Fluid Mechanics I

Dr. R. Hagmeijer

Engineering Technology
University of Twente

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Chapter 1

Introduction

1.1 The three fundamental conservation laws

Consider a group of N particles numbered by $i = 1, 2, ..., N$. The i-th particle has mass m_i, velocity $\boldsymbol{v}_i(t)$ and internal energy $\epsilon_i(t)$. Suppose that there are no external forces working on the particles, then, for the group of particles the following laws are observed again and again:

$$\sum_{i=0}^{N} \frac{d}{dt} (m_i) = 0, \tag{1.1}$$

$$\sum_{i=0}^{N} \frac{d}{dt} (m_i \boldsymbol{v}_i) = 0, \tag{1.2}$$

$$\sum_{i=0}^{N} \frac{d}{dt} \left(\frac{1}{2} m_i \|\boldsymbol{v}_i\|^2 + \epsilon_i \right) = 0. \tag{1.3}$$

In other words, the mass, momentum, and energy of the group of particles is conserved, and therefore we call these laws conservation laws. Internal forces between particles do not change the total momentum and energy because the loss by one particle is gained by one or more other particles. When there are external forces working on the particles, say $\boldsymbol{F}_i$ is the external force (vector) working on the i-th particle, then the equations become:

$$\sum_{i=0}^{N} \frac{d}{dt} (m_i) = 0, \tag{1.4}$$

$$\sum_{i=0}^{N} \frac{d}{dt} (m_i \boldsymbol{v}_i) = \sum_{i=0}^{N} \boldsymbol{F}_i, \tag{1.5}$$

$$\sum_{i=0}^{N} \frac{d}{dt} \left(\frac{1}{2} m_i \|\boldsymbol{v}_i\|^2 + \epsilon_i \right) = \sum_{i=0}^{N} \boldsymbol{F}_i \cdot \boldsymbol{v}_i. \tag{1.6}$$

We still call these laws conservation laws.

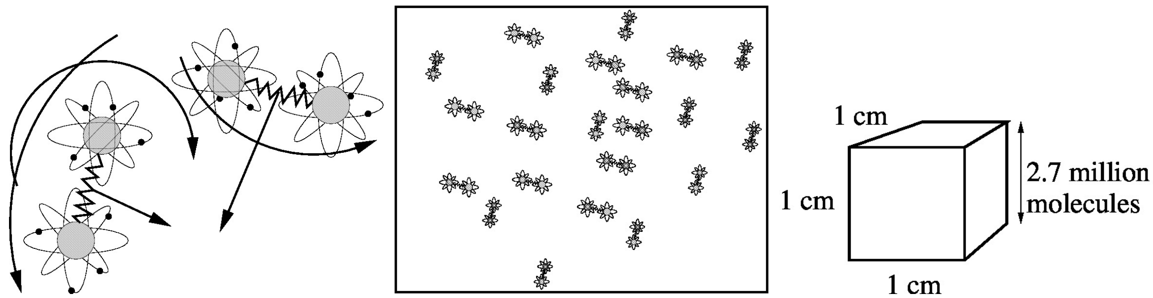


Figure 1.1: The particle model of a fluid

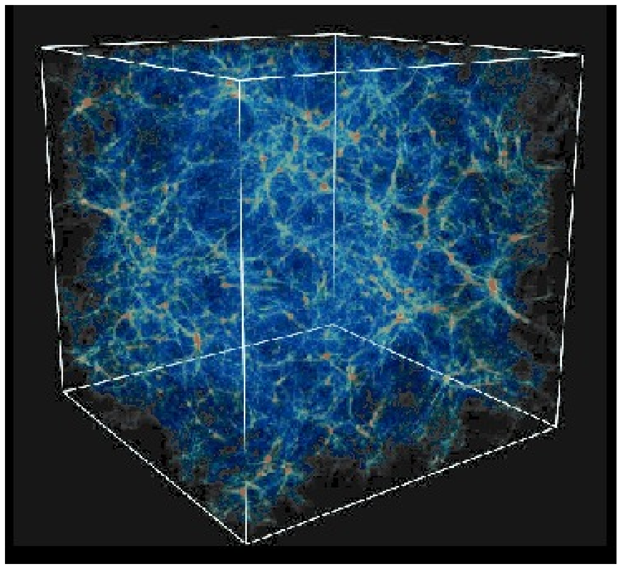


Figure 1.2: Molecular-dynamics simulation of a fluid.

1.2 Continuum model

Density At a given point in space we can construct a sphere around the point and compute the total mass of all particles within the sphere. If we then divide that mass by the volume of the sphere we have computed the amount of mass per unit volume which we will call mass density or just density. On the one hand, when we take the sphere too large, the computed density does not characterise the situation at the given point but rather the situation in a relatively large region. On the other hand, when we take the spere too small, there will be no particles within the spehere at all and the density is zero.

In order to construct the density in every point in the domain we choose the sphere sufficiently small compared to the flow domain at hand, but sufficiently large compared to the average distance between particles, assuming that this procedure still leaves a whole range of sphere sizes for which the computed density does not depend on the actual size of the sphere.

Velocity Similar to the density we can define the velocity at a given point in space by averaging the velocities of the particles over a sphere which is sufficiently small compared to the flow domain at hand, but sufficiently large compared to the average distance between particles.

Temperature The temperature at a given point in space can be computed as the average kinetic energy of particles in an appropriately sized sphere.

Pressure The pressure at a given point in space can be computed as the time-averaged force on a small area-element stuck into the flow.

1.3 Trajectories

Consider a small dust particle with time dependent position:

$$\boldsymbol{x}_p(t) = \begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix}, \tag{1.7}$$

and a surrounding fluid with velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} u(\boldsymbol{x}, t) \\ v(\boldsymbol{x}, t) \\ w(\boldsymbol{x}, t) \end{pmatrix}. \tag{1.8}$$

Note that the independent spatial coordinate $\boldsymbol{x}$ can be chosen arbirarily, whereas the spatial coordinate $\boldsymbol{x}_p(t)$ of the particle is a time-dependent variable: $\boldsymbol{x}$ and $\boldsymbol{x}_p(t)$ are different entities! When the dust particle is convected with the flow, this means that the velocity of the particle is equal to the velocity of the fluid **at the position of the particle** at all times:

$$\boxed{\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}(\boldsymbol{x}_p, t)}, \tag{1.9}$$

where we have substituted the actual particle position $\boldsymbol{x}_p(t)$ as the location where we want to know the velocity. This equation is a vector equation which in general has three components:

$$\frac{dx_p}{dt} = u(\boldsymbol{x}_p, t), \quad \frac{dy_p}{dt} = v(\boldsymbol{x}_p, t), \quad \frac{dz_p}{dt} = w(\boldsymbol{x}_p, t). \tag{1.10}$$

Example 1.1.

Consider the velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} ax + yt\cos(bz^2) \\ \exp(cxyz) \\ \sin(abx^2 z) \end{pmatrix}, \tag{1.11}$$

then the three differential equations that describe the trajectory of the particle become:

$$\begin{aligned}\frac{dx_p}{dt} &= ax_p + y_p t\cos(bz_p^2),\\ \frac{dy_p}{dt} &= \exp(cx_p y_p z_p),\\ \frac{dz_p}{dt} &= \sin(abx_p^2 z_p.\end{aligned} \tag{1.12}$$

Due to their complexity these equations can only be solved by numerical integration. There are cases however in which the velocity field is sufficiently simple and one can compute an analytical solution. For example, when the velocity field is uniform (independent of $\boldsymbol{x}$) and constant (independent of t):

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{1.13}$$

the three differential equations that describe the trajectory of the particle become simply

$$\frac{dx_p}{dt} = U, \quad \frac{dy_p}{dt} = V, \quad \frac{dz_p}{dt} = W \tag{1.14}$$

and the trajectory of a particle that passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$ is a straight line

$$\boldsymbol{x}_p = \begin{pmatrix} Ut + x_o \\ Vt + y_o \\ Wt + z_o \end{pmatrix}. \tag{1.15}$$

Example 1.2.

Consider

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}, \tag{1.16}$$

then

$$\frac{dx_p}{dt} = ax_p, \quad \frac{dy_p}{dt} = by_p, \quad \frac{dz_p}{dt} = cz_p \tag{1.17}$$

and the trajectory of a particle that passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$ is

$$\boldsymbol{x}_p(t) = \begin{pmatrix} x_o \mathrm{e}^{at} \\ y_o \mathrm{e}^{bt} \\ z_o \mathrm{e}^{ct} \end{pmatrix}. \tag{1.18}$$

Example 1.3.

Consider

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} \frac{a}{x(t+b)} \\ cy \\ ez \end{pmatrix}, \tag{1.19}$$

then the first equation becomes

$$\frac{dx_p}{dt} = \frac{a}{x_p(t+b)}. \tag{1.20}$$

This can be rewritten as

$$x_p \frac{dx_p}{dt} = \frac{a}{t+b} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}x_p^2\right) = \frac{a}{t+b}, \tag{1.21}$$

and, when the particle passes $\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$ at $t = 0$, integrated to

$$\frac{1}{2}x_p^2(t) = a\ln(t+b) - a\ln(b) + \frac{1}{2}x_o^2 \quad \Rightarrow \quad x_p(t) = \pm\sqrt{2a\ln(\frac{t+b)}{b}) + x_o^2}. \tag{1.22}$$

1.4 Vector-notation and index-notation

In the present lecture notes we will use two notations when it comes to vectors. Suppose we have a vector $\boldsymbol{u}$ with components u_1, u_2, and u_3, and a vector $\boldsymbol{n}$ with components n_1, n_2, and n_3. The inner product of these two vectors can be written in so-called vector notation as

$$\boldsymbol{u} \cdot \boldsymbol{n}. \tag{1.23}$$

It is noted that in vector notation we only write the vectors as a whole and not their components, and at the same time operations are written implicitly: you have to know what the dot means. In contrast, we also can write the inner product in index notation as

$$\sum_{j=1}^{3} u_j n_j. \tag{1.24}$$

This shows that in index notation we don't write vectors as a whole, but we write their components instead, and operations are written explicitly. As another example, consider the addition of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ resulting in a vector $\boldsymbol{w}$. In vector notation this becomes

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}, \tag{1.25}$$

whereas in index notation this becomes

$$w_i = u_i + v_i, \quad i = 1, 2, 3. \tag{1.26}$$

These and other examples are listed in the first two columns of Table (1.1)

Vector notation	Index notation	Index notation with ESC
$\boldsymbol{u}\cdot\boldsymbol{n}$	$\sum_{j=1}^{3} u_j n_j$	$u_j n_j$
$\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$	$w_i = u_i + v_i, \quad i=1,2,3$	$w_i = u_i + v_i, \quad i=1,2,3$
$A\boldsymbol{x}$	$\sum_{j=1}^{3} a_{ij} x_j, \quad i=1,2,3$	$a_{ij} x_j, \quad i=1,2,3$
$A\boldsymbol{x}\cdot\boldsymbol{u}$	$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} x_j u_i$	$a_{ij} x_j u_i$
$\boldsymbol{\nabla}\cdot\boldsymbol{u}$	$\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}$	$\frac{\partial u_j}{\partial x_j}$

Table 1.1: Vector notation versus index notation

1.5 Einstein summation convention

Vector fields play an essential role in our understanding of physics and engineering. Frequently we need to compute sums of vector components, think for example of the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \Rightarrow \quad \boldsymbol{a}\cdot\boldsymbol{b} \equiv \sum_{j=1}^{3} a_j b_j, \tag{1.27}$$

or of the multiplication of a matrix A with a vector $\boldsymbol{x}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Rightarrow \quad y_i \equiv (A\boldsymbol{x})_i \equiv \sum_{j=1}^{3} a_{ij} b_j. \tag{1.28}$$

As a result, many summation symbols will enter our formulas. Einstein [(1)] encountered this problem when dealing with theories of relativity and decided to follow a convention:

Theorem 1.1 (Einstein summation convention)**.** *When a single expression carries the same index twice, summation over the index is implied. Expressions that consist of terms that are separated by either '+', '-', or '=' are not considered single.*

With this convention we write the above two examples as

$$\sum_{j=1}^{3} a_j b_j \equiv a_j b_j, \tag{1.29}$$

and

$$\sum_{j=1}^{3} a_{ij} b_j \equiv a_{ij} b_j. \tag{1.30}$$

Note that in the second example we sum over j but not over i. Other examples are

$$\sum_{k=1}^{3} a_k \frac{\partial u_k}{\partial x_i} \equiv a_k \frac{\partial u_k}{\partial x_i}, \tag{1.31}$$

[(1)] Einstein, Albert (1916); "The Foundation of the General Theory of Relativity", Annalen der Physik

$$\sum_{i=1}^{3} u_{ii} \equiv u_{ii}, \tag{1.32}$$

$$\sum_{j=1}^{3} c_j \left(a_j + b_j\right) \equiv c_j \left(a_j + b_j\right), \tag{1.33}$$

and a counter example is

$$\sum_{j=1}^{3} \left(a_j + b_j\right) \neq a_j + b_j. \tag{1.34}$$

Finaly the reader is referred to the second and third columns of Table (1.1).

1.6 Exercises

Problem 1.1. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} x \\ -y \end{pmatrix}$

(a) Draw the curves $xy = \pm 1$ *in all four quadrants of the* $x - y$ *plane.*
(b) Draw the velocity vector at several points on the curves.
(c) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.2. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.3. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.4. *Consider the velocity field* $\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

(a) Draw the velocity vector at several points on two circles with radius 1 and 2.
(b) Compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, *and* $\frac{\partial v}{\partial y}$.

Problem 1.5. *Consider a little smoke particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$

(a) Draw the trajectory for $-1 \leq t \leq 1$.
(b) Compute the velocity vector.

(c) Construct a formula for the velocity vector as a function of $x(t)$ and $y(t)$.
(d) Draw the velocity vector at several points on the trajectory.
(e) Compute the acceleration vector.
(f) Construct a formula for the acceleration vector as a function of $x(t)$ and $y(t)$.
(g) Draw the acceleration vector at several points on the trajectory.

Problem 1.6. *Consider a little dust particle traveling along with a velocity field, and let its trajectory be given as* $\boldsymbol{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

(a) Draw the trajectory for $0 \le t \le 2\pi$.
(b) Compute the velocity vector $\boldsymbol{u}$
(c) Construct a formula for the velocity vector as a function of $x(t)$ and $y(t)$.
(d) Draw the velocity vector at several points on the trajectory.
(e) Compute the acceleration vector.
(f) Construct a formula for the acceleration vector as a function of $x(t)$ and $y(t)$.
(g) Draw the acceleration vector at several points on the trajectory.

Problem 1.7. *Using the index summation convention of Einstein, write in full:*

(a) $\frac{1}{2}u_i u_i$,
(b) $\frac{1}{2}u_i u_j$,
(c) $\frac{1}{2}u_j u_j$,
(d) $\frac{\partial u_k}{\partial x_k}$,
(e) $u_j \frac{\partial u_i}{\partial x_j}$,
(f) $u_i + u_j$,
(g) $\delta_{ij} u_i$, *where* $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
(h) $\delta_{ij} u_j$,
(i) $a_{ij} u_i u_j$.

Problem 1.8. *Rewrite the following vector-notation expressions in index notation:*

(a) $\boldsymbol{x} \cdot \boldsymbol{y}$,
(b) $\boldsymbol{\nabla} \cdot \boldsymbol{u}$,
(c) the i-th component of the vector $A\boldsymbol{x}$, where A is a 3×3 matrix,
(d) the i,j-th component of the matrix AB, where A and B are a 3×3 matrices,

Problem 1.9. *The velocity at the plane defined by the normal* $\boldsymbol{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *is* $\boldsymbol{u} = \begin{pmatrix} 15 \\ 34 \end{pmatrix}\frac{m}{s}$. *Calculate the normal and tangential velocities.*

Problem 1.10. *Given the Eulerian field*

$$\boldsymbol{u}(x, y, z, t) = 3t\boldsymbol{e}_1 + xz\boldsymbol{e}_2 + ty^2\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ *and* $\boldsymbol{e}_3$ *are the unit vectors along the coordinate axis, determine the flow acceleration.*

Problem 1.11. *A two-dimensional velocity field is described by*

$$u = \frac{x}{a + bt}, \quad v = \frac{y}{a + 2bt}.$$

Calculate the trajectories that pass by (x_o, y_o) *at* $t = 0$.

Problem 1.12. *Using polar coordinates, the velocity field in a tornado can be approximated as*

$$\boldsymbol{u} = -\frac{a}{r}\boldsymbol{e}_r + \frac{b}{r}\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_r$ *and* $\boldsymbol{e}_\theta$ *are the unit vectors in the directions* r *and* θ. *Show that the trajectories satisfy the so-called logarithmic spiral equation:*

$$r(\theta) = C \exp\left(-\frac{a}{b}\theta\right).$$

Problem 1.13. *A two-dimensional velocity field is given by*

$$u = 5ax(t + t_o), \quad v = 5ay(t - t_o).$$

Find the trajectories that pass x_o, y_o *at time* $t = 0$.

Problem 1.14. *The ideal flow around a corner placed at the origin is given by*

$$u = ax, \quad v = -ay,$$

with $a > 0$ *a constant. Determine the trajectories and draw the trajectory that passes the point* (x_o, y_o) *at time* $t = 0$ *and indicate the flow direction. Calculate the material derivative of the velocity vector.*

Problem 1.15. *The velocity field in a vortex like the one present in a cyclone, is given by:*

$$u = -\frac{Ky}{x^2 + y^2}, \quad v = \frac{Kx}{x^2 + y^2},$$

with $K > 0$. *Determine the trajectories and draw a few of them.*

Chapter 2

Mass Conservation

2.1 Mass conservation: integral formulation

Suppose we want to compute the mass M of a fluid contained in a volume V. If the mass density ρ is independent of position $\boldsymbol{x}$, the answer is simply

$$M = \rho V. \tag{2.1}$$

When the mass density varies with position $\boldsymbol{x}$ the answer has to be modified. We do this by dividing V in a number of N smaller volumes, ΔV_i, with i serving as a counter. If the volumes are sufficiently small we can take the density approximately constant within each volume, say ρ_i, and the answer becomes

$$M \approx \sum_{i=1}^{N} \rho_i \Delta V_i. \tag{2.2}$$

In the limit of infinitely small volumes and $N \to \infty$ we get [(1)]:

$$M = \lim_{N\to\infty} \sum_{i=1}^{N} \rho_i \Delta V_i \equiv \int_V \rho(\boldsymbol{x}) dV. \tag{2.3}$$

When the mass density is not only a function of position $\boldsymbol{x}$ but also function of time we simply get [(2)]:

$$M(t) = \int_V \rho(\boldsymbol{x}, t) dV. \tag{2.4}$$

We will now take a very special volume: think of a fluid in motion, for example the earth atmosphere on a windy day, and imagine that we are able to stop time for a moment. We could imagine colorizing a blob of fluid. When we let time run again we see the blob moving

[(1)] The mass M is not a function of $\boldsymbol{x}$! We have integrated over $\boldsymbol{x}$ so the answer does not contain $\boldsymbol{x}$ anymore. Compare this to $\int_a^b f(x)dx$ which only depends on a and b and not on x.

[(2)] The mass M is a function of t since ρ is a function of t and the integration is only over $\boldsymbol{x}$

around: it deforms and changes in size. We will call such volume of fluid a 'convected blob'. Our formula for the mass contained in the blob now becomes

$$M(t) = \int_{V(t)} \rho(\boldsymbol{x}, t) dV, \tag{2.5}$$

where $V(t)$ indicates that the volume of the blob is time-dependent: not only in size but also at least in shape. Evidently, each point of the blob travels with the local fluid velocity $\boldsymbol{u}$ which depends on both position $\boldsymbol{x}$ and time t. This means that also the surface of the blob travels with the local fluid velocity which, in turn, means that no fluid is leaving or entering the blob. Since we know that mass cannot be created nor destroyed (at non-relativistic speeds) we conclude that the mass contained in the blob is constant, in other words it is conserved:

$$\frac{dM}{dt} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} \int_{V(t)} \rho(\boldsymbol{x}, t) dV = 0} \tag{2.6}$$

Note that this mass conservation statement is a fundamental physical statement, it cannot be proved! It has been observed again and again, it has never been observed that mass was not conserved.

The challenge of the present chapter is to transform this fundamental physical statement into a practicle tool that can be used by scientists and engineers to make computations involving moving fluids. Of course we need mathematics to do so: mathematics has been invented, at least for a large part, to solve problems from physics and engineering.

2.2 Normal velocity, inner product

To develop a useful and practical tool to do computations for moving fluids we need to develop an expression for the velocity with which the surface of a convected blob travels in the direction normal to the surface. In Fig. (2.1) the blob is depicted, including the velocity vector $\boldsymbol{u}$ at a particular point at the surface of the blob. At the same location the so-called outer unit normal vector [3] $\boldsymbol{n}$ is depicted. Note that this vector has three properties that can be derived from its name:

(a) $\boldsymbol{n}$ is of unit length: $\|\boldsymbol{n}\| = 1$,
(b) $\boldsymbol{n}$ points outward with respect to the blob,
(c) $\boldsymbol{n}$ is oriented normal to the surface of the blob.

The normal velocity is defined as the component (part) of the velocity vector $\boldsymbol{u}$ that points in the same direction as the outward unit normal $\boldsymbol{n}$. We will call the length of this component u_n [4]. Hence, if the angle between the two vectors is ϕ, then

$$u_n = \|\boldsymbol{u}\| \cos \phi. \tag{2.7}$$

[3] The outward unit normal vector is often called 'outward unit normal'.
[4] Note that u_n is a scalar (number), not a vector

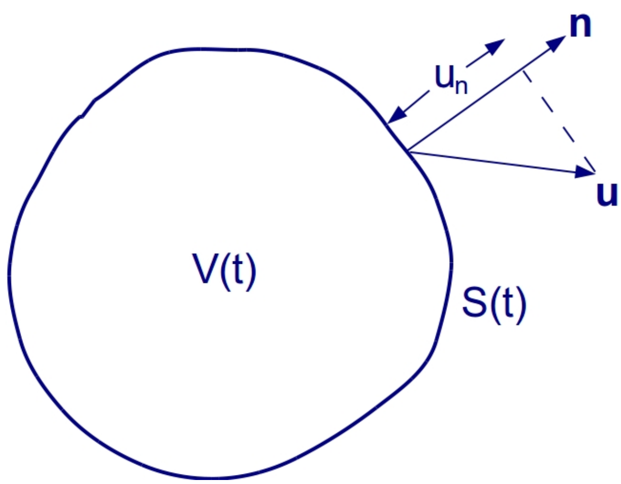


Figure 2.1: Outward unit normal vector $\boldsymbol{n}$*, velocity vector* $\boldsymbol{u}$*, and normal velocity* u_n *at a point on the surface* $S(t)$ *of a convected blob* $V(t)$*.*

The problem of course is that we don't know the value of ϕ. Luckily there is a way out by using the so-called inner product. The inner product of two vectors, say $\boldsymbol{a}$ and $\boldsymbol{b}$, is written as $\boldsymbol{a} \cdot \boldsymbol{b}$ and defined as [5]:

$$\boldsymbol{a} = (a_1, a_2, a_3)^T, \quad \boldsymbol{b} = (b_1, b_2, b_3)^T, \quad \boxed{\boldsymbol{a} \cdot \boldsymbol{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3} \tag{2.8}$$

How does this help us with finding the normal velocity? Well, the inner product defined above satisfies an extremely helpful property

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\phi \quad \Rightarrow \quad \cos\phi = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \|\boldsymbol{b}\|}, \tag{2.9}$$

where ϕ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. So, taking $\boldsymbol{a} = \boldsymbol{u}$ and $\boldsymbol{b} = \boldsymbol{n}$ we find an elegant expression for the normal velocity [6]:

$$\cos\phi = \frac{\boldsymbol{u} \cdot \boldsymbol{n}}{\|\boldsymbol{u}\| \|\boldsymbol{n}\|} \quad \Rightarrow \quad \boxed{u_n = \|\boldsymbol{u}\| \cos\phi = \boldsymbol{u} \cdot \boldsymbol{n}.} \tag{2.10}$$

2.3 Leibniz-Reynolds transport theorem

Leibniz' rule Suppose we have a time-dependent function $f(x,t)$ and we integrate it over a time-dependent interval $[a(t), b(t)]$, then the result (the area under the function between time-dependent boundaries $a(t)$ abnd $b(t)$) is a function of a, b, and t:

$$\int_{a(t)}^{b(t)} f(x,t)\, dx \equiv F(a(t), b(t), t). \tag{2.11}$$

When time advances the function F will change, see Fig. (2.3) Three effects can be observed:

[5] Note that the inner product has two vectors as input, whereas the result is a scalar
[6] Note that $\|\boldsymbol{n}\| = 1$

Figure 2.2: Gottfried Wilhelm Leibniz (left) (1646 - 1716) was a German philosopher and mathematician. He wrote in different languages, primarily in Latin, French, and German. Leibniz occupies a prominent place in the history of mathematics and the history of philosophy. He developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published.

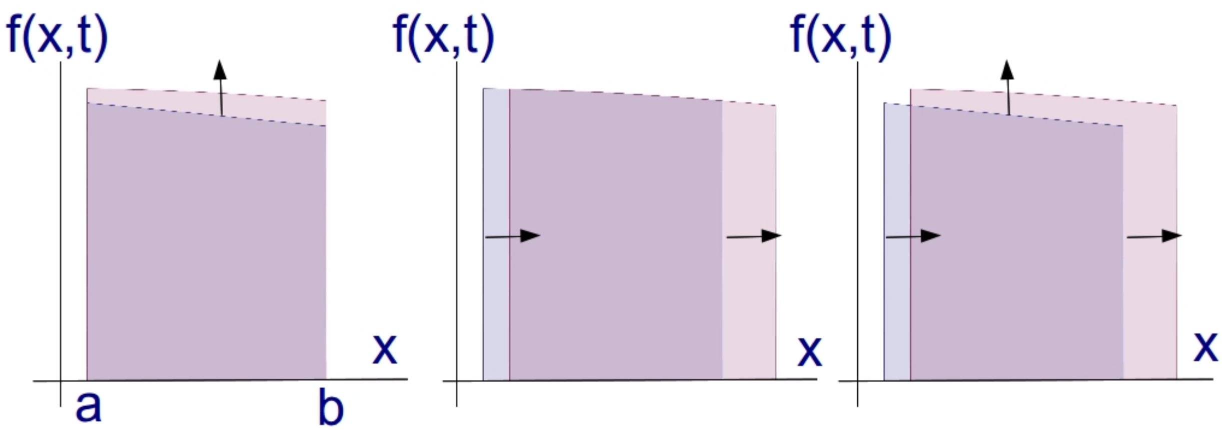


Figure 2.3: Integral variation with time: time-dependent integrand (left), time-dependent boundaries (middle), time-dependent integrand and boundaries (right).

(a) the effect of time-variation of the function $f(x,t)$ (Fig. (2.3).(left)),
(b) the effect of time-variation of the boundaty $a(t)$ (Fig. (2.3).(middle)), and
(c) the effect of time-variation of the boundaty $b(t)$ (Fig. (2.3).(middle)).

The combined effect is illustrated in (Fig. (2.3).(right).

If the time increment is small, say Δt, the time increments of f, a, and b can be estimated by means of a Tayor series expansion:

(a) $\Delta f = \frac{\partial f}{\partial t}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$,
(b) $\Delta a = \frac{da}{dt}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$, and
(c) $\Delta b = \frac{db}{dt}\Delta t + \mathcal{O}\left((\Delta t)^2\right)$.

and with these expressions the three increments of F can be estimated:

(a) the effect of time-variation of the function $f(x,t)$: $\int_{a(t)}^{b(t)} \Delta f \, dx$,
(b) the effect of time-variation of the boundaty $a(t)$: $-f(a)\Delta a$, and
(c) the effect of time-variation of the boundaty $b(t)$: $f(b)\Delta b$.

Note that we have neglected the "intersections" of these increments in the left-top and right-top corner of the area but that is allowed for smal values of Δt since these are $\mathcal{O}\left((\Delta t)^2\right)$. With these expressions and

$$\frac{\partial F}{\partial t} \equiv \lim_{\Delta t \to 0} \frac{F(a(t+\Delta t), b(t+\Delta t), t+\Delta t) - F(a(t), b(t), t)}{\Delta t} \tag{2.12}$$

we arrive at

$$\boxed{\frac{\partial F}{\partial t} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\, dx + f(b)\frac{db}{dt} - f(a)\frac{da}{dt}.} \tag{2.13}$$

This relation may be referred to as Leibniz' rule

(Leibniz-)Reynolds transport theorem The Reynolds transport theorem explains how we can compute the time derivative of an integral of a function, say $\rho(\boldsymbol{x}, t)$, that depends on time, over a convected blob $V(t)$. Let $S(t)$ denote the surface of $V(t)$, then the Reynolds transport theorem states that

$$\frac{d}{dt}\int_{V(t)} \rho(\boldsymbol{x},t)dV = \int_{V(t)} \frac{\partial \rho}{\partial t}(\boldsymbol{x},t)dV + \int_{S(t)} \rho(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t)dS. \tag{2.14}$$

We will frequently leave out the arguments $\boldsymbol{x}$ and t for reasons of readability [7]:

$$\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t}dV + \int_{S(t)} \rho\boldsymbol{u}\cdot\boldsymbol{n}dS. \tag{2.15}$$

Furthermore, we will use the Einstein summation convention:

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t}dV + \int_{S(t)} \rho u_j n_j dS} \tag{2.16}$$

[7] Which does not mean that ρ, $\boldsymbol{u}$ and $\boldsymbol{n}$ are constants!

How can we understand this theorem?

Well, on the one hand, if the blob does not move at all (one could think of a vessel containing the blob of fluid) the answer would clearly be

$$\frac{dV}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV, \tag{2.17}$$

since $\boldsymbol{x}$ and t are independent variables which alows us to interchange integration over $\boldsymbol{x}$ and differentiation with respect to t.

On the other hand, if $V(t)$ moves but ρ is constant, then the answer would be

$$\frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_{V(t)} \rho dV = \int_{S(t)} \rho u_j n_j dS. \tag{2.18}$$

This makes sense since the only way for the integral to increase in time if ρ is constant is to expand due to the motion of the surface. Evidently, if the surface moves in a direction parallel to the surface itself, this does not contribute to expansion. It is therefore clear that it is the normal velocity of the surface that determines the expansion of V. If the surface locally moves in outward normal direction the expansion contribution per unit time is $\boldsymbol{u} \cdot \boldsymbol{n} dS = u_j n_j dS$. The contribution to the increase per unit time of the volume integral over ρ is thus $\rho \boldsymbol{u} \cdot \boldsymbol{n} dS = \rho u_j n_j dS$. Finally, the overall increase per unit time of the volume integral over ρ is obtained by integrating $\rho u_j n_j$ over the complete surface $S(t)$.

So, in conclusion, we can understand both situations: on the one hand a non-moving blob but time-dependent density, and on the other hand a moving blob but time-independent density. Based on mathematical argumenst one can prove that in the general case of a moving blob ănd a time-dependent density the answer is the sum of the two answers found for the restricted cases. Such prove is left out here because it is out of the scope of these lecture notes.

2.4 Integral formulation of mass conservation

We are now ready to take the fundamental physical statement of mass conservation, Eq.(2.6), and combine it with the Reynolds transport theorem, Eq.(2.16). As a result, we obtain the integral formulation of mass conservation:

$$\boxed{\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho u_j n_j dS = 0 \quad \text{for all} \;\; (V(t), t)} \tag{2.19}$$

In summary:

(a) the first integral expresses the mass rate of change due to the density rate of change, and

(b) the second integral expresses the mass rate of change due the growth rate of the blob.

2.5 Flow rates and averaged velocity

We now want to describe the volume flow rate $\dot{v}$ through (part of) a surface S, think for example of the entrance or exit of a pump. First we define a normal vector on the surface which determines in which direction the volume flow rate is considered positive. Note that for a non-closed surface we are allowed to choose the direction of the normal vector. In case of the pump: when we define the normal vector **inward** the volume flow rate is considered **positive** when its directed **inward**. If we define the normal vector **outward** the volume flow rate is considered **positive** when its directed **outward**.

The volume flow rate through a small surface element dS is:

$$d\dot{v} = u_n dS = \boldsymbol{u} \cdot \boldsymbol{n}\, dS, \tag{2.20}$$

where u_n is the normal velocity on S. It is evident that we should use the normal velocity instead of the velocity itself since the tangential component of the velocity vector does not contribute to the volme flow rate. If we had wanted to calculate the volume flow rate from right to left we should have taken $u_n \equiv -u_j n_j$, or we should have chosen the unit normal vector in opposite direction. The **volume flow rate** through S now simply follows by integration:

$$\dot{v} = \int_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS. \tag{2.21}$$

It is easy to see that the dimension of this expression is volume per unit time since the normal vector is dimensionless and the dimension of dS is length to the power two.

From this expression we can directly calculate the **averaged normal velocity** U over S:

$$U \equiv \frac{1}{S} \int_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS, \quad S \equiv \int_S dS. \tag{2.22}$$

Similarly, one can calculate the **mass flow rate** $\dot{m}$ through the surface S, starting with the contribution of a surface element dS:

$$d\dot{m} = \rho u_n dS, \tag{2.23}$$

and then integrating over the surface:

$$\dot{m} = \int_S \rho u_j n_j dS, \tag{2.24}$$

where it again easily verified that the dimension of this expression is mass per unit time.

Finally, at solid walls the velocity vector is parallel to the wall. This means that the velocity component normal to the wall is zero, or

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ at solid walls.} \tag{2.25}$$

This observation directly leads to a daily-life experience: the volume and mass flow rates through solid walls are identically zero.

2.6 Example problem

We now return to the problem raised in the beginning of this chapter, see Fig. (2.4). For a given velocity field at the tube entrance we have to calculate the mass flow rate at the tube exit. We will do this by using the derived integral formulation of mass conservation given by Eq.(2.19). To do so we define a fluid blob in the tube that is alligned with the tube wall and with the entance and exit surfaces, see Fig. (2.4). Next we split the surface integral in

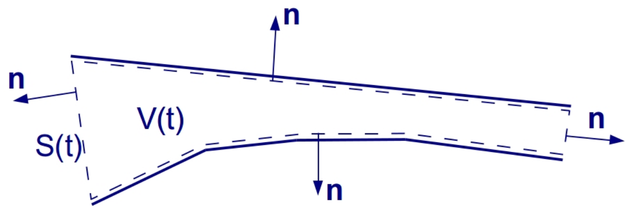


Figure 2.4: Fluid blob in tube with outward unit normals

three parts: the entrance S_1, the wall S_w, and the exit S_2:

$$\int_{S(t)} \rho u_j n_j dS = \int_{S_1(t)} \rho u_j n_j dS + \int_{S_w(t)} \rho u_j n_j dS + \int_{S_2(t)} \rho u_j n_j dS. \tag{2.26}$$

The second of these integrals is zero since integration is over a solid wall where always $u_j n_j = 0$. The third of these integrals is the mass flow rate at the exit:

$$\dot{m} = \int_{S_2(t)} \rho u_j n_j dS. \tag{2.27}$$

Therefore, substitution of these expressions into the integral formulation of mass conservation Eq.(2.19) gives

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S_1(t)} \rho u_j n_j dS + \dot{m} = 0. \tag{2.28}$$

So finally we obtain

$$\dot{m} = -\int_{V(t)} \frac{\partial \rho}{\partial t} dV - \int_{S_1(t)} \rho u_j n_j dS. \tag{2.29}$$

We see that we cannot evaluate this expression further since we need information about $\frac{\partial \rho}{\partial t}$ in the interior of $V(t)$. Suppose now that we know that the flow is steady, in other words, the flow is independent of time at all times. In that case the volume integral is zero since $\frac{\partial \rho}{\partial t}$ is zero and the result is

$$\dot{m} = -\int_{S_1(t)} \rho u_j n_j dS. \tag{2.30}$$

This makes sense because the remaining integral represents the outgoing mass flow rate at the entrance with a minus sign which is just the incoming mass flow rate at the entrance. The remaining integral still cannot be evaluated since no information is given about the mass density at the entrance.

Alternatively suppose that we know that the flow is incompressible, in other words, the mass density is a known constant. In that case the volume integral in Eq.(2.29) is again zero since $\frac{\partial \rho}{\partial t}$ is zero. The result is

$$\dot{m} = -\rho \int_{S_1(t)} u_j n_j dS. \tag{2.31}$$

2.7 Exercises

Problem 2.1. *Compute the inner product* $\boldsymbol{a} \cdot \boldsymbol{b}$ *if*

(a) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (1, 0, 0)^T$.
(b) $\boldsymbol{a} = (1, 0, 0)^T$, $\boldsymbol{b} = (0, 1, 0)^T$.
(c) $\boldsymbol{a} = (a_1, a_2, a_3)^T$, $\boldsymbol{b} = (b_1, b_2, b_3)^T$.
(d) $\boldsymbol{a} = (x, y^2, x)^T$, $\boldsymbol{b} = (y, y, z)^T$.
(e) $\boldsymbol{a} = (u, v, w)^T$, $\boldsymbol{b} = (n_1, n_2, n_3)^T$.

Problem 2.2. *Compute the inner product* $\boldsymbol{u} \cdot \boldsymbol{n}$ *if*

(a) $\boldsymbol{u} = U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.
(b) $\boldsymbol{u} = -U\boldsymbol{n}$, $\boldsymbol{n} = (n_1, n_2, n_3)^T$.

Problem 2.3. *A tube has cross-sectional area* A_a *at the entrance and cross-sectional area* A_b *at the exit, and the fluid flowing through the tube is incompressible.*

(a) If the volume flow rate at the exit is Q*, compute the average normal velocity at the exit.*
(b) If the volume flow rate at the exit is Q*, compute the average normal velocity at the entrance.*

Problem 2.4. *A channel with rectangular cross section has sides* b *and* h *at the exit. The exit cross section is plane and perpendicular to the* x*-axis, and intersects the* x*-axis at* $x = L$*. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u, v, w)^T$*, with* $u = U(1 - z/h)$, $v = \ln yz$, $w = yz^2$.
(b) $(u, v, w)^T$*, with* $u = U(1 - y/b)$, $v = \sin z$, $w = \cos y$.
(c) $(u, v, w)^T$*, with* $u = U(1 - y/b)(1 - z/h)$, $v = 0$, $w = yz$.

Problem 2.5. *A tube with circular cross section has radius* R *at the exit. The exit cross section is plane and perpendicular to the* x*-axis, and intersects the* x*-axis at* $x = L$*. Compute the average normal velocity at the exit for the following velocity vectors at the exit cross section:*

(a) $(u(r), 0, 0)^T$*, with* $u(r) = U(1 - r/R)$*, compute the average normal velocity.*
(b) $(u(r), 0, 0)^T$*, with* $u(r) = U(1 - (r/R)^2)$*, compute the average normal velocity.*

Problem 2.6. *Show how Eq.(2.19) reduces in the following two cases:*

(a) steady flow
(b) Incompressible flow

Problem 2.7. *Consider steady, incompressible flow through the device shown. Given:* U_1, A_1, U_2, A_2, A_3. *Derive an expression for the volume flow rate through port 3.*

Problem 2.8. *Incompressible oil flows steadily in a thin layer down an inclined plane with width* w. *The velocity profile is*

$$u = \frac{\rho g \sin\theta}{\mu}\left[hy - \frac{1}{2}y^2\right]. \tag{2.32}$$

Derive formulas for the volume flow rate and mass flow rate in terms of ρ, μ, g, θ, *and* h.

Problem 2.9. *Incompressible water flows steadily through a pipe of length L and radius R. Derive an expression for the uniform inlet velocity,* U, *if the velocity distribution across the outlet is given by*

$$u = V\left[1 - \frac{r^2}{R^2}\right]. \tag{2.33}$$

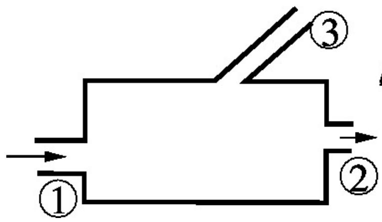
Problem 2.10. *A bend with rectangular cross section and width* w *has a linear velocity profile at port 1. The flow is uniform at ports 2 and 3. The fluid is incompressible and the flow is steady. Derive an expression for the uniform velocity at port 3.*

Problem 2.11. *Water enters a two-dimensional square channel of constant width* w, *and constant height,* h, *with uniform velocity,* U. *The channel makes a* $90°$ *bend that distorts the flow to produce the linear velocity profile shown at the exit, with* $V_{max} = 2V_{min}$. *The flow is steady and the fluid is incompressible. Derive an expression for* V_{min}.

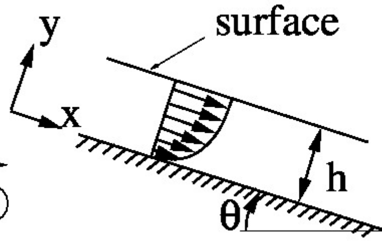
Problem 2.12. *Incompressible water flows steadily past a porous plate of width* w *and length* L. *Constant suction is applied along the plate with normal velocity* V *(towards the plate). The velocity profile at the outflow plane is:*

$$\frac{u}{U} = 3\left[\frac{y}{\delta}\right] - 2\left[\frac{y}{\delta}\right]^{3/2}. \tag{2.34}$$

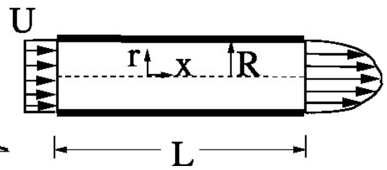
Derive a formula for the mass flow rate through the boundary at the top of the domain $(y = \delta)$.



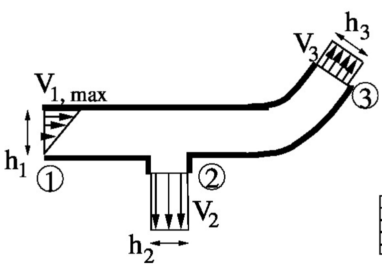
(a) Problem 2.7



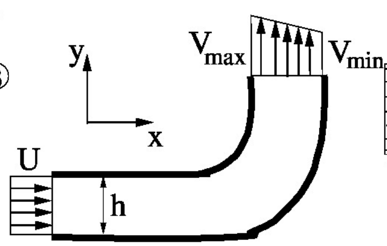
(b) Problem 2.8



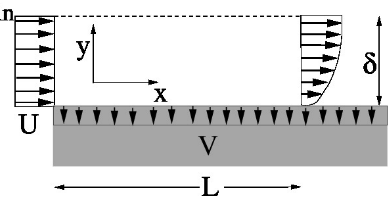
(c) Problem 2.9



(a) Problem 2.10



(b) Problem 2.11



(c) Problem 2.12

Chapter 3

Continuity Equation

3.1 Gradient, divergence

Two important and extremely useful operators in the analysis of vector fields are the gradient and the divergence.

The gradient ($\boldsymbol{\nabla}$) operates on a scalar function, say $\phi(\boldsymbol{x})$, and the result is a vector:

$$\boldsymbol{\nabla}\phi(\boldsymbol{x}) \equiv (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z})^T. \tag{3.1}$$

In words, the gradient of a scalar function consists of a vector with its components representing the angle of inclination in the three coordinate directions.

The divergence ($\boldsymbol{\nabla}\cdot$) operates on a vector function, say $\boldsymbol{u}(\boldsymbol{x})$, and the result is a scalar:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}) \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{3.2}$$

3.2 Interpretation of the velocity field divergence

Suppose we have a tiny brick-shaped blob of fluid with edges of size a, b, and c. Then the volume of is this tiny blob is $\Delta V = abc$. When the blob is convected with the fluid in a velocity field $\boldsymbol{u} = (u, v, w)^T$ it changes in size and in shape. Its size change at the moment when we start following the blob is [1]

$$\frac{d}{dt}\Delta V = \frac{d}{dt}abc = \frac{da}{dt}bc + a\frac{db}{dt}c + ab\frac{dc}{dt}. \tag{3.3}$$

We choose the tiny blob in such a way that its edges are aligned with the three cartesian axes. The time derivative of a is

$$\frac{da}{dt} = u(x + a, y, z, t) - u(x, y, z, t). \tag{3.4}$$

[1] This is just an application of the product rule of differentiation.

In approximation [2]

$$u(x+a,y,z,t) = u(x,y,z,t) + \frac{\partial u}{\partial x}a + \mathcal{O}\left(a^2\right), \tag{3.5}$$

so

$$\frac{da}{dt} = \frac{\partial u}{\partial x}a + \mathcal{O}\left(a^2\right). \tag{3.6}$$

In a similar way we derive also that

$$\frac{db}{dt} = \frac{\partial v}{\partial x}b + \mathcal{O}\left(b^2\right), \quad \frac{dc}{dt} = \frac{\partial w}{\partial x}c + \mathcal{O}\left(c^2\right). \tag{3.7}$$

Hence, when we subtitute these expressions in the time derivative of ΔV we obtain

$$\frac{d}{dt}\Delta V = \frac{\partial u}{\partial x}abc + \frac{\partial v}{\partial x}abc + \frac{\partial w}{\partial x}abc + \text{higher order terms}, \tag{3.8}$$

or

$$\frac{d}{dt}\Delta V = \frac{\partial u_j}{\partial x_j}\Delta V + \text{higher order terms}, \tag{3.9}$$

where we have used the Einstein summation convention and $(u_1,u_2,u_3)^T \equiv (u,v,w)^T$ and similarly $(x_1,x_2,x_3)^T \equiv (x,y,z)^T$. So what we have found is

$$\frac{\partial u_j}{\partial x_j} = \frac{1}{\Delta V}\frac{d}{dt}\Delta V + \text{higher order terms}. \tag{3.10}$$

This means that, in the limit of an sufficiently small blob, the divergence of the velocity field represents the relative time derivative of its volume. From this, we see immediately that the flow is incompressible if the divergence of the velocity field is zero everywhere.

3.3 Gauss' divergence theorem

To derive the differential formulations from integral formulations we need the divergence theorem of Gauss. The theorem will not be derived mathematically here since that is beyond the scope of the present lecture notes. Instead we will make the theorem plausible based on arguments from physics. We will compute the time derivative of the volume of a large convected blob, say $V(t)$, in two different ways. Gauss' divergence theorem then follows by equating the two expressions. On the one hand we can compute the volume time derivative by subdividing the large blob into a large number N of tiny blobs and take the sum:

$$\frac{dV}{dt} \approx \sum_{i=1}^{N}\frac{d}{dt}\Delta V_i \tag{3.11}$$

[2] This is just an application of Taylor's theorem.

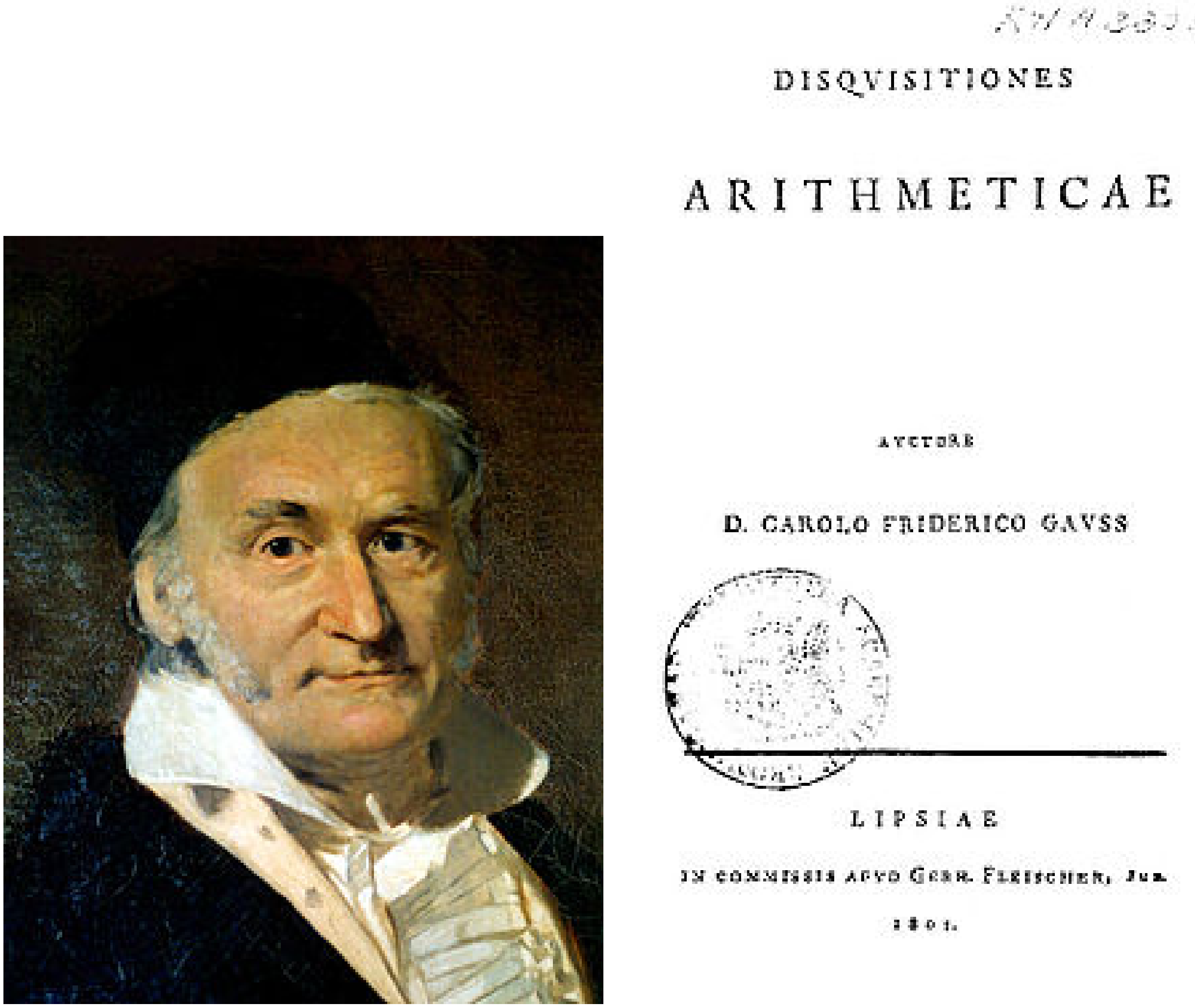


Figure 3.1: Johann Carl Friedrich Gauss (30 April 1777 23 February 1855) was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics.

where the subscript i is a counter. Using the result for the tiny blob of the previous section, Eq.(3.9), we get

$$\frac{dV}{dt} \approx \sum_{i=1}^{N} \left(\frac{\partial u_j}{\partial x_j} \right)_i \Delta V_i, \tag{3.12}$$

In the limit of $\Delta V_i \to 0$ and $N \to \infty$ this becomes an integral:

$$\frac{dV}{dt} = \int_{V(t)} \frac{\partial u_j}{\partial x_j} dV. \tag{3.13}$$

On the other hand we can compute the volume time derivative by employing the Reynolds transport theorem, Eq.(2.16):

$$\frac{dV}{dt} \equiv \frac{d}{dt}\int_{V(t)} dV = \int_{S(t)} u_j n_j dS, \tag{3.14}$$

which represents the growth rate of V at its surface.

Hence, by equating the two results for $\frac{dV}{dt}$ we obtain Gauss' divergence theorem:

$$\boxed{\int_{V(t)} \frac{\partial u_j}{\partial x_j} dV = \int_{S(t)} u_j n_j dS.} \tag{3.15}$$

Note that we have 'derived' it here for an arbitrary velocity field $\boldsymbol{u}$, so that it will hold for any vector field. [3]

3.4 Continuity equation

We derive the differential formulation of mass conservation from the integral formulation of mass conservation Eq.(2.19) by replacing the surface integral by a volume integral using Gauss' divergence theorem Eq.(3.15):

$$\int_{S(t)} \rho u_j n_j dS = \int_{V(t)} \frac{\partial \rho u_j}{\partial x_j} dV. \tag{3.16}$$

Note that here ρu_j takes the place of u_j in Eq.(3.15). With this replacement the integral form Eq.(2.19) becomes one single volume integral:

$$\int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} \right) dV = 0. \tag{3.17}$$

Since the blob V was chosen completely arbitrary, this holds for any blob, also for extremely tiny blobs. This means that

$$\boxed{\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t)}} \tag{3.18}$$

This is the differential formulation of mass conservation which is frequently called the continuity equation.

Several limiting cases are of interest. First of all, if the flow is steady, all time-derivatives ar zero, so

$$\frac{\partial \rho u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t) \tag{3.19}$$

When the flow is unsteady but incompressible ($\rho =$ constant), then $\frac{\partial \rho}{\partial t} = 0$ and we get again Eq.(3.19). But since $\rho =$ constant we can take it out of the differentiation and divide both sides by ρ (since it is larger than zero) to obtain

$$\frac{\partial u_j}{\partial x_j} = 0, \quad \text{for all } (\boldsymbol{x}, t) \tag{3.20}$$

[3] not necessarily a velocity field, Gauss derived the theorem when working on electric fields.

3.5 Exercises

Problem 3.1. *For incompressible flow, indicate for each of the following velocity fields whether they are steady/unsteady, and whether they satisfy mass conservation.*

(a) $u = x + y + z^2$, $v = x - y + z$, $w = 2xy + y^2 + 4$,
(b) $u = xyzt$, $v = -xyzt^2$, $w = \frac{1}{2}z^2(xt^2 - yt)$,
(c) $u = y^2 + 2xz$, $v = -2yz + x^2yz$, $w = \frac{1}{3}x^2z^2 + x^3y^4$.

Problem 3.2. *For a flow in the xy plane, the x component of velocity is given by* $u = ax(y - b)$.

(a) Find the y component of the velocity, v, for steady, incompressible flow.
(b) Explain why it is also valid for unsteady, incompressible flow.

Problem 3.3. *The x component of velocity in a steady, incompressible flow field in the xy plane is* $u = A/x$. *Find the simplest y component of velocity for this flow field.*

Problem 3.4. *For the following velocity fields, determine whether the continuity equation for incompressible flow is satisfied:*

(a) $\boldsymbol{u} = (ax, ay, -2az)^T$
(b) $\boldsymbol{u} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right)^T$

Problem 3.5. *For the two-dimensional velocity field* $\boldsymbol{u} = (ax, by)^T$, *and taking V with boundary S as a box defined by* $0 \le x \le p$, $0 \le y \le q$, *compute*

(a) $\int_V \frac{\partial u_j}{\partial x_j}\, dV$,
(b) $\int_S u_j n_j \, dS$.

Problem 3.6. *For the two-dimensional velocity field* $\boldsymbol{u} = (ax, by)^T$, *and taking V with boundary S as a disk defined by* $0 \le \sqrt{x^2 + y^2} \le R$, *compute*

(a) $\int_V \frac{\partial u_j}{\partial x_j}\, dV$,
(b) $\int_S u_j n_j \, dS$.

Problem 3.7. *For one-dimensional steady compresible flow* ($v = w = 0$),

(a) derive an expression for ρu *if* Φ, *the mass flow rate per unit area is given.*
(b) derive an expression for u in case the flow is incompressible.

Problem 3.8. *For one-dimensional compressible flow* ($v = w = 0$) *with constant velocity u,*

(a) show that $\rho(x, t) = \rho_o \sin(x - ut)$ *satisfies the continuity equation.*
(b) make a sketch of $\rho_o \sin(x - ut)$ *at* $t = 0$ *and* $t = 1/u$.
(c) show that $\rho(x, t) = f(x - ut)$ *satisfies the continuity equation for any function f.*

Problem 3.9. *It is known that the integral of the outward unit normal vector over an arbitrary but closed surface (3D) is the null-vector. This means that the integral of each component of the outward unit normal is zero. Proof this for the first component by taking a velocity field* $\boldsymbol{u} = (1, 0, 0)^T$ *and by using Gauss' divergence theorem.*

Chapter 4

Momentum Conservation

4.1 Forces and stresses in fluids

If we place an object in a flow field (for example wind or streaming water) we know that the object experiences a force. This force has a magnitude and it has a direction. Part of the force always points in the downstream direction, that part is called drag. The remaining part points in a direction perpendicular to the downstream direction, that part is called lift.

Both drag and lift are the result of stresses on the object's surface that are generated by the flow. At any point of the surface, the stress by the fluid on the object's surface is a vector, with both a magnitude and a direction. The part of the stress that points in the normal direction of the surface is called normal stress. The remaining part of the stress points in tangential direction of the surface is called tangential stress, but more often the shear stress.

Let us zoom in at a point on the surface of the object, see Fig. (4.1). The lengths of the normal and tangential components of the stress vector determine its length:

$$\|t\|^2 = t_n^2 + t_t^2. \tag{4.1}$$

Moreover, the length of the normal component, t_n, can easily be found from

$$t_n = \boldsymbol{t} \cdot \boldsymbol{n}. \tag{4.2}$$

Normal stress The question raised here is whether we can compute the stress vector when we know the flow field. Let us start with the most simple flow field that we know: no flow at all. We all know that in that case there is no shear stress. This is very different from the situation that we have when a solid exerts a force on another solid, think of standing still on a roof: you're hopefully not sliding down due to the shear stress exerted by the roof tiles on your shoes. So, if there is no flow, the only stress that is exerted on the surface is normal stress: pressure! Hence, in the case of no flow, we have

$$\boldsymbol{t} = -p\boldsymbol{n}. \tag{4.3}$$

Note that we have used a minus sign to indicate that the normal stress due to pressure is positive in the direction opposite to the outward unit normal.

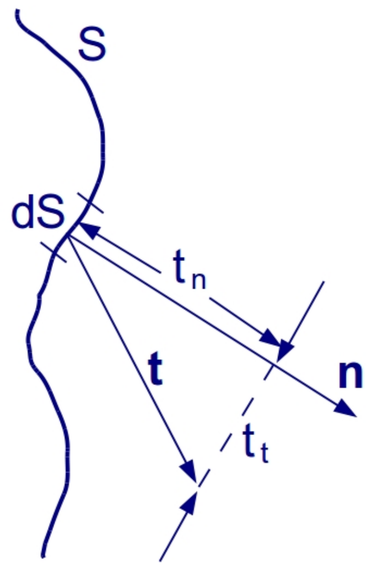


Figure 4.1: Outward unit normal vector $\boldsymbol{n}$, stress vector $\boldsymbol{t}$, normal stress t_n, and shear stress t_t at a point on the surface S of an object.

Shear stress Consider the linear shear layer flow between two plates depicted in Fig. (4.2). The velocity u in x-direction does not depend on x, but it linearly depends on y:

$$u(y) = \frac{y}{2b} U, \tag{4.4}$$

where $2b$ is the gap width between the two plates and U is the velocity of the top plate. Experimentally it has been determined that the shear stress τ experienced by both plates is

$$\tau = \mu \frac{U}{2b}, \tag{4.5}$$

where μ is a fluid-depending constant called the viscosity coefficient. Typical values are $\mu = 18 \times 10^{-6} \frac{Ns}{m^2}$ for air, $\mu = 1 \times 10^{-3} \frac{Ns}{m^2}$ for water, and $\mu = 81 \times 10^{-3} \frac{Ns}{m^2}$ for olive oil.

More generally, one can write

$$\tau = \mu \frac{\partial u}{\partial y} \tag{4.6}$$

4.2 Cauchy stress tensor

The ideas about normal stress and shear stress shown in the previous section can be extended towards a general expression for the stresses that occur in fluid flow. To this end we write the stress vector $\boldsymbol{t}$ that acts on an imaginary surface in the flow as

$$\boldsymbol{t} = S\boldsymbol{n}, \quad S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{4.7}$$

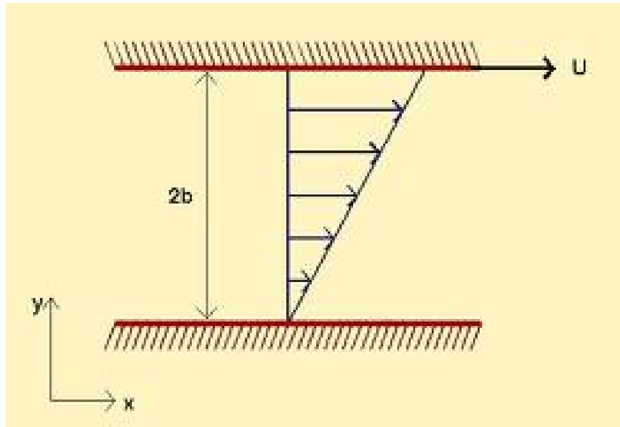


Figure 4.2: Linear shear layer between two plates, one of wich is moving.

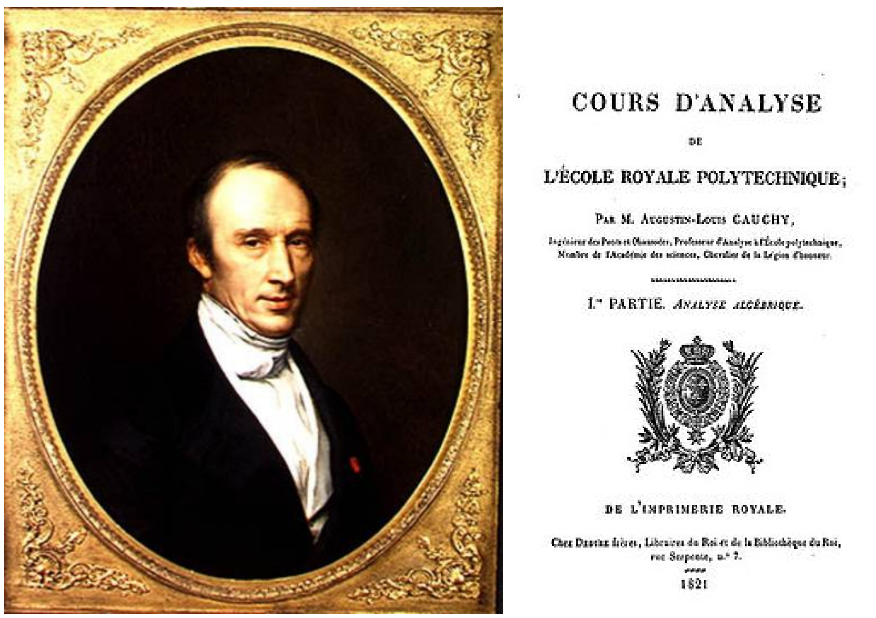


Figure 4.3: Baron Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner, rejecting the heuristic principle of the generality of algebra exploited by earlier authors. He defined continuity in terms of infinitesimals and gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. A profound mathematician, Cauchy exercised a great influence over his contemporaries and successors. His writings cover the entire range of mathematics and mathematical physics.

where $\boldsymbol{n}$ is the normal vector on the surface and the matrix S is called the Cauchy stress tensor.

The following convention is adopted: label the two adjacent sides of the surface "A" and "B". Then to calculate the stress vector caused by "A" on "B" is found by choosing the normal vector to point to "A" and to calculate $\boldsymbol{t} = S\boldsymbol{n}$. In contrast, the stress vector caused by "B" on "A" is found by choosing the normal vector to point to "B" and to calculate $\boldsymbol{t} = S\boldsymbol{n}$. Since the normal vector pointing to "A" and the normal vector pointing to "b" are each others opposite, the resulting stress vectors are also each others opposite. This observation completely agrees with Newton's third law: when an object "A" acts on another object, the other object reacts equally but in opposite direction.

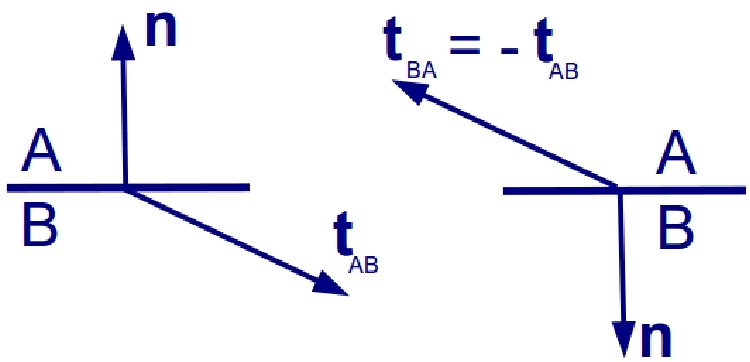


Figure 4.4: Cauchy relation convention

In index notation the stress vector is given by

$$\boxed{t_i = \sigma_{ij} n_j,} \tag{4.8}$$

which is called the Cauchy stress relation. For so-called Newtonian fluids the stress tensor is given by

$$S = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \tag{4.9}$$

or, in index notation:

$$\boxed{\sigma_{ij} = -p\delta_{ij} + \tau_{ij},} \tag{4.10}$$

with

$$\boxed{\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}.} \tag{4.11}$$

No flow or inviscid flow It is easily verified that when there is no flow, i.e. $\boldsymbol{u} = 0$, the stress relation reduces to

$$t_i = -p\delta_{ij} n_j = -p n_i, \tag{4.12}$$

which agrees with the findings above and confirms that the normal stress is pointing towards the surface and that there is no shear stress. The same reult is obtained when there is flow but the fluid is inviscid, i.e., it has a negligible viscosity ("$\mu = 0$").

Stress vector on three elemental surfaces To interpret the Cauchy stress vector let us analyse what the stress vector is on a surface aligned with the $y - z$ plane with normal vector pointing in the positive x-direction:

$$\boldsymbol{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{t} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}. \tag{4.13}$$

So, the first column of the stress tensor represents the stress vector that would act on a surface with the normal vector pointing in the positive x-direction. Similar statements can be made about the second and third columns of the stress tensor:

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{t} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}, \quad \boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{t} = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}. \tag{4.14}$$

4.3 Newton's law

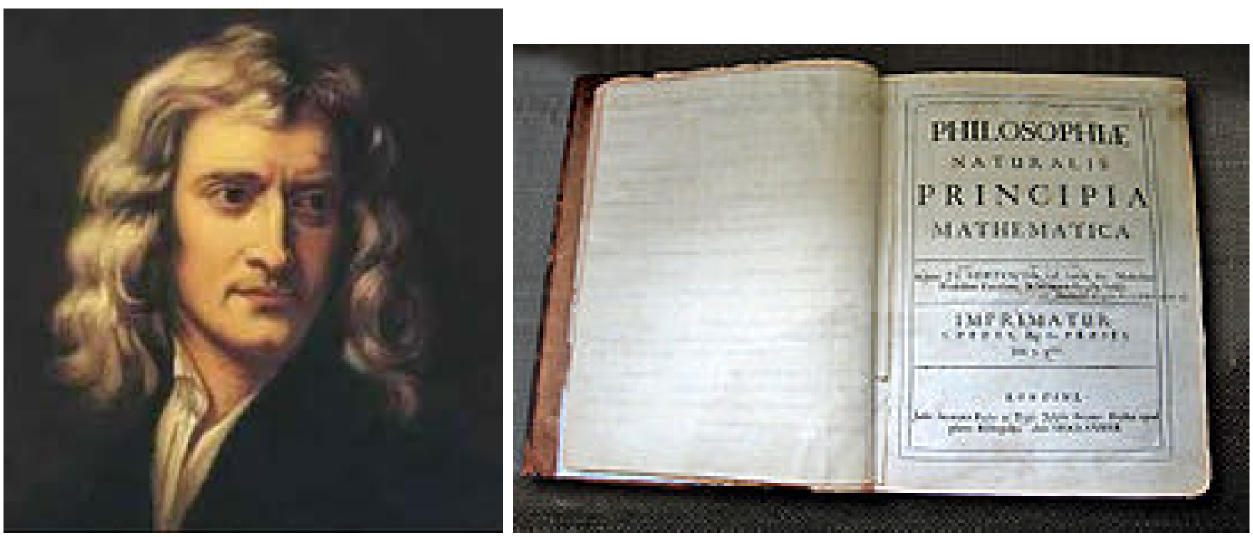


Figure 4.5: Sir Isaac Newton (1642-1727) was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian. His monograph Philosophi Naturalis Principia Mathematica, published in 1687, lays the foundations for most of classical mechanics. In this work, Newton described universal gravitation and the three laws of motion, which dominated the scientific view of the physical universe for the next three centuries.

The momentum of a point particle is governed by Newton's second law. Let the mass of the particle be m, its velocity $\boldsymbol{v}(t)$, and let a force $\boldsymbol{F}(t)$ work on the particle. Then Newton's second law states that

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}(t). \tag{4.15}$$

Note that these are in fact three equations which becomes evident when we rewrite in index notation:

$$m\frac{dv_i}{dt} = F_i(t), \quad i = 1, 2, 3. \tag{4.16}$$

The question is: how can we apply this equation to a convected blob of fluid?

Assuming that the mass of the particle is constant, the equations can also be written as

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}(t) \quad \text{or} \quad \frac{d}{dt}(mv_i) = F_i(t), \quad i = 1, 2, 3. \tag{4.17}$$

Hence, the above form states that the time derivative of the momentum of a particle, $m\boldsymbol{v}$, is equal to the force acting on the particle.

4.4 Momentum conservation: integral formulation

Newton's law can be translated to a convected blob of fluid: the time derivative of the momentum of a convected blob is equal to the force acting on the blob. So then we are left with two questions: what is the momentum of the blob, and what is the force acting on the blob?

If the blob is tiny with volume ΔV, the momentum can simply be approximated by $\rho\boldsymbol{u}\Delta V$, where $\rho\boldsymbol{u}$ is the momentum density. Just like the developments with mass conservation, the momentum of the large blob can be obtained by summation of N of these tiny-blob contributions, and in the limit of $\Delta V \to 0$ and $N \to \infty$ we obtain the blob momentum as an integral:

$$\boldsymbol{M}(t) = \int_{V(t)} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) dV. \tag{4.18}$$

The force on the blob consists of two contributions:

(a) a force $\boldsymbol{F}_S$ acting on the surface,
(b) a force $\boldsymbol{F}_V$ acting in the interior.

The force acting on the surface is caused by the surrounding fluid (or solid). On a little surface element dS we have

$$d\boldsymbol{F}_S = \boldsymbol{t} dS, \tag{4.19}$$

where $\boldsymbol{t}$ is the stress vector generated by the surroundings. The total surface force is obtained by integration:

$$\boldsymbol{F}_S = \int_{S(t)} \boldsymbol{t} dS. \tag{4.20}$$

The force acting in the interior is caused by the gravity. On a little volume element dV with mass ρdV we have

$$d\boldsymbol{F}_V = \rho\boldsymbol{g} dV, \tag{4.21}$$

where $\boldsymbol{g}$ is the gravity vector with length g, earth's gravitational constant. The total volume force is obtained by integration:

$$\boldsymbol{F}_V = \int_{V(t)} \rho \boldsymbol{g} dV. \tag{4.22}$$

Hence,

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{F} \quad \Rightarrow \quad \boxed{\frac{d}{dt}\int_{V(t)} \rho \boldsymbol{u} dV = \int_{S(t)} \boldsymbol{t} dS + \int_{V(t)} \rho \boldsymbol{g} dV} \tag{4.23}$$

Note that this momentum conservation statement is a fundamental physical statement, it cannot be proved! It has been observed again and again, it has never been observed that momentum was not conserved.

Finally, by means of the Reynolds transport theorem we obtain the integral formulation of momentum conservation(s):

$$\boxed{\boxed{\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{S(t)} \rho u_i u_j n_j dS = \int_{S(t)} t_i dS + \int_{V(t)} \rho g_i dV, \quad i = 1, 2, 3.}} \tag{4.24}$$

In summary:

(a) the first integral expresses the momentum rate of change due to the momentum density rate of change,
(b) the second integral expresses the momentum rate of change due the growth rate of the blob,
(c) the third integral expresses the momentum rate of change due the surface force, and
(d) the fourth integral expresses the momentum rate of change due the volume force.

4.5 Force by fluid on construction

The integral formulation of momentum conservation is particularly useful to calculate forces on constructions due to fluid flow. As an example consider the typical problem of a bend in a piping system as depicted in Fig. (4.7). The cross section areas at the entrance and exit are A_1 and A_2, respectively, the mean velocities at the entrance and at the exit are $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, and the pressures at the entrance and exit are p_1 and p_2, respectively. The flow is steady, the fluid mass density ρ is assumed constant, and it is assumed that viscosity and gravity effects can be neglected. The question raised here is: what is the force exerted by the fluid on the bend?

To answer this question, we construct a blob in the bend, indicated by the dashed line, and apply the integral formulation of momentum conservation (Eq.(4.24)). Since the flow is steady the first integral is zero, and since gravity effects can be neglected the last integral is also zero:

$$\int_{S(t)} \rho u_i u_j n_j dS = \int_{S(t)} t_i dS, \quad i = 1, 2, 3. \tag{4.25}$$

Figure 4.6: Examples of forces by fluids

At the instant depicted in Fig. (4.7) the surface $S(t)$ of the convected blob can be decomposed into three parts:

$$S(t) = A_1 \cup A_2 \cup A_w, \tag{4.26}$$

where A_w represents the wall of the bend. Therefore

$$\int_{A_1} \rho u_i u_j n_j dS + \int_{A_2} \rho u_i u_j n_j dS + \int_{A_w} \rho u_i u_j n_j dS =$$
$$\int_{A_1} t_i dS + \int_{A_2} t_i dS + \int_{A_w} t_i dS, \quad i = 1, 2, 3. \tag{4.27}$$

the first integral over A_w is zero because on the wall the normal velocity is zero: $u_j n_j$=0. The second integral over A_w represents the force in i-direction by the wall on the fluid, say

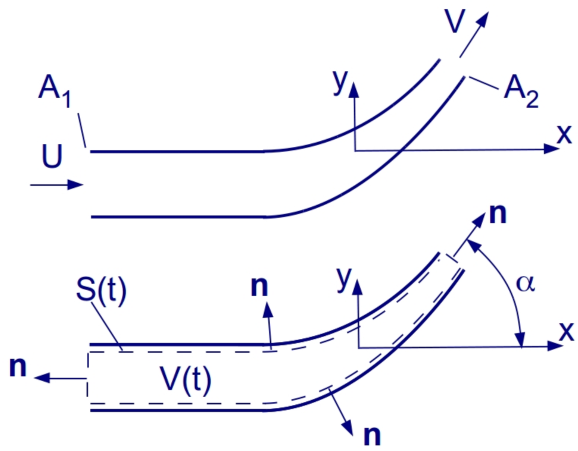


Figure 4.7: Typical problem: bend in a piping system.

F_i. Hence, the momentum equation becomes

$$\int_{A_1} \rho u_i u_j n_j dS + \int_{A_2} \rho u_i u_j n_j dS = \int_{A_1} t_i dS + \int_{A_2} t_i dS + F_i, \quad i = 1, 2, 3. \tag{4.28}$$

The force asked is the force by the fluid on the wall. We can calculate the components of this force by computing $-F_i$ with $i = 1, 2, 3$:

$$-F_i = -\int_{A_1} \rho u_i u_j n_j dS - \int_{A_2} \rho u_i u_j n_j dS + \int_{A_1} t_i dS + \int_{A_2} t_i dS, \quad i = 1, 2, 3. \tag{4.29}$$

The remaining problem is now to compute the four integrals at the right hand side.

We start by identifying the outward unit normal vectors at the entrance and exit:

$$\text{entrance: } \boldsymbol{n} = (-1, 0, 0)^T, \quad \text{exit: } \boldsymbol{n} = (\cos\alpha, \sin\alpha, 0)^T. \tag{4.30}$$

and the velocity vectors at the entrance and exit assuming uniform flow:

$$\text{entrance: } \boldsymbol{u} = (U, 0, 0)^T, \quad \text{exit: } \boldsymbol{u} = (V\cos\alpha, V\sin\alpha, 0)^T. \tag{4.31}$$

From these expressions we derive:

$$\text{entrance: } u_j n_j = -U, \quad \text{exit: } u_j n_j = V. \tag{4.32}$$

Furthermore, we use the Cauchy stress tensor to compute the stress vector $\boldsymbol{t}$ and use $\mu = 0$ since viscosity effects can be neglected:

$$t_i = \sigma_{ij} n_j = -p\delta_{ij} n_j = -pn_i. \tag{4.33}$$

With these expressions the three force components become

$$\begin{aligned}
-F_1 &= -\int_{A_1} \rho U(-U)dS - \int_{A_2} \rho V \cos\alpha(V)dS \\
&\quad + \int_{A_1} -p(-1)dS + \int_{A_2} -p\cos\alpha dS, \\
-F_2 &= -\int_{A_2} \rho V \sin\alpha(V)dS + \int_{A_2} -p\sin\alpha dS, \\
-F_3 &= 0.
\end{aligned} \tag{4.34}$$

which can be evaluated to

$$\begin{aligned}
-F_1 &= \left(\rho U^2 + p\right) A_1 - \left(\rho V^2 + p\right)\cos\alpha A_2, \\
-F_2 &= -\left(\rho V^2 + p\right)\sin\alpha A_2, \\
-F_3 &= 0.
\end{aligned} \tag{4.35}$$

Two standard checks are conducted now to verify that this answer is what we would expect and to trace any errors we may have made:

(a) are all of the terms in the answer of the right physical dimension?
(b) does the answer produce the correct limits?

The first check is easy, and it is left to the reader to show that indeed all of the terms in the answer of the physical dimension of force. The second check requires some insight to create some limiting cases for which we already now the answer. For example:

(a) due to symmetry, F_3 should be zero in all cases: ok.
(b) if $\alpha = 0$, F_2 should be zero: ok.
(c) if $\alpha = \pi$, F_2 should be zero: ok.
(d) if $\alpha = 0$, $A_1 = A_2$, and $U = V$, F_1 should be zero: ok.
(e) if $A_1 = A_2$, and $U = V$, F_1 should increase when U increases: ok.
(f) if $U = V = 0$, only pressure terms should remain: ok.
(g) when α changes sign, F_2 should change sign: ok.

4.6 Derivation of Navier-Stokes equations

We derive the differential formulation of momentum conservation from the integral formulation of momentum conservation Eq.(4.24) by replacing the surface integrals by a volume integral using Gauss' divergence theorem Eq.(3.15):

$$\int_{S(t)} \rho u_i u_j n_j dS = \int_{V(t)} \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) dV, \tag{4.36}$$

and

$$\int_{S(t)} t_i dS = \int_{S(t)} \sigma_{ij} n_j dS = \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j} dV. \tag{4.37}$$

Note that $\rho u_i u_j$ and σ_{ij} take the place of u_j in Eq.(3.15). With this replacement the integral form Eq.(4.24) becomes one single volume integral:

$$\int_{V(t)} \left(\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i \right) dV = 0, \quad i = 1, 2, 3. \tag{4.38}$$

Since the blob V was chosen completely arbitrary, this holds for any blob. This means that

$$\boxed{\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i = 0, \quad \text{for all } (\boldsymbol{x}, t)} \tag{4.39}$$

This the differential formulation of momentum conservation which is referred to as the Navier-Stokes equations.

4.7 Exercises

Problem 4.1. *By using the integral formulation of momentum conservation, show that the law of Archimedes (287 BC - 212 BC) holds: in water which is not flowing the (upward) force on a blob of water by the surrounding water is equal to the (downward) gravity force on the blob.*

Problem 4.2. *An incompressible fluid flows steadily into a T-junction of diameter D at uniform velocity U, at the opposite outlet the fluid leaves at uniform velocity V. At the lateral exit the flow leaves at unknown uniform velocity. The pressure in the T-junction is uniform: p. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.3. *An incompressible fluid flows steadily into a pipe of diameter D at uniform velocity U and pressure p_1. At the end of the pipe is a contraction of diameter d, and the fluid leaves the contraction at uniform velocity V and pressure p_2. Compute the force (in all directions) by the fluid on the pipe, neglect viscosity and gravity.*

Problem 4.4. *Incompressible water is flowing steadily through a $180°$ elbow. At the inlet the pressure is p_1 and the cross section area is A_1, at the outlet the pressure is p_2 and the cross section area is A_2. The averaged velocity at the inlet is V_1. Find the horizontal component of the force by the fluid on the elbow, neglecting viscosity and gravity.*

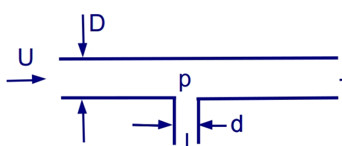
Problem 4.5. *An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h$ and width w. At the entrance the presure is $p1$ and the uniform velocity is U_1. At the exit the pressure is p_2 and the velocity distribution is*

$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2. \tag{4.40}$$

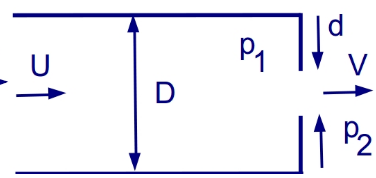
(a) *Derive an expression for the maximum velocity at the downstream section.*
(b) *Derive an expression for the force on the walls in x-direction, neglecting gravity, and neglecting viscosity at entrance and exit.*

Problem 4.6. *A small round object is tested in a wind tunnel with circular cross section with diameter D. The pressure is uniform across sections 1 and 2 and known: p_1 and p_2. At the entrance the uniform velocity is U. The velocity profile at section 2 is linear: it varies from zero at the tunnel centerline to a maximum at the tunnel wall. The viscosity effects on the wall of the wind tunnel can be neglected and the flow can be treated as incompressible.*

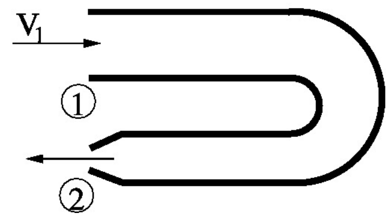
(a) *Derive an expression for the mass flow rate in the wind tunnel,*
(b) *Derive an expression for the maximum velocity at section 2*
(c) *Derive an expression for the drag of the object and its supporting vane.*



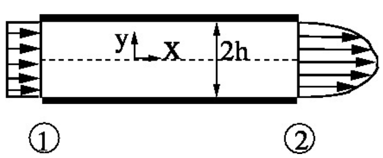
(a) Problem 4.2



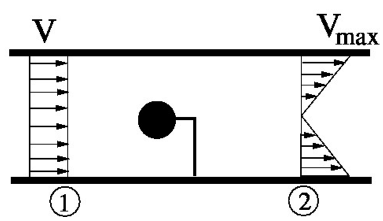
(b) Problem 4.3



(c) Problem 4.4



(a) Problem 4.5



(b) Problem 4.6

Chapter 5

Fully Developed Flow

5.1 Fully developed flow in slender ducts

Consider the flow between two infinite walls in Fig. (5.1).

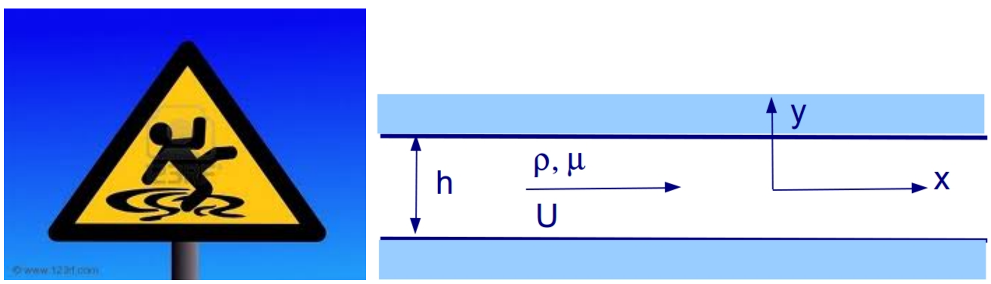


Figure 5.1: Flow between two walls: real world (left), schematic (right)

We make the following 5 assumptions:

(a) Two-dimensional flow: $\frac{\partial}{\partial z}(...) = 0$, and w=0,
(b) Incompressible flow: ρ is a constant,
(c) Fully developed flow: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial x} = 0$,
(d) Steady flow: $\frac{\partial}{\partial t}(...) = 0$, and
(e) No-slip at the boundaries: $u = 0$, $v = 0$ at $y = \pm\frac{h}{2}$.

The question is: what is the velocity field $\boldsymbol{u}(x, y)$ and what is the pressure field $p(x, y)$?

To determine the velocity and pressure fields we use the differential forms of the conservation equations for mass and momentum. The mass conservation equation, Eq.(3.18), becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.1}$$

The momentum conservation equations (the Navier-Stokes equations), Eq.(4.39), for $i = 1$ and $i = 2$ become

$$\frac{\partial}{\partial x}\left(\rho u^2 - \sigma_{11}\right) + \frac{\partial}{\partial y}\left(\rho uv - \sigma_{12}\right) = \rho g_1. \tag{5.2}$$

and

$$\frac{\partial}{\partial x}\left(\rho v u-\sigma_{21}\right)+\frac{\partial}{\partial y}\left(\rho v^{2}-\sigma_{22}\right)=\rho g_{2}. \tag{5.3}$$

From the fully developed flow assumption together with mass conservation and no-slip boundary conditions we have

$$\frac{\partial u}{\partial x}=0 \Rightarrow \frac{\partial v}{\partial y}=0 \Rightarrow v(x, y)=\text{const.} \Rightarrow v(x, y)=0. \tag{5.4}$$

Using the product rule of differentiation and the assumptions made, the velocity terms in the Navier-Stokes equations become:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\rho u^{2}\right)=\rho \frac{\partial}{\partial x}\left(u^{2}\right)=2 \rho u \frac{\partial u}{\partial x}=0, \\
&\frac{\partial}{\partial y}(\rho u v)=0, \\
&\frac{\partial}{\partial x}(\rho v u)=0, \\
&\frac{\partial}{\partial y}\left(\rho v^{2}\right)=0.
\end{aligned} \tag{5.5}$$

Using Eq.(4.10) and the assumptions made, the four stress tensor components in the Navier-Stokes equations become:

$$\begin{aligned}
\sigma_{11}&=-p+\tau_{11}=-p+2 \mu \frac{\partial u}{\partial x}=-p, \\
\sigma_{12}&=\tau_{12}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\mu \frac{\partial u}{\partial y}, \\
\sigma_{21}&=\tau_{21}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\mu \frac{\partial u}{\partial y}, \\
\sigma_{22}&=-p+\tau_{22}=-p+2 \mu \frac{\partial v}{\partial y}=-p.
\end{aligned} \tag{5.6}$$

With these expressions, and noting that $\frac{\partial^{2} u}{\partial x \partial y}=\frac{\partial^{2} u}{\partial y \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=0$, we obtain the so-called reduced Navier-Stokes equations:

$$\boxed{\frac{\partial p}{\partial x}-\mu \frac{\partial^{2} u}{\partial y^{2}}=\rho g_{1}, \qquad \frac{\partial p}{\partial y}=\rho g_{2}.} \tag{5.7}$$

5.2 Pressure gradient

The presure gradient is

$$\boldsymbol{\nabla} p \equiv\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)^{T}. \tag{5.8}$$

We will show that $\boldsymbol{\nabla} p$ is constant, in other words, is independent of x and y by showing that $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$ both are zero. From the first reduced Navier-Stokes equation we find

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\rho g_1 + \mu \frac{\partial^2 u}{\partial y^2}\right) = 0, \tag{5.9}$$

since u does not depend on x and all other terms are constants. From the second reduced Navier-Stokes equation we find

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}\left(\rho g_2\right) = 0. \tag{5.10}$$

So, indeed

$$\frac{\partial p}{\partial x} = \text{constant} \equiv \left(\frac{\partial p}{\partial x}\right)_o, \tag{5.11}$$

which means that pressure varies linearly with x. In addition, from the second reduced Navier-Stokes equation we directly find

$$\frac{\partial p}{\partial y} = \text{constant} \equiv \rho g_2, \tag{5.12}$$

which means that pressure varies linearly with y. Adding the two results we get

$$p(x, y) = p_o + \left(\frac{\partial p}{\partial x}\right)_o x + \rho g_2 y, \tag{5.13}$$

where p_o is the pressure at $x = 0$, $y = 0$. The question rising here is whether we can relate the value of $\left(\frac{\partial p}{\partial x}\right)_o$ to the velocity, viscosity, and density of the flow.

5.3 Velocity profile

Since the pressure gradient is a constant, we can integrate the first reduced Navier-Stokes equation twice with respect to y:

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right] y + c_1, \tag{5.14}$$

$$u(y) = \frac{1}{2\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right] y^2 + c_1 y + c_2. \tag{5.15}$$

To determine the two integration constants c_1 and c_2 we use the two boundary conditions at $y = \pm\frac{h}{2}$:

$$\begin{aligned} u(-\frac{h}{2}) &= \frac{1}{2\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right]\left(\frac{h}{2}\right)^2 - c_1\frac{h}{2} + c_2 = 0, \\ u(\ \frac{h}{2}) &= \frac{1}{2\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right]\left(\frac{h}{2}\right)^2 + c_1\frac{h}{2} + c_2 = 0, \end{aligned}$$

which leads to

$$c_1 = 0, \quad c_2 = -\frac{1}{2\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right]\left(\frac{h}{2}\right)^2. \tag{5.16}$$

Hence, we find for the velocity $u(y)$:

$$\boxed{u(y) = \frac{1}{2\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right]\left[y^2 - \left(\frac{h}{2}\right)^2\right].} \tag{5.17}$$

5.4 Pressure-velocity relation

The mean normal velocity through the gap between the two plates is defined as

$$U \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} u(y)dy. \tag{5.18}$$

Since the $u(y)$ is a known function, see Eq.(5.17), we can evaluate the integral, which represents the average velocity:

$$\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} u(y)dy = -\frac{1}{12}\frac{h^2}{\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right]. \tag{5.19}$$

Hence, we have found the following relation

$$U = -\frac{1}{12}\frac{h^2}{\mu}\left[\left(\frac{\partial p}{\partial x}\right)_o - \rho g_1\right] \quad \Leftrightarrow \quad \left(\frac{\partial p}{\partial x}\right)_o = -12\frac{\mu U}{h^2} + \rho g_1. \tag{5.20}$$

5.5 Stress at boundaries

Suppose we want to calculate the stress on the lower wall. The stress actually consists of a vector $\boldsymbol{t}$, the stress vector, which satisfies Eq.(4.8). We take the normal vector $\boldsymbol{n}$ pointing from the wall to the fluid since we want to know the stress caused by the fluid on the lower wall:

$$\boldsymbol{n} = (0,1,0)^T \Rightarrow t_i(x,y) = \sigma_{ij}(x,y)n_j = \sigma_{i2}(x,y) \Rightarrow \boldsymbol{t} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \tag{5.21}$$

Using the stress components calculated in Eq.(5.6) we see that the stress component in the x-direction is:

$$t_1(y) = \sigma_{12}(y) = \mu\frac{\partial u}{\partial y}. \tag{5.22}$$

Using the velocity distribution Eq.(5.15) and Eq.(5.20) we find:

$$t_1(-\frac{h}{2}) = 6\frac{\mu U}{h}. \tag{5.23}$$

In a similar way we find

$$t_2(y) = \sigma_{22}(y) = -p(y), \quad t_3(y) = \sigma_{32}(y) = 0, \tag{5.24}$$

and

$$t_2(-\frac{h}{2}) = -p, \quad t_3(-\frac{h}{2}) = 0. \tag{5.25}$$

In summary, the stress vector exerted by the fluid on the lower wall is

$$\boldsymbol{t}(-\frac{h}{2}) = (6\frac{\mu U}{h}, -p)^T. \tag{5.26}$$

In this case, the first component, t_1, which is aligned with the wall, is called the shear stress, whereas the second component, t_2, which is oriented normal to the wall, is called the normal stress.

Note that the first component does not depend on x, but, in contrast, the second component does depend linearly on x. It is left as an exercise to the reader to show and explain that the stress vector on the upper wall is

$$\boldsymbol{t}(\frac{h}{2}) = (6\frac{\mu U}{h}, p)^T. \tag{5.27}$$

5.6 Alternative boundary conditions

Two alternative boundary conditions are treated here. First, consider a moving lower wall instead of a fixed wall, and suppose that the wall velocity is U_w. Then the no-slip boundary condition on the lower wall becomes simply

$$u(x, -\frac{h}{2}) = U_w. \tag{5.28}$$

Secondly, consider the flow of a film of water, i.e., remove the upper wall and replace it by air. The shear stress exerted by the water on the air is given by $-\left(\mu\frac{\partial u}{\partial y}\right)_{\text{water}}$ while the shear stress exerted by the air on the water is given by $\left(\mu\frac{\partial u}{\partial y}\right)_{\text{air}}$. When we consider an infinitesimal volume containing both water and air, it becomes clear that both shear stresses need to be equal in absolute value to prevent an infinite acceleration of the volume:

$$\left(\mu\frac{\partial u}{\partial y}\right)_{\text{water}} = \left(\mu\frac{\partial u}{\partial y}\right)_{\text{air}}. \tag{5.29}$$

Since the viscosity coefficient of water is much larger than that of air, we have

$$\mu_{\text{water}} \gg \mu_{\text{air}} \quad \Rightarrow \quad \left(\frac{\partial u}{\partial y}\right)_{\text{water}} \ll \left(\frac{\partial u}{\partial y}\right)_{\text{air}}. \tag{5.30}$$

Therefore, when the velocity derivative in the air is moderate, we can approximate

$$\left(\frac{\partial u}{\partial y}\right)_{\text{water}} \approx 0. \tag{5.31}$$

This equation can be used as the boundary condition at the upper boundary.

5.7 Fully developed flow in slender pipes

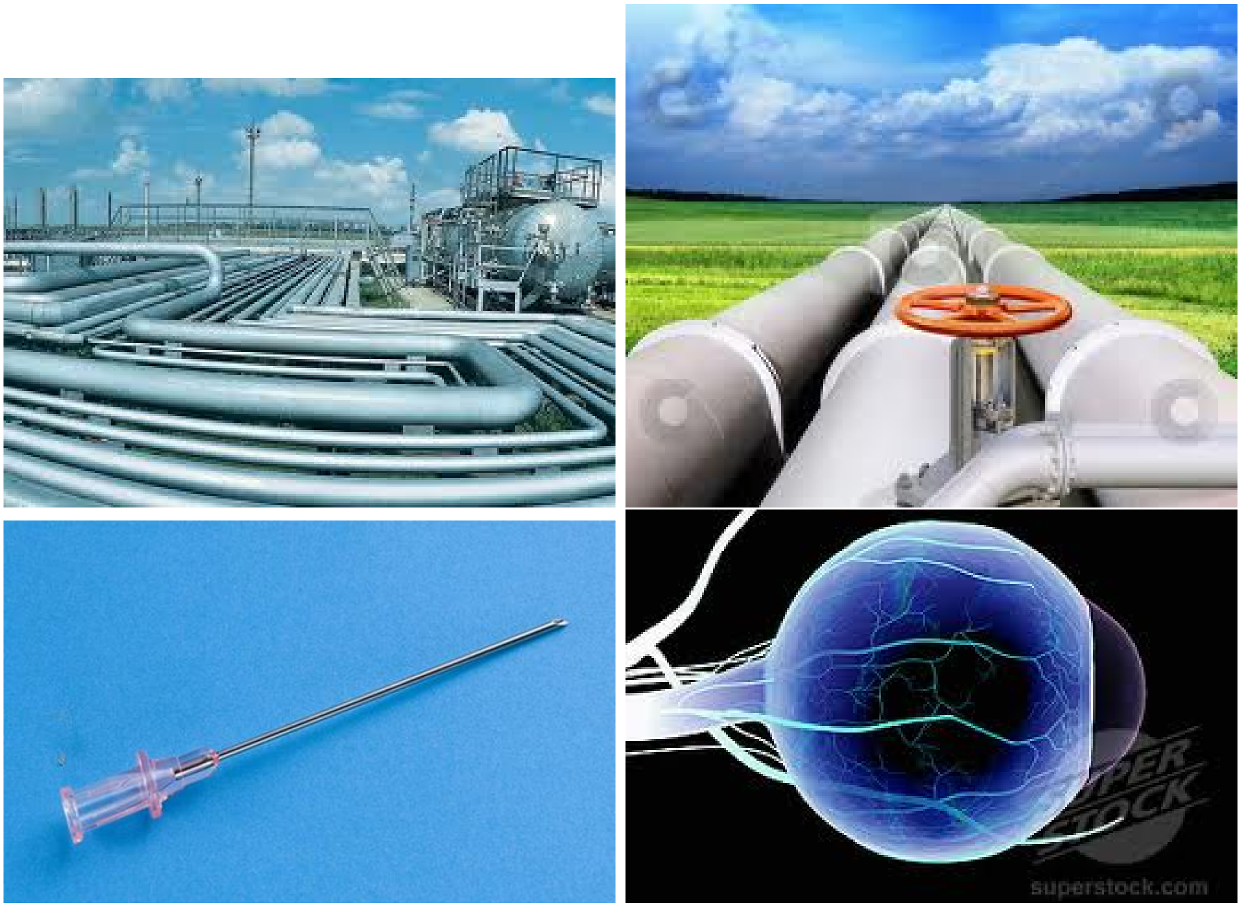


Figure 5.2: Flow in slender pipes

We can extend the developed flow analysis to pipes; think of refineries, engine cooling, hydraulics, injection needles, lung tubes, blood vessels, etc, etc. We start by transforming the partial differential equations from cartesian coordinates (x, y, z) to cylinder coordinates (r, θ, z):

$$x = x, \quad y = r\cos(\theta), \quad z = r\sin(\theta). \tag{5.32}$$

The velocities in the directions of x, r, and θ will be denoted by u, u_r, and u_θ, respectively (see Fig. (5.3):

$$\begin{aligned} u &= u, & (5.33)\\ u_r &= v\cos(\theta) + w\sin(\theta), & (5.34)\\ u_\theta &= -v\sin(\theta) + w\cos(\theta). & (5.35) \end{aligned}$$

We will make the following assumptions:

(a) Axi-symmetric flow: $\frac{\partial}{\partial\theta}(...) = 0$, $u_\theta = 0$,

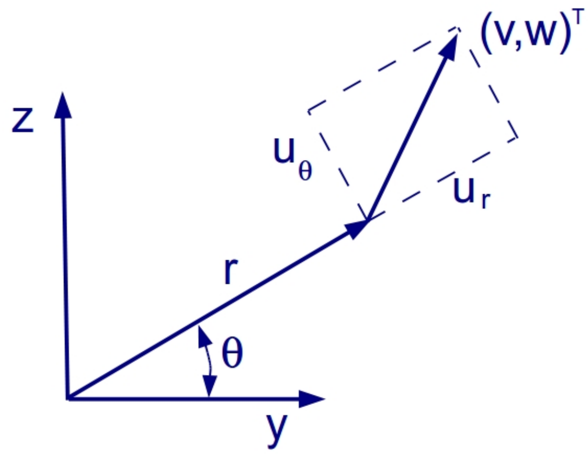


Figure 5.3: Transformation to cylindrical coordinates

(b) Incompressible flow: ρ is a constant,
(c) Fully developed flow: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u_\theta}{\partial x} = 0$,
(d) Steady flow: $\frac{\partial}{\partial t}(...) = 0$,
(e) No-slip at the boundary: $\boldsymbol{u} = 0$ at $r = R$, and
(f) Zero gravity: $g = 0$.

In this case the reduced Navier-Stokes equations become:

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right), \tag{5.36}$$

$$\frac{\partial p}{\partial r} = 0. \tag{5.37}$$

It can be shown that $\left(\frac{\partial p}{\partial x}\right)$ is constant, say $\left(\frac{\partial p}{\partial r}\right)_o$, and with the no-slip boundary condition the solution becomes

$$\boxed{u(r) = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)_o \left(R^2 - r^2\right).} \tag{5.38}$$

This solution is refered to as Hagen-Poiseuille flow.

5.8 Exercises

Problem 5.1. *Incompressible viscous oil flows steadily between stationary parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The oil viscosity is μ and the pressure drop over a distance L is Δp.*

(a) Derive an expression for the shear stress on the upper plate.
(b) Derive an expression for the volume flow rate through the channel over a width w.

Figure 5.4: The flow was discovered independently by Gotthilf Heinrich Ludwig Hagen (1797 - 1884, left), who was a German physicist and hydraulic engineer, and by Jean Louis Marie Poiseuille (1797 - 1869, right), who was a French physician and physiologist.

(c) *Compute the shear stress on the upper plate and the volume flow rate through the channel over a width w if $h = 5\,mm$, $\Delta p = -1000\,Pa$, $L = 1m$, $\mu = 0.5\,Ns/m^2$.*

Hint: *first derive an expression for the velocity field starting from the reduced Navier-Stokes equations.*

Problem 5.2. *An incompressible fluid of density ρ flows steadily between two parallel plates. The flow is laminar and fully developed, the viscosity is μ, the mean velocity is U, and the distance between the plates is h. Divide the flow into two horizontal layers, with the divide located at a distance y above the lower plate. Derive an expression for the shear stress experienced by the lower layer as a function of y, and sketch this function.*

Problem 5.3. *An hydraulic jack (NL: krik) supports a load of mass M. The diameter of the piston is D, the radial clearance between the piston and the cylinder is d, the length of the piston is L, and the viscosity of the oil is μ.*

(a) *Derive an expression for the pressure-drop in the gap between the piston and the cylinder.*
(b) *Derive a formula for the rate of leakage of hydraulic fluid past the piston. Compute the leakage rate when $M = 9000\,kg$, $D = 100\,mm$, $d = 0.05\,mm$, $L = 120\,mm$, and $\mu = 2 \times 10^{-1}\,Ns/m^2$.*

Hint: *approximate the gap between the piston and the cylinder as the gap between two flat plates (why would this be a very good approximation?). First compute the vertical pressure derivative by assuming the piston to be in equilibrium (moves extremely slowly due to the leakage).*

Problem 5.4. *Consider the steadily falling water film along a vertical wall with thickness a. The flow is incompressible, laminar, and fully developed. At the wall the velocity is zero, whereas at the outer edge of the film the shear stress is zero.*

(a) Defend the approximation assumption of zero shear stress at the film boundary.
(b) Derive an expression for $\frac{\partial p}{\partial x}$.
(c) Derive an expression for $u(y)$.

Problem 5.5. *An incompressible fluid flows steadily between two parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The upper plate moves to the right with speed U_2, the lower plate moves to the left with speed U_1. The pressure gradient in the direction of the flow is zero.*

(a) Derive an expression for the velocity distribution in the gap.
(b) Derive an expression for the volume flow rate per unit depth.

Problem 5.6. *An incompressible fluid flows steadily between two parallel plates. The flow is laminar and fully developed. The total gap width between the plates is h. The upper plate moves to the right with speed U, the lower plate is fixed.*

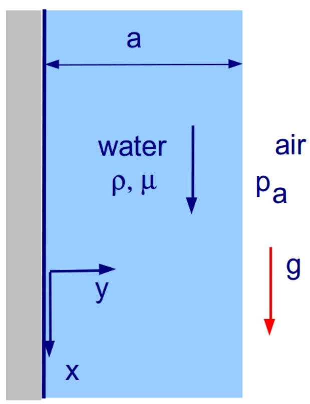
(a) Derive an expression for the pressure gradient at which the upper plate experiences zero shear stress.
(b) Derive an expression for the pressure gradient at which the lower plate experiences zero shear stress.

Problem 5.7. *The record-read head for a computer disk-drive memory storage system rides above the spinning disk on a very thin film of air (the film thickness is h). The head location is a from the disk centerline; the disk spins at angular velocity Ω. The surface area of the record-read head is A. Finally, the viscosity of air is μ and the density is ρ.*

(a) Derive an expression for the Reynolds number of the flow.
(b) Derive an expression for the shear stress.
(c) Derive an expression for the power required to overcome the viscous shear stress.
(d) Compute the values of the three expressions if $h = 0.5\,\mu m$, $a = 150\,mm$, $\Omega = 3600\,rpm$, and $A = 100\,mm^2$, $\mu = 18.0 \times 10^{-6}\,kg/ms$, and $\rho = 1.2\,kg/m^3$.

Problem 5.8. *Consider fully developed laminar incompressible flow in a pipe.*

(a) Derive an expression for the average velocity in a cross-section.
(b) Transform the previous expression to obtain a formula for the pressure gradient as a function of (amongst others) the average velocity.



(a) Problem 5.4

Chapter 6

Reynolds number

6.1 Dimensional Analysis: Introduction by Example

Consider a pendulum of length ℓ with a weight m attached to it. The pendulum is released from an an angle θ_o and the question addressed here is: how much time does it take for the pendulum to swing back and forth? We will call the time needed the period T. Before actually solving this problem one can already say a lot about how the solution will look like. It is obvious that T is a function of the parameters mentioned and the gravitational constant g:

$$T(\ell, m, g, \theta_o). \tag{6.1}$$

The period can be made non-dimensional by using ℓ and g:

$$\tilde{T} = T / \sqrt{\frac{\ell}{g}}. \tag{6.2}$$

An extremely important notion is given by **Buckingham's theorem**:

Theorem 6.1 (Buckingham)**.** *A dimensionless function depends on dimensionless arguments only.*

This means that $\tilde{T}$ only depends on combinations of ℓ, m, g and θ_o that are non-dimensional. So the question arises whether such combinations exist. To investigate this define the group

$$\Pi \equiv \ell^\alpha m^\beta g^\gamma \theta_o^\delta, \tag{6.3}$$

and try to find values for α, β, γ and δ such that Π is non-dimensional. First determine the physical dimension of all parameters:

$$[\ell] = m, \quad [m] = kg, \quad [g] = \frac{m}{s^2}, \quad [\theta_o] = 1. \tag{6.4}$$

The dimension of θ_o is 1 because it is non-dimensional. As a result the physical dimension of Π becomes:

$$[\Pi] = m^\alpha kg^\beta \left(\frac{m}{s^2}\right)^\gamma 1^\delta = m^{\alpha+\gamma} kg^\beta s^{-2\gamma}. \tag{6.5}$$

To make the group Π non-dimensional, i.e. $[\Pi] = 1$, the powers of the independent dimensions m, kg and s must be zero:

$$\alpha + \gamma = 0, \quad \beta = 0, \quad -2\gamma = 0, \quad \Rightarrow \quad \alpha = \beta = \gamma = 0. \tag{6.6}$$

Hence, the only possibility to construct Π as a non-dimensional group is taking $\Pi = \theta_o^\delta$ with δ an arbitrary number. Taking $\delta = 1$ one gets:

$$\Pi = \theta_o, \tag{6.7}$$

and the expression for the period T can now be written as

$$T(\ell, m, g, \theta_o) = \tilde{T}(\theta_o)\sqrt{\frac{\ell}{g}}. \tag{6.8}$$

From linear theory for small values of θ_o it is known that $\tilde{T}(\theta_o) = 2\pi$. For larger values of θ_o there exist no closed fomulation for $\tilde{T}$.

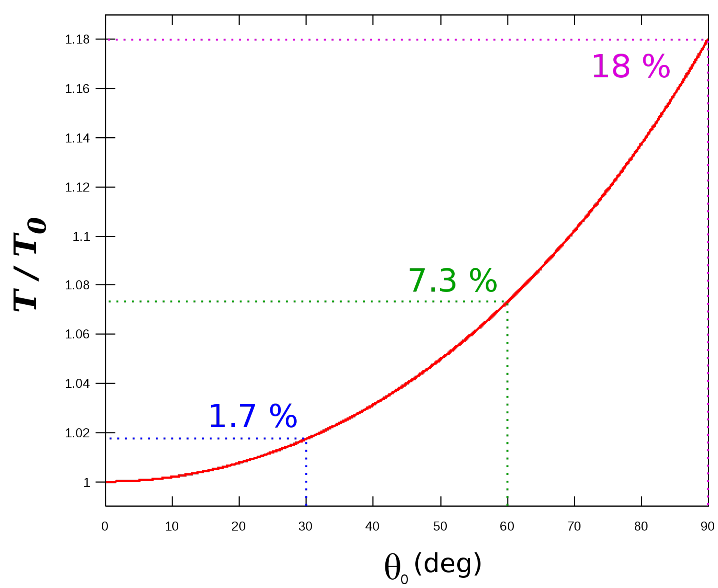


Figure 6.1: Pendulum period $\tilde{T}$ divided by 2π as a function of θ_o.

What is the benefit of this dimension analysis? Well, suppose one wants to determine the period T from experiments, then Eq.(6.1) shows that a lot of experiments are necessary with all kinds of pendula: different lengths ℓ, different weights m, different release angles θ_o and if one wants to be very accurate, different locations on earth to account for variations in the gravitational constant g. To make a high resolution data base of T would therefore be a very costly and labour intensive enterprise.

The dimension analysis however demonstrates that first of all the mass m is irrelevant for the value of T. Furthermore, it shows that one can use one pendulum at one location and only vary the release angle θ_o to make a data base of $\tilde{T}$ instead of T! The period T is then easily calculated from Eq.(6.8). This evidently means an enormous reduction of costs and effort and also makes it very easy to present the period T in reports: a simple graph of $\tilde{T}$ is sufficient.

6.2 Dimension Analysis: General Approach

The general approach to dimension analysis consists of the following steps:

(a) Identify the **dependent variable** asked for,
(b) identify M, the **number of independent parameters** that determine the variable asked for,
(c) identify N, the **number of independent physical dimensions** present,
(d) Choose N **dimensionally independent** parameters, and
(e) Scale the **dependent variable** and the remaining **independent parameters**.

In case of the pendulum these steps would result in

(a) T,
(b) ℓ, m, g, θ_o, so $M = 4$,
(c) m, kg, s, so $N = 3$,
(d) ℓ, m, g
(e) 1) $\tilde{T} = T/\sqrt{l/g}$
2) θ_o is already non-dimensional, does not havespeher to be scaled.

which then gives Eq.(6.8). It is noted that **the number of non-dimensional parameters is equal to** $M - N$.

6.3 Dimensional analysis: Reynolds number

On the one hand, looking at the fully developed flow results of the previous chapter, we could have predicted that any calculated quantity of the flow, say Q, is a function of the variables specified:

$$Q = F(\rho, U, h, \mu), \tag{6.9}$$

simply because nothing else has been specified. The quantiy Q could be for example $\left(\frac{\partial p}{\partial x}\right)_o$. It seemes that finding an expression for the pressure gradient means looking fo a function of four variables. If such function cannot be determined analytically the only way out is doing numerical experiments. Suppose we would measure the pressure derivative for 10 different values of the four variables, that would mean a set of 10^4 tests! And not only that, we would also have to test with probably a 100 different fluids to obtain satisfactory combinations af different values of density and viscosity. Luckily there is a way out of this which reduces the four-dimensional problem to a one-dimensional problem: dimensional analysis.

In the present case, the four parameters ρ, U, h, and μ contain three fundamental physical dimensions:

$$[\rho] = \frac{M}{L^3}, \quad [U] = \frac{L}{T}, \quad [h] = L, \quad [\mu] = \frac{M}{LT}, \tag{6.10}$$

where [...] means physical dimension, M denotes mass, L denotes length, and T denotes time. This means that we can multiply powers of the parameters to construct any physical dimension, as long as it is a combination of mass, length, and time:

$$[\rho^a U^b h^c \mu^d] = M^{a+d} L^{-3a+b+c-d} T^{-b-d}. \tag{6.11}$$

Suppose we want to construct in this way a physical dimension $M^x L^y T^z$, then we have to solve the following linear system of equations:

$$a + d = x, \tag{6.12}$$
$$-3a + b + c - d = y, \tag{6.13}$$
$$-b - d = z. \tag{6.14}$$

Obviously, this system has infinitely many solutions since we have four unknowns a, b, c, and d, and only three equations. Therfore, the conclusion is that we only need three parameters to construct an arbirary physical dimension. Lets drop μ and retain ρ, U, and h, then

$$[\rho^a U^b h^c] = M^a L^{-3a+b+c} T^{-b}, \tag{6.15}$$

and the linear system becomes

$$a = \qquad x, \tag{6.16}$$
$$-3a + b + c = \qquad y, \tag{6.17}$$
$$-b = \qquad z, \tag{6.18}$$

which can be written as a matrix vector multiplication,

$$A\boldsymbol{a} = \boldsymbol{x}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{6.19}$$

From linear algebra we know that this system has a unique solution if and only if the determinant of the matrix A is not zero. [(1)]

The determinant of A is 1 so the system does have a unique solution which is

$$a = x, \quad b = -z, \quad c = 3x + y + z. \tag{6.20}$$

[(1)] If the determinant of A is not zero it means that the physical dimensions of the three parameters ρ, U, and h are dimensionally independent. In other words, $A\boldsymbol{a} = 0$ has unique solution $\boldsymbol{a} = 0$ which means that the three parameters cannot be combined into a dimensionless quantity other that taking all exponents equal to zero which is obviously irrelevant.

What can we do with this? Let's take for example $Q = \left(\frac{\partial p}{\partial x}\right)_o$ and let us try to find a combination $\rho^a U^b h^c$ such that if we divide Q by this combination we get a dimensionless quantity. The dimension of $\left(\frac{\partial p}{\partial x}\right)_o$ is

$$\left[\left(\frac{\partial p}{\partial x}\right)_o\right] = \frac{M}{L^2 T^2}, \tag{6.21}$$

so

$$x = 1, \quad y = -2, \quad z = -2, \tag{6.22}$$

and, hence by Eq.(10.20) we obtain:

$$a = 1, \quad b = 2, \quad c = -1 \quad \Rightarrow \quad \rho^a U^b h^c = \frac{\rho U^2}{h}. \tag{6.23}$$

This means that when we divide $\left(\frac{\partial p}{\partial x}\right)_o$ by this combination we obtain a dimensionless quantity:

$$\left[\frac{h}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o\right] = M^0 L^0 T^0 = 1. \tag{6.24}$$

Following Buckinghams theorem, the dimensionless pressure derivative derived above depends on dimensionless quanities only. So, the question is: how many dimensionless quantities can we produce with the four parameters ρ, U, h, and μ? In other words, what are possible solutions of

$$[\rho^a U^b h^c \mu^d] = 1. \tag{6.25}$$

As mentioned before, this system has infinitely many solutions since there are four unknowns and only three equations. Therefore we treat one of the unknowns as 'known', say d, and solve for the three remaining unknowns:

$$[\rho^a U^b h^c] = [\mu^{-d}] = M^{-d} L^d T^d, \tag{6.26}$$

which leads to

$$a = -d, \quad b = -d, \quad c = -d. \tag{6.27}$$

which means that the infinitely many ways we can construct a dimensionless quantity out of the four parameters ρ, U, h, and μ is $(\rho U h/\mu)^{-d}$. The term between brackets is called the Reynolds number Re:

$$\boxed{Re \equiv \frac{\rho U h}{\mu}.} \tag{6.28}$$

So, as a result, we have derived that

$$\frac{h}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o \equiv \Phi(Re), \tag{6.29}$$

Figure 6.2: Osborne Reynolds (1842 - 1912) was a prominent innovator in the understanding of fluid dynamics. Separately, his studies of heat transfer between solids and fluids brought improvements in boiler and condenser design.

where Φ is an unknown dimensionless function and where we have absorbed the unknown value of d into the fact that Φ is unknown. Putting it slightly differently, we have revealed that the pressure derivative can be written in the following form:

$$\left(\frac{\partial p}{\partial x}\right)_o = \Phi(Re)\frac{\rho U^2}{h}. \tag{6.30}$$

The implication of this result is tremendous! By the above analysis, which is called dimension analysis, we have succeeded in reducing the four-dimensional problem of finding the pressure derivative to a one-dimensional problem of finding $\Phi(Re)$. In other words, instead of finding $\left(\frac{\partial p}{\partial x}\right)_o$ as a function of the four physical variables ρ, U, h, and μ, we have transformed the problem into finding the function Φ as a function of one variable: the Reynolds number Re. This means that we do not have to do 10,000 measurements but only 10. Or, if we require more accuracy, maybe 100. It also means that we do not have to measure with many different fluids, water will do! The same is true for the distance between the two plates, one value will do. This is due to the fact that we can vary the Reynolds number by simply varying the velocity of the flow, that's all.

It is noted that we haven't used the describing equations to perform the dimension analysis! The analysis is solely based on gathering all of the parameters that determine the quantity we are interested in and analyse their dimensions. The tricky part lies in the fact that we have to be sure that we have gathered all relevant parameters of the problem, this relies on physical insight.

For the gap flow

$$\Phi = -\frac{12}{Re}, \tag{6.31}$$

which is illustrated in Fig. (6.3)

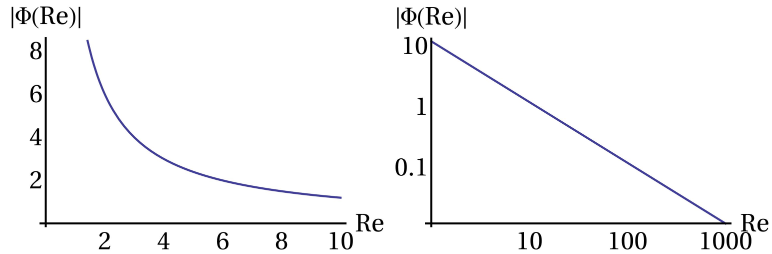


Figure 6.3: Dimensionless pressure derivative for gap flow as function of Reynolds number

For pipe flow the dimensionless pressure derivative is defined as

$$\frac{D}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)_o \equiv \Psi(Re_D), \qquad D \equiv 2R, \tag{6.32}$$

with

$$\boxed{Re_D \equiv \frac{\rho U D}{\mu}.} \tag{6.33}$$

This gives

$$\Psi = -\frac{32}{Re_D}, \tag{6.34}$$

which is illustrated in Fig. (6.7)

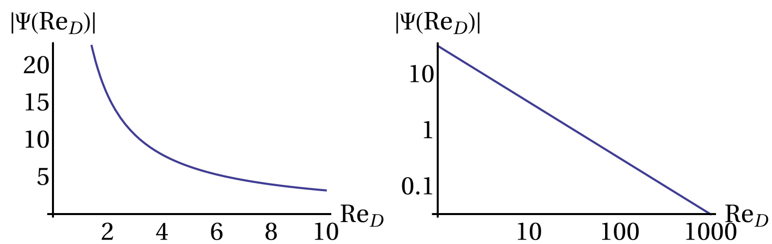


Figure 6.4: Dimensionless pressure derivative for pipe flow as function of Reynolds number

6.4 Turbulence

In the previous sections we have implicitly assumed that the flow was laminar, meaning in the case of fully developed gap flow that the flow is stationary. It is well known that when the velocity of the fluid or the gap width are increased, stationary flow does not exist anymore. Instead, the flow becomes unsteady, comprising a distribution of vortices with a large range of scales. Such flow is called turbulent. The larger vortices break up into smaller ones which again break up into even smaller ones and so on. In the end the smallest vortices vanish and their energy is released as heat. We call this energy dissipation.

The transition from laminar to turbulent flow, when looking at the dimensionless flow variables, clearly only depends on the Reynolds number: there is simply no other parameter left. The value of the Reynolds number at which transition occurs is called the critical value. Below this value, small perturbations with respect to the laminar flow solution (which are always there, imagine someone closing the door in the experiment room or a car driving by the building etc.) are damped. Above the critical value, however, perturbations are amplified and grow, leading to turbulent flow.

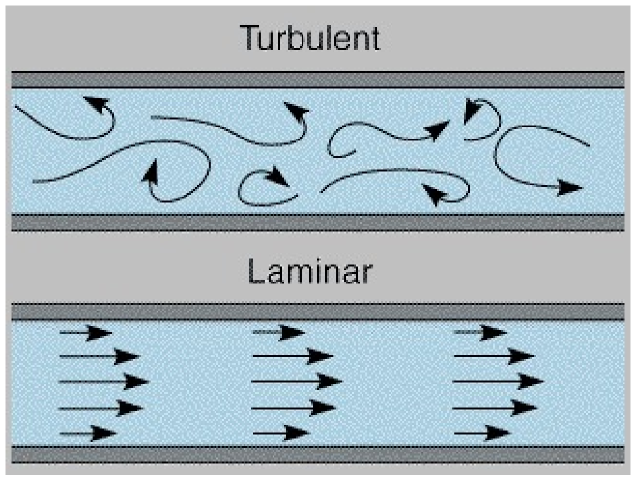


Figure 6.5: Laminar and turbulent flow regimes in a tube

6.5 Friction factor and Moody charts

In standard fluid mechanics text books it is custom to write the pressure derivative in fully developed pipe flow as

$$\left(\frac{\partial p}{\partial x}\right)_o = -f(Re_D)\,\frac{1}{2}\rho U^2/D, \tag{6.35}$$

so, in terms of the factor Ψ:

$$f(Re_D) = -2\Psi(Re_D). \tag{6.36}$$

The function $f(Re_D)$ is the called the Darcy-Weisbach friction factor. [2] Note that U is taken positive in the positive x-direction which corresponds to a negative pressure derivative. The minus sign choosen in front of the friction factor in that case ensures that f is a positive number.

Until now we have not included effects of wall roughness, we have considered perfectly smooth pipes only. Suppose now we have a pipe with a non-smooth wall characterized by

[2] The equation is named after Henry Darcy and Julius Weisbach. Henry Philibert Gaspard Darcy (1803 - 1858) was a French engineer, and Julius Ludwig Weisbach (1806 - 1871) was a German mathematician and engineer.

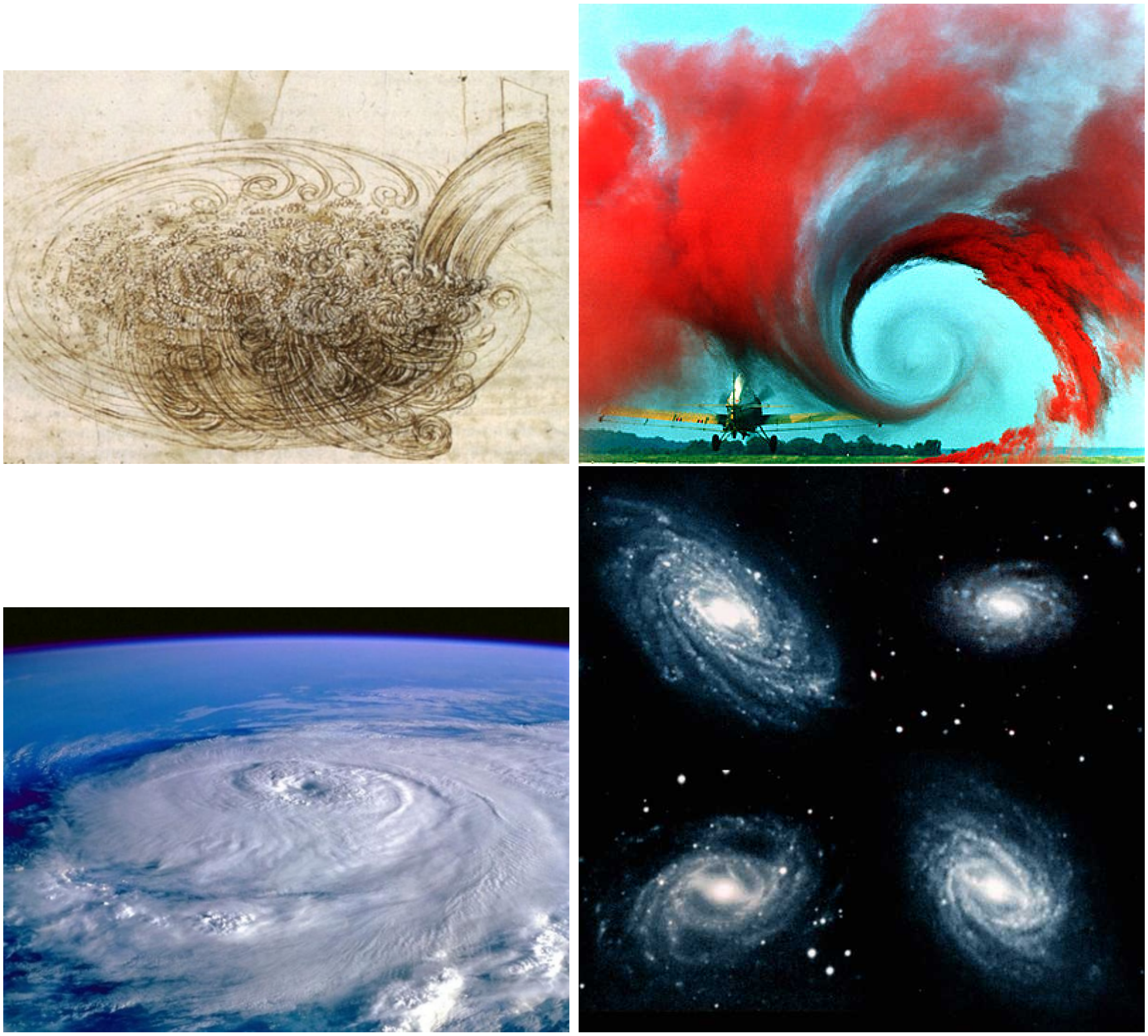


Figure 6.6: Examples of vortices on a range of scales

a mean wall roughness δ, which has the dimension of length. That means that the problem of finding the pressure derivative includes an additional parameter and dimension analysis will produce therefore an additional dimensionless number. When we scale with ρ, U, and D, the only possibility to scale δ is by dividing by D. Therefore, we define

$$\epsilon \equiv \frac{\delta}{D}. \tag{6.37}$$

as the additional dimensionless parameter. This means that the friction factor also becomes a function of ϵ:

$$\left(\frac{\partial p}{\partial x}\right)_o = -f(Re_D, \epsilon)\,\frac{1}{2}\rho U^2/D. \tag{6.38}$$

The friction factor in this form was first summarized in a single plot by Lewis Ferry Moody (1880 - 1953), and therefore the diagram is called the Moody diagram or Moody chart. [3] For laminar flow regime the friction factor is independent of ϵ,

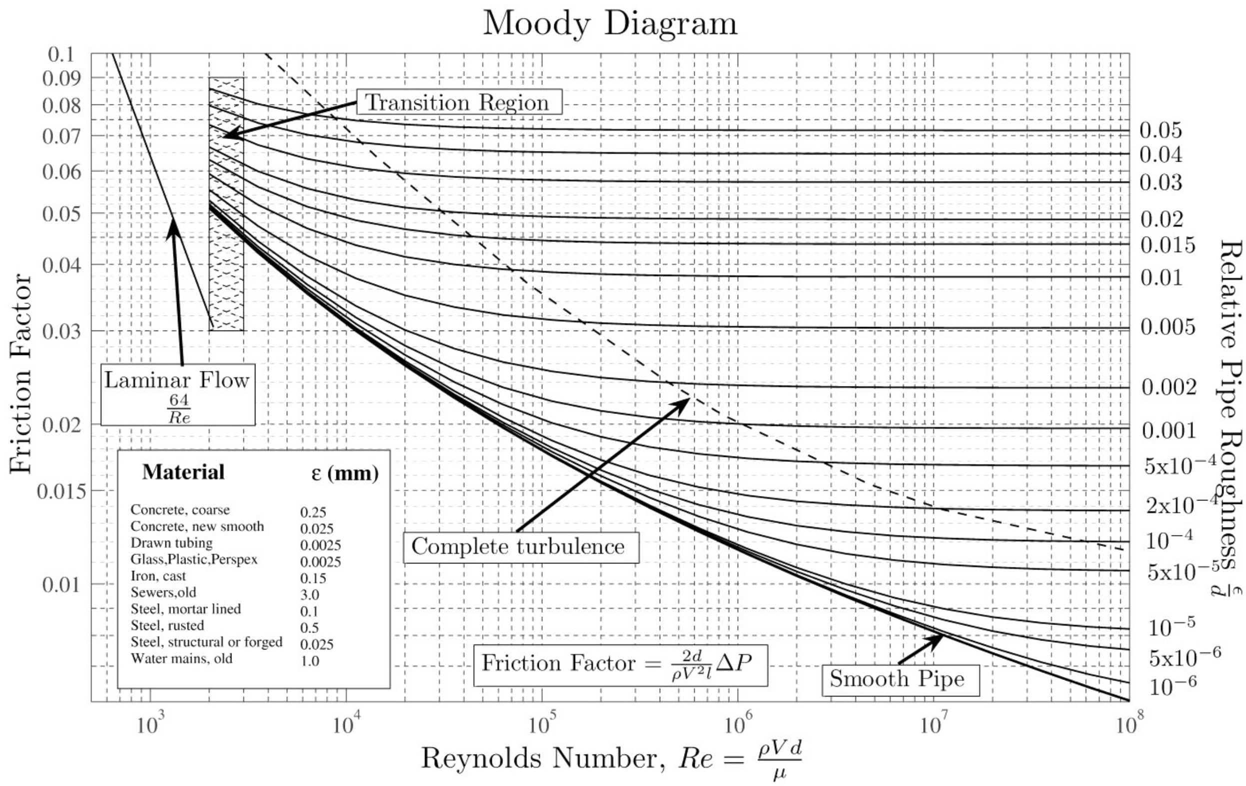


Figure 6.7: Moody chart: Darcy-Weisbach friction factor for laminar and turbulent fully developed pipe flow as function of Reynolds number Re_D and relative wall roughness ϵ.

$$f(Re_D, \epsilon) = \frac{64}{Re_D}, \quad \text{laminar flow} \tag{6.39}$$

which is just the expression found for Poiseuille flow, Eq.(5.38), whereas for the turbulent flow regime the friction factor approximately satisfies the Colebrook equation:

$$\frac{1}{\sqrt{f}} = -2\,\log_{10}\left(\frac{\epsilon}{3.7} + \frac{2.51}{Re_D\sqrt{f}}\right), \quad \text{turbulent flow.} \tag{6.40}$$

It is noted that this equation is implicit in f, in other words, it cannot be solved for f analytically.

---

[3] Lewis Ferry Moody (1880 - 1953), was an American engineer and professor, he was the first Professor of Hydraulics in the School of engineering at Princeton.

6.6 Exercises

Problem 6.1. *Determine the dimensions of force F, stress σ, power $\dot{W}$, dynamic viscosity μ and thermal conductivity k.*

Problem 6.2. *The variables which control the motion of a boat are the resistance force, F, speed V, length L, density of the liquid ρ and its viscosity μ, as well as gravity acceleration g. Obtain an expression for F using dimensional analysis.*

Problem 6.3. *It is believed that the power P of a fan depends upon the density of the liquid ρ, the volumetric flux Q, the diameter of the propeller D and the angular speed Ω. Using dimensional analysis, determine the dependence of P with respect to the other dimensionless variables.*

Problem 6.4. *In fuel injection systems, a jet of liquid breaks, forming small drops of fuel. The diameter of the resulting drops, d, supposedly depends upon the density of the liquid ρ, the viscosity μ, surface tension σ (force/length), and also upon the speed of the stream V and its diameter D. How many dimensionless parameters are required to characterize the process? Find them.*

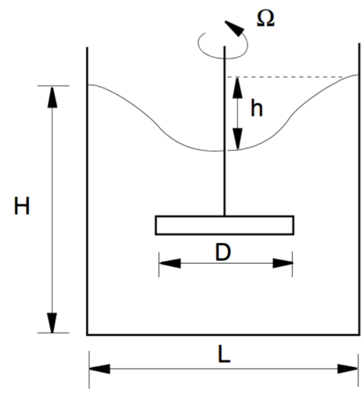
Problem 6.5. *A disc spins close to a fixed surface. The radius of the disc is R, and the space between the disk and the surface is filled with a liquid of viscosity μ. The distance between the disc and the surface is h and the disc spins at an angular velocity ω. Determine the functional relationship between the torque that acts upon the disc, T, and the other variables.*

Problem 6.6. *The drag force, F, experienced by a submarine that moves at a great depth from the surface of the water, is a function of the density ρ, viscosity μ, speed V and the transversal area of the submarine A. An expert suggests that the nondimensional relationship that allows the calculation of F is: $F = f(\frac{\rho V A}{\mu})\rho V^2 A$.*

(a) *Is the number of dimensionless parameters in the expression correct? Why?*
(b) *Are the parameters correct? If not, correct them.*
(c) *A geometrically similar model to that of the real submarine has been constructed, so that all the lengths of the model are 1/10 of those corresponding to the submarine. The model is tested in sea water. (1) The force of the real submarine when it moves at 5 m/s is to be de- termined. (2) At which speed should the model be tested?*

Problem 6.7. *An automobile must travel through standard air conditions at a speed of $100km/h$. To determine the pressure distribution, a model at a scale of 1/5 of the length of the vehicle is tested in water. Find the speed of water to be used.*
$\mu_{water} = 10^{-3}kg/(ms)$, $\rho_{water} = 1000kg/m^3$, $\mu_{air} = 1.8 \times 10^{-5}kg/(ms)$, $\rho_{air} = 1.2kg/m^3$.

Problem 6.8. *The depth of the steady central vortex h in a large tank of oil being stirred by a propeller needs to be predicted. One way is to carry out a study using a reduced scale model. Determine the conditions under which the experiment should be conducted to be considered a valid predictive tool. Note: Consider h a function of g, H, D, L and Ω.*

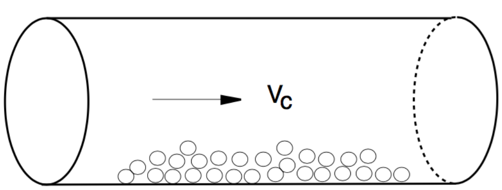


Problem 6.9. *A rectangular, thin, flat plate, with length h and width w is placed perpendicularly to a liquid flow. Imagine that the drag force D which the liquid has upon the plate is a function of w and h, the density ρ, the viscosity μ, and the speed V of the liquid coming towards the plate. Determine the set of dimensionless parameters to study the problem experimentally.*

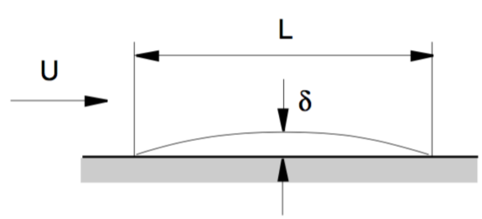
Problem 6.10. *The Reynolds number is a very important parameter for studying transport phenomena and fluid mechanics. Estimate the Reynolds number that would be characteristic of the flow around a car traveling along the highway.*
Note: $\rho_{air} \approx 1.25\, kg/m^3$, $\mu_{air} \approx 1.8 \times 10^{-5} Pa\, s$.

Problem 6.11. *A thin layer of spherical particles are lying at the bottom of a horizontal tube, as indicated in the Figure. When an incompressible liquid flows along the tube, it can be seen that at a certain critical speed the particles move and are carried along the length of the tube. We wish to study the value of this critical speed V_c. Suppose that V_c is a function of the diameter of the tube D, the particle?s diameter D_p, the liquid density ρ, the viscosity of the liquid μ, the density of the particles ρ_p and the gravity acceleration g.*

(a) Using ρ, D and g as fundamental variables, obtain the dimensionless parameters of the problem.
(b) Repeat the first question using ρ, D and μ as fundamental variables.



(a) Exercise 6.11



(b) Exercise 6.12

Problem 6.12. *During the drying process of a fine layer of liquid on a surface, the liquid evaporates and the vapor is transported in the air above the surface. We are interested in* *knowing the dependence of the drying time t upon the rest of the variables of the problem (length L, thickness of the layer δ, the liquid's vapor pressure p_v, air speed U, viscosity μ and air density ρ).*

(a) Obtain a set of dimensionless variables related to the drying time t with the rest of the variables.

(b) We wish to set up a laboratory experiment to determine the drying time of a soccerfield where $p_v = 2000\,Pa$, $L = 100\,m$, $\delta = 0.01\,m$ and $U = 2\,m/s$. In the experiment, the viscosity and the density of the air will be the same as that of the soccer field, but L will be $20\,m$ (we don't have a larger laboratory available). Calculate the values of U, δ and p_v in the experiment so that complete similarity exists with the real flow.

(c) If in the experiment the average drying time is $t = 10\,min$, calculate the drying time of the soccer field.

]

Chapter 7

Flat Plate Boundary Layer

7.1 Boundary layer equations

We consider two dimensional, incompressible, steady-state flow over a half-infinite plate, see Fig. (7.1). The Navier-Stokes equations, Eq.(4.39) in this case become:

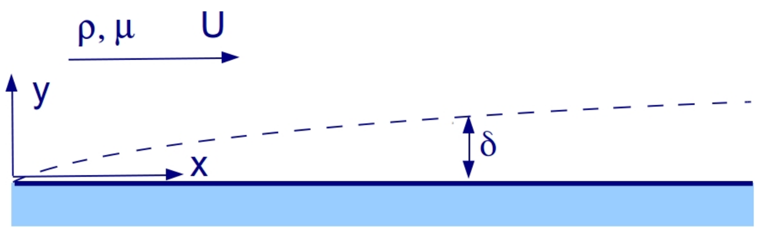


Figure 7.1: Two dimensional, incompressible, steady-state flow over a half-infinite plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{7.3}$$

The viscous flow over a flat plate satifies the following boundary conditions:

$$u(x,0) = v(x,0) = 0, \quad u(x,\infty) = U. \tag{7.4}$$

We will make two powerful assumptions based on the observation that streamlines in the boundarylayer are almost parallel with the plate:

(a) u varies much slower with x than with y, so $\|\frac{\partial^2 u}{\partial x^2}\| \ll \|\frac{\partial^2 u}{\partial y^2}\|$ in Eq.(7.2), and

(b) pressure is almost constant across the boundarylayer, so $\frac{\partial p}{\partial y} \approx 0$.

The second assumption and the fact that p is independent of x at $y = \infty$, immediately leads to the conclusion that

$$p \text{ is constant for all } x, y. \tag{7.5}$$

Furthermore, we non-dimensionalize these equations by using the only parameters we have available: U, ρ and μ. The independent variables become

$$\tilde{x} \equiv \frac{\rho U x}{\mu}, \quad \tilde{y} \equiv \frac{\rho U y}{\mu}, \tag{7.6}$$

and the dependent variables become:

$$\tilde{u} \equiv \frac{u}{U}, \quad \tilde{v} \equiv \frac{v}{U}, \quad \tilde{p} \equiv \frac{p}{\rho U^2}. \tag{7.7}$$

As a result we obtain dimensionless approximate equations first derived by Prandtl which

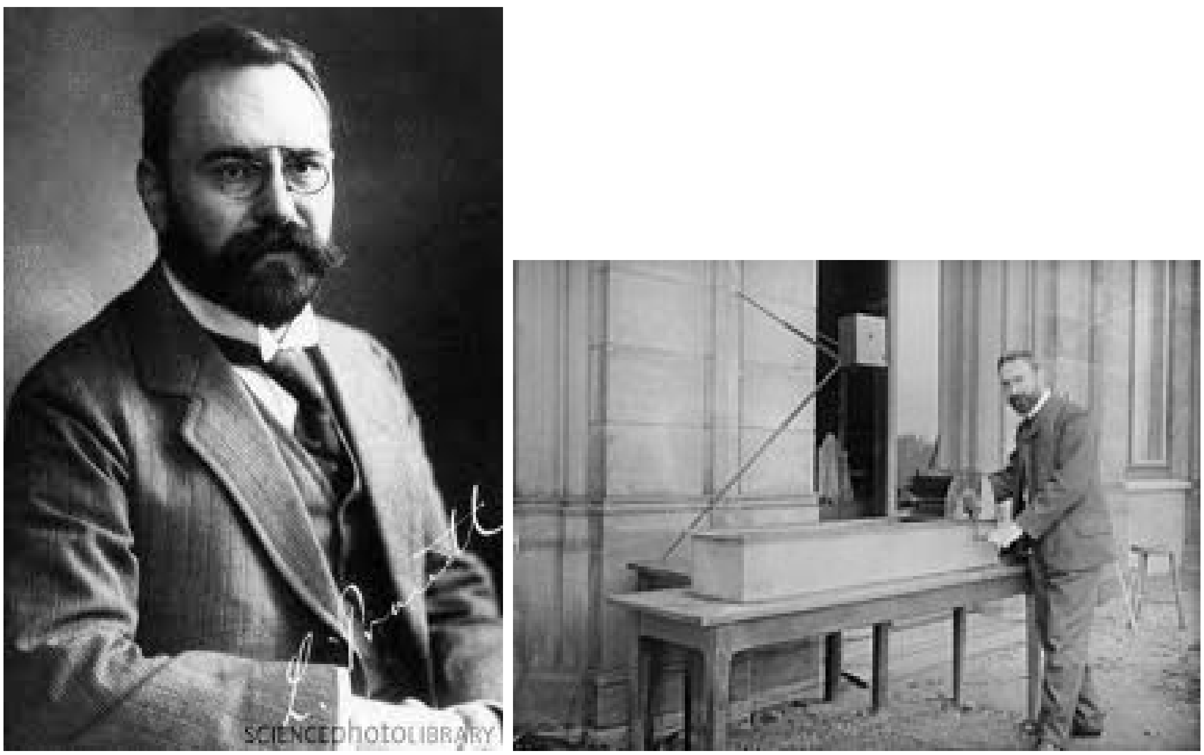


Figure 7.2: Ludwig Prandtl (1875 - 1953) was a German scientist. He was a pioneer in the development of rigorous systematic mathematical analyses which he used for underlaying the science of aerodynamics, which have come to form the basis of the applied science of aeronautical engineering. In the 1920s he developed the mathematical basis for the fundamental principles of subsonic aerodynamics in particular; and in general up to and including transonic velocities. His studies identified the boundary layer, thin-airfoils, and lifting-line theories. The Prandtl number was named after him.

are called the boundary layer equations for a flat plate:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, \\ u\frac{\partial \tilde{u}}{\partial \tilde{x}} + v\frac{\partial \tilde{u}}{\partial \tilde{y}} &= \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}. \end{aligned} \tag{7.8}$$

and dimensionless boundary conditions:

$$\tilde{u}(\tilde{x}, 0) = \tilde{v}(\tilde{x}, 0) = 0, \quad \tilde{u}(\tilde{x}, \infty) = 1. \tag{7.9}$$

7.2 Laminar flow: Blasius solution

Blasius[1] solved the boundary layer equations Eq.(7.8) with boundary conditions Eq.(7.9) by using a stream function that measures the volume flow rate between $\tilde{y} = 0$ and $\tilde{y} = \tilde{y}$:

$$\tilde{\Psi}(\tilde{x}, \tilde{y}) = \int_0^{\tilde{y}} \tilde{u}(\tilde{x}, \tilde{s}) d\tilde{s}, \tag{7.10}$$

which leads to

$$\tilde{u} = \frac{\partial \tilde{\Psi}}{\partial \tilde{y}} \equiv \tilde{\Psi}_{\tilde{y}}, \quad \tilde{v} = -\frac{\partial \tilde{\Psi}}{\partial \tilde{x}} \equiv -\tilde{\Psi}_{\tilde{x}}. \tag{7.11}$$

As a result, the first equation of Eq.(7.8) is satisfied automatically, and the second equation of Eq.(7.8) becomes an partial differential equation for $\tilde{\Psi}$ only:

$$\tilde{\Psi}_{\tilde{y}}\tilde{\Psi}_{\tilde{x}\tilde{y}} - \tilde{\Psi}_{\tilde{x}}\tilde{\Psi}_{\tilde{y}\tilde{y}} = \tilde{\Psi}_{\tilde{y}\tilde{y}\tilde{y}}, \tag{7.12}$$

By writing the unknown function $\tilde{\Psi}$ of two variables as another unknown function f of one variable

$$\tilde{\Psi}(\tilde{x}, \tilde{y}) = \sqrt{\tilde{x}} f\left(\frac{\tilde{y}}{\sqrt{\tilde{x}}}\right), \tag{7.13}$$

the partial differential equation for $\tilde{\Psi}$, Eq.(7.12), becomes an ordinary differential equation for f called the Blasius equation:

$$\boxed{ff'' + 2f''' = 0,} \tag{7.14}$$

and

$$\tilde{u}(\tilde{x}, \tilde{y}) = f'\left(\frac{\tilde{y}}{\sqrt{\tilde{x}}}\right). \tag{7.15}$$

Evidently, finding such transformation is not straight forward at all. Furthermore, the boundary conditions become

$$\boxed{f(0) = f'(0) = 0, \quad f'(\infty) = 1.} \tag{7.16}$$

[1] Paul Richard Heinrich Blasius (1883 - 1970) was a German fluid dynamics engineer.

It appears that

$$\tilde{u}(\tilde{x},\tilde{y}) = 0.99 \text{ when } \frac{\tilde{y}}{\sqrt{\tilde{x}}} \approx 5, \tag{7.17}$$

so the dimensionless boundary layer thickness, defined as $\tilde{\delta} \equiv \frac{U\delta}{\nu}$ and $\tilde{u}(\tilde{x},\tilde{\delta}) \equiv 0.99$, becomes

$$\tilde{\delta}(\tilde{x}) \approx 5\sqrt{\tilde{x}}, \tag{7.18}$$

and the dimensional boundary layer thickness becomes

$$\delta(x) \approx 5\sqrt{\frac{x\nu}{U}}. \tag{7.19}$$

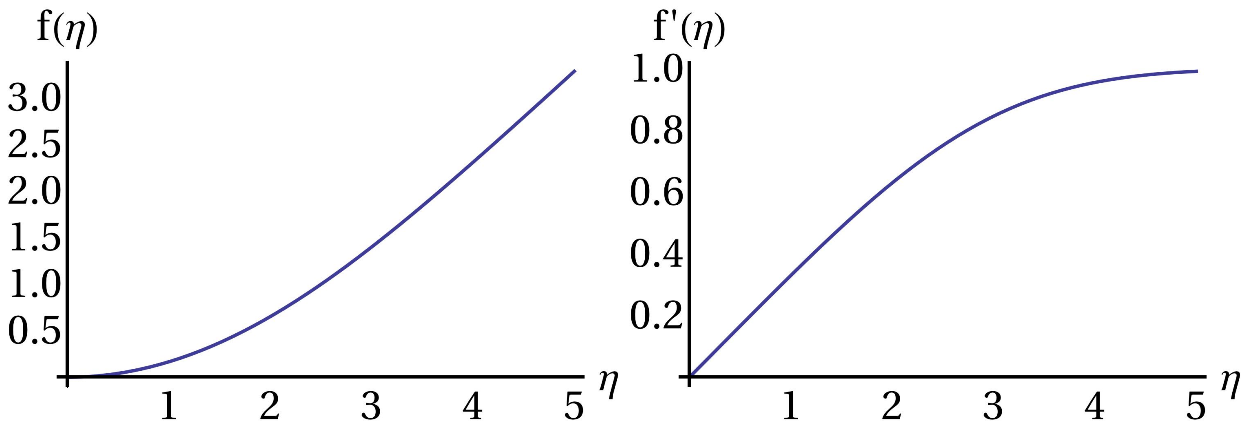


Figure 7.3: Solution of Blasius' equation

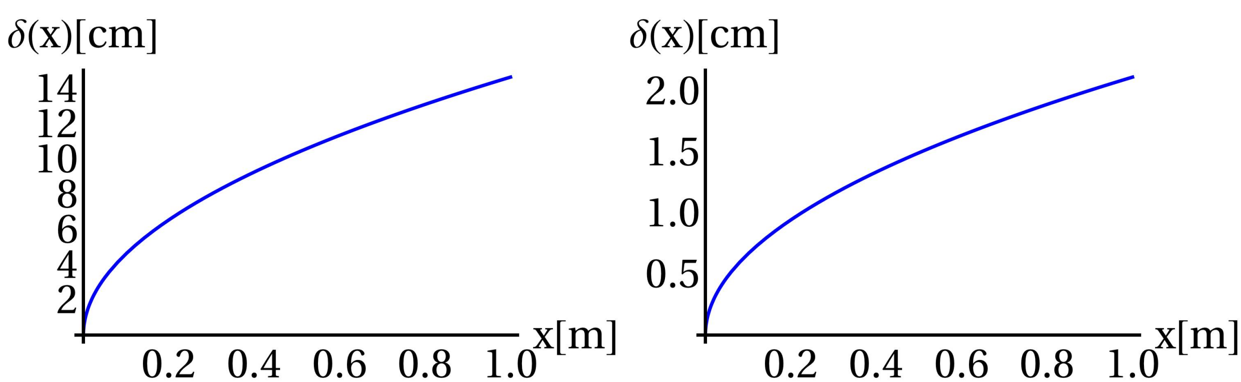


Figure 7.4: Flat plate boundary layer thickness based on Blasius' solution for water (left, $\nu = 894 \times 10^{-6}\,Pa\,s$) and air (right, $\nu = 18.6 \times 10^{-6}\,Pa\,s$) with $U = 1\,m/s$, confirming that $\delta \sim \sqrt{\nu}$. [Note: the solution for air does not hold beyond a few centimeters in downstream direction because turbulence sets in and the boundary layer would be much thicker.]

7.3 Shear stress at the wall

The stress-vector $\boldsymbol{t}$ acting on the wall is given by Cauchy's relation Eq.(4.8):

$$t_i = \sigma_{ij} n_j, \tag{7.20}$$

where in this case the normal vector is $\boldsymbol{n} = (0,1)^T$ (pointing to the acting medium!).

The shear stress at the wall is the first component of the stress vector at $\tilde{y} = y = 0$:

$$\tau_w \equiv t_1 = \sigma_{1j} n_j = \sigma_{12} = \mu \left(\frac{\partial u}{\partial y} \right)_o, \tag{7.21}$$

where the subscript o denotes $\tilde{y} = y = 0$. The Blasius solution is given in terms of dimensionless functions, therefore to actually calculate the wall shear stress we have to rewrite:

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_o = \mu \left(\frac{\partial U \tilde{u}}{\partial y} \right)_o = \mu U \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o \frac{\partial \tilde{y}}{\partial y} = \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o \rho U^2. \tag{7.22}$$

This is frequently written as

$$\tau_w = C_f \frac{1}{2} \rho U^2, \quad C_f \equiv 2 \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o. \tag{7.23}$$

The dimensionless quantity C_f is called the friction coefficient which is a little bit similar to the friction factor used in fully developed flows.

From Eq.(7.15) we see that

$$\frac{\partial \tilde{u}}{\partial \tilde{y}} = f'' \left(\frac{\tilde{y}}{\sqrt{\tilde{x}}} \right) \frac{1}{\sqrt{\tilde{x}}} \Rightarrow \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)_o = f''(0) \frac{1}{\sqrt{\tilde{x}}}. \tag{7.24}$$

The numerical solution of the Blasius equation shows that $f''(0) \approx 0.332$, so

$$C_f = \frac{0.664}{\sqrt{\tilde{x}}}. \tag{7.25}$$

7.4 Exercises

Problem 7.1. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u}$ *by using the definition of the streamfunction* $\tilde{\Psi}$, *and the definition of its partial derivative:*

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = \tilde{u} \equiv \lim_{\Delta \tilde{y} \to 0} \frac{\tilde{\Psi}(\tilde{x}, \tilde{y} + \Delta \tilde{y}) - \tilde{\Psi}(\tilde{x}, \tilde{y})}{\Delta \tilde{y}}.$$

Problem 7.2. *Show that* $\frac{\partial \tilde{\Psi}}{\partial \tilde{x}} = -\tilde{v}$ *by using mass conservation and the definition of the streamfunction* $\tilde{\Psi}$

Problem 7.3. *For sufficiently small values of* η*, Blasius' solution can be approximated by a truncated Taylor series:*

$$f(\eta) = \sum_{n=0}^{5} a_n \eta^n$$

(a) Using the boundary conditions at $\eta = 0$*, show that* $a_o = a_1 = 0$*.*
(b) Using Blasius's equation, show that $a_3 = a_4 = 0$ *and that*

$$a_5 = -\frac{1}{60} a_2^2.$$

(c) Compute a_2 *and* a_5 *given that* $f''(0) \approx 0.322$*.*
(d) Compute $f(2)$ *and* $f'(2)$*.*
(e) Plot these values in Fig. (7.4).

Problem 7.4. *Make a sketch of the skin-friction coefficient* C_f *as a function of* $\tilde{x}$*.*

Chapter 8

Equations of Euler and Bernoulli

8.1 Total derivative

The surface temperature T on the globe depends on the location, say x and y, and the time-instant, say t. Consider the following three scenarios:

(a) increase t by Δt while keeping x and y fixed,
(b) increase x by Δx while keeping t and y fixed, and
(c) increase y by Δy while keeping t and x fixed.

The three increments in temperature corresponding to these scenarios, in case of small increments, become

(a) $\Delta T \approx \frac{\partial T}{\partial t}\Delta t$,
(b) $\Delta T \approx \frac{\partial T}{\partial x}\Delta x$, and
(c) $\Delta T \approx \frac{\partial T}{\partial y}\Delta y$,

which follow from applying Taylor's theorem.

As a fourth scenario, suppose we travel over the surface of the globe. This means that our x and y coordinates are functions of time, say $x_p(t)$ and $y_p(t)$. Then by increasing time, we also increase our x and y coordinates: $\Delta x \approx \frac{dx_p}{dt}\Delta t$ and $\Delta y \approx \frac{dx_p}{dt}\Delta t$. As a consequence, the temperature increment becomes a sum of the contributions of the three scenarios:

$$\Delta T \approx \frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\frac{dx_p}{dt}\Delta t + \frac{\partial T}{\partial y}\frac{dy_p}{dt}\Delta t. \tag{8.1}$$

In the limit of $\Delta t \to 0$ we get

$$\frac{d}{dt}T\left(x_p(t), y_p(t), t\right) \equiv \lim_{\Delta t \to 0}\frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p}{dt} + \frac{\partial T}{\partial y}\frac{dy_p}{dt}, \tag{8.2}$$

which is called the total-derivative of T (with respect to t). We write d's in stead of ∂'s since $T\left(x_p(t), y_p(t), t\right)$ is a function of t only. A straight-forward generalization towards three dimensions is

$$\frac{d}{dt}T\left(x_p(t), y_p(t), z_p(t), t\right) \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p}{dt} + \frac{\partial T}{\partial y}\frac{dy_p}{dt} + \frac{\partial T}{\partial z}\frac{dz_p}{dt}, \tag{8.3}$$

which may represent the temperature in the atmosphere examined by an airplane with time-dependent position $\left(x_p(t), y_p(t), z_p(t)\right)^T$. Note that $\frac{dx_p}{dt}$, $\frac{dy_p}{dt}$, and $\frac{dz_p}{dt}$ denote the velocity components in x, y, and z-direction, respectively.

8.2 Material derivative

In fluid dynamics it is often convenient to compute the time derivative of a quantity at a position that is convected by the flow. So, this is just the total derivative introduced in the previous section but with the time-derivatives of the position taken equal to the velocity of the fluid at that position:

$$\frac{dx_p}{dt} = u\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.4}$$

$$\frac{dy_p}{dt} = v\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.5}$$

$$\frac{dz_p}{dt} = w\left(x_p(t), y_p(t), z_p(t), t\right) \tag{8.6}$$

Since this special derivative appears very frequently, a special symbol is introduced. Leaving out the arguments for readability, one writes

$$\boxed{\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z},} \tag{8.7}$$

which is called the material derivative of T, refering to the fact that the position at which the time derivative is taken is 'attached' to the material (fluid). Finally it is noted that the material derivative can be written in index notation as

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j}. \tag{8.8}$$

Steady flow When the flow is steady,

$$\frac{DT}{Dt} = u_j\frac{\partial T}{\partial x_j}. \tag{8.9}$$

Trajectory invariants Suppose that the material derivative of T is zero, then this means that that T does not vary along trajectories:

$$\frac{DT}{Dt} = 0 \Leftrightarrow T \text{ is constant along trajectories} \tag{8.10}$$

Streamline invariants Suppose that the material derivative of T is zero, and the flow is steady, then this means that that T does not vary along streamlines:

$$\frac{\partial T}{\partial t} = 0, \quad \frac{DT}{Dt} = 0 \Leftrightarrow T \text{ is constant along streamlines} \tag{8.11}$$

8.3 Euler's equation

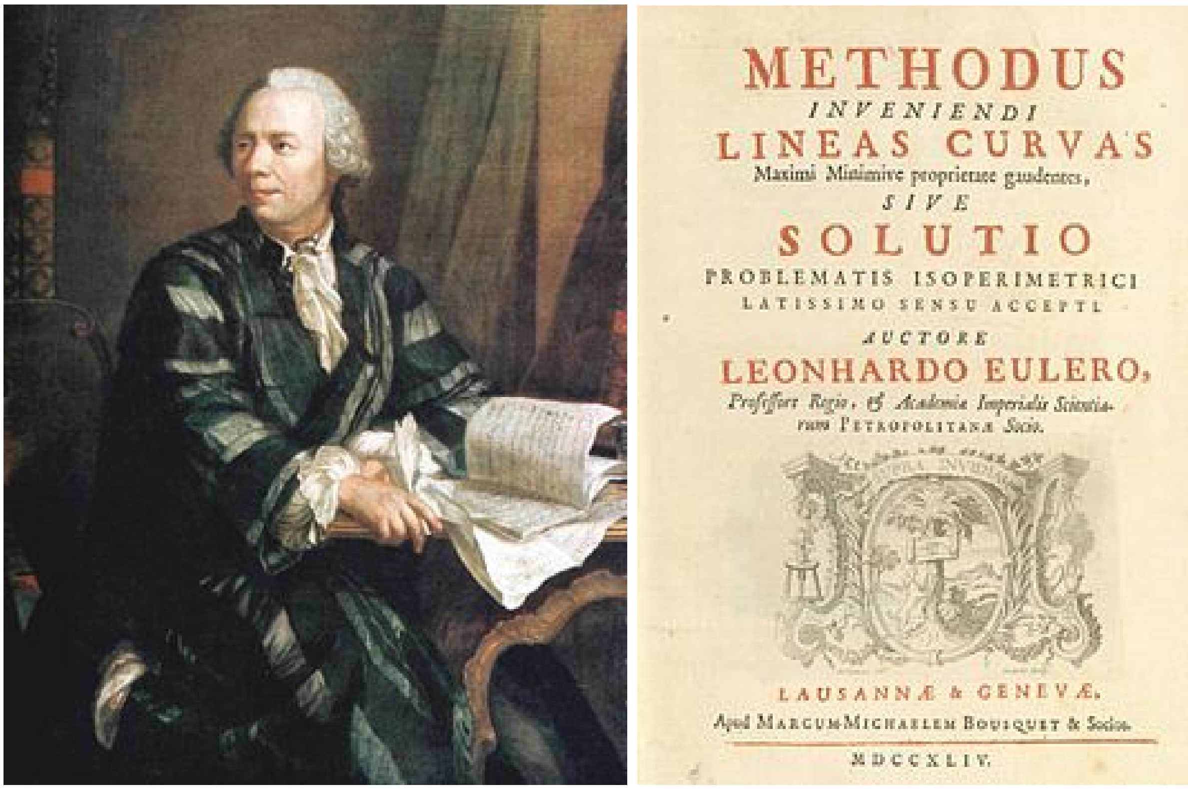


Figure 8.1: Leonhard Euler (1707-1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy.

When the flow is inviscid, i.e., $\mu = 0$, then the stress tensor Eq.(4.10) becomes

$$\sigma_{ij} = -p\delta_{ij}, \tag{8.12}$$

and the Navier-Stokes equations (momentum conservation) given by Eq.(4.39)) reduce to:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p\delta_{ij}}{\partial x_j} - \rho g_i = 0. \tag{8.13}$$

With the properties of the delta function and the summation convention one finds

$$\frac{\partial p\delta_{ij}}{\partial x_j} = \frac{\partial p}{\partial x_i}. \tag{8.14}$$

Furthermore, by means of the product rule of differentiation:

$$\frac{\partial}{\partial t}\left(\rho u_i\right) = \frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t}, \tag{8.15}$$

and

$$\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = \frac{\partial \rho u_j}{\partial x_j} u_i + \frac{\partial u_i}{\partial x_j} \rho u_j. \tag{8.16}$$

In view of the continuity equation (mass conservation) given by Eq.(3.18), the sum of the first terms at the right hand sides of these two equations is zero:

$$\frac{\partial \rho}{\partial t} u_i + \frac{\partial \rho u_j}{\partial x_j} u_i = \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right) u_i = 0, \tag{8.17}$$

So, after division by ρ, Eq.(8.13) can be written as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i. \tag{8.18}$$

The left hand side appears to be the material derivative (introduced in the previous section) of u_i, so finally we obtain Euler's equation:

$$\boxed{\frac{D u_i}{D t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i.} \tag{8.19}$$

Note that $\frac{Du_i}{Dt}$ represents the time-derivative of u_i while traveling with the flow, i.e., it represents the acceleration of the flow in i-direction. This shows that Euler's equation is closely related to Newton's second law as is expected since both equations express conservation of momentum.

8.4 Bernoulli's equation

One of the most popular equations in fluid mechanics is Bernoulli's equation, because of both its power and its simplicity. Basically, it is an equation for the material derivative of the kinetic energy of the fluid.

To derive Bernoulli's equation we start by making a number of assumptions:

(a) the flow is steady ($\frac{\partial \ldots}{\partial t} = 0$),
(b) the flow is incompressible (ρ=constant), and
(c) the flow is inviscid ($\mu = 0$).

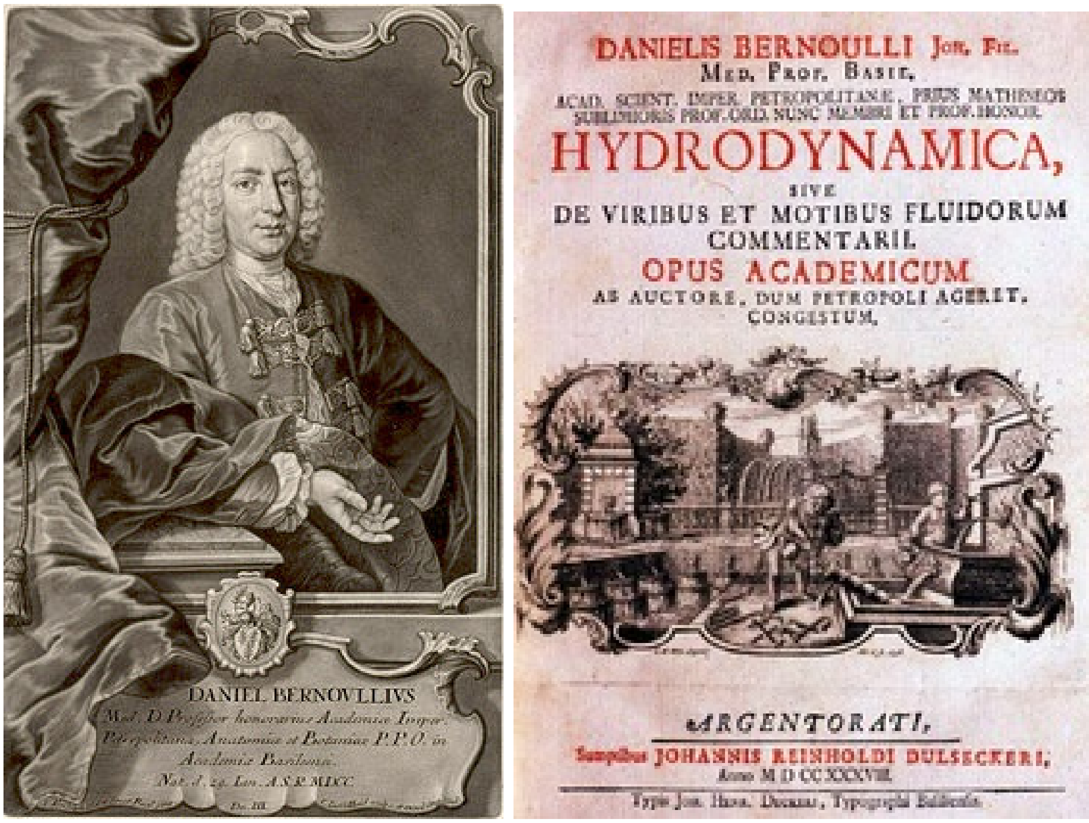


Figure 8.2: Daniel Bernoulli (1700 - 1782) was a Dutch-Swiss mathematician and was one of the many prominent mathematicians in the Bernoulli family. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics. Bernoulli's work is still studied at length by many schools of science throughout the world.

Next, we use Euler's equation Eq.(8.19), which is allowed since the flow is inviscid, and rewrite each of the terms. In view of the steady flow assumption may rewrite the first term as

$$\frac{Du_i}{Dt} = u_j \frac{\partial u_i}{\partial x_j}. \tag{8.20}$$

In view of incompressibility we may rewrite the second term as

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right). \tag{8.21}$$

Finally we rwerite the third term as:

$$g_i = -\frac{\partial g\zeta}{\partial x_i}, \tag{8.22}$$

where ζ is the height above the earth surface. This makes sense, since if ζ does not depend on x_i, then $g_i = 0$, and, on the other hand, when ζ depends on x_i only, we have $g_i = \pm g$. All intermediate situations are also covered.

Now we multiply the result by u_i:

$$u_i \left[u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i} \left(\frac{p}{\rho} \right) + \frac{\partial g\zeta}{\partial x_i} \right] = 0. \tag{8.23}$$

The first term can be rewritten as

$$u_i u_j \frac{\partial u_i}{\partial x_j} = u_j u_i \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right), \tag{8.24}$$

and in view of the summation convention the indices can be interchanged

$$u_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right) \equiv u_i \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_j u_j \right), \tag{8.25}$$

So,

$$u_i \left[\frac{\partial}{\partial x_i} \left(\frac{1}{2} u_j u_j \right) + \frac{\partial}{\partial x_i} \left(\frac{p}{\rho} \right) + \frac{\partial g\zeta}{\partial x_i} \right] = 0, \tag{8.26}$$

or

$$u_i \frac{\partial}{\partial x_i} \left[\frac{1}{2} u_j u_j + \frac{p}{\rho} + g\zeta \right] = 0. \tag{8.27}$$

Finally, in view of the flow being steady, we recognize the material derivative:

$$\boxed{\frac{D}{Dt} \left[\frac{p}{\rho} + \frac{1}{2} u_j u_j + g\zeta \right] = 0.} \tag{8.28}$$

This is Bernoulli's equation which expresses that the term between brackets is constant along streamlines.

8.5 Exercises

Problem 8.1. *Let* $T(x, y) = T_0 + ax + by$

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y$.

Problem 8.2. *Let* $T(x, y) = T_0 + ax + by + cxy$.

(a) Compute $T(x + \Delta x, y + \Delta y) - T(x, y)$.
(b) Compute $\frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y$.
(c) Under what condition do the answers of (a) and (b) coincide?

Problem 8.3. *Let* $T(x, y) = T_0 + ax + by$ *and* $x_p(t) = x_o + ut$, $y_p(t) = y_o + vt$.

(a) Compute $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.

(b) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.4. *Let* $T(x,y) = T_0 + ax + by + cxy$ *and* $x_p(t) = x_o + ut + \frac{1}{2}pt^2$, $y_p(t) = y_o + vt + \frac{1}{2}qt^2$.

(a) *Compute* $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(b) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.
(c) *Under what condition do the answers of (a) and (b) coincide?*

Problem 8.5. *Let* $T(x) = sin(ax)$ *and* $x_p(t) = ut$.

(a) *Compute* $f(t) \equiv T(x_p(t))$ *and* $\frac{df}{dt}$.
(b) *Make a sketch of* $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 1$, $u = 1$ *and* $a = 1$, $u = 2$.
(c) *Make a sketch of* $T(x_p(t))$ *on* $t \in [0, 2\pi]$ *for* $a = 2$, $u = 1$ *and* $a = 2$, $u = 2$.

Problem 8.6. *Let* $T(x,y) = xy$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) *Sketch the curve* $T(x,y) = 1$.
(b) *Compute* $f(t) \equiv T(x_p(t), y_p(t))$ *and* $\frac{df}{dt}$.
(c) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.7. *Let* $T(x,y) = xyt$ *and* $x_p(t) = e^t$, $y_p(t) = e^{-t}$.

(a) *Sketch the curve* $T(x,y) = t$.
(b) *Compute* $f(t) \equiv T(x_p(t), y_p(t), t)$ *and* $\frac{df}{dt}$.
(c) *Compute* $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx_p(t)}{dt} + \frac{\partial T}{\partial y}\frac{dy_p(t)}{dt}$.

Problem 8.8. *A baseball is thrown at speed* U *in air with pressure* p_a *and constant density* ρ. *Assume* $\mu = 0$ *and* $U = constant$.

(a) *Compute the maximum value of the pressure on the ball's surface.*
(b) *Compute the maximum value of the pressure on the ball's surface if there is a head-wind* V.
(c) *Compute the maximum value of the pressure on the ball's surface if there is a tail-wind* V.

Problem 8.9. *A stone with mass* m *is attached to a rope and swung around in a horizontal circle. The path of the stone is* $\boldsymbol{x}_p(t) = vectortwo x_p(t) y_p(t)$, *with*

$$x_p(t) = L\cos(\omega t), \quad y_p(t) = L\sin(\omega t).$$

(a) *Compute the velocity vector* $\boldsymbol{u}(t) \equiv \frac{d}{dt}\boldsymbol{x}_p(t)$.
(b) *Compute the velocity vector* $\boldsymbol{a}(t) \equiv \frac{d}{dt}\boldsymbol{u}_p(t)$.
(c) *For an arbitrary time instant* t, *sketch the vectors* $\boldsymbol{x}_p(t)$, $\boldsymbol{u}_p(t)$, *and* $\boldsymbol{a}_p(t)$.

Problem 8.10. *A flow field is specified as* $\boldsymbol{u}(\boldsymbol{x}) = \frac{U}{L}\begin{pmatrix} -y \\ x \end{pmatrix}$.

(a) Compute $\boldsymbol{a}(\boldsymbol{x}) \equiv \frac{D}{Dt}\boldsymbol{u}$.
(b) For arbitrary position $\boldsymbol{x}$*, sketch* $\boldsymbol{u}$ *and* $\boldsymbol{a}$.
(c) Compute the vector $\boldsymbol{\nabla} p$ *and add it to the sketch.*

Problem 8.11. *Explain why*

(a) Bernoulli's equation does **not** *hold in fully developed flow.*
(b) Bernoulli's equation **does** *hold in a swimming pool if nobody is swimming.*
(c) Derive, using Bernoulli's equation, an expression for the pressure in a abandoned swimming pool.

Chapter 9

Convection and Diffusion

9.1 Mass Conservation

Consider a vein or a long tube with cross-sectional area $A(x)$ with x the coordinate along the tube. Imagine the steady flow of a fluid of mass density ρ and cross-ectional averaged velocity u in the positive x-direction, then the mass flow rate (kg/s) through the tube cross section is the same at every position along the tube:

$$\frac{d}{dx}(\rho u A) = 0 \quad \Rightarrow \quad \rho u A = constant, \tag{9.1}$$

If ρ is constant this leads to $uA = constant$ and if also the cross section is constant this finally leads to $u = constant$.

If a small amount of a substance is added to the fluid it will be transported by the fluid and will also be spread over the fluid. The transportation is called **convection** and the spreading is called **diffusion**. Both mechanisms play an essential role in bio-fluid mechanics. The mass density of the added substance will be denoted by ρ_α and its concentration by c_α which can be computed from

$$c_\alpha \equiv \rho_\alpha / \rho \quad \Rightarrow \quad \rho_\alpha = \rho c_\alpha. \tag{9.2}$$

9.2 Convection

Imagine the flow of the fluid with the added substance through a tube with constant cross section A_o, one could think of a medical drug added to blood flowing through an idealized vein. Then the accumulation of the added drug in a small section of the tube of length Δx is described by the time derivative of its mass density which, if there is no spreading (i.e. no diffusion) depends on the inflow and outflow rates of the drug:

$$\frac{\partial \rho_\alpha}{\partial t} \Delta x A_o = (\rho_\alpha u A_o)_x - (\rho_\alpha u A_o)_{x+\Delta x} \tag{9.3}$$

By dividing by $A_o\Delta x$, noting that $u = U$ is a constant and by taking the limit $\Delta x \to 0$ one obtains

$$\frac{\partial \rho_\alpha}{\partial t} = -U\frac{\partial \rho_\alpha}{\partial x}, \tag{9.4}$$

which can further be simplified by using Eq.(9.2) and $\rho = constant$ which finally leads to the **convection equation**:

$$\boxed{\frac{\partial c_\alpha}{\partial t} + U\frac{\partial c_\alpha}{\partial x} = 0.} \tag{9.5}$$

The solution to the convection equation is easily guessed based on the observation that, in absense of spreading (i.e. diffusion), a given concentration profile will be just transported with velocity u without deformation. Hence it is natural to try a solution of the form $g(x - Ut)$. Substitution into the convection equation Eq.(9.5) gives:

$$g'(x - Ut)\cdot(-U) + U\,g'(x - Ut) = 0, \tag{9.6}$$

which shows that $g(x - Ut)$ is indeed a solution for any function c. If the initial condition is $c_\alpha(x, 0) = c^o_\alpha(x)$ then $g(x) = c^o_\alpha(x)$ and so $g(x - Ut) = c^o_\alpha(x - Ut)$ so the solution becomes

$$\boxed{c_\alpha(x, t) = c^o_\alpha(x - Ut)}. \tag{9.7}$$

This expression indeed reflects the intuitive idea that the concentration profile is transported to the right with speed U without deformation.

9.3 Diffusion

Imagine the same tube with a quiescent fluid with a locally added substance. From experience one knows that after a while the substance will have spreaded due to Brownian motion of the molecules.

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&&&1&&7&&21&&35&&35&&21&&7&&1&&&\\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&&\\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1&\\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}$$

Figure 9.1: Newton's binomium as a model for diffusion

At any position x the flux f of substance is proportional to **minus** the concentration derivative:

$$f = -D\frac{\partial \rho_\alpha}{\partial x}, \tag{9.8}$$

where D is the diffusion coefficient which is assumed to be constant. This means that if concentration increases in positive x-direction then the actual flux is in the negative x direction.

Again the accumulation of the added drug in a small section of the tube of length Δx is described by the time derivative of its mass density which, if there is no fluid motion (i.e. no convection) depends on the inflow and outflow rates of the drug by means of Brownian motion:

$$\frac{\partial \rho_\alpha}{\partial t}\Delta x A_o = \left(-D\frac{\partial \rho_\alpha}{\partial x}A_o\right)_x - \left(-D\frac{\partial \rho_\alpha}{\partial x}A_o\right)_{x+\Delta x} \tag{9.9}$$

By dividing by $A_o\Delta x$, noting that D is a constant and by taking the limit $\Delta x \to$ one obtains

$$\frac{\partial \rho_\alpha}{\partial t} = D\frac{\partial}{\partial x}\left(\frac{\partial \rho_\alpha}{\partial x}\right), \tag{9.10}$$

which can further be simplified by using Eq.(9.2) and $\rho = constant$ which finally leads to the **diffusion equation**.

$$\boxed{\frac{\partial c_\alpha}{\partial t} = D\frac{\partial^2 c_\alpha}{\partial x^2}.} \tag{9.11}$$

The general solution of the diffusion equation can be derived by the method of separation of variables which will not be undertaken here. Instead a fundamental solution will be derived which is of interest when initially the added substance is confined to a compact area, think of an instantaneous injection at a point $\tilde{x}$. The trial solution has the form

$$c_\alpha(x,t) = h(t)\exp\left(-\beta\frac{(x-\tilde{x})^2}{t}\right), \tag{9.12}$$

where the function $h(t)$ and the constant β are determined by solution of the trial solution into the diffusion equation:

$$h(t) = \frac{c}{\sqrt{t}}, \quad \beta = \frac{1}{4D}, \tag{9.13}$$

with c a constant. If the initial condition is $c_\alpha(x,0) = c_\alpha^o(x)$ then the general solution can be written as an integral over the fundamental solutions in Eq.(9.12):

$$\boxed{c_\alpha(x,t) = \int_{-\infty}^{\infty}\frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi Dt}}\exp\left(-\frac{(x-\tilde{x})^2}{4Dt}\right)d\tilde{x}.} \tag{9.14}$$

This solution is an integration over Gaussian distributions with standard deviation $\sqrt{2Dt}$. It is interesting that in the limit of small t the standard deviation goes to zero and the solution becomes an integration over Dirac delta functions which recovers the initial condition:

$$\lim_{t\to 0} c_\alpha(x,t) = \int_{-\infty}^{\infty} c_\alpha^o(\tilde{x})\delta(x-\tilde{x})\,d\tilde{x} = c_\alpha^o(x). \tag{9.15}$$

9.4 Convection & diffusion

In this section simultaneous convection and diffusion is considered which is described by the so-called **convection-diffusion equation**:

$$\boxed{\frac{\partial c_\alpha}{\partial t} + U\frac{\partial c_\alpha}{\partial x} = D\frac{\partial^2 c_\alpha}{\partial x^2}.} \tag{9.16}$$

To produce a solution describing an injected substance which then is both convected and diffused the above equation is first transformed to a reference fram **moving with the flow**. The idea behind it is that when one moves with the flow the velocity is zero and only the spreading mechanism (diffusion) should be present. So, after transformation one expects a diffusion equation in the new reference frame. The transformation applied is

$$c_\alpha(x,t) = \bar{c}_\alpha(\xi,\tau), \quad \xi \equiv x - Ut, \quad \tau = t. \tag{9.17}$$

Note that when ξ is constant $x - Ut$ is constant which means that x moves with time at speed U, exactly what is intended. The derivatives transform as

$$\frac{\partial c_\alpha}{\partial t} = \frac{\partial \bar{c}_\alpha}{\partial \tau} - U\frac{\partial \bar{c}_\alpha}{\partial \xi}, \quad \frac{\partial c_\alpha}{\partial x} = \frac{\partial \bar{c}_\alpha}{\partial \xi}, \quad \frac{\partial^2 c_\alpha}{\partial x^2} = \frac{\partial^2 \bar{c}_\alpha}{\partial \xi^2}, \tag{9.18}$$

which after substitution into the convection-diffusion equation Eq.(9.16) indeed gives

$$\frac{\partial \bar{c}_\alpha}{\partial \tau} = D\frac{\partial^2 \bar{c}_\alpha}{\partial \xi^2}. \tag{9.19}$$

The solution due to a local injection of substance at $(x,t) = (0,0)$ in the convection-diffusion problem corresponds to local injection at $(\xi,\tau) = (0,0)$ in the diffusion problem which gives the solution derived in the previous section:

$$\bar{c}_\alpha(\xi,\tau) = \int_{-\infty}^{\infty} \frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi D\tau}} \exp\left(-\frac{(\xi-\tilde{x})^2}{4D\tau}\right) d\tilde{x}. \tag{9.20}$$

The corresponding solution to the convection-diffusion problem is now directly available by backward transformation:

$$\boxed{c_\alpha(x,t) = \int_{-\infty}^{\infty} \frac{c_\alpha^o(\tilde{x})}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-Ut-\tilde{x})^2}{4Dt}\right) d\tilde{x}.} \tag{9.21}$$

which is an integral over spreading Gaussian distributions which is convected with speed U.

9.5 Péclet number

When a convection duffusion problem is considered, the initial condition typically exists of a given concentration profile with say a maximum value of c_α^{max} on an interval of length L.

The other two parameters involved are the convection speed U and the diffusion coefficient D. Dimension analysis shows that there exist a dimensionless number, the **Péclet number**, which expresses the realtive importance of convection compared to diffusion:

$$Pe \equiv \frac{UL}{D}. \tag{9.22}$$

As an example, consider the flow of oxygen in the human lung. The trachea is the entrance of the lung and one expects that convection is much more important than diffusion so $Pe >> 1$. On the other hand, in the alveolar region where the lung tubes end in the alveoli the flow is nearly stagnant and one expects diffusion to be much more important than convection so $Pe << 1$.

9.6 Exercises

Problem 9.1. *Consider the following convection-diffusion equation:*

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2},$$

where $\phi(x,t)$ is some property, and the units of x, t, and u are m, s, and m/s, respectively. Compute the unit of α, and check the answers for the cases $\alpha = \frac{k}{\rho C_v}$, and $\alpha = \frac{\mu}{\rho}$.

Problem 9.2. *One-dimensional sound waves are described by the wave equation*

$$\frac{\partial^2 p}{\partial t^2} - a^2\frac{\partial^2 p}{\partial x^2} = 0$$

, where p is the pressure disturbance and a the speed of sound. Show that $f(x - at)$ and $g(x + at)$ are solutions of the wave equation, with f and g arbitrary functions.

Problem 9.3. *Show, by substitution, that $T(x,t)\,(a\cos(\lambda x) + b\sin(\lambda x))\exp(-\alpha\lambda^2 t)$ is a solution of the diffusion equation*

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}.$$

Problem 9.4. *Let $T(x,t) = \bar{T}(\xi(x,t),t)$, $\xi(x,t) = x - Ut$.*

(a) Express $\frac{\partial T}{\partial t}$, $\frac{\partial T}{\partial x}$, and $\frac{\partial^2 T}{\partial x^2}$ in terms of derivatives of $\bar{T}$ with respect to ξ and t.
(b) Show that the convection-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

transforms into a diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \alpha\frac{\partial^2 \bar{T}}{\partial x^2}.$$

Chapter 10

Energy Conservation

10.1 Heat flux

The heat flux $\boldsymbol{q}$ is defined as the transport rate of thermal energy per unit area, which according to Fourier's[1] law is proportial to the negative gradient of the temperature (heat flows from warmer areas to cooler areas):

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T, \quad \text{or equivalently} \quad q_i = -k\frac{\partial T}{\partial x_i}, \tag{10.1}$$

where k is the heat conduction coefficient which is assumed constant in these notes.

10.2 Energy conservation: integral formulation

The first law of thermodynamics states that the energy change of a blob of material equals the heat added to the blob plus the work done on the blob. The energy of a convecting blob of fluid (viz. Fig. (2.1)) is simply equal to the integral of the total energy per unit volume over the blob:

$$Energy(t) \equiv \int_{V(t)} \rho(\boldsymbol{x},t)E(\boldsymbol{x},t)\,dV. \tag{10.2}$$

The amount of heat added to the blob per unit time can be written as

$$\int_{S(t)} \boldsymbol{q}\cdot(-\boldsymbol{n})dS = -\int_{S(t)} \boldsymbol{q}\cdot\boldsymbol{n}dS = -\int_{S(t)} q_j n_j dS, \tag{10.3}$$

where we took $(-\boldsymbol{n})$ instead of $\boldsymbol{n}$ since we are looking at the heat flux towards $V(t)$. The amount of work done on the blob per unit time can be written as

$$\int_{S(t)} t_i u_i dS + \int_{V(t)} \rho g_j u_j dV, \tag{10.4}$$

[1] Jean Baptiste Joseph Fourier (1768 - 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's Law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.

where the first integral denotes the work rate done by the surrounding fluid (stress), and the second integral denotes the work rate done by gravity. It is noted that the summation convention is applied everywhere, and that in the first integral of Eq.(10.4) on should sum over both i and j.

The first law of thermodynamics for the blob now becomes

$$\frac{d}{dt}\int_{V(t)} \rho E\, dV = -\int_{S(t)} q_j n_j dS + \int_{S(t)} (\sigma_{ij} n_j) u_i dS + \int_{V(t)} \rho g_j u_j dV. \tag{10.5}$$

With the Reynolds transport theorem Eq.(2.16) we can finally rewrite the term at the left-hand side and write all surface integral as a single integral:

$$\boxed{\int_{V(t)} \frac{\partial}{\partial t}(\rho E)\, dV + \int_{S(t)} \left(\rho E u_j - \sigma_{ij} u_i - k\frac{\partial T}{\partial x_j}\right) n_j dS = \int_{V(t)} \rho g_j u_j dV.} \tag{10.6}$$

In summary:

(a) the first term expresses the energy rate of change due to the total energy rate of change,
(b) the second term expresses the energy rate of change due convection (bulk motion)
(c) the third term expresses the energy rate of change due work done by stress
(d) the fourth term expresses the energy rate of change due conduction (molecules motion), and
(e) the fifth term expresses the energy rate of change due work done by gravity.

10.3 Example 1: slowly moving piston

Consider the moving piston and cylinder depicted in Fig. (10.1). The piston with area A

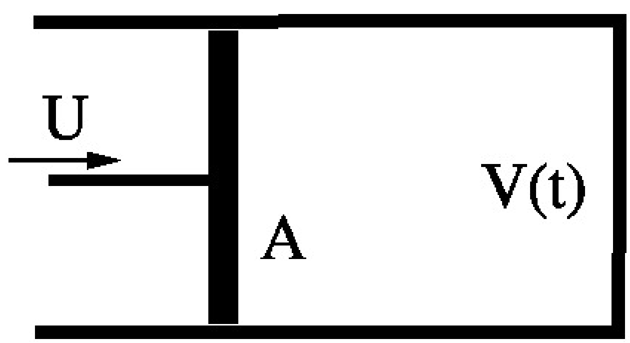

Figure 10.1: Slowly moving piston in a cylinder

is slowly moving to the right with velocity U, thereby decreasing the enclosed volume V. Suppose that V contains a perfect gas, then this gas is compressed slowly. The question is: how does the temperature change due to the movement of the piston?

To provide the answer we use the energy conservation equation. Since the piston movement is slow we assume that the flow field in V is uniform and that velocities are very small.

This means that the velocity and temperature gradients are small such that we can neglect energy losses due to friction and heat conduction. Furthermore, the total energy E as defined by Eq.(**??**), can be approximated by the specific thermal energy $e(T)$, since the quadratic velocity term is very small. Furthermore, the specific thermal energy for a perfect gas is simply C_vT, see Eq.(**??**), which leads to

$$E \approx C_v T. \tag{10.7}$$

Starting with the first term of the integral equation for energy conservation, Eq.(10.6), we approximate

$$\int_{V(t)} \frac{\partial \rho E}{\partial t}\, dV \approx V \frac{d}{dt}\left(\rho E\right) = V \frac{d}{dt}\left(\rho C_v T\right). \tag{10.8}$$

The second term of Eq.(10.6) becomes

$$\int_{S(t)} \rho E u_j n_j \, dS \approx -U \rho E A = -U A \rho C_v T = \frac{dV}{dt} \rho C_v T. \tag{10.9}$$

The third term of Eq.(10.6) becomes

$$-\int_{S(t)} \sigma_{ij} n_j u_i \, dS \approx -pUA = \rho R T \frac{dV}{dt}. \tag{10.10}$$

The fourth term of Eq.(10.6) becomes

$$-\int_{A(t)} k \frac{\partial T}{\partial x_i} n_i \, dS \approx 0. \tag{10.11}$$

The fifth term of Eq.(10.6) becomes

$$\int_{V(t)} \rho g_i u_i \, dV \approx 0. \tag{10.12}$$

With these expressions and the observation that ρV is constant (mass conservation) we find after some algebra

$$\frac{1}{T}\frac{dT}{dt} + (\gamma - 1)\frac{1}{V}\frac{dV}{dt} = 0, \qquad \gamma - 1 = R/C_v. \tag{10.13}$$

This leads to

$$\frac{d}{dt} \ln \left(T V^{\gamma - 1}\right) = 0, \tag{10.14}$$

or

$$\boxed{T V^{\gamma-1} = \text{constant}}. \tag{10.15}$$

Since $\gamma > 1$ this equation shows that T increases when V decreases. In other words, when the piston compresses the perfect gas, the temperature will increase accordingly.

By using ρV = constant and the perfect gas law Eq.(**??**) we also find three equivalent forms:

$$\boxed{T\rho^{-(\gamma-1)} = \text{constant}, \quad Tp^{-(\gamma-1)/\gamma} = \text{constant}, \quad p\rho^{-\gamma} = \text{constant}}. \tag{10.16}$$

All of these relations express that the flow is isentropic. The last expression, which relates p and ρ, is referred to as Poisson's relation.

Finally, one could raise the question how much work one has to do on the piston to compress the gas. To answer this question we need the work done on V per unit time which is just Eq.(10.20) without the minus sign. The total amount of work W is found by integrating this term over time:

$$W = \int_0^t \left\{ \int_{A(t)} \sigma_{ij} n_j u_i \, dA \right\} dt \approx -\int_0^t \rho RT \frac{dV}{dt} \, dt. \tag{10.17}$$

Since $\rho V = \rho_o V_o$ and $TV^{\gamma-1} = T_o V_o^{\gamma-1}$ and $p_o = \rho_o RT_o$, where the subscript 0 indicates the situation at $t = 0$, we obtain

$$W = -\int_0^t \rho RT \frac{dV}{dt} dt = -p_o V_o^{\gamma} \int_0^t V^{-\gamma} \frac{dV}{dt} dt, \tag{10.18}$$

which leads to

$$\boxed{W = \frac{p_o V_o}{\gamma - 1} \left\{ \left(\frac{V_o}{V(t)} \right)^{\gamma-1} - 1 \right\}.} \tag{10.19}$$

10.4 Example 2: compressor or turbine

Consider the periodic flow through the compressor depicted in Fig. (11.1). The flow is

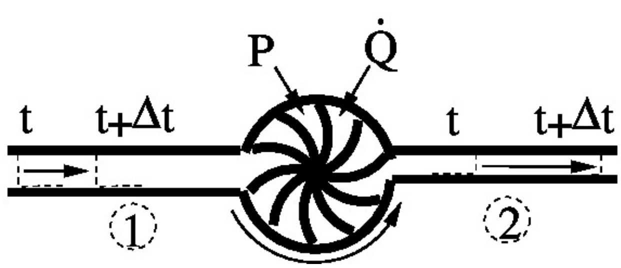

Figure 10.2: Periodic flow through a compressor.

periodic in the sense that when the rotor rotates over a small angle such that the blades shift one position, the flow field is identical to the flow field corresponding to the previous rotor position. We also assume that the flow is uniform in the inlet and outlet pipes.

The main question is: what is the relation between the work and heat input on the one hand, and the mass flow rate and pressure or temperature change on the other hand?

Figure 10.3: Various types of wind turbines.

Starting with the first two terms of the integral equation for energy conservation, Eq.(10.6), we combine them again to $\frac{d}{dt}\int_{V(t)} \rho E dV$ and approximate:

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} \rho E dV &\approx \frac{1}{\Delta t}\left\{\int_{V(t+\Delta t)} \rho E \, dV - \int_{V(t)} \rho E \, dV\right\} \\
&= \frac{1}{\Delta t}\left\{(\rho E A U \Delta t)_2 - (\rho E A U \Delta t)_1\right\} \\
&= (\rho E A U)_2 - (\rho E A U)_1 \,. \qquad (10.20)
\end{aligned}$$

Due to the quasi-steadiness of the flow we have from mass conservation

$$(\rho A U)_1 = (\rho A U)_2 = \dot{m}, \qquad (10.21)$$

where $\dot{m}$ denotes the mass flow rate. Therefore Eq.(10.20) becomes

$$\frac{d}{dt}\int_{V(t)} \rho E dV = \dot{m}\,(E_2 - E_1)\,. \qquad (10.22)$$

The third term of Eq.(10.6) can be approximated when we neglect viscous stresses at the inflow and outflow boundaries:

$$\int_{S(t)} \sigma_{ij} n_j u_i \, dS \approx \int_{A_1+A_2} -p n_i u_i \, dS + \int_{A_{blades}} \sigma_{ij} n_j u_i \, dS. \qquad (10.23)$$

The first integral becomes

$$\int_{A_1+A_2} -p n_i u_i \, dS = (pUA)_1 - (pUA)_2 = \dot{m}\left\{\left(\frac{p}{\rho}\right)_1 - \left(\frac{p}{\rho}\right)_2\right\}, \qquad (10.24)$$

whereas the second integral is just the rate of work done <u>by</u> the blades <u>on</u> the fluid:

$$\int_{A_{blades}} \sigma_{ij} n_j u_i \, dS = P. \qquad (10.25)$$

Finally, the fourth term of Eq.(10.6) is just the rate of heat supplied to the control volume:

$$-\int_{S(t)} k\frac{\partial T}{\partial x_i}n_i\,dS = \dot{Q}. \tag{10.26}$$

As a result, the energy equation becomes:

$$\dot{m}\,(E_2 - E_1) = \dot{m}\left\{\left(\frac{p}{\rho}\right)_1 - \left(\frac{p}{\rho}\right)_2\right\} + P + \dot{Q}. \tag{10.27}$$

We can finally rewrite this equation in a more compact way by using the definition of total enthalpy H:

$$\boxed{\dot{m}\,(H_2 - H_1) = P + \dot{Q}.} \tag{10.28}$$

10.5 Exercises

Problem 10.1. *Using $TV^{\gamma-1} = const$, $p = \rho RT$, and mass conservation, derive that:*

(a) $T\rho^{1-\gamma} = const$
(b) $Tp^{-\frac{\gamma-1}{\gamma}} = const$
(c) $p\rho^{-\gamma} = const$

Problem 10.2. *Starting from*

$$W = -\int_0^t p\frac{dV}{dt}dt,$$

$TV^{\gamma-1} = const$, *and mass conservation, derive that*

(a) $W = -p_oV_o^{\gamma}\int_0^t V^{-\gamma}\frac{dV}{dt}dt$,
(b) $W = \frac{p_oV_o}{\gamma-1}\left\{\left(\frac{V_o}{V(t)}\right)^{\gamma-1} - 1\right\}$,
(c) $sign(W) = sign(V_o - V(t))$.

Problem 10.3. *Starting with $p\rho^{-\gamma} = const$, derive that*

$$\frac{1}{p}\frac{dp}{dt} - \gamma\frac{1}{\rho}\frac{d\rho}{dt}.$$

Problem 10.4. *Air enters a compressor at speed U_1, temperature T_1, and leaves at U_2, T_2, and the mass flow is $\dot{m}$. The removed heat per unit mass of passing air is $\hat{e}$. Derive an expression for the power required by the compressor, assuming air can be modeled as a perfect gas.*

Chapter 11

Streamline Invariants

11.1 Energy conservation differential form

By rewriting all surface integrals in the integral form of energy conservation, Eq.(10.6), into volume integrals by means of Gauss' theorem, and then using the fact that the resulting equations holds for all possible convected blobs, one obtains the differential formulation of energy conservation:

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}\left(\rho E u_j - \sigma_{ij} u_i - k\frac{\partial T}{\partial x_j}\right) = \rho g_j u_j.} \tag{11.1}$$

11.2 Isentropic flow

Friction forces always work in opposite direction of movement. Heat flows from hot spots to cold spots. Therefore, all processes in the gas during its transport are irreversible. However, when friction and heat conduction can be neglected, all processes in the gas during its transport are reversible. Therefore we expect the gas to behave isentropically, which indeed can be derived from the conservation equations as we see below.

We start with the differential form of the conservation equations of mass, momentum and energy, while we put $\mu = 0$ and $k = 0$ to reflect that there is no friction and no heat conduction:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \tag{11.2}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij} p) = \rho g_i, \tag{11.3}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(\rho E u_j + p u_j) = \rho g_j u_j. \tag{11.4}$$

Next, we rewrite these equations in terms of material derivatives:

$$\frac{D\rho}{Dt} = -\rho\frac{\partial u_j}{\partial x_j}, \tag{11.5}$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i, \tag{11.6}$$

$$\rho\frac{DE}{Dt} = -\frac{\partial p u_j}{\partial x_j} + \rho g_j u_j. \tag{11.7}$$

Using the definition of the total energy, E, we evaluate

$$\rho\frac{DE}{Dt} = \rho\frac{De}{Dt} + \rho u_j\frac{Du_j}{Dt} = \rho\frac{De}{Dt} - u_j\frac{\partial p}{\partial x_j} + \rho g_j u_j. \tag{11.8}$$

Therefore,

$$\rho\frac{De}{Dt} - u_j\frac{\partial p}{\partial x_j} = -\frac{\partial p u_j}{\partial x_j}, \tag{11.9}$$

which leads to

$$\rho\frac{De}{Dt} + p\frac{\partial u_j}{\partial x_j} = 0. \tag{11.10}$$

We can replace the velocity divergence by means of the mass conservation equation and finally obtain, after division by ρ:

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0. \tag{11.11}$$

Noting that we know from thermodynamics that an infinitesimal increase in entropy s satisfies

$$ds = de - \frac{p}{\rho^2}d\rho, \tag{11.12}$$

we recognize that

$$\boxed{\frac{Ds}{Dt} = 0}. \tag{11.13}$$

This means that, when friction and heat conduction are neglected, the entropy is constant while travelling with the flow.

In special case of a perfect gas,

$$p = (\gamma - 1)\rho e, \tag{11.14}$$

Eq.(11.11) can be evaluated to

$$\frac{D}{Dt}\left(\rho^{1-\gamma}e\right) = 0. \tag{11.15}$$

Rewriting e in terms of ρ and p again, we get

$$\boxed{\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0.} \tag{11.16}$$

This means that $p\rho^{-\gamma}$ is constant while travelling with the flow, known as Poisson's equation.

11.3 Speed of sound, Mach number

Small Perturbations The propagation of sound is one of the phenomena that is governed by the equations of continuum mechanics. To derive the equations that govern sound propagation we neglect friction and heat conduction (which is a highly accurate approximation!) and start with the conservation equations of mass (Eq.(3.18)), momentum (Eq.(4.39)), while we replace the energy conservation equation (Eq.(11.1)) by Poisson's equation (Eq.(10.16)). We consider one-dimensional small perturbations (in x-direction) with respect to a quiescent flow (zero velocity, constant pressure and density) of a perfect gas:

$$\rho = \rho_o + \rho', \quad p = p_o + p', \quad u = u', \tag{11.17}$$

where the subscript "0" indicates constant values and where the prime indicates small perturbations. When we neglect quadratic and higher order products of perturbations we end up with the following three equations:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} &= 0, \\ \frac{\partial u}{\partial t} + \rho_o^{-1} \frac{\partial p'}{\partial x} &= 0, \\ p' = a^2 \rho', \quad a \equiv \sqrt{\gamma \frac{p_o}{\rho_o}}&, \end{aligned} \tag{11.18}$$

where γ is the (constant) ratio of specific heats.

Wave Equation By combining these equations and differentiating with respect to x and t we obtain an alternative but equivalent set of three equations:

$$\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} - a^2 \frac{\partial^2 \rho'}{\partial x^2} &= 0, \\ \frac{\partial^2 p'}{\partial t^2} - a^2 \frac{\partial^2 p'}{\partial x^2} &= 0, \\ \frac{\partial^2 u'}{\partial t^2} - a^2 \frac{\partial^2 u'}{\partial x^2} &= 0. \end{aligned} \tag{11.19}$$

We see that all three perturbations, ρ', p' and u', satisfy the same equation, which is called the **wave equation**.

It is very easy to verify that any function $f(x-at)$ satisfies the wave equation. The same holds for any function $g(x+at)$. Since the wave equation is linear the sum of these arbitrary function also is a solution:

$$p'(x,t) = f(x-at) + g(x+at). \tag{11.20}$$

A solution of the form $f(x-at)$ is constant along lines of constant $x-at$ in the x,t-plane. This means that perturbations travel with speed a to the right. Similarly, a solution of the form $g(x+at)$ is constant along lines of constant $x+at$ in the x,t-plane, which means that perturbations travel with speed a to the left.

Speed of sound As is explained above, the small perturbations which we call sound travel with speed a, which therefore must be the **speed of sound**. So, for the speed of sound in a perfect gas we have

$$\boxed{a \equiv \sqrt{\gamma \frac{p_o}{\rho_o}} = \sqrt{\gamma R T_o}}. \tag{11.21}$$

Mach number We define the Mach number M as the dimensionless ratio of the gas speed and the speed of sound:

$$\boxed{M \equiv \frac{U}{a}}, \qquad U \equiv \sqrt{u_k u_k} \tag{11.22}$$

11.4 Total temperature, -pressure, and -density

Consider stationary flow and neglect friction, heat conduction and gravity. Then the differential forms of mass and energy conservation become

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0, \tag{11.23}$$

$$\frac{\partial}{\partial x_j}(\rho H u_j) = 0, \tag{11.24}$$

Using the product rule of differentiation and division by ρ, this leads to

$$u_j \frac{\partial H}{\partial x_j} = 0, \tag{11.25}$$

which, since $\frac{\partial H}{\partial t} = 0$, can be written as

$$\frac{DH}{Dt} = 0. \tag{11.26}$$

In words, in case of stationary flow without effects of viscosity, heat conduction, and gravity, the total enthalpy is constant along stream lines.

If the gas is perfect we can rewrite H as

$$H = C_v T + \frac{1}{2} u_k u_k + RT = C_p T + \frac{1}{2} u_k u_k. \tag{11.27}$$

Using the speed of sound the total enthalpy H can be written as:

$$H = C_p T \left(1 + \frac{\gamma - 1}{2} M^2\right). \tag{11.28}$$

Total temperature If $\frac{DH}{Dt} = 0$, and C_p is constant, then we also have

$$\frac{DT_t}{Dt} = 0, \quad T_t \equiv T\left(1 + \frac{\gamma - 1}{2}M^2\right), \tag{11.29}$$

where T_t is the total temperature. This means that T_t is constant along streamlines. It is noted that T_t is a possible value of the temperature, not an actual value: it is the maximum possible value of the temperature that can be reached along the streamline at hand. Obviously this maximum value can only be reached (if reached at all!) at a point where $M = 0$, in other words, where the velocity is zero. A point where the velocity is zero is called a stagnation point, and therefore the total temperature is frequently refered to as the "stagnation temperature".

Total pressure Using the perfect gas law $p = \rho RT$, and Poisson's equation $p\rho^{-\gamma} = constant$, we can rewrite Eq.(11.29) as

$$\frac{Dp_t}{Dt} = 0, \quad p_t \equiv p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}, \tag{11.30}$$

where p_t is the total pressure, often refered to as the "stagnation pressure", which apparently also is constant along stream lines.

Total density Finally, again using the perfect gas law $p = \rho RT$ and Poisson's equation $p\rho^{-\gamma} = constant$, we can also rewrite Eq.(11.29) as

$$\frac{D\rho_t}{Dt} = 0, \quad \rho_t \equiv \rho\left(1 + \frac{\gamma - 1}{2}M^2\right)^{1/(\gamma-1)}, \tag{11.31}$$

where ρ_t is the total density, often refered to as the "stagnation density", which apparently also is constant along stream lines.

11.5 Limit of small Mach number

At this point one could raise the question whether a contradiction has been introduced if one looks at Bernoulli's equation, Eq.(8.28), with $g = 0$ and multiplied by the (constant) density:

$$\frac{D}{Dt}\left[p + \frac{1}{2}\rho u_j u_j\right] = 0. \tag{11.32}$$

and at the totasl pressure equation, Eq.(11.30):

$$\frac{D}{Dt}\left[p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}\right] = 0. \tag{11.33}$$

Both equations have been derived under exactly the same assumptions (steady flow, no effects of viscosity, heat conduction, or gravity) except for the incompresibility asumption that was used in the derivation of Bernoulli's equation. Therefore, one could raise the question whether the total pressure equation reduces to Bernoulli's equation if one assumes that the density perturbations become sufficiently small.

If the density perturbations are small, Eq.(11.31) shows that the Mach number must also be small:

$$M \ll 1. \tag{11.34}$$

Using Taylor's theorem it is easily shown that for small ϵ and any α:

$$(1+\epsilon)^{\alpha} = 1 + \alpha\epsilon + \mathcal{O}\left(\epsilon^2\right), \tag{11.35}$$

so

$$\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} = 1 + \frac{\gamma}{2}M^2 + \mathcal{O}\left(M^4\right). \tag{11.36}$$

As a result,

$$\frac{D}{Dt}\left[p\left(1+\frac{\gamma}{2}M^2\right)\right] = \mathcal{O}\left(M^4\right). \tag{11.37}$$

With the perfect gas law $p = \rho RT$ and the equation for the speed of sound, $a = \sqrt{\gamma\frac{p}{\rho}}$, one finds

$$\frac{D}{Dt}\left[p + \frac{1}{2}\rho u_j u_j\right] = \mathcal{O}\left(M^4\right). \tag{11.38}$$

This becomes Bernoulli's equation in the limit $M \to 0$.

11.6 Example: Pitot tube

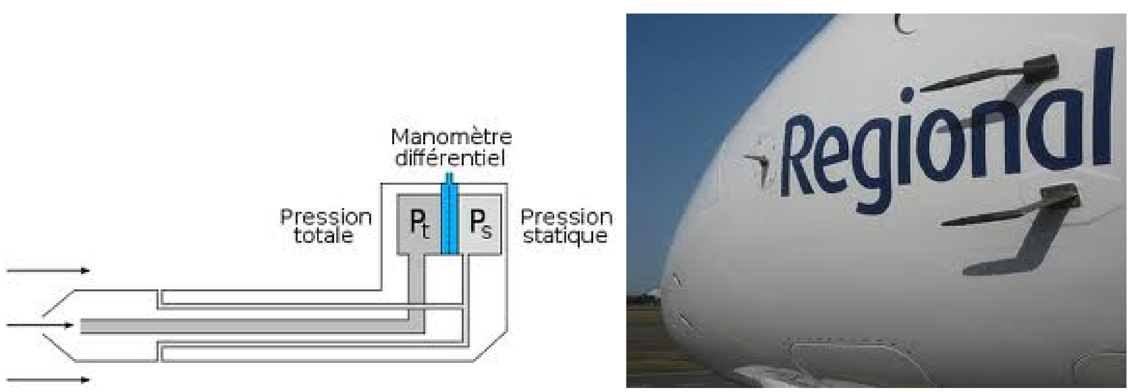

Figure 11.1: Pitot tube: basic priciple (left), and mounted under an airplane (right)

When a so-called Pitot-tube [1] is held into a flow it measures the static pressure (which is just the pressure) through a gap at the side of the probe. It also measures the total pressure through a gap at the front of the probe. The difference can be used for estimating the velocity or the Mach number.

Incompressible flow When the Mach number is small such that the flow is approximately incompressible, Bernoulli's equation gives the velocity:

$$U \equiv \sqrt{u_j u_j} = \sqrt{2\frac{p_t - p}{\rho}}. \tag{11.39}$$

Compressible flow When the Mach number is larger such that the flow is compressible, the total pressure equation gives the Mach number:

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_t}{p}\right)^{\gamma-1} - 1\right]}. \tag{11.40}$$

11.7 Exercises

Problem 11.1. *Consider a sphere in compressible flow. Far upstream from the sphere the pressure, Mach number and temperature are known: p_∞, M_∞, and T_∞. The pressure and temperature in the stagnation point are p_o and T_o.*

(a) Express p_o in terms of p_∞, M_∞.
(b) Express p_∞ in terms of p_o, M_∞.
(c) Express T_∞ in terms of T_o, M_∞.
(d) For measured p_o, T_o, p_∞, compute the velocity U_∞ far upstream of the sphere.

Problem 11.2. *(a) Show that $\frac{1}{p}\frac{Dp}{Dt} = \frac{D}{Dt}\ln p$.*
(b) What is the meaning of $\frac{Dp}{Dt}$?

Problem 11.3. *Consider steady flow with $\mu = 0$ and $k = 0$.*

(a) Show that the mass and energy equations reduce to

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\frac{\partial}{\partial x_j}(\rho u_j H) = 0$$

(b) Show that these equations lead to $\frac{DH}{Dt} = 0$.

[1] Henri Pitot (1695 - 1771) was a French hydraulic engineer and the inventor of the Pitot tube.

(c) What is the meaning of $\frac{DH}{Dt} = 0$?

Problem 11.4. *Show that in case of a perfect gas $p_t = \rho_t R T_t$.*

Problem 11.5. *For steady flow with $\mu = 0$ and $k = 0$ explain that the pressure along a streamline can not exceed the total pressure along that streamline.*

Problem 11.6. *For a thermally perfect gas show that $p = (\gamma - 1)\rho e$.*

Problem 11.7. *Let $p\rho^{\gamma} = const$, and let $p = p_o + p'$, $\rho = \rho_o + \rho'$, with p_o, ρ_o constants and p', ρ' small perturbations. Show that in the limit of $p'/p_o \to 0$, and $\rho_o/\rho' \to 0$ we have*

$$p' = a^2 \rho', \quad a^2 \equiv \gamma \frac{p_o}{\rho_o}.$$

[Hint: use Taylor series $(1+\epsilon)^{\alpha} = 1 + \alpha\epsilon$ ...]

Appendix A

Formulas available during the exam

A.1 Fluid Kinematics, particle trajectories

$$\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}\left(\boldsymbol{x}_p(t), t\right) \tag{A.1}$$

A.2 Mass Conservation

A.2.1 Integral form

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{A(t)} \rho\left(u_j n_j\right) dA = 0. \tag{A.2}$$

A.2.2 Differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{A.3}$$

A.3 Momentum Conservation

A.3.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{A(t)} \rho u_i \left(u_j n_j\right) dA = \int_{A(t)} \sigma_{ij} n_j \, dA + \int_{V(t)} \rho g_i \, dV, \quad i = 1, 2, 3. \tag{A.4}$$

A.3.2 Stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{A.5}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{A.6}$$

A.3.3 Cauchy equation

Tension vector $\boldsymbol{t}$ by medium A on medium B, $\boldsymbol{n}$ pointing to A:

$$t_i = \sigma_{ij} n_j, \quad i = 1, 2, 3. \tag{A.7}$$

A.3.4 Differential form (Navier-Stokes)

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j - \sigma_{ij}\right) = \rho g_i. \tag{A.8}$$

A.3.5 Reduced Navier-Stokes

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = \rho g_1, \quad \frac{\partial p}{\partial y} = \rho g_2. \tag{A.9}$$

A.3.6 Euler equations

Momentum conservation with $\mu = 0$:

$$\rho \frac{D u_i}{D t} = -\frac{\partial p}{\partial x_i} + \rho g_i, \quad i = 1, 2, 3. \tag{A.10}$$

A.3.7 Material derivative

Time derivative while traveling with the flow:

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j}, \quad \text{for any function } f(x, y, z, t) \tag{A.11}$$

A.3.8 Bernoulli equation

(toepassings-voorwaarden zelf onthouden!)

$$p + \frac{1}{2}\rho V^2 + \rho g \zeta = \text{constant along streamlines}, \quad V^2 = u^2 + v^2 + w^2. \tag{A.12}$$

A.4 Energy Conservation

A.4.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho E)\, dV + \int_{A(t)} \left(\rho E u_j n_j - \sigma_{ij} u_i n_j + q_j n_j\right) dA = \int_{V(t)} \rho g_j u_j \, dV, \quad i = 1, 2, 3. \tag{A.13}$$

Total energy:

$$E \equiv e + \frac{1}{2} U^2, \quad U^2 = u^2 + v^2 + w^2. \tag{A.14}$$

Enthalpy and total enthalpy:

$$h \equiv e + \frac{p}{\rho}, \quad H \equiv E + \frac{p}{\rho} \tag{A.15}$$

Thermodynamics of a perfect gas:

$$p = \rho R T, \quad e = C_v T, \quad C_p - C_v = R, \quad \gamma \equiv C_p / C_v \tag{A.16}$$

Speed of sound and Mach number:

$$a = \sqrt{\gamma R T}, \quad M \equiv U/a. \tag{A.17}$$

A.4.2 Fourier's law (heat flux)

$$q_i = -k \frac{\partial T}{\partial x_i}, \quad i = 1, 2, 3. \tag{A.18}$$

A.4.3 Compressor equation

$$\dot{m}\,(H_2 - H_1) = P + \dot{Q}. \tag{A.19}$$

A.4.4 Differential form

$$\frac{\partial}{\partial t} \rho E + \frac{\partial}{\partial x_j}\left(\rho u_j E - \sigma_{ij} u_i + q_j\right) = \rho g_j u_j. \tag{A.20}$$

A.4.5 Total temperature, -pressure and -density

$$T_t = T(1 + \frac{\gamma - 1}{2} M^2), \quad p_t = p(1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}}, \quad \rho_t = \rho(1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma - 1}}. \tag{A.21}$$

A.5 Convection and diffusion

A.5.1 Convection equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0. \tag{A.22}$$

A.5.2 Diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{A.23}$$

A.5.3 Convection-diffusion equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{A.24}$$

Appendix B

Mathematics

B.1 Inner product

Suppose we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in three-dimensional space. The inner product of these two vectors is defined as:

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} \equiv \sum_{i=1}^{3} a_i b_i \equiv a_i b_i} \tag{B.1}$$

As a result, the length of a vector, say $\boldsymbol{c}$, can be written as

$$|\boldsymbol{c}| = \sqrt{\sum_{i=1}^{3} c_i^2} = \sqrt{\boldsymbol{c} \cdot \boldsymbol{c}}. \tag{B.2}$$

What is the meaning of the inner product and how can it be used? In the plane of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we can draw the vector $\boldsymbol{b} - \boldsymbol{a}$, see Fig. (B.1), and we will compute its length, $|\boldsymbol{b} - \boldsymbol{a}|$ in two different ways to demonstrate the meaning of the inner product. First, denoting the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ by θ, we compute the length of $|\boldsymbol{b} - \boldsymbol{a}|$ by using

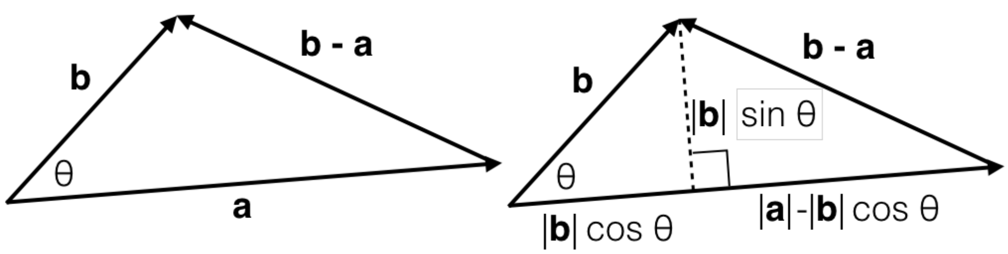

Figure B.1: Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{b} - \boldsymbol{a}$.

Pythagoras' rule:

$$\begin{aligned} |\boldsymbol{b} - \boldsymbol{a}|^2 &= (|\boldsymbol{a}| - |\boldsymbol{b}|\cos\theta)^2 + (|\boldsymbol{b}|\sin\theta)^2 \\ &= |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta + |\boldsymbol{b}|^2. \end{aligned} \tag{B.3}$$

Second, we compute the length of $|\boldsymbol{b}-\boldsymbol{a}|$ by using the inner product:

$$\begin{aligned} |\boldsymbol{b}-\boldsymbol{a}|^2 &= (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) \\ &= \boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a} \\ &= |\boldsymbol{b}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{a}|^2. \end{aligned} \tag{B.4}$$

By comparing the two answers Eq.(B.3) and Eq.(B.4) we conclude that

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta. \tag{B.5}$$

The importance of this expression is that, to calculate $\cos\theta$, we do not need to know what the angle θ is, we simply compute:

$$\boxed{\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}} \tag{B.6}$$

This situation occurs very frequently in mechanics and electrodynamics.

B.2 Functions of multiple variables

Suppose we want to describe the wind on the wheather charts of the Netherlands. It means that we will produce a top view of the country and at "each" point we will be able to draw a vector representing the wind direction and magnitude:

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{B.7}$$

Note that we write vectors in **bold face**. The quantities u and v are called "the components of the vector $\boldsymbol{u}$", where u is the "first component" or "x-component", and v is the "second component" or "y-component". The difference between $\boldsymbol{u}$ and u is thus clear: $\boldsymbol{u}$ is a vector, and u is its first component.

The location at which we measure the wind is given by an "x-position" and a "y-position" which itself is again a vector pointing form the origin to the location at hand:

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{B.8}$$

The velocity component u is a function of both x and y, in other words, it is a function of the vector $\boldsymbol{x}$. The same is true for the component v, and we write

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{pmatrix}. \tag{B.9}$$

So, the result is that we specify as input the vector $\boldsymbol{x}$ belonging to the plane $\mathbb{R}^2$, and obtain as output the vector $\boldsymbol{u}$ also belonging to the plane $\mathbb{R}^2$. In other words, each point chosen in $\mathbb{R}^2$ results in another point in $\mathbb{R}^2$. Mathematically one writes:

$$\boldsymbol{u}(\boldsymbol{x}) : \mathbb{R}^2 \mapsto \mathbb{R}^2. \tag{B.10}$$

In more complex velocity fields we may add a third dimension:

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \\ w(\boldsymbol{x}) \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{B.11}$$

where w is the "third component" or "z-component" of the vector $\boldsymbol{u}$, and one writes

$$\boldsymbol{u}(\boldsymbol{x}) : \mathbb{R}^3 \mapsto \mathbb{R}^3. \tag{B.12}$$

The function $\boldsymbol{u}(\boldsymbol{x})$ is called a vector function.

B.3 Partial derivatives

When we have a function $f(x)$ we define the derivative as the slope of the tangent:

$$\frac{df}{dx}(x) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{B.13}$$

The so-called second derivative of f is simply the derivative of the derivative:

$$\frac{d^2 f}{dx^2}(x) \equiv \lim_{\Delta x \to 0} \frac{\frac{df}{dx}(x+\Delta x) - \frac{df}{dx}(x)}{\Delta x}. \tag{B.14}$$

When f is not only a function of x but also a function of y we obtain almost exactly the same thing:

$$\frac{\partial f}{\partial x}(x,y) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \tag{B.15}$$

Note that there are three important differences:

(a) we write "curved d's" instead of straight d's,
(b) we write y in the argument list of f,
(c) we call this derivative the "x-derivative of f".

All of these differences are necessary to distinguish between the function of one variable and the function of two variables. The second derivative becomes

$$\frac{\partial^2 f}{\partial x^2}(x,y) \equiv \lim_{\Delta x \to 0} \frac{\frac{\partial f}{\partial x}(x+\Delta x, y) - \frac{\partial f}{\partial x}(x,y)}{\Delta x}. \tag{B.16}$$

In addition, we also have a so-called "y-derivatives":

$$\frac{\partial f}{\partial y}(x,y) \equiv \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}, \tag{B.17}$$

and

$$\frac{\partial^2 f}{\partial y^2}(x,y) \equiv \lim_{\Delta y \to 0} \frac{\frac{\partial f}{\partial y}(x, y+\Delta y) - \frac{\partial f}{\partial y}(x,y)}{\Delta y}, \tag{B.18}$$

which measure the slopes "in y-direction".

Finally, we also have mixed deriatives, for example

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) \equiv \lim_{\Delta y \to 0} \frac{\frac{\partial f}{\partial x}(x, y+\Delta y) - \frac{\partial f}{\partial x}(x,y)}{\Delta y}, \tag{B.19}$$

which measures the slope "in y-direction" of the x-derivative of f, and

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) \equiv \lim_{\Delta x \to 0} \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x,y)}{\Delta x}, \tag{B.20}$$

which measures the slope "in x-direction" of the y-derivative of f.

B.4 Differentiation: product rule and chain rule

Product rule Suppose we have two functions of one variable, say $f(x)$ and $g(x)$, then we could wonder what the derivative of the product is. With the definition of the derivative we find

$$\frac{d}{dx}\left(f(x)g(x)\right) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}. \tag{B.21}$$

When Δx becomes smaller and smaller the functions $f(x)$ and $g(x)$ locally start looking as straight lines. Think of the earth observed from the moon, from a satellite, and from a ship (see Fig. (B.2)).

Figure B.2: Earth observations from three different view points: gradual transition from a circle to a line.

So, when Δx is sufficiently small we may locally approximate

$$f(x) \approx ax + b, \quad g(x) \approx px + q, \quad \frac{df}{dx} \approx a, \quad \frac{dg}{dx} \approx p, \tag{B.22}$$

so

$$f(x)g(x) \approx apx^2 + (aq + pb)x + bq, \tag{B.23}$$

and the outcome of the limit above is

$$\frac{d}{dx}\left(f(x)g(x)\right) = 2apx + (aq + pb) \tag{B.24}$$

which can be rewritten as

$$\frac{d}{dx}\left(f(x)g(x)\right) = a(px+q) + (ax+b)p, \tag{B.25}$$

or

$$\boxed{\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}g + f\frac{dg}{dx},} \tag{B.26}$$

which is the product rule of differentiation

Chain rule Suppose we have two functions of one variable, say $f(x)$ and $x(t)$, then we could construct a new so-called nested function $f(x(t))$, see the illustration in Fig. (B.3). With the definition of the derivative we find

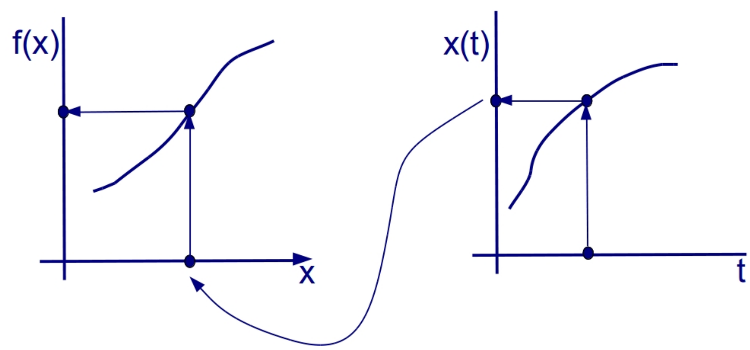

Figure B.3: A nested function $f(x(t))$.

$$\frac{d}{dx}\left(f(x(t))\right) \equiv \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t)) - f(x(t))}{\Delta t}. \tag{B.27}$$

We again can locally approximate both functions $f(x)$ and $x(t)$ as straight lines:

$$f(x) \approx ax + b, \quad x(t) \approx pt + q, \quad \frac{df}{dx} \approx a, \quad \frac{dx}{dt} \approx p, \tag{B.28}$$

so

$$f(x(t)) \approx a(pt+q) + b. \tag{B.29}$$

As a result

$$\frac{d}{dt}\left(f(x(t))\right) = ap, \tag{B.30}$$

which can be written as

$$\boxed{\frac{d}{dt}\left(f(x(t))\right) = \frac{df}{dx}\frac{dx}{dt},} \tag{B.31}$$

which is the chain rule of differentiation.