8.5 Exercises

Problem 8.1. *Let* $T(x, y) = T_0 + ax + by$

(a) *Compute* $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) *Compute* $\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$.

Problem 8.2. *Let* $T(x, y) = T_0 + ax + by + cxy$.

(a) *Compute* $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) *Compute* $\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$.
(c) *Under what condition do the answers of (a) and (b) coincide?*

Problem 8.3. *Let* $T(x, y) = T_0 + ax + by$ *and* $x_p(t) = x_o + ut$, $y_p(t) = y_o + vt$.

(a) *Compute* $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(b) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.4. *Let* $T(x, y) = T_0 + ax + by + cxy$ *and* $x_p(t) = x_o + ut + \frac{1}{2}pt^2$, $y_p(t) = y_o + vt + \frac{1}{2}qt^2$.

(a) *Compute* $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(b) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.
(c) *Under what condition do the answers of (a) and (b) coincide?*

Problem 8.5. *Let* $T(x) = sin(ax)$ *and* $x_p(t) = ut$.

(a) *Compute* $f(t) \equiv T(x_p(t))$ *and* $\frac{df}{dt}$.
(b) *Make a sketch of* $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 1$, $u = 1$ *and* $a = 1$, $u = 2$.
(c) *Make a sketch of* $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 2$, $u = 1$ *and* $a = 2$, $u = 2$.

Problem 8.6. *Let* $T(x, y) = xy$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) *Sketch the curve* $T(x, y) = 1$.
(b) *Compute* $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(c) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.7. *Let* $T(x, y) = xyt$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) *Sketch the curve* $T(x, y) = t$.
(b) *Compute* $f(t) \equiv T(x_p(t), y_p(t), t)$ *and* $\frac{df}{dt}$.
(c) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.8. *A baseball is thrown at speed* U *in air with pressure* p_a *and constant density* ρ. *Assume* $\mu = 0$ *and* $U = constant$.

(a) *Compute the maximum value of the pressure on the ball's surface.*
(b) *Compute the maximum value of the pressure on the ball's surface if there is a head-wind* V.
(c) *Compute the maximum value of the pressure on the ball's surface if there is a tail-wind* V.

Problem 8.8. *A baseball is thrown at speed U in air with pressure p_a and constant density ρ. Assume $\mu = 0$ and $U = constant$.*

(a) *Compute the maximum value of the pressure on the ball's surface.*
(b) *Compute the maximum value of the pressure on the ball's surface if there is a head-wind V.*
(c) *Compute the maximum value of the pressure on the ball's surface if there is a tail-wind V.*

Problem 8.9. *A stone with mass m is attached to a rope and swung around in a horizontal circle. The path of the stone is $\boldsymbol{x}_p(t) = vectortwox_p(t)y_p(t)$, with*

$$x_p(t) = L\cos(\omega t), \quad y_p(t) = L\sin(\omega t).$$

(a) *Compute the velocity vector $\mathbf{u}(t) \equiv \frac{d}{dt}\boldsymbol{x}_p(t)$.*
(b) *Compute the velocity vector $\mathbf{a}(t) \equiv \frac{d}{dt}\mathbf{u}_p(t)$.*
(c) *For an arbitrary time instant t, sketch the vectors $\boldsymbol{x}_p(t)$, $\mathbf{u}_p(t)$, and $\mathbf{a}_p(t)$.*

Problem 8.10. *A flow field is specified as $\mathbf{u}(\boldsymbol{x}) = \frac{U}{L}\begin{pmatrix} -y \\ x \end{pmatrix}$.*

(a) *Compute $\mathbf{a}(\boldsymbol{x}) \equiv \frac{D}{Dt}\mathbf{u}$.*
(b) *For arbitrary position $\boldsymbol{x}$, sketch $\mathbf{u}$ and $\mathbf{a}$.*
(c) *Compute the vector $\boldsymbol{\nabla} p$ and add it to the sketch.*

Problem 8.11. *Explain why*

(a) *Bernoulli's equation does* **not** *hold in fully developed flow.*
(b) *Bernoulli's equation* **does** *hold in a swimming pool if nobody is swimming.*
(c) *Derive, using Bernoulli's equation, an expression for the pressure in a abandoned swimming pool.*

9.6 Exercises

Problem 9.1. *Consider the following convection-diffusion equation:*

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha\frac{\partial^2 \phi}{\partial x^2},$$

where $\phi(x,t)$ is some property, and the units of x, t, and u are m, s, and m/s, respectively. Compute the unit of α, and check the answers for the cases $\alpha = \frac{k}{\rho C_v}$, and $\alpha = \frac{\mu}{\rho}$.

Problem 9.2. *One-dimensional sound waves are described by the wave equation*

$$\frac{\partial^2 p}{\partial t^2} - a^2\frac{\partial^2 p}{\partial x^2} = 0$$

, where p is the pressure disturbance and a the speed of sound. Show that $f(x - at)$ and $g(x + at)$ are solutions of the wave equation, with f and g arbitrary functions.

Problem 9.3. *Show, by substitution, that $T(x,t)\,(a\cos(\lambda x) + b\sin(\lambda x))\exp(-\alpha\lambda^2 t)$ is a solution of the diffusion equation*

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}.$$

Problem 9.4. *Let* $T(x,t) = \bar{T}(\xi(x,t),t)$, $\xi(x,t) = x - Ut$.

(a) *Express* $\frac{\partial T}{\partial t}$, $\frac{\partial T}{\partial x}$, *and* $\frac{\partial^2 T}{\partial x^2}$ *in terms of derivatives of* $\bar{T}$ *with respect to* ξ *and* t.

(b) *Show that the convection-diffusion equation*

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

transforms into a diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \alpha\frac{\partial^2 \bar{T}}{\partial x^2}.$$

10.5 Exercises

Problem 10.1. *Using* $TV^{\gamma-1} = const$, $p = \rho RT$, *and mass conservation, derive that:*

(a) $T\rho^{1-\gamma} = const$

(b) $Tp^{-\frac{\gamma-1}{\gamma}} = const$

(c) $p\rho^{-\gamma} = const$

Problem 10.2. *Starting from*

$$W = -\int_0^t p\frac{dV}{dt}dt,$$

$TV^{\gamma-1} = const$, *and mass conservation, derive that*

(a) $W = -p_o V_o^\gamma \int_0^t V^{-\gamma}\frac{dV}{dt}dt$,

(b) $W = \frac{p_o V_o}{\gamma-1}\left\{\left(\frac{V_o}{V(t)}\right)^{\gamma-1} - 1\right\}$,

(c) $sign(W) = sign(V_o - V(t))$.

Problem 10.3. *Starting with* $p\rho^{-\gamma} = const$, *derive that*

$$\frac{1}{p}\frac{dp}{dt} - \gamma\frac{1}{\rho}\frac{d\rho}{dt}.$$

Problem 10.4. *Air enters a compressor at speed* U_1, *temperature* T_1, *and leaves at* U_2, T_2, *and the mass flow is* $\dot{m}$. *The removed heat per unit mass of passing air is* $\hat{e}$. *Derive an expression for the power required by the compressor, assuming air can be modeled as a perfect gas.*