

Solution V.2: (Soaked towel ★★)

Task a)

To determine the mass transfer coefficient, the analogy between convective heat and mass transfer must be used. The problem can be seen as forced flow over a flat plate.

The properties of air are taken from the book of formularies:

$$\nu = 15.35 \cdot 10^{-6} \text{ m}^2/\text{s},$$

$$\text{Pr} = 0.7148,$$

$$\lambda = 25.69 \cdot 10^{-3} \text{ W/mK},$$

and

$$c_p = 1007 \text{ J/kgK}.$$

To determine the correlation that is valid, the flow regime must be determined. Using the definition of the Reynolds number:

$$\begin{aligned} \text{Re} &= \frac{uL}{\nu} \\ &= \frac{1 \left(\frac{\text{m}}{\text{s}} \right) \cdot 0.5 \text{ (m)}}{15.35 \cdot 10^{-6} \left(\frac{\text{m}^2}{\text{s}} \right)} = 3.3 \cdot 10^4, \end{aligned} \quad (\text{V.2.1})$$

it yields that the flow falls within the laminar regime. Therefore HTC.2 is applicable.

HTC

Average Nusselt number for forced laminar flow over a flat plate with isothermal surface:

$$\overline{\text{Nu}}_L = 0.664 \text{ Re}_L^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}}, \quad (\text{HTC.2})$$

for $\text{Re}_L < 2 \cdot 10^5$, $0.6 < \text{Pr} < 10$, and where $\text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$.

Using the Nusselt correlation, the Nusselt number is determined:

$$\begin{aligned} \overline{\text{Nu}}_L &= 0.664 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \\ &= 0.664 \cdot (3.3 \cdot 10^4)^{\frac{1}{2}} \cdot (0.7148)^{\frac{1}{3}} = 107 (-), \end{aligned} \quad (\text{V.2.2})$$

from which the average heat transfer coefficient is determined:

$$\begin{aligned} \overline{\alpha} &= \frac{\overline{\text{Nu}}_L \cdot \lambda}{L} \\ &= \frac{107 (-) \cdot 25.69 \cdot 10^{-3} \left(\frac{\text{W}}{\text{mK}} \right)}{0.5 \text{ (m)}} = 0.55 \left(\frac{\text{W}}{\text{m}^2\text{K}} \right). \end{aligned} \quad (\text{V.2.3})$$

Conclusion

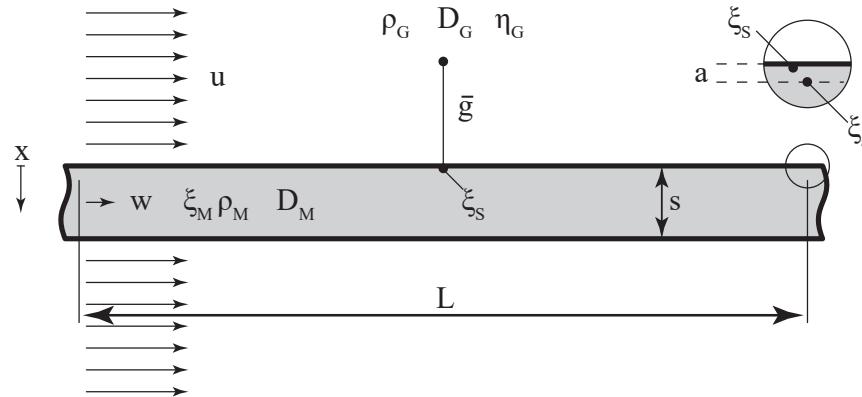
By using the Lewis relation, the mass transfer coefficient is determined:

$$\begin{aligned}\bar{g} &= \frac{\bar{\alpha}}{c_p} \\ &= \frac{0.55 \left(\frac{W}{m^2 K} \right)}{1007 \left(\frac{J}{kg K} \right)} = 5.5 \cdot 10^{-3} \left(\frac{kg}{m^2 s} \right).\end{aligned}\quad (V.2.4)$$

Thus the mass transfer coefficient $g = 5.5 \cdot 10^{-3} \left(\frac{kg}{m^2 s} \right)$.

Exercise V.3: (Metal strap ★★★)

A metal strap (M) with a sheet-thickness s is pulled at a velocity w through an oven of length L . Thereby a CO₂-containing gas atmosphere (G) flows over the strap with a velocity u . On entering the furnace, the metal strap has a homogeneous fraction of carbon ξ_M . Under quasi-steady conditions, a constant fraction of carbon ξ_S exists just below the surface of the metal strip, while carbon constantly diffuses into its interior.

**Given parameters:**

- Diffusion coefficient of carbon in metal: $D_M = 5 \cdot 10^{-10} \text{ m}^2/\text{s}$
- Diffusion coefficient of carbon in gas: $D_G = 1 \cdot 10^{-4} \text{ m}^2/\text{s}$
- Combined density of metal: $\rho_M = 8 \cdot 10^3 \text{ kg/m}^3$
- Combined density of the gases: $\rho_G = 0.8 \text{ kg/m}^3$
- Combined viscosity of the gases: $\eta_G = 5.6 \cdot 10^{-5} \text{ Pa} \cdot \text{s}$
- Flow velocity: $u = 10 \text{ m/s}$
- Sheet velocity: $w = 2 \cdot 10^{-2} \text{ m/s}$
- Initial carbon fraction: $\xi_M = 0.2 \cdot 10^{-4}$
- Surface carbon fraction: $\xi_S = 1 \cdot 10^{-4}$
- Oven length: $L = 10 \text{ m}$
- Strap thickness: $s = 50 \cdot 10^{-3} \text{ m}$
- Layer depth: $a = 0.2 \cdot 10^{-3} \text{ m}$

Hints:

- Use a co-moving coordinate system.
- Neglect diffusion in the direction of the sheet.
- The strap thickness is significantly greater than twice the depth of penetration of the carbon.

Tasks:

- a) Calculate the mean mass transfer coefficient \bar{g} on the strip's surface. Show that the assumption of a constant carbon content of ξ_s just below the surface is valid.
- b) Provide the differential equation for the carbon transport in the strap as well as the appropriate initial and boundary conditions. For this, provide a suitable equation for the carbon content in the strap $\xi(x,t)$.
- c) Calculate the carbon content of ξ_a which is established at the end of the furnace at a depth a from the strap's surface. Show that the assumption of a low penetration depth is justified.

Solution V.3: (Metal strap ★★)

Task a)

To calculate the mass transfer coefficient, Lewis' law can be applied, describing the relationship between the heat and mass transfer coefficient. First, an applicable correlation must be found. The problem can be considered as flow over a flat plate. The Reynolds number is determined from:

$$\begin{aligned} \text{Re} &= \frac{\rho_G (u - w) L}{\eta_G} \\ &= \frac{0.8 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot (10 - 2 \cdot 10^{-2}) \left(\frac{\text{m}}{\text{s}} \right) \cdot 10 \text{ (m)}}{5.6 \cdot 10^{-5} \text{ (Pa} \cdot \text{s)}} = 1.43 \cdot 10^6 \text{ (-).} \end{aligned} \quad (\text{V.3.1})$$

Since the mass fraction at the surface is uniform across the surface and $5 \cdot 10^5 < \text{Re} < 10^7$, HTC.6 can be used to determine the average heat transfer coefficient.

HTC

Average Nusselt number for forced turbulent flow over a flat plate with isothermal surface:

$$\overline{\text{Nu}}_L \approx 0.036 \text{ Pr}^{0.43} (\text{Re}_L^{0.8} - 9400), \quad (\text{HTC.6})$$

for $5 \cdot 10^5 < \text{Re}_L < 10^7$, where $\text{Re}_{x,\text{crit}} \approx 2 \cdot 10^5$.

The Schmidt number is calculated from:

$$\begin{aligned} \text{Sc} &= \frac{\eta_G}{\rho_G D_G} \\ &= \frac{5.6 \cdot 10^{-5} \text{ (Pa} \cdot \text{s)}}{0.8 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 1 \cdot 10^{-4} \left(\frac{\text{m}^2}{\text{s}} \right)} = 0.7 \text{ (-).} \end{aligned} \quad (\text{V.3.2})$$

Using Lewis' law, this can be rewritten into a correlation describing the Sherwood number:

$$\begin{aligned} \overline{\text{Sh}}_L &= 0.036 \text{ Sc}^{0.43} (\text{Re}_L^{0.8} - 9400) \\ &= 0.036 (0.7)^{0.43} \left[(1.43 \cdot 10^6)^{0.8} - 9400 \right] = 2.3 \cdot 10^3 \text{ (-).} \end{aligned} \quad (\text{V.3.3})$$

Rewriting the definition of the Sherwood number gives the average mass transfer coefficient:

$$\begin{aligned} \bar{g} &= \frac{\overline{\text{Sh}}_L \rho_G D_G}{L} \\ &= \frac{2.3 \cdot 10^3 \text{ (-)} \cdot 0.8 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 1 \cdot 10^{-4} \left(\frac{\text{m}^2}{\text{s}} \right)}{10 \text{ (m)}} = 18.4 \cdot 10^{-3} \left(\frac{\text{kg}}{\text{m}^2 \text{s}} \right). \end{aligned} \quad (\text{V.3.4})$$

Conclusion

Lastly, it must be proven that the assumption of a constant carbon content of ξ_s below the surface is valid. This means that the surface resistance is negligible compared to the diffusive resistance, which is the case if $\text{Bi} \gg 1$. Calculating the Biot number gives:

$$\begin{aligned} \text{Bi} &= \frac{\bar{g} s}{\rho_S D_S} \\ &= \frac{18.4 \cdot 10^{-3} \left(\frac{\text{kg}}{\text{m}^2 \text{s}} \right) \cdot 25 \cdot 10^{-3} \text{ (m)}}{0.8 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 1 \cdot 10^{-4} \left(\frac{\text{m}^2}{\text{s}} \right)} = 115, \end{aligned} \quad (\text{V.3.5})$$

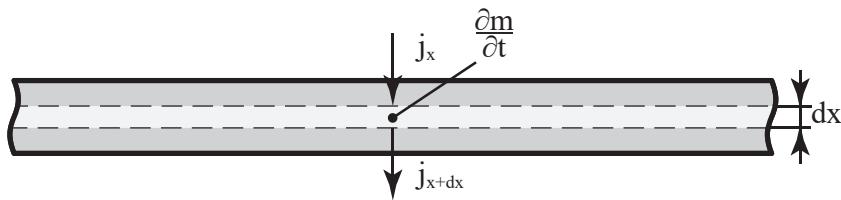
hence $Bi \gg 1$.

Task b)

The differential equation for carbon transport in the strap can be derived by setting up a mass balance around a one-dimensional infinitesimal element within the strap, or it can be directly taken from the book of formulas.

1 Setting up the balance:

Carbon diffuses in and out of the element, changing the carbon concentration over time.



The transient energy balance reads:

$$\underbrace{\frac{\partial m}{\partial t}}_{\text{Temporal change of carbon mass}} = \underbrace{j_x - j_{x+dx}}_{\text{Net rate of diffusion}}. \quad (\text{V.3.6})$$

2 Defining the elements within the balance:

The temporal change of carbon within the element yields from:

$$\frac{\partial m}{\partial t} = \rho A dx \frac{\partial \xi}{\partial t}. \quad (\text{V.3.7})$$

The incoming rate of carbon diffusion is calculated from Fick's law:

$$j_x = -\rho D_M A \frac{\partial \xi}{\partial x}, \quad (\text{V.3.8})$$

and the outgoing rate of carbon diffusion is approximated from the Taylor series expansion:

$$\begin{aligned} j_{x+dx} &= j_x + \frac{\partial j_x}{\partial x} \cdot dx \\ &= -\rho D_M A \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial x} \left(-\rho D_M A \frac{\partial \xi}{\partial x} \right) \cdot dx. \end{aligned} \quad (\text{V.3.9})$$

3 Inserting and rearranging:

$$\frac{\partial \xi}{\partial t} = D_M \frac{\partial^2 \xi}{\partial x^2}. \quad (\text{V.3.10})$$

4 Defining the boundary and/or initial conditions:

In the previous task it was proven that the surface resistance is negligible and therefore the mass

concentration at $x = 0$ can be considered constant. Hence:

$$\xi(x = 0) = \xi_S. \quad (\text{V.3.11})$$

Furthermore, since the strap thickness is significantly greater than twice the depth of penetration of the carbon, the body can be considered to be semi-infinite. Therefore the following boundary condition is applicable infinitely far away:

$$\left. \frac{\partial \xi}{\partial x} \right|_{x \rightarrow \infty} = 0, \quad \text{or} \quad \xi(x \rightarrow \infty) = \xi_M. \quad (\text{V.3.12})$$

Initially, no carbon has diffused into the strap and therefore the mass concentration must be uniform:

$$\xi(t = 0) = \xi_M. \quad (\text{V.3.13})$$

Although the boundary conditions are similar to the case of the Heisler diagrams, an equation is asked that describes the carbon content in the strap and therefore cannot be used. With the analogy of heat and mass transfer, the result of the semi-infinite plate with negligible thermal surface resistance from the book of formularies can be used.

Fundamental EQ

Temperature profile semi-infinite plate with negligible thermal surface resistance:

$$\theta^* = 1 - \operatorname{erf}\left(\frac{1}{\sqrt{4\text{Fo}}}\right),$$

where $\theta^* = \frac{T - T_0}{T_a - T_0}$, and $\text{Fo} = \frac{at}{x^2}$.

Conclusion

Hence:

$$\xi(x, t) = \xi_M + (\xi_S - \xi_M) \left[1 - \operatorname{erf}\left(\frac{1}{\sqrt{4\text{Fo}}}\right) \right], \quad (\text{V.3.14})$$

where $\text{Fo} = \frac{D_M t}{x^2}$.

Task c)

To calculate the carbon content ξ_a , which is established at a depth a from the strap's surface at the end of the furnace, the time elapsed since at which the strap leaves the furnace must first be determined. This time is calculated from:

$$\begin{aligned} t_1 &= \frac{L}{w} \\ &= \frac{10 \text{ (m)}}{2 \cdot 10^{-2} \left(\frac{\text{m}}{\text{s}} \right)} = 500 \text{ (s)}. \end{aligned} \quad (\text{V.3.15})$$

Fourier's number is calculated accordingly:

$$\begin{aligned} \text{Fo} &= \frac{D_M t_1}{a^2} \\ &= \frac{5 \cdot 10^{-10} \left(\frac{\text{m}}{\text{s}^2} \right) \cdot 500 \text{ (s)}}{(0.2 \cdot 10^{-3})^2 \text{ (m}^2\text{)}} = 6.25 \text{ (-)}. \end{aligned} \quad (\text{V.3.16})$$

Conclusion

From which the carbon content ξ_a can be determined:

$$\begin{aligned}\xi(a, t_1) &= \xi_M + (\xi_S - \xi_M) \left[1 - \operatorname{erf} \left(\frac{1}{\sqrt{4Fo}} \right) \right] \\ &= 0.2 \cdot 10^{-4} (-) + (1 \cdot 10^{-4} - 0.2 \cdot 10^{-4}) (-) \cdot \left[1 - \operatorname{erf} \left(\frac{1}{\sqrt{4 \cdot 6.25} (-)} \right) \right] = 0.82 \cdot 10^{-4} (-).\end{aligned}\quad (\text{V.3.17})$$

To show that the assumption of a low penetration depth is justified, the penetration depth must be calculated, which is taken from the book of formularies, and compared to the depth a .

Fundamental EQ

Penetration depth semi-infinite plate with negligible thermal surface resistance:

$$\delta(t) = 3.6\sqrt{at},$$

for $Bi \gg 1$.

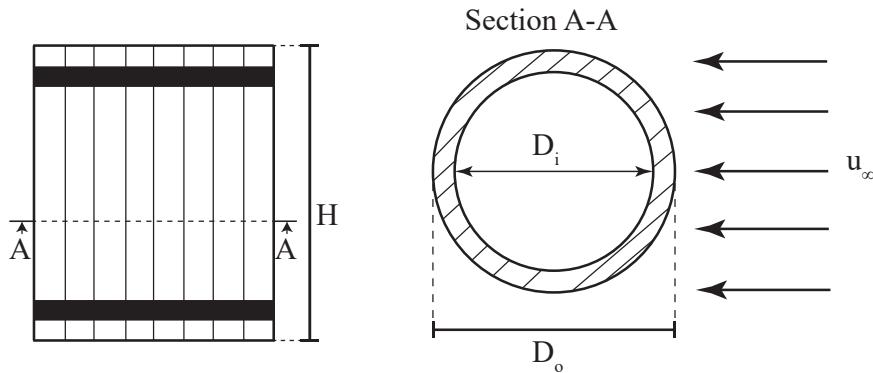
Thus

$$\begin{aligned}\delta &= 3.6\sqrt{D_M t_1} \\ &= 3.6\sqrt{5 \cdot 10^{-10} \left(\frac{\text{m}}{\text{s}^2} \right) \cdot 500 \text{ (s)}} = 25 \cdot 10^{-3} \text{ (m)},\end{aligned}\quad (\text{V.3.18})$$

which is much larger than the depth a .

Exercise V.4: (Whiskey barrel ★★)

Deep in the cellars of a distillery, Whisky is stored in wooden barrels for its maturation. The whisky is composed of an ethanol-water mixture with an ethanol mass fraction $\xi_{EW,t0}$ at the initial time t_0 (at the beginning of the storage period). The ventilation system of the cellar provides a continuous flow of fresh air (free of ethanol, $\xi_{EA,\infty} = 0$) to the barrels. During the storage period, the concentration of ethanol in the Whisky falls due to mass transport processes down to a value of $\xi_{EW,t1}$ (time t_1 , is the end of the storage).



Hint: Use the following nomenclature for the indexes:

- ξ_{EW} : Ethanol mass fraction in Whisky
- $\xi_{EW,i}$: Ethanol mass fraction in Wood, inner side of the barrel
- $\xi_{EW,o}$: Ethanol mass fraction in Wood, outer surface of the barrel
- $\xi_{EA,o}$: Ethanol mass fraction in air, at the outer surface of the barrel
- $\xi_{EA,\infty}$: Ethanol mass fraction in air, at the free flow

Given parameters:

- Initial time of the barrel storage: t_0
- End time of the barrel storage: t_1
- Length of the storage time: $\Delta t = 3 \text{ years}$
- Constant: $Z = 4.34 \cdot 10^{-7} \text{ kg/s}$
- Outer diameter: $D_o = 0.56 \text{ m}$
- Inner diameter: $D_i = 0.50 \text{ m}$
- Height: $H = 0.9 \text{ m}$
- Density of the wood: $\rho_{Wo} = 500 \text{ kg/m}^3$
- Diffusion coefficient of ethanol in wood: $D_{EWo} = 1 \cdot 10^{-11} \text{ m}^2/\text{s}$
- Density of the whisky: $\rho_W = 870 \text{ kg/m}^3$
- Dynamic viscosity of the whisky: $\eta_W = 1 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$
- Mass fraction of ethanol at initial time t_0 : $\xi_{EW,t0} = 0.54$
- Flow velocity: $u_\infty = 0.1 \text{ m/s}$

- Density of the air: $\rho_A = 1.18 \text{ kg/m}^3$
- Diffusion coefficient of ethanol in air: $D_{EA} = 1.47 \cdot 10^{-5} \text{ m}^2/\text{s}$
- Dynamic viscosity of the air: $\eta_A = 1.81 \cdot 10^{-5} \text{ Pa} \cdot \text{s}$

Hints:

- Due to the high relative humidity in the cellar, Whisky only loses ethanol.
- There is no mass transport resistance at the inner surface of the barrel.
- Ignore the Stefan flow.
- The overall density of the Whisky, the wood and the air do **not** change due to mass transport processes.
- The Whisky barrels are cylindrical and ethanol diffuses **only** through the lateral surfaces, neglecting the effect of the metal rings around the barrel.
- The Whisky is ideally mixed inside the barrel.
- All given material properties are constant.
- Jumps in concentrations are only due to density differences.

Tasks:

Hint: For the sub-problems a) to c) the system is in a (quasi-) stationary state and the concentration ξ_{EW} is known at each point in time

- a) Sketch (qualitatively) the radial concentration profile of ethanol ξ_{EW} in the Whisky at any time t within the time interval $t_0 < t < t_1$.
- b) Determine an expression for the ethanol mass flow rate taking place due to mass diffusion through the wood barrel (\dot{m}_D) and another for the mass flow produced by the convection process (\dot{m}_K) occurring at the external surface of the barrel. Additionally, calculate the mass transfer coefficient g . To do this use the known ethanol concentrations at the interfaces.
- c) Determine the unknown ethanol concentrations $\xi_{EW,i}$, $\xi_{EW,o}$, $\xi_{EA,o}$ and $\xi_{EA,\infty}$, and give an expression for the ethanol mass flow rate \dot{m}_E .
- d) By means of an instationary mass balance upon the whisky, calculate the ethanol concentration $\xi_{EW,t1}$ in the barrel, which is attained after a storage time of $\Delta t = 3$ years in the barrel. Use the simplified expression of the ethanol mass flow given by $\dot{m}_E = Z \cdot \xi_{EW}$ with $Z = 4.34 \cdot 10^{-7} \frac{\text{kg}}{\text{s}}$

Solution V.4: (Whiskey barrel **)**Task a)**

Within the barrel, the whiskey is ideally mixed and the mass transport resistance at the inner surface of the barrel is negligible, leading to a constant mass concentration.

At the interface between the whiskey and the barrel, a jump in concentration is observed. This jump in concentration is due to differences in density, as per the given hints. The domain with the lower density, which in this case is the wooden barrel, will have a higher mass concentration.

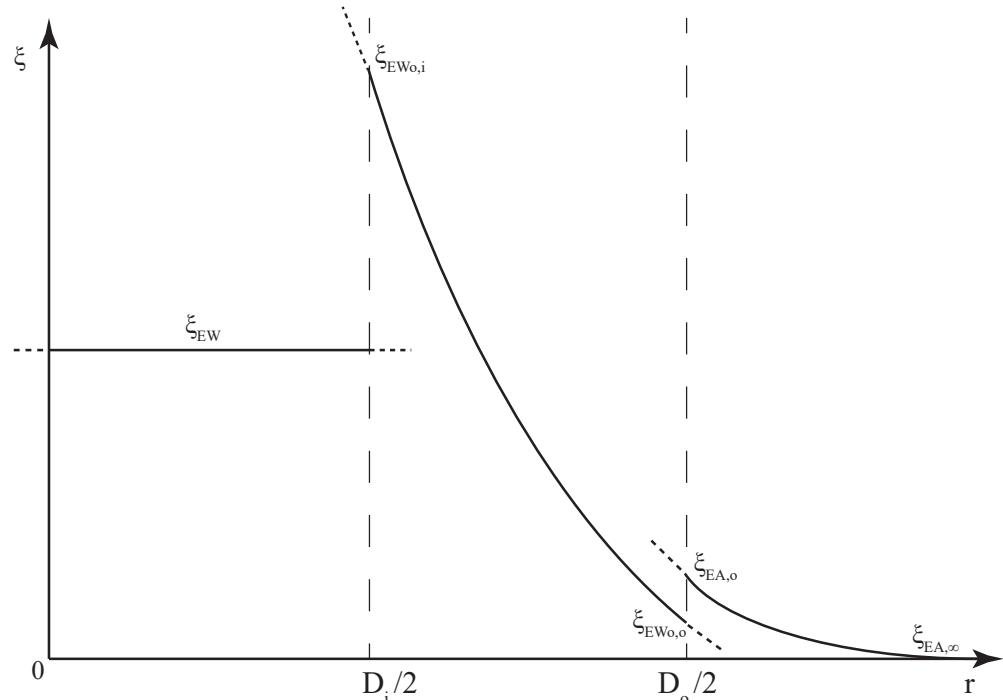
From Fick's law:

$$j = -\rho D A \frac{\partial \xi}{\partial r}, \quad (\text{V.4.1})$$

it is seen that for a constant rate of mass diffusion, density, and diffusion coefficient, with an increase in cross-sectional area in the radial direction, the mass concentration gradually decreases.

At the interface between the barrel and the air, another jump in mass concentration is observed due to the air having a smaller density, thus resulting in a larger mass fraction for the air domain.

Since the system is quasi-stationary, the ethanol concentration in the air at the outer surface of the barrel gradually decreases when moving in the radial direction, eventually approaching the ambient ethanol concentration, which is essentially zero.

Conclusion

Task b)

The ethanol mass flow rate taking place due to mass diffusion through the wood barrel can be derived from the analogy with heat transfer, using the resistance theorem.

For cylindrical coordinates, the diffusive resistance can be taken from the conductive resistance in cylindrical coordinates.

Definition

Conductive resistance of a solid cylindrical layer i :

$$R_{\text{cond.},i} = \frac{1}{2\pi L \lambda_i} \ln \frac{r_{i+1}}{r_i} \left(\frac{\text{K}}{\text{W}} \right). \quad (\text{V.4.2})$$

Hence, for mass diffusion, it is written as:

$$R_{\text{diff}} = \frac{1}{2\pi H D_{\text{EWo}} \rho_{\text{Wo}}} \ln \left(\frac{D_o}{D_i} \right). \quad (\text{V.4.3})$$

Using the resistance theorem:

$$\begin{aligned} \dot{m}_{\text{diff}} &= \frac{\Delta \xi}{R_{\text{diff}}} \\ &= 2\pi H D_{\text{EWo}} \rho_{\text{Wo}} \ln \left(\frac{D_o}{D_i} \right)^{-1} \cdot (\xi_{\text{EWo},i} - \xi_{\text{EWo},o}). \end{aligned} \quad (\text{V.4.4})$$

The mass flow produced by the convection process occurring at the external surface of the barrel can be calculated from the mass transfer coefficient, accordingly:

$$\dot{m}_{\text{conv}} = g \cdot H \pi D_o (\xi_{\text{EA},o} - \xi_{\text{EA},\infty}). \quad (\text{V.4.5})$$

The flow can be classified as external forced convection around a cylinder. Furthermore, it must be determined whether the flow is within the laminar or turbulent regime:

$$\begin{aligned} \text{Re} &= \frac{u_\infty \rho_A D_o}{\eta_A} \\ &= \frac{0.1 \left(\frac{\text{m}}{\text{s}} \right) \cdot 1.18 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 0.56 \text{ (m)}}{1.81 \cdot 10^{-5} \text{ (Pa} \cdot \text{s)}} = 3650 \text{ (-)}. \end{aligned} \quad (\text{V.4.6})$$

Hence, HTC.7 is applicable with $C = 0.683$ and $m = 0.466$.

HTC

Average Nusselt number for forced flow perpendicular to the longitudinal axis of a circular cylinder with isothermal surface:

$$\overline{\text{Nu}}_d = C \text{ Re}_d^m \text{ Pr}^{1/3}, \quad (\text{HTC.7})$$

for $\text{Pr} > 0.7$, where:

Re_d	C	m
0.4 - 4	0.989	0.330
4 - 40	0.911	0.385
40 - 4000	0.683	0.466
4000 - 40,000	0.193	0.618
40,000 - 400,000	0.0266	0.805

The Schmidt number is calculated accordingly:

$$\begin{aligned} \text{Sc} &= \frac{\eta_A}{\rho_A D_{EA}} \\ &= \frac{1.81 \cdot 10^{-5} (\text{Pa} \cdot \text{s})}{1.18 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 1.47 \cdot 10^{-5} \left(\frac{\text{m}^2}{\text{s}} \right)} = 1.04 \ (-). \end{aligned} \quad (\text{V.4.7})$$

Using Lewis' law, it can be rewritten to a correlation for the Sherwood number:

$$\begin{aligned} \overline{\text{Sh}}_d &= C \text{Re}_d^m \text{Sc}^{1/3} \\ &= 0.683 \cdot (3650)^{0.466} \ (-) \cdot 1.04^{1/3} \ (-) = 31.76 \ (-), \end{aligned} \quad (\text{V.4.8})$$

Conclusion

from which the mass transfer coefficient can be calculated:

$$\begin{aligned} g &= \frac{\overline{\text{Sh}}_d \rho_A D_{EA}}{D_o} \\ &= \frac{31.76 \ (-) \cdot 1.18 \left(\frac{\text{kg}}{\text{m}^3} \right) \cdot 1.47 \cdot 10^{-5} \left(\frac{\text{m}^2}{\text{s}} \right)}{0.56 \ (\text{m})} = 9.8 \cdot 10^{-4} \left(\frac{\text{kg}}{\text{m}^2 \text{s}} \right). \end{aligned} \quad (\text{V.4.9})$$

Task c)

Given that air is free of ethanol, the concentration far away is written as:

$$\xi_{EA,\infty} = 0. \quad (\text{V.4.10})$$

The ethanol mass fraction $\xi_{EH,i}$ can be determined by relating the mass of ethanol in the whisky $\xi_{EW} \cdot \rho_W$ to the density of the wood. According to the note, there are no mass transport resistances, and the only reason for a concentration jump is the changing density of the medium (whisky \rightarrow wood). Since the mass of ethanol at the interface must be the same, the mass fraction can be derived from a mass balance at the interface:

$$\begin{aligned} \xi_{EWo,i} \cdot \rho_{Wo} &= \xi_{EW} \cdot \rho_W \\ \Rightarrow \xi_{EWo,i} &= \xi_{EW} \cdot \frac{\rho_W}{\rho_{Wo}}. \end{aligned} \quad (\text{V.4.11})$$

The ethanol mass fraction $\xi_{EW,o}$ in the air at the outer surface of the barrel is determined using the same method:

$$\begin{aligned} \xi_{EA,o} \cdot \rho_A &= \xi_{EWo,o} \cdot \rho_{Wo} \\ \Rightarrow \xi_{EA,o} &= \xi_{EWo,o} \cdot \frac{\rho_{Wo}}{\rho_A}. \end{aligned} \quad (\text{V.4.12})$$

The ethanol mass fraction $\xi_{EWo,o}$ in the wood at the outer surface of the barrel can be determined by equating m_{conv} and m_{diff} from part b). This is valid for mass conservation for a quasi-steady state:

$$\begin{aligned} H\pi D_o g \cdot (\xi_{EW,o} - \xi_{EA,\infty}) &= 2H\pi D_{EWo} \rho_{Wo} \ln \left(\frac{D_o}{D_i} \right)^{-1} \cdot (\xi_{EH,i} - \xi_{EWo,o}) \\ \Rightarrow \xi_{EWo,o} &= \xi_{EW} \cdot \frac{2D_{EWo} \rho_W \ln \left(\frac{D_o}{D_i} \right)^{-1}}{\frac{\rho_{Wo}}{\rho_A} D_o g + 2D_{EWo} \rho_{Wo} \ln \left(\frac{D_o}{D_i} \right)^{-1}}. \end{aligned} \quad (\text{V.4.13})$$

Conclusion

The total transferred ethanol mass flow rate then results from:

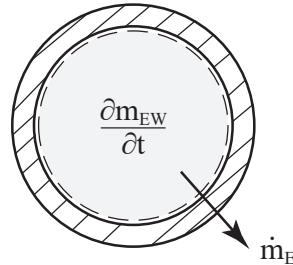
$$\begin{aligned}\dot{m}_E &= \frac{\Delta\xi}{R_{\text{diff}} + R_{\text{conv}}} \\ &= \xi_{EW,i} \frac{\pi H}{\frac{\ln(\frac{D_o}{D_i})}{2D_{EW,i}\rho_{W,i}} + \frac{\rho_A}{\rho_{W,i}g D_o}}.\end{aligned}\quad (\text{V.4.14})$$

Task d)

To calculate the ethanol concentration $\xi_{EW,t1}$ by means of an transient mass balance upon the whiskey, a global energy balance around the whiskey domain is required.

1 Setting up the balance:

The whiskey loses ethanol due to diffusion through the barrel wall. As a consequence, the mass of ethanol within the whiskey changes over time.



Hence, the transient balance reads:

$$\underbrace{\frac{\partial m_{EW}}{\partial t}}_{\substack{\text{Temporal change of} \\ \text{ethanol in whiskey}}} = \underbrace{-\dot{m}_E}_{\substack{\text{Rate of} \\ \text{ethanol loss}}}. \quad (\text{V.4.15})$$

2 Defining the elements within the balance:

The mass of ethanol in the whiskey yields from the product of the whiskey mass and the mass fraction of ethanol. Hence, the temporal change of ethanol mass is written as:

$$\frac{\partial m_{EW}}{\partial t} = \rho_W H \frac{\pi D_i^2}{4} \frac{\partial \xi_{EW}}{\partial t}, \quad (\text{V.4.16})$$

and the rate of ethanol loss is calculated from:

$$\dot{m}_E = Z \cdot \xi_{EW}. \quad (\text{V.4.17})$$

3 Inserting and rearranging:

$$\frac{1}{\xi_{EW}} \frac{\partial \xi_{EW}}{\partial t} = - \frac{4Z}{\rho_W H \pi D_i^2}. \quad (\text{V.4.18})$$

4 Defining the boundary and/or initial conditions:

Initially:

$$\xi(t=0) = \xi_{\text{EW,t0}} \quad (\text{V.4.19})$$

5 Solving the equation:

Integrating both sides of the balance:

$$\int_{\xi_{\text{EW,t0}}}^{\xi_{\text{EW}}} \frac{1}{\xi_{\text{EW}}} d\xi_{\text{EW}} = - \int_0^t \frac{4Z}{\rho_W H \pi D_i^2} dt. \quad (\text{V.4.20})$$

Solving yields

$$\xi_{\text{EW}}(t) = \xi_{\text{EW,t0}} \cdot \exp\left(-\frac{4Z}{\rho_W \pi D_i^2 H} t\right). \quad (\text{V.4.21})$$

Conclusion

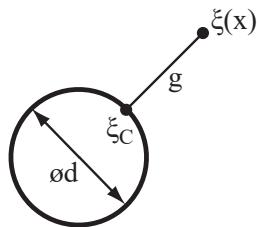
Hence:

$$\xi_{\text{EW}}(\Delta t) = \xi_{\text{EW,t0}} \cdot \exp\left(-\frac{4Z}{\rho_W \pi D_i^2 H} \Delta t\right). \quad (\text{V.4.22})$$

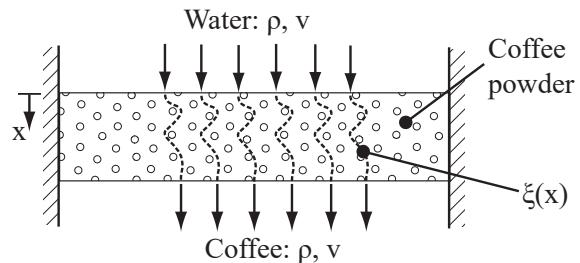
Exercise V.5: (Coffee brewing ★★)

Coffee powder consists of many coffee grains, forming a highly concentrated coffee solution during brewing. The flavors come out of the coffee grains (ξ_C) through diffusion into the passing water flow ($\xi(x)$). This diffusion process is well described by the mass-transfer coefficient g .

a) Idealized coffee grain (sphere)



b) Coffee preparation

**Given parameters:**

- Diffusion coefficient of the flavors in water: D
- Water flow velocity: v
- Total density of flavored water: ρ
- Concentration of highly concentrated coffee solution: ξ_C
- Mass transfer coefficient: g
- Diameter of a coffee grain: d
- Packing density in the coffee powder: p

Hints:

- The problem is stationary and the Stefan flow is negligible.
- All coffee grains have the same diameter and are homogeneously distributed in the powder.
- If you cannot solve part a), you may use $\dot{m}_C''' = \frac{g}{d} \cdot (\xi_C - \xi(x))$.

Tasks:

- a) Determine the mass flow out of the grains per unit volume \dot{m}_C''' , which is transferred from the coffee grains to the passing water flow.
- b) Derive the differential equation for the mass concentration of flavors $\xi(x)$ in the passing water flow. Consider the packing density p of the coffee grains within the ground coffee powder in the source term: $\dot{m}_C''' = p \cdot \dot{m}_C'''$.
- c) Write down the required differential equation and the boundary conditions, when diffusion is negligible in the flow direction, and determine the concentration profile of $\xi(x)$ for this case.
- d) Draw the concentration profiles $\xi(x)$ for two different grain diameters $d_1 < d_2$ over the length of the ground coffee powder in the given chart and mark the curves clearly.

Solution V.5: (Coffee brewing ★★★)

Task a)

The mass flow transferred from one grain is:

$$\dot{m} = g \cdot \pi d^2 \cdot (\xi_C - \xi(x)) \quad (\text{V.5.1})$$

Conclusion

The transferred mass flow per unit volume is:

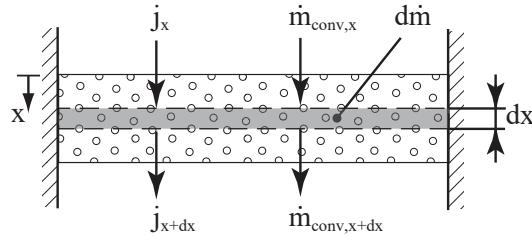
$$\begin{aligned} \dot{m}''' &= \frac{\dot{m}}{\frac{1}{6}\pi d^3} \\ &= \frac{6g}{d} (\xi_C - \xi(x)) \end{aligned} \quad (\text{V.5.2})$$

Task b)

To derive the differential equation for the mass concentration of flavors $\xi(x)$ in the passing water flow, a mass balance for an infinitesimal element must be set.

1 Setting up the balance:

The flavor is transported due to advective and diffusive transport, while the flavor coming loose from the grains can be seen as internal mass generation.



Hence, the steady-state mass balance reads:

$$0 = \underbrace{j_x - j_{x+dx}}_{\text{Net rate of diffusion}} + \underbrace{\dot{m}_{\text{conv},x} - \dot{m}_{\text{conv},x+dx}}_{\text{Net rate of advection}} + \underbrace{\frac{dm}{dx}}_{\text{Rate of substance generation}} \quad (\text{V.5.3})$$

2 Defining the elements within the balance:

The rate of diffusive transport coming in is described by Fick's law:

$$j_x = -\rho D A \frac{\partial \xi}{\partial x}, \quad (\text{V.5.4})$$

and the net rate of advective transport yields from the product of the mass flow rate and the substance mass fraction:

$$\dot{m}_{\text{conv},x} = \rho v A \xi. \quad (\text{V.5.5})$$

Both outgoing rates can be approximated by use of the Taylor series expansion:

$$\begin{aligned} j_{x+dx} &= j_x + \frac{\partial j_x}{\partial x} dx \\ &= -\rho D A \frac{\partial \xi}{\partial x} - \rho D A \frac{\partial^2 \xi}{\partial x^2} dx, \end{aligned} \quad (\text{V.5.6})$$

and

$$\begin{aligned} \dot{m}_{\text{conv},x+dx} &= \dot{m}_{\text{conv},x} + \frac{\partial \dot{m}_{\text{conv},x}}{\partial x} dx \\ &= \rho v A \xi + \rho v A \frac{\partial \xi}{\partial x} dx. \end{aligned} \quad (\text{V.5.7})$$

Lastly, the rate of substance generation yields from the volumetric term and the volume of the infinitesimal element:

$$\begin{aligned} d\dot{m} &= \dot{m}''' \cdot Adx \\ &= p \cdot \dot{m}_c''' \cdot Adx. \end{aligned} \quad (\text{V.5.8})$$

Conclusion

3 Inserting and rearranging:

$$0 = D \frac{\partial^2 \xi}{\partial x^2} - v \frac{\partial \xi}{\partial x} + \frac{\dot{m}'''}{\rho} \quad (\text{V.5.9})$$

Task c)

Neglecting diffusion in the x-direction and using the source term from a) yields the following differential equation:

$$0 = -v \frac{\partial \xi}{\partial x} + \frac{6pg}{\rho d} (\xi_C - \xi) \quad (1 \text{ P})$$

4 Defining the boundary and/or initial conditions:

Since the differential equation has been differentiated once with respect to x , only one boundary condition is required for the solution:

$$\xi(x = 0) = 0.$$

5 Solving the equation:

Substituting $\tilde{\xi} = (\xi_C - \xi)$ helps solving the differential equation:

$$\frac{\partial \tilde{\xi}}{\partial x} = -\frac{6pg}{\rho v d} \tilde{\xi}. \quad (\text{V.5.10})$$

Integrating both sides:

$$\int_{\xi_C}^{\tilde{\xi}} \frac{\partial \tilde{\xi}}{\tilde{\xi}} = \int_0^x -\frac{6pg}{\rho v d} d\tilde{x}. \quad (\text{V.5.11})$$

Conclusion

Solving the expression yields:

$$\xi = \xi_C \left(1 - \exp \left(-\frac{6pg}{\rho v d} x \right) \right). \quad (\text{V.5.12})$$

Task d)

For both mass concentration curves, the steepest gradient occurs at $x = 0$, corresponding to the initial point where the concentration difference between the water and the grains is at its maximum. This sharp increase is indicative of a high rate of mass transfer at the onset of diffusion. As the flow progresses, the concentration gradient diminishes along the flow direction, eventually stabilizing and approaching the equilibrium mass concentration of the grains, resulting in a horizontal slope on the curve.

When comparing two cases where the grain diameter d_1 is smaller than d_2 , the curve for d_1 reaches the equilibrium mass concentration more rapidly. This is attributed to the smaller grains having a larger surface area-to-volume ratio, which enhances the contact area with the water, thereby facilitating a more efficient mass transfer process.

Conclusion

