Chapter 10

Energy Conservation

10.1 Heat flux

The heat flux $\boldsymbol{q}$ is defined as the transport rate of thermal energy per unit area, which according to Fourier's[(1)] law is proportial to the negative gradient of the temperature (heat flows from warmer areas to cooler areas):

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T, \quad \text{or equivalently} \quad q_i = -k\frac{\partial T}{\partial x_i}, \tag{10.1}$$

where k is the heat conduction coefficient which is assumed constant in these notes.

10.2 Energy conservation: integral formulation

The first law of thermodynamics states that the energy change of a blob of material equals the heat added to the blob plus the work done on the blob. The energy of a convecting blob of fluid (viz. Fig. (2.1)) is simply equal to the integral of the total energy per unit volume over the blob:

$$Energy(t) \equiv \int_{V(t)} \rho(\boldsymbol{x},t)E(\boldsymbol{x},t)\,dV. \tag{10.2}$$

The amount of heat added to the blob per unit time can be written as

$$\int_{S(t)} \boldsymbol{q}\cdot(-\boldsymbol{n})dS = -\int_{S(t)} \boldsymbol{q}\cdot\boldsymbol{n}dS = -\int_{S(t)} q_j n_j dS, \tag{10.3}$$

where we took $(-\boldsymbol{n})$ instead of $\boldsymbol{n}$ since we are looking at the heat flux towards $V(t)$. The amount of work done on the blob per unit time can be written as

$$\int_{S(t)} t_i u_i dS + \int_{V(t)} \rho g_j u_j dV, \tag{10.4}$$

(1) Jean Baptiste Joseph Fourier (1768 - 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's Law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.

where the first integral denotes the work rate done by the surrounding fluid (stress), and the second integral denotes the work rate done by gravity. It is noted that the summation convention is applied everywhere, and that in the first integral of Eq.(10.4) on should sum over both i and j.

The first law of thermodynamics for the blob now becomes

$$\frac{d}{dt}\int_{V(t)} \rho E\, dV = -\int_{S(t)} q_j n_j dS + \int_{S(t)} (\sigma_{ij} n_j) u_i dS + \int_{V(t)} \rho g_j u_j dV. \tag{10.5}$$

With the Reynolds transport theorem Eq.(2.16) we can finally rewrite the term at the left-hand side and write all surface integral as a single integral:

$$\boxed{\int_{V(t)} \frac{\partial}{\partial t}(\rho E)\, dV + \int_{S(t)} \left(\rho E u_j - \sigma_{ij} u_i - k\frac{\partial T}{\partial x_j}\right) n_j dS = \int_{V(t)} \rho g_j u_j dV.} \tag{10.6}$$

In summary:

(a) the first term expresses the energy rate of change due to the total energy rate of change,
(b) the second term expresses the energy rate of change due convection (bulk motion)
(c) the third term expresses the energy rate of change due work done by stress
(d) the fourth term expresses the energy rate of change due conduction (molecules motion), and
(e) the fifth term expresses the energy rate of change due work done by gravity.

10.3 Example 1: slowly moving piston

Consider the moving piston and cylinder depicted in Fig. (10.1). The piston with area A

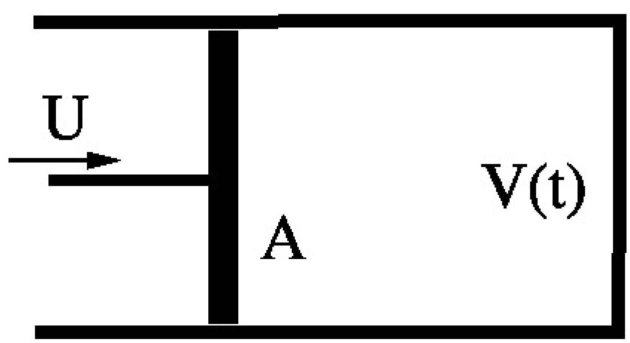

Figure 10.1: Slowly moving piston in a cylinder

is slowly moving to the right with velocity U, thereby decreasing the enclosed volume V. Suppose that V contains a perfect gas, then this gas is compressed slowly. The question is: how does the temperature change due to the movement of the piston?

To provide the answer we use the energy conservation equation. Since the piston movement is slow we assume that the flow field in V is uniform and that velocities are very small.

This means that the velocity and temperature gradients are small such that we can neglect energy losses due to friction and heat conduction. Furthermore, the total energy E as defined by Eq.(**??**), can be approximated by the specific thermal energy $e(T)$, since the quadratic velocity term is very small. Furthermore, the specific thermal energy for a perfect gas is simply $C_v T$, see Eq.(**??**), which leads to

$$E \approx C_v T. \tag{10.7}$$

Starting with the first term of the integral equation for energy conservation, Eq.(10.6), we approximate

$$\int_{V(t)} \frac{\partial \rho E}{\partial t} \, dV \approx V \frac{d}{dt} (\rho E) = V \frac{d}{dt} (\rho C_v T) \, . \tag{10.8}$$

The second term of Eq.(10.6) becomes

$$\int_{S(t)} \rho E u_j n_j \, dS \approx -U \rho E A = -U A \rho C_v T = \frac{dV}{dt} \rho C_v T. \tag{10.9}$$

The third term of Eq.(10.6) becomes

$$-\int_{S(t)} \sigma_{ij} n_j u_i \, dS \approx -pUA = \rho R T \frac{dV}{dt}. \tag{10.10}$$

The fourth term of Eq.(10.6) becomes

$$-\int_{A(t)} k \frac{\partial T}{\partial x_i} n_i \, dS \approx 0. \tag{10.11}$$

The fifth term of Eq.(10.6) becomes

$$\int_{V(t)} \rho g_i u_i \, dV \approx 0. \tag{10.12}$$

With these expressions and the observation that ρV is constant (mass conservation) we find after some algebra

$$\frac{1}{T} \frac{dT}{dt} + (\gamma - 1) \frac{1}{V} \frac{dV}{dt} = 0, \qquad \gamma - 1 = R / C_v. \tag{10.13}$$

This leads to

$$\frac{d}{dt} \ln \left(T V^{\gamma - 1} \right) = 0, \tag{10.14}$$

or

$$\boxed{T V^{\gamma - 1} = \text{constant}}. \tag{10.15}$$

Since $\gamma > 1$ this equation shows that T increases when V decreases. In other words, when the piston compresses the perfect gas, the temperature will increase accordingly.

By using ρV = constant and the perfect gas law Eq.(**??**) we also find three equivalent forms:

$$\boxed{T\rho^{-(\gamma-1)} = \text{constant}, \quad Tp^{-(\gamma-1)/\gamma} = \text{constant}, \quad p\rho^{-\gamma} = \text{constant}}. \tag{10.16}$$

All of these relations express that the flow is isentropic. The last expression, which relates p and ρ, is referred to as Poisson's relation.

Finally, one could raise the question how much work one has to do on the piston to compress the gas. To answer this question we need the work done on V per unit time which is just Eq.(10.20) without the minus sign. The total amount of work W is found by integrating this term over time:

$$W = \int_0^t \left\{ \int_{A(t)} \sigma_{ij} n_j u_i \, dA \right\} dt \approx - \int_0^t \rho RT \frac{dV}{dt} \, dt. \tag{10.17}$$

Since $\rho V = \rho_o V_o$ and $TV^{\gamma-1} = T_o V_o^{\gamma-1}$ and $p_o = \rho_o R T_o$, where the subscript 0 indicates the situation at $t = 0$, we obtain

$$W = - \int_0^t \rho RT \frac{dV}{dt} dt = -p_o V_o^{\gamma} \int_0^t V^{-\gamma} \frac{dV}{dt} dt, \tag{10.18}$$

which leads to

$$\boxed{W = \frac{p_o V_o}{\gamma - 1} \left\{ \left(\frac{V_o}{V(t)} \right)^{\gamma-1} - 1 \right\}.} \tag{10.19}$$

10.4 Example 2: compressor or turbine

Consider the periodic flow through the compressor depicted in Fig. (11.1). The flow is

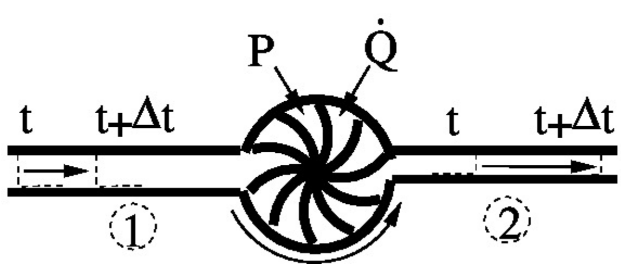

Figure 10.2: Periodic flow through a compressor.

periodic in the sense that when the rotor rotates over a small angle such that the blades shift one position, the flow field is identical to the flow field corresponding to the previous rotor position. We also assume that the flow is uniform in the inlet and outlet pipes.

The main question is: what is the relation between the work and heat input on the one hand, and the mass flow rate and pressure or temperature change on the other hand?

Figure 10.3: Various types of wind turbines.

Starting with the first two terms of the integral equation for energy conservation, Eq.(10.6), we combine them again to $\frac{d}{dt}\int_{V(t)} \rho E dV$ and approximate:

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \rho E dV &\approx \frac{1}{\Delta t}\left\{\int_{V(t+\Delta t)} \rho E\, dV - \int_{V(t)} \rho E\, dV\right\} \\ &= \frac{1}{\Delta t}\left\{(\rho EAU\Delta t)_2 - (\rho EAU\Delta t)_1\right\} \\ &= (\rho EAU)_2 - (\rho EAU)_1 \,. \end{aligned} \tag{10.20}$$

Due to the quasi-steadiness of the flow we have from mass conservation

$$(\rho AU)_1 = (\rho AU)_2 = \dot{m}, \tag{10.21}$$

where $\dot{m}$ denotes the mass flow rate. Therefore Eq.(10.20) becomes

$$\frac{d}{dt}\int_{V(t)} \rho E dV = \dot{m}\,(E_2 - E_1)\,. \tag{10.22}$$

The third term of Eq.(10.6) can be approximated when we neglect viscous stresses at the inflow and outflow boundaries:

$$\int_{S(t)} \sigma_{ij} n_j u_i\, dS \approx \int_{A_1+A_2} -p n_i u_i\, dS + \int_{A_{blades}} \sigma_{ij} n_j u_i\, dS. \tag{10.23}$$

The first integral becomes

$$\int_{A_1+A_2} -p n_i u_i\, dS = (pUA)_1 - (pUA)_2 = \dot{m}\left\{\left(\frac{p}{\rho}\right)_1 - \left(\frac{p}{\rho}\right)_2\right\}, \tag{10.24}$$

whereas the second integral is just the rate of work done <u>by</u> the blades <u>on</u> the fluid:

$$\int_{A_{blades}} \sigma_{ij} n_j u_i\, dS = P. \tag{10.25}$$

Finally, the fourth term of Eq.(10.6) is just the rate of heat supplied to the control volume:

$$-\int_{S(t)} k\frac{\partial T}{\partial x_i} n_i \, dS = \dot{Q}. \tag{10.26}$$

As a result, the energy equation becomes:

$$\dot{m}\left(E_2 - E_1\right) = \dot{m}\left\{\left(\frac{p}{\rho}\right)_1 - \left(\frac{p}{\rho}\right)_2\right\} + P + \dot{Q}. \tag{10.27}$$

We can finally rewrite this equation in a more compact way by using the definition of total enthalpy H:

$$\boxed{\dot{m}\left(H_2 - H_1\right) = P + \dot{Q}.} \tag{10.28}$$

10.5 Exercises

Problem 10.1. *Using $TV^{\gamma-1} = const$, $p = \rho RT$, and mass conservation, derive that:*

(a) $T\rho^{1-\gamma} = const$
(b) $Tp^{-\frac{\gamma-1}{\gamma}} = const$
(c) $p\rho^{-\gamma} = const$

Problem 10.2. *Starting from*

$$W = -\int_0^t p\frac{dV}{dt}dt,$$

$TV^{\gamma-1} = const$, and mass conservation, derive that

(a) $W = -p_o V_o^\gamma \int_0^t V^{-\gamma}\frac{dV}{dt}dt$,
(b) $W = \frac{p_o V_o}{\gamma-1}\left\{\left(\frac{V_o}{V(t)}\right)^{\gamma-1} - 1\right\}$,
(c) $sign(W) = sign(V_o - V(t))$.

Problem 10.3. *Starting with $p\rho^{-\gamma} = const$, derive that*

$$\frac{1}{p}\frac{dp}{dt} - \gamma\frac{1}{\rho}\frac{d\rho}{dt}.$$

Problem 10.4. *Air enters a compressor at speed U_1, temperature T_1, and leaves at U_2, T_2, and the mass flow is $\dot{m}$. The removed heat per unit mass of passing air is $\hat{e}$. Derive an expression for the power required by the compressor, assuming air can be modeled as a perfect gas.*