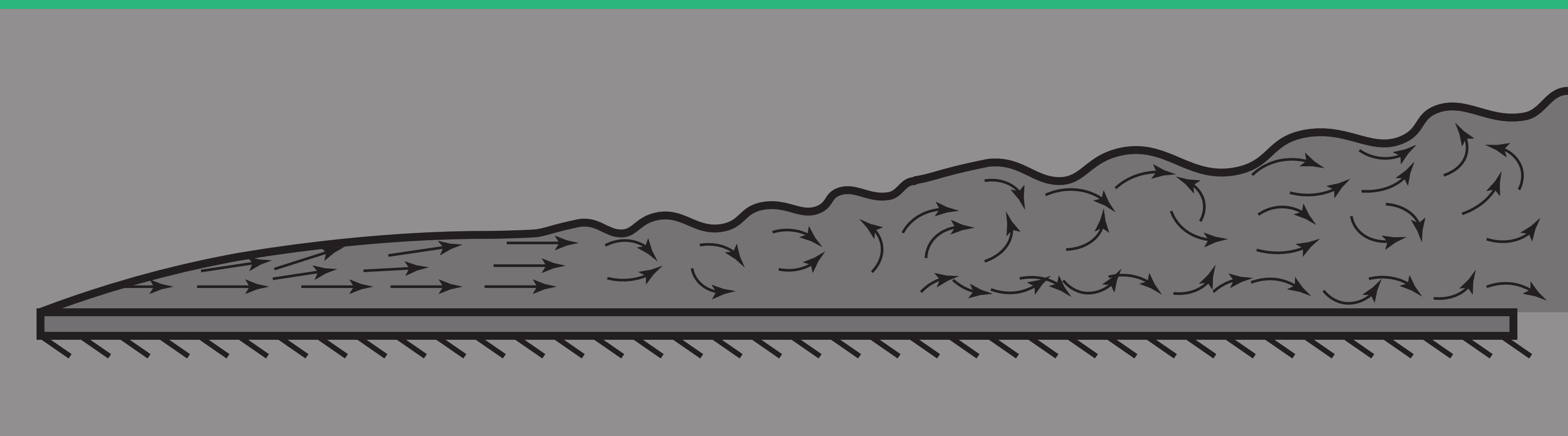
HEATQUIZ

HEAT TRANSFER

Course reader - Convection

W. Rohlfs[1],
R. Kneer[2],
D.J.G. Kuiphuis[1]

[1] Department of Thermal and Fluid Engineering, University Twente, the Netherlands
[2] Institute for Heat and Mass Transfer, RWTH Aachen University, Germany



UNIVERSITY OF TWENTE. | RWTH AACHEN UNIVERSITY

Contents

Check list

Part III - Convection

☐ Section 10	☐ Lecture L01	☐ Quiz L01	☐ Exercise III.1 ☐ Exercise III.2 ☐ Exercise III.3 ☐ Exercise III.4
☐ Section 11	☐ Lecture L02 ☐ Lecture L03	☐ Quiz L02 ☐ Quiz L03	☐ Exercise III.5
☐ Section 12	☐ Lecture L04 ☐ Lecture L05	☐ Quiz L04 ☐ Quiz L05	
☐ Section 13	☐ Lecture L06	☐ Quiz L06	☐ Exercise III.6 ☐ Exercise III.7 ☐ Exercise III.8
☐ Section 14	☐ Lecture L07 ☐ Lecture L08 ☐ Lecture L09	☐ Quiz L07 ☐ Quiz L08 ☐ Quiz L09	☐ Exercise III.9 ☐ Exercise III.10 ☐ Exercise III.11 ☐ Exercise III.12
☐ Section 15	☐ Lecture L10	☐ Quiz L10	☐ Exercise III.13
☐ Section 16	☐ Lecture L11	☐ Quiz L11	☐ Exercise III.14

Nomenclature

Symbol:	Physical constant:	Numerical value:	Unit:
c	Speed of light	299,792,458	[m/s]
g	Gravitational constant	9.81	$\left[\mathrm{m/s^2}\right]$
h	Planck's constant	$6.6261 \cdot 10^{-34}$	[Js]
σ	Stefan–Boltzmann constant	$5.6703 \cdot 10^{8}$	$\left[\mathrm{W/m^2K^4}\right]$

Symbol:	Description:	Unit:
A	Area	$\left[\mathrm{m^2}\right]$
C	Constant	[–]
E	Energy of a photon	[J]
F	Force or radiation fraction	[N] or [–]
H	Enthalpy	[J]
I	Momentum	[Ns]
L	(Characteristic) length length or radiosity	[m] or $\left[\mathrm{W/m^2}\right]$
L_{th}	Thermal entrance length	[m]
Q	Heat	[J]
R	Thermal resistance	[W/K]
T	Temperature	[K]
U	Internal energy or perimeter	[J] or [m]
V	Volume	[J]
W	Work	[J]
a	Thermal diffusivity	$\left[\mathrm{m^2/s}\right]$
c	Specific heat capacity or constant	[J/kgK] or [–]
d	Diameter	[m]
dx	Infinitesimal distance in x-direction	[m]
dy	Infinitesimal distance in y-direction	[m]
dz	Infinitesimal distance in z-direction	[m]
dr	Infinitesimal distance in r-direction	[m]
f_{e}	Arrangement factor	[–]
h	Mass-specific enthalpy	[J/kg]
k	Overall heat transfer coefficient	$\left[\mathrm{W/m^2K}\right]$
m	Mass or fin parameter	[kg] or [1/m]
n	Total number of layers	[–]
p	Pressure	$\left[\mathrm{N/m^2}\right]$
$\dot{q}''$	Heat flux density	$\left[\mathrm{W/m^2}\right]$
r	Radius	[m]
t	Time	[s]
u	Velocity (in x-direction)	[m/s]
v	Velocity (in y-direction)	[m/s]
w	Velocity (in z-direction)	[m/s]
x	Spatial coordinate	[m]
y	Spatial coordinate	[m]
z	Spatial coordinate	[m]
Δ	Difference	[–]
Φ	View factor	[–]
$\dot{\Phi}$	Heat source	[W]
Ω	Solid angle	[Str]
α	Convective heat transfer coefficient or absorptivity	$\left[\mathrm{W/m^2K}\right]$ or [–]
β	Volumetric expansion coefficient	[1/K]
δ	Wall thickness or penetration depth	[m]
ϵ	Emissivity	[–]
δ_{T}	Thermal boundary layer thickness	[m]
δ_{u}	Velocity boundary layer thickness	[m]
η	Efficiency or wavenumber	[–] or $\left[\frac{1}{\mathrm{m}}\right]$
θ	Dimensionless spatial temperature	[–]
θ^*	Dimensionless temporal temperature	[–]
λ	Thermal conductivity or wavelength	[W/mK] or [m]
μ	Dynamic viscosity	[kg/ms]
ν	Kinematic viscosity or frequency of radiation	$\left[\mathrm{m^2/s}\right]$ or $\left[\frac{1}{\mathrm{s}}\right]$
ϕ	Viewing angle	[rad]
ρ	Density or reflectivity	$\left[\mathrm{kg/m^3}\right]$ or [–]
τ	Shear stress or transmissivity	$\left[\mathrm{N/m^2}\right]$ or [–]

Superscript:	Description:
x^*	Dimensionless
x'	Distance-related or variation
x''	Area-related
x'''	Volume-related
$\dot{x}$	Time derivated
$\overline{x}$	Average
$\vec{x}$	Vector

Subscript:	Description:
x_A	Ambient A
x_a	Ambient
x_B	Ambient B
x_b	Black body
x_c	Cross-section
x_crit	Critical
x_cond	Conduction
x_conv	Convection
x_d	Hydraulic diameter as characteristic length
x_eff	Effective
x_F	Fin
x_f	Fluid
x_fl	Fluid
x_h	Hydraulic
x_in	Inlet
x_kin	Kinetic
x_L	Length as characteristic length
x_{ij}	From i to j
$x_{i\rightarrow j}$	From i to j
$x_{i\rightleftarrows j}$	Net between i and j
x_m	Mean
x_max	Maximum
x_min	Minimum
x_out	Outlet
x_p	At constant pressure
x_prop	Property
x_rad	Radiation
x_s	Solid or distance as characteristic length or surface
x_t	Turbulent
x_th	Thermal
x_v	At constant volume
x_w	Wall
x_x	Local
x_0	Incident or initial
x_1	Reference 1
x_2	Reference 2
x_3	Reference 3
x_4	Reference 4
x_α	Absorbed
x_ϵ	Emitted
x_η	Wavenumber-specific
x_λ	Wavelength-specific
x_ρ	Reflected
x_τ	Transmitted
x_∞	Upstream

Symbol:	Dimensionless number:	
Ar	Archimedes number	$\equiv \frac{\text{Bouyance forces}}{\text{Inertia forces}}$
Bi	Biot number	$\equiv \frac{\text{Thermal resistance in body}}{\text{Convective thermal resistance at surface}}$
Fo	Fourier number	$\equiv \frac{\text{Rate of diffusivity}}{\text{Rate of storage}}$
Gr	Grashof number	$\equiv \frac{\text{Bouyancy forces}}{\text{Viscous forces}}$
Nu	Nusselt number	$\equiv$ Dimensionless heat transfer coefficient
Pe	Peclet number	$\equiv \frac{\text{Rate of advection}}{\text{Rate of diffusion}}$
Pr	Prandtl number	$\equiv \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}}$
Re	Reynolds number	$\equiv \frac{\text{Inertia forces}}{\text{Viscous forces}}$
c_f	Friction coefficient	$\equiv \frac{\text{frictional head loss}}{\text{dynamic pressure}}$

Chapter elements

Here an overview is presented clarifying the individual components that form the structural organization of this text. Readers will encounter specific sections.

Phenomena

The part on **phenomena** illustrates a principle of heat transfer through tangible, real-world examples that can be readily observed. Consider scenarios such as the cooling of a cup or the sensation of warmth from the sun.

Fundamental EQ

The **fundamental EQ** part presents equations derived or experimentally determined, providing a convenient means to calculate specific parameters within the framework of heat transfer theory. Consider well-known examples such as Fourier's law or Newton's law of cooling.

Definition

The **definition** part introduces the meanings of parameters utilized in the theory of heat transfer. Common definitions within this framework include terms such as rate of heat, internal heat source, and dimensionless numbers.

Derivation

The **derivation** part is dedicated to obtaining a particular theorem or defining a parameter. Common derivations discussed include those related to conservative equations within solids and fluids. □

Approach

The part delving into the **approach** presents a systematic framework for addressing a particular problem, outlining steps such as establishing and solving an energy balance equation, for instance.

Example

The **example** part furnishes a relatively straightforward illustration demonstrating the practical application of the recently derived theory. Consider, for instance, the derivation of the temperature profile within a plane wall.

HeatQuiz

The **HeatQuiz** part offers game-based learning tasks designed to assess the comprehension of the previously discussed theory and ascertain whether sufficient knowledge of the content has been acquired to apply the theory to practical examples.

Demonstration

The **demonstration** part presents a task previously evaluated in past course exams, accompanied by a QR code offering a video solution. Therefore, it is strongly advised to attempt the task independently before consulting the video solution for the best learning experience.

HTC

The **HTC** part presents a Nusselt correlation tailored for a specific application. These correlations, determined through numerical or experimental means, serve as tools for calculating the heat transfer coefficient in particular scenarios.

Criterion

The **criterion** part introduces a set of conditions that must be satisfied for a theory to be deemed applicable.

Exercise ($\star$):

> Exercises marked with a single star $\star$ serve as foundational exercises to reinforce your fundamental understanding. While not yet at the exam level, they function as crucial stepping stones, helping you build confidence and proficiency before tackling more advanced challenges.

Exercise ($\star\star$):

> For exercises denoted with two stars $\star\star$, you can expect tasks slightly below the anticipated difficulty level of the exam or smaller assignments with a few exam points at stake.

Exercise ($\star\star\star$):

> The exercises adorned with three stars $\star\star\star$ are those that have previously appeared on exams. These tasks, often carrying significant point values, reflect the kinds of challenges that demand a higher level of mastery and are critical for thorough exam preparation.

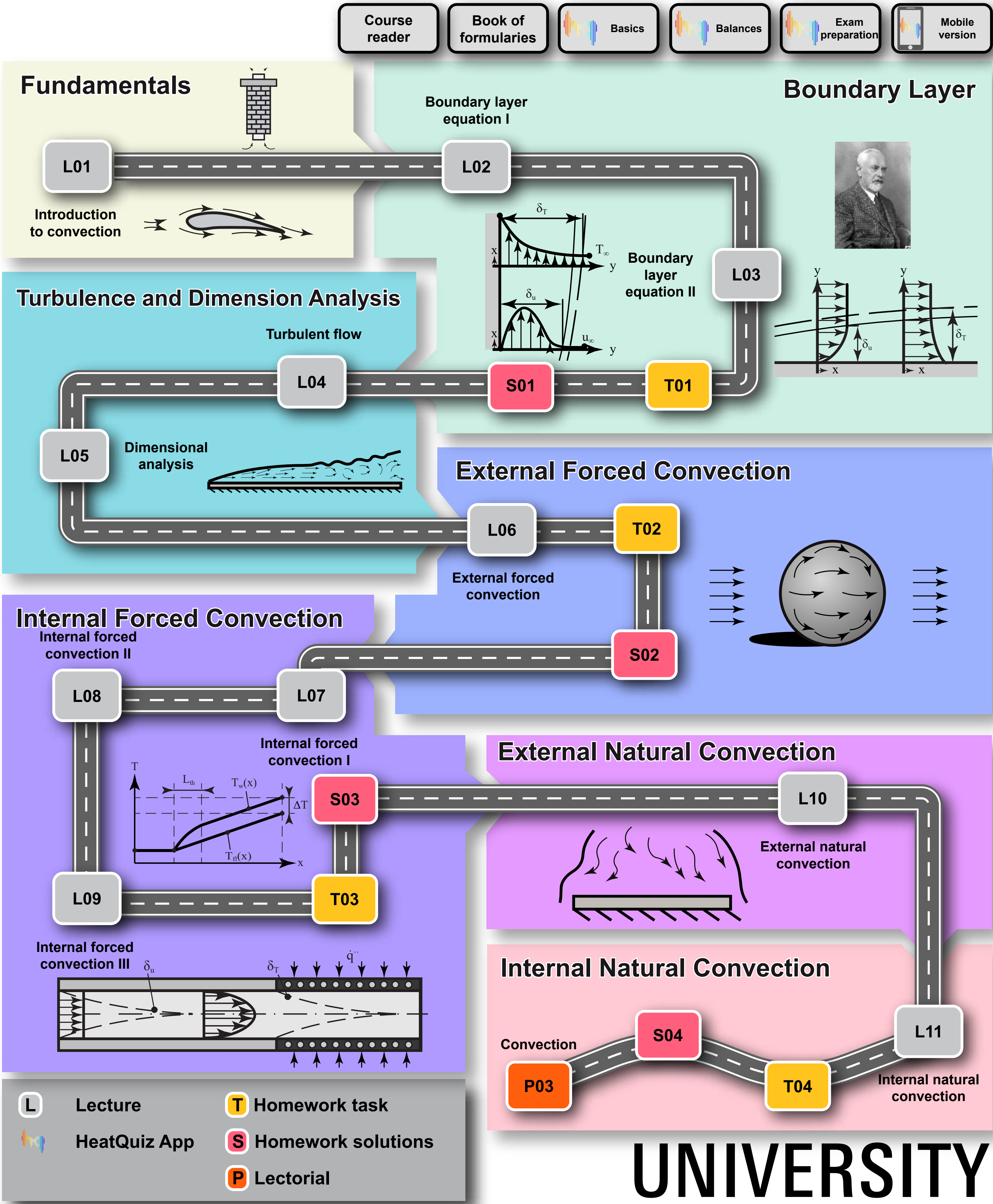
PART

III

Convection

Heat transfer: Convection
Learning path

HEATQUIZ



Prof. Dr. Dr. Wilko Rohlfs | University of Twente | w.rohlfs@utwente.nl

UNIVERSITY OF TWENTE.

Section 10

Fundamentals

L01 - Introduction to convection:

Learning goals:

- Mastering the fundamental principles of convection, with the ability to discern and articulate the distinctions between convection and advection in the context of heat and fluid flow.
- Categorizing convection problems based on their characteristics.
- Deriving the conservation equations for mass, momentum, and energy in convection.
- Understanding the parallels between momentum and energy transport in convection.

Comprehension questions:

- What is the purpose of the heat transfer coefficient and what does the coefficient describe?
- Why does Fourier's law of heat conduction also apply on the fluid side near the wall?
- What does the dimensionless Nusselt number mean?
- What is the difference between natural and forced convection?

Corresponding tutorial exercises:

- Exercise III.1 Walking man
- Exercise III.2 Thick solid plate
- Exercise III.3 Fluid flow
- Exercise III.4 Porous wall

SUBSECTION 10.1
Introduction to convection

The thermal energy transport by heat conduction was covered in the previous part. These processes take place on the molecular level and can physically be described by the thermal conductivity (material property) and Fourier's law. Conduction is driven by spatial temperature differences and occurs in any system that is out of thermal equilibrium, irrespective of the physical state (solid, liquid, or gas).

This part focuses on the principle of convection, the transport of heat by means of a fluid flow. Thereby, the fluid can be a liquid (water), a gas (air), or even granular material. The major conceptual difference between the transport mechanisms of conduction and convection is demonstrated best by a simple theoretical experiment as illustrated in figure 10.1. Two parallel horizontal plates of different temperatures are in heat exchange with each other. The enclosure is filled with gas or liquid.

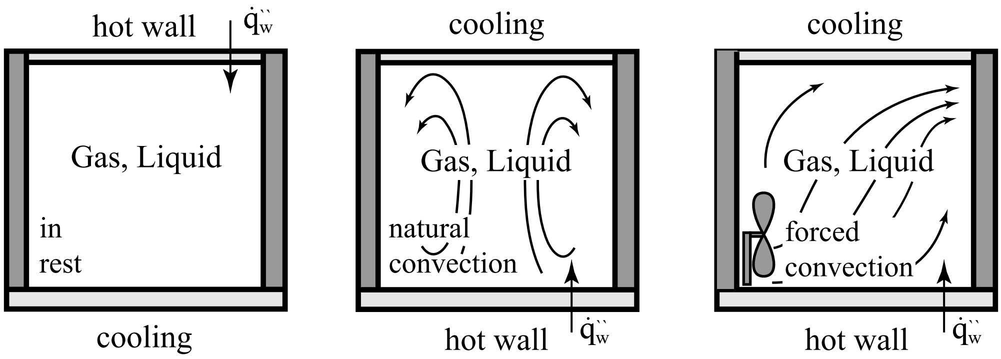


Figure 10.1. Cooling of a hot surface by conduction and convection.

In the case that the hot wall is on the top (left), the fluid remains stagnant as the density of the fluid generally decreases with temperature, think of the example of a hot air balloon. If the warmer fluid is situated above the colder fluid, a stable layer is established. This is for instance the desired configuration in warm water heat storage tanks. Heat is transported in this configuration solely by conduction from the top to the bottom.

In case the hot wall is placed on the lower side, warm fluid with a lower density on the bottom starts to rise, creating a natural fluid flow movement in the enclosure. With the fluid movement, thermal energy is transported in an accelerated way from the bottom to the top, increasing the rate of heat transfer. The process by which fluid carries thermal energy with its heat capacity from one location to another is called **advection**. Note, that due to the no-slip boundary condition at the wall and thus the absence of any motion in the wall, heat is transported close to the wall solely by conduction. Moving further away from the wall results in increased fluid motion, making heat transport by advection more dominant. However, conduction is present at any location in the enclosure, independent of fluid motion, and solely dependent on spatial temperature differences. The superposition of both mechanisms, namely conduction and advection is called convection.

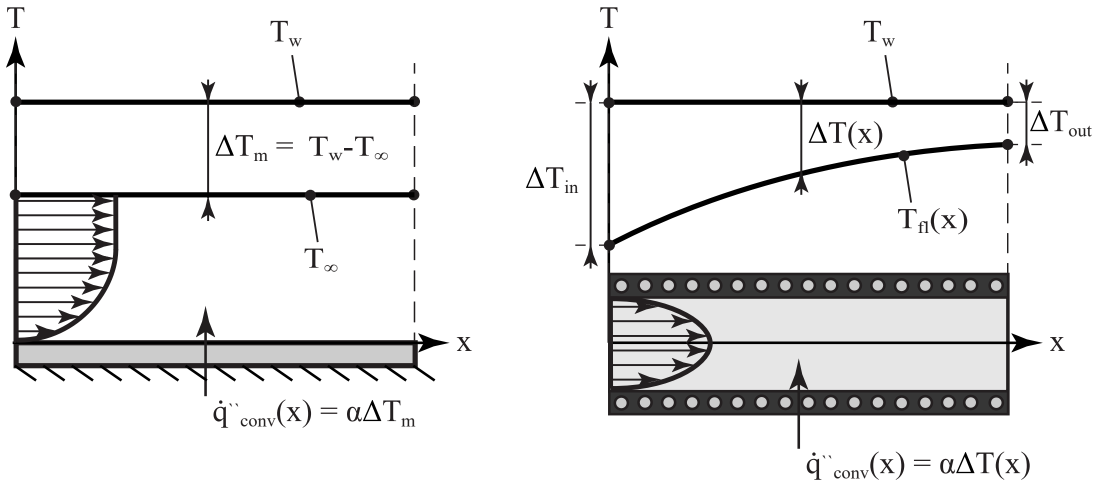
Comparing the flow configurations in the central and right sections reveals a distinct factor: the presence of an additional fan for the right case. In the central region, fluid motion occurs naturally, propelled solely by density variations. This fluid movement characterizes "natural convection". Conversely, on the right side, the fluid motion is externally induced and forced by the added fan, marking this process as "forced convection". The distinction lies in the inherent, spontaneous nature of the central flow versus the externally driven nature of the flow on the right.

10.1.1 Classification

In daily life and also in technical applications, there exist an infinite number of different configurations of convective heat transfer cases. This includes the geometry, the flow, the involved fluids, the temperatures, and so on. For engineers, the characterization and classification of the processes to adequately choose the right measures for predicting the heat transfer rate is important.

A first classification has been introduced in the previous paragraph and concerns the distinction between natural and forced convection. Another important distinction is between internal and external convection. A classical example of internal convection is the fluid flow through a heated pipe. Beginning from the entrance of the pipe, the temperature increases, first in the fluid layers close to the wall and subsequently in the entire pipe. As such, not only the mean temperature of the pipe increase but also the temperature on the centerline.

The classification between internal and external flows is important and dictates the choice of the reference temperatures. Using Figure 10.2 as an illustration, when dealing with the fully developed laminar flow with a constant wall temperature boundary condition, the method for calculating the rate of heat transfer varies. In external convection, the reference temperatures used are the constant wall temperature T_{w} and the constant fluid upstream temperature T_∞. On the other hand, when dealing with internal convection, there is no constant fluid upstream temperature because the fluid temperature $T_{\mathrm{fl}}(x)$ increases along the flow direction, and thus the temperature difference between the fluid and the wall changes as well. Instead, a mean temperature difference, called the logarithmic temperature difference, between the wall temperature $T_{\mathrm{w}}(x)$ and the fluid temperature $T_{\mathrm{fl}}(x)$ is used as a reference. The topic of the logarithmic temperature difference is introduced in Section 14.2.



(a) External convection. (b) Internal convection.

Figure 10.2. Heat transfer with a constant wall temperature for fully-developed flow.

Contrary, if the process of heat transfer from a plate is considered, the temperature increases near the wall but far away from the wall the temperature remains unaffected. As a consequence, in external convection the "far away" fluid temperature is used as a reference. Contrary, in internal convection the reference temperature describing the temperature difference between wall and fluid temperature varies within the flow direction.

A better physical understanding of this distinction will be possible after the introduction of the principle of boundary layers.

Within this course, the important distinction between the two classifications is made, as summarized in Figure 10.3.

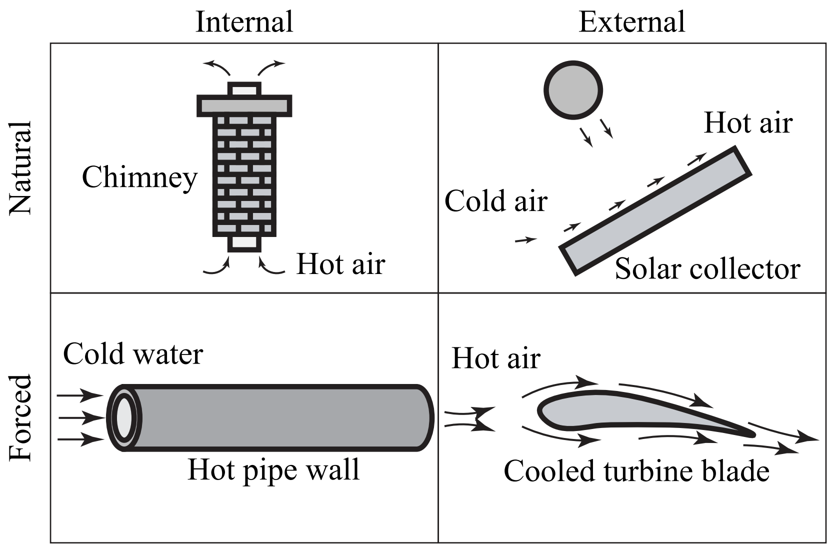


Figure 10.3. Classification of convection into internal/external convection and natural/forced convection.

SUBSECTION 10.2

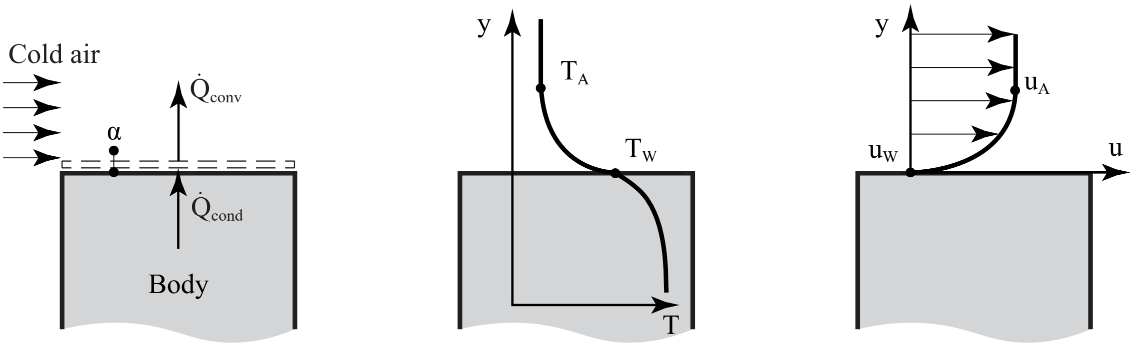
Convective heat transfer coefficient

Within the topic of conduction, it is thought that the heat flux scales linearly to the driving potential, e.g. the temperature difference. This physical description is for the transfer mechanism of conduction an excellent approximation that holds for a large number of materials and in a broad temperature band (except for cryogenic applications). For the case of convection, a first engineering approximation is again the description of the heat flux as a function of the driving potential, here the temperature difference between the wall and the fluid. The coefficient of proportionality connecting the driving temperature difference with the heat flux is the "convective heat transfer coefficient", which yields per unit area:

$$\frac{\dot{Q}_{\mathrm{conv}}}{A} = \dot{q}''_{\mathrm{conv}} = \alpha(T_{\mathrm{W}} - T_{\mathrm{A}})$$

Derivation

To deepen comprehension of the heat transfer coefficient (α), an energy balance at the interface is established. This is exemplified through the scenario of a hot plate subjected to the flow of cold air.



(a) Energy balance. (b) Temperature distribution. (c) Velocity distribution.

Figure 10.4. Heat transfer at a surface due to convection.

1 **Setting up the balance:**

First, the control volume is chosen in such a way that the lower boundary is positioned at the interface between the fluid and solid body, but within the fluid region. The upper boundary is located directly above it, making the thickness of the control volume infinitely small. Heat is conducted into the control volume from the bottom due to the no-slip boundary condition, while an equal amount of heat is convected out of the control volume through the upper boundary, which is the focus of the description. Assuming steady-state conditions, the balance reads:

$$0 = \dot{Q}_{\mathrm{cond}} - \dot{Q}_{\mathrm{conv}}$$

2 **Defining the elements within the balance:**

Due to the no-slip boundary condition and the stagnant fluid (absence of any motion), the heat transfer in the vicinity of the wall is only by conduction inside the fluid. Thus, Fourier's law applies to the fluid:

$$\dot{Q}_{\mathrm{cond}} = -A\left(\lambda_{\mathrm{f}}\frac{\partial T_{\mathrm{f}}}{\partial y}\right)_{\mathrm{W}}$$

Expressing the rate of heat transfer by convection in terms of Newton's law of cooling:

$$\dot{Q}_{\mathrm{conv}} = \alpha A\left(T_{\mathrm{W}} - T_{\mathrm{A}}\right)$$

3 **Inserting and rearranging:**

Rearranging the equation, the heat transfer coefficient is directly related to the thermal conduction in the stagnant fluid at the wall:

$$\alpha = \frac{-\left(\lambda_\mathrm{f} \frac{\partial T_\mathrm{f}}{\partial y}\right)_\mathrm{W}}{T_\mathrm{W} - T_\mathrm{A}}$$

Because the measurement of the temperature gradient in the fluid at the wall is of technical difficulty, the lower boundary of the control volume can be moved downwards into the solid area. This allows the formulation of the heat flux by

$$\dot{Q}_\mathrm{cond} = -A\left(\lambda_\mathrm{s} \frac{\partial T_\mathrm{s}}{\partial y}\right)_\mathrm{W}$$

and as such gives the expression:

$$\alpha = \frac{-\left(\lambda_\mathrm{s} \frac{\partial T_\mathrm{s}}{\partial y}\right)_\mathrm{W}}{T_\mathrm{W} - T_\mathrm{A}}$$

The temperature gradient within the wall exhibits linearity, facilitating the measurement of this parameter using two thermocouples positioned at distinct depths within the material. □

Definition

Convective heat transfer coefficient:

$$\alpha = \frac{-\left(\lambda_\mathrm{f} \frac{\partial T_\mathrm{f}}{\partial y}\right)_\mathrm{W}}{T_\mathrm{W} - T_\mathrm{A}} = \frac{-\left(\lambda_\mathrm{s} \frac{\partial T_\mathrm{s}}{\partial y}\right)_\mathrm{W}}{T_\mathrm{W} - T_\mathrm{A}} \left[\frac{\mathrm{W}}{\mathrm{m^2K}}\right] \tag{10.1}$$

10.2.1 Nusselt number

In the field of physics and engineering, a preference for non-dimensional descriptions arises to facilitate the comparative analysis of various cases. To achieve this, a dimensionless heat transfer coefficient, commonly denoted as the Nusselt number (after Wilhelm Nusselt, 1882-1957), is introduced. This parameter serves as a fundamental tool for simplifying and standardizing the assessment of heat transfer across different scenarios.

Definition

Nusselt number:

$$\mathrm{Nu} = \text{Dimensionless heat transfer coefficient} = \frac{\alpha L}{\lambda_\mathrm{f}}\ [-], \tag{10.2}$$

where λ_f is the thermal conductivity of the fluid and L is the characteristic length.

Although the Nusselt number compares convection to conduction and the heat transfer with convection exceeds heat transfer by conduction, the value of the Nusselt number can take values below unity. The interpretation that the heat transfer rate is now lower compared to pure conduction does not hold. Rather the choice of the characteristic length decides the value.

Only for the simple case illustrated in Figure 10.1, where the characteristic length L is chosen to be the distance between the two plates, a physical interpretation of the Nusselt number value is applicable. In the stagnant case (left case), heat is purely transported by conduction, and the temperature gradient in the fluid becomes:

$$-\frac{\partial T}{\partial x} = \frac{T_\mathrm{l} - T_\mathrm{u}}{L}$$

Inserting this in the Nusselt number definition yields a value of unity. With the onset of convection (center case), the heat transfer rate increases and the Nusselt number takes larger values.

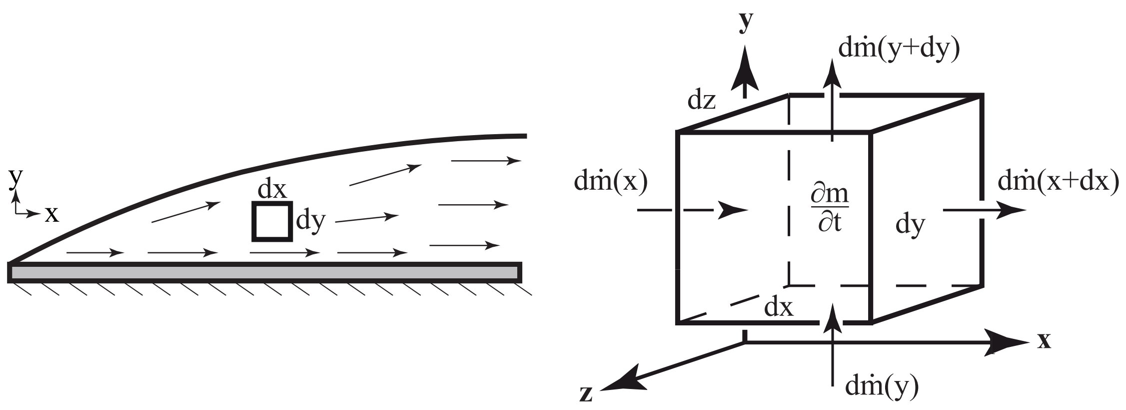
In engineering applications, the characteristic length is not necessarily in the direction of heat transfer but is often chosen for practical purposes. For a flow over a heated plate (see Section 13.2.1), the characteristic length is the size of the plate, which is perpendicular to the heat flux. A plate of very small size inevitably results in Nusselt number values below unity.

SUBSECTION 10.3

Conservation equations

The temperature field within a fluid domain follows from the solution of the energy conservation equation, also known as the 1st Law of thermodynamics. In this section, the energy balance for heat conduction, derived in the previous part, is expanded by including enthalpy flows, entering or leaving the control volume. An enthalpy flow is a flow of thermal energy carried by a fluid stream. To describe these enthalpy flows, the velocity field needs to be considered. Thus, in addition to the energy equation, the momentum equation and the continuity equation have to be formulated and solved jointly.

10.3.1 Equation of continuity



(a) Control volume located within the boundary layer.

(b) Mass balance at the control volume.

Figure 10.5. Flow over a flat plate.

Derivation

1 **Setting up the balance:**

The general mass balance around for a 2-dimensional problem with variations of the flow in x- and y-direction reads:

(Rate of change in mass) = (Rate of mass in) − (Rate of mass out) + (Rate of mass generation)

In the case of dealing with unsteady conditions and no mass generation, the mass balance yields:

$$\frac{\partial m}{\partial t} = d\dot{m}(x) - d\dot{m}(x+dx) + d\dot{m}(y) - d\dot{m}(y+dy)$$

2 **Defining the elements within the balance:**

The temporal change of mass within the control volume dV reads:

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} dx\, dy\, dz$$

Where the ingoing mass flows are defined as:

$$d\dot{m}(x) = \rho\, u\, dA_{\mathrm{x}} = \rho\, u\, dy\, dz$$

$$d\dot{m}(y) = \rho\, v\, dA_{\mathrm{y}} = \rho\, v\, dx\, dz$$

The outgoing mass flows is approximated using the Taylor-series expansion:

$$d\dot{m}(x+dx) = d\dot{m}(x) + \frac{\partial}{\partial x}\left(d\dot{m}(x)\right) \cdot dx$$

$$d\dot{m}(y+dy) = d\dot{m}(y) + \frac{\partial}{\partial y}\left(d\dot{m}(y)\right) \cdot dy$$

3 **Inserting and rearranging:**

Inserting all defined terms yields the general unsteady equation of continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

If the flow is to be steady-state, canceling the transient term yields the **steady-state equation of continuity**:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

For incompressible fluids, with ρ = constant, this equation simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

□

The equation of continuity is written in a more general form as stated in equation (10.3), where the density cancels out for incompressible fluids, as stated in equation (10.4)

Fundamental EQ

Equation of continuity:

$$\nabla \cdot \left(\rho\, \vec{u}\right) = 0 \tag{10.3}$$

Fundamental EQ

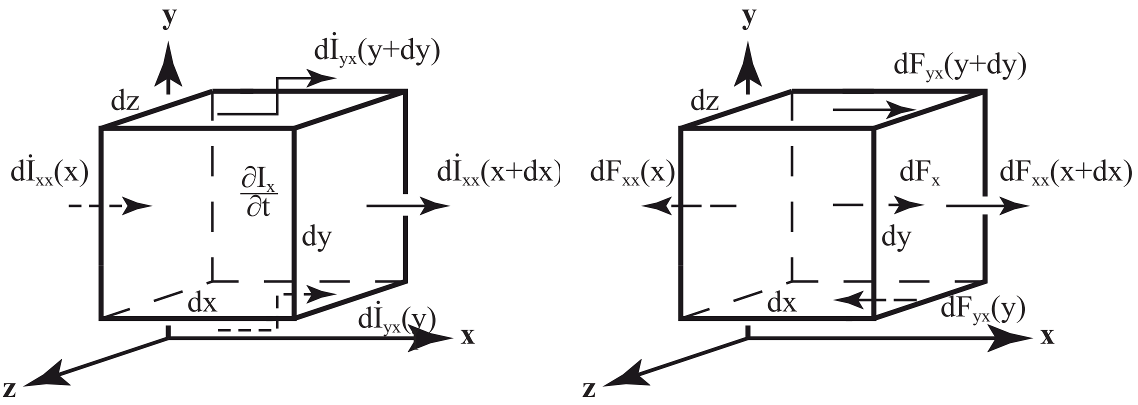
Equation of continuity for incompressible fluids:

$$\nabla \cdot \vec{u} = 0 \tag{10.4}$$

10.3.2 Equations of momentum

Momentum in the view of classical mechanics is the product of an object's velocity and mass. According to Newton's second law of motion, the force acting on an object is equal to the rate of change in momentum. In fluid mechanics where a volume element is considered, the conservation of momentum states that the change of rate in momentum equals the net rate of momentum flow and forces acting on the volume element.

Equation of momentum in x-direction



(a) Momentum flow in the direction of x. (b) Forces in the direction of x.

Figure 10.6. Momentum balance at the control volume.

Derivation

1 **Setting up the balance:**
The general momentum balance around the control volume reads:

(Rate of change in momentum) = (Net momentum flow) + (Net surface force) + (Net body force)

The unsteady momentum balance in x-direction yields:

$$\frac{\partial I_x}{\partial t} = \left(d\dot{I}_{xx}(x) - d\dot{I}_{xx}(x+dx)\right) + \left(d\dot{I}_{yx}(y) - d\dot{I}_{yx}(y+dy)\right)$$
$$+ \left(-dF_{xx}(x) + dF_{xx}(x+dx)\right) + \left(-dF_{yx}(y) + dF_{yx}(y+dy)\right) + dF_x$$

The surface forces, constituting of pressure, shear, and normal forces expressed by F_{xy}, where the first index indicates the orientation of the surface at which the force is applied (e.g. here perpendicular to the x-direction) and the second index shows the direction of the force (e.g. here in the y-direction). The term dF_x denotes a volume force, for instance, the gravitational force applied here in the x-direction.

2 **Defining the elements within the balance:**

The temporal change of momentum inside the control volume dV in the x-direction is:

$$\frac{\partial I_x}{\partial t} = \frac{\partial \rho u}{\partial t} dx\, dy\, dz$$

The incoming rate of momentum is:

$$d\dot{I}_{xx}(x) = \dot{m}(x)\, u = \rho\, u\, u\, dy\, dz$$

$$d\dot{I}_{yx}(y) = \dot{m}(y)\, u = \rho\, u\, v\, dx\, dz$$

The outgoing rate of momentum are approximated by use of the Taylor series expansion:

$$d\dot{I}_{xx}(x+dx) = d\dot{I}_{xx}(x) + \frac{\partial}{\partial x}\left(d\dot{I}_{xx}(x)\right) \cdot dx$$

$$d\dot{I}_{yx}(y+dy) = d\dot{I}_{yx}(y) + \frac{\partial}{\partial y}\left(d\dot{I}_{yx}(y)\right) \cdot dy$$

The force on the surface perpendicular to the x-direction on location x in the direction of x yields from the pressure as well as viscous forces:

$$dF_{xx}(x) = (-p(x) + \tau_{xx}(x))\, dy\, dz$$

The force on the surface perpendicular to the y-direction on location y in the direction of x yields from the viscous forces:

$$dF_{yx}(y) = \tau_{yx}(y)\, dx\, dz$$

The force on the surface perpendicular to the x-direction on position $x + dx$ in the direction of x and on the surface perpendicular to the y-direction on position $y + dy$ in the direction of x are approximated by use of the Taylor series expansion:

$$dF_{xx}(x + dx) = dF_{xx}(x) + \frac{\partial}{\partial x}(dF_{xx}(x)) \cdot dx$$

$$dF_{yx}(y + dy) = dF_{yx}(y) + \frac{\partial}{\partial y}(dF_{yx}(y)) \cdot dy$$

The gravitational force acting in the x-direction is:

$$dF_x = dm\, g_x = \rho\, dx\, dy\, dz\, g_x$$

3 **Inserting and rearranging:**

Inserting all defined terms yields the general unsteady equation of momentum in the x-direction:

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho u v}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}(x)}{\partial x} + \frac{\partial \tau_{yx}(y)}{\partial y} + \rho\, g_x$$

In fluid mechanics, see [1], the Cauchy stress relation for viscous stresses for Newtonian fluids is written as:

$$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\eta\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

so:

$$\tau_{xx} = 2\eta\frac{\partial u}{\partial x} - \frac{2}{3}\eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{yx} = \eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

inserting the Cauchy stress relation for viscous stresses, and assuming the flow is **steady-state** yields:

$$\frac{\partial \rho u u}{\partial x} + \frac{\partial \rho u v}{\partial y} = -\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x} - \frac{2}{3}\eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(\eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \rho\, g_x$$

Using the assumptions of a constant viscosity, constant density, and substitution of the continuity equation for incompressible flow in equation (10.4) yields:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial p}{\partial x}\right) + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho\, g_x$$

Task: derive the equation of momentum in y-direction yourself! □

Fundamental EQ

Equations of momentum:

$$\rho\left(\vec{u}\cdot\nabla\right)\vec{u} = -\nabla\, p + \eta\nabla^2\,\vec{u} + \rho\,\vec{g} \tag{10.5}$$

10.3.3 Equation of energy conservation

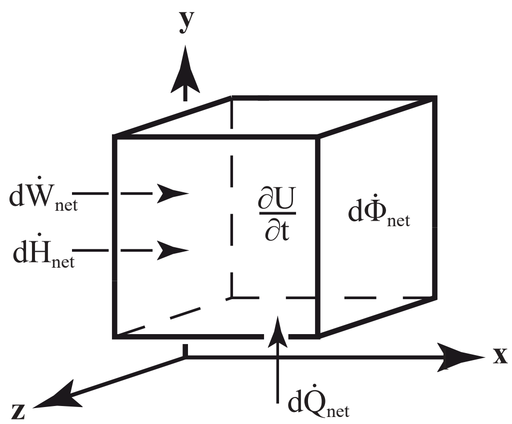


Figure 10.7. Energy balance at the control volume.

Derivation

1 **Setting up the balance:**

The 1st law of thermodynamics reads:

$$\begin{pmatrix}\text{Rate of change}\\ \text{in internal energy}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat}\\ \text{transfer in}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\end{pmatrix} + \begin{pmatrix}\text{Rate of work}\\ \text{added}\end{pmatrix} + \begin{pmatrix}\text{Rate of mass}\\ \text{flow energy in}\end{pmatrix}$$

For the given control volume this yields:

$$\frac{\partial U}{\partial t} = d\dot{Q}_{\text{net}} + d\dot{\Phi}_{\text{net}} + d\dot{W}_{\text{net}} + d\dot{H}_{\text{net}}$$

2 **Defining the elements within the balance:**

The total internal energy within the volume yields from the product of the specific enthalpy and the mass. Thus, the change of internal energy over time is:

$$\frac{\partial U}{\partial t} = \frac{\partial \rho h}{\partial t}\, dx\, dy\, dz$$

with $h = \left(e + \frac{1}{2}U^2\right)$, where e is the internal energy per unit mass and U the velocity magnitude.

The net rate of heat conduction passing through the surface of the control volume is:

$$d\dot{Q}_{\text{net}} = d\dot{Q}(x) - d\dot{Q}(x+dx) + d\dot{Q}(y) - d\dot{Q}(y+dy)$$

The rate of heat transfer entering the system:

$$d\dot{Q}(x) = -\lambda\, dy\, dz\, \frac{\partial T}{\partial x}$$

$$d\dot{Q}(y) = -\lambda\, dx\, dz\, \frac{\partial T}{\partial y}$$

The use of the Taylor series expansion can approximate the rate of heat transfer leaving the system:

$$d\dot{Q}(x+dx) = d\dot{Q}(x) + \frac{\partial}{\partial x}\left(\dot{Q}(x)\right)\cdot dx$$

$$d\dot{Q}(y+dy) = d\dot{Q}(y) + \frac{\partial}{\partial y}\left(d\dot{Q}(y)\right)\cdot dy$$

Substitution of these four terms yields the net rate of heat diffusion:

$$d\dot{Q}_{\mathrm{net}} = \left(\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)\right) dx\; dy\; dz$$

If a source term is present, the net rate of heat generation is:

$$d\dot{\Phi}_{\mathrm{net}} = \dot{\Phi}''' \; dx\; dy\; dz$$

The net rate of work added yields from the work at the surface as well as the gravitational work:

$$d\dot{W}_{\mathrm{net}} = -d\dot{W}(x) + d\dot{W}(x+dx) - d\dot{W}(y) + d\dot{W}(y+dy) + d\dot{W}_{\mathrm{g}}$$

The rate of work added at x:

$$d\dot{W}(x) = dF_{xx}(x)u + F_{xy}(x)v$$

The rate of work added at $x+dx$ yields from:

$$d\dot{W}(x+dx) = dF_{xx}(x+dx)u + F_{xy}(x+dx)v$$

The rate of work added at y is:

$$d\dot{W}(y) = F_{yy}(y)v + dF_{yx}(y)u$$

The rate of work added at $y+dy$ is described by:

$$d\dot{W}(y+dy) = F_{yy}(y+dy)v + dF_{yx}(y+dy)u$$

The rate of work added by gravity is:

$$d\dot{W}_{\mathrm{g}} = F_{g,x}u + F_{g,y}v$$

Using the found expressions, the sum of them yields the rate of work added to the system:

$$d\dot{W}_{\mathrm{net}} = \left(\frac{\partial}{\partial x}\left(-p_{xx}(x)u + \tau_{xx}u + \tau_{xy}v\right) + \frac{\partial}{\partial y}\left(-p_{yy}v + \tau_{yy}v + \tau_{yx}u\right) + \rho\left(g_x u + g_y v\right)\right) dx\; dy\; dz$$

The net rate of the enthalpy flow passing through the surface of the control volume is:

$$d\dot{H}_{\mathrm{net}} = d\dot{H}(x) - d\dot{H}(x+dx) + d\dot{H}(y) - d\dot{H}(y+dy)$$

The rate of enthalpy flowing in yields from the product of the specific enthalpy and the mass flow:

$$d\dot{H}(x) = \rho\; h\; u\; dy\; dz$$

$$d\dot{H}(y) = \rho\; h\; v\; dx\; dz$$

The rate of enthalpy leaving the system are approximated by the use of the Taylor series expansion:

$$d\dot{H}(x+dx) = d\dot{H}(x+dx) + \frac{\partial}{\partial x}\left(d\dot{H}(x)\right)\cdot dx$$

$$d\dot{H}(y+dy) = d\dot{H}(y+dy) + \frac{\partial}{\partial y}\left(d\dot{H}(y)\right)\cdot dy$$

Substitution of these four terms yields the net rate of mass energy passing through the surface:

$$d\dot{H}_{\text{net}} = -\rho h\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) dx\, dy\, dz$$

3 **Inserting and rearranging:**

Inserting all defined terms yields the general energy conservation equation:

$$\left(\frac{\partial \rho h}{\partial t} + \frac{\partial \rho h u}{\partial x} + \frac{\partial \rho h v}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \dot{\Phi}'''$$
$$+ \frac{\partial}{\partial x}\left(-p_{xx}(x)u + \tau_{xx}u + \tau_{xy}v\right) + \frac{\partial}{\partial y}\left(-p_{yy}v\tau_{yy}v + \tau_{yx}u\right) + \rho\left(g_x u + g_y v\right)$$

Under steady-state conditions, no variations over time are observed. Therefore the partial derivative concerning time cancels out.

Often, fluids do not generate any heat due to chemical reactions for example. So in most practical cases, the heat generation term cancels out.

The contributions from stresses (normal, shear, and inertial forces) can oftentimes be omitted since their values are, in most cases, negligibly small and do not contribute significantly to the net energy balance.

This yields:

$$\left(\frac{\partial \rho h u}{\partial x} + \frac{\partial \rho h v}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)$$

The specific enthalpy may be substituted by the product of the heat capacity and temperature for systems without phase changes, no chemical reactions, and, if the fluid is considered an ideal gas ($\partial h = c_p\, \partial T$) or an incompressible liquid ($\partial h = c\, \partial T$).

If, in addition, the thermal conductivity is independent of location, and the density is constant, this equation becomes:

$$\rho\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Comparing this equation with the energy equation for heat conduction, for steady-state problems without heat sources, the thermal energy transport by heat conduction is extended by an advective energy transport through enthalpy flow. □

Fundamental EQ

Equation of energy conservation:

$$\rho\left(\vec{u}\cdot\nabla T\right) = \frac{\eta}{\Pr}\nabla^2 T \tag{10.6}$$

In this context, a dimensionless number known as the Prandtl number is introduced, and the applicability is discussed in Section 11.2.1.

Definition

Prandtl number:

$$\mathrm{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\eta}{\lambda/c_p} \; [-] \tag{10.7}$$

The temperature field within the fluid flow is described by the conservation equations for mass (10.4), momentum (10.5), and energy (10.6), together with the boundary and the initial condition and appropriate estimates of the material properties. From the solution of this set of equations, especially the temperature gradient at the boundary, the heat transfer coefficient can be evaluated using equation (10.1).

Using numerical methods based on finite differences or finite volume discretizations, the system of interdependent partial differential equations can be solved. A possible tool is OpenFOAM, an open-source finite volume solver.

HeatQuiz 10.1

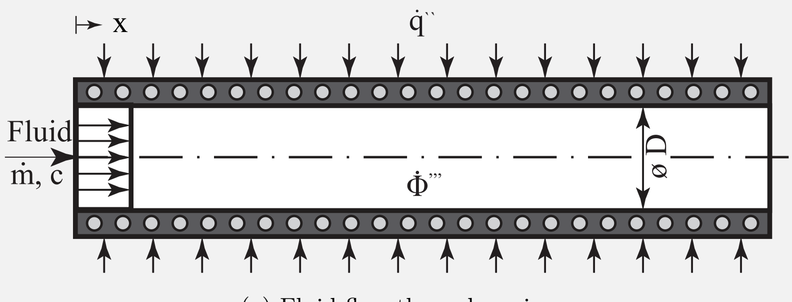
Energy balances:

Example 10.1

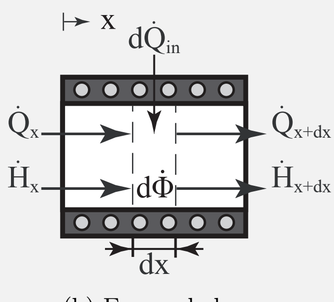
Through a very long pipe with a diameter D flows a heat-generating fluid (homogeneous and constant source strength $\dot{\Phi}'''$). In addition, a constant heat flux $\dot{q}''$ is imposed on the pipe wall. Derive the differential equation for the temperature profile in the flow direction.

Hints:

- The problem is one-dimensional in the x-direction.
- Diffusive heat transport in the direction of the flow cannot be neglected.
- The problem is steady in time.
- The thermophysical properties of the fluid are temperature-independent.
- The contributions from stresses are negligible.



(a) Fluid flow through a pipe.



(b) Energy balance.

1 **Setting up the balance:**

The steady-state energy balance for an infinitesimal element reads:

$$\cancelto{0\text{ - steady-state}}{\frac{\partial U}{\partial t}} = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} + \dot{H}_{\mathrm{x}} - \dot{H}_{\mathrm{x+dx}} + d\dot{\Phi} + d\dot{Q}_{\mathrm{in}}$$

2 Defining the elements within the balance:

The incoming rate of heat diffusion is:

$$\dot{Q}_{\mathrm{x}} = -\lambda \frac{\pi D^2}{4} \frac{\partial T_{\mathrm{f}}}{\partial x}$$

The incoming rate of enthalpy flow is:

$$\dot{H}_{\mathrm{x}} = \dot{m} c T_{\mathrm{f}}$$

For an infinitesimal element, the outgoing rate of heat diffusion and enthalpy flow are approximated by use of the Taylor series:

$$\dot{Q}_{\mathrm{x+dx}} = \dot{Q}_{\mathrm{x}} + \frac{\partial}{\partial x}\left(\dot{Q}_{\mathrm{x}}\right) \cdot dx$$

and:

$$\dot{H}_{\mathrm{x+dx}} = \dot{H}_{\mathrm{x}} + \frac{\partial}{\partial x}\left(\dot{H}_{\mathrm{x}}\right) \cdot dx$$

The heat generated within the infinitesimal element yields from the product of the element and the heat source:

$$d\dot{\Phi} = \dot{\Phi}''' \cdot dV = \dot{\Phi}''' \frac{\pi D^2}{4} dx$$

The incoming rate of heat transfer yields from the product of the heat flux and the surface area of the infinitesimal element:

$$d\dot{Q}_{\mathrm{in}} = \dot{q}'' \cdot dA_{\mathrm{s}} = \dot{q}'' \pi \, D \, dx$$

3 Inserting and rearranging:

Inserting all expressions and performing some rearranging yields the one-dimensional energy conservation equation:

$$\frac{4\dot{m}c}{\pi D^2} \frac{\partial T_{\mathrm{f}}}{\partial x} = \lambda \frac{\partial^2 T_{\mathrm{f}}}{\partial x^2} + \dot{\Phi}''' + \frac{4\dot{q}''}{D}$$

4 Defining the boundary and/or initial conditions:

To solve the differential equation above, two boundary conditions are required, since the fluid temperature has been differentiated twice concerning x. Typical boundary conditions could be the in- and outlet temperatures.

$$T(x=0) = T_{\mathrm{in}}$$

and:

$$T(x=L) = T_{\mathrm{out}}$$

5 Solving the equation:

The given differential equation is solved by utilizing the stated boundary conditions. Eventually, the temperature profile of the fluid is obtained.

Subsection 10.4
Laminar flows and turbulence

A major classification of flows is the distinction between laminar and turbulent flows. Laminar flow represents a state of fluid motion characterized by smooth, orderly, and parallel layers of streamlines. In this regime, fluid particles move in a highly organized manner, with each layer of particles sliding past the adjacent layers. Contrarily, turbulent flows are characterized by a chaotic motion of the fluid of different lengths- and timescales that enhance mixing.

To create a basic idea of turbulence, a two-dimensional flow between two plates is considered as shown in Figure 10.9. In the laminar regime, the flow is characterized by a parabolic profile with a no-slip boundary condition (zero velocity) at the wall. Now, an obstacle is positioned within the flow, introducing a small perturbation to the flow. On the microscopic scale, this perturbation can be interpreted as a fluid parcel leaving the laminar streamline. Being de-located from the original flow path, the fluid parcel needs to adopt speed e.g. change the inertia, which happens due to the action of viscous forces exerted from the neighboring fluid parcels. If the viscous forces are sufficiently high, this adaption happens sufficiently fast, such that the introduced perturbation decreases rapidly. If the flow returns to the parabolic profile, the flow is denoted as laminar.

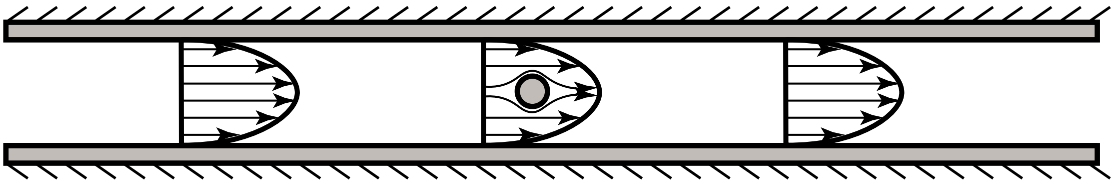


Figure 10.9. Flow with a relatively low velocity between two plates.

If viscous forces are not sufficiently strong, the de-located fluid parcel causes other fluid parcels to move away from their original streamline, further causing an enhancement of the disturbance, as illustrated in Figure 10.10. Now the flow does not return to a parabolic profile but the disturbances are enhanced. This type of flow is characterized to be turbulent.

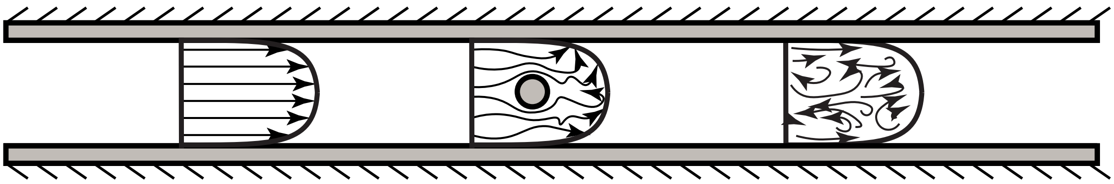


Figure 10.10. Flow with a relatively high velocity between two plates.

In conclusion, laminar flows are characterized by a dampening of any introduced disturbance while in turbulent flows disturbances are amplified. The enhanced fluid motion perpendicular to the general direction of the flow causes an enhanced mixing and transport of momentum and heat. As such, turbulent flows have a higher rate of heat transfer, but also a higher rate of shear.

Within the next sections, the solutions of the conservation equations for laminar flows are discussed. But first, to characterize flow to be laminar, the Reynolds number is often used as a threshold.

10.4.1 Reynolds number

The transition from laminar to turbulent flow depends as mentioned above on the ratio between inertia and viscous forces. These factors depend on surface geometry, surface roughness, flow velocity, surface temperature, and the nature of the fluid, among others. Osborn Reynolds in 1883 discovered that the transition from laminar to turbulence primarily depends on the relationship between the forces of inertia and viscosity within the fluid. To his honor, this ratio is termed the Reynolds number, a dimensionless quantity expressed as:

Definition

Reynolds number:

$$\mathrm{Re} = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\rho u L}{\eta} = \frac{uL}{\nu} \; [-], \tag{10.8}$$

where L is the characteristic length.

The characteristic length scale is chosen based on the problem. For internal flows, e.g. pipe- or duct flows, the characteristic length depends on the hydraulic diameter of the geometry. For the flow over a plate, the length scale is the expansion of the plate in the flow direction.

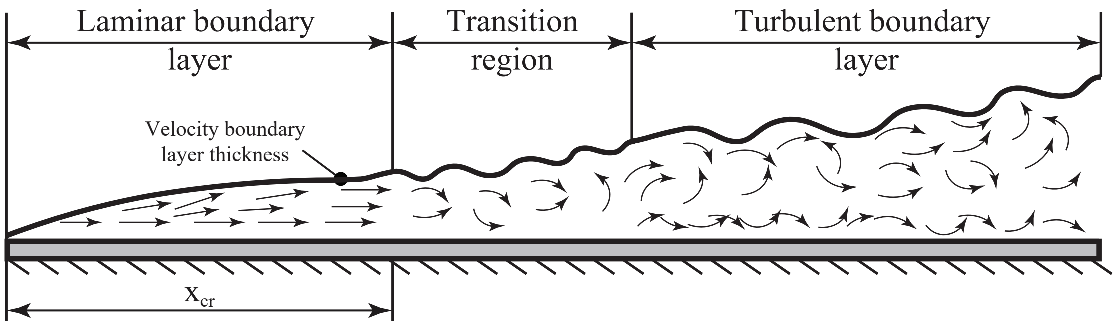


Figure 10.11. The development of the boundary layer for flow over a flat plate, and the different flow regimes. Note that there is also a fluid flow above the boundary layer but that this fluid flow is not affected by the wall.

The Reynolds number at which the transition from laminar to turbulent flow occurs, as illustrated in Figure 10.11, is termed the critical Reynolds number $\mathrm{Re}_{\mathrm{crit}}$. The specific value of the critical Reynolds number varies with different geometries and flow conditions. For instance, in the case of flow over a flat plate, the widely stated value for the critical Reynolds number is $\mathrm{Re}_{\mathrm{crit}} = 2 \cdot 10^5$, indicating the distance x_{crit} from the leading edge of the plate where this transition occurs. Note, that this critical value substantially varies with the level of perturbation (also called turbulence level) in the free stream.

Section 11

Boundary layer

L02 - Forced convection:

Learning goals:

- Comprehending the concept of a boundary layer on a flat plate within a steady laminar flow.
- Recognizing the similarities between velocity and temperature profiles within the boundary layer and understanding the resulting relationship between the heat transfer coefficient and shear stress in this context.

Comprehension questions:

- What distinguishes the Nusselt number from the Biot number?
- How does the Prandtl number relate to Boundary Layer theory?
- In the event of equivalence between the thickness of the Flow Boundary Layer and the temperature Boundary Layer ($\delta_u = \delta_T$), what is the correlation involving the Nusselt number?

L03 - Natural convection:

Learning goals:

- Gaining a comprehensive understanding of the Boundary Layer profile (both temperature and velocity) on a flat plate subjected to natural (free) convection.
- Acquiring the ability to derive and interpret the Grashof number.
- Distinguishing between the Boundary Layer profiles in the contexts of forced and free convection and comprehending their differences.

Comprehension questions:

- What is the driving potential of natural convection?
- Why are buoyancy forces negligible in forced convection?

Corresponding tutorial exercises:

- Exercise III.5 Substance container

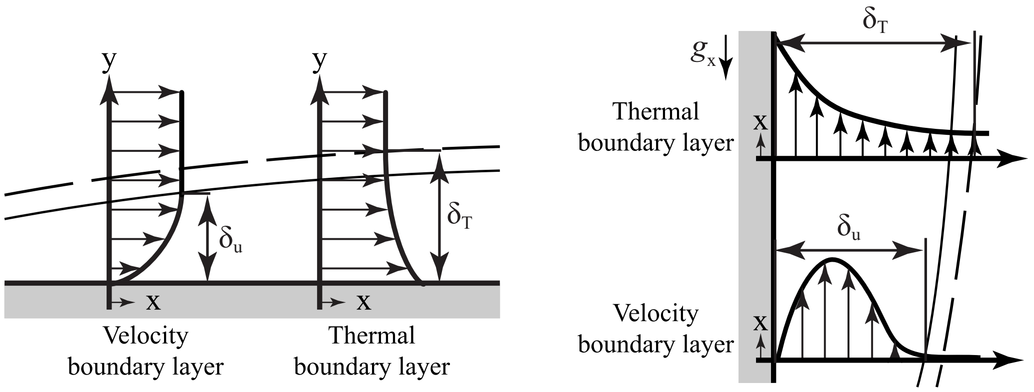
Subsection 11.1

Velocity and thermal boundary layer

The concept of boundary layers is fundamental in fluid dynamics and heat transfer, playing an important role in understanding the complex dynamics of heat exchange between a fluid and a solid surface.

The boundary layer denotes the region of the fluid flow that is affected by the presence of an object. On the object's surface, the flow velocity matches the velocity of the object (often treated as being zero), while far away from the object, the fluid flows with the so-called free stream velocity. The region in which the flow transitions between the free steam velocity to the velocity of the object is called the boundary layer. The same concept applies to the temperature, such that the thermal boundary layer denotes the region in which the temperature changes from the free stream temperature to the temperature at the object's surface. In the illustrations depicting the boundary layer, the layer's location is typically indicated within the fluid flow, situated where the velocity has nearly attained the free-stream temperature. Oftentimes "nearly" is defined as the 99% threshold of the velocity/temperature difference. The thickness of the velocity boundary layer is denoted by δ_{u}, whereas δ_{T} denotes the thermal boundary layer thickness.

Figure 11.1a illustrates the velocity and thermal boundary layer in a scenario of forced convection over a flat plate. Figure 11.1b shows the boundary layer for natural convection on a horizontal wall. Note that the velocity boundary layer reaches not only the point of highest velocity but also encounters the velocity decay back to the free-stream velocity of zero.



(a) Forced convection over a hot horizontal plate. (b) Natural convection along a hot vertical plate.

Figure 11.1. Thermal and velocity boundary layers for laminar, steady-state flow.

SUBSECTION 11.2
Forced convection

This section shows how the conservation equations for momentum and energy are reduced in their complexity within the boundary layer.

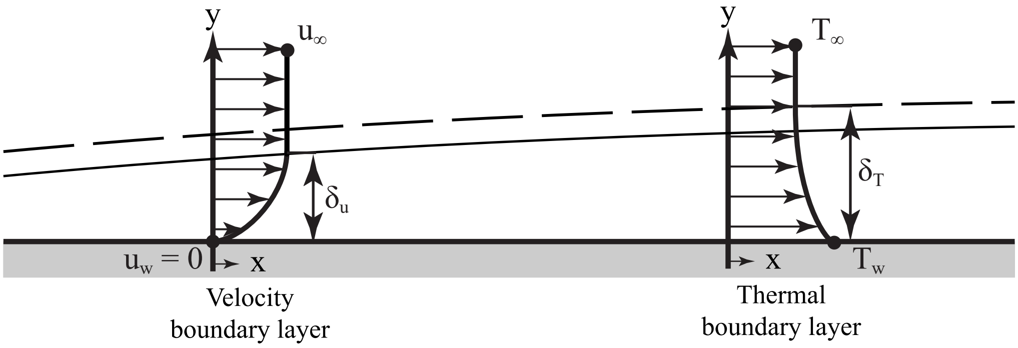


Figure 11.2. Velocity and thermal boundary layers of a surface in a stream flow.

Prandtl (1904) [2] derived the equations valid for boundary layers from the conservation laws through appropriate assumptions for the individual terms. Assumptions in this context mean that by considering the importance of different forces compared to other forces, some of these can be neglected. These assumptions, which can also be found in Schlichting and Gersten (2006) [3], state that for flow within the boundary layer, i.e. $\delta \ll L$, the following simplifications are valid:

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

The assumptions state that the change in the gradient in the flow direction (here x-coordinate) is much smaller compared to the change of the gradient in the wall-normal direction (here y-coordinate). Using these assumptions the conservation equations derived in Section 10.3 are significantly simplified.

The equation of continuity assuming a constant density, given in equation (10.4), remains the same:

Fundamental EQ

Equation of continuity for forced flow over a flat plate:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11.1}$$

For laminar flow over a flat plate, $\partial p/\partial x$ is often assumed to be negligible within the boundary layer. This results because the pressure changes in the streamwise direction are primarily balanced by viscous effects. Similarly, gravitational forces are negligible. Thus, for flow over a flat plate, where $\partial p/\partial x = 0$ and $\rho g_\mathrm{x} = 0$, the momentum equation in the x-direction yields [1]:

Fundamental EQ

Equation of momentum in x-direction for forced flow over a flat plate:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \eta \frac{\partial^2 u}{\partial y^2} \tag{11.2}$$

Lastly, the equation of energy conservation assuming constant material properties, stated in equation (10.6), simplifies to:

Fundamental EQ

Equation of energy conservation for forced flow over a flat plate:

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\mathrm{Pr}} \frac{\partial^2 T}{\partial y^2} \tag{11.3}$$

11.2.1 Prandtl number

The relative thickness of the velocity and thermal boundary layers are characterized using the dimensionless parameter known as the Prandtl number, as previously introduced in equation (10.7), and listed here again:

$$\mathrm{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\eta}{\lambda / c_p} \; [-]$$

For gases, $\mathrm{Pr} \approx 1$, indicating that both momentum and heat disperse through the fluid at roughly the same rate. However, in the case of liquid metals, $\mathrm{Pr} \ll 1$, signifying that heat diffuses rapidly relative to momentum. Conversely, in oils, $\mathrm{Pr} \gg 1$, indicating that heat diffusion occurs relatively slowly compared to momentum diffusion. Consequently, the thermal boundary layer is considerably thicker for liquid metals and substantially thinner for oils when compared to the velocity boundary layer.

11.2.2 An approximation for the boundary layer equations

Through the analysis of the boundary layer's characteristics, engineers can address problems within the realm of heat transfer. To do so, engineers require information about the velocity and temperature profiles within the boundary layer. Two concepts that care derived from these profiles are the dimensionless shear stress c_{f} (commonly known as the friction coefficient) and the dimensionless heat transfer coefficient Nu (referred to as the Nusselt number).

The definition of the friction coefficient c_{f} is as follows:

Definition

Friction coefficient:

$$\frac{c_{\mathrm{f}}}{2} = \frac{\text{frictional head loss}}{\text{dynamic pressure}} = \frac{\tau_{\mathrm{W}}}{\rho u_\infty^2} \; [-], \tag{11.4}$$

where $\tau_{\mathrm{W}} = \left(\eta \frac{\partial u}{\partial y}\right)_{\mathrm{W}}$

The definition of the Nusselt number has been presented in equation (10.2). With the substitution of the definition of the heat transfer coefficient, stated in equation (10.1), the Nusselt number yields:

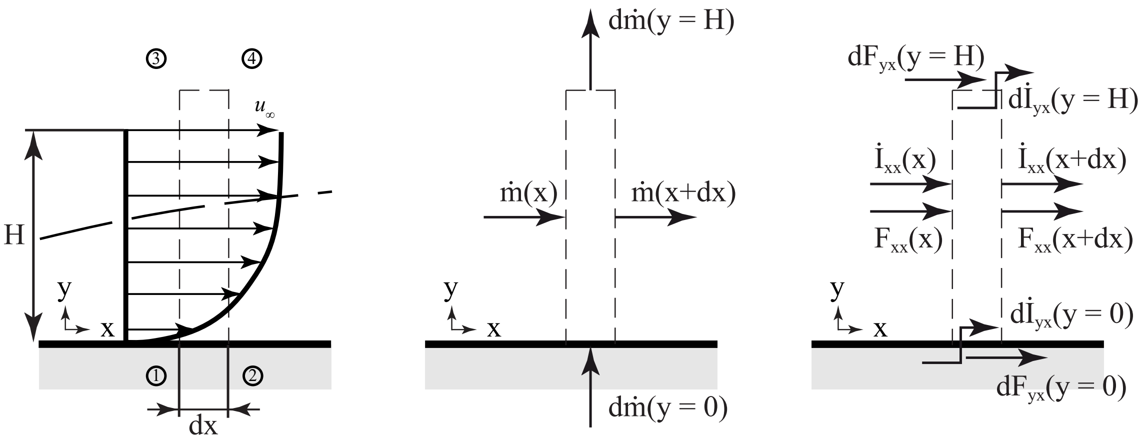
$$\mathrm{Nu} = \frac{-\left(\lambda_{\mathrm{f}} \frac{\partial T_{\mathrm{f}}}{\partial y}\right)_{\mathrm{W}}}{T_{\mathrm{W}} - T_{\mathrm{A}}} \frac{L}{\lambda_{\mathrm{f}}} \; [-]$$

Utilizing both definitions, the velocity profile or the temperature profile within a boundary layer enables us to determine the shear stress τ_{W} or the heat transfer coefficient α, respectively.

The necessary mathematical methods to **exactly** solve the boundary layer equations are sufficiently complex to fall beyond the scope of this introductory course. However, a simplified approximation for the boundary layer equations, which yields results comparable to exact solutions and shows the underlying physical principles, is discussed next.

Derivation

Consider the two-dimensional steady-state boundary layer flow over a flat plate involves using a control volume bounded by the planes $\overline{13}$, $\overline{34}$, $\overline{24}$, and the wall $\overline{12}$ to formulate the conservation equations. This control volume has an infinitesimal length dx in the x-direction. The height H is in the y-direction, and the length L is in the z-direction, where $dx \ll H$ and $dx \ll L$. Plane $\overline{34}$ is located within the potential flow outside the boundary layer.



(a) Control volume. (b) Mass balance. (c) Momentum balance.

Figure 11.3. Velocity boundary layer.

1 Setting up the balance:

The steady-state mass balance, shown in Figure 11.3b, accounting for the ingoing and outgoing mass flows reads:

$$0 = \dot{m}(x) - \dot{m}(x + dx) + d\dot{m}(y = 0) - d\dot{m}(y = H)$$

The steady-state momentum balance in x-direction, shown in Figure 11.3c, yields:

$$\begin{aligned} 0 = &\left(\dot{I}_{xx}(x) - \dot{I}_{xx}(x + dx)\right) + \left(d\dot{I}_{yx}(y = 0) - d\dot{I}_{yx}(y = H)\right) \\ &+ \left(-F_{xx}(x) + F_{xx}(x + dx)\right) + \left(-dF_{yx}(y = 0) + dF_{yx}(y = H)\right) \end{aligned}$$

2 Defining the elements within the balance:

Where the ingoing mass flows in the plane $\overline{11}$ is derived by integration:

$$\dot{m}(x) = \int_0^H \rho\, u\, L\, dy$$

The outgoing mass flow in the x-direction are approximated using the Taylor-series expansion:

$$\dot{m}(x + dx) = d\dot{m}(x) + \frac{\partial}{\partial x}\left(d\dot{m}(x)\right)\ dx$$

No mass flow passes through the wall, so:

$$d\dot{m}(y = 0) = 0$$

The outgoing mass flow in the y-direction is the product of the density, the velocity at plane $\overline{34}$ in y-direction v_∞, and the area of plane $\overline{34}$:

$$d\dot{m}(y = H) = \rho v_\infty L\, dx$$

The incoming rate of momentum is:

$$\dot{I}_{xx}(x) = \int_0^H \rho\, u^2\, L\, dy$$

Since no mass passes through the wall, the momentum flow at the wall equals zero as well:

$$d\dot{I}_{yx}(y = 0) = 0$$

The outgoing rate of momentum at $x + dx$ are approximated by use of the Taylor series expansion:

$$\dot{I}_{xx}(x + dx) = \dot{I}_{xx}(x) + \frac{\partial}{\partial x}\left(\dot{I}_{xx}(x)\right) \cdot dx$$

Where the outgoing rate of momentum at $y = H$ is written in given variables:

$$d\dot{I}_{yx}(y = H) = d\dot{m}(y = H)\, u(y = H) = \rho u_\infty v_\infty L\, dx$$

Normal forces in the planes $\overline{13}$ and $\overline{24}$ are to be neglected following the boundary layer approximation. No shear forces are present at the upper boundary plane, which is located in the potential flow. Thus:

$$F_{xx}(x) = F_{xx}(x + dx) = dF_{yx}(y = H) = 0$$

At the wall, a shear stress imposes a force on the control volume:

$$dF_{yx}(y = 0) = \tau_{\mathrm{yx}}\, L\, dx = \left(\eta \frac{du}{dy}\right)_{\mathrm{W}} L\, dx$$

3 Inserting and rearranging:

Inserting the definitions of the mass flows into the mass balance yields the integral equation of continuity:

$$\frac{\partial}{\partial x}\int_0^H \rho\, u\, dy + \rho v_\infty = 0$$

And the integral momentum equation in the x-direction:

$$\frac{\partial}{\partial x}\int_0^H \rho\, u^2\, dy + \rho u_\infty v_\infty = -\left(\eta \frac{du}{dy}\right)_{\mathrm{W}}$$

Substitution of the equation of continuity into the momentum equation yields:

$$\frac{\partial}{\partial x}\int_0^H \rho u\, (u_\infty - u)\, dy = \left(\eta \frac{du}{dy}\right)_{\mathrm{W}}$$

4 Defining the boundary and/or initial conditions:

To solve the integral equation, two boundary conditions are required. At the wall, a zero-slip condition is observed, thus:

$$u(y = 0) = 0$$

And at the boundary layer $y = \delta_{\mathrm{u}}$ the velocity is equal to that of the free flow:

$$u(y = \delta_{\mathrm{u}}) = u_\infty$$

5 **Solving the equation:**

Solving the momentum equation is only possible if the velocity profile in the boundary layer is known.

A crude approximation would be assuming a linear velocity profile:

$$u = Ay + B$$

Using the boundary conditions, yields that $A = \frac{u_\infty}{\delta_u}$ and $B = 0$:

$$u = u_\infty \frac{y}{\delta_u}$$

Substitution of this linear profile into the integral equation and assuming a constant density within the boundary layer yields:

$$\rho u_\infty \frac{d}{dx} \int_0^{\delta_u} \frac{y}{\delta_u}\left(1 - \frac{y}{\delta_u}\right) dy = \frac{\eta}{\delta_u}$$

Integration yields the function of the boundary layer thickness:

$$\delta_u = \sqrt{\frac{12\eta}{\rho u_\infty x}} = \sqrt{\frac{12}{\mathrm{Re}_x}},$$

with the Reynolds number, $\mathrm{Re}_x = \frac{\rho u_\infty x}{\eta}$, based on the length x.

With the known thickness of the boundary layer and the approximation of the linear velocity profile, the shear stress at the wall yields:

$$\tau_W = \left(\eta \frac{du}{dy}\right)_W = \eta \frac{u_\infty}{\delta_u} = \frac{1}{\sqrt{12}\sqrt{\mathrm{Re}_x}} \rho\, u_\infty^2$$

Written in dimensionless form, the local friction coefficient $c_{f,x}$ is:

$$\frac{c_{f,x}}{2} = \frac{\tau_W}{\rho u_\infty^2} \approx \frac{0.289}{\sqrt{\mathrm{Re}_x}}$$

□

Fundamental EQ

Linear velocity profile approximation of the velocity boundary layer:

$$u = u_\infty \frac{y}{\delta_u} \tag{11.5}$$

Fundamental EQ

Velocity boundary layer thickness for a linear velocity profile:

$$\delta_u = \sqrt{\frac{12}{\mathrm{Re}_x}}, \tag{11.6}$$

for $\mathrm{Re}_x < 2 \cdot 10^5$

Fundamental EQ

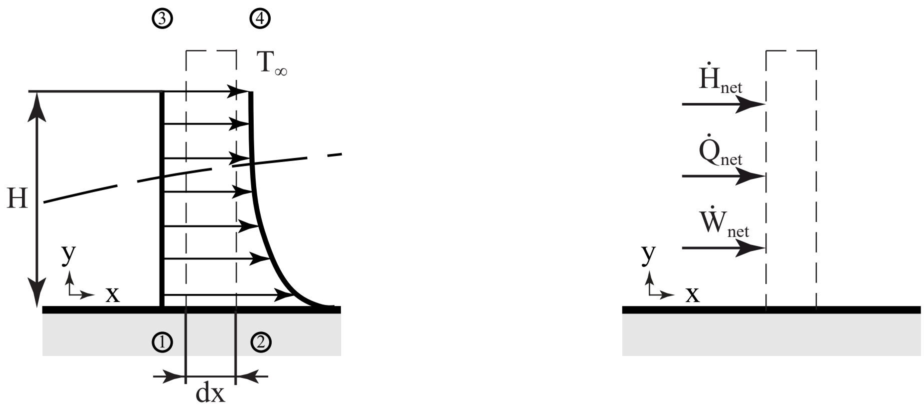
Local friction coefficient for a linear velocity profile:

$$\frac{c_{f,x}}{2} = \frac{0.289}{\sqrt{\mathrm{Re_x}}} \tag{11.7}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$

Derivation

Similarly, as for the local friction coefficient $c_{f,x}$, an expression for the local Nusselt number $\mathrm{Nu_x}$ is derived from the energy equation. The integral energy equation is derived using a similar control volume that includes the thermal boundary layer.



(a) Control volume. (b) Energy balance.

Figure 11.4. Thermal boundary layer.

1 **Setting up the balance:**

The steady-state energy balance at the control volume reads:

$$0 = \dot{Q}_{\text{net}} + \dot{W}_{\text{net}} + \dot{H}_{\text{net}}$$

2 **Defining the elements within the balance:**

The force contributions on the energy balance are much smaller than the contribution of conductive and enthalpy fluxes. Therefore, the rate of energy that yields from the forces are neglected:

$$\dot{W}_{\text{net}} = 0$$

Also, the heat conduction in the direction of the flow is negligible compared to the convective energy transport in this direction. Therefore only heat conduction at the wall has a major contribution, and thus:

$$\dot{Q}_{\text{net}} = d\dot{Q}(y = 0)$$

The use of Fourier's law can define heat conducted through the wall:

$$d\dot{Q}(y = 0) = -dx\, L \left(\lambda \frac{dT}{dy}\right)_{\mathrm{W}}$$

No mass can pass through the wall, thus the enthalpy flows at positions x, $x + dx$, and $y = H$ remain:

$$\dot{H}_{\text{net}} = \dot{H}(x) - \dot{H}(x + dx) - d\dot{H}(y = H)$$

The incoming rate of enthalpy is expressed by integration over the length H:

$$\dot{H}(x) = \int_0^H \rho\, u\, c_p(T - T_0)\; L\; dy,$$

where T_0 is the reference temperature used to determine the enthalpy reference value.

The outgoing rate of enthalpy at $x + dx$ is approximated by use of the Taylor series expansion:

$$\dot{H}(x + dx) = \dot{H}(x) + \frac{\partial}{\partial x}\left(\dot{H}(x)\right) \cdot dx$$

The outgoing rate of enthalpy at $y = H$ yields from the product of the mass flow, specific heat and temperature difference:

$$d\dot{H}(y = H) = \rho\, v_\infty\; c_p(T_\infty - T_0)\; L\; dx$$

3 Inserting and rearranging:

Inserting the definitions of total fluxes into the energy balance yields the integral equation of energy:

$$\frac{\partial}{\partial x}\int_0^H \rho\, u\, c_p(T - T_0)dy + \rho\, v_\infty\; c_p(T_\infty - T_0) = -\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}}$$

Substituting the integral equation of continuity results in:

$$\frac{d}{dx}\int_0^H \rho\, u\, c_p(T - T_\infty)\; dy = -\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}}$$

4 Defining the boundary and/or initial conditions:

To solve the integral equation, two boundary conditions are required. At the wall, a known temperature is assumed:

$$T(y = 0) = T_{\mathrm{W}}$$

And at the boundary layer $y = \delta_{\mathrm{T}}$ the temperature is equal to that of the ambient:

$$T(y = \delta_{\mathrm{T}}) = T_\infty$$

5 Solving the equation:

Solving the energy equation is only possible if the temperature profile in the boundary layer is known.

A crude approximation would be assuming a linear temperature profile:

$$T = Ay + B$$

Using the boundary conditions, yields that $A = \frac{T_\infty - T_{\mathrm{W}}}{\delta_{\mathrm{T}}}$ and $B = T_{\mathrm{W}}$:

$$T = (T_\infty - T_{\mathrm{W}})\,\frac{y}{\delta_{\mathrm{T}}} + T_{\mathrm{W}}$$

Assuming constant properties (ρ,λ,c_p) and substituting the linear velocity and temperature profile into the integral equation yields:

$$\frac{d}{dx}\int_0^H \rho\, c_p\, u_\infty \frac{y}{\delta_u}\left((T_\infty - T_W)\left(\frac{y}{\delta_T}-1\right)\right)\, dy = -\left(\lambda\frac{dT}{dy}\right)_W$$

Introducing the temperature difference θ enables to solve the integral equation:

$$\theta = T - T_W$$

Substitution of the temperature difference θ allows for integration of the energy equation:

$$\rho\, c_p\, u_\infty \frac{d}{dx}\int_0^H \frac{y}{\delta_u}\left(\theta_\infty\left(\frac{y}{\delta_T}-1\right)\right) dy = -\lambda\frac{\theta_\infty}{\delta_T}$$

After integration:

$$\delta_T d\left(\frac{\delta_T^2}{\delta_u}\right) = \frac{6\lambda}{\rho c_p u_\infty} dx$$

In case the ratio of the two boundary layer thicknesses remains constant:

$$\left(\frac{\delta_T}{\delta_u}\right)^3 \delta_u\, d\delta_u = \frac{6\lambda}{\rho c_p u_\infty} dx$$

Using the equation for the velocity boundary layer leads to a relationship for the thickness of the thermal boundary layer:

$$\frac{\delta_T}{\delta_u} = \left(\frac{\lambda}{\eta c_p}\right)^{1/3} = \frac{1}{\mathrm{Pr}^{1/3}}$$

If the thickness δ_T of the thermal boundary layer is known, the heat flux at the wall is determined from the linear temperature profile within the boundary layer:

$$\dot{q}''_W = -\lambda\left(\frac{dT}{dy}\right)_W = -\lambda\frac{\theta_\infty}{\delta_T} = -\frac{\lambda\theta_\infty \mathrm{Pr}^{1/3}}{\delta_u}$$

Substituting equation the velocity boundary layer thickness, given in equation (11.6):

$$\dot{q}''_W = \frac{\lambda}{x} 0.289\ \mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}\,(T_W - T_\infty)$$

Equaling this relationship with the Newton's law of cooling:

$$\dot{q}''_W = \alpha\,(T_W - T_\infty)$$

leads to the following equation for the heat transfer coefficient α

$$\alpha = \frac{\lambda}{x} 0.289\ \mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}$$

or in dimensionless form, the Nusselt number $\left(\mathrm{Nu} = \frac{\alpha x}{\lambda}\right)$,

$$\mathrm{Nu}_x = 0.289\ \mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}.$$

□

Fundamental EQ

Linear temperature profile approximation of the thermal boundary layer:

$$T = (T_\infty - T_\mathrm{W})\,\frac{y}{\delta_\mathrm{T}} + T_\mathrm{W} \tag{11.8}$$

Fundamental EQ

Velocity boundary layer thickness for a linear velocity profile:

$$\delta_\mathrm{T} = \left(\frac{\lambda}{\eta c_p}\right)^{1/3} \sqrt{\frac{12\eta}{\rho u_\infty \mathrm{x}}} = \frac{1}{\mathrm{Pr}^{1/3}}\sqrt{\frac{12}{\mathrm{Re_x}}}, \tag{11.9}$$

for $\mathrm{Re_x} < 2\cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$

HTC

Local Nusselt number for a linear velocity profile:

$$\mathrm{Nu_x} = 0.289\,\mathrm{Re_x^{1/2}}\mathrm{Pr}^{1/3} \tag{11.10}$$

for $\mathrm{Re_x} < 2\cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$

11.2.3 Exact solutions for the boundary layer equations

In the previous section, the results of the approximated solutions of the laminar, steady-state boundary layer equations for plane flows were shown. This section discusses the results of the exact solutions to the continuity equation, momentum in the x-direction equation, and energy equation stated in equations (11.1)-(11.3). To obtain the exact solutions of these equations, several mathematical steps and tricks need to be applied. This derivation is not discussed here due to the complexity, but the focus is on the final solution.

The solution of the momentum equation, which yields the velocity profile in the boundary layer, was first published by Blasius in 1908 [4], and is shown in Figure 11.5 as a function of the dimensionless wall distance.

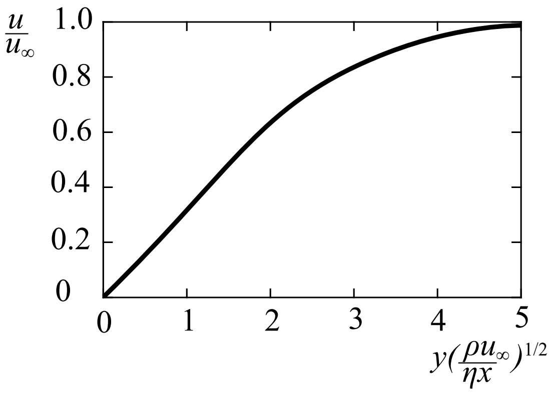


Figure 11.5. Laminar flow over a flat plate - velocity profile according to the derivation of Blasius in 1908 [4].

Solving the continuity and momentum equation yielded an expression for the velocity boundary layer:

Fundamental EQ

Velocity boundary layer thickness for flow over a flat plate:

$$\delta_\mathrm{u} = \frac{4.91}{\sqrt{\mathrm{Re_x}}}, \tag{11.11}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$

From the velocity gradient, the local friction coefficient $c_{f,x}$ is derived for laminar steady-state flow over a flat plate:

Fundamental EQ

Local friction coefficient for forced laminar flow over a flat plate:

$$\frac{c_{f,x}}{2} = \frac{\tau_W}{\rho u_\infty^2} = \frac{0.332}{\sqrt{\mathrm{Re_x}}}, \tag{11.12}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$

This relationship shows the same dependence of the friction coefficient on the Reynolds number as the approximated solution of the momentum equation, where the approximation yields to be 15% off.

The corresponding equation of energy conservation for the flat plate with constant wall temperature was solved by Pohlhausen [5] and leads to the temperature profiles shown in Figure 11.6, which are dependent on the Prandtl number.

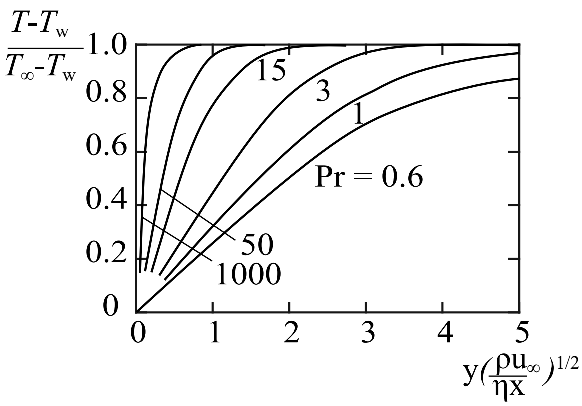


Figure 11.6. Laminar flow over a flat plate - temperature profiles according to Pohlhausen [5].

The solution of the energy conservation equation yielded the thermal boundary layer thickness:

Fundamental EQ

Thermal boundary layer thickness for flow over a flat plate:

$$\delta_T = \frac{4.91}{\sqrt{\mathrm{Re_x}}\mathrm{Pr}^{1/3}}, \tag{11.13}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$

The known temperature gradient allows to calculate the local Nusselt number $\mathrm{Nu_x}$:

$$\mathrm{Nu_x} = \underbrace{\frac{c_{f,x}}{2}}_{0.332\,\mathrm{Re_x^{-1/2}}} \mathrm{Re_x}\,\mathrm{Pr}^{1/3}$$

Rewriting gives:

HTC

Local Nusselt number for forced laminar flow over a flat plate with isothermal surface:

$$\mathrm{Nu_x} = 0.332 \mathrm{Re_x}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}}, \tag{HTC.1}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

Note, that in comparison to the Nusselt correlation obtained with the approximation of a linear temperature profile, only a minor deviation exists. Especially the dependencies concerning the dimensionless numbers (Reynolds and Prandtl) are not influenced by the simplification. With only a little bit of additional mathematical effort, e.g. assuming the temperature profile to be a polynomial of 3rd order, the approximated solution is considerably improved.

For most engineering application problems, the average friction coefficient and Nusselt number over the entire plate are used, which are determined by the following integrals:

$$\frac{\bar{c}_{\mathrm{f,L}}}{2} = \frac{1}{L} \int_0^L \frac{c_{\mathrm{f,x}}}{2}\, dx = \frac{1}{L} \int_0^L 0.332\, \mathrm{Re_x}^{-1/2}\, dx$$

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = \frac{1}{L} \int_0^L \mathrm{Nu_x}\, dx = \frac{1}{L} \int_0^L 0.332\, \mathrm{Re_x}^{1/2}\, \mathrm{Pr}^{1/3}\, dx$$

Where integration yields:

Fundamental EQ

Average friction coefficient for forced laminar flow over a flat plate:

$$\frac{\bar{c}_{\mathrm{f,L}}}{2} = 0.664\, \mathrm{Re_L}^{-1/2}, \tag{11.14}$$

for $\mathrm{Re_L} < 2 \cdot 10^5$

HTC

Average Nusselt number for forced laminar flow over a flat plate with isothermal surface:

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = 0.664\, \mathrm{Re_L}^{\frac{1}{2}} \mathrm{Pr}^{\frac{1}{3}}, \tag{HTC.2}$$

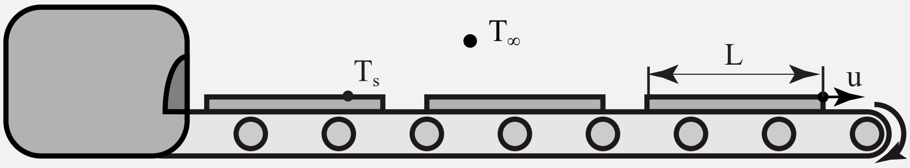
for $\mathrm{Re_L} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

The average Nusselt number is usually applied to calculate the heat transfer rate of a given area, while the local Nusselt number provides data only for a specific location (e.g. calculating the hot spot temperature).

Example 11.1 Hot and slender aluminum plates, with a length denoted as L and width as W, are in transit on a conveyor belt moving at sufficiently high velocity u, such that natural convection is irrelevant. Upon placement onto the conveyor belt, these plates exhibit a uniform surface temperature of T_s. Subsequently, the plates undergo heat dissipation to the surrounding environment characterized by an ambient temperature of T_∞. The average air properties are represented by λ, ν, and Pr. Determine the initial rate of heat transfer from the plate due to convection, expressed in terms of the given parameters.

Hints:

- $\mathrm{Re_L} < 2 \cdot 10^5$
- $0.7 < \mathrm{Pr} < 1$



1 Setting up the definition of the initial rate of heat loss by convection:

$$\dot{Q}_{\text{conv}} = \overline{\alpha} A \left(T_s - T_\infty\right)$$

2 Defining the required elements:

In order to determine the average heat transfer coefficient $\overline{\alpha}$, the expression for the average Nusselt number must be derived. The following correlation holds true for a flat plate with an isothermal surface under the conditions of $\mathrm{Re_L} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$:

$$\overline{\mathrm{Nu}}_\mathrm{L} = 0.664\, \mathrm{Re_L}^{1/2}\, \mathrm{Pr}^{1/3} = 0.664 \sqrt{\frac{uL}{\nu}} \mathrm{Pr}^{1/3}$$

Rewriting the definition of the Nusselt number yields:

$$\overline{\alpha} = \frac{\overline{\mathrm{Nu}}_\mathrm{L} \lambda}{L} = 0.664 \lambda \sqrt{\frac{u}{\nu L}} \mathrm{Pr}^{1/3}$$

The surface area subjected to convection, when neglecting the sides, is:

$$A = W \times L$$

3 Inserting and rearranging:

$$\dot{Q}_{\text{conv}} = 0.664 \lambda \sqrt{\frac{u}{\nu L}} \mathrm{Pr}^{1/3} \left(W \times L\right) \left(T_s - T_\infty\right)$$

Subsection 11.3
Natural convection

The previous section discussed laminar heat transfer between a surface and a fluid, where the flow was externally driven e.g. by a ventilator.

In the absence of external fluid-driving forces, fluid motion can evolve due to inherent internal volume forces, e.g. buoyancy caused by density differences. In the presence of gravity, a light fluid submerged in a denser fluid experiences a discernible upward force. This upward force, which the fluid imparts on an object fully or partially immersed, is referred to as the buoyancy force. The magnitude of this buoyancy force precisely matches the weight of the fluid that the object displaces.

In natural convection, velocity, and thermal boundary layers are present, as depicted in Figure 11.7. These profiles arise from the solutions of the continuity equation (10.3), the momentum equation in the x-direction (10.5), and the energy conservation equation (10.6), adapted to the specific case of the natural flow.

Both boundary conditions of the velocity boundary layer are:

- The no-slip boundary condition, where the fluid velocity matches the wall velocity.
- The stagnant fluid velocity far away from the wall ($u_\infty = 0$).

Both boundary conditions lead to the velocity profile in the boundary layer as shown in Figure 11.7.

The thermal boundary conditions are

- A given temperature or heat flux at the wall.
- The fixed fluid temperature that is far away (T_∞).

At the wall, the temperature gradient is highest, decreasing with distance to the wall. The most severe change in slope is found at the location where the velocity is highest. This follows from the intense advective flow carrying heat in the x-direction and thus reducing the wall-normal heat flux.

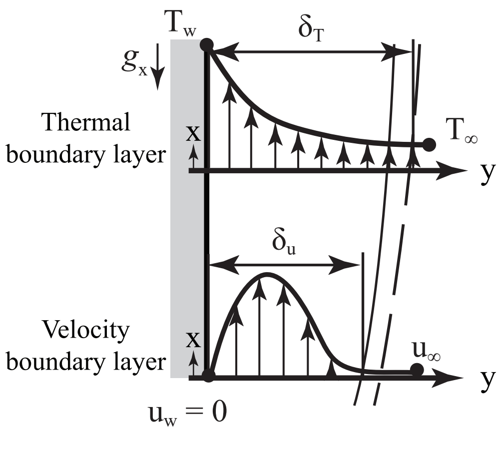


Figure 11.7. Natural convection at a vertical plate.

Concerning the conservation equations in natural convection, the equation of continuity remains the same:

Fundamental EQ

Equation of continuity for natural flow along a vertical plate:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11.15}$$

The momentum equation, as provided in equation (10.5), involves for this case the gravitational force that is pointing in the negative x-direction of the coordinate system. Consequently, the force enters the momentum equation with a negative sign.

The boundary layer assumption of Prandtl, i.e. $\frac{\delta}{L} \ll 1$ yields:

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

Furthermore, the pressure gradient $\frac{dp}{dx}$ is prescribed by the pressure change in the x-direction outside the boundary layer. In forced convection, this was assumed to be negligible, e.g. zero. In natural convection, the pressure changes are imposed due to the weight of the fluid outside of the boundary layer, i.e. $\rho_\infty g$.

$$\left(\frac{\partial p}{\partial x}\right) = -\rho_\infty g$$

Hence, using these two modifications, the momentum equation yields:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g\left(\rho_\infty - \rho\right) + \eta \frac{\partial^2 u}{\partial y^2}$$

Usually, the difference in density in the buoyancy term is written in terms of the volumetric expansion coefficient β for homogeneous media:

Definition

Volumetric expansion coefficient:

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \frac{\rho_\infty - \rho}{\rho\left(T - T_\infty\right)} \left[\frac{1}{\mathrm{K}}\right] \tag{11.16}$$

The volumetric expansion coefficient represents the volume change of a substance with temperature variations. The value is known for many fluids. For ideal gases, the following relation results from the ideal gas law:

Definition

Volumetric expansion coefficient for ideal gases:

$$\beta = \frac{1}{T} \left[\frac{1}{\mathrm{K}}\right] \tag{11.17}$$

Substitution of the definition of the volumetric expansion coefficient into the momentum equation in x-direction yields:

Fundamental EQ

Equation of momentum in the x-direction for natural flow along a vertical plate:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = g\rho\beta\left(T - T_\infty\right) + \eta \frac{\partial^2 u}{\partial y^2} \tag{11.18}$$

Assuming constant properties is justified for many cases, except for the buoyancy term, because this term represents exactly the density deviations that cause the fluid to flow.

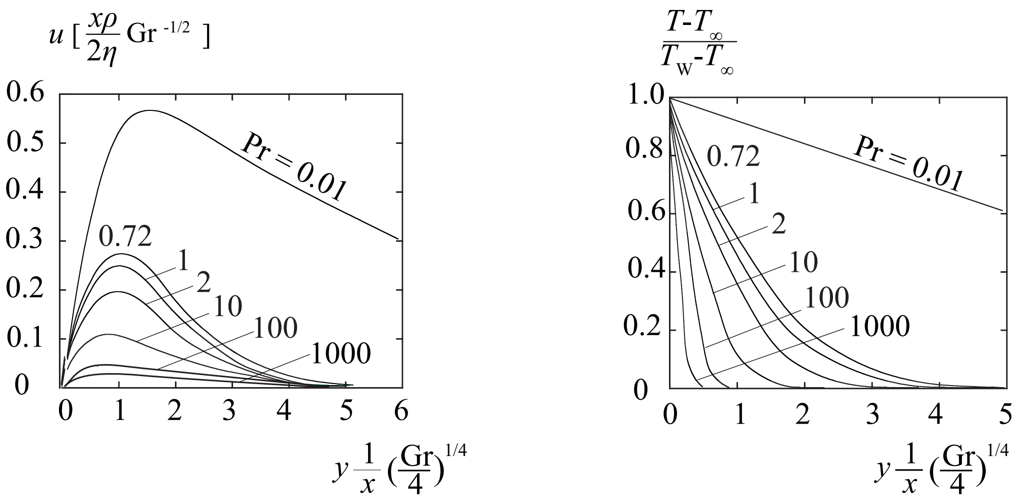
And lastly, the energy equation is the same:

Fundamental EQ

Equation of energy conservation for natural flow along a vertical plate:

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{\eta}{\mathrm{Pr}} \frac{\partial^2 T}{\partial y^2} \tag{11.19}$$

The solution renders more difficult than one of the boundary layer equations for forced convection since knowledge of the temperature field is necessary to solve the momentum equation. Yet, for the simple geometry of the vertical flat plate with constant surface temperature, an exact solution by transformation of the partial differential equations into ordinary differential equations is possible again. These ordinary differential equations have been solved before [6]. The resulting dimensionless velocity and temperature profiles are shown in a dimensionless form in Figure 11.8.



(a) Dimensionless velocity profile.

(b) Dimensionless temperature profile.

Figure 11.8. Dimensionless velocity and temperature profiles at a vertical wall for natural convection according to Ostrach [6].

In these figures, a new dimensionless number, the Grashof number is introduced. The applicability of the Grashof number is discussed in Section 11.3.1.

Definition

Grashof number:

$$\mathrm{Gr} = \frac{\text{Bouyancy forces}}{\text{Viscous forces}} = \frac{\beta g \rho^2 \left(T_\mathrm{W} - T_\infty\right) L^3}{\eta^2} = \frac{\beta g \left(T_\mathrm{W} - T_\infty\right) L^3}{\nu^2} \quad [-]\,, \tag{11.20}$$

where L is the characteristic length.

With the known temperature profile and the gradients at the wall, the local heat transfer coefficient is computed using equation (10.1) and is rewritten in a dimensionless form called the Nusselt number.

HTC

Local Nusselt number for natural laminar flow along a vertical plate with isothermal surface:

$$\mathrm{Nu}_x = 0.508 \left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}}\right)^{\frac{1}{4}} \left(\mathrm{Gr_x Pr}\right)^{\frac{1}{4}}\,, \tag{HTC.16}$$

for $\mathrm{Gr_x} \cdot \mathrm{Pr} < 4 \cdot 10^9$.

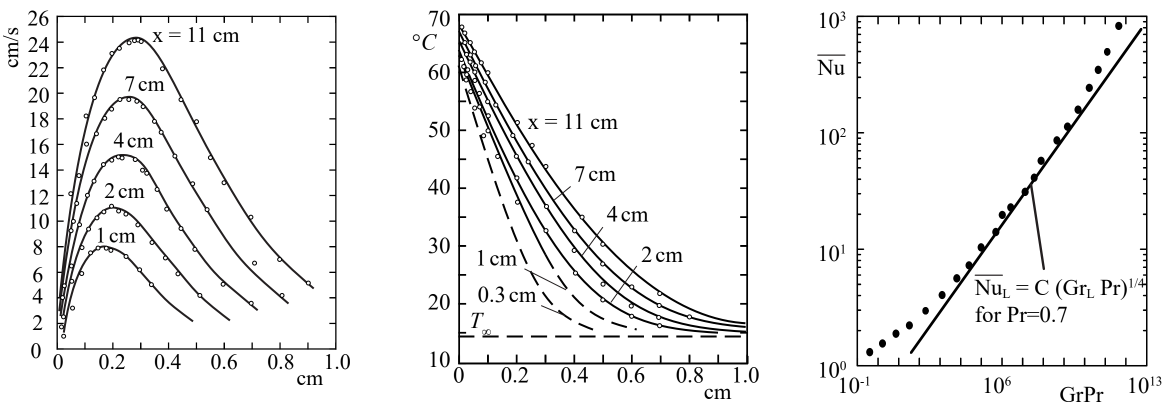
Through integration along the length of the plate, the average Nusselt number are obtained, expressed as:

HTC **Average Nusselt number for natural laminar flow along a vertical plate with isothermal surface:**

$$\overline{\mathrm{Nu}_\mathrm{L}} = C\,(\mathrm{Gr_L Pr})^{\frac{1}{4}}\,, \tag{HTC.17}$$

for $\mathrm{Gr_L} \cdot \mathrm{Pr} < 4 \cdot 10^9$, and:

Pr	0.003	0.01	0.03	0.72	1	2	10	100	1000	∞
C	0.182	0.242	0.305	0.516	0.535	0.568	0.620	0.653	0.665	0.670



(a) Velocity profiles. (b) Temperature profiles. (c) Average Nusselt number.

Figure 11.9. Calculated and measured velocity profiles temperature profiles and average Nusselt number at a vertical plate for natural convection in air, from Whitaker [7].

In Figure 11.9 the calculated values of the velocity profiles, temperature profiles, and average Nusselt number at a vertical, heated plate in air are compared to experimental values. Comparing these profiles for air shows a very good accordance and proves the validity of the simplified assumptions used during the derivation of the conservation laws. For the local mean values of the Nusselt number, the theoretical values conform to the experimental values only in the middle range of (GrPr)-values. For $\mathrm{GrPr} < 10^3$ these boundary layer approximations are no longer valid. In the range of $\mathrm{GrPr} > 10^9$, where heat transfer is intensified by the shift from laminar to turbulent flow. When turbulent flow starts to dominate, turbulence effects play a more significant role in terms of momentum and heat exchange.

These two effects, which are also observed during forced convection, make the theoretical solution of the conservation equations significantly more difficult. If the boundary layer assumptions are dropped, instead of the parabolic differential equations, elliptical differential equations have to be solved, and numerical methods or eventually experiments are necessary. In addition, when the flow shifts from laminar to turbulent, models have to be developed that include the impact of turbulence on the momentum and heat exchange. The turbulent flow characteristics are discussed in Section 12.1.

11.3.1 Grashof number

In forced convection, the flow behavior is primarily dictated by the Reynolds number, which characterizes the ratio of inertial forces to viscous forces acting on the fluid. In natural convection flow characteristics are governed by the dimensionless Grashof number, which represents the ratio of buoyancy forces to viscous forces within the fluid. The Grashof number was given in equation (11.20), and is listed here again:

$$\mathrm{Gr} = \frac{\text{Bouyancy forces}}{\text{Viscous forces}} = \frac{\beta g \rho^2 \left(T_\mathrm{W} - T_\infty\right) L^3}{\eta^2} = \frac{\beta g \left(T_\mathrm{W} - T_\infty\right) L^3}{\nu^2} \quad [-]$$

The role that the Reynolds number plays in forced convection is analogous to the role played by the Grashof number in natural convection. Consequently, the Grashof number serves as the primary criterion for determining whether fluid flow in natural convection is laminar or turbulent. As an example, consider vertical plates where the critical Grashof number, denoted as $\mathrm{Gr}_\mathrm{crit}$, is approximately $4 \cdot 10^9$. Consequently, the flow undergoes a transition to a turbulent state when the Grashof number Gr exceeds $4 \cdot 10^9$.

Section 12

Turbulence and dimensional analysis

L04 - Turbulent flow:

Learning goals:

- Recognizing the conditions leading to the onset of turbulent flow.
- Grasping the macroscopic implications of turbulent fluctuations in mass and heat transport.

Comprehension questions:

- How does turbulence affect heat transfer?

L05 - Dimensional analysis:

Learning goals:

- Developing a foundational understanding of dimensional analysis.
- Grasping the physical significance of pertinent dimensionless numbers used to characterize convection problems.
- Gaining the capability to differentiate various convective heat transfer scenarios based on flow and boundary conditions.

Comprehension questions:

- What are the fundamental principles conveyed by dimensional analysis, and what factors must be considered to ensure identical solutions for two distinct problems?
- Which specific dimensionless numbers play a crucial role in empirically derived heat transfer laws?

Subsection 12.1
Turbulent flow

The previous sections have focused on heat transfer from a wall to a laminar flowing fluid. This allowed for a straightforward solution of the conservation equations in the case of simple geometries. In the case of turbulent flows, when flow instabilities lead to the generation of vortex structures, the same equations remain valid, however, their solution becomes more complicated. Because the turbulent structures are of very small spatial and temporal scales, the numerical solution of conservation equations becomes expensive in terms of computational resources.

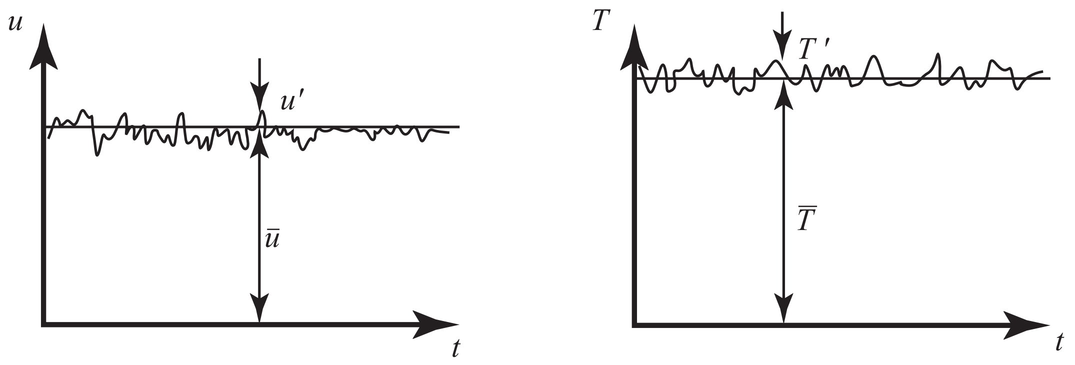
One possible solution to overcome the difference in time- and length scale is to perform localized averaging of the conservation equations. For this, the velocity, pressure, and temperature field is segregated into an average value, e.g. $\bar{u}$ for the velocity, and a fluctuation u'. The spatial and temporal changes of the average value are on a larger length- and time scale, which allows for an adequate computation using a coarser resolution.

Definition

Velocity in the x-direction for average steady-state turbulent flow:

$$u = \overline{u} + u' \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{12.1}$$

However, the small-scale fluctuations also influence the large-scale structures as they intensify the transport of momentum and heat. To identify the contribution of the fluctuations, the conservation equations are rewritten, replacing the field variables u, p, T, with the respective combinations, e.g. $\bar{u}+u'$. A reformulation and averaging of the equation reveal one additional term that does not cancel out in the averaged conservation equation, namely the product of the fluctuations $\overline{u'v'}$. This term describes the transport of momentum due to small-scale velocity fluctuations. Linking the concept of momentum diffusion to this turbulent transport process, the momentum transport by $\overline{u'v'}$ can be interpreted as additional shear stress within the liquid. Now, by reconsidering the formulation of shear stresses in the momentum conservation equation through viscosity and a velocity gradient, the turbulent shear stress is reformulated using a concept known as "turbulent viscosity". The same principle applies to heat transport through fluctuations in the velocity and temperature field.



(a) Velocity profile as a function of time. (b) Temperature profile as a function of time.

Figure 12.1. On the average steady state, turbulent flow.

Definition

Turbulent shear stress

$$\tau_{\mathrm{tur}} = -\rho\overline{u'v'} = \eta_{\mathrm{t}} \frac{\partial \overline{u}}{\partial y} \; [\mathrm{Pa}], \tag{12.2}$$

where η_{t} refers to the turbulent viscosity.

Definition

Turbulent heat diffusion

$$\dot{q}''_{\text{tur}} = \rho c_p \overline{v'T'} = -\lambda_{\text{t}} \frac{\partial \overline{T}}{\partial y} \left[\frac{\text{W}}{\text{m}^2}\right], \tag{12.3}$$

where λ_{t} refers to the turbulent thermal conductivity.

With the definition of the turbulent shear stress and the turbulent heat diffusion, the continuity equations for the averaged quantities are rewritten. A detailed explanation is given in Appendix A. This results:

Fundamental EQ

Equation of continuity for incompressible turbulent flow:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{12.4}$$

Fundamental EQ

Equation of momentum in the x-direction for turbulent flow:

$$\rho \left(\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y}\right) = -\left(\frac{\partial \overline{p}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta_{\text{eff}} \frac{\partial \overline{u}}{\partial y}\right), \tag{12.5}$$

with $\eta_{\text{eff}} = \eta + \eta_t$.

Fundamental EQ

Equation of energy conservation for turbulent flow:

$$\rho c_p \left(\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial x}\right) = \frac{\partial}{\partial y}\left(\lambda_{\text{eff}} \frac{\partial \overline{T}}{\partial y}\right), \tag{12.6}$$

with $\lambda_{\text{eff}} = \lambda + \lambda_t$

Contrary to viscosity and thermal conductivity, turbulent viscosity, and turbulent thermal conductivity are not material properties. These values depend on the location within the flow as well as on the history. Close to walls, the turbulent fluctuations reduce in intensity, which also leads to a reduction in the value of the turbulent transport properties. In a pipe flow configuration, for instance, the turbulent fluctuations are highest in the center, far away from the walls. Due to the higher apparent viscosity, the velocity profile is nearly flat in the center with steep gradients close to the wall. Because of the steep gradients, both friction and heat transfer are intensified in turbulent flows.

New conservation equations are required to describe the spatial and temporal evolution of these quantities. These conservation equations include the production, advection, and destruction of turbulent fluctuations. The conservation equations are called "turbulence models". Unfortunately, not one generally valid model approach has been identified, such that there exists a variety of model approaches that work better and worse compared to the application and flow type.

Because of the complexity, this course will not go into the details of turbulence modeling, and the interested reader is advised to check the relevant literature.

SUBSECTION 12.2
Dimensional analysis

The ongoing development of computer technology allows now for fast numerical simulations of many heat transfer and fluid flow problems. Nevertheless, initializing and running the computational simulation still requires a severe amount of time and effort. Consequently, a faster approach is the use of empirical heat transfer correlations for an initial design. These laws are a result of measurements, which are shown graphically or in the form of empirical formulas.

This section demonstrates that there is no imperative need to replicate experiments when alterations are made to fluid properties, geometric configurations, or flow conditions. Previous discussions on analytical solutions for heat transfer in forced and natural convection have revealed that the outcomes are succinctly expressed through a set of key characteristic numbers.

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\mathrm{Pr}) \text{ for forced convection}$$
$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Gr},\mathrm{Pr}) \text{ for natural convection}$$

In instances where a solution to the conservation laws remains elusive, the method of dimensional analysis can frequently be employed to minimize the array of parameters influencing heat transfer. The relationships among these coefficients are derived through targeted measurements.

Derivation To determine the characteristic numbers, the conservation equations for natural convection along a vertical plate, as derived in Section 11.3, are rewritten in a dimensionless form.

1 **Setting up the balance:**

Refer to Section 11.3.

2 **Defining the elements within the balance:**

Refer to Section 11.3.

3 **Inserting and rearranging:**

To reduce complexity, the two-dimensional, laminar boundary layer equations with constant properties in the form of equation (11.15) to (11.19), with an additional pressure gradient $\left(\frac{dp}{dx}\right)_{\mathrm{kin}}$ in the momentum equation is regarded. Note, while deriving the momentum equation for the vertical flat plate (11.18) only the pressure gradient, describing the main fluid $\left(\frac{dp}{dx}\right)_{\mathrm{pot}}$ was required, and the pressure gradient caused by the acceleration of the fluid $\left(\frac{dp}{dx}\right)_{\mathrm{kin}}$, as in a tube flow, was omitted. Hence the more general case is discussed here.

Equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum equation:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\left(\frac{dp}{dx}\right)_{\mathrm{kin}} + \eta \frac{\partial^2 u}{\partial y^2} - g_{\mathrm{x}} \rho \beta \left(T - T_\infty\right)$$

Energy equation:

$$\rho\, u\, c_p \frac{\partial T}{\partial x} + \rho\, v\, c_p \frac{\partial T}{\partial y} = \lambda \frac{\partial^2 T}{\partial y^2}$$

All the variables in these equations are required to be non-dimensionalized using appropriate characteristic parameters that depend on the specific case at hand. For instance, the coordinates x and y are normalized relative to a characteristic length L, which can be the diameter d for objects like spheres or tubes, or the length L of a flat plate for scenarios involving flow over a plate.

Similarly, the velocity components u and v are divided by the relevant characteristic velocity u_∞, which is the mean velocity for flow in tubes and the free stream velocity for flow over a plate. Similar to pressure, this quantity is normalized to twice the dynamic pressure, and temperature is non-dimensionalized using a characteristic temperature difference, like the temperature difference between the surface and the fluid at a point further away from the object.

Introducing these dimensionless variables:

$$\mathrm{x}^* = \frac{\mathrm{x}}{L};\ \ \mathrm{y}^* = \frac{\mathrm{y}}{L};\ \ u^* = \frac{u}{u_\infty};\ \ v^* = \frac{v}{u_\infty};\ \ p^* = \frac{p}{\rho u_\infty^2};\ \ \theta^* = \frac{T - T_\mathrm{W}}{T_\infty - T_\mathrm{W}}$$

and substituting the dimensionless variables into the conservation equations yields the dimensionless conservation equations:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{dp^*}{dx^*}\right)_\mathrm{kin} + \left(\frac{\eta}{\rho u_\infty L}\right)\frac{\partial^2 u^*}{\partial y^{*2}} + \left(\frac{g_x \beta L (T_\mathrm{W} - T_\infty)}{u_\infty^2}\right)\theta^*$$

$$u^*\frac{\partial \theta^*}{\partial x^*} + v^*\frac{\partial \theta^*}{\partial y^*} = \left(\frac{\lambda}{\rho c_p u_\infty L}\right)\frac{\partial^2 \theta^*}{\partial y^{*2}}$$

In addition to the dimensionless variables, these equations contain dimensionless quantities in the brackets. A closer look at these expressions, reveals that these are the dimensionless numbers from which some of them have been discussed before.

The dimensionless momentum equation contains two of these terms.

First, the Reynolds number:

$$\mathrm{Re} = \frac{\rho u_\infty L}{\eta}$$

And the Archimedes number:

$$\mathrm{Ar} = \frac{\mathrm{Gr}}{\mathrm{Re}^2} = \frac{g_\mathrm{x} \beta L (T_\mathrm{W} - T_\infty)}{u_\infty^2}$$

The dimensionless energy equation contains the Peclet number:

$$\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr} = \left(\frac{\rho u_\infty L}{\eta}\right)\left(\frac{\eta c_p}{\lambda}\right)$$

Note that at this point the Archimedes number and Peclet number have not been discussed before, but will be later in Sections 12.2.1 and 12.2.2 respectively.

Substitution of these dimensionless numbers yields:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{dp^*}{dx^*}\right)_\mathrm{kin} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\theta^*$$

$$u^*\frac{\partial \theta^*}{\partial x^*} + v^*\frac{\partial \theta^*}{\partial y^*} = \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial^2 \theta^*}{\partial y^{*2}}$$

In various scenarios, for instance, when a different type of fluid flows through a pipe with a varied diameter, if the dimensionless numbers align or a combination thereof remains consistent, the solutions to the differential equations will similarly align. However, also the boundary conditions must correspond between the two scenarios. The solution represent the dimensionless profiles of velocity $\vec{u}$, temperature θ^*, and pressure p^*.

Consequently, the velocity and temperature profiles are expressed as functions of these common parameters.

$$\vec{u}=\vec{u}\ (\mathrm{x}^*,\mathrm{y}^*,\mathrm{Re},\mathrm{Gr},\mathrm{Pr})$$

and

$$\theta^* = \theta^*\ (\mathrm{x}^*,\mathrm{y}^*,\mathrm{Re},\mathrm{Gr},\mathrm{Pr})$$

The heat transferred is derived from the gradient of the temperature profile in the fluid at the surface of the body.

An energy balance at the surface yields:

$$-\left(\lambda\frac{\partial T}{\partial y}\right)_{\mathrm{W}} = \alpha\left(T_{\mathrm{W}} - T_{\infty}\right)$$

Rewriting gives:

$$\alpha = \frac{-\left(\lambda\frac{dT}{dy}\right)_{\mathrm{W}}}{\left(T_{\mathrm{W}} - T_{\infty}\right)},$$

which is written in the dimensionless heat transfer coefficient, the Nusselt Number:

$$\mathrm{Nu} = \frac{\alpha L}{\lambda} = -\left(\frac{d\theta^*}{dy^*}\right)_{\mathrm{W}}$$

The Nusselt number is thus equal to the dimensionless temperature gradient and hence, can be expressed in terms of the dimensionless numbers, in a similar manner as the temperature field

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\mathrm{Gr},\mathrm{Pr})$$

Consequently, problems sharing similar geometries and boundary conditions are formally represented by four dimensionless numbers, a principle that holds even in the context of turbulent flow, as long as the turbulence levels remain relatively comparable. □

HTC **General Nusselt number function:**

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\mathrm{Gr},\mathrm{Pr}) \tag{12.7}$$

For many practical purposes, equation (12.7), is further simplified.

For forced convection, the inertial and friction forces exceed the buoyant forces resulting from the temperature changes. Neglecting the buoyancy term in the equation of motion leads to:

HTC **General Nusselt number function forced convection:**

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Re},\ \mathrm{Pr}) \tag{12.8}$$

For natural convection, the velocity field is caused by temperature differences, and a velocity u_∞ induced by external sources is not present. The inertial forces may be neglected, hence:

HTC

General Nusselt number function natural convection:

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Gr}, \mathrm{Pr}) \tag{12.9}$$

Note that dimensional analysis cannot state the form of the functional relationship between the Nu number and the Re, Gr, Pr numbers. This relationship can either be found from the analytical solutions, if these are expressed in characteristic numbers, or from experiments. Often, experimental and numerical simulation results are expressed in the form of power law, i.e.:

$$\mathrm{Nu} = C\,\mathrm{Re}^{\mathrm{m}}\mathrm{Pr}^{\mathrm{n}}\mathrm{Gr}^{\mathrm{p}}$$

12.2.1 Archimedes number

The Archimedes number Ar is a dimensionless parameter that plays a role in fluid dynamics, specifically in the context of buoyant forces and natural convection. This dimensionless number represents the ratio of buoyancy forces to viscous forces acting within a fluid. In essence, the Archimedes number quantifies the relative significance of gravitational or buoyant effects compared to the dissipative effects of viscosity in a fluid. When $\mathrm{Ar} \gg 1$, buoyant forces dominate, and the fluid motion is predominantly driven by these buoyancy effects. Conversely, when $\mathrm{Ar} \ll 1$, viscous forces are the primary driving force, and the fluid behavior is characterized by viscosity-dominant flow.

Definition

Archimedes number:

$$\mathrm{Ar} = \frac{\mathrm{Gr}}{\mathrm{Re}^2} = \frac{\text{Bouyance forces}}{\text{Inertia forces}} = \frac{g_{\mathrm{x}}\beta L(T_{\mathrm{W}} - T_\infty)}{u_\infty^2} \quad [-], \tag{12.10}$$

where L is the characteristic length.

12.2.2 Peclet number

The Peclet number, typically denoted as Pe, is a dimensionless parameter used in various fields of science and engineering, particularly in the study of heat and mass transfer. This number provides insights into the relative significance of convective transport compared to diffusive transport in a given system.

A high Peclet number implies that advective transport dominates over diffusive transport, signifying that the motion of the fluid carries heat or mass more efficiently than molecular diffusion. Conversely, a low Peclet number indicates that diffusive transport is more influential, with molecular diffusion being the primary mode of heat or mass transfer.

Definition

Peclet number:

$$\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr} = \frac{\text{Rate of advection}}{\text{Rate of diffusion}} = \frac{\rho c_p u L}{\lambda} \quad [-], \tag{12.11}$$

where L is the characteristic length.

12.2.3 The Reynolds analogy

The Reynolds analogy is a result of dimensional analysis in heat transfer that establishes a relationship between heat transfer and momentum transfer in a flowing fluid. The analogy provides a framework for analyzing and predicting the simultaneous transport of heat and momentum within a fluid medium.

Derivation As an example, consider the case of turbulent flow over a flat plate. As presented before, the dimensionless momentum and energy conservation equation within the boundary layer are:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{dp^*}{dx^*}\right)_{\text{kin}} + \frac{1}{\text{Re}}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\text{Gr}}{\text{Re}^2}\theta^*$$

$$u^*\frac{\partial \theta^*}{\partial x^*} + v^*\frac{\partial \theta^*}{\partial y^*} = \frac{1}{\text{Re Pr}}\frac{\partial^2 \theta^*}{\partial y^{*2}}$$

For forced convection over a flat plate $\left(\frac{dp^*}{dx^*}\right)_{\text{kin}} = 0$, and $\text{Gr} \ll \text{Re}^2$.

Moreover, in the instance where Pr = 1, an approximation often applicable to gases, the conservation equations for both momentum and energy share identical forms:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial \theta^*}{\partial x^*} + v^*\frac{\partial \theta^*}{\partial y^*} = \frac{1}{\text{Re}}\frac{\partial^2 \theta^*}{\partial y^{*2}}$$

Also, the boundary conditions of these equations are similar.

At the wall:

$$u^*(y=0) = \frac{0}{u_\infty} = 0 \qquad \theta^*(y=0) = \frac{T_{\text{W}} - T_{\text{W}}}{T_\infty - T_{\text{W}}} = 0$$

and within the free steam:

$$u^*(y\to\infty) = \frac{u_\infty}{u_\infty} = 1 \qquad \theta^*(y\to\infty) = \frac{T_\infty - T_{\text{W}}}{T_\infty - T_{\text{W}}} = 1$$

Hence, the solution for the dimensionless velocity, u^*, and temperature profile, θ^*, must be equivalent.

Recalling the definitions of the friction coefficient and the Nusselt number, these are expressed in a dimensionless manner:

$$\frac{c_{\text{f}}}{2} = \frac{\tau_{\text{W}}}{\rho u_\infty^2} = \frac{\eta}{\rho u_\infty L}\left(\frac{\partial u^*}{\partial y^*}\right)_{\text{W}} = \frac{1}{\text{Re}}\left(\frac{\partial u^*}{\partial y^*}\right)_{\text{W}}$$

$$\text{Nu} = \frac{-\lambda_{\text{f}}\left(\frac{\partial T_{\text{f}}}{\partial y}\right)_{\text{W}}}{T_{\text{W}} - T_\infty}\frac{L}{\lambda_{\text{f}}} = -\frac{\lambda_{\text{f}}\left(T_\infty - T_{\text{W}}\right)}{L\left(T_{\text{W}} - T_\infty\right)}\frac{L}{\lambda_{\text{f}}}\left(\frac{\partial \theta^*}{\partial y^*}\right)_{\text{W}} = \left(\frac{\partial \theta^*}{\partial y^*}\right)_{\text{W}}$$

Because the solution for u^* and θ^* is equivalent, it is stated that:

$$\left(\frac{\partial \theta^*}{\partial y^*}\right)_{\text{W}} = \left(\frac{\partial u^*}{\partial y^*}\right)_{\text{W}}$$

This yields the Reynolds analogy:

$$\mathrm{Nu} = \frac{c_\mathrm{f}}{2}\mathrm{Re}$$

□

HTC

Reynolds analogy:

$$\mathrm{Nu} = \frac{c_\mathrm{f}}{2}\mathrm{Re} \tag{12.12}$$

for $\mathrm{Pr} = 1$, $\left(\frac{dp^*}{dx^*}\right)_\mathrm{kin} = 0$, and $\mathrm{Gr} \ll \mathrm{Re}^2$

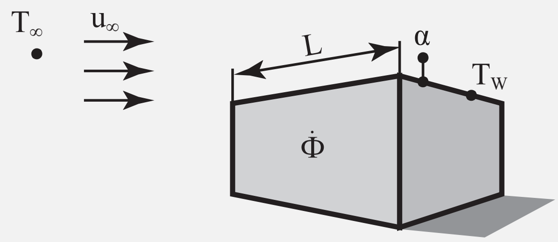
If fluid dynamical quantities (velocity profile, friction coefficient) are known, the heat transfer parameter (temperature profile, heat transfer coefficient) can be estimated or vice versa.

Example 12.1

The roof of a building (with $L = 10$ m and $A = 100$ m^2) is exposed to a flow ($\eta = 1.778 \times 10^{-5}$ Pa · s, $\rho = 1.246$ kg/m^3, $c_\mathrm{p} = 1006$ J/kgK, $\lambda = 0.024$ W/mK) parallel to the length, with a velocity $u_\infty = 3$ m/s. The temperature of the roof is $T_\mathrm{W} = 15$ °C, and the air temperature far away is $T_\infty = 10$ °C. The drag force on the roof is measured to be $D = 3.49$ N. Determine the rate of heat generated within the center.

Hints:

- Heat losses from the walls are negligible.
- Assume steady-state operating conditions.
- Radiative losses are negligible.



① **Setting up the balance:**

When setting up an energy balance around the building, all heat being generated is being lost through the roof by convection:

$$0 = \dot{\Phi} - \dot{Q}_\mathrm{conv}$$

where $\dot{\Phi}$, the rate of heat generation, has to be determined.

② **Defining the elements within the balance:**

The rate of heat loss is expressed by use of Newton's law of cooling:

$$\dot{Q}_\mathrm{conv} = \alpha A\,(T_\mathrm{w} - T_\infty)$$

First, the Reynolds number needs to be checked to determine the flow regime:

$$\mathrm{Re}_\mathrm{L} = \frac{\rho u_\infty L}{\eta} = 2.1 \cdot 10^6$$

Because $\mathrm{Re_L} > 2 \cdot 10^5$, the flow is of turbulent type. For this flow regime, until here, no Nusselt correlation has been given.

The roof is considered to be a flat plate, and therefore $\left(\frac{dp^*}{dx^*}\right)_{\mathrm{kin}} = 0$, and $\mathrm{Gr} \ll \mathrm{Re}^2$.

To see whether the Reynolds analogy can be used, the Prandlt number needs to be checked.

$$\mathrm{Pr} = \frac{c_\mathrm{p}\eta}{\lambda} = 0.75$$

The Prandtl number is relatively close to 1, and thus the Reynolds analogy may offer a good approximation.

The shear stress yields from:

$$\tau_\mathrm{w} = \frac{D}{A} = 0.035 \text{ Pa}$$

From this, the dimensionless friction coefficient is defined:

$$\frac{c_\mathrm{f}}{2} = \frac{\tau_\mathrm{W}}{\rho u_\infty^2} = 3.1 \cdot 10^{-3}$$

Using the Reynolds analogy, the Nusselt number is determined:

$$\mathrm{Nu} = \frac{c_\mathrm{f}}{2}\mathrm{Re} = 6.5 \cdot 10^3$$

Rewriting the definition of the Nusselt number yields the heat transfer coefficient:

$$\alpha = \frac{\mathrm{Nu}\lambda}{L} = 15.7 \text{ W/m}^2\text{K}$$

3 **Inserting and rearranging:**

Finally, the heat generated by the building yields from rewriting the energy balance:

$$\dot{\Phi} = \alpha A \left(T_\mathrm{w} - T_\infty\right) = 7.9 \text{ kW}$$

Section 13

External forced convection

L06 - External forced convection:

Learning goals:

- Acquiring knowledge and comprehension of dimensionless numbers.
- Gaining an overview of diverse application scenarios and the corresponding correlations.

Comprehension questions:

- Which dimensionless numbers must be taken into account in forced convection, and how is the suitability of a correlation assessed?
- At which temperature should the material properties influencing dimensionless numbers be determined?
- What distinguishes local from average heat transfer in the context of a flat plate subjected to heating or cooling?

Corresponding tutorial exercises:

- Exercise III.6 Moving train
- Exercise III.7 Transverse flow
- Exercise III.8 Heating of a cylinder

Subsection 13.1
Heat transfer correlations

The following sections describe the heat transfer correlations for flat plates, cylinders that are in a flow parallel and perpendicular to their longitudinal axis, bundles of smooth tubes in an orthogonal flow, and flow around spheres. These correlations are used to calculate for example the rate of heat loss of various real-world examples. Note that these correlations serve only a portion of the correlation that exists in the literature.

The heat flux from a wall with the surface area A and temperature T_{W} to a fluid with the temperature T_{f} is:

$$\dot{Q}_{\mathrm{W}} = \alpha A \left(T_{\mathrm{W}} - T_{\mathrm{f}}\right)$$

To determine the heat transfer coefficient α, Nusselt number correlations for several important applications need to be used. The upcoming sections delve into these correlations.

But before, the applicability of the given correlations always should be checked for each case according to a set of criteria.

Approach 13.1 **Checking the applicability of heat transfer correlations:**

1. **Classification:** The flow of the given problem must be classified. In some cases, forced convection dominates, while in others, the absence of this forced driving force leads to the dominance of natural convection. Additionally, a classification of whether the convection exhibits an external or internal behavior is essential.

2. **Object:** The geometry of the given problem must be analogous to the geometry of the experiment from which the heat transfer law is estimated.

3. **Regime:** The flow profile must be checked using the critical Reynolds and Grashof numbers in the range describing the shift from laminar to turbulent flow. A law valid for laminar flow cannot be applied to turbulent flow.

4. **Thermal boundary conditions**: The thermal boundary conditions of the investigated case should be compared to those under which the applied heat transfer law was derived. Valid laws, for example, for flow in tubes with constant wall temperature can be used only with restrictions for cases with a constant heat flux rate on the wall.

5. **Temperature changes**: For cases where high-temperature changes occur in the system, care should be taken to use the same reference temperature at which the properties were determined. In most of the heat transfer laws, the properties are to be determined at a mean temperature. In special cases, where the variable properties are to be described, the correlations are extended to include additional terms, either those of temperature ratios or viscosity ratios.

6. **Characteristic numbers**: Verify that the field of characteristic numbers in which the law was determined matches one of the problems under investigation. This is especially important for laws that have been determined experimentally, which describe the functional relationships only approximately. In some cases, some specific characteristics are used.

7. **Fluid temperature**: In the case of internal forced convection ensure that the caloric mean value is used to determine the temperature. More information about this topic follows in Section 14.

Literature encompasses a wide array of Nusselt correlations. The suitability of a particular Nusselt correlation for diverse objects is contingent upon various factors. Figure 13.1 offers a roadmap to assist in selecting the appropriate Nusselt correlation from among the numerous correlations presented in this course reader.

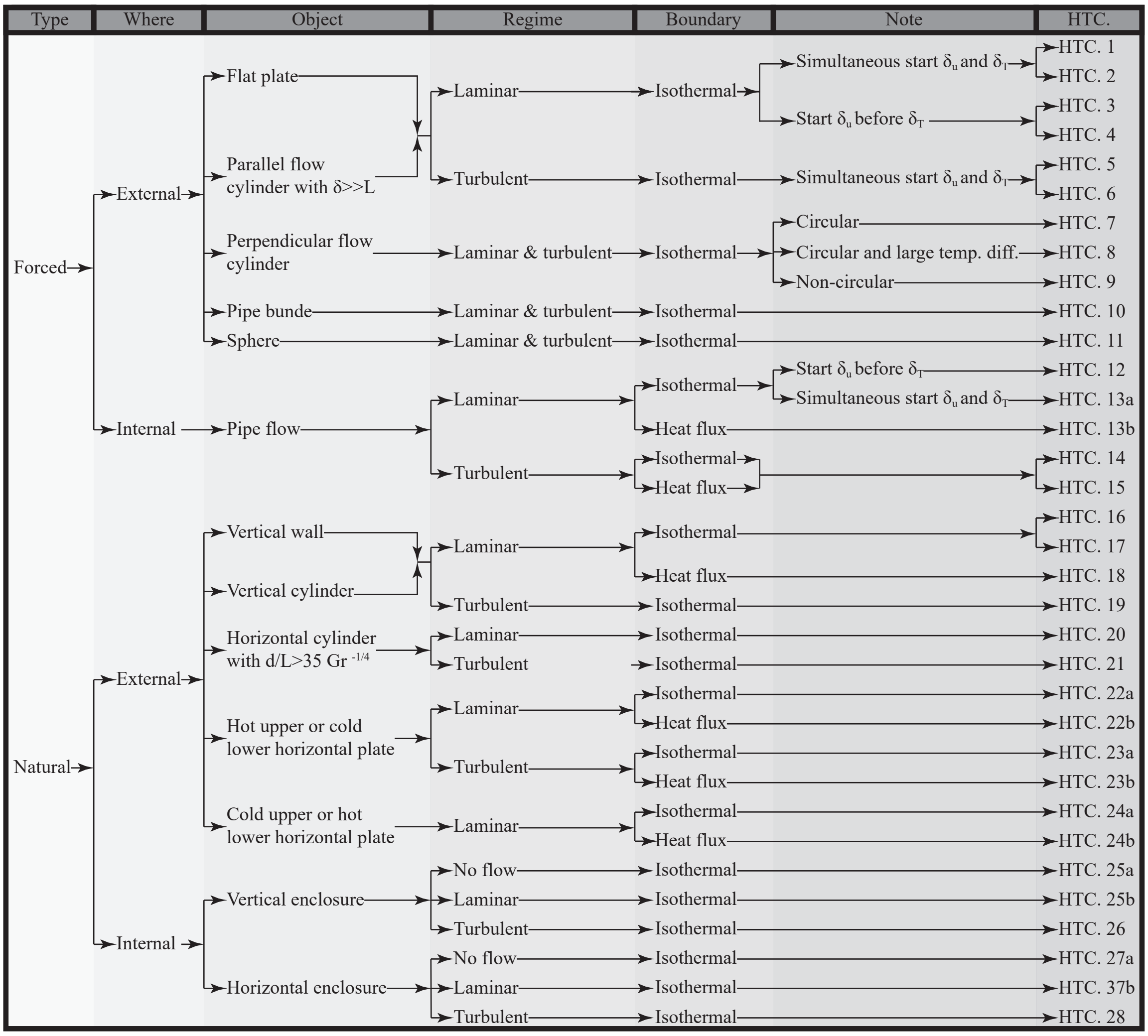


Figure 13.1. Heat transfer correlations roadmap.

The fluid properties used in the correlations are to be evaluated at, if not stated otherwise, the average film temperature.

Definition

Fluid property temperature external forced convection:

$$T_{\text{prop}} = \frac{T_{\text{W}} + T_{\infty}}{2} \text{ [K]} \tag{13.1}$$

Subsection 13.2

Introduction to external forced convection

External forced convection is a fundamental flow configuration in fluid dynamics and heat transfer that occurs when a fluid, such as air or a liquid, flows over the surface of a solid object. In this context, "external" signifies that the flow is in contact with the exterior of the solid, and "forced" convection implies that the motion of the fluid is driven by an external force, typically generated by mechanical means like pumps, fans, or natural winds.

This specific section provides various Nusselt correlations that can be applied to different scenarios where external convection plays an important role. Think hereby of flow over a flat plate or flow around a sphere.

13.2.1 Flow over a flat plate

The flat plate example serves a practical role in various applications, such as airflow over a solar panel or the roof of a train. Different correlations exist depending on the flow regime or the location where the thermal boundary layer begins. This discussion primarily focuses on three scenarios: laminar flow where both the hydrodynamic and thermal inflows initiate simultaneously, laminar flow with the thermal inflow beginning after the hydrodynamic inflow, and turbulent flow with both the hydrodynamic and thermal inflows starting concurrently. All these cases involve a flat plate with an isothermal surface.

The shift from laminar to turbulent flow, as illustrated in Figure 13.2, depends on the degree of turbulence of the undisturbed flow, the surface roughness, etc. For flow over flat surfaces the critical Reynolds number, i.e. the beginning of the eddy area is expected at $1 \cdot 10^5$ to $2 \cdot 10^5$. In extremely low turbulence wind tunnels, critical Reynolds numbers in the order of $\mathrm{Re}_{x,crit} \approx 1 \cdot 10^6$ are reached.

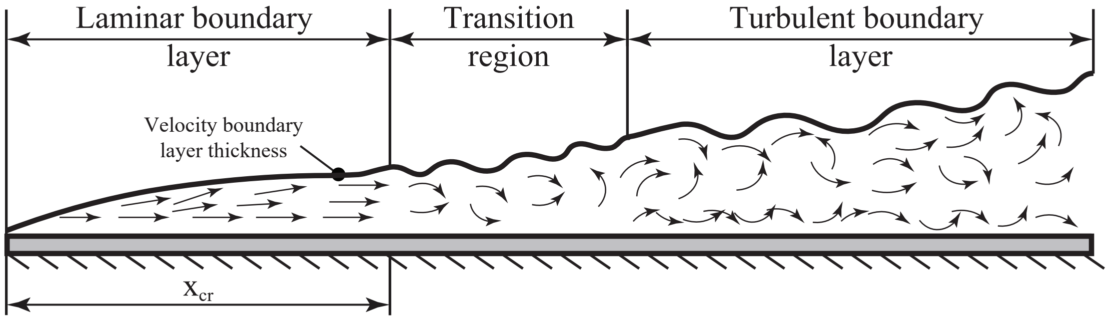


Figure 13.2. Laminar to the turbulent transition of the boundary layer of a plate.

Simultaneous hydrodynamic and thermal inflow, isothermal surface, laminar flow

The heat transfer correlations, given in HTC.1 and HTC.2, represent "exact" solutions of the conservation equations for laminar flow, as previously explained. By employing local Nusselt number correlations, the average Nusselt number has been derived. These equations apply when both hydrodynamic and thermal inflows originate from the same position, and the flow remains laminar throughout, with the entire surface maintained at an isothermal condition.

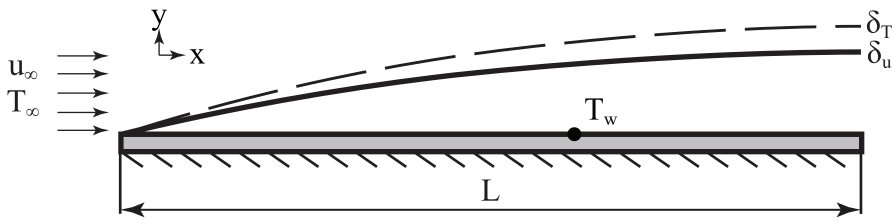


Figure 13.3. Laminar boundary layer of a plate with isothermal surface.

HTC **Local Nusselt number for forced laminar flow over a flat plate with isothermal surface:**

$$\mathrm{Nu_x} = 0.332\mathrm{Re_x}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}, \tag{HTC.1}$$

for $\mathrm{Re_x} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

HTC **Average Nusselt number for forced laminar flow over a flat plate with isothermal surface:**

$$\overline{\mathrm{Nu}}_\mathrm{L} = 0.664\,\mathrm{Re_L}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}, \tag{HTC.2}$$

for $\mathrm{Re_L} < 2 \cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

Hydrodynamical inflow, heated or cooled from position $x = x_0$, isothermal surface, laminar flow

In certain scenarios, the velocity boundary layer starts to develop before the thermal boundary layer does. This situation arises when the temperature of an initial portion of a flat plate is considered equal to that of the flow, thereby preventing the formation of a thermal boundary layer. However, at a subsequent position x_0, the plate experiences heating, and a thermal boundary layer forms. In the context of laminar flow over a surface that undergoes isothermal heating, and under the condition of laminar flow, correlations HTC.3 and HTC.4 are valid.

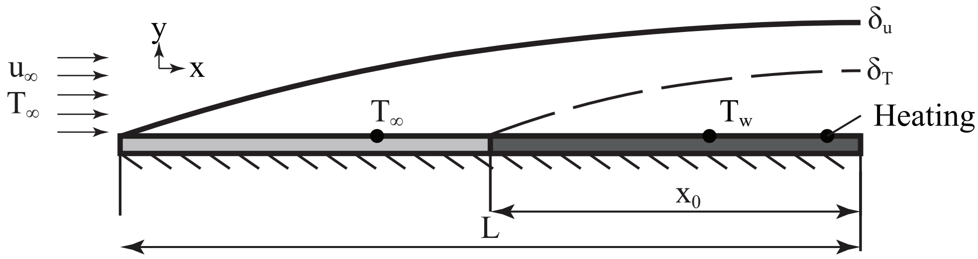


Figure 13.4. Laminar boundary layer of a plate, heated from $x = x_0$ with isothermal surface.

HTC **Local Nusselt number for forced laminar flow over a flat plate with isothermal surface and first hydrodynamic inflow:**

$$\mathrm{Nu_x} = 0.332\,\mathrm{Re_x}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}\left[1-\left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}, \tag{HTC.3}$$

for $\mathrm{Re_x} < 2\cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

HTC **Average Nusselt number for forced laminar flow over a flat plate with isothermal surface and first hydrodynamic inflow:**

$$\overline{\mathrm{Nu}}_\mathrm{L} = 0.664\,\mathrm{Re_L}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}\frac{\left[1-\left(\frac{x_0}{L}\right)^{\frac{3}{4}}\right]^{\frac{2}{3}}}{\left[1-\frac{x_0}{L}\right]}, \tag{HTC.4}$$

for $\mathrm{Re_L} < 2\cdot 10^5$ and $0.6 < \mathrm{Pr} < 10$.

Simultaneous hydrodynamic and thermal inflow, isothermal surface, turbulent flow

When the flow exhibits a relatively high velocity, the transitioning into a turbulent state occurs at a critical location, frequently occurring when $\mathrm{Re_x} \approx 2\cdot 10^5$ is surpassed. In such instances, there is relied on correlations HTC.5 and HTC.6, both of which are derived from experimental data and remain valid under turbulent flow conditions while simultaneously initiating the development of the velocity and thermal boundary layer and having an isothermal surface.

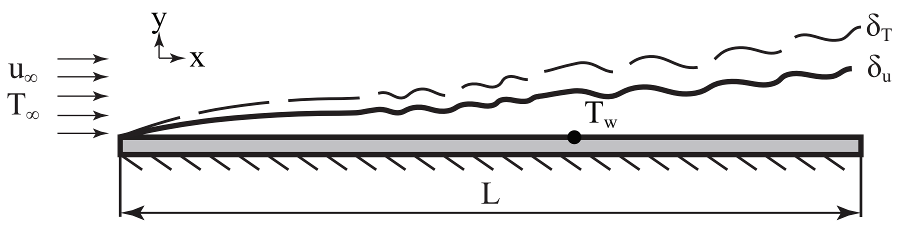


Figure 13.5. Turbulent boundary layer of a plate with isothermal surface.

HTC **Local Nusselt number for forced turbulent flow over a flat plate with isothermal surface:**

$$\mathrm{Nu_x} = 0.0296\,\mathrm{Re_x}^{0.8}\,\mathrm{Pr}^{0.43}, \tag{HTC.5}$$

for $5\cdot 10^5 < \mathrm{Re_x} < 10^7$.

HTC **Average Nusselt number for forced turbulent flow over a flat plate with isothermal surface:**

$$\overline{\mathrm{Nu}}_\mathrm{L} \approx 0.036\,\mathrm{Pr}^{0.43}\left(\mathrm{Re_L}^{0.8} - 9400\right), \tag{HTC.6}$$

for $5\cdot 10^5 < \mathrm{Re_L} < 10^7$.

13.2.2 Flow parallel along to the longitudinal axis of a cylindrical rod

In other real-world examples, the flow passes along the longitudinal axis of a cylinder. If the body's diameter significantly exceeds the thickness of the boundary layer, cylinders in longitudinal flow can be approximated as flat plates. Under these conditions, correlations HTC.1 through HTC.6 are applicable.

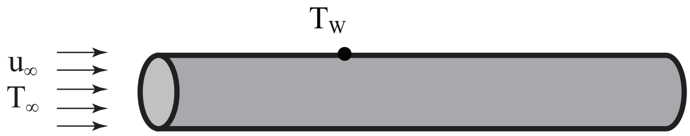


Figure 13.6. Flow parallel along to the longitudinal axis of a cylinder.

Criterion

Criterion for approximating a parallel flow along a cylinder's longitudinal axis as flow over a flat plate:

$$d \gg \delta \tag{13.2}$$

If applicable, HTC.1 to HTC.6 can be used.

13.2.3 Flow perpendicular to the longitudinal axis of a cylindrical rod

In situations where fluid flow is directed perpendicularly to the longitudinal axis of a cylinder, as is the case in cross-flow over heat exchanger tubes, the local Nusselt number exhibits significant variations influenced by the circumferential angle. These variations result from the transition from laminar to turbulent flow and the detachment of the flow from the cylinder surface.

To calculate the average Nusselt number for circular cylinders under these conditions, correlation HTC.7 [8] proves to be a suitable choice. For situations characterized by more substantial temperature differences, correlation HTC.8 [9] accounts for the impact of temperature-dependent properties. In instances where the cross-section of the cylinder deviates from a simple circular shape, such as squares or hexagons, correlation HTC.9 [8] provides an appropriate solution for analyzing heat transfer in non-circular cylinders subjected to cross-flow conditions.

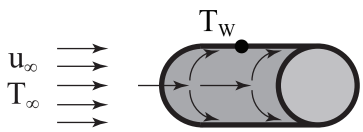


Figure 13.7. Flow perpendicular to the longitudinal axis of a cylinder.

For cross-flow, the maximum height d concerning the cylinder's cross-section located in the flow is taken as the characteristic length:

Definition

Characteristic length cylinder in cross flow:

$$L = d \ [\mathrm{m}] \tag{13.3}$$

HTC **Average Nusselt number for forced flow perpendicular to the longitudinal axis of a circular cylinder with isothermal surface:**

$$\overline{\mathrm{Nu}}_\mathrm{d} = C\,\mathrm{Re_d}^m \mathrm{Pr}^{0.4}, \tag{HTC.7}$$

where:

$\mathrm{Re_d}$	C	m
0.4 - 4	0.989	0.330
4 - 40	0.911	0.385
40 - 4000	0.683	0.466
4000 - 40000	0.193	0.618
40000 - 400000	0.0266	0.805

HTC **Average Nusselt number for forced flow perpendicular to the longitudinal axis of a circular cylinder with isothermal surface:**

$$\overline{\mathrm{Nu}}_\mathrm{d} = \left[0.40\,\mathrm{Re_d}^{\frac{1}{2}} + 0.06\,\mathrm{Re_d}^{\frac{2}{3}}\right]\mathrm{Pr}^{0.4}\left(\frac{\eta_\infty}{\eta_\mathrm{w}}\right)^{\frac{1}{4}}, \tag{HTC.8}$$

for $\mathrm{Re} \in \mathbb{R}$, and where $T_\mathrm{prop} = T_\infty$.

HTC **Average Nusselt number for forced flow perpendicular to the longitudinal axis of a non-circular cylinder with isothermal surface:**

$$\overline{\mathrm{Nu}}_\mathrm{d} = C\,\mathrm{Re_d}^m \mathrm{Pr}^{0.4}, \tag{HTC.9}$$

where:

Cross-section	$\mathrm{Re_d}$	C	m
	$5 \cdot 10^3 - 10^5$	0.246	0.588
	$5 \cdot 10^3 - 10^5$	0.102	0.675
	$5 \cdot 10^3 - 1.94 \cdot 10^4$	0.160	0.638
	$1.95 \cdot 10^4 - 10^5$	0.0385	0.782
	$5 \cdot 10^3 - 10^5$	0.153	0.638
	$5 \cdot 10^3 - 10^5$	0.228	0.731

13.2.4 Bundles of smooth tubes in an orthogonal flow

In heat exchangers, bundles of smooth pipes are frequently encountered, arranged either in-line or staggered configurations. Given the variation in fluid temperatures from row to row within the tube bundle, a representative mean temperature difference ΔT_m needs to be considered when computing heat transfer.

Fundamental EQ

Heat transfer rate in ducts:

$$\dot{Q} = \overline{\alpha} A \Delta T_\mathrm{m}, \tag{13.4}$$

where ΔT_m is a representative temperature difference between the temperature at the wall T_W and the energetically averaged caloric mean temperature of the fluid T_fl.

For many practical purposes where the duct has an isothermal surface along the direction of the flow, the logarithmic temperature difference (LMTD) is a good approximation. The derivation of this logarithmic temperature difference is explained in Section 14.2.

Definition

LMTD for flow along ducts with isothermal surface:

$$\Delta T_\mathrm{m} = \Delta T_\mathrm{ln} = \frac{\Delta T_\mathrm{in} - \Delta T_\mathrm{out}}{\ln \frac{\Delta T_\mathrm{in}}{\Delta T_\mathrm{out}}} \ [\mathrm{K}], \tag{13.5}$$

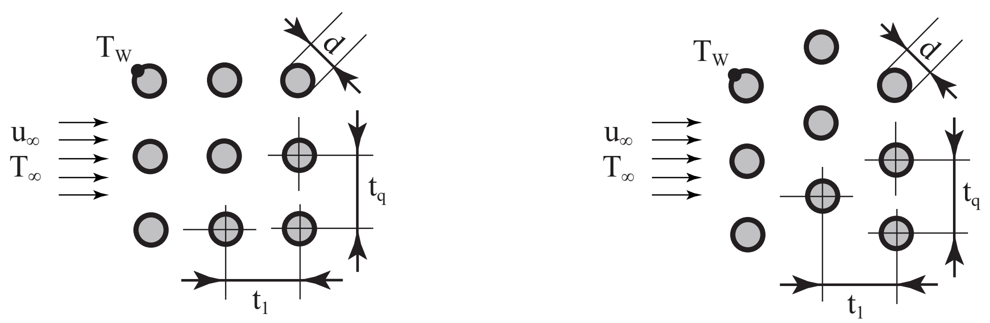
where $\Delta T_\mathrm{in} = T_\mathrm{in} - T_\mathrm{w}$ and $\Delta T_\mathrm{out} = T_\mathrm{out} - T_\mathrm{w}$.

In some cases, the duct does not have an isothermal surface temperature along the direction of the flow, but a constant heat flux is impressed. If the length of the pipe is sufficiently long enough, the mean difference between the wall and fluid temperature should be used used:

Definition

Mean temperature difference for flow along ducts with a constant impressed heat flux:

$$\Delta T_\mathrm{m} = (T_\mathrm{W} - T_\mathrm{fl})_\mathrm{m} \ [\mathrm{K}] \tag{13.6}$$



(a) In-line arrangement.   (b) Staggered arrangement.

Figure 13.8. Arrangements of tubes for flow at right angles to tube bundle.

In a wide variety of industrial applications bundles of smooth tubes with an isothermal surface temperature are used. For this case, the correlation HTC.10 for heat exchange on the outer surfaces of the tubes [10] is appropriate to be applied.

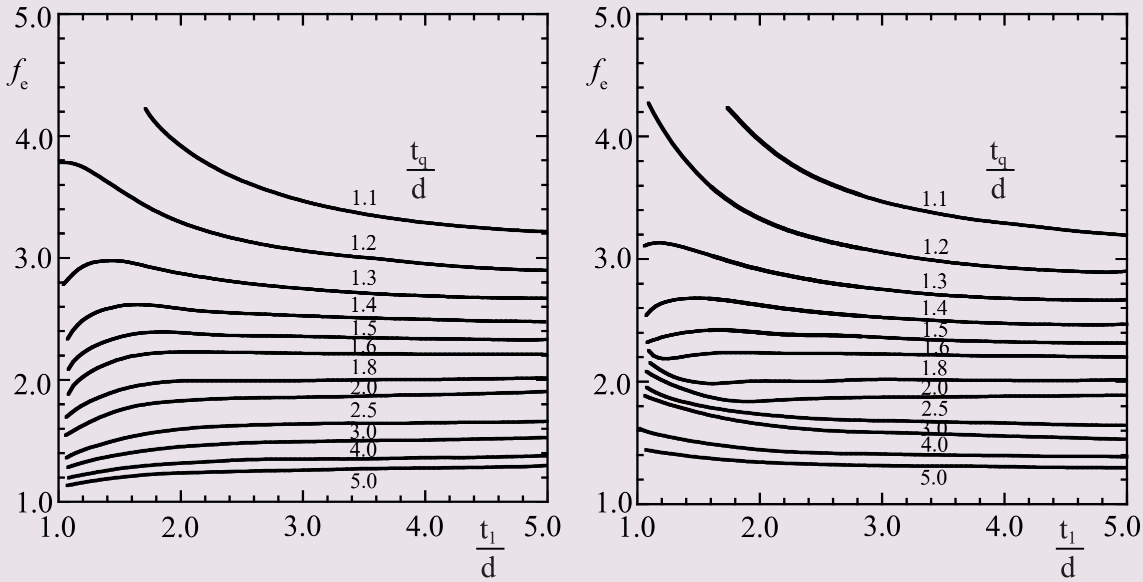
HTC

Average Nusselt number for forced flow perpendicular to the longitudinal axis of a bundle of smooth tubes:

$$\overline{\mathrm{Nu}}_\mathrm{d} = 0.287\, \mathrm{Re}_\mathrm{d}^{0.6} \mathrm{Pr}^{0.36} \cdot f_\mathrm{e}, \tag{HTC.10}$$

for $\mathrm{Re} \in \mathbb{R}$, and where $T_{\mathrm{prop}} = \frac{T_{\mathrm{out}}+T_{\mathrm{in}}}{2}$.

The tube arrangement factor f_{e} depends on the relative longitudinal distance/diameter ratio t_{l}/d, the relative transverse distance/diameter ratio t_{q}/d of the tubes, as well as the tube arrangement. This factor can be retrieved from:



(a) In-line arrangement (b) Staggered arrangement.

13.2.5 Flow around a sphere

Drawing an analogy to the scenario of cylinders exposed to flow perpendicular to their longitudinal axis, the experimental correlation HTC.11 finds applicability in the context of fluid flow around spheres [9].

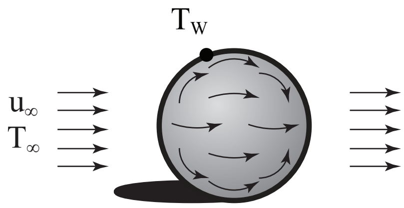


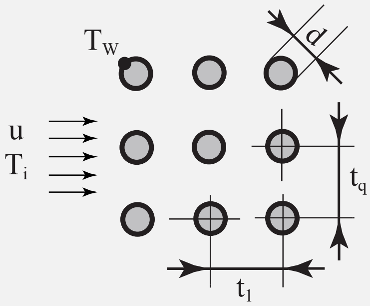
Figure 13.10. Flow around a sphere.

HTC **Average Nusselt number for forced flow around a sphere:**

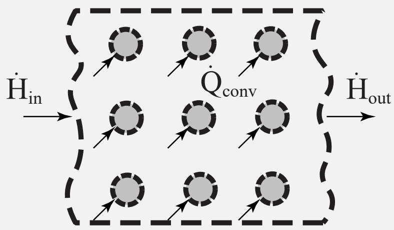
$$\overline{\mathrm{Nu}}_{\mathrm{d}} = 2 + \left(0.4\,\mathrm{Re_d}^{1/2} + 0.06\,\mathrm{Re_d}^{2/3}\right)\mathrm{Pr}^{0.4}\left(\frac{\eta_\infty}{\eta_{\mathrm{w}}}\right)^{1/4}, \tag{HTC.11}$$

for $3.5 < \mathrm{Re_d} < 8 \cdot 10^4$, $0.7 < \mathrm{Pr} < 380$, and where $T_{\mathrm{prop}} = T_\infty$.

Example 13.1 Hot water (with average properties c_p = 4.2 kJ/kgK, ρ = 965 kg/m^3, λ = 0.68 W/mK, Pr = 1.85, and $\eta = 0.3 \cdot 10^{-3}$ kg/m · s) flows into a pipe bundle heat exchanger at a velocity of 0.01 m/s, mass flow rate $\dot{m}$ = 42 kg/s and an average inlet temperature of 95 °C. The pipe bundles are arranged in an in-line configuration with 12 columns and 12 rows, featuring a uniform surface temperature of 10 °C. Each tube in the bundle has a diameter of 20 cm, a length of 1 m, a longitudinal spacing of 36 cm, and a transverse spacing of 40 cm. Determine the average outlet temperature, denoted as T_{out}, of the hot water after passing through the heat exchanger.



(a) Heat exchanger geometry.



(b) Energy balance.

1 Setting up the balance:

The energy balance around the fluid reads:

$$0 = \dot{H}_{\text{in}} - \dot{H}_{\text{out}} - \dot{Q}_{\text{conv}}$$

2 Defining the elements within the balance:

The ingoing rate of enthalpy flow is defined as:

$$\dot{H}_{\text{in}} = \dot{m} c_p T_{\text{in}} = 42 \cdot 4200 \cdot 95 = 16.6 \text{ MW}$$

The determination of both the outgoing enthalpy flow and the rate of heat lost by convection depends on the unknown outlet temperature. Consequently, an initial approximation must be made regarding this temperature. In this iterative process, there is initially iterated with an assumed value of T_{out} = 77 °C. The validity of this initial guess is assessed by verifying whether the energy balance equation holds in equilibrium. If this condition is not met, an alternative guess for the outlet temperature must be considered.

The outgoing rate of enthalpy flow is defined as:

$$\dot{H}_{\text{out}} = \dot{m} c_p T_{\text{out}} = 42 \cdot 4200 \cdot 77 = 13.5 \text{ MW}$$

The rate of heat lost by convection from the fluid to the pipes is written as:

$$\dot{Q}_{\text{conv}} = \overline{\alpha} A_s \Delta T_m$$

To determine the average heat transfer coefficient, the Reynolds number, tube arrangement factor, and average Nusselt number need to be determined first.

The Reynolds number yields from:

$$\text{Re}_d = \frac{\rho u d}{\eta} = \frac{965 \cdot 0.01 \cdot 0.2}{0.3 \cdot 10^{-3}} = 6.43 \cdot 10^3$$

Using Figure 13.9a, in combination with the ratios $t_q/d = 0.36/0.2 = 1.8$ and $t_1/d = 0.4/0.2 = 2$ yields that:

$$f_e \approx 2$$

With this given, the average Nusselt number is determined from the following correlation:

$$\overline{\mathrm{Nu}}_d = 0.287\,\mathrm{Re}_d^{0.6}\mathrm{Pr}^{0.36} \cdot f_e = 0.28 \cdot \left(6.43 \cdot 10^3\right)^{0.6} \cdot 1.85^{0.36} \cdot 2 = 138.1$$

Rewriting the definition of the average Nusselt number yields:

$$\overline{\alpha} = \frac{\overline{\mathrm{Nu}}_d \lambda}{d} = \frac{138.1 \cdot 0.68}{0.2} = 469.5\ \mathrm{W/m^2K}$$

The surface area of the pipes yields from the product of the outer surface area of a single pipe, the number of rows, and the number of columns:

$$A_s = \pi d L \cdot (12 \times 12) = \pi \cdot 0.2 \cdot 1 \cdot (12 \times 12) = 90.5\ \mathrm{m}^2$$

Lastly, the logarithmic temperature difference yields from:

$$\Delta T_m = \frac{\Delta T_{in} - \Delta T_{out}}{\ln \frac{\Delta T_{in}}{\Delta T_{out}}} = \frac{(95-10)-(77-10)}{\ln \frac{(95-10)}{(77-10)}} = 75.6\ ^\circ\mathrm{C}$$

So the rate of heat lost by convection equals:

$$\dot{Q}_{conv} = \overline{\alpha} A_s \Delta T_m = 469.5 \cdot 90.5 \cdot 75.6 = 3.2\ \mathrm{MW}$$

3 **Inserting and rearranging:**

Checking whether the balance is in equilibrium, so:

$$\dot{H}_{in} - \dot{H}_{out} - \dot{Q}_{conv} = 16.6 - 13.5 - 3.2 = -0.1\ \mathrm{MW}$$

For this purpose, the level of accuracy is considered acceptable, given that the discrepancy is only 0.6% of the total energy flow entering the system. If a higher degree of precision is required, a second estimate, such as $T_{out} = 76.7\ ^\circ\mathrm{C}$, may be proposed. This adjustment would result in a deviation of just 0.01%, relative to the total energy entering the system.

Section 14

Internal forced convection

L07 - Introduction:

Learning goals:

- Grasping the fundamental distinctions between external and internal flows.
- Comprehending the hydrodynamic and thermal inlet characteristics.
- Developing the ability to compute the caloric mean temperature.
- Developing the capability to compute local temperatures, heat fluxes, and average heat transfer coefficients.

Comprehension questions:

- What distinctions exist between external and internal flows?
- Define the concepts of hydrodynamic and thermal entrance length and explain their significance.
- Elaborate on the concept of the caloric mean temperature and describe its calculation method.
- How does the local heat transfer coefficient vary within a laminar pipe flow?

L08 - LMTD:

Learning goals:

- Understanding the concept of the logarithmic mean temperature difference (LMTD).
- Developing the ability to calculate and apply the LMTD.

Comprehension Question:

- What is the meaning of the LMTD, and in what situations is it essential to apply this concept?

L09 - Heat transfer correlations:

Learning goals:

- Proficiency in calculating the heat transfer coefficient in laminar flows under fully developed conditions.
- Discerning between various flow configurations and selecting the appropriate correlation for heat transfer coefficients (HTC).

Comprehension questions:

- Why does the HTC remain constant in the fully developed region of an internal flow?
- What are the key steps involved in calculating the HTC in the fully developed region?
- What factors can lead to a loss of self-similarity in heat transfer behavior?

Corresponding tutorial exercises:

- Exercise III.9 Pipe flow with a constant heat flux
- Exercise III.10 Insulated pipe
- Exercise III.11 Heating of a pipe
- Exercise III.12 Flow through a grid

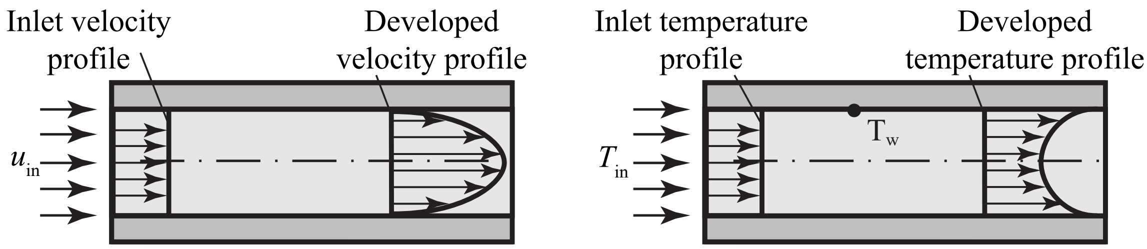
Subsection 14.1

Introduction to internal forced convection

Most fluids, especially liquids, are transported in circular pipes. This is because pipes with a circular cross-section can withstand large pressure differences between the inside and the outside without undergoing significant distortion.

Although the fluid flow theory is reasonably well understood, analytical solutions are obtained only for a few simple cases such as fully developed laminar flow in a circular pipe. Therefore, one must rely on experimental results and empirical relations for most fluid flow problems rather than closed-form analytical solutions. Noting that the experimental results are obtained under carefully controlled laboratory conditions and that no two systems are exactly alike, one must not be naive to view the results obtained as “exact.”

The developed fluid velocity profile in a pipe changes from zero at the wall because of the no-slip condition to a maximum at the pipe center, as illustrated in Figure 14.1. When analyzing heat transfer related to fluid flow, for convenience the average velocity of u_m, and "some average" fluid temperature, specifically called the caloric mean temperature T_m are used. The average velocity in heating and cooling applications may change somewhat because of changes in density with temperature, but in general, this effect is negligible. For this reason, assessing fluid properties at the caloric mean temperature and considering them constant is recommended. This approach is adopted due to the advantages associated with the use of constant properties, despite the inherent compromise in accuracy.



(a) Velocity profile. (b) Temperature profile.

Figure 14.1. Development of flow in a tube.

The average velocity for an incompressible fluid are determined from an integral expression:

Definition

Average fluid velocity:

$$u_\mathrm{m} = \frac{\int\limits_{A_\mathrm{c}} \rho u \, dA_\mathrm{c}}{\int\limits_{A_\mathrm{c}} \rho \, dA_\mathrm{c}} \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{14.1}$$

In heat transfer analysis, the caloric mean temperature is used as the reference temperature for the fluid at a specific position within the pipe.

Derivation

The rate of enthalpy flowing through a cross-section can be defined in two manners.

First by use of the definition of enthalpy flow:

$$\dot{H} = \int\limits_{A_\mathrm{c}} \rho u c_\mathrm{p} T \, dA_\mathrm{c}$$

Secondly, by using introducing the caloric mean temperature:

$$\dot{H} = \dot{m} c_\mathrm{p} T_\mathrm{m} = T_\mathrm{m} \int_{A_\mathrm{c}} \rho u c_\mathrm{p} \, dA_\mathrm{c}$$

Equalling both expressions and rewriting yields:

$$\rightarrow T_\mathrm{m} = \frac{\int_{A_\mathrm{c}} \rho u c_\mathrm{p} T \, dA_\mathrm{c}}{\int_{A_\mathrm{c}} \rho u c_\mathrm{p} dA_\mathrm{c}}$$

□

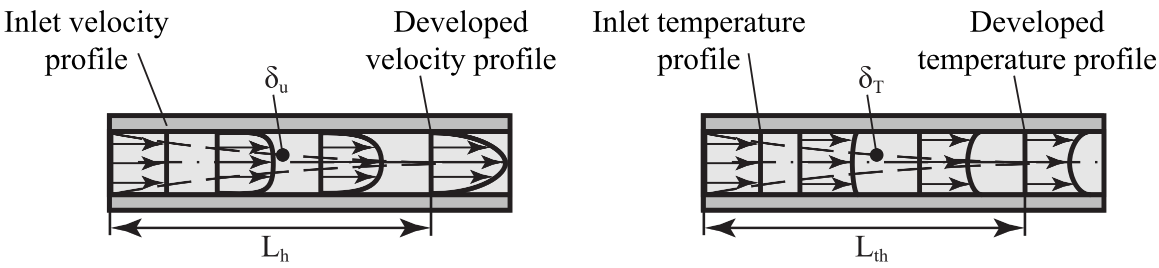
Definition

Caloric mean temperature:

$$T_\mathrm{m} = \frac{\int_{A_\mathrm{c}} \rho u c_\mathrm{p} T \, dA_\mathrm{c}}{\int_{A_\mathrm{c}} \rho u c_\mathrm{p} dA_\mathrm{c}} \; [\mathrm{K}] \qquad (14.2)$$

14.1.1 Entry regions

Consider a scenario where a fluid enters a circular pipe with a consistent velocity. Due to the no-slip condition, the fluid particles in direct contact with the inner pipe wall come to a complete halt. This stationary layer also induces a gradual decrease in the velocity of fluid particles in the neighboring layers due to friction. To compensate for this reduction in velocity, the fluid's velocity at the midpoint of the pipe must increase to maintain a consistent mass flow rate through the pipe. Consequently, a velocity gradient forms along the length of the pipe.



(a) Hydrodynamic inlet. (b) Thermal inlet.

Figure 14.2. Entry regions of fluid flow in a pipe.

The boundary layer is the region in the flow where the influence of viscous shearing forces, a result of fluid viscosity, becomes prominent. This boundary layer effectively divides the flow in a pipe into two distinct regions: the boundary layer region, characterized by significant viscous effects and velocity changes, and the core region, where frictional effects are negligible, and the radial velocity remains nearly constant.

As fluid flows along the pipe, the thickness of the boundary layer increases in the direction of flow until the center of the pipe is reached, filling the entire pipe. This is depicted in Figure 14.2a, where the velocity profile becomes fully developed. The segment from the pipe inlet to the point at which the velocity profile achieves full development is termed the hydrodynamic entrance region, and the respective entrance length is referred to as the hydrodynamic entry length, denoted as L_h. Beyond the entrance region, one calls the flow hydrodynamically fully developed. In this region, the velocity profile exhibits a parabolic shape in laminar flow and appears somewhat flatter or fuller in turbulent flow due to eddy motion and more mixing in the radial direction.

Now, consider a scenario where a fluid, initially at a uniform temperature, enters a circular tube with the surface maintained at a different temperature. In this case, the fluid particles in contact with the tube's surface equal the surface temperature, initiating convective heat transfer within the tube. This process leads to the development of a thermal boundary layer along the tube.

The thickness of this thermal boundary layer also increases when moving along the tube until the tube's center is reached, filling the entire tube, as demonstrated in Figure 14.2b. The segment of flow where the thermal boundary layer evolves and reaches the tube's center is designated as the thermal entrance region, with the respective length denoted as the thermal entry length, represented as L_{th}. Flow within the thermal entrance region is described as thermally developing flow since this is where the temperature profile takes shape. Beyond the thermal entrance region, the thermally fully developed region is encountered, where the dimensionless temperature profile remains constant. The state of flow characterized by both hydrodynamic and thermal development, where both velocity and dimensionless temperature profiles remain unaltered, is referred to as fully developed flow.

Typically, the hydrodynamic entry length is defined as the distance from the tube entrance where the wall shear stress reaches approximately 2 percent of the fully developed value. In laminar flow, the hydrodynamic and thermal entry lengths are approximated as follows:

Definition

Laminar hydrodynamic entry length:

$$L_{\mathrm{h}} \approx 0.05\,\mathrm{Re_d}\,d\ [\mathrm{m}] \tag{14.3}$$

Definition

Laminar thermodynamic entry length:

$$L_{\mathrm{th}} \approx 0.05\,\mathrm{Re_d Pr}\,d\ [\mathrm{m}] \tag{14.4}$$

Within turbulent flow, the mixing resulting from random fluctuations typically dominates the influence of molecular diffusion. Consequently, the hydrodynamic and thermal entry lengths tend to be approximately equal in size and remain largely unaffected by the Prandtl number. Notably, the entry length is considerably shorter in turbulent flow, as one would anticipate, and the reliance on the Reynolds number is comparatively less pronounced. In numerous real-world tube flows, the impact of entrance effects becomes inconsequential after traversing approximately ten tube diameters. Consequently, the hydrodynamic and thermal entry lengths are commonly considered to be:

Definition

Turbulent hydrodynamic entry length:

$$L_{\mathrm{h}} \approx 10\,d\ [\mathrm{m}] \tag{14.5}$$

Definition

Turbulent thermodynamic entry length:

$$L_{\mathrm{th}} \approx 10\,d\ [\mathrm{m}] \tag{14.6}$$

In the scientific literature, you can find correlations for Nusselt numbers in the entrance regions. Nevertheless, in practical applications of forced convection, the length of the tubes is often several times larger than the length of either entrance region. As a result, the flow through these tubes is commonly assumed to be fully developed throughout the entire tube length. This simplified approach yields reasonably accurate results for long tubes and provides conservative estimates for shorter ones.

The above observations strictly apply to turbulent flow. In laminar flow, Nusselt numbers exhibit significantly lower values than in turbulent flow, the distance required for the Nusselt number to stabilize is typically much longer, and the flow is susceptible to the thermal boundary conditions.

SUBSECTION 14.2

Logarithmic temperature difference

For flow within ducts, such as tubes, channels, or nozzles, the expression for the rate of heat transfer by convection has been previously provided in equation (13.4), and is reiterated below for reference:

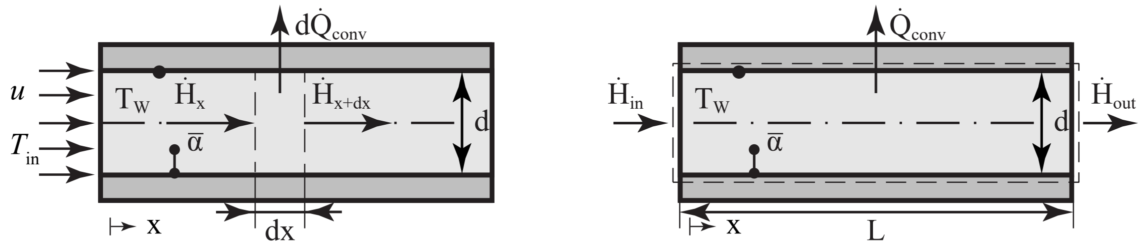
$$\dot{Q} = \overline{\alpha} A \Delta T_\mathrm{m}$$

where ΔT_m is a representative temperature difference between the temperature at the wall T_W and the energetically averaged caloric mean temperature of the fluid T_fl. In case of an isothermal pipe surface area and great changes in the fluid temperature along the flow path, the logarithmic temperature difference (LMTD) should be used.

The concept of the LMTD was previously introduced in Section 13.2.4, where the definition was provided in equation (13.5). However, this definition may have seemed abrupt or unexplained. Therefore, this section is dedicated to elucidating the derivation of the LMTD definition.

Derivation

A fluid with average specific heat capacity c_p flows into a long pipe featuring a diameter d, length L, and an isothermal surface at temperature T_w. The fluid exhibits a mass flow rate of $\dot{m}$ and enters the pipe with an average inlet temperature of T_in. Within the pipe, the fluid undergoes heat dissipation to the surroundings through convection characterized by the average coefficient $\overline{\alpha}$.



(a) Infinitesimal energy balance.

(b) Global energy balance.

1 **Setting up the balance:**

For the derivation of the LMTD, two energy balances are set.

First the local energy balance:

$$\dot{H}_\mathrm{x} - \dot{H}_\mathrm{x+dx} - d\dot{Q}_\mathrm{conv}$$

Secondly the global energy balance around the entire pipe:

$$\dot{H}_\mathrm{in} - \dot{H}_\mathrm{out} - \dot{Q}_\mathrm{conv}$$

2 **Defining the elements within the balance:**

In- and outgoing rate of enthalpy flows for the infinitesimal energy balance:

$$\dot{H}_\mathrm{x} = \dot{m} c_\mathrm{p} T_\mathrm{m}(x)$$

$$\dot{H}_\mathrm{x} = \dot{H}_\mathrm{x} + \frac{\partial}{\partial x}\left(\dot{H}_\mathrm{x}\right) \cdot dx$$

The rate of heat loss by convection for the infinitesimal energy balance is written as:

$$d\dot{Q}_\mathrm{conv} = \overline{\alpha}\, \pi\, d\, dx \cdot (T_\mathrm{m}(x) - T_\mathrm{w})$$

In- and outgoing rate of enthalpy flows for the global energy balance, with the yet unknown outlet temperature T_{out}:

$$\dot{H}_{\mathrm{in}} = \dot{m} c_{\mathrm{p}} T_{\mathrm{in}}$$

$$\dot{H}_{\mathrm{out}} = \dot{m} c_{\mathrm{p}} T_{\mathrm{out}}$$

The rate of heat loss by convection for the global energy balance is stated as:

$$\dot{Q}_{\mathrm{conv}} = \overline{\alpha}\, \pi\, d\, L \Delta T_{\mathrm{m}}$$

3 Inserting and rearranging:

Inserting the fluxes into the infinitesimal energy balance and doing some rewriting yields:

$$\frac{\partial T(x)}{\partial x} = \frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \cdot (T_{\mathrm{m}}(x) - T_{\mathrm{w}})$$

Similarly for the global energy balance:

$$0 = \dot{m} c_{\mathrm{p}} (T_{\mathrm{in}} - T_{\mathrm{out}}) - \overline{\alpha}\, \pi\, d\, L \Delta T_{\mathrm{m}}$$

4 Defining the boundary and/or initial conditions:

To solve the differential equation, derived from the infinitesimal element, one boundary condition is required. The inlet temperature is given, which yields:

$$T_{\mathrm{m}}(x = 0) = T_{\mathrm{in}}$$

5 Solving the equation:

To solve the differential equation, the temperature difference θ is introduced:

$$\theta = T_{\mathrm{m}}(x) - T_{\mathrm{w}}$$

Substitution into the differential equation derived from the infinitesimal element yields:

$$\frac{\partial \theta}{\partial x} = \frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \cdot \theta$$

Rewriting gives:

$$\int \frac{1}{\theta} \partial \theta = \int \frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \partial x$$

Integration and rewriting yields:

$$\theta = A \cdot \exp\left(-\frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \cdot x \right)$$

The boundary condition, $\theta(x = 0) = T_{\mathrm{in}} - T_{\mathrm{w}}$, is used to find that $A = \theta_{\mathrm{in}} = T_{\mathrm{in}} - T_{\mathrm{w}}$.

Writing the equation back into the form of $T_{\mathrm{m}}(x) = \dots$ gives:

$$T_{\mathrm{m}}(x) = (T_{\mathrm{in}} - T_{\mathrm{w}}) \cdot \exp\left(-\frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \cdot x \right) + T_{\mathrm{w}}$$

The outlet temperature yields from $T_{\mathrm{m}}(x = L)$:

$$T_{\mathrm{out}} = (T_{\mathrm{in}} - T_{\mathrm{w}}) \cdot \exp\left(-\frac{\overline{\alpha} \pi d}{\dot{m} c_{\mathrm{p}}} \cdot L \right) + T_{\mathrm{w}}$$

Now the outlet temperature is known, the expression found for the outlet temperature is rewritten to get $\dot{m}c_\mathrm{p}$ free:

$$\dot{m}c_\mathrm{p} = -\frac{\overline{\alpha}\pi dL}{\ln\left(\frac{T_\mathrm{out}-T_\mathrm{w}}{T_\mathrm{in}-T_\mathrm{w}}\right)}$$

After, substitution of $\dot{m}c_\mathrm{p}$ into the global energy balance and rewriting yields:

$$\Delta T_\mathrm{m} = -\frac{T_\mathrm{in} - T_\mathrm{out}}{\ln\left(\frac{T_\mathrm{out}-T_\mathrm{w}}{T_\mathrm{in}-T_\mathrm{w}}\right)}$$

Lastly, substituting $\Delta T_\mathrm{in} = T_\mathrm{in} - T_\mathrm{w}$ and $\Delta T_\mathrm{out} = T_\mathrm{out} - T_\mathrm{w}$ yields the definition of the LMTD:

$$\Delta T_\mathrm{m} = \frac{\Delta T_\mathrm{in} - \Delta T_\mathrm{out}}{\ln\left(\frac{\Delta T_\mathrm{in}}{\Delta T_\mathrm{out}}\right)}$$

□

This derivation shows that for a pipe wall with a constant heat flux being applied, the LMTD offers a good estimate for evaluating the mean temperature difference between the wall and fluid.

HeatQuiz 14.1

Energy balances:

SUBSECTION 14.3
Heat transfer correlations

Many Nusselt correlations are a result of integral measurement on a system and thus yield only the averaged heat transfer coefficients $\overline{\alpha}$ of a heat transporting medium. The same applies to the average heat flux:

$$\dot{Q} = \overline{\alpha} A \Delta T_\mathrm{m}$$

where both the wall temperature as well as the fluid temperature may depend on the location.

For many practical purposes when the duct has an isothermal surface along the direction of the flow, the LMTD is a good approximation.

$$\Delta T_\mathrm{m} = \Delta T_\mathrm{ln} = \frac{\Delta T_\mathrm{in} - \Delta T_\mathrm{out}}{\ln \frac{\Delta T_\mathrm{in}}{\Delta T_\mathrm{out}}}$$

where $\Delta T_\mathrm{in} = T_\mathrm{in} - T_\mathrm{w}$ and $\Delta T_\mathrm{out} = T_\mathrm{out} - T_\mathrm{w}$.

In some cases, the duct does not have an isothermal surface temperature along the direction of the flow, but a constant heat flux is impressed. If the length of the pipe is sufficiently long enough, the mean difference between the wall and fluid temperature is used:

$$\Delta T_\mathrm{m} = (T_\mathrm{W} - T_\mathrm{fl})_\mathrm{K}$$

Furthermore, the Nusselt correlations that are used to determine the average heat transfer coefficient $\overline{\alpha}$, require the Reynolds number $\mathrm{Re_d}$ to be determined using the mean fluid velocity and the hydraulic diameter.

To do so, the properties used for the dimensionless numbers are to be determined, if not stated otherwise, at the following mean fluid temperature over the tube length:

Definition

Fluid property temperature internal forced convection:

$$T_\mathrm{prop} = \frac{T_\mathrm{out} + T_\mathrm{in}}{2} \text{ [K]} \tag{14.7}$$

In practice, non-circular pipes are usually used in applications such as the heating and cooling systems of buildings where the pressure difference is relatively small, the manufacturing and installation costs are lower, and the available space is limited for ductwork. For a fixed surface area, the circular tube gives the most heat transfer for the least pressure drop, which explains the overwhelming popularity of circular tubes in heat transfer equipment. In the analysis of internal convection, the hydraulic diameter is used as the characteristic length for determining dimensionless numbers such as the Reynolds number:

Definition

Characteristic length of pipes:

$$d_\mathrm{h} = 4 \frac{\text{cross-section area}}{\text{wetted perimeter}} = 4 \frac{A_\mathrm{c}}{U} \text{ [m]} \tag{14.8}$$

14.3.1 Laminar flow in circular tubes

Fluid flow in a tube exhibits varying characteristics, ranging from laminar to turbulent, dependent upon the flow conditions. When fluid moves at low velocities, the flow tends to be streamlined, falling under the category of laminar flow. However, as the velocity surpasses a critical threshold, the flow undergoes a transition into turbulence. Worth noting is that this shift from laminar to turbulent flow is not abrupt; rather, this shift manifests within a range of velocities where the flow oscillates between laminar and turbulent states before fully embracing turbulence. In practice, the majority of pipe flows tend to be turbulent. Laminar flow, on the other hand, prevails when highly viscous fluids, like oils, traverse through small-diameter circular tubes or narrow channels. As a practical guideline, the flow within a tube is typically considered laminar when the Reynolds number is below 2300, and hence, $\mathrm{Re}_{d,crit} \approx 2300$ serves as a criterion for distinguishing between laminar and turbulent flow.

Isothermal surface, hydrodynamically developed flow at the start of the heated or cooled section of a tube

When fluid initially enters a pipe with a wall temperature matching that of the fluid, the primary alteration occurs in the velocity profile $u(r)$ which evolves along the flow direction until a hydrodynamically developed state is reached. However, upon exposure to heating from the pipe with a uniformly heated surface, a shift in the temperature profile ensues, initiating and advancing along the flow direction until reaching the state of being fully thermally developed.

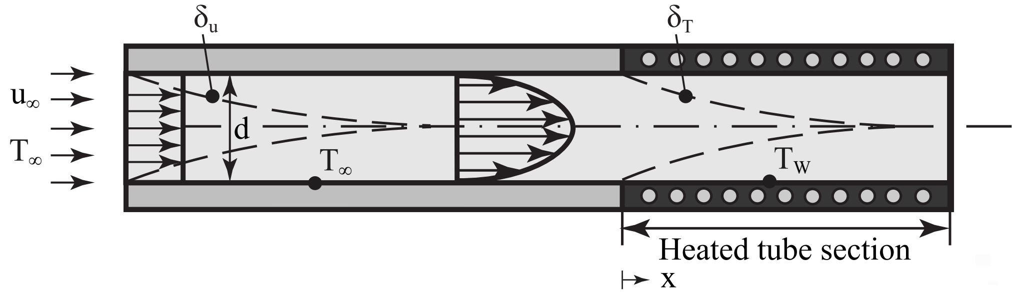


Figure 14.4. Hydrodynamically developed laminar flow at the start of the heated or cooled section of a tube with an isothermal surface.

Figure 14.5 depicts the analytical solution for the local and average Nusselt numbers in this particular scenario, assuming effects due to distinction in dynamic viscosity between the fluid at the wall and the mean temperature to be negligible. As the diagram illustrates, when the flow reaches thermal full development, the Nusselt number converges to a constant value of 3.66.

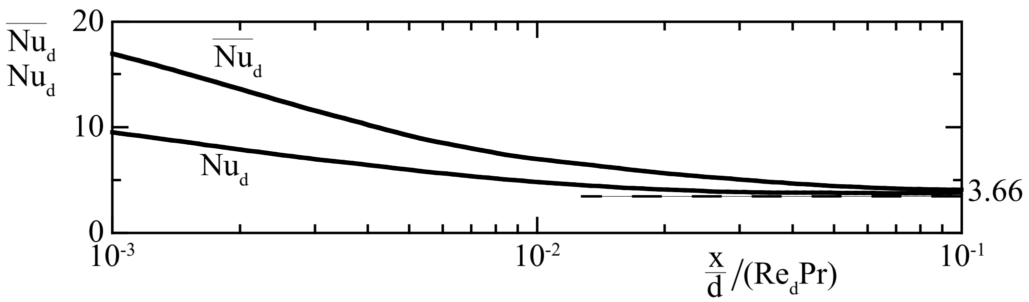


Figure 14.5. Local and average Nusselt number for laminar hydrodynamically developed tube flow with $\eta = \eta_w$ [11].

In some cases, large differences between the fluid properties at the average temperature and the surface temperature of the wall are observed, which have a significant impact on the average Nusselt number. In this case, HTC.12 provides a good estimate for the average Nusselt number $\overline{\text{Nu}}_\text{d}$:

HTC **Average Nusselt number for forced laminar flow within a circular pipe being hydrodynamic developed at the start of the isothermal heated or cooled section:**

$$\overline{\text{Nu}}_\text{d} = \left(3.66 + \frac{0.19\left(\text{Re}_\text{d}\text{Pr}\frac{d}{L}\right)^{0.8}}{1 + 0.117\left(\text{Re}_\text{d}\text{Pr}\frac{d}{L}\right)^{0.467}}\right)\left(\frac{\eta}{\eta_\text{W}}\right)^{0.14}, \tag{HTC.12}$$

for $\text{Re}_\text{d} < 2300$.

Isothermal surface, simultaneous hydrodynamic and thermal inflow

In certain situations, both the thermal and velocity boundary layers start developing simultaneously. In these cases, the initial rate of heat transfer at the beginning of the pipe is slightly higher compared to scenarios where the flow first becomes hydrodynamically developed and starts to thermally develop.

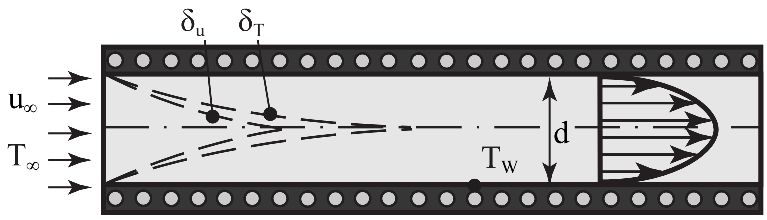


Figure 14.6. Heated tube with simultaneous hydrodynamic and thermal laminar inflow.

The conservation equations for this problem can also be solved analytically. The outcomes are influenced by the Prandtl number of the fluid, and the resulting average Nusselt number is shown in Figure 14.7. Just like the correlation discussed previously, where there was dealt with the hydrodynamically developed flow before the heated section, the average Nusselt number gradually approaches 3.66 as the fluid travels a sufficient distance within the pipe.

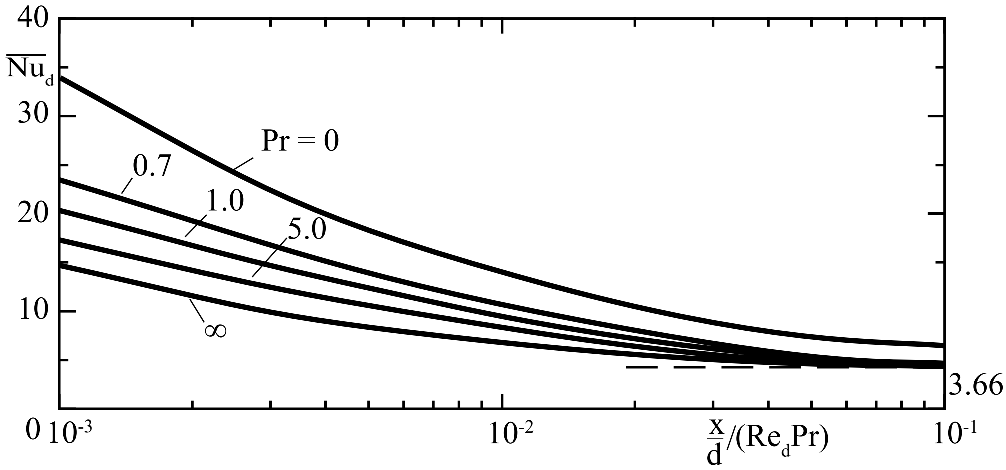


Figure 14.7. Average Nusselt number for laminar flow in tubes and simultaneous hydrodynamic and thermal laminar inflow.

The solution shown in Figure 14.7 can be approximated by HTC.13a:

HTC **Average Nusselt number for forced laminar flow within an isothermal circular pipe Simultaneous hydrodynamic and thermal start:**

$$\overline{\mathrm{Nu}}_{\mathrm{d}} = \left(3.66 + \frac{0.0677\left(\mathrm{Re_d} Pr \frac{\mathrm{d}}{\mathrm{L}}\right)^{1.33}}{1+0.1Pr\left(\mathrm{Re_d}\frac{\mathrm{d}}{\mathrm{L}}\right)^{0.83}}\right)\left(\frac{\eta}{\eta_{\mathrm{W}}}\right)^{0.14}, \qquad \text{(HTC.13a)}$$

for $\mathrm{Re_d} < 2300$.

Impressed heat flow

In some cases the hydrodynamically developed flow is being heated by a constant heat flux $\dot{q}''$, as shown in Figure 14.8. In this case, the caloric mean fluid temperature and the wall temperature increase both linearly along the flow direction when the flow is thermally developed.

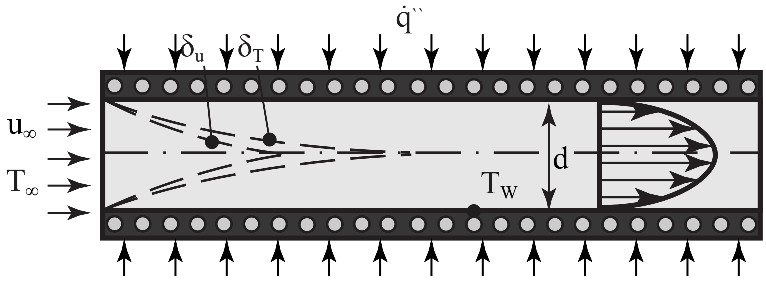
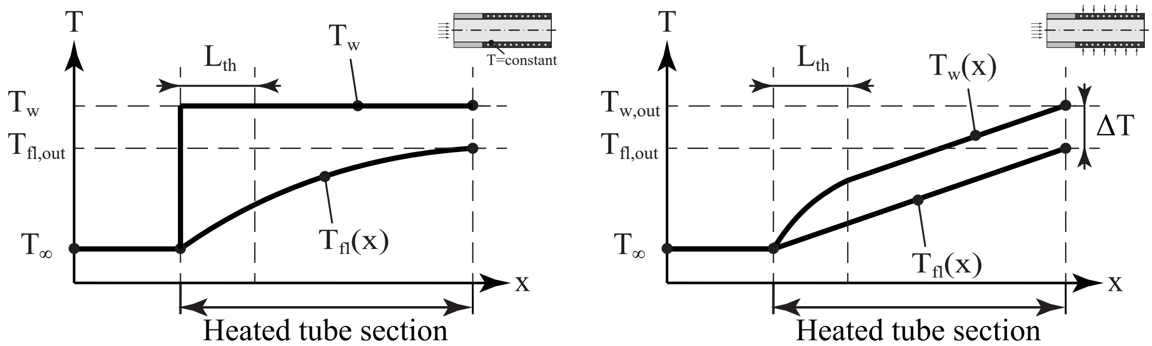


Figure 14.8. Hydrodynamically developed laminar flow at the start of the heated or cooled section of a tube with a constant heat flux $\dot{q}''$.

Figure 14.9 provides temperature profiles for two cases: one where a fluid is heated by a pipe with a constant isothermal surface temperature and the other with a pipe having a constant heat flux.



(a) For a tube with an isothermal surface temperature.

(b) For a tube with a constant heat flux.

Figure 14.9. Temperature profile caloric mean fluid temperature and wall temperature.

In the case of the isothermal surface temperature, shown in Figure 14.9a, the temperature difference and, consequently, the temperature gradient between the wall temperature and the fluid decrease as the flow progresses. This leads to a reduction in the rate of heat transfer along the direction of the flow.

For pipe flow subjected to constant heat flux, the wall temperature initially rises slightly faster than the fluid temperature until the fluid becomes thermally developed, as sketched in Figure 14.9b. After this point, the wall temperature increases at the same rate as the fluid temperature, resulting in a constant heat flux transferred from the fluid to the wall. As the temperature difference between the fluid and the wall temperature remains consistent along the flow direction, slightly larger Nusselt numbers are observed.

For pipes for which the thermal inlet length is much smaller than the overall pipe length, the heat transfer coefficients increase by about 20% if a constant flux is imposed. In this case, the average Nusselt number is written as:

HTC

Average Nusselt number for forced laminar flow within a circular pipe being fully developed with impressed heat flow:

$$\overline{\mathrm{Nu}}_{\mathrm{d}} = 4.36, \tag{HTC.13b}$$

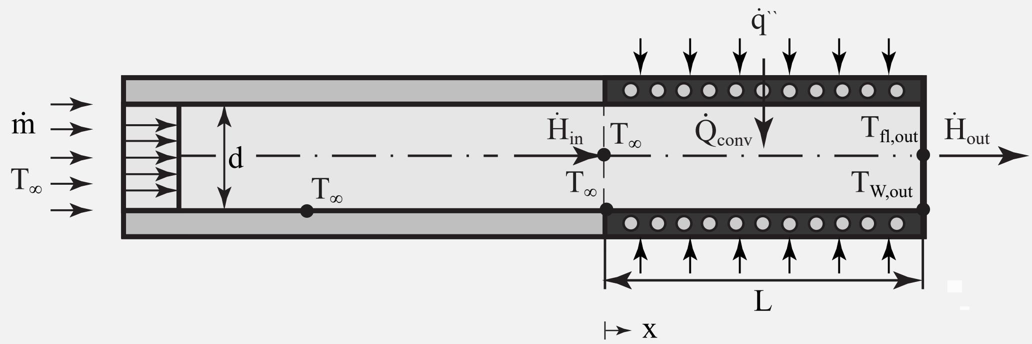
for $\mathrm{Re_d} < 2300$ and $L_{\mathrm{th}} \ll L$.

Example 14.1

Some fluid (mass flow $\dot{m}$, specific heat capacity c_{p}, thermal conductivity λ) flows through a very long pipe with diameter d, is hydrodynamically developed before entering the heated section and is heated with a constant heat flux $\dot{q}''$. The fluid enters at an ambient temperature T_∞. Determine the fluid $T_{\mathrm{fl,out}}$ and surface temperature $T_{\mathrm{w,out}}$ of the pipe at the end of the pipe.

Hints:

- Neglect axial heat diffusion.
- Neglect any difference in fluid properties in the radial direction of the pipe.
- $L_{\mathrm{th}} \ll L$
- $\mathrm{Re} \approx 10^2$



1 **Setting up the balance:**

$$0 = \dot{H}_{\mathrm{in}} - \dot{H}_{\mathrm{out}} + \dot{Q}_{\mathrm{conv}}$$

2 **Defining the elements within the balance:**

The in- and outgoing rate of enthalpy flows are expressed as:

$$\dot{H}_{\text{in}} = \dot{m} c_{\text{p}} T_{\infty}$$

$$\dot{H}_{\text{out}} = \dot{m} c_{\text{p}} T_{\text{fl,out}}$$

Rate of heat transfer by convection from the pipe wall to the fluid:

$$\dot{Q}_{\text{conv}} = \dot{q}'' dL$$

and:

$$\dot{Q}_{\text{conv}} = \overline{\alpha} \pi d L (T_{\text{w}}(x) - T_{\text{fl}}(x))_{\text{m}}$$

But since $L \ll L_{\text{th}}$, one can consider $(T_{\text{w}}(x) - T_{\text{fl}}(x))$ to be constant along the pipe, so the constant temperature difference $\Delta T = (T_{\text{w}} - T_{\text{fl}})$ is introduced:

$$\dot{Q}_{\text{conv}} = \overline{\alpha} \pi d L \Delta T$$

The average convection coefficient yields from the average Nusselt number applicable for laminar flow subjected to impressed heat flow:

$$\overline{\alpha} = \frac{\overline{\text{Nu}}_{\text{d}} \lambda}{d} = 4.36 \frac{\lambda}{d}$$

3 **Inserting and rearranging:**

Inserting the definition of the enthalpy flows and the first definition of the rate of heat transfer by convection, and rewriting yields the fluid outlet temperature:

$$T_{\text{fl,out}} = T_{\infty} + \frac{\dot{q}'' dL}{\dot{m} c_{\text{p}}}$$

Equal the two definitions of the rate of heat transfer by convection gives:

$$\dot{q}'' dL = 4.36 \frac{\lambda}{d} \pi d L \Delta T$$

Where rewriting gives

$$\Delta T = \frac{\dot{q}'' d}{4.36 \lambda}$$

Inserting $\Delta T = T_{\text{w,out}} - T_{\text{fl,out}}$ and rewriting yields:

$$T_{\text{w,out}} = \frac{\dot{q}'' d}{4.36 \lambda} + T_{\text{fl,out}} = \dot{q}'' \left(\frac{d}{4.36 \lambda} + \frac{dL}{\dot{m} c_{\text{p}}} \right) + T_{\infty}$$

14.3.2 Turbulent flow in (non-)circular tubes

As previously mentioned, $Re_{d,crit} \approx 2300$ serves as a criterion to differentiate between laminar and turbulent flow. Turbulent flow is often preferred in practical applications due to the higher heat transfer coefficients. Many correlations for heat transfer coefficients in turbulent flow are derived from experimental investigations because dealing with turbulent flow theoretically can be quite challenging.

While prior correlations addressing laminar flow in tubes were confined to tubes featuring exclusively circular cross-sections, the ensuing correlations provided for turbulent flow apply to pipes exhibiting diverse cross-sectional geometries.

Isothermal surface or impressed heat flow, simultaneous hydrodynamic and thermal inflow

In the case of very short pipes, where both the thermal and velocity boundary layers develop simultaneously, the thermal and hydrodynamic entrances have a notable impact on the average heat transfer coefficient. This scenario is depicted in Figure 14.10.

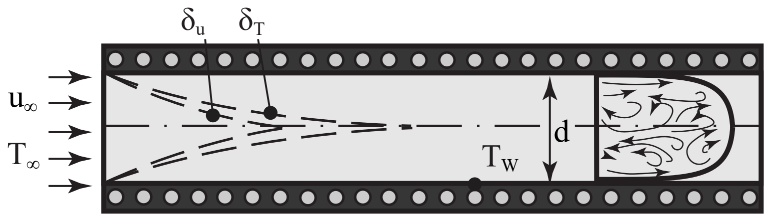


Figure 14.10. Heated tube with simultaneous hydrodynamic and thermal turbulent inflow.

An empirical relationship for the average Nusselt number for turbulent pipe flows heated from the beginning based on many experiments is stated as HTC.14 [12]:

HTC **Average Nusselt number for forced turbulent flow within a pipe Simultaneous hydrodynamic and thermal start:**

$$\overline{Nu}_d = 0.0235\left(Re_d^{0.8} - 230\right)\left(1.8Pr^{0.3} - 0.8\right)\left(1 + \left(\frac{d}{L}\right)^{\frac{2}{3}}\right)\left(\frac{\eta}{\eta_W}\right)^{0.14}, \qquad \text{(HTC.14)}$$

for $Re_d > 2300$, $0.6 < Pr < 500$ and $\frac{L}{d} > 1$.

The entry lengths in turbulent flow are generally quite brief, often spanning just 10 to 40 tube diameters in length. Consequently, the Nusselt number calculated for fully developed turbulent flow is roughly applied to the entire tube. This direct method offers reasonable estimations for pressure drop and heat transfer in long tubes while being more conservative for shorter ones. In many cases, HTC.15 is used as a satisfactory approximation:

HTC **Average Nusselt number for forced turbulent flow within a pipe being fully developed:**

$$\overline{Nu}_d = 0.027Re_d^{0.8}Pr^{\frac{1}{3}}\left(\frac{\eta}{\eta_W}\right)^{0.14}, \qquad \text{(HTC.15)}$$

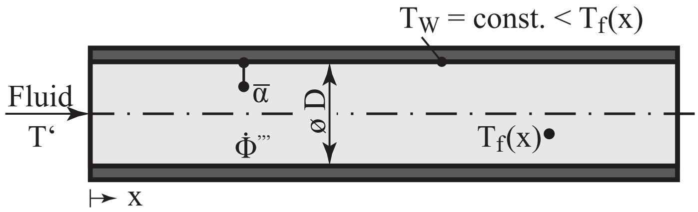
for $3000 < Re_d < 10^5$ and $\frac{L}{d} > 40$.

HeatQuiz 14.2

Energy balances: **Energy balances:**

Demonstration 14.1

Through a very long pipe with a diameter D flows a heat-generating fluid (homogeneous and constant source strength $\dot{\Phi}''' > 0$). In addition, the pipe has a uniform, constant wall temperature T_{W}.



Given parameters:

- Pipe diameter: D
- Average heat transfer coefficient: $\overline{\alpha}$
- Material properties of the fluid: λ, c, ρ, ν, Pr
- AVerage velocity of the the fluid: u
- Inlet temperature of the fluid: T'
- Heat source strength: $\dot{\Phi}''' > 0$

Hints:

- The problem is steady in time.
- The problem is one-dimensional in the x-direction.
- The thermophysical properties of the fluid are temperature-independent.

Tasks:

a) Derive the differential equations for the temperature profile in the flow direction, with and without neglecting the diffusive heat transport in the direction of the flow.

b) Find the heat transfer coefficient $\overline{\alpha}^*$ between the pipe's inner wall and the flow, when the temperature $T_{\mathrm{f}}(x) = T_{\mathrm{f}}'$ in the direction of the flow remains constant.

Video solution:

Section 15

External natural convection

L10 - External natural convection:

Learning goals:

- Acquiring familiarity with correlations applicable to natural convection scenarios.

Comprehension questions:

- Which dimensionless numbers are essential to consider when applying heat transfer laws?
- What constitutes the driving force in natural convection?
- Differentiate between the two distinct cases for horizontal plates and elucidate their distinctions from vertical plates.

Corresponding tutorial exercises:

- Exercise III.13 Horizontal and vertical wall

Subsection 15.1
Introduction to external natural convection

Natural convection plays a significant role in various engineering and environmental processes. Unlike forced convection, which relies on external means such as fans or pumps to move fluids, natural convection results from density differences caused by temperature gradients in a fluid. When a fluid is heated, the density decreases and the fluid rises, while cooler, denser fluid descends. This motion creates a self-sustaining flow pattern, leading to heat transfer between the fluid and the surroundings.

External natural convection specifically refers to this phenomenon occurring on the outer surface of solid objects exposed to surrounding fluids, typically air or a liquid. This type of convection finds applications in numerous real-world scenarios, from understanding the cooling of electronic components to predicting temperature variations in Earth's oceans and atmosphere. Natural convection stands as an important area of exploration in fluid dynamics and heat transfer, where intricate interactions between fluid properties, geometry, and temperature gradients frequently come into play.

Subsection 15.2
Heat transfer correlations

Heat transfer through natural convection on a surface is contingent on several factors, including the surface's geometry, orientation, temperature distribution across the surface, and the thermophysical characteristics of the involved fluid.

Even though an understanding of the natural convection mechanism is possessed, the intricate nature of fluid motion poses a significant challenge in deriving straightforward analytical relationships for heat transfer. While some analytical solutions do exist for natural convection, they tend to be specific to uncomplicated geometries and necessitate simplifying assumptions. As a result, except for certain straightforward scenarios, heat transfer correlations in natural convection predominantly rely on experimental investigations.

All properties, if not stated otherwise, are to be determined at the mean boundary layer temperature:

Definition

Fluid property temperature external natural convection:

$$T_{\text{prop}} = \frac{T_{\text{W}} + T_{\infty}}{2} \ [\text{K}] \tag{15.1}$$

15.2.1 Vertical plate

The transition from laminar to turbulent flow in external natural convection along vertical plates is a phenomenon that occurs as the Grashof number, surpasses a particular threshold, as illustrated in Figure 15.1.

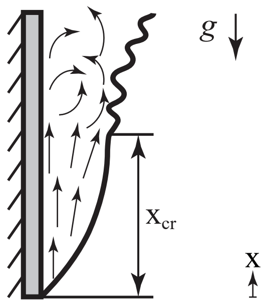


Figure 15.1. The development of the boundary layer for flow along a vertical plate.

In practical applications, the criterion for distinguishing between laminar and turbulent flow can vary depending on the specific context and requirements. Commonly employed thresholds are the Grashof number Gr or the product of the Grashof and Prandtl number $\mathrm{Gr} \cdot \mathrm{Pr}$. In the turbulent flow regime, the heat transfer characteristics are notably different from laminar flow, with significantly increased mixing and convective heat transfer coefficients, leading to enhanced heat transfer rates and different temperature distributions along the surface.

Isothermal surface, laminar flow

As previously discussed in Section 11.3, a theoretical relationship for the Nusselt number is established by examining the conservation equations governing natural convection along a vertical plate subjected to laminar flow conditions. Figure 15.2 illustrates a vertical plate with an isothermal surface, a common configuration for such analyses

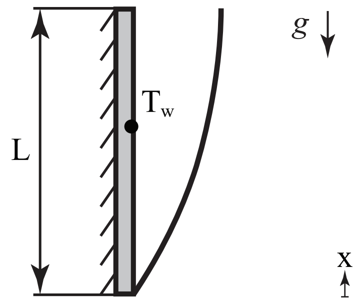


Figure 15.2. Vertical plate with an isothermal surface and laminar natural flow.

The local Nusselt number that applies to this scenario is represented by HTC.16, while the corresponding average Nusselt number is described in HTC.17.

HTC **Local Nusselt number for natural laminar flow along a vertical plate with isothermal surface:**

$$\mathrm{Nu}_x = 0.508 \left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}} \right)^{\frac{1}{4}} (\mathrm{Gr_x Pr})^{\frac{1}{4}}, \tag{HTC.16}$$

for $\mathrm{Gr_x} \cdot \mathrm{Pr} < 4 \cdot 10^9$.

HTC

Average Nusselt number for natural laminar flow along a vertical plate with isothermal surface:

$$\overline{\mathrm{Nu}}_\mathrm{L} = C\,(\mathrm{Gr_L Pr})^{\frac{1}{4}}\,, \tag{HTC.17}$$

for $\mathrm{Gr_L} \cdot \mathrm{Pr} < 4 \cdot 10^9$, and:

Pr	0.003	0.01	0.03	0.72	1	2	10	100	1000	∞
C	0.182	0.242	0.305	0.516	0.535	0.568	0.620	0.653	0.665	0.670

Constant heat flux, laminar flow

Rather than considering a vertical wall with an isothermal surface, a different scenario arises when a constant heat flux is applied to the wall, as depicted in Figure 15.3. In such situations, when determining the appropriate Nusselt correlation, the local Grashof number Gr_x^* should be modified:

Definition

Modified Grashof number for natural flow along a vertical plate with impressed heat flow:

$$\mathrm{Gr}_\mathrm{x}^* = \mathrm{Gr_x Nu_x} = \frac{\rho^2 g \beta \dot{q}_\mathrm{W}'' x^4}{\lambda \eta^2} \quad [-] \tag{15.2}$$

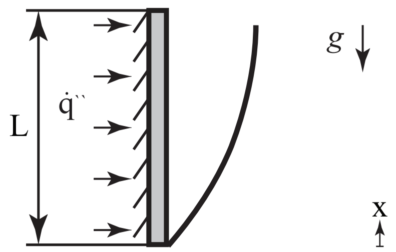


Figure 15.3. Vertical plate with a constant heat flux and laminar natural flow.

The correlation for the local Nusselt number in this specific context has been established through experimental work [13], and is detailed in HTC.18.

HTC

Local Nusselt number for natural laminar flow along a vertical plate with impressed heat flow:

$$\mathrm{Nu_x} = 0.60\,(\mathrm{Gr_x^* Pr})^{\frac{1}{5}}\,, \tag{HTC.18}$$

for $10^5 < \mathrm{Gr}_\mathrm{x}^* < 10^{11}$.

Isothermal surface, turbulent flow

In some scenarios, the transition to turbulent flow can lead to a substantial increase in the rate of heat transfer, as depicted in Figure 15.4. This figure illustrates turbulent natural convection along a vertical plate with an isothermal surface.

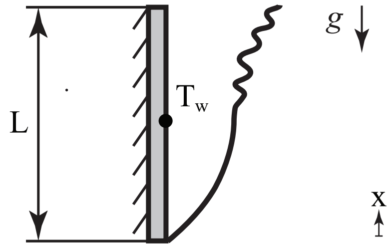


Figure 15.4. Vertical plate with an isothermal surface and turbulent natural flow.

For turbulent flow along a vertical plate with an isothermal surface, an estimate for the average Nusselt Number is obtained using HTC.19.

HTC **Average Nusselt number for natural turbulent flow along a vertical plate with isothermal surface:**

$$\overline{Nu}_L = 0.13\left(Gr_L Pr\right)^{\frac{1}{3}}, \tag{HTC.19}$$

for $10^9 < Gr_L \cdot Pr < 10^{12}$

15.2.2 Vertical cylinder

When the diameter of the cylinder is considerably larger than the thickness of the developed boundary layer, the relationships applicable to vertical plates, HTC.16 to HTC.19, are extended to this scenario. This concept is depicted in Figure 15.5.

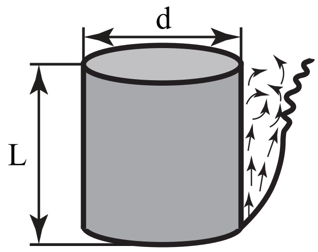


Figure 15.5. Natural convection along a vertical cylinder.

In this scenario, the cylinder is treated as a vertical wall, and consequently, the characteristic length becomes the length of the cylinder. To apply these correlations, the following criterion should be satisfied:

Criterion **Criterion for approximating a vertical cylinder as a vertical plate:**

$$\frac{d}{L} > 35 \cdot Gr_L^{-\frac{1}{4}} \tag{15.3}$$

If applicable, HTC.16 to HTC.19 can be used.

15.2.3 Horizontal cylinder

In the realm of natural convection around a cylinder, numerous real-world examples showcase the significance. This scenario is schematically shown in Figure 15.6. One such instance is the cooling of electrical cables suspended in air, where the ambient air temperature directly impacts the heat dissipation from the cables. Additionally, the cooling of pipes or tubes transporting fluids serves as another example. Here, natural convection is instrumental in transferring heat from the outer surface of the tubes to the surrounding air.

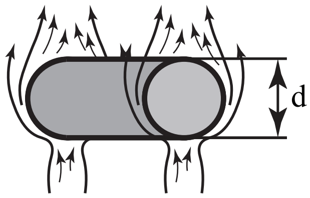


Figure 15.6. Natural convection along a horizontal cylinder.

When analyzing a cylinder with an isothermal surface, the approximations listed in HTC.20 and HTC.21 [14], can be employed based on the prevailing flow regime.

HTC

Average Nusselt number for natural laminar flow around a horizontal cylinder with isothermal surface:

$$\overline{\mathrm{Nu}}_{\mathrm{d}} = 0.53\,(\mathrm{Gr_d Pr})^{\frac{1}{4}}\,, \tag{HTC.20}$$

for $10^4 < \mathrm{Gr_d Pr} < 10^9$.

HTC

Average Nusselt number for natural turbulent flow around a horizontal cylinder with isothermal surface:

$$\overline{\mathrm{Nu}}_{\mathrm{d}} = 0.13\,(\mathrm{Gr_d Pr})^{\frac{1}{3}}\,, \tag{HTC.21}$$

for $10^9 < \mathrm{Gr_d Pr} < 10^{12}$.

15.2.4 Horizontal plate

Natural convection around horizontal plates is a phenomenon with diverse real-world applications. One prominent example is the cooling of electronic components, such as printed circuit boards, where heat dissipation from the surface of the boards is influenced by the surrounding air temperature. Another application lies in the cooling of flat surfaces in buildings, where natural convection aids in regulating the temperature of walls and roofs. Additionally, solar collectors, which absorb sunlight through horizontal surfaces, experience natural convection that influences the overall efficiency of energy absorption and dissipation.

For these cases, the characteristic length to be taken is the ratio between the surface area and the perimeter of the horizontal plate:

Definition

Characteristic length horizontal plates:

$$L = \frac{\text{Surface area}}{\text{Perimeter}} = \frac{A}{U}\ [\mathrm{m}] \tag{15.4}$$

Heated upper plate side or cooled lower side

When addressing a horizontal plate with a heated upper side or a cooled lower side, as illustrated in Figure 15.7, the applicability of HTC.22a to HTC.23b comes into play. The relevant Nusselt correlation is contingent upon the prevailing flow regime and boundary conditions, which may involve an isothermal surface or a consistently applied heat flux.

(a) Heated upper side. (b) Cooled lower side.

Figure 15.7. Natural convection along a horizontal plate.

HTC **Average Nusselt number for natural laminar flow over a horizontal plate with a heated upper or cooled lower:**

Isothermal surface:

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = 0.54\,(\mathrm{Gr}_{\mathrm{L}}\mathrm{Pr})^{\frac{1}{4}}\,, \qquad \text{(HTC.22a)}$$

for $2\cdot 10^4 < \mathrm{Gr}_{\mathrm{L}}\mathrm{Pr} < 8\cdot 10^6$.

Impressed heat flow:

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = 0.13\,(\mathrm{Gr}_{\mathrm{L}}\mathrm{Pr})^{\frac{1}{3}}\,, \qquad \text{(HTC.22b)}$$

for $\mathrm{Gr}_{\mathrm{L}}\mathrm{Pr} < 2\cdot 10^8$.

HTC **Average Nusselt number for natural turbulent flow over a horizontal plate with a heated upper or cooled lower:**

Isothermal surface:

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = 0.15\,(\mathrm{Gr}_{\mathrm{L}}\mathrm{Pr})^{\frac{1}{3}}\,, \qquad \text{(HTC.23a)}$$

for $8\cdot 10^6 < \mathrm{Gr}_{\mathrm{L}}\mathrm{Pr} < 10^{11}$.

Impressed heat flow:

$$\overline{\mathrm{Nu}}_{\mathrm{L}} = 0.16\,(\mathrm{Gr}_{\mathrm{L}}\mathrm{Pr})^{\frac{1}{3}}\,, \qquad \text{(HTC.23b)}$$

for $2\cdot 10^8 < \mathrm{Gr}_{\mathrm{L}}\mathrm{Pr} < 10^{11}$.

Cooled upper plate side or heated lower side

Similarly, correlations are available for laminar natural convection flow along a cooled upper plate side or a heated lower plate side, as illustrated in Figure 15.8. In this context, HTC.24a and HTC.24b can be applied, depending on the boundary conditions involving either an isothermal surface or a consistently applied heat flux.

(a) Heated lower side. (b) Cooled upper side.

Figure 15.8. Natural convection along a horizontal plate.

HTC **Average Nusselt number for natural laminar flow over a horizontal plate with a cooled upper or heated lower:**

Isothermal surface:

$$\overline{\mathrm{Nu}}_\mathrm{L} = 0.27\,(\mathrm{Gr_L Pr})^{\frac{1}{4}}\,, \tag{HTC.24a}$$

for $10^5 < \mathrm{Gr_L Pr} < 10^{10}$.

Impressed heat flow:

$$\overline{\mathrm{Nu}}_\mathrm{L} = 0.58\,(\mathrm{Gr_L Pr})^{\frac{1}{5}}\,, \tag{HTC.24b}$$

for $10^6 < \mathrm{Gr_L Pr} < 10^{11}$.

SECTION 16

Internal natural convection

L11 - Internal natural convection:

Learning goals:

- Comprehension of the impact of heated and cooled surfaces within confined environments.
- Developing decision-making skills for both vertical and horizontal configurations.
- Acquiring a comprehensive understanding of various application scenarios.

Comprehension questions:

- Explain why heat is primarily transferred between two horizontally oriented surfaces within a fluid layer via conduction when the upper plate is heated.
- Identify the exceptional circumstance that contradicts the rule described in the previous question.

Corresponding tutorial exercises:

- Exercise III.14 PV-T panel

SUBSECTION 16.1
Introduction to internal natural convection

Internal natural convection refers to the spontaneous movement of fluid within enclosed spaces or channels. This phenomenon is prevalent in various practical scenarios, ranging from the cooling of machinery components to the regulation of temperature gradients in geological formations. The study of internal natural convection is integral to fluid dynamics and heat transfer research due to the intricate interplay among fluid characteristics, geometrical considerations, and temperature differentials.

(a) Enclosure with vertical walls.

(b) Enclosure with horizontal walls.

Figure 16.1. Internal natural convection.

SUBSECTION 16.2
Heat transfer correlations

Enclosures are commonly encountered in real-world applications, and the study of heat transfer within them holds practical significance. Analyzing heat transfer in enclosed spaces is complex due to the inherent characteristic that the fluid within the enclosure does not typically remain static. In a vertical enclosure, the fluid near the hotter surface ascends, while the fluid near the cooler surface descends, inducing a rotational motion within the enclosure that augments heat transfer. The typical flow patterns in vertical and horizontal rectangular enclosures are illustrated in Figure 16.1.

Unless otherwise specified, all properties are to be evaluated at a mean boundary layer temperature, which is the average of the wall temperatures T_1 and T_2 within the enclosure:

Definition

Fluid property temperature internal natural convection:

$$T_{\text{prop}} = \frac{T_1 + T_2}{2} \; [\text{K}] \tag{16.1}$$

For these cases the distance between the enclosures s is taken as the characteristic length:

Definition

Characteristic length enclosures:

$$L = s \; [\text{m}] \tag{16.2}$$

For enclosures, the following Grashof number needs to be used:

Definition

Grashof number enclosures:

$$\text{Gr}_s = \frac{\beta g \rho^2 \left(T_{\max} - T_{\min}\right) s^3}{\eta^2} \; [-] \tag{16.3}$$

16.2.1 Enclosure with vertical walls

In an enclosure with vertical walls, depicted in Figure 16.2, the nature of natural convection flow intricately depends on the Grashof number. Variations in the Grashof number dictate the predominant heat transfer mechanism. When $\mathrm{Gr_s} < 2 \cdot 10^3$, the flow is predominantly conductive, with heat transfer primarily occurring through the material and no fluid movement. In the laminar regime, $2 \cdot 10^3 < \mathrm{Gr_s} < 2 \cdot 10^4$, a blend of conductive and convective heat transfer is observed. However, for $2 \cdot 10^5 < \mathrm{Gr_s} < 10^7$, buoyancy forces take precedence, leading to turbulent convective flow.

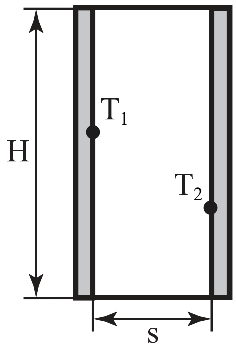


Figure 16.2. Vertical, enclosed fluid layers.

For these different flow regimes HTC.25a to HTC.26 are used to determine the average Nusselt number for an enclosure with vertical isothermal walls.

HTC **Average Nusselt number for natural laminar flow within a vertical enclosure with isothermal surfaces:**

Negligible flow:

$$\overline{\mathrm{Nu}}_\mathrm{s} = 1, \qquad \text{(HTC.25a)}$$

for $\mathrm{Gr_s} < 2 \cdot 10^3$ and $3.1 < H/s < 42.2$.

Laminar flow:

$$\overline{\mathrm{Nu}}_\mathrm{s} = 0.20 \left(\frac{H}{s}\right)^{-\frac{1}{9}} (\mathrm{Gr_s Pr})^{\frac{1}{4}}, \qquad \text{(HTC.25b)}$$

for $2 \cdot 10^3 < \mathrm{Gr_s} < 2 \cdot 10^4$ and $3.1 < H/s < 42.2$.

HTC **Average Nusselt number for natural turbulent flow within a vertical enclosure with isothermal surfaces:**

$$\overline{\mathrm{Nu}}_\mathrm{s} = 0.071 \left(\frac{H}{s}\right)^{-\frac{1}{9}} (\mathrm{Gr_s Pr})^{\frac{1}{3}}, \qquad \text{(HTC.26)}$$

for $2 \cdot 10^5 < \mathrm{Gr_s} < 10^7$ and $3.1 < H/s < 42.2$.

16.2.2 Enclosure with horizontal walls

The mode of heat transfer within a horizontal enclosure, depicted in Figure 16.3, is contingent upon the positioning of the hotter plate. As depicted in Figure 16.1b the colder plate is situated at the top, and convection currents arise within the enclosure, given that the lighter fluid consistently remains below the denser fluid. There is a tendency for the lighter fluid to displace the denser fluid and ascend to the top, where the lighter fluid comes into contact with the cooler plate and undergoes cooling. If the hotter plate is situated above the colder plate, heat transfer occurs solely through conduction, resulting in the Nusselt number being equal to 1. Conversely, when the hotter plate is located at the bottom, the denser fluid rests below the lighter fluid.

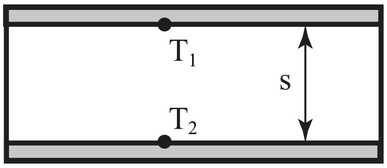


Figure 16.3. Horizontal, enclosed fluid layers.

For horizontal surfaces, the specific threshold for the transition between laminar and turbulent flow varies based on the system's geometry and conditions. HTC.27a to HTC.28 provide the average Nusselt correlations for different flow regimes.

HTC

Average Nusselt number for natural laminar flow within a horizontal enclosure with isothermal surfaces:

Negligible flow:

$$\mathrm{Nu_s} = 1, \tag{HTC.27a}$$

for $\mathrm{Gr_s} < 2 \cdot 10^3$ or $T_1 > T_2$.

Laminar flow:

$$\overline{\mathrm{Nu}}_\mathrm{s} = 0.21\,(\mathrm{Gr_s Pr})^{\frac{1}{4}}\,, \tag{HTC.27b}$$

for $10^4 < \mathrm{Gr_s} < 3.2 \cdot 10^5$.

HTC

Average Nusselt number for natural turbulent flow within a horizontal enclosure with isothermal surfaces:

$$\overline{\mathrm{Nu}}_\mathrm{s} = 0.075\,(\mathrm{Gr_s Pr})^{\frac{1}{3}}\,, \tag{HTC.28}$$

for $3.2 \cdot 10^5 < \mathrm{Gr_s} < 10^7$.

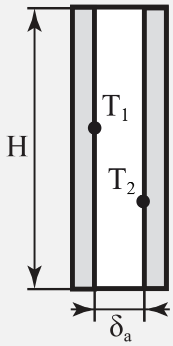
HeatQuiz 16.1

Temperature profiles:

Example 16.1 Consider a double-plane window. Two thermocouples are being placed, which monitor $T_1 = 45$ °C and $T_2 = 20$ °C.

Within the enclosure, a fluid is positioned with the following material properties: $\lambda = 0.005$ W/mK, $\rho = 1.2$ kg/m^3, and $\nu = 1.5 \times 10^{-5}$ m^2/s. The thickness of the enclosure is $\delta_a = 5$ mm, and the surface area through which heat travels yields to be 20 cm × 20 cm.

Determine the rate of heat transfer and compare it to the scenario where convective effects are disregarded, and only conduction is significant.



1 **Setting up the balance:**

In the case of not neglecting convection, the energy balance reads:

$$\dot{Q} = \overline{\alpha} A \left(T_1 - T_2\right)$$

In the case of solely assuming conduction, the balance reads:

$$\dot{Q} = \lambda A \frac{T_1 - T_2}{\delta_a}$$

2 **Defining the elements within the balance:**

All parameters, except the heat transfer coefficient, have been provided. To determine this coefficient, calculating the Grashof number is required, which subsequently enables the determination of a Nusselt correlation applicable to the situation.

The temperature to determine the fluid at properties reads:

$$T_{\text{prop}} = \frac{T_1 + T_2}{2} = 32.5 \text{ °C}$$

The volume expansion for an ideal gas is determined by the following expression:

$$\beta = \frac{1}{T_{\text{prop}}} = 0.0033 \text{ K}^{-1}$$

The Grashof number yields from:

$$\text{Gr}_s = \frac{\beta g \left(T_1 - T_2\right) \delta_a^3}{\nu^2} = 446$$

To determine the average heat transfer coefficient, HTC.25a can be used, reasoning from the fact that $\text{Gr}_s < 2 \cdot 10^3$, and $3.1 < H/\delta_a < 42.2$. Thus $\overline{\text{Nu}}_s = 1$.

Rewriting the expression of the Nusselt number yields:

$$\overline{\alpha} = \frac{\overline{\mathrm{Nu}}_\mathrm{s}\lambda}{\delta_\mathrm{a}} = 5\ \mathrm{W/m^2K}$$

3 **Inserting and rearranging:**

The expression, including the effects of convection yields:

$$\dot{Q} = \overline{\alpha} A \left(T_1 - T_2\right) = 5\ \mathrm{W}$$

From the expression, excluding the effects of convection yields:

$$\dot{Q} = \lambda A \frac{T_1 - T_2}{\delta_\mathrm{a}} = 5\ \mathrm{W}$$

The fact that both expressions yield the same results is because $\overline{\mathrm{Nu}}_\mathrm{s} = 1$, and thus $\overline{\alpha} = \frac{\lambda}{\delta_\mathrm{a}}$.

For a low Grashof number, the buoyancy-driven forces are relatively weak compared to viscous forces, leading to negligible natural convection effects. In such cases, the fluid's motion driven by temperature differences becomes negligible, and heat transfer predominantly occurs through conduction. The low Grashof number signifies that the buoyancy-induced flow is insufficient to significantly influence the convective heat transfer, making conduction the dominant mode of heat transport in the system.

References

[1] H. Schlichting and K. Gersten, *Grenzschichttherorie.* Springer, Berlin, 2006.

[2] L. Prandtl, “Verhandlungen des dritten internationalen mathematiker-kongresses,” *Heidelberg, Leipeizig*, pp. 484–491, 1904.

[3] H. Schlichting and K. Gersten, *Grundzüge Der Grenzschicht-Theorie.* Springer, 2006.

[4] H. Blasius, “Grenzschichten in Flüssigkeiten mit kleiner Reibung,” *Z. Math. Phys.*, vol. 56, pp. 1–37, 1908.

[5] K. Pohlhausen, “Zur näherungsweisen integration der differentialgleichung der laminaren grenzschicht,” *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 1, no. 4, 1921.

[6] S. Ostrach, “An Analysis of Laminar Free Convection Flow and Heat Transfer about a Flat Plate Parallel to the Direction of the Generating Body Force,” *NACA Rep. 1111*, 1953.

[7] S. Whitaker, *Fundamental Principles of Heat Transfer.* Pergamom Press Inc,. New York, 1974.

[8] M. Jakob, *Heat Transfer.* John Wiley & Sons, New York, 1949.

[9] S. Whitaker, *Fundamental Pricnciples of Heat Transfer.* Pergamom Press Inc,. New York, 1976.

[10] U. Witte, *Dampferzeugertechnik.* Vulkan-Verlag, Essen, 1974.

[11] W. M. Rohsenow and J. P. Harnett, *Handbook of Heat Transfer.* McGraw-Hill Book Company, New York, 1973.

[12] V. VDI-Wärmeatlas, “Verlag gmbh,” 1997.

[13] J. P. Holman, *Heat Transfer.* McGraw-Hill Book Company, 4 ed., 1976.

[14] F. J. Bayley, J. M. Owen, and A. B. Turner, *Heat Transfer.* William Clowes & Sons, Ltd., London, 1972.

PART
V

Appendix

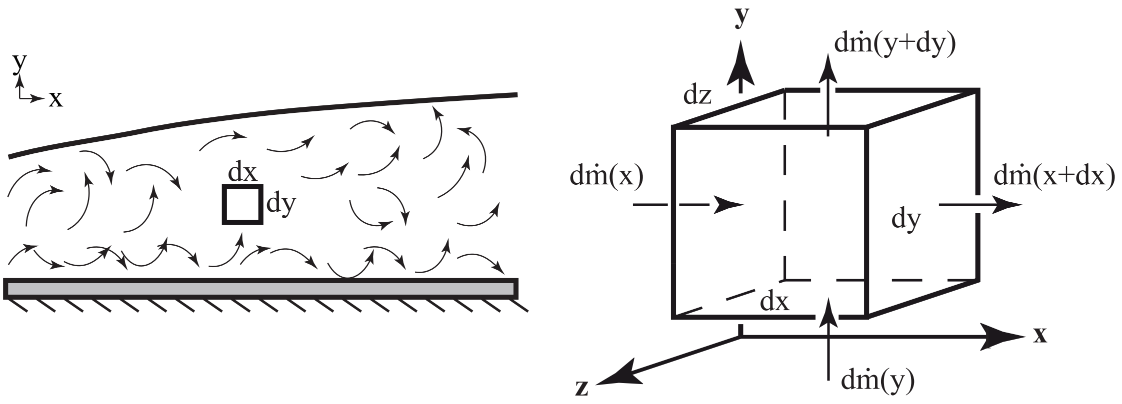
Section A

Turbulence models derivation

Equation of continuity

Derivation

For on the average steady-state flow without generation of mass, the difference between mass flowing in and out must vanish.



(a) Control volume located within the boundary layer.

(b) Mass balance at the control volume.

Figure A.1. Turbulent flow over a flat plate.

1 Setting up the balance:

2 Defining the elements within the balance:

See Section 10.3

3 Inserting and rearranging:

Recall:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

Substituting the definitions of the velocity components for turbulent flow in the x- and y-directions, and assuming ρ = constant, the following expression is obtained:

$$\frac{\partial \overline{u} + u^{'}}{\partial x} + \frac{\partial \overline{v} + v^{'}}{\partial y} = 0$$

Now the continuity equation will be averaged. The ensemble rules of averaging need to be employed, keeping in mind that the average of products of fluctuating quantities will not in general vanish.

When dealing with, on average, steady-state turbulent flow it yields the average of the fluctuation cancels out, thus $\overline{u^{'}} = 0$ and $\overline{v^{'}} = 0$. The equation simplifies to:

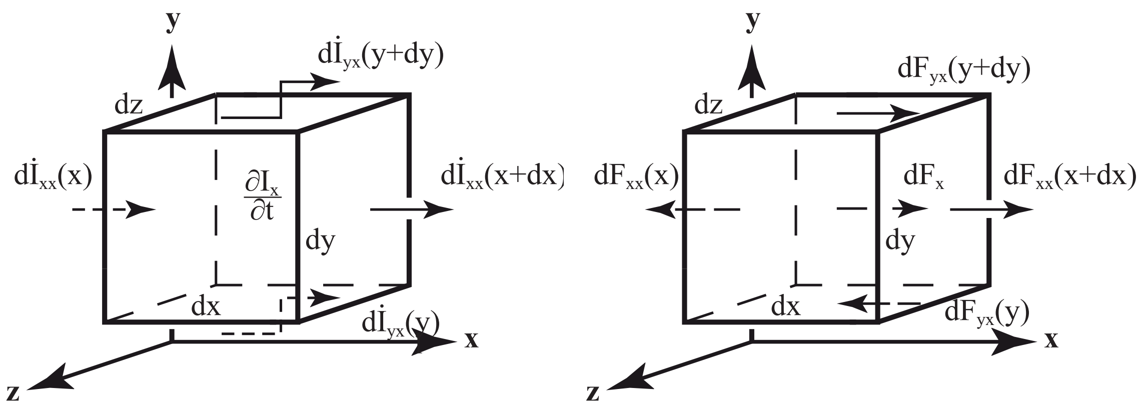
$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

□

Equations of momentum

The momentum equation states that the difference between the momentum in- and outflow of the volume element is equal to the external forces acting on the volume element, whereas inertial forces and surface forces can be relevant.

Equation of momentum in x-direction



(a) Momentum flow in the direction of x.

(b) Forces in the direction of x.

Figure A.2. Momentum balance at the control volume.

Derivation

1 **Setting up the balance:**

Refer to Section 10.3

2 **Defining the elements within the balance:**

Refer to Section 10.3

3 **Inserting and rearranging:**

Recall:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\left(\frac{\partial p}{\partial x}\right)+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho\, g_x$$

Inserting the definition of parameters u, v, and p yields:

$$\rho\left(\left(\overline{u}+u'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial x}+\left(\overline{v}+v'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial y}\right)=-\left(\frac{\partial\left(\overline{p}+p'\right)}{\partial x}\right)+\eta\left(\frac{\partial^2\left(\overline{u}+u'\right)}{\partial x^2}+\frac{\partial^2\left(\overline{u}+u'\right)}{\partial y^2}\right)+\rho\, g_x$$

Taking the two terms of the left-hand side:

$$\left(\overline{u}+u'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial x}=\frac{\partial\left(\overline{u}+u'\right)\left(\overline{u}+u'\right)}{\partial x}-\left(\overline{u}+u'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial x}$$

$$\left(\overline{v}+v'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial y}=\frac{\partial\left(\overline{u}+u'\right)\left(\overline{v}+v'\right)}{\partial y}-\left(\overline{u}+u'\right)\frac{\partial\left(\overline{v}+v'\right)}{\partial y}$$

Inserting the two expressions above into the momentum equation it yields:

$$\rho\left(\frac{\partial\left(\overline{u}+u^{'}\right)\left(\overline{u}+u^{'}\right)}{\partial x}+\frac{\partial\left(\overline{u}+u^{'}\right)\left(\overline{v}+v^{'}\right)}{\partial y}-\underbrace{\cancel{\left(\overline{u}+u^{'}\right)\left(\frac{\partial\left(\overline{u}+u^{'}\right)}{\partial x}+\frac{\partial\left(\overline{v}+v^{'}\right)}{\partial y}\right)}}_{0\,-\,\text{equation of continuity}}\right)$$

$$=-\left(\frac{\partial\left(\overline{p}+p^{'}\right)}{\partial x}\right)+\eta\left(\frac{\partial^2\left(\overline{u}+u^{'}\right)}{\partial x^2}+\frac{\partial^2\left(\overline{u}+u^{'}\right)}{\partial y^2}\right)+\rho\, g_x$$

Now the momentum equation will be averaged. The ensemble rules of averaging need to be employed, keeping in mind that the average of products of fluctuating quantities will not in general vanish. After averaging, the momentum equation becomes:

$$\rho\left(\frac{\partial\overline{u}\overline{u}}{\partial x}+\frac{\partial\overline{u}\overline{v}}{\partial y}+\frac{\partial\overline{u^{'}u^{'}}}{\partial x}+\frac{\partial\overline{u^{'}v^{'}}}{\partial y}\right)=-\left(\frac{\partial\overline{p}}{\partial x}\right)+\eta\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}\right)+\rho\, g_x$$

Using the chain rule for the first two terms on the left hand side of the momentum equation yields:

$$\frac{\partial\overline{u}\overline{u}}{\partial x}=\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial\overline{u}}{\partial x}$$

$$\frac{\partial\overline{u}\overline{v}}{\partial y}=\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{u}\frac{\partial\overline{v}}{\partial y}$$

Inserting the two terms above into the momentum equation yields:

$$\rho\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{u}\overbrace{\cancel{\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}\right)}}^{0\,-\,\text{equation of continuity}}+\frac{\partial\overline{u^{'}u^{'}}}{\partial x}+\frac{\partial\overline{u^{'}v^{'}}}{\partial y}\right)=-\left(\frac{\partial\overline{p}}{\partial x}\right)+\eta\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}\right)+\rho\, g_x$$

Rewriting yields:

$$\rho\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}\right)=-\left(\frac{\partial\overline{p}}{\partial x}\right)+\eta\frac{\partial}{\partial x}\left(\frac{\partial\overline{u}}{\partial x}+\rho\overline{u^{'}u^{'}}\right)+\eta\frac{\partial}{\partial y}\left(\frac{\partial\overline{u}}{\partial y}+\rho\overline{u^{'}v^{'}}\right)+\rho\, g_x$$

It was stated in Section 10.3, that for $\delta \ll L$ it yields that $\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial x^2}$, also the effect of gravity is negligible, so the equation simplifies to:

$$\rho\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}\right)=-\left(\frac{\partial\overline{p}}{\partial x}\right)+\eta\frac{\partial}{\partial y}\left(\frac{\partial\overline{u}}{\partial y}+\rho\overline{u^{'}v^{'}}\right)$$

The equation can be written in analog form, where $-\rho\overline{u^{'}v^{'}}=\eta_{\mathrm{t}}\frac{\partial\overline{u}}{\partial y}$ and $\eta_{\mathrm{eff}}=\eta+\eta_{\mathrm{t}}$ is introduced:

$$\rho\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}\right)=-\left(\frac{\partial\overline{p}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta_{\mathrm{eff}}\frac{\partial\overline{u}}{\partial y}\right)$$

□

Equation of energy conservation

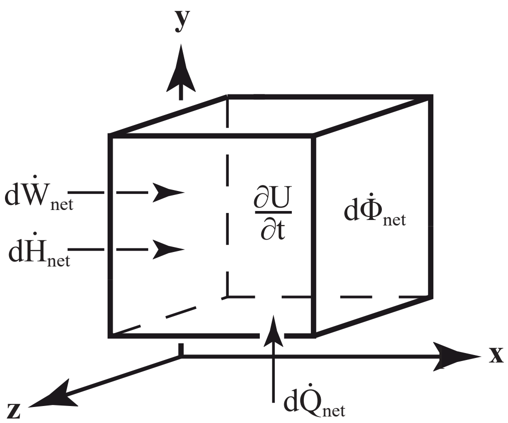


Figure A.3. Energy balance at the control volume.

Derivation

1 **Setting up the balance:**
Refer to Section 10.3

2 **Defining the elements within the balance:**

Refer to Section 10.3

3 **Inserting and rearranging:**

Recall:

$$\rho\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

Inserting the definition of parameters u, v, and p yields:

$$\rho\left(\left(\overline{u}+u^{'}\right)\frac{\partial\left(\overline{T}+T^{'}\right)}{\partial x}+\left(\overline{v}+v^{'}\right)\frac{\partial\left(\overline{T}+T^{'}\right)}{\partial y}\right)=\frac{\lambda}{c_p}\left(\frac{\partial^2\left(\overline{T}+T^{'}\right)}{\partial x^2}+\frac{\partial^2\left(\overline{T}+T^{'}\right)}{\partial y^2}\right)$$

Taking the two terms of the left-hand side:

$$\left(\overline{u}+u^{'}\right)\frac{\partial\left(\overline{T}+T^{'}\right)}{\partial x}=\frac{\partial\left(\overline{T}+T^{'}\right)\left(\overline{u}+u^{'}\right)}{\partial x}-\left(\overline{T}+T^{'}\right)\frac{\partial\left(\overline{u}+u^{'}\right)}{\partial x}$$

$$\left(\overline{v}+v^{'}\right)\frac{\partial\left(\overline{T}+T^{'}\right)}{\partial y}=\frac{\partial\left(\overline{T}+T^{'}\right)\left(\overline{v}+v^{'}\right)}{\partial y}-\left(\overline{T}+T^{'}\right)\frac{\partial\left(\overline{v}+v^{'}\right)}{\partial y}$$

Inserting the two expressions above into the momentum equation yields

$$\rho\left(\frac{\partial\left(\overline{T}+T^{'}\right)\left(\overline{u}+u^{'}\right)}{\partial x}+\frac{\partial\left(\overline{T}+T^{'}\right)\left(\overline{v}+v^{'}\right)}{\partial y}-\left(\overline{T}+T^{'}\right)\underbrace{\cancel{\left(\frac{\partial\left(\overline{u}+u^{'}\right)}{\partial x}+\frac{\partial\left(\overline{v}+v^{'}\right)}{\partial y}\right)}}_{=0\,\rightarrow\,\text{equation of continuity}}\right)$$

$$=\frac{\lambda}{c_p}\left(\frac{\partial^2\left(\overline{T}+T^{'}\right)}{\partial x^2}+\frac{\partial^2\left(\overline{T}+T^{'}\right)}{\partial y^2}\right)$$

Now the energy conservation equation will be averaged. The ensemble rules of averaging need to be employed, keeping in mind that the average of products of fluctuating quantities will not in general vanish. After averaging, the energy conservation equation becomes:

$$\rho\left(\frac{\partial \overline{uT}}{\partial x}+\frac{\partial \overline{vT}}{\partial y}+\frac{\partial \overline{u'T'}}{\partial x}+\frac{\partial \overline{v'T'}}{\partial y}\right)=\frac{\lambda}{c_p}\left(\frac{\partial^2 \overline{T}}{\partial x^2}+\frac{\partial^2 \overline{T}}{\partial y^2}\right)$$

Using the chain rule for the first two terms on the left hand side of the momentum equation yields:

$$\frac{\partial \overline{uT}}{\partial x}=\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{T}\frac{\partial \overline{u}}{\partial x}$$

$$\frac{\partial \overline{vT}}{\partial x}=\overline{v}\frac{\partial \overline{T}}{\partial x}+\overline{T}\frac{\partial \overline{v}}{\partial x}$$

Inserting the two terms above into the energy conservation equation yields

$$\rho\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial x}+\overline{T}\cancelto{0\,-\,\text{equation of continuity}}{\left(\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}\right)}+\frac{\partial \overline{u'T'}}{\partial x}+\frac{\partial \overline{v'T'}}{\partial y}\right)=\frac{\lambda}{c_p}\left(\frac{\partial^2 \overline{T}}{\partial x^2}+\frac{\partial^2 \overline{T}}{\partial y^2}\right)$$

Rewriting yields

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial x}\right)=\lambda\frac{\partial}{\partial x}\left(\frac{\partial \overline{T}}{\partial x}+\rho c_p\overline{u'T'}\right)+\lambda\frac{\partial}{\partial y}\left(\frac{\partial \overline{T}}{\partial y}+\rho c_p\overline{v'T'}\right)$$

It was stated in Section 10.3, that for $\delta \ll L$ it yields that $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$, so the equation simplifies to:

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial x}\right)=\lambda\frac{\partial}{\partial y}\left(\frac{\partial \overline{T}}{\partial y}+\rho c_p\overline{v'T'}\right)$$

The equation can be written in analog form, where $\rho c_p\overline{v'T'}=-\lambda_{\mathrm{t}}\frac{\partial \overline{T}}{\partial y}$ and $\lambda_{\mathrm{eff}}=\lambda+\lambda_{\mathrm{t}}$ is introduced:

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial x}\right)=\frac{\partial}{\partial y}\left(\lambda_{\mathrm{eff}}\frac{\partial \overline{T}}{\partial y}\right)$$

By assuming constant density, constant specific heat capacity, and substituting the definitions of u, v, T, along with some additional algebraic manipulations, the following expression is obtained:
□

PART
VI

Exercises

.

SECTION III

Convection exercises

Exercise III.1 (Walking man ⋆)**:**

A man has a body surface area of A and a skin temperature of T_s, with an average surface temperature of the clothed person of T_c. The convection heat transfer coefficient α for a clothed man walking in the air with temperature T_A is expressed as:

$$\alpha = C \cdot \sqrt{V},$$

for $0.5 < V < 5$ m/s, and where $C = 8.2\ \frac{\mathrm{J}}{\mathrm{m}^{2.5}\mathrm{s}^{0.5}\mathrm{K}}$, and V is the relative velocity of the man with respect to the air.

Given parameters:

- Surface area of the body: $A = 1.8\ \mathrm{m}^2$
- Thermal conductivity of the skin: $\lambda_\mathrm{s} = 0.25$ W/mK
- Thermal conductivity of clothes: $\lambda_\mathrm{s} = 0.03$ W/mK
- Thermal conductivity of the air: $\lambda_\mathrm{a} = 0.026$ W/mK
- Skin temperature of the man: $T_\mathrm{s} = 33$ °C
- Surface temperature of the clothed man: $T_\mathrm{c} = 30$ °C
- Air temperature: $T_\mathrm{A} = 15$ °C

Hints:

- Assume steady-state operating conditions.
- Assume the heat transfer coefficient to be constant over the entire surface.

Tasks:

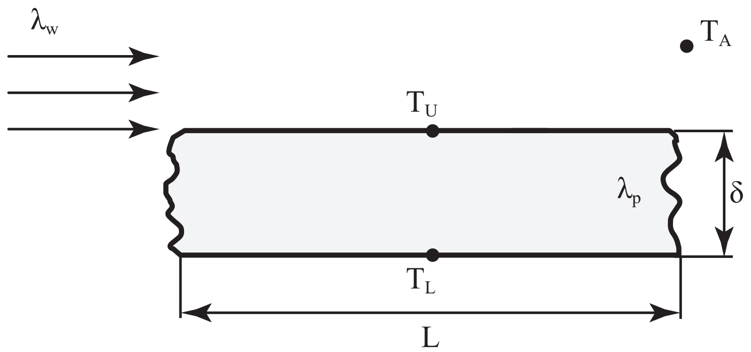
a) Determine the rate of heat loss from the man by convection while walking in still air at a speed of 1 m/s.

b) Determine the rate of heat loss from the man walking in the air when walking in the same direction of the wind with a velocity of 1.5 m/s, while the wind is blowing at a velocity of 2 m/s.

c) Determine the rate of heat loss and the relative velocity from the man while walking in still air with a Nusselt number of Nu = 510, and a characteristic length of $L = 1$ m.

Exercise III.2 (Thick solid plate ⋆):

The top surface of a thick solid plate is cooled by water flowing. The upper and lower surfaces of the solid plate are maintained at constant temperatures T_U and T_L respectively.



Given parameters:

- Thickness of the plate: δ
- Length of the plate: L
- Thermal conductivity of the plate: λ_p
- Thermal conductivity of the water: λ_w
- Upper surface temperature of the plate: T_U
- Lower surface temperature of the plate: T_L
- Ambient temperature: T_A

Hints:

- Assume steady-state operating conditions.
- Assume the heat transfer coefficient to be constant over the entire surface.
- $T_\mathrm{L} > T_\mathrm{U}$

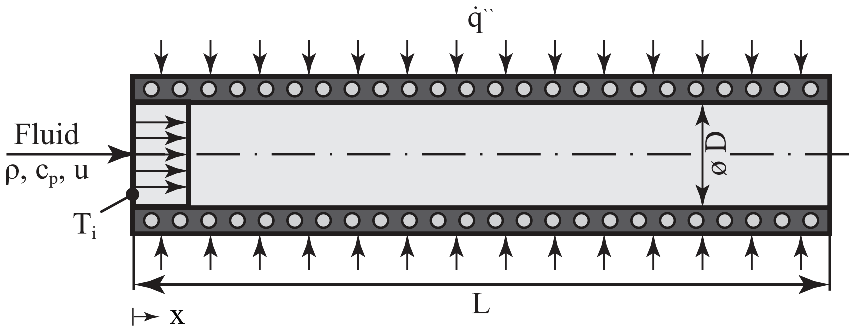
Tasks:

a) Determine the Nusselt number in terms of the given variables, using the length L of the plat as the characteristic length.

b) Determine the temperature gradient inside the water at the interface in terms of the given variables.

Exercise III.3 (Pipe flow ⋆⋆):

A fluid flows through a long cylindrical tube. A constant heat flux density $\dot{q}''$ is imposed on the fluid.



Given parameters:

- Diameter of the pipe: D
- Length of the plate: L
- Heat flux density: $\dot{q}''$
- Density of the fluid: ρ
- Specific heat capacity of the fluid: c_p
- Average velocity of the fluid: u
- Fluid inlet temperature: T_i

Hints:

- Assume one-dimensional heat transfer in the axial direction.
- Assume steady-state operating conditions.
- Conduction in the fluid is negligible.

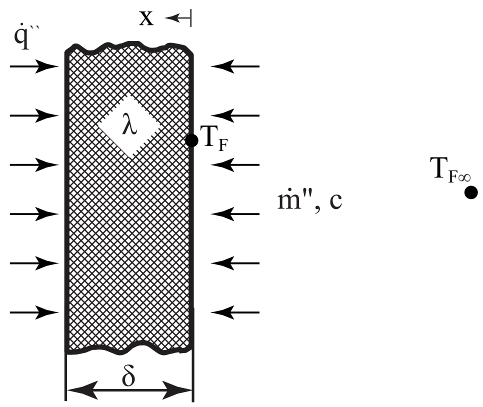
Tasks:

a) Determine the temperature profile of the fluid.

b) Determine the temperature of the fluid at 75% of the pipe length.

Exercise III.4 (Porous wall ⋆ ⋆ ⋆)**:**

The surface of a porous wall, impermeable to radiation, absorbs a radiative heat flux. For cooling purposes, a coolant is circulated through the wall with an inlet temperature is T_F.



Given parameters:

- Imposed radiative heat flux: $\dot{q}'' = 150 \cdot 10^3$ W/m^2
- Wall thickness: $\delta = 50$ mm
- Wall Thermal conductivity: $\lambda = 8$ W/mK
- Coolant specific heat capacity: $c = 1000$ J/kgK
- Coolant inlet temperature: $T_F = -15$ °C
- Coolant area specific mass flux: $\dot{m}'' = 0.6$ kg/m$^2 \cdot$ s

Hints:

- Within the wall, conduction of the imposed radiative heat flux is negligible.
- The local fluid and wall temperatures can be assumed to be identical.

Tasks:

a) Determine the temperature profile $T(x)$ for the porous wall.

b) Determine the maximum temperature T_{max} reached within the wall.

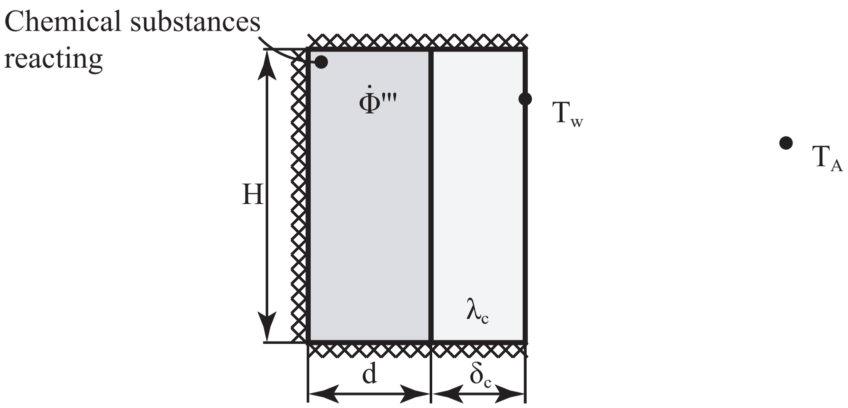
c) Determine the heat flux $\dot{q}''_F$ per unit area, which is transmitted into the fluid at $x = 0$.

d) Which temperature $T_{F,\infty}$ does the fluid reach far away from the wall?

e) Sketch the temperature profiles for two different mass fluxes and mark each curve.

Exercise III.5 (Substance container ⋆)**:**

Imagine you are involved in the design of a chemical substance container. These containers house substances that generate heat during chemical reactions. The top and back are adiabatically insulated. During this reaction heat is dissipated to the surrounding air.



Given parameters:

- Height of the container: $H = 80$ cm
- Depth of the container: $d = 50$ cm
- Wall thickness of the container: $\delta_c = 10$ cm
- Thermal conductivity of the wall: $\lambda_c = 0.3$ W/mK
- Thermal conductivity of the air: $\lambda = 0.025$ W/mK
- Prandtl number of the air: $\mathrm{Pr} = 0.72$
- Kinematic viscosity of the air: $\nu = 1.5 \cdot 10^{-5}$ m^2/s
- Outside temperature of the wall: $T_w = 30$ °C
- Temperature of the ambient air: $T_A = 20$ °C

Hints:

- Assume one-dimensional heat transfer.
- Assume steady-state operating conditions.

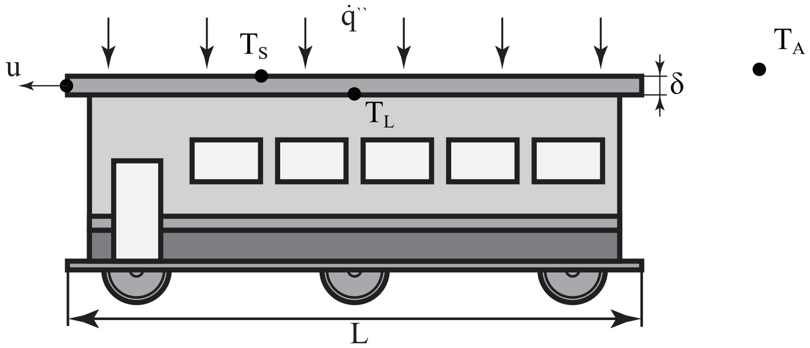
Tasks:

a) Determine the interface temperature between the chemical substances and their container.

b) Determine the heat generated by the substances per unit volume $\dot{\Phi}'''$.

Exercise III.6 (Moving train ⋆):

Consider the roof surface of a passenger car on a moving train. This surface is exposed to solar radiation, with an incident heat flux denoted as $\dot{q}''$. The ambient air temperature is represented by T_A.



Given parameters:

- Velocity of the train: $u = 50$ km/h
- Length of the train roof: $L = 10$ m
- Width of the train roof: $W = 3$ m
- Thickness of the train roof: $\delta = 20$ cm
- Thermal conductivity of the train roof: $\lambda = 0.03$ W/mK
- Lower temperature of the train roof: $T_L = 16$ °C
- Ambient temperature: $T_A = 15$ °C
- Solar irradiation: $\dot{q}'' = 288$ W/m^2
- Properties of air:

T [°C]	ρ [kg/m^3]	c [kJ/kg·K]	λ [W/mK]	ν [m^2/s]	Pr [–]
0	1.275	1.006	$24.18 \cdot 10^{-3}$	$13.52 \cdot 10^{-6}$	0.7179
20	1.188	1.007	$25.69 \cdot 10^{-3}$	$15.35 \cdot 10^{-6}$	0.7148
40	1.112	1.007	$27.16 \cdot 10^{-3}$	$17.26 \cdot 10^{-6}$	0.7122
80	0.9859	1.008	$30.01 \cdot 10^{-3}$	$21.35 \cdot 10^{-6}$	0.7083
100	0.9329	1.009	$31.39 \cdot 10^{-3}$	$23.51 \cdot 10^{-6}$	0.7073

Hints:

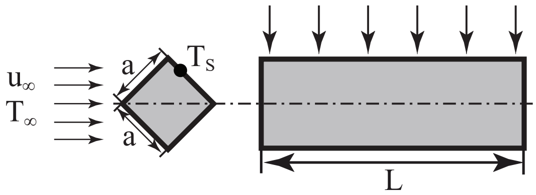
- Assume steady-state heat transfer to be one-dimensional
- Neglect radiation heat exchange with the surroundings.

Tasks:

a) Determine the equilibrium temperature of the top surface T_s.

Exercise III.7 (Transverse flow ⋆⋆):

Air flows transversely across a beam of length L, with a square cross-sectional area, as can be seen in the figure. The circles with crosses indicate streamlines that are moving away from the observer.



Given parameters:

- Beam geometrical dimensions: a,L
- Material properties of the air: η,ρ, Pr ,λ
- Velocity of the crossflow: u_∞
- Temperatures of the surface and ambient: T_s, T_∞

Hints:

- Assume steady-state conditions
- $10^4 \leq \mathrm{Re} \leq 10^5$
- Heat loss from the sides is negligible.

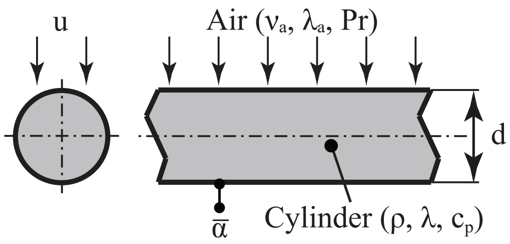
Tasks:

a) Provide an expression for the rate of heat loss in terms of the given parameters.

b) Determine the percentual change of the heat transfer coefficient if we had a similar-form rod with four times the crosswise width at double flow velocity.

Exercise III.8 (Heating of a cylinder ★★):

A long cylinder is kept at a homogeneous temperature T_0. This cylinder is suddenly, at time $t = 0$, exposed to a warm air flow with temperature T_a and a transverse velocity of u.



Given parameters:

- Diameter of the cylinder: $d = 0.055$ m
- Density of the cylinder: $\rho = 1500$ kg/m^3
- Thermal conductivity of the cylinder: $\lambda = 0.119$ W/mK
- Heat capacity of the cylinder: $c_p = 1000$ J/kgK
- Initial homogenous temperature of the cylinder: $T_0 = 10$ °C
- Critical temperature of the cylinder: $T_c = 38$ °C
- Temperature of the air flow: $T_a = 40$ °C
- Velocity of the air: $u = 0.1$ m/s
- Thermal conductivity of the air: $\lambda_a = 25.7 \cdot 10^{-3}$ W/mK
- Kinematic viscosity of the air: $\nu_a = 15.35 \cdot 10^{-6}$ m^2/s
- Prandtl number of the air: Pr = 0.71

Hints:

- The average heat transfer coefficient $\overline{\alpha}$ is steady in time.
- The material properties can be taken as constant.

Tasks:

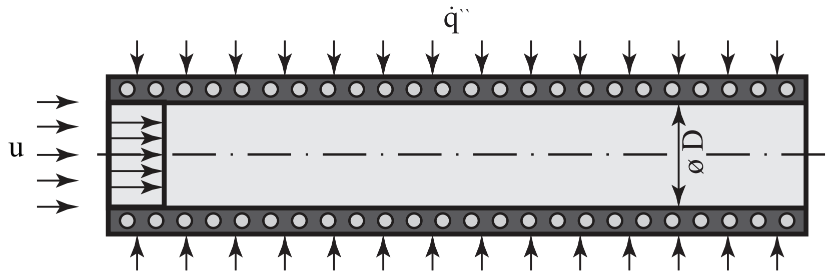
a) Determine the initial rate of heat transfer per unit length of the cylinder.

b) At $t = t_c$ the cylinder reaches the critical temperature T_c at its hottest point. Sketch qualitatively the temperature distribution at time-points $t = 0$ and $t = t_c$.

c) Determine the time t_c, until the cylinder reaches the critical temperature T_c at its hottest point.

Exercise III.9 (Pipe flow with a constant heat flux ⋆⋆)**:**

A fluid flows through a long cylindrical tube. A constant heat flux density $\dot{q}''$ is imposed on the fluid.



Given parameters:

- Average axial velocity: u
- Heat flux density: $\dot{q}''$
- Fluid density: ρ
- Fluid thermal capacity: c_p
- Fluid thermal conductivity: λ
- Inner pipe diameter: D

Hint:

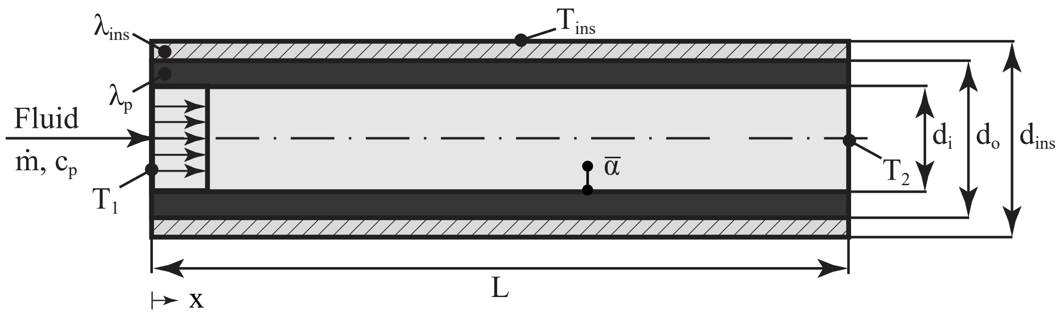
- Axial heat diffusion cannot be neglected.

Tasks:

a) Derive the transient differential energy equation for the mean fluid temperature along the axial direction.

Exercise III.10 (Insulated pipe ⋆⋆)**:**

A pipe is being heated by a stationary flow. The outer surface of the pipe has an insulation layer with its external side kept at a constant temperature T_{ins}.



Given parameters:

- Temperature of the fluid at the inlet: T_1
- Temperature of outer surface area of the pipe: T_{ins}
- Convective heat transfer coefficient: $\overline{\alpha}$
- Mass flow of the fluid: $\dot{m}$
- Specific heat capacity of the fluid: c_{p}
- Inner diameter of the pipe: d_{i}
- Outer diameter of the pipe excluding insulation: d_{o}
- Outer diameter of the pipe including insulation: d_{ins}
- Length of the pipe: L
- Thermal conductivity of the pipe wall: λ_{p}
- Thermal conductivity of the insulation layer: λ_{ins}

Tasks:

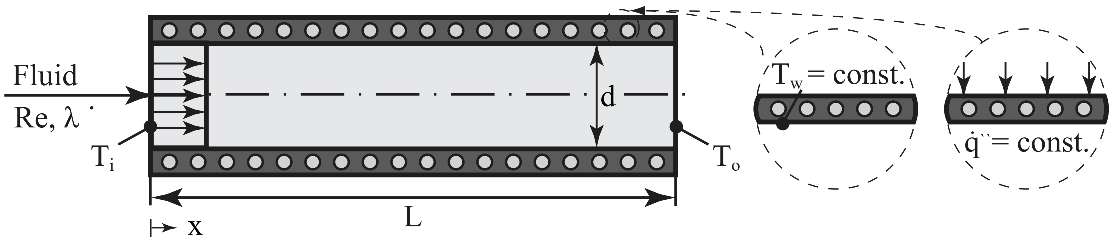
a) Find an expression for the exit temperature T_2 in terms of given parameters.

Exercise III.11 (Heating of a pipe ⋆⋆⋆):

A fluid is flowing through a pipe. The flow is thermally and hydrodynamically developed. A heat flow that is transferred from the wall by convection is heating the fluid from T_i to T_o. For this purpose,

- in case 1: a constant, homogeneous **wall temperature** T_w
- in case 2: a constant, homogeneous **heat flux** $\dot{q}''$

is impressed.



Given parameters:

- Temperature of the fluid at the inlet: T_i
- Temperature of the fluid at the outlet: T_o
- Wall temperature (case 1): T_w
- Heat flux (case 2): $\dot{q}''$
- Length of the pipe: L
- Inner diameter of the pipe: d
- Reynolds number of the flow: Re < 2300
- Density of the fluid: λ
- Dynamic viscosity of the fluid: η
- Conductivity of the fluid: λ

Hints:

- Heat conduction in the direction of the fluid flow is negligible.
- Difference in fluid properties in the radial direction is negligible.

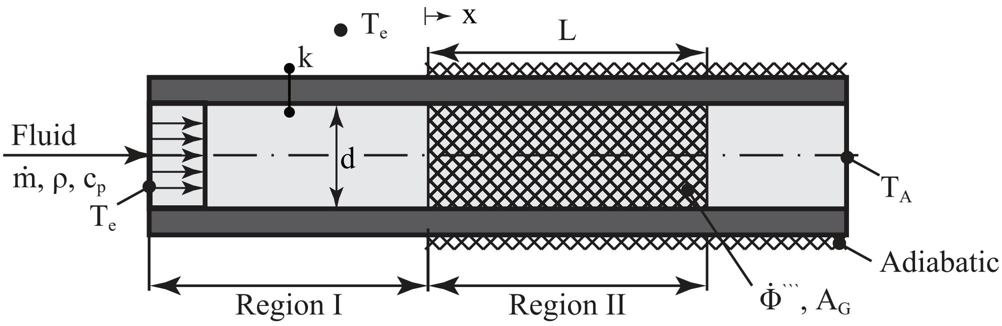
Tasks:

a) Write out the global energy balance in terms of given parameters, determine the mean heat transfer coefficient $\overline{\alpha}$ and provide the respective mean temperature difference ΔT_m between the inner wall of the tube and the fluid for both cases.

b) Draw qualitatively for both cases the profile of the wall temperature T_w and the mean fluid temperature T_fl.

Exercise III.12 (Flow through a grid ⋆⋆⋆):

Water flows through a long tube that has adiabatic walls from a certain location $x = 0$. The area upstream of $x = 0$ is named region I. Between the point $x = 0$, and $x = L$ (region II) a very fine-meshed, electrically heated grid is located in the flow. Well ahead of the grid, the flow has the ambient temperature T_e, and downstream of the grid, the temperature T_A.



Given parameters:

• Water temperature before the grid:	T_e
• Environment temperature:	T_e
• Water temperature after the grid:	T_A
• Mass flow rate:	$\dot{m}$
• Thermal conductivity:	λ
• Specific heat capacity:	c_p
• Diameter of the pipe/grid:	d
• Length of the grid:	L
• Average heat flux on the surface of the grid:	$\dot{q}''$
• Heat transfer area of the grid:	A_G
• Overall heat transfer coefficient between water and environment, based on the inner pipe wall area	k

Hint:

- The problem is steady and one-dimensional.
- The electrically heated mesh is so fine that a homogeneous heat flux is introduced.
- The volume of the fine-meshed grid can be neglected.

Tasks:

a) Determine the volumetric heat release $\dot{\Phi}'''$ created by the electrically heated grid.

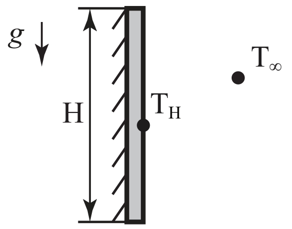
b) Derive the differential equations for the temperature profile of the water in the pipe in regions I and II. It is unknown whether heat diffusion is negligible, and thus should be included in the equation.

c) Provide all the coupling or boundary conditions required for the solution of the problem (regions I and II).

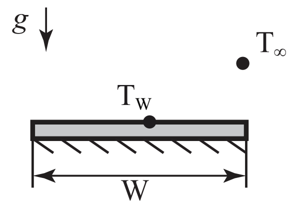
d) Sketch the temperature profiles of the water in the pipe with and without consideration of the diffusive heat transport.

Exercise III.13 (Horizontal and vertical wall ⋆⋆)**:**

Two heat-emitting surfaces (case 1: H height, case 2: W width) with the respective wall temperatures T_{H} and T_{W} are given. The quiescent environment has a temperature T_∞.



(a) Case 1



(b) Case 2

Given parameters:

- Prandtl Number: $\mathrm{Pr} = 1$
- Value range for laminar boundary layer: $1 \cdot 10^5 < \mathrm{Gr_L Pr} < 1 \cdot 10^6$
- Geometrical ratio: $W = 2 \cdot H$
- Length of both plates: L
- Temperatures: $T_{\mathrm{H}} = 2 \cdot T_{\mathrm{W}} = 4 \cdot T_\infty$

Hint:

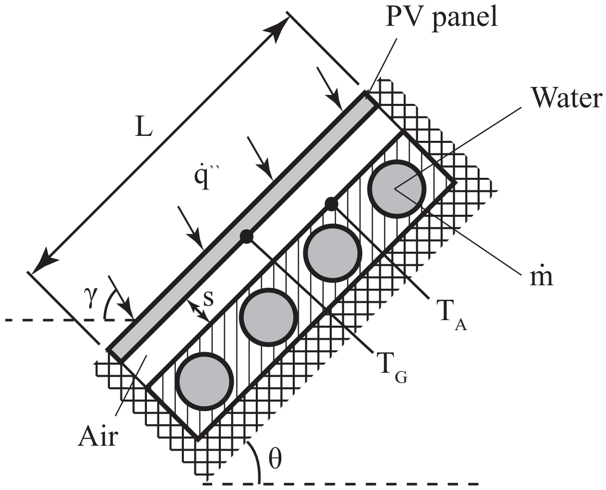
- The difference in average fluid properties between both cases is negligible.
- $L \gg W$

Tasks:

a) Determine the ratio of the convective losses from the surfaces.

Exercise III.14 (PV-T Panel ⋆⋆):

PV-T panels, generating thermal and electrical power, are frequently inclined towards the sun to enhance their efficiency. The tilt angle, denoted as θ, plays a crucial role in determining the effectiveness of the solar panel. Radiation is incident upon a PV-T collector at an angle γ, and it possesses a constant heat density represented by $\dot{q}''$. Within the tube collectors, water flows in, entering at a temperature T_{in}, and exits at a temperature T_{out}.



Given parameters:

- Collector height: $L = 0.8$ m
- Collector width: $W = 3$ m
- Space between absorber plate and glass cover: $s = 2$ cm
- Heat flux density: $\dot{q}'' = 1000$ W/m^2
- Heat flux angle: $\gamma = 60^o$
- Glass cover temperature: $T_G = 40$ °C
- Absorber plate temperature: $T_A = 100$ °C
- Air average density: $\rho = 1.05$ kg/m^3
- Air average thermal conductivity: $\lambda = 0.029$ W/mK
- Air average kinematic viscosity: $\nu = 1.9 \cdot 10^{-5}$ m^2/s
- Air average Prandtl number: Pr = 0.71
- Water inlet temperature: $T_{\text{in}} = 10$ °C
- Water mass flow rate: $\dot{m} = 0.01$ kg/s
- Water average specific heat capacity: $c_p = 4.2$ kJ/kgK
- Water outlet temperature:

$$T_{\text{out}} = T_{\text{in}} \left(2 + 2 \cdot \sin\left(2\gamma - \theta\right)\right)$$

Hints:

- All incident radiation is absorbed by the collector.
- The back side of the absorber is heavily insulated.
- The process can be assumed to be steady-state.

Tasks:

a) Determine the overall efficiency of the PV-T panel for $\theta = 0^o$.

b) Reason what you expect to be the optimal tilt angle for the PV-T panel to have the highest generation of useful energy.

c) Draw the temperature profile for the domain $0 \leq x \leq \infty$.