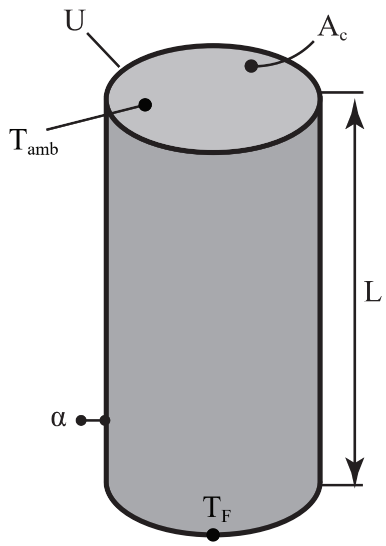
**Exercise II.7** (Pin-fin cooling on gas turbine blades ⋆⋆)**:**

A rod fin is used for pin-fin cooling on gas turbine blades. The rod-fin of length $L$, with a tip temperature equal to the ambient $T_{\mathrm{amb}}$, is given.




**Given parameters:**

- Fin geometry: $U, A_{\mathrm{c}}, L$
- Fin material properties: $\lambda$
- Surface heat transfer coefficient: $\alpha$
- Fin base temperature and environment temperature: $T_{\mathrm{F}}, T_{\mathrm{amb}}$

**Tasks:**

a) Derive the heat conduction equation for the given problem.

b) Derive the function of the temperature profile inside the fin.

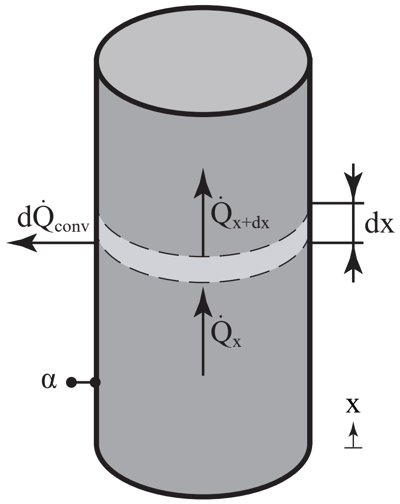
c) Give the expression for the rate of heat loss in terms of the given parameters.

**Solution II.7** (Pin-fin cooling on gas turbine blades ★★):

Task a)

① **Setting up the balance:**

Before doing the calculations, it is necessary to have a grasp of the system. In this particular scenario, we are working with a fin that conducts heat through its body and dissipates it through convection to the surrounding environment.




The heat conduction equation inside the fin can be derived based on the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} - d\dot{Q}_{\mathrm{conv}} \tag{II.7.1}$$

② **Defining the elements within the balance:**

The ingoing rate of heat transfer can be derived from Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} \tag{II.7.2}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer can be approximated by use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx \end{aligned} \tag{II.7.3}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element can be written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{amb}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{amb}}\right) \end{aligned} \tag{II.7.4}$$

Conclusion

### 3 Inserting and rearranging:

Inserting the definitions of the heat fluxes into the energy balance and doing some rewriting yields the heat conduction equation:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c} \cdot (T(x) - T_{amb}) \tag{II.7.5}$$

Task b)

### 4 Defining the boundary and/or initial conditions:

The differential equation can be solved with the application of two boundary conditions, a consequence of the fact that the temperature has undergone two differentiations with respect to $x$.

The temperature at the fin base is:

$$T(x = 0) = T_F \tag{II.7.6}$$

The temperature at the fin head is:

$$T(x = L) = T_{amb} \tag{II.7.7}$$

### 5 Solving the equation:

As we are dealing with a 2nd order differential equation, it should be homogenized to solve this equation.

The differential equation for a steady-state fin can be solved by introducing the temperature difference $\theta$ and the fin parameter $m$:

$$\theta = T(x) - T_{amb} \tag{II.7.8}$$

$$m^2 = \frac{\alpha U}{\lambda A_c} \tag{II.7.9}$$

Substitution into the differential equation, given in equation (II.7.5), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2 \theta \tag{II.7.10}$$

Furthermore, the boundary conditions can be rewritten in terms of the temperature difference $\theta$ as well:

$$\theta(x = 0) = T_F - T_{amb} = \theta_F \tag{II.7.11}$$

$$\theta(x = L) = T_{amb} - T_{amb} = 0 \tag{II.7.12}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x) \tag{II.7.13}$$

The constants A and B can be determined by using the boundary conditions. Whereas, constant A yields from:

$$\theta(x = 0) = A \cdot \cosh(0) + B \cdot \sinh(0) = \theta_F \tag{II.7.14}$$

$$\Rightarrow A = \theta_F$$

and constant B yields from:

$$\theta(x=L) = A \cdot \cosh(mL) + B \cdot \sinh(mL) = 0 \qquad \text{(II.7.15)}$$
$$\Rightarrow B = -A \cdot \frac{1}{\tanh(mL)}$$

Plugging the constants into the standard solution yields the equation for the fin temperature:

$$\theta(x) = \theta_F \cdot \left(\cosh(m \cdot x) - \frac{\sinh(m \cdot x)}{\tanh(mL)}\right) \qquad \text{(II.7.16)}$$

Conclusion

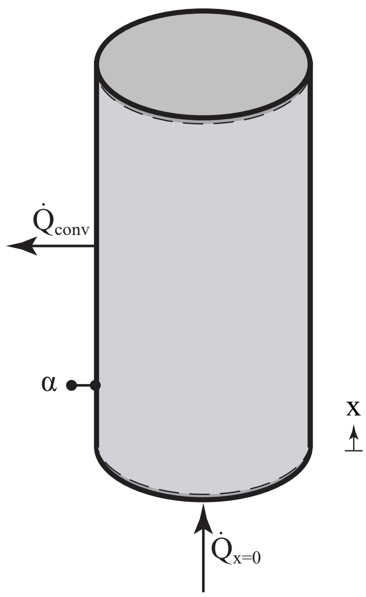
Substitution of temperature difference $\theta$ back and rewriting yields the temperature profile inside the fin:

$$T(x) = T_{amb} + (T_F - T_{amb}) \cdot \left(\cosh(m \cdot x) - \frac{\sinh(m \cdot x)}{\tanh(mL)}\right) \qquad \text{(II.7.17)}$$

Task c)

1 **Setting up the balance:**

To determine the total rate of heat loss, an energy balance over the entire fin is required. It conducts heat from its base at $x = 0$, which is then lost by convection to the ambient.




Therefore the energy balance reads:

$$0 = \dot{Q}_{x=0} - \dot{Q}_{conv} \qquad \text{(II.7.18)}$$

### 2 Defining the elements within the balance:

From the energy balance it yields that $\dot{Q}_{x=0} = \dot{Q}_{\text{conv}}$. Both can be expressed therefore by use of Fourier's law:

$$\dot{Q}_{x=0} = \dot{Q}_{\text{conv}} = -\lambda A_{\text{c}} \cdot \left.\frac{\partial T}{\partial x}\right|_{x=0} \tag{II.7.19}$$

where:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = (T_{\text{F}} - T_{\text{amb}}) \cdot \left( m \cdot \sinh(0) - \frac{m \cdot \cosh(0)}{\tanh(mL)} \right) = -\frac{(T_{\text{F}} - T_{\text{amb}}) \cdot m}{\tanh(mL)} \tag{II.7.20}$$

Conclusion

### 3 Inserting and rearranging:

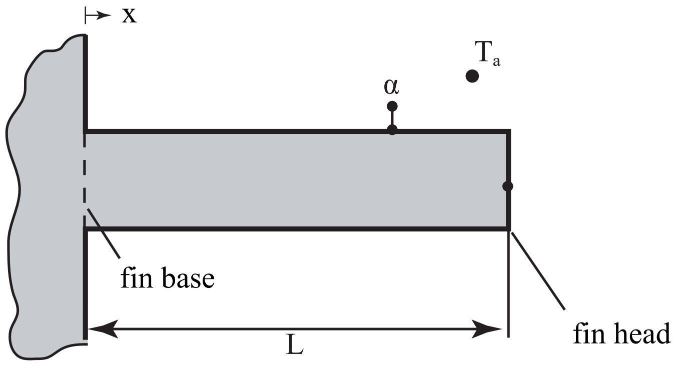
Substitution of all terms yields the rate of heat loss in terms of the given parameters:

$$\dot{Q}_{x=0} = \dot{Q}_{\text{conv}} = \lambda A_{\text{c}} \frac{(T_{\text{F}} - T_{\text{amb}}) \cdot m}{\tanh(mL)} \tag{II.7.21}$$

where $m = \sqrt{\frac{\alpha U}{\lambda A_{\text{c}}}}$ had been defined before.

**Exercise II.8** (New fin material ⋆⋆):

An electric motor manufacturer is using fins for cooling purposes. He is considering changing the material used for the fins from copper to aluminum. Because the length $L$ of the fin is also modified, the temperature at the fin head remains identical for both materials. However, he does not understand the impact of such a change on the performance of cooling.




**Given parameters:**

- Thermal conductivity of copper: $\lambda_{\mathrm{C}}$
- Thermal conductivity of aluminium: $\lambda_{\mathrm{A}}$

**Hints:**

- The cross-section and the thickness remain unchanged.
- There is no change in the convective heat transfer coefficient.
- The temperature at the fin base does not change.
- For both fins, the heat flow through the head is negligible.

**Tasks:**

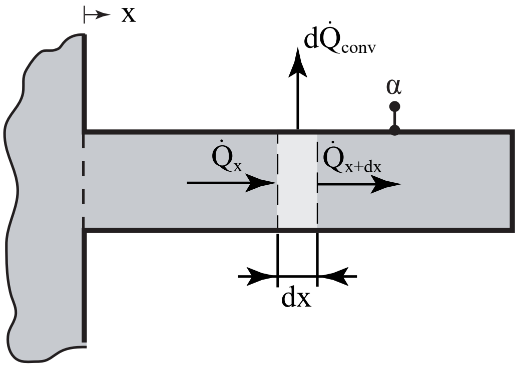
a) Determine the ratio between the heat flow of the aluminum and the copper fin in terms of given parameters.

**Solution II.8** (New fin material ⋆⋆):

Task a)

1 **Setting up the balance:**

Before doing the calculations, it is necessary to have a grasp of the system. In this particular scenario, we are working with a fin that conducts heat through its body and dissipates it through convection to the surrounding environment.




The heat conduction equation inside the fin can be derived based on the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} - d\dot{Q}_{\mathrm{conv}} \tag{II.8.1}$$

2 **Defining the elements within the balance:**

The ingoing rate of heat transfer can be derived from Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} \tag{II.8.2}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer can be approximated by use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx \end{aligned} \tag{II.8.3}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element can be written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{a}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{a}}\right) \end{aligned} \tag{II.8.4}$$

### 3 Inserting and rearranging:

Inserting the definitions of the heat fluxes into the energy balance and doing some rewriting yields the heat conduction equation:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c} \cdot (T(x) - T_a) \tag{II.8.5}$$

### 4 Defining the boundary and/or initial conditions:

The differential equation can be solved with the application of two boundary conditions, a consequence of the fact that the temperature has been differentiated twice with respect to $x$.

The temperature at the fin base is:

$$T(x = 0) = T_B \tag{II.8.6}$$

Heat flow through the head is negligible, and thus $\dot{Q}_{x=L} = -\lambda A_c \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0$ yields

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \tag{II.8.7}$$

### 5 Solving the equation:

As we are dealing with a 2nd order differential equation, it should be homogenized to solve this equation.

The differential equation for a steady-state fin can be solved by introducing the temperature difference $\theta$ and the fin parameter $m$:

$$\theta = T(x) - T_a \tag{II.8.8}$$

$$m^2 = \frac{\alpha U}{\lambda A_c} \tag{II.8.9}$$

Substitution into the differential equation, given in equation (II.9.14), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2 \theta \tag{II.8.10}$$

Furthermore, the boundary conditions can be rewritten in terms of the temperature difference $\theta$ as well:

$$\theta(x = 0) = T_B - T_{amb} = \theta_B \tag{II.8.11}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0 \tag{II.8.12}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x) \tag{II.8.13}$$

The constants A and B can be determined by using the boundary conditions. Whereas, constant A yields from:

$$\begin{aligned} \theta(x = 0) = A \cdot \cosh(0) + B \cdot \sinh(0) = \theta_B \\ \Rightarrow A = \theta_B \end{aligned} \tag{II.8.14}$$

And constant B yields from:

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = A \cdot m \cdot \sinh(mL) + B \cdot m \cdot \cosh(mL) = 0 \qquad \text{(II.8.15)}$$
$$\Rightarrow B = -A \cdot \tanh(mL)$$

Plugging in the constants yields the equation for the fin temperature:

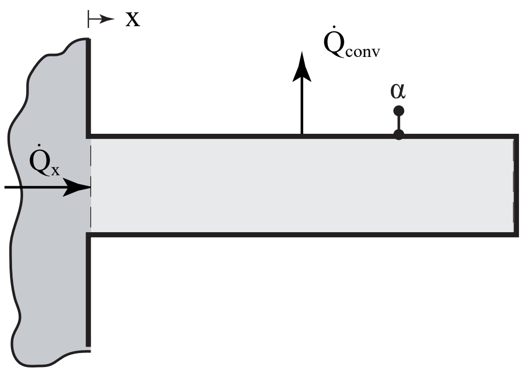
$$\theta(x) = \theta_\mathrm{B} \left(\cosh(m \cdot x) - \tanh(mL)\sinh(m \cdot x)\right) \qquad \text{(II.8.16)}$$
$$= \theta_\mathrm{B} \cdot \frac{\cosh(m \cdot (L-x))}{\cosh(mL)}$$

Substitution of temperature difference $\theta$ back and rewriting yields the temperature profile inside the fin:

$$T(x) = T_\mathrm{a} + (T_\mathrm{B} - T_\mathrm{a})\left(\cosh(m \cdot x) - \tanh(mL)\sinh(m \cdot x)\right) \qquad \text{(II.8.17)}$$

1 **Setting up the balance:**

To determine the total rate of heat loss, an energy balance over the entire fin is required. It conducts heat from its base at $x = 0$, which is then lost by convection to the ambient.




Therefore the energy balance reads:

$$0 = \dot{Q}_{x=0} - \dot{Q}_\mathrm{conv} \qquad \text{(II.8.18)}$$

2 **Defining the elements within the balance:**

From the energy balance it yields that $\dot{Q}_{x=0} = \dot{Q}_\mathrm{conv}$. Both can be expressed therefore by use of Fourier's law:

$$\dot{Q}_{x=0} = \dot{Q}_\mathrm{conv} = -\lambda A_\mathrm{c} \cdot \left.\frac{\partial T}{\partial x}\right|_{x=0} \qquad \text{(II.8.19)}$$

where:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = (T_\mathrm{B} - T_\mathrm{a}) \cdot \left(m \cdot \sinh(0) - m \cdot \tanh(mL) \cdot \cosh(0)\right) \qquad \text{(II.8.20)}$$
$$= -(T_\mathrm{B} - T_\mathrm{a}) \cdot m \cdot \tanh(mL)$$

**3 Inserting and rearranging:**

Substitution of all terms yields the rate of heat loss in terms of the given parameters:

$$\dot{Q}_{x=0} = \dot{Q}_{\mathrm{conv}} = \lambda A_{\mathrm{c}} m \tanh\left(mL\right)\left(T_{\mathrm{B}} - T_{\mathrm{a}}\right) \tag{II.8.21}$$

where $m = \sqrt{\frac{\alpha U}{\lambda A_{\mathrm{c}}}}$ had been defined before.

The rate of heat losses for the aluminium and copper fin can be expressed respectively as:

$$\dot{Q}_{\mathrm{A}} = \lambda_{\mathrm{A}} A_{\mathrm{c}} m_{\mathrm{A}} \tanh\left(m_{\mathrm{A}} L_{\mathrm{A}}\right)\left(T_{\mathrm{B}} - T_{\mathrm{a}}\right) \tag{II.8.22}$$

and:

$$\dot{Q}_{\mathrm{c}} = \lambda_{\mathrm{c}} A_{\mathrm{c}} m_{\mathrm{c}} \tanh\left(m_{\mathrm{c}} L_{\mathrm{c}}\right)\left(T_{\mathrm{B}} - T_{\mathrm{a}}\right) \tag{II.8.23}$$

However, the lengths are not the same. The ratio of the length of both fins can be found by the tip temperature:

$$T(x = L_{\mathrm{A}}) = T_{\mathrm{a}} + \left(T_{\mathrm{B}} - T_{\mathrm{a}}\right)\left(\cosh\left(m_{\mathrm{A}} L_{\mathrm{A}}\right) - \tanh\left(m_{\mathrm{A}} L_{\mathrm{A}}\right)\sinh\left(m_{\mathrm{A}} L_{\mathrm{A}}\right)\right) \tag{II.8.24}$$

$$T(x = L_{\mathrm{C}}) = T_{\mathrm{a}} + \left(T_{\mathrm{B}} - T_{\mathrm{a}}\right)\left(\cosh\left(m_{\mathrm{C}} L_{\mathrm{C}}\right) - \tanh\left(m_{\mathrm{C}} L_{\mathrm{C}}\right)\sinh\left(m_{\mathrm{C}} L_{\mathrm{C}}\right)\right) \tag{II.8.25}$$

Equalling the two expressions for the tip temperature of the aluminum and copper fin yields:

$$m_{\mathrm{A}} L_{\mathrm{A}} = m_{\mathrm{C}} L_{\mathrm{C}} \tag{II.8.26}$$

Finally, determining the ratio between the heat flows, substituting the fin parameter, and canceling all identical terms, including the relationship presented in equation (II.8.26), we obtain:

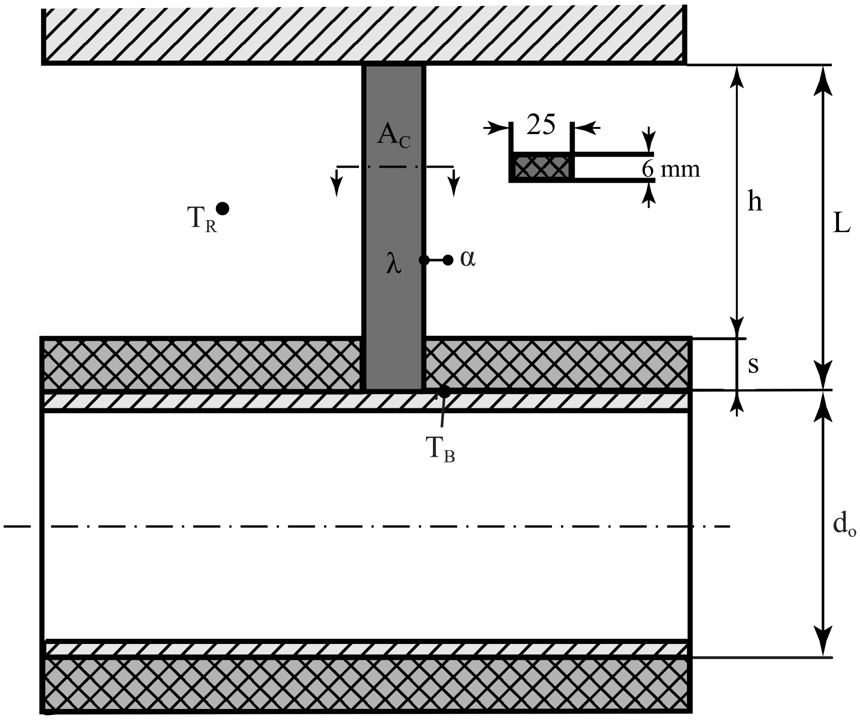
$$\frac{\dot{Q}_{\mathrm{A}}}{\dot{Q}_{\mathrm{C}}} = \frac{\lambda_{\mathrm{A}} A_{\mathrm{c}} \sqrt{\frac{\alpha U}{\lambda_{\mathrm{A}} A_{\mathrm{c}}}}\theta_{\mathrm{B}} \tanh\left(m_{\mathrm{A}} L_{\mathrm{A}}\right)}{\lambda_{\mathrm{C}} A_{\mathrm{c}} \sqrt{\frac{\alpha U}{\lambda_{\mathrm{C}} A_{\mathrm{c}}}}\theta_{\mathrm{B}} \tanh\left(m_{\mathrm{C}} L_{\mathrm{C}}\right)} = \sqrt{\frac{\lambda_A}{\lambda_C}} \tag{II.8.27}$$

**Conclusion**

If the electric motor manufacturer transitions from using copper to aluminum as the material, the impact on the heat flow of aluminum with respect to the copper situation will be $\sqrt{\frac{\lambda_A}{\lambda_C}}$.

**Exercise II.9** (Pipe fastening ⋆ ⋆ ⋆):

A pipe containing brine is insulated with cork and fastened to the ceiling with steel bands welded to the pipe.




**Given parameters:**

- Outer diameter of the pipe: $d_\text{o} = 50$ mm
- Insulation thickness: $s = 40$ mm
- Cross-section of the steel band: $A_\text{c} = 25 \times 6$ mm
- Length of the steel band: $L = 290$ mm
- Heat transfer coefficient at the steel band's surface: $\alpha = 6$ W/m$^2$K
- Thermal conductivity of the steel band: $\lambda = 58$ W/mK
- Temperature outer wall of the brine pipe: $T_\text{B} = -23.5$ °C
- Temperature of the room: $T_\text{R} = 20$ °C

**Hints:**

- The temperature distribution in the steel band's cross-section is homogeneous.
- The heat fluxes from the steel bands into both the ceiling and the insulation are negligible.

**Tasks:**

a) Calculate the heat $\dot{Q}$ from one steel band absorbed by the brine.

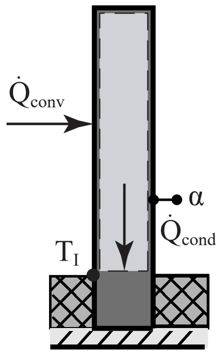
b) Up to which height $h_0$ does frost form on the steel ban ($h_0$ is the distance from the surface of the pipe's insulation layer), if the steam content of the air in the surrounding room is above the saturation vapor pressure for the maximum steel band temperature?

**Solution II.9** (Pipe fastening ⋆ ⋆ ⋆):

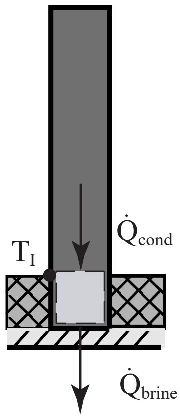
Task a)

1 **Setting up the balance:**

The steel band can be viewed as two interconnected systems in series. The initial segment of the steel band, referred to as the fin part, absorbs heat due to convection. This heat is then conducted until a specific section of the steel band is reached where no convection can occur, and only heat conduction, termed the plane wall part, takes place.




(a) Fin part




(b) Plane wall part

The respective energy balances of the fin and plane wall part read:

$$0 = \dot{Q}_{\text{conv}} - \dot{Q}_{\text{cond}} \tag{II.9.1}$$

and:

$$0 = \dot{Q}_{\text{cond}} - \dot{Q}_{\text{brine}} \tag{II.9.2}$$

2 **Defining the elements within the balance:**

Heat transfer from the steel band into the ceiling and the insulation is negligible, indicating that the fin's head is adiabatic, and all heat conducted through the fin is transferred to the plane wall part.

For a fin with an adiabatic head, the heat transfer rate can be expressed as:

$$\dot{Q}_{\text{conv}} = \lambda A_{\text{c}} m \theta_{\text{I}} \tanh\left(mh\right) \tag{II.9.3}$$

with:

$$m = \sqrt{\frac{\alpha U}{\lambda A}} = \sqrt{\frac{2\alpha\left(a+b\right)}{\lambda ab}} \tag{II.9.4}$$

$$= \sqrt{\frac{6\ [\text{W/m}^2\text{K}] \cdot 2 \cdot (0.025 + 0.006)\ \ [\text{m}]}{58\ [\text{W/mK}] \cdot 0.025\ [\text{m}] \cdot 0.006\ [\text{m}]}} = 6.54 \left[\frac{1}{\text{m}}\right]$$

and:

$$\theta = T_{\text{R}} - T(x) \tag{II.9.5}$$

Note that $\theta \neq T(x) - T_R$, because otherwise the direction of the heat flow will be incorrect

However, $\dot{Q}_{conv}$ incorporates the interface temperature $T_I$, which is still unknown. Hence, a second definition for the rate of heat transfer is necessary to determine this temperature.

We can describe the heat conduction through the plane wall part by use of Fourier's law:

$$\begin{aligned} \dot{Q}_{brine} &= -\lambda A_c \frac{\partial T}{\partial x} \\ &= \lambda A_c \frac{T_I - T_B}{s} \end{aligned} \tag{II.9.6}$$

From both balances it can be found that $\dot{Q}_{conv} = \dot{Q}_{cond} = \dot{Q}_{brine}$

3 **Inserting and rearranging:**

Inserting the found definitions into the energy balances and doing some rewriting yields:

$$\begin{aligned} T_I &= \frac{T_R sm \tanh(mh) + T_B}{sm \tanh(mh) + 1} \\ &= \frac{20\ [°C] \cdot 0.04\ [m] \cdot 6.54 \left[\frac{1}{m}\right] \cdot \tanh\left(6.54 \left[\frac{1}{m}\right] \cdot (0.29 - 0.04)\ \ [m]\right) - 23.5\ [°C]}{0.04\ [m] \cdot 6.54 \left[\frac{1}{m}\right] \cdot \tanh\left(6.54 \left[\frac{1}{m}\right] \cdot (0.29 - 0.04)\ \ [m]\right) + 1} = -15\ [°C] \end{aligned} \tag{II.9.7}$$

Now that we have established the temperature $T_I$ at the interface between the fin system and the plane wall system, we can substitute it into one of the fluxes to ascertain the rate of heat transfer towards the brine:

$$\begin{aligned} \dot{Q}_{brine} &= \lambda A_c \frac{T_I - T_B}{s} \\ &= 56\ [W/mK] \cdot (0.025 \cdot 0.006)\ \ [m^2] \cdot \frac{(-15 + 23.5)\ \ [K]}{0.04\ [m]} = 1.85\ W \end{aligned} \tag{II.9.8}$$

Conclusion

The heat from one steel band absorbed by the brine is 1.85 W.

Task b)

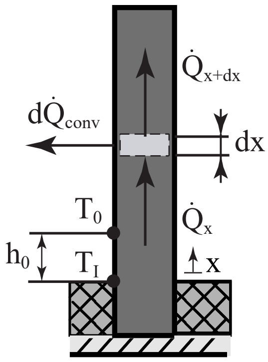
Frost can be formed at a temperature below 0 °C, therefore the following condition needs to be satisfied to determine the height $h_0$ up to which frost forms.

$$T(x = h_0) = T_0 = 0\,[^{\circ}\mathrm{C}] \tag{II.9.9}$$

To find this height, the temperature profile inside the fin part needs to be determined.

1 **Setting up the balance:**

In this scenario, we are working with a fin that conducts heat through its body and gains heat through convection from the surrounding environment.




The heat conduction equation inside the fin can be derived based on the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} - d\dot{Q}_{\mathrm{conv}} \tag{II.9.10}$$

Note that the energy balance assumes heat conduction in the positive x-direction and heat loss by convection. However, the steel band will receive heat due to convection and conduct it in the negative x-direction towards the brine. This direction will be considered when defining the fluxes.

2 **Defining the elements within the balance:**

The ingoing rate of heat transfer can be derived from Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} \tag{II.9.11}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer can be approximated by use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx \end{aligned} \tag{II.9.12}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element can be written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{R}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{R}}\right) \end{aligned} \tag{II.9.13}$$

3 **Inserting and rearranging:**

Inserting the definitions of the heat fluxes into the energy balance and doing some rewriting yields the heat conduction equation:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c} \cdot (T(x) - T_R) \tag{II.9.14}$$

4 **Defining the boundary and/or initial conditions:**

The differential equation can be solved with the application of two boundary conditions, a consequence of the fact that the temperature has undergone two differentiations to $x$.

The temperature of the steel band at the surface of the pipe's insulation layer is:

$$T(x=0) = T_I \tag{II.9.15}$$

Heat flow through the ceiling is negligible, and thus $\dot{Q}_{x=h} = -\lambda A_c \left.\frac{\partial T}{\partial x}\right|_{x=h} = 0$ yields

$$\left.\frac{\partial T}{\partial x}\right|_{x=h} = 0 \tag{II.9.16}$$

5 **Solving the equation:**

As we are dealing with a $2^{nd}$ order differential equation, it should be homogenized to solve this equation.

The differential equation for a steady-state fin can be solved by introducing the temperature difference $\theta$ and the fin parameter $m$:

$$\theta = T(x) - T_R \tag{II.9.17}$$

$$m^2 = \frac{\alpha U}{\lambda A_c} \tag{II.9.18}$$

Substitution into the differential equation, given in equation (II.9.14), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2 \theta \tag{II.9.19}$$

Furthermore, the boundary conditions can be rewritten in terms of the temperature difference $\theta$ as well:

$$\theta(x=0) = T_I - T_{amb} = \theta_I \tag{II.9.20}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0 \tag{II.9.21}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x) \tag{II.9.22}$$

The constants A and B can be determined by using the boundary conditions. Whereas, constant A yields from:

$$\theta(x=0) = A \cdot \cosh(0) + B \cdot \sinh(0) = \theta_I \tag{II.9.23}$$

$$\Rightarrow A = \theta_I$$

And constant B yields from:

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=h} = A\cdot m\cdot\sinh(mh) + B\cdot m\cdot\cosh(mh) = 0 \tag{II.9.24}$$
$$\Rightarrow B = -A\cdot\tanh(mh)$$

Plugging in the constants yields the equation for the fin temperature:

$$\theta(x) = \theta_{\mathrm{I}}\left(\cosh(m\cdot x) - \tanh(mh)\sinh(m\cdot x)\right) \tag{II.9.25}$$
$$= \theta_{\mathrm{I}}\cdot\frac{\cosh(m\cdot(h-x))}{\cosh(mh)}$$

Substitution of temperature difference $\theta$ back and rewriting yields the temperature profile inside the fin:

$$T(x) = T_{\mathrm{R}} + (T_{\mathrm{I}} - T_{\mathrm{R}})\cdot\frac{\cosh(m\cdot(h-x))}{\cosh(mh)} \tag{II.9.26}$$

Substitution of the frost temperature into the temperature profile of the fin and rewriting yields:

$$T(x=h_0) = T_{\mathrm{R}} + (T_{\mathrm{I}} - T_{\mathrm{R}})\cdot\frac{\cosh(m\cdot(h-h_0))}{\cosh(mh)} = T_0 \tag{II.9.27}$$
$$\Rightarrow h_0 = h - \frac{1}{m}\cosh^{-1}\left[\frac{T_0 - T_{\mathrm{R}}}{T_{\mathrm{I}} - T_{\mathrm{R}}}\cosh(mh)\right]$$
$$= 0.25\ [\mathrm{m}] - \frac{1}{0.25\ [\mathrm{m}]}\cosh^{-1}\left[\frac{(0-20)\ {}^\circ\mathrm{C}}{(-15-20)\ {}^\circ\mathrm{C}}\cosh\left(6.54\left[\frac{1}{\mathrm{m}}\right]\cdot 0.25\ [\mathrm{m}]\right)\right] = 0.1\ [\mathrm{m}]$$

Conclusion

Up to 0.1 m from the surface of the pipe's insulation layer frost will form on the steel ban.