1.6 Exercises

Problem 1.1. *Consider the velocity field* $\mathbf{u}(\boldsymbol{x}) = \begin{pmatrix} x \\ -y \end{pmatrix}$

(a) Draw the curves $xy = \pm 1$ *in all four quadrants of the* $x - y$ *plane.*
(b) Draw the velocity vector at several points on the curves.
(c) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.2. *Consider the velocity field* $\mathbf{u}(\boldsymbol{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.3. *Consider the velocity field* $\mathbf{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.4. *Consider the velocity field* $\mathbf{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.5. *Consider a little smoke particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$

(a) Draw the trajectory for $-1 \leq t \leq 1$.
(b) Compute the velocity vector.

Problem 1.6. *Consider a little dust particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

(a) Draw the trajectory for $0 \leq t \leq 2\pi$.
(b) Compute the velocity vector $\boldsymbol{u}$
(c) Construct a formula for the velocity vector as a function of $x(t)$ *and* $y(t)$.
(d) Draw the velocity vector at several points on the trajectory.
(e) Compute the acceleration vector.
(f) Construct a formula for the acceleration vector as a function of $x(t)$ *and* $y(t)$.
(g) Draw the acceleration vector at several points on the trajectory.

Problem 1.7. *Using the index summation convention of Einstein, write in full:*

(a) $\frac{1}{2}u_i u_i$,
(b) $\frac{1}{2}u_i u_j$,
(c) $\frac{1}{2}u_j u_j$,

(d) $\frac{\partial u_k}{\partial x_k}$,
(e) $u_j \frac{\partial u_i}{\partial x_j}$,
(f) $u_i + u_j$,
(g) $\delta_{ij} u_i$, *where* $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
(h) $\delta_{ij} u_j$,
(i) $a_{ij} u_i u_j$.

Problem 1.8. *Rewrite the following vector-notation expressions in index notation:*

(a) $\boldsymbol{x} \cdot \boldsymbol{y}$,
(b) $\boldsymbol{\nabla} \cdot \boldsymbol{u}$,
(c) *the i-th component of the vector $A\boldsymbol{x}$, where A is a 3×3 matrix,*
(d) *the i, j-th component of the matrix AB, where A and B are a 3×3 matrices,*

Problem 1.9. *The velocity at the plane defined by the normal* $\boldsymbol{n} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *is* $\boldsymbol{u} = \begin{pmatrix} 15 \\ 34 \end{pmatrix} \frac{m}{s}$. *Calculate the normal and tangential velocities.*

Problem 1.10. *Given the Eulerian field*

$$\boldsymbol{u}(x, y, z, t) = 3t\boldsymbol{e}_1 + xz\boldsymbol{e}_2 + ty^2\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the unit vectors along the coordinate axis, determine the flow acceleration.

Problem 1.11. *A two-dimensional velocity field is described by*

$$u = \frac{x}{a + bt}, \quad v = \frac{y}{a + 2bt}.$$

Calculate the trajectories that pass by (x_o, y_o) at $t = 0$.

Problem 1.12. *Using polar coordinates, the velocity field in a tornado can be approximated as*

$$\boldsymbol{u} = -\frac{a}{r}\boldsymbol{e}_r + \frac{b}{r}\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the directions r and θ. Show that the trajectories satisfy the so-called logarithmic spiral equation:

$$r(\theta) = C \exp\left(-\frac{a}{b}\theta\right).$$

Problem 1.13. *A two-dimensional velocity field is given by*

$$u = 5ax(t + t_o), \quad v = 5ay(t - t_o).$$

Find the trajectories that pass x_o, y_o at time $t = 0$.

Problem 1.14. *The ideal flow around a corner placed at the origin is given by*

$$u = ax, \quad v = -ay,$$

with $a > 0$ a constant. Determine the trajectories and draw the trajectory that passes the point (x_o, y_o) at time $t = 0$ and indicate the flow direction. Calculate the material derivative of the velocity vector.

Problem 1.15. *The velocity field in a vortex like the one present in a cyclone, is given by:*

$$u = -\frac{Ky}{x^2 + y^2}, \quad v = \frac{Kx}{x^2 + y^2},$$

with $K > 0$. Determine the trajectories and draw a few of them.

Alternatively suppose that we know that the flow is incompressible, in other words, the mass density is a known constant. In that case the volume integral in Eq.(2.29) is again zero since $\frac{\partial \rho}{\partial t}$ is zero. The result is

$$\dot{m} = -\rho \int_{S_1(t)} u_j n_j dS. \tag{2.31}$$

2.7 Exercises

Problem 2.1. *Compute the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ if*

(a) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (1, 0, 0)^T$.
(b) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (0, 1, 0)^T$.
(c) $\boldsymbol{a} = (a_1, a_2, a_3)^T$, $\boldsymbol{b} = (b_1, b_2, b_3)^T$.
(d) $\boldsymbol{a} = (x, y^2, x)^T$, $\boldsymbol{b} = (y, y, z)^T$.
(e) $\boldsymbol{a} = (u, v, w)^T$, $\boldsymbol{b} = (n_1, n_2, n_3)^T$.

Problem 2.2. *Compute the inner product $\boldsymbol{u} \cdot \boldsymbol{n}$ if*

(a) $\boldsymbol{u} = U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.
(b) $\boldsymbol{u} = -U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.

Problem 2.3. *A tube has cross-sectional area A_a at the entrance and cross-sectional area A_b at the exit, and the fluid flowing through the tube is incompressible.*

(a) If the volume flow rate at the exit is Q, compute the average normal velocity at the exit.
(b) If the volume flow rate at the exit is Q, compute the average normal velocity at the entrance.

Problem 2.4. *A channel with rectangular cross section has sides b and h at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u, v, w)^T$, *with* $u = U(1 - z/h)$, $v = \ln yz$, $w = yz^2$.
(b) $(u, v, w)^T$, *with* $u = U(1 - y/b)$, $v = \sin z$, $w = \cos y$.
(c) $(u, v, w)^T$, *with* $u = U(1 - y/b)(1 - z/h)$, $v = 0$, $w = yz$.

Problem 2.5. *A tube with circular cross section has radius R at the exit. The exit cross section is plane and perpendicular to the x-axis, and intersects the x-axis at $x = L$. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u(r), 0, 0)^T$, with $u(r) = U(1 - r/R)$, compute the average normal velocity.
(b) $(u(r), 0, 0)^T$, with $u(r) = U(1 - (r/R)^2)$, compute the average normal velocity.

Problem 2.6. *Show how Eq.(2.19) reduces in the following two cases:*

(a) steady flow
(b) Incompressible flow

$$\boxed{\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho u_j n_j dS = 0 \quad \text{for all } (V(t), t)} \tag{2.19}$$

Problem 2.7. *Consider steady, incompressible flow through the device shown. Given: U_1, A_1, U_2, A_2, A_3. Derive an expression for the volume flow rate through port 3.*

Problem 2.8. *Incompressible oil flows steadily in a thin layer down an inclined plane with width w. The velocity profile is*

$$u = \frac{\rho g \sin\theta}{\mu}\left[hy - \frac{1}{2}y^2\right]. \tag{2.32}$$

Derive formulas for the volume flow rate and mass flow rate in terms of ρ, μ, g, θ, and h.

Problem 2.9. *Incompressible water flows steadily through a pipe of length L and radius R. Derive an expression for the uniform inlet velocity, U, if the velocity distribution across the outlet is given by*

$$u = V\left[1 - \frac{r^2}{R^2}\right]. \tag{2.33}$$

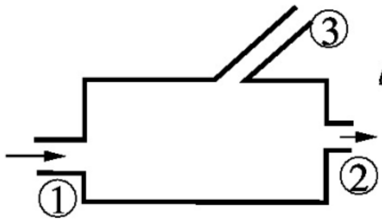
Problem 2.10. *A bend with rectangular cross section and width w has a linear velocity profile at port 1. The flow is uniform at ports 2 and 3. The fluid is incompressible and the flow is steady. Derive an expression for the uniform velocity at port 3.*

Problem 2.11. *Water enters a two-dimensional square channel of constant width w, and constant height, h, with uniform velocity, U. The channel makes a $90°$ bend that distorts the flow to produce the linear velocity profile shown at the exit, with $V_{max} = 2V_{min}$. The flow is steady and the fluid is incompressible. Derive an expression for V_{min}.*

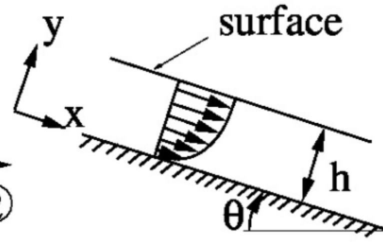
Problem 2.12. *Incompressible water flows steadily past a porous plate of width w and length L. Constant suction is applied along the plate with normal velocity V (towards the plate). The velocity profile at the outflow plane is:*

$$\frac{u}{U} = 3\left[\frac{y}{\delta}\right] - 2\left[\frac{y}{\delta}\right]^{3/2}. \tag{2.34}$$

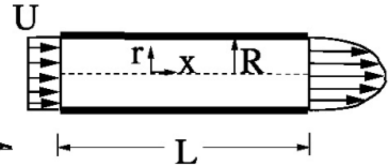
Derive a formula for the mass flow rate through the boundary at the top of the domain ($y = \delta$).



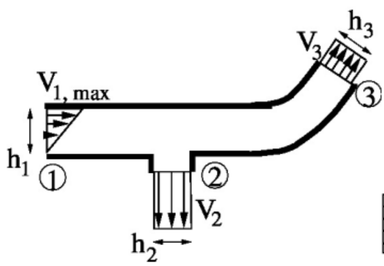
(a) Problem 2.7



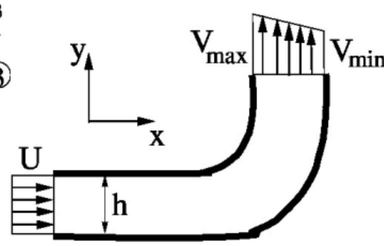
(b) Problem 2.8



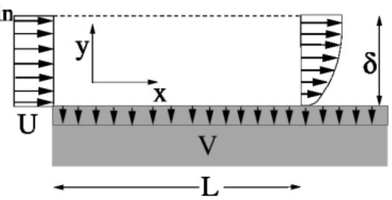
(c) Problem 2.9



(a) Problem 2.10



(b) Problem 2.11



(c) Problem 2.12

3.5 Exercises

Problem 3.1. *For incompressible flow, indicate for each of the following velocity fields whether they are steady/unsteady, and whether they satisfy mass conservation.*

(a) $u = x + y + z^2$, $v = x - y + z$, $w = 2xy + y^2 + 4$,
(b) $u = xyzt$, $v = -xyzt^2$, $w = \frac{1}{2}z^2(xt^2 - yt)$,
(c) $u = y^2 + 2xz$, $v = -2yz + x^2yz$, $w = \frac{1}{3}x^2z^2 + x^3y^4$.

Problem 3.2. *For a flow in the xy plane, the x component of velocity is given by* $u = ax(y - b)$.

(a) Find the y component of the velocity, v, for steady, incompressible flow.
(b) Explain why it is also valid for unsteady, incompressible flow.

Problem 3.3. *The x component of velocity in a steady, incompressible flow field in the xy plane is* $u = A/x$. *Find the simplest y component of velocity for this flow field.*

Problem 3.4. *For the following velocity fields, determine whether the continuity equation for incompressible flow is satisfied:*

(a) $\boldsymbol{u} = (ax, ay, -2az)^T$
(b) $\boldsymbol{u} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right)^T$

Problem 3.5. *For the two-dimensional velocity field* $\boldsymbol{u} = (ax, by)^T$, *and taking V with boundary S as a box defined by* $0 \leq x \leq p$, $0 \leq y \leq q$, *compute*

(a) $\int_V \frac{\partial u_j}{\partial x_j}\, dV$,
(b) $\int_S u_j n_j \, dS$.

Problem 3.6. *For the two-dimensional velocity field* $\boldsymbol{u} = (ax, by)^T$, *and taking V with boundary S as a disk defined by* $0 \leq \sqrt{x^2 + y^2} \leq R$, *compute*

(a) $\int_V \frac{\partial u_j}{\partial x_j}\, dV$,
(b) $\int_S u_j n_j \, dS$.

Problem 3.7. *For one-dimensional steady compresible flow* ($v = w = 0$),

(a) derive an expression for ρu *if* Φ, *the mass flow rate per unit area is given.*
(b) derive an expression for u in case the flow is incompressible.

Problem 3.8. *For one-dimensional compressible flow* ($v = w = 0$) *with constant velocity u,*

(a) show that $\rho(x, t) = \rho_o \sin(x - ut)$ *satisfies the continuity equation.*
(b) make a sketch of $\rho_o \sin(x - ut)$ *at* $t = 0$ *and* $t = 1/u$.
(c) show that $\rho(x, t) = f(x - ut)$ *satisfies the continuity equation for any function f.*

Problem 3.9. *It is known that the integral of the outward unit normal vector over an arbitrary but closed surface (3D) is the null-vector. This means that the integral of each component of the outward unit normal is zero. Proof this for the first component by taking a velocity field* $\boldsymbol{u} = (1, 0, 0)^T$ *and by using Gauss' divergence theorem.*

4.7 Exercises

Problem 4.1. *By using the integral formulation of momentum conservation, show that the law of Archimedes (287 BC - 212 BC) holds: in water which is not flowing the (upward) force on a blob of water by the surrounding water is equal to the (downward) gravity force on the blob.*

Problem 4.2. *An incompressible fluid flows steadily into a T-junction of diameter D at uniform velocity U, at the opposite outlet the fluid leaves at uniform velocity V. At the lateral exit the flow leaves at unknown uniform velocity. The pressure in the T-junction is uniform: p. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.3. *An incompressible fluid flows steadily into a pipe of diameter D at uniform velocity U and pressure p_1. At the end of the pipe is a contraction of diameter d, and the fluid leaves the contraction at uniform velocity V and pressure p_2. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.4. *Incompressible water is flowing steadily through a $180°$ elbow. At the inlet the pressure is p_1 and the cross section area is A_1, at the outlet the pressure is p_2 and the cross section area is A_2. The averaged velocity at the inlet is V_1. Find the horizontal component of the force by the fluid on the elbow, neglecting viscosity and gravity.*

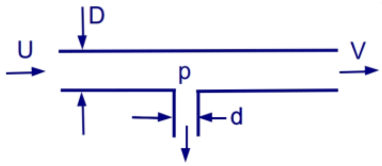
Problem 4.5. *An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h$ and width w. At the entrance the presure is $p1$ and the uniform velocity is U_1. At the exit the pressure is p_2 and the velocity distribution is*

$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2. \tag{4.40}$$

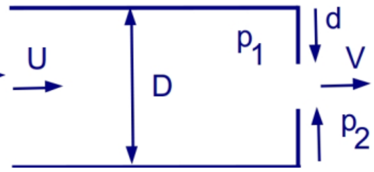
(a) Derive an expression for the maximum velocity at the downstream section.
(b) Derive an expression for the force on the walls in x-direction, neglecting gravity, and neglecting viscosity at entrance and exit.

Problem 4.6. *A small round object is tested in a wind tunnel with circular cross section with diameter D. The pressure is uniform across sections 1 and 2 and known: p_1 and p_2. At the entrance the uniform velocity is U. The velocity profile at section 2 is linear: it varies from zero at the tunnel centerline to a maximum at the tunnel wall. The viscosity effects on the wall of the wind tunnel can be neglected and the flow can be treated as incompressible.*

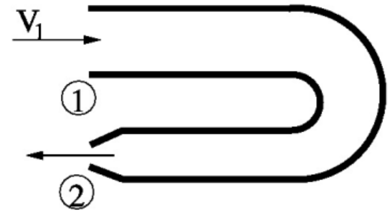
(a) Derive an expression for the mass flow rate in the wind tunnel,
(b) Derive an expression for the maximum velocity at section 2
(c) Derive an expression for the drag of the object and its supporting vane.



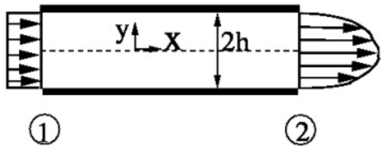
(a) Problem 4.2



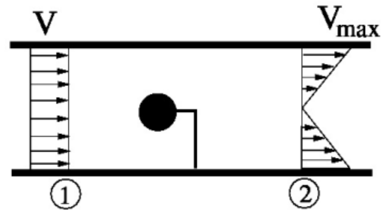
(b) Problem 4.3



(c) Problem 4.4



(a) Problem 4.5



(b) Problem 4.6

5.8 Exercises

Problem 5.1. *Incompressible viscous oil flows steadily between stationary parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The oil viscosity is μ and the pressure drop over a distance L is Δp.*

(a) Derive an expression for the shear stress on the upper plate.
(b) Derive an expression for the volume flow rate through the channel over a width w.
(c) Compute the shear stress on the upper plate and the volume flow rate through the channel over a width w if $h = 5\,mm$, $\Delta p = -1000\,Pa$, $L = 1m$, $\mu = 0.5\,Ns/m^2$.

Hint: *first derive an expression for the velocity field starting from the reduced Navier-Stokes equations.*

Problem 5.2. *An incompressible fluid of density ρ flows steadily between two parallel plates. The flow is laminar and fully developed, the viscosity is μ, the mean velocity is U, and the distance between the plates is h. Divide the flow into two horizontal layers, with the divide located at a distance y above the lower plate. Derive an expression for the shear stress experienced by the lower layer as a function of y, and sketch this function.*

Problem 5.3. *An hydraulic jack (NL: krik) supports a load of mass M. The diameter of the piston is D, the radial clearance between the piston and the cylinder is d, the length of the piston is L, and the viscosity of the oil is μ.*

(a) Derive an expression for the pressure-drop in the gap between the piston and the cylinder.
(b) Derive a formula for the rate of leakage of hydraulic fluid past the piston. Compute the leakage rate when $M = 9000\,kg$, $D = 100\,mm$, $d = 0.05\,mm$, $L = 120\,mm$, and $\mu = 2 \times 10^{-1}\,Ns/m^2$.

Hint: *approximate the gap between the piston and the cylinder as the gap between two flat plates (why would this be a very good approximation?). First compute the vertical pressure derivative by assuming the piston to be in equilibrium (moves extremely slowly due to the leakage).*

Problem 5.4. *Consider the steadily falling water film along a vertical wall with thickness a. The flow is incompressible, laminar, and fully developed. At the wall the velocity is zero, whereas at the outer edge of the film the shear stress is zero.*

(a) Defend the approximation assumption of zero shear stress at the film boundary.
(b) Derive an expression for $\frac{\partial p}{\partial x}$.
(c) Derive an expression for $u(y)$.

Problem 5.5. *An incompressible fluid flows steadily between two parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The upper plate moves to the right with speed U_2, the lower plate moves to the left with speed U_1. The pressure gradient in the direction of the flow is zero.*

(a) Derive an expression for the velocity distribution in the gap.
(b) Derive an expression for the volume flow rate per unit depth.

Problem 5.6. *An incompressible fluid flows steadily between two parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The upper plate moves to the right with speed U, the lower plate is fixed.*

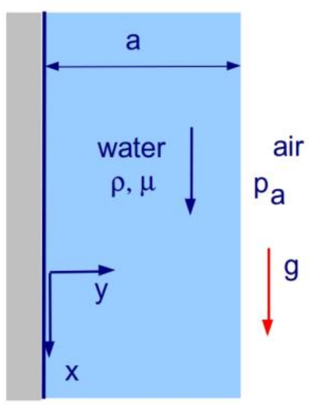
(a) *Derive an expression for the pressure gradient at which the upper plate experiences zero shear stress.*
(b) *Derive an expression for the pressure gradient at which the lower plate experiences zero shear stress.*

Problem 5.7. *The record-read head for a computer disk-drive memory storage system rides above the spinning disk on a very thin film of air (the film thickness is h). The head location is a from the disk centerline; the disk spins at angular velocity Ω. The surface area of the record-read head is A. Finally, the viscosity of air is μ and the density is ρ.*

(a) *Derive an expression for the Reynolds number of the flow.*
(b) *Derive an expression for the shear stress.*
(c) *Derive an expression for the power required to overcome the viscous shear stress.*
(d) *Compute the values of the three expressions if $h = 0.5\,\mu m$, $a = 150\,mm$, $\Omega = 3600\,rpm$, and $A = 100\,mm^2$, $\mu = 18.0 \times 10^{-6}\,kg/ms$, and $\rho = 1.2\,kg/m^3$.*

Problem 5.8. *Consider fully developed laminar incompressible flow in a pipe.*

(a) *Derive an expression for the average velocity in a cross-section.*
(b) *Transform the previous expression to obtain a formula for the pressure gradient as a function of (amongst others) the average velocity.*



(a) Problem 5.4

6.6 Exercises

Problem 6.1. *Determine the dimensions of force F, stress σ, power $\dot{W}$, dynamic viscosity μ and thermal conductivity k.*

Problem 6.2. *The variables which control the motion of a boat are the resistance force, F, speed V, length L, density of the liquid ρ and its viscosity μ, as well as gravity acceleration g. Obtain an expression for F using dimensional analysis.*

Problem 6.3. *It is believed that the power P of a fan depends upon the density of the liquid ρ, the volumetric flux Q, the diameter of the propeller D and the angular speed Ω. Using dimensional analysis, determine the dependence of P with respect to the other dimensionless variables.*

Problem 6.4. *In fuel injection systems, a jet of liquid breaks, forming small drops of fuel. The diameter of the resulting drops, d, supposedly depends upon the density of the liquid ρ, the viscosity μ, surface tension σ (force/length), and also upon the speed of the stream V and its diameter D. How many dimensionless parameters are required to characterize the process? Find them.*

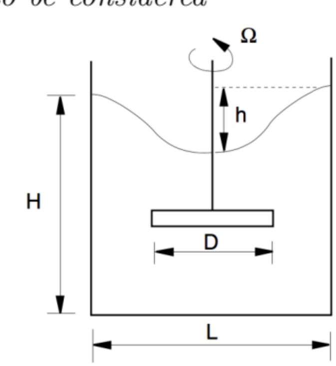
Problem 6.5. *A disc spins close to a fixed surface. The radius of the disc is R, and the space between the disk and the surface is filled with a liquid of viscosity μ. The distance between the disc and the surface is h and the disc spins at an angular velocity ω. Determine the functional relationship between the torque that acts upon the disc, T, and the other variables.*

Problem 6.6. *The drag force, F, experienced by a submarine that moves at a great depth from the surface of the water, is a function of the density ρ, viscosity μ, speed V and the transversal area of the submarine A. An expert suggests that the nondimensional relationship that allows the calculation of F is: $F = f(\frac{\rho V A}{\mu})\rho V^2 A$.*

(a) *Is the number of dimensionless parameters in the expression correct? Why?*
(b) *Are the parameters correct? If not, correct them.*
(c) *A geometrically similar model to that of the real submarine has been constructed, so that all the lengths of the model are 1/10 of those corresponding to the submarine. The model is tested in sea water. (1) The force of the real submarine when it moves at 5 m/s is to be de- termined. (2) At which speed should the model be tested?*

Problem 6.7. *An automobile must travel through standard air conditions at a speed of $100 km/h$. To determine the pressure distribution, a model at a scale of 1/5 of the length of the vehicle is tested in water. Find the speed of water to be used.*
$\mu_{water} = 10^{-3} kg/(ms)$, $\rho_{water} = 1000 kg/m^3$, $\mu_{air} = 1.8 \times 10^{-5} kg/(ms)$, $\rho_{air} = 1.2 kg/m^3$.

Problem 6.8. *The depth of the steady central vortex h in a large tank of oil being stirred by a propeller needs to be predicted. One way is to carry out a study using a reduced scale model. Determine the conditions under which the experiment should be conducted to be considered a valid predictive tool. Note: Consider h a function of g, H, D, L and Ω.*

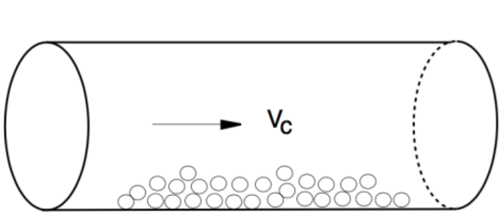


Problem 6.9. *A rectangular, thin, flat plate, with length h and width w is placed perpendicularly to a liquid flow. Imagine that the drag force D which the liquid has upon the plate is a function of w and h, the density ρ, the viscosity μ, and the speed V of the liquid coming towards the plate. Determine the set of dimensionless parameters to study the problem experimentally.*

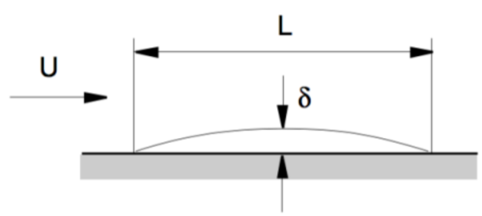
Problem 6.10. *The Reynolds number is a very important parameter for studying transport phenomena and fluid mechanics. Estimate the Reynolds number that would be characteristic of the flow around a car traveling along the highway.*
Note: $\rho_{air} \approx 1.25\, kg/m^3$, $\mu_{air} \approx 1.8 \times 10^{-5} Pa\, s$.

Problem 6.11. *A thin layer of spherical particles are lying at the bottom of a horizontal tube, as indicated in the Figure. When an incompressible liquid flows along the tube, it can be seen that at a certain critical speed the particles move and are carried along the length of the tube. We wish to study the value of this critical speed V_c. Suppose that V_c is a function of the diameter of the tube D, the particle?s diameter D_p, the liquid density ρ, the viscosity of the liquid μ, the density of the particles ρ_p and the gravity acceleration g.*

(a) Using ρ, D and g as fundamental variables, obtain the dimensionless parameters of the problem.
(b) Repeat the first question using ρ, D and μ as fundamental variables.



(a) Exercise 6.11



(b) Exercise 6.12

Problem 6.12. *During the drying process of a fine layer of liquid on a surface, the liquid evaporates and the vapor is transported in the air above the surface. We are interested in knowing the dependence of the drying time t upon the rest of the variables of the problem (length L, thickness of the layer δ, the liquid's vapor pressure p_v, air speed U, viscosity μ and air density ρ).*

(a) Obtain a set of dimensionless variables related to the drying time t with the rest of the variables.
(b) We wish to set up a laboratory experiment to determine the drying time of a soccerfield where $p_v = 2000\, Pa$, $L = 100\, m$, $\delta = 0.01\, m$ and $U = 2\, m/s$. In the experiment, the viscosity and the density of the air will be the same as that of the soccer field, but L will be $20\, m$ (we don't have a larger laboratory available). Calculate the values of U, δ and p_v in the experiment so that complete similarity exists with the real flow.
(c) If in the experiment the average drying time is $t = 10\, min$, calculate the drying time of the soccer field.

7.4 Exercises

Problem 7.1. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u}$ *by using the definition of the streamfunction* $\tilde{\Psi}$, *and the definition of its partial derivative:*

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u} \equiv \lim_{\Delta \tilde{y} \to 0} \frac{\tilde{\Psi}(\tilde{x}, \tilde{y} + \Delta \tilde{y}) - \tilde{\Psi}(\tilde{x}, \tilde{y})}{\Delta \tilde{y}}.$$

Problem 7.2. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{x}} = -\tilde{v}$ *by using mass conservation and the definition of the streamfunction* $\tilde{\Psi}$

Problem 7.3. *For sufficiently small values of* η, *Blasius' solution can be approximated by a truncated Taylor series:*

$$f(\eta) = \sum_{n=0}^{5} a_n \eta^n$$

(a) Using the boundary conditions at $\eta = 0$, *show that* $a_o = a_1 = 0$.
(b) Using Blasius's equation, show that $a_3 = a_4 = 0$ *and that*

$$a_5 = -\frac{1}{60} a_2^2.$$

(c) Compute a_2 *and* a_5 *given that* $f''(0) \approx 0.322$.
(d) Compute $f(2)$ *and* $f'(2)$.
(e) Plot these values in Fig. (7.4).

Problem 7.4. *Make a sketch of the skin-friction coefficient* C_f *as a function of* $\tilde{x}$.

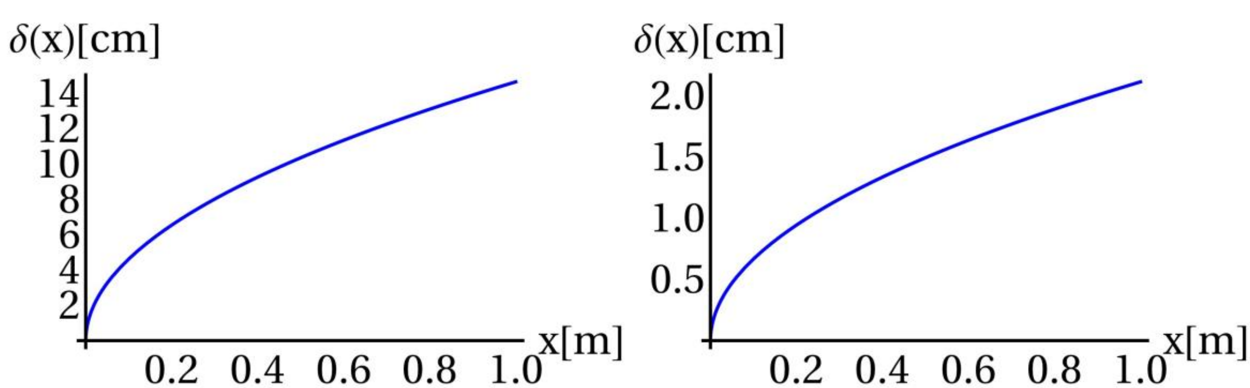


Figure 7.4: Flat plate boundary layer thickness based on Blasius' solution for water (left, $\nu = 894 \times 10^{-6}\, Pa\, s$*) and air (right,* $\nu = 18.6 \times 10^{-6}\, Pa\, s$*) with* $U = 1\, m/s$*, confirming that* $\delta \sim \sqrt{\nu}$*. [Note: the solution for air does not hold beyond a few centimeters in downstream direction because turbulence sets in and the boundary layer would be much thicker.]*

8.5 Exercises

Problem 8.1. *Let* $T(x, y) = T_0 + ax + by$

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$.

Problem 8.2. *Let* $T(x, y) = T_0 + ax + by + cxy$.

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$.
(c) Under what condition do the answers of (a) and (b) coincide?

Problem 8.3. *Let* $T(x, y) = T_0 + ax + by$ *and* $x_p(t) = x_o + ut$, $y_p(t) = y_o + vt$.

(a) Compute $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(b) Compute $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.4. *Let* $T(x, y) = T_0 + ax + by + cxy$ *and* $x_p(t) = x_o + ut + \frac{1}{2}pt^2$, $y_p(t) = y_o + vt + \frac{1}{2}qt^2$.

(a) Compute $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(b) Compute $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.
(c) Under what condition do the answers of (a) and (b) coincide?

Problem 8.5. *Let* $T(x) = sin(ax)$ *and* $x_p(t) = ut$.

(a) Compute $f(t) \equiv T(x_p(t))$ *and* $\frac{df}{dt}$.
(b) Make a sketch of $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 1$, $u = 1$ *and* $a = 1$, $u = 2$.
(c) Make a sketch of $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 2$, $u = 1$ *and* $a = 2$, $u = 2$.

Problem 8.6. *Let* $T(x, y) = xy$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) Sketch the curve $T(x, y) = 1$.
(b) Compute $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(c) Compute $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.7. *Let* $T(x, y) = xyt$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) Sketch the curve $T(x, y) = t$.
(b) Compute $f(t) \equiv T(x_p(t), y_p(t), t)$ *and* $\frac{df}{dt}$.
(c) Compute $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.8. *A baseball is thrown at speed* U *in air with pressure* p_a *and constant density* ρ. *Assume* $\mu = 0$ *and* $U = constant$.

(a) Compute the maximum value of the pressure on the ball's surface.
(b) Compute the maximum value of the pressure on the ball's surface if there is a head-wind V.
(c) Compute the maximum value of the pressure on the ball's surface if there is a tail-wind V.

Problem 8.8. *A baseball is thrown at speed U in air with pressure p_a and constant density ρ. Assume $\mu = 0$ and $U = constant$.*

(a) *Compute the maximum value of the pressure on the ball's surface.*
(b) *Compute the maximum value of the pressure on the ball's surface if there is a head-wind V.*
(c) *Compute the maximum value of the pressure on the ball's surface if there is a tail-wind V.*

Problem 8.9. *A stone with mass m is attached to a rope and swung around in a horizontal circle. The path of the stone is $\boldsymbol{x}_p(t) = vectortwox_p(t)y_p(t)$, with*

$$x_p(t) = L\cos(\omega t), \quad y_p(t) = L\sin(\omega t).$$

(a) *Compute the velocity vector $\mathbf{u}(t) \equiv \frac{d}{dt}\boldsymbol{x}_p(t)$.*
(b) *Compute the velocity vector $\boldsymbol{a}(t) \equiv \frac{d}{dt}\boldsymbol{u}_p(t)$.*
(c) *For an arbitrary time instant t, sketch the vectors $\boldsymbol{x}_p(t)$, $\boldsymbol{u}_p(t)$, and $\boldsymbol{a}_p(t)$.*

Problem 8.10. *A flow field is specified as $\mathbf{u}(\boldsymbol{x}) = \frac{U}{L}\begin{pmatrix} -y \\ x \end{pmatrix}$.*

(a) *Compute $\boldsymbol{a}(\boldsymbol{x}) \equiv \frac{D}{Dt}\boldsymbol{u}$.*
(b) *For arbitrary position $\boldsymbol{x}$, sketch $\boldsymbol{u}$ and $\boldsymbol{a}$.*
(c) *Compute the vector $\boldsymbol{\nabla} p$ and add it to the sketch.*

Problem 8.11. *Explain why*

(a) *Bernoulli's equation does* **not** *hold in fully developed flow.*
(b) *Bernoulli's equation* **does** *hold in a swimming pool if nobody is swimming.*
(c) *Derive, using Bernoulli's equation, an expression for the pressure in a abandoned swimming pool.*

9.6 Exercises

Problem 9.1. *Consider the following convection-diffusion equation:*

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha\frac{\partial^2 \phi}{\partial x^2},$$

where $\phi(x,t)$ is some property, and the units of x, t, and u are m, s, and m/s, respectively. Compute the unit of α, and check the answers for the cases $\alpha = \frac{k}{\rho C_v}$, and $\alpha = \frac{\mu}{\rho}$.

Problem 9.2. *One-dimensional sound waves are described by the wave equation*

$$\frac{\partial^2 p}{\partial t^2} - a^2\frac{\partial^2 p}{\partial x^2} = 0$$

, where p is the pressure disturbance and a the speed of sound. Show that $f(x - at)$ and $g(x + at)$ are solutions of the wave equation, with f and g arbitrary functions.

Problem 9.3. *Show, by substitution, that $T(x,t)\,(a\cos(\lambda x) + b\sin(\lambda x))\exp(-\alpha\lambda^2 t)$ is a solution of the diffusion equation*

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}.$$

Problem 9.4. *Let* $T(x,t) = \bar{T}(\xi(x,t),t)$, $\xi(x,t) = x - Ut$.

(a) Express $\frac{\partial T}{\partial t}$, $\frac{\partial T}{\partial x}$, *and* $\frac{\partial^2 T}{\partial x^2}$ *in terms of derivatives of* $\bar{T}$ *with respect to* ξ *and* t.
(b) Show that the convection-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

transforms into a diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \alpha\frac{\partial^2 \bar{T}}{\partial x^2}.$$

10.5 Exercises

Problem 10.1. *Using* $TV^{\gamma-1} = const$, $p = \rho RT$, *and mass conservation, derive that:*

(a) $T\rho^{1-\gamma} = const$
(b) $Tp^{-\frac{\gamma-1}{\gamma}} = const$
(c) $p\rho^{-\gamma} = const$

Problem 10.2. *Starting from*

$$W = -\int_0^t p\frac{dV}{dt}dt,$$

$TV^{\gamma-1} = const$, *and mass conservation, derive that*

(a) $W = -p_o V_o^\gamma \int_0^t V^{-\gamma}\frac{dV}{dt}dt$,
(b) $W = \frac{p_o V_o}{\gamma-1}\left\{\left(\frac{V_o}{V(t)}\right)^{\gamma-1} - 1\right\}$,
(c) $sign(W) = sign(V_o - V(t))$.

Problem 10.3. *Starting with* $p\rho^{-\gamma} = const$, *derive that*

$$\frac{1}{p}\frac{dp}{dt} - \gamma\frac{1}{\rho}\frac{d\rho}{dt}.$$

Problem 10.4. *Air enters a compressor at speed* U_1, *temperature* T_1, *and leaves at* U_2, T_2, *and the mass flow is* $\dot{m}$. *The removed heat per unit mass of passing air is* $\hat{e}$. *Derive an expression for the power required by the compressor, assuming air can be modeled as a perfect gas.*

11.7 Exercises

Problem 11.1. *Consider a sphere in compressible flow. Far upstream from the sphere the pressure, Mach number and temperature are known: p_∞, M_∞, and T_∞. The pressure and temperature in the stagnation point are p_o and T_o.*

(a) Express p_o in terms of p_∞, M_∞.
(b) Express p_∞ in terms of p_o, M_∞.
(c) Express T_∞ in terms of T_o, M_∞.
(d) For measured p_o, T_o, p_∞, compute the velocity U_∞ far upstream of the sphere.

Problem 11.2. *(a) Show that $\frac{1}{p}\frac{Dp}{Dt} = \frac{D}{Dt}\ln p$.*
(b) What is the meaning of $\frac{Dp}{Dt}$?

Problem 11.3. *Consider steady flow with $\mu = 0$ and $k = 0$.*

(a) Show that the mass and energy equations reduce to

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\frac{\partial}{\partial x_j}(\rho u_j H) = 0$$

(b) Show that these equations lead to $\frac{DH}{Dt} = 0$.
(c) What is the meaning of $\frac{DH}{Dt} = 0$?

Problem 11.4. *Show that in case of a perfect gas $p_t = \rho_t R T_t$.*

Problem 11.5. *For steady flow with $\mu = 0$ and $k = 0$ explain that the pressure along a streamline can not exceed the total pressure along that streamline.*

Problem 11.6. *For a thermally perfect gas show that $p = (\gamma - 1)\rho e$.*

Problem 11.7. *Let $p\rho^\gamma = const$, and let $p = p_o + p'$, $\rho = \rho_o + \rho'$, with p_o, ρ_o constants and p', ρ' small perturbations. Show that in the limit of $p'/p_o \to 0$, and $\rho_o/\rho' \to 0$ we have*

$$p' = a^2\rho', \quad a^2 \equiv \gamma\frac{p_o}{\rho_o}.$$

[Hint: use Taylor series $(1+\epsilon)^\alpha = 1 + \alpha\epsilon$ ...]