Example Examination
Fluid Mechanics I
Module 7, 201500321, 2015

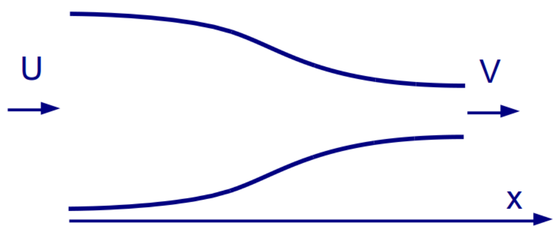
Problem 1 [2 points.]
Given is the flow field

$$u(x,y,z,t) = \frac{1+t}{e^x}, \quad v(x,y,z,t) = \frac{sin(t)}{y^2}, \quad w(x,y,z,t) = zt, \tag{1}$$

where u, v and w are the velocity components in x, y and z direction respectively.

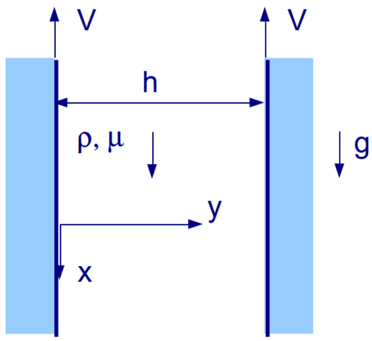
- *Compute the trajectory $x_p(t)$, $y_p(t)$, $z_p(t)$ of a dust particle that is located in the point $x = 0$, $y = 1$, $z = 1$ at $t = 0$.*

Problem 2 [2 points.]
An incompressible fluid with density ρ enters a construction through an opening of area size A at uniform velocity U and uniform pressure p_{in}, and leaves the construction through an opening of area size $\frac{1}{2}A$ at uniform velocity $V = 2U$ and uniform pressure p_{out}. Viscosity and gravity can be neglected and the flow is steady.



- *Compute the force in x-direction by the fluid on the pipe.*

Problem 3 [2 POINTS.]
Incompressible viscous water with density ρ falls steadily between two vertical parallel plates, both plates move upward with speed V. The flow is laminar and fully developed. The gap width between the plates is h, gravity acceleration is g, the viscosity is μ and the pressure is constant: p_o.



(a) *Derive an expression for the velocity profile $u(y)$, starting from the reduced Navier-Stokes equations, with $y = 0$ on the left plate.*
(b) *Derive an expression for the shear stress by the flow on the left plate.*

Problem 4 [1 POINT.]
Consider the following temperature distribution:

$$T(x,t) = [a\cos(\lambda x) - b\sin(\lambda x)]\exp(-\beta t).$$

Determine the parameter β such that $T(x,t)$ satisfies the following diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}.$$

Problem 5 [2 POINTS.]
In the steady, inviscid and compressible flow around an object of an ideal (perfect) gas with given γ and gas constant R, and given temperature T_∞ and Mach number M_∞ at infinity, the velocity is measured at some point not far from the object: u.

(a) *Compute the temperature T at that point.*
(b) *Show that the total enthalpy is constant along streamlines.*

Appendix A

Formulas available during the Exam

A.1 Introduction

A.1.1 Particle trajecories

$$\frac{dx_p}{dt} = u(x_p(t), y_p(t), z_p(t), t), \quad \frac{dy_p}{dt} = v(x_p(t), y_p(t), z_p(t), t), \quad \frac{dz_p}{dt} = w(x_p(t), y_p(t), z_p(t), t). \tag{A.1}$$

A.1.2 Streamlines

$$\frac{dx_p}{ds} = u(x_p(s), y_p(s), z_p(s), t), \quad \frac{dy_p}{ds} = v(x_p(s), y_p(s), z_p(s), t), \quad \frac{dz_p}{ds} = w(x_p(s), y_p(s), z_p(s), t). \tag{A.2}$$

A.2 Mass Conservation

A.2.1 Integral form

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{A(t)} \rho\,(u_j n_j)\, dA = 0. \tag{A.3}$$

A.2.2 Differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{A.4}$$

A.3 Momentum Conservation

A.3.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{A(t)} \rho u_i\,(u_j n_j)\, dA = \int_{A(t)} \sigma_{ij} n_j\, dA + \int_{V(t)} \rho g_i\, dV, \quad i = 1, 2, 3. \tag{A.5}$$

A.3.2 Stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{A.6}$$

$$\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{A.7}$$

A.3.3 Cauchy equation

Tension vector $\boldsymbol{t}$ by medium A on medium B, $\boldsymbol{n}$ pointing to A:

$$t_i = \sigma_{ij} n_j, \quad i = 1, 2, 3. \tag{A.8}$$

A.3.4 Differential form (Navier-Stokes)

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j - \sigma_{ij}) = \rho g_i. \tag{A.9}$$

A.3.5 Reduced Navier-Stokes

$$\frac{\partial p}{\partial x} - \mu\frac{\partial^2 u}{\partial y^2} = \rho g_1, \quad \frac{\partial p}{\partial y} = \rho g_2. \tag{A.10}$$

A.3.6 Material derivative

Time derivative while traveling with the flow:

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_j\frac{\partial f}{\partial x_j}, \quad \text{for any function } f(x, y, z, t) \tag{A.11}$$

Exam

Fluid Mechanics I

(WB module-7 201700127)

- Electronic devices are not allowed to be used or to be present on your desk during the exam (this includes cell-phones and calculators).

- You can not use a red pencil or pen

Problem 1 [2 points.]

The drag force, F, experienced by a submarine that moves at a great depth from the surface of the water, is a function of the density ρ, viscosity μ, speed V and the transversal area of the submarine A. An expert suggests that the nondimensional function $\tilde{f}$ that allows the calculation of F is: $\tilde{f}(\frac{\rho V A}{\mu}) = \frac{F}{2\rho V A}$.

(a) *Is this expression correct? Why or why not?*

(b) *If not, correct it.*

Problem 2 [3 points.]

An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h$ and width w. The x-axis points in the streamwise direction, the y-axis points in the height direction, with $y = 0$ in the middle of the channel. At the entrance the presure is p_1 and the uniform velocity is U. At the exit the pressure is p_2 and the velocity distribution is:

$$\frac{u}{V} = 1 - \left(\frac{y}{h}\right)^2. \tag{1}$$

(a) *Derive an expression for V.*

(b) *Derive an expression for the force by the fluid on the walls in x-direction, neglecting gravity, and neglecting viscosity at entrance and exit.*

Hint: $\int_{-h}^{h} \left(1 - \left(\frac{y}{h}\right)^2\right)^2 dy = \frac{16}{15}h$

Problem 3 [3 POINTS.]

An incompressible viscous fluid flows steadily between two parallel plates. The bottom plate is stationary and the top plate moves to the right with velocity V. The flow is laminar and fully developed. The total gap width between the plates is h. The x-axis points downstream and the y-axis starts at the bottom plate. Gravity can be neglected, the viscosity is μ and the given pressure derivative is constant: $\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_o$.

(a) Derive an expression for the velocity field.

(b) Derive an expression for the shear stress on the top plate.

(c) Compute the value of V when the shear stress on the top plate is zero and make a sketch of the velocity field for that case.

Problem 4 [2 POINTS.]

Consider steady parallel flow of a perfect gas (with gas constant R and ratio of specific heats γ) around a sphere and neglect viscosity, heat conduction, and gravity. The temperature and Mach number far uptream are given: T_∞ and M_∞, respectively.

(a) The temperature of the gas at the top of the sphere, T_1, is measured. Compute the Mach number at the top of the sphere.

(b) Derive the value of the velocity at the bottom of the sphere.

Appendix A

Formulas available during the Exam

A.1 Fluid kinematics, particle trajectories

$$\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}\left(\boldsymbol{x}_p(t), t\right) \tag{A.1}$$

A.2 Mass Conservation

A.2.1 Integral form

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{A(t)} \rho\left(u_j n_j\right) dA = 0. \tag{A.2}$$

A.2.2 Differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{A.3}$$

A.3 Momentum Conservation

A.3.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{A(t)} \rho u_i\left(u_j n_j\right) dA = \int_{A(t)} \sigma_{ij} n_j\, dA + \int_{V(t)} \rho g_i\, dV, \quad i = 1, 2, 3. \tag{A.4}$$

A.3.2 Stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{A.5}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{A.6}$$

A.3.3 Cauchy equation

Tension vector $\boldsymbol{t}$ by medium A on medium B, $\boldsymbol{n}$ pointing to A:

$$t_i = \sigma_{ij} n_j, \quad i = 1, 2, 3. \tag{A.7}$$

A.3.4 Differential form (Navier-Stokes)

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j - \sigma_{ij}\right) = \rho g_i. \tag{A.8}$$

A.3.5 Reduced Navier-Stokes

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = \rho g_1, \quad \frac{\partial p}{\partial y} = \rho g_2. \tag{A.9}$$

A.3.6 Euler equations

Momentum conservation with $\mu = 0$:

$$\rho \frac{D u_i}{D t} = -\frac{\partial p}{\partial x_i} + \rho g_i, \quad i = 1, 2, 3. \tag{A.10}$$

A.3.7 Material derivative

Time derivative while traveling with the flow:

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j}, \quad \text{for any function } f(x, y, z, t) \tag{A.11}$$

A.3.8 Bernoulli equation

(toepassings-voorwaarden zelf onthouden!)

$$p + \frac{1}{2}\rho V^2 + \rho g \zeta = \text{constant along streamlines}, \quad V^2 = u^2 + v^2 + w^2. \tag{A.12}$$

A.4 Energy Conservation

A.4.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho E)\, dV + \int_{A(t)} \left(\rho E u_j n_j - \sigma_{ij} u_i n_j + q_j n_j\right) dA = \int_{V(t)} \rho g_j u_j \, dV, \quad i = 1, 2, 3. \tag{A.13}$$

Total energy:

$$E \equiv e + \frac{1}{2}U^2, \quad U^2 = u^2 + v^2 + w^2. \tag{A.14}$$

Enthalpy and total enthalpy:

$$h \equiv e + \frac{p}{\rho}, \quad H \equiv E + \frac{p}{\rho} \tag{A.15}$$

Thermodynamics of a perfect gas:

$$p = \rho R T, \quad e = C_v T, \quad C_p - C_v = R, \quad \gamma \equiv C_p / C_v \tag{A.16}$$

Speed of sound and Mach number:

$$a = \sqrt{\gamma R T}, \quad M \equiv U/a. \tag{A.17}$$

A.4.2 Fourier's law (heat flux)

$$q_i = -k \frac{\partial T}{\partial x_i}, \quad i = 1, 2, 3. \tag{A.18}$$

A.4.3 Compressor equation

$$\dot{m} \left(H_2 - H_1 \right) = P + \dot{Q}. \tag{A.19}$$

A.4.4 Differential form

$$\frac{\partial}{\partial t} \rho E + \frac{\partial}{\partial x_j} \left(\rho u_j E - \sigma_{ij} u_i + q_j \right) = \rho g_j u_j. \tag{A.20}$$

A.4.5 Total temperature, -pressure and -density

$$T_t = T(1 + \frac{\gamma - 1}{2} M^2), \quad p_t = p(1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}}, \quad \rho_t = \rho(1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma - 1}}. \tag{A.21}$$

A.5 Convection and diffusion

A.5.1 Convection equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0. \tag{A.22}$$

A.5.2 Diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{A.23}$$

A.5.3 Convection-diffusion equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{A.24}$$

Exam
Fluid Mechanics I
(WB module-7 201700127)

- Electronic devices are not allowed to be used or to be present on your desk during the exam (this includes cell-phones and calculators).
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Problem 1 [2 points.] *The ideal flow around a corner placed at the origin is given by*

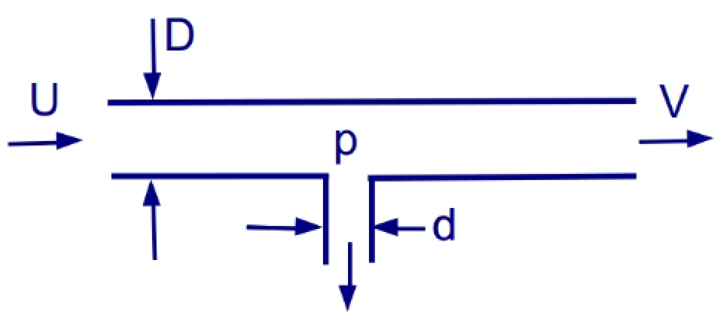
$$u = ax, \quad v = -ay,$$

with $a > 0$ a constant.

(a) Determine the trajectories and draw the trajectory that passes the point (x_o, y_o) at time $t = 0$ and indicate the flow direction.
(b) Calculate the material derivative of the velocity vector.

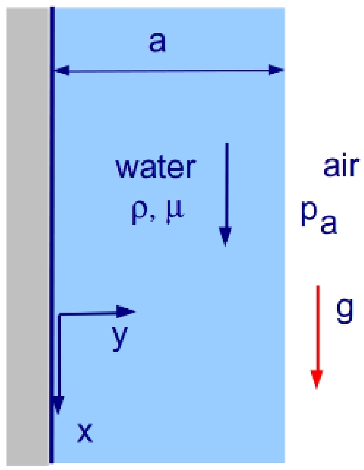
Problem 2 [3 points.] *An incompressible fluid flows steadily into a T-junction of diameter D at uniform velocity U, at the opposite outlet the fluid leaves at uniform velocity V. At the lateral exit the flow leaves at unknown uniform velocity W. The pressure in the T-junction is uniform: p. Neglect viscosity and gravity.*

(a) Compute the unknown velocity W.
(b) Compute the force (in all directions) by the fluid on the pipe.
(c) Compute the mass flow rate through the vertical exit.



Problem 3 [3 POINTS.] *Consider the steadily falling water film along a vertical wall with thickness a. The flow is incompressible, laminar, and fully developed. At the wall the velocity is zero, whereas at the outer edge of the film the shear stress is zero.*

(a) Defend the approximation assumption of zero shear stress at the film boundary.
(b) Derive an expression for $\frac{\partial p}{\partial x}$.
(c) Derive an expression for $u(y)$.



Problem 4 [2 POINTS.] *For steady flow with $\mu = 0$ and $k = 0$ explain that the pressure along a streamline can not exceed the total pressure along that streamline.*

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$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}. \tag{A.24}$$

PLEASE READ THE FOLLOWING PARAGRAPH CAREFULLY

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

Copy the text below (handwritten) to your answer-sheet:
I have made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

ALLOWED/NOT ALLOWED SOURCES

(a) Allowed are: the reader and the tutorial-answers.
(b) You are allowed to write on a tablet as long as it is readable.
(c) Internet is not allowed, except for handing in the answers.
(d) Communication with other human beings is not allowed, in any form.

HAND-IN

(a) Make photo's of your work, put them to a single PDF-file.
(example: JPG-pictures put into a Word document, saved as a PDF-file)
(b) Use the following format: s1234567_emailaddress.pdf, where

s1234567 = your student number
emailaddress = your student emailadres without @utwente.student.nl

(c) Then submit the PDF-file in CANVAS, Module 7, Assignments. Your deadline is 12.30.

DEADLINES

(a) Students without dispensation: the deadline is 12.30.
(b) Students with dispensation: you have 15 more minutes, the deadline is 12.45. Please include a photograph of your proof of dispensation.

(Note: the exam-duration is actually 3 hours. The reason that all students have 45 minutes extra time is to prevent CANVAS from getting overloaded due to submission of 250 PDF's. Students with dispensation have 15 minutes extra.

CONFERENCE

A conference will run in parallel to enable students to ask questions about the assignments. **Only** when you have a question you log in and **type** your question. Never log in when another student is already logged in.

EXAMINATION
FLUID MECHANICS I
(WB MODULE-7 201700127)
7 APRIL 2020: 8.45 - 12.30 (OR 12.45, SEE PAGE -1-)
(you are not allowed use a red pencil or pen)

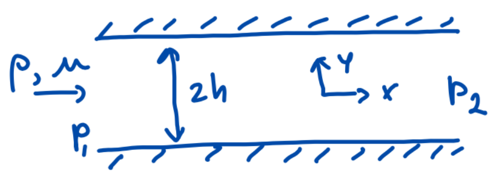
Problem 1 [2 POINTS.]
A two-dimensional velocity field is given by $u = -by$ and $v = bx$, where x and y are the cartesian coordinates.

(a) *Is this field representing the flow of an incompressible fluid?*
(b) *Compute $\frac{Du}{Dt}$ as a function of x and y.*
(c) *Compute $\frac{d}{dt}\left(r_p^2\right)$, with $r_p(t)^2 \equiv x_p(t)^2 + y_p(t)^2$, and $x_p(t), y_p(t)$ the coordinates of a particle moving with the flow.*

$$\textit{Hint:} \quad x_p\frac{dx_p}{dt} + y_p\frac{dy_p}{dt} = \frac{1}{2}\frac{dx_p^2}{dt} + \frac{1}{2}\frac{dy_p^2}{dt} = \frac{1}{2}\frac{dr_p^2}{dt}.$$

Problem 2 [3 POINTS.]
An incompressible fluid with density ρ and viscosity μ flows steadily through a two-dimensional channel of height $2h$, width w, and length L. The x-axis points in the streamwise direction, the y-axis points in the height direction, with $y = 0$ in the middle of the channel. At the entrance the pressure is p_1 and at the exit the pressure is p_2. Gravity can be neglegted. The flow is fully developed and the velocity distribution is:

$$u(y) = V\left(1 - \left(\frac{y}{h}\right)^2\right), \quad V \textit{ is known.}$$

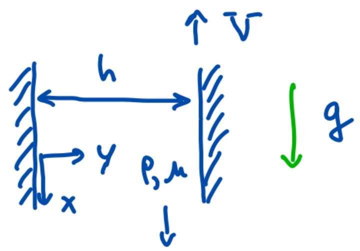


(a) *What is the meaning of V?.*
(b) *Compute the average velocity U as a factor times V.*
(c) *Derive an expression for the force by the fluid on the walls in x-direction by using the integral-formulation of momentum conservation.*

Hint: $\int_{-h}^{h}\left(1 - \left(\frac{y}{h}\right)^2\right)^2 dy = \frac{16}{15}h.$

Problem 3 [3 POINTS.]

An incompressible viscous fluid flows steadily between two vertical parallel plates. The left plate is stationary and the right plate moves upwards with velocity V. The flow is laminar and fully developed. The total gap width between the plates is h, the width (z-direction) is w. The x-axis points downstream and the y-axis starts at the left plate. The fluid viscosity is μ and the pressure deriva tive is zero: $\frac{\partial p}{\partial x} = 0$. Gravity points downward.



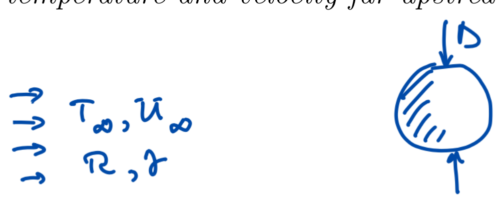
(a) Derive an expression for the velocity field.

(b) Derive an expression for the shear stress by the fluid on the right plate.

(c) Give a formula for the average velocity, the volume flow rate, and the mass flow rate. If your answers contain an integral, you don't have to calculate the integral.

Problem 4 [2 POINTS.]

Consider steady flow of a perfect gas (with gas constant R and ratio of specific heats γ) around a sphere of diameter D, and neglect viscosity, heat conduction, and gravity. The temperature and velocity far upstream are given: T_∞ and U_∞, respectively.



(a) Is mass (kg) one of the physical dimensions that is present in this problem?

(b) Use dimension analysis to derive the dimensionless parameters on which the stagnation temperature T_o depends.

(c) Derive an exact expression for the stagnation temperature T_o and compare it to the answer of (b).

Hint: the unit of R is $\frac{J}{kg\,K} = \frac{m^2}{K\,s^2}$.

Appendix A

Formulas available during the Exam

A.1 Fluid kinematics, particle trajectories

$$\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}\left(\boldsymbol{x}_p(t), t\right) \tag{A.1}$$

A.2 Mass Conservation

A.2.1 Integral form

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{A(t)} \rho\left(u_j n_j\right) dA = 0. \tag{A.2}$$

A.2.2 Differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{A.3}$$

A.3 Momentum Conservation

A.3.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{A(t)} \rho u_i \left(u_j n_j\right) dA = \int_{A(t)} \sigma_{ij} n_j\, dA + \int_{V(t)} \rho g_i\, dV, \quad i = 1, 2, 3. \tag{A.4}$$

A.3.2 Stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{A.5}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{A.6}$$

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Tension vector $\boldsymbol{t}$ by medium A on medium B, $\boldsymbol{n}$ pointing to A:

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$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j - \sigma_{ij}) = \rho g_i. \tag{A.8}$$

A.3.5 Reduced Navier-Stokes

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = \rho g_1, \quad \frac{\partial p}{\partial y} = \rho g_2. \tag{A.9}$$

A.3.6 Euler equations

Momentum conservation with $\mu = 0$:

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i, \quad i = 1, 2, 3. \tag{A.10}$$

A.3.7 Material derivative

Time derivative while traveling with the flow:

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j}, \quad \text{for any function } f(x, y, z, t) \tag{A.11}$$

A.3.8 Bernoulli equation

(you have to remember the validity conditions yourself!)

$$p + \frac{1}{2}\rho V^2 + \rho g \zeta = \text{constant along streamlines}, \quad V^2 = u^2 + v^2 + w^2. \tag{A.12}$$

A.4 Energy Conservation

A.4.1 Integral form

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho E)\, dV + \int_{A(t)} (\rho E u_j n_j - \sigma_{ij} u_i n_j + q_j n_j)\, dA = \int_{V(t)} \rho g_j u_j\, dV, \quad i = 1, 2, 3. \tag{A.13}$$

Total energy:

$$E \equiv e + \frac{1}{2}U^2, \quad U^2 = u^2 + v^2 + w^2. \tag{A.14}$$

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A.4.5 Total temperature, -pressure and -density

$$T_t = T(1 + \frac{\gamma - 1}{2}M^2), \quad p_t = p(1 + \frac{\gamma - 1}{2}M^2)^{\frac{\gamma}{\gamma-1}}, \quad \rho_t = \rho(1 + \frac{\gamma - 1}{2}M^2)^{\frac{1}{\gamma-1}}. \tag{A.21}$$

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A.5.1 Convection equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0. \tag{A.22}$$

A.5.2 Diffusion equation

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}. \tag{A.23}$$

A.5.3 Convection-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}. \tag{A.24}$$

Examination

Fluid Mechanics I

(WB module-7, 202000138)

11 April 2022: 8.45 - 11.45 (12.30)

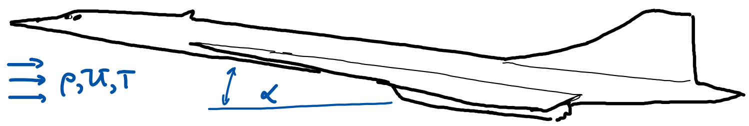
- Electronic devices are not allowed to be used or to be present on your desk during the exam (this includes cell-phones and calculators).

- You are *not* allowed to use a red pencil or pen

- Success!

Problem 1 [2 points.]

The lift force L on the Concorde airplane at high speeds primarily depends on air-density ρ, air-speed U, air-temperature T, angle-of-attack α, and finally on the specific gas-constant of air, R $(J\,kg^{-1}\,K^{-1})$. The size of the airplane is characterized by the wingspan b.

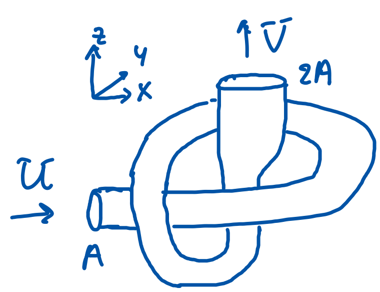


(a) *How many independent physical dimensions are present in this problem?*

(b) *The lift coeficient, $C_L \equiv L/(\frac{1}{2}\rho U^2 b^2)$, depends on two dimensionless parameters. Determine these two paramters.*

Problem 2 [3 points.]

An incompressible fluid flows steadily through the tube depicted in the figure. At the entrance and exit viscosity can be neglected, and the velocity fields and pressures are uniform. At the entrance the fluid enters at speed U in the x-direction, and at the exit it leaves at speed V in the z-direction. The cross-sectional areas of the entrance and exit are A and $2A$, respectively. Gravity can be neglected.



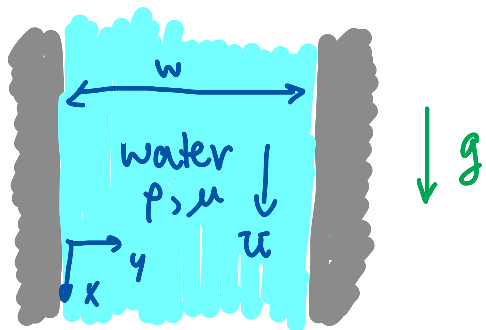
(a) *Give the velocity vectors and outward unit normal vectors at the entrance and at the exit in three components.*

(b) *Compute $u_j n_j$ at the entrance and at the exit.*

(c) *Compute the horizontal force F by the fluid on the tube.*

Problem 3 [3 POINTS.]
Water of constant density ρ and viscosity μ flows steadily and fully developed through a vertical gap of width w under the influence of gravity. Pressure is constant throughout the fluid.

(a) *Determine the velocity field $u(y)$.*
(b) *Compute the average speed of the fluid.*
(c) *Compute the shear stress by the fluid on the right wall.*



Problem 4 [2 POINTS.]
Consider the steady flow of a perfect gas (with gas-constant R and constant ratio of specific heats γ) around a sphere. Neglect viscosity, heat conduction, and gravity. Under these conditions the conservation equations of mass and energy reduce to:

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0, \quad \frac{\partial}{\partial x_j}(\rho u_j H) = 0.$$

The temperature, pressure, and Mach number far uptream are given: T_∞ , p_∞ and M_∞.

(a) *Show that H is constant along streamlines.*
(b) *Compute the density ρ_o in the stagnation point of the sphere.*

Appendix A

Formulas available during the Exam

A.1 Fluid kinematics, particle trajectories

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A.2 Mass Conservation

A.2.1 Integral form

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{A(t)} \rho\left(u_j n_j\right) dA = 0. \tag{A.2}$$

A.2.2 Differential form

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$$\rho \frac{D u_i}{D t} = -\frac{\partial p}{\partial x_i} + \rho g_i, \quad i = 1, 2, 3. \tag{A.10}$$

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Time derivative while traveling with the flow:

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j}, \quad \text{for any function } f(x, y, z, t) \tag{A.11}$$

A.3.8 Bernoulli equation

(toepassings-voorwaarden zelf onthouden!)

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Examination

Fluid Mechanics I

(WB module-7, 202000138)

31 May 2022: 8.45 - 11.45 (12.30)

- Electronic devices or books/notes are not allowed to be present on your desk.
- You are not allowed to write with a red pencil or pen, but you can use one in a drawing.
- Check the dimensions and signs of your answers!

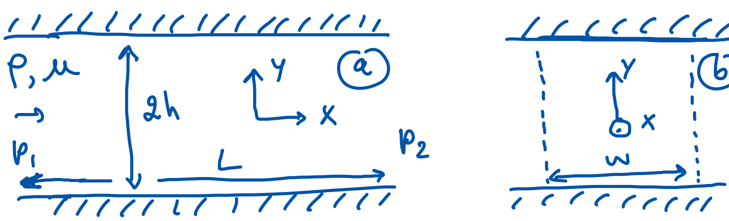
Problem 1 [2 points.]
A two-dimensional velocity field is given by $u = ax^2$ and $v = -2axy$, where a is a positive constant and x and y are the cartesian coordinates. A very small particle follows the fluid exactly along the trajectory $(x_p(t), y_p(t))^T$.

(a) Compute the divergence of the velocity field. What does the answer tell you?

(b) Compute $\frac{Dv}{Dt}$ as a function of x and y.

(c) Compute $\frac{dy_p}{dt}$ and $\frac{d^2y_p}{dt^2}$ as functions of $x_p(t)$ and $y_p(t)$, and explain the relation between $\frac{Dv}{Dt}$ and $\frac{d^2y_p}{dt^2}$.

Problem 2 [3 points.]
An incompressible fluid with density ρ and viscosity μ flows steadily through a two-dimensional channel of height $2h$ and length L, see figure (a). The x-axis points in the streamwise direction, the y-axis points in the height direction, with $y = 0$ in the middle of the channel. At the entrance the pressure is p_1 and at the exit the pressure is p_2. We consider a portion of the channel of width w, see figure (b) where the x-axis points towards the observer. Gravity can be neglegted. The flow is fully developed and the velocity distribution is $u(y) = V\left(1 - \left(\frac{y}{h}\right)^2\right)$,



with V a known constant.

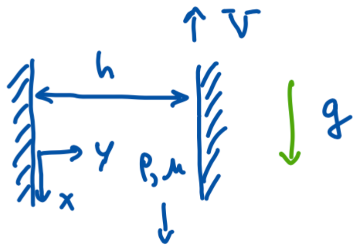
(a) Compute the ratio of the average speed and the maximum speed.

(b) What is the meaning of the integral $\int_A \sigma_{ji} n_j dA$ if A is the surface of a blob of fluid?

(c) Compute the force F by the fluid on the walls of the portion of the channel depicted in the two figures without using the integral-formulation of momentum conservation.

Problem 3 [3 POINTS.]

An incompressible viscous fluid flows steadily between two vertical parallel plates. The left plate is stationary and the right plate moves upward with velocity V. The flow is laminar and fully developed. The total gap width between the plates is h, the width (z-direction) is w. The x-axis points downstream and the y-axis starts at the left plate. The fluid viscosity is μ and the pressure satisfies $p(x, y) = ax + b$, with a and b constant. Gravity points downward.



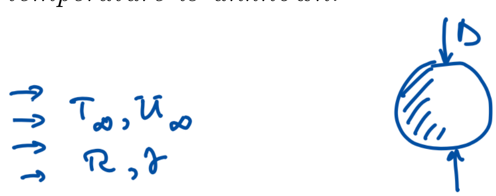
(a) Derive an expression for the velocity field.

(b) Derive an expression for the shear stress by the fluid on the right plate.

(c) Give formulas for the average velocity U, the volume flow rate Q, and the mass flow rate $\dot{m}$. If your answers contain an integral, you don't have to calculate the integral, just give the formula's.

Problem 4 [2 POINTS.]

Consider steady flow of a perfect gas around a sphere of diameter D. The gas constant is R and the ratio of specific heats is γ. Viscosity, heat conduction, and gravity can be neglected. The temperature and velocity far upstream are T_∞ and U_∞, respectively. The stagnation temperature is unknown.



(a) Which physical dimensions are present in this problem?

(b) The dimensionless ratio T_o/T_∞ depends on dimensionless parameters. Use dimension analysis to derive these dimensionless parameters.

Hint: the unit of R is $\frac{J}{kg\,K}$

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