Tutorial T04 – Elasticity – stress and strain

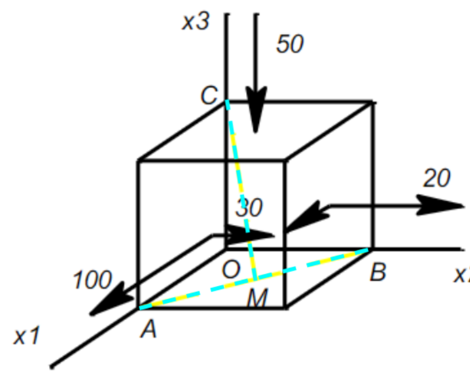
December 10, 2024

Answer the following questions as they could come up in an exam.

1 Stress basics (geometry and stress vector)

... based on section 3.1 (Exercise V1 in old material before 2022)

Given:
$E = 200\,\text{GPa}$, $\nu = 0.25$
$OA = OB = a$ and $OC = \frac{1}{2}\sqrt{2}a$



Questions:

a) Find normal stress σ_{ABC} and shear stress τ_{ABC} acting on the area ABC.
b) What are the components of the strain-tensor ε_{ij}?
c) What are the eigen-strains?

<u>Answers:</u>

a) The stress Tensor is: $\left[\sigma_{ij}\right] = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix}$ MPa,
given the arrows, using symmetry, direction of arrows (sign), and non-existing (zero).

First, find the normal to the plane: *by taking the cross-product of two line vectors (in the plane).*

$$\vec{AC} \times \vec{AB} = \begin{pmatrix} -a \\ 0 \\ \frac{a}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} = \frac{-a^2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

Normalizing the vector using the normality condition ($\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$), one can find b:

$$b^2 \left(1^2 + 1^2 + \sqrt{2}^2\right) = 1 \Longrightarrow b = \pm\frac{1}{2}$$

After using the cross-product, with the vectors in random order we pay close attention to the fact that the normal is facing outside the plane. With the normal you indicate which side the material is. To make the normal point away from the material, we can choose b positive. *(Alternatively, one* *could work with the previous factor $-a^2/\sqrt{2}$, for which b is just an abbreviation.)*

Cauchy: Stress or traction vector: $p_i = \sigma_{ij} n_j$, so that:

$$\longrightarrow \big[p\big] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \big[\sigma\big] \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & -50 \end{bmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \begin{bmatrix} 65 \\ 25 \\ -25\sqrt{2} \end{bmatrix}$$

The normal stress on the plane ABC is: $\sigma = \big[\hat{n}\big]^T \cdot \big[p\big] = 20\,\mathrm{MPa}$

The shear stress on the plane ABC, using Pythagoras, is:

$$\tau^2 = p^2 - \sigma^2 = \big[p_1^2 + p_2^2 + p_3^2\big] - \sigma^2 = 6100 - 400 = 5700\,\mathrm{MPa}^2,$$

so that $\tau = 75.5\,\mathrm{MPa}$.

b)

Hooke's law for strain $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij})$, with $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, allows to obtain: $\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$, $\varepsilon_{12} = \frac{\sigma_{12}}{2G}$, with $G = \frac{E}{2(1+\nu)}$, and - similarly - the other components.

$$\big[\varepsilon\big] = \begin{bmatrix} 5.375 & 1.875 & 0 \\ 1.875 & 0.375 & 0 \\ 0 & 0 & -4 \end{bmatrix} 10^{-4} = \frac{1}{8} \begin{bmatrix} 43 & 15 & 0 \\ 15 & 3 & 0 \\ 0 & 0 & -32 \end{bmatrix} 10^{-4}$$

c)

Principal strains are computed, like for stress, solving:

$$\det(\varepsilon_{ij} - \varepsilon\delta_{ij}) = \begin{bmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon \end{bmatrix} = 0$$

$$\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0$$

$$E_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = (14/8)\,10^{-4}$$

$$E_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2$$

$$= \left(\frac{1}{8}\,10^{-4}\right)^2 \big(43 \times 3 - 32 \times 3 - 32 \times 43 - 15^2 - 0 - 0\big) = -24.5\,10^{-8}$$

$$E_3 = \det(\varepsilon) = ... = 6\,10^{-12}$$

with solutions: $\varepsilon_I = 6\,10^{-4}$, $\varepsilon_{II} = -0.25\,10^{-4}$, $\varepsilon_{III} = -4\,10^{-4}$, sorted.

(Subscripts as in ε_{III} are used to appear different from ε_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

The third eigen-value can be read off directly from strain tensor ($\varepsilon_{III} = \varepsilon_{33}$, due to the zero values in rows and columns); the others still have to be found, from the characteristic equation (by decomposition or polynomial division), or from the invariants.

2 Stress tensor basics

... based on sections 3.1-3.3. (Exercise V2 in old material before 2022)

Given:

- Linear elastic isotropic material with modulus $E = 2 \cdot 10^5$ N/mm^2
- The stress cube, below, in units of N/mm^2
- One principal stress is: $8\,\text{N/mm}^2$

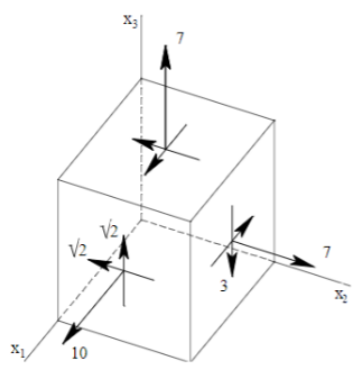


Figure 1: Stress cube → write down the stress matrix

Questions:
a) Find the other principal (eigen) stresses
b) Find the eigen-directions and plot these in a graph.
c) What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?

Answers:
a) The stress tensor from the cube is:

$$[\sigma_{ij}] = \begin{bmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{bmatrix} \text{MPa}$$

Note that the first index denotes the direction of the normal to the according surface on which this stress component works, while the second index gives the direction of the stress component.

Next, get the characteristic equation from:

$$\det(\sigma_{ij} - \sigma\delta_{ij}) = \begin{bmatrix} 10-\sigma & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7-\sigma & -3 \\ \sqrt{2} & -3 & 7-\sigma \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

with invariants:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 24\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = 176\text{MPa}^2$$

$$I_3 = \det(\sigma) = 384\text{MPa}^3$$

Here, the characteristic equation is not easily solvable; one way is to use the given eigen-value, $\sigma = 8\,N/mm^2$, and polynomial division (units dropped for simplicity, but must be added for final answer). Take the characteristic equation and divide by $(\sigma - 8)$:

$$(\sigma^3 - 24\sigma^2 + 176\sigma - 384)\backslash(\sigma - 8) = \sigma^2 - 16\sigma + 48$$

$$\underline{(\sigma^3 - 8\sigma^2)}$$

$$-16\sigma^2 + 176\sigma - 384$$

$$\underline{(-16\sigma^2 + 128\sigma)}$$

$$+\ 48\sigma - 384$$

$$\underline{(+\ 48\sigma - 384)}$$

$$\%$$

The result is a second order polynomial, which can be solved as:
$\sigma_{1,2} = (16 \pm \sqrt{16^2 - 4 \times 48})/2 = 12$ and $4\,\text{MPa}$.

Therefore, the sorted eigen-values are: $\sigma_I = 12\,\text{MPa}$, $\sigma_{II} = 8\,\text{MPa}$, $\sigma_{III} = 4\,\text{MPa}$.
(Subscripts as in σ_{III} are used to appear different from σ_3, since after sorting they do not have the meaning of coordinate! Both versions are correct, its matter of taste.)

b) Direction of $\sigma_I = 12\text{MPa}$
There are various ways to solve for eigen-vectors, here is one example ...
Insert values, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11} - \sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_I \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$-2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - 5n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - 5n_3 = 0$$

The eigen-direction associated to the first, largest eigen-value:

$$\Longrightarrow \hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

Direction of $\sigma_{II} = 8$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$2n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 - n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 - n_3 = 0$$

The eigen-direction associated to the second, intermediate eigen-value:

$$\Longrightarrow \hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$$

Direction of $\sigma_{III} = 4$MPa

$$\begin{bmatrix} \sigma_{11} - \sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{III} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$6n_1 - \sqrt{2}n_2 + \sqrt{2}n_3 = 0$$

$$-\sqrt{2}n_1 + 3n_2 - 3n_3 = 0$$

$$\sqrt{2}n_1 - 3n_2 + 3n_3 = 0$$

The eigen-direction associated to the third, smallest eigen-value:

$$\Longrightarrow \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

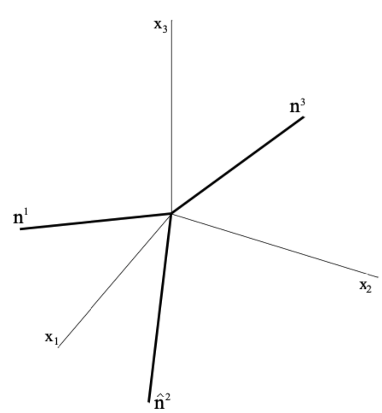


Figure 2: Sketch of the eigen-vectors (with coordinate axes) in bold.

The directions are unspecified, indicated by the plus-minus from taking a square-root; all three direction vectors are normalized (check it, if enough time in exam), $(n_i)^2 = 1$; furthermore, all three normal (eigen) vectors must be pair-wise perpendicular on each other, i.e. $n_i^{(a)} n_i^{(b)} = 0$, for all $a, b = I, II, III$ with $a \neq b$. This perpendicularity allows to obtain, alternatively, one eigen-vector by a cross-product, e.g. above $\hat{n}^{(III)} = \hat{n}^{(I)} \times \hat{n}^{(II)}$.

c)
What is the maximal shear strain for a given volumetric strain of $\varepsilon_V = 0.6\,10^{-4}$?
Since ν is not given, we need an additional relation:

$$\varepsilon_V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0.6 \times 10^{-4}$$

and thus:

$$\varepsilon_V = \varepsilon_{kk} = \frac{1}{E}\left((1+\nu)\sigma_{kk} - \nu\sigma_{mm}\delta_{kk}\right) = \frac{1}{E}\left((1+\nu)\sigma_{kk} - 3\nu\sigma_{kk}\right) = \frac{1-2\nu}{E}\sigma_{kk}$$

to determine the unknown Poisson ratio:

$$\rightarrow \nu = \frac{1}{2} - \frac{E\varepsilon_V}{2\sigma_{kk}} = \frac{1}{2} - \frac{200\,\mathrm{GPa}\,0.6 \times 10^{-4}}{2 \times 24\,\mathrm{MPa}} = \frac{1}{2} - \frac{20 \times 0.6}{2 \times 24} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

And, finally, the maximum deformation angle and shear strain are:

$$\varepsilon_{shear,max} = \gamma_{max}/2 = \tau_{max}/2G = (\sigma_I - \sigma_{III})\,2(1+\nu)/(4E) = 8(5/2)/(800000) = (5/2) \times 10^{-5}$$

Alternative calculation can be done by computing the eigen-strains, and then from that the maximum shear strain.

3 Stress tensor basics

... based on section 3,4 (Exercise V3 in old material before 2022)

Given:

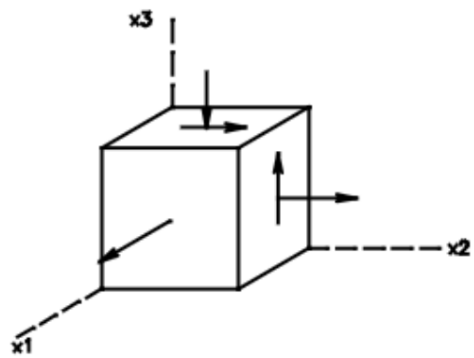


Figure 3: Stress cube, empty → fill it

The stress-state is described by the matrix: $\begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix}$ N/mm^2,
with $E = 2 \cdot 10^5$ N/mm^2, and $\nu = 0.25$.

Questions:
a) Compute the principal stresses
b) Compute the eigen-directions
c) Compute the maximal shear-stress
d) Give the unit vector normal to the plane on which the maximal shear stress works and its orientation in x'_p, i.e. the coordinate system defined by the eigen-directions.
e) Give the orientation of the plane on which the maximal shear stress works in a graphic/sketch.
f) What is the strain in the direction of the normal vector from question d).

Answers:
a)
The sorted eigen-values are: $\sigma_I = 60\,\text{MPa}$, $\sigma_{II} = 40\,\text{MPa}$, $\sigma_{III} = -40\,\text{MPa}$.
The first eigenvalue can be directly seen from the stress matrix; the others are taken from the second order polynomial remaining from the characteristic equation (no details shown here).

b)
Without calculation necessary (due to the special structure of this plane stress):

$$\hat{n}^{(I)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The other eigen-directions are obtained from $(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$, with normalization $n_j^2 = 1$:

$$\hat{n}^{(II)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{n}^{(III)} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \pm\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

Insert values, for example σ_{II}, solve the system of equations, and normalize the solution.

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad n_1^2+n_2^2+n_3^2=1$$

$$-20n_2 + 20\sqrt{3}n_3 = 0$$

$$20\sqrt{3}n_2 - 60n_3 = 0$$

$n_2 = \sqrt{3}n_3$ and $n_2 = (3/\sqrt{3})n_3 = \sqrt{3}n_3$ (identical due to dependency)

$$n_2^2 + n_3^2 = (1+3)n_3^2 = 1$$

$$n_3 = \sqrt{1/4} = \pm 1/2 = \pm 0.5$$

This results in the eigen-direction associated to the second, intermediate eigen-value, as given above. *The third eigenvalue calculation is similar (not shown).*

c)
The maximum shear stress is: $\tau_{max} = (\sigma_I - \sigma_{III})/2 = 50$ MPa.

d)
The maximal shear stress acts on a surface rotated by 45^o from the x'_1 and x'_3 directions, related to eigen-directions of σ_I and σ_{III}, respectively, see sketch.
In this coordinate system, the normalized unit vector is obtained from the $(1,0,1)$ direction, but still has to be normalized, so that: $\hat{n}^{\tau_{max}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

This vector also can be obtained from: $\hat{n}^{(I)'} + \hat{n}^{(III)'}$, in the *same* coordinate system, with coordinate unit vectors: $x'_1 = \hat{n}^{(I)'} = (1,0,0)$ and $x'_3 = \hat{n}^{(III)'} = (0,0,1)$, and normalization.
Note: The solution is different in the original coordinate system, where the normal unit vectors (without prime) from answer b) are to be used.

e) Graphic/sketch

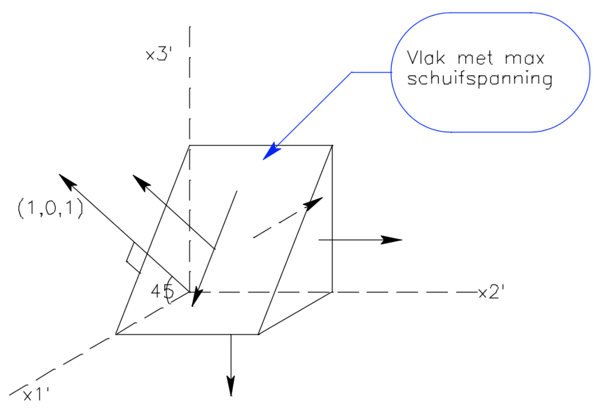


Figure 4: Sketch of the normal to the plane with maximal shear stress (in Dutch: "vlak met maximale schuifspanning"), in the coordinate system x'_p of the eigen-directions of stress, with perpendicular (sorted) intermediate stress eigen-direction x'_2.

f)

The strain in the direction of the normal vector from question d) can be obtained in various ways. Here, we compute the eigen-strains directly from the eigen-stresses using the law of Hooke: $\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right)$ where in the eigen-system, all non-diagonal elements are zero (using arabic number convention):

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{2\,10^{-5}}\left[(5/4)\sigma_1 - (1/4)\sigma_{kk}\right] = \frac{1}{2\,10^{-5}}\left[(5/4)60 - (1/4)(60+40-40)\right] = 3\,10^{-4} \\
\varepsilon_2 &= ... = \frac{1}{2\,10^{-5}}\left[(5/4)40 - (1/4)(60+40-40)\right] = 1.75\,10^{-4} \\
\varepsilon_3 &= ... = \frac{1}{2\,10^{-5}}\left[(5/4)(-40) - (1/4)(60+40-40)\right] = -3.25\,10^{-4}
\end{aligned}
$$

Now one can use a rotation matrix with 45 degrees about the 2-direction

$$
[R] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}
$$

to compute the strain components on the surface of maximal shear stress:

$$
[\varepsilon''] = [R][\varepsilon'][R]^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = ...
$$

For the 11-component in the new coordinate system, pointing in the normal vector direction of d), i.e., the direction of the plane on which the maximal shear stress acts, we only need to compute:

$$
\varepsilon''_{11} = (1/\sqrt{2})(1/\sqrt{2})\varepsilon_1 + (1/\sqrt{2})(1/\sqrt{2})\varepsilon_3 = (1/2)(\varepsilon_1 + \varepsilon_3) = (1/2)(3 - 3.25)\,10^{-4} = -0.125\,10^{-4}
$$

which is the normal strain (on this plane), while the maximal shear strain is simply:

$$
\varepsilon_{shear,max} = \varepsilon''_{13} = (1/2)(\varepsilon_1 - \varepsilon_3) = 3.125\,10^{-4}
$$

which is the shear strain (on this plane).

Alternatively (not shown), use the circle of Mohr for strain.

Exercise 4 will be continued in tutorial T05
Exercise 5 will be continued in tutorial T05

6 Stress and transformation

... based on sections 3, 4, 5.1 (Exercise V4 in old material before 2022)

Given:
$E = 2 \cdot 10^{11}\,\text{Pa}$, $\nu = 0.25$

Stress-state in point P: $\big[\sigma\big] = \begin{bmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{bmatrix}$ MPa

Questions:
a) Show that the principal stresses are 8, 16 and 24 MPa.
Compute the directional cosines (transformation matrix entries) of the smallest eigen-stress.
... based on sections 4,5.1
b) Compute the volumetric (isotropic) strain.
c) What is the largest angle-change (not shear-strain) in P?
... based on section 5.1
d) Which material property is implicitly used/assumed in Hooke's law?

Answers:
a)
From $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$, the characteristic equation follows as:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 48\sigma^2 + 704\sigma - 3072 = 0\,.$$

Given the eigenvalues, σ, one can test their validity by inserting one by one; or one can factorize the equation, e.g. by polynomial division; or one computes the invariants from the eigen-values and confirms the characteristic equation. *Watch the signs in the definitions.*
Sorting the eigen-values is convention and part of the answer:
$\sigma_I = 24\,\text{MPa}$, $\sigma_{II} = 16\,\text{MPa}$, and $\sigma_{III} = 8\,\text{MPa}$.
it allows to refer a certain eigen-value, e.g. the smallest and its eigen-direction.

The eigen-direction of the minor eigen-stress, $\sigma_{III} = 8\,\text{MPa}$ is obtained by solving:

$$\begin{bmatrix} \sigma_{11}-\sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and normalization: } n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

so that (dropping the superscript for brevity):

$11n_1 - 5n_2 - \sqrt{6}n_3 = 0 \rightarrow n_1 = (5/11)n_2 + (\sqrt{6}/11)n_3$

$-5n_1 + 11n_2 - \sqrt{6}n_3 = 0 \rightarrow n_2 = (5/11)n_1 + (\sqrt{6}/11)n_3$

$-\sqrt{6}n_1 - \sqrt{6}n_2 + 2n_3 = 0 \rightarrow n_3 = (\sqrt{6}/2)n_1 + (\sqrt{6}/2)n_2$

Subtracting line 2 from 1 yields: $n_1 - n_2 = (5/11)(n_2 - n_1) \rightarrow n_1 = n_2$

Inserting into line 3 yields: $n_3 = \sqrt{6}n_1,$ so that: $\begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\, c \begin{bmatrix} 1 \\ 1 \\ \sqrt{6} \end{bmatrix}$

where the unknown $c = 1/\sqrt{8} = \sqrt{2}/4$ is obtained from normalization, resulting in:

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{bmatrix}$$

b)
For the volumetric (isotropic) strain, we can use the short-cut (not the full strain tensor), as: $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \left[\frac{1-2\nu}{E}\right]\sigma_{kk} = 12\ 10^{-5}$.

c)
The largest change of angle is: $\gamma_{max} = \frac{\tau_{max}}{G} = \frac{1}{2}\frac{\sigma_I - \sigma_{III}}{G} = 10^{-4}$, using $G = \frac{E}{2(1+\nu)} = (4/5)\ 10^5\,\text{MPa}$, where the largest shear strain is just half of that: $\varepsilon_{max} = \gamma_{max}/2$.

d)
Isotropic (direction independent) material behavior is intrinsically assumed in the law of Hooke.

7 Displacement, strain and stress relation

... based on sections 3,4,5 (Exercise V7 in old material before 2022)

Given is the displacement field:
$u_1 = x_1 x_3$, $u_2 = -x_1 x_2$, and $u_3 = x_1^2 - x_3^2$
and material properties $E = 2\,\mathrm{GPa}$ and $\nu = 0.25$
Consider the units, drop them in calculations to save space, but give them in end-results!

Questions:

a) Compute the stress tensor (components).
b) In the point $(x_1, x_2, x_3) = (0, 0, z_0)$ compute the eigen-stresses and maximum shear stress.

Answers:
a)
To compute the components of the stress tensor,
first, the strain tensor has to be computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

resulting in the components:

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = x_3 & \gamma_{12} &= 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 - x_2 = -x_2 \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = -x_1 & \gamma_{23} &= 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 + 0 = 0 \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} = -2x_3 & \gamma_{31} &= 2\varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2x_1 + x_1 = 3x_1
\end{aligned}$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -x_1 - x_3$$

Next, obtain σ_{kk} from Hooke's law:

$$\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\}$$

$$\begin{aligned}
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} & & & \delta_{jj} &= 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} & &\Longrightarrow & \sigma_{kk} &= \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}$$

Substitute σ_{kk} into Hooke's law and rearrange for σ_{ij}:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

Compute the terms for the stress tensor using the previously obtained equation:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\left[x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 + x_3)\\
\sigma_{22} &= \frac{E}{1+\nu}\left[-x_1 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-x_1 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-3x_1 - x_3)\\
\sigma_{33} &= \frac{E}{1+\nu}\left[-2x_3 + \frac{\nu}{1-2\nu}(-x_1 - x_3)\right] = \frac{8}{5}\left[-2x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right] = \frac{4}{5}(-x_1 - 5x_3)\\
\sigma_{12} &= \frac{E}{1+\nu}\left[-\frac{1}{2}x_2\right] = -\frac{4}{5}x_2\\
\sigma_{23} &= 0\\
\sigma_{31} &= \frac{E}{1+\nu}\left[\frac{3}{2}x_1\right] = \frac{12}{5}x_1
\end{aligned}$$

b)
The principal stresses in point $(x, y, z) = (0, 0, z_0)$ are identical to the diagonal components, here only (if all shear stresses are zero); and the maximal shear stress τ_{max} are:

$$\begin{aligned}
\sigma_1 &= \sigma_{11} = \frac{4}{5}(-x_1 + x_3) = \frac{4}{5}z_0 & \sigma_{12} &= -\frac{4}{5}x_2 = 0\\
\sigma_2 &= \sigma_{22} = \frac{4}{5}(-3x_1 - x_3) = -\frac{4}{5}z_0 & \sigma_{23} &= 0\\
\sigma_3 &= \sigma_{33} = \frac{4}{5}(-x_1 - 5x_3) = -4z_0 & \sigma_{31} &= \frac{12}{5}x_1 = 0
\end{aligned}$$

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 2.4z_0\ \mathrm{GPa\,m^{-1}}$$

for a given length z_0 with units [m].

8 Displacement, strain and stress equilibrium

... based on sections 3,4,5 (Exercise V8 in old material before 2022)

In a homogenous body made of a linear elastic, isotropic material, the displacement field is given:

$u_1 = \frac{p}{E} a \left[\frac{x_2}{a} + 2\frac{x_1 x_2}{a^2} - \frac{x_2^2}{a^2} \right]$

$u_2 = \frac{p}{E} a \left[\frac{x_1}{a} + \alpha\frac{x_1^2}{a^2} + \beta\frac{x_1 x_2}{a^2} - 2\frac{x_2^2}{a^2} \right]$

$u_3 = 0$

with coordinates x_1 and x_2, and constant coefficients p, E, a, and $\nu = 0.25$.

Questions:

In the absence of volume forces, compute the magnitude of the parameters α and β, using the information that the stress field is in mechanical equilibrium.

Answers:

A homogeneous body made from a linear elastic isotropic material ($E = 2\,\mathrm{GPa}$ and $\nu = 0.25$) is in a deformed state in mechanical equilibrium, in the absence of a volume force, with parameters reference stress p, and length a, and unknown coefficients α and β, according to the displacement field:

$$
\begin{aligned}
u_1 &= \frac{pa}{E}\left[\left(\frac{x_2}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - \left(\frac{x_2}{a}\right)^2\right] \\
u_2 &= \frac{pa}{E}\left[\left(\frac{x_1}{a}\right) + \alpha\left(\frac{x_1}{a}\right)^2 + \beta\left(\frac{x_1}{a}\right)\left(\frac{x_2}{a}\right) - 2\left(\frac{x_2}{a}\right)^2\right] \\
u_3 &= 0
\end{aligned}
$$

The stress components, to be computed from Hooke's law, require the strain components, computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,.$$

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{pa}{E}\left[2\left(\frac{1}{a}\right)\left(\frac{x_2}{a}\right)\right] = \frac{p}{E}\left[2\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \frac{pa}{E}\left[\beta\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 4\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right] = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] \\
\varepsilon_{33} &= 0
\end{aligned}
$$

and the trace:

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{p}{E}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]$$

$$
\begin{aligned}
\gamma_{12} = 2\varepsilon_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\
&= \frac{pa}{E}\left[\left(\frac{1}{a}\right) + 2\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) - 2\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right) + \left(\frac{1}{a}\right) + 2\alpha\left(\frac{x_1}{a}\right)\left(\frac{1}{a}\right) + \beta\left(\frac{x_2}{a}\right)\left(\frac{1}{a}\right)\right] \\
&= 2\frac{p}{E}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\gamma_{23} = 2\varepsilon_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \\
\gamma_{31} = 2\varepsilon_{31} &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0
\end{aligned}
$$

Obtain σ_{kk} with Hooke's law:

$$
\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\}
$$

$$
\begin{aligned}
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} & & \delta_{jj} = 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} & \implies \quad & \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}
$$

Substitute σ_{kk} into Hooke's and rearrange for σ_{ij}:

$$
\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \qquad \implies \qquad \sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]
$$

The stress-equilibrium equations (by partial differentiation) are then obtained as:

$$
\begin{aligned}
\sigma_{ij} &= \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right] \\
\sigma_{11} &= \frac{E}{1+\nu}\left[\left[2\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{E}{1+\nu}\frac{1}{1-2\nu}\left[\left[2(1-3\nu)\left(\frac{x_2}{a}\right)\right] + \left[\nu\beta\left(\frac{x_1}{a}\right)\right]\right] \\
\sigma_{22} &= \frac{E}{1+\nu}\left[\left[\beta\left(\frac{x_1}{a}\right) - 4\left(\frac{x_2}{a}\right)\right] + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{E}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta(1-\nu)\left(\frac{x_1}{a}\right)\right] + \left[(-4+6\nu)\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{33} &= \frac{E}{1+\nu}\left[0 + \frac{\nu}{1-2\nu}\left[\beta\left(\frac{x_1}{a}\right) - 2\left(\frac{x_2}{a}\right)\right]\right] \\
&= \frac{E}{1+\nu}\frac{1}{1-2\nu}\left[\left[\beta\nu\left(\frac{x_1}{a}\right) - 2\nu\left(\frac{x_2}{a}\right)\right]\right] \\
\sigma_{12} &= \frac{E}{1+\nu}\left[\varepsilon_{12}\right] = \frac{E}{1+\nu}\left[1 + (\alpha+1)\left(\frac{x_1}{a}\right) + \left(\frac{\beta}{2} - 1\right)\left(\frac{x_2}{a}\right)\right] \\
\sigma_{23} &= 0 \\
\sigma_{31} &= 0
\end{aligned}
$$

In absence of a volume force, the three equilibrium equations for $j = 1, 2, 3$ are:

$$
\sigma_{ij,i} + 0 = (\sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3}) + 0 = 0
$$

Using partial differentiation:

$$\sigma_{11,1} = \frac{\partial}{\partial x_1}\sigma_{11} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] \qquad \sigma_{12,1} = \frac{\partial}{\partial x_1}\sigma_{12} = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right]; \quad \sigma_{13,1} = 0$$

$$\sigma_{22,2} = \frac{\partial}{\partial x_2}\sigma_{22} = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] \quad \sigma_{23,2} = \frac{\partial}{\partial x_2}\sigma_{23} = 0; \quad \sigma_{21,2} = \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right]$$

$$\sigma_{33,3} = \frac{\partial}{\partial x_3}\sigma_{33} = 0 \qquad \sigma_{31,3} = \frac{\partial}{\partial x_3}\sigma_{31} = 0; \quad \sigma_{32,3} = 0$$

Computing the three equations:

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0 = \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[\nu\beta\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\left[\left(\frac{\beta}{2}-1\right)\left(\frac{1}{a}\right)\right] + 0 \tag{1}$$

$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} = 0 = \frac{p}{1+\nu}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{1+\nu}\frac{1}{1-2\nu}\left[(-4+6\nu)\left(\frac{1}{a}\right)\right] + 0 \tag{2}$$

$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0 \tag{3}$$

Inserting the value for $\nu = 0.25$:

$$(1) \implies \frac{p}{\frac{5}{4}}\left[\frac{1}{2}\beta + \left(\frac{\beta}{2}-1\right)\right]\left(\frac{1}{a}\right) = 0 \qquad \implies \qquad p\frac{4}{5}\left[\beta-1\right]\left(\frac{1}{a}\right) = 0$$

$$(2) \implies \frac{p}{\frac{5}{4}}\left[(\alpha+1)\left(\frac{1}{a}\right)\right] + \frac{p}{\frac{5}{4}}2\left[(-\frac{5}{2})\left(\frac{1}{a}\right)\right] = 0 \qquad \implies \qquad p\frac{4}{5}\left[(\alpha+1)-5\right]\left(\frac{1}{a}\right) = 0$$

Solving for the two unknowns (α, β):

$$(1) \implies \beta = 1$$
$$(2) \implies \alpha = 4$$

This allows for mechanical equilibrium in all points in the body, because the equilibrium equations do not depend on the position. Note that either of the two equations – both have to be valid at the same time – provides one unknown.

9 Displacement, strain, stress

... based on sections 3,4,5 (Exercise V9 in old material before 2022)

Within a homogeneous body made of a linear elastic, isotropic material the displacement field:

$$u_1 = \frac{1}{3}(1-2\nu)x_1^3 - (3-2\nu)x_1x_2^2 - 3x_2 - 3x_3$$

$$u_2 = (1-2\nu)x_1^2x_2 + \frac{1}{3}(1+2\nu)x_2^3 + 3x_1 - 4x_3$$

$$u_3 = 3x_1 + 4x_2$$

the elasticity modulus E, and the Poisson-ratio ν are given.

Questions:

... based on section 4
a) Compute the components of the strain tensor.
... based on section 5.1
b) Compute the components of the stress tensor.
... based on section 3
c) Confirm that the stress-field is in equilibrium in the absence of volume forces.

Answers:
a)
The strain tensor is computed from the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right)$$

with the components:

$$\begin{aligned}
\varepsilon_{11} &= (1-2\nu)x_1^2 - (3-2\nu)x_2^2 \\
\varepsilon_{22} &= (1-2\nu)x_1^2 + (1+2\nu)x_2^2 \\
\varepsilon_{33} &= 0 \\
\varepsilon_{kk} &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 2(1-2\nu)\left(x_1^2 - x_2^2\right)
\end{aligned}$$

and the (symmetric) non-diagonal elements:

$$\begin{aligned}
\gamma_{12} &= 2\varepsilon_{12} = -2(3-2\nu)x_1x_2 - 3 + 2(1-2\nu)x_1x_2 + 3 = -4x_1x_2 \\
\gamma_{23} &= 2\varepsilon_{23} = -4 + 4 = 0 \\
\gamma_{13} &= 2\varepsilon_{13} = -3 + 3 = 0
\end{aligned}$$

b)
Use Hooke's law (strain-stress):

$$\begin{aligned}
\varepsilon_{ij} &= \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\} \\
\varepsilon_{kk} &= \frac{1}{E}\{(1+\nu)\sigma_{kk} - \nu\delta_{jj}\sigma_{kk}\} & \delta_{jj} &= 1+1+1 = 3 \\
\varepsilon_{kk} &= \frac{1}{E}(1-2\nu)\sigma_{kk} \qquad \Longrightarrow & \sigma_{kk} &= \frac{E}{1-2\nu}\varepsilon_{kk}
\end{aligned}$$

From the stress-strain relation compute the stress tensor components:

$$\sigma_{ij} = \frac{E}{1+\nu}\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\}$$

$$
\begin{aligned}
\sigma_{11} &= \frac{E}{1+\nu}\{\varepsilon_{11}+\frac{\nu}{1-2\nu}\delta_{11}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{(1-2\nu)x_1^2-(3-2\nu)x_2^2+2\nu\left(x_1^2-x_2^2\right)\} = \frac{E}{1+\nu}\{x_1^2-3x_2^2\}\\
\sigma_{22} &= \frac{E}{1+\nu}\{\varepsilon_{22}+\frac{\nu}{1-2\nu}\delta_{22}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{(1-2\nu)x_1^2+(1+2\nu)x_2^2+2\nu\left(x_1^2-x_2^2\right)\} = \frac{E}{1+\nu}\{x_1^2+x_2^2\}\\
\sigma_{33} &= \frac{E}{1+\nu}\{\varepsilon_{33}+\frac{\nu}{1-2\nu}\delta_{33}\varepsilon_{kk}\} = \frac{E}{1+\nu}\{0+2\nu\left(x_1^2-x_2^2\right)\} = \frac{E}{1+\nu}2\nu\{x_1^2-x_2^2\}\\
\sigma_{12} &= \sigma_{21} = \frac{E}{1+\nu}\{\frac{1}{2}(-4x_1x_2)\} = \frac{E}{1+\nu}\{-2x_1x_2\}\\
\sigma_{23} &= \sigma_{32} = \sigma_{13} = \sigma_{31} = 0
\end{aligned}
$$

c)
Given an absence of a volume force, $f_j = 0$, the three stress-equilibrium equations ($j = 1, 2, 3$) are:

$$\sigma_{ij,i} + f_j = (\sigma_{1j,1} + \sigma_{2j,2} + \sigma_{3j,3}) + 0 = 0$$

Using partial differentiation:

$$
\begin{aligned}
&\sigma_{11,1} = \frac{\partial}{\partial x_1}\sigma_{11} = \frac{E}{1+\nu}2x_1 \quad \sigma_{21,2} = \frac{\partial}{\partial x_2}\sigma_{21} = \frac{E}{1+\nu}(-2x_1) \quad \sigma_{31,3} = \frac{\partial}{\partial x_3}\sigma_{31} = 0;\\
&\sigma_{22,2} = \frac{\partial}{\partial x_2}\sigma_{22} = \frac{E}{1+\nu}2x_2 \quad \sigma_{12,1} = \frac{\partial}{\partial x_1}\sigma_{12} = \frac{E}{1+\nu}(-2x_2) \quad \sigma_{32,3} = \frac{\partial}{\partial x_3}\sigma_{32} = 0;\\
&\sigma_{33,3} = \frac{\partial}{\partial x_3}\sigma_{33} = 0 \quad \sigma_{13,1} = \frac{\partial}{\partial x_1}\sigma_{13} = 0 \quad \sigma_{23,2} = \frac{\partial}{\partial x_2}\sigma_{23} = 0;
\end{aligned}
$$

and inserting the elements into the equations:

$$
\begin{aligned}
\sigma_{i1,i} &= \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = 0;\\
\sigma_{i2,i} &= \sigma_{22,2} + \sigma_{12,1} + \sigma_{32,3} = 0;\\
\sigma_{i3,i} &= \sigma_{33,3} + \sigma_{13,1} + \sigma_{23,2} = 0;
\end{aligned}
$$

confirms stress equilibrium.

10 Strain-stress relations

... based on sections 4,5.1 (Exercise V5 in old material before 2022)

In a Cartesian coordinate system, at point P, the strain tensor is given as:

$$[\varepsilon_{ij}] = \frac{5}{8}\begin{bmatrix} -1 & -15 & 5\sqrt{2} \\ -15 & -1 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 14 \end{bmatrix} \; 10^{-5}$$

Questions:

a) For a material with modulus $E = 2.10^5$ MPa and Poisson ratio $\nu = 0.25$, compute the eigen-stresses and the eigen-directions.
b) Explain/argue why the eigen-directions of stress and strain are identical for a homogeneous, isotropic material.

Answers:
a)
First, the stress tensor is determined using Hooke's law $\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$.
Insert and solve for stress in the normal way ... for each component.
For example:

$\sigma_{12} = \frac{E}{1+\nu}(\epsilon_{12}) = -15\,\text{MPa}$,
$\sigma_{11} = \frac{E}{1+\nu}\left(\epsilon_{11} + \frac{\nu}{1-2\nu}\epsilon_{kk}\right) = 5\,\text{MPa}$, with $\epsilon_{kk} = (5/8)\,12\,10^{-5}$,
etc.

Another way to do this (often done for anisotropic materials and in finite element implementations – not needed for linear isotropic elasticity, to solve this rather simple problem, just for completeness) is to assemble the independent stress- and strain-tensor components in vectors and express the corresponding stiffness-matrix in moduli, using the Lamé-coeffcients (for brevity):

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad\qquad 2\mu = 2G = \frac{E}{1+\nu}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

$$[\sigma_{ij}] = \begin{bmatrix} 5 & -15 & 5\sqrt{2} \\ -15 & 5 & -5\sqrt{2} \\ 5\sqrt{2} & -5\sqrt{2} & 20 \end{bmatrix} \text{MPa}$$

From the stress, the characteristic equation, invariants, and principal stresses can be computed:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 30\,\text{MPa}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{31}^2 - \sigma_{32}^2 = -100\,\text{MPa}^2$$

$$I_3 = \det(\sigma) = -3000\,\text{MPa}^3$$

In this case, solution by decomposition is possible, as:

$$\sigma^2\,(\sigma - 30) - 100\,(\sigma - 30) = 0$$

Therefore: $\sigma_I = 30\text{MPa}$, $\sigma_{II} = 10\text{MPa}$, $\sigma_{III} = -10\text{MPa}$.

Principal directions can be calculated as:
Direction of $\sigma_I = 30\text{MPa}$

$$\begin{bmatrix} \sigma_{11}-\sigma_I & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_I & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_I \end{bmatrix} \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(I)} + n_2^{(I)} + n_3^{(I)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(I)} \\ n_2^{(I)} \\ n_3^{(I)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{II} = 10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11}-\sigma_{II} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{II} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{II} \end{bmatrix} \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad n_1^{(II)} + n_2^{(II)} + n_3^{(II)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(II)} \\ n_2^{(II)} \\ n_3^{(II)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

Direction of $\sigma_{III} = -10\,\text{MPa}$

$$\begin{bmatrix} \sigma_{11}-\sigma_{III} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_{III} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_{III} \end{bmatrix} \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad n_1^{(III)} + n_2^{(III)} + n_3^{(III)} = 1$$

$$\Longrightarrow \begin{bmatrix} n_1^{(III)} \\ n_2^{(III)} \\ n_3^{(III)} \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}$$

b)
For an isotropic material, one has the eigendirections of stress-and strain identical. Note that the term with δ_{ij} in the law of Hooke has no direction (is isotropic = direction-independent); thus the direction is carried by the terms ε_{ij} and σ_{ij} and thus the directions are equal for stress and strain.