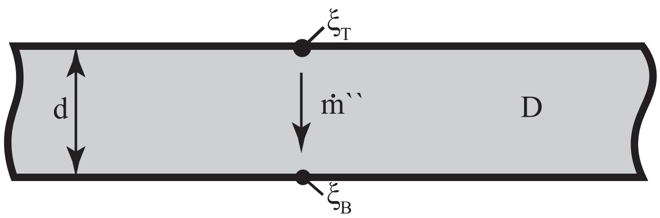
SECTION V

Mass transfer solutions

Exercise V.1: (Wet wood ⋆⋆)

The mass fraction of water at the top of an indefinitely wide wooden plate is ξ_T. A Mass flux $\dot{m}''$ of water is passing through the wood each second and is led away at the bottom.



Given parameters:

- Coefficient of diffusion of water in wood: $D = 1.2 \cdot 10^{-9}\ \frac{\text{m}^2}{\text{s}}$
- Total density of the moist wood: $\rho = 650\ \text{kg/m}^3$
- Thickness of the wooden plate: $d = 0.1\ \text{m}$
- Mass fraction of water at the top: $\xi_T = 0.2$
- Led away water mass flux: $\dot{m}'' = 1.4 \cdot 10^{-6}\ \text{kg/m}^2\text{s}$

Hints:

- Assume that the total density ρ of the moist wood is constant.
- Assume steady-state one-dimensional mass transport in the direction of the thickness of the wooden plate.

Tasks:

a) Sketch the mass fraction profile within the wet wood.

b) Determine the value of the mass fraction ξ_B at the bottom of the plate.

c) Derive the mass fraction distribution within the wet wood from the mass transport equation.

Solution V.1: (Wet wood ⋆⋆)

Task a)

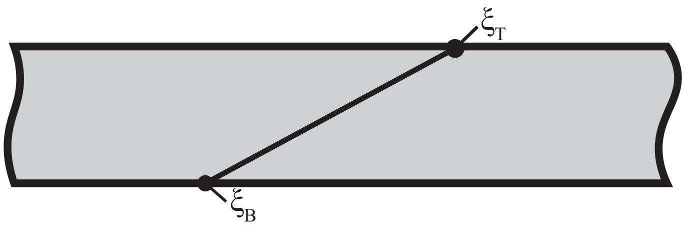
The mass transport in this case is solely governed by diffusion through the wood. From Fick's law:

$$j = -\rho D A \frac{\partial \xi}{\partial x}, \tag{V.1.1}$$

it yields that for a constant rate of mass transport and cross-sectional area, the mass fraction gradient mush behave constant.

Conclusion

Hence, the mass fraction profile within the wet wood displays a linear behavior.



Task b)

Given that the mass fraction profile within the wet wood is linear, the temperature gradient can be determined by the difference in mass fraction between the top and bottom of the wood, divided by the thickness of the plate. Since mass transport in this scenario is solely governed by diffusion, the diffusion rate equals the total mass flux. By reformulating this expression, the mass fraction at the bottom of the plate can be calculated.

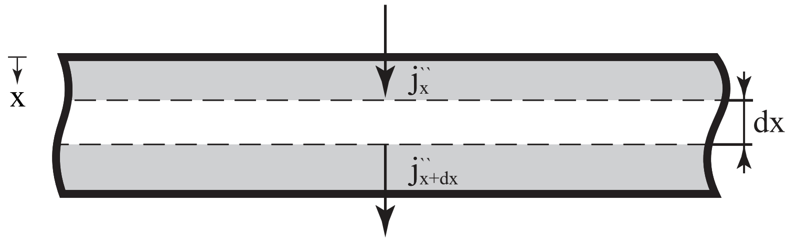
$$\begin{aligned} j'' &= -\rho D \frac{\xi_T - \xi_B}{d} = \dot{m}'' \\ \Rightarrow \xi_B &= -\frac{\dot{m}'' \cdot d}{\rho D} + \xi_T \\ &= -\frac{1.4 \cdot 10^{-6} \left(\frac{kg}{m^2 \cdot s}\right) \cdot 0.1 \ (m)}{650 \left(\frac{kg}{m^3}\right) \cdot 1.2 \cdot 10^{-9} \left(\frac{m^2}{s}\right)} + 0.2 \ (-) = 0.021 \ (-). \end{aligned} \tag{V.1.2}$$

Task c)

The mass transport equation can be derived from a substance balance within the infinitesimal element within the wood.

1 Setting up the balance:

Substance diffuses into the element and out of the element.



Hence, the steady-state one-dimensional balance is written as:

$$0 = \underbrace{\dot{j}''_{\mathrm{x}} + \dot{j}''_{\mathrm{x+dx}}}_{\text{Net rate of diffusion}}. \tag{V.1.3}$$

2 Defining the elements within the balance:

The incoming rate of substance diffusion is governed by Fick's law:

$$\dot{j}''_{\mathrm{x}} = -\rho D \frac{\partial \xi}{\partial x}, \tag{V.1.4}$$

where the outgoing rate is approximated by the use of the Taylor series expansion:

$$\begin{aligned} \dot{j}''_{\mathrm{x}} &= \dot{j}''_{\mathrm{x}} + \frac{\partial \dot{j}''_{\mathrm{x}}}{\partial x} dx \\ &= -\rho D \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x}\left(-\rho D \frac{\partial \xi}{\partial x}\right) dx. \end{aligned} \tag{V.1.5}$$

3 Inserting and rearranging:

$$0 = \frac{\partial^2 \xi}{\partial x^2}. \tag{V.1.6}$$

4 Defining the boundary and/or initial conditions:

Since the differential equation has been differentiated twice with respect to x, two boundary conditions are required.

The first boundary condition yields the mass fraction given atop of the wood:

$$\xi(x = 0) = \xi_{\mathrm{T}}, \tag{V.1.7}$$

and the second boundary condition from the mass fraction at the bottom of the wood:

$$\xi(x = d) = \xi_{\mathrm{B}}. \tag{V.1.8}$$

5 Solving the equation:

By integrating the differential equation twice, the following expression for the mass fraction distribution is found:

$$\xi(x) = Ax + B, \tag{V.1.9}$$

where using the boundary conditions yields that $A = \frac{\xi_{\mathrm{B}} - \xi_{\mathrm{T}}}{d}$ and $B = \xi_{\mathrm{T}}$.

Conclusion

Hence, the mass fraction distribution is written as:

$$\xi(x) = \frac{\xi_{\mathrm{B}} - \xi_{\mathrm{T}}}{d} x + \xi_{\mathrm{T}}. \tag{V.1.10}$$