HEATQUIZ

Formulary Heat Transfer: Complete

Version 1 from 2022

from 13th January 2022

Black body radiation

$$\dot{q}''_{\lambda,\mathrm{b}} = \frac{c_1 \lambda^{-5}}{\exp\left[c_2/(\lambda T)\right] - 1} \qquad \text{(Planck's distribution law)}$$

$$\dot{q}''_{\mathrm{b}} = \int\limits_{\lambda=0}^{\infty} \dot{q}''_{\lambda\mathrm{b}} d\lambda = \sigma T^4 \qquad \text{(Stefan-Boltzmann's law)}$$

$$\lambda_{\max} T = 2898\,\mu\mathrm{m\,K} \qquad \text{(Wien's law of displacement)}$$

with the constants

$$\sigma = 5.67\ 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2\,K^4}} \qquad \text{(Stefan-Boltzmann constant)}$$

$$c_1 = 3.741\ 10^{-16}\,\mathrm{W\,m^2}$$

$$c_2 = 1.439\ 10^{-2}\,\mathrm{m\,K}$$

λT in µm K	1000.0	1250.0	1500.0	1750.0	2000.0	2500.0
$F(\lambda)$	0.00031	0.00308	0.01283	0.03363	0.06663	0.16115
λT in µm K	3000.0	3500.0	4000.0	5000.0	6000.0	8000.0
$F(\lambda)$	0.27322	0.38250	0.48085	0.63315	0.73715	0.85556

Distribution of black body radiation: $F(\lambda) = \int_0^{\lambda} \dot{q}''_{\lambda\mathrm{b}} d\lambda \,/\, \sigma T^4$

Properties of radiating bodies

- spectral properties

$$\left.\begin{aligned} \rho\left(\lambda\right) &\equiv \frac{\dot{q}''_{\lambda\rho}}{\dot{q}''_{\lambda\mathrm{o}}} \\ \alpha\left(\lambda\right) &\equiv \frac{\dot{q}''_{\lambda\alpha}}{\dot{q}''_{\lambda\mathrm{o}}} \\ \tau\left(\lambda\right) &\equiv \frac{\dot{q}''_{\lambda\tau}}{\dot{q}''_{\lambda\mathrm{o}}} \end{aligned}\right\} \text{with } \rho(\lambda)+\alpha(\lambda)+\tau(\lambda)=1$$

here: $\dot{q}''_{\lambda\mathrm{o}}$ impacting spectral heat flux

$$\varepsilon\left(\lambda\right) \equiv \frac{\dot{q}''_{\lambda\varepsilon}}{\dot{q}''_{\lambda\mathrm{b}}}$$

$$\alpha(\lambda) = \varepsilon(\lambda) \qquad \text{(Kirchhoff's law)}$$

- spectrally averaged

$$\varepsilon \equiv \frac{\dot{q}''_{\varepsilon}}{\dot{q}''_{\mathrm{b}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\varepsilon}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{b}}\,d\lambda} \qquad \alpha \equiv \frac{\dot{q}''_{\alpha}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\alpha}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda}$$

$$\rho \equiv \frac{\dot{q}''_{\rho}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\rho}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda} \qquad \tau \equiv \frac{\dot{q}''_{\tau}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\tau}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda}$$

- special cases

Radiation properties independent of wavelength:

$$\rho+\alpha+\tau=1 \quad \text{and} \quad \alpha=\varepsilon \qquad \text{(Grey body)}$$

$$\alpha=1 \quad \text{and} \quad \alpha=\varepsilon=1 \qquad \text{(Black body)}$$

Spectral radiative properties

$$\rho(\lambda)+\alpha(\lambda)=1 \qquad \text{(Solid body impermeable for radiation)}$$

$$\alpha(\lambda)+\tau(\lambda)=1 \qquad \text{(Gas)}$$

Radiative heat exchange

$$\dot{Q}_{i\to j} = \dot{Q}_i \Phi_{ij} \qquad \text{(Radiative heat flow)}$$

$$\dot{Q}_i = \dot{q}_i'' A_i = \dot{Q}_{i,\mathrm{b}}\, \varepsilon_i + \underbrace{\sum_j \dot{Q}_{j\to i}\, \rho_i}_{\text{Reflection}} + \underbrace{\sum_k \dot{Q}_{k\to i}\, \tau_i}_{\text{Transmission}} \qquad \text{(Surface brightness)}$$

$$\text{with} \quad \dot{Q}_{i,\mathrm{b}} = \dot{q}_{i,\mathrm{b}}'' A_i \qquad \text{(Black body radiation)}$$

$$\Phi_{ij} = \frac{1}{A_i} \int_{A_j} \int_{A_i} \frac{\cos\varphi_i\, \cos\varphi_j}{\pi r^2}\, dA_i\, dA_j \qquad \text{(View factor)}$$

$$A_i \Phi_{ij} = A_j \Phi_{ji} \qquad \text{(Reciprocity relationship)}$$

$$\sum_j \Phi_{ij} = 1 \qquad \text{(Sum rule)}$$

$$\dot{Q}_{\mathrm{i,net}} = \dot{Q}_i - \sum_j \dot{Q}_{j\to i} \qquad \text{(Net radiative heat flow)}$$

$$\dot{Q}_{1\rightleftharpoons 2} = \dot{Q}_{1\to 2} - \dot{Q}_{2\to 1} \qquad \text{(Radiative heat exchange)}$$

$$\begin{aligned} \dot{Q}_{1\rightleftharpoons 2} &= A_1 \Phi_{12}\, \sigma \left[(T_1)^4 - (T_2)^4 \right] \qquad \text{(Between two black bodies)} \\ &= A_2 \Phi_{21}\, \sigma \left[(T_1)^4 - (T_2)^4 \right] \end{aligned}$$

$$\dot{q}_{1\rightleftharpoons 2}'' = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} \sigma \left(T_1^4 - T_2^4 \right) \qquad \text{(Between two grey plates)}$$

- Plates are plane, parallel and infinitely long

$$\dot{Q}_{1\rightleftharpoons 2} = \frac{A_1}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1 \right)} \sigma \left(T_1^4 - T_2^4 \right) \qquad \text{(Between two grey bodies)}$$

- Body 2 encloses body 1 ($A_2 > A_1$)
- Body 1 convex ($\Phi_{11} = 0$)

View factors of simple geometries

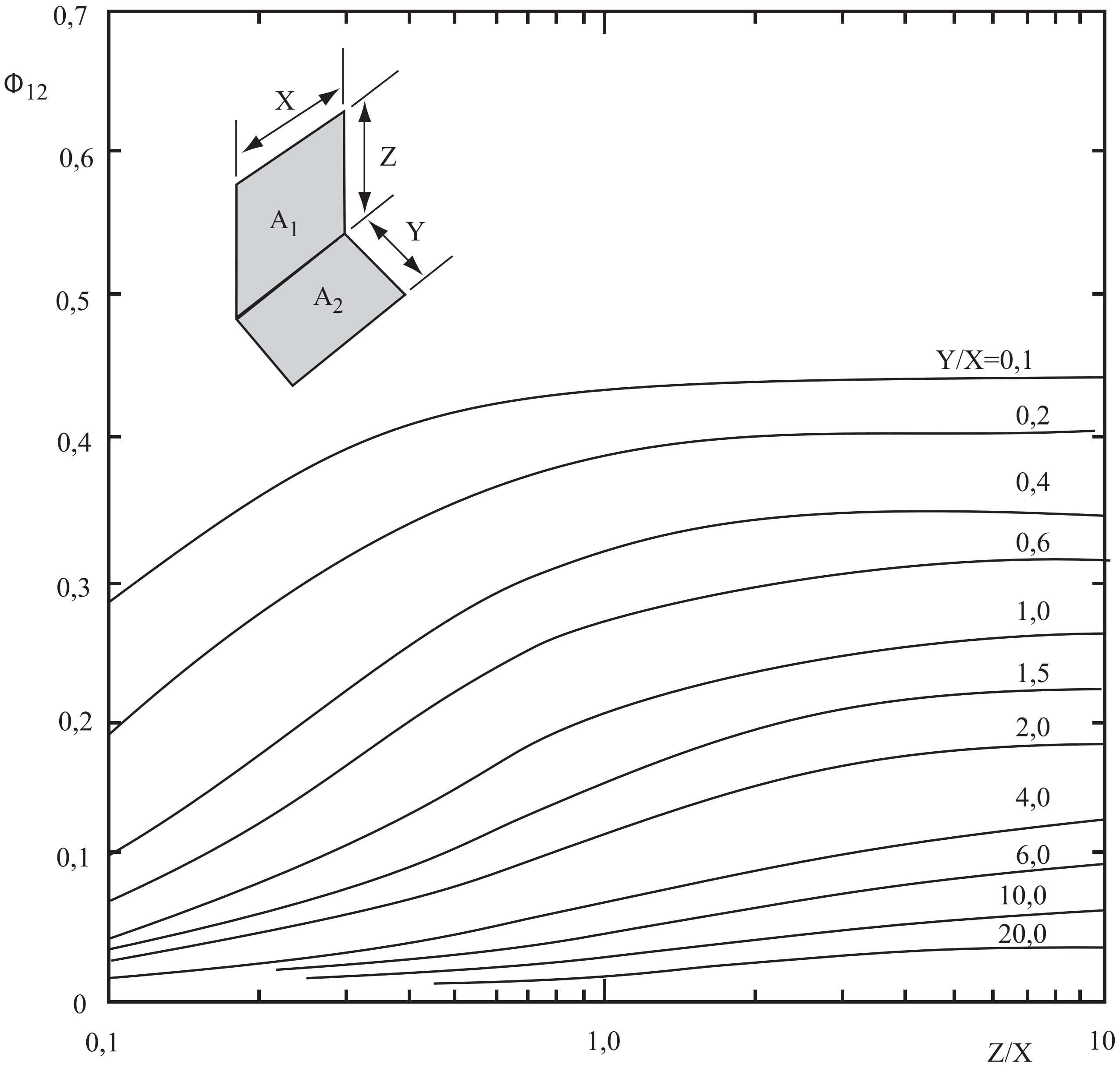


Diagramm 1: View factor of the radiation transfer between perpendicular, rectangular plates

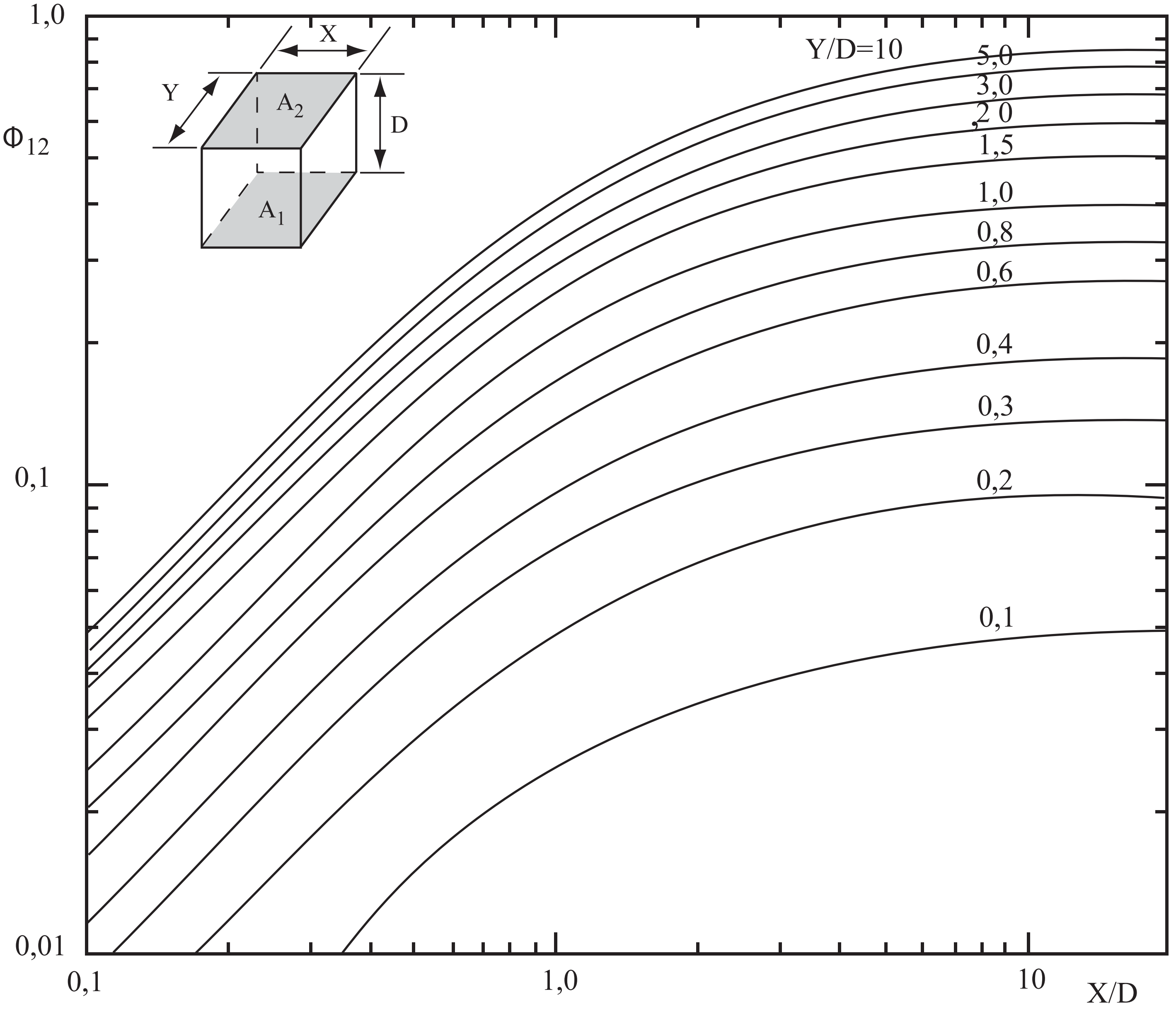


Diagramm 2: View factor of the radiation transfer between parallel, rectangular plates

Heat conduction

$$\dot{q}'' = -\lambda \frac{\partial T}{\partial x} \qquad \text{(Fourier's law)}$$

Heat transport equation

- Cartesian coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) \right] + \dot{\Phi}'''$$

- Cylindrical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) \right] + \dot{\Phi}'''$$

- Spherical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \Phi}\left(\lambda \frac{\partial T}{\partial \Phi}\right) \right] + \dot{\Phi}'''$$

Steady state heat conduction in walls without heat sources

$$R = \frac{T_\mathrm{A} - T_\mathrm{B}}{\dot{Q}} \qquad \text{where} \qquad R = \sum_i R_i \qquad \text{(Heat resistance)}$$

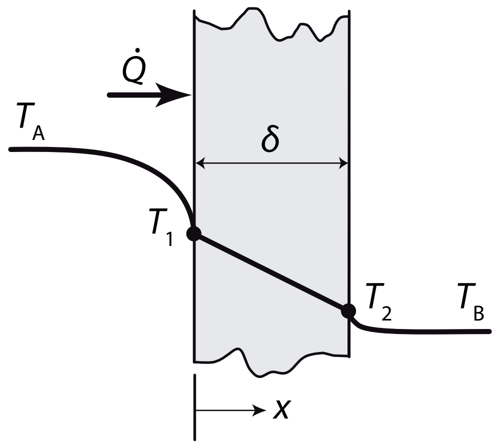
- Plane wall

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0 \qquad \text{with BC} \qquad \begin{aligned} T(x=0) &= T_1 \\ T(x=\delta) &= T_2 \end{aligned}$$

$$T = T_1 + \frac{T_2 - T_1}{\delta} x \qquad \text{(Temperature profile)}$$

$$\dot{Q} = -\lambda A \frac{\mathrm{d}T}{\mathrm{d}x} = \lambda A \frac{T_1 - T_2}{\delta} \qquad \text{(Heat flow rate)}$$

$$R = \frac{\delta}{\lambda A} \qquad \text{(Heat resistance)}$$



- Wall consisting of n layers

$$\dot{Q} = \lambda_1 \frac{A}{\delta_1} (T_1 - T_2) = \lambda_2 \frac{A}{\delta_2} (T_2 - T_3) = \cdots = \lambda_n \frac{A}{\delta_n} (T_n - T_{n+1})$$

$$\dot{Q} = \frac{A}{\sum\limits_{i=1}^{n} \frac{\delta_i}{\lambda_i}} (T_1 - T_{n+1}) \qquad \text{(Without conv. heat transfer)}$$

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_A} + \sum\limits_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_B}} (T_A - T_B) \qquad \text{(With conv. heat transfer)}$$

- Thick-walled tube

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0 \qquad \text{with BC} \qquad \begin{array}{l} T(r = r_1) = T_1 \\ T(r = r_2) = T_2 \end{array}$$

$$\begin{aligned} T &= T_1 + \ln\left(\frac{r}{r_1}\right) \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(Temperature profile)} \\ &= T_2 + \ln\left(\frac{r}{r_2}\right) \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \end{aligned}$$

$$\dot{Q} = 2\pi\lambda L \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(Heat flow)}$$

$$R = \frac{1}{2\pi\lambda L} \ln\frac{r_2}{r_1} \quad \text{mit} \quad r_2 > r_1 \qquad \text{(Heat resistance)}$$

- Thick-walled tube consisting of n layers

$$\dot{Q} = 2\pi r L \left(-\lambda_i \frac{\mathrm{d}T}{\mathrm{d}r} \right) = const.$$

$$\dot{Q} = \frac{T_1 - T_{n+1}}{\frac{1}{2\pi L} \sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{r_{i+1}}{r_i}} \qquad \text{(Without conv. heat transfer)}$$

$$\dot{Q} = \frac{2\pi L}{\frac{1}{\alpha_A r_1} + \sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{r_{i+1}}{r_i} + \frac{1}{\alpha_B r_{n+1}}} (T_A - T_B) \qquad \text{(With conv. heat transfer)}$$

Fins

$$\theta = T - T_\mathrm{a} \qquad \text{(Temperature difference)}$$

$$\eta_\mathrm{F} = \frac{\dot{Q}_\mathrm{F}}{\dot{Q}_\mathrm{max}} = \frac{\dot{Q}_\mathrm{F}}{A_0\, \alpha\, \theta_\mathrm{b}} = \frac{\text{transferred heat}}{\text{maximum transferable heat}} \qquad \text{(Efficiency of the fin)}$$

here: A_0 Heat transferring surface

θ_b Fin base temperature

Rod fins and plane fins

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \underbrace{\frac{\alpha U}{\lambda A_c}}_{=m^2} \theta = 0 \quad \text{with} \quad \begin{array}{ll} \text{BC1:} & \theta(x=0) = \theta_\mathrm{b} \\ \text{BC2:} & \text{may vary, see the following:} \end{array}$$

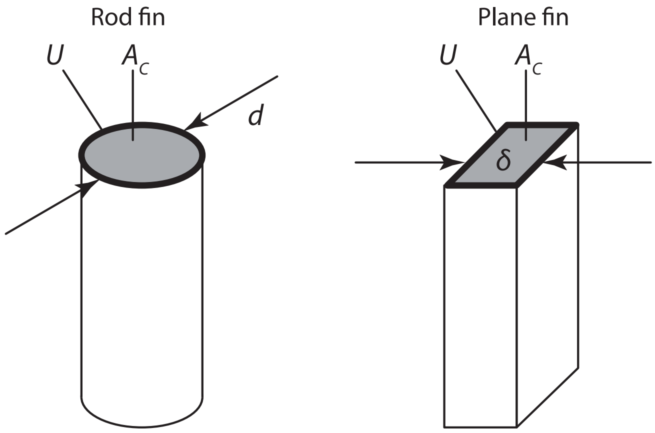
(Differential equation for fins)

$$\theta(x) = A \cosh(mx) + B \sinh(mx) \qquad \text{(Method of solution)}$$

$$\dots = C \exp(mx) + D \exp(-mx)$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{4\alpha}{\lambda d}} \qquad \text{(Rod fin)}$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{2\alpha}{\lambda \delta}} \qquad \text{(Plane fin)}$$



Boundary condition 2:

- Fins with adiabatic head:

$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = 0$$

$$\theta = \theta_\mathrm{b} \frac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left[mL\right]} \qquad \text{(Temperature profile)}$$

$$\dot{Q} = \lambda\, A_\mathrm{c}\, m\, \theta_\mathrm{b}\, \tanh\left(mL\right) \qquad \text{(Heat flow through the fin)}$$

$$\eta = \frac{\tanh(mL)}{mL} \qquad \text{(Efficiency of the fin)}$$

- Fins with head at ambient temperature (long fins):

$$\text{BC2:} \quad \theta(x = L) = 0$$

- Fins transferring heat at the fin head:

$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = \alpha\, \theta(x = L)$$

One-dimensional, unsteady state heat conduction

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2} \qquad \text{(Differential equation)}$$

$$\frac{\partial \theta^*}{\partial t} = a\,\frac{\partial^2 \theta^*}{\partial x^2} \qquad \text{with} \qquad \theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0}$$

- Semi-infinite plate with negligible heat transfer resistance:

$$\mathrm{Bi} = \frac{\alpha\, L}{\lambda} \gg 1$$

$$\left.\begin{matrix} t = 0 \\ 0 < x < \infty \end{matrix}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC1)}$$

$$\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} \quad T = T_\mathrm{a} \qquad \theta^* = 1 \qquad \text{(BC2)}$$

$$\left.\begin{matrix} t > 0 \\ x \to \infty \end{matrix}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC3)}$$

$$\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right) \quad \text{with} \quad \mathrm{Fo} = \frac{at}{x^2} \qquad \text{(Temperature profile)}$$

$$\left.\dot{q}''\right|_{x=0} = \sqrt{\frac{\lambda c \rho}{\pi t}}\,(T_\mathrm{a} - T_0) \qquad \text{(Heat flux)}$$

$$\delta(t) \approx 3{,}6\,\sqrt{at} \qquad \text{(Temperature penetration depth)}$$

- Semi-infinite plate, **non** negligible heat transfer resistance:

$$\left.\begin{array}{r} t > 0 \\ x = 0 \end{array}\right\} \quad \alpha \left(T_\mathrm{a} - T(x=0)\right) = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} \tag{BC1}$$

$$\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right) \cdots \qquad \text{(Temperature profile)}$$

$$\cdots - \left[\exp\left(\mathrm{Bi}_x + \mathrm{Fo}\,\mathrm{Bi}_x^2\right)\right] \left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}} + \sqrt{\mathrm{Fo}}\,\mathrm{Bi}_x\right)\right]$$

$$\text{with} \quad \mathrm{Bi}_x = \frac{\alpha x}{\lambda}$$

$$\mathrm{Fo} = \frac{a t}{x^2}$$

- Semi-infinite plate, periodically changing surface temperature:

$$\left.\begin{array}{r} t > 0 \\ x = 0 \end{array}\right\} \quad T\left(x=0\right) = T_\mathrm{m} + \left(T_\mathrm{max} - T_\mathrm{m}\right)\,\cos\left(2\pi\, t/\tau\right) \tag{BC1}$$

$$\theta^* = \frac{T - T_\mathrm{m}}{T_\mathrm{max} - T_\mathrm{m}} = \exp\left(-\sqrt{\frac{\pi x^2}{a\tau}}\right)\,\cos\left(\frac{2\pi}{\tau}t - \sqrt{\frac{\pi x^2}{a\tau}}\right) \qquad \text{(Temperature profile)}$$

One-dimensional, unsteady heat conduction in simple bodies

$$\frac{T_\mathrm{m} - T_\mathrm{a}}{T_0 - T_\mathrm{a}} \qquad \text{(Dimensionless temperature in the middle of a body)}$$

$$\frac{T - T_\mathrm{a}}{T_\mathrm{m} - T_\mathrm{a}} \qquad \text{(Dimensionless temperature at position } x \text{ or } r\text{)}$$

$$\frac{Q}{Q_0} \quad \text{mit} \quad Q_0 = m\,c\,\left(T_0 - T_\mathrm{a}\right) \qquad \text{(Dimensionless heat loss)}$$

Determination of temperature profile and heat flow for unsteady conditions

$\rightarrow$ Figures 3 - 11

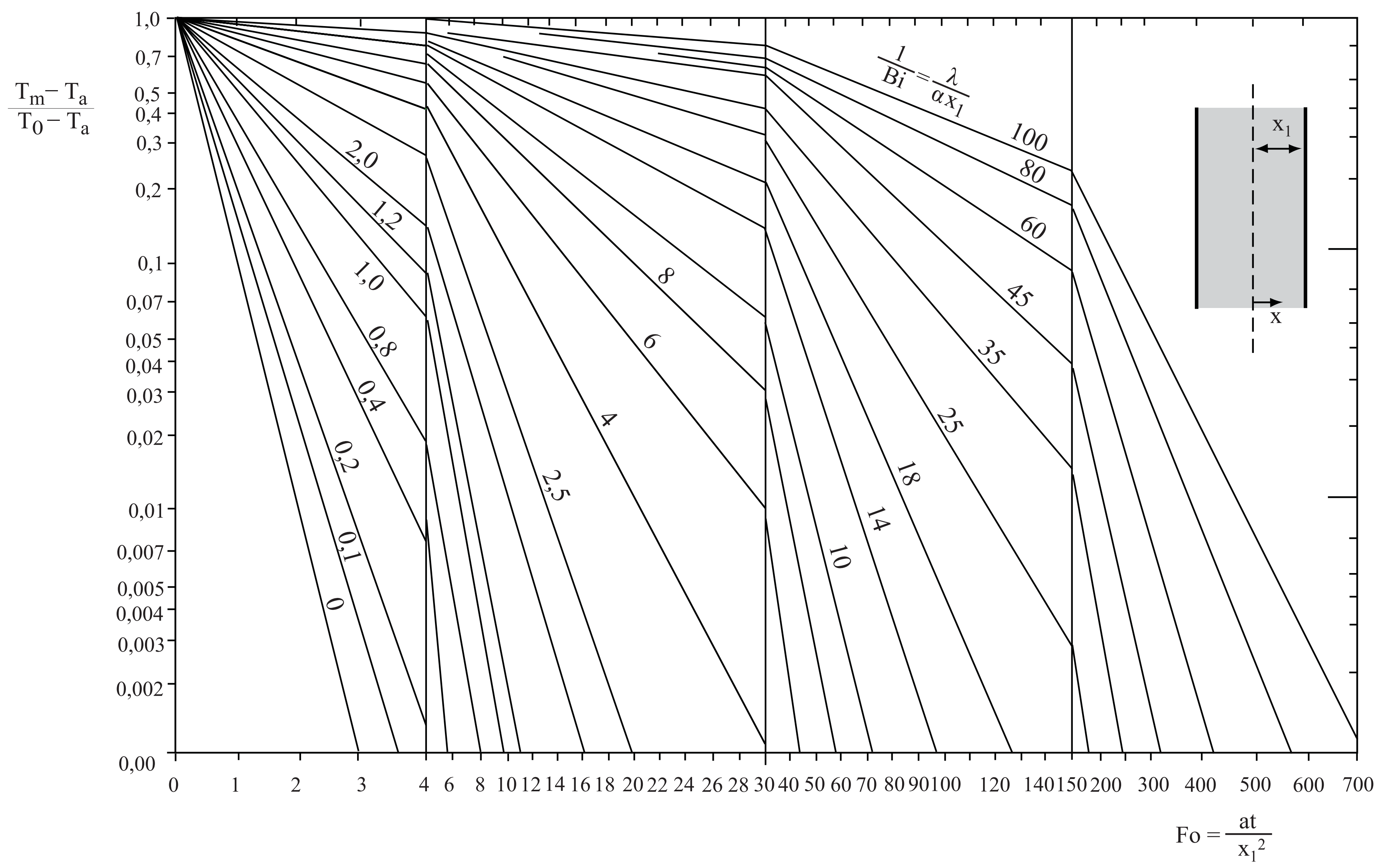


Diagramm 3: Mid-plane temperature of a plate with thickness $2x_1$

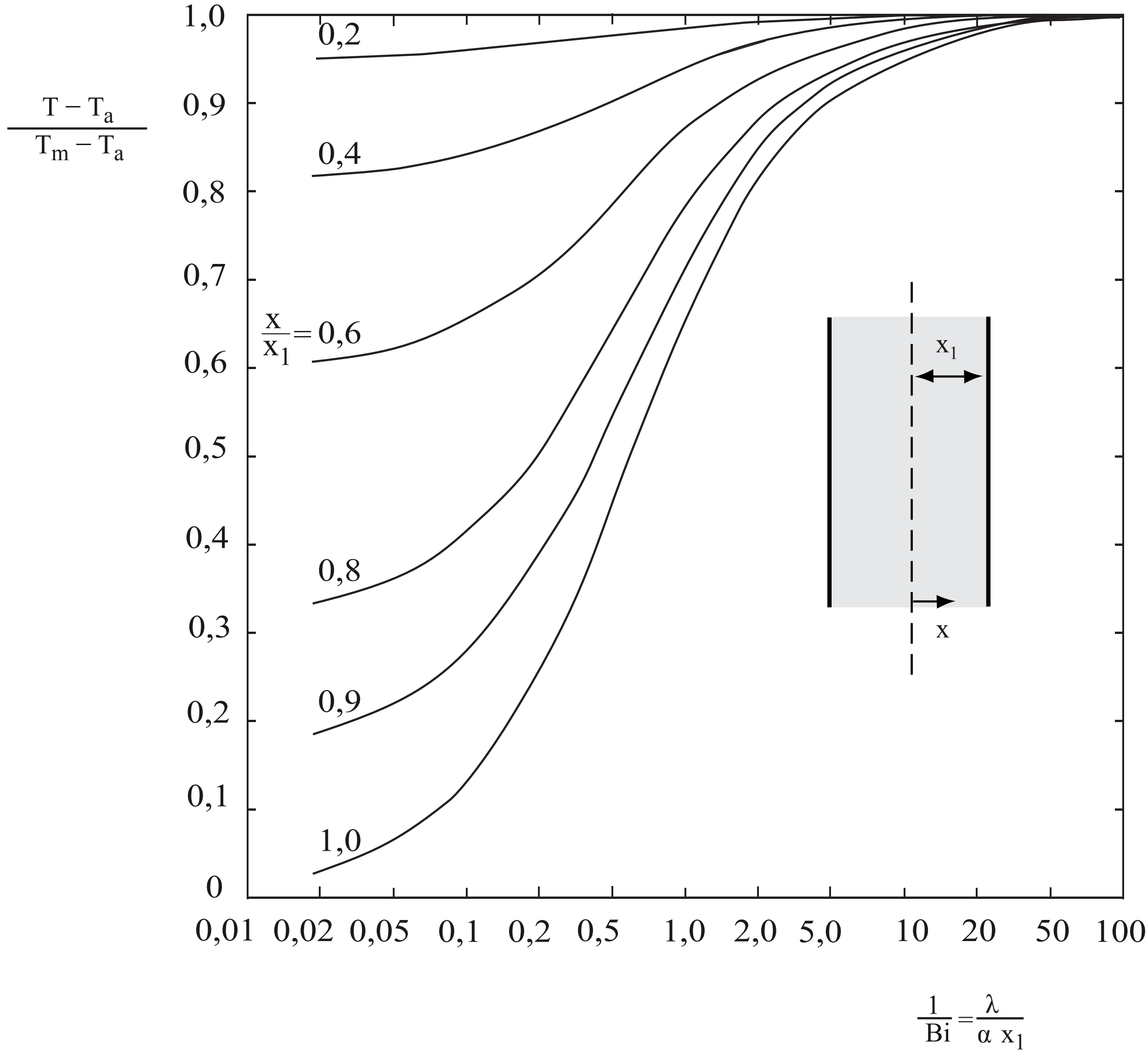


Diagramm 4: Temperature distribution in a plate (valid for Fo > 0,2)

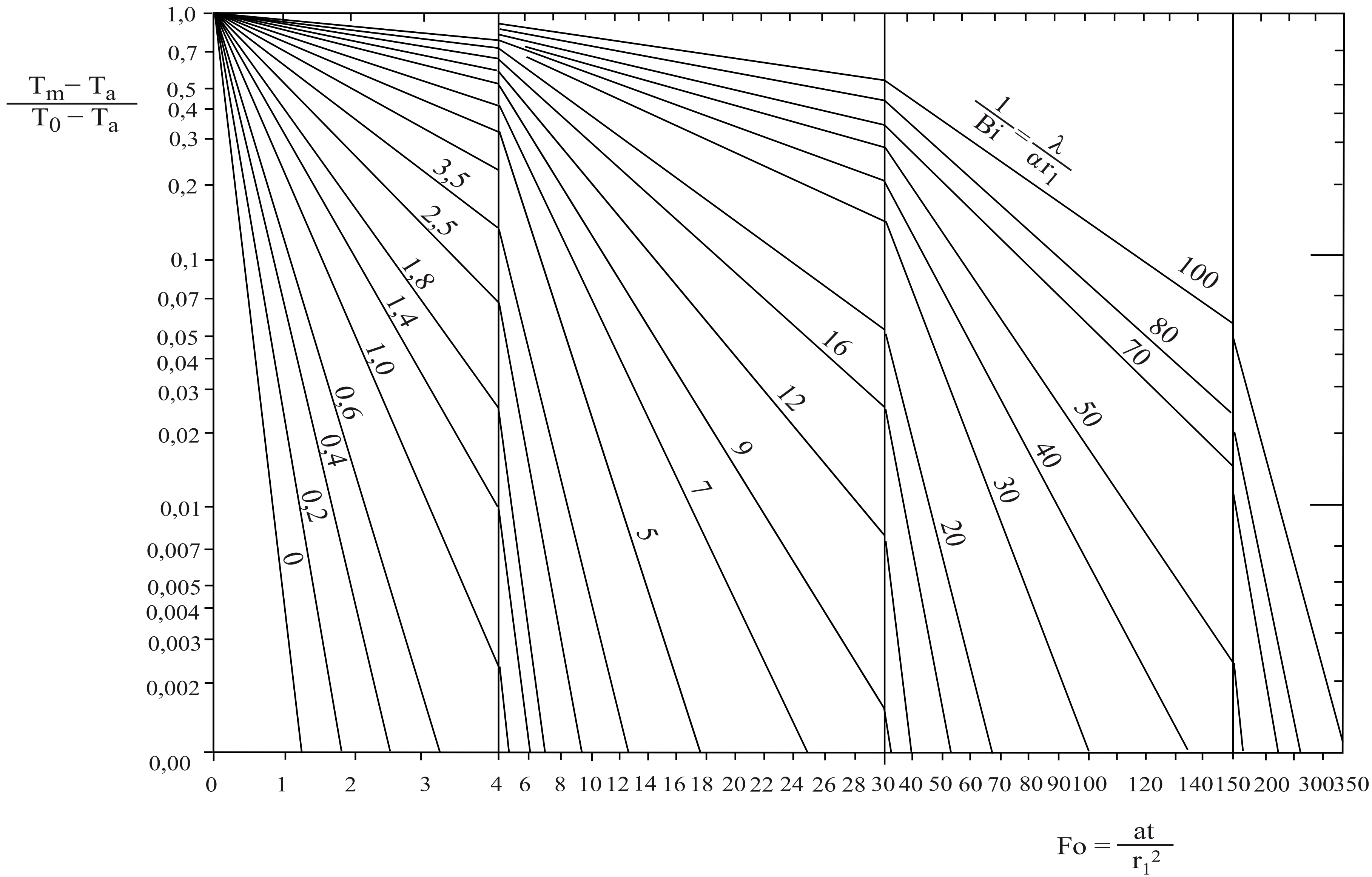


Diagramm 5: Temperature along the axis of a cylinder with radius r_1

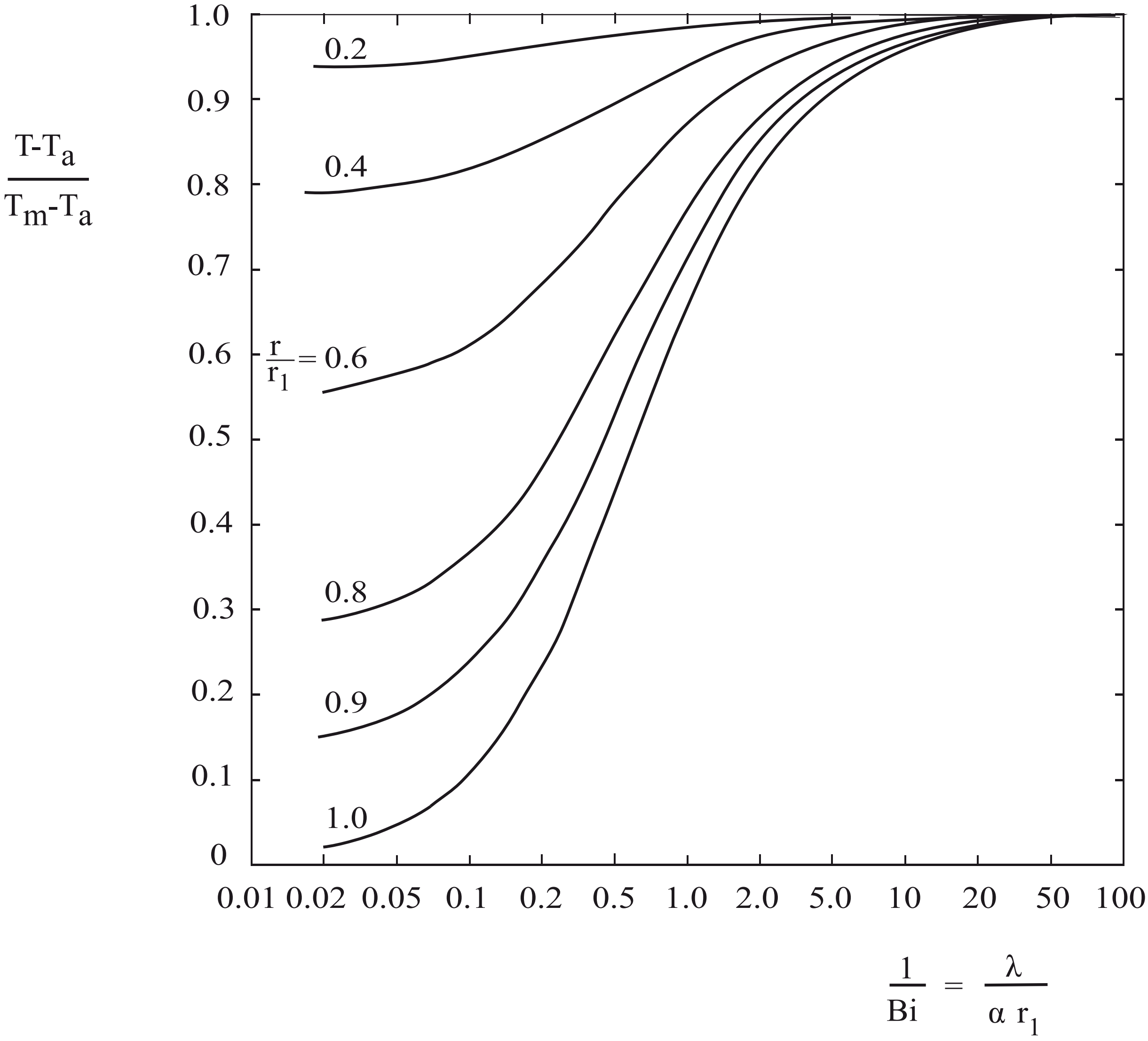


Diagramm 6: Temperature distribution in a cylinder (valid for Fo > 0,2)

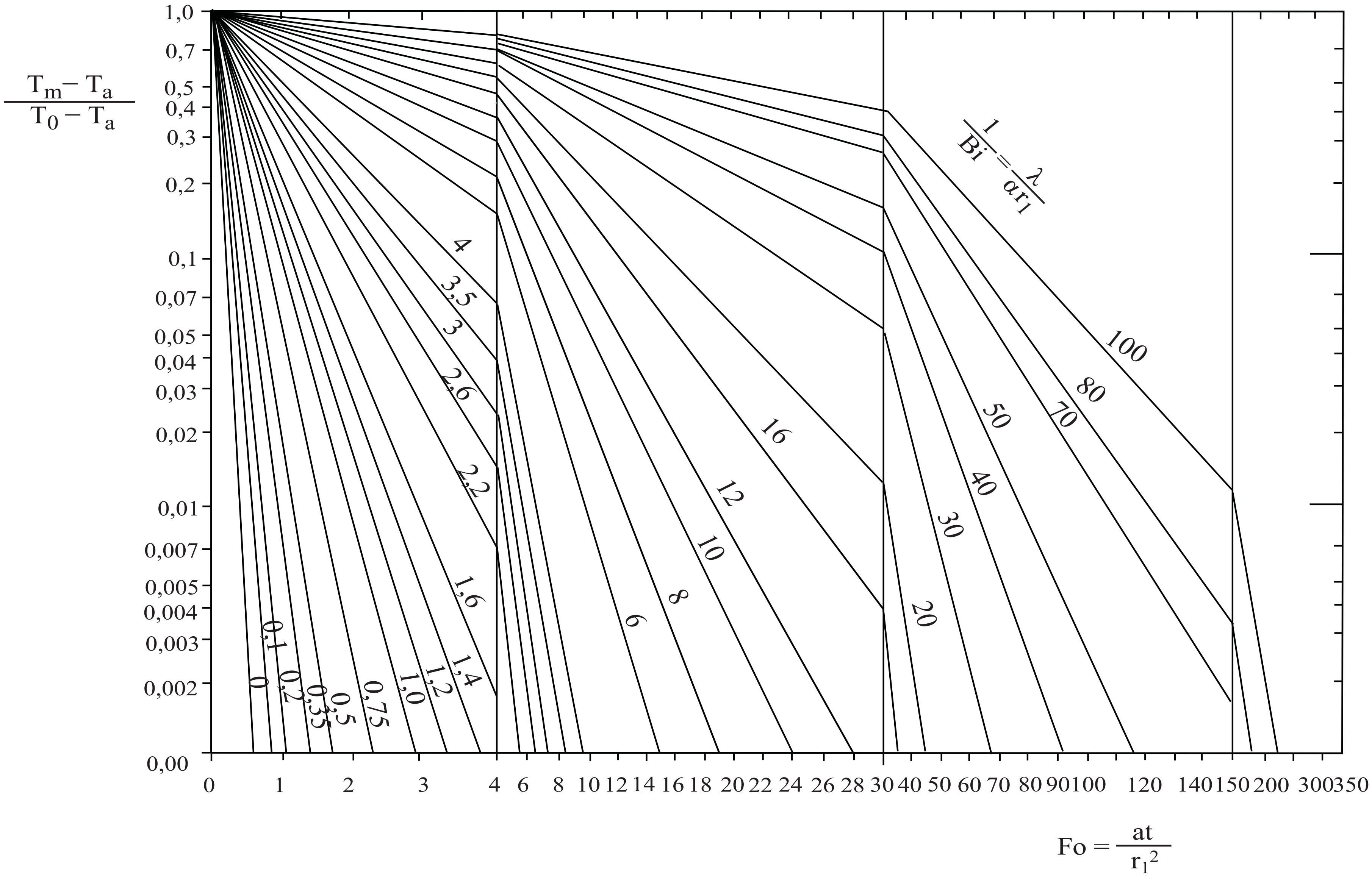


Diagramm 7: Temperature in the centre of a sphere with radius r_1

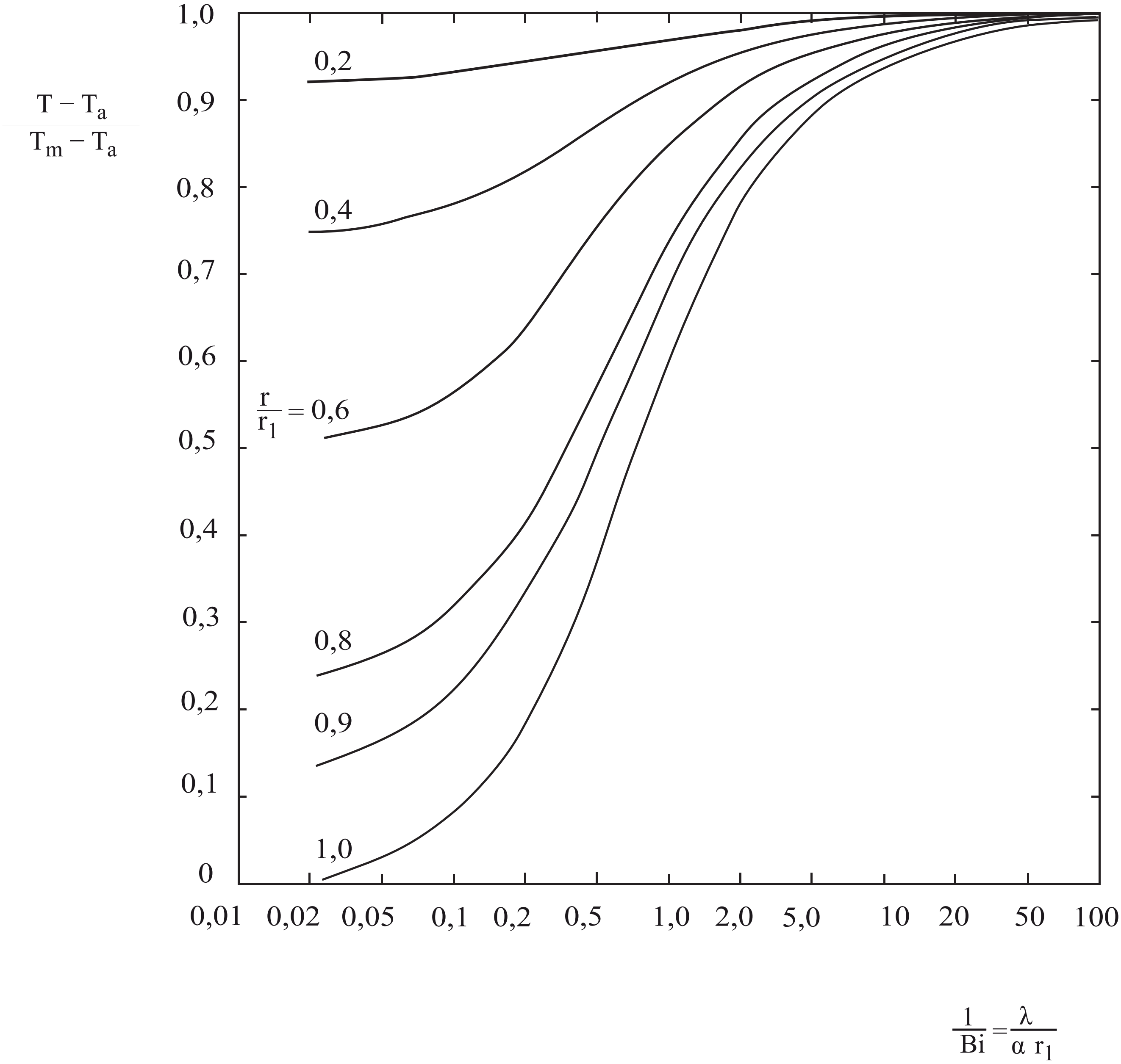


Diagramm 8: Temperature distribution in a sphere (valid for Fo > 0,2)

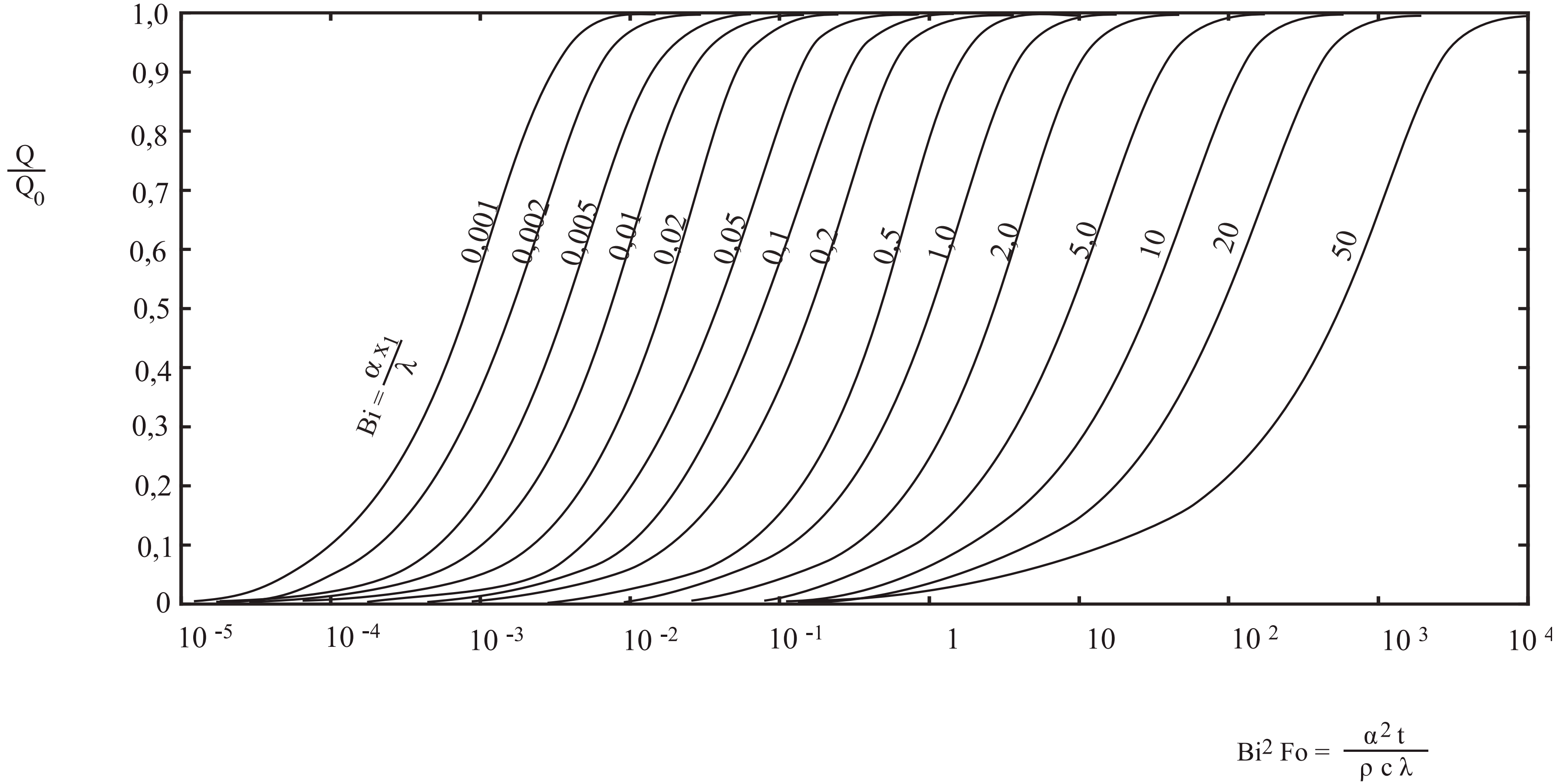


Diagramm 9: Heat loss of a plate

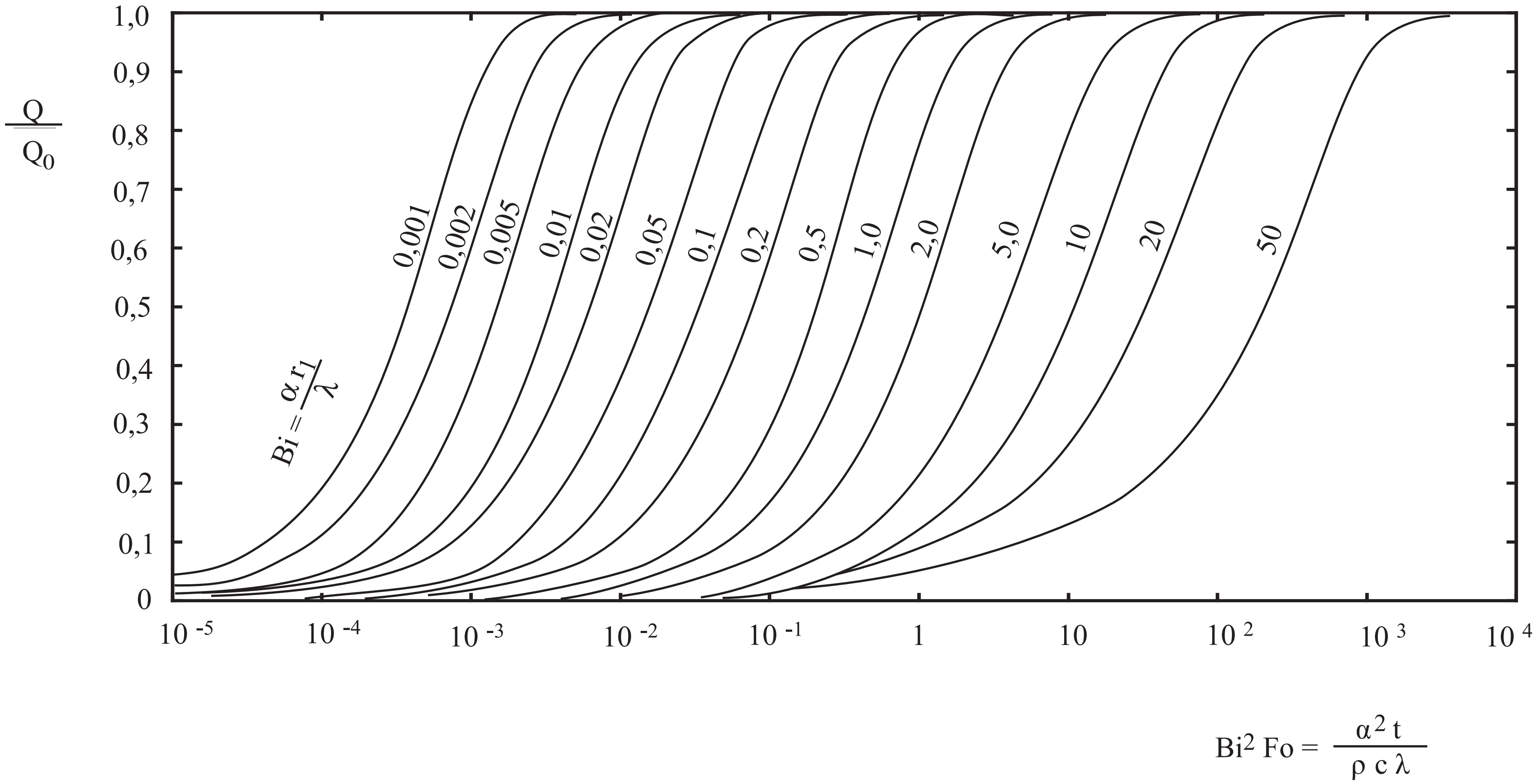


Diagramm 10: Heat loss of a cylinder

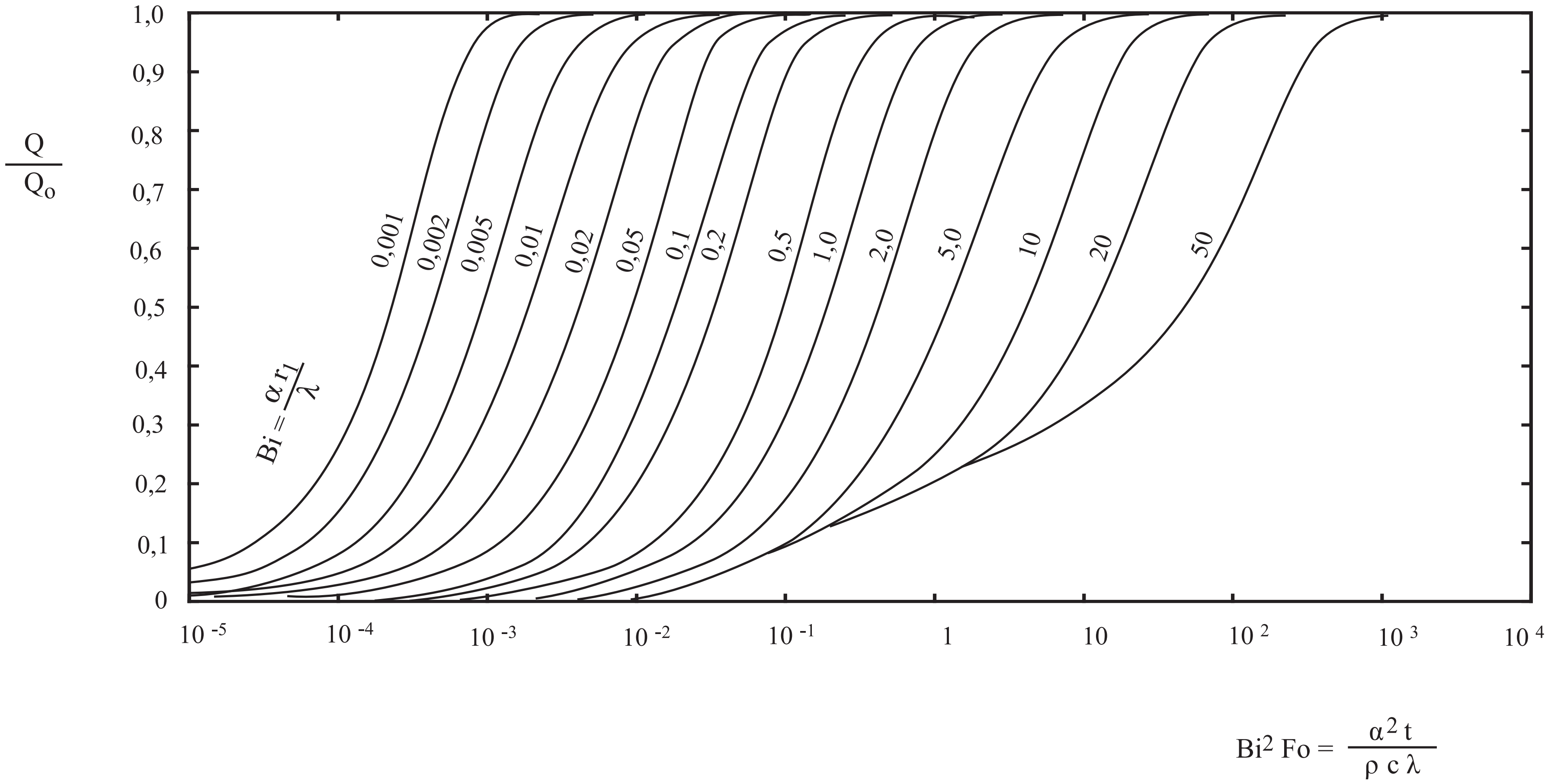


Diagramm 11: Heat loss of a sphere

4. Convection

$$\rho u c_\mathrm{p} \frac{\partial T}{\partial x} + \rho v c_\mathrm{p} \frac{\partial T}{\partial y} + \rho w c_\mathrm{p} \frac{\partial T}{\partial z} = \cdots \qquad \text{(Equation of energy conservation)}$$

$$\cdots = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}'''$$

Convective heat transfer

$$\frac{\dot{Q}}{A} = \dot{q}''_\mathrm{w} = \alpha (T_\mathrm{w} - T_\mathrm{fl}) \qquad \text{(Convective heat flux)}$$

$$W = \frac{1}{\alpha\, A} \qquad \text{(Heat resistance)}$$

$$\alpha = \frac{-\left(\lambda \, \frac{dT}{dy}\right)_{\mathrm{Fluid,\,w}}}{T_\mathrm{w} - T_\mathrm{fl}} \qquad \text{(Heat transfer coefficient)}$$

$$\overline{\alpha} = \frac{1}{L} \int_0^L \alpha\,(x)\ \mathrm{d}x \qquad \text{(Average h.t. coefficient)}$$

Boundary layer equations (Approximation with linear velocity profile)

$$\frac{\delta_u}{x} \approx \sqrt{\frac{12\,\eta}{\rho\, u_\infty x}} = \sqrt{\frac{12}{\mathrm{Re}_x}} \qquad \text{(Thickness of the velocity boundary layer)}$$

$$\frac{\delta_T}{\delta_u} \approx \left(\frac{\lambda}{\eta c_\mathrm{p}}\right)^{1/3} = \frac{1}{\mathrm{Pr}^{1/3}} \qquad \text{(Thickness of the temperature boundary layer)}$$

5. Heat transfer correlations

$$\Delta T_{\mathrm{ln}} = (T_{\mathrm{w}} - T_{\mathrm{fl}})_{\mathrm{m}} = \frac{\Delta T_{\mathrm{I}} - \Delta T_{\mathrm{O}}}{\ln \frac{\Delta T_{\mathrm{I}}}{\Delta T_{\mathrm{O}}}} \qquad \text{(Logarithmic temperature difference)}$$

$$\dot{Q}_{\mathrm{m}} = \bar{\alpha}\, A\, (T_{\mathrm{w}} - T_{\mathrm{fl}})_{\mathrm{m}} \qquad \text{(Average heat flow)}$$

Forced convection flow along surfaces

$$\mathrm{Nu}_x = f\,(\mathrm{Re}_x, \mathrm{Pr}, \ldots) \qquad \text{(Nusselt-correlation)}$$

$$T_{\mathrm{prop.}} = \frac{T_{\mathrm{w}} + T_\infty}{2} \qquad \text{(Temperature for determination of properties)}$$

- **Flat plate – laminar flow, isothermal surface (1)**

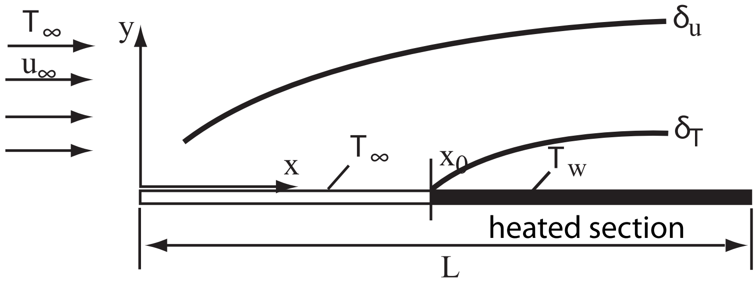
  ($0.6 < \mathrm{Pr} < 10$ and $\mathrm{Re}_x < \mathrm{Re}_{x,\,\mathrm{crit}} \approx 2 \cdot 10^5$)

$$\mathrm{Nu}_x = 0.332\, \mathrm{Re}_x^{1/2} \mathrm{Pr}^{1/3} \qquad \text{(HTC.1)}$$

$$\overline{\mathrm{Nu}}_L = 0.664\, \mathrm{Re}_L^{1/2} \mathrm{Pr}^{1/3} \qquad \text{(HTC.2)}$$

- **Flat plate – laminar boundary layer flow, isothermal surface (2)**

  Heating or cooling starts at $x = x_0$



($0.6 < \mathrm{Pr} < 10$ and $\mathrm{Re}_x < \mathrm{Re}_{x,\,\mathrm{crit}} \approx 2 \cdot 10^5$)

$$\mathrm{Nu}_x = 0.332\, \mathrm{Re}_x^{1/2} \mathrm{Pr}^{1/3} \left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \text{(HTC.3)}$$

$$\overline{\mathrm{Nu}}_L = \frac{L}{L - x_0} \frac{1}{\lambda} \int_{x_0}^{L} \alpha(x) \mathrm{d}x$$

$$= 0.664 \, \mathrm{Re}_L^{1/2} \mathrm{Pr}^{1/3} \frac{\left[1 - \left(\frac{x_0}{L}\right)^{3/4}\right]^{2/3}}{\left[1 - \frac{x_0}{L}\right]} \tag{HTC.4}$$

- **Flat plate – turbulent boundary layer flow, isothermal surface**

  ($\mathrm{Re}_{L,\,\mathrm{crit}} \approx 2 \cdot 10^5$ and $5 \cdot 10^5 < \mathrm{Re} < 10^7$)

$$\mathrm{Nu}_x = 0.0296 \, \mathrm{Re}_x^{0.8} \mathrm{Pr}^{0.43} \tag{HTC.5}$$

$$\overline{\mathrm{Nu}}_L \approx 0.036 \, \mathrm{Pr}^{0.43} \left(\mathrm{Re}_L^{0.8} - 9400\right) \tag{HTC.6}$$

- **Cylinders in a flow parallel to their longitudinal axis**

  If the diameter of the body is much greater compared to the thickness of the boundary layer, cylinders in longitudinal flow can be regarded as flat plates.

- **Cylinders in a flow perpendicular to their longitudinal axis**

$$\overline{\mathrm{Nu}}_d = C \, \mathrm{Re}_d^m \mathrm{Pr}^{0.4} \tag{HTC.7}$$

Re_d	C	m
0.4 – 4	0.989	0.330
4 – 40	0.911	0.385
40 – 4000	0.683	0.466
4000 – 40000	0.193	0.618
40000 – 400000	0.0266	0.805

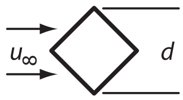
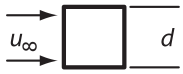
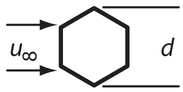
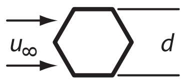
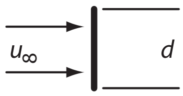
HTC.8 can be used as an alternative to HTC.7:

$$\overline{\mathrm{Nu}}_d = \left[0.40 \, \mathrm{Re}_d^{1/2} + 0.06 \, \mathrm{Re}_d^{2/3}\right] \mathrm{Pr}^{0.4} \left(\frac{\eta_\infty}{\eta_\mathrm{w}}\right)^{1/4} \tag{HTC.8}$$

here: $T_{\mathrm{prop.}} = T_\infty$

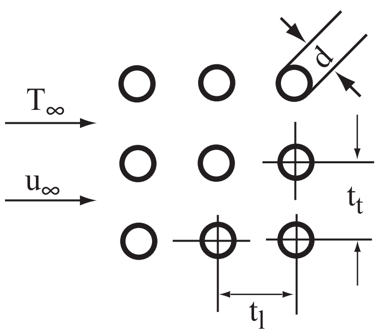
- **Mean heat transfer for non circular cylinders**

$$\overline{\mathrm{Nu}}_d = C\ \mathrm{Re}_d^m \mathrm{Pr}^{0.4} \tag{HTC.9}$$

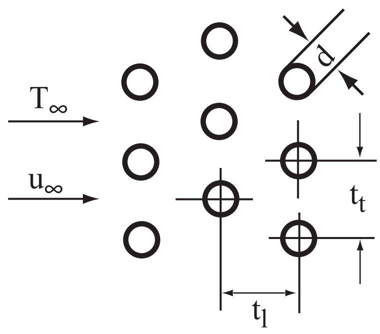
Geometry	Re_d	C	m
	$5 \cdot 10^3 - 1 \cdot 10^5$	0.246	0.588
	$5 \cdot 10^3 - 1 \cdot 10^5$	0.102	0.675
	$5 \cdot 10^3 - 1.95 \cdot 10^4$	0.160	0.638
	$1.95 \cdot 10^4 - 1 \cdot 10^5$	0.0385	0.782
	$5 \cdot 10^3 - 1 \cdot 10^5$	0.153	0.638
	$4 \cdot 10^3 - 1.5 \cdot 10^4$	0.228	0.731

- **Flow perpendicular to plain tube bundles**

$$\overline{\mathrm{Nu}}_d = 0.287\ \mathrm{Re}_d^{0.6}\ \mathrm{Pr}^{0.36}\ f_\mathrm{e} \tag{HTC.10}$$



in-line arrangement



staggered arrangement

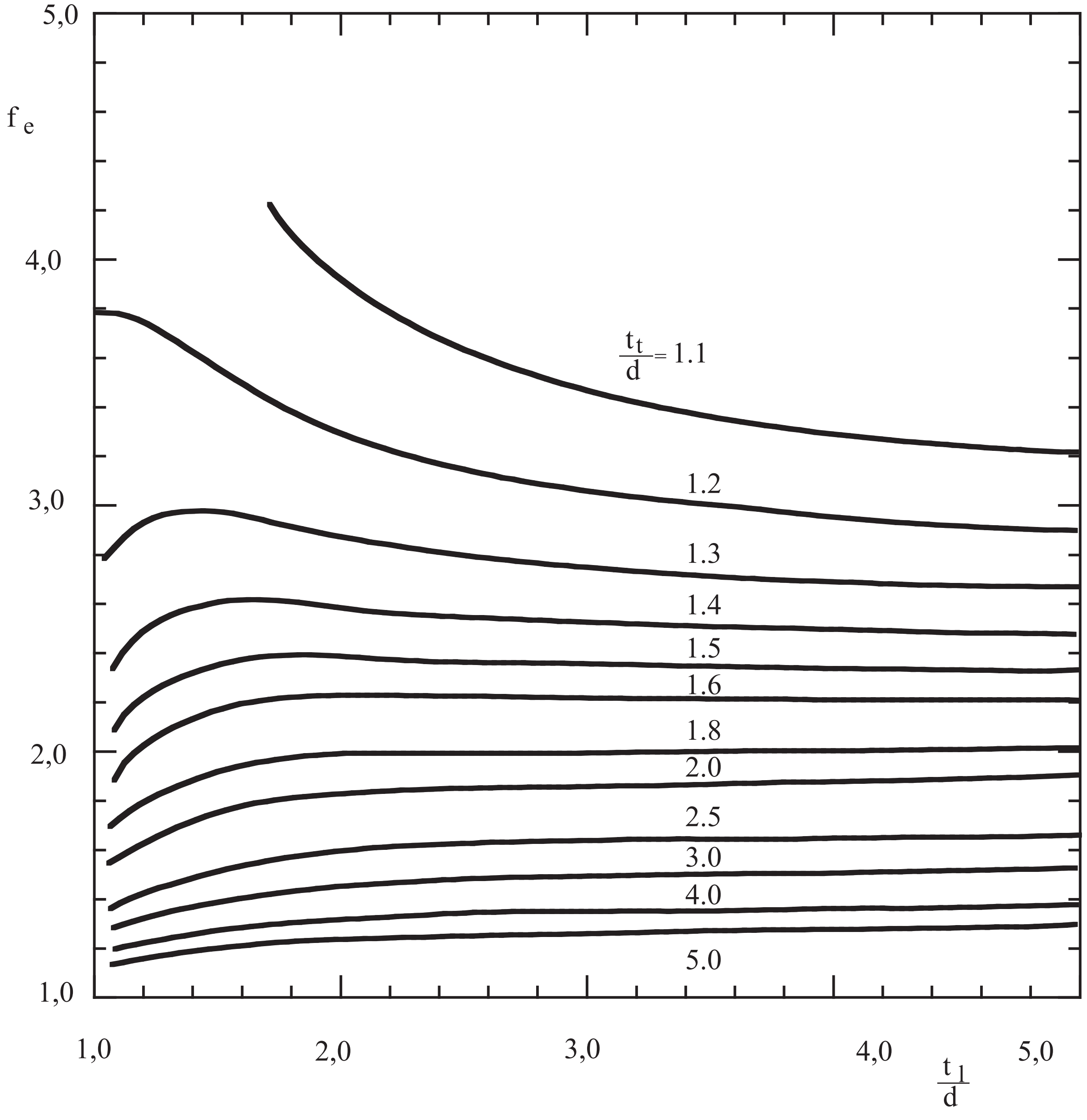


Figure 1: Arrangement factor, in-line arrangement

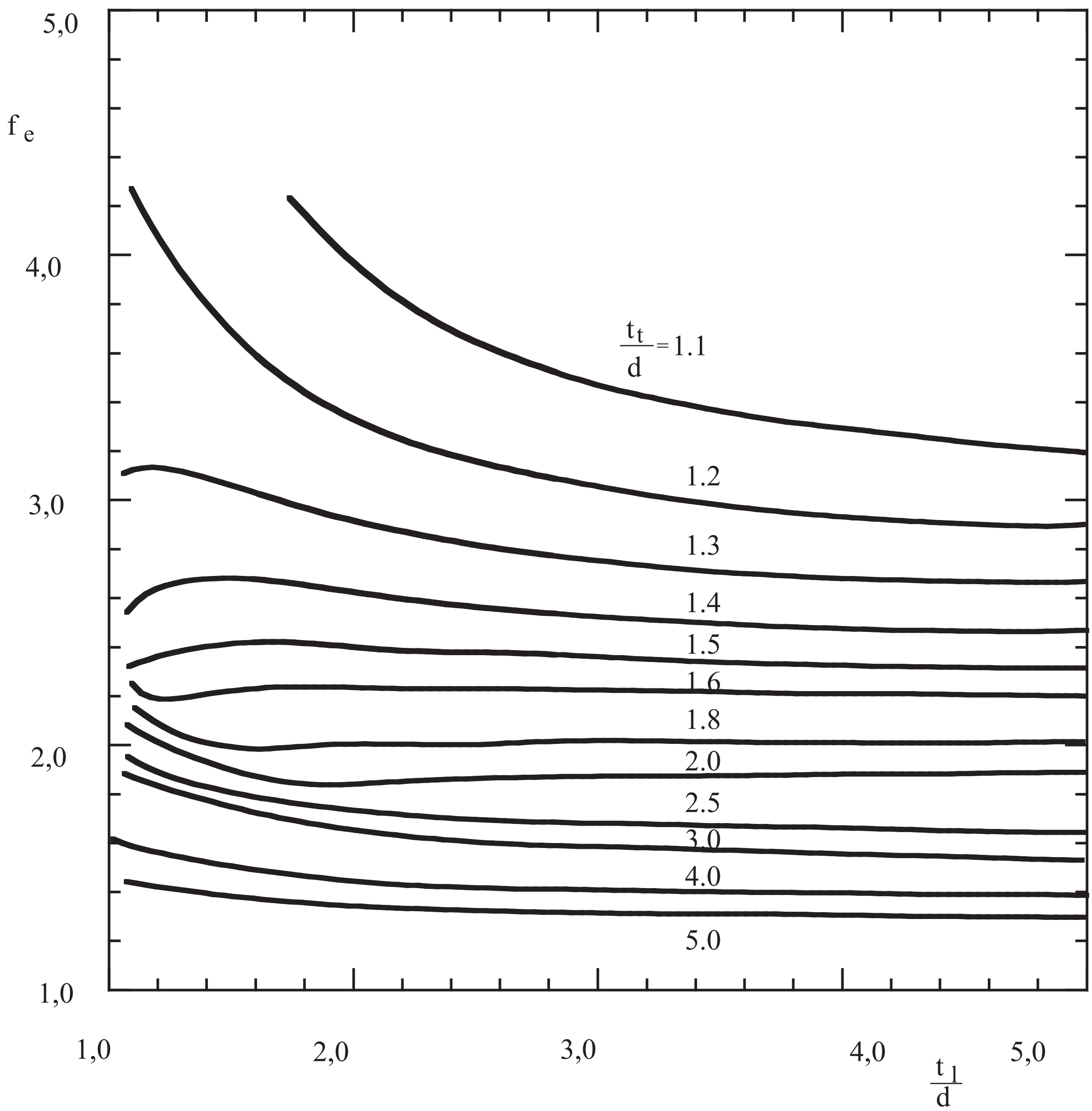


Figure 2: Arrangement factor, staggered arrangement

- **Heat transfer from a surrounding fluid to spheres in the stream**

  $(0.7 < \mathrm{Pr} < 380$ and $3.5 < \mathrm{Re}_d < 8 \cdot 10^4)$

$$\overline{\mathrm{Nu}}_d = 2 + \left(0.4\ \mathrm{Re}_d^{1/2} + 0.06\ \mathrm{Re}_d^{2/3}\right)\ \mathrm{Pr}^{0.4}\ \left(\frac{\eta_\infty}{\eta_\mathrm{w}}\right)^{1/4} \qquad \text{(HTC.11)}$$

here: $T_{\mathrm{prop.}} = T_\infty$

Forced convection in tubes, internal flow

$$\mathrm{Nu}_x = f\left(\mathrm{Re}_x, \mathrm{Pr}, \ldots\right) \qquad \text{(Nusselt-correlation)}$$

$$T_{\mathrm{mat}} = \frac{T_{\mathrm{fl,\,O}} + T_{\mathrm{fl,\,I}}}{2} \qquad \text{(Material property determination temperature)}$$

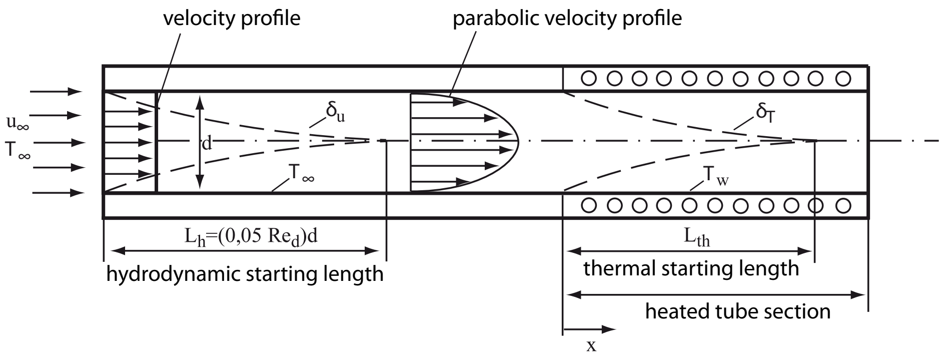
$$d_h = 4\frac{A_\mathrm{c}}{U} \qquad \text{(Hydraulic mean diameter)}$$

here: A_c cross-section area

U wetted perimeter

- **Laminar flow in tubes – isothermal surface (1)**

  Fully developed flow at the start of the heat transferring section of a tube $(\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300)$



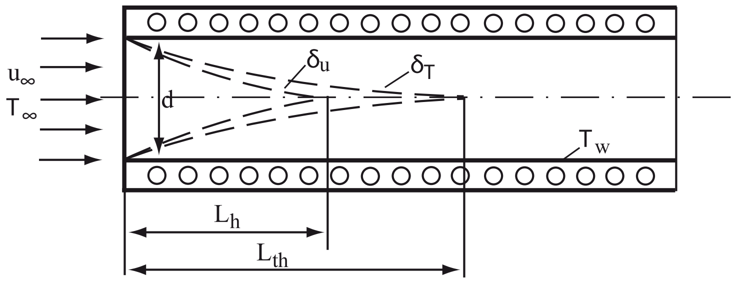
$$\overline{\mathrm{Nu}}_d = \left(3.66 + \frac{0.19\left(\mathrm{Re}_d\mathrm{Pr}\frac{d}{L}\right)^{0.8}}{1 + 0.117\left(\mathrm{Re}_d\mathrm{Pr}\frac{d}{L}\right)^{0.467}}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0.14} \qquad \text{(HTC.12)}$$

$$\frac{L_{\mathrm{th}}}{d} \approx 0.05\ \mathrm{Re}_d\mathrm{Pr} \qquad \text{(Thermal starting length)}$$

After L_{th}, the Nusselt number has a final value of $\overline{\mathrm{Nu}}_\infty = 3.66\left(\eta/\eta_\mathrm{W}\right)^{0.14}$.

- **Laminar flow in tubes – isothermal surface (2)**

  Simultaneous hydrodynamic and thermal start ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$)



$$\overline{\mathrm{Nu}}_d = \left(3.66 + \frac{0.0677\left(\mathrm{Re}_d \mathrm{Pr}\frac{d}{L}\right)^{1.33}}{1 + 0.1\mathrm{Pr}\left(\mathrm{Re}_d \frac{d}{L}\right)^{0.83}}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0.14} \qquad \text{(HTC.13)}$$

$$\frac{L_\mathrm{th}}{d} \approx 0.05\,\mathrm{Re}_d\mathrm{Pr} \qquad \text{(Thermal starting length)}$$

  After L_th, the Nusselt number has a final value of $\overline{\mathrm{Nu}}_\infty = 3.66\,(\eta/\eta_\mathrm{W})^{0.14}$.

- **Laminar flow in tubes – impressed heat flow**

  If instead of the wall temperature, the heat flow at the wall remains constant, then the heat transfer coefficients have values increased by 20%.

- **Turbulent flow in tubes – isothermal surface**

  Simultaneous hydrodynamic and thermal start
  ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$, $\mathrm{Re}_d > 2300$, $0.6 < \mathrm{Pr} < 500$ and $L/d > 1$)

$$\overline{\mathrm{Nu}}_d = 0.0235\left(\mathrm{Re}_d^{0.8} - 230\right)\left(1.8\mathrm{Pr}^{0.3} - 0.8\right)\left(1 + \left(\frac{d}{L}\right)^{2/3}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0.14} \qquad \text{(HTC.14)}$$

  Simplified Nusselt law for the *fully developed* turbulent pipe flow
  ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$, $3000 < \mathrm{Re}_d < 10^5$ and $L/d > 40$)

$$\overline{\mathrm{Nu}}_d = 0.027\mathrm{Re}_d^{0.8}\mathrm{Pr}^{1/3}\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0.14} \qquad \text{(HTC.15)}$$

- **Turbulent flow in tubes – impressed heat flow**

  The heat transfer coefficients at impressed heat flows are comparable to the coefficients obtained at constant wall temperatures.

Natural convection

$$\mathrm{Nu}_x = f\left(\mathrm{Gr}_x, \mathrm{Pr}, \ldots\right) \qquad \text{(Nusselt-correlation)}$$

$$T_{\text{prop.}} = \frac{T_\text{w} + T_\infty}{2} \qquad \text{(Temperature for determination of properties)}$$

$$\beta = \frac{1}{T_\infty} \qquad \text{(Isobaric expansion coefficient)}$$

- **Vertical plate – laminar boundary layer flow, isothermal surface**

$$\mathrm{Nu}_x = 0.508 \left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}}\right)^{1/4} \left(\mathrm{Gr}_x \mathrm{Pr}\right)^{1/4} \qquad \text{(HTC.16)}$$

$$\overline{\mathrm{Nu}}_L = C \left(\mathrm{Gr}_L \mathrm{Pr}\right)^{1/4} \qquad \text{(HTC.17)}$$

$$\text{for} \quad \mathrm{Gr}_L \mathrm{Pr} < \mathrm{Gr}_{L,crit} \mathrm{Pr} = 4 \cdot 10^9$$

Pr	0.003	0.01	0.03	0.72	1	2	10	100	1000	∞
C	0.182	0.242	0.305	0.516	0.535	0.568	0.620	0.653	0.665	0.670

- **Vertical plate – laminar boundary layer flow, impressed heat flow**

$$\mathrm{Nu}_x = 0.60 \left(\mathrm{Gr}_x^* \mathrm{Pr}\right)^{1/5} \qquad \text{(HTC.18)}$$

$$\text{for} \quad 10^5 < \mathrm{Gr}_x^* < 10^{11}$$

$$\text{with} \quad \mathrm{Gr}_x^* = \mathrm{Gr}_x \mathrm{Nu}_x = \frac{\rho^2 g \beta \dot{q}_w'' x^4}{\lambda \eta^2}$$

- **Vertical plate – turbulent boundary layer flow, isothermal surface**

$$\overline{\mathrm{Nu}}_L = 0.13 \left(\mathrm{Gr}_L Pr\right)^{1/3} \qquad \text{(HTC.19)}$$

$$\text{for} \quad 10^9 < \left(\mathrm{Gr}_L \mathrm{Pr}\right) < 10^{12}$$

- **Vertical cylinder – laminar and turbulent boundary layer flow**

  For diameter-length-ratios of $d/L > 35\,\mathrm{Gr}_L^{-1/4}$, the relationships valid for the vertical plate can be applied.

- **Horizontal cylinder – isothermal surface**

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_d = 0.53\,(\mathrm{Gr}_d\mathrm{Pr})^{1/4} \tag{HTC.20}$$
$$\text{for} \quad 10^4 < \mathrm{Gr}_d\mathrm{Pr} < 10^9$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_d = 0.13\,(\mathrm{Gr}_d\mathrm{Pr})^{1/3} \tag{HTC.21}$$
$$\text{for} \quad 10^9 < \mathrm{Gr}_d\mathrm{Pr} < 10^{12}$$

- **Horizontal plate – isothermal surface**

  Free upper side with $T_\mathrm{W} > T_\infty$ or free lower side with $T_\mathrm{W} < T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0.54\,(\mathrm{Gr}_L\mathrm{Pr})^{1/4} \tag{HTC.22a}$$
$$\text{for} \quad 10^4 < \mathrm{Gr}_L\mathrm{Pr} < \cdot 10^7$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_L = 0.15\,(\mathrm{Gr}_L\mathrm{Pr})^{1/3} \tag{HTC.23a}$$
$$\text{for} \quad 10^7 < \mathrm{Gr}_L\mathrm{Pr} < 10^9$$

  Free upper side with $T_\mathrm{W} < T_\infty$ or free lower side with $T_\mathrm{W} > T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0.27\,(\mathrm{Gr}_L\mathrm{Pr})^{1/4} \tag{HTC.24a}$$
$$\text{for} \quad 10^5 < \mathrm{Gr}_L\mathrm{Pr} < 10^{10}$$

- **Horizontal plate – impressed heat flow**

  Free upper side with $T_\mathrm{W} > T_\infty$ or free lower side with $T_\mathrm{W} < T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0.16\,(\mathrm{Gr}_L\mathrm{Pr})^{1/4} \tag{HTC.22b}$$
$$\text{for} \quad \mathrm{Gr}_L\mathrm{Pr} < 2 \cdot 10^8$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_L = 0.13\,(\mathrm{Gr}_L\mathrm{Pr})^{1/3} \tag{HTC.23b}$$
$$\text{for} \quad 5 \cdot 10^8 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

Free upper side with $T_\mathrm{W} < T_\infty$ or free lower side with $T_\mathrm{W} > T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0.58\,(\mathrm{Gr}_L\mathrm{Pr})^{1/5} \tag{HTC.24b}$$
$$\text{for} \quad 10^6 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

- **Fluid layers between isothermal, vertical walls**

  Height/distance ratio $3.1 < H/s < 42.2$

$$\text{heat conduction only:} \qquad \overline{\mathrm{Nu}}_s = 1 \quad \text{for} \quad \mathrm{Gr}_s < 2 \cdot 10^3$$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_s = 0.20\,(H/s)^{-1/9}\,(\mathrm{Gr}_s\mathrm{Pr})^{1/4} \tag{HTC.25}$$
$$\text{for} \quad 2 \cdot 10^3 < \mathrm{Gr}_s < 2 \cdot 10^4$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_s = 0.071\,(H/s)^{-1/9}\,(\mathrm{Gr}_s\mathrm{Pr})^{1/3} \tag{HTC.26}$$
$$\text{for} \quad 2 \cdot 10^5 < \mathrm{Gr}_s < 10^7$$

- **Fluid layers between isothermal, horizontal walls**

$$\text{heat conduction only:} \qquad \overline{\mathrm{Nu}}_s = 1 \quad \text{for} \quad \mathrm{Gr}_s < 2 \cdot 10^3$$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_s = 0.21\,(\mathrm{Gr}_s\mathrm{Pr})^{1/4} \tag{HTC.27}$$
$$\text{for} \quad 2 \cdot 10^3 < \mathrm{Gr}_s < 3.2 \cdot 10^5$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_s = 0.075\,(\mathrm{Gr}_s\mathrm{Pr})^{1/3} \tag{HTC.28}$$
$$\text{for} \quad 3.2 \cdot 10^5 < \mathrm{Gr}_s < 10^7$$

If heated from above, the relationships for heat conduction only are valid.

Appendix A – Properties of various materials

Tabelle 1: Emissivity of various solids (Total emissivity ε, Emissivity in normal direction of the surface ε_n)

Surface	T K	ε_n	ε	**Surface**	T K	ε_n	ε
Metals				Zinc, highly polished poliert	500		0,045
Alumium, plain	443	0,039	0,049		600		0,055
…polished	373	0,095		Iron plate, galvanized			
…heavily oxidized	366	0,2		…plain	301	0,228	
	777	0,31		…grey oxidized	297	0,276	
Aluminum oxide	550	0,63		Tin, non oxidized	298		0,043
	1100	0,26			373	0,05	
	1089	0,052		**Non-Metals**			
Chromium, polished	423	423	423	Asbestos, paper	296	0,96	
				…Papier	311	0,93	
Gold. highly polished	500	0,018			644	0,94	
	900	900		Concrete, rough	273 – 366		0,94
Copper, polished	293	0,03		Roofing felt	294	0,91	
…struck	293	0,037		Gips	293	0,8 – 0,9	
…black oxidized	293	0,78		Glas	293	0,94	
…oxidized	403	0,76		Quartz (7 mm thick)	555	0,93	
Inconel, rolled	1089		0,69		1111	0,47	
…sandblasted	1089		0,79	Rubber	293	0,92	
Iron and steel,				Wood			
…highly polished	450	0,052		Oak, planed	273 – 366		0,9
…polished	700	0,144		Beech	343	0,94	0,91
	1300	0,377		Ceramics			
…sanded	293	0,242		White Al_2O_3	366		0,9
Cast iron, polished	473	0,21		Carbon			
Cast steel, polished	1044	0,52		…not oxidized	298		0,81
	1311	0,56			773		0,79
Iron sheet				…Fibers	533		0,95
…heavy rusty	292	0,685		…Graphite	373		0,76
…rolled	294	0,657		Corundum, rough	353	0,85	0,84
Cast iron,				Coating, colors:			
…oxidized at 866 K	472	0,64		Oil paint black	366		0,92
	872	0,78		…green	366		0,95
Steel,				…red	366		0,97
…oxidized at 866 K	472	0,79		…white	373		0,94
	872	0,79		Coating. white	373	0,925	
Brass, not oxidized	298	0,035		…flat black	353		0,97
	373	0,035		Bakelite coating	353	0,935	
…oxidized	473	0,61		Mennig color	373	0,93	
	873	0,59		Radiator (acc. to VDI-74)	373	0,925	
	1673	0,17		Enamel, white on iron	292	0,897	
Nickel, not oxidized	298	0,045		Marble			
	373	0,06		light grey. polished	273 – 366		0,9
	873	0,478		Paper	273		0,92
…oxidized	473	0,37			366		0,94
Platinum	422	0,022		Porcelain, white	295		0,924
	1089	0,123		Clay, glassy	298		0,9
Mercury,				…flat	298		0,93
…not oxidized	298	0,1		Water	273	0,95	
	373	0,12			373	0,96	
Silver, polished	311	0,022		Ice, smooth with water	273	0,966	0,92
	644	0,031		…rough surface	273	0,985	
Wolfram	298		0,024	Bricks, red	273 – 366		0,93
	1273		0,15				
	1773		0,23				

Appendix B – Mathematical formulary

Error function

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\xi=0}^{\xi=\eta} \exp(-\xi^2) d\xi$$

$$\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\xi=\eta}^{\xi=\infty} \exp(-\xi^2) d\xi$$

Important characteristics

$$\mathrm{erf}(\infty) = 1 \quad \mathrm{erf}(-\eta) = -\mathrm{erf}(\eta) \quad \frac{d}{d\eta}\left[\mathrm{erf}(\eta)\right] = \frac{2}{\sqrt{\pi}} \exp(-\eta^2)$$

Tabelle 2: Evaluation of the Error function

η	$\mathrm{erf}(\eta)$	$\mathrm{erfc}(\eta)$	$2/\sqrt{\pi}\exp(-\eta^2)$
0	0	1	1.128
0.05	0.056	0.944	1.126
0.1	0.112	0.888	1.117
0.15	0.168	0.832	1.103
0.2	0.223	0.777	1.084
0.25	0.276	0.724	1.060
0.3	0.329	0.671	1.031
0.35	0.379	0.621	0.998
0.4	0.428	0.572	0.962
0.45	0.475	0.525	0.922
0.5	0.520	0.480	0.879
0.55	0.563	0.437	0.834
0.6	0.604	0.396	0.787
0.65	0.642	0.378	0.740
0.7	0.678	0.322	0.691
0.75	0.711	0.289	0.643
0.8	0.742	0.258	0.595
0.85	0.771	0.229	0.548
0.9	0.797	0.203	0.502
0.95	0.821	0.179	0.458
1	0.843	0.157	0.415
1.1	0.880	0.120	0.337
1.2	0.910	0.090	0.267
1.3	0.934	0.066	0.208
1.4	0.952	0.048	0.159
1.5	0.966	0.034	0.119
1.6	0.976	0.024	0.087
1.7	0.984	0.016	0.063
1.8	0.989	0.011	0.044
1.9	0.993	0.007	0.030
2	0.995	0.005	0.021

Bessel functions

Tabelle 3: Evaluation of the Bessel functions of 1. and 2. mode

x	$I_0(x)$	$I_1(x)$	$2/\pi \cdot K_0(x)$	$2/\pi \cdot K_1(x)$
0	1	0	∞	∞
0.2	1.0100	0.1005	1.1160	3.0410
0.4	1.0404	0.2040	0.7095	1.3910
0.6	1.0920	0.3137	0.4950	0.8294
0.8	1.1665	0.4329	0.3599	0.5486
1	1.2661	0.5652	0.2680	0.3832
1.2	1.3937	0.7147	0.2028	0.2768
1.4	1.5534	0.8861	0.1551	0.2043
1.6	1.7500	1.0848	0.1197	0.1532
1.8	1.9896	1.3172	0.9290 10^{-1}	0.1163
2	2.2796	1.5906	0.7251	0.8904 10^{-1}
2.2	2.6291	1.9141	0.5683	0.6869
2.4	3.0493	2.2981	0.4470	0.5330
2.6	3.5533	2.7554	0.3527	0.4156
2.8	4.1573	3.3011	0.2790	0.3254
3	4.8808	3.9534	0.2212	0.2556
3.2	5.7472	4.7343	0.1757	0.2014
3.4	6.7848	5.6701	0.1398	0.1592
3.6	8.0277	6.7028	0.1114	0.1261
3.8	9.5169	8.1404	0.8891 10^{-2}	0.9999 10^{-2}
4	11.302	9.7595	0.7105	0.7947
4.2	13.443	11.706	0.5684	0.6327
4.4	16.010	14.046	0.4551	0.5044
4.6	19.093	16.863	0.3648	0.4027
4.8	22.794	20.253	0.2927	0.3218
5	27.240	24.336	0.2350	0.2575
5.2	32.584	29.254	0.1888	0.2062
5.4	39.009	35.182	0.1518	0.1653
5.6	46.738	42.328	0.1221	0.1326
5.8	56.038	50.946	0.9832 10^{-3}	0.1064
6	67.234	61.342	0.7920	0.8556 10^{-3}
6.2	80.718	73.886	0.6382	0.6879
6.4	96.962	89.026	0.5146	0.5534
6.6	116.54	107.31	0.4151	0.4455
6.8	140.14	129.38	0.3350	0.3588
7	168.59	156.04	0.2704	0.2891
7.2	202.92	188.25	0.2184	0.2331
7.4	244.34	227.18	0.1764	0.1880
7.6	294.33	274.22	0.1426	0.1517
7.8	354.69	331.10	0.1153	0.1424
8	427.56	399.87	0.9325 10^{-4}	0.9891 10^{-4}
8.2	515.59	483.05	0.7543	0.7991
8.4	621.94	583.66	0.6104	0.6458
8.6	750.46	705.38	0.4941	0.5220
8.8	905.80	852.66	0.4000	0.4221
9	1,093.0	1.030.90	0.3239	0.3415
9.2	1,320.7	1.246.70	0.2624	0.2763
9.4	1,595.3	1,507.90	0.2126	0.2236
9,6	1,927.5	1,824.10	0.1722	0.1810
9,8	2,329.4	2,207.10	0.1396	0.1465
10	2,815.7	2,671.00	0.1131	0.1187

Particular functions

$$\sin(x \pm y) = \sin(x) \cdot \cos(y) \pm \cos(x) \cdot \sin(y)$$

$$\cos(x \pm y) = \cos(x) \cdot \cos(y) \mp \sin(x) \cdot \sin(y)$$

$$\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2}$$

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2}$$

$$\sinh(x \pm y) = \sinh(x) \cdot \cosh(y) \pm \cosh(x) \cdot \sinh(y)$$

$$\cosh(x \pm y) = \cosh(x) \cdot \cosh(y) \pm \sinh(x) \cdot \sinh(y)$$

$$\sinh(2x) = 2 \cdot \sinh(x) \cdot \cosh(x)$$

$$\cosh(2x) = \sinh^2(x) + \cosh^2(x) = 2\cosh^2(x) - 1$$

$$\mathrm{artgh}(x) = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{mit} \quad (|x| < 1)$$

$$\mathrm{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\mathrm{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad \text{mit} \quad (|x| \geq 1)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp\left(-\eta^2\right) d\eta \quad \text{mit} \quad \mathrm{erf}(\infty) = 1$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$\exp\left(\ln(x)\right) = x$$

$$\lg(x) = \frac{\ln(x)}{\ln(10)}$$

Series

- Arithmetic series

$$s = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d)$$

- Geometric Series

$$s = a + aq + aq^2 + \cdots + aq^{n-1} = \sum_{\nu=0}^{n-1} aq^\nu = a\,\frac{1-q^n}{1-q}$$

$$\text{Infinite series:} \qquad s = \sum_{\nu=0}^{\infty} aq^\nu = a\,\frac{1}{1-q} \quad \text{for} \quad |q| < 1$$

- Taylor-series

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

- Power series of particular functions

$$\begin{aligned}
(1 \pm x)^m &= 1 \pm mx + \frac{m(m-1)}{2!}x^2 \pm \frac{m(m-1)(m-2)}{3!}x^3 + ... \\
&= \sum_{\nu=0}^{m} \binom{m}{\nu} x^\nu \quad \text{with} \quad \binom{m}{\nu} = \frac{m!}{\nu!(m-\nu)!} \\
\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - ... = \sum_{\nu=0}^{\infty} (-1)^{\nu+1}\,\frac{x^\nu}{\nu} \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{\nu=0}^{\infty} (-1)^\nu\,\frac{x^{2\nu+1}}{(2\nu+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{\nu=0}^{\infty} (-1)^\nu\,\frac{x^{2\nu}}{2\nu!}
\end{aligned}$$

Differentiation

- Product rule
$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=\left[f\cdot g\right]'=f'\cdot g+g'\cdot f$$
- Quotient rule
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\left[\frac{f}{g}\right]'=\frac{f'\cdot g-g'\cdot f}{g^2}$$
- Chain rule
$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=\left[f\left(g(x)\right)\right]'=\frac{df}{dg}\cdot\frac{dg}{dx}$$
- Total differential of function $z=f(x,y)$
$$dz=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$
- Derivation of a composite function of more than one variable $z=f(x,y)$ with $x(t)$ and $y(t)$
$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Derivatives of elementary functions

$$\begin{array}{rcll}
\text{Function} & \xrightarrow{d/dx(\ldots)} & \text{Derivative} & \\
x^n & & n\,x^{n-1} & \\
\exp(x) & & \exp(x) & \\
\ln(x) & & \dfrac{1}{x} & \\
a^x & & a^x\ln(a) & \\
\sin(x) & & \cos(x) & \\
\cos(x) & & -\sin(x) & \\
\tan(x) & & \dfrac{1}{\cos^2(x)} & =\quad 1+\tan^2(x) \\
\cot(x) & & -\dfrac{1}{\sin^2(x)} & =-(1+\cot^2(x))
\end{array}$$

$$
\begin{array}{rcll}
\text{Function} & \xrightarrow{d/dx(...)} & \text{Derivative} & \\
\sinh(x) & & \cosh(x) & \\
\cosh(x) & & \sinh(x) & \\
\tanh(x) & & \dfrac{1}{\cosh^2(x)} & = \quad 1 - \tanh^2(x) \\
\coth(x) & & -\dfrac{1}{\sinh^2(x)} & \\
\lg(x) & & \dfrac{1}{\ln(10)}\dfrac{1}{x} &
\end{array}
$$

Indefinite expressions

Rule of Bernoulli de l'Hospital

- Indefinite expressions of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

- Indefinite expressions of form $0 \cdot \infty$ with $f(x_0) = 0$ and $g(x_0) = \infty$

$$\lim_{x \to x_0} f(x) \cdot g(x) = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \lim_{x \to x_0} \frac{f'(x)}{\left(\frac{1}{g(x)}\right)'}$$

- Indefinite expressions of form $\infty - \infty$

$$\lim_{x \to x_0} \left(f(x) - g(x)\right) = \lim_{x \to x_0} \frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)'}{\left(\frac{1}{g(x)} \cdot \frac{1}{f(x)}\right)'}$$

Integration

- Indefinite Integral

$$\int f(x)dx = F(x) + C \quad \text{with the primitive function} \quad F(x)$$

- Definite Integral

$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx = -\int\limits_b^a f(x)dx = F(b) - F(a)$$

- Differentiation of an integral by its upper limit

$$\frac{d}{dx}\int\limits_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int\limits_a^x f(x,t)dt = \int\limits_a^x \frac{\partial f(x,t)}{\partial x}dt + f(x,t)$$

Integration rules

- Integration by parts

$$\int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)dx$$

- Substitution formula

$$\int f(x)dx \quad \text{Substitution of} \quad x = g(t) \quad \to \quad t = h(x)$$

  - Indefinite Integration

$$\int f(x)dx = \int f(g(t))\cdot g'(t)dt + C$$

  - Definite Integration

$$\int\limits_a^b f(x)dx = \int\limits_{h(a)}^{h(b)} f(g(t))\cdot g'(t)dt$$

Primitives (antiderivatives) of elementary functions

Integrand	$\to$	Primitive	Integrand	$\to$	Primitive
$f(x)$	$\to$	$F(x) = \int f(x)dx$	$f(x)$	$\to$	$F(x) = \int f(x)dx$
x^n		$\dfrac{x^{n+1}}{n+1}$	$\exp(x)$		$\exp(x)$
$\dfrac{1}{x}$		$\ln\lvert x\rvert$	a^x		$\dfrac{a^x}{\ln(a)}$
$\sin(x)$		$-\cos(x)$	$\sinh(x)$		$\cosh(x)$
$\cos(x)$		$\sin(x)$	$\cosh(x)$		$\sinh(x)$
$\dfrac{1}{\sin^2(x)}$		$-\mathrm{ctan}(x)$	$\dfrac{1}{\sinh^2(x)}$		$-\mathrm{ctanh}(x)$
$\tan(x)$		$-\ln\lvert\cos(x)\rvert$	$\tanh(x)$		$\ln\lvert\cosh(x)\rvert$
$\cot(x)$		$\ln\lvert\sin(x)\rvert$	$\coth(x)$		$\ln\lvert\sinh(x)\rvert$
$\dfrac{1}{a^2+x^2}$		$\dfrac{1}{a}\mathrm{arc}\tan\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2+x^2}}$		$\mathrm{arc}\sinh\dfrac{x}{a}$
					$= \ln\lvert x+\sqrt{x^2+a^2}\rvert$
$\dfrac{1}{a^2-x^2}$		$\dfrac{1}{a}\mathrm{arc}\tanh\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2-x^2}}$		$\mathrm{arc}\sin\dfrac{x}{a}$
$(\lvert x\rvert < a)$		$= \dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$			
$\dfrac{1}{x^2-a^2}$		$-\dfrac{1}{a}\mathrm{arc}\coth\dfrac{x}{a}$	$\dfrac{1}{\sqrt{x^2-a^2}}$		$\mathrm{arc}\cosh\dfrac{x}{a}$
$(\lvert x\rvert > a)$		$= \dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$			$= \ln\lvert x+\sqrt{x^2-a^2}\rvert$
$\dfrac{1}{a^4-x^4}$		$\dfrac{1}{4a^3}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert\cdots$	$\sqrt{x^2\pm a^2}$		$\dfrac{x}{2}\sqrt{x^2\pm a^2}\cdots$
		$+\dfrac{1}{2a^3}\mathrm{arc}\tan\dfrac{x}{a}$			$\pm\dfrac{a^2}{2}\ln\lvert x+\sqrt{x^2\pm a^2}\rvert$

plus an integration constant C in every case

Integration methods

- Rational functions

$\int \frac{Q(x)}{P(x)}dx$ with the polynomials $Q(x)$, $P(x)$ $\rightarrow$ expansion into partial fraction

- Irrational functions

$$R(\ldots) = \text{rational function of } (\ldots)$$

$$\int R(\sinh(x), \cosh(x))dx \quad \rightarrow \quad \text{Substitute} \quad t = \tanh\left(\frac{x}{2}\right)$$

$$\sinh(x) = \frac{2t}{1-t^2}$$

$$\cosh(x) = \frac{1+t^2}{1-t^2}$$

$$dx = \frac{2}{1-t^2}dt$$

$$\int R(\sinh(x)) \cdot \cosh(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \sinh(x)$$

$$\int R(\cosh(x)) \cdot \sinh(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \cosh(x)$$

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{1/n}\right] dx \quad \rightarrow \quad \text{Substitute} \quad t = \left(\frac{ax+b}{cx+d}\right)^{1/n}$$

$$\int R\left(x, \sqrt{ax^2+2bx+c}\right) dx \quad \text{with} \quad ac - b^2 \neq 0$$

$$ac - b^2 > 0\ ,\ a > 0 \quad \rightarrow \quad \text{Substitute} \quad \sinh(t) = \frac{ax+b}{\sqrt{ac-b^2}}$$

$$ac - b^2 < 0\ ,\ a > 0 \quad \rightarrow \quad \text{Substitute} \quad \cosh(t) = \frac{ax+b}{\sqrt{b^2-ac}}$$

$$ac - b^2 < 0\ ,\ a < 0 \quad \rightarrow \quad \text{Substitute} \quad \sinh(t) = \frac{ax+b}{\sqrt{b^2-ac}}$$

- Trigonometric functions

$$\int R(\sin(x), \cos(x))dx \quad \rightarrow \quad \text{Substitute} \quad t = \tan\left(\frac{x}{2}\right)$$
$$\sin(x) = \frac{2t}{1+t^2}$$
$$\cos(x) = \frac{1-t^2}{1+t^2}$$
$$dx = \frac{2}{1+t^2}dt$$

$$\int R(\sin(x)) \cdot \cos(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \sin(x)$$
$$\int R(\cos(x)) \cdot \sin(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \cos(x)$$

- Other transcendental functions

$$\int R(\exp(x))\, dx \quad \rightarrow \quad \text{Substitute} \quad t = \exp(x)$$
$$\int R(\ln(x))\, dx \quad \rightarrow \quad \text{Substitute} \quad t = \ln x$$
$$\cdots \quad \rightarrow \quad \text{partial Integration}$$

Application of single or multiple integration by parts on integrals of the following form:

$$\int \exp(ax)\sin(bx)dx^* \qquad \int P(x)\exp(ax)$$
$$\int \exp(ax)\cos(bx)dx^* \qquad \int P(x)\ln(bx)$$
$$\int \exp(ax)\sinh(bx)dx^* \qquad \int P(x)\sin(bx)$$
$$\int \exp(ax)\sinh(bx)dx^* \qquad \int P(x)\cos(bx)$$
$$\int P(x)\sinh(bx)$$
$$\int P(x)\cosh(bx)$$

*) The multiple application of the integration by parts leads to the original integral: Solving an algebraic equation

Differential equations

The order of a differential equation is the one of the highest included derivative in the equation. If a specific initial state is determined besides the already given ordinary boundary conditions of a differential equation, the problem is called Initial value problem.

Ordinary differential equations of first order

- Separable types

  Type: $y' = f(x) \cdot g(y)$ $\rightarrow$ Solution: $\int \frac{dy}{g(y)} = \int f(x)dx + C$

  Type: $y' = f(ax + by + c)$ $\rightarrow$ Substitute: $z = ax + by + c$

  $\cdots$ $\rightarrow$ separable equation: $z' = a + b\ f(z)$

- Linear differential equation

  The dependent variable y and its derivatives appear in its first power only. Coefficients $f(x)$ are not constant, inhomogeneous if $g(x) \neq 0$

  Type: $y' + y\ f(x) = g(x)$

  $\rightarrow$ Solution: $y = \exp\left(-\int f(x)dx\right)\left(C + \int g(x)\exp\left(\int f(x)dxdx\right)\right)$

- Bernoulli differential equation

  Type: $y' + y\ f(x) = g(x)y^n$ $\rightarrow$ Substitute: $z = y^{1-n}$

  $z' = (1-n)y^{-n}y'$

  $\cdots$ $\rightarrow$ linear equation: $z' + (1-n)f(x)z \cdots$

  $\cdots = (1-n)g(x)$

Ordinary differential equations of higher order

Ordinary differential equations of high order always can be converted into a system of ordinary differential equations of first order. If a ordinary differential equation y_1 is of order n, auxiliary functions are introduced as follows:

$$\begin{aligned} y_1^{'} &= y_2 \\ y_2^{'} &= y_3 \\ &\vdots \\ y_{n-1}^{'} &= y_n \\ y_n^{'} &= f\left(x, y_1, y_2, y_3, \ldots, y_n\right) \end{aligned}$$

Therefor a system of n ordinary differential equations of first order is set, which can be solved like described in the text above.

Linear differential equations of second order with constant coefficients

$$\text{Type:} \quad y'' + a_1 y' + a_0 y = f(x) \quad \rightarrow \quad \text{Solution:} \quad y = y_\mathrm{H} + y_\mathrm{P}$$

Homogeneous solution

Complimentary function $y_\mathrm{H} = \exp(\mu x)$, determination of roots of characteristic polynomial which is derived for the homogeneous solution:

$$\mu^2 + a_1 \mu + a_0 = 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

Case-by-case analysis of the roots:

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) > 0 \quad \rightarrow \quad \mu_1 \neq \mu_2$ (two real roots)

  Approach: $y_\mathrm{H} = C_1 \exp(\mu_1 x) + C_2 \exp(\mu_2 x)$

  $$a_1 = 0,\ a_0 < 0 \quad \rightarrow \quad \mu_{1/2} = \pm\sqrt{-a_0},\ \mu = \sqrt{-a_0}$$

  Approach: $y_\mathrm{H} = C_1 \sinh(\mu x) + C_2 \cosh(\mu x)$

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) = 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2}$ (one double root)

  Approach: $y_\mathrm{H} = \exp\left(-\frac{a_1}{2}\, x\right)\ (C_1 + C_2\, x)$

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) < 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2} \pm i \sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}$ (conj. complex root)

  Approach: $y_\mathrm{H} = \exp(-\frac{a_1}{2}\, x) \cdots$

  $$\cdots \left(C_1 \cos\left(x\sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}\right) + C_2 \sin\left(x\sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}\right)\right)$$

  $$a_1 = 0,\ a_0 > 0 \quad \rightarrow \quad \mu_{1/2} = \pm\, i\, \sqrt{a_0},\ \mu = \sqrt{a_0}$$

  Approach: $y_\mathrm{H} = C_1 \sin(\mu x) + C_2 \cos(\mu x)$

Particular solution – Particular form of perturbation term

$$f(x) = \exp(kx)\ (P_n(x)\cos(\omega x) + Q_n(x)\sin(\omega x))$$

Approach in form of perturbation term:

$$y_\mathrm{P} = \exp(kx)\ (M_n(x)\cos(\omega x) + N_n(x)\sin(\omega x)) \cdot x^m$$

- For $\omega \neq 0$ always both polynomials M_n and N_n have to be considered in the approach of a solution, for $\omega = 0$ N_n is omitted.
- The polynomials M_n and N_n always have to be completely considered in the approach of a solution, i.e. no coeffizient is expected to be zero.

Case-by-case analysis for m on basis of roots of the characteristic polynomial:

$$\begin{array}{rrcl} & \mu_1 = \mu_2 \overset{!}{=} k & \rightarrow & m = 2 \\ & \mu_{1/2} \overset{!}{=} k \pm i\omega & \rightarrow & m = 1 \\ \text{apart from that:} & \mu_{1/2} \neq k \pm i\omega & \rightarrow & m = 0 \end{array}$$

Application of the perturbation term $f(x)$ and the Ansatz y_P to the differential equation. The coefficients of the polynomials $M_n(x)$ and $N_n(x)$ are calculated by comparison of coefficients.

Particular solution – In general

Solution of the precipitation integral

$$y_\mathrm{P}(x) = \int_0^x y_{\mathrm{H},0}(x - \xi) \cdot f(\xi) d\xi$$

Here $y_{\mathrm{H},0}$ is a particular homogeneous solution with the initial-/boundary conditions $y_\mathrm{H}(0) = 1$ and $y'_\mathrm{H}(0) = 1$.