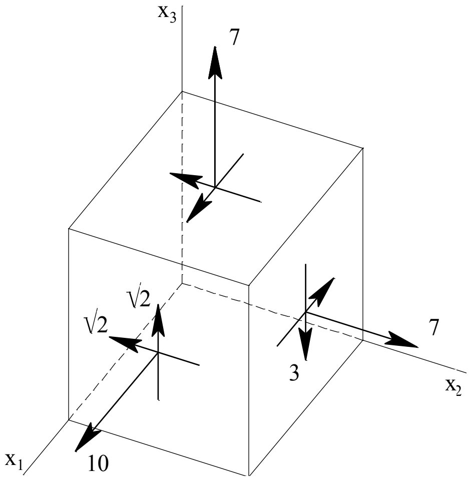
Exercise V-2



Given:
Linear elastic isotropic material
$E=2\cdot 10^5$ N/mm^2
One principal stress is given: 8 N/mm^2

Questions:
a) the other principal (eigen) stresses
b) the eigen-directions and a sketch
c) the maximal shear-strain for a given volumetric strain $\varepsilon_V = 0{,}6{\cdot}10^{-4}$
d) the equivalent stresses according to Tresca and von Mises

Solutions:

a) The principal stresses can be computed as the eigen-values of the stress-matrix/tensor. This means one has to find the solutions to: $\det([\sigma]-\sigma[I])=0$

$$\det([\sigma]-\sigma[I]) = \det\begin{bmatrix}\sigma_{11}-\sigma & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma\end{bmatrix} = \det\begin{bmatrix}10-\sigma & -\sqrt{2} & \sqrt{2}\\ -\sqrt{2} & 7-\sigma & -3\\ \sqrt{2} & -3 & 7-\sigma\end{bmatrix} = 0$$

Note that the first index denotes the direction of the normal to the accoding surface on which this stress component is working, while the second index gives the direction in which the stress component works.

The determinant yields the characteristic equation:

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$, with invariants: $I_1 = \sigma_{11}+\sigma_{22}+\sigma_{33} = 24\ N/mm^2$

$$I_2 = \sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}\sigma_{21}-\sigma_{23}\sigma_{32}-\sigma_{31}\sigma_{13} = 176\left(N/mm^2\right)^2$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33}-\sigma_{11}\sigma_{23}\sigma_{32}-\sigma_{12}\sigma_{21}\sigma_{33}+\sigma_{12}\sigma_{23}\sigma_{31}+\sigma_{13}\sigma_{21}\sigma_{32}-\sigma_{13}\sigma_{31}\sigma_{22} = 384\left(N/mm^2\right)^3$$

so that:

$$\sigma^3 - 24\sigma^2 + 176\sigma - 384 = 0$$

This equation has three solutions, but cannot be solved in general; however, one is known alredy, 8 *N/mm*^{2}, thus after decomposition, the equation will look like:

$$(\sigma-8)(a\sigma^2 + b\sigma + c) = 0$$

with unknown *a*, *b* and *c*. To get these coefficients, divide the whole characteristic equation by: $\sigma - 8$ using polynomial division:

$$\sigma - 8 / \sigma^3 - 24\sigma^2 + 176\sigma - 384 \backslash \sigma^2 - 16\sigma + 48$$

$$\begin{array}{r} \underline{\sigma^3 - 8\sigma^2\ -} \\ -16\sigma^2 + 176\sigma \\ \underline{-16\sigma^2 + 128\sigma\ -} \\ 48\sigma - 384 \\ \underline{48\sigma - 384\ -} \\ 0 \end{array}$$

The equation decomposed into its three factors is now:

$$(\sigma-8)(\sigma^2-16\sigma+48)=(\sigma-8)(\sigma-4)(\sigma-12)=0,$$

from which we get the principal stresses:

$$\sigma_1 = 12\,MPa$$
$$\sigma_2 = 8\,MPa \qquad\qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$$
$$\sigma_3 = 4\,MPa$$

b) The eigen-directions are obtained by inserting one eigen-stress into $([\sigma]-\sigma[I])$, and multiplying the matrix with a unit-direction vector with three unknowns to solve:

Hoofdrichting 1:

$$\begin{bmatrix} -2 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -5 & -3 \\ \sqrt{2} & -3 & -5 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From the rows of the matrix, one gets the following three relations:

$$-2\hat{n}_1 - \sqrt{2}\hat{n}_2 + \sqrt{2}\hat{n}_3 = 0 \qquad [1]$$
$$-\sqrt{2}\hat{n}_1 - 5\hat{n}_2 - 3\hat{n}_3 = 0 \qquad [2]$$
$$\sqrt{2}\hat{n}_1 - 3\hat{n}_2 - 5\hat{n}_3 = 0 \qquad [3]$$

From [3] follows:

$$\hat{n}_1 = \tfrac{1}{2}\sqrt{2}(3\hat{n}_2 + 5\hat{n}_3) \qquad [4]$$

Inserted in [1] gives:

$$\hat{n}_2 = -\hat{n}_3 \qquad [5]$$

[5] back in [4] gives $\hat{n}_1$ expressed in terms of $\hat{n}_3$:

$$\hat{n}_1 = \sqrt{2}\hat{n}_3$$

Using the normalisation condition: $\hat{n}_1^{\,2} + \hat{n}_2^{\,2} + \hat{n}_3^{\,2} = 1$ ne gets the value of $\hat{n}_3$

$$\left(\sqrt{2}^{\,2} + 1 + 1\right)\hat{n}_3^{\,2} = 1 \qquad \Leftrightarrow \qquad \hat{n}_3 = \pm\tfrac{1}{2}$$

The first eigen-direction (associated to the first, major principal stress) is thus:

$$\{\hat{n}_i\}^1 = \pm\frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ -1 \\ 1 \end{Bmatrix}$$

Hoofdrichting 2:

$$\begin{bmatrix} 2 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -1 & -3 \\ \sqrt{2} & -3 & -1 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Similar to above, after inerting the second eigen-stress:

$$2\hat{n}_1 - \sqrt{2}\hat{n}_2 + \sqrt{2}\hat{n}_3 = 0 \qquad [6]$$
$$-\sqrt{2}\hat{n}_1 - \hat{n}_2 - 3\hat{n}_3 = 0 \qquad [7]$$
$$\sqrt{2}\hat{n}_1 - 3\hat{n}_2 - \hat{n}_3 = 0 \qquad [8]$$

From [8] follows

$$\hat{n}_1 = \tfrac{1}{2}\sqrt{2}\left(3\hat{n}_2 + \hat{n}_3\right) \qquad [9]$$

Inserting into [6] gives

$$\hat{n}_2 = -\hat{n}_3 \qquad [10]$$

[10] back in [9] gives $\hat{n}_1$ expressed in terms of $\hat{n}_3$:

$$\hat{n}_1 = -\sqrt{2}\hat{n}_3$$

Again, using the normalization to get $\hat{n}_3$ results in:

$$\left(\left(-\sqrt{2}\right)^2 + (-1)^2 + 1^2\right)\hat{n}_3^{\,2} = 1 \qquad \Leftrightarrow \qquad \hat{n}_3 = \pm\tfrac{1}{2}$$

The second eigen-direction is thus:

$$\{\hat{n}_i\}^2 = \pm\frac{1}{2}\begin{Bmatrix} -\sqrt{2} \\ -1 \\ 1 \end{Bmatrix} = \pm\frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ 1 \\ -1 \end{Bmatrix}$$

Hoofdrichting 3:

$$\begin{bmatrix} 6 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 3 & -3 \\ \sqrt{2} & -3 & 3 \end{bmatrix} \cdot \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

As above, after inserting the third eigen-stress:

$$6\hat{n}_1 - \sqrt{2}\hat{n}_2 + \sqrt{2}\hat{n}_3 = 0 \qquad [11]$$
$$-\sqrt{2}\hat{n}_1 + 3\hat{n}_2 - 3\hat{n}_3 = 0 \qquad [12]$$
$$\sqrt{2}\hat{n}_1 - 3\hat{n}_2 + 3\hat{n}_3 = 0 \qquad [13]$$

Uit [13] volgt dat

$$\hat{n}_1 = \tfrac{1}{2}\sqrt{2}\left(3\hat{n}_2 - 3\hat{n}_3\right) \qquad [14]$$

Inserted in [11] yields

$$\hat{n}_2 = \hat{n}_3 \qquad [15]$$

[15] back in [14] gives

$$\hat{n}_1 = 0$$

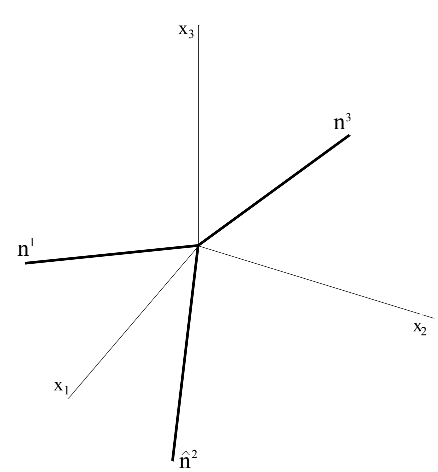
And the normalisation gives for $\hat{n}_3$

$$\left(0^2 + 1^2 + 1^2\right)\hat{n}_3^{\,2} = 1 \qquad \Leftrightarrow \qquad \hat{n}_3 = \pm\sqrt{\tfrac{1}{2}} = \pm\tfrac{1}{2}\sqrt{2}$$

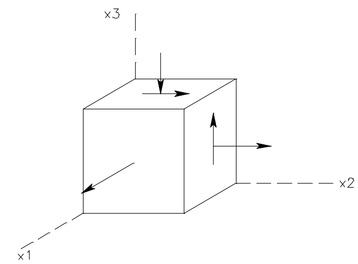
The third eigen-direction is thus:

$$\{\hat{n}_i\}^3 = \pm\frac{1}{2}\sqrt{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

Plotting the three direction-vectors:



Exercise V.3



Given:

a stress-state:

$$\sigma_{ij} = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20 \end{bmatrix} MPa$$

$E = 2 \cdot 10^5$ MPa $\qquad \nu = 0.25$

Solutions

a) Compute the principal stresses

From the stress matrix, one can see directly that one of the eigenvalues must be equal to σ_{11}. The shear-stresses on the '1-surface' are namely all zero ($\sigma_{12} = \sigma_{13} = 0$). Also plot the stresses in the stress-cube picture. From the linear algebra procedure follows:

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \Rightarrow$$

$$\det\begin{pmatrix} 60-\sigma & 0 & 0 \\ 0 & 20-\sigma & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-\sigma \end{pmatrix} = (60-\sigma)\left\{(20-\sigma)(-20-\sigma) - \left(20\sqrt{3}\right)^2\right\} =$$

$$(60-\sigma)\left\{-400 + \sigma^2 - 1200\right\} = (60-\sigma)\left\{\sigma^2 - 1600\right\} = 0$$

$$\Rightarrow \sigma_1 = 60 \text{ MPa}, \ \sigma_2 = 40 \text{ MPa}, \ \sigma_3 = -40 \text{ MPa}$$

b) Compute the eigen-directions

$$\left(\sigma_{ij} - \sigma\delta_{ij}\right)\hat{n}_j = 0 \ \ \& \ \ \hat{n}_j\hat{n}_j = 1$$

1st principal stress inserted:

$$\begin{bmatrix} 60-60 & 0 & 0 \\ 0 & 20-60 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-60 \end{bmatrix}\left\{\hat{n}_j^1\right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -80 \end{bmatrix}\left\{\hat{n}_j^1\right\} = 0 \Rightarrow \left\{\hat{n}_j^1\right\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

(compare to the sketch of the stress-cube.)

2nd principal stress inserted:

$$\begin{bmatrix} 60-40 & 0 & 0 \\ 0 & 20-40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20-40 \end{bmatrix}\left\{\hat{n}_j^2\right\} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & -20 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -60 \end{bmatrix}\left\{\hat{n}_j^2\right\} = 0$$

$$20 \cdot n_1^2 = 0 \Rightarrow n_1^2 = 0$$

$$-20 \cdot n_2^2 + 20\sqrt{3} \cdot n_3^2 = 0 \Rightarrow n_3^2 = \frac{20 \cdot n_2^2}{20\sqrt{3}} = \frac{n_2^2}{\sqrt{3}}$$

$$20\sqrt{3} \cdot n_2^2 - 60 \cdot n_3^2 = 0 \Rightarrow n_3^2 = \frac{20\sqrt{3} \cdot n_2^2}{60} = \frac{\sqrt{3} \cdot n_2^2}{3} = \frac{n_2^2}{\sqrt{3}}$$ (Equal to the above, due to dependence

$$\hat{n}_j\hat{n}_j = 1 \Rightarrow 0^2 + \left(n_2^2\right)^2 + \left(\frac{n_2^2}{\sqrt{3}}\right)^2 = 1 \Rightarrow \left(n_2^2\right)^2 = \frac{1}{1\frac{1}{3}} = \frac{3}{4} \Rightarrow n_2^2 = \frac{\sqrt{3}}{2} \Rightarrow n_3^2 = \frac{1}{2} \Rightarrow \left\{\hat{n}_j^2\right\} = \frac{1}{2}\begin{Bmatrix} 0 \\ \sqrt{3} \\ 1 \end{Bmatrix}$$

3rd stress eigenvalue inserted:

$$\begin{bmatrix} 60+40 & 0 & 0 \\ 0 & 20+40 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & -20+40 \end{bmatrix} \left\{ \hat{n}_j^3 \right\} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 60 & 20\sqrt{3} \\ 0 & 20\sqrt{3} & 20 \end{bmatrix} \left\{ \hat{n}_j^3 \right\} = 0$$

$$100 \cdot n_1^3 = 0 \Rightarrow n_1^3 = 0$$

$$60 \cdot n_2^3 + 20\sqrt{3} \cdot n_3^3 = 0 \Rightarrow n_3^3 = -\frac{60 \cdot n_2^3}{20\sqrt{3}} = -\sqrt{3} n_2^3$$

Equal to the above, due to dependence

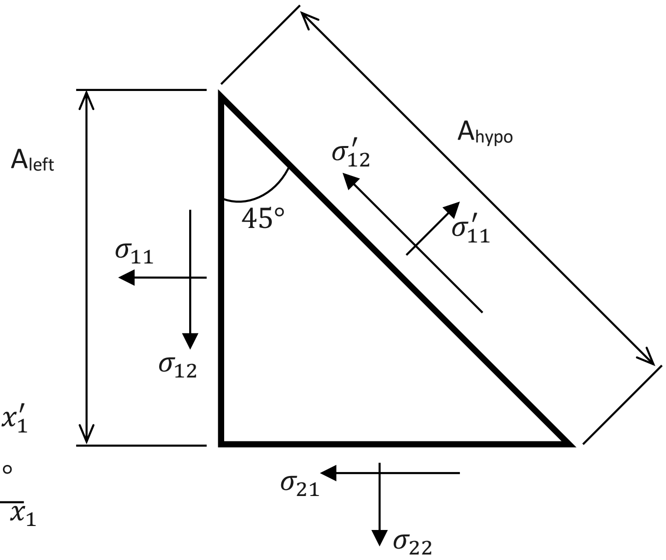
$$\hat{n}_j \hat{n}_j = 1 \Rightarrow 0^2 + \left(n_2^3\right)^2 + \left(-\sqrt{3} n_2^3\right)^2 = 1 \Rightarrow \left(n_2^3\right)^2 = \frac{1}{4} \Rightarrow n_2^3 = \frac{1}{2} \Rightarrow n_3^2 = -\frac{\sqrt{3}}{2} \Rightarrow \left\{ \hat{n}_j^2 \right\} = \frac{1}{2} \begin{Bmatrix} 0 \\ 1 \\ -\sqrt{3} \end{Bmatrix}$$

c) The maximal shear-stress

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{1}{2}\left(60 - -40\right) = 50 \text{ MPa}$$

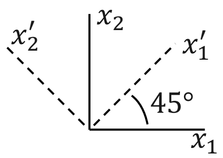
Exercise V-10

Problem:



Given:

- In a point P of a body we have a plane-stress state with: $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Given are these (mixed) stress components:
  $\sigma_{11} = 92$ MPa
  $\sigma'_{11} = 194$ MPa
  $\sigma'_{12} = -42$ MPa
- The material is linear elastic with:
  $E = 2 \cdot 10^5$ MPa
  $\nu = 0.25$



Questions:

- Give the stress tensor in the original $x_1x_2x_3$-system.
- Give the stress tensor in the new $x'_1x'_2x_3$ coordinate system, as obtained by a rotation of the coordinates about 45° around the x_3-axis, as sketched above.
- Compute the eigen-stresses and the eigen-directions.
- Give the strain tensor in the $x'_1x'_2x_3$ coordinate system.
- Compute the specific elastic energy in point P.

Solutions:

- There are two ways to solve this problem. The triangle above represents all stresses on all sides, but only part of the stress components are given.
  By considering force equilibrium and using the respective stress components divided by the side-lengths of the triangle (which also has a third dimension outside the plane, not shown). Assume the sides have unit-length, then the hypotenuse has according to Phytagoras length $\sqrt{2}$. Further assume the thickness also to be unit-length. The ratio between sides and hypothenuse is then:

$$\frac{A_{left}}{A_{hypo}} = \frac{A_l}{A_h} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

  With this we can get:.
  - Force-balance in x_1-direction:

$$A_h\sigma'_{11}\cos(45°) - A_h\sigma'_{12}\sin(45°) - A_l\sigma_{11} - A_l\sigma_{21} =$$
$$\sigma'_{11}\cos(45°) - \sigma'_{12}\sin(45°) - \frac{A_l}{A_h}\sigma_{11} - \frac{A_l}{A_h}\sigma_{21} =$$
$$\frac{1}{2}\sqrt{2}\sigma'_{11} - \frac{1}{2}\sqrt{2}\sigma'_{12} - \frac{1}{2}\sqrt{2}\sigma_{11} - \frac{1}{2}\sqrt{2}\sigma_{21} = 0$$

    So that: $\sigma_{21} \equiv \sigma_{12} = \sigma'_{11} - \sigma'_{12} - \sigma_{11} = 194 - -42 - 92 = 144$ MPa
  - Force-balance in x_2-direction:

$$A_h\sigma'_{11}\sin(45°) + A_h\sigma'_{12}\cos(45°) - A_l\sigma_{12} - A_l\sigma_{22} =$$
$$\sigma'_{11}\sin(45°) + \sigma'_{12}\cos(45°) - \frac{A_l}{A_h}\sigma_{12} - \frac{A_l}{A_h}\sigma_{22} =$$
$$\frac{1}{2}\sqrt{2}\sigma'_{11} + \frac{1}{2}\sqrt{2}\sigma'_{12} - \frac{1}{2}\sqrt{2}\sigma_{12} - \frac{1}{2}\sqrt{2}\sigma_{22} = 0$$

    so that: $\sigma_{22} = \sigma'_{11} + \sigma'_{12} - \sigma_{12} = 194 + -42 - 144 = 8$ MPa

The stress tensor in the $x_1x_2x_3$-system is thus:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

- The stress tensor in the $x_1'x_2'x_3$-system is obtained by rotating it from the $x_1x_2x_3$-system by $45°$. For this we have to build the rotationmatrix $R_{pi} = \cos(x_p, x_i)$. The stress tensor in the new system is then: $\sigma'_{pq} = R_{pi}R_{qj}\sigma_{ij}$ or in matrix-vector notation: $[\sigma'] = [R][\sigma][R]^T$:

$$[\sigma'] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & 144 & 0 \\ 144 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 194 & -42 & 0 \\ -42 & -94 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

The alternative for calculation uses the unknowns in this transformation relation:

$$[\sigma'] = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 92 & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

and, after spelling them out allows to solve without the geometry and force-balance:

$$[\sigma'] = \frac{1}{2}\sqrt{2} \begin{bmatrix} 92 + \sigma_{12} & \sigma_{12} + \sigma_{22} & 0 \\ -92 + \sigma_{12} & -\sigma_{12} + \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

and second step:

$$[\sigma'] = \frac{1}{2} \begin{bmatrix} 92 + 2\sigma_{12} + \sigma_{22} & -92 + \sigma_{22} & 0 \\ -92 + \sigma_{22} & 92 - 2\sigma_{12} + \sigma_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 194 & -42 & 0 \\ -42 & \sigma'_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

which yields from the non-diagonal: $\sigma_{22} = 8$, inserted in the 11-component: $\sigma_{12} = 144$, and inserted into 22: $\sigma'_{22} = -94$, as above. Thus the tensor transformation equation, given only the rotation angle, ignoring the geometry/area considerations, leads to the same result.

- The principal stresses and eigen-directions can now be computed the usual way with: $(\sigma_{ij} - \sigma\delta_{ij})\hat{n}_j = 0$ and $\hat{n}_j\ \hat{n}_j = 1$ or in vector-matrix notation: $([\sigma] - \sigma[I])\{\hat{n}\} = \{0\}$ and $\|\hat{n}\| = 1$ for the directions; while the principal stresses follow from $([\sigma] - \sigma[I]) = 0$, for which the determinant has to vanish:

$$det([\sigma] - \sigma[I]) = \begin{vmatrix} 92 - \sigma & 144 & 0 \\ 144 & 8 - \sigma & 0 \\ 0 & 0 & -\sigma \end{vmatrix} = [(92 - \sigma) \cdot (8 - \sigma) - 144^2] \cdot (-\sigma) = 0$$

The second term gives one eigenvalue, while first term is a polynomial of second order:

$$\sigma^2 - 100\sigma + 736 - 144^2 = 0$$

which has the solutions:

$$\sigma^{(1)} = \frac{100 + \sqrt{100^2 - 4 \cdot (736 - 144^2)}}{2} = 200 \text{ MPa}$$

$$\sigma^{(3)} = \frac{100 - \sqrt{100^2 - 4 \cdot (736 - 144^2)}}{2} = -100 \text{ MPa}$$

$$\sigma^{(3)} = 0 \text{ MPa}$$

The zero eigenvalue can already be deduced from the information that we have a plane-stess state.

The eigen-direction for the first eigen-value can be computed as:

$(92-200)n_1^{(1)} + 144 \cdot n_2^{(1)} = 0 \quad \rightarrow \quad -3 \cdot n_1^{(1)} + 4 \cdot n_2^{(1)} = 0$
$144 \cdot n_1^{(1)} + (8-200)n_2^{(1)} = 0 \quad \rightarrow \quad 3 \cdot n_1^{(1)} - 4 \cdot n_2^{(1)} = 0$
$-200 \cdot n_3^{(1)} = 0 \quad \rightarrow \quad n_3^{(1)} = 0$

After normalization, $\left|n^{(1)}\right| = 1$, this results in:

$$n^{(1)} = \frac{n^{(1)}}{\left|n^{(1)}\right|} = \begin{Bmatrix} 0.8 \\ 0.6 \\ 0 \end{Bmatrix}$$

Similar calculations – not details given here – yield:

$n^{(3)} = \begin{Bmatrix} 0.6 \\ -0.8 \\ 0 \end{Bmatrix}$ and (without calculation, due to plane-stress in 3-direction) $n^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

- The strain components can be calculated using the linear elastic material law:

$$\varepsilon_{ij} = \frac{1}{E}\{(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\} \quad \text{of} \quad \varepsilon' = \frac{1}{E}\{(1+\nu)[\sigma'] - \nu[I]tr[\sigma']\}$$

which result in: $\varepsilon' = \begin{bmatrix} 1087.5 & -262.5 & 0 \\ -262.5 & -712.5 & 0 \\ 0 & 0 & -125 \end{bmatrix} \cdot 10^{-6}$

- The specific elastic energy is then:

$$\pi_{el} = \frac{1}{2}\left(\sigma'_{11}\varepsilon'_{1,1} + 2\sigma'_{12}\varepsilon'_{1,2} + \sigma'_{22}\varepsilon'_{2,2} + 0 + 0 + 0\right) = 0.15 \text{ MPa}$$