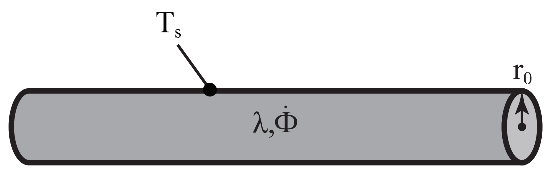
Exercise II.10: (Resistance wire ⋆)

A long homogeneous resistance wire is used to heat the air in a room by the passage of an electric current. Heat is generated in the wire uniformly at a constant rate $\dot{\Phi}'''$ as a result of resistance heating.



Given parameters:

- Outer radius of the wire: $r_0 = 5$ mm
- Heat generation in the wire: $\dot{\Phi}''' = 5 \cdot 10^7$ W/m^3
- Temperature of the outer surface of the wire: $T_s = 180$ °C
- Thermal conductivity of the wire: $\lambda = 6$ W/mK

Hints:

- The problem is one-dimensional in radial direction.
- Assume steady-state conditions.

Tasks:

a) Derive the heat conduction equation by setting up an energy balance.

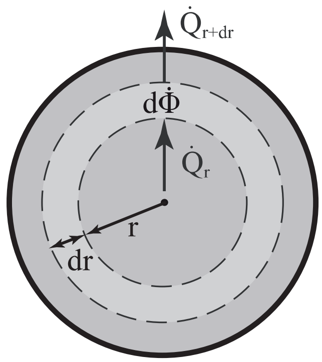
b) Determine the temperature at $r_1 = 3.5$ mm.

Solution II.10: (Pipe fastening ⋆⋆⋆)

The problem can be considered one-dimensional in the radial direction and assumed to be in a steady state. To derive the heat conduction equation, an energy balance must be established for an infinitesimal element within the cross-section of the wire needs to be established.

Task a)

1 Setting up the balance:



The energy balance for the infinitesimal element within the wire reads:

$$0 = \underbrace{\dot{Q}_{\mathrm{r}} - \dot{Q}_{\mathrm{r+dr}}}_{\text{Net rate of diffusion}} + \underbrace{d\dot{\Phi}.}_{\text{Internal heat generation}} \tag{II.10.1}$$

2 Defining the elements within the balance:

The ingoing conductive heat flux is described by use of Fourier's law:

$$\begin{aligned} \dot{Q}_{\mathrm{r}} &= -\lambda A(r)\frac{\partial T}{\partial r} \\ &= -\lambda 2\pi r L\frac{\partial T}{\partial r}. \end{aligned} \tag{II.10.2}$$

The outgoing heat flux is approximated for an infinitesimal element by the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{r+dr}} &= \dot{Q}_{\mathrm{r}} + \frac{\partial \dot{Q}_{\mathrm{r}}}{\partial r}\cdot dr \\ &= -\lambda 2\pi r L\frac{\partial T}{\partial r} + \frac{\partial}{\partial r}\left(-\lambda 2\pi r L\frac{\partial T}{\partial r}\right)\cdot dr. \end{aligned} \tag{II.10.3}$$

The heat generated within the infinitesimal element is expressed as:

$$d\dot{\Phi} = \dot{\Phi}''' \cdot dV, \tag{II.10.4}$$

where dV is:

$$\begin{aligned} dV &= \pi L\left((r+dr)^2 - r^2\right) = \pi L\left(2r\cdot dr + dr^2\right) \\ &\approx 2\pi r L \cdot dr. \end{aligned} \tag{II.10.5}$$

Note that $dr^2 \ll 2r\cdot dr$ and therefore is negligible.

Conclusion

3 Inserting and rearranging:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right) + \dot{\Phi}'''. \tag{II.10.6}$$

Task b)

To find the temperature at $r = 3.5$mm, the temperature profile function must be obtained by solving the heat conduction equation.

4 Defining the boundary and/or initial conditions:

To solve the heat conduction equation, two boundary conditions are needed, given that the temperature has been differentiated twice with respect to r. At the center, symmetry is present, and consequently, the slope of the temperature profile should be horizontal, indicating a zero-slope gradient.

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0. \tag{II.10.7}$$

Moreover, the temperature at the interface of the wire has been specified, providing the second boundary condition.

$$T(r = r_0) = T_s. \tag{II.10.8}$$

5 Solving the equation:

Integrating the heat conduction equation once yields:

$$\frac{\partial T}{\partial r} = -\frac{\dot{\Phi}'''}{2\lambda} r + C_1, \tag{II.10.9}$$

and twice gives:

$$T(r) = -\frac{\dot{\Phi}'''}{4\lambda} r^2 + C_1 r + C_2. \tag{II.10.10}$$

Using the boundary condition of symmetry:

$$\begin{aligned} \left.\frac{\partial T}{\partial r}\right|_{r=0} &= -\frac{\dot{\Phi}'''}{2\lambda} r + C_1 = 0 \\ \Rightarrow C_1 &= 0, \end{aligned} \tag{II.10.11}$$

and the boundary condition of the known surface temperature gives:

$$\begin{aligned} T(r = r_0) &= -\frac{\dot{\Phi}'''}{4\lambda} r_0^2 \; C_2 = T_s \\ \Rightarrow C_2 &= T_s + \frac{\dot{\Phi}'''}{4\lambda} r_0^2. \end{aligned} \tag{II.10.12}$$

Substitution of the definitions of both constants gives the radial temperature profile:

$$T(r) = T_s + \frac{\dot{\Phi}'''}{4\lambda}\left(r_0^2 - r^2\right). \tag{II.10.13}$$

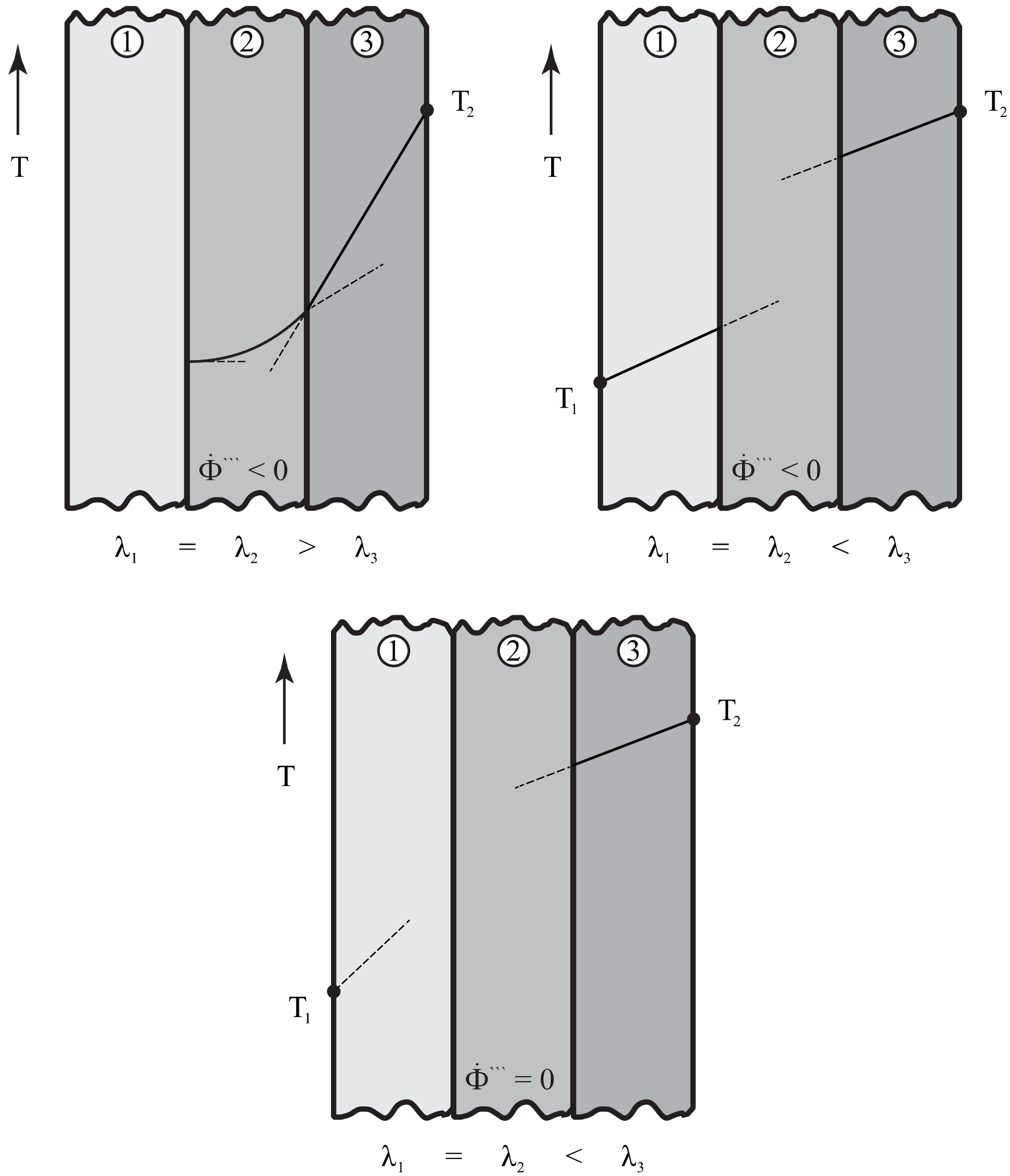
From which the temperature at $r_1 = 3.5$ mm is found:

$$\begin{aligned} T(r=r_1) &= -\frac{\dot{\Phi}'''}{4\lambda} r_1^2 \; + T_s + \frac{\dot{\Phi}'''}{4\lambda} r_0^2 \\ &= 180 \;\; (^{\circ}\mathrm{C}) \;\; + \frac{5 \cdot 10^7 \; \left(\frac{\mathrm{W}}{\mathrm{m}^3}\right)}{4 \cdot 6 \; \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} \cdot \left(0.005^2 - 0.0035^2\right) \; \left(\mathrm{m}^2\right) = 207 \quad (^{\circ}\mathrm{C}) \, . \end{aligned} \tag{II.10.14}$$

Conclusion

The temperature at $r_1 = 3.5$ mm is thus 207 °C.

Exercise II.11: (Multi-layer walls with source ⋆⋆)



Tasks:

a) Complete the temperature profiles in the three-layered walls.

Solution II.11: (walls with source ⋆⋆)

Task a.1)

The temperature profile of layer 1 from the top left wall must be drawn horizontally.

First, the gradient at the interface between layers 1 and 2, as becomes evident by the energy balance at the interface must be determined:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 \tag{II.11.1}$$

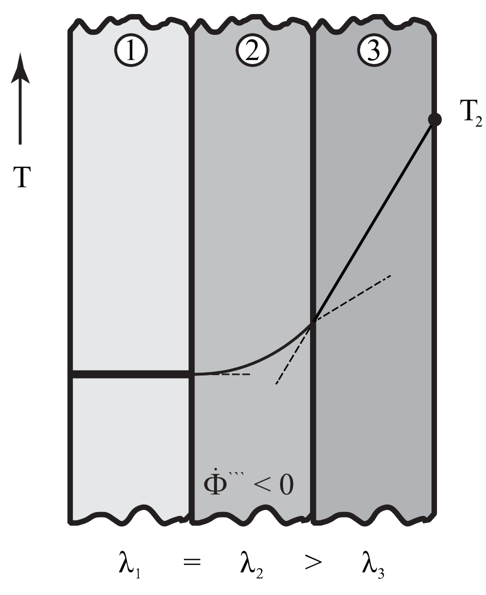
Given that $\lambda_1 = \lambda_2$, at the interface $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$, resulting in a horizontal gradient at that position.

Moreover, no heat generation occurs in that layer, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x}, \tag{II.11.2}$$

if the rate of heat transfer within the layer remains constant, with a consistent area and thermal conductivity, the gradient must be the same throughout the entire layer, thus remaining horizontal within the entire layer.

Conclusion



Task a.2)

The temperature profile for layer 2 of the top right wall must be drawn with a decreasing temperature gradient when moving from right to left while having the same gradient as layer 1 at their interface and a steeper gradient than layer 3 at their interface.

First, the gradients at the interfaces between layers 1 and 2 and layers 2 and 3, which become evident from the energy balances at the interface need to be determined. The energy balance at the interface between layers 1 and 2 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 . \tag{II.11.3}$$

Given $\lambda_1 = \lambda_2$, the interface between both layers yields $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$, resulting in a gradient of layer 2 just as steep as that of layer 1 at that position.

The energy balance at the interface between layers 2 and 3 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_3 . \tag{II.11.4}$$

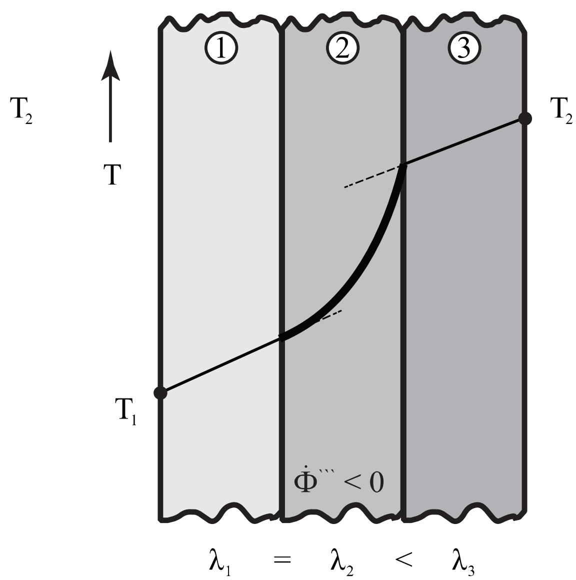
Given $\lambda_2 < \lambda_3$, the interface between both layers yields $\left(\frac{\partial T}{\partial x}\right)_2 > \left(\frac{\partial T}{\partial x}\right)_3$, resulting in a gradient of layer 2 being steeper than that of layer 3 at that position.

Moreover, layer 2 acts as a sink, thus removal occurs in that layer, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x}, \tag{II.11.5}$$

the rate of heat transfer within the layer decreases, with a consistent area and thermal conductivity, a decrease in gradient occurs when moving along the direction that heat travels in (right to left).

Conclusion



Task a.3)

The temperature profile for layers 1 and 2 of the bottom wall must be drawn with a constant temperature gradient when moving from right to left while having the specified gradient of layer 1 at the interface with the ambient and a steeper gradient than layer 3 at their interface.

First, the gradients at the interfaces between layers 1 and 2 and layers 2 and 3 must be determined, which both become evident from the energy balances at the interface. The energy balance at the interface between layers 1 and 2 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 . \tag{II.11.6}$$

Given that $\lambda_1 = \lambda_2$, the interface reveals $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$, resulting in a gradient of layer 2 just as steep as that of layer 1 at that position.

The energy balance at the interface between layers 2 and 3 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_3 . \tag{II.11.7}$$

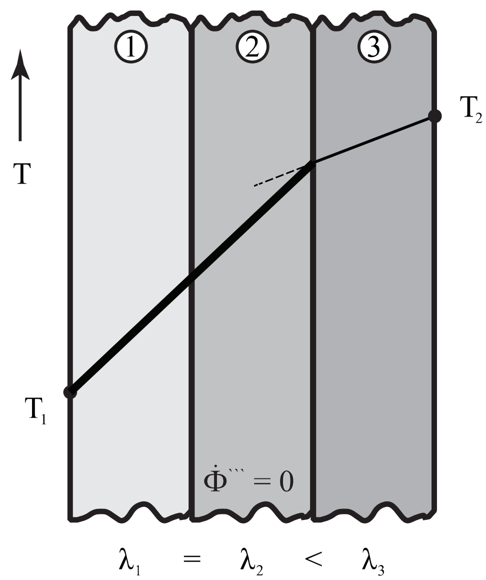
Given that $\lambda_2 < \lambda_3$, at the interface $\left(\frac{\partial T}{\partial x}\right)_2 > \left(\frac{\partial T}{\partial x}\right)_3$, resulting in a gradient of layer 2 that is steeper than that of layer 3 at that position.

Moreover, no heat generation occurs in layers 1 and 2, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x}, \tag{II.11.8}$$

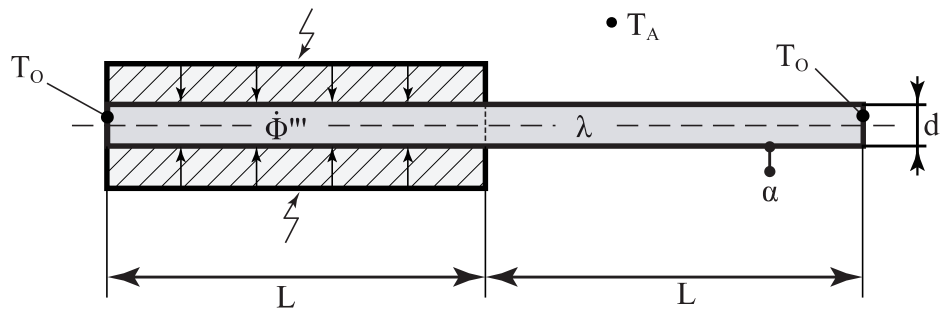
if the rate of heat transfer within the layers remains constant, with a consistent area and thermal conductivity, the gradient remains the same throughout both layers.

Conclusion



Exercise II.12: (Copper rod ⋆⋆⋆)

Both ends of a copper rod with a length L and a diameter d are kept at the same temperature T_O. The left half of the rod is insulated against all radial heat losses. An electric heating element generates Joule's heat of heat flux density $\dot{\Phi}'''$. The right half of the rod is subjected to a flow of the ambient air with an air temperature of T_A, yielding a heat transfer coefficient α. The thermal conductivity of the rod is given as λ.



Given parameters:

- Length of the rod: $L = 1$ m
- Diameter of the rod: $d = 5.2$ mm
- Temperature of the ends of both rods: $T_O = 120$ °C
- Temperature of the ambient: $T_A = 100$ °C
- Heat transfer coefficient: $\alpha = 6$ W/m^{2}K
- Thermal conductivity of the rod: $\lambda = 372$ W/mK

Hint:

- Place the origin of the coordinate system in the middle of the rod.

Tasks:

a) Derive the equation for the temperature profile in the rod by setting up an energy balance.

b) Determine an expression for $\dot{\Phi}'''$ such that the temperature in the center of the rod is also T_O, similar to the temperatures at its ends.

c) Calculate the value for $\dot{\Phi}'''$ for the conditions postulated in b).

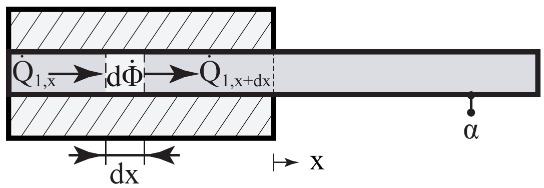
d) Determine the extremes of the temperature distribution for the given values. Give their position and values, additionally, sketch the temperature profile.

Solution II.12: (Copper rod ⋆⋆⋆)

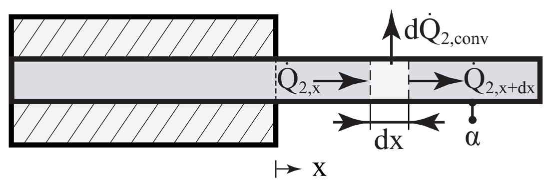
Task a)

The temperature profile is obtained from the conduction equation within an element. Note that both segments of the rod require a different conduction equation to be solved. The left segment, denoted as part 1, involves heat generation, while the right segment, denoted as part 2, does not include this term but experiences heat loss through convection, acting as a fin. Consequently, distinct conduction equations must be formulated and solved for both segments to accurately model their respective behaviors.

1 **Setting up the balance:**



(a) Part 1



(b) Part 2

The energy balance of part 1 reads:

$$0 = \underbrace{\dot{Q}_{1,\mathrm{x}} - \dot{Q}_{1,\mathrm{x+dx}}}_{\text{Net rate of diffusion}} + \underbrace{d\dot{\Phi},}_{\text{Internal heat generation}} \tag{II.12.1}$$

and the energy balance of part 2 reads:

$$0 = \underbrace{\dot{Q}_{2,\mathrm{x}} - \dot{Q}_{2,\mathrm{x+dx}}}_{\text{Net rate of diffusion}} - \underbrace{d\dot{Q}_{\mathrm{conv}}.}_{\text{Convective losses}} \tag{II.12.2}$$

2 **Defining the elements within the balance:**

For both rods the incoming conducted heat is expressed by Fourier's law:

$$\dot{Q}_{1,\mathrm{x}} = -\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x}, \tag{II.12.3}$$

and:

$$\dot{Q}_{2,\mathrm{x}} = -\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x}. \tag{II.12.4}$$

Similarly, the outgoing conducted heat is approximated by the use of the Taylor series expansion for an infinitesimal element:

$$\begin{aligned} \dot{Q}_{1,\mathrm{x+dx}} &= \dot{Q}_{1,\mathrm{x}} + \frac{\partial \dot{Q}_{1,\mathrm{x}}}{\partial x} \cdot dx \\ &= -\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x}\right) \cdot dx, \end{aligned} \tag{II.12.5}$$

and:

$$\dot{Q}_{2,\mathrm{x+dx}} = \dot{Q}_{2,\mathrm{x}} + \frac{\partial \dot{Q}_{2,\mathrm{x}}}{\partial x} \cdot dx \\ = -\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x}\right) \cdot dx. \tag{II.12.6}$$

The generated heat within the infinitesimal element of part 1 results from the product of the volumetric source term and the volume of the infinitesimal element:

$$d\dot{\Phi} = \dot{\Phi}''' dV \\ = \dot{\Phi}''' \frac{\pi d^2}{4} dx. \tag{II.12.7}$$

Lastly, the rate of heat loss by convection is determined from the product of the heat transfer coefficient, the surface area of the infinitesimal element, and the temperature difference between the element and the ambient:

$$d\dot{Q}_{\mathrm{conv}} = \alpha \cdot dA_{\mathrm{s}} \cdot (T_2(x) - T_{\mathrm{A}}) \\ = \alpha \pi d \cdot dx \, (T_2(x) - T_{\mathrm{A}}). \tag{II.12.8}$$

3 **Inserting and rearranging:**

For part 1:

$$\frac{\partial^2 T_1}{\partial x^2} = -\frac{\dot{\Phi}'''}{\lambda}, \tag{II.12.9}$$

and for part 2:

$$0 = \frac{\partial^2 T_2}{\partial x^2} - \frac{4\alpha}{\lambda d} (T_2(x) - T_{\mathrm{A}}). \tag{II.12.10}$$

4 **Defining the boundary and/or initial conditions:**

To determine the temperature profiles inside the system, in total 4 boundary and/or coupling conditions are required. The temperature at the left side of the system is known:

$$T_1(x = -L) = T_{\mathrm{O}}, \tag{II.12.11}$$

and at the right side:

$$T_2(x = L) = T_{\mathrm{O}}. \tag{II.12.12}$$

But also, the temperatures at the interface between the left and right sides must be equal to each other:

$$T_1(x = 0) = T_2(x = 0). \tag{II.12.13}$$

Lastly, the energy balance at the interface $\dot{Q} = -\lambda A \left.\frac{\partial T_1}{\partial x}\right|_{x=0} = -\lambda A \left.\frac{\partial T_2}{\partial x}\right|_{x=0}$ gives that:

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=0} = \left.\frac{\partial T_2}{\partial x}\right|_{x=0}. \tag{II.12.14}$$

5 Solving the equation:

Conclusion

Integrating the differential equation for part 1 once yields:

$$\frac{\partial T_1}{\partial x} = -\frac{\dot{\Phi}'''}{\lambda} \cdot x + c_1, \tag{II.12.15}$$

and twice gives:

$$T_1(x) = -\frac{\dot{\Phi}'''}{2 \cdot \lambda} \cdot x^2 + c_1 \cdot x + c_2. \tag{II.12.16}$$

The differential equation for part 2 is a linear second-order differential equation with constant coefficients. Therefore, this differential equation is solved with the same approach for a fin system. Where the first step is introducing the spatial temperature difference θ_2

$$\theta_2 = T_2(x) - T_\mathrm{A}, \tag{II.12.17}$$

and the fin parameter m:

$$m = \sqrt{\frac{4\alpha}{\lambda d}}. \tag{II.12.18}$$

Substitution into the differential equation of part 2 gives:

$$0 = \frac{\partial^2 \theta_2}{\partial x^2} - m \cdot \theta_2, \tag{II.12.19}$$

which has the standard solution:

$$\theta_2 = c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx). \tag{II.12.20}$$

Therefore:

$$T_2(x) = T_\mathrm{A} + c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx). \tag{II.12.21}$$

To determine the constants c_1, c_2, c_3, and c_4 the set boundary conditions are to be used, which gives:

$$c_1 = \frac{1}{L} \cdot \frac{(T_\mathrm{O} - T_\mathrm{A}) - \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda} L^2\right) \cdot \cosh(mL)}{\frac{\sinh(mL)}{mL} + \cosh(mL)}, \tag{II.12.22}$$

$$c_2 = T_\mathrm{A} + \frac{(T_\mathrm{O} - T_\mathrm{A}) + \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda} L^2\right) \cdot \frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)}, \tag{II.12.23}$$

$$c_3 = \frac{c_1}{m}, \tag{II.12.24}$$

and:

$$c_4 = c_2 - T_\mathrm{A}. \tag{II.12.25}$$

With the substitution of these constants in both temperature functions, the temperature profile in the entire rod is defined for a given heat source density $\dot{\Phi}'''$.

Task b)

To determine $\dot{\Phi}'''$ for $T(x=0) = T_\mathrm{O}$, the expressions for $T_1(x=0)$ and $T_2(x=0)$ must be determined.

First $T_1(x=0)$:

$$T_1(x=0) = T_\mathrm{O}, \tag{II.12.26}$$

and $T_2(x=0)$:

$$T_2(x=0) = T_\mathrm{A} + \frac{T_\mathrm{O} - T_\mathrm{A} + \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda}L^2\right)\frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)}. \tag{II.12.27}$$

Conclusion

Equalling both expressions and rewriting yields:

$$\begin{aligned}\dot{\Phi}''' &= \frac{2\lambda m}{L} \cdot (T_\mathrm{O} - T_\mathrm{A}) \cdot \frac{\cosh(mL) - 1}{\sinh(mL)} \\ &= \frac{2\lambda m}{L} \cdot (T_\mathrm{O} - T_\mathrm{A}) \cdot \tanh\left(\frac{mL}{2}\right).\end{aligned} \tag{II.12.28}$$

Task c)

To find the numerical value of the volumetric heat source, first, the fin parameter needs to be determined:

$$\begin{aligned}m &= \sqrt{\frac{4\alpha}{\lambda d}} \\ &= \sqrt{\frac{4 \cdot 6\ \left(\frac{\mathrm{W}}{\mathrm{m^2K}}\right)}{372\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right) \cdot 0.0052\ (\mathrm{m})}} = 3.52\ \left(\mathrm{m^{-1}}\right).\end{aligned} \tag{II.12.29}$$

Substitution into the previously determined expression for the volumetric heat source results:

$$\begin{aligned}\dot{\Phi}''' &= \frac{2\lambda m}{L} \cdot (T_\mathrm{O} - T_\mathrm{A}) \cdot \tanh\left(\frac{mL}{2}\right) \\ &= \frac{2 \cdot 372\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right) \cdot 3.52\ \left(\mathrm{m^{-1}}\right)}{1\ (\mathrm{m})} \cdot (120 - 100)\ (\mathrm{K}) \cdot \tanh\left(\frac{3.52\ \left(\mathrm{m^{-1}}\right) \cdot 1\ (\mathrm{m})}{2}\right) = 49.4\ \left(\frac{\mathrm{kW}}{\mathrm{m^3}}\right).\end{aligned} \tag{II.12.30}$$

Conclusion

The volumetric heat generation within the left part is about 49.4 $\frac{\mathrm{kW}}{\mathrm{m^3}}$

Task d)

Given the temperature of T_O at the left tip, center, and right tip, coupled with the homogeneous heat generation in the left part and the heat loss due to convection in the right part, the maximum temperature is located within the left part, while the minimum temperature is in the right part.

More specifically, due to symmetry within both parts, the location of the maximum temperature is located at

$$x_\mathrm{max} = -\frac{L}{2}. \tag{II.12.31}$$

and the location of the minimum temperature is:

$$x_{\mathrm{min}} = \frac{L}{2}. \tag{II.12.32}$$

Substitution of these positions into the derived temperature profiles gives the maximum temperature

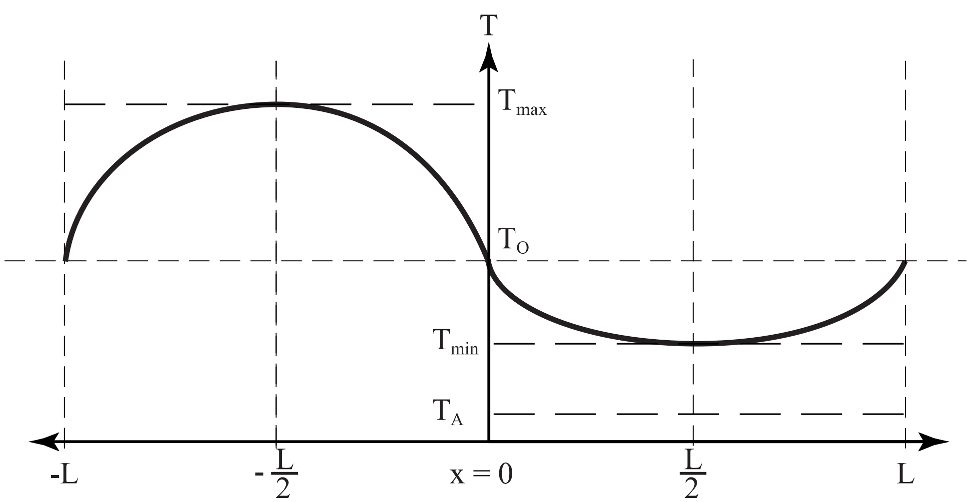
$$\begin{aligned} T_{\mathrm{max}} &= T_1\left(x = -\frac{L}{2}\right) \\ &= T_{\mathrm{O}} + \frac{\dot{\Phi}''' L^2}{8\lambda} \\ &= 120\ (^{\circ}\mathrm{C}) + \frac{49{,}400\ \left(\frac{\mathrm{W}}{\mathrm{m}^3}\right) \cdot 1^2\ \left(\mathrm{m}^2\right)}{8 \cdot 372\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)} = 137\ (^{\circ}\mathrm{C}), \end{aligned} \tag{II.12.33}$$

and the minimum temperature:

$$\begin{aligned} T_{\mathrm{min}} &= T_2\left(x = \frac{L}{2}\right) \\ &= T_{\mathrm{A}} + \frac{T_{\mathrm{O}} - T_{\mathrm{A}}}{\cosh\left(\frac{mL}{2}\right)} \\ &= 100\ (^{\circ}\mathrm{C}) + \frac{(120 - 100)\ \ (^{\circ}\mathrm{C})}{\cosh\left(\frac{3.52\ \left(\mathrm{m}^{-1}\right) \cdot 1\ (\mathrm{m})}{2}\right)} = 107\ (^{\circ}\mathrm{C}). \end{aligned} \tag{II.12.34}$$

Conclusion

Which results in the following temperature profile:



where the maximum temperature is thus 137 °C and the minimum temperature 107 °C.