

Materials: Elasticity Theory

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Elasticiteitstheorie – Elasticity Theory course codes 201600060 (PM) and MOD06 201900126

script & sheets, based on:
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Overview:

Index notation, matrix-vector notation, vectors and tensors
Stress, strain, (material behavior: elasticity, plasticity, viscosity)

1) Index notation and summation convention

Set of quantities: $a_1, a_2, a_3, \dots, a_n \longrightarrow$
 a_i , with $i = 1, 2, 3, \dots, n$

In 2- or 3-dim. space a_i is the i^{th} component
of the vector \underline{a} , with $n = 2$ or 3

also: a_i is the i^{th} element of the (column)-vector $\{a\}$
(just a column of numbers?)

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Difference:

vector as geometric, physical quantity
(e.g. distance or velocity with norm [unit] and direction)

$\underline{a} = \vec{a}$ - or vector as matrix with one column $\{a\}$

summation convention

inner product (scalar product) of two vectors:

$$\{x\}^T \{y\} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i$$

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shorter (Einstein) notation:

$$\{x\}^T \{y\} = (\underline{x} \cdot \underline{y} =) x_i y_i = x_p y_p = x_q y_q$$

where i , p and q are so-called *dummy-indices*.

Those are double, in products, and have to be summed over.

So called *free indices* occur only once in product-terms.

System of equations:

$$\begin{aligned}y_1 &= A_{11} x_1 + A_{12} x_2 + \cdots + A_{1n} x_n \\y_2 &= A_{21} x_1 + A_{22} x_2 + \cdots + A_{2n} x_n \\&\vdots \quad \vdots \quad \vdots \quad \vdots \\y_m &= A_{m1} x_1 + A_{m2} x_2 + \cdots + A_{mn} x_n\end{aligned}$$

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The i^{th} equation is:

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shortly: $y_i = A_{ij} x_j \quad \begin{cases} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{cases}$

or $\{y\} = [A] \{x\}$

or $\underline{y} = \underline{\underline{A}} \cdot \underline{x}$

Matrix $[A]$ with entries A_{ij}

(where i is the row-index, and j is the column-index)

Example: $S = \{x\}^T \cdot \{x\} = \left(\sum_{i=1}^n \right) x_i x_i$

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Kronecker delta: $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Thus: $S = x_i x_i = \delta_{ij} x_i x_j = \{x\}^T [I] \{x\}$

Also: $x_i = \delta_{ij} x_j = \delta_{ip} x_p$

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BUT: $\delta_{ii} \neq 1$

instead, note: $\delta_{ii} = \delta_{11} + \delta_{22} + \dots + \delta_{nn} = 1 + 1 + \dots + 1 = n$

Differentiation:

Consider a function: $F(x_1, x_2, \dots, x_n)$

Partial derivative: $\frac{\partial F}{\partial x_i} = F_{,i}$

The index $, i$ means the partial derivative with respect to x_i !

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Total differential of F :

$$dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \dots + \frac{\partial F}{\partial x_n} dx_n = \left(\sum_{i=1}^n \right) \frac{\partial F}{\partial x_i} dx_i = F_{,i} dx_i$$

2) Scalars, vectors and Cartesian tensors

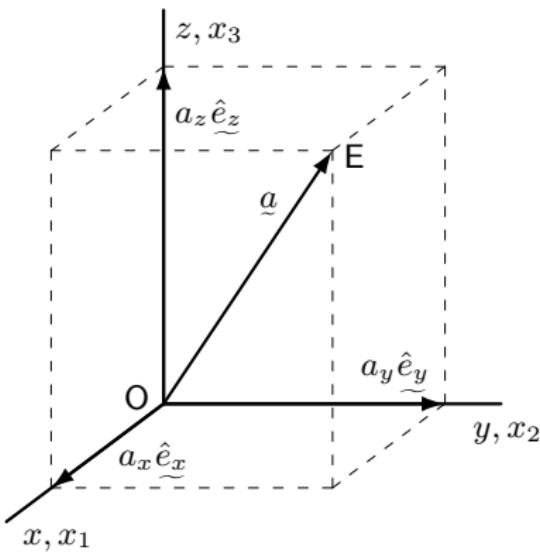
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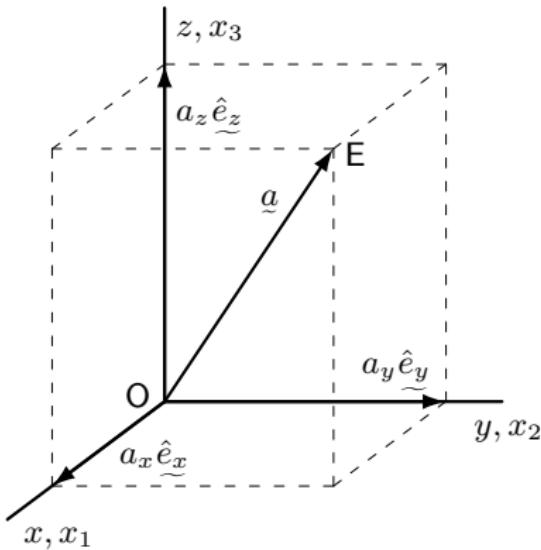
Vector fields, functions of position, quantity with norm [unit]
and a direction, e.g., force, displacement, velocity, acceleration, ...

Cartesian coordinate system



$$\begin{aligned}\underline{a} &= a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \\ &= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 = \\ &= \sum_{i=1}^3 a_i \hat{e}_i = a_i \hat{e}_i\end{aligned}$$

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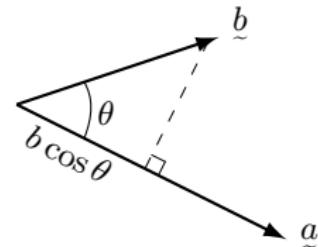
Norm (length) of \underline{a} :

$$|\underline{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i a_i}$$

inner product

(scalar product, “dot”-product):

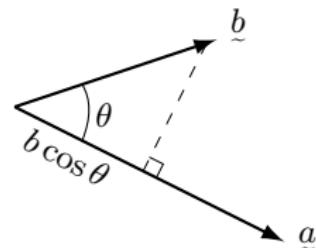
$$\underline{a} \cdot \underline{b} = a b \cos \theta$$



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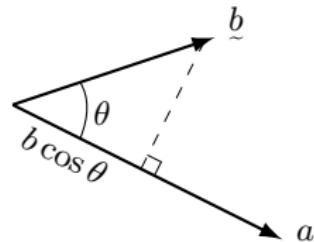
Also:

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$$

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For unit-direction-vectors:

$$\underline{\hat{e}_x} \cdot \underline{\hat{e}_x} = \dots = 1 \quad \text{or} \quad \underline{\hat{e}_1} \cdot \underline{\hat{e}_1} = \underline{\hat{e}_2} \cdot \underline{\hat{e}_2} = \underline{\hat{e}_3} \cdot \underline{\hat{e}_3} = 1$$

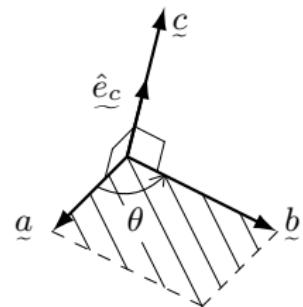
$$\underline{\hat{e}_x} \cdot \underline{\hat{e}_y} = \dots = 0 \quad \text{or} \quad \underline{\hat{e}_1} \cdot \underline{\hat{e}_2} = \underline{\hat{e}_2} \cdot \underline{\hat{e}_3} = \underline{\hat{e}_3} \cdot \underline{\hat{e}_1} = 0$$

$$\Rightarrow \underline{\hat{e}_i} \cdot \underline{\hat{e}_j} = \delta_{ij} \quad \text{in a way that e.g.:}$$

$$\underline{a} \cdot \underline{b} = a_i \underline{\hat{e}_i} \cdot b_j \underline{\hat{e}_j} = a_i b_j \underline{\hat{e}_i} \cdot \underline{\hat{e}_j} = a_i b_j \delta_{ij} = a_i b_i$$

Outer product or vector-product
(“cross”-product)

$$\underline{a} * \underline{b} = (|\underline{a}| |\underline{b}| \sin \theta) \hat{e}_c = \underline{c}$$



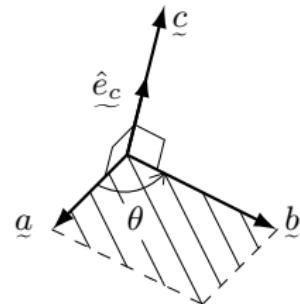
Outer product or vector-product

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$$\underline{a} * \underline{b} = \underline{a} \times \underline{b} = (|\underline{a}| |\underline{b}| \sin \theta) \hat{\underline{e}}_c = \underline{c}$$

Also:

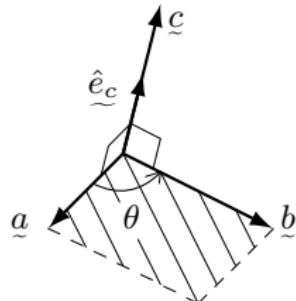
$$\begin{aligned}\underline{a} * \underline{b} &= \det \begin{pmatrix} \hat{\underline{e}}_1 & \hat{\underline{e}}_2 & \hat{\underline{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \\ &= (a_2 b_3 - a_3 b_2) \hat{\underline{e}}_1 + (a_3 b_1 - a_1 b_3) \hat{\underline{e}}_2 + (a_1 b_2 - a_2 b_1) \hat{\underline{e}}_3 = \\ &= c_1 \hat{\underline{e}}_1 + c_2 \hat{\underline{e}}_2 + c_3 \hat{\underline{e}}_3 = c_i \hat{\underline{e}}_i = \underline{c}\end{aligned}$$



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Also:

$$\begin{aligned}
 \underline{a} * \underline{b} &= \det \left(\begin{bmatrix} \hat{\underline{e}}_1 & \hat{\underline{e}}_2 & \hat{\underline{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \right) = \\
 &= (a_2 b_3 - a_3 b_2) \hat{\underline{e}}_1 + (a_3 b_1 - a_1 b_3) \hat{\underline{e}}_2 + (a_1 b_2 - a_2 b_1) \hat{\underline{e}}_3 = \\
 &= c_1 \hat{\underline{e}}_1 + c_2 \hat{\underline{e}}_2 + c_3 \hat{\underline{e}}_3 = c_i \hat{\underline{e}}_i = \underline{c}
 \end{aligned}$$

permutation symbol: $c_i = \varepsilon_{ijk} a_j b_k$ ($\underline{c} = \varepsilon_{ijk} a_j b_k \hat{\underline{e}}_i$) with:

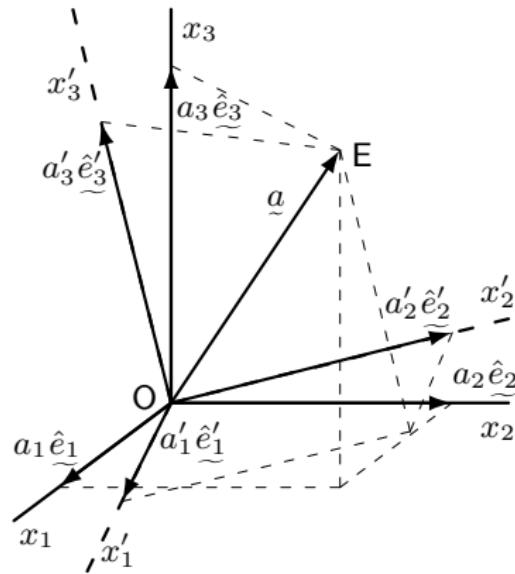
$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 ; \quad \varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1 ; \quad \text{note: } \varepsilon_{ijk} = 0$$

Rotation of axes: (Cartesian coordinates)

Rotations from “old” x_i -coordinates
to “new” x'_p -coordinates.

For an arbitrary vector holds:

$$\underline{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 = \\ = a'_1 \hat{e}'_1 + a'_2 \hat{e}'_2 + a'_3 \hat{e}'_3$$



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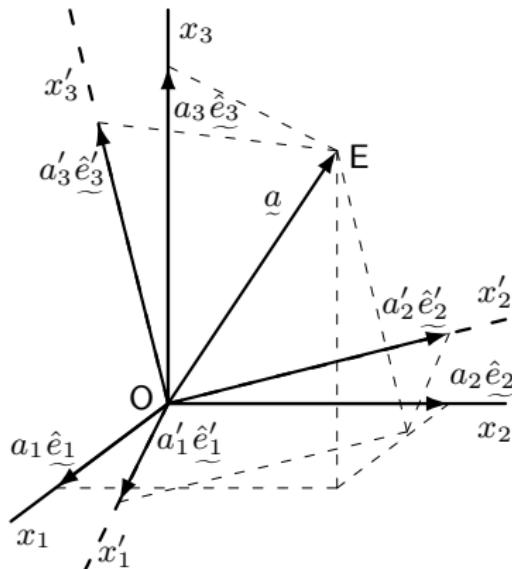
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In index notation: $\underline{\underline{a}} = a_i \hat{e}_i = a'_p \hat{e}'_p$

Direction-cosines:

$$R_{pi} = \cos(x'_p, x_i) \Rightarrow$$

$$\begin{aligned}a'_1 &= a_1 \cos(x'_1, x_1) + a_2 \cos(x'_1, x_2) + a_3 \cos(x'_1, x_3) = \\ &= a_i \cos(x'_1, x_i) = R_{1i} a_i \quad (\text{change now } 1 \rightarrow p)\end{aligned}$$



For $p = 1, 2, 3$:

$$\begin{aligned} a'_p &= a_1 \cos(x'_p, x_1) + a_2 \cos(x'_p, x_2) + a_3 \cos(x'_p, x_3) = \\ &= a_i \cos(x'_p, x_i) = R_{pi} a_i \end{aligned}$$

Summation over *second* index of R .

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Summation over *second* index of R .

Inverse relation, from “new” to “old”:

$$\begin{aligned} a_j &= a'_1 \cos(x_j, x'_1) + a'_2 \cos(x_j, x'_2) + a'_3 \cos(x_j, x'_3) = \\ &= a'_q \cos(x_j, x'_q) = R_{qj} a'_q \end{aligned}$$

Note: summation over the *first* index of R !

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Note: summation over the *first* index of R !

If this transformation rule holds, it is a vector (line with a length and direction): a so-called first-order-tensor!

Written in matrix notation $a'_p = R_{pi} a_i$ becomes :

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In *matrix-vector-notation* :
 $\{a'\} = [R]\{a\}$

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The inverse transform $a_j = R_{qj} a'_q$ in matrix notation:

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With $[R]^T$ being the transposed of $[R]$. Matrix mirroring with respect to the main diagonal (top-left \rightarrow bottom-right).

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Properties of the (transformation matrix) rotation matrix $[R]$.

Substitute $\{a\} = [R]^T\{a'\}$ in $\{a'\} = [R]\{a\}$. This gives:

$$\{a'\} = [R]\{a\} = [R][R]^T\{a'\} \implies [R][R]^T = [I]$$

Also: $[R][R]^{-1} = [I]$ (definition of inverted matrix), so:

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And $[R][R]^T = [I]$:

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\Rightarrow *Normality condition* and *orthogonality condition*

Orthogonality rotation matrix $[R]$ using index notation:

Derived before: $a'_p = R_{pi} a_i$ and $a_i = R_{qi} a'_q$

So it holds: $a'_p = R_{pi} a_i = R_{pi} R_{qi} a'_q$ also: $a'_p = \delta_{pq} a'_q$

From which follows: $R_{pi} R_{qi} = \delta_{pq}$ or $R_{ip} R_{jp} = \delta_{ij}$

Equivalent with: $[R] [R]^T = [I]$

(Note: $[A] [B] = [C] \iff A_{ip} B_{pj} = C_{ij}$)

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Reverse: $a_j = R_{qj} a'_q = R_{qj} R_{qi} a_i$ also: $a_j = \delta_{ij} a_i$

From which follows: $R_{qi} R_{qj} = \delta_{ij}$ or $R_{pi} R_{pj} = \delta_{ij}$

Equivalent with: $[R]^T [R] = [I]$

Second order tensor

Consider 2 vectors, \underline{a} and \underline{b} with components a_i and b_j

Dyadic product (per definition): $\underline{S} = \underline{a} \underline{b}$ with $S_{ij} = a_i b_j$

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Rotation of axis: $S'_{ij} = a'_i b'_j$

Substitute transformation rule for vectors:

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Quantities that satisfies this transformation rule are called *second order tensors*. NOTE, not every second order tensor can be written as a dyadic product of two vectors.

Second order tensor: $S'_{ij} = R_{ip} R_{jq} S_{pq}$

Can be written as: $S'_{ij} = R_{ip} S_{pq} R_{jq} = R_{ip} S_{pq} R_{qj}^T$

In matrix-vector-notation: $[S'] = [R] [S] [R]^T$

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In matrix-vector-notation: $[S'] = [R] [S] [R]^T$

Inverse: $S_{ij} = R_{pi} R_{qj} S'_{pq} = R_{ip}^T S'_{pq} R_{qj}$

In matrix-vector-notation: $[S] = [R]^T [S'] [R]$

Overview tensors (\Rightarrow means transforms to)

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0. If $\Phi(x_r) \Rightarrow \Phi'(x_r)$ by $\Phi = \Phi'$
then *zero order tensor field or scalar field*
1. If $a_i(x_r) \Rightarrow a'_p(x_r)$ by $a'_p = R_{pi} a_i$
then *first order tensor field or vector field*

Overview tensors (\Rightarrow means transforms to)

0. If $\Phi(x_r) \Rightarrow \Phi'(x_r)$ by $\Phi = \Phi'$

then *zero order tensor field or scalar field*

1. If $a_i(x_r) \Rightarrow a'_p(x_r)$ by $a'_p = R_{pi} a_i$

then *first order tensor field or vector field*

2. If $S_{ij}(x_r) \Rightarrow S'_{pq}(x_r)$ by $S'_{pq} = R_{pi} R_{qj} S_{ij}$

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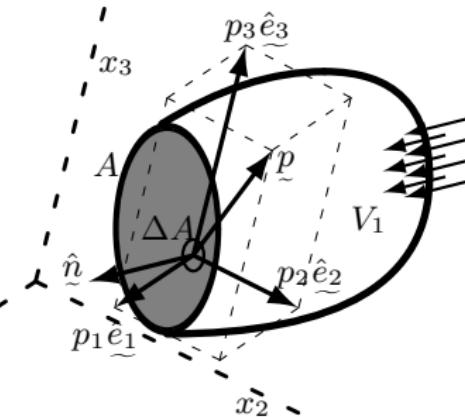
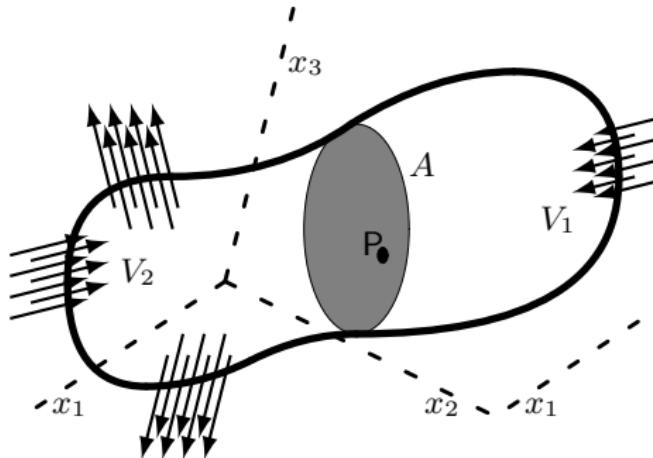
3. If $S_{ijk}(x_r) \Rightarrow S'_{pqt}(x_r)$ by $S'_{pqt} = R_{pi} R_{qj} R_{tk} S_{ijk}$

then *third order tensor field*

4. etc. etc. etc.

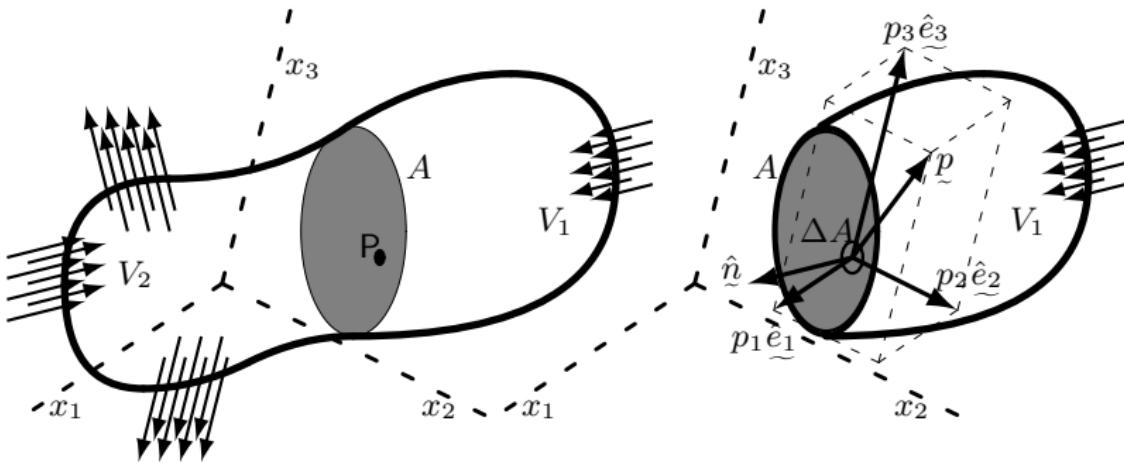
3) Stresses

3.1) Stress state in a point



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3.1) Stress state in a point



Per definition, stress vector $\tilde{p} = \lim_{\Delta A \rightarrow 0} \left(\frac{\Delta F}{\Delta \tilde{A}} \right)$

A few remarks:

- ▶ A stress vector \underline{p} has in general an arbitrary orientation

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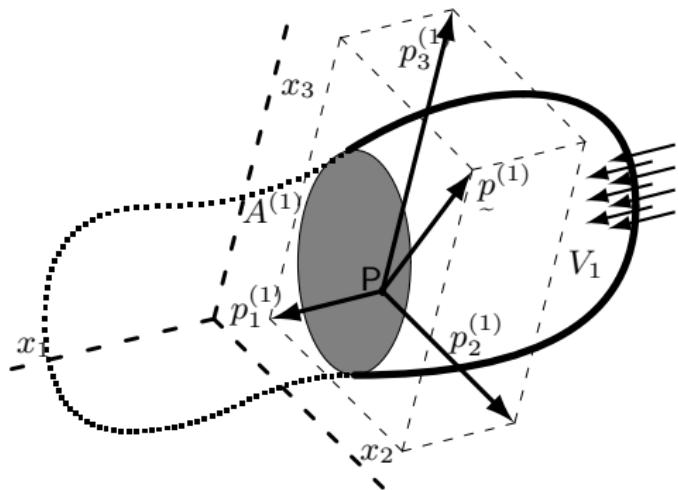
- ▶ A stress vector $\underline{\underline{p}}$ has in general an arbitrary orientation
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- ▶ A stress vector $\underline{\underline{p}}$ can be decomposed in a normal and a tangential component w.r.t. the orientation of ΔA
- ▶ A stress vector $\underline{\underline{p}}$ can be decomposed along a Cartesian coordinate system $\underline{\underline{p}} = p_i \hat{e}_i$

Cartesian
stress- components σ_{ij} :

$$\sigma_{11} = p_1^{(1)}$$

$$\sigma_{12} = p_2^{(1)}$$

$$\sigma_{13} = p_3^{(1)}$$



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$$\sigma_{11} = p_1^{(1)}$$

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$$\sigma_{13} = p_3^{(1)}$$

Cross-section

perpendicular to x_2 -axis:

$$\sigma_{21} = p_1^{(2)}$$

$$\sigma_{22} = p_2^{(2)}$$

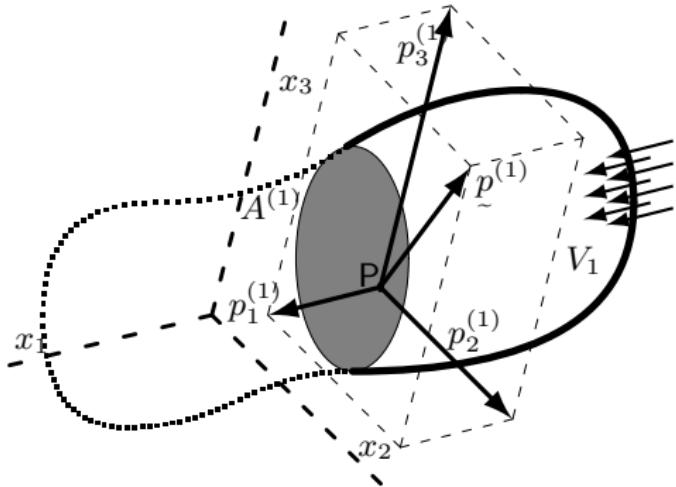
$$\sigma_{23} = p_3^{(2)}$$

Cross-section perpendicular to x_3 -axis:

$$\sigma_{31} = p_1^{(3)}$$

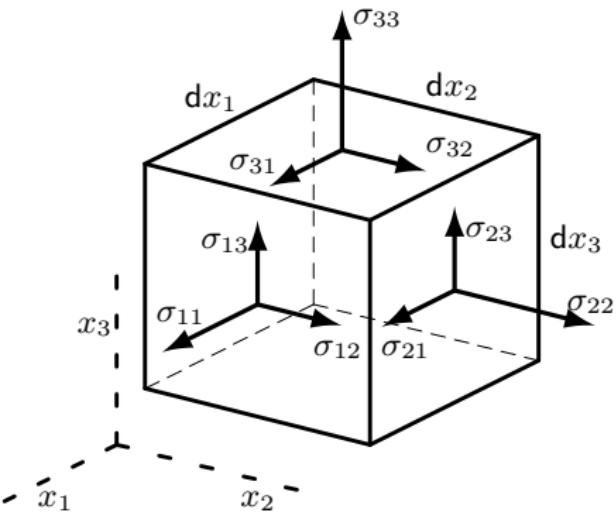
$$\sigma_{32} = p_2^{(3)}$$

$$\sigma_{33} = p_3^{(3)}$$



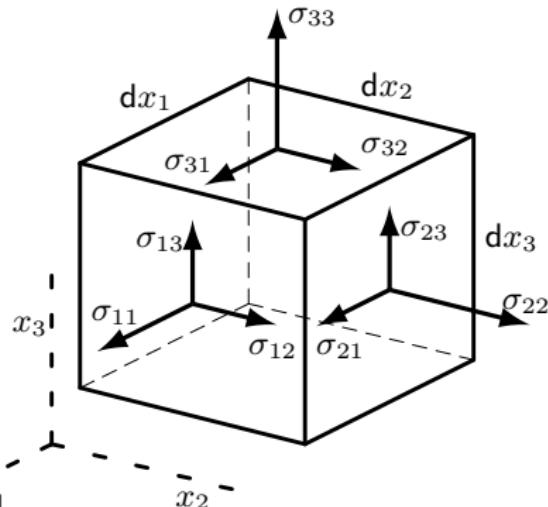
Where

σ_{ij} is the stress component in the directions of x_j acting on a surface with its normal pointing in x_i -direction.



Where

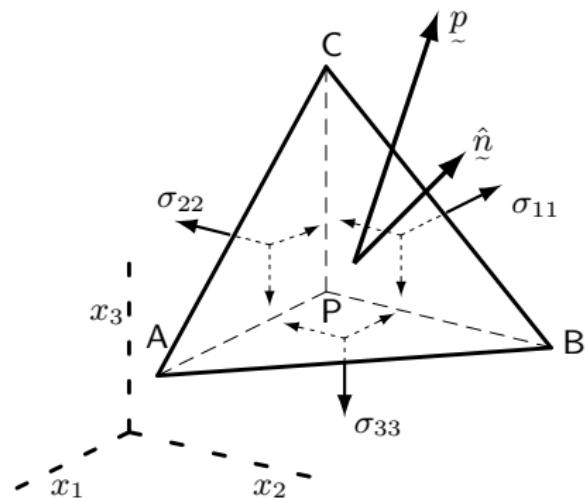
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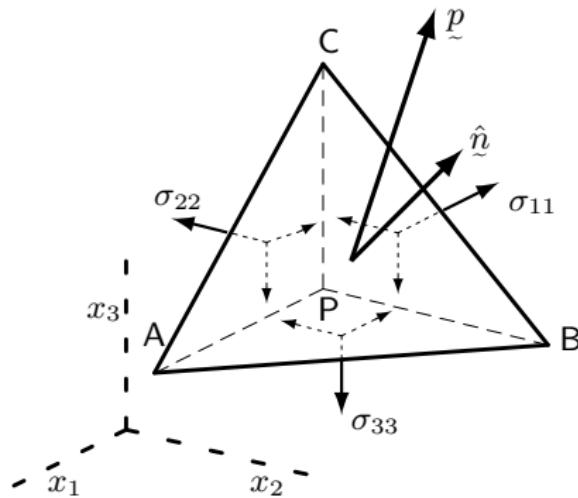
Displayed in a stress matrix:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

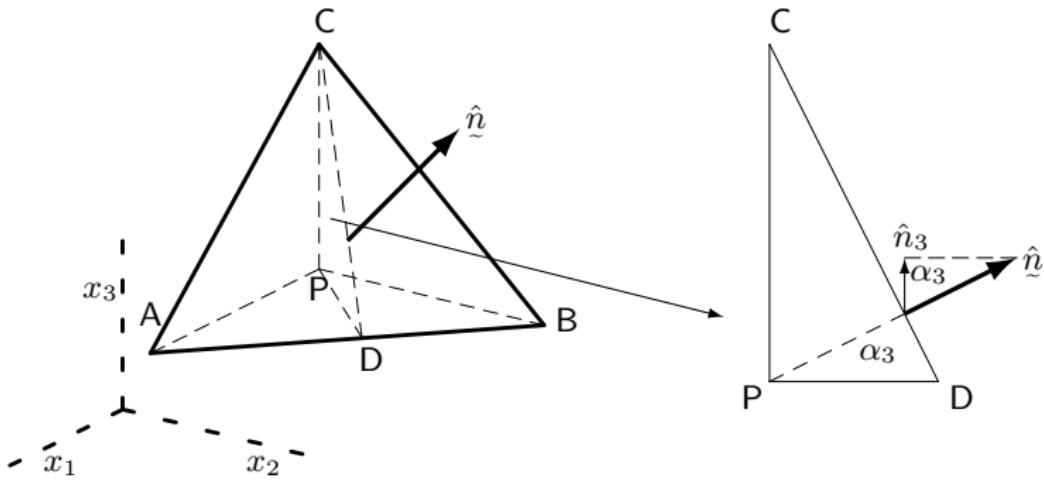
3.2) Stress-vector on arbitrary cross-section

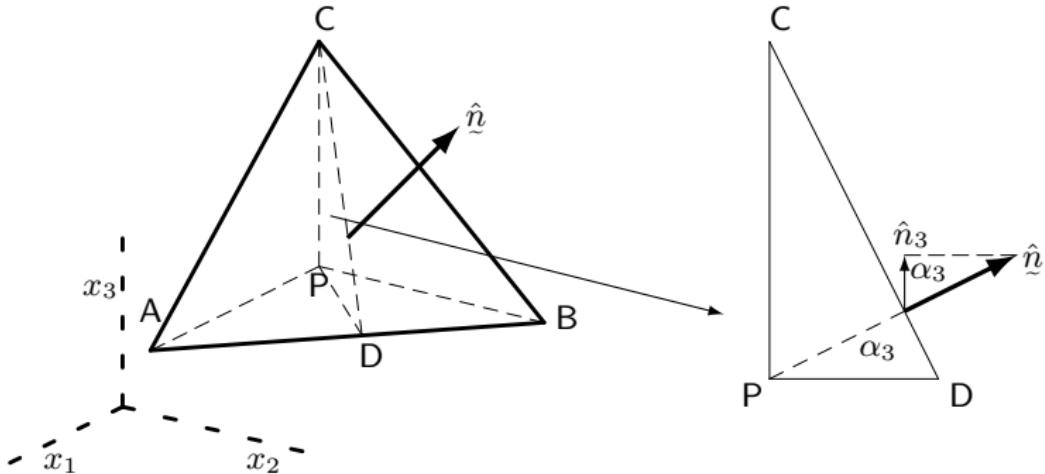


3.2) Stress-vector on arbitrary cross-section



Equilibrium: $p_1 \Delta A_{ABC} - \sigma_{11} \Delta A_{PBC} - \sigma_{21} \Delta A_{PCA} - \sigma_{31} \Delta A_{PAB} = 0$
 $\Rightarrow p_1 = \sigma_{11} \frac{\Delta A_{PBC}}{\Delta A_{ABC}} + \sigma_{21} \frac{\Delta A_{PCA}}{\Delta A_{ABC}} + \sigma_{31} \frac{\Delta A_{PAB}}{\Delta A_{ABC}}$





Now it holds:

$$\frac{\Delta A_{PAB}}{\Delta A_{ABC}} = \frac{\frac{1}{2} |PD| |AB|}{\frac{1}{2} |CD| |AB|} = \frac{|PD|}{|CD|} = \cos \alpha_3 = \frac{\hat{n}_3}{|\hat{n}|} = \hat{n}_3 \quad (|\hat{n}| = 1)$$

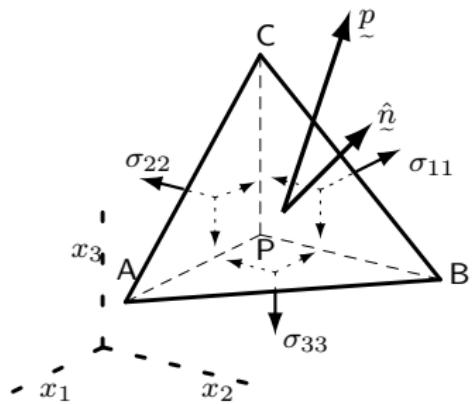
And similar for the other two area ratios.

Combined this gives:

$$\frac{\Delta A_{PBC}}{\Delta A_{ABC}} = \cos(\hat{n}, x_1) = \hat{n}_1$$

$$\frac{\Delta A_{PCA}}{\Delta A_{ABC}} = \cos(\hat{n}, x_2) = \hat{n}_2$$

$$\frac{\Delta A_{PAB}}{\Delta A_{ABC}} = \cos(\hat{n}, x_3) = \hat{n}_3$$

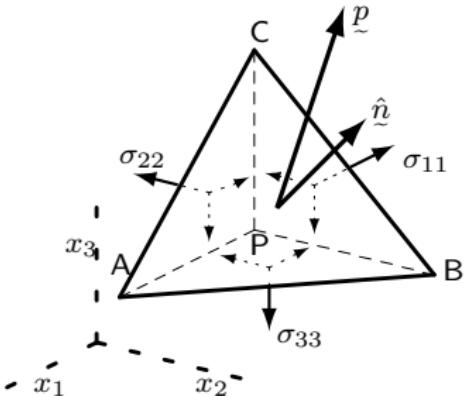


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$$\frac{\Delta A_{PAB}}{\Delta A_{ABC}} = \cos(\hat{n}, x_3) = \hat{n}_3$$



Equilibrium in x_1 -direction ($i = 1, 2, 3$):

$$p_1 \Delta A = \sigma_{11} \Delta A_{PBC} + \sigma_{21} \Delta A_{PCA} + \sigma_{31} \Delta A_{PAB}$$

$$\Rightarrow p_1 = \sigma_{11} \hat{n}_1 + \sigma_{21} \hat{n}_2 + \sigma_{31} \hat{n}_3 = \sigma_{j1} \hat{n}_j$$

Equilibrium in x_i -direction:

$$\Rightarrow p_i = \sigma_{1i} \hat{n}_1 + \sigma_{2i} \hat{n}_2 + \sigma_{3i} \hat{n}_3 = \sigma_{ji} \hat{n}_j$$

Cauchy's formula: $p_i = \sigma_{ji} \hat{n}_j$. The summation takes place over the index j , and index i can be either 1, 2 or 3.

$$\underline{\underline{p}} = \hat{\underline{\underline{n}}} \cdot \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \cdot \hat{\underline{\underline{n}}} \quad \text{or} \quad \{p\} = \{\hat{n}\}^T [\sigma] = [\sigma]^T \{\hat{n}\}$$

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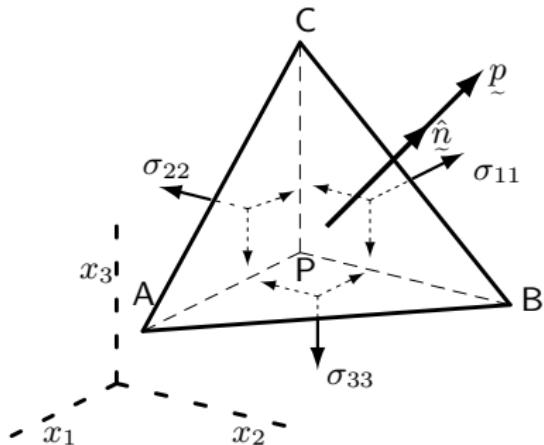
It will be shown later that the stress matrix is symmetric, so $\sigma_{ij} = \sigma_{ji}$ or $\underline{\sigma} = \underline{\sigma}^T$ or $[\sigma] = [\sigma]^T$. From that:

$$\begin{aligned} p_i &= \sigma_{ij} \hat{n}_j \\ \underline{p} &= \underline{\sigma} \cdot \underline{\hat{n}} \quad \text{or} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \\ \{p\} &= [\sigma] \{\hat{n}\} \end{aligned}$$

3.3) Principal stresses and principal directions

Question:

Is it possible to orientate the area ABC in such a way that the direction of the stress vector is the same as the direction of the normal acting on this area.



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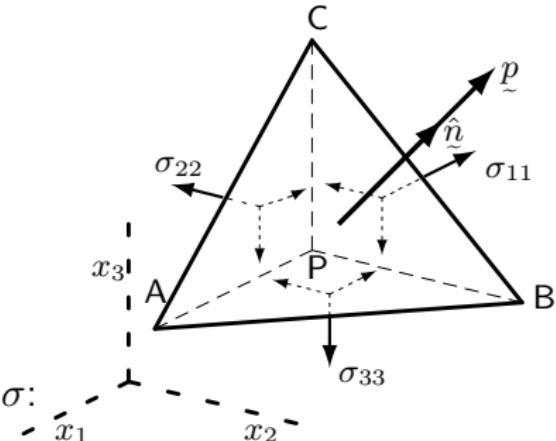
Is it possible to orientate the area ABC in such a way that the direction of the stress vector is the same as the direction of the normal acting on this area.

In that case it must hold that:

$$\tilde{p} = \text{factor} \times \hat{n}.$$

Substitute “factor” with the symbol σ :

$$\tilde{p} \stackrel{?}{=} \sigma \hat{n} \quad \text{or} \quad p_i \stackrel{?}{=} \sigma \hat{n}_i$$



Are there multiple orientations of area ABC possible such that:

$$\underline{\underline{p}} \stackrel{?}{=} \sigma \hat{\underline{\underline{n}}} \quad \text{or} \quad p_i \stackrel{?}{=} \sigma \hat{n}_i$$

Cauchy's formula gives: $p_i = \sigma_{ij} \hat{n}_j$

Combined: $p_i = \sigma_{ij} \hat{n}_j \stackrel{?}{=} \sigma \hat{n}_i = \sigma \delta_{ij} \hat{n}_j$

Written as a matrix ($[\sigma]\{\hat{n}\} = \sigma\{\hat{n}\}$):

$$\{p\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \stackrel{?}{=} \sigma \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$

$[\sigma]\{\hat{n}\} = \sigma\{\hat{n}\}$ can be written as: $([\sigma] - \sigma[I])\{\hat{n}\} = \{0\}$

$$\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of algebraic equations.

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$$\begin{bmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of algebraic equations.

This also yields a solution $\{\hat{n}\} \neq \{0\}$ when the determinant of the coefficient matrix is equal to zero.

$$\det \begin{pmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{pmatrix} = 0$$

$$\det \begin{pmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{pmatrix} = 0$$

Written as *characteristic equation*:

$$\boxed{\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0} \quad \text{with}$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii}$$

$$I_2 = \sigma_{11} \sigma_{22} - \sigma_{12} \sigma_{21} + \dots = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji})$$

$$I_3 = \det([\sigma])$$

From this the *principal stresses*: σ_1 , σ_2 and σ_3 are obtained for which holds that: $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Principal stresses are the *eigenvalues* of the stress matrix $[\sigma]$.

With eigenvalues come *eigenvectors*. Here the *principal directions*.

Obtain first principal directions, related to σ_1 :

$$\begin{bmatrix} (\sigma_{11} - \sigma_1) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma_1) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a *dependent* system of equations!!! e.g.:

$$\begin{bmatrix} (\sigma_{11} - \sigma_1) & \sigma_{12} \\ \sigma_{21} & (\sigma_{22} - \sigma_1) \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \end{bmatrix} = \begin{bmatrix} -\sigma_{13} \\ -\sigma_{23} \end{bmatrix} \hat{n}_3 \Rightarrow \begin{aligned} \hat{n}_1 &= a \hat{n}_3 \\ \hat{n}_2 &= b \hat{n}_3 \end{aligned}$$

Found: $\hat{n}_1 = a \hat{n}_3$ and $\hat{n}_2 = b \hat{n}_3$.

Normal vector is an unit vector, so $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$

With this the first principal direction is determined.

The same could be done for the second and third principal directions

All three principal directions are perpendicular to each other!

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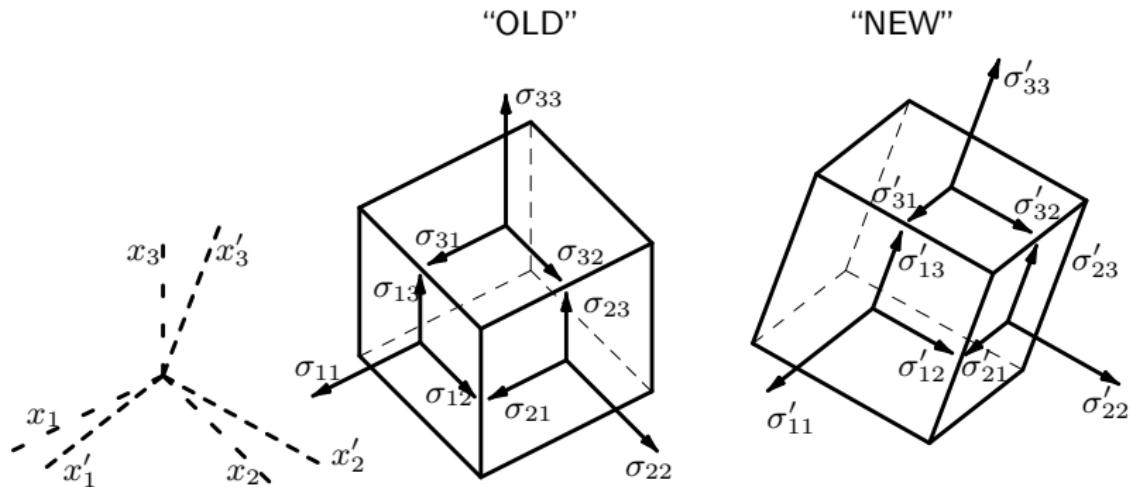
The same could be done for the second and third principal directions

All three principal directions are perpendicular to each other!

Preferably a coordinate system coinciding with the principal directions:

$$[\sigma]_{\text{ps}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \Rightarrow \begin{aligned} \text{Stress invariants:} \\ I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 \\ I_3 &= \sigma_1 \sigma_2 \sigma_3 \end{aligned}$$

3.4) Rotation matrix



Directional cosines: $R_{pi} = \cos(x'_p, x_i)$

Surface

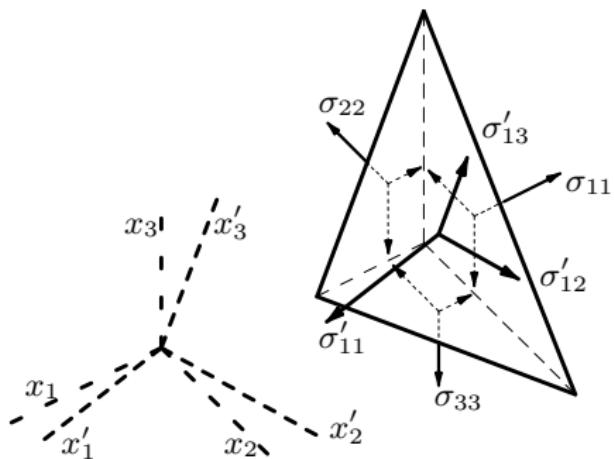
perpendicular to x'_1 -axis.

Normal vector $\hat{n}^{(1)}$

with: $\hat{n}_i^{(1)} = \cos(x'_1, x_i) = R_{1i}$

Cauchy's formula:

$p_j^{(1)} = \sigma_{ij} \hat{n}_i^{(1)} = \sigma_{ij} R_{1i}$



Surface

perpendicular to x'_1 -axis.

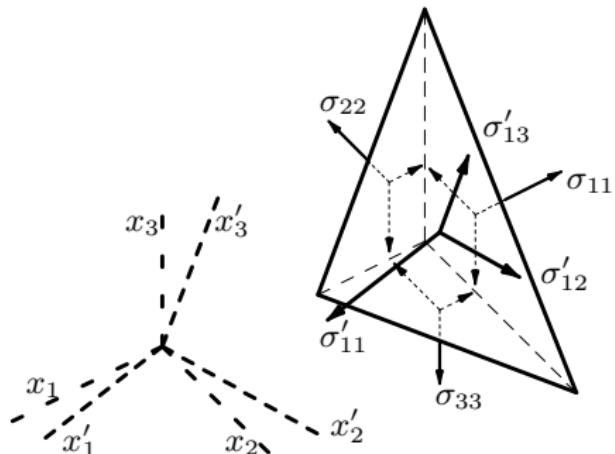
Normal vector $\hat{n}^{(1)}$

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Cauchy's formula:

$$p_j^{(1)} = \sigma_{ij} \hat{n}_i^{(1)} = \sigma_{ij} R_{1i}$$

After rotation:



$$\sigma'_{1q} = p'_q^{(1)} = R_{qj} p_j^{(1)} = R_{qj} \sigma_{ij} R_{1i} = R_{1i} R_{qj} \sigma_{ij}$$

Same for the x'_2 - and x'_3 -directions. So:

$$\sigma'_{pq} = R_{pi} R_{qj} \sigma_{ij}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During rotation of coordinate axes it holds:

$$\sigma'_{pq} = R_{pi} R_{qj} \sigma_{ij} \quad \text{or} \quad [\sigma'] = [R] [\sigma] [R]^T$$

Elaborated:

$$\begin{aligned}\sigma'_{pq} &= R_{p1} R_{q1} \sigma_{11} + R_{p1} R_{q2} \sigma_{12} + R_{p1} R_{q3} \sigma_{13} + \\&+ R_{p2} R_{q1} \sigma_{21} + R_{p2} R_{q2} \sigma_{22} + R_{p2} R_{q3} \sigma_{23} + \\&+ R_{p3} R_{q1} \sigma_{31} + R_{p3} R_{q2} \sigma_{32} + R_{p3} R_{q3} \sigma_{33}\end{aligned}$$

The stress components σ_{ij} turn out to be the components of a second order tensor. During transformation it holds that:

$$\sigma'_{pq} = R_{pi} R_{qj} \sigma_{ij} \quad \text{or} \quad [\sigma'] = [R] [\sigma] [R]^T$$

Elaborated:

$$\begin{aligned}\sigma'_{pq} &= R_{p1} R_{q1} \sigma_{11} + R_{p1} R_{q2} \sigma_{12} + R_{p1} R_{q3} \sigma_{13} + \\ &+ R_{p2} R_{q1} \sigma_{21} + R_{p2} R_{q2} \sigma_{22} + R_{p2} R_{q3} \sigma_{23} + \\ &+ R_{p3} R_{q1} \sigma_{31} + R_{p3} R_{q2} \sigma_{32} + R_{p3} R_{q3} \sigma_{33}\end{aligned}$$

Also the inverse transformation it holds (“new” to “old”):

$$\sigma_{ij} = R_{pi} R_{qj} \sigma'_{pq} \quad \text{or} \quad \sigma_{pq} = R_{ip} R_{jq} \sigma'_{ij} \quad \text{or} \quad [\sigma] = [R]^T [\sigma'] [R]$$

3.5) Equilibrium equations

In reader 3-dim., here 2-dim.:

Instead

of a single point now *a volume*.

Stresses are a function of x_i .

Coordinates of the points:

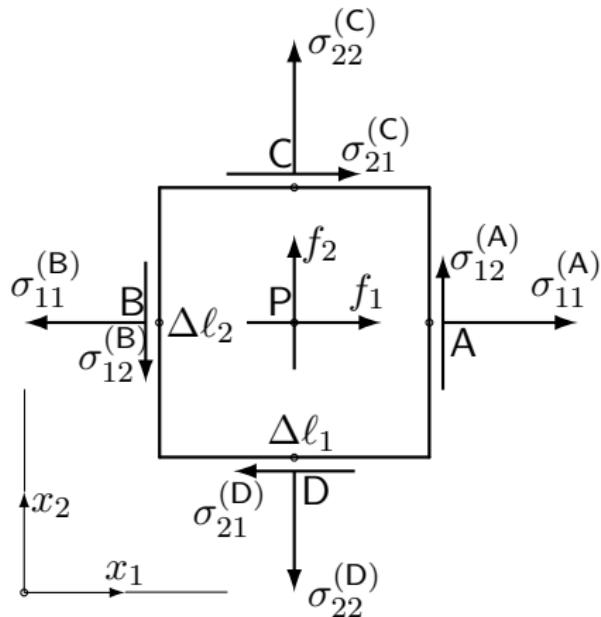
$$P : (x_1, x_2)$$

$$A : (x_1 + \frac{1}{2}\Delta\ell_1, x_2)$$

$$B : (x_1 - \frac{1}{2}\Delta\ell_1, x_2)$$

$$C : (x_1, x_2 + \frac{1}{2}\Delta\ell_2)$$

$$D : (x_1, x_2 - \frac{1}{2}\Delta\ell_2)$$



f_1 and f_2 are volume forces or body forces, forces per unit volume.

3.5) Equilibrium equations

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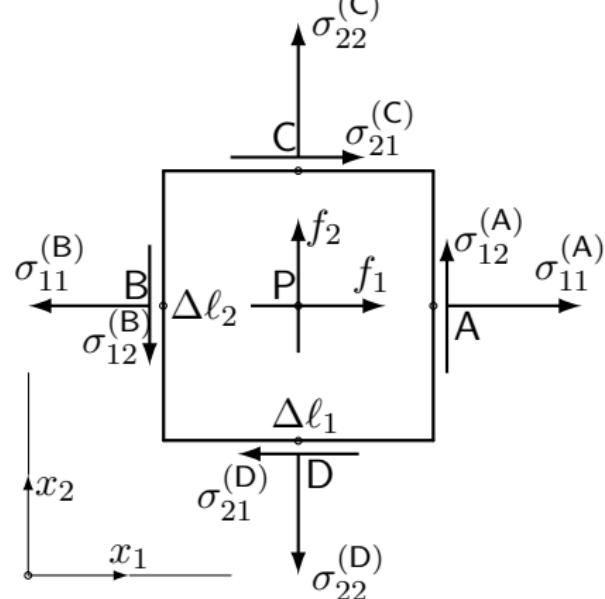
$$P : (x_1, x_2)$$

$$A : \left(x_1 + \frac{1}{2}\Delta\ell_1, x_2\right)$$

$$B : \left(x_1 - \frac{1}{2}\Delta\ell_1, x_2\right)$$

$$C : \left(x_1, x_2 + \frac{1}{2}\Delta\ell_2\right)$$

$$D : \left(x_1, x_2 - \frac{1}{2}\Delta\ell_2\right)$$



Force equilibrium in x_1 -direction (volume size in x_3 -direction is $\Delta\ell_3$):

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \Delta\ell_2 \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 + f_1 \Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = 0$$

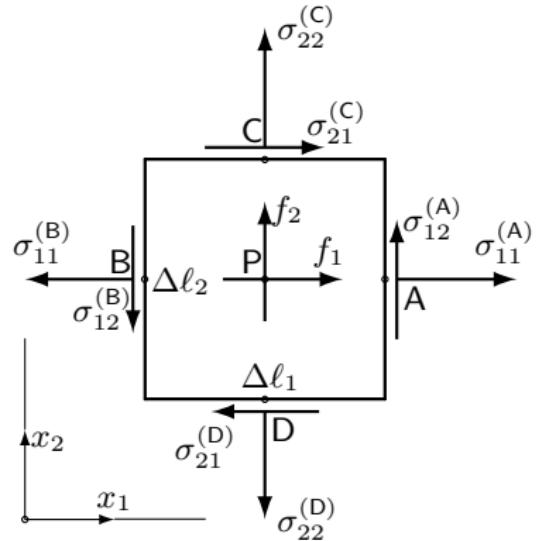
$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \Delta\ell_2 \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 + f_1 \Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = 0$$

$$\sigma_{11}^{(A)} = \sigma_{11}(x_1 + \frac{1}{2}\Delta\ell_1, x_2) =$$

$$= \sigma_{11}(x_1, x_2) + \frac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1} \right) + \text{H.O.T.}$$

$$\sigma_{11}^{(B)} = \sigma_{11}(x_1 - \frac{1}{2}\Delta\ell_1, x_2) =$$

$$= \sigma_{11}(x_1, x_2) - \frac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1} \right) + \text{H.O.T.}$$



$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \Delta\ell_2 \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 + f_1 \Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = 0$$

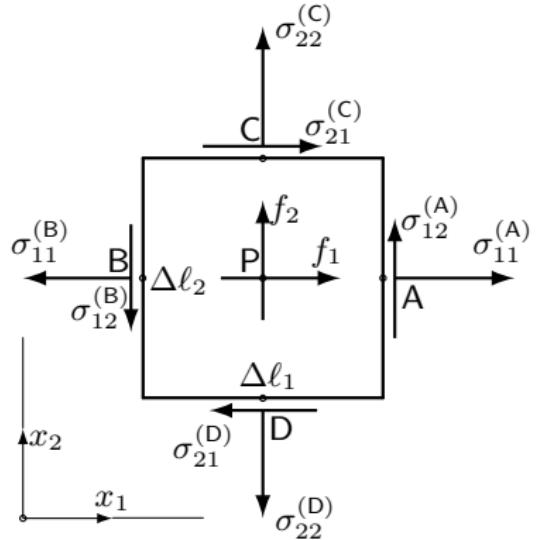
$$\begin{aligned}\sigma_{11}^{(A)} &= \sigma_{11}(x_1 + \frac{1}{2}\Delta\ell_1, x_2) = \\&= \sigma_{11}(x_1, x_2) + \frac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1} \right) + \text{H.O.T.}\end{aligned}$$

$$\begin{aligned}\sigma_{11}^{(B)} &= \sigma_{11}(x_1 - \frac{1}{2}\Delta\ell_1, x_2) = \\&= \sigma_{11}(x_1, x_2) - \frac{1}{2}\Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1} \right) + \text{H.O.T.}\end{aligned}$$

So (neglecting Higher Order Terms):

$$\sigma_{11}^{(A)} - \sigma_{11}^{(B)} = \Delta\ell_1 \left(\frac{\partial \sigma_{11}}{\partial x_1} \right)$$

$$\text{Idem: } \sigma_{21}^{(C)} - \sigma_{21}^{(D)} = \Delta\ell_2 \left(\frac{\partial \sigma_{21}}{\partial x_2} \right)$$



Equilibrium of forces in x_1 -direction:

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \Delta\ell_2 \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 + f_1 \Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(A)} - \sigma_{11}^{(B)} = \Delta\ell_1 \frac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(C)} - \sigma_{21}^{(D)} = \Delta\ell_2 \frac{\partial\sigma_{21}}{\partial x_2}$

Results in (after dividing by $\Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

Shorter: $\sum_{j=1}^{\dots} \frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \Rightarrow \sigma_{j1,j} + f_1 = 0$

Equilibrium of forces in x_1 -direction:

$$(\sigma_{11}^{(A)} - \sigma_{11}^{(B)}) \Delta\ell_2 \Delta\ell_3 + (\sigma_{21}^{(C)} - \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 + f_1 \Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = 0$$

Derived: $\sigma_{11}^{(A)} - \sigma_{11}^{(B)} = \Delta\ell_1 \frac{\partial\sigma_{11}}{\partial x_1}$ and $\sigma_{21}^{(C)} - \sigma_{21}^{(D)} = \Delta\ell_2 \frac{\partial\sigma_{21}}{\partial x_2}$

Results in (after dividing by $\Delta\ell_1 \Delta\ell_2 \Delta\ell_3 = \Delta V$):

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + f_1 = 0$$

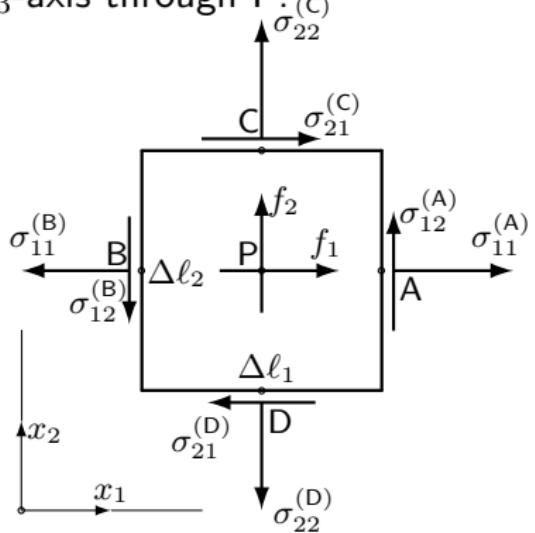
Shorter: $\sum_{j=1}^{\dots} \frac{\partial\sigma_{j1}}{\partial x_j} + f_1 = 0 \Rightarrow \sigma_{j1,j} + f_1 = 0$

The same for other directions, expanded to 3-dim.:

$$\boxed{\sigma_{ji,j} + f_i = 0} \quad \left(\sum_{j=1}^3 \frac{\partial\sigma_{ji}}{\partial x_j} + f_i = 0 \text{ for } i = 1, 2, 3 \right)$$

Equilibrium of moments around the x_3 -axis through P:

$$(\sigma_{12}^{(A)} + \sigma_{12}^{(B)}) \Delta\ell_2 \Delta\ell_3 \frac{1}{2} \Delta\ell_1 + \\ - (\sigma_{21}^{(C)} + \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 \frac{1}{2} \Delta\ell_2 = 0 \\ \implies (\sigma_{12}^{(A)} + \sigma_{12}^{(B)}) - (\sigma_{21}^{(C)} + \sigma_{21}^{(D)}) = 0$$



Equilibrium of moments around the x_3 -axis through P:

$$(\sigma_{12}^{(A)} + \sigma_{12}^{(B)}) \Delta\ell_2 \Delta\ell_3 \frac{1}{2} \Delta\ell_1 + \\ - (\sigma_{21}^{(C)} + \sigma_{21}^{(D)}) \Delta\ell_1 \Delta\ell_3 \frac{1}{2} \Delta\ell_2 = 0 \\ \Rightarrow (\sigma_{12}^{(A)} + \sigma_{12}^{(B)}) - (\sigma_{21}^{(C)} + \sigma_{21}^{(D)}) = 0$$

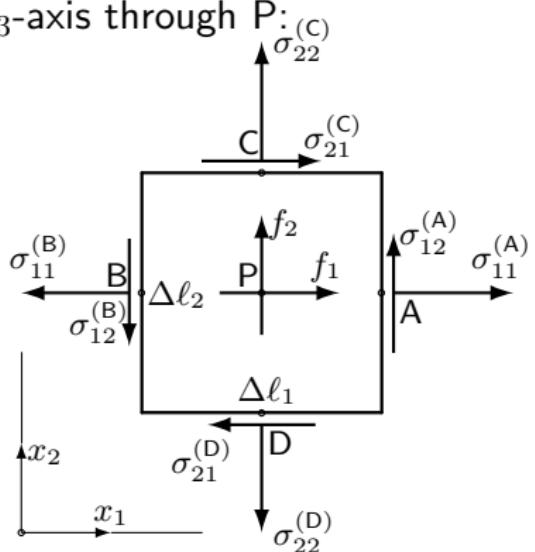
$$\sigma_{12}^{(A)} = \sigma_{12} + \frac{1}{2} \Delta\ell_1 \left(\frac{\partial \sigma_{12}}{\partial x_1} \right) + \text{H.O.T.}$$

$$\sigma_{12}^{(B)} = \sigma_{12} - \frac{1}{2} \Delta\ell_1 \left(\frac{\partial \sigma_{12}}{\partial x_1} \right) + \text{H.O.T.}$$

$$\sigma_{21}^{(C)} = \sigma_{21} + \frac{1}{2} \Delta\ell_2 \left(\frac{\partial \sigma_{21}}{\partial x_2} \right) + \text{H.O.T.}$$

$$\sigma_{21}^{(D)} = \sigma_{21} - \frac{1}{2} \Delta\ell_2 \left(\frac{\partial \sigma_{21}}{\partial x_2} \right) + \text{H.O.T.}$$

$$\Rightarrow 2 \sigma_{12} - 2 \sigma_{21} = 0 \Rightarrow \sigma_{12} = \sigma_{21}$$



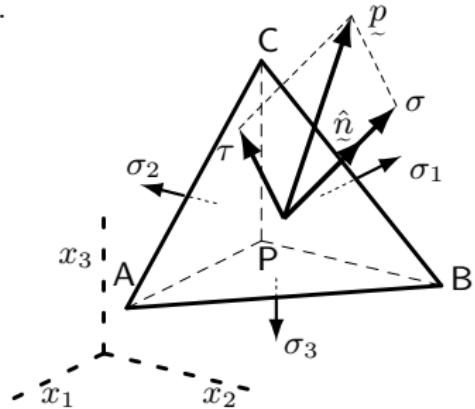
in general:

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

3.6) Maximum shear stresses, extrema

How to obtain maximum shear stress τ_{\max} .

Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i -coordinate system such that it will coincide with the principal directions.



3.6) Maximum shear stresses, extrema

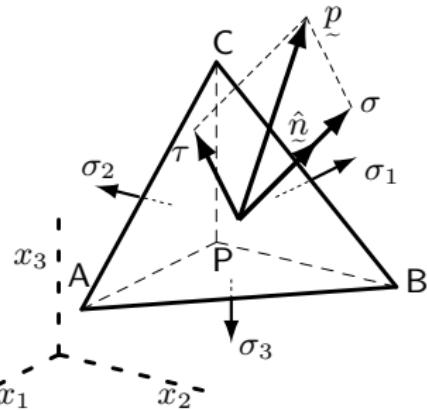
How to obtain maximum shear stress τ_{\max} .

Consider the stress state in a point. Calculate the principal stresses σ_i with corresponding principal directions. Orientate the x_i -coordinate system such that it will coincide with the principal directions.

The stress matrix will be:

$$[\sigma_h] = [\sigma]_{x_i} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Apply an arbitrary orientated surface, with normal vector \tilde{n} .



The components of the stress vector \underline{p} to the surface (**Cauchy**):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \hat{n}_1 \\ \sigma_2 \hat{n}_2 \\ \sigma_3 \hat{n}_3 \end{bmatrix}$$

The components of the stress vector \underline{p} to the surface ([Cauchy](#)):

$$\{p\} = [\sigma_h]\{\hat{n}\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \hat{n}_1 \\ \sigma_2 \hat{n}_2 \\ \sigma_3 \hat{n}_3 \end{bmatrix}$$

The magnitude, σ , of the normal stress on the surface is:

$$\sigma = \{\hat{n}\}^T \{p\} = [\hat{n}_1 \ \hat{n}_2 \ \hat{n}_3] \begin{bmatrix} \sigma_1 \hat{n}_1 \\ \sigma_2 \hat{n}_2 \\ \sigma_3 \hat{n}_3 \end{bmatrix} = \sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2$$

The components of the stress vector \underline{p} to the surface ([Cauchy](#)):

$$\{\underline{p}\} = [\sigma_h]\{\hat{n}\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \hat{n}_1 \\ \sigma_2 \hat{n}_2 \\ \sigma_3 \hat{n}_3 \end{bmatrix}$$

The magnitude, [\$\sigma\$](#) , of the normal stress on the surface is:

$$\sigma = \{\hat{n}\}^T \{\underline{p}\} = [\hat{n}_1 \ \hat{n}_2 \ \hat{n}_3] \begin{bmatrix} \sigma_1 \hat{n}_1 \\ \sigma_2 \hat{n}_2 \\ \sigma_3 \hat{n}_3 \end{bmatrix} = \sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2$$

The magnitude, [\$\tau\$](#) , of the shear stress on the surface:

$$\tau^2 = |\underline{p}|^2 - \sigma^2 = \sigma_1^2 \hat{n}_1^2 + \sigma_2^2 \hat{n}_2^2 + \sigma_3^2 \hat{n}_3^2 - (\sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2)^2$$

Found: $\tau^2 = \sigma_1^2 \hat{n}_1^2 + \sigma_2^2 \hat{n}_2^2 + \sigma_3^2 \hat{n}_3^2 - (\sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2)^2$

Eliminate the directional coefficient \hat{n}_3 with: $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2 \implies$

$$\begin{aligned}\tau^2 &= (\sigma_1^2 - \sigma_3^2) \hat{n}_1^2 + (\sigma_2^2 - \sigma_3^2) \hat{n}_2^2 + \sigma_3^2 + \\ &\quad - \{(\sigma_1 - \sigma_3) \hat{n}_1^2 + (\sigma_2 - \sigma_3) \hat{n}_2^2 + \sigma_3\}^2\end{aligned}$$

Extrema can be found with: $\frac{\partial \tau^2}{\partial \hat{n}_1} = \frac{\partial \tau^2}{\partial \hat{n}_2} = 0$

Found: $\tau^2 = \sigma_1^2 \hat{n}_1^2 + \sigma_2^2 \hat{n}_2^2 + \sigma_3^2 \hat{n}_3^2 - (\sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2)^2$
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Extrema can be found with: $\frac{\partial \tau^2}{\partial \hat{n}_1} = \frac{\partial \tau^2}{\partial \hat{n}_2} = 0$

The following solutions are found:

$$\hat{n}_1 = 0 \quad \text{with} \quad \begin{cases} 1) & \hat{n}_2 = 0 & \hat{n}_3 = \pm 1 \\ 2) & \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} \end{cases} \quad \circlearrowright$$

The first solution is $\tau = 0$; this is trivial and not interesting

The second solution, $\hat{n}_1 = 0$; $\hat{n}_2 = \pm \frac{1}{2}\sqrt{2}$; $\hat{n}_3 = \mp \frac{1}{2}\sqrt{2}$ gives:

$$\tau^2 = (\sigma_2^2 - \sigma_3^2) \frac{1}{2} + \sigma_3^2 - \{(\sigma_2 - \sigma_3) \frac{1}{2} + \sigma_3\}^2 = \frac{1}{4} (\sigma_2 - \sigma_3)^2$$

An extreme case for τ is found at: $\frac{1}{2} |\sigma_2 - \sigma_3|$.

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In a similar way the following solutions can be found:

$$\begin{array}{llll} \hat{n}_1 = 0 & \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} & \tau = \frac{1}{2} |\sigma_2 - \sigma_3| \\ \hat{n}_1 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_2 = \mp \frac{1}{2}\sqrt{2} & \hat{n}_3 = 0 & \tau = \frac{1}{2} |\sigma_1 - \sigma_2| \\ \hat{n}_1 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_2 = 0 & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} & \tau = \frac{1}{2} |\sigma_1 - \sigma_3| \end{array}$$

The second solution, $\hat{n}_1 = 0$; $\hat{n}_2 = \pm \frac{1}{2}\sqrt{2}$; $\hat{n}_3 = \mp \frac{1}{2}\sqrt{2}$ gives:

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In a similar way the following solutions can be found:

$$\begin{array}{llll} \hat{n}_1 = 0 & \hat{n}_2 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} & \tau = \frac{1}{2} |\sigma_2 - \sigma_3| \\ \hat{n}_1 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_2 = \mp \frac{1}{2}\sqrt{2} & \hat{n}_3 = 0 & \tau = \frac{1}{2} |\sigma_1 - \sigma_2| \\ \hat{n}_1 = \pm \frac{1}{2}\sqrt{2} & \hat{n}_2 = 0 & \hat{n}_3 = \mp \frac{1}{2}\sqrt{2} & \tau = \frac{1}{2} |\sigma_1 - \sigma_3| \end{array}$$

If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then $\boxed{\tau_{\max} = \frac{1}{2} (\sigma_1 - \sigma_3)}$

This τ_{\max} originates from a 45° rotation around the x_2 -axis

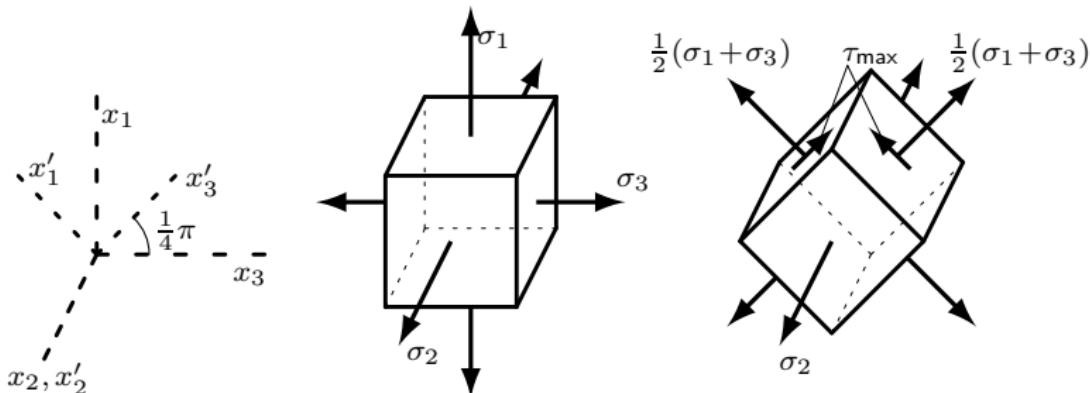
45° rotation around x_2 -axis: $[R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$

Then: $[\sigma'] = [R] [\sigma_h] [R]^T = \begin{bmatrix} \frac{1}{2}(\sigma_1 + \sigma_3) & 0 & \frac{1}{2}(\sigma_1 - \sigma_3) \\ 0 & \sigma_2 & 0 \\ \frac{1}{2}(\sigma_1 - \sigma_3) & 0 & \frac{1}{2}(\sigma_1 + \sigma_3) \end{bmatrix}$

45° rotation around x_2 -axis:

$$[R] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Then: $[\sigma'] = [R] [\sigma_h] [R]^T = \begin{bmatrix} \frac{1}{2}(\sigma_1 + \sigma_3) & 0 & \frac{1}{2}(\sigma_1 - \sigma_3) \\ 0 & \sigma_2 & 0 \\ \frac{1}{2}(\sigma_1 - \sigma_3) & 0 & \frac{1}{2}(\sigma_1 + \sigma_3) \end{bmatrix}$



Extrema from the **normal stress** on a surface.

Found previously: $\sigma = \sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2$

Elimination of \hat{n}_3 using $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$

$$\sigma = (\sigma_1 - \sigma_3) \hat{n}_1^2 + (\sigma_2 - \sigma_3) \hat{n}_2^2 + \sigma_3$$

Extrema when: $\frac{\partial \sigma}{\partial \hat{n}_1} = \frac{\partial \sigma}{\partial \hat{n}_2} = 0$

$$\text{Resulting in: } (\sigma_1 - \sigma_3) 2\hat{n}_1 = (\sigma_2 - \sigma_3) 2\hat{n}_2 = 0$$

Extrema from the **normal stress** on a surface.

Found previously: $\sigma = \sigma_1 \hat{n}_1^2 + \sigma_2 \hat{n}_2^2 + \sigma_3 \hat{n}_3^2$

Elimination of \hat{n}_3 using $\hat{n}_3^2 = 1 - \hat{n}_1^2 - \hat{n}_2^2$

$$\sigma = (\sigma_1 - \sigma_3) \hat{n}_1^2 + (\sigma_2 - \sigma_3) \hat{n}_2^2 + \sigma_3$$

Extrema when: $\frac{\partial \sigma}{\partial \hat{n}_1} = \frac{\partial \sigma}{\partial \hat{n}_2} = 0$

$$\text{Resulting in: } (\sigma_1 - \sigma_3) 2\hat{n}_1 = (\sigma_2 - \sigma_3) 2\hat{n}_2 = 0$$

Solution (general case): $\hat{n}_1 = \hat{n}_2 = 0 ; \hat{n}_3 = 1$

This is exactly the 3th principal direction corresponding to σ_3 . σ_1 and σ_2 can be found with elimination of \hat{n}_1 resp. \hat{n}_2 . So, the principal stresses are the extrema of the normal stress.

When $\sigma_1 \geq \sigma_2 \geq \sigma_3$ it must hold that: $\sigma_1 \geq \sigma \geq \sigma_3$.

3.7) Miscellaneous, including Mohr's circle(s)

Consider the following special stress state: $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$

NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a **Principal stress** (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

3.7) Miscellaneous, including Mohr's circle(s)

Consider the following special stress state: $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$

NOTE, that $\sigma_{21} = \sigma_{12}$ and that σ_{33} must be a **Principal stress** (because $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 0$).

Calculate principal stresses:

$$\begin{aligned} & \det \left(\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma & 0 \\ 0 & 0 & \sigma_{33} - \sigma \end{bmatrix} \right) = 0 \\ \Rightarrow & (\sigma_{33} - \sigma) \{(\sigma_{11} - \sigma)(\sigma_{22} - \sigma) - \sigma_{12}^2\} = 0 \\ \Rightarrow & (\sigma_{33} - \sigma) \{\sigma^2 - (\sigma_{11} + \sigma_{22})\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0 \end{aligned}$$

Found: $(\sigma_{33} - \sigma)\{\sigma^2 - (\sigma_{11} + \sigma_{22})\sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2\} = 0$

Solutions:

$$\begin{aligned}\sigma_{1,2} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \frac{1}{2}\sqrt{(\sigma_{11} + \sigma_{22})^2 - 4\sigma_{11}\sigma_{22} + 4\sigma_{12}^2} = \\ &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2} \\ \sigma_3 &= \sigma_{33}\end{aligned}$$

The order of the principal stresses is STILL random!

$$\text{Found: } (\sigma_{33} - \sigma) \{ \sigma^2 - (\sigma_{11} + \sigma_{22}) \sigma + \sigma_{11}\sigma_{22} - \sigma_{12}^2 \} = 0$$

Solutions:

$$\begin{aligned}\sigma_{1,2} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \frac{1}{2}\sqrt{(\sigma_{11} + \sigma_{22})^2 - 4\sigma_{11}\sigma_{22} + 4\sigma_{12}^2} = \\ &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2} \\ \sigma_3 &= \sigma_{33}\end{aligned}$$

The order of the principal stresses is STILL random!

The principal directions can be found with:

$$\begin{bmatrix} \sigma_{11} - \sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma_i & 0 \\ 0 & 0 & \sigma_{33} - \sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1, 2, 3$$

$$\begin{bmatrix} \sigma_{11} - \sigma_i & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} - \sigma_i & 0 \\ 0 & 0 & \sigma_{33} - \sigma_i \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for resp. } i = 1, 2, 3$$

Consecutively resulting in, for $i = 1, 2, 3$:

$$\{\hat{n}\}^{(1)} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} ; \quad \{\hat{n}\}^{(2)} = \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix} ; \quad \{\hat{n}\}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$a_1^2 + a_2^2 = 1 \Rightarrow a_1 = \frac{\sigma_{12}}{w} ; \quad a_2 = \frac{\sigma_i - \sigma_{11}}{w} ; \quad w = \sqrt{(\sigma_i - \sigma_{11})^2 + \sigma_{12}^2}$$

What happens with a rotation of angle α around the x_3 -axis?

Rotation matrix: $[R] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

What happens with a rotation of angle α around the x_3 -axis?

Rotatiematrix: $[R] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Calculation (using $\sigma'_{21} = \sigma'_{12}$):

$$\begin{aligned} [\sigma'] &= \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & 0 \\ \sigma'_{21} & \sigma'_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = [R][\sigma][R]^T = \\ &= \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

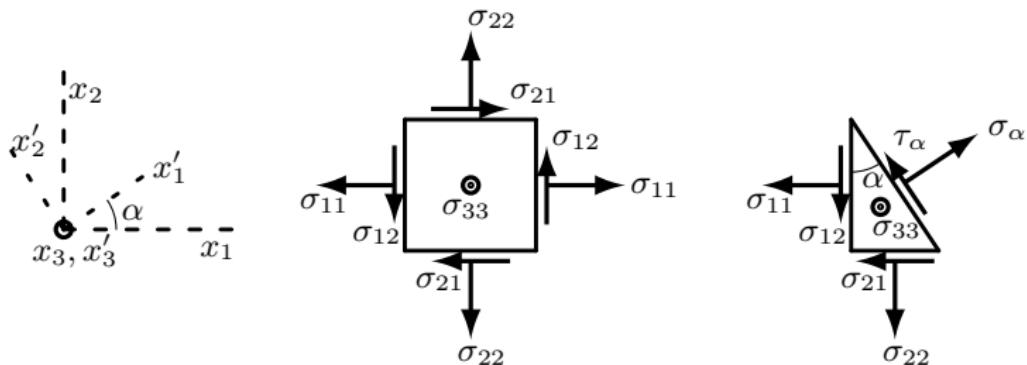
Solving matrices:

$$\begin{aligned}\sigma_\alpha &= \sigma'_{11} = \sigma_{11}(\cos \alpha)^2 + 2\sigma_{12} \sin \alpha \cos \alpha + \sigma_{22}(\sin \alpha)^2 = \\&= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\alpha + \sigma_{12} \sin 2\alpha \\ \tau_\alpha &= \sigma'_{12} = (\sigma_{22} - \sigma_{11}) \sin \alpha \cos \alpha + \sigma_{12} \{(\cos \alpha)^2 - (\sin \alpha)^2\} = \\&= -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha\end{aligned}$$

Solving matrices:

$$\begin{aligned}\sigma_\alpha &= \sigma'_{11} = \sigma_{11}(\cos \alpha)^2 + 2\sigma_{12} \sin \alpha \cos \alpha + \sigma_{22}(\sin \alpha)^2 = \\ &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\alpha + \sigma_{12} \sin 2\alpha \\ \tau_\alpha &= \sigma'_{12} = (\sigma_{22} - \sigma_{11}) \sin \alpha \cos \alpha + \sigma_{12} \{(\cos \alpha)^2 - (\sin \alpha)^2\} = \\ &= -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha\end{aligned}$$

Visualisation:



$$\begin{aligned}\sigma_\alpha &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\alpha + \sigma_{12} \sin 2\alpha \\ \tau_\alpha &= -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha\end{aligned}$$

The equation of a circle is hidden in the expressions of σ_α and τ_α :

$$\{\sigma_\alpha - \frac{1}{2}(\sigma_{11} + \sigma_{22})\}^2 + \tau_\alpha^2 = \frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2$$

$$\sigma_\alpha = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\alpha + \sigma_{12} \sin 2\alpha$$

$$\tau_\alpha = -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha$$

The equation of a circle is hidden in the expressions of σ_α and τ_α :

$$\{\sigma_\alpha - \frac{1}{2}(\sigma_{11} + \sigma_{22})\}^2 + \tau_\alpha^2 = \frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2$$

Mohr's circle

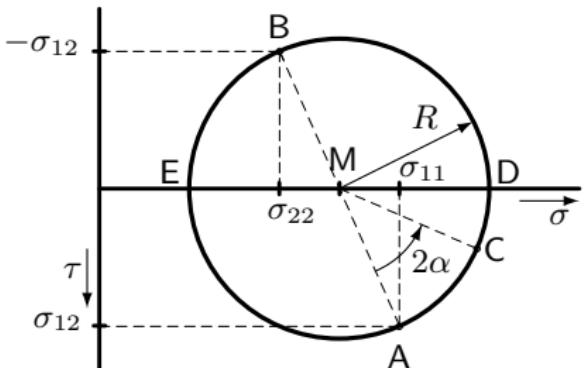
Midpoint M: $\left\{\frac{1}{2}(\sigma_{11} + \sigma_{22}), 0\right\}$

Radius: $R = \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2}$

Omit index α at σ and τ !

A: $\alpha = 0$ $(\sigma_{11}, \sigma_{12})$

B: $\alpha = \frac{1}{2}\pi$ $(\sigma_{22}, \sigma_{21})$



C: $\alpha = ??$ $(\sigma_\alpha, \tau_\alpha)$

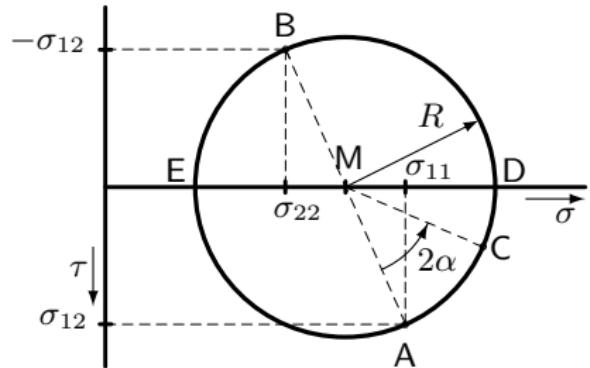
D: $\alpha = \alpha_1$ $(\sigma_1, 0)$

E: $\alpha = \alpha_2$ $(\sigma_2, 0)$

$(2\alpha_1 =$
angle AMD, $2\alpha_2 =$ angle AME)

$$\tan(2\alpha_1) = \frac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$$

$$\alpha_2 = \alpha_1 \pm \frac{1}{2}\pi$$



C: $\alpha = ??$ $(\sigma_\alpha, \tau_\alpha)$

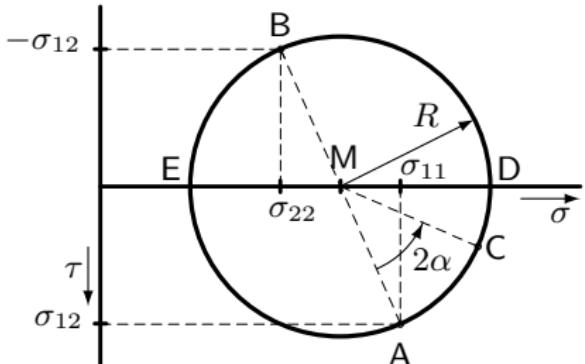
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$$\tan(2\alpha_1) = \frac{\sigma_{12}}{\frac{1}{2}(\sigma_{11} - \sigma_{22})}$$

$$\alpha_2 = \alpha_1 \pm \frac{1}{2}\pi$$



$$\sigma_1 = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + R = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2}$$

$$\sigma_2 = \frac{1}{2}(\sigma_{11} + \sigma_{22}) - R = \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2}$$

Principal stresses are the same as found earlier!!

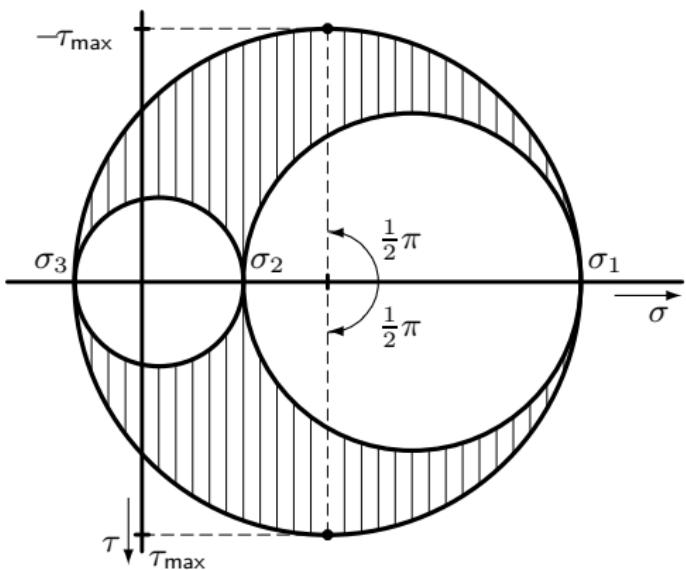
Arbitrary, three dimensional stress state.

Note: $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

Angle with first principal direction:

$$\frac{1}{2} (\pm \frac{1}{2}\pi) = \pm \frac{1}{4}\pi$$

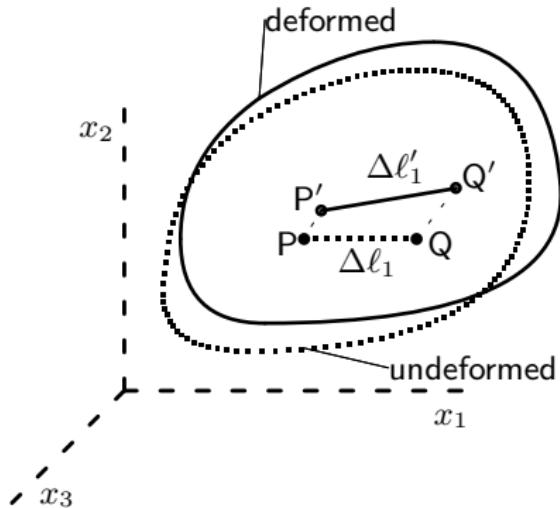
Points from an arbitrary chosen coordinate system will end up in the striped area!



4) Deformation

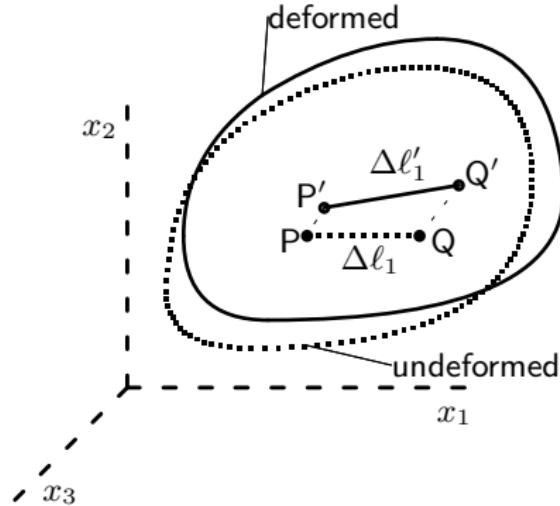
4.1) Strain

Consider a line segment PQ parallel with the x_1 -axis.



4) Deformation

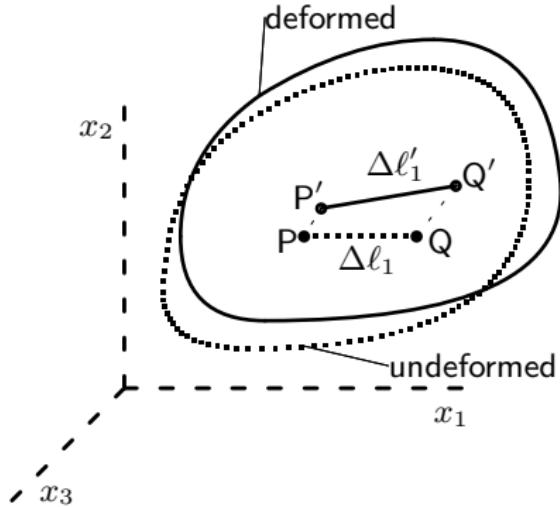
4.1) Strain



Consider a line segment PQ parallel with the x_1 -axis.
The coordinates of P are x_1, x_2, x_3
Translation of P to P'
is up with components $u_i(x_1, x_2, x_3)$.

4) Deformation

4.1) Strain



Consider a line segment PQ parallel with the x_1 -axis.

The

coordinates of P are x_1, x_2, x_3

Translation of P to P' is u_P

with components $u_i(x_1, x_2, x_3)$.

The coordinates

of Q are $x_1 + \Delta\ell_1, x_2, x_3$

Translation of Q to Q' is u_Q with components $u_i(x_1 + \Delta\ell_1, x_2, x_3)$.

with: $u_i(x_1 + \Delta\ell_1, x_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_1 \frac{\partial u_i}{\partial x_1} + \text{H.O.T.}$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{pmatrix} \quad Q': \begin{pmatrix} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \quad Q': \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1} \Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1} \Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1} \Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\Delta\ell'_1 = \sqrt{\left(x_1^{(Q')} - x_1^{(P')}\right)^2 + \left(x_2^{(Q')} - x_2^{(P')}\right)^2 + \left(x_3^{(Q')} - x_3^{(P')}\right)^2} =$$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \quad Q': \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1} \Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1} \Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1} \Delta\ell_1 \end{pmatrix}$$

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Coordinates of the translated points:

$$P': \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{pmatrix} \quad Q': \begin{pmatrix} x_1 + \Delta\ell_1 + u_1 + \frac{\partial u_1}{\partial x_1} \Delta\ell_1 \\ x_2 + u_2 + \frac{\partial u_2}{\partial x_1} \Delta\ell_1 \\ x_3 + u_3 + \frac{\partial u_3}{\partial x_1} \Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned}\Delta\ell'_1 &= \sqrt{\left(x_1^{(Q')} - x_1^{(P')}\right)^2 + \left(x_2^{(Q')} - x_2^{(P')}\right)^2 + \left(x_3^{(Q')} - x_3^{(P')}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1 + \frac{\partial u_1}{\partial x_1} \Delta\ell_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1} \Delta\ell_1\right)^2 + \left(\frac{\partial u_3}{\partial x_1} \Delta\ell_1\right)^2} = \\ &= \Delta\ell_1 \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\end{aligned}$$

Coordinates of the translated points:

$$P': \begin{pmatrix} x_1+u_1 \\ x_2+u_2 \\ x_3+u_3 \end{pmatrix} \quad Q': \begin{pmatrix} x_1+\Delta\ell_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1 \\ x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta\ell_1 \\ x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta\ell_1 \end{pmatrix}$$

Length of line segment PQ in the deformed shape:

$$\begin{aligned} \Delta\ell'_1 &= \sqrt{\left(x_1^{(Q')}-x_1^{(P')}\right)^2 + \left(x_2^{(Q')}-x_2^{(P')}\right)^2 + \left(x_3^{(Q')}-x_3^{(P')}\right)^2} = \\ &= \sqrt{\left(\Delta\ell_1+\frac{\partial u_1}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\Delta\ell_1\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\Delta\ell_1\right)^2} = \\ &= \Delta\ell_1 \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} \\ &\approx \Delta\ell_1 \left(1 + \frac{\partial u_1}{\partial x_1}\right) \quad \text{if } \frac{\partial u_i}{\partial x_1} \ll 1 \end{aligned}$$

The strain in x_1 -direction is:

$$\varepsilon_{11} = \frac{\Delta\ell'_1 - \Delta\ell_1}{\Delta\ell_1} \approx \frac{\Delta\ell_1 (1 + \frac{\partial u_1}{\partial x_1} - 1)}{\Delta\ell_1} = \frac{\partial u_1}{\partial x_1}$$

The strain in x_1 -direction is:

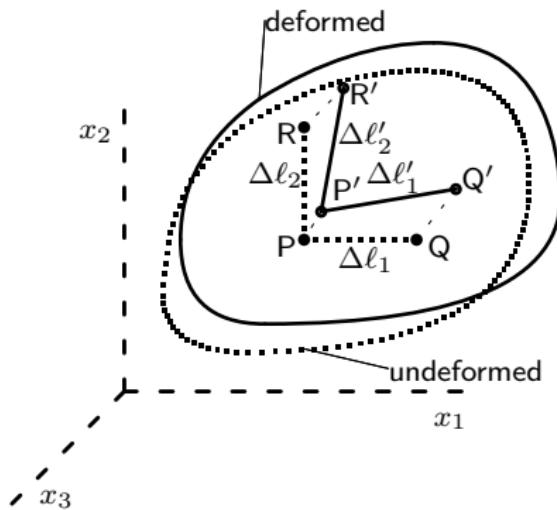
$$\varepsilon_{11} = \frac{\Delta\ell'_1 - \Delta\ell_1}{\Delta\ell_1} \approx \frac{\Delta\ell_1 (1 + \frac{\partial u_1}{\partial x_1} - 1)}{\Delta\ell_1} = \frac{\partial u_1}{\partial x_1}$$

Same for the other directions:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

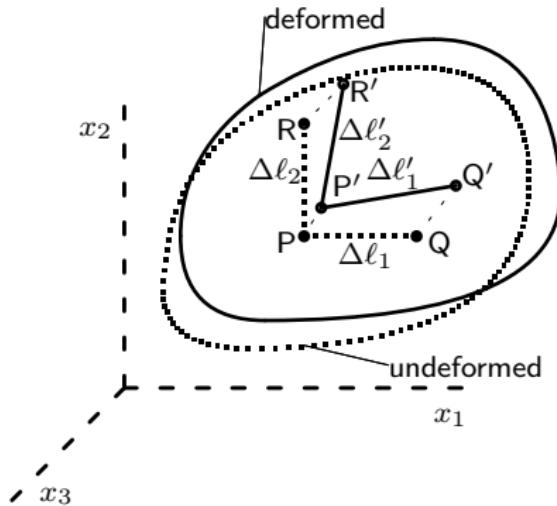
$$\text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \text{ for } \begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$$

4.2) Shear strain



Consider line segments PQ and PR in resp. x_1 - and x_2 -directions.

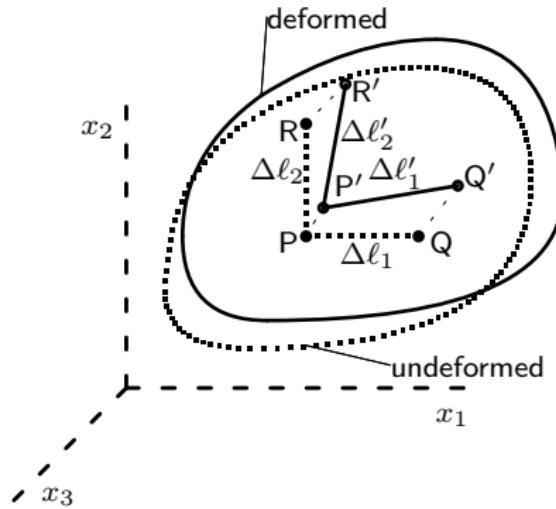
4.2) Shear strain



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The coordinates
of point R are $x_1, x_2 + \Delta\ell_2, x_3$

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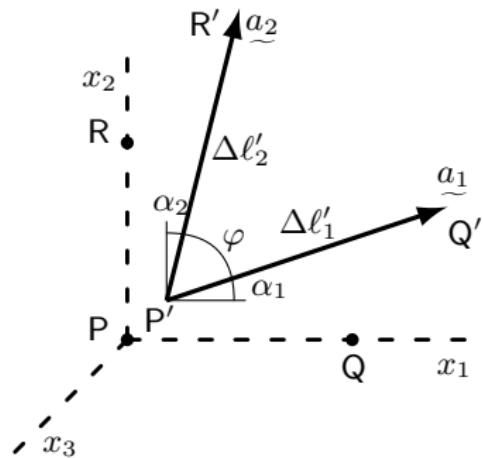
Besides

the translations of $P \rightarrow P'$ and $Q \rightarrow Q'$ we also have \underline{u}_R from $R \rightarrow R'$,

the components $u_i(x_1, x_2 + \Delta\ell_2, x_3)$.

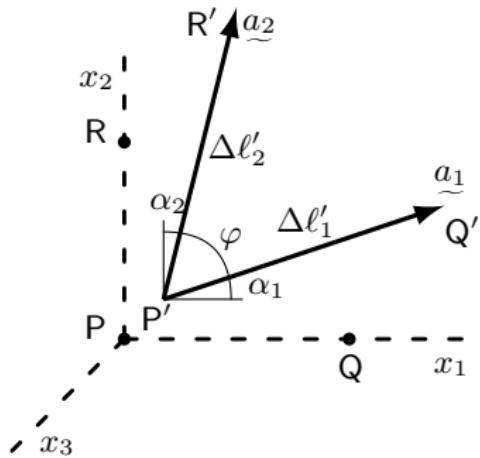
$$\text{with: } u_i(x_1, x_2 + \Delta\ell_2, x_3) = u_i(x_1, x_2, x_3) + \Delta\ell_2 \frac{\partial u_i}{\partial x_2} + \text{H.O.T.}$$

For the coordinates of the points P' , Q' and R' , see reader.



The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90° angle.
So: $\gamma_{12} = \frac{1}{2}\pi - \varphi$

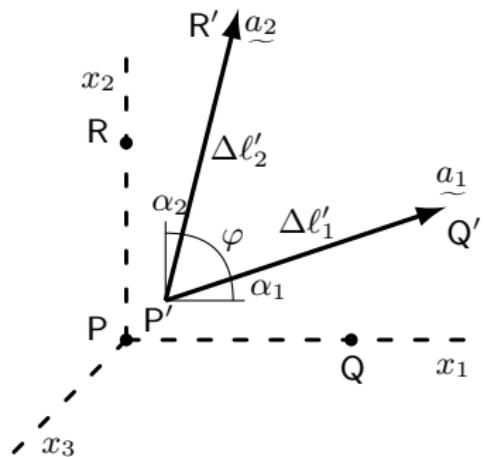
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The shear strain γ_{12} is the change in angle between PQ and PR compared to their original 90° angle.
So: $\gamma_{12} = \frac{1}{2}\pi - \varphi$

$$\begin{aligned}\underline{\underline{a}_1} \cdot \underline{\underline{a}_2} &= |\underline{\underline{a}_1}| |\underline{\underline{a}_2}| \cos(\varphi) \\ &= |\underline{\underline{a}_1}| |\underline{\underline{a}_2}| \cos\left(\frac{1}{2}\pi - \gamma_{12}\right) \\ &= |\underline{\underline{a}_1}| |\underline{\underline{a}_2}| \sin(\gamma_{12})\end{aligned}$$

For the coordinates of the points P' , Q' and R' , see reader.



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From which:

$$\gamma_{12} = \arcsin \left(\frac{\underline{\underline{a}}_1 \cdot \underline{\underline{a}}_2}{|\underline{\underline{a}}_1| |\underline{\underline{a}}_2|} \right)$$

After “some” calculations:

$$\boxed{\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}} \quad \text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for } \begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$$

After "some" calculations:

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

if $\frac{\partial u_i}{\partial x_j} \ll 1$ for $\begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$

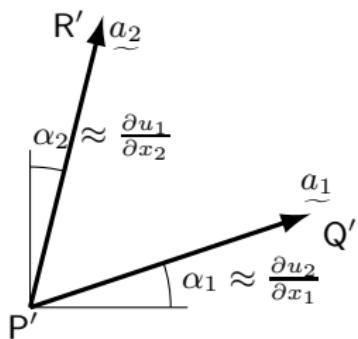
Shear strain $\gamma_{12} = \alpha_1 + \alpha_2$

From the figure: $\tan(\alpha_1) = \frac{\frac{\partial u_2}{\partial x_1} \Delta \ell_1}{\Delta \ell_1 + \frac{\partial u_1}{\partial x_1} \Delta \ell_1}$

As approximation: $\alpha_1 \approx \frac{\partial u_2}{\partial x_1}$

And: $\alpha_2 \approx \frac{\partial u_1}{\partial x_2}$

In general it holds that $\alpha_1 \neq \alpha_2 !!!$



Results for 3 directions:

$$\gamma_{12} = \gamma_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\gamma_{13} = \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \quad \text{if } \frac{\partial u_i}{\partial x_j} \ll 1 \quad \text{for } \begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$$

$$\gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

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$$\gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

Summarised in index notation:

$$\boxed{\gamma_{ij} = \gamma_{ji} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}} \quad \text{with } i \neq j$$

4.3) Strain tensor

Strains and shear strains from the theory of *small deformations*:

$$\begin{aligned}\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} & \gamma_{12} = \gamma_{21} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} & \gamma_{13} = \gamma_{31} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} & \gamma_{23} = \gamma_{32} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\end{aligned}$$

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$$\begin{aligned}\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} & \gamma_{12} = \gamma_{21} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} & \gamma_{13} = \gamma_{31} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} & \gamma_{23} = \gamma_{32} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\end{aligned}$$

If $i \neq j$ is used for $\varepsilon_{ij} = \frac{1}{2} \gamma_{ij}$ then:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

for $\begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$

Notation $\left(\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i}) \right)$:

$$\underline{\varepsilon} = \varepsilon_{ij} \hat{e}_i \hat{e}_j \quad [\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{bmatrix}$$

Notation $\left(\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i}) \right)$:

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Outdated notation

$(x_1 \rightarrow x, \ x_2 \rightarrow y \text{ and } x_3 \rightarrow z, \ u_1 \rightarrow u, \ u_2 \rightarrow v \text{ and } u_3 \rightarrow w)$:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \ (= \varepsilon_x) \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \ (= \varepsilon_y) \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \ (= \varepsilon_z)$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \varepsilon_{yx} = \frac{1}{2}\gamma_{yx}$$

$$\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \varepsilon_{zx} = \frac{1}{2}\gamma_{zx}$$

$$\varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \varepsilon_{zy} = \frac{1}{2}\gamma_{zy}$$

Question: Are ε_{ij} the components of a second order tensor?

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Apply rotation of coordinate axes, $x_i \rightarrow x'_p$:

$$\begin{aligned}x'_p &= x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) = \\&= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i\end{aligned}$$

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For the deformation components it holds:

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} \frac{\partial x'_q}{\partial x_j} + \frac{\partial u_j}{\partial x'_p} \frac{\partial x'_p}{\partial x_i} \right) = \\&= \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi} \right)\end{aligned}$$

Question: Are ε_{ij} the components of a second order tensor?

Apply rotation of coordinate axes, $x_i \rightarrow x'_p$:

$$\begin{aligned}x'_p &= x_1 \cos(x'_p, x_1) + x_2 \cos(x'_p, x_2) + x_3 \cos(x'_p, x_3) = \\&= x_1 R_{p1} + x_2 R_{p2} + x_3 R_{p3} = R_{pi} x_i\end{aligned}$$

For the deformation components it holds:

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} \frac{\partial x'_q}{\partial x_j} + \frac{\partial u_j}{\partial x'_p} \frac{\partial x'_p}{\partial x_i} \right) = \\&= \frac{1}{2} \left(\frac{\partial u_i}{\partial x'_q} R_{qj} + \frac{\partial u_j}{\partial x'_p} R_{pi} \right)\end{aligned}$$

Transformation components of displacement vector: $u_i = R_{pi} u'_p \implies$

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2} \left(R_{pi} \frac{\partial u'_p}{\partial x'_q} R_{qj} + R_{qj} \frac{\partial u'_q}{\partial x'_p} R_{pi} \right) = R_{pi} R_{qj} \frac{1}{2} \left(\frac{\partial u'_p}{\partial x'_q} + \frac{\partial u'_q}{\partial x'_p} \right) = \\&= R_{pi} R_{qj} \varepsilon'_{pq}\end{aligned}$$

Summarized: (from “new” to “old” and vice versa):

$$\varepsilon_{ij} = R_{pi} R_{qj} \varepsilon'_{pq}$$

and

$$\varepsilon'_{pq} = R_{pi} R_{qj} \varepsilon_{ij}$$

In matrix-vector-notation:

$$[\varepsilon] = [R]^T [\varepsilon'] [R]$$

and

$$[\varepsilon'] = [R] [\varepsilon] [R]^T$$

Summarized: (from “new” to “old” and vice versa):

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In matrix-vector-notation:

$$[\varepsilon] = [R]^T [\varepsilon'] [R]$$

and

$$[\varepsilon'] = [R] [\varepsilon] [R]^T$$

So ε_{ij} are indeed components of a second order tensor. As well as the stress tensor, this tensor is symmetric; for the deformation matrix it holds:
 $[\varepsilon]^T = [\varepsilon]$.

Also see the reader regarding the arbitrary oriented line segment PQ.

4.4) Principal strains and principal directions

Second order tensor, so principal directions follow from:

$$\det \begin{pmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon \end{pmatrix} = \det([\varepsilon] - \varepsilon[I]) = 0$$

Resulting in the characteristic equation: $\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$

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Resulting in the characteristic equation: $\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$

The *deformation invariants* are:

$$E_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii}$$

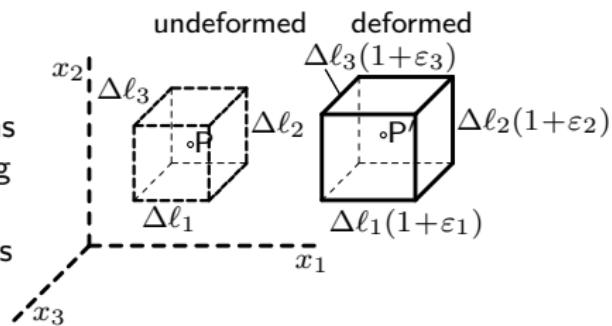
$$E_2 = \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21} + \textcircled{O} = \frac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji})$$

$$E_3 = \det([\varepsilon])$$

After solving we obtain the principal strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and the 3 orthogonal principal directions. The invariants can be expressed in terms of principal strains.

4.5) Volumetric strain

Consider a x_i -coordinate system oriented in such a way that it coincides with the principal directions of the deformed state. A corresponding infinitesimal volume element does not have any shear strain. The surfaces remain perpendicular to each other during deformation.



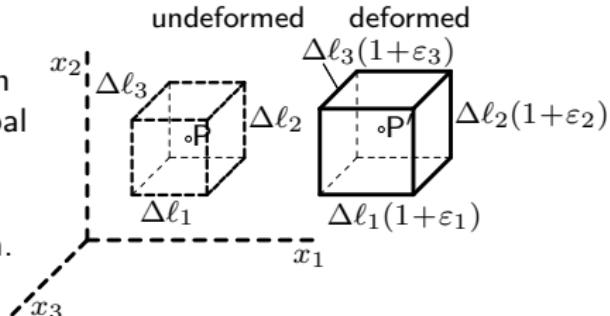
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A corresponding infinitesimal volume element does not have any shear strain.

The surfaces remain perpendicular to each other during deformation.



$$\text{Volume undeformed: } V = \Delta\ell_1 \Delta\ell_2 \Delta\ell_3$$

$$\text{Volume deformed: } V' = \Delta\ell_1(1 + \varepsilon_1)\Delta\ell_2(1 + \varepsilon_2)\Delta\ell_3(1 + \varepsilon_3)$$

Volumetric strain:

$$\varepsilon_V = \frac{V' - V}{V} = (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - 1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \text{H.O.T.}$$

For small strains we ignore H.O.T. (also including the third invariant E_3).

$$\Rightarrow \varepsilon_V = E_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad \text{or} \quad \boxed{\varepsilon_V = \varepsilon_{kk} = E_1}$$

4.6) Engineering strain/stress vs. true strain/stress

Note that:

engineering strain $\varepsilon = \frac{\Delta L}{L_0}$ is defined relative to the original length, L_0 ,
while stress $\sigma = \frac{F}{A_0}$ is defined relative to the original cross-section, A_0 .

Neither can be assumed constant during deformation,
so that a true strain increment becomes $\delta\varepsilon_t = \frac{\delta L}{L(t)}$,
defined relative to the momentary (at time t) size of the sample, $L(t)$,
while the true stress becomes $\sigma_t = \frac{F(t)}{A(t)}$, with momentary cross-section $A(t)$.

Integrating the true strain increment $\delta\varepsilon_t$ from 0 to ε_t ,
and the right hand side from L_0 to $L(t)$,
using $\ln L(t) - \ln L(0) = \ln(L(t)/L_0)$, with $L(t) = L_0 + \Delta L$,
yields the true strain $\varepsilon_t = \ln(1 + \varepsilon)$.

5) Material behavior

5.1) Hooke's law

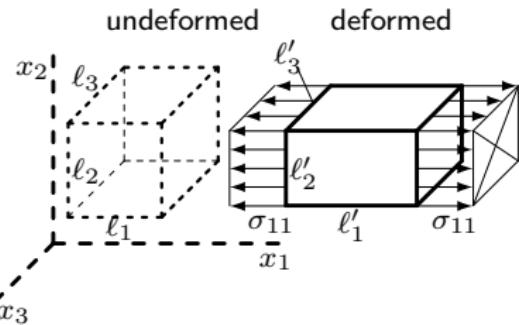
Linear elastic, isotropic material behavior.

$$\text{Strain: } \varepsilon_{11} = \frac{\ell'_1 - \ell_1}{\ell_1}$$

$$\text{Relation stress and strain: } \varepsilon_{11} = \frac{\sigma_{11}}{E}$$

E is the

Young's modulus (modulus of elasticity)



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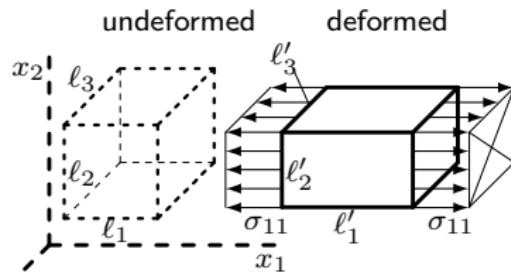
E is the

Young's modulus (modulus of elasticity)

$$\text{Lateral strain: } \varepsilon_{22} = \frac{\ell'_2 - \ell_2}{\ell_2} \leq 0 \quad \text{en} \quad \varepsilon_{33} = \frac{\ell'_3 - \ell_3}{\ell_3} \leq 0$$

$$\text{Relation lateral strain and stresses: } \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$

ν is the Poisson's ratio



Isotropic material behavior!!!

So, with a single normal stress σ_{11} (other stress components are zero!):

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad \text{and} \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$

The shear strain seems to remain equal to zero.

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The shear strain seems to remain equal to zero.

Also, when we apply a single normal stress σ_{22} , again with the other stress components equal to zero:

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{33} = -\nu \frac{\sigma_{22}}{E}$$

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Likewise for normal stress σ_{33} :

$$\varepsilon_{33} = \frac{\sigma_{33}}{E} \quad \text{and} \quad \varepsilon_{11} = \varepsilon_{22} = -\nu \frac{\sigma_{33}}{E}$$

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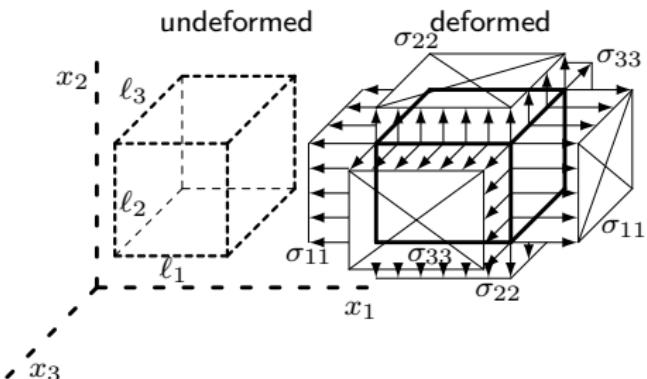
Also a combined stress state is possible. Superposition principle.

Result:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E}$$

$$\varepsilon_{22} = -\nu \frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E}$$

$$\varepsilon_{33} = -\nu \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}$$



This is Hooke's law in the case that only normal stresses are present!

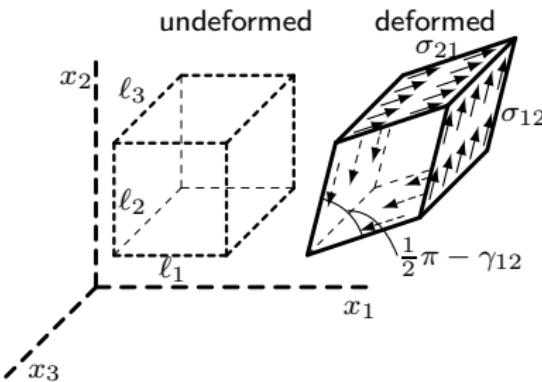
Shear, shear stresses and shear strain:

$$2\varepsilon_{12} = \gamma_{12} = \frac{\sigma_{12}}{G} = \frac{\sigma_{21}}{G}$$

$$2\varepsilon_{13} = \gamma_{13} = \frac{\sigma_{13}}{G} = \frac{\sigma_{31}}{G}$$

$$2\varepsilon_{23} = \gamma_{23} = \frac{\sigma_{23}}{G} = \frac{\sigma_{32}}{G}$$

The material constant G is the *shear modulus*.



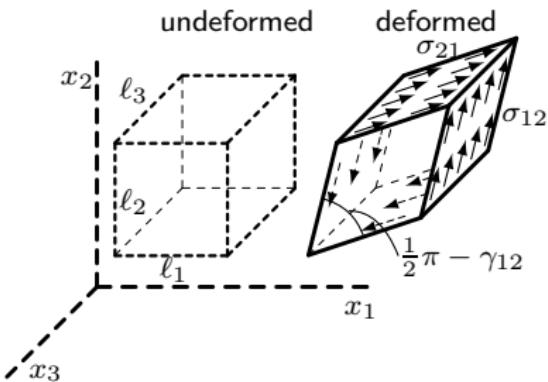
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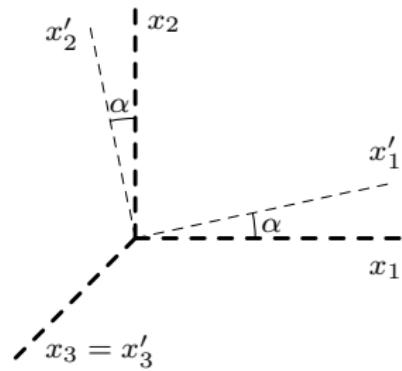
NOTE: With isotropic material behavior E , G and ν are correlated with each other. This correlation can be found via a coordinate system rotation.

Linear stress state

$$[\sigma]_{x_i} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

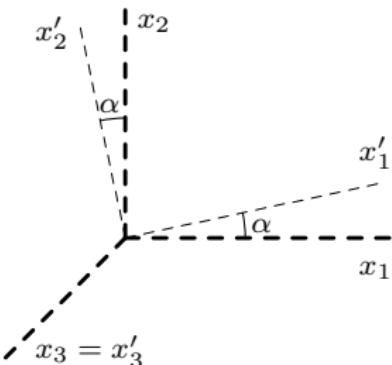
From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$



Linear stress state

$$[\sigma]_{x_i} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



From Hooke's law follows:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} \quad ; \quad \varepsilon_{22} = \varepsilon_{33} = -\nu \frac{\sigma_{11}}{E}$$

so:

$$[\varepsilon] = \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu\sigma_{11} & 0 \\ 0 & 0 & -\nu\sigma_{11} \end{bmatrix} \quad \text{and} \quad [R] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve for σ'_{12} with $\sigma'_{12} = R_{1p} R_{q2} \sigma_{pq}$ or with $[\sigma'] = [R][\sigma][R]^T$:

$$\begin{bmatrix} \diamond & \sigma'_{12} & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \diamond & -\sin \alpha & \diamond \\ \diamond & \cos \alpha & \diamond \\ \diamond & 0 & \diamond \end{bmatrix}$$

All non-relevant values for the matrix multiplication are shown as a \diamond .

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All non-relevant values for the matrix multiplication are shown as a \diamond . Or shorter:

$$\sigma'_{12} = [\cos \alpha \quad \sin \alpha \quad 0] \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \\ 0 \end{bmatrix} = -\sigma_{11} \cos \alpha \sin \alpha$$

The same holds for shear strain ε'_{12} . From $[\varepsilon'] = [R][\varepsilon][R]^T$ follows:

$$\begin{aligned}\varepsilon'_{12} &= \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \end{bmatrix} \frac{1}{E} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\nu \sigma_{11} & 0 \\ 0 & 0 & -\nu \sigma_{11} \end{bmatrix} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \\ 0 \end{bmatrix} = \\ &= -\frac{(1+\nu)}{E} \sigma_{11} \cos \alpha \sin \alpha\end{aligned}$$

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Due to isotropic behavior it must hold that:

$$\varepsilon'_{12} = \frac{1}{2} \gamma'_{12} = \frac{1}{2} \frac{\sigma'_{12}}{G} = -\frac{1}{2G} \sigma_{11} \cos \alpha \sin \alpha \quad (\sigma'_{12} = -\sigma_{11} \cos \alpha \sin \alpha)$$

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When we compare both expressions, the following important equation is found:

$$G = \frac{E}{2(1+\nu)}$$

NOTE: This equation only holds for *isotropic* material behavior!

Summary of Hooke's law:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) ; \quad \gamma_{12} = 2\varepsilon_{12} = \frac{1}{G}\sigma_{12} \quad (\text{also } 12 \rightarrow 21)$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11}) ; \quad \gamma_{13} = 2\varepsilon_{13} = \frac{1}{G}\sigma_{13} \quad (\text{also } 13 \rightarrow 31)$$

$$\varepsilon_{33} = \frac{1}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) ; \quad \gamma_{23} = 2\varepsilon_{23} = \frac{1}{G}\sigma_{23} \quad (\text{also } 23 \rightarrow 32)$$

Hooke's law in index notation. For strains:

$$\begin{aligned}\varepsilon_{11} &= \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E} \sigma_{11} - \frac{\nu}{E} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \\ &= \frac{(1+\nu)}{E} \left(\sigma_{11} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right)\end{aligned}$$

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$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} = \frac{(1+\nu)}{E} \left(\sigma_{33} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right)$$

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$$\begin{aligned}\varepsilon_{11} &= \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E} = \frac{(1+\nu)}{E} \sigma_{11} - \frac{\nu}{E} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \\ &= \frac{(1+\nu)}{E} \left(\sigma_{11} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right)\end{aligned}$$

$$\begin{aligned}\varepsilon_{22} &= \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E} - \nu \frac{\sigma_{11}}{E} = \frac{(1+\nu)}{E} \left(\sigma_{22} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right) \\ \varepsilon_{33} &= \frac{\sigma_{33}}{E} - \nu \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} = \frac{(1+\nu)}{E} \left(\sigma_{33} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right)\end{aligned}$$

So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} \left(\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right) \quad \text{if } i = j$$

For shear strains:

$$\begin{aligned}\varepsilon_{12} &= \frac{\sigma_{12}}{2G} = \frac{(1+\nu)}{E} \sigma_{12} & ; \quad \varepsilon_{13} &= \frac{\sigma_{13}}{2G} = \frac{(1+\nu)}{E} \sigma_{13} \\ \varepsilon_{23} &= \frac{\sigma_{23}}{2G} = \frac{(1+\nu)}{E} \sigma_{23}\end{aligned}$$

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So:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} \sigma_{ij} \text{ if } i \neq j \text{ and } \varepsilon_{ij} = \frac{(1+\nu)}{E} \left(\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk} \right) \text{ if } i = j$$

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Combined they result in Hooke's law in index notation ($\sigma \rightarrow \varepsilon$) :

$$\boxed{\varepsilon_{ij} = \frac{(1+\nu)}{E} \left(\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk} \delta_{ij} \right) = \frac{1}{2G} \left(\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk} \delta_{ij} \right)}$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

The *inverse*, stresses expressed in deformations.

We found: $\varepsilon_{ij} = \frac{(1+\nu)}{E} \left(\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk} \delta_{ij} \right)$

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A summation of normal strains gives:

$$\begin{aligned}\varepsilon_{kk} &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{(1+\nu)}{E} \left(\sigma_{11} + \sigma_{22} + \sigma_{33} - 3 \frac{\nu}{(1+\nu)} \sigma_{kk} \right) = \\ &= \frac{(1+\nu)}{E} \left(\sigma_{kk} - 3 \frac{\nu}{(1+\nu)} \sigma_{kk} \right) = \frac{(1-2\nu)}{E} \sigma_{kk}\end{aligned}$$

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Rewriting Hooke's law:

$$\sigma_{ij} = \frac{E}{(1+\nu)} \varepsilon_{ij} + \frac{\nu}{(1+\nu)} \sigma_{kk} \delta_{ij} = \frac{E}{(1+\nu)} \varepsilon_{ij} + \frac{\nu}{(1+\nu)} \frac{E}{(1-2\nu)} \varepsilon_{kk} \delta_{ij}$$

Resulting in: Hooke's law ($\varepsilon \rightarrow \sigma$) :

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk} \delta_{ij} \right) = 2G \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk} \delta_{ij} \right)$$

for $i = 1, 2, 3$; $j = 1, 2, 3$.

Resulting in; Hooke's law ($\varepsilon \rightarrow \sigma$) :

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for $i = 1, 2, 3$; $j = 1, 2, 3$.

Notation according to Lamé:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

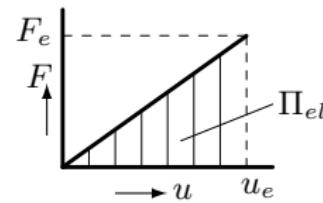
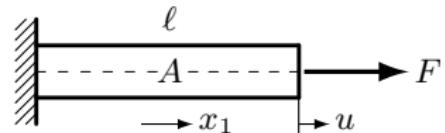
With the so called Lamé's (material) constants:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{en} \quad \mu = G = \frac{E}{2(1+\nu)}$$

5.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \, du = \frac{1}{2} F_e u_e$$



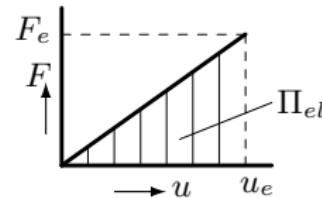
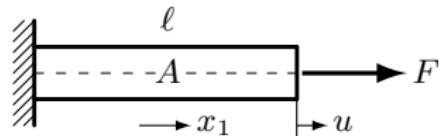
5.2) Elastic energy

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$$\Pi_{el} = \int_0^e F \, du = \frac{1}{2} F_e u_e$$

At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$



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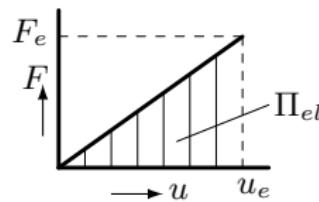
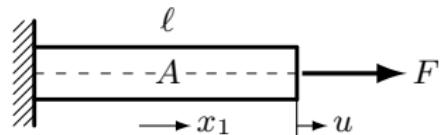
$$\Pi_{el} = \int_0^e F \, du = \frac{1}{2} F_e u_e$$

At the end:

$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$

Such that:

$$\Pi_{el} = \frac{1}{2} A \sigma_{11} \ell \varepsilon_{11} = \frac{1}{2} \sigma_{11} \varepsilon_{11} V$$



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Work:

$$\Pi_{el} = \int_0^{u_e} F \, du = \frac{1}{2} F_e u_e$$

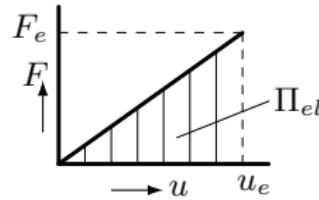
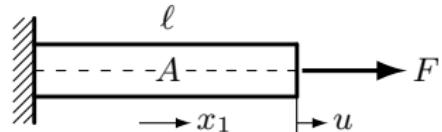
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Specific internal elastic energy: $\pi_{el} = \frac{1}{2} \sigma_{11} \varepsilon_{11}$



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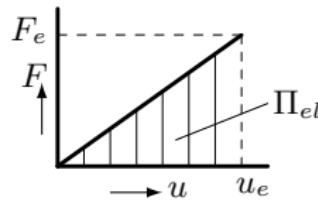
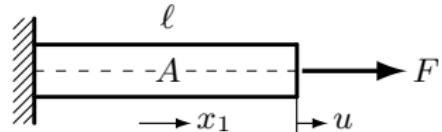
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Specific internal elastic energy: $\pi_{el} = \frac{1}{2} \sigma_{11} \varepsilon_{11}$

Contribution to π_{el} during shearing: $\frac{1}{2} \sigma_{12} \gamma_{12}$



An arbitrary three dimensional stress state:

$$\pi_{el} = \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{33}\varepsilon_{33} + \frac{1}{2}\sigma_{12}\gamma_{12} + \frac{1}{2}\sigma_{13}\gamma_{13} + \frac{1}{2}\sigma_{23}\gamma_{23}$$

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Written differently (with $\gamma_{ij} = 2\varepsilon_{ij} = 2\varepsilon_{ji}$):

$$\begin{aligned}\pi_{el} &= \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{12}\varepsilon_{12} + \frac{1}{2}\sigma_{13}\varepsilon_{13} + \\ &+ \frac{1}{2}\sigma_{21}\varepsilon_{21} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{23}\varepsilon_{23} + \\ &+ \frac{1}{2}\sigma_{31}\varepsilon_{31} + \frac{1}{2}\sigma_{32}\varepsilon_{32} + \frac{1}{2}\sigma_{33}\varepsilon_{33}\end{aligned}$$

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So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}}$$

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So, for the *specific elastic energy* holds (summation convention!!):

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Elastic energy of a body with volume V :

$$\Pi_{el} = \int_V \pi_{el} \, dV = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \, dV$$

By filling in Hooke's law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$, either the stresses or the strains can be eliminated.

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The *specific elastic volume-change energy* is:

$$\boxed{\pi_{el,vol} = \frac{1}{2}\sigma_m\varepsilon_V} \quad \text{with}$$

$$\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk} \quad \text{and} \quad \varepsilon_V = \varepsilon_{kk}$$

With Hooke's law we have seen that:

$$\varepsilon_V = \varepsilon_{kk} = \frac{(1-2\nu)}{E} \sigma_{kk} = \frac{3(1-2\nu)}{E} \sigma_m = \frac{\sigma_m}{C} \quad (\sigma_m = C \varepsilon_V)$$

Where C is the bulk modulus.

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Where C is the bulk modulus. Filling in $\pi_{el_{vol}}$:

$$\pi_{el_{vol}} = \frac{1}{2} \sigma_m \varepsilon_V = \frac{1}{2} \sigma_m \frac{3(1-2\nu)}{E} \sigma_m = \frac{1}{2} \frac{(1-2\nu)}{3E} \sigma_{hh} \sigma_{kk}$$

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Turns out to be invariant for coordinate system transformations.

The *specific elastic shape-change (or distortional) energy*:

$$\begin{aligned}\pi_{el_{shape}} &= \pi_{el} - \pi_{el_{vol}} = \dots = \\ &= \frac{1}{6} \frac{(1+\nu)}{E} \{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}\end{aligned}$$

5.3) Deviatoric stresses and strains

Deviatoric stresses: $\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$

Deviatoric strains: $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_V \delta_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij}$

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Also the deviatoric stresses have principal stresses and principal directions. The corresponding coefficients in the characteristic equation

$$\hat{\sigma}^3 - J_2 \hat{\sigma} - J_3 = 0$$

i.e., the deviatoric invariants are $J_1 = 0$, $J_2 > 0$ and J_3 .

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It turns out that: **Law of Hooke:** $\hat{\sigma}_{ij} = 2G \hat{\varepsilon}_{ij}$

$$\pi_{el_shape} = \frac{1}{6} \frac{(1+\nu)}{E} \{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\} = \frac{(1+\nu)}{E} J_2$$

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And: $(\pi_{el} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} ; \quad \pi_{el_vol} = \frac{1}{2} \sigma_m \varepsilon_V)$

$$\boxed{\pi_{el_shape} = \frac{1}{2} \frac{(1+\nu)}{E} \hat{\sigma}_{ij} \hat{\sigma}_{ij} = \frac{1}{2} \frac{1}{2G} \hat{\sigma}_{ij} \hat{\sigma}_{ij} = \frac{1}{2} \hat{\sigma}_{ij} \hat{\varepsilon}_{ij}}$$

5.4) Failure criteria

1) Tresca, bases on maximum shear stress.

$$\sigma_{\text{eq}} = 2 \tau_{\max} = \max\{|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\}$$

Or, if $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

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2) Von Mises, based on the maximum specific elastic distortional energy.

$$\begin{aligned}\sigma_{\text{eq}} &= \sqrt{3 \frac{E}{1+\nu} \pi_{el_shape}} = \sqrt{3 J_2} = \\ &= \sqrt{\frac{1}{2} \{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\}}\end{aligned}$$

Shear stress criteria

Tresca, Coulomb, Quest, Mohr

Maximum shear stress is leading.

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Comparison 3-dim stress state with 1-dim stress state in a tensile test. $(\sigma_1 = \sigma_{\text{eq}} ; \sigma_2 = \sigma_3 = 0)$

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \quad (3\text{-dim})$$

$$\tau_{\max} = \frac{1}{2}\sigma_{\text{eq}} \quad (1\text{-dim})$$

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$$\rightarrow \sigma_{\text{eq}} = \sqrt{\frac{1}{2} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \}}$$

$$\sigma_{\text{eq}} \leq \bar{\sigma} \quad (\text{allowable stress})$$

Special case

Determine the equivalent stresses of a beam with circular cross section. A bending and torsional moment are acting on the beam (Hibbeler pg.195 (5.7) $T\rho/J$):

$$\sigma_{11} = \frac{M_b R}{I_b} \quad \text{with} \quad I_b = \frac{\pi}{4} R^4$$

$$\sigma_{12} = \frac{M_w R}{I_p} \quad \text{with} \quad I_p = \frac{\pi}{2} R^4 = 2 I_b$$

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The complete stress state becomes:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Principal stresses?}$$

$$\det \begin{pmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{pmatrix} = 0 \rightarrow$$

$$\det \begin{pmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma & 0 \\ 0 & 0 & -\sigma \end{pmatrix} = 0 \rightarrow$$

Solving determinant: $(-\sigma)\{(\sigma_{11} - \sigma)(-\sigma) - \sigma_{12}^2\} = 0 \rightarrow$

$$\sigma_1 = 0 ; \quad \sigma^2 - \sigma_{11}\sigma - \sigma_{12}^2 = 0 \rightarrow \sigma_{2,3} = \frac{1}{2}\sigma_{11} \pm \frac{1}{2}\sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}$$

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The solutions in the right order:

$$\sigma_1 = \frac{1}{2}\left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right); \sigma_2 = 0; \sigma_3 = \frac{1}{2}\left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right)$$

We just found:

$$\sigma_1 = \frac{1}{2} \left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2} \right); \sigma_2 = 0; \sigma_3 = \frac{1}{2} \left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2} \right)$$

Tresca: $\sigma_{\text{eq}} = \sigma_1 - \sigma_3 = \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}$

von Mises: $\sigma_{\text{eq}} = \sqrt{\frac{1}{2} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \}} =$
 $= \sqrt{\sigma_{11}^2 + 3\sigma_{12}^2}$

Note the difference!!

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$$\sigma_1 = \frac{1}{2} \left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2} \right); \sigma_2 = 0; \sigma_3 = \frac{1}{2} \left(\sigma_{11} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2} \right)$$

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Note the difference!! Expressed in the bending and torsional moment:

Tresca: $\sigma_{\text{eq}} = \frac{R}{I_b} \sqrt{M_b^2 + M_w^2}$

von Mises: $\sigma_{\text{eq}} = \frac{R}{I_b} \sqrt{M_b^2 + \frac{3}{4}M_w^2}$

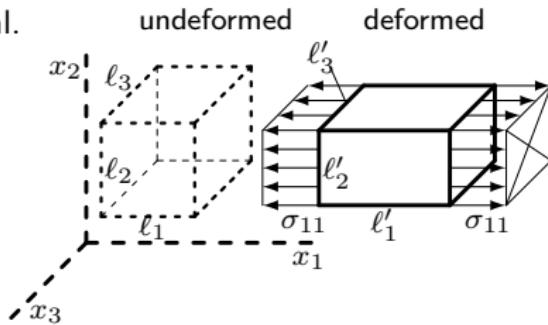
6) General material behavior

6.1) Hooke's law \Rightarrow viscosity

Linear elastic, isotropic viscous material.

In the special case of 1D:

$$\text{Strain: } \varepsilon_{11} = \frac{\ell'_1 - \ell_1}{\ell_1}$$



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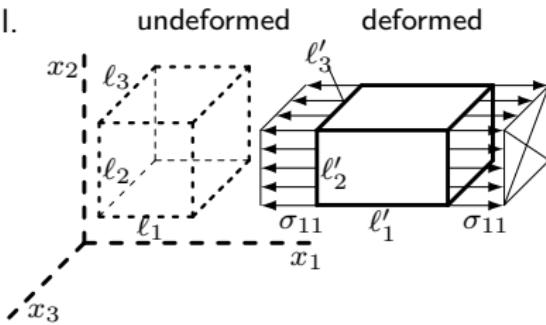
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$$\text{Strain: } \varepsilon_{11} = \frac{\ell'_1 - \ell_1}{\ell_1}$$

$$\text{Strain-rate: } \dot{\varepsilon}_{11} = \frac{\ell'_1 - \ell_1}{\ell_1 \Delta t}$$

Relation between stress and strain:

$$\sigma_{11} = E\varepsilon_{11} + \eta\dot{\varepsilon}_{11}$$



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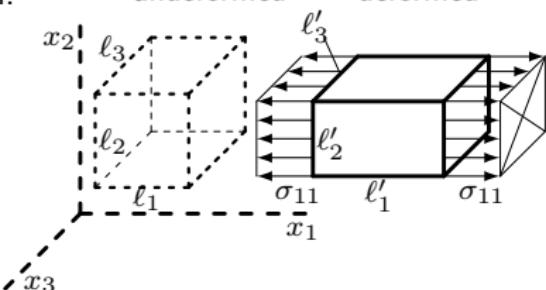
etc. ...

E is the Young's modulus (modulus of elasticity) and η is the viscosity.

Also a combined stress state is possible. Superposition principle.

Resulting in; Hooke's law + viscous stress

undeformed deformed



($\varepsilon \rightarrow \sigma$) :

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)} \left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \varepsilon_{kk} \delta_{ij} \right) + \eta \dot{\varepsilon}_{ij}}$$

for $i = 1, 2, 3 ; j = 1, 2, 3$.

6.2) Elastic energy

Work:

$$\Pi_{el} = \int_0^e F \, du = \frac{1}{2} F_e u_e$$

At the end:

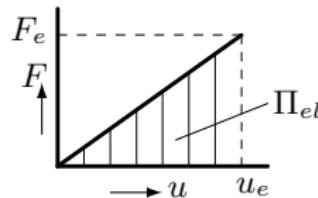
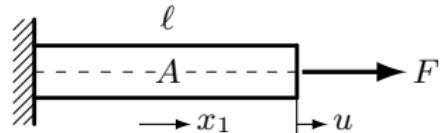
$$\sigma_{11} = \frac{F_e}{A} \quad \text{en} \quad \varepsilon_{11} = \frac{u_e}{\ell}$$

Such that:

$$\Pi_{el} = \frac{1}{2} A \sigma_{11} \ell \varepsilon_{11} = \frac{1}{2} \sigma_{11} \varepsilon_{11} V$$

Specific internal elastic energy: $\pi_{el} = \frac{1}{2} \sigma_{11} \varepsilon_{11}$

Contribution to π_{el} during shearing: $\frac{1}{2} \sigma_{12} \gamma_{12}$



An arbitrary three dimensional stress state:

$$\pi_{el} = \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{33}\varepsilon_{33} + \frac{1}{2}\sigma_{12}\gamma_{12} + \frac{1}{2}\sigma_{13}\gamma_{13} + \frac{1}{2}\sigma_{23}\gamma_{23}$$

Written differently (with $\gamma_{ij} = 2\varepsilon_{ij} = 2\varepsilon_{ji}$):

$$\begin{aligned}\pi_{el} &= \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{12}\varepsilon_{12} + \frac{1}{2}\sigma_{13}\varepsilon_{13} + \\ &+ \frac{1}{2}\sigma_{21}\varepsilon_{21} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{23}\varepsilon_{23} + \\ &+ \frac{1}{2}\sigma_{31}\varepsilon_{31} + \frac{1}{2}\sigma_{32}\varepsilon_{32} + \frac{1}{2}\sigma_{33}\varepsilon_{33}\end{aligned}$$

So, for the *specific elastic energy* holds (summation convention!!):

$$\boxed{\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}}$$

Elastic energy of a body with volume V :

$$\Pi_{el} = \int_V \pi_{el} \, dV = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \, dV$$

Visco-elastic energy:

Hooke's and (simplest) Newtonian viscous law in the equation of the specific elastic energy $\pi_{el} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$, and
 $\pi_{visc} = \int \sigma_{ij}^{visc} d\varepsilon_{ij} = \eta\dot{\varepsilon}_{ij} \int d\varepsilon_{ij}$, for constant η and ε_{ij} .

Elastic and viscous energy of a body with volume V :

$$\Pi_{el} = \int_V \pi_{el} dV = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij} dV \quad \text{and} \quad \Pi_{visc} = \int_V \pi_{visc} dV$$

1D only: Elastic and viscous energy of a body with volume V :

$$\begin{aligned}\Pi &= \Pi_{el} + \Pi_{visc} = \int_V (\pi_{el} + \pi_{visc}) dV \\ &= \int_V \left(\frac{1}{2}E\varepsilon_{11}^2 + \eta\dot{\varepsilon}_{11}\varepsilon_{11} \right) dV\end{aligned}$$