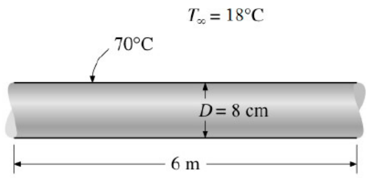
Solutions lecture 4

4.1 Cooling of a hot water pipe

Analysis

In this situation, natural convection will occur. Given data are: $T_s = 70\,°\mathrm{C}$, $T_\infty = 18\,°\mathrm{C}$, length is 6 metres and the diameter 8.0 cm. A sketch is presented below:



Approach

Assumptions

Route to solution

This natural convection problem can be solved by using the following steps:

1. Determine the average temperature.
2. Determine the Grashof number.
3. Determine the Rayleigh number.
4. Choose the right correlation based on geometry and the Rayleigh number.
5. Determine the Nusselt number.
6. Derive the value of h.
7. Substitute in Newton's cooling law.

Elaboration

We start with determining the average temperature:

$$T_f = \frac{T_s + T_\infty}{2} = \frac{70 + 18}{2} = \frac{88}{2} = 44\,°\mathrm{C}$$

For natural convection, the Grasshof number needs to be calculated:

$$\mathrm{Gr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

where $9.81\,\mathrm{m\,s^{-2}}$ is the gravitational constant, β is the thermal expansion coefficient:

$$\beta = \frac{2}{T_s + T_\infty} = \frac{2}{342.15 + 291.15} = 3.15 \times 10^{-3}\,\mathrm{K^{-1}}$$

Note that the temperatures need to be substituted in **Kelvin**, not Celsius. Furthermore, D is the aforementioned diameter, and ν is the kinematic viscosity. The value for this is taken at 45 °C, to be $1.750 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. Substituting these values:

$$\mathrm{Gr} = \frac{g\beta(T_s - T_\infty)D^3}{\nu^2} = \frac{9.81 \cdot (3.15 \cdot 10^{-3}) \cdot (70 - 18) \cdot 0.08^3}{(1.750 \cdot 10^{-5})^2} = 2.67 \cdot 10^6[-]$$

At 45 °C, the Prandtl number is $\mathrm{Pr} = 0.7241[-]$. With this, the Rayleigh number can be determined:

$$\mathrm{Ra} = \mathrm{Gr} \cdot \mathrm{Pr} = 2.67 \cdot 10^6 \cdot 0.7241 = 1.94 \cdot 10^6[-]$$

Now, since the pipe is horizontal and cylindrical, we can use the following correlation:

$$\mathrm{Nu} = \left(0.6 + \frac{0.387 \cdot \mathrm{Ra}_D^{\frac{1}{6}}}{\left(1 + \left(\frac{0.559}{\mathrm{Pr}}\right)^{\frac{9}{16}}\right)^{\frac{8}{27}}} \right)^2$$

Substitution of all variables:

$$\mathrm{Nu} = \left(0.6 + \frac{0.387 \cdot (1.94 \cdot 10^6)^{\frac{1}{6}}}{\left(1 + \left(\frac{0.559}{0.7241}\right)^{\frac{9}{16}}\right)^{\frac{8}{27}}} \right)^2 = 17.58[-]$$

The Nusselt number is defined as:

$$\mathrm{Nu} = \frac{hD}{k}$$

At 45 °C, the value of k is $k = 0.026\,99\,\mathrm{W\,m^{-1}\,K^{-1}}$. Rewriting and substituting gives:

$$h = \frac{\mathrm{Nu}_D k}{D} = \frac{17.58 \cdot 0.02699}{0.08} = 5.93\,\mathrm{W\,m^{-2}\,K^{-1}}$$

We can now substitute all values in Newton's cooling law:

$$\dot{Q} = hA\Delta T$$

where the area is $A = \pi DL = \pi \cdot 0.08 \cdot 6 = 1.51\,\mathrm{m^2}$

$$\dot{Q} = h \cdot A(T_s - T\infty) = 5.39 \cdot 1.51 \cdot (70 - 18) = 466\,\mathrm{W}$$

Evaluation

Check your answer:

- Does the answer have the correct dimensions?
- Is the answer in the right order of magnitude?
- Did you answer all the questions that were asked?

4.2 Convection of heat from a coffee machine

Analysis

This plate loses 90 W of heat, of which is 52.4% due to radiation and the other 47.6% due to natural convection. The situation is sketched below.

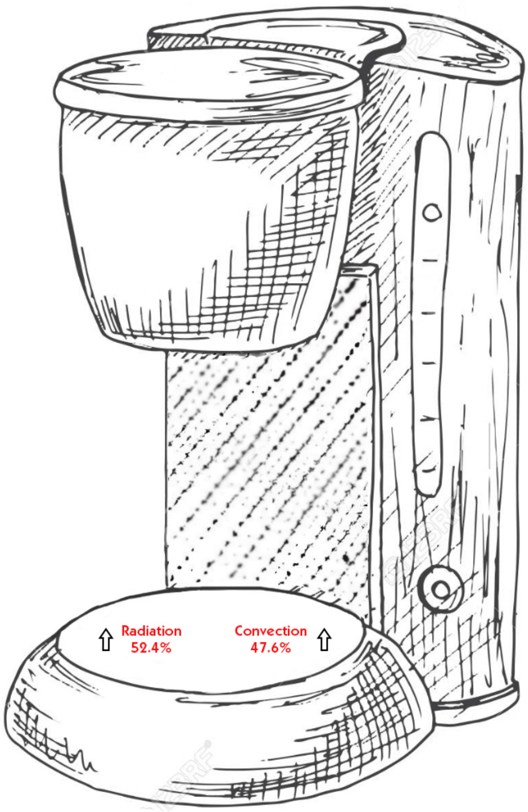


Figure 4.1: Heat loss of a coffee machine plate

This gives a total of $0.476 \cdot 90 = 42.84$ W of natural convection. For natural convection, the following relation is valid:

$$\dot{Q} = hA\Delta T$$

However, h is dependent on the average temperature, which is subsequently dependent on the temperature T_s of the plate, considering the fact that his is codependent for the film temperature of which the value will be based

$$T_f = \frac{T_s + T_\infty}{2}$$

We need to determine the temperature by means of iteration. The maximum temperature of the plate is given, 250 °C. A first initial guess for the temperature of the plate is 180 °C.

Approach

Assumptions

Route to solution

1. Determine the average temperature

2. Determine the Grashof number
3. Determine the Rayleigh number
4. Choose the right correlation based on geometry and the Rayleigh number
5. Determine the Nusselt number
6. Derive the value of h
7. Substitute in Newton's cooling law
8. If necessary, iterate

Elaboration

The average temperature is:

$$T_f = \frac{T_s + T_\infty}{2} = \frac{180 + 20}{2} = 100\,^\circ\mathrm{C}$$

The Grashof number is defined as:

$$\mathrm{Gr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

in which $g = 9.81\,\mathrm{m\,s^{-2}}$ is the gravitational constant, β is the thermal expansion coefficient:

$$\beta = \frac{1}{T_f} = \frac{1}{100 + 273} = 2.68 \times 10^{-3}\,\mathrm{K^{-1}}$$

Note that the temperatures in β has the unit of Kelvin, not Celsius. Furthermore L_c is the characteristic length, which, for a horizontal cylindrical flat plate with diameter D, is

$$L_c = \frac{A_s}{p} = \frac{\frac{\pi}{4}D^2}{\pi D} = \frac{D}{4} = 0.04\mathrm{m}$$

At 100 °C, air has the following properties: $k = 0.030\,95\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $\nu = 2.306 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. Substituting all these values give:

$$\mathrm{Gr} = \frac{9.81 \cdot (5.38 \cdot 10^{-3} \cdot (180 - 20) \cdot 0.04^3)}{(2.306 \cdot 10^{-5})^2} = 5.063 \cdot 10^5[-]$$

At 100 °C, the Prandtl number is $\mathrm{Pr} = 0.7111[-]$. With this, the Rayleigh number can be determined to be:

$$\mathrm{Ra} = \mathrm{Gr} \cdot \mathrm{Pr} = 5.063 \cdot 10^5 \cdot 0.7111 = 3.60 \cdot 10^5[-]$$

Now, the situation concerns an upper part of a horizontal hot flat plate with $10^4 < \mathrm{Ra} < 10^7$. For this, the Nusselt number is

$$\mathrm{Nu} = 0.54\mathrm{Ra}^{\frac{1}{4}} = 0.54 \cdot (3.60 \cdot 10^5)^{\frac{1}{4}} = 13.2[-]$$

The Nusselt number is defined as

$$\mathrm{Nu} = \frac{hL_c}{k}$$

Substitution of all variables gives

$$h = \frac{Nu \cdot k}{L_c} = \frac{13.2 \cdot 0.03095}{0.04} = 10.2\,\mathrm{W\,m^{-2}\,K^{-1}}$$

Substitution of this result into Newton's cooling law, where the surface is $A = \frac{\pi}{4}D^2$:

$$\dot{Q} = hA\Delta T = 12.1 \cdot \frac{\pi}{4} \cdot 0.16^2 \cdot (180 - 20) = 32.8\,\mathrm{W}$$

As stated earlier, this 38.9 W is not in accordance with the 42.84 W we calculated earlier. This means that the plate will be hotter than 180 °C. This means we need to reiterate.

Let's say that the plate is 220 °C. The average temperature is then:

$$T_f = \frac{T_s + T_\infty}{2} = \frac{220 + 20}{2} = 120\,^\circ\mathrm{C}$$

The Grashof number is defined as:

$$\mathrm{Gr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

in which $g = 9.81\,\mathrm{m\,s^{-2}}$ is the gravitational constant, β is the thermal expansion coefficient:

$$\beta = \frac{1}{T_f} = \frac{1}{120 + 273} = 2.54 \times 10^{-3}\,\mathrm{K^{-1}}$$

At 120 °C, air has the following properties: $k = 0.032\,35\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $\nu = 2.522 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. Substituting all these values give:

$$\mathrm{Gr} = \frac{9.81 \cdot (2.54 \cdot 10^{-3} \cdot (220 - 20) \cdot 0.04^3)}{(2.522 \cdot 10^{-5})^2} = 5.014 \cdot 10^5 [-]$$

At 120 °C, the Prandtl number is $\mathrm{Pr} = 0.7073[-]$. With this, the Rayleigh number can be determined to be:

$$\mathrm{Ra} = \mathrm{Gr} \cdot \mathrm{Pr} = 5.014 \cdot 10^5 \cdot 0.7073 = 3.55 \cdot 10^5 [-]$$

Now, the situation concerns an upper part of a horizontal hot flat plate with $10^4 < \mathrm{Ra} < 10^7$. For this, the Nusselt number is

$$\mathrm{Nu} = 0.54\mathrm{Ra}^{\frac{1}{4}} = 0.54 \cdot (3.55 \cdot 10^5)^{\frac{1}{4}} = 13.2[-]$$

The Nusselt number is defined as

$$\mathrm{Nu} = \frac{hL_c}{k}$$

Substitution of all variables gives

$$h = \frac{Nu \cdot k}{L_c} = \frac{13.2 \cdot 0.03235}{0.04} = 10.7\,\mathrm{W\,m^{-2}\,K^{-1}}$$

Substitution of this result into Newton's cooling law:

$$\dot{Q} = hA\Delta T = 10.7 \cdot \frac{\pi}{4} \cdot 0.16^2 \cdot (220 - 20) = 42.9\,\mathrm{W}$$

This value is quite in accordance with the earlier presented 42.84 W. This indicates that the plate will be around 220 °C.

Evaluation

Check your answer:

- Does the answer have the correct dimensions?
- Is the answer in the right order of magnitude?
- Did you answer all the questions that were asked?

4.3 Heat convection parameters

For forced convection, it was provided in lecture 3, that Nu is a function of Re and Pr:

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr})$$
$$\frac{hL_c}{k} = f\left(\frac{\rho U L_c}{\mu}, \mathrm{Pr}\right)$$

This means that the group of parameters in which the heat transfer coefficient h is processed, is a function of ρ, Y, L_c, μ and Pr. We can rewrite this to:

$$h = f(\rho, U, L_c, \mu, \mathrm{Pr}, k)$$

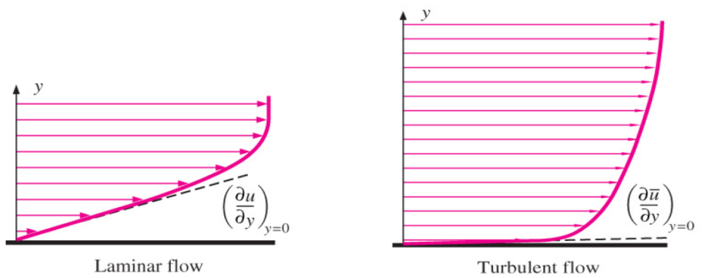
These variables will be discussed from here:

- Density ρ: Combined with the velocity U, the density forms the impulse of the fluid. The impulse can be seen as an indicator for the impact force of the oncoming particles. Is this value high, the fluid particles just above the surface will be hardly affected by the surface they are flowing above and by the particles close to the surface area. The particles close to the surface will stagnate by the surface and by the viscosity and will affect particles further away from the surface.

  Having a high density will result in a high impulse and therefore the boundary layer will grow less fast and remain thinner. A small boundary layer means a small distance for the heat to cover and subsequently an easier heat transfer. h will increase if the density increases.

  At a certain point the impulse will be that much larger than the viscous effects that the laminar flow will transform into turbulent flow. In that case the h will increase drastically as the gradient close to the surface will be a lot larger.

  For a turbulent flow, looking very close to the surface a very small boundary layer will appear, meaning a large value of h. It can be concluded that a turbulent flow results in a larger h.



- Velocity U : Just like the density, the velocity is determining the impulse (ρU). So all statements for above are valid as well if ρ is replaced by U.

- Characteristic length L_c: the larger the surface area, the more the boundary layer has opportunity to grow. At the end of the characteristic length, the boundary layer will be thick with locally low h values, especially compared with the leading edge. The larger the L_c, the smaller the average value of h.

- Dynamic viscosity μ: the dynamic viscosity ensures the no slip boundary condition at the surface is felt in the fluid above the surface. The higher the viscosity, the larger the influence and the more thick the boundary layer will grow. The viscosity has an inverted effect compared to the density and the velocity. This is also quite obvious if the definition of the Reynolds number is considered.

  If the dynamic viscosity is high enough, the impulse will be dominated by the viscous effects and this will result in a laminar flow. The Reynolds number will be under the critical value for that specific geometry.

- Prandtl number Pr: the Prandtl number indicates how, given a certain velocity boundary layer (as well present without thermal effects), the thermal boundary layer will look like as a temperature difference is present between the surface and the free flow. A thermal boundary layer larger than (Pr ¡ 1), thinner than (Pr ¿ 1) or as thick as (Pr = 1) the velocity boundary layer.

  If in a flow the Prandtl number is enhanced, without altering velocity profile, the thermal boundary layer will get thinner. Using the statement at the density variable, the value of h will increase.

- Thermal conductivity k: If a fluid conducts heat better, the value of k will be higher. Convection can only be present if the particles are able to conduct heat. Convection can be stated as conduction being enhanced by the continuous supply of fresh particles and the take away of old particles. Therefore, if you calculate the convective heat transfer, the conductive heat transfer is already included and there is no need to calculate the conductive heat transfer independently.

For natural convection, the following correlations are valid:

$$\mathrm{Nu} = f(\mathrm{Ra}) = g(\mathrm{Gr}, \mathrm{Pr})$$

$$\frac{hL_c}{k} = f(g, \beta, (T_s - T_\infty), L_c, \nu, \mathrm{Pr}, k)$$

Giving for h:

$$h = f(g, \beta, (T_s - T_\infty), L_c, \nu, \mathrm{Pr}, k)$$

The gravitational constant g is always the same, and therefore has no influence on h. L_c, Pr and k have the same effect as in the case of forced convection. The temperature difference can be seen as the driver behind the phenomenon and therefore fulfills the same role as U in the case of forced convection. A higher value will result in a higher value for h.

- Thermal expansion coefficient β: This coefficient is larger the more a fluid expands with a certain increase in temperature. The larger the expansion, the more the decrease in density and therefore a larger flow velocity and a higher value for h.
- Kinematic viscosity ν: the kinematic viscosity is the dynamic viscosity divided by the density. Therefore it has the same effect as the effects of dynamic viscosity and density.

A last important remark is the influence of the geometry on the value of h. The geometry determines how the flow pattern will look like and which correlation is valid. Also, the roughness of the surface plays a large role: the rougher the surface the quicker the transition from laminar to turbulent flow.

4.4 Light bulb temperature

Analysis

Consider a 25 W lightbulb with a light-efficiency of 10 %. The lightbulb has a diameter of 8.0 cm, and an outside temperature of 25 °C. When assuming all heat is lost due to natural convection, determine the surface temperature of the lightbulb.

Approach

Assumptions

- Only natural convection
- Light bulb can be modelled as a sphere

Route to solution

1. Determine the average temperature
2. Determine the Grashof number
3. Determine the Rayleigh number
4. Choose the right correlation based on geometry and the Rayleigh number
5. Determine the Nusselt number
6. Derive the value of h
7. Substitute in Newton's cooling law
8. If necessary, iterate

Elaboration

We take an initial guess for the surface temperature, $T_s = 100$ °C. The average temperature is:

$$T_f = \frac{T_s + T_\infty}{2} = \frac{100 + 25}{2} = 62.5\,°\mathrm{C}$$

The Grashof number is defined as:

$$\mathrm{Gr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

in which $g = 9.81\,\mathrm{m\,s^{-2}}$ is the gravitational constant, β is the thermal expansion coefficient:

$$\beta = \frac{2}{T_f} = \frac{2}{298 + 373} = 2.98 \times 10^{-3}\,\mathrm{K^{-1}}$$

Note that the temperatures in β has the unit of Kelvin, not Celsius.

At 60 °C, air has the following properties: $k = 0.028\,08\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $\nu = 1.896 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. Substituting all these values give:

$$\mathrm{Gr} = \frac{9.81 \cdot (2.98 \cdot 10^{-3} \cdot (100 - 25) \cdot 0.08^3}{(1.896 \cdot 10^{-5})^2} = 3.12 \cdot 10^6[-]$$

At 60 °C, the Prandtl number is $\mathrm{Pr} = 0.7202[-]$. With this, the Rayleigh number can be determined to be:

$$\mathrm{Ra} = \mathrm{Gr} \cdot \mathrm{Pr} = 3.12 \cdot 10^6 \cdot 0.7202 = 2.25 \cdot 10^6[-]$$

Now, the situation concerns a sphere with $\mathrm{Ra}_D \leq 10^{11}$, where $Pr \geq 0.7$. For this, the Nusselt number is:

$$\mathrm{Nu} = 2 + \frac{0.589\mathrm{Ra}^{1/4}}{[1 + (0.469/\mathrm{Pr})^{9/16}]^{4/9}} = 19.6$$

The Nusselt number is defined as

$$\mathrm{Nu} = \frac{hL_c}{k}[-]$$

Substitution of all variables gives

$$h = \frac{Nu \cdot k}{D} = \frac{19.6 \cdot 0.02808}{0.08} = 6.89\,\mathrm{W\,m^{-2}\,K^{-1}}$$

Substitution of this result into Newton's cooling law, where the surface is $A = \pi D^2$:

$$\dot{Q} = hA\Delta T = 6.89 \cdot \pi \cdot D^2 \cdot (100 - 25) = 10.4W$$

As stated earlier, this 10.4 W is not in accordance with the 22.5 W we need to have earlier. This means that the surface is hotter than 100°C. Reverse calculation shows a ΔT of 162 °C. After some iterations, a value of $T_s = 168°C$ is found. With this value, the following results are obtained:

$$T_f = 96.5\,^{\circ}\mathrm{C} \approx 100\,^{\circ}\mathrm{C}$$

$$\beta = 2.7 \times 10^{-3}\,\mathrm{K^{-1}}$$

At 100 °C, air has the following properties: $k = 0.030\,95\,\mathrm{W\,m^{-1}\,K^{-1}}$, $\nu = 2.306 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$ and $\mathrm{Pr} = 0.7202[-]$.

$$\mathrm{Gr} = 3.66 \cdot 10^6[-]$$

$$\mathrm{Ra} = 2.60 \cdot 10^6[-]$$

$$\mathrm{Nu} = 20.25[-]$$

$$h = 7.84\,\mathrm{W\,m^{-2}\,K^{-1}}$$

$$\dot{Q} = 22.53\,\mathrm{W}$$

$$T = 167.8\,^{\circ}\mathrm{C}$$

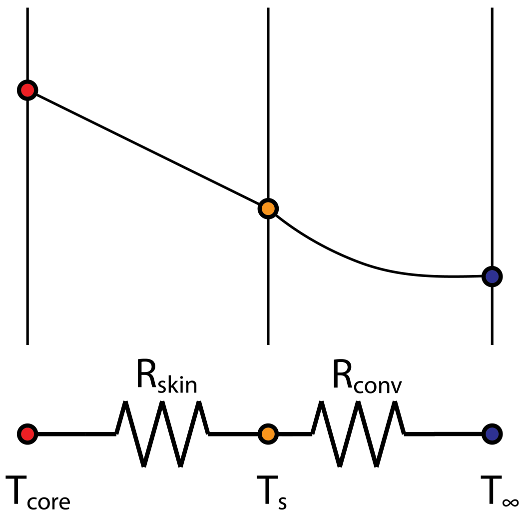
4.5 Cooling of a human head - Hand in

a) The skin temperature and heat loss must be calculated. The point of here is to find the state of equilibrium, in which the temperature of the main surface (T_s) is such that the heat flow through conduction in the skull is equal to the heat flow through convection around the skull. The law of conservation of energy is then fulfilled: one continuous heat flow 'moves' first via conduction and then via convection and on that route there can be no energy is lost or generated.

The procedure is therefore to choose a T_s and to calculate the heat flows through conduction and convection. If they do not match, then the calculation will have to be repeated with a newly estimated T_s in such a way that the largest heat flow of both will become smaller (and automatically the smaller larger).

This explanation of the approach should be clearly reflected in the elaborations and is at least as important as the elaboration itself. The exact way in which the sum is worked out may differ. A fairly mathematical criterion is formulated below. It is also conceivable (and perhaps more transparent) that the two heat flows are determined completely independently of each other and then compared. It is also possible to work with heat resistances and then it is possible, for example, that not both partial heat flows are compared, but, for example, the total heat flow with one of the two partial heat flows (by convection or conduction, one of the two is sufficient).

In short, several routes are possible and they are usually not necessarily right or wrong; what is important is that it is clearly explained what exactly is happening and what the thinking behind it is.



The figure above gives an idea of what the thermal resistance network looks like.
Important is that the rate of heat transfer through each resistor is equal to each other:

$$\dot{Q}_{\text{skin}} = \dot{Q}_{\text{conv}}$$

$$\frac{T_{\text{core}} - T_{\text{s}}}{R_{\text{skin}}} = \frac{T_{\text{s}} - T_{\infty}}{R_{\text{conv}}}$$

$$\frac{T_{\text{core}} - T_{\text{s}}}{\frac{D_2 - D_1}{2\pi k_{\text{skin}} D_1 D_2}} = \frac{T_{\text{s}} - T_{\infty}}{\frac{1}{h\pi D_2^2}}$$

An assumption for T_{s} should be made in order to determine h.

The final surface temperature should close to T_{s} =35.59 °C.

The average fluid properties at a temperature of 17.795 °C are:

$$\rho = 1.2133 \text{ kg/m}^3$$

$$\nu = 1.4957 \cdot 10^{-5} \text{ m}^2/\text{s}$$

$$\text{Pr} = 0.7193$$

$$k = 0.025 \text{ W/mK}$$

$$\beta = \frac{1}{17.795 + 273.15} = 0.0034 \text{ K}^{-1}$$

With this given, the Grashof number Gr can be determined, where the characteristic length equals the diameter of the head.

$$\text{Gr} = \frac{g \cdot \beta \cdot (T_\text{s} - T_\infty) \cdot D_2^3}{\nu^2} = \frac{9.81 \cdot 0.0034 \cdot (35.59 - 0) \cdot 0.145^3}{(1.4957 \cdot 10^{-5})^2} = 1.6353 \cdot 10^7$$

And the Rayleigh number Ra:

$$\text{Ra} = \text{Gr} \cdot \text{Pr} = 1.6353 \cdot 10^7 \cdot 0.7193 = 1.1763 \cdot 10^7$$

As the Rayleigh number Ra has been determined, an applicable correlation for the Nusselt number Nu can be used to calculate this number:

$$\text{Nu} = 2 + \frac{0.589 \cdot \text{Ra}^{1/4}}{\left[1 + (0.469/\text{Pr})^{9/16}\right]^{4/9}} = 2 + \frac{0.589 \cdot \left(1.6353 \cdot 10^7\right)^{1/4}}{\left[1 + (0.469/0.7193)^{9/16}\right]^{4/9}} = 28.65$$

Having determined the Nusselt number Nu, the heat transfer coefficient h can be determined:

$$h = \frac{\text{Nu}k}{D_2} = \frac{28.65 \cdot 0.025}{0.145} = 4.94 \text{ W/m}^2\text{K}$$

Having determined the heat transfer coefficient, the rate of heat transfer conducted through the skin and due to convection can be calculated:

$$\dot{Q}_\text{skin} = \frac{T_\text{core} - T_\text{s}}{\frac{D_2 - D_1}{2\pi k_\text{skin} D_1 D_2}} = \frac{37 - 35.59}{\frac{0.145 - 0.14}{2\pi \cdot 0.3 \cdot 0.14 \cdot 0.145}} = 10.79 \text{ W}$$

$$\dot{Q}_\text{conv} = \frac{35.59 - 0}{\frac{1}{3.94 \cdot \pi \cdot 0.145^2}} = 11.61 \text{ W}$$

Finally, it must be checked whether at this temperature found the h would not be appreciably different than at the assumed temperature. If so, the whole approach must be repeated with a new estimate. If not, the answer can be regarded as converged. The latter is the case here, but it is not further demonstrated here. This is desirable in submitted work.