$$\sum_{i=1}^{3} u_{ii} \equiv u_{ii}, \tag{1.32}$$

$$\sum_{j=1}^{3} c_j \left(a_j + b_j\right) \equiv c_j \left(a_j + b_j\right), \tag{1.33}$$

and a counter example is

$$\sum_{j=1}^{3} \left(a_j + b_j\right) \neq a_j + b_j. \tag{1.34}$$

Finaly the reader is referred to the second and third columns of Table (1.1).

1.6 Exercises

Problem 1.1. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} x \\ -y \end{pmatrix}$

(a) Draw the curves $xy = \pm 1$ *in all four quadrants of the* $x - y$ *plane.*
(b) Draw the velocity vector at several points on the curves.
(c) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.2. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.3. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.4. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.5. *Consider a little smoke particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} e^{t} \\ e^{-t} \end{pmatrix}$

(a) Draw the trajectory for $-1 \leq t \leq 1$.
(b) Compute the velocity vector.

Problem 1.10. *Given the Eulerian field*

$$\boldsymbol{u}(x, y, z, t) = 3t\boldsymbol{e}_1 + xz\boldsymbol{e}_2 + ty^2\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the unit vectors along the coordinate axis, determine the flow acceleration.

Problem 1.11. *A two-dimensional velocity field is described by*

$$u = \frac{x}{a + bt}, \qquad v = \frac{y}{a + 2bt}.$$

Calculate the trajectories that pass by (x_o, y_o) at $t = 0$.

Problem 1.12. *Using polar coordinates, the velocity field in a tornado can be approximated as*

$$\boldsymbol{u} = -\frac{a}{r}\boldsymbol{e}_r + \frac{b}{r}\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the directions r and θ. Show that the trajectories satisfy the so-called logarithmic spiral equation:

$$r(\theta) = C \exp\left(-\frac{a}{b}\theta\right).$$

Problem 1.13. *A two-dimensional velocity field is given by*

$$u = 5ax(t + t_o), \qquad v = 5ay(t - t_o).$$

Find the trajectories that pass x_o, y_o at time $t = 0$.

Problem 1.14. *The ideal flow around a corner placed at the origin is given by*

$$u = ax, \qquad v = -ay,$$

with $a > 0$ a constant. Determine the trajectories and draw the trajectory that passes the point (x_o, y_o) at time $t = 0$ and indicate the flow direction. Calculate the material derivative of the velocity vector.

Problem 1.15. *The velocity field in a vortex like the one present in a cyclone, is given by:*

$$u = -\frac{Ky}{x^2 + y^2}, \qquad v = \frac{Kx}{x^2 + y^2},$$

with $K > 0$. Determine the trajectories and draw a few of them.

Alternatively suppose that we know that the flow is incompressible, in other words, the mass density is a known constant. In that case the volume integral in Eq.(2.29) is again zero since $\frac{\partial \rho}{\partial t}$ is zero. The result is

$$\dot{m} = -\rho \int_{S_1(t)} u_j n_j dS. \tag{2.31}$$

2.7 Exercises

Problem 2.1. *Compute the inner product* $\boldsymbol{a} \cdot \boldsymbol{b}$ *if*

(a) $\boldsymbol{a} = (1,0,0)^T$, $\boldsymbol{b} = (1,0,0)^T$.
(b) $\boldsymbol{a} = (1,0,0)^T$, $\boldsymbol{b} = (0,1,0)^T$.
(c) $\boldsymbol{a} = (a_1,a_2,a_3)^T$, $\boldsymbol{b} = (b_1,b_2,b_3)^T$.
(d) $\boldsymbol{a} = (x,y^2,x)^T$, $\boldsymbol{b} = (y,y,z)^T$.
(e) $\boldsymbol{a} = (u,v,w)^T$, $\boldsymbol{b} = (n_1,n_2,n_3)^T$.

Problem 2.2. *Compute the inner product* $\boldsymbol{u} \cdot \boldsymbol{n}$ *if*

(a) $\boldsymbol{u} = U\boldsymbol{n}$, $\boldsymbol{n} = (n_1,n_2,n_3)^T$.
(b) $\boldsymbol{u} = -U\boldsymbol{n}$, $\boldsymbol{n} = (n_1,n_2,n_3)^T$.

Problem 2.3. *A tube has cross-sectional area* A_a *at the entrance and cross-sectional area* A_b *at the exit, and the fluid flowing through the tube is incompressible.*

(a) If the volume flow rate at the exit is Q*, compute the average normal velocity at the exit.*
(b) If the volume flow rate at the exit is Q*, compute the average normal velocity at the entrance.*

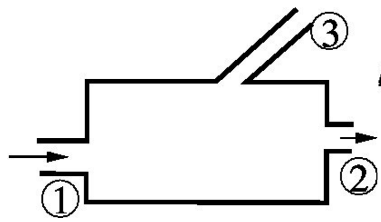
Problem 2.4. *A channel with rectangular cross section has sides* b *and* h *at the exit. The exit cross section is plane and perpendicular to the* x*-axis, and intersects the* x*-axis at* $x = L$*. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u,v,w)^T$, *with* $u = U(1 - z/h)$, $v = \ln yz$, $w = yz^2$.
(b) $(u,v,w)^T$, *with* $u = U(1 - y/b)$, $v = \sin z$, $w = \cos y$.
(c) $(u,v,w)^T$, *with* $u = U(1 - y/b)(1 - z/h)$, $v = 0$, $w = yz$.

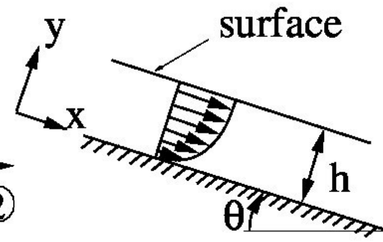
Problem 2.5. *A tube with circular cross section has radius* R *at the exit. The exit cross section is plane and perpendicular to the* x*-axis, and intersects the* x*-axis at* $x = L$*. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u(r),0,0)^T$, *with* $u(r) = U(1 - r/R)$, *compute the average normal velocity.*
(b) $(u(r),0,0)^T$, *with* $u(r) = U(1 - (r/R)^2)$, *compute the average normal velocity.*

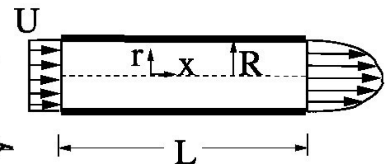
Problem 2.6. *Show how Eq.(2.19) reduces in the following two cases:*



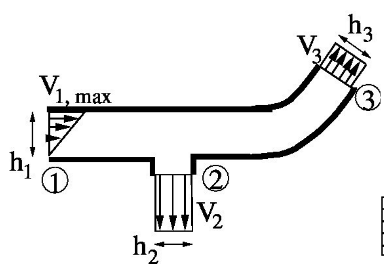
(a) Problem 2.7



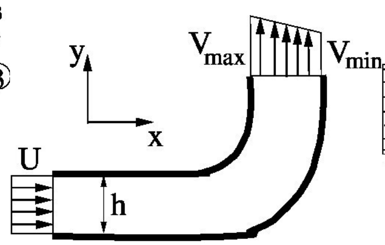
(b) Problem 2.8



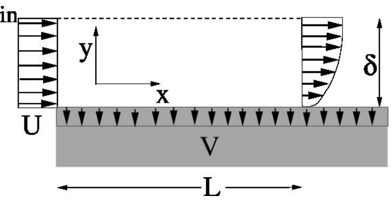
(c) Problem 2.9



(a) Problem 2.10



(b) Problem 2.11



(c) Problem 2.12