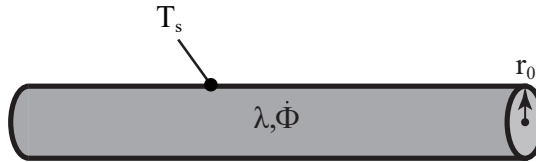**Exercise 21.10** (Resistance wire ★):

A long homogeneous resistance wire is used to heat the air in a room by the passage of an electric current. Heat is generated in the wire uniformly at a constant rate $\dot{\Phi}'''$ as a result of resistance heating.

Given parameters:

- Outer radius of the wire: $r_0 = 5$ mm

- Heat generation in the wire: $\dot{\Phi}''' = 5 \cdot 10^7$ W/m^3

- Temperature of the outer surface of the wire: $T_s = 180$ °C

- Thermal conductivity of the wire: $\lambda = 6$ W/mK

Hints:

- The problem is one-dimensional in radial direction.

- Assume steady-state conditions.

Tasks:

a) Derive the heat conduction equation by setting up an energy balance.

b) Determine the temperature at $r_1 = 3.5$ mm.

> **Solution 21.10** (Pipe fastening ★ ★ ★)**:**

The problem can be treated as one-dimensional in the radial direction and to be a steady state. To derive the heat conduction equation, an energy balance must be established for an infinitesimal element within the cross-section of the wire needs to be established.

> Task a)

① **Setting up the balance:**
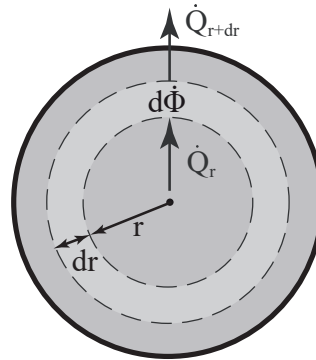
The energy balance for the infinitesimal element within the wire reads:

$$0 = \dot{Q}_{\mathrm{r}} - \dot{Q}_{\mathrm{r+dr}} + d\dot{\Phi} \tag{21.10.1}$$

② **Defining the elements within the balance:**

The ingoing conductive heat flux can be described by use of Fourier's law:

$$\dot{Q}_{\mathrm{r}} = -\lambda A(r)\frac{\partial T}{\partial r} \tag{21.10.2}$$

$$= -\lambda 2\pi r L \frac{\partial T}{\partial r}$$

The outgoing heat flux can be approximated for an infinitesimal element by the Taylor series expansion:

$$\dot{Q}_{\mathrm{r+dr}} = \dot{Q}_{\mathrm{r}} + \frac{\partial \dot{Q}_{\mathrm{r}}}{\partial r} \cdot dr \tag{21.10.3}$$

$$= -\lambda 2\pi r L \frac{\partial T}{\partial r} + \frac{\partial}{\partial r}\left(-\lambda 2\pi r L \frac{\partial T}{\partial r}\right) \cdot dr$$

The heat generated within the infinitesimal element can be expressed as:

$$d\dot{\Phi} = \dot{\Phi}''' \cdot dV \tag{21.10.4}$$

where dV is:

$$dV = \pi L\left((r+dr)^2 - r^2\right) = \pi L\left(2r \cdot dr + dr^2\right) \tag{21.10.5}$$

$$\approx 2\pi r L \cdot dr$$

Note that $dr^2 \ll 2r \cdot dr$ and therefore is negligible.

> **Conclusion**

③ Inserting and rearranging:

Substitution of the definition of the heat fluxes into the energy balance and rewriting of yields:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right) + \dot{\Phi}''' \tag{21.10.6}$$

> **Task b)**

To find the temperature at $r = 3.5\text{mm}$, the temperature profile function must be obtained by solving the heat conduction equation.

④ Defining the boundary and/or initial conditions:

To solve the heat conduction equation, two boundary conditions are needed, given that the temperature has been differentiated twice with respect to r.

At the center, symmetry is present, and consequently, the slope of the temperature profile should be horizontal, indicating a zero-slope gradient.

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \tag{21.10.7}$$

Moreover, the temperature at the interface of the wire has been specified, providing the second boundary condition.

$$T(r = r_0) = T_{\mathrm{s}} \tag{21.10.8}$$

⑤ Solving the equation:

Integrating the heat conduction equation once yields:

$$\frac{\partial T}{\partial r} = -\frac{\dot{\Phi}'''}{2\lambda}r + C_1 \tag{21.10.9}$$

and twice gives:

$$T(r) = -\frac{\dot{\Phi}'''}{4\lambda}r^2 + C_1 r + C_2 \tag{21.10.10}$$

Using the boundary condition of symmetry it yields that:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = -\frac{\dot{\Phi}'''}{2\lambda}r + C_1 = 0 \tag{21.10.11}$$

$$\Rightarrow C_1 = 0$$

and from the boundary condition of the known surface temperature, it can be found that:

$$T(r = r_0) = -\frac{\dot{\Phi}'''}{4\lambda}r_0^2\, C_2 = T_{\mathrm{s}} \tag{21.10.12}$$

$$\Rightarrow C_2 = T_{\mathrm{s}} + \frac{\dot{\Phi}'''}{4\lambda}r_0^2$$

Substitution of the definitions of both constants yields the radial temperature profile:

$$T(r) = T_s + \frac{\dot{\Phi}'''}{4\lambda}\left(r_0^2 - r^2\right) \tag{21.10.13}$$

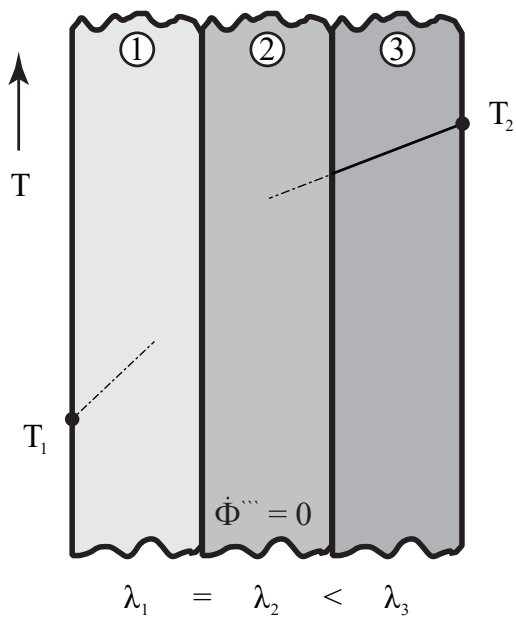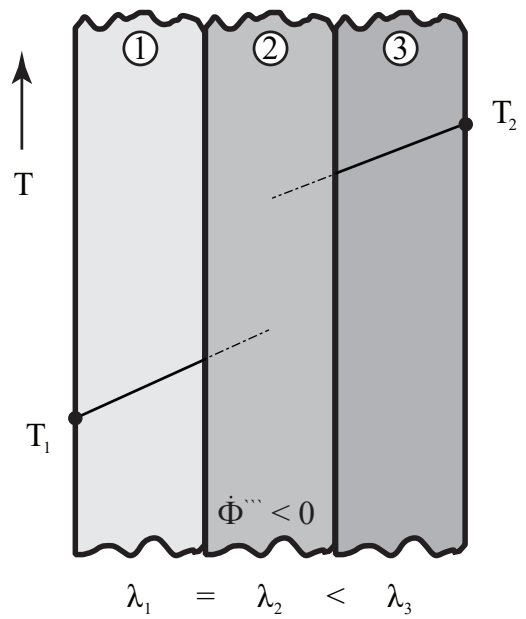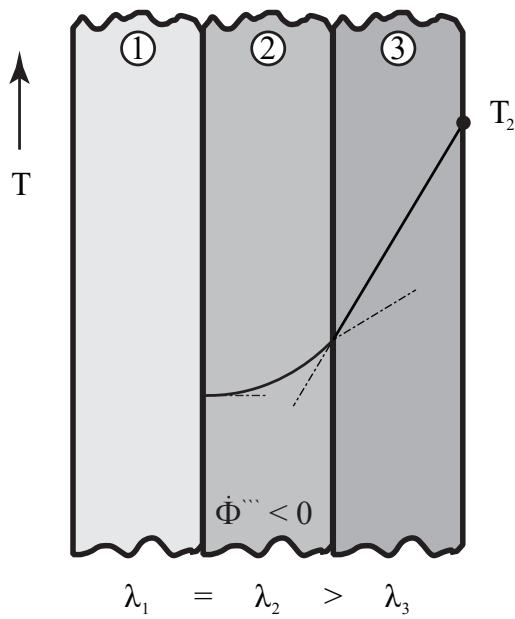**HEATQUIZ**   **UNIVERSITY OF TWENTE.**   **Institute of Heat and Mass Transfer**   **RWTH AACHEN UNIVERSITY**

From which the temperature at $r_1 = 3.5$ mm can be found:

$$T(r = r_1) = -\frac{\dot{\Phi}'''}{4\lambda}r_1^2 + T_s + \frac{\dot{\Phi}'''}{4\lambda}r_0^2 \qquad (21.10.14)$$

$$= 180 \; [^{\circ}\text{C}] + \frac{5 \cdot 10^7 \; [\text{W/m}^3]}{4 \cdot 6 \; [\text{W/mK}]} \cdot \left(0.005^2 - 0.0035^2\right) \; [\text{m}^2] = 207 \; [^{\circ}\text{C}]$$

Conclusion

The temperature at $r_1 = 3.5$ mm is thus 207 °C.

CONDUCTION SOLUTIONS

Exercise 21.11 (Multi-layer walls with source ★★):

Tasks:

a) Complete the temperature profiles in the three-layered walls.

Solution 21.11 (Multi-layer walls with source ★★)**:**

Task a.1)

The temperature profile of layer 1 from the top left multi-layer wall should be drawn horizontally.

First, one needs to determine the gradient at the interface between layers 1 and 2, as becomes evident by the energy balance at the interface:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 \tag{21.11.1}$$

Given that $\lambda_1 = \lambda_2$, it follows that $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$ at the interface between both layers, resulting in a horizontal gradient at that position.

Moreover, no heat generation occurs in that layer, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x} \tag{21.11.2}$$

it can be observed that if the rate of heat transfer within the layer remains constant, with a consistent area and thermal conductivity, the gradient should be the same throughout the entire layer, thus remaining horizontal within the entire layer.

Conclusion
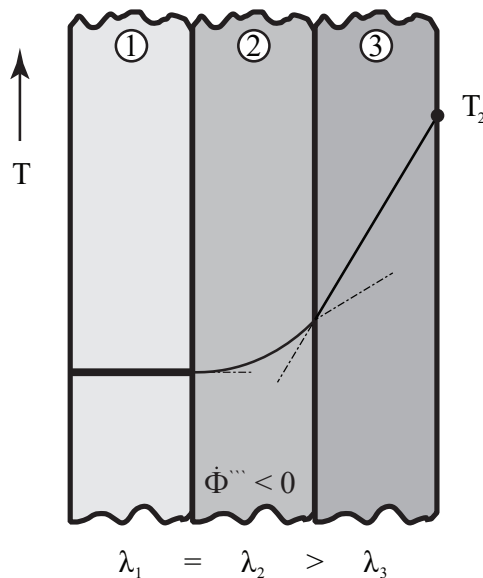
Task a.2)

The temperature profile for layer 2 of the top right multi-layer wall should be drawn with a decreasing temperature gradient when moving from right to left while having the same gradient as layer 1 at their interface and a steeper gradient than layer 3 at their interface.

First, one needs to determine the gradients at the interfaces between layers 1 and 2 and layers 2 and 3, which become evident from the energy balances at the interface. The energy

balance at the interface between layers 1 and 2 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 \tag{21.11.3}$$

Given that $\lambda_1 = \lambda_2$, it follows that $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$ at the interface between both layers, resulting in a gradient of layer 2 just as steep as that of layer 1 at that position.

The energy balance at the interface between layers 2 and 3 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_3 \tag{21.11.4}$$

Given that $\lambda_2 < \lambda_3$, it follows that $\left(\frac{\partial T}{\partial x}\right)_2 > \left(\frac{\partial T}{\partial x}\right)_3$ at the interface between both layers, resulting in a gradient of layer 2 that should be steeper than that of layer 3 at that position.

Moreover, layer 2 acts as a sink, thus removal occurs in that layer, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x} \tag{21.11.5}$$

it can be observed that if the rate of heat transfer within the layer decreases, with a consistent area and thermal conductivity, the gradient should decrease when moving along the direction that heat travels in (right to left).

Conclusion
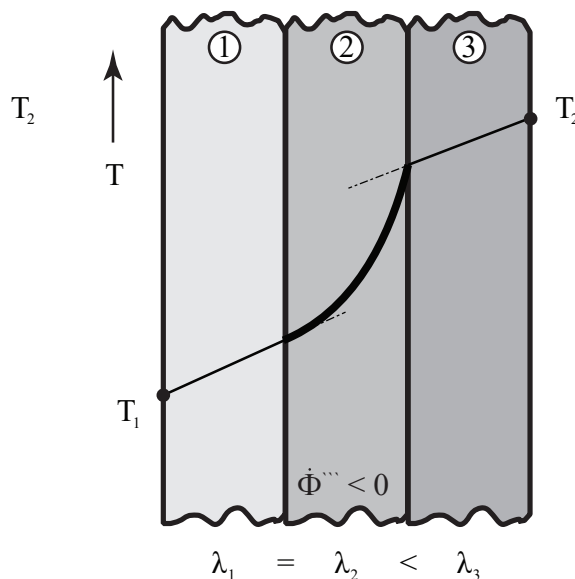
Task a.3)

The temperature profile for layers 1 and 2 of the bottom multi-layer wall should be drawn with a constant temperature gradient when moving from right to left while having the specified gradient of layer 1 at its interface with the ambient and a steeper gradient than layer 3 at their interface.

First, one needs to determine the gradients at the interfaces between layers 1 and 2 and layers 2 and 3, which become evident from the energy balances at the interface. The energy

balance at the interface between layers 1 and 2 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_1 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 \tag{21.11.6}$$

Given that $\lambda_1 = \lambda_2$, it follows that $\left(\frac{\partial T}{\partial x}\right)_1 = \left(\frac{\partial T}{\partial x}\right)_2$ at the interface between both layers, resulting in a gradient of layer 2 just as steep as that of layer 1 at that position.

The energy balance at the interface between layers 2 and 3 reads:

$$\dot{Q} = \left(-\lambda A \frac{\partial T}{\partial x}\right)_2 = \left(-\lambda A \frac{\partial T}{\partial x}\right)_3 \tag{21.11.7}$$

Given that $\lambda_2 < \lambda_3$, it follows that $\left(\frac{\partial T}{\partial x}\right)_2 > \left(\frac{\partial T}{\partial x}\right)_3$ at the interface between both layers, resulting in a gradient of layer 2 that should be steeper than that of layer 3 at that position.

Moreover, no heat generation occurs in layers 1 and 2, and according to Fourier's law:

$$\dot{Q} = -\lambda A \frac{\partial T}{\partial x} \tag{21.11.8}$$

it can be observed that if the rate of heat transfer within the layers remains constant, with a consistent area and thermal conductivity, the gradient should be the same throughout both layers.

Conclusion
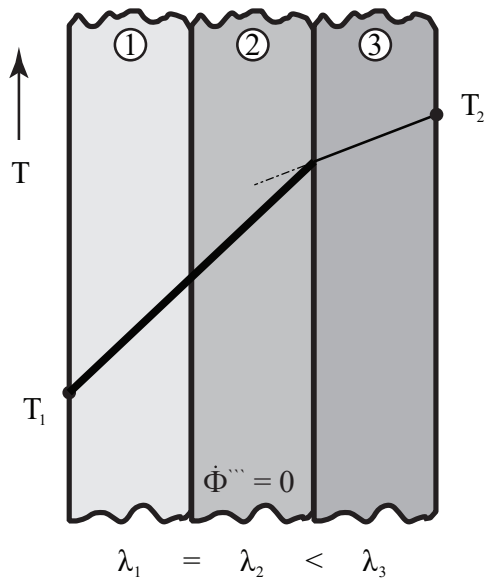
Exercise 21.12 (Copper rod ★ ★ ★):

Both ends of a copper rod with a length L and a diameter d are kept at the same temperature T_O. The left half of the rod is insulated against all radial heat losses. An electric heating element generates Joule's heat of heat flux density $\dot{\Phi}'''$. The right half of the rod is subjected to a flow of the ambient air with an air temperature of T_A, yielding a heat transfer coefficient α. The thermal conductivity of the rod is given as λ.
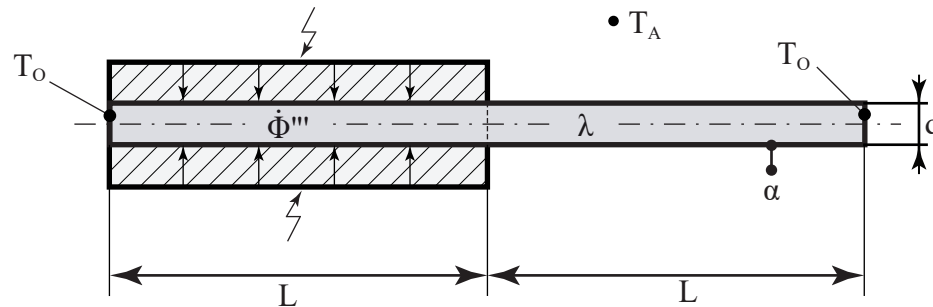
Given parameters:

- Length of the rod: $L = 1$ m

- Diameter of the rod: $d = 5.2$ mm

- Temperature of the ends of both rods: $T_O = 120$ °C

- Temperature of the ambient: $T_A = 100$ °C

- Heat transfer coefficient: $\alpha = 6$ W/m²K

- Thermal conductivity of the rod: $\lambda = 372$ W/mK

Hint:

- Place the origin of the coordinate system in the middle of the rod.

Tasks:

a) Derive the equation for the temperature profile in the rod by setting up an energy balance.

b) Determine an expression for $\dot{\Phi}'''$ such that the temperature in the center of the rod is also T_O, similar to the temperatures at its ends.

c) Calculate the value for $\dot{\Phi}'''$ for the conditions postulated in b).

d) Determine the extremes of the temperature distribution for the given values. Give their position and values, additionally, sketch the temperature profile.

Solution 21.12 (Copper rod ★ ★ ★):

Task a)

As observed earlier, the temperature profile can be obtained from the conduction equation within an element. However, it is crucial to note that both segments of the rod require a different conduction equation to be solved. The left segment, denoted as part 1, involves heat generation, while the right segment, denoted as part 2, does not include this term but experiences heat loss through convection, acting as a fin. Consequently, distinct conduction equations must be formulated and solved for both segments to accurately model their respective behaviors.

1 **Setting up the balance:**

(a) Part 1   (b) Part 2
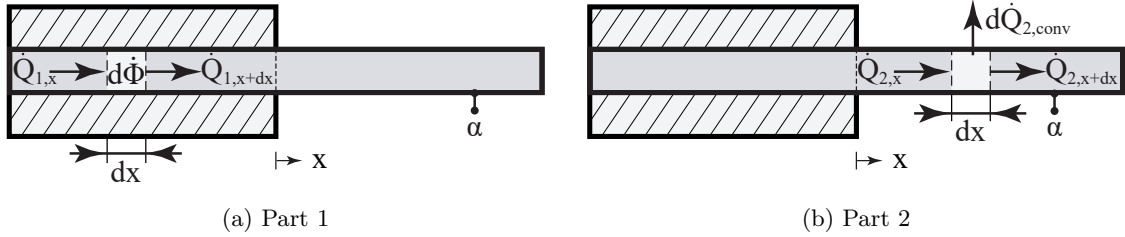
The energy balance of part 1 reads:

$$0 = \dot{Q}_{1,\text{x}} - \dot{Q}_{1,\text{x+dx}} + d\dot{\Phi} \tag{21.12.1}$$

and the energy balance of part 2 reads:

$$0 = \dot{Q}_{2,\text{x}} - \dot{Q}_{2,\text{x+dx}} - d\dot{Q}_{\text{conv}} \tag{21.12.2}$$

Note that the energy balance assumes heat conduction in the positive x-direction and heat loss by convection. However, the steel band will receive heat due to convection and conduct it in the negative x-direction towards the brine. This direction will be considered when defining the fluxes.

2 **Defining the elements within the balance:**

For both rods the incoming conducted heat can be expressed by Fourier's law:

$$\dot{Q}_{1,\text{x}} = -\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x} \tag{21.12.3}$$

and:

$$\dot{Q}_{2,\text{x}} = -\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x} \tag{21.12.4}$$

Similarly, the outgoing conducted heat can be approximated by the use of the Taylor series expansion for an infinitesimal element:

$$\dot{Q}_{1,\text{x+dx}} = \dot{Q}_{1,\text{x}} + \frac{\partial \dot{Q}_{1,\text{x}}}{\partial x} \cdot dx \tag{21.12.5}$$

$$= -\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda \frac{\pi d^2}{4} \frac{\partial T_1}{\partial x}\right) \cdot dx$$

and:

$$\dot{Q}_{2,\text{x+dx}} = \dot{Q}_{2,\text{x}} + \frac{\partial \dot{Q}_{2,\text{x}}}{\partial x} \cdot dx \tag{21.12.6}$$

$$= -\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda \frac{\pi d^2}{4} \frac{\partial T_2}{\partial x}\right) \cdot dx$$

The generated heat within the infinitesimal element of part 1 yields from the product of the volumetric source term and the volume of the infinitesimal element:

$$d\dot{\Phi} = \dot{\Phi}''' dV \tag{21.12.7}$$

$$= \dot{\Phi}''' \frac{\pi d^2}{4} dx$$

Lastly, the rate of heat loss by convection yields from the product of the heat transfer coefficient, the surface area of the infinitesimal element, and the temperature difference between the element and the ambient:

$$d\dot{Q}_{\text{conv}} = \alpha \cdot dA_\text{s} \cdot (T_2(x) - T_\text{A}) \tag{21.12.8}$$

$$= \alpha\pi d \cdot dx \, (T_2(x) - T_\text{A})$$

3 **Inserting and rearranging:**

Substituting these definitions into the energy balances and rearranging yields the differential equation for part 1:

$$\frac{\partial^2 T_1}{\partial x^2} = -\frac{\dot{\Phi}'''}{\lambda} \tag{21.12.9}$$

and for part 2:

$$0 = \frac{\partial^2 T_2}{\partial x^2} - \frac{4\alpha}{\lambda d}(T_2(x) - T_\text{A}) \tag{21.12.10}$$

4 **Defining the boundary and/or initial conditions:**

To determine the temperature profiles inside the system, in total 4 boundary and/or coupling conditions are required.

The temperature at the left side of the system is known:

$$T_1(x = -L) = T_\text{O} \tag{21.12.11}$$

and at the right side:

$$T_2(x = L) = T_\text{O} \tag{21.12.12}$$

But also, the temperatures at the interface between the left and right sides should be equal to each other:

$$T_1(x = 0) = T_2(x = 0) \tag{21.12.13}$$

Lastly, from the energy balance at the interface $\dot{Q} = -\lambda A \left.\frac{\partial T_1}{\partial x}\right|_{x=0} = -\lambda A \left.\frac{\partial T_2}{\partial x}\right|_{x=0}$ it yields that:

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=0} = \left.\frac{\partial T_2}{\partial x}\right|_{x=0} \tag{21.12.14}$$

⑤ **Solving the equation:**

Conclusion

Integrating the differential equation for part 1 once yields:

$$\frac{\partial T_1}{\partial x} = -\frac{\dot{\Phi}'''}{\lambda} \cdot x + c_1 \tag{21.12.15}$$

and twice gives:

$$T_1(x) = -\frac{\dot{\Phi}'''}{2 \cdot \lambda} \cdot x^2 + c_1 \cdot x + c_2 \tag{21.12.16}$$

The differential equation for part 2 is a linear second-order differential equation with constant coefficients. Therefore, this differential equation can be solved with the same approach as for a fin system. Where the first step is introducing the spatial temperature difference θ_2

$$\theta_2 = T_2(x) - T_A \tag{21.12.17}$$

and the fin parameter m:

$$m = \sqrt{\frac{4\alpha}{\lambda d}} \tag{21.12.18}$$

Substitution into the differential equation of part 2 gives:

$$0 = \frac{\partial^2 \theta_2}{\partial x^2} - m \cdot \theta_2 \tag{21.12.19}$$

which has its standard solution:

$$\theta_2 = c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx) \tag{21.12.20}$$

And therefore:

$$T_2(x) = T_A + c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx) \tag{21.12.21}$$

To determine the constants c_1, c_2, c_3 and c_4 the set boundary conditions should be used, which eventually yields:

$$c_1 = \frac{1}{L} \cdot \frac{(T_O - T_A) - \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right) \cdot \cosh(mL)}{\frac{\sinh(mL)}{mL} + \cosh(mL)} \tag{21.12.22}$$

$$c_2 = T_A + \frac{(T_O - T_A) + \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right) \cdot \frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)} \tag{21.12.23}$$

$$c_3 = \frac{c_1}{m} \tag{21.12.24}$$

$$c_4 = c_2 - T_A \tag{21.12.25}$$

With the substitution of these constants in both temperature functions, the temperature profile in the entire rod is defined for a given heat source density $\dot{\Phi}'''$.

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Task b)

To find $\dot{\Phi}'''$ for which $T(x=0)=T_O$, we should find the expressions for $T_1(x=0)$ and $T_2(x=0)$.

From the previously determined temperature profiles, it can be found that:

$$T_1(x=0) = T_O \tag{21.12.26}$$
$$c_2 = T_O$$
$$T_A + \frac{T_O - T_A + \left(T_O - T_A + \frac{\dot{\Phi}'''}{2\lambda}L^2\right)\frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)} = T_O$$

Conclusion

Rewriting yields:

$$\dot{\Phi}''' = \frac{2\lambda m}{L} \cdot (T_O - T_A) \cdot \frac{\cosh(mL)-1}{\sinh(mL)} \tag{21.12.27}$$
$$= \frac{2\lambda m}{L} \cdot (T_O - T_A) \cdot \tanh\left(\frac{mL}{2}\right)$$

Task c)

To find the numerical value of the volumetric heat source, first, the fin parameter needs to be determined:

$$m = \sqrt{\frac{4\alpha}{\lambda d}} \tag{21.12.28}$$
$$= \sqrt{\frac{4 \cdot 6 \ [\text{W/m}^2\text{K}]}{372 \ [\text{W/mK}] \cdot 0.0052 \ [\text{m}]}} = 3.52 \left[\frac{1}{\text{m}}\right]$$

Substitution into the previously found expression for the volumetric heat source yields:

$$\dot{\Phi}''' = \frac{2\lambda m}{L} \cdot (T_O - T_A) \cdot \tanh\left(\frac{mL}{2}\right) \tag{21.12.29}$$
$$= \frac{2 \cdot 372 \ [\text{W/mK}] \cdot 3.52 \left[\frac{1}{\text{m}}\right]}{1 \ [1]} \cdot (120-100) \ [\text{K}] \cdot \tanh\left(\frac{3.52 \left[\frac{1}{\text{m}}\right] \cdot 1 \ [\text{m}]}{2}\right) = 49.4 \ [\text{kW/m}^3]$$

Conclusion

The volumetric heat generation within the left part is about 49.4 kW/m^3

Task d)

It can be reasoned that given the temperature of T_O at the left tip, center, and right tip, coupled with the homogeneous heat generation in the left part and the heat loss due to convection in the right part, the maximum temperature will be located within the left part, while the minimum temperature will be in the right part.

More specifically, due to symmetry within both parts, the location of the maximum temperature

will be at

$$x_{\max} = -\frac{L}{2} \qquad (21.12.30)$$

and the location of the minimum temperature will be:

$$x_{\min} = \frac{L}{2} \qquad (21.12.31)$$

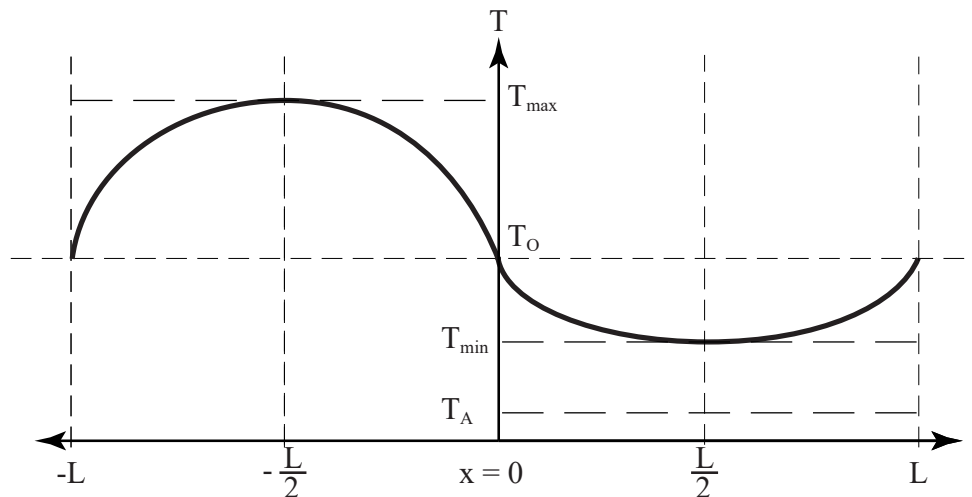Substitution of these positions into the derived temperature profiles yields the maximum temperature

$$T_{\max} = T_1\left(x = -\frac{L}{2}\right) \qquad (21.12.32)$$

$$= T_{\mathrm{O}} + \frac{\dot{\Phi}''' L^2}{8\lambda}$$

$$= 120 \,[^{\circ}\mathrm{C}] + \frac{49{,}400 \,\left[\mathrm{W/m^3}\right] \cdot 1^2 \,\left[\mathrm{m^2}\right]}{8 \cdot 372 \,[\mathrm{W/mK}]} = 137 \,[^{\circ}\mathrm{C}]$$

and the minimum temperature:

$$T_{\min} = T_2\left(x = \frac{L}{2}\right) \qquad (21.12.33)$$

$$= T_{\mathrm{A}} + \frac{T_{\mathrm{O}} - T_{\mathrm{A}}}{\cosh\left(\frac{mL}{2}\right)}$$

$$= 100 \,[^{\circ}\mathrm{C}] + \frac{(120 - 100) \,[^{\circ}\mathrm{C}]}{\cosh\left(\frac{3.52 \,\left[\frac{1}{\mathrm{m}}\right] \cdot 1 \,[\mathrm{m}]}{2}\right)} = 107 \,[^{\circ}\mathrm{C}]$$

Conclusion

Which results in the following temperature profile:

where the maximum temperature is thus 137 °C and the minimum temperature 107 °C.