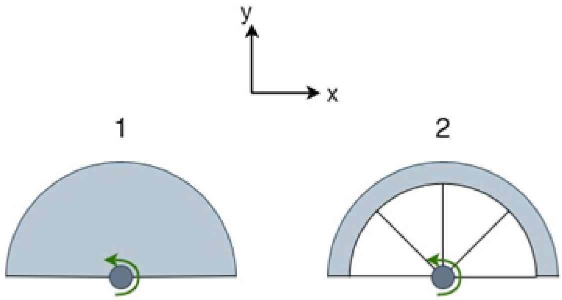
Mass Moment of Inertia Depending on Axis of Rotation



A half-disk (1) and half-ring (2) of equal masses and identical outer radii can be rotated about a point as depicted in the figure.
Which of the statements about the mass moments of inertia I_{xx}, I_{yy} and I_{zz} are correct for both objects?
Hint **: The rotations shown in the figure correspond to the mass moments of inertia I_{zz} .**

Using known expressions:

$$I_{xx} = \int r_x^2 \, \mathrm{d}m \tag{1}$$

$$I_{yy} = \int r_y^2 \, \mathrm{d}m \tag{2}$$

$$I_{zz} = \int r_z^2 \, \mathrm{d}m \quad \Rightarrow \quad I_{zz} = I_{xx} + I_{yy} \tag{3}$$

Solution:

Since working with circles it is easier to convert the equations to polar coordinates. I_{xx} and I_{yy} become:

$$I_{xx} = \rho \cdot t \int_0^R \int_0^\pi r^2 \cdot r \, \mathrm{d}\theta \mathrm{d}r \tag{4}$$

$$I_{yy} = \rho \cdot t \int_0^R \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} r^2 \cdot r \; \mathrm{d}\theta \mathrm{d}r \tag{5}$$

Where ρ is the density and t is the thickness of the objects. First we work Equation 4 out for the half-disk. This results in.

$$I_{xx} = \rho \cdot t \int_0^R \pi r^3 \; \mathrm{d}r \tag{6}$$

$$I_{xx} = \rho \cdot t \cdot \left[\pi \cdot \frac{1}{4} r^4\right]_0^R \tag{7}$$

$$I_{xx} = \rho \cdot t \cdot \pi \cdot \frac{1}{4} R^4 \tag{8}$$

The mass of a half-disk is $m = \rho \cdot t \cdot \pi R^2$, inserting this in Equation 8 results in.

$$I_{xx} = \frac{1}{4} m R^2 \tag{9}$$

Now we work out Equation 5 out for the half-disk. This results in.

$$I_{yy} = \rho \cdot t \int_0^R r^3 \, [\theta]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \; \mathrm{d}r \tag{10}$$

$$I_{yy} = \rho \cdot t \int_0^R r^3 (\frac{1}{2}\pi - -\frac{1}{2}\pi) \; \mathrm{d}r \tag{11}$$

$$I_{yy} = \rho \cdot t \int_0^R \pi r^3 \; \mathrm{d}r \tag{12}$$

$$I_{yy} = \rho \cdot t \cdot \left[\pi \cdot \frac{1}{4} r^4\right]_0^R \tag{13}$$

$$I_{yy} = \rho \cdot t \cdot \pi \cdot \frac{1}{4} R^4 \tag{14}$$

Again, we use $m = \rho \cdot t \cdot \pi R^2$.

$$I_{yy} = \frac{1}{4} m R^2 \tag{15}$$

Inserting I_{xx} and I_{yy} in Equation 3 gives:

$$I_{zz} = I_{xx} + I_{yy} \quad \Rightarrow \quad I_{zz} = \frac{1}{4} m R^2 + \frac{1}{4} m R^2 = \frac{1}{2} m R^2 \tag{16}$$

Thus for the half-disk the following relation is true:

$$I_{xx} = I_{yy} < I_{zz} \tag{17}$$

The half-ring can be seen as a larger half-disk (with radius R_2 and mass m_2) minus a smaller half-disk with radius (with radius R_1 and mass m_1). This results in the following relations for I_{xx} and I_{yy}

$$I_{xx} = I_{yy} = \frac{1}{4} m_2 {r_2}^2 - \frac{1}{4} m_1 {r_1}^2 \tag{18}$$

$$I_{xx} = I_{yy} = \frac{1}{4}(\rho \cdot t \cdot \pi {r_2}^2) \cdot {r_2}^2 - \frac{1}{4}(\rho \cdot t \cdot \pi {r_1}^2) \cdot {r_1}^2 \tag{19}$$

$$I_{xx} = I_{yy} = \frac{1}{4}\rho \cdot t \cdot \pi \cdot ({r_2}^4 - {r_1}^4) \tag{20}$$

Using algebra we can decompose $({r_2}^4 - {r_1}^4)$ in $({r_2}^2 - {r_1}^2)({r_2}^2 + {r_1}^2)$. Inserting this gives us.

$$I_{xx} = I_{yy} = \frac{1}{4}\rho \cdot t \cdot \pi \cdot ({r_2}^2 - {r_1}^2)({r_2}^2 + {r_1}^2) \tag{21}$$

Where the mass of the ring is $m = \rho \cdot t \cdot \pi \cdot ({r_2}^2 - {r_1}^2)$.

$$I_{xx} = I_{yy} = \frac{1}{4} m \cdot ({r_2}^2 + {r_1}^2) \tag{22}$$

Inserting this in Equation 3 gives:

$$I_{zz} = I_{xx} + I_{yy} \quad \Rightarrow \quad I_{zz} = \frac{1}{4} m \cdot ({r_2}^2 + {r_1}^2) + \frac{1}{4} m \cdot ({r_2}^2 + {r_1}^2) = \frac{1}{2} m \cdot ({r_2}^2 + {r_1}^2) \tag{23}$$

Thus $I_{xx} = I_{yy} < I_{zz}$ for both the half-disk and half-ring.