Tutorial T03 – Elasticity – Stress

November 26, 2024

Answer the following questions as they could come up in an exam.

Exercises 1,2,4 will be continued in tutorial T04

3 Stress tensor basics

... based on section 3,4 (Exercise V3 in old material before 2022)

Given:

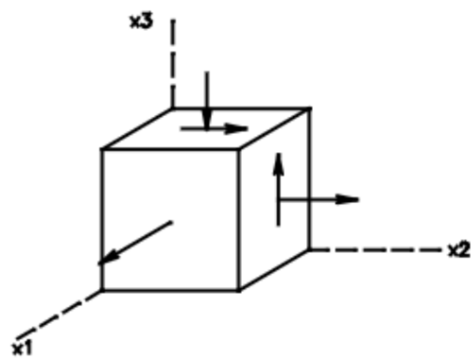

Figure 1: Stress cube, empty → fill it

The stress-state is described by the matrix: $\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix}$ N/mm^2,
with $E = 2 \cdot 10^5$ N/mm^2, and $\nu = 0.25$.

Questions:
a) Compute the principal stresses
b) Compute the eigen-directions
c) Compute the maximal shear-stress
d) Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x'_p, i.e. the coordinate system defined by the eigen-directions.
e) Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.

Answers:
a)
The sorted eigen-values are: $\sigma_I = 60\,\mathrm{MPa}$, $\sigma_{II} = 40\,\mathrm{MPa}$, $\sigma_{III} = -40\,\mathrm{MPa}$.
The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).

b)

Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example σ_{II}, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad n_1^2 + n_2^2 + n_3^2 = 1$$

$-20n_2 + 20\sqrt{3}n_3 = 0$

$20\sqrt{3}n_2 - 60n_3 = 0$

$n_2 = \sqrt{3}n_3$ and $n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3$ (identical due to dependency)

$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$

$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above. *The third eigenvalue calculation is similar (not shown).*

c)
The maximum shear stress is: $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50$ MPa.

d)
The maximal shear stress acts on a surface rotated by 45^o from the x_1' and x_3' directions, related to eigen-directions of σ_I and σ_{III}, respectively, see sketch.
In this coordinate system, the normalized unit vector is obtained from the $(1, 0, 1)$ direction,

but still has to be normalized, so that: $\hat{n}^{\tau_{max}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

e) Graphic/sketch

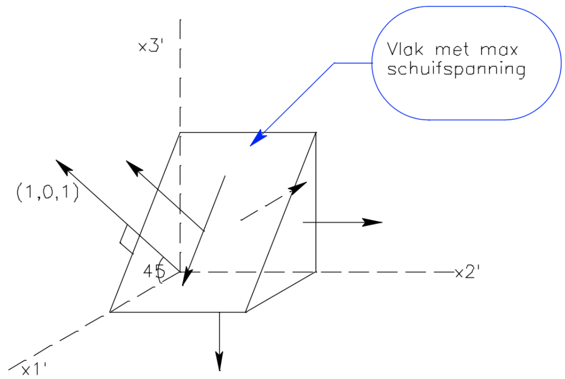

Figure 2: Sketch of the normal to the plane with maximal shear stress (in Dutch: "vlak met maximale schuifspanning"), in the coordinate system x'_p of the eigen-directions of stress, with perpendicular (sorted) intermediate stress eigen-direction x'_2.

5 Stress equilibrium

... based on sections 3,5 (Exercise V12 in old material before 2022)

In a linear elastic ($E = 2 \cdot 10^5$ MPa, $\nu = 0.25$) body under load, the stress-field is given (with four free parameters), with respect to the Cartesian $x_1 - x_2 - x_3$ coordinate system as:

$\sigma_{11}(x_1, x_2, x_3) = \sigma_0 \left[20 + \alpha_1(\frac{x_1}{L}) - 10(\frac{x_2}{L}) + \alpha_2(\frac{x_1}{L})^2\right]$

$\sigma_{22}(x_1, x_2, x_3) = \sigma_0 \left[10 + 8(\frac{x_1}{L}) + \beta_1(\frac{x_2}{L}) + \beta_2(\frac{x_2}{L})^2\right]$

$\sigma_{12}(x_1, x_2, x_3) = \sigma_0 \left[12 - 10(\frac{x_1}{L}) + 7(\frac{x_2}{L}) - 8(\frac{x_1}{L})(\frac{x_2}{L})\right]$

$\sigma_{13}(x_1, x_2, x_3) = \sigma_{23}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = 0$, and
with reference stress $\sigma_0 = 1\,\mathrm{MPa}$ and reference length $L = 1\,\mathrm{m}$.

Note: Question (a) is general, symbolic, with variables x_1, x_2, x_3 and coefficients α_1, α_2, β_1, β_2; only from question (b) on, use the single, chosen point $\mathrm{P}(x_1 = 0, x_2 = 0, x_3 = 0)$.

Questions:
... based on section 3
a) Does the stress field agree with the stress-equilibrium equations in absence of volume-forces? Which relations have to be valid for the free coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ due to stress equilibrium?
b) Compute the eigen-stresses in point P using linear algebra, mathematics – not the graphical circle of Mohr procedure.
Describe and name the state of stress in point P (and in all other points in the body).
c) Compute the eigen-direction of the major eigen-stress.
d) Draw the relevant circle of Mohr and confirm graphically the results of (b) and (c); explain.

<u>**Answers:**</u>
a)

Given was the plane stress-field, independent of x_3, in absence of body forces $f_i = 0$:

$$\sigma_{11}(x_1,x_2) = \sigma_0\left[20 + \alpha_1\frac{x_1}{L} - 10\frac{x_2}{L} + \alpha_2\left(\frac{x_1}{L}\right)^2\right]$$
$$\sigma_{22}(x_1,x_2) = \sigma_0\left[10 + 8\frac{x_1}{L} + \beta_1\frac{x_2}{L} + \beta_2\left(\frac{x_2}{L}\right)^2\right]$$
$$\sigma_{12}(x_1,x_2) = \sigma_0\left[12 - 10\frac{x_1}{L} + 7\frac{x_2}{L} - 8\frac{x_1}{L}\frac{x_2}{L}\right]$$

Using the respective stress-equilibrium equations, in this case two, one obtains:

$$\frac{d}{dx_1}\sigma_{11}(x_1,x_2) + \frac{d}{dx_2}\sigma_{12}(x_1,x_2) = \sigma_0\left[\frac{\alpha_1}{L} + 2\alpha_2\frac{x_1}{L^2}\right] + \sigma_0\left[\frac{7}{L} - 8\frac{x_1}{L^2}\right] = 0$$
$$\frac{d}{dx_1}\sigma_{12}(x_1,x_2) + \frac{d}{dx_2}\sigma_{22}(x_1,x_2) = \sigma_0\left[\frac{-10}{L} - 8\frac{x_2}{L^2}\right] + \sigma_0\left[\frac{\beta_1}{L} + 2\beta_2\frac{x_2}{L^2}\right] = 0$$

From these equations, one gets the coefficients that solve them: $\alpha_1 = -7$, $\alpha_2 = 4$, $\beta_1 = 10$, $\beta_2 = 4$.

Because the field equations must be valid for all constants and points x_1, x_2, x_3, independently, one can group them accordingly: The constant terms from the first and second equations provide α_1 and β_1, respectively, while the x_1 and x_2 groups provide α_2 and β_2.

b)

The stress Tensor in point $P = (x_1 = 0, x_2 = 0, x_3 = 0)$ is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 20 & 12 & 0 \\ 12 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ MPa

From this stress tensor, the characteristic equation is:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 30\sigma^2 + 56\sigma - 0 = (\sigma^2 - 30\sigma + 56)(\sigma - 0) = 0\,.$$

Knowing/recognizing that one eigen-value is zero, i.e. also $I_3 = 0$, the principal stresses can be computed from the second order polynomial as: $\sigma_I = 28\,\text{MPa}$, $\sigma_{II} = 2\,\text{MPa}$, $\sigma_{III} = 0\,\text{MPa}$. This is a plane-stress state with all stresses on the x_3-surface equal to zero, which also has consequences for the eigen-directions ...

c)
The principal directions can be calculated the usual way, where $\hat{\mathbf{n}}^{(III)} = (0,0,1)$ is directly visible from the tensor, due to the zero shear stresses in the x_3-direction.

The eigen-direction of the major stress $\sigma_I = 28\,\text{MPa}$ is obtained solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and normalization: } n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

so that: $-8n_1^{(I)} + 12n_2^{(I)} = 0 \rightarrow n_1^{(I)} = (3/2)n_2^{(I)}$ and thus: $[(9/4)+1]n_2^{(I)} = 1 \rightarrow n_2^{(I)} = 2/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \\ 0 \end{bmatrix}$$

The eigen-direction of the intermediate stress, $\sigma_{II} = 2\,\text{MPa}$ was not asked, just for completeness:

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

so that: $18n_1^{(II)} + 12n_2^{(II)} = 0 \rightarrow n_1^{(II)} = -(2/3)n_2^{(II)}$ and thus: $[(4/9)+1]n_2^{(II)} = 1 \rightarrow n_2^{(II)} = 3/\sqrt{13}$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm \begin{bmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix}$$

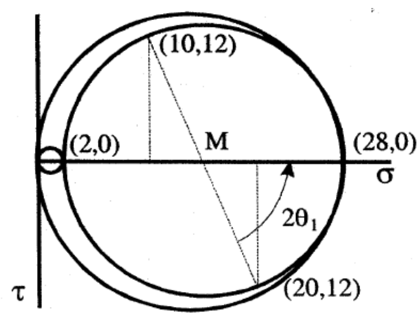

Figure 3: Sketch of a Mohr circle, focus is on the right, inner circle.

d)
Mohr's circle
Consider only the two non-zero eigenvalues that characterise the plane-stress state in point P.

The circle centre is: $M = \sigma_{avg} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{20+10}{2} = 15\,\text{MPa}$,
and its radius is: $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{20-10}{2}\right)^2 + (12)^2} = 13\,\text{MPa}$.

The eigenvalues are therefore:
$\sigma_I = M + R = 28\,\text{MPa}$, $\sigma_{II} = C - R = 2\,\text{MPa}$.

The eigen-directions are:
$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{24}{10} = 2.4 \Longrightarrow \theta_I = (1/2)\arctan(2.4) = 67.38°/2 = 33.69°$, which corresponds to the orientation of the first eigenvector relative to the horizontal $\theta_I = \arcsin(2/\sqrt{13}) = \arccos(3/\sqrt{13})$; and $\theta_{II} = (180° + 67.3°)/2 = 247.3°/2 = 123.7° = \arccos(-2/\sqrt{13})$.

The maximum shear stress is just the radius: $\tau^{max} = R = 13$ MPa

6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:
$E = 2 \cdot 10^{11}$ Pa, $\nu = 0.25$

Stress-state in point P: $\left[\sigma\right] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa

Questions:
a) Show that the principal stresses are 8, 16 and 24 MPa.
Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.

<u>Answers:</u>
a)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0\,.$$

Given the eigenvalues, σ, one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*
Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 24\,\text{MPa}$, $\sigma_{II} = 16\,\text{MPa}$, and $\sigma_{III} = 8\,\text{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.
The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8\,\text{MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$$

$$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$$

$$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$

Inserting into line 3 yields: $n_3 = \sqrt{6}n_1$, so that: $\begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\, c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$