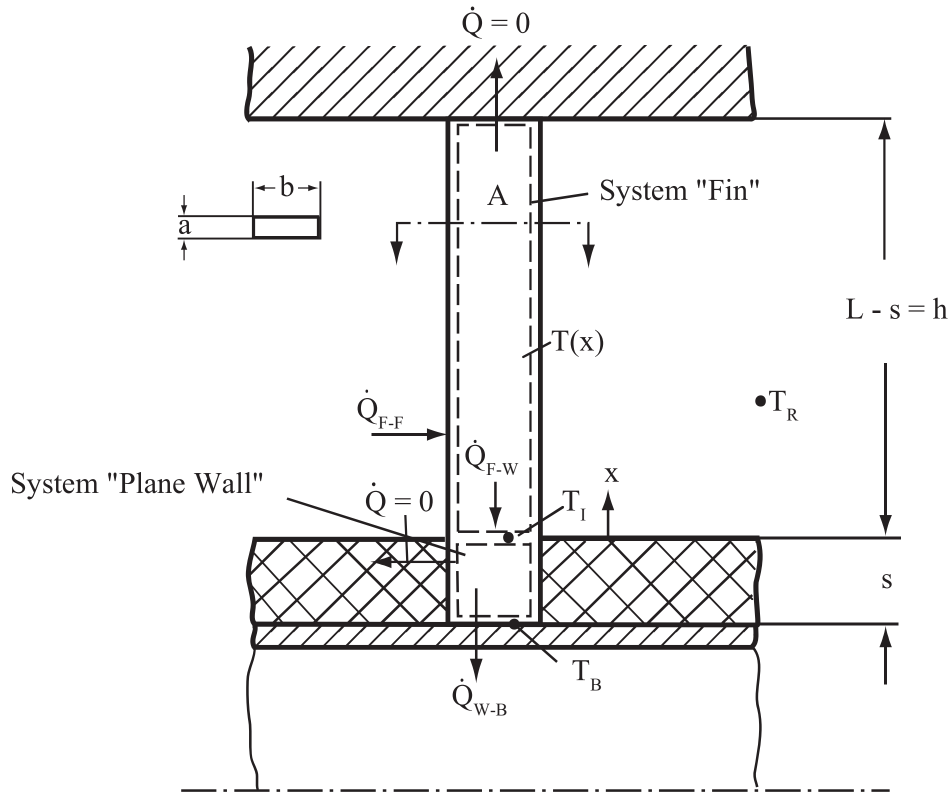
2.8 Pipe fastening (Alternative solution)

1. Problem type

   Steady-state heat transfer through a prismatic fin cross-section with heat transfer through a plane wall.

2. System boundary



   Due to the different transfer mechanisms two systems need to be investigated. These systems are coupled through the fin base temperature.

3. Heat balances

System **Fin**

$$+\dot{Q}_{\mathrm{F-F}} - \dot{Q}_{\mathrm{F-W}} = 0 \qquad \dot{Q}_{\mathrm{F-F}} = \dot{Q}_{\mathrm{F-W}} \tag{2.87}$$

System **Plane wall**

$$+\dot{Q}_{\mathrm{F-W}} - \dot{Q}_{\mathrm{W-B}} = 0 \qquad \dot{Q}_{\mathrm{F-W}} = \dot{Q}_{\mathrm{W-B}} \tag{2.88}$$

4. Description of balance quantities

Heat flux through the **fin** under the assumption that the fin's head is adiabatic

$$\dot{Q}_{\mathrm{F-W}} = \lambda A_{\mathrm{c}} m \tanh\left(m\left(L-s\right)\right) \tag{2.89}$$

with

$$m = \sqrt{\frac{\alpha \cdot U}{\lambda \cdot A}} = \sqrt{\frac{\alpha \cdot 2\left(a+b\right)}{\lambda \cdot a \cdot b}} \tag{2.90}$$

$$\theta = T_{\mathrm{R}} - T_{\mathrm{I}} \tag{2.91}$$

Heat flux through the **Plane Wall**

$$\dot{Q}_{\mathrm{W-B}} = \frac{\lambda}{s} \cdot A_{\mathrm{c}} \cdot \left(T_{\mathrm{I}} - T_{\mathrm{B}}\right) \tag{2.92}$$

5. Solution

- Heat flux into the brine

  With $\dot{Q}_{\mathrm{F-W}} = \dot{Q}_{\mathrm{W-B}}$ follows

$$\lambda A_{\mathrm{c}} m \left(T_{\mathrm{R}} - T_{\mathrm{I}}\right) \tanh\left(m\left(L-s\right)\right) = \frac{\lambda}{s} \cdot A_{\mathrm{c}} \cdot \left(T_{\mathrm{I}} - T_{\mathrm{B}}\right) \tag{2.93}$$

  Rearranging results in:

$$T_{\mathrm{I}} = \frac{T_{\mathrm{R}} s m \tanh\left(m\left(L-s\right)\right) + T_{\mathrm{B}}}{s m \tanh\left(m\left(L-s\right)\right) + 1} = -15.013\ ^{\circ}\mathrm{C} \tag{2.94}$$

- Numerical values:

$$m = 6.54\ {}^{1}/_{\mathrm{m}} \tag{2.95}$$

  Results in:

$$\dot{Q} = \frac{\lambda}{s} \cdot A_{\mathrm{c}} \cdot \left(T_{\mathrm{I}} - T_{\mathrm{B}}\right) = 1.85\ \mathrm{W} \tag{2.96}$$

- Frost covered length of steel fastening h_0

  Conditions for h_0:

$$T(x = h_0) = T_0 = 0\,^\circ\mathrm{C} \tag{2.97}$$

  The determination of h_0 thus necessitates knowledge of the temperature profile within the fin.
  For the given conditions it reads

$$\frac{T(x) - T_\mathrm{R}}{T_\mathrm{I} - T_\mathrm{R}} = \frac{\cosh(m(h-x))}{\cosh(m \cdot h)} \tag{2.98}$$

  With aforementioned condition follows

$$\cosh(m(h - h_0)) = \frac{T_0 - T_\mathrm{R}}{T_\mathrm{I} - T_\mathrm{R}} \cdot \cosh(m \cdot h) = z \tag{2.99}$$

$$\begin{aligned} h_0 &= h - \frac{1}{m} \cdot (z) \\ &= h - \frac{1}{m} \cdot \ln\left(z + \sqrt{z^2 - 1}\right) \end{aligned} \tag{2.100}$$

  the fin base temperature T_B remains unknown in this equation. It is easily obtained from eq. 2.92:

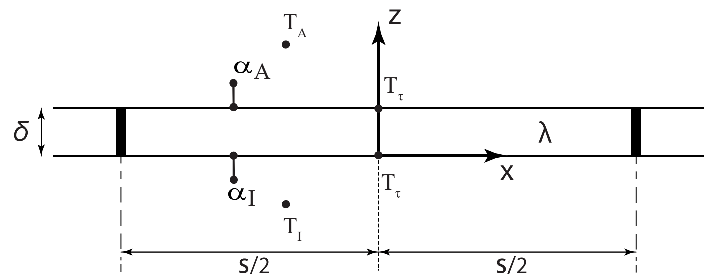
$$T_\mathrm{I} = T_\mathrm{B} + \frac{\dot{Q} \cdot s}{\lambda \cdot A} \tag{2.101}$$

- Numerical values:

$$\begin{aligned} z &= \frac{-20}{-35} \cdot \cosh(1.635) = 1.52 \\ h_0 &= 0.25 - \frac{1}{6.54} \ln(2.665) = 0.1\,\mathrm{m} \\ h_0 &= 10\,\mathrm{cm} \end{aligned}$$

2.9 Foggy rear window

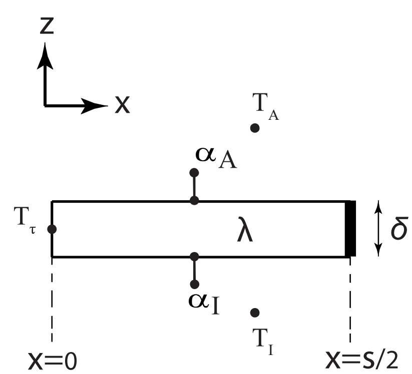
Problem type:
Steady-state one-dimensional heat transfer through a fin in x-direction.



Note that the assumption for one-dimensional heat transfer can be verified by determining the Biot number in z-direction.

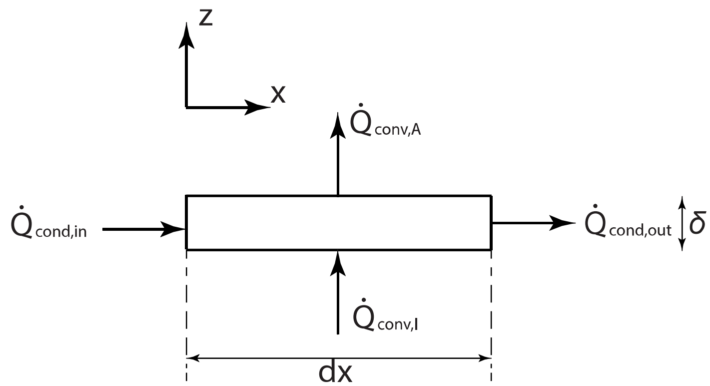
$$\mathrm{Bi_z} = \frac{R_\mathrm{cond}}{R_\mathrm{conv}} = \frac{\delta/\lambda}{1/\alpha_A + 1/\alpha_I} = \frac{\frac{0.005\ [\mathrm{m}]}{1.16\ [\mathrm{W/mK}]}}{\frac{1}{30\ [\mathrm{W/m^2K}]} + \frac{1}{3\ [\mathrm{W/m^2K}]}} = 0.012 << 1 \tag{2.102}$$

The given problem can be simplified to be a fin problem:



In order to determine the required power $\dot{q}'$, the rate of heat transfer in x-direction should be determined using Fourier's law.

Energy balance of an infinitesimal element:



$$\dot{Q}_{\text{cond,in}} + \dot{Q}_{\text{conv,I}} - \dot{Q}_{\text{cond,out}} - \dot{Q}_{\text{conv,A}} = 0 \tag{2.103}$$

$$\Rightarrow \lambda \cdot \delta \cdot \frac{d^2T}{dx^2} - [\alpha_{\text{A}} \cdot (T - T_{\text{A}}) + \alpha_{\text{I}} \cdot (T - T_{\text{I}})] = 0 \tag{2.104}$$

Rearranging:

$$\lambda \cdot \delta \cdot \frac{d^2T}{dx^2} - \left[T \cdot (\alpha_{\text{I}} + \alpha_{\text{A}}) - (\alpha_{\text{I}} \cdot T_{\text{I}} + \alpha_{\text{A}} \cdot T_{\text{A}}) \cdot \frac{(\alpha_{\text{I}} + \alpha_{\text{A}})}{(\alpha_{\text{I}} + \alpha_{\text{A}})}\right] = 0 \tag{2.105}$$

Homogenization parameters:

$$\alpha^* = (\alpha_{\text{I}} + \alpha_{\text{A}}) \tag{2.106}$$

$$T^* = \frac{(\alpha_{\text{I}} \cdot T_{\text{I}} + \alpha_{\text{A}} \cdot T_{\text{A}})}{(\alpha_{\text{I}} + \alpha_{\text{A}})} \tag{2.107}$$

$$\theta = T - T^* \tag{2.108}$$

$$m^2 = \frac{\alpha^*}{\lambda \cdot \delta} \tag{2.109}$$

Results in:

$$\lambda \cdot \delta \cdot \frac{d^2T}{dx^2} - \alpha^* \cdot (T - T^*) \tag{2.110}$$

$$\lambda \cdot \delta \cdot \frac{d^2\theta}{dx^2} - \alpha^* \cdot \theta = 0 \tag{2.111}$$

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \tag{2.112}$$

$$\rightarrow \theta\,(x) = A \cdot \cosh\,(m \cdot x) + B \cdot \sinh\,(m \cdot x) \tag{2.113}$$

Boundary conditions:

At x = 0 (specified temperature):

$$T\left(x=0\right)=T_{\tau}\rightarrow\theta\left(x=0\right)=T_{\tau}-T^{*} \tag{2.114}$$

At x = 0 (thermal symmetry):

$$\frac{dT}{dx}|_{x=0}=0\rightarrow\frac{d\theta}{dx}|_{x=0}=0 \tag{2.115}$$

Thus:

$$\theta\left(x=0\right)=T_{\tau}-T^{*}\rightarrow A=T_{\tau}-T^{*} \tag{2.116}$$

$$\frac{d\theta}{dx}|_{x=0}=0\rightarrow B=0 \tag{2.117}$$

Results in:

$$\theta\left(x\right)=\left(T_{\tau}-T^{*}\right)\cdot\cosh\left(m\cdot x\right) \tag{2.118}$$

$$\rightarrow T(x)=\left(T_{\tau}-T^{*}\right)\cdot\cosh\left(m\cdot x\right)+T^{*} \tag{2.119}$$

Determining $\dot{q}'$:

At x = s/2 (specified heat flux)

$$\lambda\cdot\delta\cdot L\cdot\frac{dT}{dx}|_{x=s/2}=\frac{\dot{q}'\cdot L}{2} \tag{2.120}$$

$$\rightarrow\dot{q}'=2\cdot\delta\cdot\lambda\cdot\sqrt{\frac{\alpha_{\mathrm{I}}+\alpha_{\mathrm{A}}}{\lambda\cdot\delta}}\cdot\sinh\left(\sqrt{\frac{\alpha_{\mathrm{I}}+\alpha_{\mathrm{A}}}{\lambda\cdot\delta}}\cdot\frac{s}{2}\right)\cdot\left(T_{\tau}-\frac{\alpha_{\mathrm{I}}\cdot T_{\mathrm{I}}+\alpha_{\mathrm{A}}\cdot T_{\mathrm{A}}}{\alpha_{\mathrm{I}}+\alpha_{\mathrm{A}}}\right)=8\ [\mathrm{W/m}] \tag{2.121}$$

2.10 New fin material

1. Problem type:
   Steady-state, one-dimensional heat conduction through a fin with an adiabatic head.

a) Calculate the ratio between the heat flow of the aluminium and copper fin.

Aluminium fin:

Heat flux:

$$\dot{Q}_{\mathrm{A}} = \lambda_{\mathrm{A}} \cdot A_{\mathrm{c}} \cdot m_{\mathrm{A}} \cdot \theta_{\mathrm{B}} \cdot \tanh\left(m_{\mathrm{A}} \cdot L_{\mathrm{A}}\right) \tag{2.122}$$

Fin parameter (plane fin):

$$m_{\mathrm{A}} = \sqrt{\frac{2 \cdot \alpha}{\lambda_{\mathrm{A}} \cdot \delta}} \tag{2.123}$$

Temperature profile:

$$\theta_{\mathrm{A}}(x) = \theta_{\mathrm{B}} \cdot \frac{\cosh\left[m_{\mathrm{A}} \cdot (L_{\mathrm{A}} - x)\right]}{\cosh\left[m_{\mathrm{A}} \cdot L_{\mathrm{A}}\right]} \tag{2.124}$$

Head temperature:

$$\Rightarrow \theta_{\mathrm{A}}(L) = \frac{\theta_{\mathrm{B}}}{\cosh\left[m_{\mathrm{A}} \cdot L_{\mathrm{A}}\right]} \tag{2.125}$$

Copper fin:

Heat flux:

$$\dot{Q}_{\mathrm{C}} = \lambda_{\mathrm{C}} \cdot A_{\mathrm{c}} \cdot m_{\mathrm{C}} \cdot \theta_{\mathrm{B}} \cdot \tanh\left(m_{\mathrm{C}} \cdot L_{\mathrm{C}}\right) \tag{2.126}$$

Fin parameter (plane fin):

$$m_{\mathrm{c}} = \sqrt{\frac{2 \cdot \alpha}{\lambda_{\mathrm{c}} \cdot \delta}} \tag{2.127}$$

Temperature profile:

$$\theta_{\mathrm{C}}(x) = \theta_{\mathrm{B}} \cdot \frac{\cosh\left[m_{\mathrm{C}} \cdot (L_{\mathrm{C}} - x)\right]}{\cosh\left[m_{\mathrm{C}} \cdot L_{\mathrm{C}}\right]} \tag{2.128}$$

Head temperature:

$$\Rightarrow \theta_{\mathrm{C}}(L) = \frac{\theta_{\mathrm{B}}}{\cosh\left[m_{\mathrm{C}} \cdot L_{\mathrm{C}}\right]} \tag{2.129}$$

Length ratio:

Head temperatures remains unchanged:

$$\theta_{\mathrm{A}}(L) = \theta_{\mathrm{C}}(L) \tag{2.130}$$

$$\Rightarrow m_{\mathrm{A}} \cdot L_{\mathrm{A}} = m_{\mathrm{C}} \cdot L_{\mathrm{C}} \tag{2.131}$$

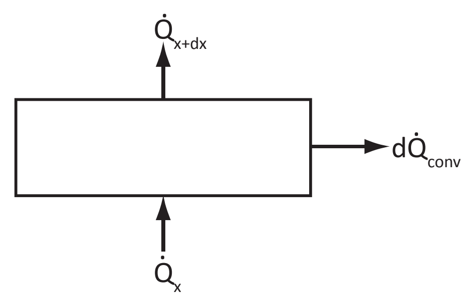
Heat flux ratio:

$$\Rightarrow \frac{\dot{Q}_{\mathrm{A}}}{\dot{Q}_{\mathrm{C}}} = \frac{\lambda_{\mathrm{A}} \cdot A_{\mathrm{c}} \cdot \sqrt{\frac{2 \cdot \alpha}{\lambda_{\mathrm{A}} \cdot \delta}} \cdot \theta_{\mathrm{B}} \cdot \tanh\left(m_{\mathrm{A}} \cdot L_{\mathrm{A}}\right)}{\lambda_{\mathrm{C}} \cdot A_{\mathrm{c}} \cdot \sqrt{\frac{2 \cdot \alpha}{\lambda_{\mathrm{c}} \cdot \delta}} \cdot \theta_{\mathrm{B}} \cdot \tanh\left(m_{\mathrm{C}} \cdot L_{\mathrm{C}}\right)} = \frac{\lambda_A^{1/2}}{\lambda_C^{1/2}} \tag{2.132}$$

2.11 Spherical fin

1. Problem type:
   Steady-state, one-dimensional heat conduction through a fin subjected to convection.

a) Derive the differential equation of the temperature in this fin as a function of x.



The balance at the fin element reads as follows:

$$\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} = d\dot{Q}_{\mathrm{conv}} \tag{2.133}$$

The left-hand side of the balance can be described as follows:

$$\dot{Q}_{\mathrm{x}} = -\lambda \cdot A_{\mathrm{Q}} \cdot \frac{dT}{dx} \tag{2.134}$$

$$\dot{Q}_{\mathrm{x+dx}} = \dot{Q}_{\mathrm{x}} + \frac{d(\dot{Q}_{\mathrm{x}})}{dx} \cdot dx \tag{2.135}$$

$$\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} = \lambda \cdot \frac{d}{dx}\left(A_{\mathrm{Q}} \cdot \frac{dT}{dx}\right) \tag{2.136}$$

The right-hand side of the balance can be described as follows:

$$d\dot{Q}_{\mathrm{conv}} = \alpha \cdot C \cdot dx \cdot (T - T_{\mathrm{amb}}) \tag{2.137}$$

The radius r as a function of x can be described using Pythagoras' theorem:

$$r(x) = \sqrt{R^2 - x^2} \tag{2.138}$$

The area A_Q can be described as follows:

$$A_\mathrm{Q} = \pi \cdot r(x)^2 = \pi \cdot (R^2 - x^2) \tag{2.139}$$

The circumference C can be described as follows:

$$C = 2 \cdot \pi \cdot r(x) = 2 \cdot \pi \cdot \sqrt{(R^2 - x^2)} \tag{2.140}$$

Plugging in all the terms yields the differential equation:

$$\lambda \cdot \frac{d}{dx}\left((R^2 - x^2) \cdot \frac{dT}{dx}\right) = 2 \cdot \alpha \cdot \sqrt{(R^2 - x^2)} \cdot (T - T_\mathrm{amb}) \tag{2.141}$$

2.12 Rod fin

1. Problem type:
   Steady-state, one-dimensional heat conduction through a fin subjected to convection.

a) Give the expression for the heat flux as a function of the given variables.

The following equation for the fin temperature can be taken from the formulary:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x) \tag{2.142}$$

with

$$\theta(x) = T(x) - T_{\text{amb}} \tag{2.143}$$

and

$$m^2 = \frac{\alpha \cdot U}{\lambda \cdot A_{\text{Q}}} \tag{2.144}$$

The constants A and B should be determined using the boundary conditions.

The temperature at the fin base is T_{b}:

$$\theta(x = 0) = A = T_{\text{b}} - T_{\text{amb}} = \theta_{\text{b}} \tag{2.145}$$

The temperature at the fin head is T_{amb}:

$$\theta(x = L) = A \cdot \cosh(m \cdot L) + B \cdot \sinh(m \cdot L) = T_{\text{amb}} - T_{\text{amb}} = 0 \tag{2.146}$$

$$\Rightarrow B = -A \cdot \frac{1}{\tanh(m \cdot L)} \tag{2.147}$$

Plugging in the constants yields the equation for the fin temperature:

$$\theta(x) = \theta_{\text{b}} \cdot \left(\cosh(m \cdot x) - \frac{\sinh(m \cdot x)}{\tanh(m \cdot L)} \right) \tag{2.148}$$

Dissipated heat through the fin:

$$\dot{Q} = -\lambda \cdot A_{\text{C}} \cdot \frac{d\theta}{dx}|_{x=0} \tag{2.149}$$

with

$$\frac{d\theta}{dx}|_{x=0} = \theta_{\text{b}} \cdot \left(m \cdot \sinh(m \cdot 0) - \frac{m \cdot \cosh(m \cdot 0)}{\tanh(m \cdot L)} \right) = -\frac{\theta_{\text{b}} \cdot m}{\tanh(m \cdot L)} \tag{2.150}$$