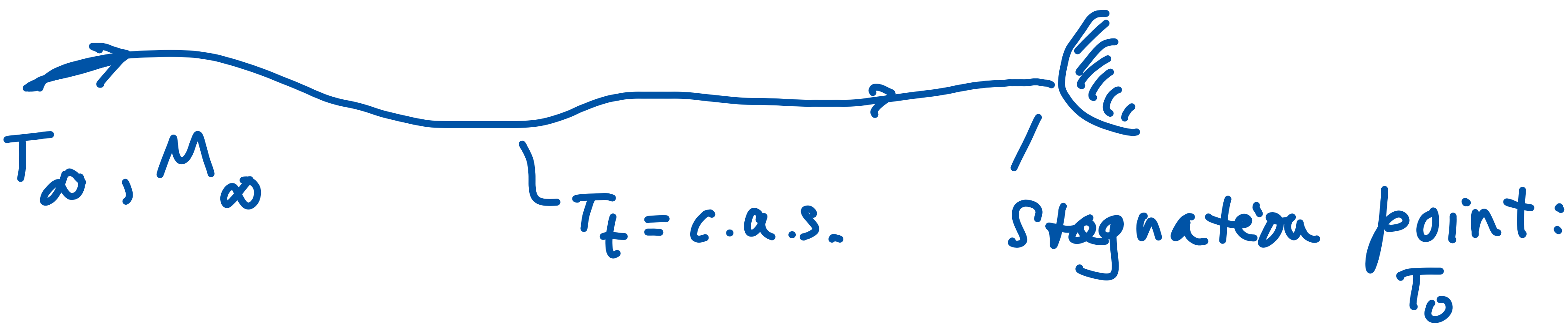
$$\frac{\gamma R}{C_p} = \gamma - 1 \qquad \frac{u_k u_k}{\gamma R T} = \frac{|\vec{u}|^2}{c^2} = \left(\frac{|\vec{u}|}{c} \right)^2$$

$\frac{|\vec{u}|}{c} \equiv M$ Mach number.

$M < 1$ subsonic flow

$M > 1$ supersonic flow

$$\Rightarrow \boxed{\begin{array}{c} c_p T_t = c.a.s. \quad \Rightarrow \quad T_t = c.a.s. \\ \underbrace{T_t}_{\text{total temperature.}} \equiv T\left(1 + \frac{\gamma-1}{2} M^2\right) \end{array}}$$



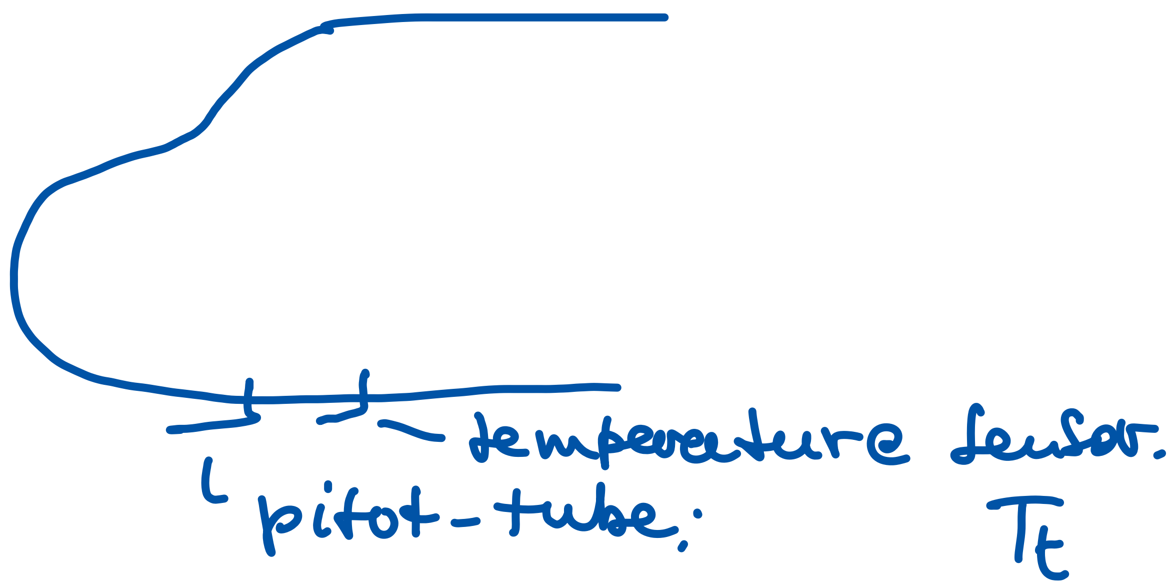
$$\Rightarrow \quad \underbrace{T_\infty\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)}_{\text{total temperature.}} = T_0 (1 + 0) = T_0$$

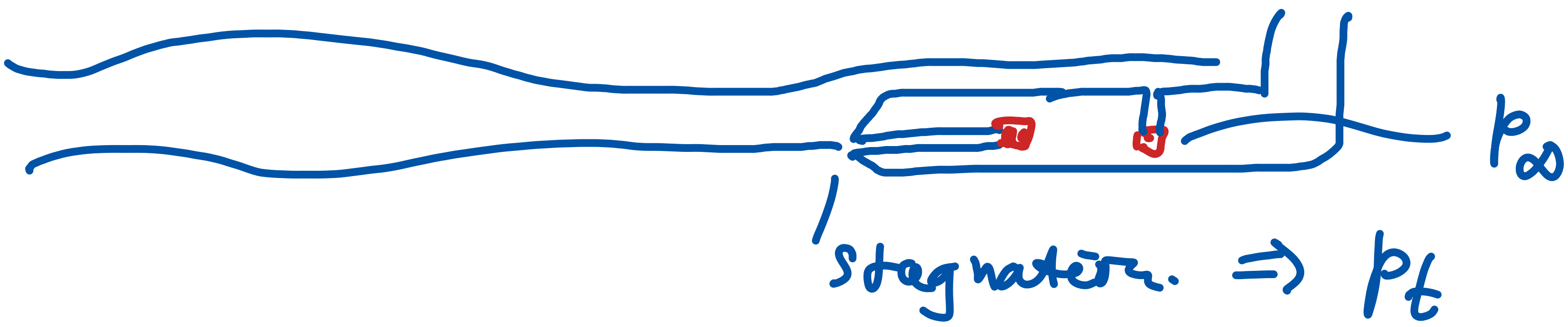
$$T_0 = T_\infty + \underbrace{\frac{\gamma-1}{2} M_\infty^2 T_\infty}_{>0} \quad \Rightarrow \quad T_0 > T_\infty$$

$$p = \rho R T \qquad p \rho^{-\gamma} = c.a.s.$$

$$\Rightarrow \quad p\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} = c.a.s. \equiv p_t$$

$$\rho\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{1}{\gamma-1}} = c.a.s. \equiv \rho_t$$





From T_t, p_t, p_∞ one can compute the velocity far upstream (∞)

How does $p\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} = \text{c.a.s.}$ $\leftarrow \rho \neq \text{const.}$

compare with $p + \frac{1}{2}\rho u_k u_k = \text{c.a.s.}$?

$\uparrow$ Bernoulli (ρ = const

Assume $M^2 \ll 1$

Taylor series: $(1+\epsilon)^\alpha = 1 + \alpha\epsilon + \mathcal{O}(\epsilon^2)$

$$\Rightarrow \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} = 1 + \frac{\gamma}{\gamma-1}\cdot\frac{\gamma-1}{2}M^2 + \mathcal{O}(M^4)$$

$$\Rightarrow p(\quad)^{\frac{\gamma}{\gamma-1}} = p\left(1+\frac{\gamma}{2}\frac{u_k u_k}{\gamma RT}\right) + \cdots$$

$$= p + \frac{\cancel{\gamma}}{2}\frac{u_k u_k}{\cancel{\gamma}\cancel{RT}}\rho\cancel{RT} + \cdots$$

$$\boxed{= p + \frac{1}{2}\rho u_k u_k + O(M^4)}$$

if $M \to 0$: Bernoulli