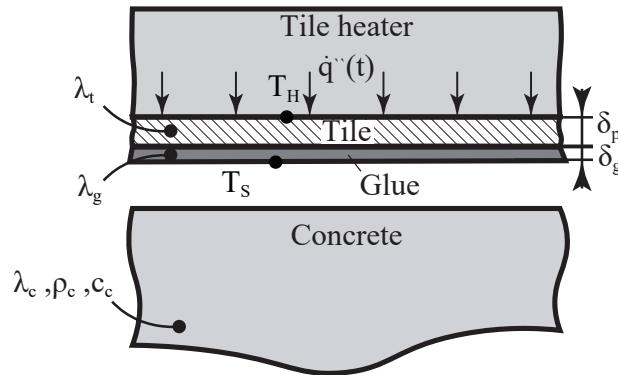


Exercise II.15 (Tile setting ★★):

A tile setter employs a modern technique for tile installation, involving preheating the tile and glue before affixing them to the concrete. The tile and glue are heated until they reach a steady-state condition, achieving a uniform heating temperature T_H and a constant heat flux \dot{q}''_0 . Once these conditions are met, the tile setter places the heated tile and glue on the concrete, maintaining a constant temperature T_S throughout the process. After reaching a critical temperature T_{crit} at a distance δ_{crit} within the concrete, the heater is removed. Initially, the concrete used to be at a homogeneous temperature T_0

**Given parameters:**

- Steady-state heat flux: $\dot{q}''_0 = 7.5 \text{ kW/m}^2$
- Thickness of the pile: $\delta_p = 10 \text{ mm}$
- Thickness of the glue: $\delta_g = 2 \text{ mm}$
- Conductivity of the pile: $\lambda_p = 1.0 \text{ W/mK}$
- Conductivity of the glue: $\lambda_g = 0.35 \text{ W/mK}$
- Conductivity of the concrete: $\lambda_c = 2.3 \text{ W/mK}$
- Heat capacity of the concrete: $c_c = 1,000 \text{ J/kgK}$
- Density of the concrete: $\rho_c = 2,400 \text{ kg/m}^3$
- Initial temperature of the concrete: $T_0 = 20 \text{ }^\circ\text{C}$
- Heating temperature of the tile heater: $T_H = 200 \text{ }^\circ\text{C}$
- Critical temperature: $T_{\text{crit}} = 35 \text{ }^\circ\text{C}$
- Critical distance: $\delta_{\text{crit}} = 10 \text{ mm}$

Hints:

- Heat will never penetrate entirely through the concrete.

Tasks:

- Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the concrete. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.
- Determine the time t_{crit} at which the heater can be removed.
- Illustrate the concrete's temperature profile that depicts both temporal and spatial variations.

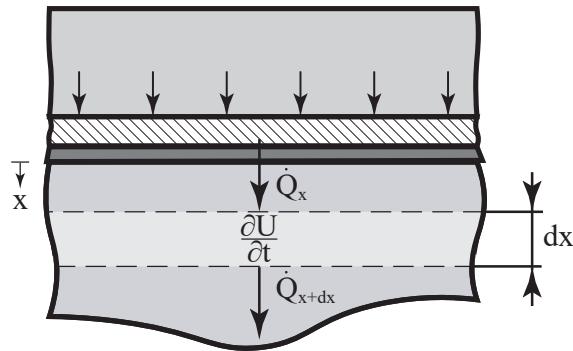
Solution II.15 (Tile setting **):

Task a)

As observed earlier, the temperature profile can be obtained from the conduction equation within an element.

1 Setting up the balance:

Within the concrete, we will observe a spatial as well as a temporal temperature difference.



The conduction equation can be derived from an infinitesimal element within the concrete. This energy balance reads:

$$\frac{\partial U}{\partial t} = \dot{Q}_x - \dot{Q}_{x+dx} \quad (\text{II.15.1})$$

2 Defining the elements within the balance:

Change of internal energy over the course of time:

$$\begin{aligned} \frac{\partial U}{\partial t} &= dm \cdot c_c \frac{\partial T}{\partial t} \\ &= \rho_c c_c A dx \frac{\partial T}{\partial t} \end{aligned} \quad (\text{II.15.2})$$

The ingoing conductive flux can be defined by use of Fourier's law:

$$\dot{Q}_x = -\lambda_c A \frac{\partial T}{\partial x} \quad (\text{II.15.3})$$

and the outgoing can be approximated by use of the Taylor series expansion, as we are dealing with an infinitesimal element:

$$\begin{aligned} \dot{Q}_{x+dx} &= \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} \\ &= \frac{\partial}{\partial x} \left(-\lambda_c A \frac{\partial T}{\partial x} \right) \cdot dx \end{aligned} \quad (\text{II.15.4})$$

Conclusion

3 Inserting and rearranging:

Substitution and rewriting gives:

$$\frac{\partial T}{\partial t} = a_c \frac{\partial^2 T}{\partial x^2} \quad (\text{II.15.5})$$

where the thermal diffusivity is defined as $a = \frac{\lambda}{\rho c_p}$

4 Defining the boundary and/or initial conditions:

For solving the differential equation, we necessitate one initial condition and two boundary conditions. This is due to the fact that the temperature T has undergone differentiation once concerning time (t) and twice concerning space (x).

Initially, the temperature within the concrete has a homogeneous temperature:

$$\left. \begin{array}{l} t = 0 \\ 0 < x < \infty \end{array} \right\} T = T_0 \quad (\text{II.15.6})$$

It is stated that heat will never penetrate entirely through the concrete. This implies that at all times, far away from the surface, the temperature will still be equal to the initial temperature. Therefore, the concrete can be observed as a semi-infinite body and the temperature for $x \rightarrow \infty$ will remain to be T_0 at all times:

$$\left. \begin{array}{l} t > 0 \\ x \rightarrow \infty \end{array} \right\} T = T_0 \quad (\text{II.15.7})$$

When $t > 0$ the temperature at $x = 0$ is remained at a constant temperature T_S :

$$\left. \begin{array}{l} t > 0 \\ x = 0 \end{array} \right\} T = T_S \quad (\text{II.15.8})$$

However, to determine this temperature T_S , an energy balance around the tile and the glue needs to be set:

$$\begin{aligned} 0 &= \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} && (\text{II.15.9}) \\ 0 &= \dot{q}_0'' A - \frac{T_H - T_S}{\frac{\delta_p}{\lambda_p A} + \frac{\delta_g}{\lambda_g A}} \\ \Rightarrow T_S &= T_H - \dot{q}_0'' \left(\frac{\delta_p}{\lambda_p} + \frac{\delta_g}{\lambda_g} \right) \\ &= 200 [\text{°C}] - 7,500 [\text{W/m}^2] \left(\frac{0.01 [\text{m}]}{1.0 [\text{W/mK}]} + \frac{0.002 [\text{m}]}{0.35 [\text{W/mK}]} \right) = 82 [\text{°C}] \end{aligned}$$

From the differential equation, the boundary conditions, and the initial condition, we find that we are dealing with a semi-infinite body with a constant surface temperature.

Task b)

⑤ Solving the equation:

The solution to the given differential equation can be written as:

$$\theta^* = 1 - \operatorname{erf} \left[\frac{x}{\sqrt{4at}} \right] \quad (\text{II.15.10})$$

where:

$$\theta^* = \frac{T(x,t) - T_0}{T_S - T_0} \quad (\text{II.15.11})$$

To determine, the critical time, the equation should be rewritten:

$$\begin{aligned} \frac{T_{\text{crit}} - T_0}{T_S - T_0} &= 1 - \operatorname{erf} \left[\frac{\delta_{\text{crit}}}{\sqrt{\frac{4\lambda_c}{\rho_c c_c} t_{\text{crit}}}} \right] \\ \Rightarrow t_{\text{crit}} &= \frac{\rho_c c_c}{4\lambda_c} \cdot \left(\frac{\delta_{\text{crit}}}{\operatorname{erf}^{-1} \left[1 - \frac{T_{\text{crit}} - T_0}{T_S - T_0} \right]} \right)^2 \\ &= \frac{2,400 \text{ [kg/m}^3\text{]} \cdot 1,000 \text{ [J/kgK]}}{4 \cdot 2.3 \text{ [W/mK]}} \cdot \left(\frac{0.01 \text{ [m]}}{\operatorname{erf}^{-1} \left[1 - \frac{(35-20)}{(82-20)} \text{ [}^\circ\text{C]} \right]} \right)^2 = 38 \text{ [s]} \end{aligned} \quad (\text{II.15.12})$$

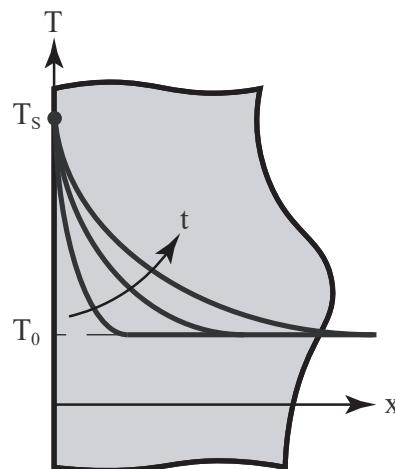
Conclusion

The heater can be removed after 38 seconds.

Task c)

Initially, the temperature within the concrete is uniformly at temperature T_0 . As soon as $t > 0$, the surface temperature is held constant at T_S . The gradient of the temperature profile diminishes in the direction of penetration until it reaches the initial temperature, where the profile has a horizontal slope. Additionally, with time, the temperature within the body rises.

Conclusion



Exercise II.16 (Heating and quenching of a sphere ★★★):

A sphere, initially at a homogeneous temperature of T_0 , is put into an oven. The oven temperature remains constant at a homogeneous temperature of T_o .

Given parameters:

- Initial temperature of the sphere: $T_0 = 25 \text{ } ^\circ\text{C}$
- Intermediate temperature of the sphere: $T_h = 150 \text{ } ^\circ\text{C}$
- Oven temperature: $T_o = 200 \text{ } ^\circ\text{C}$
- Quenching temperature: $T_q = 30 \text{ } ^\circ\text{C}$
- Heat transfer coefficient: $\alpha = 110 \text{ W/m}^2\text{K}$
- Radius of the sphere: $r_1 = 1.5 \text{ cm}$
- Thermal conductivity of the sphere: $\lambda = 1.52 \text{ W/mK}$
- Density the sphere: $\rho = 1.45 \cdot 10^3 \text{ kg/m}^3$
- Specific heat capacity the sphere: $c_p = 0.88 \text{ kJ/kg} \cdot \text{K}$

Hints:

- Heat radiation can be neglected.
- It always remains that $Fo > 0.2$

Tasks:

- a) Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the sphere. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.
- b) Determine the temperature of the center of the sphere after 3 minutes.

After some time the sphere has a hot homogeneous temperature T_h and is being quenched. During this process, the quenching temperature is constant at T_q . Further, in time, the center of the sphere has a temperature of $54 \text{ } ^\circ\text{C}$ and the surface has a temperature of $44.4 \text{ } ^\circ\text{C}$.

- c) Determine the time instant when the center of the sphere has a temperature of $54 \text{ } ^\circ\text{C}$ and the surface has a temperature of $44.4 \text{ } ^\circ\text{C}$.
- d) Determine the amount of heat dissipated at this time instant.

Solution II.16 (Heating and quenching of a sphere ★★★):**Task a)**

The temperature profile within the solid body can be derived from the conduction equation, considering both spatial and temporal aspects.

In practical scenarios, the temperature profile within the sphere is never homogeneous. However, the Biot number is a useful parameter for assessing whether the temperature distribution within the sphere can be approximated as homogeneous. If the Biot number allows for such an assumption, the lumped capacity model becomes a viable approach to solving the problem.

The characteristic length of the Biot number can be written as:

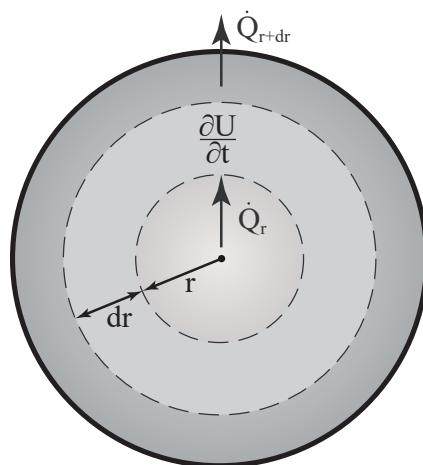
$$\begin{aligned} L &= \frac{V}{A} = \frac{\frac{4\pi r_1^3}{3}}{4\pi r_1^2} \\ &= \frac{r_1}{3} \\ &= \frac{0.015 \text{ [m]}}{3} = 0.005 \text{ [m]} \end{aligned} \quad (\text{II.16.1})$$

Substitution into the definition of the Biot number yields:

$$\begin{aligned} \text{Bi} &= \frac{\alpha L}{\lambda} \\ &= \frac{\alpha r_1}{3\lambda} \\ &= \frac{110 \text{ [W/m}^2\text{K]} \cdot 0.015 \text{ [m]}}{3 \cdot 1.52 \text{ [W/mK]}} = 0.36 \text{ [-]} \end{aligned} \quad (\text{II.16.2})$$

As the Biot number is not significantly smaller than 1, it is advisable to consider both spatial and temporal variations in the temperature of the sphere to ensure a high level of accuracy in the results.

① Setting up the balance:



Due to symmetry, the problem can be simplified to one-dimensional in the radial direction. As the temperature is not uniform within the body, an infinitesimal element within the sphere is

considered as the control volume to derive the necessary differential equation. The energy balance for this element is given by:

$$\frac{\partial U}{\partial t} = \dot{Q}_r - \dot{Q}_{r+dr} \quad (\text{II.16.3})$$

2 Defining the elements within the balance:

The temporal change of internal energy within the infinitesimal element can be written as:

$$\begin{aligned} \frac{\partial U}{\partial t} &= dm \cdot c_p \frac{\partial T}{\partial t} \\ &= \rho \cdot dV \cdot c_p \cdot \frac{\partial T}{\partial t} \end{aligned} \quad (\text{II.16.4})$$

where the volume of the infinitesimal element yields from:

$$\begin{aligned} dV &= \frac{4\pi}{3} [(r + dr)^3 - r^3] \\ &= \frac{4\pi}{3} [3r^2 \cdot dr + 3r \cdot dr^2 + dr^3] \\ &\approx 4\pi r^2 \cdot dr \end{aligned} \quad (\text{II.16.5})$$

note that volume of the infinitesimal element can be approximated to be $4\pi r^2 \cdot dr$, because $3r \cdot dr^2 \ll 3r^2 \cdot dr$ and $dr^3 \ll 3r^2 \cdot dr$.

The incoming rate of heat conduction can be written as:

$$\begin{aligned} \dot{Q}_r &= -\lambda A(r) \frac{\partial T}{\partial r} \\ &= -4\lambda\pi r^2 \frac{\partial T}{\partial r} \end{aligned} \quad (\text{II.16.6})$$

For an infinitesimal element, the outgoing rate of heat conduction can be approximated by use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{r+dr} &= \dot{Q}_r + \frac{\partial \dot{Q}_r}{\partial r} \cdot dr \\ &= -4\lambda\pi r^2 \frac{\partial T}{\partial r} + \frac{\partial}{\partial r} \left(-4\lambda\pi r^2 \frac{\partial T}{\partial r} \right) \cdot dr \end{aligned} \quad (\text{II.16.7})$$

Conclusion

3 Inserting and rearranging:

Substitution and rewriting gives:

$$\frac{\partial T}{\partial t} = \frac{a}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \quad (\text{II.16.8})$$

where the thermal diffusivity is defined as $a = \frac{\lambda}{\rho c_p}$

4 Defining the boundary and/or initial conditions:

To solve the obtained differential equation, two boundary conditions are needed, as the temperature has been differentiated twice with respect to r , and one initial condition is necessary because the temperature has been differentiated once to t .

Initially, the entire temperature of the sphere is to be at T_0 , therefore the initial condition can be written as:

$$T(t = 0) = T_0 \quad (\text{II.16.9})$$

Furthermore, since the temperature profile of the sphere is symmetrical, implying a horizontal slope at the center, the first boundary condition can be expressed as:

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad (\text{II.16.10})$$

Lastly, the second boundary condition can be found from an energy balance at the surface of the sphere, which states that $\dot{Q} = -\lambda A \left. \frac{\partial T}{\partial r} \right|_{r=r_1} = \alpha A (T(r = r_1) - T_O)$. Rewriting yields:

$$\left. \frac{\partial T}{\partial r} \right|_{r=r_1} = -\frac{\alpha}{\lambda} (T(r = r_1) - T_O)$$

The differential equation, initial condition, and boundary conditions align with the temperature profiles described by the Heisler diagrams. Hence, there is no necessity to solve this equation, and the Heisler diagrams can be directly applied without undergoing the mathematical derivation.

Task b)

5 Solving the equation:

To determine the temperature at the center of the sphere after 3 minutes the chart on the next page can be used which describes the temperature of the center in a sphere. It is necessary to calculate the inverse of the Biot number and the Fourier number at that specific time instant.

One should note that the Biot number defined in the Heisler diagrams is slightly different from how we defined it before and it is a factor 3 larger in the case of spheres. This factor has been accounted for within the results of the chart.

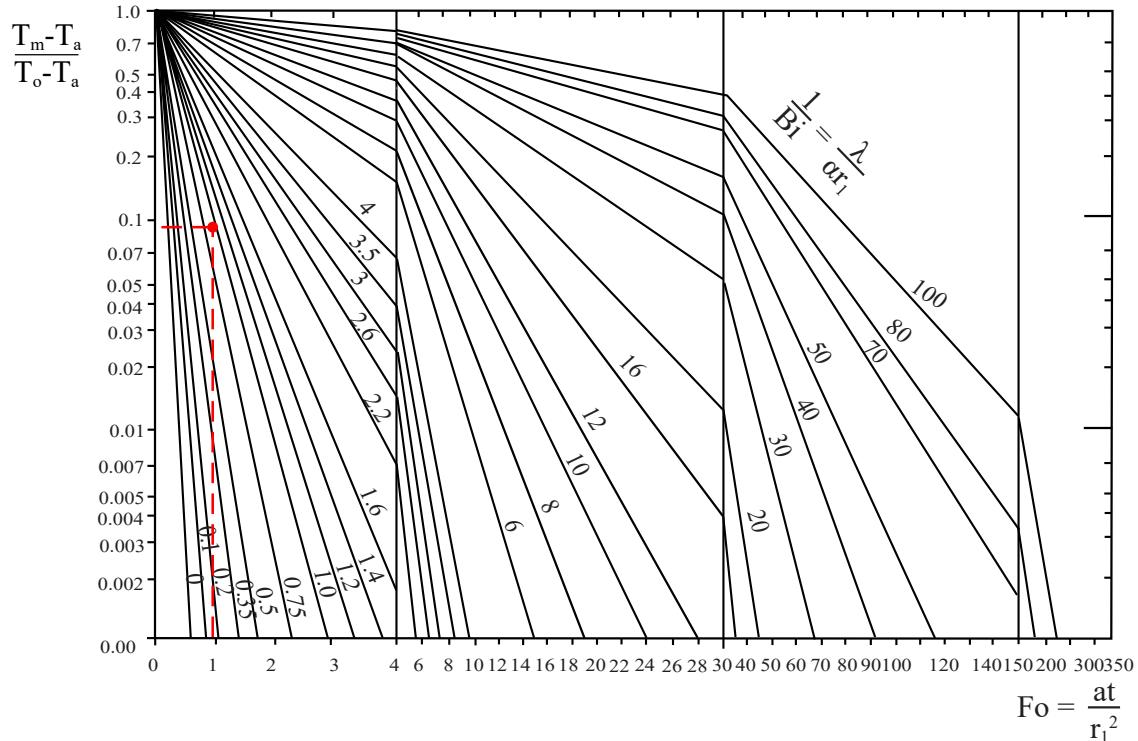
The inverse of the Biot number for the Heisler diagrams can be found from:

$$\begin{aligned} \frac{1}{Bi} &= \frac{\lambda}{\alpha r_1} \\ &= \frac{1.52 [\text{W/mK}]}{110 [\text{W/m}^2\text{K}] \cdot 0.015 [\text{m}]} = 0.92 [-] \end{aligned} \quad (\text{II.16.11})$$

The Fourier number yields from:

$$\begin{aligned} Fo &= \frac{at_1}{r_1^2} = \frac{\lambda t_1}{\rho c_p r_1^2} \\ &= \frac{1.52 [\text{W/mK}] \cdot 180 [\text{s}]}{1450 [\text{kg/m}^3] \cdot 880 [\text{J/kgK}] 0.015^2 [\text{m}^2]} = 0.95 [-] \end{aligned} \quad (\text{II.16.12})$$

Using the Heisler diagram for the temperature in the center of a sphere:



From the chart, it yields the:

$$\frac{T_m - T_a}{T_o - T_a} \approx 0.09 \quad (\text{II.16.13})$$

Note that in our given case, the ambient temperature T_a is the oven temperature T_O . Substitution of this and rewriting yields:

$$\begin{aligned} T_m &= 0.09 (T_0 - T_O) + T_O \\ &= 0.09 (25 - 200) \text{ [°C]} + 200 \text{ [°C]} = 184 \text{ [°C]} \end{aligned} \quad (\text{II.16.14})$$

Conclusion

The temperature of the center of the sphere after 3 minutes equals 184 °C.

Task c)

Now the sphere is being quenched, the ambient temperature and the convection coefficient have been changed, and therefore also the Biot number. The chart on the next page can be used to determine the inverse of the Biot number when knowing the center and the surface temperature of the sphere.

To do so, the dimensionless temperature difference needs to be determined, where the

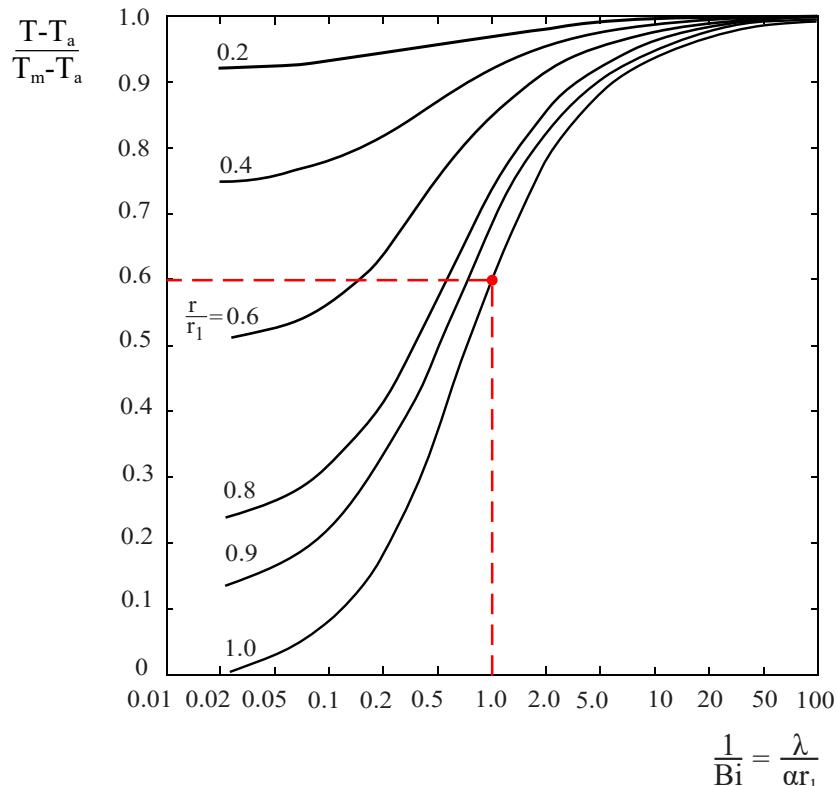
ambient temperature T_a is equal to the quenching temperature T_q :

$$\begin{aligned}\frac{T - T_a}{T_m - T_a} &= \frac{T - T_q}{T_m - T_q} \\ &= \frac{(44.4 - 30) \text{ [°C]}}{(54 - 30) \text{ [°C]}} = 0.6 \text{ [-]}\end{aligned}\quad (\text{II.16.15})$$

and the ratio between radii:

$$\frac{r_1}{r_2} = 1 \text{ [-]} \quad (\text{II.16.16})$$

With this, the following diagram can be read:



Which yields the inverse of the Biot number:

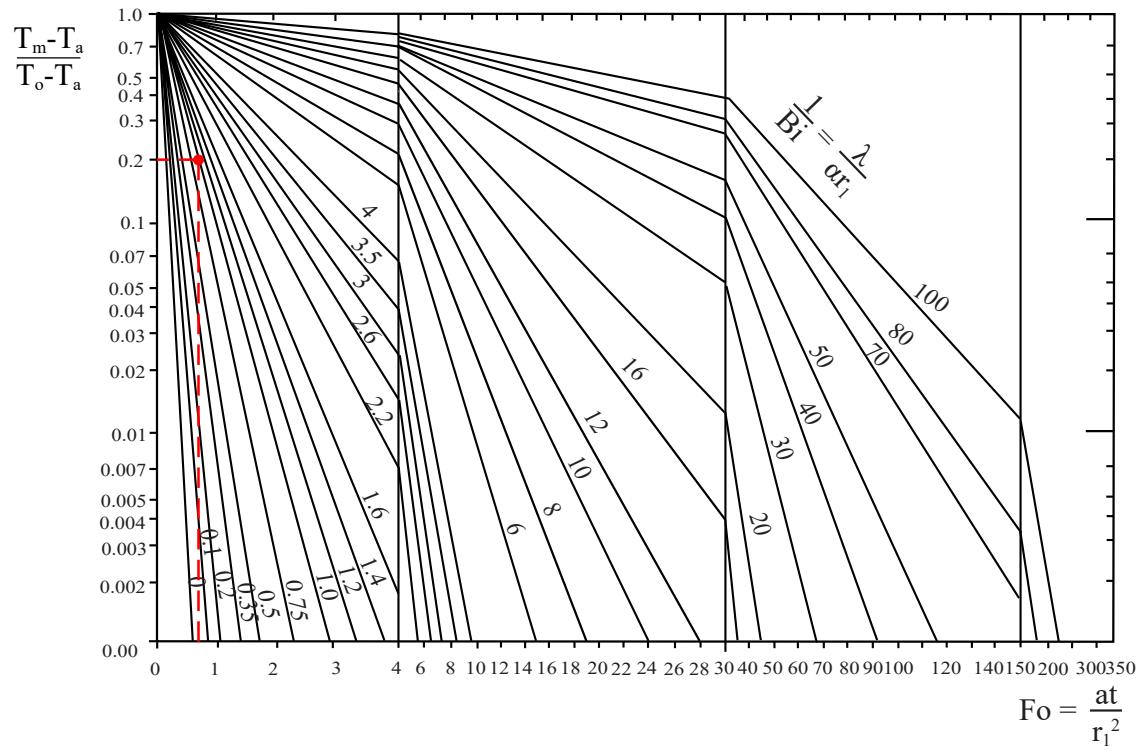
$$\frac{1}{Bi} \approx 1 \text{ [-]} \quad (\text{II.16.17})$$

Now the inverse of the Biot number is known, the chart describing the center temperature of the sphere over the course of time can be used again. Now to determine the Fourier number at the specific time instant.

To do so, besides the Biot number, the dimensionless temperature difference is needed as well. In this case, the ambient temperature T_a is equal to the quenching temperature T_q .

$$\begin{aligned}\frac{T_m - T_a}{T_0 - T_a} &= \frac{T_m - T_q}{T_0 - T_q} \\ &= \frac{(54 - 30) [\text{°C}]}{(150 - 30) [\text{°C}]} = 0.2 [-]\end{aligned}\quad (\text{II.16.18})$$

Reading the Heisler diagram as used in the previous task again yields the Fourier number at this specific time instant t_2 .



Reading the chart leads to:

$$Fo \approx 0.7 \quad (\text{II.16.19})$$

Lastly, rewriting the definition of the Fourier number leads to:

$$\begin{aligned}Fo &= \frac{at}{r_1^2} = \frac{\lambda t}{\rho c_p r_1^2} \\ \Rightarrow t &= \frac{Fo \rho c_p r_1^2}{\lambda} \\ &= \frac{0.7 [-] \cdot 1450 [\text{kg/m}^3] \cdot 880 [\text{J/kgK}] \cdot 0.015^2 [\text{m}^2]}{1.52 [\text{W/mK}]} = 132 [\text{s}]\end{aligned}\quad (\text{II.16.20})$$

Conclusion

After 2.2 minutes the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

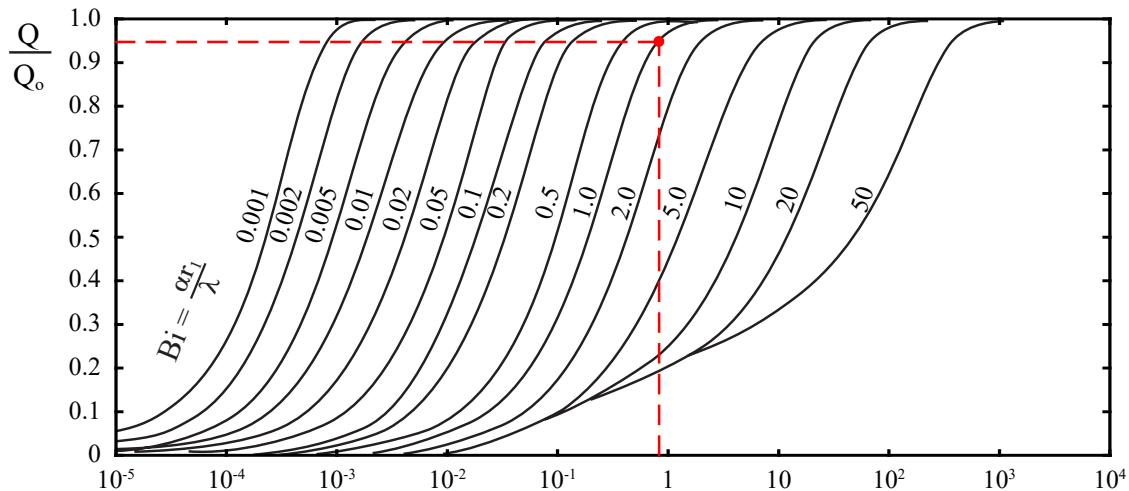
Task d)

To determine the amount of heat dissipated after 2.2 minutes, the chart below can be used. To do so, first, the product of the Biot number squared and the Fourier number needs to be determined.

This yields from:

$$\text{Bi}^2 \cdot \text{Fo} = 1^2 [-] \cdot 0.7 [-] = 0.7 [-] \quad (\text{II.16.21})$$

Reading the Heisler diagram describing the heat loss by a sphere:



$$\text{Bi}^2 \text{Fo} = \frac{\alpha^2 t}{\rho c \lambda}$$

Which yields:

$$\frac{Q}{Q_0} \approx 0.95 \quad (\text{II.16.22})$$

Rewriting and substituting the definition of the initial heat within the sphere yields:

$$\begin{aligned} Q &= 0.95Q_0 \\ &= 0.95\rho \frac{4\pi r_1^2}{3} c_p \cdot (T_0 - T_q) \\ &= 0.95 \cdot 1.45 \cdot 10^3 [\text{kg/m}^3] \cdot \frac{4\pi \cdot 0.015^2 [\text{m}^2]}{3} \cdot 880 [\text{J/kgK}] \cdot (150 - 30) [\text{K}] = 137 [\text{kJ}] \end{aligned} \quad (\text{II.16.23})$$

Conclusion

Thus 137 kJ of heat has been dissipated after 2.2 minutes by the quenching process.