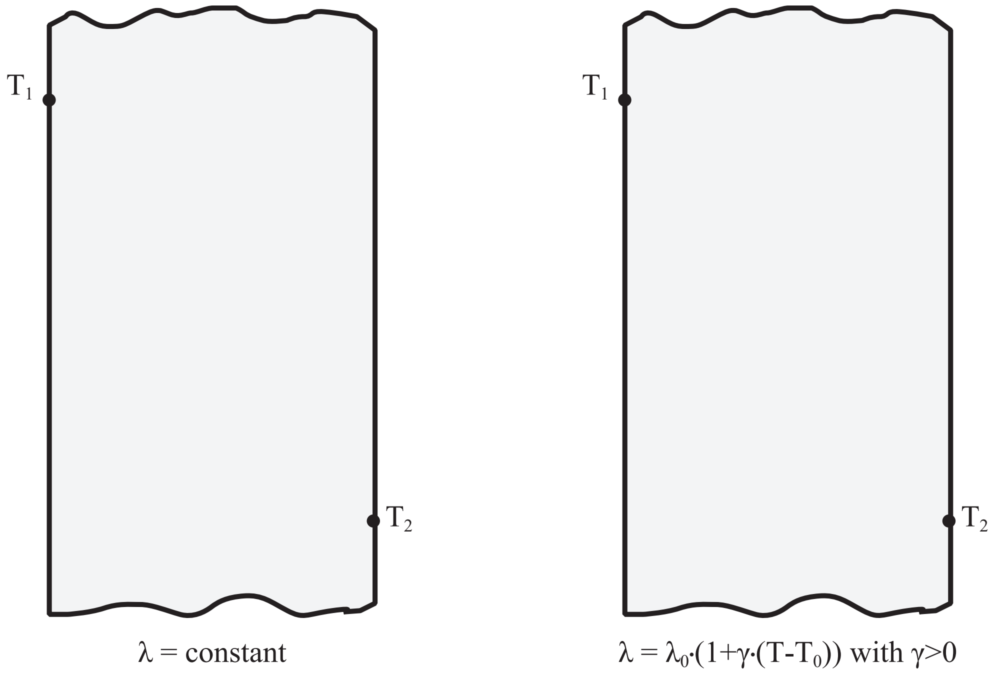
SECTION II

Conduction solutions

Exercise II.1: (Temperature profiles in planar walls ⋆)

Both sides of a planar wall are heated to a constant temperature of T_1 and T_2, where $T_1 > T_2$.



Tasks:

a) Sketch the steady-state temperature profile for a constant thermal conductivity.

b) Sketch the steady-state temperature profile for the conductivity being temperature-dependent:

$$\lambda = \lambda_0 (1 + \gamma (T - T_0)), \text{ with } \gamma > 0,$$

where λ_0 is the thermal conductivity at the temperature T_0.

Solution II.1: (Temperature profiles in planar walls ⋆)

Task a)

In the context of planar wall systems without heat generation, the heat transfer rate $\dot{Q}$ remains uniform across the entirety of the wall, irrespective of its position within the structure. Recall Fourier's law:

$$\dot{Q} = \lambda A \frac{\partial T}{\partial x}, \tag{II.1.1}$$

which indicates that in the scenario of a planar wall featuring a consistent cross-sectional area A and unchanging thermal conductivity λ, the temperature gradient $\frac{\partial T}{\partial x}$ must be constant. As a result, a continuous linear temperature profile should be drawn.

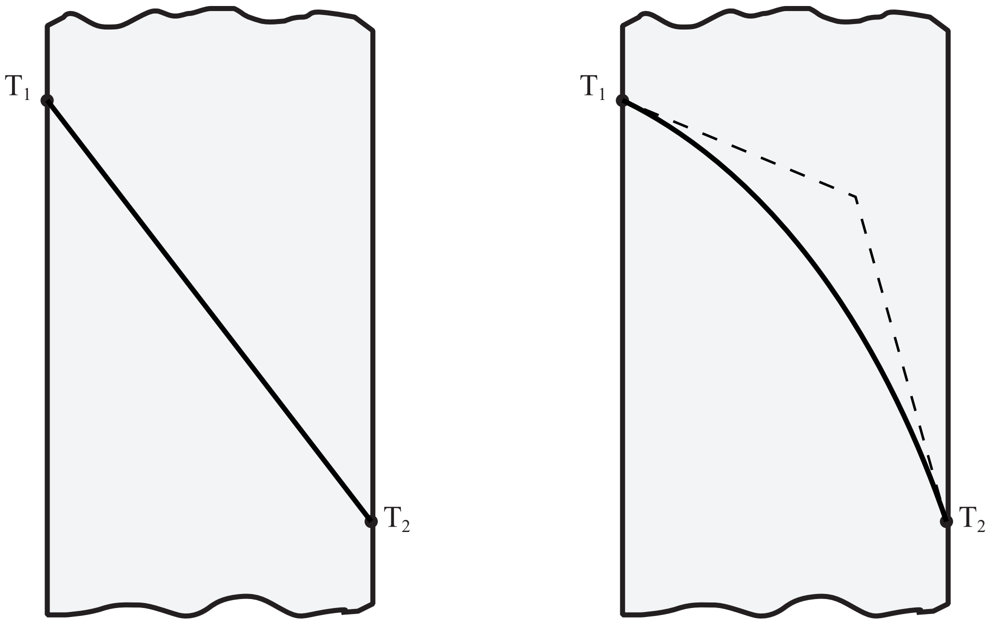
Task b)

In the context of this task, when traversing from the hot to the cold side, the temperature within the planar wall diminishes. Recall the equation for thermal conductivity λ:

$$\lambda = \lambda_0(1 + \gamma(T - T_0)) \text{ with } \gamma > 0. \tag{II.1.2}$$

This function states that as the temperature decreases, the thermal conductivity also diminishes. Consequently, when progressing from the hot side to the cold side, the temperature gradient $\frac{\partial T}{\partial x}$ should increase. Consequently, a temperature profile exhibiting an elevating gradient must be depicted.

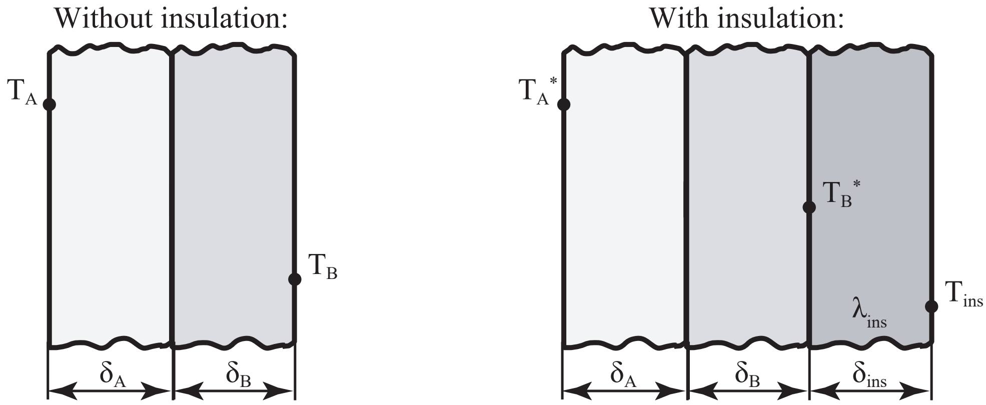
Conclusion



λ = constant　　　　$\lambda = \lambda_0 \cdot (1+\gamma \cdot (T-T_0))$ with $\gamma>0$

Exercise II.2: (Onion layer principle ⋆⋆)

A solar panel manufacturer makes use of heat processing applications that include preheating, curing, heat treating, and finishing. The manufacturer has an old and a new type of industrial oven. The newer one has an additional insulation layer.



Given parameters:

Old oven:

- Surface temperature of layer A: $T_A = 260$ °C
- Surface temperature of layer B: $T_B = 32$ °C
- Thickness of layer A: $\delta_A = 125$ mm
- Thickness of layer B: $\delta_B = 200$ mm

New oven:

- Surface temperature of layer A: $T_A^* = 305$ °C
- Surface temperature of layer B: $T_B^* = 219$ °C
- Surface temperature of insulation layer: $T_{ins} = 27$ °C
- Thickness of layer A: $\delta_A = 125$ mm
- Thickness of layer B: $\delta_B = 200$ mm
- Thickness of insulation layer: $\delta_{ins} = 25$ mm
- Thermal conductivity of insulation layer: $\lambda_{ins} = 0.075$ W/mK

Hint:

- Assume steady-state conditions.

Tasks:

a) Determine the heat flux per unit area $\dot{q}''$ for the situations without and with the insulating layer.

Solution II.2: (Onion layer principle ★★)

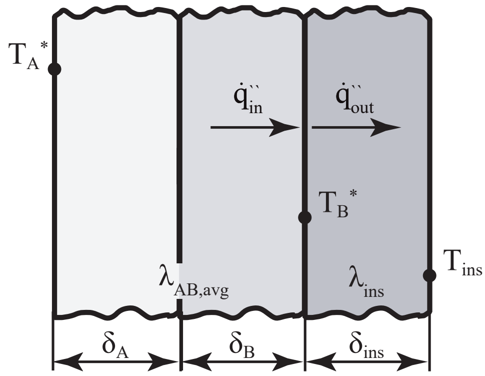
Task a)

The heat flux is determined by setting up an energy balance at an interface within both systems. Because the thermal conductivity of layers A and B are unknown, their average value needs to be determined. This is done by determining the heat flux for the insulated case first and rewriting this expression.

System with insulation:

1 **Setting up the balance:**

The problem is recognized as a multi-layer wall problem without any heat generation. Consequently, the heat transfer rate remains consistent across every layer of the wall, regardless of its position. In essence, what enters the system exits it in a balanced manner.



The energy balance at the interface between layer B and the insulation layer reads:

$$0 = \underbrace{\dot{q}''_{\text{in}} - \dot{q}''_{\text{out}}}_{\substack{\text{Net rate of}\\ \text{diffusion}}}. \tag{II.2.1}$$

2 **Defining the elements within the balance:**

The outgoing heat flux can be calculated using Fourier's law. The temperature gradient in a plane wall without any generation of heat exhibits a linear characteristic and, hence can be ascertained from the temperature variance across the insulation layer throughout its thickness:

$$\begin{aligned} \dot{q}''_{\text{out}} &= -\lambda_{\text{ins}} \left(\frac{\partial T}{\partial x}\right)_{\text{ins}} \\ &= -\lambda_{\text{ins}} \frac{T_{\text{ins}} - T_{\text{B}}^{*}}{\delta_{\text{ins}}} \\ &= -0.075 \left(\frac{\text{W}}{\text{mK}}\right) \cdot \frac{(27 - 219)\ (\text{K})}{0.025\ (\text{m})} = 576 \left(\frac{\text{W}}{\text{m}^2}\right). \end{aligned} \tag{II.2.2}$$

The incoming heat flux is computed employing Fourier's law. Given the unknown thermal conductivity of layers A and B, alongside known surface temperatures, the heat flux can be represented in relation to their average thermal conductivity, multiplied by the temperature difference over the total thickness of both layers:

$$\dot{q}''_{\text{in}} = -\lambda_{\text{AB,avg}} \left(\frac{\partial T}{\partial x}\right)_{\text{AB,avg}} = -\lambda_{\text{AB,avg}} \cdot \frac{T_B^* - T_A^*}{\delta_A + \delta_B}. \tag{II.2.3}$$

3 Inserting and rearranging:

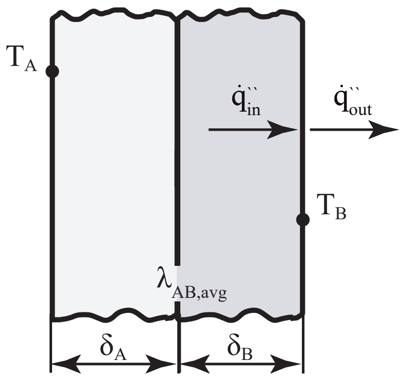
$$\lambda_{\text{AB,avg}} = \dot{q}''_{\text{out}} \frac{(\delta_A + \delta_B)}{T_A^* - T_B^*} = 576 \left(\frac{\text{W}}{\text{m}^2}\right) \cdot \frac{(0.125 + 0.200)\ (\text{m})}{(305 - 219)\ (\text{K})} = 2.18 \left(\frac{\text{W}}{\text{mK}}\right). \tag{II.2.4}$$

The combined average thermal conductivity of layers A and B has been calculated to be 2.18 $\frac{\text{W}}{\text{mK}}$, as this parameter is necessary to determine the heat flux for the system without insulation.

System without insulation:

1 Setting up the balance:

The problem is recognized as a multi-layer wall problem without any heat generation.



The energy balance at the interface between layer B and the ambient reads:

$$0 = \underbrace{\dot{q}''_{\text{in}} - \dot{q}''_{\text{out}}}_{\text{Net rate of diffusion}}. \tag{II.2.5}$$

2 Defining the elements within the balance:

The heat flux through each layer remains constant for a plane wall. Therefore, the incoming heat flux is expressed using Fourier's law with the average thermal conductivity of layers A and B:

$$\begin{aligned}\dot{q}''_{\mathrm{in}} &= -\lambda_{\mathrm{AB,avg}} \left(\frac{\partial T}{\partial x}\right)_{\mathrm{AB,avg}} \\ &= -\lambda_{\mathrm{AB,avg}} \frac{T_{\mathrm{B}} - T_{\mathrm{A}}}{(\delta_{\mathrm{A}} + \delta_{\mathrm{B}})} \\ &= -2.18 \left(\frac{\mathrm{W}}{\mathrm{mK}}\right) \cdot \frac{(32 - 260)\ (\mathrm{K})}{(0.125 + 0.200)\ (\mathrm{m})} = 1530 \left(\frac{\mathrm{W}}{\mathrm{m}^2}\right).\end{aligned} \tag{II.2.6}$$

3 **Inserting and rearranging:**

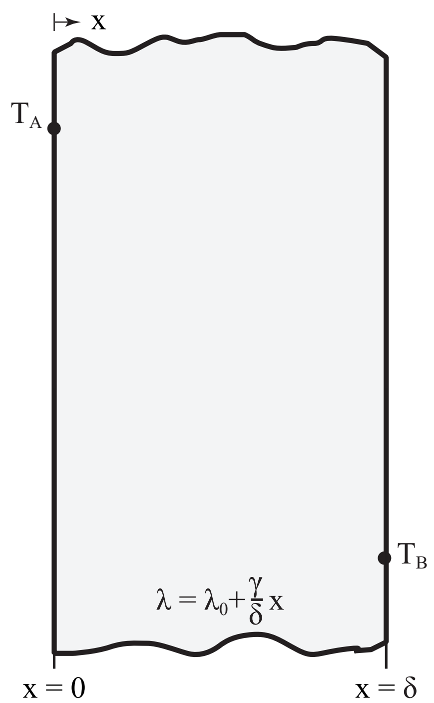
$$\dot{q}''_{\mathrm{in}} = 1530 \left(\frac{\mathrm{W}}{\mathrm{m}^2}\right). \tag{II.2.7}$$

Conclusion

The heat flux per unit area for the system with insulation is 576 $\frac{\mathrm{W}}{\mathrm{m}^2}$, and for the system without insulation 1530 $\frac{\mathrm{W}}{\mathrm{m}^2}$.

Exercise II.3: (Heat conduction equation ⋆⋆⋆)

Both sides of a planar wall are heated to a constant temperature of T_A and T_B, respectively; where $T_A > T_B$.



Given parameters:

- Thermal conductivity as a function of the position in the wall:

$$\lambda(x) = \lambda_0 + \frac{\gamma}{\delta} \cdot x.$$

Hints:

- Assume one-dimensional steady-state heat transfer in x-direction.

Tasks:

a) Derive the function of the temperature profile inside the plane wall.

b) Sketch the temperature profile inside the plane wall in the x-direction.

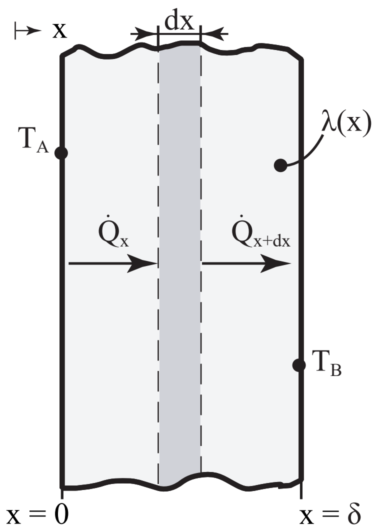
Solution II.3: (Heat conduction equation ⋆⋆⋆)

Task a)

The heat conduction equation for a system is derived by setting up the energy balance of an infinitesimal element within the respective domain.

1 **Setting up the balance:**

The provided problem constitutes a steady-state plane wall scenario without heat generation. Consequently, it exhibits the characteristic of maintaining a uniform heat transfer rate across the entire wall.



The one-dimensional steady-state energy balance for an infinitesimal element is described as:

$$0 = \underbrace{\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}}_{\text{Net rate of diffusion}} . \tag{II.3.1}$$

2 **Defining the elements within the balance:**

Fourier's law is used to describe the ingoing flux:

$$\dot{Q}_{\mathrm{x}} = -\lambda(x) A \frac{\partial T}{\partial x} . \tag{II.3.2}$$

For an infinitesimal element, the outgoing conductive heat flux is approximated by the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \cdot dx \\ &= -\lambda(x) A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-\lambda(x) A \frac{\partial T}{\partial x} \right) \cdot dx . \end{aligned} \tag{II.3.3}$$

3 **Inserting and rearranging:**

$$0 = \frac{\partial}{\partial x}\left(\left[\lambda_0 + \frac{\gamma}{\delta}x\right] \cdot \frac{\partial T}{\partial x}\right). \tag{II.3.4}$$

4 Defining the boundary and/or initial conditions:

To solve the given differential equation, two boundary conditions are required since the temperature has been differentiated twice to x. The following two boundary conditions are known from the given figure:

$$T(x=0) = T_\mathrm{A}, \tag{II.3.5}$$

and:

$$T(x=\delta) = T_\mathrm{B}. \tag{II.3.6}$$

5 Solving the equation:

The function of the temperature profile is derived by solving the heat conduction equation. Integration of the heat conduction equation once yields:

$$\begin{aligned} \left[\lambda_0 + \frac{\gamma}{\delta}x\right] \cdot \frac{\partial T}{\partial x} &= C_1 \\ \Rightarrow \frac{\partial T}{\partial x} &= \frac{C_1}{\lambda_0 + \frac{\gamma}{\delta}x}. \end{aligned} \tag{II.3.7}$$

Second time integrating:

$$T(x) = \frac{C_1\delta}{\gamma} \ln\left|\lambda_0 + \frac{\gamma}{\delta}x\right| + C_2. \tag{II.3.8}$$

Substitution of the first boundary condition results in:

$$\begin{aligned} T(x=0) &= \frac{C_1\delta}{\gamma} \ln|\lambda_0| + C_2 = T_\mathrm{A} \\ \Rightarrow C_2 &= T_\mathrm{A} - \frac{C_1\delta}{\gamma} \ln|\lambda_0|, \end{aligned} \tag{II.3.9}$$

and substitution of the second boundary condition:

$$\begin{aligned} T(x=\delta) &= \frac{C_1\delta}{\gamma} \ln\left|\lambda_0 + \frac{\gamma}{\delta}\cdot\delta\right| + C_2 = T_\mathrm{B} \\ \Rightarrow C_1 &= \frac{\gamma\,(T_\mathrm{B} - T_\mathrm{A})}{\delta \ln\left|1 + \frac{\gamma}{\lambda_0}\right|}. \end{aligned} \tag{II.3.10}$$

Conclusion

Substitution of C_1 and C_2 into the temperature profile, stated in equation (II.3.8), and rewriting gives:

$$T(x) = \frac{(T_\mathrm{B} - T_\mathrm{A})}{\ln\left|1 + \frac{\gamma}{\lambda_0}\right|} \cdot \left(\ln\left|\lambda_0 + \frac{\gamma}{\delta}x\right| - \ln|\lambda_0|\right) + T_\mathrm{A}. \tag{II.3.11}$$

Task b)

In the context of planar wall systems without heat generation, the heat transfer rate $\dot{Q}$ remains uniform across the entirety of the wall, irrespective of its position within the structure. Recall the thermal conductivity function:

$$\lambda(x) = \lambda_0 + \frac{\gamma}{\delta} \cdot x, \tag{II.3.12}$$

which indicates an increase in thermal conductivity when moving in the positive x-direction. According to Fourier's law:

$$\dot{Q} = -\lambda(x) \cdot A \cdot \frac{\partial T}{\partial x}, \tag{II.3.13}$$

the temperature gradient $\frac{\partial T}{\partial x}$ should decrease along the x-direction due to the constant rate of heat transfer $\dot{Q}$ and the cross-sectional area A for a plane wall.

Conclusion