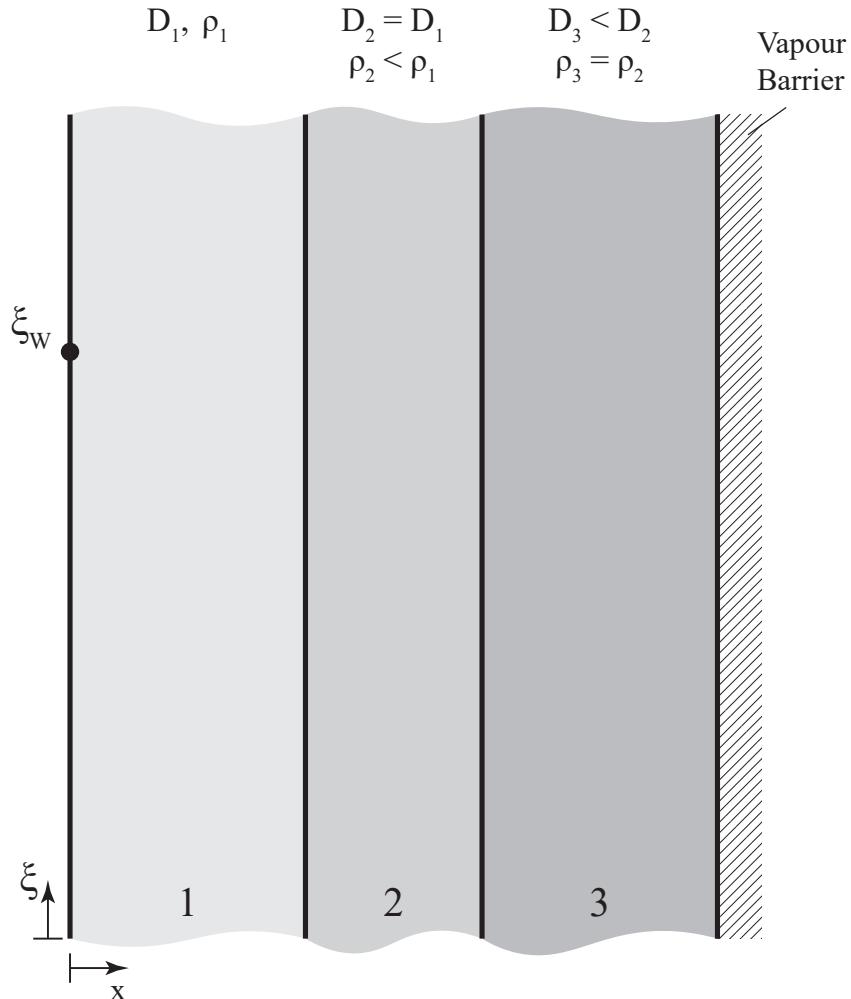


Exercise V.6: (Pipe burst ★★)

Due to a pipe burst in a multilayered wall of a bathroom at the time t_0 , water starts to diffuse through the wall until it reaches an impermeable layer (vapour barrier).

**Hints:**

- The properties of the individual wall layers are given in the illustration.
- ξ_W represents the mass fraction of water at the wall-side of the bathroom wall.
- At time t , the steady state has not been reached yet.
- Saturation does not occur at any point inside the wall.

Tasks:

- Sketch the water mass fraction profile across the different layers for a determined point in time $t > t_0$.

Solution V.6: (Pipe burst **)**Task a)**

In layer 1, the mass fraction at the wall side, denoted as ξ_W , is provided. Since saturation has not yet occurred anywhere within the wall, the profile is transient, resulting in a gradual decrease in slope as one moves in the positive x direction.

At the interface between layer 1 and layer 2, a jump in concentration is observed due to the lower density of layer 2. Given that the total mass of water is the same on both sides of the interface, the mass fraction is higher for a material with lower density.

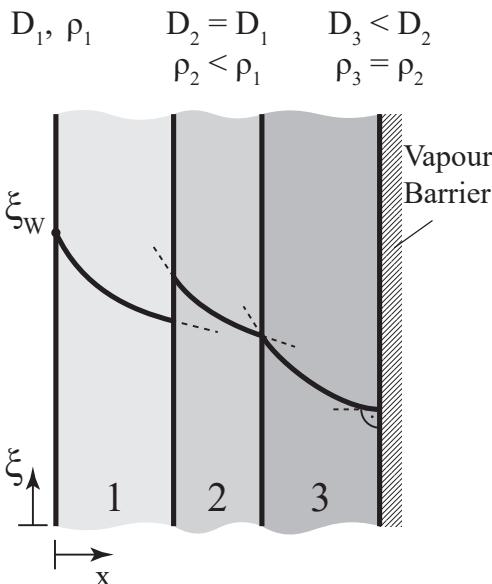
Similar to layer 1, layer 2 also exhibits a gradual decrease in slope because less mass has penetrated into the wall.

At the interface between layer 2 and layer 3, no jump in concentration is observed since both layers have the same density. However, a kink in the concentration gradient is noted due to the smaller diffusivity of layer 3. According to Fick's law:

$$j = -\rho D A \frac{\partial \xi}{\partial x}, \quad (\text{V.6.1})$$

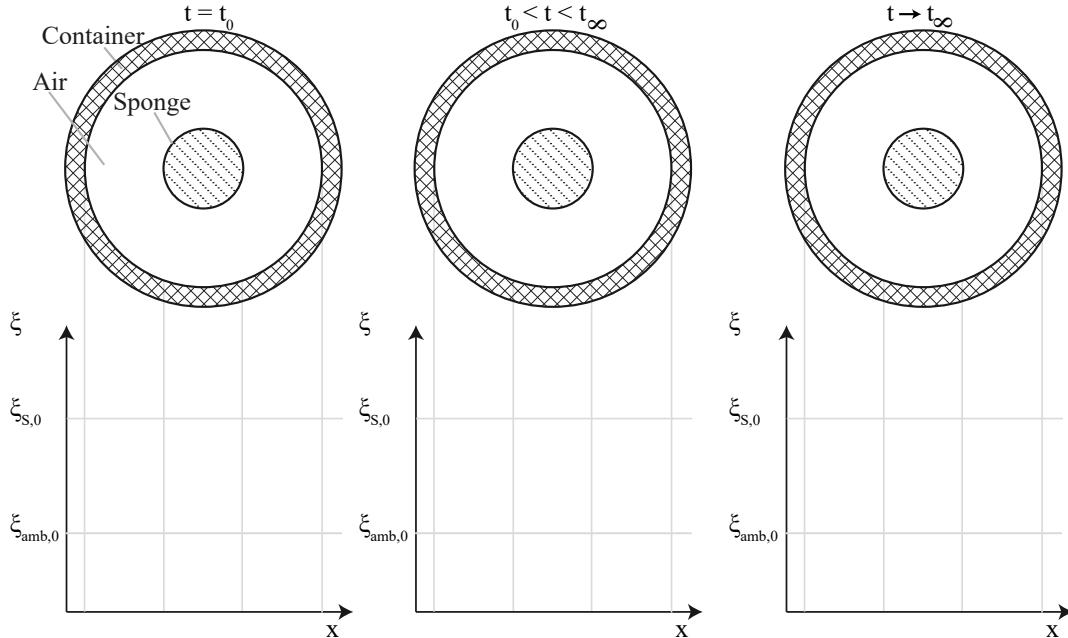
for the same rate of diffusion across the interface, the same density, and the same cross-sectional area, the gradient must be steeper for a material with lower diffusivity.

As in layers 1 and 2, layer 3 shows a gradual decrease in slope until reaching the vapor barrier, which is impermeable and thus prevents further mass transfer. Consequently, the concentration gradient in layer 3 ends with a horizontal slope.

Conclusion

Exercise V.7: (Wet sponge **)

A long, cylindric sponge is placed in the center of a long, cylindric container. Following that, the container is sealed compound-impermeable. At the time $t = t_0$ the sponge is homogeneously charged with water with a mass fraction ξ_S ($t = t_0$) = $\xi_{S,0}$, while the air within the container exhibits a mass fraction ξ_{amb} ($t = t_0$) = $\xi_{amb,0}$.

**Hints:**

- Convective transfer can be neglected.

Tasks:

- Draw the radial distribution of the mass fraction in both cylinder and sponge using the provided diagrams for the specified points in time.

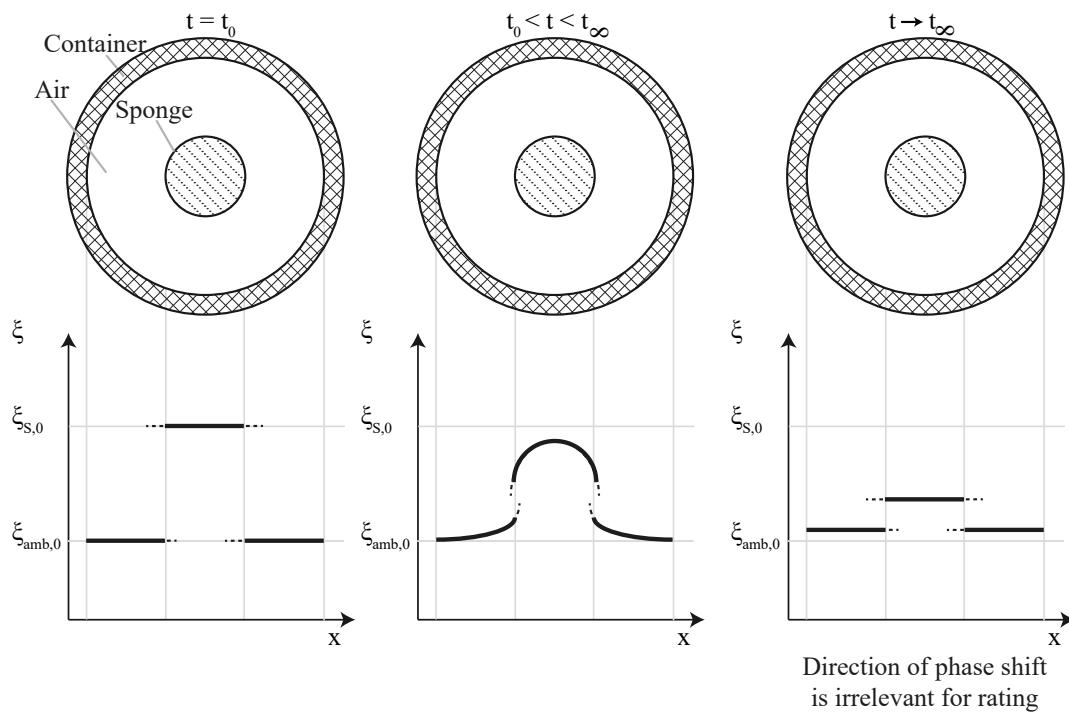
Solution V.7: (Wet sponge **)**Task a)**

Initially, the sponge is uniformly saturated with water, characterized by a mass fraction ξ_S at time $t = t_0$, denoted as $\xi_{S,0}$. Simultaneously, the air within the container has a mass fraction of water ξ_{amb} at $t = t_0$, denoted as $\xi_{amb,0}$.

As time progresses ($t_0 < t < t_\infty$), due to the lower mass concentration of water in the air, mass starts to diffuse into the air of the container. This diffusion reduces the total mass of water in the sponge and increases the total water in the air. At the center of the sponge, a horizontal concentration gradient is observed due to symmetry conditions. Moving radially outwards, an increase in the gradient is observed since a more significant portion of the water near the interface has diffused.

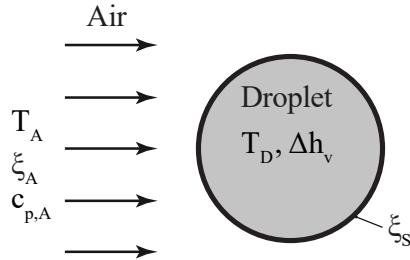
At the phase interface, a phase shift is seen due to differences in solubility, with both the phase concentrations having reached their equilibrium concentration (the direction of the phase shift is irrelevant). Moving further radially outward, the water has not fully penetrated throughout the entire domain and therefore approaches the initial ambient concentration of the air gradually.

At $t \rightarrow t_\infty$, a steady state is reached, with both domains showing a uniform concentration again at their equilibrium concentration that was first reached at the phase boundary.

Conclusion

Exercise V.8: (Evaporating droplet **★★**)

A fuel droplet with constant and homogeneous temperature T_D evaporates in hot air $T_A > T_D$.

**Given parameters:**

- Mass fraction of fuel in air: ξ_A
- Mass fraction of fuel to droplet surface: ξ_s
- Ambient temperature: $T_A > T_D$
- Enthalpy of vaporization: Δh_v
- Heat capacity: c_p,A

Hints:

- Thermal radiation effects are neglectable.
- The Lewis number is equal to one ($Le = 1$).

Tasks:

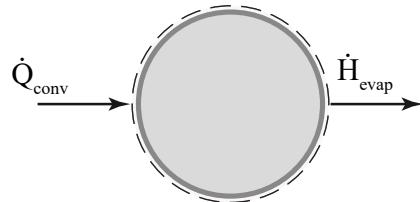
- a) Determine the droplet temperature as a function of the given variables.

Solution V.8: (Evaporating droplet **)**Task a)**

To determine the droplet temperature as a function of the given variables, a global energy balance around the droplet must be established.

1 Setting up the balance:

If the droplet's temperature is temporal constant and homogeneous, the energy required for evaporation and the energy supplied by convection are identical. In the consequent calculation, the change in inner energy is therefore not included. The heat transport by convection to the droplet surface must balance the energy required for evaporation.



Hence, the energy balance reads:

$$0 = \underbrace{\dot{Q}_{\text{conv}}}_{\substack{\text{Convective} \\ \text{transport}}} - \underbrace{\dot{H}_{\text{evap}}}_{\substack{\text{Latent heat} \\ \text{of vaporization}}} \quad (\text{V.8.1})$$

2 Defining the elements within the balance:

The incoming rate of heat transfer from the hot air is calculated as:

$$\dot{Q}_{\text{conv}} = \alpha \frac{\pi D^2}{4} (T_A - T_D). \quad (\text{V.8.2})$$

The latent heat of vaporization yields from the product of the enthalpy of vaporization and the rate of vaporization:

$$\dot{H}_{\text{evap}} = \Delta h_v \dot{m}_{\text{evap}}. \quad (\text{V.8.3})$$

The rate of evaporation can be taken from the book of formularies.

Theorem**Stefan flow mass transfer rate:**

$$\dot{m}_A'' = g \cdot \frac{\xi_{A,S} - \xi_{A,\infty}}{1 - \xi_{A,S}}.$$

Hence,

$$\dot{m}_{\text{evap}} = g \frac{\pi D^2}{4} \cdot \frac{\xi_S - \xi_A}{1 - \xi_S}. \quad (\text{V.8.4})$$

Where the mass transfer coefficient is approximated using Lewis' law:

Theorem**Lewis' law for gases:**

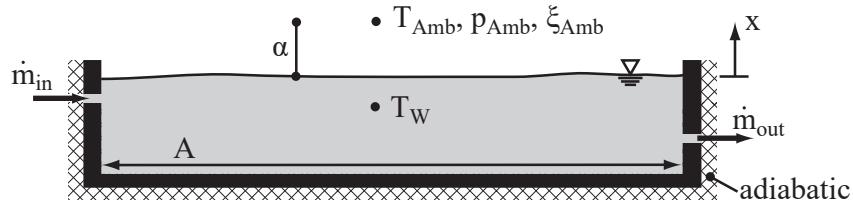
$$g = \frac{\alpha}{c_p},$$

for $Le \approx 1$.**Conclusion****3 Inserting and rearranging:**

$$T_D = T_A - \frac{\Delta h_v}{c_{p,A}} \frac{\xi_S - \xi_A}{1 - \xi_S}. \quad (\text{V.8.5})$$

Exercise V.9: (Thermal bath ★★)

During cold ambient temperatures, the operating company of a thermal bath has to compensate for the heat and water losses of its outdoor pool to guarantee a constant water temperature.

**Given parameters:**

- Ambient temperature: T_{amb}
- Ambient pressure: p_{Amb}
- Water concentration in the ambient air: ξ_{Amb}
- Water surface: A
- Molar mass of air: M_{Air}
- Molar mass of water: M_{W}
- Saturation vapour pressure of water at T_w : $p_{\text{sw}}(T_w)$
- Enthalpy of evaporation of water at T_w : $\Delta h_v(T_w)$
- Mass flow of the inflowing water: \dot{m}_{in}
- Convective heat transfer coefficient: α
- Thermal conductivity of humid air: λ
- Specific heat capacity of humid air: $c_{\text{p,Air}}$
- Specific heat capacity of liquid water: $c_{\text{p,W}}$
- Lewis-number: $\text{Le} = 1$

Hints:

- The water temperature in the pool is homogeneous.
- Radiation is negligible.

Tasks:

- Determine the evaporating water mass flow \dot{m} !
- Determine the concentration gradient at the air-side of the water surface $\frac{\partial \xi}{\partial x}|_{x=0}$.
- What temperature T_{in} of the inflowing water is needed?

Hint: Consider the mass flow of evaporating water \dot{m} to be known and to be negligible compared to the inflow (i.e.: $\dot{m}_{\text{in}} \approx \dot{m}_{\text{out}}$).

Solution V.9: (Thermal bath ★★★)

Task a)

Conclusion

Considering Stefan correction yields the evaporating mass flow:

$$\dot{m} = gA \frac{\xi_O - \xi_{\text{Amb}}}{1 - \xi_O}. \quad (\text{V.9.1})$$

Using Lewis' law ($\text{Le} = 1$), the mass transfer coefficient can be determined:

$$g = \frac{\alpha}{c_{\text{pAir}}}. \quad (\text{V.9.2})$$

Since there is saturation at the phase boundary, the mass fraction at the surface can be determined as follows:

$$\xi_O = \frac{\rho_O}{\rho_O + \rho_{\text{Air}}} = \frac{1}{1 + \frac{p_{\text{Air}}}{p_{\text{w}}} \cdot \frac{M_{\text{Air}}}{M_{\text{w}}}} = \frac{1}{1 + \frac{p_{\text{Amb}} - p_{\text{sW}}(T_{\text{w}})}{p_{\text{sW}}(T_{\text{w}})} \cdot \frac{M_{\text{Air}}}{M_{\text{w}}}}. \quad (\text{V.9.3})$$

Task b)

The passing mass flow can be determined using the downstream gradient at the phase boundary:

$$g \cdot A \cdot \frac{\xi_O - \xi_{\text{Amb}}}{1 - \xi_O} = -\rho D \cdot A \cdot \left. \frac{\partial \xi}{\partial x} \right|_{x=0} \cdot \frac{1}{1 - \xi_O}. \quad (2 \text{ P})$$

Conclusion

Rearranging and considering the analogy yields the wanted gradient:

$$\left. \frac{\partial \xi}{\partial x} \right|_{x=0} = -\frac{g}{\rho D} \cdot (\xi_O - \xi_{\text{Amb}}) \quad (\text{V.9.4})$$

$$= -\frac{\text{Sh}}{L} \cdot (\xi_O - \xi_{\text{Amb}}) \quad (\text{V.9.5})$$

$$= -\frac{\text{Nu}}{L} \cdot (\xi_O - \xi_{\text{Amb}}) \quad (\text{V.9.6})$$

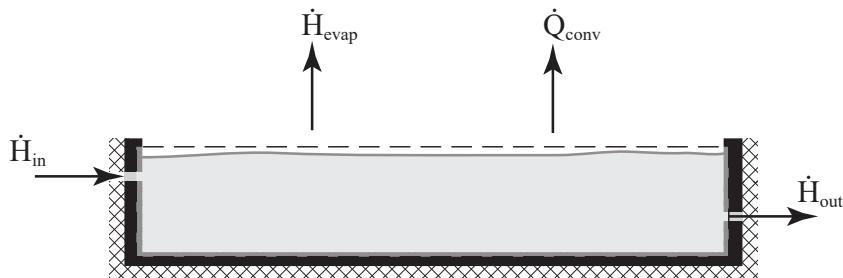
$$= -\frac{\alpha}{\lambda} \cdot (\xi_O - \xi_{\text{Amb}}). \quad (\text{V.9.7})$$

Task c)

The inlet temperature needed for the water flowing in can be determined by setting up an energy balance around the entire pool.

1 Setting up the balance:

Energy is transported by the movement of the mass flowing in and out of the bath. Furthermore, the bath loses heat to the ambient due to convection, while the evaporation process also consumes energy. The inflow of hot water is to realize a constant water temperature, so there is no temporal change in inner energy.



Hence, the energy balance around the pool yields:

$$0 = -\underbrace{\dot{Q}_{\text{conv}}}_{\text{Convective losses}} - \underbrace{\dot{H}_{\text{evap}}}_{\text{Latent enthalpy for phase change}} + \underbrace{\dot{H}_{\text{in}} - \dot{H}_{\text{out}}}_{\text{Net rate of advection}}. \quad (\text{V.9.8})$$

2 Defining the elements within the balance:

The heat loss due to convection is calculated as:

$$\dot{Q}_{\text{conv}} = \alpha A (T_W - T_{\text{Amb}}). \quad (\text{V.9.9})$$

The enthalpy of evaporation is:

$$\dot{H}_{\text{evap}} = \dot{m} \Delta h_V (T_W). \quad (\text{V.9.10})$$

The enthalpy flows are written as:

$$\dot{H}_{\text{in}} = \dot{m}_{\text{in}} c_{\text{pW}} T_{\text{in}} - T_W, \quad (\text{V.9.11})$$

and:

$$\dot{H}_{\text{out}} = \dot{m}_{\text{in}} c_{\text{pW}} T_W. \quad (\text{V.9.12})$$

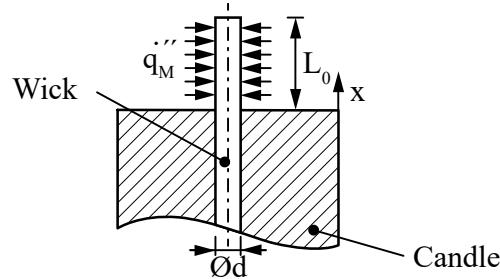
Conclusion

3 Inserting and rearranging:

$$T_{\text{in}} = T_W + \frac{\alpha A (T_W - T_{\text{Amb}}) + \dot{m} \Delta h_V}{\dot{m}_{\text{in}} c_{\text{pW}}}. \quad (\text{V.9.13})$$

Exercise V.10: (Burning candle ★★★)

The lights go out at the "Antarctic Research Alliance" station. So the scientists employ candles in order to see in the dark.



Given parameters:

- Net heat flux from the ignition source → wick: \dot{q}_M''
- Gravitational constant: g
- Wick diameter: d
- Wick density: $\rho_{Wi} \ll \rho_W$
- Wick length **after** ignition: L
- Wick local Sherwood number **after** ignition: Sh_x
- Air density: ρ_A
- Air viscosity: η_A
- Air heat capacity: $c_{p,A}$
- Air thermal conductivity: λ_A
- Air ambient temperature: T_A
- Wax density: ρ_W
- Wax diffusion coefficient in ambient air: D_{WA}
- Wax diffusion coefficient in ambient wick: D_{WW}
- Wax enthalpy of fusion: h_W
- Wax initial volume fraction in the wick at $t = 0$: ψ_W
- Wax mass fraction in ambient air: $\xi_A = 0$

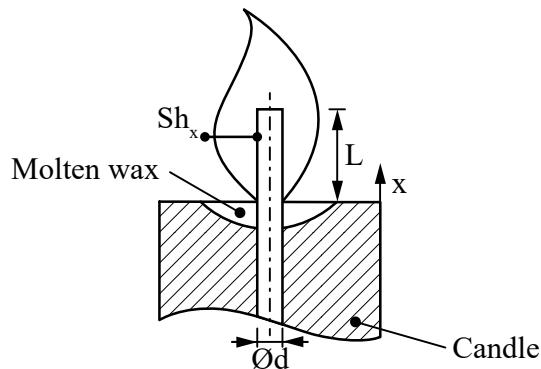
Hints:

- During the melting of the wax, assume a homogeneous and time-constant temperature.
- Assume all given material values as constant.
- The reference density of the mass fraction is similar to the wax density.
- Adopt the Lewis number, $Le = 1$.

Tasks:

- a) To ignite a candle, all the wax in the candle wick has to be melted. For this, the exposed wick is heated with a uniform-constant radiative heat flux \dot{q}_M'' . Determine the time t_M that it takes to melt all the wax, after reaching its melting temperature.

After ignition, the candle burns under steady conditions. The wick now stands in a small pool of liquid wax which diffuses upwards through the wick and later evaporates along the wick and burns at the flame. This flame generates a local Sherwood number Sh_x along the length of the exposed wick. Furthermore, the wick adopts a steady length L , defined by the distance between the wax pool surface and the wick tip. The tip of the wick is defined by the position at which there is no more wax left on the wick, so it burns away.



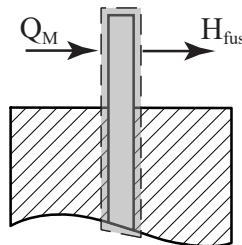
- b) Derive an expression for the wax local mass transfer coefficient g_x from the wick to the environment, in the function of the x .
- c) Define a meaningful Biot number Bi_x , that can be used to justify the assumption that the mass fraction ξ of wax along the wick (along the coordinate x) is a one-dimensional case and assume that its value is much smaller than one, therefore: $Bi_x \ll 1$. Derive the respective differential equation and provide the required boundary conditions.
- d) Provide the integral equation for the total mass flow of evaporated wax \dot{m}_W , that goes to the flame.
- e) Sketch the wax mass fraction along the wick, and provide the initial and final coordinates for the distribution.

Solution V.10: (Burning candle ★★★)**Task a)**

To determine the time that it takes to melt all the wax **in the wick** after reaching the melting temperature can be done by setting up a global energy balance around the wick.

1 Setting up the balance:

During the ignition process a constant net heat flux from the ignition source to the wick is provided. Since the wax is already at its melting temperature according to the problem statement, only the enthalpy of fusion for the phase change of the wax needs to be provided, and thus no change in inner energy.



Hence, the energy balance around the wick reads:

$$0 = \underbrace{Q_s}_{\text{Net heat from source to wick}} - \underbrace{H_{\text{fus}}}_{\text{Enthalpy of fusion}}. \quad (\text{V.10.1})$$

2 Defining the elements within the balance:

The net heat from the source to the wick is calculated as follows,

$$Q_s = \dot{q}_s'' \cdot d \pi L_0 t_s, \quad (\text{V.10.2})$$

and the enthalpy of fusion by:

$$H_{\text{fus}} = h_w = \psi_w \cdot \rho_w \cdot \frac{\pi d^2}{4} L_0. \quad (\text{V.10.3})$$

Conclusion**3 Inserting and rearranging:**

$$t_s = \frac{h_w \cdot \psi_w \cdot d \cdot \rho_w}{4 \cdot \dot{q}_s''}. \quad (\text{V.10.4})$$

Task b)

Conclusion

Rewriting the definition of the Sherwood number Sh_x , the mass transfer coefficient is obtained:

$$\begin{aligned} Sh &= \frac{g_x x}{\rho_L D_L} \\ \Rightarrow g_x &= \frac{Sh_x \rho_L D_L}{x}. \end{aligned} \quad (V.10.5)$$

Task c)

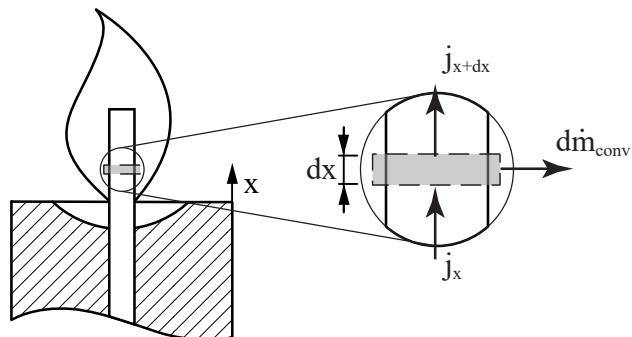
Using the local mass transfer coefficient g_x from the previous part of the problem, the Biot number is obtained analogously to the "thermal" version as the ratio of diffusive to convective resistance:

$$Bi_x = \frac{g_x \cdot d/2}{\rho_w D_{WW}}. \quad (V.10.6)$$

The required differential equation can be obtained from setting up a substance balance for an infinitesimal element within the wick.

1 Setting up the balance:

Since according to the problem statement $Bi_x \ll 1$, the gradient of the mass fraction in the radial direction is negligible, so mass only diffuses in and out in the axial direction of the wick, while it transfers mass to the environment by convection.



Hence, the steady-state substance balance within the element reads:

$$0 = \underbrace{j_x - j_{x+dx}}_{\text{Net rate of mass diffusion}} - \underbrace{d\dot{m}_{conv}}_{\text{Convective mass transport}}. \quad (V.10.7)$$

2 Defining the elements within the balance:

The ingoing rate of mass diffusion is calculated according to Fick's law:

$$j_x = -\rho_w D_{WW} \frac{\pi d^2}{4} \frac{\partial \xi}{\partial x}, \quad (V.10.8)$$

where the outgoing rate is approximated using the Taylor series expansion:

$$\begin{aligned} j_{x+dx} &= j_x + \frac{\partial j_x}{\partial x} \cdot dx \\ &= -\rho_W D_{WW} \frac{\pi d^2}{4} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial} \left(-\rho_W D_{WW} \frac{\pi d^2}{4} \frac{\partial \xi}{\partial x} \right) \cdot dx. \end{aligned} \quad (\text{V.10.9})$$

The convective term, including the Stefan flow factor, is calculated as:

$$\begin{aligned} d\dot{m}_{\text{conv}} &= g_x \pi d dx \underbrace{\frac{\xi - \xi_U}{1 - \xi}}_{\approx 1}^{=0} \\ &= g_x \pi d dx \xi. \end{aligned} \quad (\text{V.10.10})$$

Conclusion

3 Inserting and rearranging:

$$0 = \rho_W D_{WW} \frac{d}{4} \frac{\partial^2 \xi}{\partial x^2} - g_x \xi. \quad (\text{V.10.11})$$

4 Defining the boundary and/or initial conditions:

The necessary boundary conditions are given within the problem description: On the one hand, the wick is in a puddle of liquid wax, hence:

$$\xi|_{x=0} = 1, \quad (\text{V.10.12})$$

and the tip of the wick contains no burning wax:

$$\xi|_{x=L} = 0. \quad (\text{V.10.13})$$

Task d)

Conclusion

Using the local mass transfer coefficient g_x , the mass flow rate evaporating over the total length L of the wick is

$$\begin{aligned} \dot{m}_W &= \int_0^L \dot{m}'_{\text{conv}}(x) dx \\ &= \int_0^L g_x \cdot \pi d \cdot \xi dx. \end{aligned} \quad (\text{V.10.14})$$