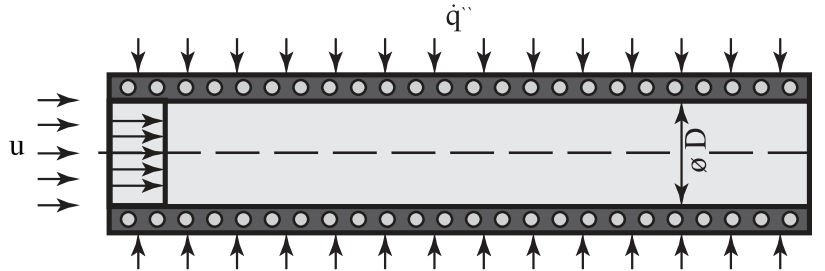


Exercise III.9: (Pipe flow with a constant heat flux **)

A fluid flows through a long cylindrical tube. A constant heat flux density \dot{q}'' is imposed on the fluid.

**Given parameters:**

- Average axial velocity: u
- Heat flux density: \dot{q}''
- Fluid density: ρ
- Fluid thermal capacity: c_p
- Fluid thermal conductivity: λ
- Inner pipe diameter: D

Hint:

- Axial heat diffusion cannot be neglected.

Tasks:

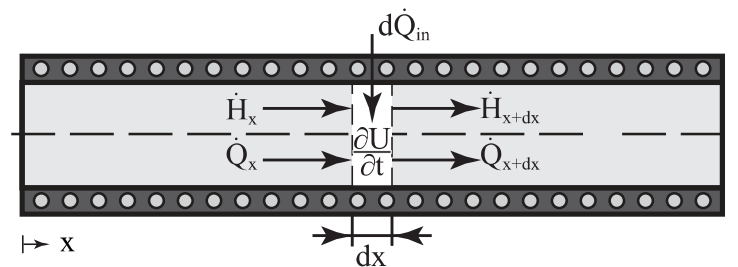
- a) Derive the transient differential energy equation for the mean fluid temperature along the axial direction.

Solution III.9: (Pipe flow with a constant heat flux ★★)**Task a)**

The temperature distribution is derived from the energy conservation equation for a specific system. The energy balance for an infinitesimal element within a specified domain must be derived and solved to derive this distribution.

1 Setting up the balance:

Energy transport due to fluid motion, energy transport from the environment, and energy transport due to diffusion all contribute to the balance as the fluid undergoes heating within the pipe.



The transient energy balance for an infinitesimal element including both advection and diffusion within the pipe reads:

$$\underbrace{\frac{\partial U}{\partial t}}_{\text{Temporal change of inner energy}} = \underbrace{\dot{H}_x - \dot{H}_{x+dx}}_{\text{Net rate of advection}} + \underbrace{\dot{Q}_x - \dot{Q}_{x+dx}}_{\text{Net rate of diffusion}} + \underbrace{d\dot{Q}_{in}}_{\text{External heating}} \quad (\text{III.9.1})$$

2 Defining the elements within the balance:

The temporal change of inner energy is written as:

$$\begin{aligned} \frac{\partial U}{\partial t} &= dm \, c_p \frac{\partial T}{\partial t} \\ &= \rho \frac{\pi D^2}{4} \cdot dx \, c_p \frac{\partial T}{\partial t} \end{aligned} \quad (\text{III.9.2})$$

The rate of heat transport entering due to the motion of the fluid is stated as:

$$\begin{aligned} \dot{H}_x &= \dot{m} c_p T(x) \\ &= \rho u \frac{\pi D^2}{4} c_p T(x). \end{aligned} \quad (\text{III.9.3})$$

and the rate of heat transfer transport leaving due to the motion of the fluid is approximated using the Taylor series expansion:

$$\begin{aligned} \dot{H}_{x+dx} &= \dot{H}_x + \frac{\partial \dot{H}_x}{\partial x} \cdot dx \\ &= \rho u \frac{\pi D^2}{4} c_p T(x) + \frac{\partial}{\partial x} \left(\rho u \frac{\pi D^2}{4} c_p T(x) \right) \cdot dx. \end{aligned} \quad (\text{III.9.4})$$

The rate of heat diffusion entering due to the motion of the fluid is defined by the use of Fourier's law:

$$\begin{aligned}\dot{Q}_x &= -\lambda A_c \frac{\partial T}{\partial x} \\ &= -\lambda \frac{\pi D^2}{4} \frac{\partial T}{\partial x},\end{aligned}\tag{III.9.5}$$

and the rate of heat diffusion leaving due to the motion of the fluid is approximated by the use of the Taylor series expansion:

$$\begin{aligned}\dot{Q}_{x+dx} &= \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} \cdot dx \\ &= -\lambda \frac{\pi D^2}{4} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-\lambda \frac{\pi D^2}{4} \frac{\partial T}{\partial x} \right) \cdot dx.\end{aligned}\tag{III.9.6}$$

The incoming rate of heat transfer from the uniform heat flux is written as:

$$\begin{aligned}d\dot{Q}_{\text{in}} &= \dot{q}'' \cdot dA_s \\ &= \dot{q}'' \cdot \pi D \cdot dx.\end{aligned}\tag{III.9.7}$$

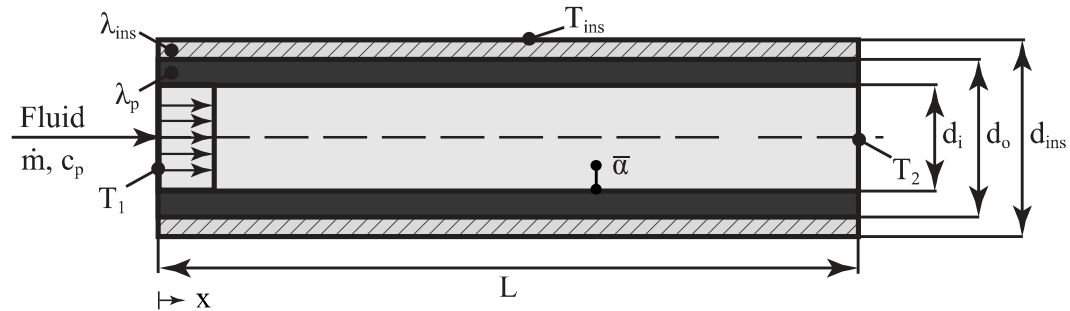
Conclusion

3 Inserting and rearranging:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \frac{\lambda}{\rho c_p} \frac{\partial^2 T}{\partial x^2} + \frac{4\dot{q}''}{\rho c_p D}.\tag{III.9.8}$$

Exercise III.10: (Insulated pipe ★★)

A pipe is being heated by a stationary flow. The outer surface of the pipe has an insulation layer with its external side kept at a constant temperature T_{ins} .



Given parameters:

- Temperature of the fluid at the inlet: T_1
- Temperature of outer surface area of the pipe: T_{ins}
- Convective heat transfer coefficient: $\bar{\alpha}$
- Mass flow of the fluid: \dot{m}
- Specific heat capacity of the fluid: c_p
- Inner diameter of the pipe: d_i
- Outer diameter of the pipe excluding insulation: d_o
- Outer diameter of the pipe including insulation: d_{ins}
- Length of the pipe: L
- Thermal conductivity of the pipe wall: λ_p
- Thermal conductivity of the insulation layer: λ_{ins}

Tasks:

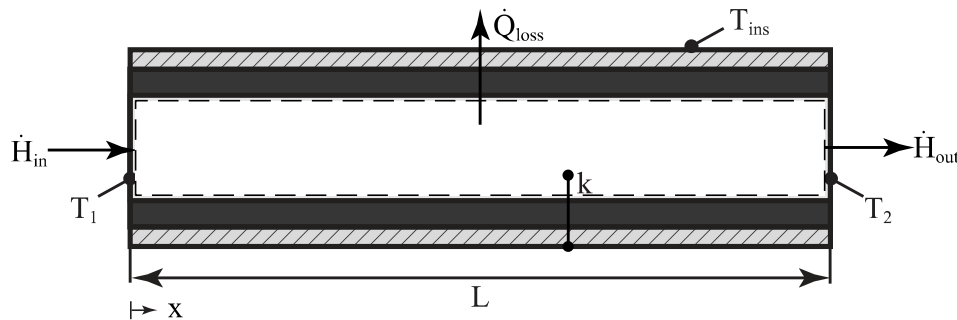
- a) Find an expression for the exit temperature T_2 in terms of given parameters.

Solution III.10: (Insulated pipe ★★)**Task a)**

The exit temperature distribution is derived from a global energy balance around the pipe.

① Setting up the balance:

The problem involves hot fluid flowing through a pipe, where heat is lost to the environment through the pipe wall via conduction. The difference in energy between the fluid entering and leaving the pipe equals the heat loss to the surroundings.



The steady-state global energy balance around the pipe reads:

$$0 = \dot{H}_{\text{in}} - \dot{H}_{\text{out}} - \dot{Q}_{\text{loss}}. \quad (\text{III.10.1})$$

② Defining the elements within the balance:

The rate of heat transport entering due to the motion of the fluid is stated as:

$$\dot{H}_{\text{in}} = \dot{m} c_p T_1, \quad (\text{III.10.2})$$

and the rate of heat transfer transport leaving as:

$$\dot{H}_{\text{out}} = \dot{m} c_p T_2. \quad (\text{III.10.3})$$

The rate of heat loss through the wall is written as the product of the overall heat transfer coefficient, the surface area, and the logarithmic temperature difference:

$$\begin{aligned} \dot{Q}_{\text{loss}} &= k A_s \Delta T_m \\ &= k \pi d L \Delta T_m, \end{aligned} \quad (\text{III.10.4})$$

where the logarithmic temperature difference is defined as:

$$\Delta T_m = \frac{T_1 - T_2}{\ln \left(\frac{T_2 - T_{\text{ins}}}{T_1 - T_{\text{ins}}} \right)}, \quad (\text{III.10.5})$$

and the overall heat transfer coefficient as:

$$k = \left[\frac{1}{\alpha} + \frac{d_i}{2} \left(\frac{1}{\lambda_p} \ln \left(\frac{d_o}{d_i} \right) + \frac{1}{\lambda_{\text{ins}}} \ln \left(\frac{d_{\text{ins}}}{d_o} \right) \right) \right]^{-1}. \quad (\text{III.10.6})$$

Conclusion

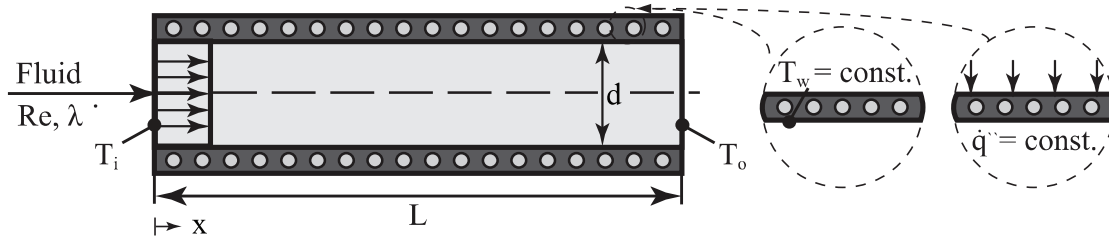
3 Inserting and rearranging:

$$T_2 = T_1 - \frac{1}{\dot{m}c_p} \left[\frac{1}{\bar{\alpha}} + \frac{d_i}{2} \left(\frac{1}{\lambda_p} \ln \left(\frac{d_o}{d_i} \right) + \frac{1}{\lambda_{\text{ins}}} \ln \left(\frac{d_{\text{ins}}}{d_o} \right) \right) \right]^{-1} \pi d L \frac{T_1 - T_2}{\ln \left(\frac{T_2 - T_{\text{ins}}}{T_1 - T_{\text{ins}}} \right)}. \quad (\text{III.10.7})$$

Exercise III.11: (Heating of a pipe ★ ★ ★)

A fluid is flowing through a pipe. The flow is thermally and hydrodynamically developed. A heat flow that is transferred from the wall by convection is heating the fluid from T_i to T_o . For this purpose

- in case 1: a constant, homogeneous **wall temperature** T_w is impressed,
- in case 2: a constant, homogeneous **heat flux** \dot{q}'' is impressed.

**Given parameters:**

- | | |
|---|-------------|
| • Temperature of the fluid at the inlet: | T_i |
| • Temperature of the fluid at the outlet: | T_o |
| • Wall temperature (case 1): | T_w |
| • Heat flux (case 2): | \dot{q}'' |
| • Length of the pipe: | L |
| • Inner diameter of the pipe: | d |
| • Reynolds number of the flow: | $Re < 2300$ |
| • Density of the fluid: | λ |
| • Dynamic viscosity of the fluid: | η |
| • Conductivity of the fluid: | λ |

Hints:

- Heat conduction in the direction of the fluid flow is negligible.
- Difference in fluid properties in the radial direction is negligible.

Tasks:

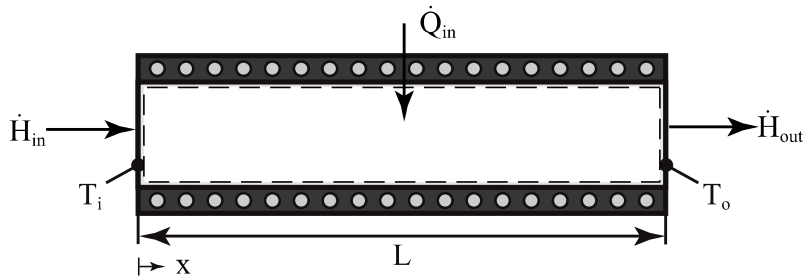
- Write out the global energy balance in terms of given parameters, determine the mean heat transfer coefficient $\bar{\alpha}$ and provide the respective mean temperature difference ΔT_m between the inner wall of the tube and the fluid for both cases.
- Draw qualitatively for both cases the profile of the wall temperature T_w and the mean fluid temperature T_f .

Solution III.11: (Heating of a pipe ★★★)**Task a)**

First, the global energy balance around the entire pipe must be defined.

① Setting up the balance:

The scenario entails a cold fluid moving within a pipe, whereby heat is absorbed from the surroundings. In case 1, the situation involves a uniform and constant hot wall temperature, while in case 2, a consistently applied heat flux is involved. The energy difference between the fluid entering and exiting the pipe equals the absorbed heat.



The energy balance around the entire pipe for both cases reads:

$$0 = \underbrace{\dot{H}_{\text{in}} - \dot{H}_{\text{out}}}_{\text{Net rate of advection}} + \underbrace{\dot{Q}_{\text{in}}}_{\text{External heating}} \quad (\text{III.11.1})$$

② Defining the elements within the balance:

The rate of heat transport entering due to the motion of the fluid is stated as:

$$\begin{aligned} \dot{H}_{\text{in}} &= \dot{m} c_p T_i \\ &= \rho u \frac{\pi d^2}{4} c_p T_i, \end{aligned} \quad (\text{III.11.2})$$

where the fluid velocity is found by rewriting the expression of the Reynolds number:

$$\begin{aligned} \text{Re} &= \frac{\rho u d}{\eta} \\ \Rightarrow u &= \frac{\eta \text{Re}}{\rho d}. \end{aligned} \quad (\text{III.11.3})$$

The rate of heat leaving is written as:

$$\dot{H}_{\text{out}} = \frac{\pi \eta \text{Re} d}{4} c_p T_o. \quad (\text{III.11.4})$$

The rate of heat transfer through the wall is written as the product of the mean heat transfer coefficient, the surface area, and the logarithmic temperature difference:

$$\dot{Q}_{\text{in}} = \bar{\alpha} \pi d \Delta T_m. \quad (\text{III.11.5})$$

For case 1, where there is a laminar flow within an isothermal circular pipe, HTC 13a is used. This correlation shows that if $L \gg L_{th}$, $\overline{Nu}_{d,\infty} = 3.66$, and thus:

$$\begin{aligned}\overline{Nu}_{d,1} &= \frac{\bar{\alpha}_1 d}{\lambda} = 3.66 \\ \Rightarrow \bar{\alpha}_1 &= \frac{3.66\lambda}{d},\end{aligned}\quad (III.11.6)$$

where the logarithmic temperature difference is used as the respective mean temperature difference:

$$\Delta T_{m,1} = \frac{T_i - T_o}{\ln\left(\frac{T_o - T_w}{T_i - T_w}\right)}.\quad (III.11.7)$$

For case 2, where laminar flow within a circular pipe with a constant heat flux is imposed, HTC 13b is used. This correlation states that if $L \gg L_{th}$, $\overline{Nu}_{d,\infty} = 4.36$, and thus:

$$\begin{aligned}\overline{Nu}_{d,2} &= \frac{\bar{\alpha}_2 d}{\lambda} = 4.36 \\ \Rightarrow \bar{\alpha}_2 &= \frac{4.36\lambda}{d}.\end{aligned}\quad (III.11.8)$$

The respective mean temperature difference used for this case is the difference between the in- and outlet temperature:

$$\Delta T_{m,2} = T_o - T_i,\quad (III.11.9)$$

which can alternatively be written as:

$$\Delta T_{m,2} = \frac{\dot{q}''}{\bar{\alpha}_2}.\quad (III.11.10)$$

Conclusion

3 Inserting and rearranging:

For case 1:

$$0 = \frac{\pi\eta Red}{4} c_p (T_i - T_o) + 3.66\pi\lambda \frac{T_i - T_o}{\ln\left(\frac{T_o - T_w}{T_i - T_w}\right)}.\quad (III.11.11)$$

For case 2:

$$0 = \frac{\pi\eta Red}{4} c_p (T_i - T_o) + 4.36\pi\lambda (T_o - T_i).\quad (III.11.12)$$

Task b)

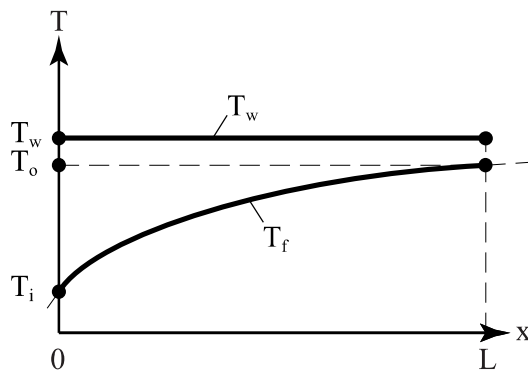
For case 1, the pipe's wall is maintained at a constant temperature. As the fluid flows through the pipe, heat is continuously absorbed from the wall. Initially, the fluid temperature at the inlet differs from the wall temperature. However, as the fluid progresses along the pipe's length, it absorbs more heat, causing the fluid temperature to increase gradually.

The temperature profile within the pipe tends to approach the wall temperature asymptotically. In other words, the fluid temperature will keep rising, and depending on the length of the pipe and the flow rate, the fluid temperature will eventually converge to the wall temperature. This convergence occurs because heat transfer between the fluid and the wall balances out over the length of the pipe, leading to a stabilized temperature distribution.

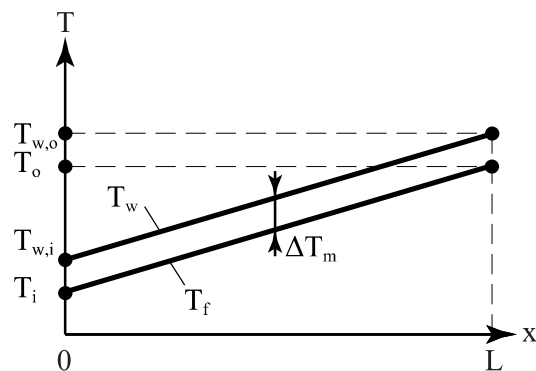
For case 2, there is a constant heat flux imposed on the pipe's wall. This means that the rate at which heat is transferred to the fluid remains constant along the length of the pipe. As the fluid flows, heat is absorbed from the wall at a steady rate.

Similar to the constant wall temperature case, the fluid temperature increases as the fluid moves along the pipe. However, in this scenario, the wall temperature also increases at the same rate as the fluid temperature due to the constant heat flux boundary condition.

Conclusion



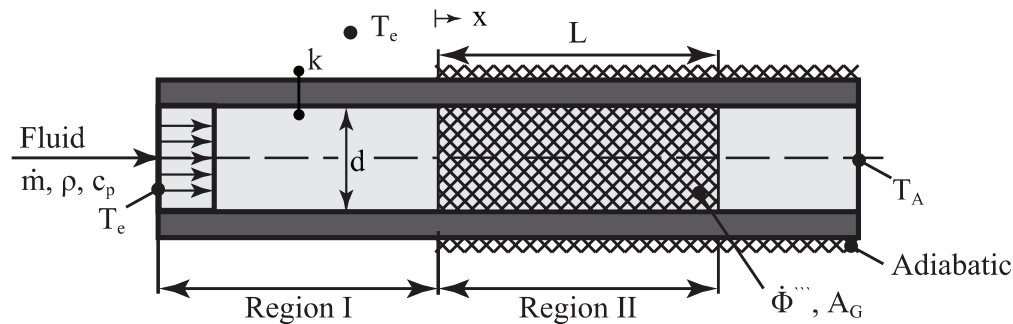
(a) Case 1



(b) Case 2

Exercise III.12: (Flow through a grid ★★★)

Water flows through a long tube that has adiabatic walls from a certain location $x = 0$. The area upstream of $x = 0$ is named region I. Between the point $x = 0$, and $x = L$ (region II) a very fine-meshed, electrically heated grid is located in the flow. Well ahead of the grid, the flow has the ambient temperature T_e , and downstream of the grid, the temperature T_A .

**Given parameters:**

- Water temperature before the grid: T_e
- Environment temperature: T_e
- Water temperature after the grid: T_A
- Mass flow rate: \dot{m}
- Thermal conductivity: λ
- Specific heat capacity: c_p
- Diameter of the pipe/grid: d
- Length of the grid: L
- Average heat flux on the surface of the grid: \dot{q}''
- Heat transfer area of the grid: A_G
- Overall heat transfer coefficient between water and environment, based on the inner pipe wall area: k

Hint:

- The problem is steady and one-dimensional.
- The electrically heated mesh is so fine that a homogeneous heat flux is introduced.
- The volume of the fine-meshed grid can be neglected.

Tasks:

- a) Determine the volumetric heat release $\dot{\Phi}'''$ created by the electrically heated grid.
- b) Derive the differential equations for the temperature profile of the water in the pipe in regions I and II. It is unknown whether heat diffusion is negligible, and thus should be included in the equation.
- c) Provide all the coupling or boundary conditions required for the solution of the problem (regions I and II).
- d) Sketch the temperature profiles of the water in the pipe with and without consideration of the diffusive heat transport.

Solution III.12: (Flow through a grid ★★)**Task a)**

The fluid flows through a very fine-meshed electrically heated grid. This grid can be considered to act as a source term, which generates heat due to the Joule heating effect. When an electric current passes through the grid, resistance within the material causes it to heat up. This heat is transferred to the fluid flowing around the grid.

To determine the volumetric heat release $\dot{\Phi}''' \left(\frac{\text{W}}{\text{m}^3} \right)$, the generated heat within region II $\dot{\Phi}$ (W) must be divided by the volume of region II $V_{\text{II}} \left(\text{m}^3 \right)$.

The total heat generated $\dot{\Phi}$ (W) by this grid yields from the product of the average heat flux on the surface of the $\dot{q}'' \left(\frac{\text{W}}{\text{m}^2} \right)$ grid and the heat transfer area of the grid $A_G \left(\text{m}^2 \right)$:

$$\dot{\Phi} = \dot{q}'' A_G. \quad (\text{III.12.1})$$

The volume of region II $V_{\text{II}} \left(\text{m}^3 \right)$ is found from the product of the cross-sectional area of the pipe $\frac{\pi D^2}{4} \left(\text{m}^2 \right)$ and the length of region II $L \left(\text{m} \right)$:

$$V_{\text{II}} = \frac{\pi d^2}{4} L, \quad (\text{III.12.2})$$

Conclusion

which gives the volumetric heat release:

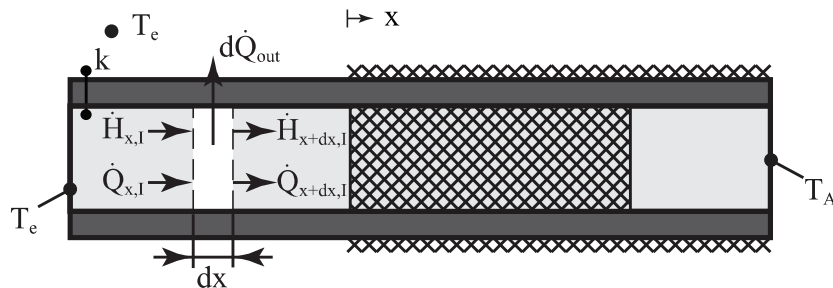
$$\dot{\Phi}''' = \frac{4\dot{q}'' A_G}{\pi d^2 L}. \quad (\text{III.12.3})$$

Task b)

The temperature profile is derived from the energy conservation equation for a specific system. To derive the temperature profile for regions I and II, the energy balance for an infinitesimal element within both domains must be derived and solved to derive these profiles.

① Setting up the balance:

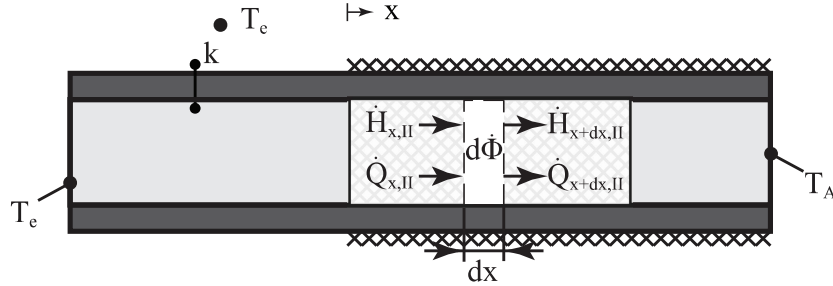
Within the initial section of the pipe, heat dissipation to the surroundings occurs when a temperature gradient exists between the fluid and its environment. This phenomenon arises particularly when heat diffusion opposes the direction of fluid flow. Consequently, the energy balance within this region encompasses the net rates of advection and diffusion, along with external losses.



The steady-state energy balance for an infinitesimal element in region I reads:

$$0 = \underbrace{\dot{H}_{x,I} - \dot{H}_{x+dx,I}}_{\text{Net rate of advection}} + \underbrace{\dot{Q}_{x,I} - \dot{Q}_{x+dx,I}}_{\text{Net rate of diffusion}} - \underbrace{d\dot{Q}_{\text{out}}}_{\text{External loss}}. \quad (\text{III.12.4})$$

In the second region, the pipe is adiabatically insulated, eliminating any losses to the environment. Here, the fluid flows through a grid structure, where heat generation becomes a significant factor in the energy balance. This equilibrium encompasses the terms of advection, diffusion, and heat generation as the fluid traverses this section.



The steady-state energy balance for an infinitesimal element in region II is written as:

$$0 = \underbrace{\dot{H}_{x,II} - \dot{H}_{x+dx,II}}_{\text{Net rate of advection}} + \underbrace{\dot{Q}_{x,II} - \dot{Q}_{x+dx,II}}_{\text{Net rate of diffusion}} + \underbrace{d\dot{\Phi}}_{\text{Heat generation}}. \quad (\text{III.12.5})$$

2 Defining the elements within the balance:

The ingoing conductive heat flux is described by use of Fourier's Law, which reads:

$$\begin{aligned} \dot{Q}_x &= -\lambda A_c \frac{\partial T}{\partial x} \\ &= -\lambda \frac{\pi d^2}{4} \frac{\partial T}{\partial x}. \end{aligned} \quad (\text{III.12.6})$$

The outgoing conductive heat flux for an infinitesimal element is approximated by use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{x+dx} &= \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} \cdot dx \\ &= -\lambda \frac{\pi d^2}{4} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-\lambda \frac{\pi d^2}{4} \frac{\partial T}{\partial x} \right) \cdot dx. \end{aligned} \quad (\text{III.12.7})$$

The rate of ingoing enthalpy flow is described as:

$$\dot{H}_x = \dot{m} c_p T. \quad (\text{III.12.8})$$

Like the outgoing conductive heat flux, the outgoing enthalpy flux is approximated using the Taylor series expansion. This expansion provides an approximation that is expressed as follows:

$$\begin{aligned} \dot{H}_{x+dx} &= \dot{H}_x + \frac{\partial \dot{H}_x}{\partial x} \cdot dx \\ &= \dot{m} c_p T(x) + \frac{\partial}{\partial x} (\dot{m} c_p T(x)) \cdot dx. \end{aligned} \quad (\text{III.12.9})$$

For region I, where is gained from the ambient, the flux is described as the product of the overall heat transfer coefficient k , the surface area of the element, and the temperature difference of the fluid and the ambient:

$$\begin{aligned} d\dot{Q}_{\text{out}} &= k dA_s (T_I(x) - T_e) \\ &= k \pi d dx (T_I(x) - T_e). \end{aligned} \quad (\text{III.12.10})$$

For region II, where heat is generated, the heat generated within the infinitesimal element is written as:

$$\begin{aligned} d\dot{\Phi} &= \dot{\Phi}''' dV \\ &= \frac{\dot{q}'' A_G}{L} dx. \end{aligned} \quad (\text{III.12.11})$$

Conclusion

3 Inserting and rearranging:

For region I:

$$\dot{m} c_p \frac{\partial T_I}{\partial x} = -\lambda \frac{\pi d^2}{4} \frac{\partial^2 T_I}{\partial x^2} - k \pi d (T_I(x) - T_e),$$

and for region II:

$$\dot{m} c_p \frac{\partial T_{II}}{\partial x} = -\lambda \frac{\pi d^2}{4} \frac{\partial^2 T_{II}}{\partial x^2} + \frac{\dot{q}'' A_G}{L}.$$

Task c)

4 Defining the boundary and/or initial conditions:

Conclusion

Since both differential equations have been differentiated twice with respect to space x , four unique boundary/coupling conditions are required to solve this set of equations.

The fluid entering region I has a constant temperature, denoted as T_e , which remains unchanged at that point. Therefore, one of the following two boundary conditions must be considered for region I:

$$T_I(x \rightarrow -\infty) = T_e, \quad (\text{III.12.12})$$

alternatively, given that $T_I(x \rightarrow -\infty) = T_e$, no heat transfer from the ambient environment will occur. Consequently, a horizontal gradient within the temperature profile is observed. Thus, this boundary condition can also be expressed as:

$$\left. \frac{\partial T_I}{\partial x} \right|_{x \rightarrow -\infty} = 0. \quad (\text{III.12.13})$$

The remaining three boundary conditions required to solve the equations must be picked from the latter four.

The temperature of regions I and II at the interface are equal to each other:

$$T_I(x=0) = T_{II}(x=0), \quad (\text{III.12.14})$$

similarly the the rate of heat transfer at this interface:

$$\left. \frac{\partial T_I}{\partial x} \right|_{x=0} = \left. \frac{\partial T_{II}}{\partial x} \right|_{x=0}. \quad (\text{III.12.15})$$

At $x = L$ the temperature in region II equals T_A :

$$T_{II}(x = L) = T_A. \quad (\text{III.12.16})$$

Lastly, no heat is transferred anymore at that position. So, therefore:

$$\left. \frac{\partial T_{II}}{\partial x} \right|_{x=L} = 0. \quad (\text{III.12.17})$$

Task d)

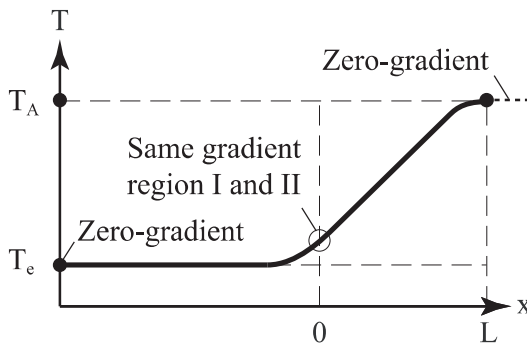
In the case of including diffusive heat transport, heat is transferred in two directions. Advective transport, driven by the motion of the fluid, occurs in the flow direction, while diffusive transport moves in the direction where the temperature decreases, which is opposite to the flow direction.

Within a significant portion of region I, the fluid temperature remains constant and equal to the inlet temperature T_e , as minimal heat exchange occurs with the environment. However, near the entrance of region II, the effect of diffusive heat transport becomes significant, causing the fluid to heat up before entering this region. This effect doesn't permeate throughout the entirety of region I since the heat is lost to the environment.

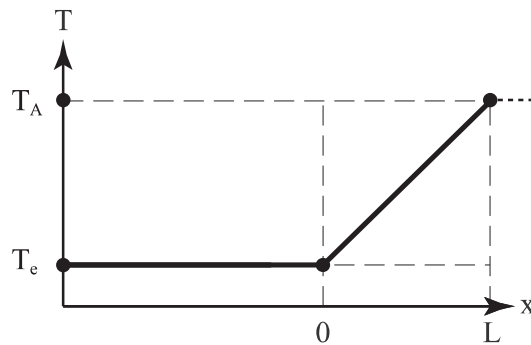
As the fluid enters region II, the temperature gradually increases until reaching the end of this region. At this point, the slope of the temperature profile gradually decreases as the effect of diffusion becomes less significant, eventually reaching temperature T_A . Beyond this point, no further changes in temperature occur, and the temperature remains constant.

In the scenario without diffusive heat transport, the effect of diffusion near the entrance of region II is absent. Consequently, the temperature within region I remains at T_e throughout this entire region. Upon entering region II, the temperature increases steadily at a constant rate until reaching the outlet of region II. After this point, no further changes in temperature are observed.

Conclusion



(a) With diffusive heat transport.



(b) Without diffusive heat transport.