HEATQUIZ

HEAT TRANSFER

Course reader - Conduction

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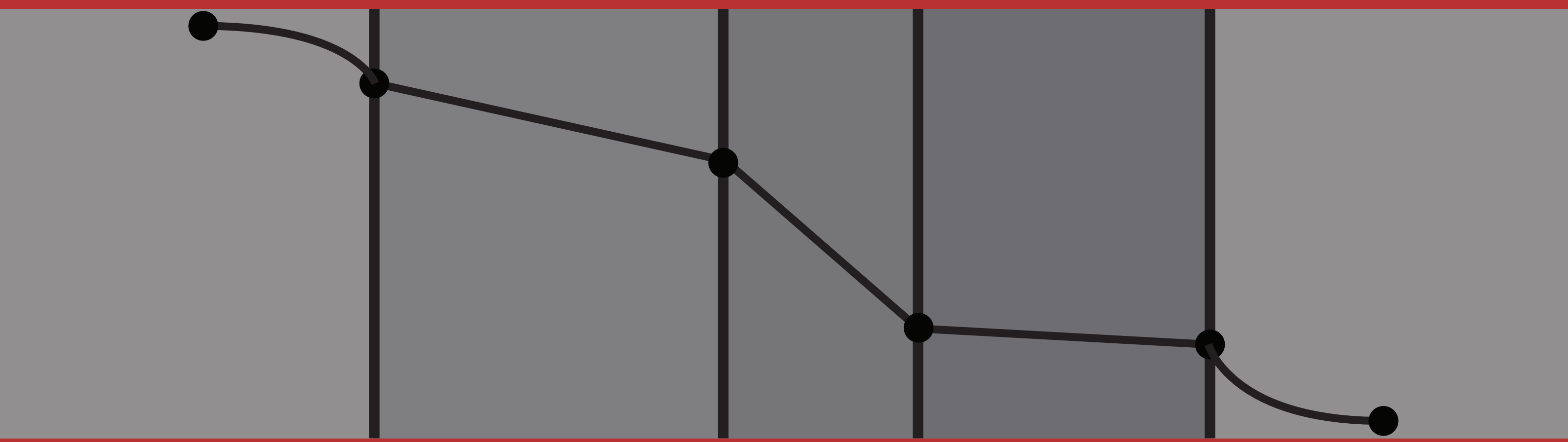

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Nomenclature

Symbol	Physical constant	Numerical value	Unit:
N_{A}	Avogadro constant	$6.022 \cdot 10^{23}$	(1/mol)
R_{m}	Universal gas constant	8.314	(J/K mol)
c	Speed of light	299,792,458	(m/s)
g	Gravitational acceleration	9.81	$\left(\mathrm{m/s^2}\right)$
h	Planck's constant	$6.6261 \cdot 10^{-34}$	(Js)
σ	Stefan–Boltzmann constant	$5.6703 \cdot 10^{8}$	$\left(\mathrm{W/m^2K^4}\right)$

Symbol	Description	Unit
A	Area	$\left(\mathrm{m^2}\right)$
C	Constant or molar concentration	(–) or $\left(\mathrm{mol/m^3}\right)$
D	Diffusion coefficient	$\left(\mathrm{m^2/s}\right)$
E	Energy of a photon	(J)
F	Force or radiation fraction	(N) or (–)
H	Total enthalpy	(J)
H^{cc}	Molar based Henry coefficient	(–)
H^{*}	Mass based Henry coefficient	(–)
I	Momentum	(Ns)
K_{N}	Mass Nernst coefficient	(–)
L	(Characteristic) length or radiosity	(m) or $\left(\mathrm{W/m^2}\right)$
L_{h}	Hydrodynamic entrance length	(m)
L_{th}	Thermal entrance length	(m)
M	Molar mass	(kg/mol)
N	Number of particles	(–)
Q	Heat	(J)
R	Thermal resistance	(K/W)
T	Temperature	(K)
U	Internal energy or perimeter	(J) or (m)
V	Volume	$\left(\mathrm{m^3}\right)$
W	Work	(J)
a	Thermal diffusivity	$\left(\mathrm{m^2/s}\right)$
c	Specific heat capacity or constant	(J/kgK) or (–)
d	Diameter	(m)
dx	Infinitesimal distance in x-direction	(m)
dy	Infinitesimal distance in y-direction	(m)
dz	Infinitesimal distance in z-direction	(m)
dr	Infinitesimal distance in r-direction	(m)
f_{e}	Arrangement factor	(–)
g	Convective mass transfer coefficient	$\left(\mathrm{kg/m^2s}\right)$
h	Mass-specific enthalpy	(J/kg)
$\dot{j}''$	Diffusive mass flux density	$\left(\mathrm{W/m^2}\right)$
k	Overall heat transfer coefficient	$\left(\mathrm{W/m^2K}\right)$
m	Mass or fin parameter	(kg) or (1/m)
$\dot{m}''$	Mass flux density	$\left(\mathrm{W/m^2}\right)$
n	Number of layers or number of moles	(–)
$\dot{n}''$	Molar flux density	$\left(\mathrm{mol/m^2s}\right)$
p	Pressure	$\left(\mathrm{N/m^2}\right)$
$\dot{q}''$	Heat flux density	$\left(\mathrm{W/m^2}\right)$
r	Radius	(m)
t	Time	(s)
u	Velocity component (in x-direction)	(m/s)
v	Velocity component (in y-direction)	(m/s)
w	Velocity component (in z-direction)	(m/s)
x	Spatial coordinate	(m)
y	Spatial coordinate	(m)
z	Spatial coordinate	(m)
Δ	Difference	(–)
Φ	View factor	(–)
$\dot{\Phi}$	Heat source	(W)
Ω	Solid angle	(Str)
α	Convective heat transfer coefficient or absorptivity	$\left(\mathrm{W/m^2K}\right)$ or (–)
β	Volumetric expansion coefficient	(1/K)
δ	Wall thickness or penetration depth	(m)
ϵ	Emissivity	(–)
δ_{T}	Thermal boundary layer thickness	(m)
δ_{u}	Velocity boundary layer thickness	(m)

η	Efficiency or wavenumber	$(-)$ or $(1/\mathrm{m})$
θ	Dimensionless spatial temperature	$(-)$
θ^*	Dimensionless temporal temperature	$(-)$
λ	Thermal conductivity or wavelength	$(\mathrm{W/mK})$ or (m)
μ	Dynamic viscosity	$(\mathrm{kg/ms})$
ν	Kinematic viscosity or frequency of radiation	$(\mathrm{m^2/s})$ or $(1/\mathrm{s})$
ξ	Mass fraction	$(-)$
ϕ	Viewing angle	(rad)
ρ	Density or reflectivity	$(\mathrm{kg/m^3})$ or $(-)$
τ	Shear stress or transmissivity	$(\mathrm{N/m^2})$ or $(-)$
ψ	Molar fraction	$(-)$

Superscript	**Description**
x^*	Dimensionless
x'	Distance-related or derivation
x''	Area-related
x'''	Volume-related
$\dot{x}$	Rate
$\overline{x}$	Average
$\vec{x}$	Vector

Subscript	**Description**
x_{A}	Ambient A
x_{a}	Ambient
x_{adv}	Advection
x_{B}	Ambient B
x_{b}	Black body
x_{c}	Cross-section
x_{crit}	Critical
x_{cond}	Conduction
x_{conv}	Convection
x_{d}	Hydraulic diameter as characteristic length
x_{diff}	Diffusion
x_{eff}	Effective
x_{evap}	Evaporation
x_{F}	Fin
x_{f}	Fluid
x_{fl}	Fluid
x_{h}	Hydraulic
x_{in}	Inlet
x_{kin}	Kinetic
x_{L}	Length as characteristic length
x_{ij}	From i to j
$x_{i \to j}$	From i to j
$x_{i \rightleftarrows j}$	Net between i and j
x_{m}	Mean
x_{max}	Maximum
x_{min}	Minimum
x_{out}	Outlet or outgoing
x_{p}	At constant pressure
x_{prop}	Property
x_{rad}	Radiation
x_{s}	Solid or distance as characteristic length or surface
x_{t}	Turbulent
x_{th}	Thermal
x_{v}	At constant volume
x_{w}	Wall
x_{x}	Local
x_0	Incident or initial
x_1	Reference 1
x_2	Reference 2
x_3	Reference 3
x_4	Reference 4
x_α	Absorbed
x_ϵ	Emitted
x_η	Wavenumber-specific
x_λ	Wavelength-specific
x_ρ	Reflected
x_τ	Transmitted
x_∞	Upstream

Symbol	Dimensionless number	Definition
Ar	Archimedes number	$\equiv \frac{\text{Bouyance forces}}{\text{Inertia forces}}$
Bi	Biot number	$\equiv \frac{\text{Thermal resistance in body}}{\text{Convective thermal resistance at surface}}$
Fo	Fourier number	$\equiv \frac{\text{Rate of diffusivity}}{\text{Rate of storage}}$
Gr	Grashof number	$\equiv \frac{\text{Bouyancy forces}}{\text{Viscous forces}}$
Le	Lewis number	$\equiv \frac{\text{Molecular diffusivity of heat}}{\text{Molecular diffusivity of mass}}$
Nu	Nusselt number	$\equiv$ Dimensionless heat transfer coefficient
Pe	Peclet number	$\equiv \frac{\text{Rate of advection}}{\text{Rate of diffusion}}$
Pr	Prandtl number	$\equiv \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}}$
Re	Reynolds number	$\equiv \frac{\text{Inertia forces}}{\text{Viscous forces}}$
Sc	Schmidt number	$\equiv \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of mass}}$
Sh	Sherwood number	$\equiv$ Dimensionless mass transfer coefficient
c_f	Friction coefficient	$\equiv \frac{\text{Frictional head loss}}{\text{Dynamic pressure}}$

Reader elements

Here, an overview is presented clarifying the individual components that form the structural organization of this text. Readers will encounter specific sections.

Definition

The **definition** part introduces the meanings of parameters utilized in the theory of heat transfer. Common definitions within this framework include terms such as rate of heat, internal heat source, and dimensionless numbers.

Fundamental EQ

The **fundamental EQ** part presents equations derived or experimentally determined, providing a convenient means to calculate specific parameters within the framework of heat transfer theory. Consider well-known examples such as Fourier's law or Newton's law of cooling.

HTC

The **HTC** part presents a Nusselt correlation tailored for a specific application. These correlations, determined numerically or experimentally, serve to calculate the heat transfer coefficient in particular scenarios.

Criterion

The **criterion** part introduces a set of conditions that must be satisfied for a theory to be deemed applicable.

Approach

The **approach** part presents a systematic framework for addressing a particular problem, outlining steps such as establishing and solving an energy balance equation.

Phenomena

The **phenomena** part illustrates a principle of heat transfer through tangible, real-world examples that can readily be observed. Consider scenarios such as cooling a cup or warming up in the sunshine.

HeatQuiz

The **HeatQuiz** part offers game-based learning tasks designed to assess the comprehension of the previously discussed theory and ascertain whether sufficient knowledge of the content has been acquired to apply the theory to practical examples.

Derivation

The **derivation** part is dedicated to obtaining a particular theorem or defining a parameter. Common derivations discussed include those related to conservative equations within solids and fluids.

Example

The **example** part furnishes a relatively straightforward illustration of the practical application of the recently derived theory. Consider, for instance, the derivation of the temperature profile within a plane wall.

Demonstration

The **demonstration** part presents a task previously evaluated in past course exams, accompanied by a QR code offering a video solution. Therefore, attempting the task independently before consulting the video solution for the best learning experience is strongly advised.

Exercise ($\star$)

> Exercises marked with a single star $\star$ serve as foundational exercises to reinforce the fundamental understanding. While not yet at the exam level, they function as crucial stepping stones, helping to build confidence and proficiency before tackling more advanced challenges.

Exercise ($\star\star$)

> For exercises denoted with two stars $\star\star$, the tasks are slightly below the anticipated difficulty level of the exam or smaller assignments with a few exam points at stake.

Exercise ($\star\star\star$)

> The exercises with three stars $\star\star\star$ are those that appeared on previous exams. These tasks, often carrying significant point values, reflect the challenges that demand a higher level of mastery and are critical for exam preparation.

PART

I

Introduction to Heat Transfer

Learning goals:

- Conceptual understanding of the relation between heat transfer and thermodynamics.
- Conceptual understanding of the principles of energy transfer.
- Conceptual understanding of balances to solve engineering problems.

Comprehension questions:

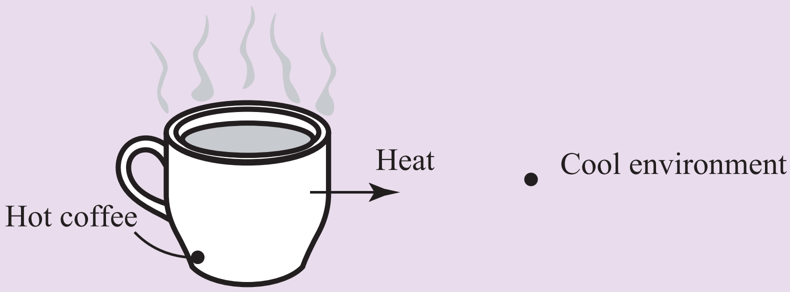
- What is the relationship between thermodynamics and heat transfer?
- What are the mechanisms of heat transfer?
- How can energy balances be used to derive governing equations for heat transfer?
- What are some practical applications of heat transfer in engineering?

Section 1

Thermodynamics and heat transfer

Heat transfer, as a fundamental concept and sub-discipline of thermodynamics, plays a vital role in understanding the processes leading to thermal equilibrium. While classical thermodynamics primarily focuses on describing equilibrium states, heat transfer delves into the mechanisms and pathways that bring systems to this state. Unlike the concepts taught in the thermodynamics lectures before, which disregard time in the formulated equilibrium descriptions, heat transfer processes are intimately linked with the quantity of time. These processes occur due to spatial variations in thermodynamic potentials, specifically temperature differences, which drive heat flows and subsequently lead to temporal changes in temperature.

Phenomena 1.1 Consider preparing a cup of coffee or tea. According to the principles of thermodynamic equilibrium, the temperature of the beverage eventually reaches the same value as the room temperature. However, the time to reach the desired drinking temperature is a key focus of this course.



To understand this principle, recall the second law of thermodynamics. This law states that heat naturally moves from areas of higher temperature to those of lower temperature in a process called heat transfer. This phenomenon is attributed to the concept of entropy, which represents the tendency of systems to evolve towards a state of higher disorder. When the hot coffee comes into contact with the cooler environment, the kinetic energy of the high-energy molecules in the liquid is transferred to the lower-energy molecules in the environment, resulting in a more random distribution of energy and achieving a state of equilibrium.

Proficiency in describing heat transfer processes is indispensable for engineers across a wide range of scientific and engineering disciplines. In today's world, the miniaturization of electrical components is often constrained by the challenge of effectively dissipating heat. Electrical engines, power electronics, and batteries, which form the core components of an electrical drive train, heavily rely on efficient heat removal to ensure optimal performance and longevity. Additionally, the greenhouse effect and resulting climate change are consequences of the atmosphere's transmissivity to thermal radiation, which is dependent on CO_2 concentration. Advancements in home insulation and the implementation of modern heating systems, such as heat pumps, necessitate improved insulation materials or enhanced pathways for efficient heat transfer.

At the core, the description of heat transfer processes relies on the concept of balances, i.e. energy balances in conjunction with mass and momentum balances. Changes in the inner energy in time within a specified control volume are related to the energy flows across the system's boundaries. Describing the internal changes (transient processes) as well as the fluxes with the proper consecutive laws is an essential part of the heat transfer course. For the description of fluxes, the three basic mechanisms for the transportation of heat are conduction, convection, and radiation.

SUBSECTION 1.1
Heat transfer mechanisms

Heat conduction is a fundamental mode of heat transfer that occurs through direct contact between substances on a molecular level, spanning not only solids but also liquids and gases. This process involves the transfer of thermal energy from regions of higher temperature to regions of lower temperature, driven by the exchange of kinetic energy between molecules. The rate of heat conduction depends on various factors, such as the temperature gradient, the thermal conductivity of the material, and the cross-sectional area available for heat flow. Efficient heat conductors, such as metals, exhibit high thermal conductivity, while insulators, like ceramics or nonmetals, have low conductivity, impeding the transfer of heat.

Figure 1.1. Conductive heat transport.

Phenomena 1.2

Imagine you have a metal spoon resting in a cup of hot soup. As you hold the handle of the spoon, you begin to feel the heat spreading throughout the handle.

In this example, the phenomenon of conduction is observed. Conduction is the transfer of heat energy through direct contact between objects that are at different temperatures. When you hold the metal spoon, the heat from the hot soup is conducted through the metal and transferred to your hand.

To better understand conduction, think of heat as the kinetic energy of molecules. In hot objects, these molecules move faster and have more energy. When the hot soup comes into contact with the metal spoon, the high-energy molecules collide with the cooler molecules in the spoon's handle and transfer some of their kinetic energy.

Heat convection is the transfer of heat energy from one point to another through the movement of a fluid or a gas. This transfer of energy occurs due to the movement of fluid or gas into a region of lower/higher temperature. Convection can occur in two ways: natural convection, which is driven by buoyancy forces due to temperature differences, and forced convection, which is driven by an external force such as a fan or a pump. Convection is an important mechanism of heat transfer in many engineering systems, from the cooling of engines to the design of heat exchangers.

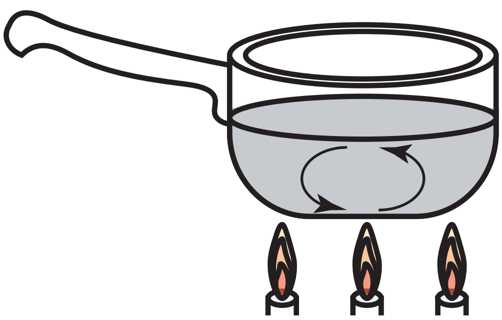


Figure 1.2. Convective heat transport.

Phenomena 1.3 Imagine you're riding your bicycle on a hot summer day. When you stop at a red light, you feel the heat around you intensify, creating an uncomfortable feeling. But as soon as you start moving at a slow speed, you feel cooler despite the physical effort of pedaling.

In this example, the transfer of heat from your body to the surrounding environment is influenced by the airflow created as you ride. When you're moving, the air rushes past your skin, taking away heat from your body and quickly moving away. This movement of air is called convection.

Convection is directly linked to the flow of fluids (like air or water). In this case, the movement of the air around you helps to cool your body down. The concept of convection shows a strong connection between the study of fluid mechanics and understanding heat transfer.

By studying both, fluid mechanics and heat transfer, you develop the ability to understand how air and other fluids move and how they affect the temperature of objects. This knowledge helps to design cooling systems, understand weather patterns, and improve understanding of how heat transfer behaves in different situations.

Radiation is the transfer of heat energy through electromagnetic waves, without the need for a material medium. This transfer of energy occurs due to the emission of electromagnetic waves by a hot object, which are absorbed by another object at a lower temperature. The rate of radiation is dependent on the temperature and emissivity of the objects involved, as well as the distance between them. Radiation is an important mechanism of heat transfer in many engineering systems, from the design of solar panels to the cooling of satellites.

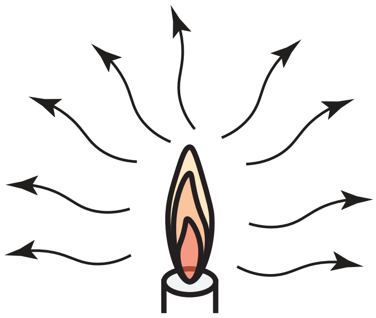


Figure 1.3. Radiative heat transport.

Phenomena 1.4 Imagine you are sitting outside on a sunny day, feeling the warmth of the sun on your skin. As the sun's rays reach the Earth's surface, they release energy in the form of electromagnetic waves, including infrared radiation. This process is known as radiative heat transfer.

Radiative heat transfer is the transfer of heat energy through electromagnetic waves without the need for any physical medium or direct contact. In this case, the sun acts as the heat source, and the energy is transferred from the sun to your body through the air.

To understand radiative heat transfer, consider that all objects above absolute zero temperature emit electromagnetic radiation in the form of photons. The hotter the object, the more intense the radiation is emitted. When the sun's rays reach your skin, they are absorbed, and the energy of the photons is transferred to your body's molecules, causing an increase in temperature and making you feel warm.

Unlike conduction and convection, which require physical contact or the movement of particles, radiative heat transfer can occur through a vacuum or any transparent medium, such as air or glass. This is why you can still feel the warmth of the sun on a clear day even though the space between the sun and the Earth is a near-vacuum.

Radiative heat transfer plays a crucial role in various natural and human-made processes. For example, the primary way the Earth receives energy from the sun is by radiation, which drives weather patterns and supports life on Earth. Similarly, radiative heat transfer is essential in designing and understanding heat exchange mechanisms in space, where conduction and convection are absent.

By studying radiative heat transfer, scientists and engineers can develop technologies that harness solar energy for power generation, and create more efficient heating and cooling systems.

Subsection 1.2

Heat transfer mechanisms from molecular point of view

Heat transfer mechanisms, such as conduction and radiation, can be understood and explained at the molecular level.

Heat conduction from a molecular point of view:

At the molecular level, heat conduction occurs through the interaction between neighboring molecules. In a solid material, such as a metal, the atoms or molecules are closely packed and bonded together. When there is a temperature difference within the material, the molecules at the higher temperature have higher kinetic energy. These energetic molecules collide with their neighboring molecules, transferring a portion of their energy. During these collisions, the higher kinetic energy molecules impart some of their energy to the adjacent molecules with lower kinetic energy. This process continues throughout the material, creating a chain reaction of energy transfer. As a result, heat is conducted from the region of higher temperature to the region of lower temperature.

The ability of a material to conduct heat is determined by the molecular structure and bonding. Materials with closely packed atoms or molecules and strong intermolecular forces, such as metals, are efficient heat conductors. In contrast, materials with more loosely arranged molecules or weaker intermolecular forces, like insulators, impede the transfer of heat.

Heat radiation from a molecular point of view:

Heat radiation is the transfer of thermal energy through electromagnetic waves. Unlike conduction or convection, radiation does not require a material medium to propagate. Instead, radiative transport can occur in a vacuum or through transparent substances.

At the molecular level, heat radiation involves the emission and absorption of electromagnetic waves, specifically in the form of infrared radiation. Molecules have vibrating and rotating motions due to their thermal energy. These molecular motions result in the emission of electromagnetic waves, including infrared radiation.

When two objects are at different temperatures, the molecules in the hotter object have higher thermal energy, leading to more intense molecular vibrations and rotations. Consequently, these molecules emit a higher intensity of infrared radiation. The cooler object absorbs more radiation than it emits, causing the molecules to gain energy and increase their kinetic motion.

Heat radiation can also occur through reflection and transmission. Some materials can reflect a significant portion of incident radiation. Others, like transparent substances, allow radiation to pass through with minimal absorption, enabling heat transfer.

In summary, heat conduction involves the transfer of thermal energy through molecular collisions, whereas heat radiation occurs through the emission, absorption, reflection, and transmission of electromagnetic waves, particularly infrared radiation. Both processes contribute to the overall transfer of heat in different contexts and materials.

Section 2

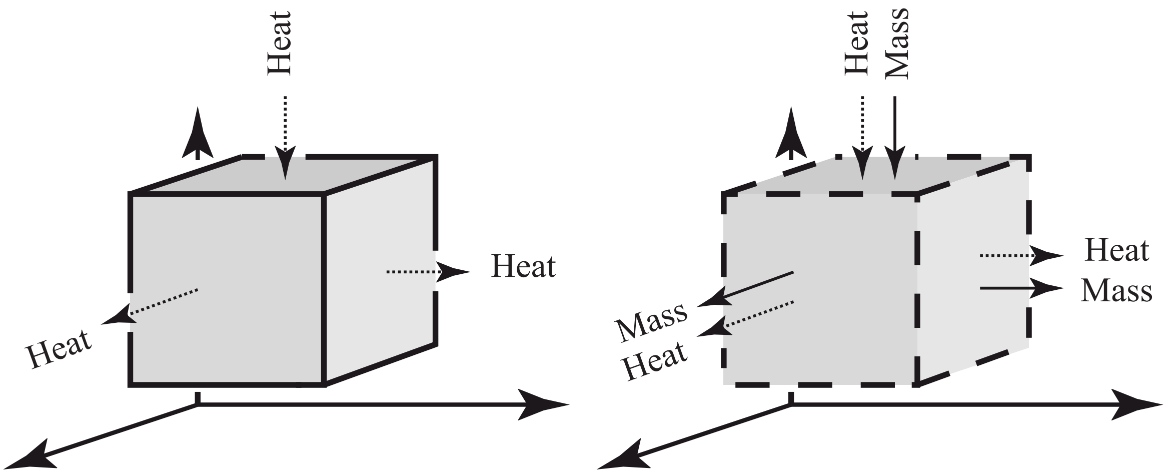
Systems and control volumes

A system refers to a specific amount of matter or a designated area in space that is selected for analysis. The region or mass located outside of the system is known as the surroundings. A boundary, either tangible or conceptual, separates the system from the surroundings. This boundary may be stationary or moveable, always representing the surface that connects the system to the surroundings. Worth noting that, from a mathematical standpoint, the boundary is considered to have zero thickness and cannot contain any mass or occupy any space in the surrounding area.

Depending on the selection of a specific mass or volume in space for examination, systems fall into either open or closed categories. A closed system contains a fixed quantity of mass and does not permit any matter to cross the boundary. Neither input nor output of mass is possible within a closed system, as depicted in Figure 2.1. However, energy can transfer in the form of work or heat across the boundary of a closed system, and the volume of a closed system does not have to be constant (imagine a helium balloon that rises into the sky, thereby expanding in volume but remains constant in mass).

An open system is a region in space that is deliberately chosen for analysis. This region typically encompasses a device that involves the flow of mass, such as a turbine, compressor, or nozzle. To examine the flow through these devices more effectively, the region within the device is commonly selected as the control volume. In an open system, both mass and energy can cross the boundary of the control volume, allowing for a more comprehensive analysis of the system's behavior.

A water bottle can exemplify open and closed systems. With the cap secure, the bottle is closed, barring fluid exchange. The liquid remains constant, isolating the interior. Removing the cap opens the system, allowing liquid exchange with the surroundings. This showcases how a water bottle shifts between open and closed, impacting matter and energy exchange across the system's boundaries.



(a) Closed system. (b) Open system.

Figure 2.1. Different control volumes and their boundaries.

Section 3
Balances

As an engineer, you likely learned about the concept of "balance" in your prior courses, such as mechanics (statics and dynamics). A balance is a versatile tool that applies to various situations, including mass, species, forces, or energy. When applied to dynamics, a balance can lead to an equation of motion.

To visualize the concept of energy balance, imagine a shoe box. Energy can flow in or out of the box through the defined boundaries. In subsequent chapters, you will learn more about how these energy flows can be described. Additionally, thermal energy can be generated inside the box, for example, by an electrical device that converts electric energy to thermal energy. All incoming and outgoing fluxes across the system boundaries, together with the heat sources/sinks inside the system, will affect the inner energy of the shoe box. If there is a positive net energy remaining in the box, the temperature inside increases, and if there is a negative net energy, the temperature will decrease.

Note, while considering one specific form of energy (heat), sources or sinks of this energy form can exist without violating the basic thermodynamic principle that energy in general cannot be created or destroyed.

Fundamental EQ

General energy balance:

$$\begin{aligned}\textbf{Temporal change in inner energy} &= \\ \textbf{ingoing fluxes} - \textbf{outgoing fluxes} &+ \textbf{sources} - \textbf{sinks}.\end{aligned} \tag{3.1}$$

The temporal change in inner energy $\frac{\partial U}{\partial t}$ may be linked to a change in temperature by $U = mcT$, with m denoting the system's mass, and c the specific heat capacity. The unit of this term is $\left(\frac{\mathrm{J}}{\mathrm{s}}\right)$ or (W). In analogy, all heat fluxes crossing the systems boundaries as well as the heat source and sinks need to be described in absolute values (W), rather than in area-specific $\left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)$ or volume-specific $\left(\frac{\mathrm{W}}{\mathrm{m}^3}\right)$ quantities respectively. A system that does not change its inner energy in time is called a steady-state system.

Fundamental EQ

General energy balance steady-state system:

$$0 = \textbf{ingoing fluxes} - \textbf{outgoing fluxes} + \textbf{sources} - \textbf{sinks}. \tag{3.2}$$

The opposite of a steady system is an unsteady system.

Subsection 3.1
Systematic approach to solve engineering problems

In this textbook, we aim to teach you a systematic problem-solving approach in which the general structure is described as follows:

Approach 3.1

Solving complex problems by use of balances:

- For a chosen/defined control volume for the energy balance
  - Ensure that the chosen element is representative of the entire domain, as seen in the analysis of fluid flow through a pipeline. Consider a lengthy pipe through which fluid flows steadily. When modeling the fluid behavior, ensure the infinitesimal element is located within the fluid itself rather than the pipe wall.

1 **Setting up the balance:**

- Define changes in the internal energy.
  - The temporal change of fluid temperature over the course of time.
- Define fluxes across the boundaries.
  - Diffusive heat transfer due to temperature gradients.
  - Advective heat transfer due to fluid motion.
  - External mechanisms, such as radiative heat transport from the sun.
- Define internal heat sources and sinks.
  - Chemical reactor where the fluid flows through a reaction vessel.

2 **Defining the elements within the balance:**

- Define the fluxes based on constitutive laws for conduction, convection, and radiation.
  - Fourier's law of heat diffusion: $\dot{q}'' = -\lambda \frac{\partial T}{\partial x}$.
  - Newton's law of cooling: $\dot{q}'' = \alpha \left(T_{\mathrm{w}} - T_{\mathrm{f}}\right)$.
  - Stefan-Boltzmann Law: $\dot{q}'' = \sigma \epsilon T^4$.

3 **Inserting and rearranging:**

- Substitute all definitions and rewrite the differential equation in a clear manner.

4 **Defining the boundary and/or initial conditions:**

- Find the required conditions for solving the given differential equation.

5 **Solving the equation:**

- Using the boundary and initial conditions, find a mathematical expression for the required function.

Please be aware that several aspects mentioned in the provided approach have yet to be explored and may seem unexpected. The reader will explore these facets of heat transfer in the upcoming sections. The primary point to emphasize at this juncture is that a significant portion of the presented problems can be resolved by appropriately defining the domain and following the prescribed five steps.

By breaking down the problem into smaller steps, we can more easily analyze and understand the behavior of a system. This systematic approach is not only useful in heat transfer but also in many other fields of engineering and science. This approach allows us to tackle complex problems with better confidence and efficiency, leading to more effective and optimized designs.

An essential aspect of solving a differential equation is to apply the appropriate boundary or initial conditions that govern the behavior of the system being modeled. While this process may appear straightforward, errors occurring due to the complex nature of many differential equations are not uncommon. To minimize the potential for such errors, a systematic approach for determining the boundary or initial conditions necessary to solve the differential equation is recommended.

Approach 3.2

Finding the required boundary conditions:

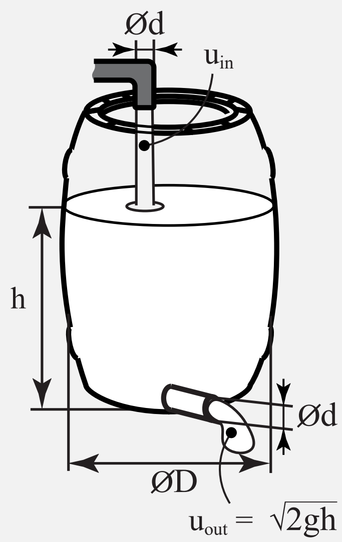
First, determine the number of conditions required to solve the equation. For partial differential equations, the number of initial and/or boundary conditions required depends on the number of independent variables and the order of the derivatives with respect to each variable.

For each required condition:

1. Is the temperature at the given location/beginning known?
   - If yes, state this condition.
   - If no, try step 2.
2. Are there symmetric or adiabatic conditions at the location/beginning?
   - If yes, state this condition.
   - If no, try step 3.
3. Is a heat transfer rate at the given location/beginning known?
   - If yes, derive this condition from a local energy balance.
   - If no, try a different location.

To illustrate the above-described systematic solution process, consider a fluid dynamic example that is solvable with a minor physics background.

Example 3.1 A fluid flows into the top of a barrel at a constant average velocity u_{in}. At the same time, a portion leaves the barrel through a pipe at a velocity u_{out}. Initially, the height of the fluid is equal to h_0. Find an expression for the height $h(t)$ of the fluid over time.



Hint:

- $u_{\text{in}} < u_{\text{out}}$.

1 Setting up the balance:

An incoming rate of mass from the tap is observed, while simultaneously, an outgoing mass flow due to the pipe at the bottom of the barrel is noted. If these rates are not identical, a temporal change of mass within the barrel will occur. Therefore, the mass balance is expressed as follows:

$$\frac{dm}{dt} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}.$$

Therein, the change in mass within the barrel is equal to the difference between the incoming and outgoing mass fluxes. Note, the dot above the m indicates that not the mass but the mass flux is considered.

2 Defining the elements within the balance:

Mass inside the barrel:

$$m = \rho V = \rho h(t)\frac{\pi D^2}{4}.$$

Incoming rate of mass:

$$\dot{m}_{\text{in}} = \rho u_{\text{in}} A_{\text{c}} = \rho u_{\text{in}} \frac{\pi d^2}{4}.$$

Outgoing rate of mass:

$$\dot{m}_{\text{out}} = \rho u_{\text{out}} \frac{\pi d^2}{4} = \rho\sqrt{2gh(t)}\frac{\pi d^2}{4}.$$

The concept of the outgoing mass flux is slightly more intricate because the pressure buildup in the barrel caused by the water level is considered. In fluid mechanics, you will discover that the velocity of the outflow has a square-root relationship with the water level. Important is to note that the water level, denoted as h, changes over time.

3 Inserting and rearranging:

$$\frac{dh}{dt} = \left(\frac{d}{D}\right)^2 \left(u_{\text{in}} - \sqrt{2gh(t)}\right).$$

4 Defining the boundary and/or initial conditions:

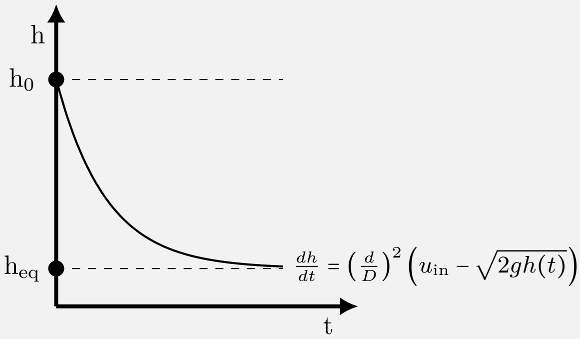
The height h has been differentiated once with respect to t and as such an initial water level (h) will be required to provide a solution to the problem.

Initially:

$$h(t = 0) = h_0.$$

5 Solving the equation:

Using an ODE solver, the following relationship is found:



Where at some point in time $u_{\text{in}} = u_{\text{out}}$ and the height of the fluid $h(t)$ reaches an equilibrium position which makes sense from a physical point of view.

Section 4
Important definitions and notations

Energy, also in the form of heat, is often measured in units such as Joules (J) or kilowatt-hours (kWh), depending on the context. Note, that energy is a scalar quantity without a direction.

On the other hand, the rate of energy is a vector quantity, which has both magnitude and direction. The rate of energy, also known as power P, refers to the amount of energy E transferred per unit of time Δt in units such as Jouler per seconds $\left(\frac{\mathrm{J}}{\mathrm{s}}\right)$, better known as Watts (W).

The heat transfer rate $\dot{Q}$, referring to the amount of heat Q transferred per unit of time Δt, is measured in the unit of Watts (W).

Definition

Rate of heat:

$$\dot{Q} = \frac{Q}{\Delta t} \ (\mathrm{W}) . \tag{4.1}$$

Furthermore, the heat transfer rate $\dot{Q}$ per unit area A is called heat flux $\dot{q}''$ $\left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)$. Thereby, area-specific values are denoted by two dashes $''$.

Definition

Heat flux:

$$\dot{q}'' = \frac{\dot{Q}}{A} \left(\frac{\mathrm{W}}{\mathrm{m}^2}\right) . \tag{4.2}$$

Internal energy U is a system's total energy due to the particles' motion and interactions. The inner energy U of a system is directly proportional to both its mass m and specific enthalpy h. The internal energy is conserved in closed systems. Heat balances require accounting for changes in internal energy over time, along with external heat transfer and work done on the system, to describe the temperature distribution and behavior of a system.

Definition

Temporal change of inner energy:

$$\frac{\Delta U}{\Delta t} = m\frac{\Delta h}{\partial t} \approx mc\frac{\Delta T}{\Delta t} \ (\mathrm{W}) . \tag{4.3}$$

Note: $h \approx cT$ at constant pressure p, constant specific heat capacity c and constant density ρ. A typical applicable case is water flow through a pipe.

Internal heat generation $\dot{\Phi}$ is a key factor to consider when modeling heat transfer problems. This parameter can significantly impact the temperature distribution and overall behavior of a system. This is important in fields such as nuclear engineering, where the heat generated by radioactive decay must be carefully accounted for in the design and operation of reactors. Similarly, in materials science, the heat generated by friction during manufacturing processes can affect the quality and performance of the final product. Thereby, the volume-specific heat generation $\dot{\Phi}'''$ $\left(\frac{\mathrm{W}}{\mathrm{m}^3}\right)$ is a result of the total heat generation $\dot{\Phi}$ and the systems volume V. Volume-specific values are denoted by three dashes $'''$.

Definition

Volum specific internal heat generation:

$$\dot{\Phi}''' = \frac{\dot{\Phi}}{V} \left(\frac{\mathrm{W}}{\mathrm{m}^3}\right) . \tag{4.4}$$

Enthalpy flow, or advective transport, is another important concept in the study of heat transfer, representing the amount of energy transferred between a system and the surroundings as heat, with the addition of any work done by or on the system. Enthalpy flow is used for analyzing open systems, where energy transport due to mass transfer is of significance. The energy transported by the motion of mass $\dot{H}$ is directly proportional to the mass flow $\dot{m}$ and the specific enthalpy h.

Definition

Enthalpy flow:

$$\dot{H} = \dot{m}h = \dot{m}cT \ \ (\mathrm{W})\,. \tag{4.5}$$

To illustrate the difference between energy and power, consider the example of a light bulb. The energy consumed by a light bulb over a certain period, say an hour, is measured in kilowatt-hours. The power, or rate of energy consumption, of the light bulb is measured in watts or kilowatts. In this case, energy is the total amount of work that the light bulb consumed over time, while power is the rate at which work is performed.

In the context of heat transfer, energy balances are particularly important. An energy balance involves accounting for the inputs and outputs of thermal energy within a system. The goal is to ensure that the amount of energy entering the system is equal to the amount of energy leaving the system and that the energy within the system is conserved.

The principle of energy conservation, states that energy cannot be created or destroyed but can only be transformed from one form to another. The increase of inner energy in time of a control volume is the difference between the incoming and outflowing energy fluxes and the heat produced within the body.

Fundamental EQ

Law of conservation of energy for open systems:

$$\frac{dU}{dt} = \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}} + \sum \dot{H}_{\mathrm{in}} - \sum \dot{H}_{\mathrm{out}} + \dot{\Phi}. \tag{4.6}$$

Fundamental EQ

Law of conservation of energy for closed systems:

$$\frac{dU}{dt} = \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}} + \dot{\Phi}. \tag{4.7}$$

By applying this principle, governing equations for heat transfer that describe the behavior of thermal energy within a system can be derived.

One of the most important applications of energy balances in heat transfer is the development of the conduction equation. This equation describes the rate of heat transfer through a material by conduction and is derived by applying an energy balance to a small volume of material. The conduction equation allows the calculation of temperature profiles within a material as well as the heat transfer rate.

Example 4.1 A sphere with diameter d, density ρ, and specific heat capacity c_{p}, initially at temperature T_0, is subjected to a constant heat flux $\dot{q}''$. Heat losses are negligible, and the sphere's temperature is considered homogeneous. What is the sphere's temperature at time t_1?

1 Setting up the balance:

$$\frac{dU}{dt} = \dot{Q}_{\mathrm{in}} - \dot{Q}_{\mathrm{out}}.$$

2 Defining the elements within the balance:

Internal energy:

$$U = mc_{\mathrm{p}}T(t) = \rho\frac{\pi d^3}{6}c_{\mathrm{p}}T(t).$$

Rate of heat transferred to the sphere:

$$\dot{Q}_{\mathrm{in}} = \dot{q}''A_{\mathrm{s}} = \dot{q}''\pi d^2.$$

Rate of heat loss from the sphere:

$$\dot{Q}_{\mathrm{out}} = 0.$$

3 Inserting and rearranging:

$$\frac{dT}{dt} = \frac{6\dot{q}''}{\rho c_{\mathrm{p}} d}.$$

4 Defining the boundary and/or initial conditions:

The temperature T has been differentiated once with respect to t. To solve the differential equation, one initial condition is required.

Initially at $t = 0$:

$$T(t = 0) = T_0.$$

5 Solving the equation:

Integration yields:

$$T(t) = \frac{6\dot{q}''}{\rho c_{\mathrm{p}} d} t + c_1.$$

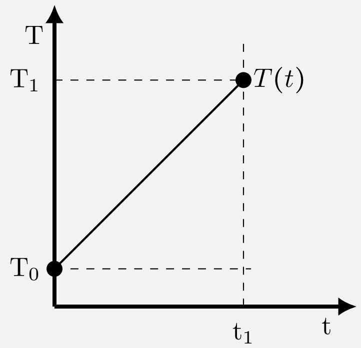
The initial condition yields that $c_1 = T_0$

Which gives the temperature function over the course of time:

$$T(t) = \frac{6\dot{q}''}{\rho c_{\mathrm{p}} d} t + T_0.$$

Thus, the sphere's temperature at t_1 equals:

$$T(t_1) = \frac{6\dot{q}''}{\rho c_{\mathrm{p}} d} t_1 + T_0.$$



PART

II

Conduction

SECTION 5

Fundamentals

L01 - Introduction to Fourier's law:

Learning goals:

- Conceptual understanding of steady-state and transient heat conduction.
- Application of heat conduction principles to systems with sources and sinks.
- Proficient calculation of heat flow within objects.
- Insightful understanding of temperature distribution within objects.

Comprehension questions:

- What is the driving potential of heat conduction?
- Which three influencing variables determine a heat flow transferred by heat conduction according to Fourier's law?
- Why must the temperature gradient in a positive coordinate system have a negative sign?
- Which material property is decisive for heat conduction?

L02 - Steady conservation equation:

Learning goals:

- Proficient formulation of energy balances for diverse scenarios.
- Application of Taylor Series Expansion to derive differential equations from energy balances.
- Definition of boundary conditions for energy balances.

Comprehension questions:

- What is the steady-state temperature profile for a homogeneous, one-dimensional, flat wall without heat sources?
- Under which conditions does Poisson's equation become Laplace's equation?

L03 - Unsteady conservation equation:

Learning goals:

- Conceptual understanding of internal energy and differentiation of kinetic from potential energy
- Ability to discriminate between specific heat at constant temperature and constant pressure
- Ability to proficiently formulate energy balances for diverse scenarios
- Ability to define boundary conditions for energy balances
- Ability to proficiently solve differential equations for simple cases

Comprehension questions:

- Under which conditions does Poisson's equation become Laplace's equation?
- What is the steady state temperature profile for a homogeneous, one-dimensional, flat wall without heat sources?

Corresponding tutorial exercises:

- Exercise II.1 Temperature profiles in planar walls
- Exercise II.2 Onion layer principle
- Exercise II.3 Heat conduction equation

Subsection 5.1
Fourier's law

In 1822, Fourier presented his research on heat transfer in his seminal work titled "Théorie analytique de la chaleur" (The Analytical Theory of Heat), where he derived his reasoning from Newton's law of cooling. According to Fourier, the heat flow between two neighboring molecules is directly proportional to the infinitesimally small temperature difference between them. This insight led him to propose a linear relationship between the rate of heat flow and the temperature gradient.

Figure 5.1. Jean Baptiste Joseph Fourier (1768–1830) [1].

The proportionality constant in this relationship, known as thermal conductivity $\lambda\ \left(\frac{\mathrm{W}}{\mathrm{mK}}\right)$, characterizes the heat-conducting ability of a material, indicating the ease with which heat is transferred. The unit results from connecting the rate of heat transfer $\dot{Q}$ (W) with the temperature gradient $\frac{\partial T}{\partial x}\ \left(\frac{\mathrm{K}}{\mathrm{m}}\right)$ and the area $A\ \left(\mathrm{m}^2\right)$, which is perpendicular to the direction of the heat flow.

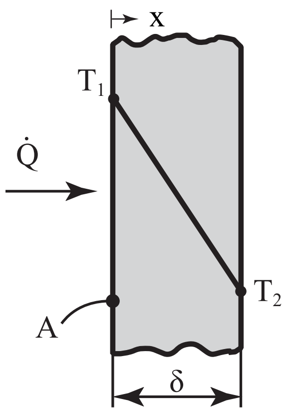


Figure 5.2. Heat flow through a plane wall.

For the rate of heat transfer per unit area, the following equation applies, with the convention that T_1 represents the higher temperature:

$$\frac{\dot{Q}}{A} = \dot{q}'' = \lambda \frac{T_1 - T_2}{\delta}.$$

While the determination of the heat flow rate is correct, the definition of T_1 being the higher temperature is arbitrary and does not provide a straightforward calculation method. In line with the second law of thermodynamics, heat naturally flows from regions of higher temperature to regions of lower temperature. In this context, temperature acts as a driving potential, much like height in a gravity field.

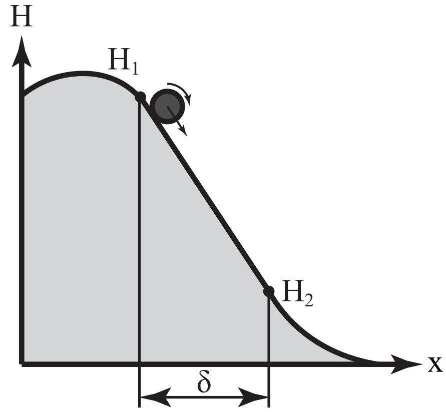


Figure 5.3. Ball rolling downhill.

To better understand this concept, consider the analogy of a ball placed on top of a hill. The ball effortlessly rolls downhill due to the force of gravity. Similarly, in Fourier's law, heat flows from regions of higher temperatures to regions of lower temperatures, propelled by the temperature difference.

Now, consider the temperature gradient $\frac{\partial T}{\partial x}$ as the slope of the hill. Figure 5.2 shows that $\frac{\partial T}{\partial x} < 0$. From a physical standpoint, a negative sign should be incorporated into the general expression of Fourier's law. This negative sign indicates the direction of heat flow, accounting for the fact that heat naturally moves from a higher temperature to a lower temperature.

Mathematically, this is written as:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x}.$$

Returning to the hill analogy, when the slope of the hill is steeper (corresponding to a larger temperature gradient), the ball rolls faster and covers more distance within a given time. In the context of heat flow, this analogy suggests that a higher rate of heat flow occurs when there is a larger temperature difference between two points.

Following this principle, a clear definition of the heat flux is given by Fourier's law.

Fundamental EQ

Fourier's law:

$$\dot{q}'' = -\lambda \frac{\partial T}{\partial x}. \tag{5.1}$$

SUBSECTION 5.2

Conservation equations

The conservation equations for heat, mass, and momentum serve as the fundamental framework for calculations in fluid mechanics and heat transfer. These equations form the backbone of Computational Fluid Dynamics codes, enabling insights into flow characteristics and heat transfer performance.

Each conservation equation operates on the principle of balance. Within this section, there is a focus on developing the conservation equation specifically for heat within a static (non-moving) closed control volume in a Cartesian coordinate system. The control volume is assumed to be sufficiently small, with dimensions of dx, dy, and dz, and the state variables, such as temperature, remain constant within the control volume. To derive the conservation equation, the cumulative effect of all heat fluxes crossing the boundaries of the control volume, the temperature variations over time within the control volume, and the presence of a heat source or heat sink are considered.

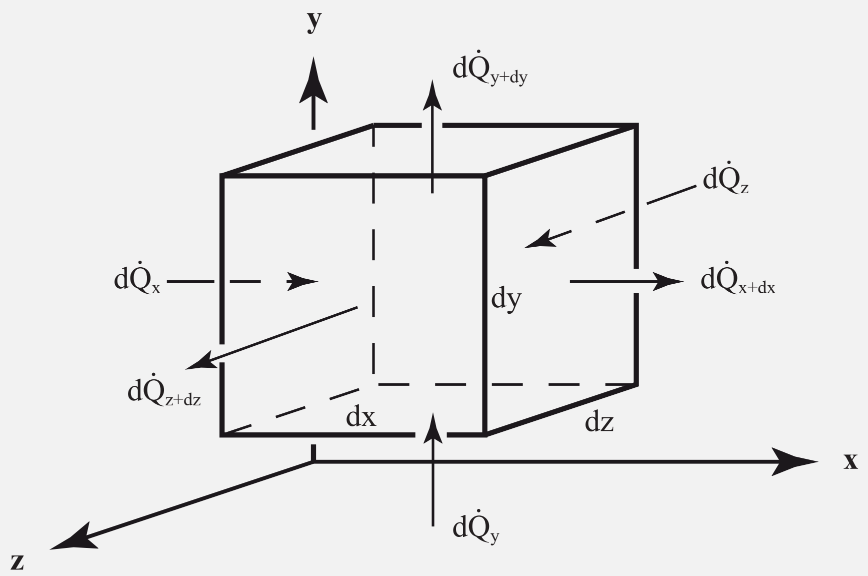


Figure 5.4. Energy balance of a static control volume.

Derivation

1 Setting up the balance:

$$\underbrace{\frac{\partial U}{\partial t}}_{1} = \underbrace{d\dot{Q}_x - d\dot{Q}_{x+dx} + d\dot{Q}_y - d\dot{Q}_{y+dy} + d\dot{Q}_z - d\dot{Q}_{z+dz}}_{2} + \underbrace{d\dot{\Phi}}_{3},$$

where 1 covers the temporal change of the temperature in the control volume, 2 describes the heat fluxes across the boundaries in all three spatial directions and 3 accounts for a heat source within the control volume.

Note, that the direction of all heat fluxes is assumed to be in the positive direction of the coordinate system.

Keep in mind that the flux $d\dot{Q}_x$ represents only a fraction of the total flux $\dot{Q}_x$ in the x

direction. This specific flux is related to the flow passing through the partial cross-sectional area $dydz$, not the complete cross-section A_x. Similarly, $d\dot{\Phi}$ describes the heat generation within the infinitesimal element, which constitutes a minute portion of the total heat generated.

2 Defining the elements within the balance:

The incoming energy flow at the location x across the surface area $dydz$ is due to the heat transfer mechanism of conduction. Using Fourier's Law, see equation (5.1), the incoming heat flow rate in x-direction $d\dot{Q}_x$ is:

$$d\dot{Q}_x = -\lambda \frac{\partial T}{\partial x} \underbrace{dydz}_{\text{area}} .$$

The Taylor series expansion principle is used for the outgoing heat flux. Therein a small change in the heat flow rate is assumed, which can be described linearly:

$$\begin{aligned} d\dot{Q}_{x+dx} &= d\dot{Q}_x + \frac{\partial}{\partial x}\left(d\dot{Q}_x\right) dx \\ &= -\lambda \frac{\partial T}{\partial x} dydz + \frac{\partial}{\partial x}\left(-\lambda \frac{\partial T}{\partial x}\right) dxdydz. \end{aligned}$$

Note that in such local balances using Taylor series expansion, the term $d\dot{Q}_x$ vanishes as this term is included within the ingoing and outgoing fluxes. Only the local change of the heat flux remains. Similar expressions are obtained for the other two directions of the coordinate system, respectively.

The heat released or absorbed, for instance by an electric heater or a chemical reaction, in the element is:

$$d\dot{\Phi} = \dot{\Phi}''' \underbrace{dx\,dy\,dz}_{\text{volume}},$$

where $\dot{\Phi}'''$ denotes the specific intensity of the source term in $\left(\frac{\mathrm{W}}{\mathrm{m}^3}\right)$.

The temporal change of internal energy of the element, assuming a constant mass within the control volume is described as:

$$\frac{\partial U}{\partial t} = \frac{d(mc_{\mathrm{v}}T)}{dt} = \rho\, c_{\mathrm{v}} dxdydz \frac{\partial T}{\partial t},$$

where ρ $\left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)$ is the density and c_{v} $\left(\frac{\mathrm{J}}{\mathrm{kg \cdot K}^3}\right)$ the specific heat capacity at constant volume (the index v is neglected for solids).

3 Inserting and rearranging:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' .$$

The law of conservation of energy provides the differential equation governing the temperature field in Cartesian coordinates. Likewise, analogous differential equations are derived for systems described in cylindrical and spherical coordinates, as stated in equations (5.2) - (5.4).

Fundamental EQ

Equation of energy conservation for solids in Cartesian coordinates (x,y,z,t):

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' . \tag{5.2}$$

Fundamental EQ

Equation of energy conservation for solids in cylindrical coordinates (r,θ,z,t):

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right) + \dot{\Phi}'''. \tag{5.3}$$

Fundamental EQ

Equation of energy conservation for solids in spherical coordinates (r,θ,ϕ,t):

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \theta}\left(\lambda\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\lambda\frac{\partial T}{\partial \phi}\right) + \dot{\Phi}'''. \tag{5.4}$$

In most cases, the thermal conductivity λ is considered to be constant, which simplifies equations (5.2) - (5.4). Hence, equation (5.2) are written as:

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{\Phi}'''}{\lambda},$$

where a is the so-called thermal diffusivity.

Definition

Thermal diffusivity:

$$a = \frac{\lambda}{\rho c}\left(\frac{m^2}{s}\right). \tag{5.5}$$

The solution of equations (5.2) - (5.4) including the boundary conditions gives the temperature field of the body. The ability of a material to 'let heat pass through it' increases with increasing thermal diffusivity. This is due to a high thermal conductivity λ or a low heat capacity ρc of the material.

The following sections present the solutions of the differential equations (5.2) - (5.4) for specific cases and applications. The temperature field determined for a specific problem under consideration of its boundary conditions is used in equation (5.1) to determine the heat flux through a surface.

Furthermore, equation (5.2) is in literature termed Poisson's and Laplace's equation. Where Poisson's equation is the steady-state variant of the conservation equation and Laplace's equation is the steady-state variant of the conservation equation when no heat sources are present.

Fundamental EQ

Poisson's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{\Phi}'''}{\lambda} = 0. \tag{5.6}$$

Fundamental EQ

Laplace's equation:

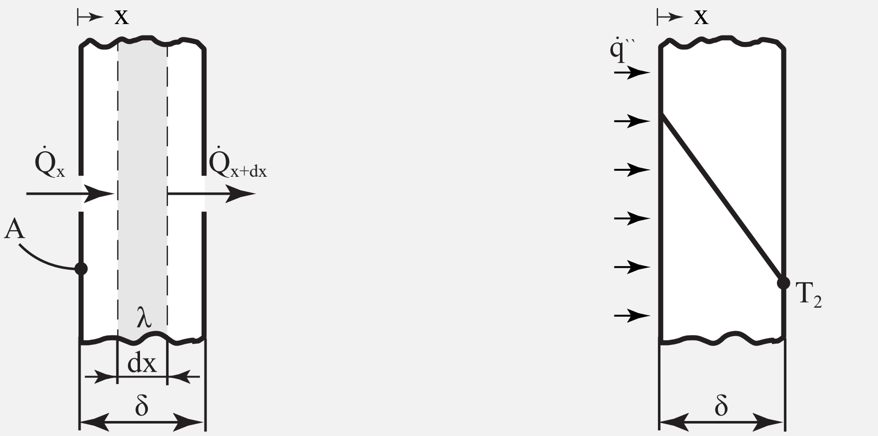
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{5.7}$$

HEATQUIZ 5.1

Temperature profiles:

Energy balances:

Example 5.1 Determine the temperature profile within a plane wall of thickness δ and cross-sectional area A. At the left side of the wall, a constant flux $\dot{q}''$ is imposed, while on the right side, a constant temperature of T_2 is prescribed. Assume one-dimensional steady-state conditions without sources or sinks.



(a) Energy balance. (b) Temperature distribution.

1 Setting up the balance:

$$\cancelto{0\text{ - steady-state}}{\frac{\partial U}{\partial t}} = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}.$$

The energy balance states that the ingoing heat flux and the outgoing heat flux are identical.

2 Defining the elements within the balance:

The ingoing rate of heat transfer described by Fourier's law, see equation (5.1):

$$\dot{Q}_{\mathrm{x}} = -\lambda A \frac{\partial T}{\partial x}.$$

The outgoing rate of heat transfer can be approximated by use of the Taylor series expansion:

$$\dot{Q}_{\mathrm{x+dx}} = \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \cdot dx = -\lambda A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A \frac{\partial T}{\partial x}\right) \cdot dx.$$

3 Inserting and rearranging:

$$\frac{\partial^2 T}{\partial x^2} = 0.$$

The equation states that the curvature (e.g. second derivative) of the temperature profile is zero. Therefore, the solution to the differential equation acts as a straight line with a constant slope. This ensures a constant heat flux.

4 Defining the boundary and/or initial conditions:

The temperature T has been differentiated twice with respect to x. Thus, to solve the differential equation, two boundary conditions are required.

At the left side, at $x = 0$, a constant heat flux is prescribed. A local energy balance $\left(-\lambda \frac{\partial T}{\partial x}\big|_{x=0} = \dot{q}''\right)$ yields:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{\dot{q}''}{\lambda}.$$

At the right side, at $x = \delta$, the temperature T_2 is given:

$$T(x = \delta) = T_2.$$

5 Solving the equation:

Integrating the derived differential equation once gives:

$$\frac{\partial T}{\partial x} = A.$$

Integrating twice results:

$$T(x) = Ax + B.$$

From the boundary conditions is is found that $A = -\frac{\dot{q}''}{\lambda}$, and $B = T_2 + \frac{\dot{q}''}{\lambda}\delta$.

This finally results in the following linear temperature profile:

$$T(x) = \frac{\dot{q}''}{\lambda}(\delta - x) + T_2.$$

Section 6

Multi-layer walls

L04 - Heat conduction in a multilayer plane wall:

Learning goals:

- Conceptual understanding of the temperature profile within a multilayer wall under steady-state conditions.
- Analyzing the combination of thermal resistors connected in series to determine total resistance.

Comprehension questions:

- What is the course of the temperature profile in a flat wall without heat sources and sinks in a steady state?
- Under what conditions does the assumption hold that the heat flow is constant throughout all layers?
- How is the thermal resistance of a plane wall defined? How can the thermal resistance be calculated for a wall of n layers?

L05 - Heat conduction in a cylindrical coordinate system:

Learning goals:

- Conceptual understanding of schematic profiles for temperature and heat flux in case of varying cross-sectional areas.
- Ability to derive the differential equation through energy balances.
- Proficient solving of differential equations.
- Ability to extend the equation to encompass multiple resistors.
- Ability to solve the problem using an engineering approach.

Comprehension questions:

- What is the course of the temperature profile for cylindrical bodies?
- How does the temperature profile of a cylindrical body differ from the temperature profile of a plane wall? What is the reason for this?
- Under which conditions can the curvature of the cylinder and thus the change of the area inside the cylinder wall be neglected?

L06 - Heat conduction in a multilayer plane wall with convection:

Learning goals:

- Conceptual understanding of the temperature profiles within multilayer plane walls considering convective resistances on the surfaces.
- Ability to apply the concept of total resistance to multilayer plane walls with convection.
- Ability to calculate heat flow through multilayer plane walls with convection.

Comprehension questions:

- What is the curvature of the temperature profile on the fluid side due to convection?
- What influence does the additional consideration of convection have on the total heat transfer?

L07 - Heat conduction in a multilayer pipe wall with convective resistances:

Learning goals:

- Understanding the change in surface area in a multilayer pipe wall.
- Grasping the concept of the temperature profile in a multilayer pipe wall.
- Developing the ability to calculate the total thermal resistance in a multilayer pipe wall.
- Developing the ability to calculate heat flow in a multilayer pipe wall.

Comprehension questions:

- How does the curved surface of a pipe affect the temperature gradient at constant heat flow and constant thermal conductivity?
- What reference area and reference radius must be considered when calculating the total heat transfer coefficient k for a pipe wall problem?

L08 - Example: Pipe in the heating system:

Learning goals:

- Mastering the procedure for calculating thermal resistances and heat flows in a pipe wall.

Comprehension questions:

- Which simplifying assumptions are valid when calculating the heat flow through a pipe wall?
- Which resistance determines the heat transfer (coefficient)?

Corresponding tutorial exercises:

- Exercise II.4 Window insulation
- Exercise II.5 Ice layer
- Exercise II.6 Warm-water pipe

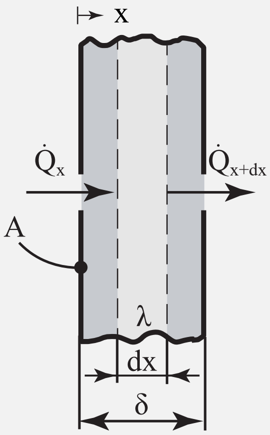
SUBSECTION 6.1
Multi-layer wall

The preceding section examined the temperature distribution in a uniform material with a constant thermal conductivity. Nevertheless, in various practical and everyday scenarios, such as a thermally insulated house wall, multiple materials with distinct thermal properties are arranged in series.

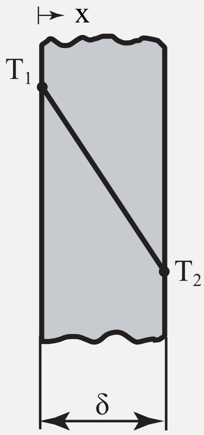
In this subsection, the central conceptual idea is a constant rate of heat transfer that enters the wall on one side, travels through the material, and exits on the other side. To facilitate the analysis, the thermal resistance theorem simplifies the process by assuming a constant heat transfer rate, steady-state conditions, uniform thermal conductivity within each layer, and the absence of internal heat sources or sinks. This framework enables the prediction of total thermal resistance and heat transfer rate through the layers, proving invaluable for optimizing energy efficiency and insulation in a design.

Derivation

To derive the temperature profile in a multi-layer wall, first, the focus will lie on deriving the equation for a single layer by solving the energy conservation equation for this domain.



(a) Energy balance.



(b) Temperature distribution.

Figure 6.1. Conduction through a flat wall.

1 Setting up the balance:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}.$$

2 Defining the elements within the balance:

Ingoing rate of heat transfer:

$$\dot{Q}_{\mathrm{x}} = -\lambda A \frac{\partial T}{\partial x}.$$

Outgoing rate of heat transfer (approximated by use of the Taylor series expansion):

$$\begin{aligned}\dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x}}}{\partial x} \cdot dx \\ &= -\lambda A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A \frac{\partial T}{\partial x}\right) \cdot dx.\end{aligned}$$

3 Inserting and rearranging:

$$\frac{\partial^2 T}{\partial x^2} = 0.$$

4 Defining the boundary and/or initial conditions:

The temperature T has been differentiated twice with respect to x and to solve the differential equation, two boundary conditions are required.

At $x = 0$:

$$T(x = 0) = T_1.$$

At $x = \delta$:

$$T(x = \delta) = T_2.$$

5 Solving the equation:

Integrating twice gives:

$$T(x) = Ax + B.$$

From the boundary conditions yields $B = T_1$ and $A = \frac{T_2 - T_1}{\delta}$.

Which results in the following temperature function:

$$T(x) = T_1\left(1 - \frac{x}{\delta}\right) + T_2\,\frac{x}{\delta}.$$

Using Fourier's law, as stated in equation (5.1), the rate of heat transfer is determined:

$$\begin{aligned}\dot{Q} &= -\lambda A\frac{\partial T}{\partial x}\\ &= \lambda A\frac{T_1 - T_2}{\delta}.\end{aligned}$$

Fundamental EQ

Temperature profile of a plane wall with known temperatures T_1 and T_2:

$$T(x) = T_1\left(1 - \frac{x}{\delta}\right) + T_2\,\frac{x}{\delta}, \tag{6.1}$$

where $T_1 > T_2$.

Fundamental EQ

Rate of heat transfer through a plane wall with known temperatures T_1 and T_2:

$$\dot{Q} = \lambda A\frac{T_1 - T_2}{\delta}, \tag{6.2}$$

where $T_1 > T_2$.

Derivation

In the case of a multi-layer wall, i.e. consisting of many layers with different thicknesses and materials, the energy balance must be calculated for each section in turn. The heat entering section 1, flows unchanged out of section 3 for steady-state, one-dimensional cases.

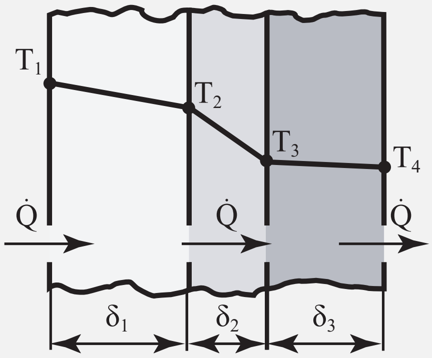


Figure 6.2. Conduction through a multi-layer wall.

1 **Setting up the balance:**

$$\dot{Q}_1 = \dot{Q}_2 = \dot{Q}_3 = \dot{Q}.$$

2 **Defining the elements within the balance:**

Using equation 6.2, the rate of heat transfer through each layer is written as:

$$\dot{Q}_1 = \lambda_1 \frac{A}{\delta_1}(T_1 - T_2), \qquad \dot{Q}_2 = \lambda_2 \frac{A}{\delta_2}(T_2 - T_3), \qquad \text{and} \qquad \dot{Q}_3 = \lambda_3 \frac{A}{\delta_3}(T_3 - T_4).$$

3 **Inserting and rearranging:**

Which yields the following expression for the total rate of heat transfer:

$$\dot{Q} = \frac{A}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \frac{\delta_3}{\lambda_3}}(T_1 - T_4).$$

The derived equation describing the rate of heat transfer draws parallels with Ohm's law in electrical conductors, stating that the electric current is directly proportional to voltage and inversely proportional to resistance. The derived expression states that the rate of heat flow through a multi-layer wall is directly proportional to the temperature difference and inversely proportional to the cumulative thermal resistances across the layers.

The derived expression can be generalized for a system comprising n layers and is commonly referred to as the thermal resistance theorem.

Fundamental EQ

Rate of heat transfer through a solid multi-layer wall without convection:

$$\dot{Q} = \frac{1}{\sum_{i=1}^{n} R_{\mathrm{cond},i}}(T_1 - T_{n+1}). \tag{6.3}$$

Definition

Conductive resistance of a solid plane layer i:

$$R_{\mathrm{cond},i} = \frac{\delta_i}{A\lambda_i} \left(\frac{\mathrm{K}}{\mathrm{W}}\right). \tag{6.4}$$

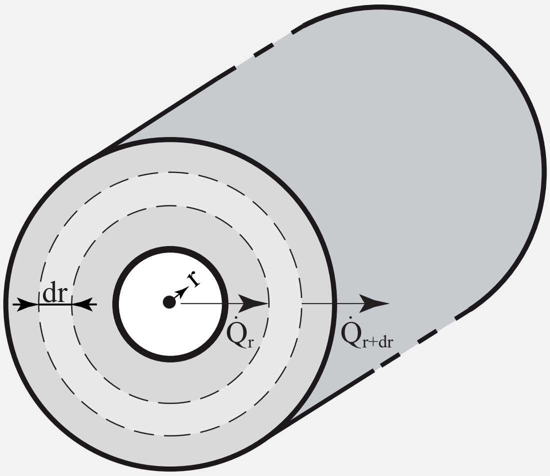
SUBSECTION 6.2

Cylindrical coordinates multi-layer pipe wall

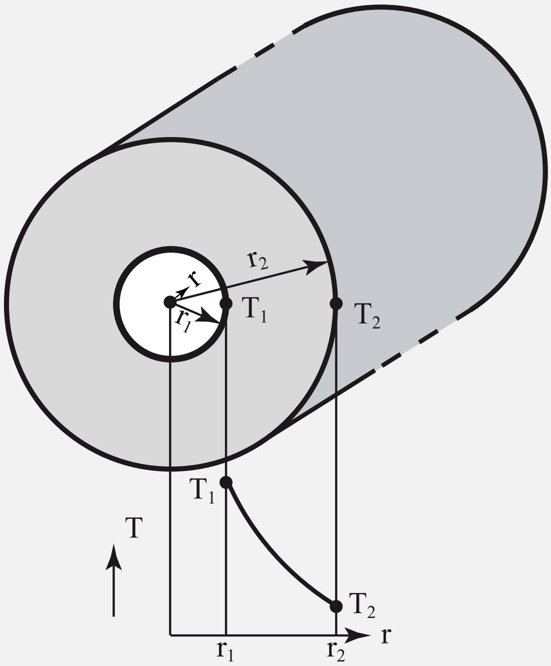
Similarly to planar wall systems, the thermal resistance theorem applies to cylindrical systems, such as pipe walls. However, a notable distinction in the cylindrical coordinate system is the varying cross-sectional area in the radial direction. Specifically, the area for the heat flux at $r + dr$ is larger compared to the area for the heat flux at r. If the rate of heat transfer remains constant across the wall, the temperature gradient must decrease in absolute value in the outwards direction.

Derivation

To establish the temperature distribution within a multi-pipe wall, the initial step involves concentrating on determining the profile for an individual cylindrical layer through the solution of energy conservation principles.



(a) Energy balance.



(b) Radial temperature distribution.

Figure 6.3. Conduction through a tube wall.

1 Setting up the balance:

$$0 = \dot{Q}_r - \dot{Q}_{r+dr}.$$

2 Defining the elements within the balance:

Ingoing rate of heat transfer at r through the variable area of $2\pi rL$:

$$\dot{Q}_r = -\lambda\, 2\pi r L \frac{\partial T}{\partial r}.$$

Outgoing heat flow through the area $2\pi(r+dr)L$ at the location $r+dr$:

$$\begin{aligned}\dot{Q}_{r+dr} &= \dot{Q}_r + \frac{\partial \dot{Q}_r}{\partial r} dr \\ &= -\lambda\, 2\pi r L \frac{\partial T}{\partial r} - \lambda\, 2\pi L \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) dr.\end{aligned}$$

3 Inserting and rearranging:

$$\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = 0.$$

4 Defining the boundary and/or initial conditions:

The differential equation is of second order with respect to r. For a solution of the differential equation, two boundary conditions are required.

The temperature at $r = r_1$:

$$T(r = r_1) = T_1,$$

and the temperature at $r = r_2$:

$$T(r = r_2) = T_2.$$

5 Solving the equation:

Integrating twice gives:

$$T = A \ \ln\left(\frac{r}{r_0}\right) + B,$$

where r_0 is the reference radius.

Using the boundary conditions gives that $A = \frac{T_2 - T_1}{\ln}\left(\frac{r_2}{r_1}\right)$ and $B = T_1 - \frac{\ln\left(\frac{r_1}{r_0}\right)(T_2 - T_1)}{\ln\left(\frac{r_2}{r_1}\right)}$.

The temperature in the wall of the tube proves to have logarithmic character, which is stated by:

$$T = T_1 + \ln\left(\frac{r}{r_1}\right) \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)},$$

or alternatively:

$$T = T_2 + \ln\left(\frac{r}{r_2}\right) \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}.$$

By using Fourier's Law, equation (5.1), the rate of heat transfer through the walls is determined:

$$\begin{aligned}\dot{Q} &= -\lambda A \frac{\partial T}{\partial r} = -\lambda 2\pi r L \frac{\partial T}{\partial r} \\ &= 2\pi L \lambda \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)}.\end{aligned}$$

Fundamental EQ

Temperature profile of a cylindrical wall with known temperatures T_1 and T_2:

$$T(r) = T_1 + \ln\left(\frac{r}{r_1}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}, \tag{6.5}$$

where $T_1 > T_2$.

Fundamental EQ

Rate of heat transfer through a cylindrical wall with known temperatures T_1 and T_2:

$$\dot{Q} = 2\pi L\lambda\frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)}, \tag{6.6}$$

where $T_1 > T_2$.

Derivation

For a multi-layered pipe, comprising various layers with varying thicknesses and materials, the energy balance needs to be computed sequentially for each section. In steady-state, one-dimensional scenarios, the heat entering section 1 remains unaltered as the same heat transfer rate leaves section 3.

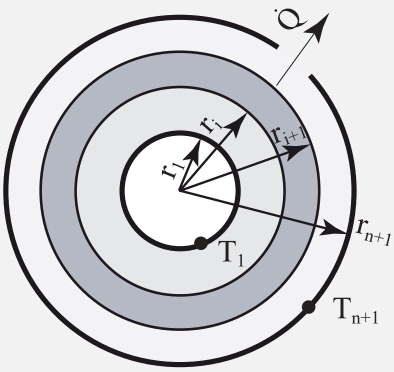


Figure 6.4. Conduction through a multi-layer pipe.

1 Setting up the balance:

$$\dot{Q}_1 = \dot{Q}_2 = \dot{Q}_3 = \dot{Q}.$$

2 Defining the elements within the balance:

Using equation (6.6), the rate of heat transfer through each layer is:

$$\dot{Q}_1 = 2\pi L\lambda_1\frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)}, \qquad \dot{Q}_2 = 2\pi L\lambda_2\frac{T_2 - T_3}{\ln\left(\frac{r_3}{r_2}\right)}, \qquad \text{and} \qquad \dot{Q}_3 = 2\pi L\lambda_3\frac{T_3 - T_4}{\ln\left(\frac{r_4}{r_3}\right)}.$$

3 Inserting and rearranging:

Which gives the following expression for the total rate of heat transfer:

$$\dot{Q} = \frac{2\pi L}{\frac{1}{\lambda_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\lambda_2}\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{\lambda_3}\ln\left(\frac{r_4}{r_3}\right)}\left(T_1 - T_4\right).$$

Rewriting the equation in terms of thermal resistances, as stated in equation (6.3), the above equation becomes:

$$\dot{Q} = \frac{T_1 - T_4}{R_{\text{cond},1} + R_{\text{cond},2} + R_{\text{cond},3}},$$

where:

$$R_{\text{cond.},i} = \frac{1}{2\pi L\lambda_i}\ln\frac{r_{i+1}}{r_i}.$$

Definition

Conductive resistance of a solid cylindrical layer i:

$$R_{\text{cond.},i} = \frac{1}{2\pi L\lambda_i}\ln\frac{r_{i+1}}{r_i}\ \left(\frac{\text{K}}{\text{W}}\right). \tag{6.7}$$

HEATQUIZ 6.1

Temperature profiles:

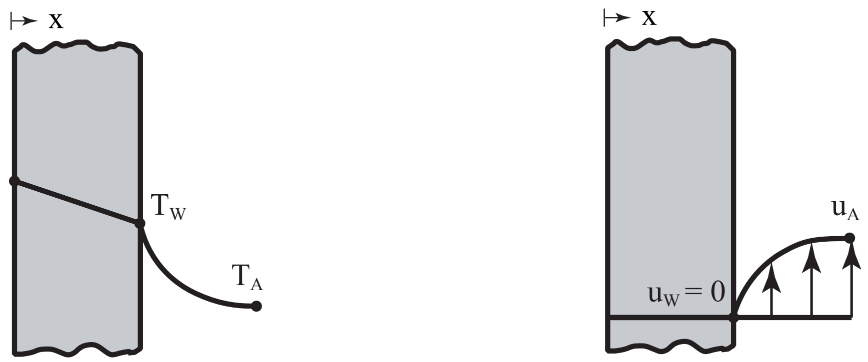
Subsection 6.3
Convective transport

Although this part of the course focuses on heat transfer through conduction, convective heat transfer is introduced in this section briefly since convective transport is often used to define a boundary condition for conduction problems.

Recalling the brief introduction to convection in the introduction, the transport of thermal energy from a body into a moving fluid is not solely taken over by conduction. Fluid movement additionally transports heat in the direction of flow and in this way enhances the overall heat transport process.

To visualize the process of convection, consider a hot plate that is cooled by a flow of cold air. While initially cold molecules approach the wall, they become warmer and in this manner carry the energy away from the hot plate. The mechanism by which fluid particles carry heat and transport that with the main flow is called "advection". An important conceptional understanding involves the flow field in the very near region of the wall. Here, the so-called "no-slip boundary condition" states that the fluid velocity approaches the value of zero, shown in Figure 6.5b. As such, heat transport there can only occur by the process of conduction. With distance from the wall, the fluid velocity increases, increasing the heat transport by advection. Thus, the importance of the advective heat transport increases while the conductive heat transport decreases. This has also an influence on the temperature distribution. Close to the wall, the temperature gradient is high while with increasing distance, the temperature gradient decreases. Later, when learning about thermal boundary layers, the temperature profiles in the fluid will become more obvious.

When examining the interface between a flat plate and the surrounding medium, a kink in the temperature profile becomes apparent, as depicted in Figure 6.5a. This kink occurs because, in general, the thermal conductivity of fluids is considerably lower than that of solids. Moving in the positive x-direction away from the wall, the temperature gradually approaches the ambient temperature. The temperature difference between the fluid near the interface and the fluid further away decreases moving further into the medium. Consequently, a decrease in the temperature gradient is observed until the ambient temperature is reached. At this point, the temperature profile becomes nearly horizontal, indicating that the heat transfer is dominated by advection, and the fluid has attained thermal equilibrium with the surroundings.



(a) Temperature distribution. (b) Velocity distribution.

Figure 6.5. Heat transfer from a hot plate to air by convection.

To describe the heat transfer, Fourier's law can only be used in regions that are solely affected by conduction, e.g. at the wall:

$$\dot{Q} = -A\left(\lambda_{\mathrm{fluid}}\frac{\partial T}{\partial x}\right)_{\mathrm{fluid,wall}} = -A\left(\lambda_{\mathrm{solid}}\frac{\partial T}{\partial x}\right)_{\mathrm{solid,wall}} . \tag{6.8}$$

There exists a direct relation between the temperature gradient in the wall and the gradient within the fluid. This relationship is inverse to the ratio of the respective thermal conductivities. As the thermal conductivity of fluids is in most cases lower compared to the thermal conductivity of solids, the temperature gradient in the fluid is steeper than in the wall.

Instead of equation (6.8), normally an empirical assumption for the convective heat transfer is used, which is known to be Newton's law of cooling.

Fundamental EQ

Newton's law of cooling:

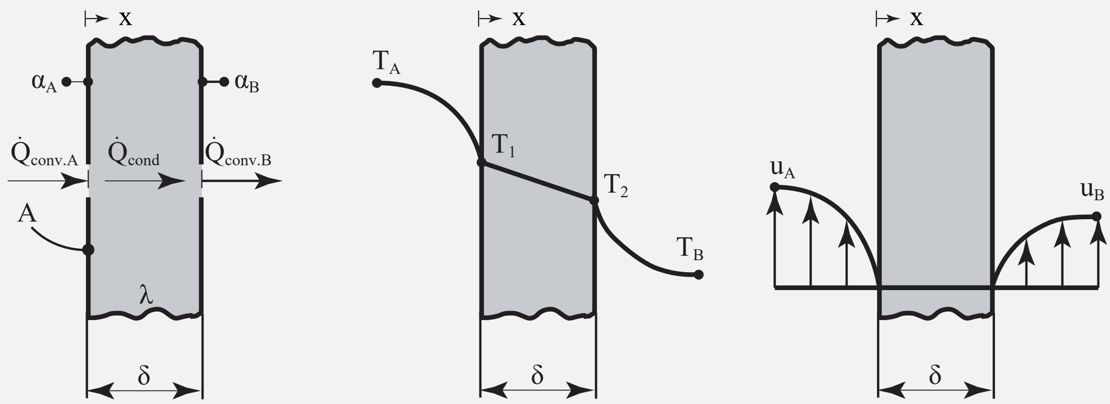
$$\dot{Q} = \alpha A\left(T_{\mathrm{W}} - T_{\mathrm{A}}\right) . \tag{6.9}$$

The heat transfer coefficient α is defined as the rate of heat transfer between a solid surface and a fluid per unit surface area per unit temperature difference $\left(\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}\right)$. The definition is more thoroughly reviewed in Section **??**. Remember, for now, this parameter is influenced by various factors, including fluid properties, velocity, and geometry, among others. This complexity renders the precise determination of heat transfer coefficients challenging.

Subsection 6.4

Multi-layer wall with convection

This section discusses the heat transfer through a wall with additional resistances on the outside, specifying as a boundary condition the temperature away from the wall. To simplify calculations, the thermal resistance theorem is employed. This theorem relies on a few assumptions: the heat transfer rate remains steady, the system maintains a constant state, the materials within each layer distribute heat uniformly, heat transfer is one-dimensional and there are no internal heat sources or drains. By using this method, the overall thermal resistance is determined and enhances the heat transfer passing through the layers.



(a) Energy balance. (b) Temperature distribution. (c) Velocity distribution.

Figure 6.6. Heat transfer through with wall at convective boundary conditions.

Derivation

1 Setting up the balance:

In a steady state, the energy balances read at the interfaces between fluids and solids read:

$$\dot{Q}_{\text{conv,A}} = \dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv,B}} = \dot{Q}.$$

2 Defining the elements within the balance:

Applying Newton's law of cooling, as given in equation (6.9), the heat flux from fluid A to the wall through convection can be deduced. Note, the fluid and wall temperatures are flipped in this context. If not, the result would indicate a negative heat transfer rate. The rate of heat transfer by convection is written as:

$$\dot{Q}_{\text{conv,A}} = A\alpha_{\text{A}}\left(T_{\text{A}} - T_1\right),$$

conduction of heat through the wall:

$$\dot{Q}_{\text{cond}} = \lambda A \frac{T_1 - T_2}{\delta},$$

and from the wall to fluid B by convection:

$$\dot{Q}_{\text{conv,B}} = A\alpha_{\text{B}}\left(T_2 - T_{\text{B}}\right).$$

3 **Inserting and rearranging:**

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_\mathrm{A}} + \frac{\delta}{\lambda} + \frac{1}{\alpha_\mathrm{B}}} \left(T_\mathrm{A} - T_\mathrm{B}\right).$$

Rewriting the equation to the form of the thermal resistance analogy, including the convective resistances, the expression becomes:

$$\dot{Q} = \frac{T_\mathrm{A} - T_\mathrm{B}}{R_\mathrm{conv,A} + R_\mathrm{cond} + R_\mathrm{conv,B}}.$$

Where the 'heat transfer resistance' for convection is introduced:

$$R_{\mathrm{conv},j} = \frac{1}{A\alpha_j}.$$

Similar to the multi-layer wall without convection, the rate of heat transfer for the multi-layer wall with convection consisting of n layers can also be expressed in a general form.

Fundamental EQ

Rate of heat transfer through a solid multi-layer wall with convection:

$$\dot{Q} = \frac{1}{R_{\mathrm{conv},A} + \sum_{i=1}^{n} R_{\mathrm{cond},i} + R_{\mathrm{conv},B}} \left(T_\mathrm{A} - T_\mathrm{B}\right). \tag{6.10}$$

Definition

Convective resistance at a solid plane layer j:

$$R_{\mathrm{conv},j} = \frac{1}{A\alpha_j} \quad \left(\frac{\mathrm{K}}{\mathrm{W}}\right). \tag{6.11}$$

6.4.1 Overall heat transfer coefficient

Introducing the overall heat transfer coefficient k provides a convenient way to work with total thermal resistance, developing an analogy to the convective heat transfer coefficient α. This coefficient encapsulates the combined impact of conduction and convection across different mediums and interfaces within a thermal system.

Derivation

To understand the definition of this coefficient, the energy balance for a plane wall subjected to convection on both sides is derived.

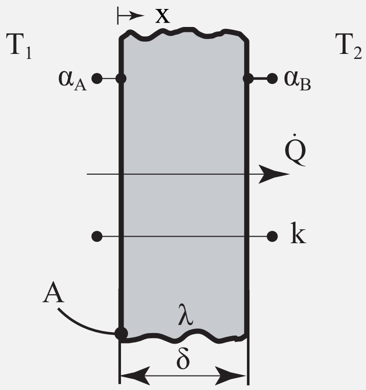


Figure 6.7. Heat transfer through a wall with convective boundary conditions.

1 Setting up the balance:

The energy balance at an interface between the solid and the ambient reads:

$$0 = \dot{Q}_{\mathrm{in}} - \dot{Q}_{\mathrm{out}}.$$

2 Defining the elements within the balance:

The ingoing rate of heat transfer is defined by the overall heat transfer coefficient k, which is analogous to the convective heat transfer coefficient:

$$\dot{Q}_{\mathrm{in}} = kA\,(T_1 - T_2)\,.$$

The outgoing rate of heat transfer by use of the total thermal resistance:

$$\dot{Q}_{\mathrm{out}} = \frac{T_1 - T_2}{R_{\mathrm{conv,A}} + R_{\mathrm{cond}} + R_{\mathrm{conv,B}}},$$

where these resistances are defined as:

$$R_{\mathrm{conv,A}} = \frac{1}{A\alpha_{\mathrm{A}}}, \qquad R_{\mathrm{cond}} = \frac{\delta}{A\lambda}, \qquad \text{and} \qquad R_{\mathrm{conv,B}} = \frac{1}{A\alpha_{\mathrm{B}}}.$$

3 Inserting and rearranging:

Inserting and rewriting yields the definition of the overall heat transfer coefficient:

$$k = \left(\frac{1}{\alpha_{\mathrm{A}}} + \frac{\delta}{\lambda} + \frac{1}{\alpha_{\mathrm{B}}}\right)^{-1}.$$

The rate of heat transfer can be written as the product of the overall heat transfer coefficient, heat transfer area, and respective temperature difference.

Fundamental EQ

Rate of heat transfer through a solid multi-layer wall with convection:

$$\dot{Q} = kA\left(T_{\mathrm{A}} - T_{\mathrm{B}}\right). \tag{6.12}$$

The overall heat transfer coefficient yields from terms, such as the convective heat transfer coefficients, wall thicknesses, and thermal conductivities of these walls.

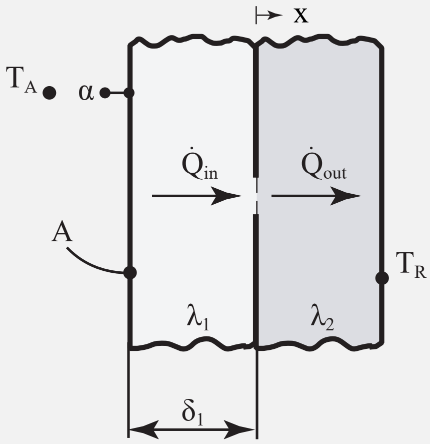
Definition

Overall heat transfer coefficient solid multi-layer plane wall system:

$$k = \left(\frac{1}{\alpha_{\mathrm{A}}} + \sum_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_{\mathrm{B}}}\right)^{-1} \quad \left(\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}\right). \tag{6.13}$$

The overall heat transfer coefficient offers a more convenient approach compared to calculating individual thermal resistances within a system. By encapsulating the combined effects of conduction and convection in a single parameter, the analysis is simplified and a straightforward measure of the overall heat transfer efficiency is provided. This convenience proves especially valuable in complex systems with a substantial number of layers or interfaces, sparing the need for intricate calculations of each individual resistance.

Example 6.1 A wall with surface area A has two solid layers. The left side of the wall is in contact with a hot fluid with temperature T_A, while the right side has a temperature of T_R. The thickness of the left layer is δ_1, and the thermal conductivity is λ_1. The thickness of the right layer is unknown, but the thermal conductivity is λ_2. Find an expression for the thickness δ_2 of the first layer.



1 Setting up the balance:

$$\underbrace{\frac{\partial U}{\partial t}}_{0\text{ - steady-state}} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}.$$

2 Defining the elements within the balance:

Ingoing rate of heat transfer:

$$\dot{Q}_{\text{in}} = kA\left(T_A - T_M\right),$$

where T_M equals the temperature of the wall at the interface between both solid layers.

The overall heat transfer coefficient is defined as:

$$k = \left(\frac{1}{\alpha} + \frac{\delta_1}{\lambda_1}\right)^{-1},$$

and the outgoing rate of heat transfer as:

$$\dot{Q}_{\text{out}} = -\lambda_2 A\frac{\partial T}{\partial x} = \lambda_2 A\frac{T_M - T_R}{\delta_2}.$$

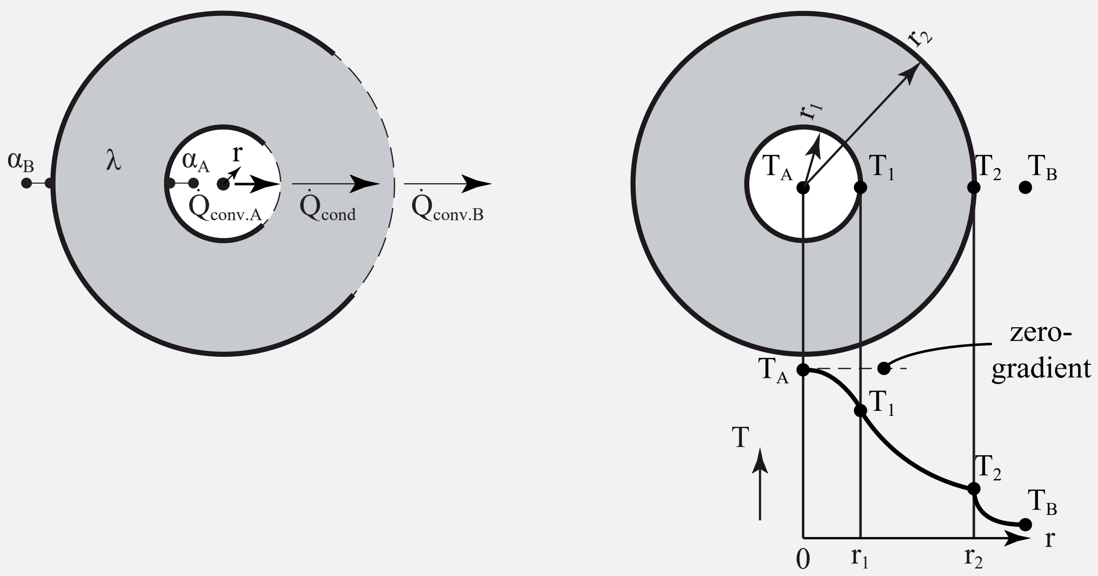
3 Inserting and rearranging:

$$\delta_2 = \frac{\lambda_2}{k}\frac{T_M - T_R}{T_A - T_M}$$

SUBSECTION 6.5

Multi-layer pipe wall with convection

The derived relationships from the previous section discussing a multi-layer wall with convection are used to solve problems where single or multi-layered walls are used. The same holds for pipe walls having multiple layers surrounded by fluids. As a practical example, the calculation of the heat loss of an insulated hot water pipe is used.



(a) Energy balance.   (b) Radial temperature distribution.

Figure 6.8. Heat transfer through a pipe at convective boundary conditions.

Derivation

1 Setting up the balance:

In steady-state, the energy balances at the two fluid/solid interfaces read:

$$\dot{Q}_{\text{conv,A}} = \dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv,B}} = \dot{Q}.$$

2 Defining the elements within the balance:

From Newton's law of cooling, equation (6.9), the heat flux from fluid A to the pipe by convection is found:

$$\dot{Q}_{\text{conv,A}} = A_{\text{A}}\alpha_{\text{A}}\left(T_{\text{A}} - T_1\right) = \left(2\pi r_1 L\right)\alpha_{\text{A}}\left(T_{\text{A}} - T_1\right).$$

Using Fourier's law, the rate of heat conduction is described:

$$\dot{Q}_{\text{cond}} = 2\pi\lambda L \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)},$$

and using Newton's law of cooling, the rate of heat transport from the wall to fluid B by convection is written as:

$$\dot{Q}_{\text{conv,B}} = A_{\text{B}}\alpha_{\text{B}}\left(T_2 - T_{\text{B}}\right) = \left(2\pi r_2 L\right)\alpha_{\text{B}}\left(T_2 - T_{\text{B}}\right).$$

3 **Inserting and rearranging:**

$$\dot{Q} = \frac{2\pi L}{\frac{1}{\alpha_A r_1} + \frac{1}{\lambda}\ln\frac{r_2}{r_1} + \frac{1}{\alpha_B r_2}} (T_A - T_B) \, .$$

Rewriting in analogy to the thermal resistance theorem yields:

$$\dot{Q} = \frac{T_A - T_B}{R_{\mathrm{conv,A}} + R_{\mathrm{cond}} + R_{\mathrm{conv,B}}} .$$

Definition

Convective resistance at a solid cylindrical layer j:

$$R_{\mathrm{conv},j} = \frac{1}{A_j \alpha_j} \left(\frac{\mathrm{K}}{\mathrm{W}}\right), \tag{6.14}$$

with $A_A = 2\pi r_1 L$ for convection on the inside and $A_B = 2\pi r_{n+1} L$ for convection on the outside.

6.5.1 Overall heat transfer coefficent

A general expression including the overall heat transfer coefficient k for a solid multi-layer pipe system is applicable as well to express the rate of heat transfer, as stated before:

$$\dot{Q} = kA\,(T_A - T_B) \, ,$$

where any surface area A may be used. In practice often the outer radius r_{n+1} is used. The overall heat transfer coefficient for cylindrical systems yields from the convective heat transfer coefficients, thermal conductivities of the different layers, and their respective radii.

Definition

Overall heat transfer coefficient solid multi-layer cylindrical wall system:

$$k = \left(\frac{1}{\alpha_A}\frac{r}{r_1} + r\sum_{i=1}^{n}\frac{1}{\lambda_i}\ln\frac{r_{i+1}}{r_i} + \frac{1}{\alpha_B}\frac{r}{r_{n+1}}\right)^{-1} \left(\frac{\mathrm{W}}{\mathrm{m^2K}}\right), \tag{6.15}$$

where any reference radius r may be used.

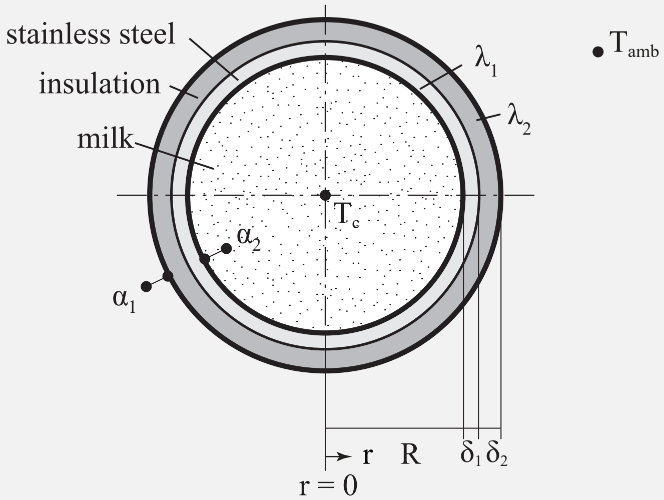
For thin-walled tubes equation (6.13) for flat plates may also be used, for the area $A = 2\pi r_m L$ is substituted, where r_m describes the arithmetic mean radius. Thus, for problems with heat transfer coefficients of equal order of magnitude on both sides of the fluid, a good practice is to calculate the mean value of the inner and outer radius. If the heat transfer coefficients greatly deviate from each other, the radius with the lower heat transfer coefficient should be selected.

HEATQUIZ 6.2

Temperature profiles:

Demonstration 6.1

A dairy farm stores milk in a cylindrical tank. The tank is filled with liters of milk at a temperature T_c. The ambient temperature outside the tank is T_{amb}. The tank walls consist of a layer of stainless steel with insulation wrapped around. The temperature of the milk at the interface is not allowed to have a temperature below 23 °C.



Given parameters:

- Dimensions of the cylinder: $R = 1$ m, $\delta_1 = 5$ cm, $L = 4$ m
- Conductivities of the layers: $\lambda_1 = 16$ W/mK, $\lambda_2 = 0.02$ W/mK
- Heat transfer coefficients: $\alpha_1 = 10$ W/m^{2}K, $\alpha_2 = 5$ W/m^{2}K
- Temperatures: $T_c = 25$ °C, $T_{amb} = 10$ °C

Hints:

- Radiation is to be neglected.
- The problem is steady-state.

Tasks:

a) What is the minimum thickness δ_2 of the insulation layer?

b) Derive a function for the temperature $T(r)$ in the stainless steel layer.

c) Draw qualitatively the temperature profile $T(r)$ for the domain $0 < r < \infty$.

Video solution:

SECTION 7
Fins

L09 - Introduction to the topic of fins:

Learning goals:

- Develop a conceptual understanding of fins and their applications.
- Ability to describe the relevant heat transfer processes in fins.
- Ability to describe and draw the characteristic temperature profiles in fins.
- Ability to develop energy balances for fins.
- Ability to derive and solve the differential equation governing heat transfer in fins.

Comprehension questions:

- What are fins and what are they used for?
- Which heat flows are considered in the derivation of the fin differential equation?
- What is the temperature profile in a fin (from physical consideration)?

L10 - Biot number:

Learning goals:

- Ability to identify and characterize the dominant thermal resistance using relevant dimensionless numbers.
- Ability to determine the dimensionless Biot number with the appropriate length scale and thermal properties.

Comprehension questions:

- What information does the Biot number provide?
- Which assumptions may be made for $\mathrm{Bi} \ll 1$?
- For a classical fin problem, is the Biot number high or low?

L11 - Solution to the fin equation:

Learning goals:

- Ability to homogenize the differential equation for fins.
- Ability to develop the general solution for the differential equation.
- Interpreting the fin parameter m across various fin geometries.
- Ability to recognize and apply different constraints in the context of fin problems.

Comprehension questions:

- Which approach can be used to solve the inhomogeneous differential equation for fins?
- Which parameters affect the fin parameter m?
- Which common boundary conditions can be used to solve the temperature profile in the fin?

L12 - Fin efficiency:

Learning goals:

- Conceptual understanding of the fin efficiency.
- Ability to use the equation for fin efficiency in the design of fins.

Comprehension questions:

- Which relation describes the fin efficiency coefficient?
- What is the assumption for the theoretical maximum transmittable heat of a fin?
- How can the fin efficiency be increased?

Corresponding tutorial exercises:

- Exercise II.7 Pin-fin cooling on gas turbine blades
- Exercise II.8 New fin material
- Exercise II.9 Pipe fastening

Subsection 7.1

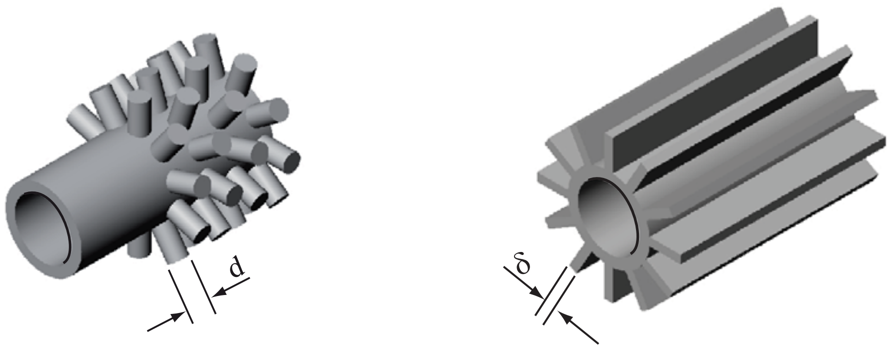
Introduction to fins

Fins are a widely used element in various heat transfer applications, playing a crucial role in improving heat transfer between a solid surface and a fluid medium. Essentially, fins are extended surfaces that are attached to the solid surface, effectively enlarging the available surface area for heat transfer. They find application in a diverse range of systems such as heat exchangers, radiators, air conditioning units, electronic cooling devices, and more.

The underlying principle behind fins is straightforward. When a solid surface comes into contact with a fluid, such as air or water, heat transfer occurs at the interface between the solid and fluid. However, the rate of heat transfer is often limited by the low thermal conductivity of the fluid in the vicinity of the wall, where fluid velocity is typically low. By introducing fins to the surface, the effective surface area for heat transfer is increased, resulting in a significant enhancement of the heat transfer rate.

Various types of fins exist for heat transfer applications, each with an own set of advantages and disadvantages. Common types include rectangular fins, triangular fins, annular fins, and pin fins. The selection of fin type depends on the specific application and design requirements. Fins are typically placed in areas where the highest heat transfer resistances are present on the heat-emitting or heat-absorbing surface, taking forms such as cylindrical rod fins, plane fins, circular fins, etc.

Note, that this discussion will primarily focus on fin geometries that can be approximated as one-dimensional, allowing for analytical approaches. For fins with two-dimensional or three-dimensional configurations, numerical methods are typically employed to describe their behavior accurately.



(a) Rod fins. (b) Plane fins.

Figure 7.1. Tubes with finned surfaces.

Derivation

The concept of fins relies on the concept that heat is conducted from the base to the tip of the fin. As the heat progresses along the length of the fin, convective heat transport causes heat dissipation.

To formulate the steady-state energy conservation equation for fins and establish an expression for the temperature distribution within the fin, as well as the rate of heat transfer, certain assumptions must be made. These assumptions are necessary to simplify the problem into a one-dimensional analysis:

- The temperature shape in a fin is one-dimensional, i.e. the temperature varies only in the direction of the fin length (x-direction in Figure 7.2), but not in the radial direction.
- The heat transfer coefficient α is known and constant over the entire fin surface.

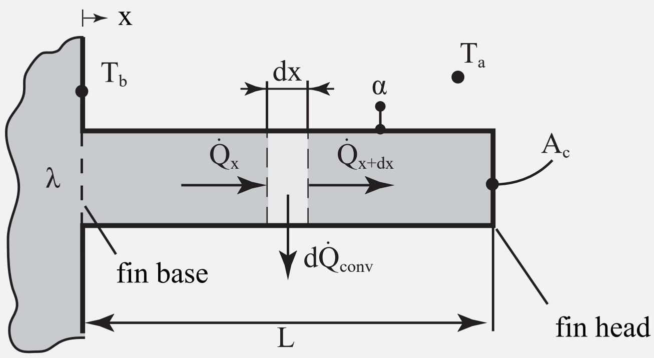


Figure 7.2. Energy balance of a fin.

1 Setting up the balance:

The energy balance applied to the fin element with a cross-sectional area A_c reveals that the disparity between the incoming and outgoing heat flux must be counterbalanced through convection to the surrounding environment:

$$\underbrace{\cancel{\frac{\partial U}{\partial t}}}_{0\text{ - steady-state}} = \dot{Q}_x - \dot{Q}_{x+dx} - d\dot{Q}_\mathrm{conv}.$$

2 Defining the elements within the balance:

Applying Fourier's law, as presented in equation (5.1), in combination with the technique of Taylor series expansion, the conductive heat fluxes are formulated as follows:

$$\dot{Q}_x = -\lambda A_\mathrm{c}\frac{\partial T}{\partial x},$$

and:

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x}dx = -\lambda A_\mathrm{c}\left(\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2}dx\right).$$

Using Newton's law of cooling, as given in equation (6.9), the convection flux from the element with circumference U and length dx to the surrounding is:

$$\begin{aligned} d\dot{Q}_{\text{conv}} &= \alpha \, dA_{\text{s}} \, (T - T_{\text{a}}) \\ &= \alpha U \,(T - T_{\text{a}})\, dx. \end{aligned}$$

3 **Inserting and rearranging:**

$$\lambda A_{\text{c}} \frac{\partial^2 T}{\partial x^2} - \alpha U \,(T - T_{\text{a}}) = 0.$$

The differential equation above is an inhomogeneous equation because of the constant term $\alpha U T_{\text{a}}$. The solution of this differential equation is presented in Section 7.3.

Fundamental EQ

Inhomogeneous fin equation:

$$\lambda A_{\text{c}} \frac{d^2 T}{dx^2} - \alpha U \,(T - T_{\text{a}}) = 0. \tag{7.1}$$

Subsection 7.2
Biot number

The Biot number, denoted as Bi, holds significance in the field of heat transfer quantifying the relative influence of internal thermal resistance compared to external thermal resistance. This dimensionless parameter is defined by the ratio of the internal heat transfer resistance through conduction, denoted as R_λ, to the external heat transfer resistance through convection, denoted as R_α.

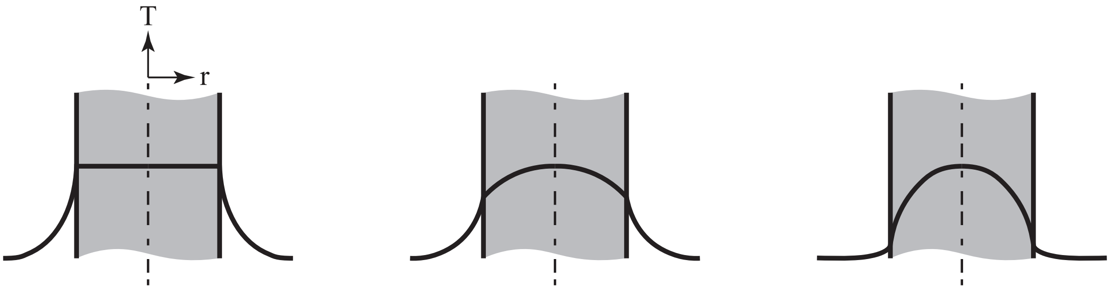
The internal resistance, R_λ, inversely scales with the solid's thermal conductivity λ_s and is directly proportional to the relevant length scale in the direction of heat transfer. Note, this length scale L is geometry-dependent. For instance, in the case of a plate subjected to heat transfer from both sides, the length equals half of the plate's thickness. On the other hand, for a rod or sphere, the characteristic length corresponds to the radius. An alternative definition of the length scale involves the ratio of volume to surface area (V/A), both of which are equally prevalent in textbooks.

The external resistance, R_α, inversely varies with the heat transfer coefficient α. Thus, $R_\alpha \propto 1/\alpha$.

Definition **Biot number:**

$$\mathrm{Bi} = \frac{\text{Conductive thermal resistance in body}}{\text{Convective thermal resistance at surface}} = \frac{\alpha L}{\lambda_s} \quad (-)\,, \tag{7.2}$$

where λ_s is the thermal conductivity of the solid and $L = V/A$.



(a) Bi ≪ 1. (b) Bi ≈ 1. (c) Bi ≫ 1.

Figure 7.3. Temperature profile in a cylinder in the radial direction for different Biot numbers.

In situations where Bi ≪ 1, the heat transfer resistance outside the body significantly surpasses the resistance within the solid body. This circumstance characterizes the "external resistance control" regime. Conversely, when Bi ≫ 1, the resistance within the solid body outweighs the resistance outside, placing the system in the "internal resistance control" domain.

The Biot number offers insight into the interplay between a material's internal resistance to heat transfer and the external resistance at the surface. This parameter helps discern whether conduction or convection predominates in the overall heat transfer process. When the Biot number is notably smaller than 1 (Bi ≪ 1), heat efficiently travels through the material, resulting in minor temperature gradients. Consequently, the temperature gradients within the material become negligible in comparison to the temperature difference outside the material, as depicted in Figure 7.3a. This condition permits simplifications in heat transfer analysis, facilitating the use of one-dimensional approximations even for problems that would otherwise demand two- or three-dimensional considerations.

In most fin-type scenarios, a common simplification is to treat the material as if heat exclusively travels along the length of the fin, which, for rod fins, is perpendicular to the radial direction.

This simplification arises from the considerably smaller temperature variation in the radial direction compared to the heat transfer direction, rendering this heat flow to be negligible. Consequently, the heat transfer problem has an analytical solution.

Example 7.1 Prove that the simplification of the temperature profile of a rod fin being one-dimensional in the direction of the height is justified in the case of $\alpha = 10$ W/m^{2}K, $\lambda = 385$ W/mK, $d = 3$ cm and $H = 10$ cm.

The biot number is determined by:

$$\mathrm{Bi} = \frac{R_\lambda}{R_\alpha} = \frac{\alpha}{\lambda} L.$$

The characteristic length is found from the ratio of the volume and the surface area:

$$L = \frac{V}{A} = \frac{\frac{\pi d^2}{4} H}{\pi d H} = \frac{d}{4} = 0.0075 \text{ m}.$$

Inserting and rearranging:

$$\mathrm{Bi} = \frac{\alpha}{\lambda} L = \frac{10}{385} \cdot 0.0075 = 1.95 \cdot 10^{-4}.$$

Bi $\ll 1$ implies that the change in temperature in the radial direction is negligible. Therefore the fin is considered to be a one-dimensional system with only a change in temperature in the height direction.

SUBSECTION 7.3
Solution to the fin equation

The fin equation (7.1), is an inhomogeneous 2nd order differential equation. An inhomogeneous differential equation is a type of differential equation where the right-hand side is not equal to zero, thus without a forcing term or source. In other words, the equation describes a relationship between a function and its derivatives, along with an additional non-zero function or constant on the right side that affects the behavior of the solution. This term can represent external influences, boundary conditions, or initial conditions that impact the solution to the differential equation.

One approach to solving the equation is to transform the inhomogeneous equation into a homogeneous equation by introducing a temperature difference θ. This temperature difference includes the non-zero term.

Derivation

Recall the inhomogeneous fin equation:

$$\lambda A_{\mathrm{c}} \frac{\partial^2 T}{\partial x^2} - \alpha U \left(T - T_{\mathrm{a}}\right) = 0.$$

Introducing the temperature difference

$$\theta = T - T_{\mathrm{a}},$$

yields that the second derivative with respect to x is written as:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}.$$

Note, the non-zero but constant forcing term drops out when differentiating the equation. Rewriting the inhomogeneous fin equation with θ yields the homogeneous fin equation:

$$\frac{\partial^2 \theta}{\partial x^2} - \underbrace{\frac{\alpha U}{\lambda A_{\mathrm{c}}}}_{\mathrm{m}^2}\theta = 0,$$

where m represents the fin parameter $\mathrm{m} = \left(\frac{\alpha U}{\lambda A_{\mathrm{c}}}\right)^{\frac{1}{2}}$ with unit $\left(\frac{1}{\mathrm{m}}\right)$.

By use of an educational/sophisticated guess of the solution in the form of $\theta(x) = e^{sx}$ the standard solution is found:

$$\theta = A\, e^{\mathrm{m}x} + B\, e^{-\mathrm{m}x} = A^*\, \sinh(\mathrm{m}x) + B^* \cosh(\mathrm{m}x).$$

Exercise: Prove that this standard solution is true

Definition

Fin parameter:

$$\mathrm{m} = \sqrt{\frac{\alpha U}{\lambda A_{\mathrm{c}}}} \left(\frac{1}{\mathrm{m}}\right), \tag{7.3}$$

where for a rod fin $\mathrm{m} = \sqrt{\frac{4\alpha}{\lambda d}}$, and for a plane fin $\mathrm{m} = \sqrt{\frac{2\alpha}{\lambda \delta}}$.

Fundamental EQ

Homogeneous fin equation:

$$\frac{d^2\theta}{dx^2} - \mathrm{m}^2\theta = 0, \tag{7.4}$$

where $\theta = T(x) - T_\mathrm{a}$.

Fundamental EQ

General solution of the fin equation:

$$\theta = A\, e^{\mathrm{mx}} + B\, e^{-\mathrm{mx}} = A^*\ \sinh(\mathrm{mx}) + B^* \cosh(\mathrm{mx}). \tag{7.5}$$

Derivation

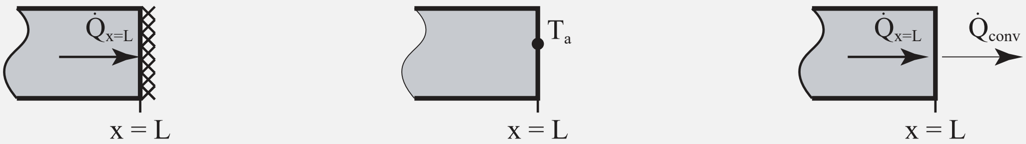
To obtain a solution for the temperature profile of the fin, the integration constants need to be determined. The fin equation has been differentiated twice with respect to x. Thus two boundary conditions are required. Usually, boundary conditions are defined at the base and tip of the fin.

Boundary conditions at the base of the fin:

Generally, the temperature at the base of the fin is known $T(x = 0) = T_\mathrm{b}$, and this boundary condition is written in the dimensionless form as:

$$\theta = \theta_\mathrm{b}.$$

Boundary conditions at the tip of the fin:



(a) Fin with an adiabatic tip. (b) Tip temperature equal to ambient. (c) Convective heat transfer at the tip.

Figure 7.4. Common boundary conditions at a fin tip.

Some possible boundary conditions at the tip of the fins are:

a) **Adiabatic tip:** For sufficiently long fins heat flow out of the top of the fin is negligible in comparison to the total dissipated heat, hence the tip of the fin is regarded to be adiabatic. From a local energy balance at the fin tip $-\lambda A_\mathrm{c} \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0$, which yields the dimensionless form:

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0.$$

b) **Tip temperature equal to the ambient:** An alternative scenario arises when the fin's length is significantly long enough that the temperature at the tip closely aligns with the ambient temperature, denoted as $T(x = L) = T_\mathrm{a}$. This situation leads to the following conditions:

$$\theta\,(x = L) = 0.$$

c) **Convective heat transfer at the tip:** Lastly, in some scenario's the fin has a finite height and a substantial amount of heat is lost from the tip as well. A local energy balance: $-\lambda A_\mathrm{c} \left.\frac{\partial T}{\partial x}\right|_{x=L} = \alpha A_\mathrm{c}\,(T(x = L) - T_\mathrm{a})$, gives that:

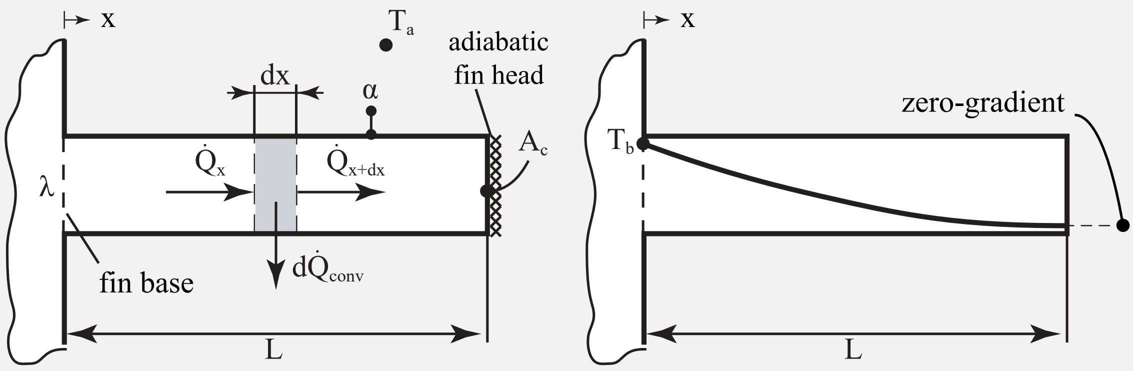
$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = -\frac{\alpha}{\lambda}\theta\,(x = L)\,.$$

HEATQUIZ 7.1

Temperature profiles: **Energy balances:**

Example 7.2

Derive the rate of heat transfer from a fin with base temperature T_b and an adiabatic head.



(a) Energy balance. (b) Temperature distribution.

1 Setting up the balance:

The steady-state energy balance for an infinitesimal element within the fin reads:

$$0 = \dot{Q}_x - \dot{Q}_{x+dx} - d\dot{Q}_{\text{conv}}.$$

2 Defining the elements within the balance:

By employing Fourier's law, as expressed in equation (5.1), in conjunction with the Taylor series expansion, the conductive heat fluxes are written as:

$$\dot{Q}_x = -\lambda A_c \frac{\partial T}{\partial x},$$

and:

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} dx = -\lambda A_c \left(\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2} dx \right).$$

Using Newton's law of cooling, as provided in equation (6.9), the convection flux from the element with circumference U and length dx to the surroundings is established as follows:

$$d\dot{Q}_{\text{conv}} = \alpha U \left(T - T_a \right) dx.$$

3 Inserting and rearranging:

$$\frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c}(T - T_a) = 0.$$

Introducing the temperature difference and fin parameter to solve the fin equation:

$$\theta = T - T_a, \qquad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}, \qquad \text{and} \qquad \mathrm{m}\sqrt{\frac{\alpha U}{\lambda A_c}}.$$

Inserting yields:

$$\frac{\partial^2 \theta}{\partial x^2} - \mathrm{m}^2\theta = 0,$$

with the standard solution:

$$\theta = A^* \sinh(\mathrm{mx}) + B^* \cosh(\mathrm{mx}).$$

4 Defining the boundary and/or initial conditions:

Since the differential equation has undergone two differentiations with respect to x, the need for two boundary conditions arises.

At $x = 0$ the temperature is known $T(x = 0) = T_b$:

$$\theta(x = 0) = T_b - T_a = \theta_b.$$

At $x = L$ there is no heat transfer $\left(\dot{Q} = -\lambda A_c \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0\right)$, which gives:

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0.$$

5 Solving the equation:

Using the boundary conditions the integration constants are determined, which gives $A^* = -\theta_b \frac{\sinh(\mathrm{mL})}{\cosh(\mathrm{mL})} = -\theta_b \tanh(\mathrm{mL})$ and $B^* = \theta_b$.

This results in the following dimensionless temperature function:

$$\begin{aligned}\theta &= \theta_b\left[\cosh(\mathrm{mx}) - \tanh(\mathrm{mL}) \sinh(\mathrm{mx})\right] \\ &= \theta_b \frac{\cosh\left[\mathrm{m}\left(L - \mathrm{x}\right)\right]}{\cosh\left(\mathrm{mL}\right)}.\end{aligned}$$

In Figure 7.6 the dimensionless temperature shape of a rod fin with a circular cross-section, for different fin materials with $\mathrm{m} = \left(\frac{4\alpha}{\lambda d}\right)^{\frac{1}{2}}$ is shown.

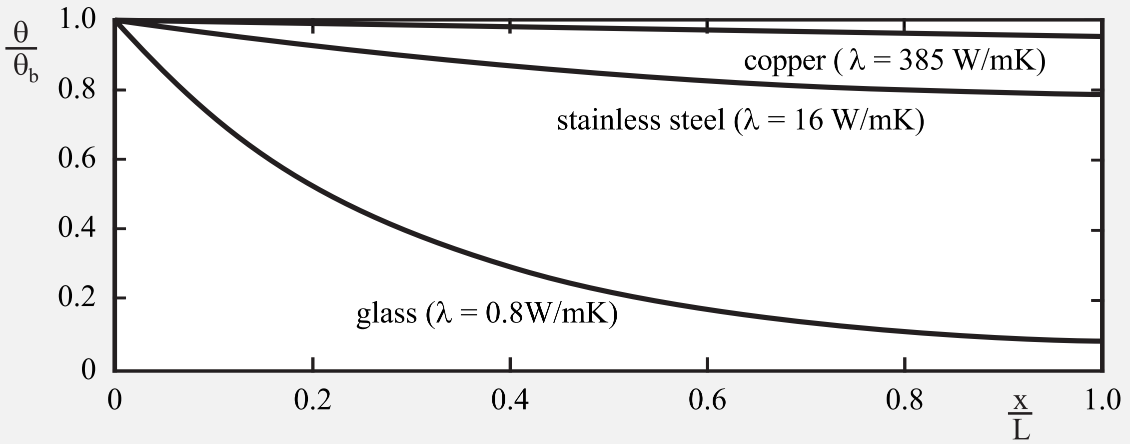
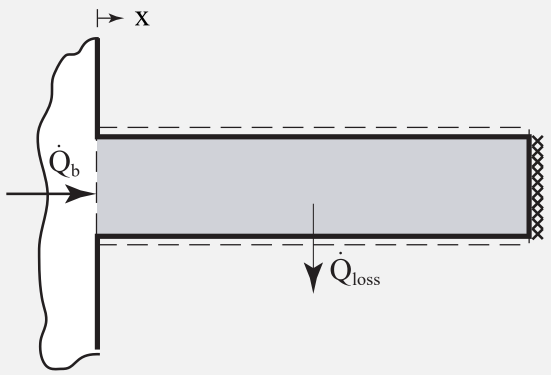


Figure 7.6. Change of temperature over the length of a rod fin with d = 8 mm, L = 40 mm, and α = 10 $\frac{\mathrm{W}}{\mathrm{m^2K}}$.



The temperature profile provides insight into calculating the overall heat transfer from the fin to the surroundings. A global energy balance along the fin demonstrates that, under steady-state conditions, the heat entering the fin's base through conduction equals the heat being transferred to the surroundings via convection.

1 Setting up the balance:

$$\overbrace{\cancel{\frac{\partial U}{\partial t}}}^{0\text{ - steady-state}} = \dot{Q}_{\mathrm{b}} - \mathrm{Q}_{\mathrm{loss}}.$$

2 Defining the elements within the balance:

Defining the conductive flux by use of Fourier's law, as stated in equation (5.1), and rewriting yields:

$$\dot{Q}_{\mathrm{loss}} = \dot{Q}_{\mathrm{b}} = -\lambda A_{\mathrm{c}} \left.\frac{\partial \theta}{\partial x}\right|_{\mathrm{x=0}}.$$

3 Inserting and rearranging:

Substitution of the derivative temperature function θ with respect to x yields the rate of heat loss:

$$\dot{Q}_{\mathrm{loss}} = \lambda A_{\mathrm{c}} \mathrm{m} \theta_{\mathrm{b}} \tanh(\mathrm{mL}) .$$

Fundamental EQ

Temperature profile of a fin with base temperature T_{b} and an adiabatic tip:

$$\theta = \theta_{\mathrm{b}} \frac{\cosh\left[\mathrm{m}\left(L - \mathrm{x}\right)\right]}{\cosh(\mathrm{mL})} . \tag{7.6}$$

Fundamental EQ

Rate of heat transfer for a fin with base temperature T_{b} and an adiabatic tip:

$$\dot{Q} = \lambda A_{\mathrm{c}} \mathrm{m} \theta_{\mathrm{b}} \tanh(\mathrm{mL}) . \tag{7.7}$$

Subsection 7.4
Fin efficiency

Fin efficiency is used for analyzing heat transfer systems with fins. The value is defined as the ratio of actual heat transferred by the fin to the maximum possible if the entire fin surface matches the base temperature. This efficiency indicates how efficiently the material of the fin is used.

Definition

Fin efficiency:

$$\eta_\mathrm{F} = \frac{\text{transferred heat}}{\text{maximum transferable heat}} = \frac{\dot{Q}}{\dot{Q}_\mathrm{max}} \quad (-)\,. \tag{7.8}$$

Example 7.3

Determine the fin efficiency for a fin with base temperature T_b and an adiabatic tip.

To determine the efficiency of the fin, equation (7.8) is used:

$$\eta_\mathrm{F} = \frac{\dot{Q}}{\dot{Q}_\mathrm{max}}.$$

The rate of heat transfer from the fin has been determined in example 7.2, and is listed in equation (7.7):

$$\dot{Q} = \lambda A_\mathrm{b} \mathrm{m} \theta_\mathrm{b} \tanh\left(\mathrm{mL}\right).$$

The maximum transferable heat is the rate of heat transfer if the entire surface of the fin is at the same temperature as the base:

$$\dot{Q}_\mathrm{max} = \alpha\, U\, L\, \left(T_\mathrm{b} - T_\mathrm{a}\right) = \alpha\, U\, L\, \theta_\mathrm{b}.$$

Inserting and rearranging:

$$\eta_\mathrm{F} = \frac{\tanh(mL)}{mL}.$$

The function of the fin efficiency is shown in Figure 7.7. Obviously, the efficiency cannot be optimized solely over the length of the fin, since factors such as the mass of the fin or volume require also consideration.

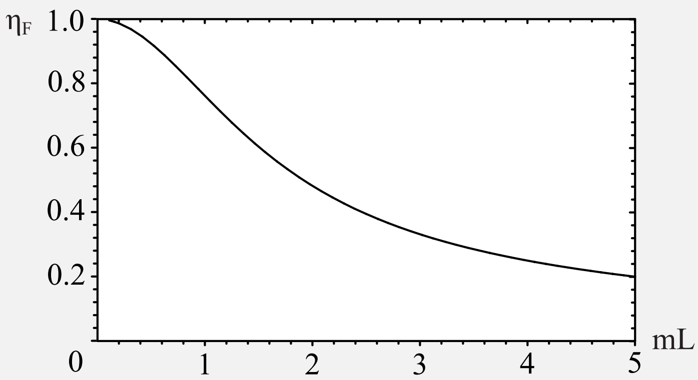


Figure 7.7. Efficiency coefficient of a rod fin.

Section 8

Heat sources and sinks

L13 - Heat sources and sinks:

Learning goals:

- Proficient understanding of the influence of heat sources/sinks on the temperature profile.
- Ability to incorporate heat sources/sinks in the energy balance and deduce the differential equation.
- Ability to reformulate and solve the differential equation
- Ability to calculate the maximum and minimum temperatures within a body and the corresponding heat fluxes.

Comprehension questions:

- What is the resulting temperature profile for cylindrical bodies with a heat source?
- What are the possible boundary conditions that can exist on the surface of a cylinder?
- How is the generated heat distributed over the cylinder's surface?
- How can the minimum and maximum temperatures within a body be determined?

Corresponding tutorial exercises:

- Exercise II.10 Resistance wire
- Exercise II.11 Multi-layer walls with source
- Exercise II.12 Copper rod

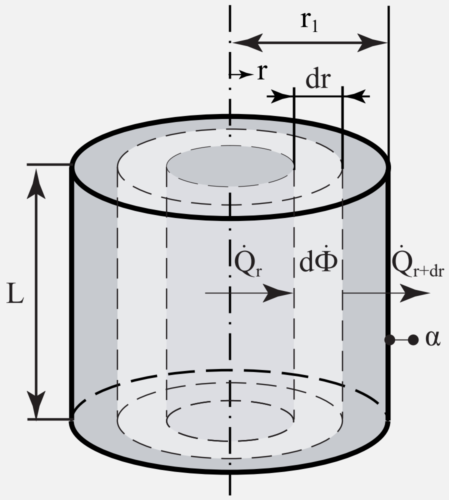
SUBSECTION 8.1

Introduction to heat sources and sinks

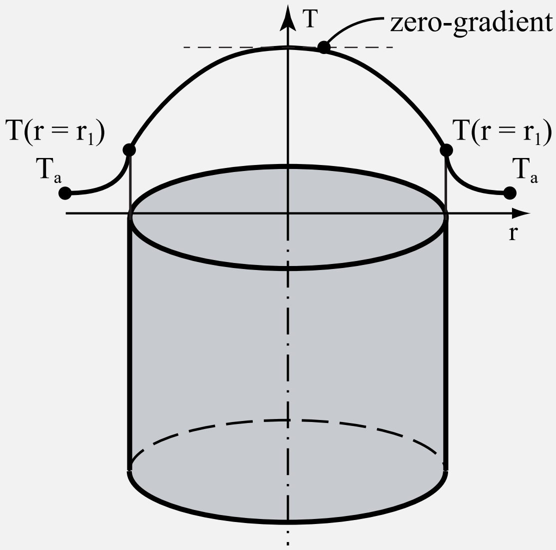
Numerous heat transfer scenarios encompass the presence of internally generated heat. This phenomenon may arise from factors like chemical reactions (as in combustion), electric Joule heating, viscous dissipation, or the absorption of radiation. The upcoming section focuses on a cylindrical object, where there is a consistent heat generation rate per unit volume denoted as $\dot{\Phi}'''$ with unit $\left(\frac{\mathrm{W}}{\mathrm{m}^3}\right)$. Subsequently, the analysis is extended to encompass plates and spheres.

Derivation

As an example, a cylindrical rod subjected to a homogeneous heat generation is considered. In the following steps, the temperature distribution will be determined.



(a) Energy balance.



(b) Radial temperature distribution.

Figure 8.1. Heat conduction in a cylinder with heat sources.

1 **Setting up the balance:**

Energy balance at the infinitesimal element between r and $r+dr$ with a volume of $dV = 2\pi r dr L$ yields:

$$0 = \dot{Q}_{\mathrm{r}} - \dot{Q}_{\mathrm{r+dr}} + d\dot{\Phi},$$

where $\dot{Q}_{\mathrm{r}}$ and $\dot{Q}_{\mathrm{r+dr}}$ symbolize the conductive fluxes entering and exiting the infinitesimal element, while $d\dot{\Phi}$ denotes the heat generated within this infinitesimal component.

2 **Defining the elements within the balance:**

Ingoing:

$$\dot{Q}_{\mathrm{r}} = -\lambda 2\pi r L \frac{\partial T}{\partial r},$$

and outgoing conductive flux:

$$\dot{Q}_{\mathrm{r+dr}} = -\lambda 2\pi r L \frac{\partial T}{dr} + \frac{\partial}{\partial r}\left(-\lambda 2\pi r L \frac{\partial T}{dr}\right) \cdot dr.$$

The heat generated in the infinitesimal element:

$$\begin{aligned} d\dot{\Phi} &= \dot{\Phi}''' dV \\ &= \dot{\Phi}''' 2\pi r dr L. \end{aligned}$$

3 **Inserting and rearranging:**

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{\Phi}'''}{\lambda} = 0.$$

4 **Defining the boundary and/or initial conditions:**

Since the equation has been differentiated twice with respect to r, two boundary conditions are necessary for the solution.

At the center, symmetrical conditions yield a horizontal slope in the temperature profile at $r = 0$:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0.$$

At $r = r_1$ a local energy balance $\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}}$:

$$\dot{Q}_{\text{cond}} = 2\pi r_1 L \alpha \left(T(r = r_1) - T_{\text{a}}\right),$$

where $T(r = r_1)$ is yet an unknown temperature.

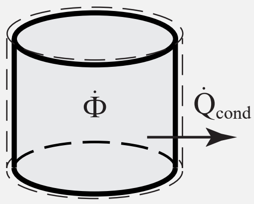


Figure 8.2. Global energy balance around the cylinder.

Since the dissipated heat by conduction for steady-state conditions must be produced by the heat sources within the cylinder, a global energy balance gives:

$$0 = \dot{\Phi} - \dot{Q}_{\text{cond}}.$$

Defining the fluxes and rewriting yields:

$$\dot{Q}_{\text{cond}} = \pi r_1^2 L \dot{\Phi}'''.$$

Rewriting the boundary conditions obtained from the local energy balance at $r = r_1$ results in:

$$T(r = r_1) = T_{\text{a}} + \frac{r_1 \dot{\Phi}'''}{2\alpha}.$$

5 **Solving the equation:**

By integrating twice with respect to r:

$$T(r) = -\frac{\dot{\Phi}'''}{4\lambda} r^2 + A \ln(r) + B.$$

Employing the boundary conditions yields that $A = 0$ and $B = T_\mathrm{a} + \frac{\dot{\Phi}''' r_1}{2\alpha} + \frac{\dot{\Phi}''' r_1^2}{4\lambda}$. Substitution results:

$$T = T_\mathrm{a} + \frac{\dot{\Phi}''' r_1^2}{4\lambda} \left(1 + \frac{2\lambda}{\alpha r_1} - \left(\frac{r}{r_1} \right)^2 \right).$$

The temperature profile for steady-state 1-D heat conduction in bodies with source is given in a general form valid both for the plane and spherical geometries.

Fundamental EQ

General temperature profile steady-state 1-D heat conduction in bodies with source:

$$T = T_\mathrm{a} + \frac{\dot{\Phi}''' s^2}{2(n+1)\lambda} \left(1 + \frac{2\lambda}{\alpha s} - \left(\frac{\xi}{s} \right)^2 \right), \tag{8.1}$$

where the parameter ξ, the characteristic length s and the control parameter n are to be included using the following table:

	Plate*)	Cylinder	Sphere
ξ	x	r	r
s	δ	r_1	r_1
n	0	1	2

*) For plates x should be with reference to the plane of symmetry and δ is half the plate thickness.

HEATQUIZ 8.1

Temperature profiles: **Energy balances:**

Example 8.1 Determine the maximum and surface temperature in the cylindrical fuel rod.

Using the derived temperature profile:

$$T = T_{\mathrm{a}} + \frac{\dot{\Phi}''' r_1^2}{4\lambda}\left(1 + \frac{2\lambda}{\alpha r_1} - \left(\frac{r}{r_1}\right)^2\right).$$

From the derivative, the maxima and minima result from:

$$\frac{\partial T}{\partial x} = -\frac{\dot{\Phi}''' r_1}{8\lambda} r = 0$$

$$\rightarrow r = 0.$$

Which yields the maximum temperature in the cylindrical body at $r = 0$:

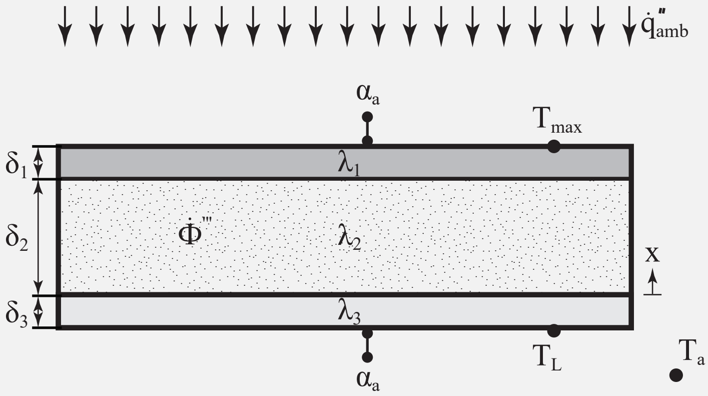
$$T_{\max} = T_{\mathrm{a}} + \frac{\dot{\Phi}''' r_1^2}{4\lambda}\left(1 + \frac{2\lambda}{\alpha r_1}\right).$$

The surface temperature at $r = r_1$ is:

$$T = T_{\mathrm{a}} + \frac{\dot{\Phi}''' r_1}{2\alpha}.$$

Demonstration 8.1

A solid layer is infinitely large and reactive with a homogeneous, constant source strength $\dot{\Phi}''' > 0$. The layer is located between two parallel plates. The upper plate is exposed to radiation from the surroundings with a constant radiative heat flux of $\dot{q}''_{\text{amb}}$. The maximum temperature is located at the outer face of the upper plate T_{max}. The surrounding temperature is T_{a}. The temperature at the outer face of the lower plane reaches T_{L}.



Given parameters:

- Thickness of the plates: δ_1, δ_2, δ_3
- Conductivity of the plates: $\lambda_1 > \lambda_2 > \lambda_3$
- Convective heat transfer coefficient: α_{a}
- Source strength of the reactive layer: $\dot{\Phi}'''$
- Radiative heat flux from the surrounding: $\dot{q}''_{\text{amb}}$
- Maximum temperature: T_{max}
- Temperature of the surrounding: T_{a}
- Temperature on the lower surface: $T_{\text{L}} > T_{\text{a}}$

Tasks:

a) Derive an analytical expression of the reactive layer's temperature $T(x)$.

b) Draw the qualitative temperature profile $T(x)$ in the plates, the layer, and the surrounding.

Video solution:

Section 9

Unsteady heat conduction

L14 - Introduction to unsteady conduction:

Learning goals:

- Conceptual understanding of unsteady heat conduction.
- Ability to non-dimensionalize unsteady heat transfer problems.
- Ability to explore the significance of dimensionless numbers.

Comprehension questions:

- Under which condition is the temperature within a body to be assumed as homogeneous? Which dimensionless number is used for this purpose?
- What describes the Fourier number?

L15 - Example fever:

Learning goals:

- Develop a practical problem-solving approach.

Comprehension questions:

- For safety reasons, mercury thermometers are no longer offered in the markets. Thermometers filled with alcohol are also hardly used anymore. Why? What are the disadvantages of these measuring instruments?
- The standard devices currently in use are digital thermometers. How is the internal body temperature determined?

L16 - Semi-infinite plate:

Learning goals:

- Conceptual understanding of the boundary conditions in a semi-infinite body with an affected wall temperature.
- Ability to solve problems using the error function table.

Comprehension questions:

- What is the concept behind a semi-infinite body and what are the limitations?
- Which two dimensionless numbers describe the unsteady temperature course within a (semi-infinite) body with relevant convective resistance?
- What is thermal penetration depth?

L17 - Dimensionless numbers and Heisler diagrams:

Learning goals:

- Knowing the significance of dimensionless numbers, particularly Fourier and Biot numbers, in analyzing unsteady heat transport.
- Conceptual understanding of Heisler diagrams for evaluating core body temperature, local temperature profile, and heat flow.
- Ability to apply Heisler diagrams effectively to real-world scenarios.

Comprehension questions:

- Which two dimensionless numbers describe a body's unsteady heat transfer problem with additional external thermal resistance?
- What tool allows the determination of the temperature profile or the amount of heat transferred for extended plates, long cylinders, or spheres?

Corresponding tutorial exercises:

- Exercise II.13 Cooling of a copper rod
- Exercise II.14 The temperature delay
- Exercise II.15 Tile setting
- Exercise II.16 Heating and quenching of a sphere

SUBSECTION 9.1

Introduction to unsteady conduction

Previous chapters have dealt with conduction processes, for which a steady state has been reached. The obtained temperature field was not a function of time.

In this section, the changing states of heating and cooling until the system reaches its equilibrium temperature are explored. These processes are described by equations (5.2), (5.3), or (5.4), and they adhere to the initial and boundary conditions specific to the given problem.

Since a general solution to these equations is not possible, only some typical examples that are often used in practice can be described. This section focuses on three types of examples that are used to describe one-dimensional transient conduction.

Figure 9.1 illustrates the classification of one-dimensional transient conduction. The primary classification and the method of solution depend on whether the temperature within a body is homogeneous. When homogeneous, the lumped capacity model discussed in Section 9.2 can be applied to solve the problem. If not, the choice depends on whether heat fully penetrates through the entire body. In such cases, the semi-infinite body approach or Heisler diagrams, as discussed in Sections 9.3 and 9.4 respectively, are employed to address these problems.

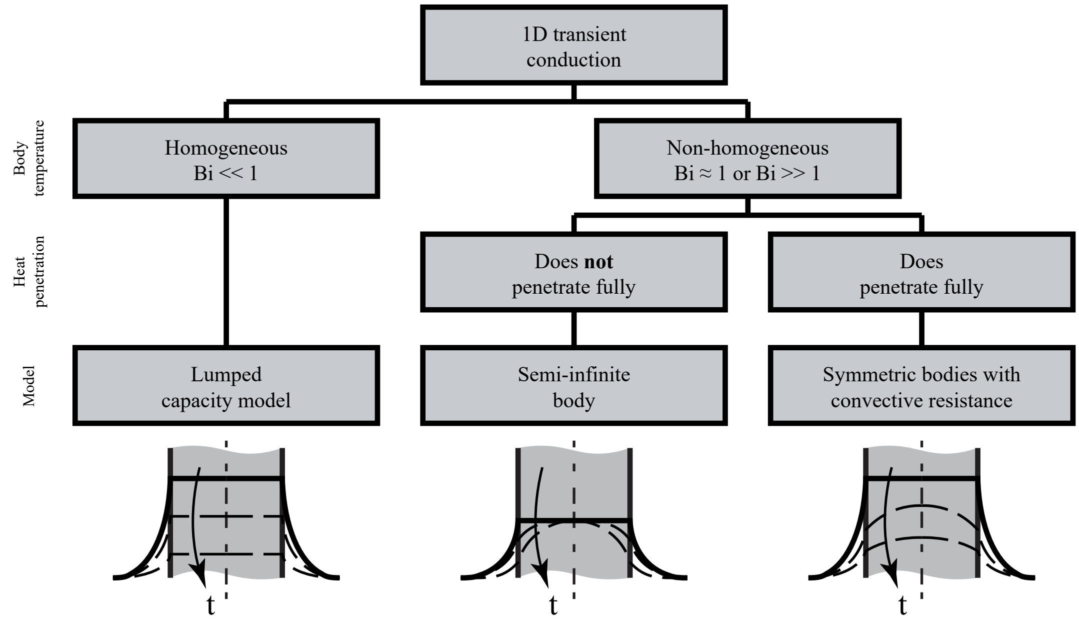


Figure 9.1. Classification one-dimensional transient heat conduction.

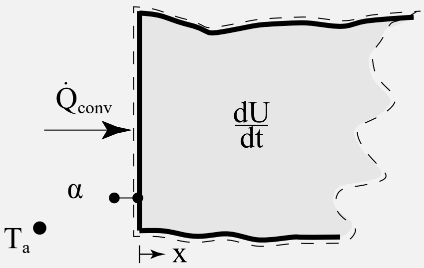
Subsection 9.2
Lumped capacity model

If a body has a high thermal conductivity so that the thermal resistance of the body is small compared to the heat transfer resistance between the body and the surrounding fluid, a nearly homogeneous temperature is established at each point in time during heating or cooling. This is described by $\mathrm{Bi} \ll 1$.

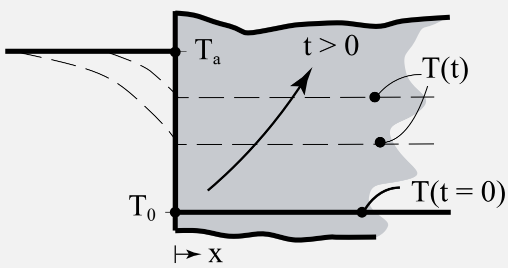
Therefore, there is no need to apply the differential equation for the temperature field as in equations (5.2) - (5.4). A simpler approach involves formulating the energy balance for the entire body. This balance asserts that the internal energy of the body, a function of time, alters as a result of convection-based heat transfer from its surface to the surroundings.

Lumped body analysis is a simplification technique used in heat transfer studies, particularly when the temperature variation within a solid object is negligible. This approach treats the entire object as a single entity with uniform temperature, being especially useful for systems with rapid heat transfer or high thermal conductivity. By focusing on the overall energy balance rather than intricate temperature distribution, lumped body analysis provides insights into transient processes like heating or cooling. This model involves a first-order ordinary differential equation that relates the body's internal energy change to heat transfer, facilitating practical engineering decisions in scenarios where detailed temperature gradients are less significant.

Derivation In this derivation, consider a body with a uniform temperature that undergoes heating through surrounding convection—a scenario applicable to cases where the Biot number is much less than 1. At the outset, the body is initially at a temperature of T_0.



(a) Energy balance.



(b) Temperature distribution.

Figure 9.2. Heating of a lumped body.

1 Setting up the balance:

$$\frac{\partial U}{\partial t} = \dot{Q}_{\text{conv}}.$$

2 Defining the elements within the balance:

Temporal change of internal energy:

$$\frac{\partial U}{\partial t} = \rho c V \frac{\partial T}{\partial t}.$$

Rate of heat transfer by convection:

$$\dot{Q}_{\text{conv}} = \alpha A \left(T_{\text{a}} - T\right).$$

3 Inserting and rearranging:

$$\rho c V \frac{\partial T}{\partial t} + \alpha A \left(T - T_a\right) = 0.$$

4 Defining the boundary and/or initial conditions:

To solve the differential equation, one initial condition is required:

$$T(t = 0) = T_0.$$

5 Solving the equation:

By defining the dimensionless temperature as $\theta^* = \frac{(T - T_0)}{(T_a - T_0)}$, where T represents the time-dependent body temperature, the substitution results in:

$$\frac{\partial \theta^*}{\partial t} + \frac{\alpha}{\rho c}\frac{A}{V}\left(\theta^* - 1\right) = 0.$$

Further reformulation of the equation gives:

$$\frac{\partial \theta^*}{\left(\theta^* - 1\right)} = -\frac{\alpha}{\rho c}\frac{A}{V}\partial t.$$

Integrating the differential equation and using the initial condition $T(t = 0) = T_0$, i.e. $\theta^* = 0$:

$$\frac{T - T_0}{T_a - T_0} = 1 - \exp\left(-\frac{\alpha}{\rho c}\frac{A}{V}t\right).$$

Introducing two dimensionless numbers, namely, the Biot number:

$$\mathrm{Bi} = \frac{\alpha L}{\lambda},$$

and the Fourier number,

$$\mathrm{Fo} = \frac{\lambda t}{\rho c L^2} = \frac{a t}{L^2},$$

where L is the characteristic length, resulting from the ratio $\frac{V}{A}$.

The expression can be stated in the following form:

$$\frac{T - T_0}{T_a - T_0} = 1 - \exp\left(-\mathrm{Bi} \cdot \mathrm{Fo}\right).$$

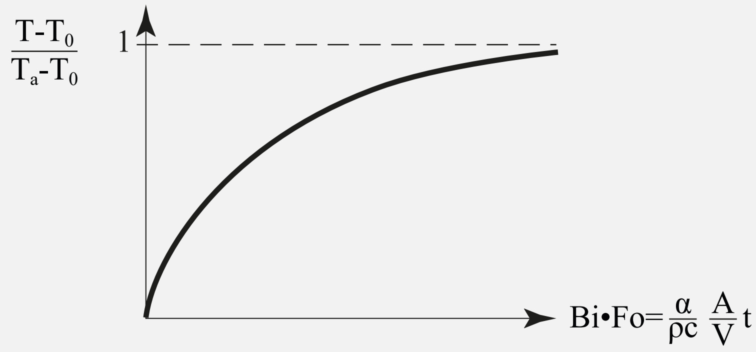


Figure 9.3. The temperature of the body as a function of the time.

Fundamental EQ

Lumped capacity model:

$$\theta^* = 1 - \exp\left(-\frac{\alpha}{\rho c}\frac{A}{V}t\right), \tag{9.1}$$

which is valid for $\mathrm{Bi} \ll 1$, and where $\theta^* = \frac{T - T_0}{T_a - T_0}$.

In other words, the temporal evolution of the dimensionless body temperature written in the form of equation (9.6) is general and the same for all problems, for which equal Biot numbers describe the boundary conditions. Moreover, this is valid for all problems where the Biot number is small, $\mathrm{Bi} \ll 1$.

Example: Fever

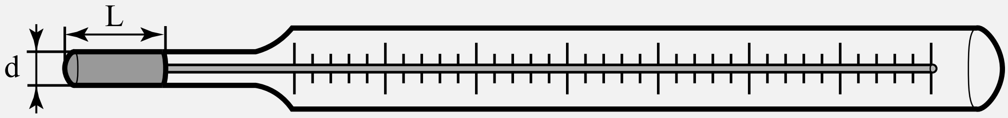
Phenomena 9.1

Imagine you are feeling under the weather and you would like to monitor your body temperature. In such cases, a clinical thermometer might be the method of choice. Traditionally, these thermometers were constructed using a mercury-filled capillary as the temperature indicator. When you use the thermometer, you will notice the mercury volume rises with an increase in temperature.

The functioning of a mercury thermometer is based on the principle of linear expansion as the temperature increases. As heat is transferred from your body to the thermometer, the mercury inside expands. This expansion is directly proportional to the temperature rise, allowing for an accurate measurement of your body's heat.

The design of a mercury thermometer incorporates various heat transfer mechanisms, including convection and conduction. As the thermometer comes into contact with your body, heat is absorbed and the mercury expands accordingly.

Example 9.1 A child is not feeling well, prompting his mother to investigate the possibility of fever. To assess his condition, she chooses to measure his body temperature in two distinct instances using a mercury thermometer.



The mercury thermometers employed for the measurements possess an initial uniform temperature of $T_0 = 20$ °C, a length $L = 30$ mm, and a diameter $d = 4$ mm.

Mercury, the medium within the thermometer, is characterized by the following properties: specific heat $c_{\mathrm{Hg}} = 140$ J/kgK, density $\rho_{\mathrm{Hg}} = 14 \cdot 10^3$ kg/m^3, and thermal conductivity $\lambda_{\mathrm{Hg}} = 9$ W/mK.

The measurements yielded the following results:

- $t_1 = 40$ s $\rightarrow$ $T_1 = 34$ °C,
- $t_2 = 100$ s $\rightarrow$ $T_2 = 39$ °C.

a) How high is the fever?

Currently, a small Biot number, Bi $\ll 1$, is assumed. This implies a homogenous temperature in the thermometer.

Hence, the lumped capacity model is applicable:

$$\frac{T - T_0}{T_{\mathrm{a}} - T_0} = 1 - \exp\left(-\frac{\alpha}{\rho_{\mathrm{Hg}} c_{\mathrm{Hg}}} \frac{A}{V} t\right),$$

the derivation of this temperature profile has been established in Section 9.2.

For the sake of simplicity, parameter $m = \frac{\alpha}{\rho_{\mathrm{Hg}} c_{\mathrm{Hg}}} \frac{A}{V}$ is substituted, leading to:

$$\frac{T - T_0}{T_{\mathrm{a}} - T_0} = 1 - \exp(-mt).$$

The equation contains two unknown parameters: The ambient body temperature T_{a}, which should be determined, and the parameter m.

To obtain the value of m, an educated guess for the ambient body temperature T_{a} needs to be made. Given the two measurements, assuming that the body temperature will approximate $T_{\mathrm{a}} = 40$ °C is reasonable. Since m remains consistent across both measurements, correct estimation of T_{a} will result in similar values.

For measurement 1:

$$\frac{T_1 - T_0}{T_{\mathrm{a}} - T_0} = 1 - \exp(-m_1 t_1)$$

$$\Rightarrow m_1 = -\frac{1}{t_1} \ln\left(1 - \frac{T_1 - T_0}{T_{\mathrm{a}} - T_0}\right) = 0.0301,$$

and for measurement 2:

$$\frac{T_2 - T_0}{T_\mathrm{a} - T_0} = 1 - \exp\left(-m_2 t_2\right)$$
$$\Rightarrow m_2 = -\frac{1}{t_2}\ln\left(1 - \frac{T_2 - T_0}{T_\mathrm{a} - T_0}\right) = 0.0300.$$

This confirms the correctness of the educated guess for T_a. In scenarios where this does not hold, an alternative value for T_a could be tested iteratively until a congruence between both parameters m is achieved.

b) What is the value for the heat transfer coefficient α?

To determine the heat transfer coefficient α, first, the surface area A should be determined:

$$A = \pi d L,$$

as well as volume V of the mercury tip:

$$V = \frac{\pi d^2}{4} L.$$

Inserting into the definition of m and rewriting:

$$m = \frac{\alpha A}{\rho_\mathrm{Hg} c_\mathrm{Hg} V}$$
$$\Rightarrow \alpha = \frac{m \rho_\mathrm{Hg} c_\mathrm{Hg} d}{4} = 58.8\ \mathrm{W/m^2K}.$$

c) How long does the measurement take to determine the temperature with an accuracy of 0.1 K?

The measuring time t_m is reached when $T = 39.9$ K.

Inserting and rearranging:

$$\frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \exp\left(-m t_\mathrm{m}\right)$$
$$\Rightarrow t_\mathrm{m} = -\frac{1}{m}\ln\left(1 - \frac{T - T_0}{T_\mathrm{a} - T_0}\right) = 176.6\ \mathrm{s}.$$

d) Is the assumption of using the lumped capacity model valid?

To use the lumped capacity model, the condition $\mathrm{Bi} \ll 1$ should be satisfied. The characteristic length is derived from $L = V/A$, allowing to express the Biot number as:

$$\mathrm{Bi} = \frac{\alpha \cdot L}{\lambda_\mathrm{Hg}} = \frac{\alpha \cdot d}{2\lambda_\mathrm{Hg}} = 0.013 \ll 1.$$

Thus, the lumped capacity model is applicable.

SUBSECTION 9.3
Semi-infinite bodies

Situations where the internal thermal resistance of a body dominates the heat transfer process require a different solution approach compared to the "lumped body" solution. Spatial variations of the temperature inside the body call for resolving the body's temperature profile. As such, the differential equation is composed of a time $\frac{\partial T}{\partial t}$ and spatial derivatives $\frac{\partial^2 T}{\partial x^2}$, which is called a partial differential equation (PDE). The solution to the partial differential equation depends on the initial condition for the temperature field as well as on the boundary conditions. Although an infinite number of different configurations exist, a few basic solutions may already help to evaluate a larger number of heat transfer problems.

One such solution is called the semi-infinite plate. At the heart of the semi-infinite body model lies the idea that the temperature change induced on the body has not yet fully penetrated deep below its surface. Instead, the thermal disturbance remains confined to a shallow region, allowing for a simplified one-dimensional analysis. To better grasp this concept, we explore the notion of penetration depth δ.

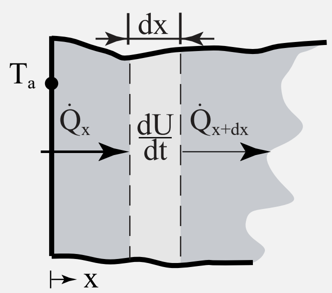
The penetration δ depth refers to the distance from the surface of the body at which the temperature change becomes negligible or almost insignificant. In other words, the penetration depth represents the extent to which the heat has propagated into the material, and beyond this depth, the temperature remains nearly unaffected by the external changes.

In bodies with large Biot numbers, the penetration depth tends to be relatively small. Thus, the thermal disturbance caused by external conditions, such as sudden heating or cooling, doesn't propagate very far into the body before the influence becomes negligible.

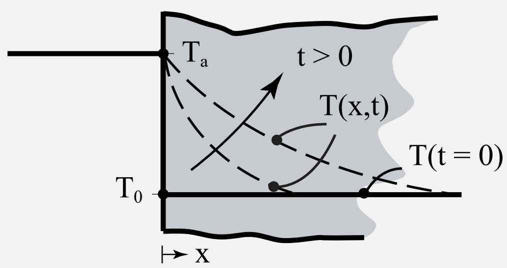
Semi-infinite body with a given temperature at the surface (Bi $\gg$ 1)

Derivation

The first scenario explores a semi-infinite plate with a steady surface temperature, which is the case if Bi $\gg$ 1. A real-world application of this concept is found in food processing. Imagine a scenario where a slice of food, such as a piece of meat or a baked product, is removed from an oven or cooked on one side. The outer layer of the food faces the high heat of the oven or cooking surface, while the interior starts at a lower temperature.



(a) Energy balance.



(b) Temperature distribution.

Figure 9.4. Heating of a semi-finite plate with a constant surface temperature.

1 **Setting up the balance:**

$$\frac{\partial U}{\partial t} = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}.$$

2 Defining the elements within the balance:

Temporal change of internal energy:

$$\frac{\partial U}{\partial t} = mc\frac{\partial T}{\partial t} = \rho c A dx \frac{\partial T}{\partial t}.$$

Ingoing conductive flux:

$$\dot{Q}_{\mathrm{x}} = -\lambda A \frac{\partial T}{\partial x},$$

and outgoing conductive flux:

$$\dot{Q}_{\mathrm{x+dx}} = -\lambda A \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A \frac{\partial T}{\partial x}\right) \cdot dx.$$

3 Inserting and rearranging:

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2}.$$

4 Defining the boundary and/or initial conditions:

For solving the differential equation, one initial condition and two boundary conditions are required. This is because the temperature T has undergone differentiation once concerning time (t) and twice concerning space (x).

Initially, the temperature within the body has a homogeneous temperature:

$$\left.\begin{matrix} t = 0 \\ 0 < x < \infty \end{matrix}\right\} T = T_0.$$

When $t > 0$ the temperature at $x = 0$ is remained at a constant temperature T_{a}:

$$\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} T = T_{\mathrm{a}}.$$

When dealing with a semi-infinite body, heat will never penetrate through the entire body and the temperature for $x \to \infty$ will remain to be T_0 at all times:

$$\left.\begin{matrix} t > 0 \\ x \to \infty \end{matrix}\right\} T = T_0.$$

5 Solving the equation:

Introducing the dimensionless temperature difference $\theta^* = \frac{T - T_{\mathrm{o}}}{T_{\mathrm{a}} - T_{\mathrm{o}}}$, and the thermal diffusivity $a = \frac{\lambda}{\rho\, c}$, the differential equation is rewritten to:

$$\frac{\partial \theta^*}{\partial t} = a\left(\frac{\partial^2 \theta^*}{\partial x^2}\right).$$

This case is solvable by introducing one independent variable $\eta(x,t)$, for which the partial differential equation reduces to an ordinary differential equation so that the following is valid:

$$\theta^*(\mathrm{x,t}) = \theta^*\left[\eta(\mathrm{x,t})\right].$$

The variable η can be found by smart people who visited courses from Anthony Thornton. For η, the following formulation is suitable:

$$\eta = \frac{x}{\sqrt{4at}} = \frac{1}{\sqrt{4\mathrm{Fo}}}.$$

Thus the ordinary differential equation can be further simplified to:

$$\frac{\partial^2\theta^*}{\partial\eta^2} + 2\eta\frac{\partial\theta^*}{\partial\eta} = 0.$$

The second-order differential equation requires two boundary conditions in the new coordinates η and θ^*. Transforming the initial and boundary conditions:

$$\begin{array}{llll} \text{I.C. 1:} & \eta = \infty & : & \theta^* = 0, \\ \text{B.C. 1:} & \eta \to 0 & : & \theta^* = 1, \quad \text{and} \\ \text{B.C. 2:} & \eta \to \infty & : & \theta^* = 0. \end{array}$$

Since I.C. 1 equals B.C. 2 after transformation, and they do not contradict each other, a similar solution is possible using the variable η.

The solution of this second-order differential equation is carried out with the usual methods of substitution. For a more detailed derivation refer to literature as [2].

Solving the differential equation by use of the boundary conditions yields the dimensionless temperature profile:

$$\theta^* = 1 - \mathrm{erf}(\eta) = 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4at}}\right).$$

The second term on the right side of the equation above is called the error function $\mathrm{erf}(\eta)$, which is a standard mathematical function.

The temperature profile is shown in Figure 9.5.

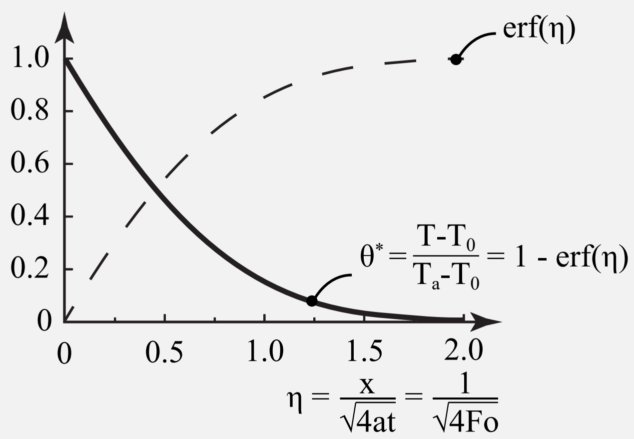


Figure 9.5. Temperature characteristics in a semi-infinite body - constant surface temperature.

The diagram shows that the initial temperature difference reduces to 1% of its initial value at $\eta = 1.8$.

This value is often used to define the penetration depth or thickness of the temperature boundary layer

$$\delta(t) = x(\eta = 1.8) = 3.6\sqrt{at}.$$

From the temperature profile, the heat transfer rate at the surface is determined by use of Fourier's law as stated in equation (5.1):

$$\dot{q}''\Big|_{x=0} = -\lambda \frac{\partial T}{\partial x}\Big|_{x=0}.$$

Which gives:

$$\dot{q}''\Big|_{x=0} = \frac{\lambda}{\sqrt{\pi a t}}\left(T_\mathrm{a} - T_\mathrm{o}\right).$$

In other words, the heat transfer rate drops steadily with time, so that at time t the total amount of heat transferred is described by:

$$\int_{t=0}^{t} \dot{q}''\Big|_{x=0} dt = 2\frac{\lambda}{\sqrt{\pi a t}}\left(T_\mathrm{a} - T_\mathrm{o}\right).$$

Fundamental EQ

Temperature profile semi-infinite plate with negligible thermal surface resistance:

$$\theta^* = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}}\right), \tag{9.2}$$

for $\mathrm{Bi} \gg 1$, where $\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0}$, and $\mathrm{Fo} = \frac{at}{x^2}$.

Fundamental EQ

Penetration depth semi-infinite plate with negligible thermal surface resistance:

$$\delta(t) = 3.6\sqrt{at}, \tag{9.3}$$

for $\mathrm{Bi} \gg 1$.

Fundamental EQ

Heat flux semi-infinite plate with negligible thermal surface resistance:

$$\dot{q}''\Big|_{x=0} = \frac{\lambda}{\sqrt{\pi a t}}\left(T_\mathrm{a} - T_0\right), \tag{9.4}$$

for $\mathrm{Bi} \gg 1$.

9.3.1 Fourier number

The Fourier number plays a role in understanding heat transfer processes, particularly in transient conduction scenarios. This dimensionless number named after the renowned mathematician and physicist Joseph Fourier, who made significant contributions to the study of heat transfer. The Fourier number is defined as the ratio of thermal diffusivity, which represents a material's ability to conduct heat relative to its capacity to store heat, multiplied by time. In simpler terms, the Fourier number quantifies the relative importance of the time required for heat conduction to occur within a solid compared to the time required for thermal energy storage within that material.

A high Fourier number indicates that the heat conduction process is occurring rapidly compared to the rate of thermal energy storage, suggesting a swift change in the temperature distribution within the material. On the other hand, a low Fourier number implies that thermal energy storage within the material is substantial compared to the rate of heat conduction, signifying a slower evolution of the temperature distribution over time. The Fourier number is particularly useful in scenarios where the temperature within a solid is changing dynamically, such as during the heating or cooling of a material. Engineers and scientists often use this dimensionless parameter to analyze and predict the behavior of heat transfer processes, aiding in the design and optimization of various thermal systems.

Definition

Fourier number:

$$\mathrm{Fo} = \frac{\text{Rate of diffusivity}}{\text{Rate of storage}} = \frac{at}{L^2} \; (-), \tag{9.5}$$

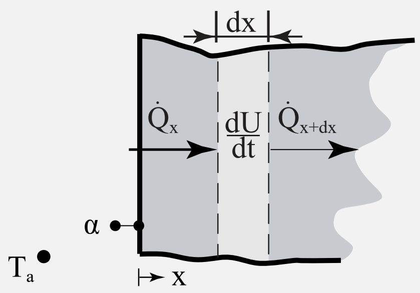
where L is the characteristic length.

Semi-infinite plate with non-negligible heat transfer resistance

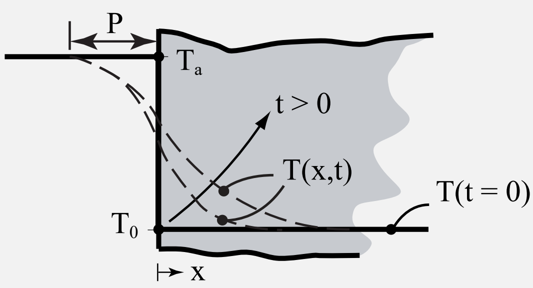
In the previous derivation, the temperature at the body surface is assumed to match the surrounding temperature for all $t > 0$. This is valid only if the heat transfer resistance is small compared to the thermal resistance within the body, or if $\mathrm{Bi} \gg 1$.

Derivation

This scenario delves into situations involving semi-infinite bodies where the temperature difference with the surrounding environment is notably significant compared to the internal temperature difference within the body. To illustrate, think of the cooling process for a just-baked item like a pie or casserole. Once out of a hot oven, the food is placed in a room with a lower temperature. This dynamic showcases the emphasis, as the food's outer layers cool swiftly due to the cooler room, while the interior maintains a higher temperature from the initial cooking.



(a) Energy balance.



(b) Temperature distribution.

Figure 9.6. Heating of a semi-finite plate with a non-negligible heat transfer resistance.

1 Setting up the balance:

See derivation semi-infinite plate with a given surface temperature.

2 Defining the elements within the balance:

See derivation semi-infinite plate with a given surface temperature.

3 Inserting and rearranging:

Still, the same differential equation remains valid:

$$\frac{\partial^2 \theta^*}{\partial \eta^2} + 2\eta \frac{\partial \theta^*}{\partial \eta} = 0.$$

4 **Defining the boundary and/or initial conditions:**

If the heat transfer resistance cannot be neglected, a surface temperature is observed which is between the surrounding temperature and the initial temperature of the body.

Initially, the temperature within the body has a homogeneous temperature:

$$\left.\begin{array}{c} t = 0 \\ 0 < x < \infty \end{array}\right\} T = T_0.$$

A new boundary condition is introduced for $x = 0$, which yields from an energy balance at the interface:

$$\left.\begin{array}{c} t > 0 \\ x = 0 \end{array}\right\} \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\alpha}{\lambda}\left(T(x=0) - T_\mathrm{a}\right).$$

This relationship is presented illustratively, as shown in the Figure 9.6b. The extrapolations of all gradient lines intersect at point P, which is given by the coordinates T_a and $-\frac{\lambda}{\alpha}$.

Because there is dealt with a semi-infinite body, heat will never penetrate through the entire body and the temperature for $x \to \infty$ will remain to be T_0 at all times:

$$\left.\begin{array}{c} t > 0 \\ x \to \infty \end{array}\right\} T = T_0.$$

5 **Solving the equation:**

With these initial and boundary conditions, the computation load for solving the differential equation is much larger than before. A detailed discussion is provided in Schneider [3].

The following equation describes the solved temperature profile:

$$\theta^* = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}}\right) - \left[\exp\left(\mathrm{Bi} + \mathrm{Fo}\,\mathrm{Bi}^2\right)\right]\left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}} + \sqrt{\mathrm{Fo}} \cdot \mathrm{Bi}\right)\right],$$

where $\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0}$, $\mathrm{Fo} = \frac{at}{x^2}$, and $\mathrm{Bi} = \frac{\alpha x}{\lambda}$.

The solution of this equation is shown in Figure 9.7. The previously discussed case of negligible heat transfer resistances is thus a special case, $\sqrt{\mathrm{Fo}}\mathrm{Bi} \to \infty$, as shown in the diagram.

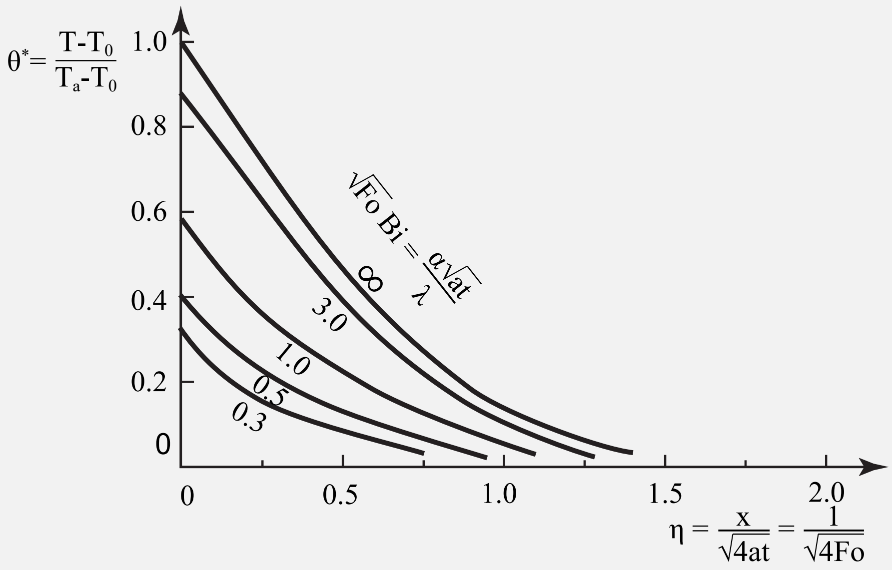


Figure 9.7. The temperature as a function of time of a semi-infinite body with finite convective heat transfer resistance.

Fundamental EQ

Temperature profile semi-infinite plate with non-negligible thermal surface resistance:

$$\theta^* = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}}\right) - \left[\exp\left(\mathrm{Bi} + \mathrm{Fo}\,\mathrm{Bi}^2\right)\right]\left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}} + \sqrt{\mathrm{Fo}}\cdot\mathrm{Bi}\right)\right] \tag{9.6}$$

where $\theta^* = \frac{T-T_0}{T_a-T_0}$, $\mathrm{Fo} = \frac{at}{x^2}$, and $\mathrm{Bi} = \frac{\alpha x}{\lambda}$.

SUBSECTION 9.4

Dimensionless numbers and Heisler diagrams

In the previous sections, several examples were given, which had simple analytical solutions to the conduction equation, as stated in equation (5.2).

However, in cases of systems with complicated geometries or boundary and initial conditions, analytical solutions do not suffice. If so, numerical solutions using the method of finite differences are nowadays available.

One of the major disadvantages of such methods is the fact that, in general, the derived temperature field is dependent on a large number of parameters:

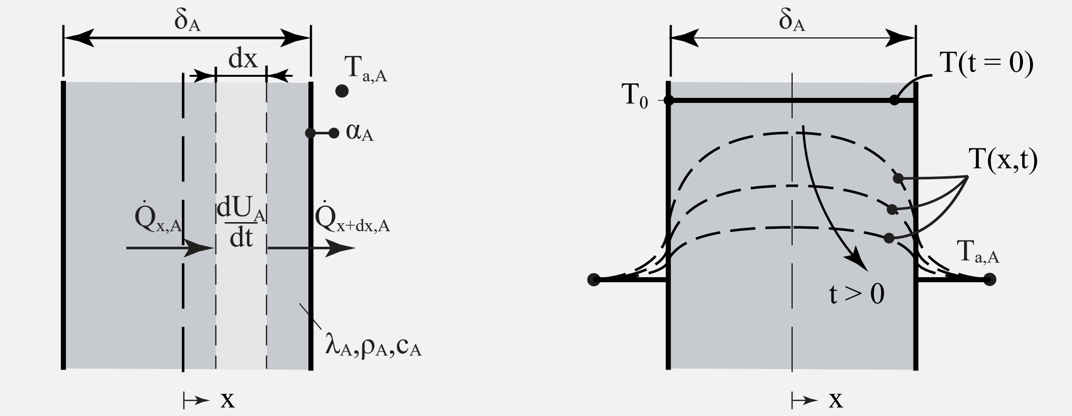
$$T = T\,(\mathrm{x,y,z,t},\rho,\mathrm{c},\lambda,\text{initial and boundary conditions})\,,$$

and that if one of the parameters changes, the entire calculation has to be redone.

Next, a simple example of unsteady state heat conduction in a plate shows that the number of dependent parameters is considerably reduced by introducing dimensionless numbers, some of which have already been discussed in previous examples.

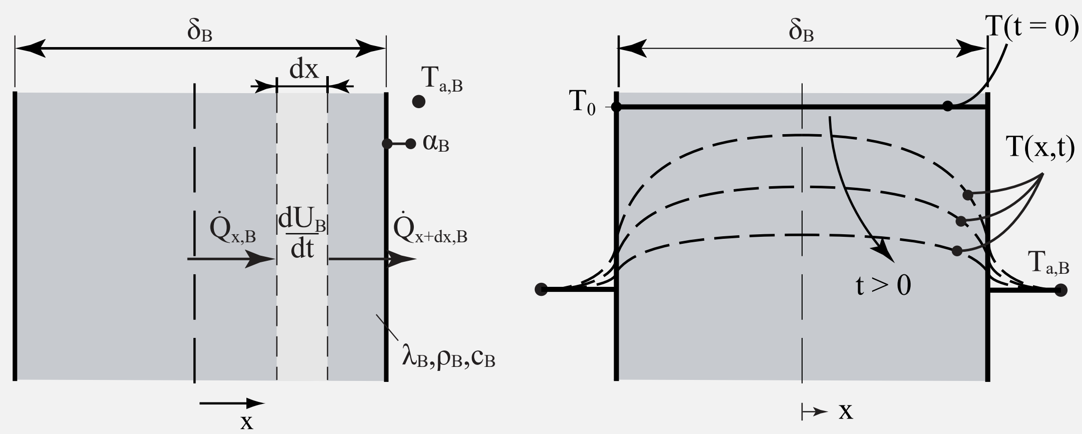
Derivation

Two plates are considered, with lengths much larger than their thickness, which are brought at a given time in another environment with different temperatures. The thickness of the plate, its initial temperature, the new temperature of the surroundings, the material properties, and the boundary conditions are labeled A for the first and B for the second plate.



(a) Energy balance. (b) Temperature distribution.

Figure 9.8. Unsteady heat transfer for plate configuration A.



(a) Energy balance. (b) Temperature distribution.

Figure 9.9. Unsteady heat transfer for plate configuration B.

1 **Setting up the balance:**

System A

$$\frac{\partial U_A}{\partial t} = \dot{Q}_{x,A} - \dot{Q}_{x+dx,A},$$

System B

$$\frac{\partial U_B}{\partial t} = \dot{Q}_{x,B} - \dot{Q}_{x+dx,B}.$$

2 **Defining the elements within the balance:**

System A

$$\frac{\partial U_{\mathrm{A}}}{\partial t} = \rho_{\mathrm{A}}\, c_{\mathrm{A}}\, A\, dx\, \frac{\partial T_{\mathrm{A}}}{\partial t},$$

$$\dot{Q}_{\mathrm{x,A}} - \dot{Q}_{\mathrm{x+dx,A}} = \frac{\partial}{\partial x}\left(\lambda_{\mathrm{A}} A \frac{\partial T_{\mathrm{A}}}{\partial x}\right)\cdot dx,$$

System B

$$\frac{\partial U_{\mathrm{B}}}{\partial t} = \rho_{\mathrm{B}}\, c_{\mathrm{B}}\, A\, dx\, \frac{\partial T_{\mathrm{B}}}{\partial t}.$$

$$\dot{Q}_{\mathrm{x,B}} - \dot{Q}_{\mathrm{x+dx,B}} = \frac{\partial}{\partial x}\left(\lambda_{\mathrm{B}} A \frac{\partial T_{\mathrm{B}}}{\partial x}\right)\cdot dx.$$

3 Inserting and rearranging:

System A

$$\frac{\partial T_{\mathrm{A}}}{\partial t} = \left(\frac{\lambda}{\rho c}\right)_{\mathrm{A}} \frac{\partial^2 T_{\mathrm{A}}}{\partial x^2},$$

System B

$$\frac{\partial T_{\mathrm{B}}}{\partial t} = \left(\frac{\lambda}{\rho c}\right)_{\mathrm{B}} \frac{\partial^2 T_{\mathrm{B}}}{\partial x^2}.$$

4 Defining the boundary and/or initial conditions:

With the initial condition:

$$T(t = 0) = T_0.$$

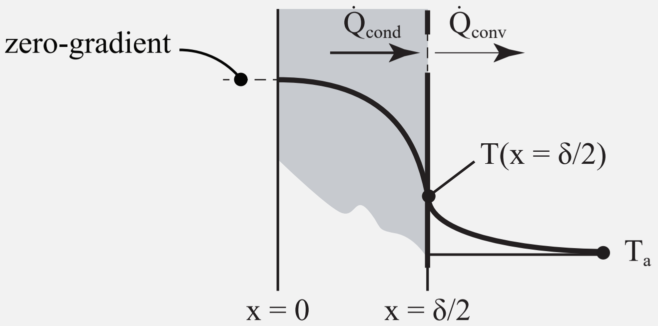


Figure 9.10. Boundary conditions at the surface.

In the middle symmetry if found, so the first boundary condition is stated as:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0.$$

At the interface between the solid and fluid, heat is transferred from conduction through the solid to the fluid by convection. An energy balance at this interface yields the second boundary condition:

$$\lambda \left.\frac{\partial T}{\partial x}\right|_{x=\frac{\delta}{2}} = \alpha\left(T\left(x = \frac{\delta}{2}\right) - T_{\mathrm{a}})\right).$$

5 Solving the equation:

The differential equations can be written in a dimensionless form. To this purpose, reference values were selected that characterize the system. The geometry of both plates is described by the plate thickness δ. If the heat conduction process is periodic in time the process is similar if only the period of oscillations τ is different for both plates. The temperature, or the temperature difference $\theta = T - T_{\mathrm{a}}$, is referenced to the characteristic temperature difference $\theta_0 = T_0 - T_{\mathrm{a}}$ of the system.

Hence, the dimensionless variables are:

$$\mathrm{x}^* = \frac{x_\mathrm{A}}{\delta_\mathrm{A}} = \frac{x_\mathrm{B}}{\delta_\mathrm{B}},$$
$$\mathrm{t}^* = \frac{t_\mathrm{A}}{\tau_\mathrm{A}} = \frac{t_\mathrm{B}}{\tau_\mathrm{B}}, \text{ and}$$
$$\theta^* = \frac{\theta_\mathrm{A}}{\theta_{\mathrm{A}_0}} = \frac{\theta_\mathrm{B}}{\theta_{\mathrm{B}_0}}.$$

By substituting them into the differential equations for the temperature field they read as follows:

System A

$$\frac{\partial\theta^*}{\partial t^*} = \left(\frac{a_\mathrm{A}\tau_\mathrm{A}}{\delta_\mathrm{A}^2}\right)\frac{\partial^2\theta^*}{\partial x^{*2}},$$

System B

$$\frac{\partial\theta^*}{\partial t^*} = \left(\frac{a_\mathrm{B}\tau_\mathrm{B}}{\delta_\mathrm{B}^2}\right)\frac{\partial^2\theta^*}{\partial x^{*2}}.$$

Thus, the differential equations are identical, if the Fourier numbers:

$$\mathrm{Fo} = \frac{a\tau}{\delta^2} = \frac{\lambda}{\rho c}\frac{\tau}{\delta^2},$$

of both systems are equal. The equivalence of the parameters which are included in the Fourier number is not necessary. The dimensionless temperature fields are only then equal, if the boundary conditions of both systems are equal, too.

In the dimensionless form, the initial and boundary conditions are:

System A

$$\theta^* = 1,$$
$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{\mathrm{x}=0} = 0,$$
$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{x=\frac{\delta_\mathrm{A}}{2}} = -\frac{\alpha_\mathrm{A}\delta_\mathrm{A}}{\lambda_\mathrm{A}}\left|\theta^*_{x=\frac{\delta_\mathrm{A}}{2}}\right.,$$

System B

$$\theta^* = 1,$$
$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{\mathrm{x}=0} = 0,$$
$$\left.\frac{\partial\theta^*}{\partial x^*}\right|_{x=\frac{\delta_\mathrm{B}}{2}} = -\frac{\alpha_\mathrm{B}\delta_\mathrm{B}}{\lambda_\mathrm{B}}\left|\theta^*_{x=\frac{\delta_\mathrm{B}}{2}}\right..$$

Hence, the boundary conditions are identical if the Biot numbers:

$$\mathrm{Bi} = \frac{\alpha\delta}{\lambda}$$

of both systems are equal.

If the system had to be extended to consider three-dimensional bodies, the dimensionless temperature field has to be described by the following dimensionless parameters:

$$\begin{aligned}\frac{T-T_\mathrm{a}}{T_0-T_\mathrm{a}} &= \frac{T-T_\mathrm{a}}{T_0-T_\mathrm{a}}\left(\frac{\mathrm{x}}{\delta_1},\frac{\mathrm{y}}{\delta_2},\frac{\mathrm{z}}{\delta_3},\frac{t}{\tau},\left(\frac{a\tau}{\delta^2}\right)_{1,2,3},\left(\frac{\alpha\delta}{\lambda}\right)_{1,2,3}\right)\\ &= \frac{T-T_\mathrm{a}}{T_0-T_\mathrm{a}}\left(\mathrm{x}^*,\mathrm{y}^*,\mathrm{z}^*,\mathrm{t}^*,\mathrm{Fo}_{1,2,3},\mathrm{Bi}_{1,2,3}\right).\end{aligned}$$

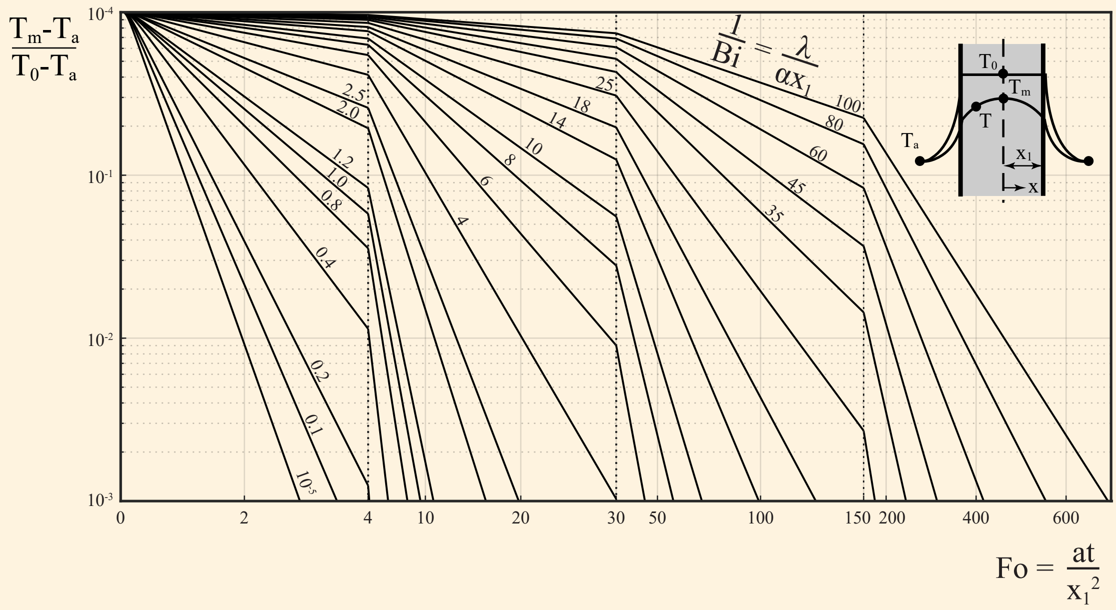
The solution to this is presented in so-called Heisler diagrams, which are shown on the subsequent pages, presenting the temperature as a function of time and space and the heat lost as a function of time.

The previously mentioned examples are derived using these parameters, respectively. The analytical or numerical solutions of the differential equations are often presented in diagrams using these dimensionless parameters. Three examples are considered which are useful to approximate many practical problems. These examples will show the temperature profile and the transferred heat during cooling (or heating) of extended plates, long cylinders, and spheres, whose surrounding temperatures have been abruptly changed at a given time. Although these one-dimensional problems can be solved analytically, because of their complicated computations, the diagrams given by [4] are recommended.

Similar diagrams, listed in the book of formularies, show the mid-plane temperature as a function of the time for a plate, cylinder, and sphere, together with additional diagrams used for the determination of the temperatures at other points of the body. Appropriately interpolating these diagrams leads to heat loss as a function of the time of the body.

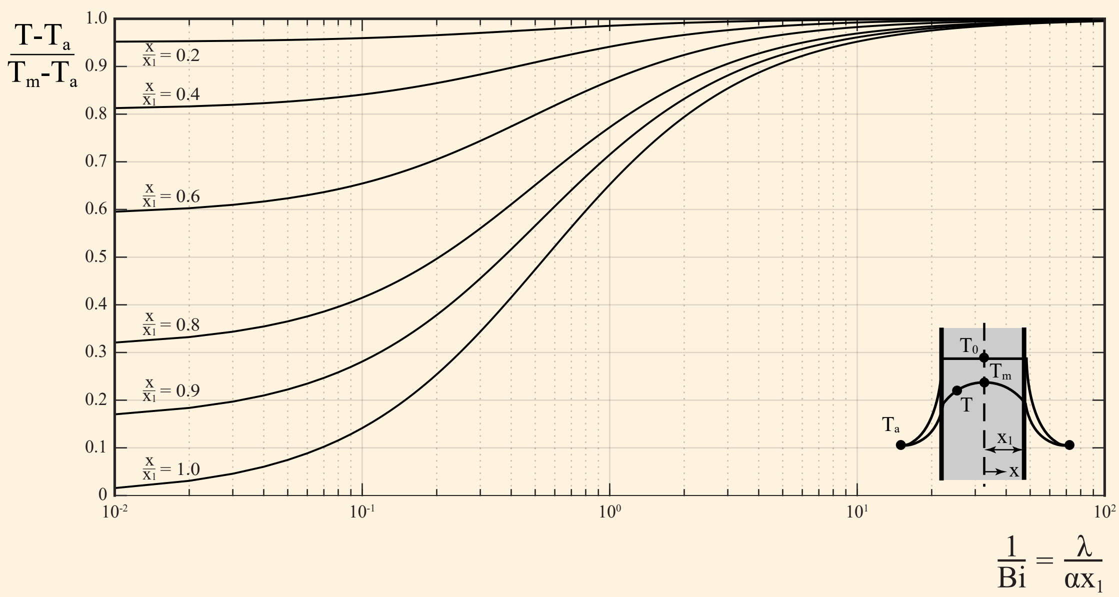
Fundamental EQ

Temperature profile mid-plane of a plate with thickness $2x_1$ [5]:



Fundamental EQ

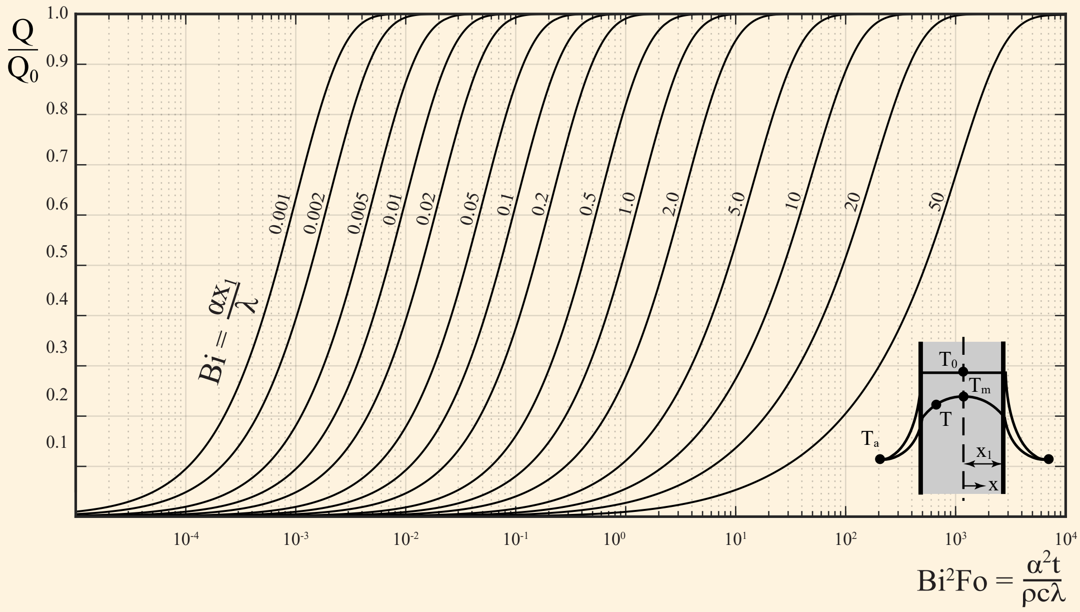
Temperature distribution in a plate with thickness $2x_1$ [5]:



which is for $Fo > 0.2$.

Fundamental EQ

Heat loss of a plate [5]:



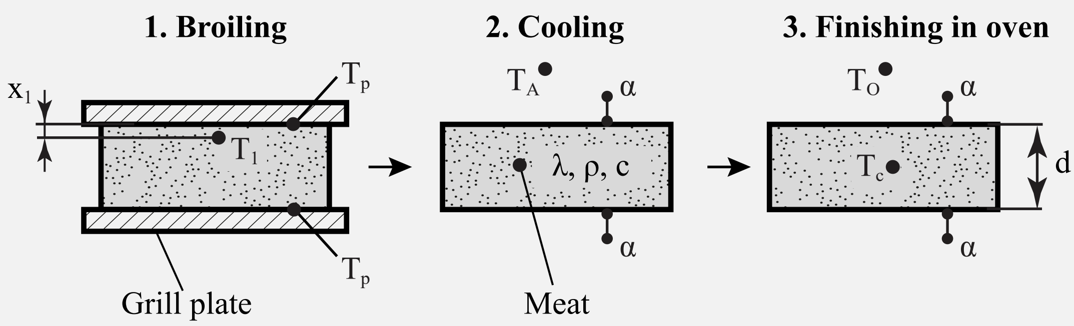
where $Q = mc\,(T(t) - T_a)$ and $Q_0 = mc\,(T_0 - T_a)$.

HEATQUIZ 9.1

Temperature profiles: **Energy balances:**

Demonstration 9.1

To obtain the perfect steak, a piece of meat is prepared in three steps: First, the steak is briefly broiled between two hot grill plates. After, the steak cools until ambient temperature throughout is reached. Eventually, the steak is finished in the oven at a low temperature until the piece is done.



Given parameters:

- Ambient temperature: $T_A = 25\ °C$
- Temperature of grill plates: $T_P = 800\ °C$
- Final temperature at location 1: $T_{1,E} = 250\ °C$
- Oven temperature: $T_O = 110\ °C$
- Maximum core temperature: $T_{C,max} = 59\ °C$
- Density: $\rho = 930\ kg/m^3$
- Specific thermal conductivity: $\lambda = 0.6\ W/mK$
- Specific heat capacity: $c_p = 2900\ J/kgK$
- Meat thickness: $d = 3\ cm$
- Distance of point 1 from the surface: $x_1 = 3\ mm$
- Heat transfer coefficient in the oven: $\alpha = 20\ W/m^2K$

Hints:

- In all subproblems, the steak is to be regarded as a flat plate with one-dimensional heat conduction in the vertical direction only.
- All material properties are constant and isotropic.
- Grill plate and oven temperature are constant in time and space.
- Neglect the heat transfer resistance between the grill plate and the piece of meat.
- Neglect heat transfer by radiation in all subproblems.

Tasks:

a) Determine the time t_1 for the very brief broiling. The broiling is to be interrupted when the meat's temperature at location 1 close to the surface ($x_1 = 3$ mm) reaches a temperature of $T_{1,E} = 250\ °C$. Before broiling the meat has a homogeneous temperature of T_A. During broiling, a thermal penetration depth of $\delta \ll d/2$ is assumed.

b) Determine the time t_2 for finishing in the oven. The meat is done when a core temperature $T_{C,max} = 59\ °C$ in the middle of the meat is reached. Heat transfer from the oven to the meat happens exclusively through convection with the heat transfer coefficient α.

c) Qualitatively sketch the temporal evolution of the core temperature $T_C(t)$ during broiling, cooling, and finishing in the oven.

d) Qualitatively sketch the temporal evolution of the temperature $T_1(t)$ at location 1 during broiling, cooling, and finishing in the oven until the piece is done.

Video solution:

References

[1] S. M. et Franklin, "Portraits et Histoire des Hommes Utiles, Collection de Cinquante Portrait." http://web.mit.edu/2.51/www/fourier.jpg, 1839. Accessed: 2023-07-16.

[2] Y. A. Cengel, A. J. Ghajar, and M. Kanoglu, "Heat and mass transfer: fundamentals and applications," 2011.

[3] P. Schneider, *Conduction Heat Transfer.* Addion-Wesley Publishing Comp., 1955.

[4] M. Heisler, "Temperature charts for induction and constant temperature heating," *Trans. ASME*, pp. 227–236, 1947.

[5] A. L. Maßmeyer, S. Pielsticker, H. Askarizadeh Ravizi, and R. Kneer, "Übernahmeungenauigkeiten von Heisler- und Gröber-Diagrammen in der Standardliteratur zur Wärme- und Stoffübertragung," Jahrestreffen der DECHEMA/VDI-Fachgruppen Wärme- und Stoffübertragung und Trocknungstechnik, Magdeburg (Germany), 11 Mar 2024 - 13 Mar 2024, Mar 2024. Veröffentlicht auf dem Publikationsserver der RWTH Aachen University.

PART

VII

Exercises

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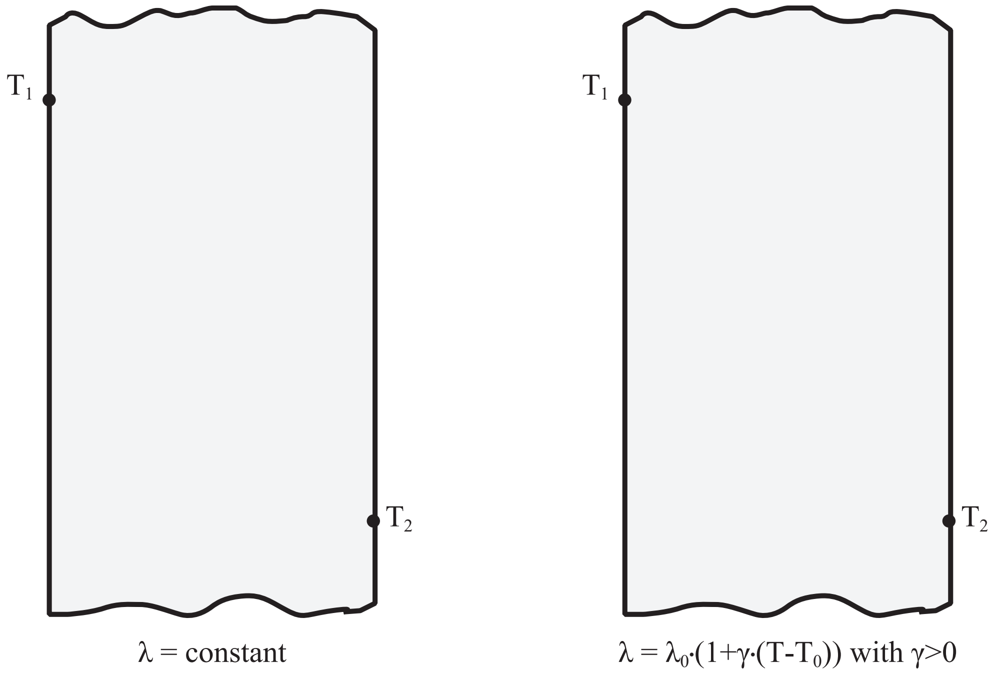
Section II
Übungen zur Wärmeleitung

Exercise II.1: (Temperature profiles in planar walls ⋆)

Both sides of a planar wall are heated to a constant temperature of T_1 and T_2, where $T_1 > T_2$.



Tasks:

a) Sketch the steady-state temperature profile for a constant thermal conductivity.

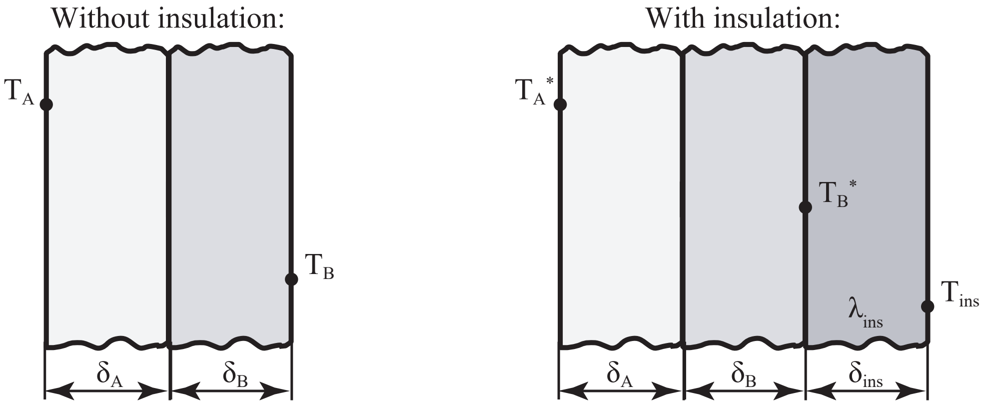
b) Sketch the steady-state temperature profile for the conductivity being temperature-dependent:

$$\lambda = \lambda_0(1 + \gamma(T - T_0)), \text{ with } \gamma > 0,$$

where λ_0 is the thermal conductivity at the temperature T_0.

Exercise II.2: (Onion layer principle ⋆⋆)

A solar panel manufacturer makes use of heat processing applications that include preheating, curing, heat treating, and finishing. The manufacturer has an old and a new type of industrial oven. The newer one has an additional insulation layer.



Given parameters:

Old oven:

- Surface temperature of layer A: T_A = 260 °C
- Surface temperature of layer B: T_B = 32 °C
- Thickness of layer A: δ_A = 125 mm
- Thickness of layer B: δ_B = 200 mm

New oven:

- Surface temperature of layer A: T_A^* = 305 °C
- Surface temperature of layer B: T_B^* = 219 °C
- Surface temperature of insulation layer: T_{ins} = 27 °C
- Thickness of layer A: δ_A = 125 mm
- Thickness of layer B: δ_B = 200 mm
- Thickness of insulation layer: δ_{ins} = 25 mm
- Thermal conductivity of insulation layer: λ_{ins} = 0.075 W/mK

Hint:

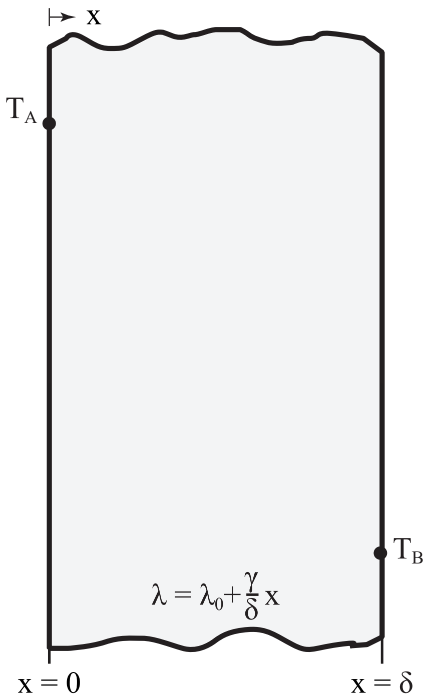
- Assume steady-state conditions.

Tasks:

a) Determine the heat flux per unit area $\dot{q}''$ for the situations without and with the insulating layer.

Exercise II.3: (Heat conduction equation ⋆⋆⋆)

Both sides of a planar wall are heated to a constant temperature of T_A and T_B, respectively; where $T_A > T_B$.



Given parameters:

- Thermal conductivity as a function of the position in the wall:

$$\lambda(x) = \lambda_0 + \frac{\gamma}{\delta} \cdot x.$$

Hints:

- Assume one-dimensional steady-state heat transfer in x-direction.

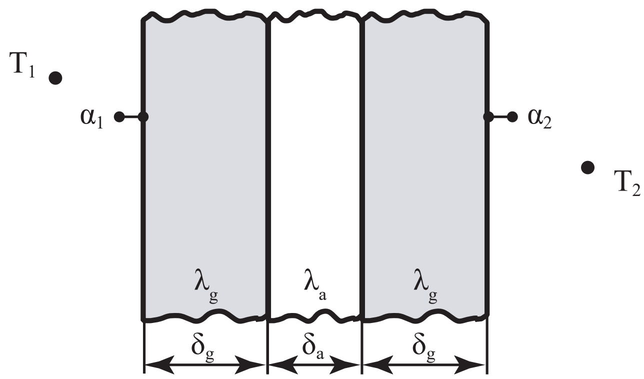
Tasks:

a) Derive the function of the temperature profile inside the plane wall.

b) Sketch the temperature profile inside the plane wall in the x-direction.

Exercise II.4: (Window insulation ⋆)

Consider a 1.2-m-height and 2-m-wide double-pane window consisting of two layers of glass separated by a stagnant air space. Convection occurs at the inside and outside of the pane window. Disregard any heat transfer by radiation.



Given parameters:

- Conductivity of glass: $\lambda_g = 0.78$ W/mK
- Conductivity of air: $\lambda_a = 0.026$ W/mK
- Thickness of glass layer: $\delta_g = 3$ mm
- Thickness of air layer: $\delta_a = 15$ mm
- Inside convection coefficient: $\alpha_1 = 10$ W/m^{2}K
- Outside convection coefficient: $\alpha_2 = 25$ W/m^{2}K
- Inside temperature: $T_1 = 22$ °C
- Outside temperature: $T_2 = -7$ °C

Tasks:

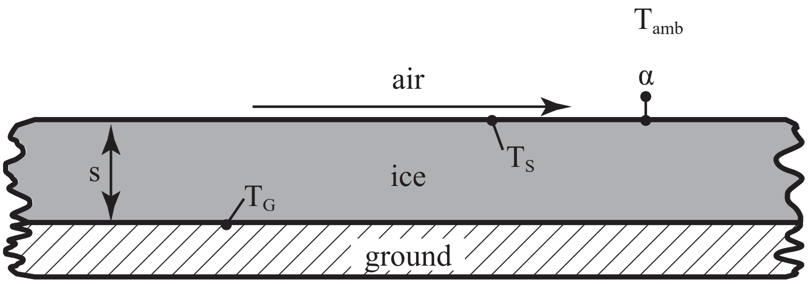
a) Determine the steady heat transfer rate through this double-pane window and the temperature of its inner surface.

b) Compare your results with a three-layer glass (3-mm-thickness) with two stagnant air spaces filled with krypton ($\delta_k = 8\,\text{mm}$, $\lambda_k = 0.00949\,{}^{\text{W}}/_{\text{m K}}$).

c) Discuss the reason for choosing a three-layer glass and scrutinize all assumptions made in tasks a) and b).

Exercise II.5: (Ice layer ⋆⋆)

During a cold winter day, the ground is covered with an ice layer of thickness s. Air is flowing over the ice layer. The problem is one-dimensional and steady-state. No layer of water is forming on top of the ice.



Given parameters:

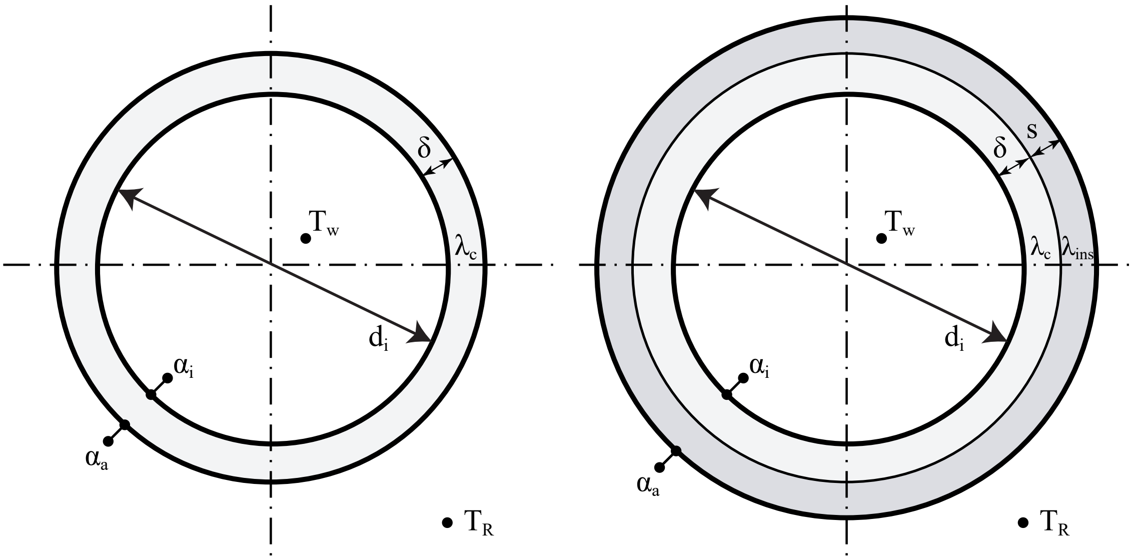
- Conductivity of ice: $\lambda = 2.2$ W/mK
- Heat transfer coefficient at the ice surface: $\alpha = 10$ W/m^{2}K
- Temperature of the air: $T_{\mathrm{amb}} = 5$ °C
- Temperature of the ice at the surface: $T_{\mathrm{S}} = -3$ °C
- Temperature of the ice at the ground: $T_{\mathrm{G}} = -10$ °C
- Temperature of the air: $T_{\mathrm{amb}} = 5$ °C

Tasks:

a) Determine the thickness s of the ice layer.

Exercise II.6: (Warm-water pipe ⋆ ⋆ ⋆)

In a room, a copper warm-water pipe is utilized to contain water. This copper pipe features an inner diameter of d_i and a wall thickness denoted as δ. During a chilly winter day, insulation measures are taken, involving the addition of an extra insulation layer with a thickness of s.



Given parameters:

- Heat transfer coefficient at the inner side of the pipe: α_i = 2300 W/m^{2}K
- Heat transfer coefficient at the outer side of the pipe: α_a = 6 W/m^{2}K
- Temperature of the room: T_R = 20 °C
- Temperature of the water: T_W = 80 °C
- Conductivity of copper: λ_c = 372 W/mK
- Conductivity of insulation material: λ_ins = 0.042 W/mK
- Inner diameter of the copper pipe: d_i = 6 mm
- Thickness of the copper pipe: δ = 1 mm
- Thickness of the insulation layer: s = 4 mm

Hints:

- Changes to the heat transfer coefficient at the outer side of the pipe as a function of the diameter are disregarded.

Tasks:

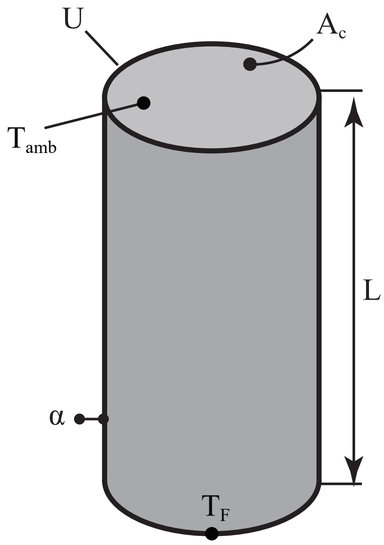
a) Calculate the heat transferred per unit length of the pipe, denoted as $\dot{q}'$, for both an uninsulated pipe and an insulated pipe. What noteworthy observations can be made from your findings?

b) Qualitatively sketch the heat emission profile $\dot{q}'$ as a function of the insulation thickness for different thermal conductivities of the insulation material. Explain the underlying physical principles.

c) Calculate the required thermal conductivity for the insulating material to always achieve a reduction in heat loss, regardless of the thickness of the insulation.

Exercise II.7: (Pin-fin cooling on gas turbine blades ⋆⋆)

A rod fin is used for pin-fin cooling on gas turbine blades. The rod-fin of length L, with a tip temperature equal to the ambient T_{amb}, is given.



Given parameters:

- Fin geometry: U, A_c, L
- Fin material properties: λ
- Surface heat transfer coefficient: α
- Fin base temperature and environment temperature: $T_\text{F}, T_{\text{amb}}$

Tasks:

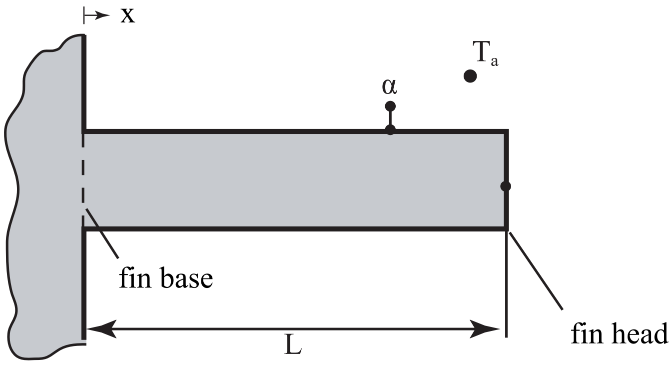
a) Derive the heat conduction equation for the given problem.

b) Derive the function of the temperature profile inside the fin.

c) Give the expression for the rate of heat loss in terms of the given parameters.

Exercise II.8: (New fin material ⋆⋆)

An electric motor manufacturer is using fins for cooling purposes. He is considering changing the material used for the fins from copper to aluminum. Because the length L of the fin is also modified, the temperature at the fin head remains identical for both materials. However, he does not understand the impact of such a change on the performance of cooling.



Given parameters:

- Thermal conductivity of copper: λ_C
- Thermal conductivity of aluminium: λ_A

Hints:

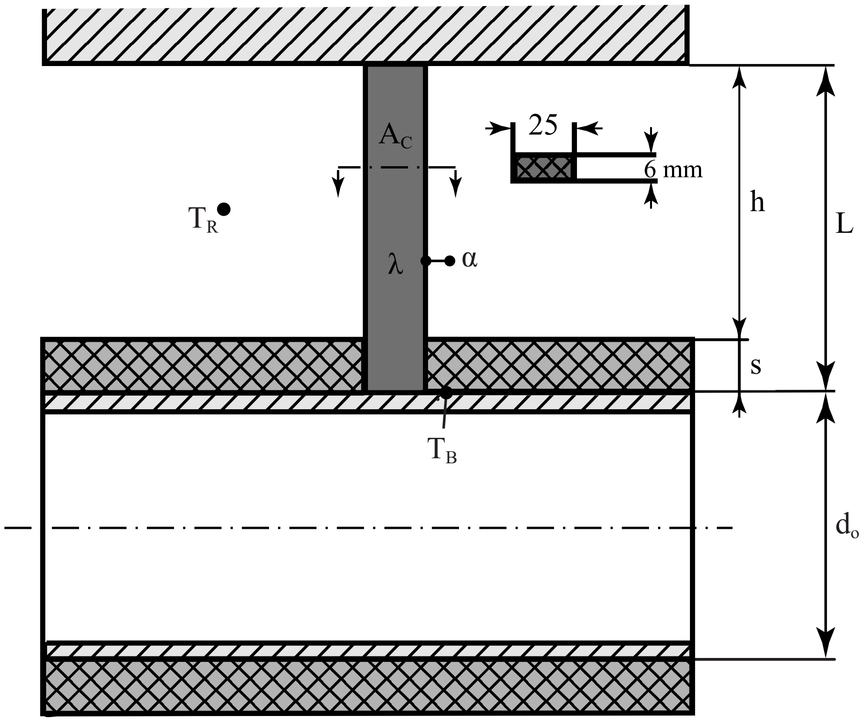
- The cross-section and the thickness remain unchanged.
- There is no change in the convective heat transfer coefficient.
- The temperature at the fin base does not change.
- For both fins, the heat flow through the head is negligible.

Tasks:

a) Determine the ratio between the heat flow of the aluminum and the copper fin in terms of given parameters.

Exercise II.9: (Pipe fastening ⋆ ⋆ ⋆)

A pipe containing brine is insulated with cork and fastened to the ceiling with steel bands welded to the pipe.



Given parameters:

- Outer diameter of the pipe: $d_o = 50$ mm
- Insulation thickness: $s = 40$ mm
- Cross-section of the steel band: $A_c = 25 \times 6$ mm
- Length of the steel band: $L = 290$ mm
- Heat transfer coefficient at the steel band's surface: $\alpha = 6$ W/m^{2}K
- Thermal conductivity of the steel band: $\lambda = 58$ W/mK
- Temperature outer wall of the brine pipe: $T_B = -23.5$ °C
- Temperature of the room: $T_R = 20$ °C

Hints:

- The temperature distribution in the steel band's cross-section is homogeneous.
- The heat fluxes from the steel bands into both the ceiling and the insulation are negligible.

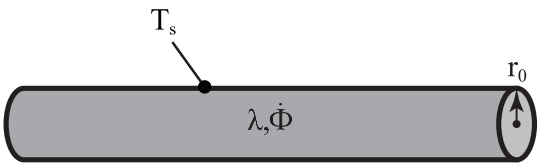
Tasks:

a) Calculate the heat $\dot{Q}$ from one steel band absorbed by the brine.

b) Up to which height h_0 does frost form on the steel ban (h_0 is the distance from the surface of the pipe's insulation layer), if the steam content of the air in the surrounding room is above the saturation vapor pressure for the maximum steel band temperature?

Exercise II.10: (Resistance wire ⋆)

A long homogeneous resistance wire is used to heat the air in a room by the passage of an electric current. Heat is generated in the wire uniformly at a constant rate $\dot{\Phi}'''$ as a result of resistance heating.



Given parameters:

- Outer radius of the wire: $r_0 = 5$ mm
- Heat generation in the wire: $\dot{\Phi}''' = 5 \cdot 10^7$ W/m^3
- Temperature of the outer surface of the wire: $T_s = 180$ °C
- Thermal conductivity of the wire: $\lambda = 6$ W/mK

Hints:

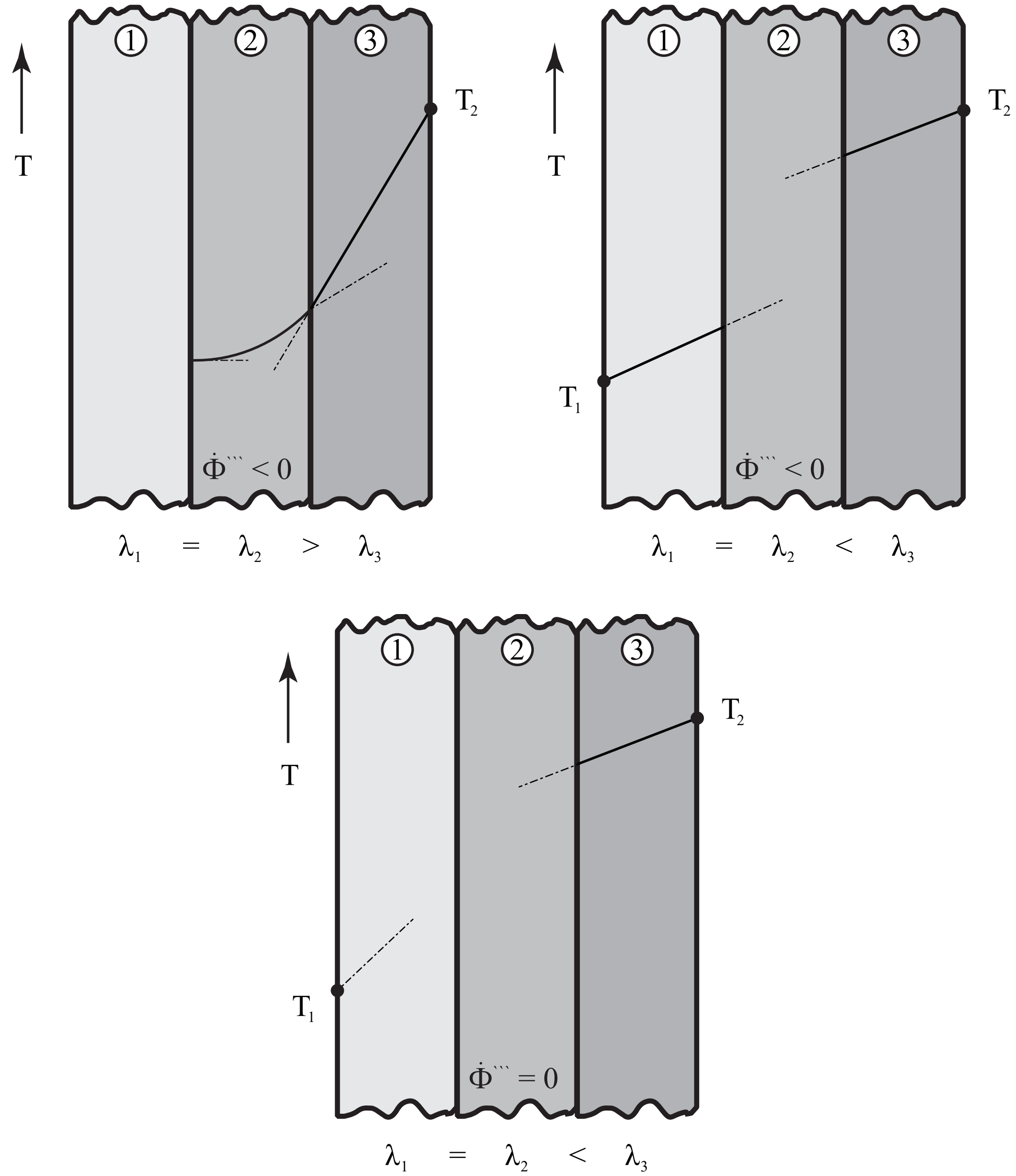
- The problem is one-dimensional in radial direction.
- Assume steady-state conditions.

Tasks:

a) Derive the heat conduction equation by setting up an energy balance.

b) Determine the temperature at $r_1 = 3.5$ mm.

Exercise II.11: (Multi-layer walls with source ⋆⋆)

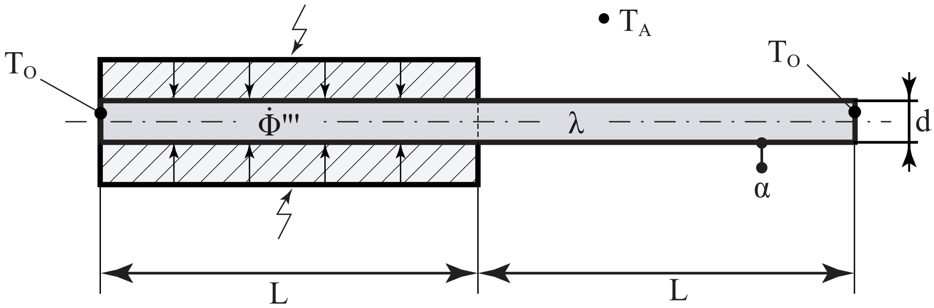


Tasks:

a) Complete the temperature profiles in the three-layered walls.

Exercise II.12: (Copper rod ⋆ ⋆ ⋆)

Both ends of a copper rod with a length L and a diameter d are kept at the same temperature T_O. The left half of the rod is insulated against all radial heat losses. An electric heating element generates Joule's heat of heat flux density $\dot{\Phi}'''$. The right half of the rod is subjected to a flow of the ambient air with an air temperature of T_A, yielding a heat transfer coefficient α. The thermal conductivity of the rod is given as λ.



Given parameters:

- Length of the rod: $L = 1$ m
- Diameter of the rod: $d = 5.2$ mm
- Temperature of the ends of both rods: $T_O = 120$ °C
- Temperature of the ambient: $T_A = 100$ °C
- Heat transfer coefficient: $\alpha = 6$ W/m^{2}K
- Thermal conductivity of the rod: $\lambda = 372$ W/mK

Hint:

- Place the origin of the coordinate system in the middle of the rod.

Tasks:

a) Derive the equation for the temperature profile in the rod by setting up an energy balance.

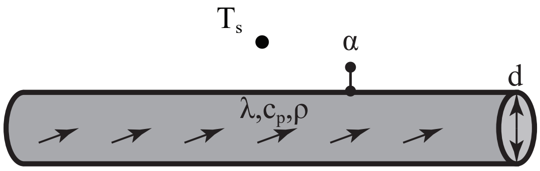
b) Determine an expression for $\dot{\Phi}'''$ such that the temperature in the center of the rod is also T_O, similar to the temperatures at its ends.

c) Calculate the value for $\dot{\Phi}'''$ for the conditions postulated in b).

d) Determine the extremes of the temperature distribution for the given values. Give their position and values, additionally, sketch the temperature profile.

Exercise II.13: (Cooling of a copper rod ⋆⋆)

A long copper rod is initially at a uniform temperature T_0. It is now exposed to an air stream at T_∞ with a heat transfer coefficient α.



Given parameters:

- Diameter of the copper rod: $d = 2$ cm
- Initial temperature: $T_0 = 100$ °C
- Air stream temperature: $T_\infty = 20$ °C
- Heat transfer coefficient: $\alpha = 200$ W/m^{2}K
- Thermal conductivity of copper: $\lambda = 399$ W/mK
- Specific heat capacity of copper: $c_p = 382$ J/kgK
- Density of copper: $\rho = 8930$ kg/m^3

Hints:

- Heat radiation can be neglected.
- Setup an energy balance.

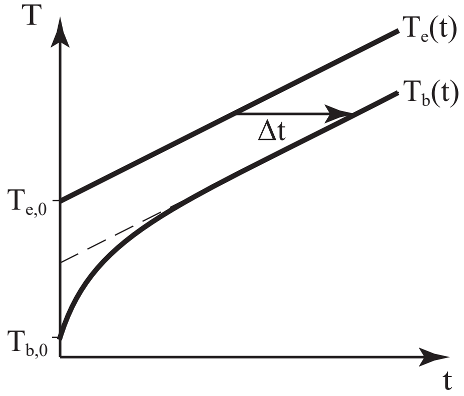
Tasks:

a) Determine how long will it take for the copper rod to cool to a temperature of $T_1 = 25$ °C.

b) Sketch the temperature profile over the course of time.

Exercise II.14: (The temperature delay ⋆⋆)

A body with a temperature of T_b is located within an environment with the linearly rising temperature T_e and heats up accordingly to the diagram below. As $t \to \infty$, the temperature of the body follows that of the environment with a constant time delay Δt.



Given parameters:

- Heat transfer coefficient of the body: α
- Surface of the body: A
- Mass of the body: m
- Heat capacity of the body: c_p
- Temperature of the environment: $T_\mathrm{e}\,(t)$

Hints:

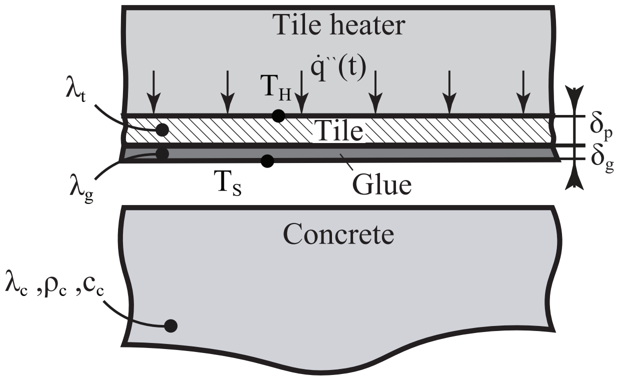
- The temperature is uniform within the body
- The environment, and its temperature, are not affected by the body.
- Heat radiation can be neglected.
- Setup an energy balance.

Tasks:

a) Determine this delay Δt.

Exercise II.15: (Tile setting ⋆⋆)

A tile setter employs a modern technique for tile installation, involving preheating the tile and glue before affixing them to the concrete. The tile and glue are heated until they reach a steady-state condition, achieving a uniform heating temperature T_H and a constant heat flux $\dot{q}''_0$. Once these conditions are met, the tile setter places the heated tile and glue on the concrete, maintaining a constant temperature T_S throughout the process. After reaching a critical temperature T_{crit} at a distance δ_{crit} within the concrete, the heater is removed. Initially, the concrete used to be at a homogeneous temperature T_0



Given parameters:

- Steady-state heat flux: $\dot{q}''_0$ = 7.5 kW/m^2
- Thickness of the pile: δ_p = 10 mm
- Thickness of the glue: δ_g = 2 mm
- Conductivity of the pile: λ_p = 1.0 W/mK
- Conductivity of the glue: λ_g = 0.35 W/mK
- Conductivity of the concrete: λ_c = 2.3 W/mK
- Heat capacity of the concrete: c_c = 1,000 J/kgK
- Density of the concrete: ρ_c = 2,400 kg/m^3
- Initial temperature of the concrete: T_0 = 20 °C
- Heating temperature of the tile heater: T_H = 200 °C
- Critical temperature: T_{crit} = 35 °C
- Critical distance: δ_{crit} = 10 mm

Hints:

- Heat will never penetrate entirely through the concrete.

Tasks:

a) Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the concrete. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.

b) Determine the time t_{crit} at which the heater can be removed.

c) Illustrate the concrete's temperature profile that depicts both temporal and spatial variations.

Exercise II.16: (Heating and quenching of a sphere ⋆⋆⋆)

A sphere, initially at a homogeneous temperature of T_0, is put into an oven. The oven temperature remains constant at a homogeneous temperature of T_o.

Given parameters:

- Initial temperature of the sphere: $T_0 = 25\ ^\circ\mathrm{C}$
- Intermediate temperature of the sphere: $T_h = 150\ ^\circ\mathrm{C}$
- Oven temperature: $T_o = 200\ ^\circ\mathrm{C}$
- Quenching temperature: $T_q = 30\ ^\circ\mathrm{C}$
- Heat transfer coefficient: $\alpha = 110\ \mathrm{W/m^2K}$
- Radius of the sphere: $r_1 = 1.5\ \mathrm{cm}$
- Thermal conductivity of the sphere: $\lambda = 1.52\ \mathrm{W/mK}$
- Density the sphere: $\rho = 1.45 \cdot 10^3\ \mathrm{kg/m^3}$
- Specific heat capacity the sphere: $c_p = 0.88\ \mathrm{kJ/kg \cdot K}$

Hints:

- Heat radiation can be neglected.
- It always remains that $\mathrm{Fo} > 0.2$.

Tasks:

a) Derive the differential equation and establish the boundary and/or initial conditions to determine the temperature profile of the sphere. Based on your findings, identify the method that can be employed to determine the temperature at a particular position and time.

b) Determine the temperature of the center of the sphere after 3 minutes.

After some time the sphere has a hot homogeneous temperature T_h and is being quenched. During this process, the quenching temperature is constant at T_q. Further, in time, the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

c) Determine the time instant when the center of the sphere has a temperature of 54 °C and the surface has a temperature of 44.4 °C.

d) Determine the amount of heat dissipated at this time instant.