PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = 2x[n+2] + 6x[n] + 2x[n-2]. \tag{1}$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal. Explain your answers.

(b) Obtain an expression for the frequency response of this system.

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(d) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 10 - 10\cos(0.5\pi(n-1))$$

Hint: Use the frequency response and superposition to solve this problem.

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SOLUTION

a.i) The system is linear because:

$$x_1[n] \to y_1[n] = 2x_1[n+2] + 6x_1[n] + 2x_1[n-2]$$

$$x_2[n] \to y_2[n] = 2x_2[n+2] + 6x_2[n] + 2x_2[n-2]$$

$$c_1 x_1[n] + c_2 x_2[n] \to y_3[n] = 2\left(c_1 x_1[n+2] +\right.$$

$$\left. + c_2 x_2[n+2]\right) + 6\left(c_1 x_1[n] + c_2 x_2[n]\right) +$$

$$+ 2\left(c_1 x_1[n-2] + c_2 x_2[n-2]\right) =$$

$$= c_1\left(2x_1[n+2] + 6x_1[n] + 2x_1[n-2]\right) +$$

$$+ c_2\left(2x_2[n+2] + 6x_2[n] + 2x_1[n-2]\right) =$$

$$= c_1 y_1[n] + c_2 y_2[n].$$

ii) The system is time-invariant because:

$$x[n-n_0] \to 2x[n+2-n_0] + 6x[n-n_0] +$$

$$+ 2x[n-2-n_0] = y[n-n_0].$$

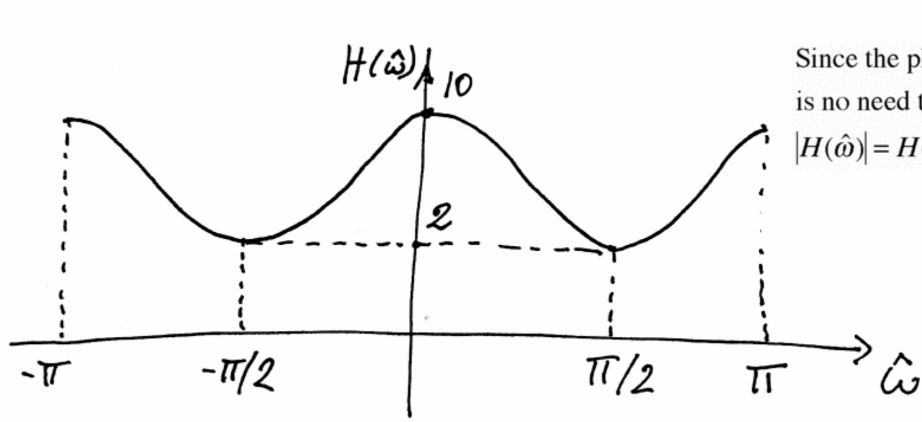
iii) The system is not causal because $y[n]$ depends on $x[n+2]$.

b) $h[n] = 2\delta[n+2] + 6\delta[n] + 2\delta[n-2]$

$$\Rightarrow H(\hat{\omega}) = 2e^{j2\hat{\omega}} + 6 + 2e^{-j2\hat{\omega}} =$$

$$= 6 + 4\cos(2\hat{\omega})$$

c)



Since the phase is zero, there is no need to plot the phase: $|H(\hat{\omega})| = H(\hat{\omega})$

d) $H(0) = 10 \qquad H(0.5\pi) = 2 = H(-0.5\pi)$

$$10 \longrightarrow H(0)\cdot 10 = 100$$

$$10\cos\left[\tfrac{\pi}{2}(n-1)\right] = 5e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}n} + 5e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{2}n}$$

$$\longrightarrow 5e^{-j\frac{\pi}{2}}H(0.5\pi)e^{j\frac{\pi}{2}n} + 5e^{j\frac{\pi}{2}}H(-0.5\pi)e^{-j\frac{\pi}{2}n} =$$

$$= 10e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}n} + 10e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{2}n} =$$

$$= 20\cos\left[\tfrac{\pi}{2}(n-1)\right]$$

$$y_1[n] = 100 - 20\cos\left[\tfrac{\pi}{2}(n-1)\right]$$

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PROBLEM:

The intention of the following MATLAB program is to filter a sinusoid via the `conv` function. However, the cosine signal has a starting point at $n = 0$; so we assume that it is zero for $n < 0$.

```
omegahat = pi/2;
nn = [ 0:4000 ];
xn = 6*cos(omegahat*nn - pi/2);
bb = ones(1,6)/6;
yn = conv( bb, xn );
```

(a) Determine a formula for $\mathcal{H}(\hat{\omega})$ for this FIR filter.

(b) Make a plot of the magnitude of $\mathcal{H}(\hat{\omega})$ and label *all* the frequencies where $|\mathcal{H}(\hat{\omega})|$ is zero. Use `freqz(bb, 1, ww)` in MATLAB, where `ww` is a vector of frequencies that defines a dense grid for $\hat{\omega}$.

(c) Use convolution to determine a formula (or table) for $y[n]$, the signal contained in the vector `yn`. Give the individual values for $n = 0, 1, 2, 3, 4, 5$, and then provide a general formula for $y[n]$ that is correct for $6 \leq n \leq 4000$. This formula should give numerical values for the amplitude, phase and frequency of $y[n]$. Hint: the formula is a sinusoid for $n \geq 6$.

(d) Give at least one different value of `omegahat` such that the output is guaranteed to be zero, for $n \geq 6$.

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SOLUTION

a) $h[n] = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$

$$H(\hat{\omega}) = \sum_{k=0}^{5} h[k] e^{j\hat{\omega}k}$$

$$= e^{-j\frac{5}{2}\hat{\omega}} \frac{1}{6} \frac{\sin(3\hat{\omega})}{\sin(\hat{\omega}/2)}$$

b) SEE ATTACHED PLOT

c) ONCE PAST INITIAL TRANSITION EFFECTS ("END-EFFECTS")
THE OUTPUT FREQUENCY LOOKS LIKE A SCALED + SHIFTED
COPY OF THE INPUT FREQUENCY.

BECAUSE $\hat{\omega} = \frac{\pi}{2}$, THERE ARE 4 SAMPLES PER 1 PERIOD

MATLAB INDEX → 1 2 3 4 5 6 7 8 9 10 11

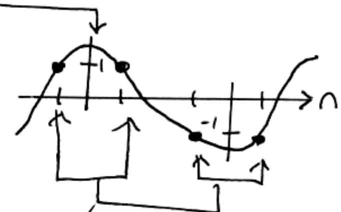
in ← $x[n] = [\,0 \quad 6 \quad 0 \quad -6 \quad 0 \quad \underbrace{6 \quad 0 \quad -6 \quad 0}_{\text{PERIOD = 4 SAMPLES}} \quad 6 \quad 0 \quad -6 \quad 0 \ldots]$

$\boxed{h}$

out → $y[n] = [\,\underbrace{0 \quad 1 \quad 1 \quad 0 \quad 0}_{\text{TRANSIENT RESPONSE}} \quad \underbrace{1 \quad 1 \quad -1 \quad -1}_{\text{STEADY STATE}} \quad 1 \quad 1 \quad -1 \quad -1 \ldots]$

SINUSOID

$$y[n \geq 6] = \sqrt{2} \cos\left(\frac{1}{2}\pi n - \frac{3}{4}\pi\right)$$



IF $A\cos(45°) = 1$
THEN $A = \sqrt{2}$

I GUESSED + CONFIRMED

SO, MUST BE $0° \pm 45°$ or $180° \pm 45°$

– OR –

SOLVE
$y[7] = A\cos\left(\frac{1}{2}\pi 7 - \phi\right) = 1$
$y[8] = A\cos\left(\frac{1}{2}\pi 8 - \phi\right) = -1$
$y[9] = A\cos\left(\frac{1}{2}\pi 9 - \phi\right) = -1$
$y[10] = A\cos\left(\frac{1}{2}\pi 10 - \phi\right) = 1$ FOR A AND ϕ
etc...

d) SEE PLOT OF $H(\hat{\omega})$: ZERO FOR ANY $\hat{\omega} = k \cdot \frac{1}{3}\pi$ $\quad k \in$ INTEGER

– OR –

WHEN IS ($|$NUMERATOR OF $H(\hat{\omega})| = \sin(3\hat{\omega})$) = ZERO?

WHEN $\sin(\hat{\omega}3) = 0$

$$\hat{\omega}3 = k\pi$$
$$\hat{\omega} = k\frac{\pi}{3}$$

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RUNNING AVERAGE FILTER DERIVATION

SEMI-INFINITE GEOMETRIC SERIES

$$(a^1 + a^2 + a^3 + a^4 + \dots) = \sum_{k=1}^{\infty} a^k = \sum_{k=0}^{\infty} a^{k+1} = a\sum_{k=0}^{\infty} a^k$$

$$+\,(a^0) = \longrightarrow + \; a^0 = 1$$

$$(a^0 + a^1 + a^2 + a^3 + a^4 + \dots) = \sum_{k=0}^{\infty} a^k = 1 + a\sum_{k=0}^{\infty} a^k$$

$$\sum_{k=0}^{\infty} a^k - a\sum_{k=0}^{\infty} a^k = 1$$

$$(1-a)\sum_{k=0}^{\infty} a^k = 1$$

$$\boxed{\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad \text{FOR } |a|<1}$$

SEMI-INFINITE

FINITE GEOMETRIC SERIES

$0 \longrightarrow \infty$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

$M \longrightarrow \infty$

$$\sum_{\substack{k=M \\ k-M=0}}^{\infty} a^{k-M} = a^{-M}\sum_{k=M}^{\infty} a^k = \frac{1}{1-a}$$

$$\sum_{k=M}^{\infty} a^k = \frac{a^M}{1-a}$$

$0 \quad M-1$

FINITE

$$\sum_{k=0}^{M-1} a^k = \sum_{k=0}^{\infty} a^k - \sum_{k=M}^{\infty} a^k = \boxed{\frac{1-a^M}{1-a} = \sum_{k=0}^{M-1} a^k}$$

$h[n] \rightarrow H(\hat{\omega})$

$$H(\hat{\omega}) = \sum_{k=0}^{L-1} h[n]\, e^{-j\hat{\omega}k} = \left\{ \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right) = \frac{1}{L}\sum_{k=0}^{L-1} e^{j\hat{\omega}k} \right\} \text{ FOR L-ELEMENT RUNNING AVERAGE}$$

$$= \frac{1}{L}\,\frac{e^{-j\hat{\omega}\frac{L}{2}}}{e^{-j\hat{\omega}2}}\,\frac{\left(e^{j\hat{\omega}\frac{L}{2}} - e^{-j\hat{\omega}\frac{L}{2}}\right)}{\left(e^{j\hat{\omega}/2} - e^{j\hat{\omega}/2}\right)} = \frac{e^{j\hat{\omega}(L-1)/2}}{L}\cdot\frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)}$$

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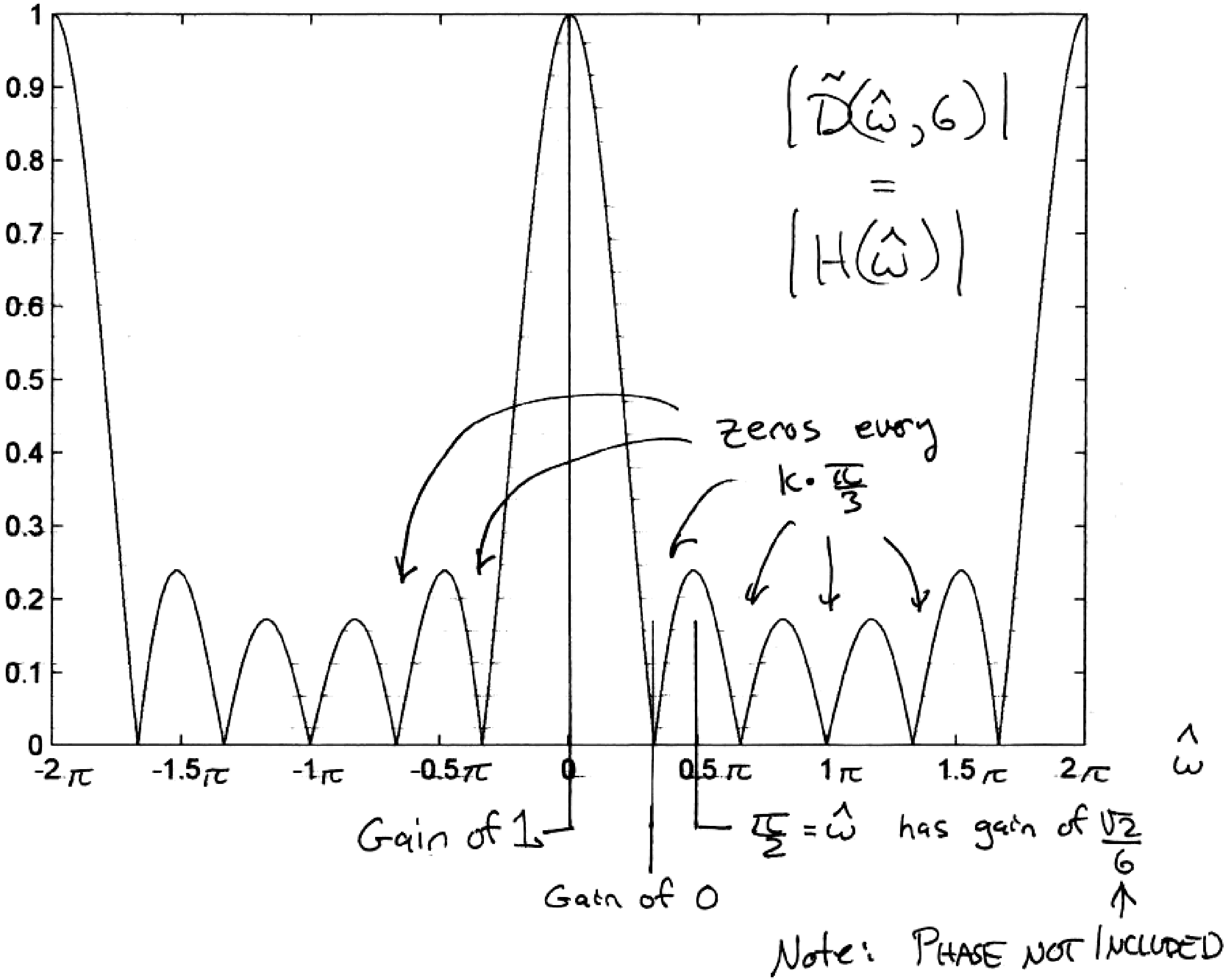

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$H(\hat{\omega}) = (1 + 0.8e^{-j\hat{\omega}})(1 - e^{-j\pi/2}e^{-j\hat{\omega}})(1 - e^{j\pi/2}e^{-j\hat{\omega}}). \tag{1}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. *Hint: Multiply out the factors to obtain a sum of powers of* $e^{-j\hat{\omega}}$.

(b) What is the impulse response of this system?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?

(d) Use superposition to determine the output of this system when the input is

$$x[n] = 3 + \delta[n-3] + e^{j0.5\pi n} \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results.

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SOLUTION

Chap6: Analyze an LTI System Defined by a Factored Frequency Response
stusol/sp_01/pset07_8.pdf

a)

$$H(\hat{\omega}) = (1 + 0.8e^{-j\hat{\omega}})(1 - \cancel{e^{j\frac{\pi}{2}}} e^{-j\hat{\omega}} + - \cancel{e^{-j\frac{\pi}{2}}} e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}) =$$

$$= 1 + 0.8e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + 0.8e^{-j3\hat{\omega}}$$

$$y[n] = x[n] + 0.8\,x[n-1] + x[n-2] + + 0.8\,x[n-3]$$

b)

$$h[n] = \delta[n] + 0.8\,\delta[n-1] + \delta[n-2] + + 0.8\,\delta[n-3].$$

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c) If $x[n] = X_0 e^{j\hat{\omega} n}$, then $y[n] = X_0 H(\hat{\omega}) e^{j\hat{\omega} n}$, so $y[n] \equiv 0$ if $H(\hat{\omega}) = 0$. From the factored form of $H(\hat{\omega})$ given in eqn. (3), $H(\hat{\omega}) = 0$ when:

$$1 - e^{-j\frac{\pi}{2}} e^{-j\hat{\omega}} = 1 - e^{-j(\hat{\omega} + \frac{\pi}{2})} = 0 \Rightarrow \hat{\omega} = -\frac{\pi}{2}$$

$$1 - e^{j\frac{\pi}{2}} e^{-j\hat{\omega}} = 1 - e^{-j(\hat{\omega} - \frac{\pi}{2})} = 0 \Rightarrow \hat{\omega} = \frac{\pi}{2}$$

$\left(1 + 0.8 e^{-j\hat{\omega}} \neq 0 \right.$ for all values of $\hat{\omega}$ because $\left. |0.8 e^{-j\hat{\omega}}| = 0.8 < 1. \right)$

So $H(\hat{\omega}) = 0$ for $\hat{\omega} = \pm \frac{\pi}{2}$.

d) $3 \to H(0) \cdot 3 = 3.6 \cdot 3 = 10.8$

$$\delta[n-3] \to h[n-3] = \delta[n-3] + 0.8\,\delta[n-4] + \delta[n-5] + 0.8\,\delta[n-6]$$

$$e^{j0.5\pi n} \to H(\tfrac{\pi}{2}) e^{j0.5\pi n} = 0.$$

So $y[n] = 10.8 + \delta[n-3] + 0.8\,\delta[n-4] + \delta[n-5] + 0.8\,\delta[n-6]$.

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PROBLEM:

Consider again the cascade system in Figure 1 with

$$h_1[n] = \delta[n] - \delta[n-1] \quad \text{and} \quad h_2[n] = u[n] - u[n-5].$$

(a) Determine $H_1(\hat{\omega})$, the frequency response of the first system.

(b) Determine $H_2(\hat{\omega})$, the frequency response of the second system.

(c) By convolution, show that $h[n] = h_1[n] * h_2[n] = \delta[n] - \delta[n-5]$ (see part part (c) of Problem 7.5 with $\alpha = 1$). From $h[n]$ determine $H(\hat{\omega})$ the frequency response of the overall system (from $x[n]$ to $y[n]$).

(d) Show that your result in part (c) is the product of the results in parts (a) and (b); i.e., $H_1(\hat{\omega})H_2(\hat{\omega}) = H(\hat{\omega})$.

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SOLUTION

a) $H_1(\hat{\omega}) = 1 - e^{-j\hat{\omega}}$

b) $H_2(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$

(c) $h[n] = \delta[n] - \delta[n-5]$

Therefore, the filter coefficients are: $\{b_k\} = \{1, 0, 0, 0, 0, -1\}$

$$\Rightarrow \mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = 1 - e^{-j\hat{\omega}5}$$

d)
$$H_1(\hat{\omega}) H_2(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$$
$$- e^{-j\hat{\omega}}\left(1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}\right)$$
$$= 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$$
$$- e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}} - e^{-j4\hat{\omega}} - e^{-j5\hat{\omega}}$$
$$= 1 - e^{-j5\hat{\omega}} = H(\hat{\omega}).$$

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PROBLEM:

Suppose that three systems are hooked together in "cascade." In other words, the output of $\mathcal{S}_1$ is the input to $\mathcal{S}_2$, and the output of $\mathcal{S}_2$ is the input to $\mathcal{S}_3$. The three systems are specified as follows:

$$\mathcal{S}_1: \qquad y_1[n] = x_1[n] + x_1[n-2]$$

$$\mathcal{S}_2: \qquad y_2[n] = 7x_2[n-5] + 7x_2[n-6]$$

$$\mathcal{S}_3: \qquad \mathcal{H}_3(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$$

NOTE: the output of $\mathcal{S}_i$ is $y_i[n]$ and the input is $x_i[n]$.

The objective in this problem is to determine the equivalent system that is a single operation from the input $x[n]$ (into $\mathcal{S}_1$) to the output $y[n]$ which is the output of $\mathcal{S}_3$. Thus $x[n]$ is $x_1[n]$ and $y[n]$ is $y_3[n]$.

(a) Determine the difference equation for $\mathcal{S}_3$.

(b) Determine the frequency response of the first two systems: $\mathcal{H}_i(\hat{\omega})$ for $i = 1, 2$.

(c) Determine the frequency response of the overall cascaded system.

(d) Write *one difference equation* that defines the overall system in terms of $x[n]$ and $y[n]$ only.

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SOLUTION

a) $H_3(\hat\omega) = e^{-j\hat\omega} - e^{-j2\hat\omega} \Rightarrow h = [0, 1, -1]$

$$= e^{-j\hat\omega\frac{3}{2}}\left(e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}\right) = e^{-j\hat\omega\frac{3}{2}}\, 2j\sin(\hat\omega/2)$$

b) $H_1(\hat\omega) = e^{j\hat\omega 0} + e^{-j\hat\omega 2} = e^{-j\hat\omega}\left(e^{-j\hat\omega} + e^{j\hat\omega}\right)$ $\qquad h_1 = [1, 0, 1]$

$$= e^{-j\hat\omega}\, 2\cos(e\hat\omega)$$

$H_2(\hat\omega) = 7e^{-j\hat\omega 5} + 7e^{-j\hat\omega 6} = 7e^{-j\omega 5.5}\left(e^{-j\frac{\hat\omega}{2}} + e^{j\frac{\hat\omega}{2}}\right)$

$$= 7e^{-j\omega 5.5}\, 2\cos\left(\frac{\hat\omega}{2}\right) \qquad h_2 = [0,0,0,0,0,7,7]$$

c) $H_\Sigma = H_1 \cdot H_2 \cdot H_3 = \left(e^{j\hat\omega 0} + e^{-j\hat\omega 2}\right)\left(7e^{-j\hat\omega 5} + 7e^{-j\hat\omega 6}\right)\left(e^{-j\hat\omega} - e^{-j2\hat\omega}\right)$

= 8 TERMS
COLLECT CONJUGATES
INVERSE EULER → COSINE FORM

-OR-

d) IT'S PROBABLY FASTER TO CONVOLVE THEN $h_\Sigma \xrightarrow{n \to \hat\omega} H_\Sigma$ (IN THIS CASE).

$h_\Sigma[n] = h_1[n] * h_2[n] * h_3[n]$

$= [1\ 0\ 1] * [0\ 0\ 0\ 0\ 0\ 7\ 7] * [0\ 1\ -1]$

$= [0\ 0\ 0\ 0\ 0\ 7\ 7\ 7\ 7] * [0\ 1\ -1]$

$= [0\ 0\ 0\ 0\ 0\ 0\ 7\ 0\ 0\ 0\ -7] \rightarrow y[n] = 7(x[n-6] - x[n-10])$

$H_\Sigma(\hat\omega) = 7e^{-j\hat\omega 6} - 7e^{-j\hat\omega 10}$

$= 7e^{-j\hat\omega 8}\left(e^{j\hat\omega 2} - e^{-j\hat\omega 2}\right)$

$= 7e^{j\hat\omega 8}\, 2j\sin(2\hat\omega)$

$= 14e^{j\hat\omega 8}\, e^{j\frac{\pi}{2}}\sin(2\hat\omega)$

$= 14e^{j(\hat\omega 8 + \frac{\pi}{2})}\sin(2\hat\omega)$

$= \underbrace{e^{j(\hat\omega 8 + \frac{\pi}{2})}}_{\text{PHASE}}\ \underbrace{14\sin(2\hat\omega)}_{\text{MAGNITUDE}}$

Note: 6 SAMPLE DELAY BEFORE OUTPUT RESPONDS TO INPUT

THIS IS MANIFEST BY THE PHASE IN THE FREQUENCY RESPONSE

$\Phi \propto$ delay

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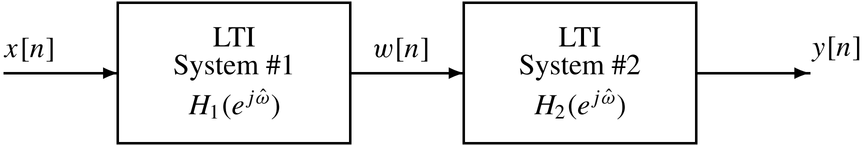
PROBLEM:

Consider the following cascade system:



Both systems are 4-point running average systems, i.e.,

$$H_2(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) = \frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)}\, e^{-j3\hat{\omega}/2}$$

(a) Determine the frequency response of the overall system from the input $x[n]$ to the output $y[n]$.

(b) Plot the magnitude and phase of the overall frequency response for $-\pi \le \hat{\omega} \le \pi$.

(c) What is the total time delay (in samples) for the overall system?

(d) If the input is $x[n] = e^{j\hat{\omega}n}$, for which values of $\hat{\omega}$ will $y[n] = 0$?

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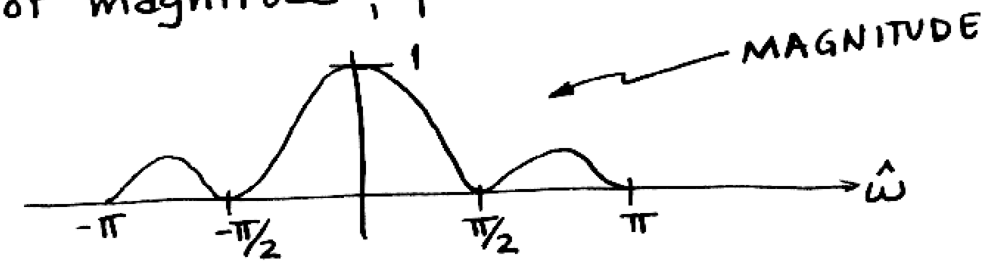
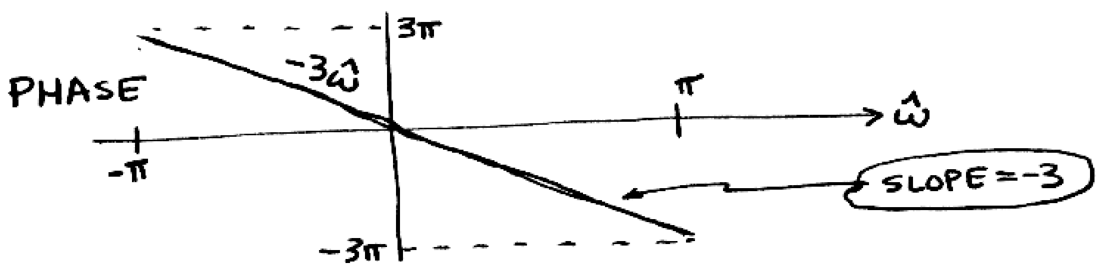
SOLUTION

(a) $H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) \cdot H_2(e^{j\hat{\omega}}) = H_1^2(e^{j\hat{\omega}})$

$$= \left[\frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)} \right]^2 e^{-j3\hat{\omega}}$$

(b) Plot magnitude & phase





(c) Delay = − SLOPE of PHASE = 3 samples

(d) $y[n] = 0$ when $H(e^{j\hat{\omega}}) = 0$

$\Rightarrow$ at $\hat{\omega} = \pm\pi/2, \pi$

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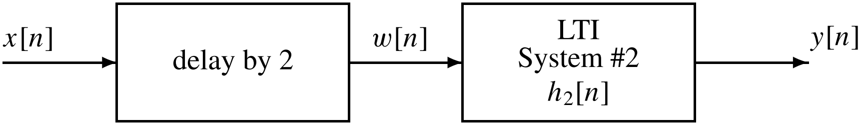
PROBLEM:

Consider the following cascade system:



(a) Find and plot the magnitude of the frequency response of the first filter $|\mathcal{H}_1(\hat{\omega})|$.

(b) If the overall impulse response of the cascade is

$$h_{eq}[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

determine the impulse response of the second filter $h_2[n]$.

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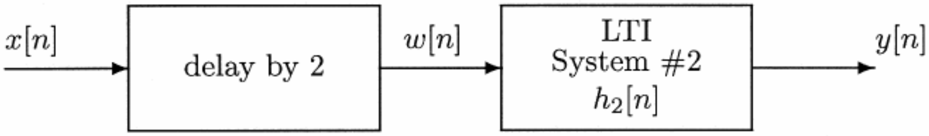
SOLUTION

Consider the following cascade system:

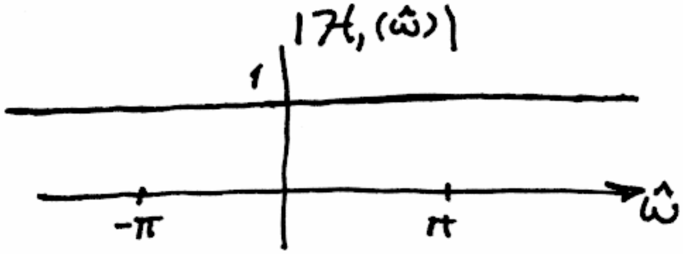


(a) Find and plot the magnitude of the frequency response of the first filter $|\mathcal{H}_1(\hat{\omega})|$.

$$h_1[n] = \delta[n-2]$$

$$\mathcal{H}_1(\hat{\omega}) = e^{-j2\hat{\omega}}$$

$$\boxed{|\mathcal{H}_1(\hat{\omega})| = 1}$$



(b) If the overall impulse response of the cascade is

$$h_{eq}[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

determine the impulse response of the second filter $h_2[n]$.

$$\delta[n-2] * h_2[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

$$\Longrightarrow \boxed{h_2[n] = \delta[n-1] + \tfrac{1}{2}\delta[n-2]}$$

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PROBLEM:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

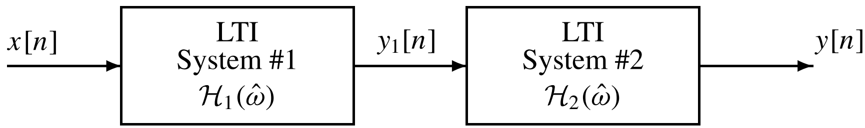

Figure 1: Cascade connection of two LTI systems.

(a) Show that if the input is $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, then the corresponding output of the overall system is

$$y[n] = \mathcal{H}_2(\hat{\omega})\mathcal{H}_1(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n} = \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

where $\mathcal{H}_1(\hat{\omega})$ is the frequency response of the first system and $\mathcal{H}_2(\hat{\omega})$ is the frequency response of the second system. That is, show that the overall frequency response of a cascade of two LTI system is the product of the individual frequency responses, and therefore the cascade system is equivalent to a single system with frequency response $\mathcal{H}(\hat{\omega}) = \mathcal{H}_2(\hat{\omega})\mathcal{H}_1(\hat{\omega})$.

(b) Use the result of part (a) to show that the order of the systems is not important; i.e., show that for the same input $x[n]$ into the systems of Figs. 1 and 2, the overall outputs are the same ($w[n] = y[n]$).

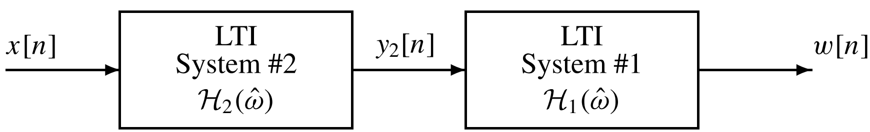

Figure 2: Equivalent system to system of Figure 1.

(c) Suppose that System #1 is described by the difference equation $y_1[n] = x[n] + x[n-2]$. and System #2 is described by the frequency response function $\mathcal{H}_2(\hat{\omega}) = (1 - e^{-j\hat{\omega}2})$. Determine the frequency response function of the overall cascade system.

(d) Sketch the frequency response (magnitude and phase) of the overall cascade system for $-\pi \leq \hat{\omega} \leq \pi$.

(e) Obtain a single difference equation that relates $y[n]$ to $x[n]$ in Fig. 1 and $w[n]$ to $x[n]$ in Fig. 2.

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PROBLEM:

A discrete-time system is known to be LTI, and had been measured at two frequencies, $\hat{\omega} = 0$ and $\hat{\omega} = \pi$.

$$H(e^{j0}) = \tfrac{1}{2} \qquad \text{and} \qquad H(e^{j\pi}) = 2e^{-j11\pi}$$

(a) Determine the output when the input is

$$x_1[n] = \begin{cases} 8 & \text{for } n \text{ even} \\ 4 & \text{for } n \text{ odd} \end{cases}$$

(b) If we also know that $H(e^{j\pi/2}) = H^*(e^{-j\pi/2}) = 3e^{-j11\pi/2}$, then determine the output $y_2[n]$ when the input is a triangle-shaped signal with a period of four:

$$x_2[n] = \{\ldots, 1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 3,\ 2,\ \ldots\}$$

i.e., $x_2[n+4] = x_2[n]$ with $x_2[0] = 1$, $x_2[1] = 2$, $x_2[2] = 3$, $x_2[3] = 2$, $x_2[4] = 1$, and so on.

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$H(e^{j\hat{\omega}}) = (1 - e^{-j\hat{\omega}})(1 + e^{-j\pi/4}e^{-j\hat{\omega}})(1 + e^{j\pi/4}e^{-j\hat{\omega}}) \tag{1}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.
Hint: Multiply out the factors to obtain a sum of powers of $e^{-j\hat{\omega}}$.

(b) What is the impulse response of this system?

(c) If the input is a complex exponential of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for which values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?
Hint: In this part, the answer is easy to obtain if you use the factored form of Eq. (1).

(d) Use superposition to determine the output of this system when the input is

$$x[n] = 3 + \delta[n-2] + \cos(0.5\pi n + \pi/4) \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results. This is what it means to apply the principle of *Superposition*.

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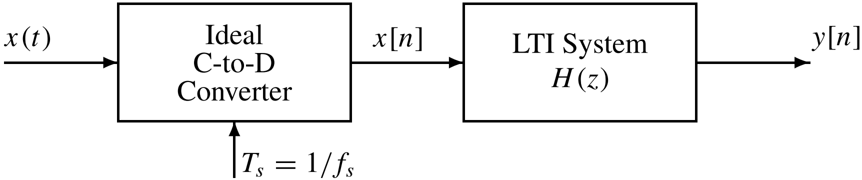
PROBLEM:



The input to the C-to-D converter in the above system is

$$x(t) = 100 + 50\cos(1000\pi t - \pi/4)$$

The sampling frequency is $f_s = 2000$ samples/second. The LTI system is an L-point moving averager defined by the equation

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

(a) Is it possible to find a value of L such that $y[n] = A$ for $-\infty < n < \infty$, where A is a constant? If so, give a rough outline of your plan for finding L.

(b) Determine the *minimum* value of L such that the cosine term is removed as specified in part (a). Also determine the value of the constant A for your system in part (a).

$L_{min} =$

$A =$

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SOLUTION

(a) After sampling $x[n] = 100 + 50\cos(\frac{\pi}{2}n - \pi/4)$

Since we want $y[n] = A$ = constant, it is clear that we want $\mathcal{H}(\pm\frac{\pi}{2}) = 0$.

For L-point moving averager $H(z) = \dfrac{1 - z^{-L}}{L(1 - z^{-1})}$

or $\mathcal{H}(\hat{\omega}) = \dfrac{\sin \omega L/2}{L \sin(\omega/2)} e^{-j\omega(\frac{L-1}{2})}$

$\mathcal{H}(\hat{\omega}) = 0$ at $\omega = \frac{2\pi}{L}k$ k an integer $\neq L$

So we want $\frac{2\pi}{L} = \frac{\pi}{2} \Rightarrow \boxed{L = 4}$

Recall $y[n] = 100\mathcal{H}(0) + 50|\mathcal{H}(\frac{\pi}{2})|\cos(\frac{\pi}{2}n - \frac{\pi}{4} + \angle\mathcal{H}(\pi/2))$

Since $\mathcal{H}(\frac{\pi}{2}) = 0$ and $\mathcal{H}(0)$ for 4-point averager,

$\boxed{A = 100}$

PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = -3Gx[n-1] + 6Gx[n-2] - 3Gx[n-3]$$

where G is a constant to be determined.

(a) When the input is the signal, $x_1[n] = 1 + (-1)^n$, the output is $y_1[n] = 60(-1)^{n+1}$. Determine the value of G, and then determine the output when the input is

$$x_2[n] = \begin{cases} 5 & \text{for } n \text{ even} \\ 25 & \text{for } n \text{ odd} \end{cases}$$

Use linearity and time invariance to simplify your work.

(b) Obtain an expression for the frequency response of this system, using G from part (a).

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint:* Use symmetry to simplify your expression before determining the magnitude and phase.

(d) For the system above, determine the output $y_1[n]$ when the input is

$$x_1[n] = 4 + 8\cos(0.5\pi n + \pi/2)$$

Hint: Use the frequency response and superposition to solve this problem.

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PROBLEM:

We have shown that an LTI system can be represented in several equivalent ways. In each part below, you are given one representation of an LTI system and you are to provide the other representations requested. (Frequency response formulas can be given in any convenient form. You do **NOT** have to simplify them.)

(a) Frequency response:

Impulse response:

Difference equation: $y[n] = x[n] + 2x[n-1] + x[n-2]$

(b) Frequency response: $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(2\cos(\hat{\omega}))$

Impulse response:

Difference equation:

(c) Frequency response:

Impulse response:

MATLAB Implementation: `y = conv([0,1,0,-1],x)`

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SOLUTION

We have shown that an LTI system can be represented in several equivalent ways. In each part below, you are given one representation of an LTI system and you are to provide the other representations requested. (Frequency response formulas can be given in any convenient form. You do **NOT** have to simplify them.)

(a) Frequency response: $\mathcal{H}(\hat{\omega}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$

Impulse response: $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

Difference equation: $y[n] = x[n] + 2x[n-1] + x[n-2]$

(b) Frequency response: $\mathcal{H}(\hat{\omega}) = e^{\ j\hat{\omega}}(2\cos(\hat{\omega}))$ $= e^{j\hat{\omega}}\left(e^{j\hat{\omega}} + e^{-j\hat{\omega}}\right) = e^{j2\hat{\omega}} + 1$

Impulse response: $h[n] = \delta[n+2] + \delta[n]$

Difference equation: $y[n] = x[n+2] + x[n]$

(b) Frequency response: $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(2\cos(\hat{\omega}))$ $= e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + e^{-j\hat{\omega}}\right) = 1 + e^{-j2\hat{\omega}}$

Impulse response: $h[n] = \delta[n] + \delta[n-2]$

Difference equation: $y[n] = x[n] + x[n-2]$

(c) Frequency response: $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$

Impulse response: $h[n] = \delta[n-1] - \delta[n-3]$

MATLAB Implementation: `y = conv([0,1,0,-1],x)`

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\pi/4}e^{-j\hat{\omega}})(1 - e^{j\pi/4}e^{-j\hat{\omega}}). \tag{1}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.

(b) What is the output if the input is $x[n] = \delta[n]$?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?

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SOLUTION

a) First, expand out $H(\hat{\omega})$

$$H(\hat{\omega}) = (1+e^{-j\hat{\omega}})(1-e^{-j\pi/4}e^{-j\hat{\omega}})(1-e^{j\pi/4}e^{-j\hat{\omega}})$$
$$H(\hat{\omega}) = (1+e^{-j\hat{\omega}})(1-e^{j\pi/4}e^{-j\hat{\omega}}-e^{-j\pi/4}e^{-j\hat{\omega}}+e^{-j2\hat{\omega}})$$
$$H(\hat{\omega}) = (1+e^{-j\hat{\omega}})(1-e^{-j\hat{\omega}}(e^{j\pi/4}+e^{-j\pi/4})+e^{-j2\hat{\omega}})$$
$$H(\hat{\omega}) = (1+e^{-j\hat{\omega}})(1-\sqrt{2}e^{-j\hat{\omega}}+e^{-j2\hat{\omega}})$$
$$H(\hat{\omega}) = 1-\sqrt{2}e^{-j\hat{\omega}}+e^{-j2\hat{\omega}}+e^{-j\hat{\omega}}-\sqrt{2}e^{-j2\hat{\omega}}+e^{-j3\hat{\omega}}$$
$$H(\hat{\omega}) = 1+(1-\sqrt{2})e^{-j\hat{\omega}}+(1-\sqrt{2})e^{-j2\hat{\omega}}+e^{-j3\hat{\omega}}$$

Considering the form $H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$, the coefficients are $b_k = \{1, 1-\sqrt{2}, 1-\sqrt{2}, 1\}$

with $M = 3$. From these coefficients, $y[n]$ can be written as

$$y[n] = \sum_{k=0}^{3} b_k x[n-k] = x[n]+(1-\sqrt{2})x[n-1]+(1-\sqrt{2})x[n-2]+x[n-3].$$

b) If the input is $x[n] = \delta[n]$, then the output is

$$y[n] = \delta[n]+(1-\sqrt{2})\delta[n-1]+(1-\sqrt{2})\delta[n-2]+\delta[n-3]$$

c) With an input of $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, the output will be of the form $y[n] = H(\hat{\omega})(Ae^{j\phi}e^{j\hat{\omega}n})$. Therefore, $y[n] = 0$ for all n only when $H(\hat{\omega}) = 0$. By consider the roots of $H(\hat{\omega})$, we know that $H(\hat{\omega}) = 0$ when: i) $(1+e^{-j\hat{\omega}}) = 0$, ii) $(1-e^{-j\pi/4}e^{-j\hat{\omega}}) = 0$ or iii) $(1-e^{j\pi/4}e^{-j\hat{\omega}}) = 0$. Solving each of these gives

i) $e^{-j\hat{\omega}} = -1$ which is true when $\hat{\omega} = \pi + 2\pi l$

ii) $e^{-j\hat{\omega}} = e^{j\pi/4}$ which is true when $\hat{\omega} = -\pi/4 + 2\pi l$

iii) $e^{-j\hat{\omega}} = e^{-j\pi/4}$ which is true when $\hat{\omega} = \pi/4 + 2\pi l$

Therefore, $y[n] = 0$ for all n when $\hat{\omega} = \{\pi+2\pi l, \pi/4+2\pi l, -\pi/4+2\pi l\}$, l an integer.

For the specific range of $-\pi \le \hat{\omega} \le \pi$, the possible $\hat{\omega}$ are $-\pi$, π, $\pi/4$, and $-\pi/4$, but, note that π and $-\pi$ result in the same frequency.

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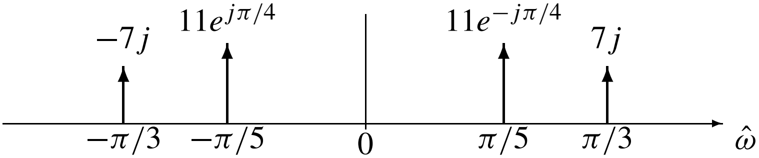
PROBLEM:

An FIR filter is characterized by the following frequency response:

$$H(e^{j\hat{\omega}}) = \frac{\sin(3\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j3\hat{\omega}}$$

(a) If the input to the filter is a signal with the following spectrum, determine a formula for the input signal, $x[n]$ for $-\infty < n < \infty$.



(b) Using the input signal from part (a), determine the output, $y[n]$ for $-\infty < n < \infty$.

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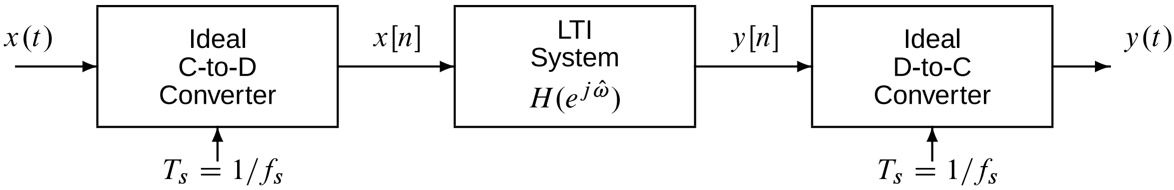
PROBLEM:

Consider the following system for discrete-time filtering of a continuous-time signal:



In this problem, assume that the frequency response of the discrete-time system is

$$H(e^{j\hat{\omega}}) = 1 + e^{-j2\hat{\omega}}$$

(a) Make a plot of the frequency response magnitude for $H(e^{j\hat{\omega}})$ over the frequency range $-\pi < \hat{\omega} \leq \pi$.

(b) In this part, assume that the input is

$$x(t) = 500 + 400\cos(800\pi t) \quad \text{for} \quad -\infty < t < \infty$$

For a sampling rate of $f_s = 1000$ samples/sec, draw the spectrum of $x[n]$, the discrete-time signal after the C-to-D converter.

(c) For the same $x(t)$ as in the previous part, and the same sampling rate, determine a simple formula for the output $y(t)$ for $-\infty < t < \infty$.

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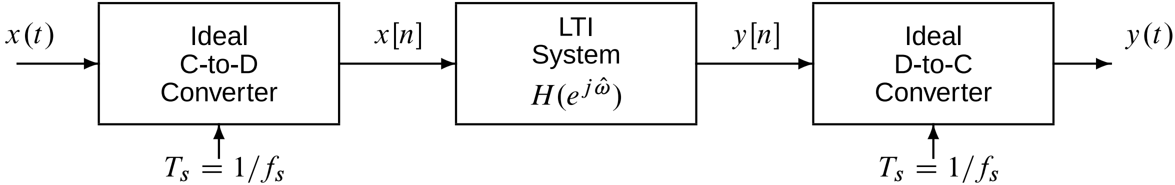
PROBLEM:

Consider the following system for discrete-time filtering of a continuous-time signal:



In this problem, assume that the impulse response of the discrete-time system is

$$h[n] = \tfrac{1}{2}\delta[n-4] + \delta[n-5] + \tfrac{1}{2}\delta[n-6]$$

(a) Determine the frequency response formula, $H(e^{j\hat{\omega}})$, for the LTI system.

(b) For a sampling rate of $f_s = 800$ samples/sec, determine the frequency of an input sinusoid of the form $x(t) = \cos(\omega t)$ such that the resulting output will be $y(t) = \cos(\omega t + \phi)$, i.e., the output amplitude and frequency are the same as the input.

(c) In this part, assume that the input is

$$x(t) = 99 + 88\cos(500\pi t) \quad \text{for} \quad -\infty < t < \infty$$

For a sampling rate of $f_s = 800$ samples/sec, determine the output $y(t)$ for $-\infty < t < \infty$.

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PROBLEM:

For the *aliased sinc* function:

$$\text{asinc}(\hat{\omega}, 11) = \frac{\sin(5\frac{1}{2}\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

(a) Make a plot of asinc$(\hat{\omega}, 11)$ over the range $-4\pi \leq \hat{\omega} \leq +4\pi$. Label all the zero crossings.

(b) Determine the period of asinc$(\hat{\omega}, 11)$. Is it equal to 2π; why, or why not?

(c) Find the maximum value of the function.

NOTE: the *aliased sinc* function is defined via: $\text{asinc}(\hat{\omega}, L) = \dfrac{\sin(L\hat{\omega}/2)}{\sin(\frac{1}{2}\hat{\omega})}$

In MATLAB consult help on `diric` for more information.

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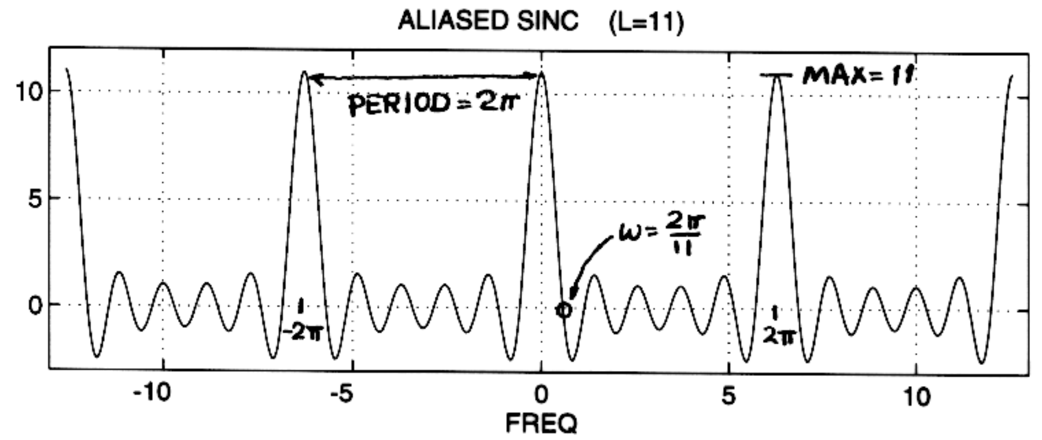
SOLUTION

(a)



ZEROS AT $\omega = \frac{2\pi l}{10}$

(b) PERIOD IS 2π

$$\text{asinc}(\hat{\omega}+2\pi, 11) = \frac{\sin(5.5(\hat{\omega}+2\pi))}{\sin(\frac{1}{2}(\hat{\omega}+2\pi))} = \frac{\sin(5.5\hat{\omega}+11\pi)}{\sin(\frac{1}{2}\hat{\omega}+\pi)}$$

$$= \frac{-\sin(5.5\hat{\omega})}{-\sin(\frac{1}{2}\hat{\omega})} = \frac{\sin(5.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} = \text{asinc}(\hat{\omega}, 11)$$

(c) AT $\hat{\omega} = 0, 2\pi, 4\pi$ etc we get

$$\lim_{\hat{\omega}\to 0} \frac{\sin(5.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} \longrightarrow \frac{5.5\hat{\omega}}{\frac{1}{2}\hat{\omega}} = 11$$

BECAUSE $\sin\theta \approx \theta$ when $\theta \to 0$

or, use L'Hôpital's Rule. (TAKE DERIV OF NUMERATOR & DENOM)

$$\lim_{\hat{\omega}\to 0} \frac{\sin(5.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} = \lim_{\hat{\omega}\to 0} \frac{5.5\cos 5.5\hat{\omega}}{\frac{1}{2}\cos\frac{1}{2}\hat{\omega}} = \frac{5.5}{1/2} = 11$$

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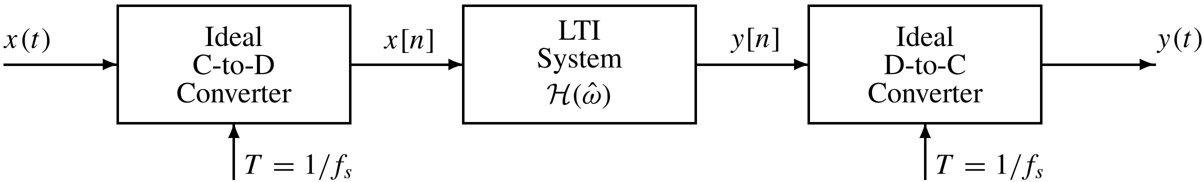
PROBLEM:

The input to the C-to-D converter in the figure below is

$$x(t) = 3 + 5\cos(4000\pi t) + 6\cos(6000\pi t - \pi/2)$$

The frequency response for the digital filter (LTI system) is $\mathcal{H}(\hat{\omega}) = \dfrac{\sin(3\hat{\omega})}{6\sin(\frac{1}{2}\hat{\omega})} e^{-j2.5\hat{\omega}}$

The sampling frequency is $f_s = 12000$ samples/second.



(a) For the *Dirichlet* function: $\tilde{\mathcal{D}}(\hat{\omega}, 6) = \dfrac{\sin(3\hat{\omega})}{6\sin(\frac{1}{2}\hat{\omega})}$. make a plot of $\tilde{\mathcal{D}}(\hat{\omega}, 6)$ over the range $-2\pi \le \hat{\omega} \le +2\pi$. Label all the zero crossings.

(b) Determine the period of $\tilde{\mathcal{D}}(\hat{\omega}, 6)$. Is it equal to 2π; why, or why not?

(c) Find the maximum value of the function $\tilde{\mathcal{D}}(\hat{\omega}, 6)$.

(d) Determine an expression for $y(t)$, the output of the D-to-C converter (as a sum of sinusoids).

In MATLAB consult help on `diric` for more information about computing the *Dirichlet* function; also there is a function called `dirich()` on the web site.

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SOLUTION

a) SEE PLOTS OF $\tilde{D}(\hat{\omega},6)$ and $|\tilde{D}(\hat{\omega},6)|$

ZERO CROSSINGS ARE MINS IN $|\tilde{D}(\hat{\omega},6)|$

b) PERIOD FOR NUMERATOR $\{\sin(3\hat{\omega})\}$ IS: $3\hat{\omega} = 2\pi$ or $\hat{\omega} = \frac{2}{3}\pi$

PERIOD FOR DENOMINATOR $\{\sin(\frac{1}{2}\hat{\omega})\}$ IS: $\frac{1}{2}\hat{\omega} = 2\pi$ or $\hat{\omega} = 4\pi$

BECAUSE $\frac{2}{3}\pi$ FITS AN INTEGER NUMBER OF TIMES INTO 4π

USE 4π = PERIOD

c) $\max(\tilde{D}(\hat{\omega},6))$ OCCURS WHEN THE DENOMINATOR $\to 0$

OR WHEN $\sin(\frac{1}{2}\hat{\omega}) = 0$

$$\tfrac{1}{2}\hat{\omega} = k\pi$$

$$\hat{\omega} = k\cdot 2\pi \qquad \hat{\omega} = [-6\pi, -4\pi, -2\pi, 0, 2\pi, 4\pi \ldots]$$

d) $x(t) = 3 + 5\cos(4000\pi t) + 6\cos(6000\pi t - \pi/2)$

COMPLEX AMPLITUDE

f		$\hat{f}$		$\hat{\omega}$	
3 kHz	=	$f_s \frac{1}{4}$	=	$\frac{\pi}{2}$	$6\angle -\frac{\pi}{2}$
2 kHz	=	$f_s \frac{1}{6}$	=	$\frac{\pi}{3}$	$5\angle 0$
0 Hz	=	$f_s 0$	=	0	$3\angle 0$

$H(\hat{\omega} = 0 = DC) = 1$

$H(\hat{\omega} = \frac{\pi}{3}) = 0 \cdot e^{-j\frac{5}{2}\cdot\frac{\pi}{3}} = 0$

$H(\hat{\omega} = \frac{\pi}{2}) = -\frac{\sqrt{2}}{6} e^{-j\frac{5}{2}\cdot\frac{\pi}{2}} = \frac{\sqrt{2}}{6} e^{-j\frac{\pi}{4}} = \frac{\sqrt{2}}{6}\angle -\frac{\pi}{4}$

If we low pass filter to allow only $\left(\begin{matrix}\hat{\omega} < \pi \\ f < 6\text{kHz}\end{matrix}\right)$ rad/samp in $y(t)$:

$$y(t) = 1\cdot 3 + 0\cdot 5\cos(4000\pi t) + \left(\tfrac{\sqrt{2}}{6}\angle -\tfrac{\pi}{4}\right)\cos\left(6000\pi t - \tfrac{\pi}{2}\right)\cdot 6$$

$$= 3 + \sqrt{2}\cos(6000\pi t - \tfrac{3}{4}\pi)$$

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RUNNING AVERAGE FILTER DERIVATION

SEMI-INFINITE GEOMETRIC SERIES

$$\left(a^1 + a^2 + a^3 + a^4 + \ldots\right) = \sum_{k=1}^{\infty} a^k = \sum_{k=0}^{\infty} a^{k+1} = a\sum_{k=0}^{\infty} a^k$$

$$+\left(a^0\right) = \longrightarrow + \; a^0 = 1$$

$$\left(a^0 + a^1 + a^2 + a^3 + a^4 + \ldots\right) = \sum_{k=0}^{\infty} a^k = 1 + a\sum_{k=0}^{\infty} a^k$$

$$\sum_{k=0}^{\infty} a^k - a\sum_{k=0}^{\infty} a^k = 1$$

$$(1-a)\sum_{k=0}^{\infty} a^k = 1$$

$$\boxed{\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \qquad \text{FOR } |a| < 1}$$

SEMI-INFINITE

FINITE GEOMETRIC SERIES

$0 \longrightarrow \infty$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

$M \longrightarrow \infty$

$$\sum_{\substack{k=M \\ k-M=0}}^{\infty} a^{k-M} = a^{-M}\sum_{k=M}^{\infty} a^k = \frac{1}{1-a}$$

$$\sum_{k=M}^{\infty} a^k = \frac{a^M}{1-a}$$

$0 \qquad M-1$

FINITE

$$\sum_{k=0}^{M-1} a^k = \sum_{k=0}^{\infty} a^k - \sum_{k=M}^{\infty} a^k = \boxed{\frac{1-a^M}{1-a} = \sum_{k=0}^{M-1} a^k}$$

$h[n] \rightarrow H(\hat{\omega})$

$$H(\hat{\omega}) = \sum_{k=0}^{L-1} h[n]\, e^{-j\hat{\omega}k} = \left\{ \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right) = \frac{1}{L}\sum_{k=0}^{L-1} e^{j\hat{\omega}k} \right\}$$

FOR L-ELEMENT RUNNING AVERAGE

$$= \frac{1}{L}\; \frac{e^{-j\hat{\omega}\frac{L}{2}}}{e^{j\hat{\omega}2}}\; \frac{\left(e^{j\hat{\omega}\frac{L}{2}} - e^{-j\hat{\omega}\frac{L}{2}}\right)}{\left(e^{j\hat{\omega}/2} - e^{j\hat{\omega}/2}\right)} = \frac{e^{j\hat{\omega}(L-1)/2}}{L} \cdot \frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)}$$

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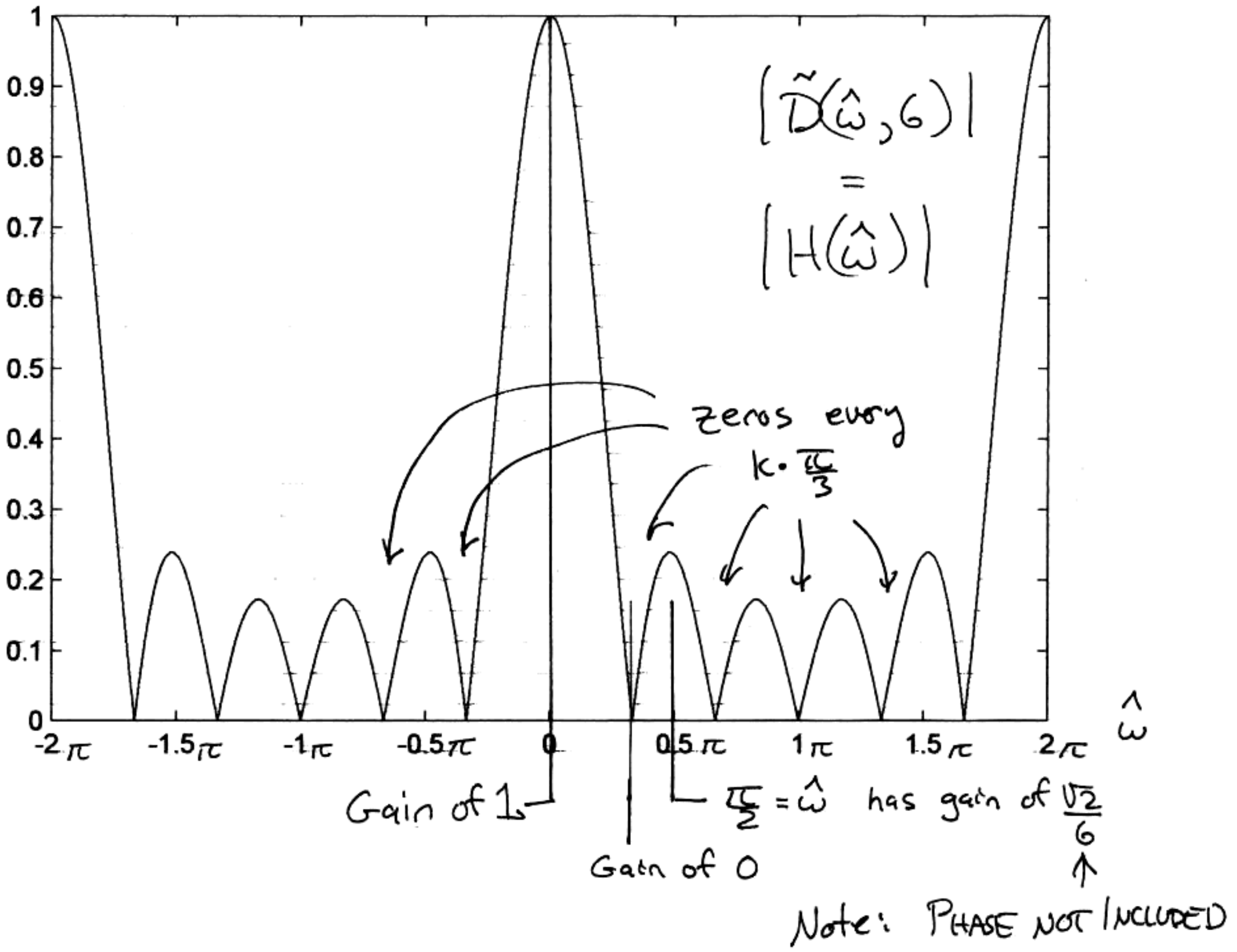
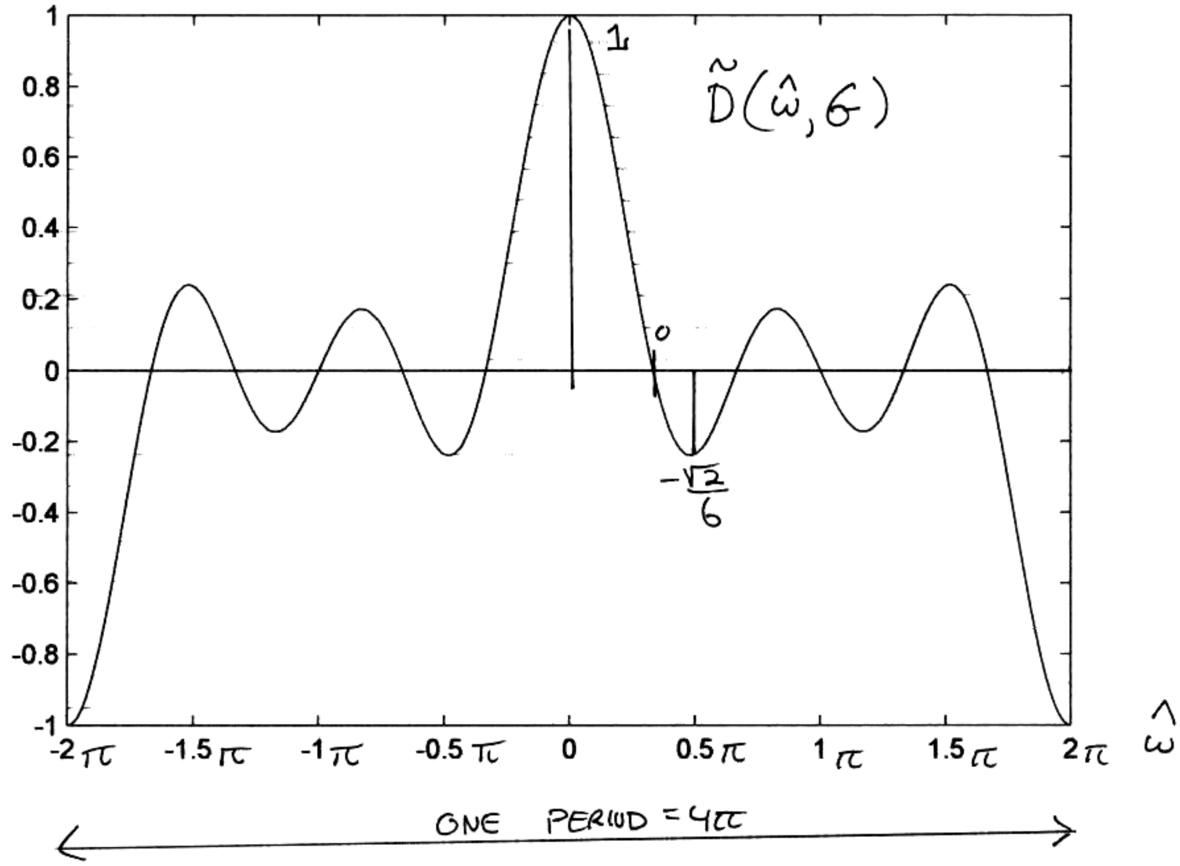

$|\tilde{D}(\hat{\omega},6)| = |H(\hat{\omega})|$
zeros every $k\cdot\frac{\pi}{3}$
Gain of 1
Gain of 0
$\frac{\pi}{2}=\hat{\omega}$ has gain of $\frac{\sqrt{2}}{6}$
Note: PHASE NOT INCLUDED
$\hat{\omega}$


$\tilde{D}(\hat{\omega},6)$
1
0
$-\frac{\sqrt{2}}{6}$
$\hat{\omega}$
ONE PERIOD = 4π

PROBLEM:

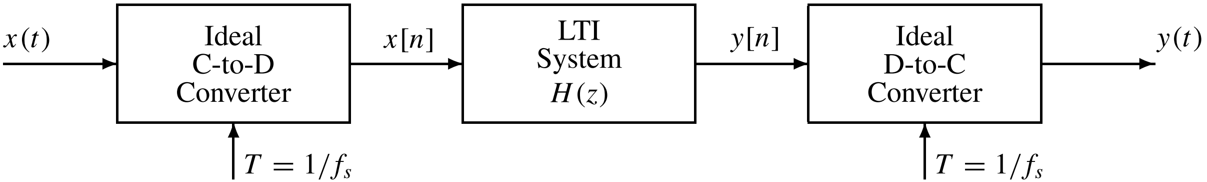
The input to the C-to-D converter in the figure below is

$$x(t) = 10 + 4\cos(4000\pi t - \pi/8) + 6\cos(11000\pi t - \pi/3)$$

The system function of the LTI system is

$$H(z) = (1 + z^{-2})$$

If $f_s = 8000$ samples/second, determine an expression for $y(t)$, the output of the D-to-C converter.



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SOLUTION

$$\mathcal{H}(\hat{\omega}) = 1 + e^{-j2\hat{\omega}}$$

Consider each term separately:

$$10 \xrightarrow{C/D} 10 \xrightarrow{LTI} 10 \cdot \mathcal{H}(0) = 10 \cdot 2 = 20 \xrightarrow{D/C} 20$$

$4\cos\left(4000\pi t - \frac{\pi}{8}\right)$:

$$\omega = \pm 4000\pi \xrightarrow{C/D} \hat{\omega} = \frac{\omega}{f_s} = \pm\frac{\pi}{2}$$

$$\mathcal{H}\left(\pm\frac{\pi}{2}\right) = 1 + e^{\mp j2\frac{\pi}{2}} = 1 + e^{\mp j\pi} = 0$$

Therefore this term is removed by the filter.

$$6\cos\left(11000\pi t - \frac{\pi}{3}\right) = 3e^{-j\frac{\pi}{3}}e^{j11000\pi t} +$$
$$+ 3e^{j\frac{\pi}{3}}e^{-j11000\pi t}$$

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$$\omega = \pm 11000\pi \xrightarrow{C/D} \hat{\omega} = \pm \frac{11}{8}\pi$$

(Warning: These values of $\hat{\omega}$ are outside the $[-\pi, \pi]$ interval, therefore aliasing occurs during D/C conversion).

$$\mathcal{H}\left(\pm \frac{11}{8}\pi\right) = 1 + e^{\mp j 2 \frac{11}{8}\pi} = 1 + e^{\mp j \frac{11}{4}\pi} =$$

$$= 0.7654\, e^{\mp j\, 1.1781} = 0.7654\, e^{\mp j \frac{3\pi}{8}}$$

$$3 e^{-j\frac{\pi}{3}} e^{j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{-j\frac{17}{24}\pi} e^{j\frac{11}{8}\pi n} \xrightarrow{D/C}$$

$$\xrightarrow{D/C} 2.296\, e^{-j\frac{17}{24}\pi} e^{-j5000\pi t}$$

(because of aliasing, $\omega = (\hat{\omega} - 2\pi) f_s = -5000\pi$)

$$3 e^{j\frac{\pi}{3}} e^{-j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{j\frac{17}{24}\pi} e^{-j\frac{11}{8}\pi n} \xrightarrow{D/C}$$

$$\xrightarrow{D/C} 2.296\, e^{j\frac{17}{24}\pi} e^{j5000\pi t}$$

(because of aliasing, $\omega = (\hat{\omega} + 2\pi) f_s = 5000\pi$).

$$y(t) = 20 + 2.296\, e^{j(5000\pi t + \frac{17}{24}\pi)} + 2.296\, e^{-j(5000\pi t + \frac{17}{24}\pi)} =$$

$$= 20 + 4.592 \cos\left(5000\pi t + \frac{17}{24}\pi\right).$$

PROBLEM:

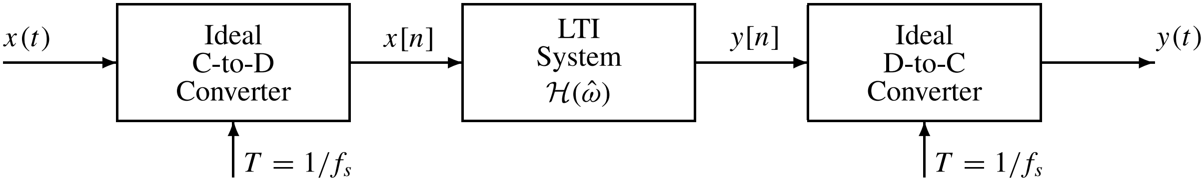
The input to the C-to-D converter in the figure below is

$$x(t) = 3 + 4\cos(3000\pi t + \pi/2) + 12\cos(20000\pi t - 2\pi/3)$$

The frequency response for the digital filter (LTI system) is

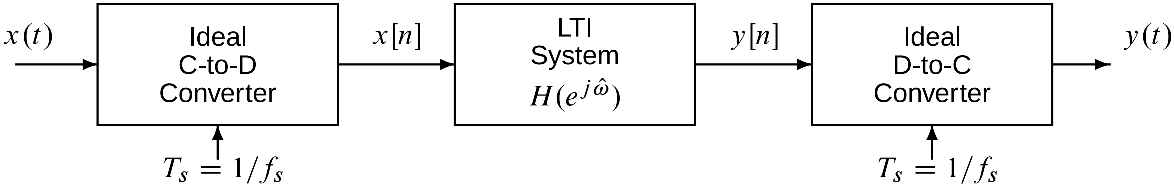
$$\mathcal{H}(\hat{\omega}) = \frac{\sin(4.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j4\hat{\omega}}$$

If $f_s = 10000$ samples/second, determine an expression for $y(t)$, the output of the D-to-C converter.



PROBLEM:

Consider the following system for discrete-time filtering of a continuous-time signal:



In this problem, assume that the frequency response of the discrete-time system is

$$H(e^{j\hat{\omega}}) = 1 + e^{-j\hat{\omega}}$$

(a) Make a plot of the frequency response magnitude for $H(e^{j\hat{\omega}})$ over the frequency range $-\pi < \hat{\omega} \leq \pi$.

(b) For a sampling rate of $f_s = 300$ samples/sec, determine the frequency of an input sinusoid of the form $x(t) = \cos(\omega t)$ such that the resulting output will be zero.

(c) In this part, assume that the input is

$$x(t) = 10 + 20\cos(100\pi t) \quad \text{for} \quad -\infty < t < \infty$$

For a sampling rate of $f_s = 300$ samples/sec, determine the output $y(t)$ for $-\infty < t < \infty$.

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PROBLEM:

A linear time-invariant discrete-time system is described by the difference equation

$$y[n] = x[n] + 2x[n-1] + 3x[n-2] - x[n-4].$$

(a) Draw a block diagram that represents this system in terms of unit-delay elements, coefficient multipliers, and adders as in Figure 5.13 in the text.

(b) Determine the impulse response $h[n]$ for this system. Express your answer as a sum of scaled and shifted unit impulse sequences.

(c) Use convolution to determine the output due to the input

$$x[n] = \delta[n] - \delta[n-1] + \delta[n-2]$$

Plot the output sequence $y[n]$ for $-3 \leq n \leq 10$.

(d) Now consider another LTI system whose impulse response is

$$h_d[n] = \delta[n] - \delta[n-1] + \delta[n-2].$$

Use convolution again to determine $y_d[n] = x_d[n] * h_d[n]$, the output of this system when the input is

$$x_d[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] - \delta[n-4].$$

How does your answer compare to the answer in part (c)? This example illustrates the general commutative property of convolution; i.e., $x[n] * h[n] = h[n] * x[n]$.

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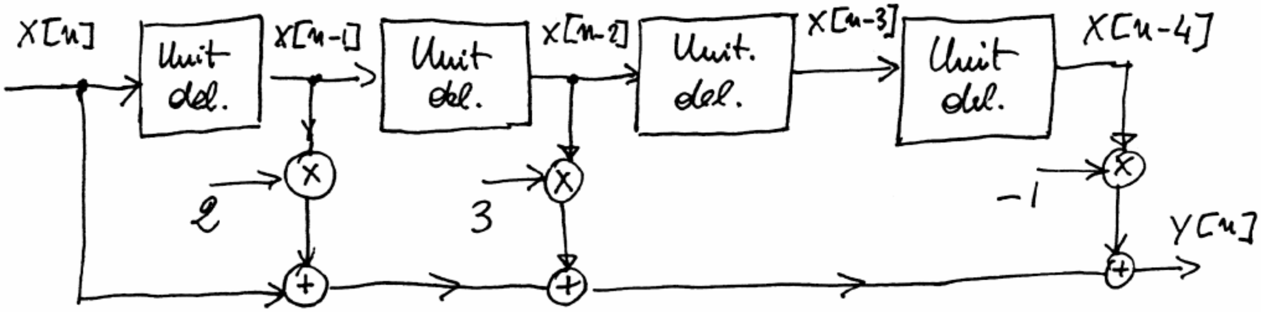
SOLUTION

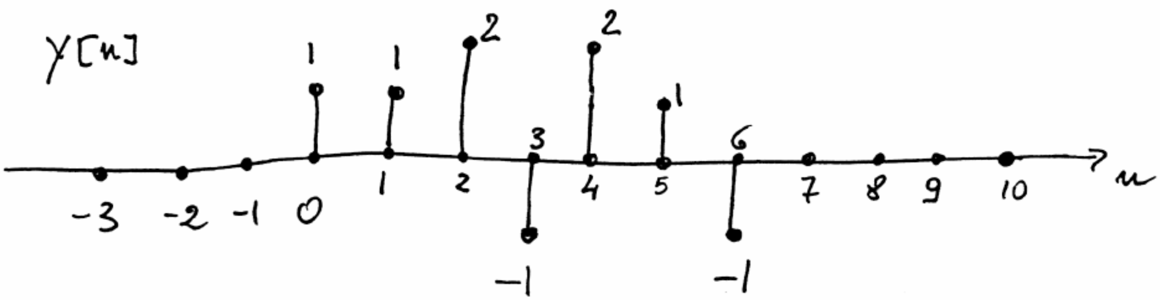
a)



b) $h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] - \delta[n-4]$

c)

1	2	3	0	-1		
1	-1	1				
1	2	3	0	-1		
	-1	-2	-3	0	1	
		1	2	3	0	-1
1	1	2	-1	2	1	-1



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d)

1	-1	1				
1	2	3	0	-1		
1	-1	1				
	2	-2	2			
		3	-3	3		
			0	0	0	
				-1	1	-1
1	1	2	-1	2	1	-1

Same as in (c).

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{j\pi/4}e^{-j\hat{\omega}})(1 - e^{-j\pi/4}e^{-j\hat{\omega}})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.

(b) What is the output if the input is $x[n] = \delta[n]$?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \leq \hat{\omega} \leq \pi$ will $y[n] = 0$ for all n?

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SOLUTION

(a) Multiply out $\mathcal{H}(\hat{\omega})$

$$\mathcal{H}(\hat{\omega}) = (1+e^{-j\hat{\omega}})(1-e^{j\pi/4}e^{-j\hat{\omega}})(1-e^{-j\pi/4}e^{-j\hat{\omega}})$$

$$= (1+e^{-j\hat{\omega}})(1-\underbrace{(e^{j\pi/4}+e^{-j\pi/4})}_{=2\cos\pi/4=\sqrt{2}}e^{-j\hat{\omega}}+e^{-j2\hat{\omega}})$$

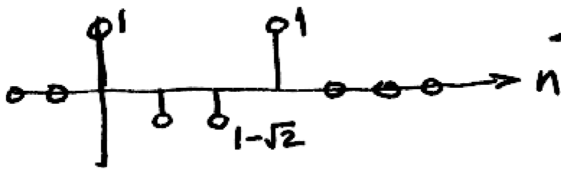
$$\mathcal{H}(\hat{\omega}) = \underset{b_0}{1} + \underset{b_1}{(1-\sqrt{2})}e^{-j\hat{\omega}} + \underset{b_2}{(1-\sqrt{2})}e^{-j2\hat{\omega}} + \underset{b_3=1}{e^{-j3\hat{\omega}}}$$

Use filter coeffs to write difference eqn:

$$y[n] = x[n] + (1-\sqrt{2})x[n-1] + (1-\sqrt{2})x[n-2] + x[n-3]$$

(b) when $x[n]=\delta[n]$, the output is the impulse response:

$$h[n] = \delta[n] + (1-\sqrt{2})\delta[n-1] + (1-\sqrt{2})\delta[n-2] + \delta[n-3]$$



PLOT of $h[n]$

(c) $y[n]$ will be zero for frequencies where $\mathcal{H}(\hat{\omega})=0$.

Look at each factor of $\mathcal{H}(\hat{\omega})$

$$(1+e^{-j\hat{\omega}})=0 \Rightarrow e^{j\hat{\omega}}=-1 \Rightarrow \hat{\omega}=\pi$$

$$(1-e^{j\pi/4}e^{-j\hat{\omega}})=0 \Rightarrow e^{j\hat{\omega}}=e^{j\pi/4} \Rightarrow \hat{\omega}=\pi/4$$

$$(1-e^{-j\pi/4}e^{-j\hat{\omega}})=0 \Rightarrow e^{j\hat{\omega}}=e^{-j\pi/4} \Rightarrow \hat{\omega}=-\pi/4$$

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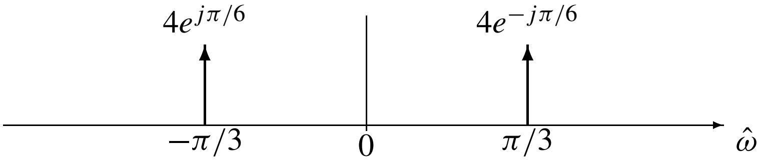
PROBLEM:

An FIR filter is characterized by the following frequency response:

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}\cos(\hat{\omega})$$

(a) If the input to the filter is a signal with the following spectrum, determine the output, $y[n]$ for $-\infty < n < \infty$.



(b) Determine the difference equation that relates the input and output.

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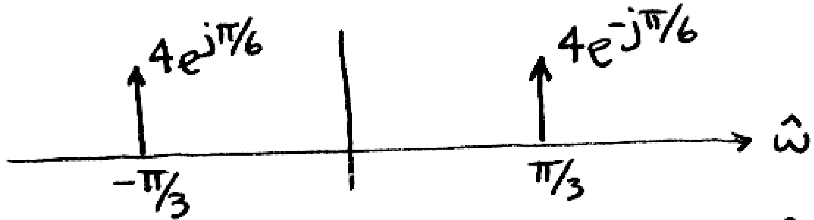
SOLUTION

(a)



The input signal consists of two freq. components:

$$x[n] = \underbrace{4e^{j\pi/6}}_{\text{MULTIPLY BY } \mathcal{H}(-\pi/3)} e^{-j\pi n/3} + \underbrace{4e^{-j\pi/6}}_{\text{MULTIPLY BY } \mathcal{H}(\pi/3)} e^{j\pi n/3}$$

$$\mathcal{H}(\pi/3) = e^{-j\pi/3}\cos(\pi/3) = \tfrac{1}{2}e^{-j\pi/3}$$

$$\mathcal{H}(-\pi/3) = e^{+j\pi/3}\cos(-\pi/3) = \tfrac{1}{2}e^{j\pi/3}$$

Thus, the output is

$$y[n] = \left(\tfrac{1}{2}e^{j\pi/3}\right)\left(4e^{j\pi/6}\right)e^{-j\pi n/3} + \left(\tfrac{1}{2}e^{-j\pi/3}\right)\left(4e^{-j\pi/6}\right)e^{j\pi n/3}$$

$$= 2e^{j\pi/2}e^{-j\pi n/3} + 2e^{-j\pi/2}e^{j\pi n/3}$$

$$= 4\cos\left(\tfrac{\pi n}{3} - \tfrac{\pi}{2}\right) \quad \text{or} \quad 4\sin\left(\tfrac{\pi n}{3}\right)$$

(b) $$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}\left(\tfrac{1}{2}e^{j\hat{\omega}} + \tfrac{1}{2}e^{-j\hat{\omega}}\right)$$

$$= \underbrace{\tfrac{1}{2}}_{b_0} + \underbrace{\tfrac{1}{2}}_{b_2}e^{-j2\hat{\omega}}$$

$$y[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-2]$$

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PROBLEM:

Circle the correct answer to each of these short answer questions:

1. If the impulse response of an FIR filter is defined with a scaling parameter β

$$h[n] = \beta(2\delta[n] - \delta[n-1] + 2\delta[n-2])$$

Determine β so that the DC value of the frequency response $H(e^{j\hat{\omega}})$ will be equal to one.

   (a) $\beta = 1$
   (b) $\beta = 1/2$
   (c) $\beta = 1/3$
   (d) $\beta = 1/4$
   (e) $\beta = 1/5$

2. For the following MATLAB code: `yy = firfilt( [1,0,0,0,-5], xx )` pick the correct difference equation for the filter being implemented.

   (a) $y[n] = \delta[n] - 5\delta[n-1]$
   (b) $y[n] = \delta[n] - 5\delta[n-4]$
   (c) $y[n] = x[n] - 5x[n-1]$
   (d) $y[n] = x[n-4]$
   (e) $y[n] = x[n] - 5x[n-4]$

3. The MATLAB statement: `xx = [ cos(0.13*pi*(0:2000)), cos(0.17*pi*(0:2000)) ];`,

   (a) Defines `xx` as the sum of two sinusoids played simultaneously.
   (b) Defines `xx` as the concatenation of two sinusoids played in succession.
   (c) Defines `xx` as a frequency response.
   (d) Defines `xx` as a spectrogram.

4. If a filter is defined by the MATLAB operation: `yy = firfilt(0.2*ones(1,5),xx)`, then the filter is:

   (a) a highpass FIR filter.
   (b) a lowpass FIR filter.
   (c) a highpass IIR filter.
   (d) a lowpass IIR filter.
   (e) an allpass filter, i.e., its frequency response magnitude is constant.

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SOLUTION

Circle the correct answer to each of these short answer questions:

1. If the impulse response of an FIR filter is defined with a scaling parameter β

$$h[n] = \beta(2\delta[n] - \delta[n-1] + 2\delta[n-2])$$

Determine β so that the DC value of the frequency response $H(e^{j\hat{\omega}})$ will be equal to one.

(a) $\beta = 1$
(b) $\beta = 1/2$
(c) $\beta = 1/3$
(d) $\beta = 1/4$
(e) $\beta = 1/5$

$H(e^{j\hat{\omega}}) = 2\beta - \beta e^{-j\hat{\omega}} + 2\beta e^{-j2\hat{\omega}}$

$H(e^{j0}) = 1 = 2\beta - \beta + 2\beta = 3\beta$

$\Rightarrow \beta = 1/3$

2. For the following MATLAB code: `yy = firfilt( [1,0,0,0,-5], xx )` pick the correct difference equation for the filter being implemented.

(a) $y[n] = \delta[n] - 5\delta[n-1]$
(b) $y[n] = \delta[n] - 5\delta[n-4]$
(c) $y[n] = x[n] - 5x[n-1]$
(d) $y[n] = x[n-4]$
(e) $y[n] = x[n] - 5x[n-4]$

$b_0 = 1$
$b_4 = -5$

3. The MATLAB statement: `xx = [ cos(0.13*pi*(0:2000)), cos(0.17*pi*(0:2000)) ];`,

(a) Defines **xx** as the sum of two sinusoids played simultaneously.
(b) Defines **xx** as the concatenation of two sinusoids played in succession.
(c) Defines **xx** as a frequency response.
(d) Defines **xx** as a spectrogram.

4. If a filter is defined by the MATLAB operation: `yy = firfilt(0.2*ones(1,5),xx)`, then the filter is:

(a) a highpass FIR filter.
(b) a lowpass FIR filter.
(c) a highpass IIR filter.
(d) a lowpass IIR filter.
(e) an allpass filter, i.e., its frequency response magnitude is constant.

$b_k = \{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\}$

5pt Running Average which is Lowpass

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$H(e^{j\hat{\omega}}) = \left(1 + e^{-j2\hat{\omega}}\right)\left(1 + e^{-j4\pi/3}e^{-j\hat{\omega}}\right)\left(1 + e^{-j2\pi/3}e^{-j\hat{\omega}}\right) \quad (1)$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.
Hint: Multiply out the factors to obtain a sum of powers of $e^{-j\hat{\omega}}$.

(b) Determine the impulse response of this system, and make a stem plot. Notice that $h[n]$ is finite length.

(c) If the input is a complex exponential of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for which values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?
Hint: In this part, the answer is easy to obtain if you use the factored form of Eq. (1).

(d) Use superposition to determine the output of this system when the input is

$$x[n] = 3 + 7\delta[n-1] + 13\cos(0.5\pi n - \pi/4) \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results. This is what it means to apply the principle of *Superposition*.

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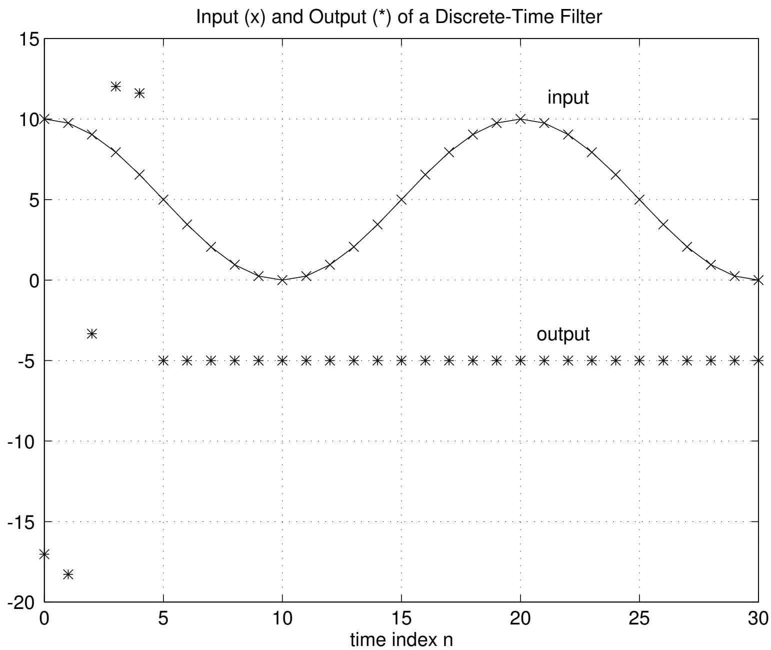
PROBLEM:

The following figure was created by the following MATLAB statements:

```
n=0:30;
x=A+B*cos(omega*n);
y=conv(x,h);
plot(n,x(n+1),'-',n,x(n+1),'x',n,y(n+1),'*')
...
```

where `A`, `B`, `omega`, and `h` are previously defined.



(a) What is the length of the vector `h`?

(b) Are there any zeros on the unit circle? (*yes* *no*)
If so, where are they?

(c) What is the "d.c. gain" of the discrete-time filter?

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SOLUTION

(a) After $n=5$ the output is constant (i.e., -5)

The input is zero for $n<0$, which may not be obvious.

$$y[n] = \sum_{k=0}^{L-1} h[k]\,x[n-k] \longleftarrow \text{CONVOLUTION}$$

when $n=5$, we must have $n-k \geq 0$ so that then zeros in $x[n]$ are not used.

$$\Rightarrow \quad n-(L-1) \geq 0$$

$$5-(L-1) \geq 0 \quad \Rightarrow \quad 6 \geq L$$

so $\boxed{L=6}$ is the length of $h[n]$

(b) The frequency of the cosine can be determined by measuring the period of the plot $\Rightarrow$ period = 20 samples

$$\Rightarrow \quad \hat{\omega} = 2\pi/20 = \pi/10$$

The cosine is removed by the filter, so there are zeros at $\hat{\omega} = \pm\pi/10$

zeros at $\boxed{z = e^{\pm j\pi/10}}$ on the unit circle.

(c) $\text{D.C\_out} = H(e^{j0}) \cdot \text{D.C\_in}$

D.C_in is the constant offset of the input, i.e. 5

D.C_out is the level of the output, -5

$$-5 = H(e^{j0}) \cdot 5 \quad \Rightarrow \boxed{\mathcal{H}(0) = -1}$$

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PROBLEM:

The intention of the following MATLAB program is to filter a sinusoid via the `conv` function, but the cosine signal has a starting point at $n = 0$; we assume that it is zero for $n < 0$.

```
omega = pi/2;
nn = [ 0:4000 ];
xn = cos(omega*nn - pi/2);
bb = [ 1 0 0 0 1 ];
yn = conv( bb, xn );
```

(a) Determine $\mathcal{H}(\hat{\omega})$ for the FIR filter.

(b) Make a plot of the magnitude of $\mathcal{H}(\hat{\omega})$ and label *all* the frequencies where $|\mathcal{H}(\hat{\omega})|$ is zero.

(c) Determine a formula for $y[n]$, the signal contained in the vector `yn`. Give the individual values for $n = 0, 1, 2, 3$, and then provide a general formula for $y[n]$ that is correct for $4 \leq n \leq 4000$. This formula should give numerical values for the amplitude, phase and frequency of $y[n]$.

(d) Give at least one value of `omega` such that the output is guaranteed to be zero, for $n \geq 4$.

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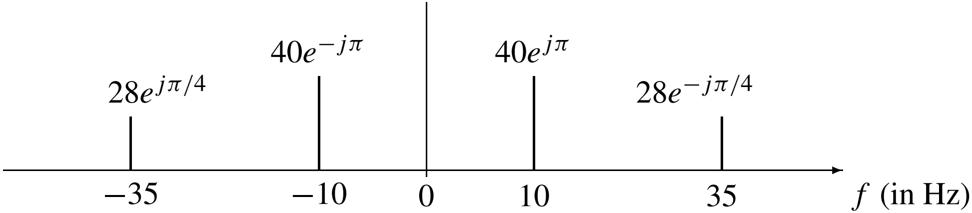
PROBLEM:

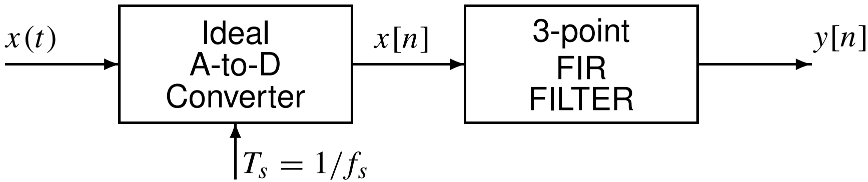
A signal $x(t)$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x(t)$. Make sure to express $x(t)$ as a real-valued signal.

(b) If the signal is sampled at a rate of $f_s = 25$ Hz, sketch the "digital" spectrum of this signal. Indicate the complex phasor value at each frequency. Only the range $-\pi < \hat{\omega} \le \pi$ needs to be shown.

(c) If the length-3 FIR filter (below) has filter coefficients $\{b_k\} = \{1, b_1, 1\}$, show that $b_1 = -2\cos(0.8\pi) = 1.618$ will make the output signal $y[n]$ equal to zero.



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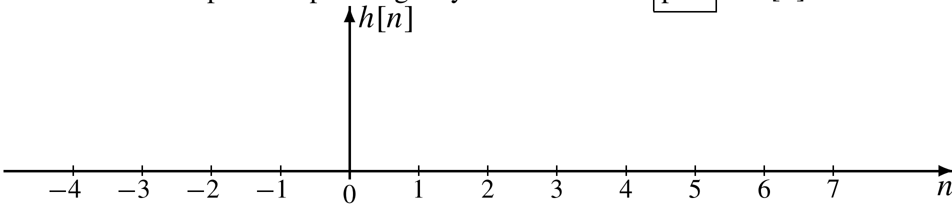
PROBLEM:

The following FIR filter is specified by the filter coefficients $\{b_k\} = \{0, 2, 3, 2\}$

(a) Determine the impulse response: give your answer as a $\boxed{\text{plot}}$ of $h[n]$ vs. n.

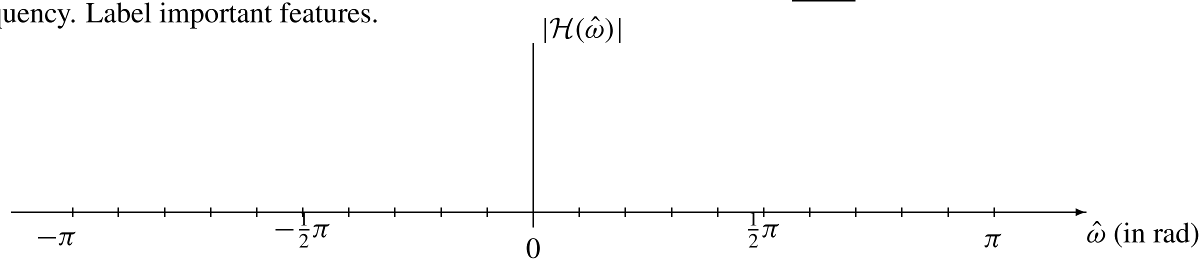


Plot zero values also

(b) Determine the frequency response, $\mathcal{H}(\hat{\omega})$, and select one of the following as the correct answer:

(A) $2\cos\hat{\omega} + 3e^{-j(2\hat{\omega}-\pi)}$ **(B)** $(4\cos\hat{\omega}+3)e^{-j\hat{\omega}}$ **(C)** $(3+4\cos\hat{\omega})e^{-j2\hat{\omega}}$ **(D)** $2\cos\hat{\omega}+3$

(c) Determine the magnitude of $\mathcal{H}(\hat{\omega})$ and present your answer as a a $\boxed{\text{plot}}$ of the magnitude vs. frequency. Label important features.



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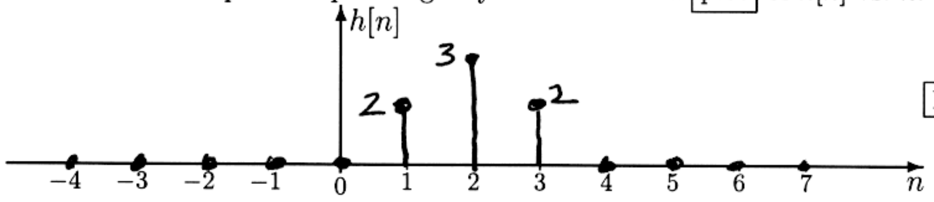
SOLUTION

The following FIR filter is specified by the filter coefficients $\{b_k\} = \{0, 2, 3, 2\}$

$y[n] = 2x[n-1] + 3x[n-2] + 2x[n-3]$
$h[n] = 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3]$

(a) Determine the impulse response: give your answer as a $\boxed{\text{plot}}$ of $h[n]$ vs. n.



Plot zero values also

$h[n] = 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3]$

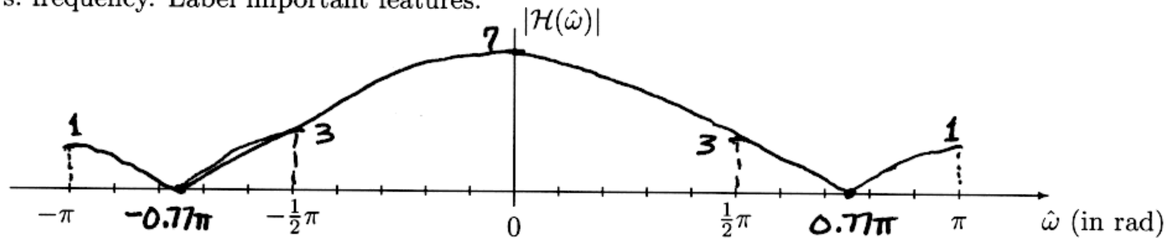
(b) Determine the frequency response, $\mathcal{H}(\hat\omega)$, and select one of the following as the correct answer:

(A) $2\cos\hat\omega + 3e^{-j(2\hat\omega-\pi)}$ **(B)** $(4\cos\hat\omega + 3)e^{-j\hat\omega}$ **(C)** $(3 + 4\cos\hat\omega)e^{-j2\hat\omega}$ **(D)** $2\cos\hat\omega + 3$

$$\mathcal{H}(\hat\omega) = 2e^{-j\hat\omega} + 3e^{-j2\hat\omega} + 2e^{-j3\hat\omega}$$
$$= e^{-j2\hat\omega}\left(2e^{+j\hat\omega} + 3 + 2e^{-j\hat\omega}\right)$$
$$= e^{-j2\hat\omega}\left(4\cos\hat\omega + 3\right)$$

C

(c) Determine the magnitude of $\mathcal{H}(\hat\omega)$ and present your answer as a a $\boxed{\text{plot}}$ of the magnitude vs. frequency. Label important features.



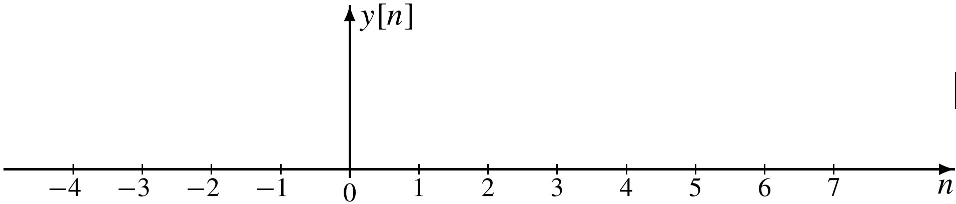
$|\mathcal{H}(\hat\omega)| = |3 + 4\cos\hat\omega|$

zero crossing at
$\cos\hat\omega = -\frac{3}{4} \Rightarrow \hat\omega = 0.77\pi$
$= -0.77\pi$

at $\hat\omega = 0 \rightarrow 3 + 4 = 7$
$\hat\omega = \pi \rightarrow 3 - 4 = -1 \leftarrow$ MAG $= 1$
$\hat\omega = \pi/2 \rightarrow 3 + 0 = 3$

PROBLEM:

(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{0, 2, 3, 2\}$, make a $\boxed{\text{plot}}$ of the output when the input is the unit step signal: $x[n] = u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$



Plot zero values also

(b) Suppose that the frequency response of a different FIR filter is

$$\mathcal{H}(\hat{\omega}) = \frac{\sin(9\hat{\omega}/2)}{\sin(\frac{1}{2}\hat{\omega})} e^{-j9\hat{\omega}}$$

If the input signal is $x[n] = 3 + 2\cos(0.2\pi n + 0.3\pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

$y[n] =$

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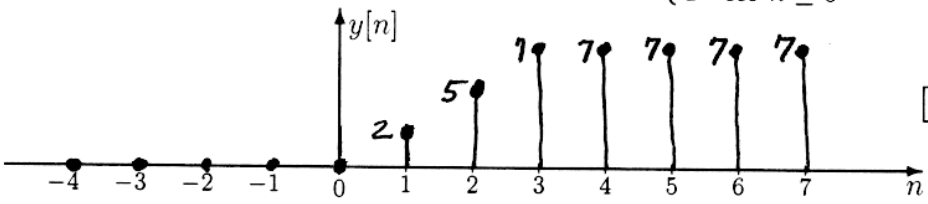
SOLUTION

(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{0, 2, 3, 2\}$, make a $\boxed{\text{plot}}$ of the output when the input is the unit step signal: $x[n] = u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$



Plot zero values also

n	<0	0	1	2	3	4	≥5 ...
x[n]	0	1	1	1	1	1	1 ...
y[n]	0	0	2	5	7	7	7 ...

$y[0] = 2x[-1] + 3x[-2] + 2x[-3] = 0$

$y[2] = 2+3+0 = 5$

$y[1] = 2x[0] + 3x[-1] + 2x[-2] = 2$

$y[3] = 2x[2] + 3x[1] + 2x[0] = 2+3+2 = 7$

(b) Suppose that the frequency response of a different FIR filter is

$$\mathcal{H}(\hat{\omega}) = \frac{\sin(9\hat{\omega}/2)}{\sin(\frac{1}{2}\hat{\omega})} e^{-j9\hat{\omega}}$$

If the input signal is $x[n] = 3 + 2\cos(0.2\pi n + 0.3\pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

$$\boxed{y[n] = 27 + 2\cos(0.2\pi n - 1.5\pi)}$$

x[n] has two frequency components: $\hat{\omega} = 0$ & $\hat{\omega} = 0.2\pi$

$\mathcal{H}(0) = 9e^{j0} = 9$

$\mathcal{H}(0.2\pi) = \dfrac{\sin(0.9\pi)}{\sin(0.1\pi)} e^{-j1.8\pi} = 1e^{-j1.8\pi}$

$y[n] = (9)(3) + (2)(1)\cos(0.2\pi n + \underbrace{0.3\pi - 1.8\pi}_{-1.5\pi})$

PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}})$$

(a) Write the difference equation for the FIR filter that gives the relation between the input $x[n]$ and the output $y[n]$. Give numerical values for the filter coefficients.

(b) What is the output of this FIR filter if the input is $x[n] = \delta[n]$?

(c) Evaluate the frequency response $\mathcal{H}(\hat{\omega})$ at the frequencies $\hat{\omega} = \pi$ and $\hat{\omega} = \pi/3$.

(d) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \leq \hat{\omega} \leq \pi$ will $y[n] = 0$ for all n?

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PROBLEM:

The following MATLAB code will compute a time response and the frequency response of a digital filter:

```
bb = [ 3, 3 ];     aa = [ 1, -1/3 ];
xn = [1, -1, 1, -1, 1];
yn = filter( bb, aa, xn );
subplot(2,1,1), stem( [0:4], yn );    %--- TIME RESPONSE
w = -pi : (pi/100) : pi;
H = freqz( bb, aa, w );
subplot(2,1,2), plot( w, abs(H) )      %--- FREQUENCY RESPONSE (MAG)
```

(a) Make the plot of `yn` that will be done by the MATLAB `stem` function (in line #4).

(b) Again referring to the MATLAB code above, make an approximate sketch of the magnitude response versus $\hat{\omega}$ over the range $-\pi \leq \hat{\omega} \leq \pi$. Label the sketch where $|H(e^{j\hat{\omega}})|$ is at its peak value and where it is zero.

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PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3]$$

(a) Find the frequency response $\mathcal{H}(\hat{\omega})$, and then express it as a mathematical formula, in polar form (magnitude and phase).

(b) Plot the magnitude and phase of $\mathcal{H}(\hat{\omega})$ as a function of $\hat{\omega}$ for $-\pi < \hat{\omega} < \pi$. Do this by hand, but you could check your answer by using the MATLAB function `freqz`.

(c) Find all frequencies, $\hat{\omega}$, for which the response to the input $e^{j\hat{\omega}n}$ is zero.

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PROBLEM:

A linear time-invariant system is described by the FIR difference equation

$$y[n] = x[n] - 3x[n-1] + 9x[n-2] - 3x[n-3] + x[n-4]$$

(a) Write a simple formula for the magnitude of the frequency response $|H(e^{j\hat{\omega}})|$. Express your answer in terms of real-valued functions only.

(b) Derive a simple formula for the phase of the frequency response $\angle H(e^{j\hat{\omega}})$.

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SOLUTION

A linear time-invariant system is described by the FIR difference equation

$$y[n] = x[n] - 3x[n-1] + 9x[n-2] - 3x[n-3] + x[n-4]$$

(a) Write a simple formula for the magnitude of the frequency response $|H(e^{j\hat{\omega}})|$. Express your answer in terms of real-valued functions only.

$$H(e^{j\hat{\omega}}) = 1 - 3e^{-j\hat{\omega}} + 9e^{-j2\hat{\omega}} - 3e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$$

$$= e^{-j2\hat{\omega}}\left(e^{j2\hat{\omega}} - 3e^{j\hat{\omega}} + 9 - 3e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)$$

$$= e^{-j2\hat{\omega}}\left(2\cos 2\hat{\omega} - 6\cos\hat{\omega} + 9\right)$$

$\Rightarrow$

$$|H(e^{j\hat{\omega}})| = 2\cos 2\hat{\omega} - 6\cos\hat{\omega} + 9$$

(b) Derive a simple formula for the phase of the frequency response $\angle H(e^{j\hat{\omega}})$.

$\angle H(e^{j\hat{\omega}}) = -2\hat{\omega}$ from part (a)

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PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = 2x[n] + 4x[n-1] - 3x[n-2] + x[n-3] - 3x[n-4] + 4x[n-5] + 2x[n-6]$$

(a) Write a simple formula for the magnitude of the frequency response $|H(e^{j\hat{\omega}})|$. Express your answer in terms of real-valued functions only.

(b) Derive a simple formula for the phase of the frequency response $\angle H(e^{j\hat{\omega}})$.

(c) *Impulse Response:* Determine the response of this system to a unit impulse input; i.e., find the output $y[n] = h[n]$ when the input is $x[n] = \delta[n]$. Plot $h[n]$ as a function of n.

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SOLUTION

(a) Exploit symmetry: i.e, $b_0 = b_6$, $b_1 = b_5$, etc.

Filter coeffs are $\{2, 4, -3, 1, -3, 4, 2\}$

$$H(\hat{\omega}) = \sum_{k=0}^{6} b_k e^{-jk\omega} = 2 + 4e^{-j\hat{\omega}} - 3e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} \quad e^{-j4\hat{\omega}} + 4e^{-j5\hat{\omega}} + 2e^{-j6\hat{\omega}}$$

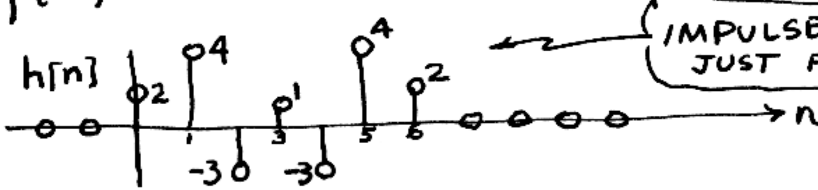
FACTOR OUT $e^{-j3\hat{\omega}}$:

$$H(\hat{\omega}) = e^{-j3\hat{\omega}}\left(2e^{+j3\hat{\omega}} + 4e^{+j2\hat{\omega}} - 3e^{j\hat{\omega}} + 1 - 3e^{-j\hat{\omega}} + 4e^{-j2\hat{\omega}} + 2e^{-j3\hat{\omega}}\right)$$

$$= e^{-j3\hat{\omega}}\Big(\underbrace{4\cos 3\hat{\omega} + 8\cos 2\hat{\omega} - 6\cos\hat{\omega} + 1}_{\text{MAGNITUDE (BUT MAY HAVE MINUS SIGN)}}\Big).$$

(b) $\varphi(\hat{\omega}) = -3\hat{\omega}$

(c) IMPULSE RESPONSE IS JUST FILTER COEFFS



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PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = 2x[n] + 4x[n-1] + 2x[n-2]$$

(a) Find the frequency response $H(e^{j\hat{\omega}})$, and then express it as a mathematical formula, in polar form (magnitude and phase).

(b) $H(e^{j\hat{\omega}})$ is a periodic function of $\hat{\omega}$; determine the period.

(c) Plot the magnitude and phase of $H(e^{j\hat{\omega}})$ as a function of $\hat{\omega}$ for $-\pi < \hat{\omega} < 3\pi$. Do this by hand and then check with the MATLAB function `freqz`.

(d) Find all frequencies, $\hat{\omega}$, for which the output response to the input $e^{j\hat{\omega}n}$ is zero.

(e) When the input to the system is $x[n] = \sin(\pi n/10)$ determine the functional form for the output signal $y[n]$.

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SOLUTION

$$y[n] = 2x[n] + 4x[n-1] + 2x[n-2].$$

(a) The filter coefficients are: $\{b_k\} = \{2, 4, 2\}$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{2} b_k e^{-jk\hat{\omega}} = 2 + 4e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}}$$

FACTOR OUT $e^{-j\hat{\omega}}$:

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}\left(2e^{j\hat{\omega}} + 4 + 2e^{-j\omega}\right)$$

$$= e^{-j\hat{\omega}}\left(4 + 4\cos\hat{\omega}\right)$$

THIS TERM IS THE MAGNITUDE; IT IS NEVER NEGATIVE

$$\varphi(\hat{\omega}) = -\hat{\omega}$$

$$M(\hat{\omega}) = |\mathcal{H}(\hat{\omega})| = 4 + 4\cos\hat{\omega}$$

(b) $H(\hat{\omega})$ ALWAYS HAS PERIOD EQUAL TO 2π

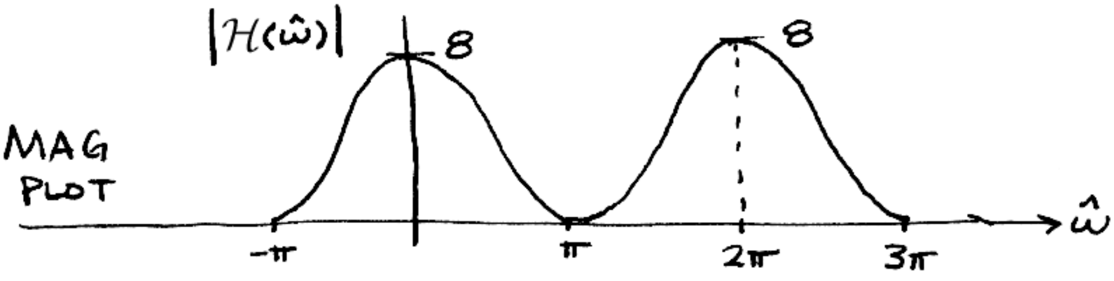
Proof: $\mathcal{H}(\hat{\omega} + 2\pi) = e^{-j(\hat{\omega}+2\pi)}\left(4 + 4\cos(\hat{\omega} + 2\pi)\right)$

$$= e^{-j\hat{\omega}}\left(4 + 4\cos\hat{\omega}\right) = \mathcal{H}(\hat{\omega}).$$

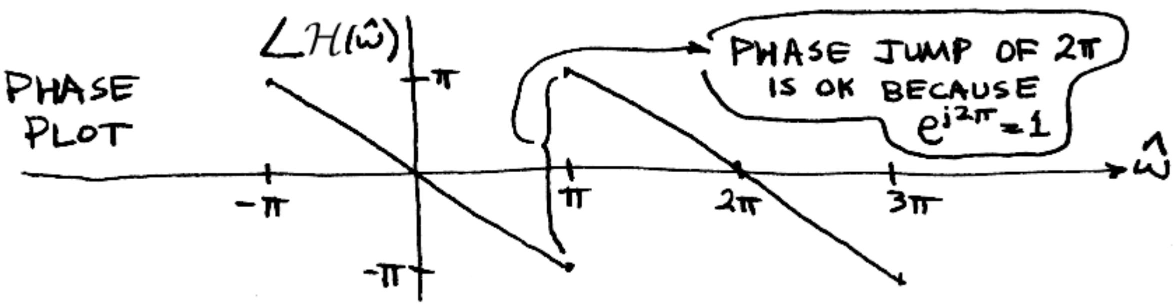
Because $e^{j2\pi} = 1$ & cosine has period $= 2\pi$

(c)

MAG PLOT



PHASE PLOT



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Prob (cont)

(d) $$e^{j\hat{\omega}n} \longrightarrow \boxed{\mathcal{H}(\hat{\omega})} \longrightarrow y[n] = \mathcal{H}(\hat{\omega})e^{j\hat{\omega}n}$$

OUTPUT WILL BE ZERO WHEN $\mathcal{H}(\hat{\omega}) = 0$

THIS HAPPENS WHEN $|\mathcal{H}(\hat{\omega})| = 0$

FROM THE PLOT IN PART (c) THIS HAPPENS WHEN $\boxed{\hat{\omega} = \pi}$ or $-\pi$.

THUS $\boxed{e^{j\pi n}}$ PRODUCES ZERO OUTPUT.

NOTE: $e^{j\pi n} = e^{-j\pi n}$ SO $\hat{\omega} = +\pi$ & $\hat{\omega} = -\pi$ ARE ACTUALLY THE SAME FREQ.

(e) $x[n] = \sin(\pi n/10) = \frac{1}{2j}\left(e^{j\pi n/10} - e^{-j\pi n/10}\right)$.

USE LINEARITY: i.e., find responses for $e^{+j\pi n/10}$ and for $e^{-j\pi n/10}$ and then subtract.

when input is: OUTPUT WILL BE:

$$\frac{1}{2j}e^{j\pi n/10} \longrightarrow \frac{1}{2j}\mathcal{H}\left(\frac{\pi}{10}\right)e^{j\pi n/10}$$

$$= \frac{1}{2}e^{-j\pi/2}\left(7.8e^{-j0.1\pi}\right)e^{j\pi n/10}$$

$$= 3.9e^{-j0.6\pi}e^{j\pi n/10}$$

$$-\frac{1}{2j}e^{-j\pi n/10} \longrightarrow \frac{1}{2}e^{+j0.5\pi}\left(\mathcal{H}(-\pi/10)\right)e^{-j\pi n/10}$$

$$= 3.9e^{+j0.6\pi}e^{-j\pi n/10}$$

$\therefore$ $y[n]$ = SUM of these two

$$= 3.9e^{-j0.6\pi}e^{j\pi n/10} + 3.9e^{+j0.6\pi}e^{-j\pi n/10}$$

$$y[n] = 7.8\cos(\pi n/10 - 0.6\pi)$$

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PROBLEM:

Suppose that a LTI system has a frequency response function equal to

$$\mathcal{H}(\hat{\omega}) = 2 + 3e^{-j\hat{\omega}} + 3e^{-j3\hat{\omega}} + 2e^{-j4\hat{\omega}}$$

(a) Determine the difference equation that relates the output $y[n]$ of the system to the input $x[n]$.

(b) Determine and plot the *impulse response*.

(c) Determine the output when the input is a pulse:

$$p[n] = \begin{cases} 1 & \text{for } 0 \leq n \leq 3 \\ 0 & n < 0 \end{cases}$$

Use *convolution* for a quick solution.

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SOLUTION

$$\mathcal{H}(\hat{\omega}) = \underset{b_0}{2} + \underset{b_1}{3}e^{-j\hat{\omega}} + \underset{b_3}{3}e^{-j3\hat{\omega}} + \underset{b_4}{2}e^{-j4\hat{\omega}}$$

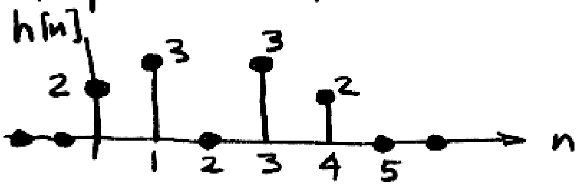
$b_2 = 0$

(a) $y[n] = 2x[n] + 3x[n-1] + 3x[n-3] + 2x[n-4]$

(b) $h[n] = 2\delta[n] + 3\delta[n-1] + 3\delta[n-3] + 2\delta[n-4]$

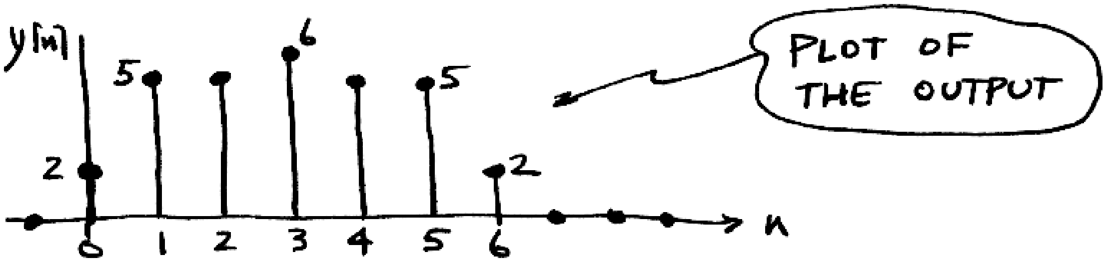
because we just use $x[n] = \delta[n]$

The impulse response reads out the filter coeffs.



(C) Use "convolution" which can be done with the following table:

2	3	0	3	2		
1	1	1				
2	3	0	3	2		
	2	3	0	3	2	
		2	3	0	3	2
2	5	5	6	5	5	2
$n=0$						$n=6$.



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PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = x[n] - 2x[n-1] + x[n-2]$$

(a) Find the frequency response $H(\hat{\omega})$, and then express it as a mathematical formula, in polar form (magnitude and phase).

(b) Plot the magnitude and phase of $H(\hat{\omega})$ as a function of $\hat{\omega}$ for $-\pi < \hat{\omega} < \pi$. Do this by hand and with the MATLAB function `freqz`.

(c) Find all frequencies, ω, for which the response to the input $e^{j\omega n}$ is zero.

(d) When the input to the system is $x[n] = \sin(\pi n/100)$ determine the functional form for the output signal $y[n]$.

(e) *Impulse Response:* Determine the response of this system to a unit impulse input; i.e., find the output $y[n] = h[n]$ when the input is $x[n] = \delta[n]$. Plot $h[n]$ as a function of n.

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SOLUTION

$$y[n] = x[n] - 2x[n-1] + x[n-2] \qquad \mathbf{b} = [1 \;\; -2 \;\; 1]$$

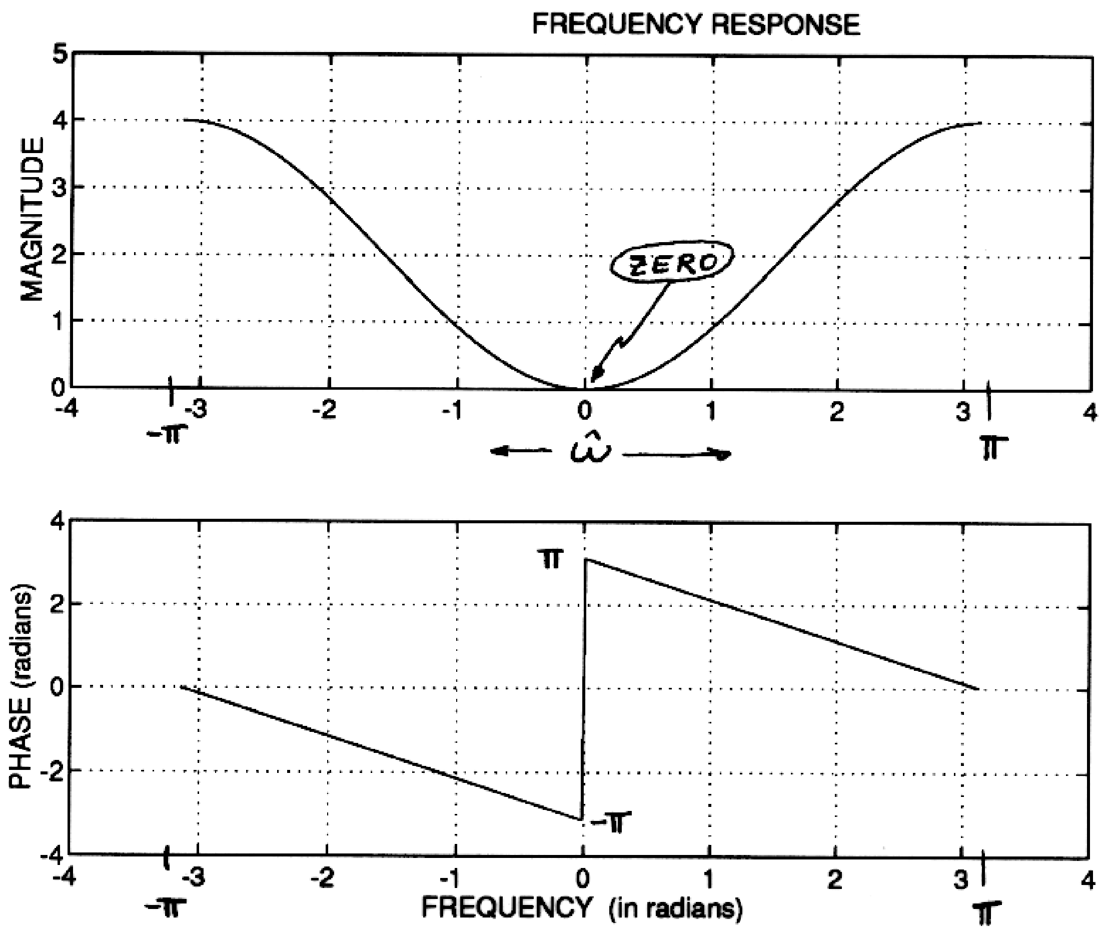
(a) $$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{2} b_k e^{-j\hat{\omega}k} = 1 - 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

Can simplify $\mathcal{H}(\hat{\omega})$

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} \left\{ e^{j\hat{\omega}} - 2 + e^{-j\hat{\omega}} \right\}$$

$$= \underbrace{(2\cos\hat{\omega} - 2)}_{\text{THIS TERM IS THE MAGNITUDE EXCEPT IT MAY HAVE A MINUS SIGN}} \overbrace{e^{-j\hat{\omega}}}^{\text{PHASE}}$$

(b)



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Prob (cont.)

(c) $\mathcal{H}(\hat{\omega}) = 0$ only when $\hat{\omega} = 0$

$\therefore \quad x[n] = e^{j0n} = 1$ will give zero output

(d) $\mathcal{H}(\hat{\omega})$ at $\hat{\omega} = \pi/100$ is $\mathcal{H}(\pi/100) = 0.001\, e^{j0.99\pi}$

Since freq response changes mag & phase

$x[n] = \sin(\pi n/100) = \cos(\pi n/100 - \pi/2)$

$\Rightarrow y[n] = 0.001 \cos(\pi n/100 - 0.5\pi + 0.99\pi)$

$y[n] = 0.001 \cos(\pi n/100 + 0.49\pi)$

or $\quad y[n] = 0.001 \sin(\pi n/100 + 0.99\pi)$

Another approach:

$$\sin(\pi n/100) = \frac{1}{2j} e^{j\pi n/100} - \frac{1}{2j} e^{-j\pi n/100}$$

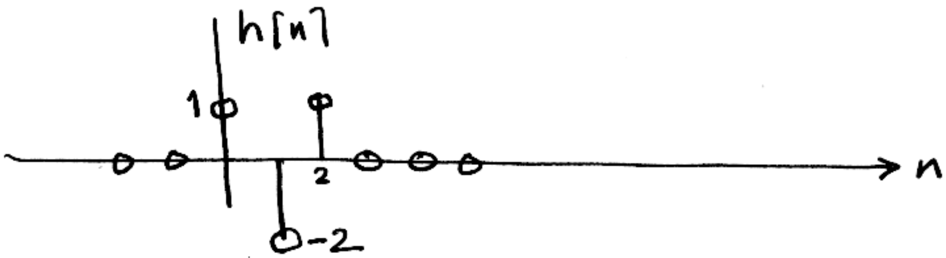
$$y[n] = \mathcal{H}\left(\tfrac{\pi}{100}\right) \frac{1}{2j} e^{j\pi n/100} - \frac{1}{2j} \mathcal{H}\left(-\tfrac{\pi}{100}\right) e^{-j\pi n/100}$$

$$= \frac{1}{2j}\left(0.001\, e^{j\left(\frac{\pi n}{100} + .99\pi\right)} - 0.001\, e^{-j\left(\frac{\pi n}{100} + .99\pi\right)}\right).$$

$$= 0.001 \sin\left(\frac{\pi n}{100} + .99\pi\right)$$

NOTE: $\mathcal{H}\left(-\frac{\pi}{100}\right) = 0.001\, e^{-j0.99\pi}$

(e) $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$.



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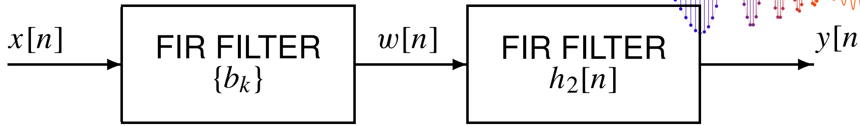
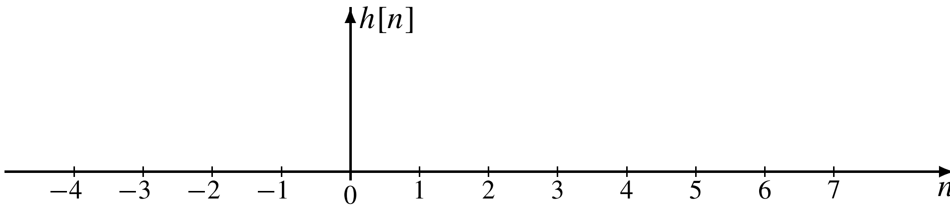
PROBLEM:



(a) If the filter coefficients of the first FIR filter are $\{b_k\} = \{0, 1, -2, 1\}$, and the impulse response of the second FIR filter is $h_2[n] = \delta[n] + 2\delta[n-2] + \delta[n-3]$, use convolution to determine the impulse response of the overall system, $h[n]$. Give your answer as a plot below.



Label Carefully

Plot zero values also

(b) Suppose that the overall frequency response of the cascade system (using different FIR filters from those in part (a)) is

$$\mathcal{H}(\hat{\omega}) = (2 + 2\cos(\hat{\omega}))e^{-j\hat{\omega}}$$

If the input signal is $\quad x[n] = 10 + 6\cos(0.5\pi n + \pi/3) \quad$ for $-\infty < n < \infty$, determine a simple mathematical expression for the overall output signal $y[n]$.

$y[n] =$

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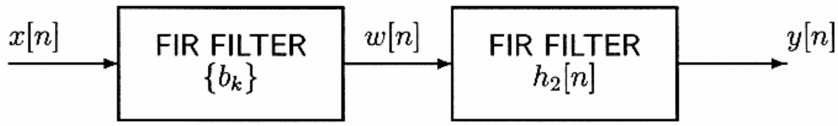
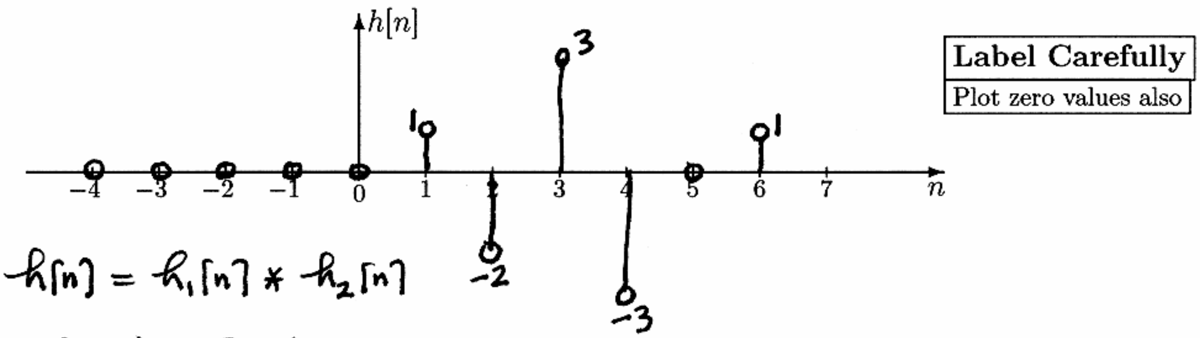
SOLUTION



(a) If the filter coefficients of the first FIR filter are $\{b_k\} = \{0, 1, -2, 1\}$, and the impulse response of the second FIR filter is $h_2[n] = \delta[n] + 2\delta[n-2] + \delta[n-3]$, use convolution to determine the impulse response of the overall system, $h[n]$. Give your answer as a plot below.



$h[n] = h_1[n] * h_2[n]$

$$
\begin{array}{ccccccc}
0 & 1 & -2 & 1 & & & \\
1 & 0 & 2 & 1 & & & \\
\hline
0 & 1 & -2 & 1 & & & \\
 & 0 & 0 & 0 & 0 & & \\
 & & 0 & 2 & -4 & 2 & \\
 & & & 0 & 1 & -2 & 1 \\
\hline
0 & 1 & -2 & 3 & -3 & 0 & 1 \\
\uparrow & & & \uparrow & & & \uparrow \\
n=0 & & & n=3 & & & n=6
\end{array}
$$

(b) Suppose that the overall frequency response of the cascade system (using different FIR filters from those in part (a)) is

$$\mathcal{H}(\hat{\omega}) = (2 + 2\cos(\hat{\omega}))e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 10 + 6\cos(0.5\pi n + \pi/3)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the overall output signal $y[n]$.

$$\boxed{y[n] = 40 + 12\cos(\tfrac{\pi}{2}n - \pi/6)}$$

The input signal has freqs: $\hat{\omega} = 0$ & $\hat{\omega} = 0.5\pi = \pi/2$

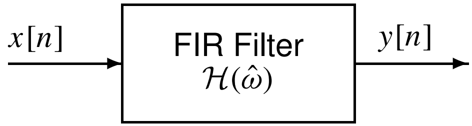
$\mathcal{H}(\hat{\omega})\big|_{\hat{\omega}=0} = (2 + 2\cos(0))e^{-j0} = 4$

$\mathcal{H}(\hat{\omega})\big|_{\hat{\omega}=\pi/2} = (2 + 2\cos(\pi/2))e^{-j\pi/2} = 2e^{-j\pi/2}$

$$y[n] = 10\cdot 4 + 6\cdot 2\cos\Big(\frac{\pi}{2}n + \underbrace{\frac{\pi}{3} - \frac{\pi}{2}}_{-\pi/6}\Big)$$

PROBLEM:

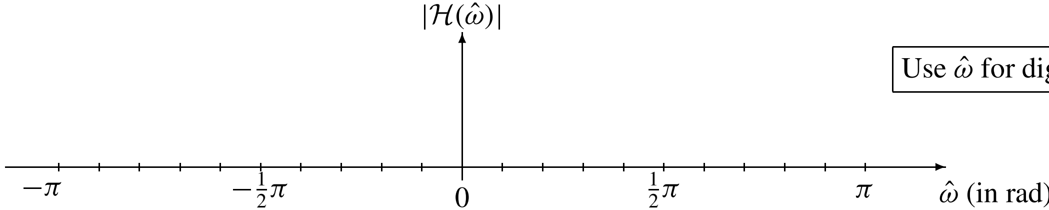
Consider the following system diagram



where $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j5\hat{\omega}}$.

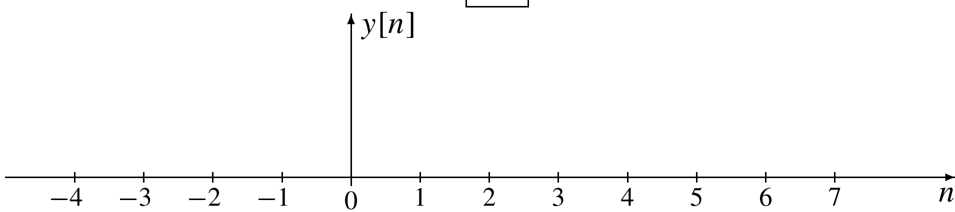
(a) Write the frequency response $\mathcal{H}(\hat{\omega})$ in $\boxed{\text{polar}}$ form.

(b) $\boxed{\text{Plot}}$ the magnitude vs. frequency of $\mathcal{H}(\hat{\omega})$. Label important features.



Use $\hat{\omega}$ for digital freq.

(c) For the input $x[n] = 2\delta[n] - \delta[n-2]$, $\boxed{\text{plot}}$ the output signal $y[n]$.



Label Carefully

Plot zero values also

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SOLUTION

Consider the following system diagram

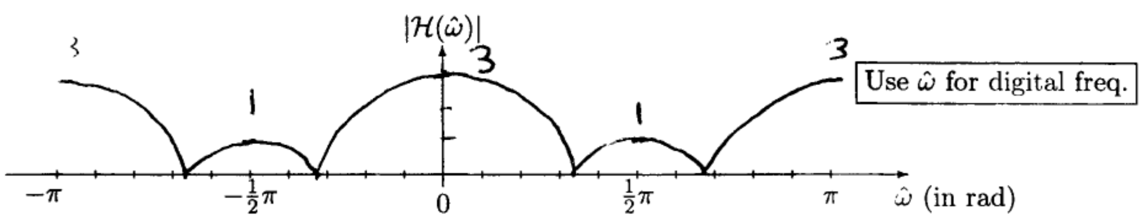
where $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j5\hat{\omega}}$.

(a) Write the frequency response $\mathcal{H}(\hat{\omega})$ in $\boxed{\text{polar}}$ form.

$$\mathcal{H}(\hat{\omega}) = e^{-j3\hat{\omega}}\left(e^{j2\hat{\omega}} + 1 + e^{-j2\hat{\omega}}\right)$$

$$= (1 + 2\cos 2\hat{\omega})\, e^{-j3\hat{\omega}}$$

(b) $\boxed{\text{Plot}}$ the magnitude vs. frequency of $\mathcal{H}(\hat{\omega})$. Label important features.

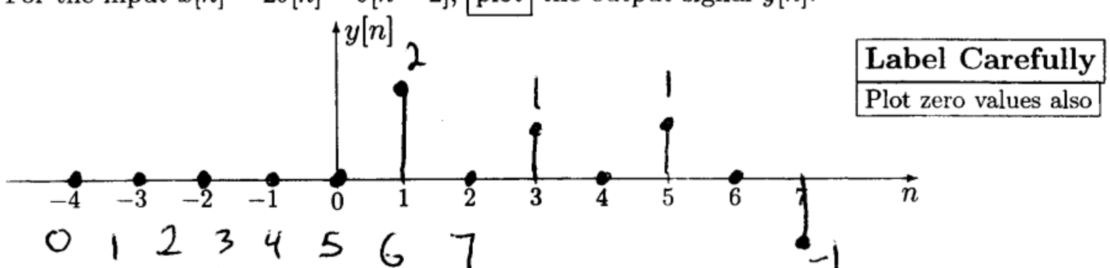


Zeros at $1 + 2\cos 2\hat{\omega} = 0$

$$\cos 2\hat{\omega} = -\frac{1}{2} \Rightarrow \hat{\omega} = \frac{4\pi}{6},$$

$$\hat{\omega} = \frac{2\pi}{3}, \frac{\pi}{3} \text{ are zeros}$$

(c) For the input $x[n] = 2\delta[n] - \delta[n-2]$, $\boxed{\text{plot}}$ the output signal $y[n]$.



n	0	1	2	3	4	5	6	7
$h[n]$	0	1	0	1	0	1	0	0
$x[n]$	2	0	-1	0	0	0	0	0
	0	2	0	2	0	2	0	0
	0	0	0	-1	0	-1	0	-1
	0	2	0	1	0	1	0	-1

PROBLEM:

The complex-valued frequency response for an L-point moving average filter is

$$H(\omega) = \frac{1}{L}\sum_{k-0}^{L-1} e^{-j\omega k} = \frac{1}{L}\,\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}$$

(a) Derive a formula for the phase of $H(\omega)$ and make a plot.

(b) Derive a formula for the magnitude of $H(\omega)$ and make a plot.

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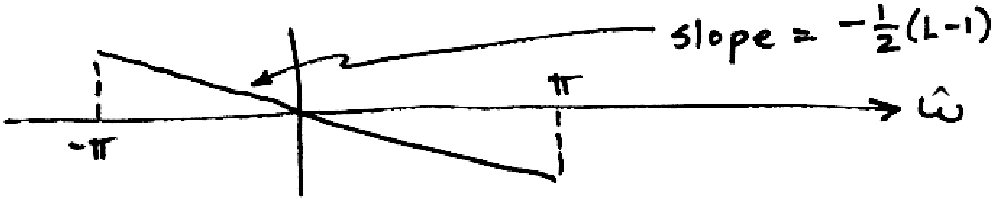
SOLUTION

(a) $$\frac{1}{L}\,\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}} = \frac{1}{L}\,\frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}}\,\frac{\left(e^{+j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2}\right)}{\left(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}\right)}$$

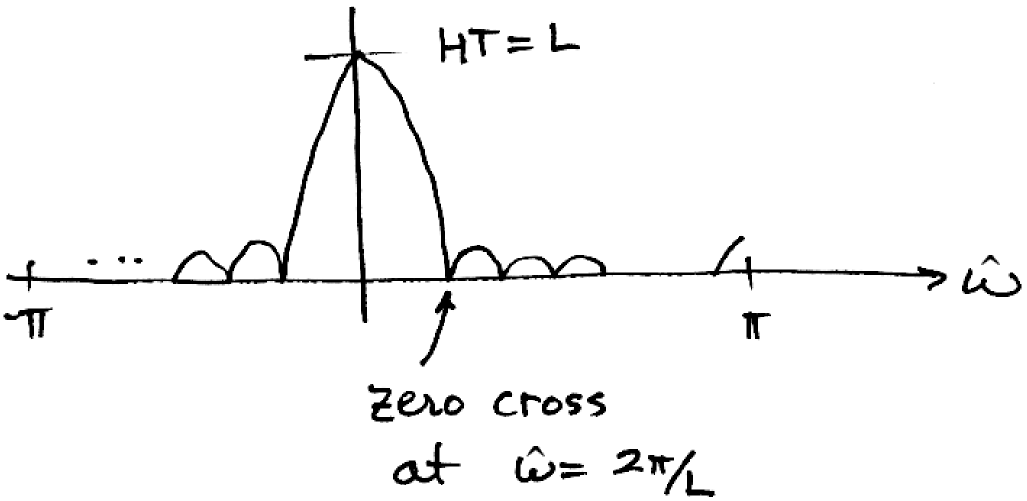
$$= e^{-j\hat{\omega}(L-1)/2}\cdot\frac{1}{L}\,\frac{\sin \hat{\omega}L/2}{\sin \hat{\omega}/2}$$

phase vs. $\hat{\omega}$ $= -\hat{\omega}(L-1)/2$



(b) Magnitude.

$$\left|\frac{1}{L}\,\frac{\sin\frac{\hat{\omega}L}{2}}{\sin\frac{\hat{\omega}}{2}}\right|$$



PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = x[n] + 3x[n-1] + 3x[n-2] + x[n-3]$$

(a) Find the frequency response $\mathcal{H}(\hat{\omega})$, and then express it as a mathematical formula, in polar form (magnitude and phase).

(b) Plot the magnitude and phase of $\mathcal{H}(\hat{\omega})$ as a function of $\hat{\omega}$ for $-\pi < \hat{\omega} < \pi$. Do this by hand, but you could check your answer by using the MATLAB function `freqz`.

(c) When the input to the system is $x[n] = \exp(j\pi n/2)$ determine the functional form for the output signal $y[n]$. Find numerical values for the magnitude and phase of $y[n]$.

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SOLUTION

$$y[n] = x[n] + 3x[n-1] + 3x[n-2] + x[n-3]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

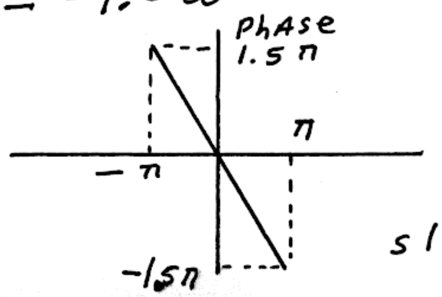
$$\mathcal{H}(\hat\omega) = \sum_{k=0}^{M} b_k e^{-j\hat\omega k}$$

$$\mathcal{H}(\hat\omega) = 1 + 3e^{-j\hat\omega} + 3e^{-j\hat\omega 2} + e^{-j\hat\omega 3}$$

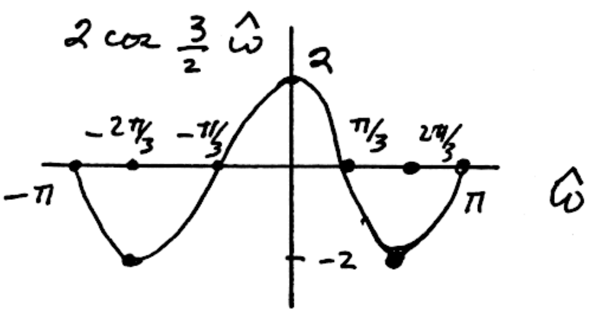
$$\mathcal{H}(\hat\omega) = e^{-j\hat\omega 1.5}\left[e^{+j\hat\omega 1.5} + 3e^{j\hat\omega .5} + 3e^{-j\hat\omega .5} + e^{-j\hat\omega 1.5}\right]$$

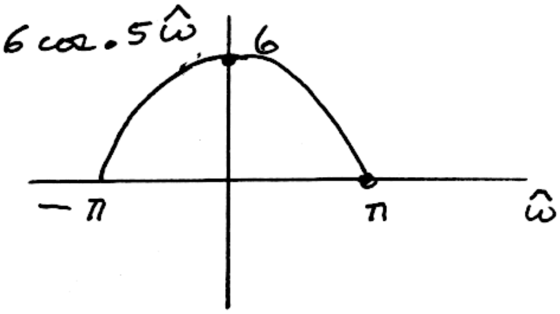
$$\mathcal{H}(\hat\omega) = e^{-j\hat\omega 1.5}\left[2\cos 1.5\hat\omega + 6\cos .5\hat\omega\right]$$

$$\text{Phase} = -1.5\hat\omega = \angle \mathcal{H}(\hat\omega)$$

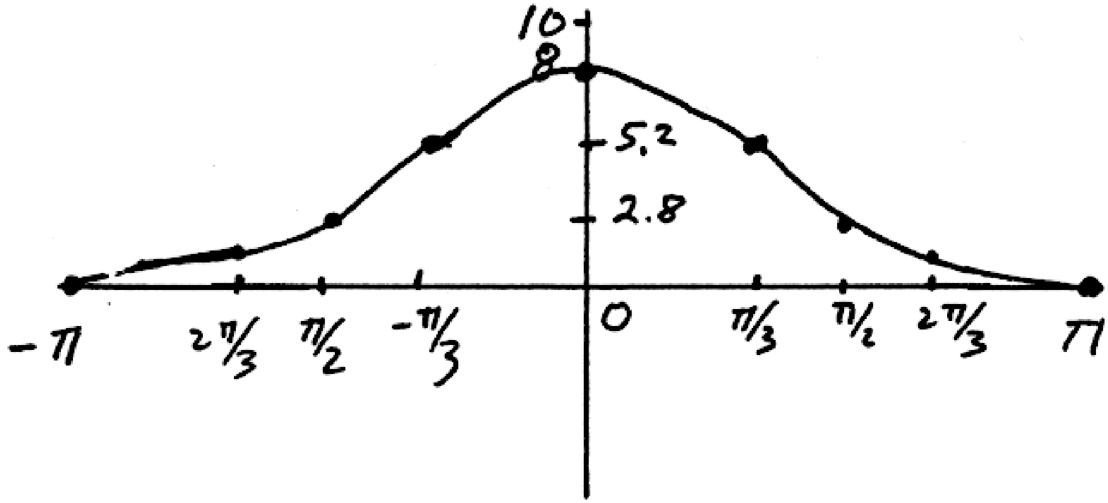


$$\text{Magnitude} = \left|2\cos 1.5\hat\omega + 6\cos .5\hat\omega\right|$$

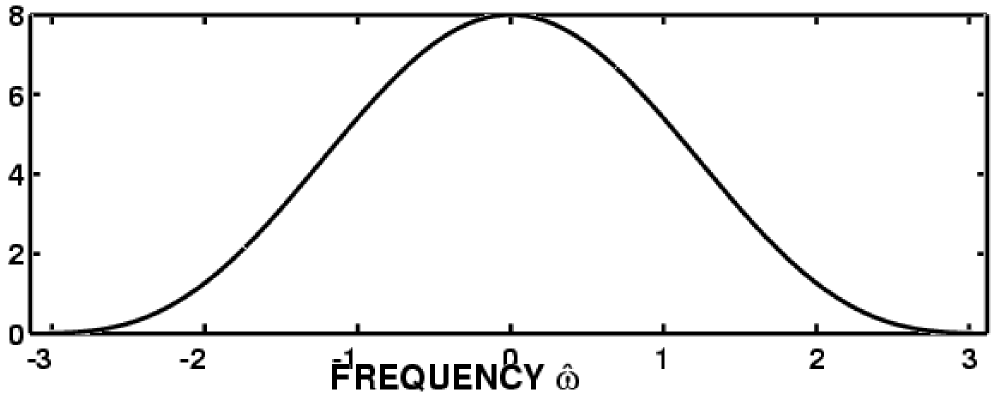




$$\text{SUM} = 2\cos\tfrac{3}{2}\hat{\omega} + 6\cos\tfrac{1}{2}\hat{\omega}$$



SINCE FUNCTION IS POSITIVE, ABOVE PLOT IS MAGNITUDE.



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(c) $x[n] = e^{j\pi n/2}$ for all n

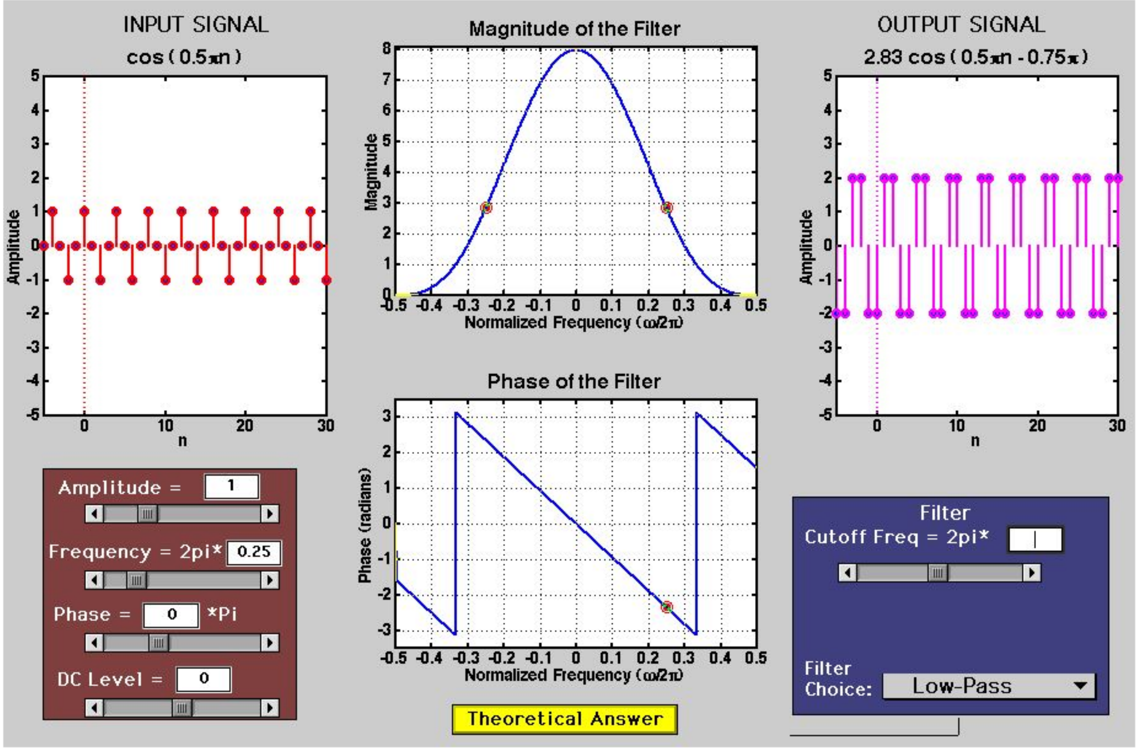
$$y[n] = e^{j\pi n/2} + 3e^{j\pi(n-1)/2} + 3e^{j\pi(n-2)/2} + e^{j\pi(n-3)/2}$$

$$= e^{j\pi n/2}\left(1 + 3e^{-j\pi/2} + 3e^{-j\pi} + e^{-j3\pi/2}\right)$$

$$= e^{j\pi n/2}(\underbrace{1 - j3 - 3 + j}_{= -2-j2 = 2\sqrt{2}\, e^{-j3\pi/4} = 2.83\, e^{-j3\pi/4}})$$

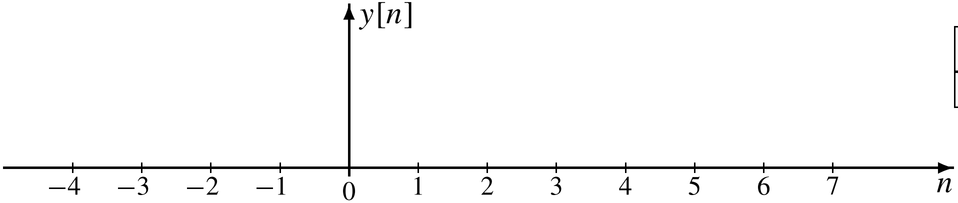
$$y[n] = 2\sqrt{2}\, e^{-j3\pi/4} e^{j\pi n/2}$$

For this problem, the frequency response is evaluated at $\hat{\omega} = \pi/2$. A plot generated from the MATLAB GUI "ltidemo" is attached.



PROBLEM:

(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{0, 1, -1, 1\}$, make a $\boxed{\text{plot}}$ of the output when the input is the unit step signal: $x[n] = u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$



Label Carefully

Plot zero values also

(b) Suppose that the frequency response of a different FIR filter is

$$\mathcal{H}(\hat{\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 1 + 3\cos(\pi n + \pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

$y[n] =$

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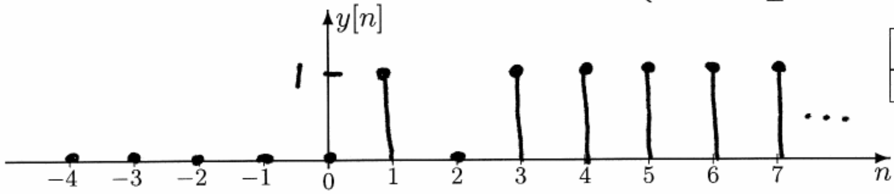
SOLUTION

(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{0, 1, -1, 1\}$, make a $\boxed{\text{plot}}$ of the output when the input is the unit step signal: $x[n] = u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$



Label Carefully

Plot zero values also

$x[n-1]$:	0	0	0	0	1	1	1	1	1	1	...
$-x[n-2]$:		0	0	0	0	−1	−1	−1	−1	−1	...
$x[n-3]$:			0	0	0	0	1	1	1	1	...
$y[n]$:	0	0	0	0	1	0	1	1	1	1	...

(b) Suppose that the frequency response of a different FIR filter is

$$\mathcal{H}(\hat{\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 1 + 3\cos(\pi n + \pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

$\boxed{y[n] = 1}$

$\mathcal{H}(0) = 1 \qquad \mathcal{H}(\pi) = 0$

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PROBLEM:

(a) Determine the frequency response of the FIR system:

$$y[n] = 10x[n-2] - 10x[n-5]$$

Give your answer as a formula ***in the following form:*** $H(e^{j\hat{\omega}}) = je^{-j\alpha\hat{\omega}}\beta\sin(\lambda\hat{\omega})$ by finding numerical values for α, β and λ.

$\alpha =$	$\beta =$	$\lambda =$

(b) For the system in part (a), determine the output signal $y[n]$ when the input signal is

$$x[n] = \sqrt{7} + 100\cos(0.1\pi n)$$

(c) Write a few lines of MATLAB code that would compute the specific values of the frequency response needed in part (b).

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PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = -3x[n-2] + 6x[n-4] - 3x[n-6]. \tag{1}$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal. Explain your answers.

(b) Obtain an expression for the frequency response of this system.

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(d) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 4 + 8\cos(0.5\pi n + \pi/2)$$

Hint: Use the frequency response and superposition to solve this problem.

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SOLUTION

$$y[n] = -3x[n-2] + 6x[n-4] - 3x[n-6]$$

Part A

System is **linear**. To see this, consider two inputs $x_1[n]$ and $x_2[n]$, the corresponding outputs $y_1[n]$ and $y_2[n]$ are given by:

$$\begin{aligned} y_1[n] &= -3x_1[n-2] + 6x_1[n-4] - 3x_1[n-6] \\ y_2[n] &= -3x_2[n-2] + 6x_2[n-4] - 3x_2[n-6] \end{aligned}$$

and if we plug in $x[n] = a_1x_1[n] + a_2x_2[n]$ as the input, the resulting output is

$$\begin{aligned} y[n] &= -3(a_1x_1[n-2] + a_2x_2[n-2]) \\ &\quad +6(a_1x_1[n-4] + a_2x_2[n-4]) \\ &\quad -3(a_1x_1[n-6] + a_2x_2[n-6]) \\ &= a_1\big(-3x_1[n-2] + 6x_1[n-4] - 3x_1[n-6]\big) + \\ &\quad a_2\big(-3x_2[n-2] + 6x_2[n-4] - 3x_2[n-6]\big) \\ &= a_1y_1[n] + a_2y_2[n]. \end{aligned}$$

System is **time-invariant** since plugging in a delayed version $x[n-n_0]$ of the input $x[n]$ yields a delayed version of the corresponding output.

$$y[n-n_0] = -3x[n-n_0-2] + 6x[n-n_0-4] - 3x[n-n_0-6]$$

System is **causal** since the output $y[n]$ at time n only depends upon the input at times previous to n: $x[n-2]$, $x[n-4]$, $x[n-6]$.

Part B

The impulse response $h[n]$ is obtained by setting $x[n] = \delta[n]$:

$$h[n] = -3\delta[n-2] + 6\delta[n-4] - 3\delta[n-6]$$

with a corresponding frequency response of

$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= -3e^{-j2\hat{\omega}} + 6e^{-j4\hat{\omega}} - 3e^{-j6\hat{\omega}} \\ &= e^{-j4\hat{\omega}}\Big(-3e^{j2\hat{\omega}} + 6 - 3e^{-j2\hat{\omega}}\Big) \\ &= e^{-j4\hat{\omega}}\Big(6 - 6\cos 2\hat{\omega}\Big) = \boxed{6e^{-j4\hat{\omega}}\Big(1 - \cos 2\hat{\omega}\Big)} \end{aligned}$$

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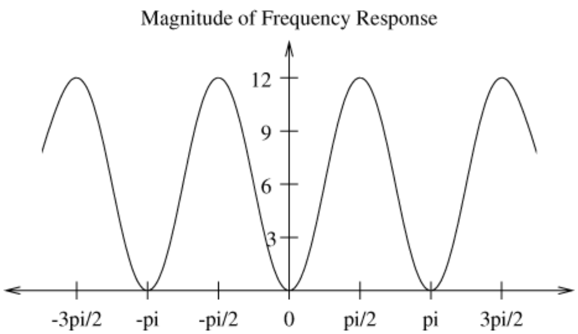
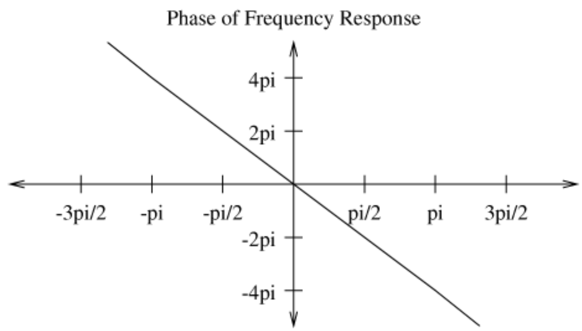
Part C

$$|\mathcal{H}(\hat{\omega})| = 6\Big(1 - \cos 2\hat{\omega}\Big) \qquad \angle\mathcal{H}(\hat{\omega}) = -4\hat{\omega}$$





Part D

$$x_1[n] = 4e^{j0n} + 4e^{j\pi/2}e^{j0.5\pi n} + 4e^{-j\pi/2}e^{-j0.5\pi n}$$

Using the above frequency response, we may right

$$\begin{aligned} y_1[n] &= 4e^{j0n}\mathcal{H}(0) + 4e^{j\pi/2}e^{j0.5\pi n}\mathcal{H}(0.5\pi) + 4e^{-j\pi/2}e^{-j0.5\pi n}\mathcal{H}(-0.5\pi) \\ &= 4(0) + 4e^{j\pi/2}e^{j0.5\pi n}(12) + 4e^{-j\pi/2}e^{-j0.5\pi n}(12) \\ &= \boxed{96\cos(0.5\pi n + \pi/2)} \end{aligned}$$

PROBLEM:

A discrete-time system is defined by the input/output relation (given as a difference equation)

$$y[n] = -Gx[n-2] + 2Gx[n-3] - Gx[n-4]$$

where G is a real-valued constant to be determined.

(a) Obtain an expression (in terms of G) for the frequency response of this system. Simplify into the "magnitude-phase" form: $H(e^{j\hat{\omega}}) = e^{j\psi(\hat{\omega})}M(\hat{\omega})$, where $M(\hat{\omega})$ and $\psi(\hat{\omega})$ are real.

(b) When the input is the signal, $x_1[n] = (-1)^n$, the output is $y_1[n] = 20(-1)^{n+1}$. Determine the value of G.

$G = \boxed{}$

(c) For the system above, set $G = 1$ and then determine the output $y_2[n]$ when the input is

$$x_2[n] = 2\cos(0.5\pi n - \pi/3)$$

Fill in the boxes below:

$$y_2[n] = \boxed{}\cos(\boxed{}\, n + \boxed{})$$

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PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = 2x[n] - 5x[n-1] + 2x[n-2]. \tag{1}$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal. Explain your answers.

(b) Obtain an expression for the frequency response of this system.

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(d) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.5\pi(n-1))$$

Hint: use the linearity and time-invariance properties.

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SOLUTION

$$y[n] = 2x[n] - 5x[n-1] + 2x[n-2]$$

a) Linearity test:

Let $x[n] = a_1x_1[n] + a_2x_2[n]$, then

$$y[n] = 2(a_1x_1[n] + a_2x_2[n]) - 5(a_1x_1[n-1] + a_2x_2[n-1]) + 2(a_1x_1[n-2] + a_2x_2[n-2])$$
$$y[n] = 2a_1x_1[n] + 2a_2x_2[n] - 5a_1x_1[n-1] - 5a_2x_2[n-1] + 2a_1x_1[n-2] + 2a_2x_2[n-2]$$
$$y[n] = 2a_1x_1[n] - 5a_1x_1[n-1] + 2a_1x_1[n-2] + 2a_2x_2[n] - 5a_2x_2[n-1] + 2a_2x_2[n-2]$$
$$y[n] = a_1(2x_1[n] - 5x_1[n-1] + 2x_1[n-2]) + a_2(2x_2[n] - 5x_2[n-1] + 2x_2[n-2])$$
$$y[n] = a_1y_1[n] + a_2y_2[n]$$

Yes, $y[n]$ is linear.

Time-invariance test:

We have to show that a delay in $x[n]$ by n_0, $x[n-n_0]$, results in a delay in $y[n]$ to $y[n-n_0]$.

To do this, we start by letting $w[n]$ be the output when the input $x[n]$ is delayed by n_0 giving

$$w[n] = 2x[n-n_0] - 5x[n-1-n_0] + 2x[n-2-n_0]$$

Then if we delay $y[n]$ by n_0, we obtain

$$y[n-n_0] = 2x[n-n_0] - 5x[n-1-n_0] + 2x[n-2-n_0]$$

We can see that

$$y[n-n_0] = w[n]$$

so $y[n]$ is a time-invariant system.

Causal:

$y[n]$ is causal since it depends only on $x[n]$, $x[n-1]$ and $x[n-2]$ which are in the present or in the past.

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b) Since $y[n]$ is an LTI system, the frequency response is of the form

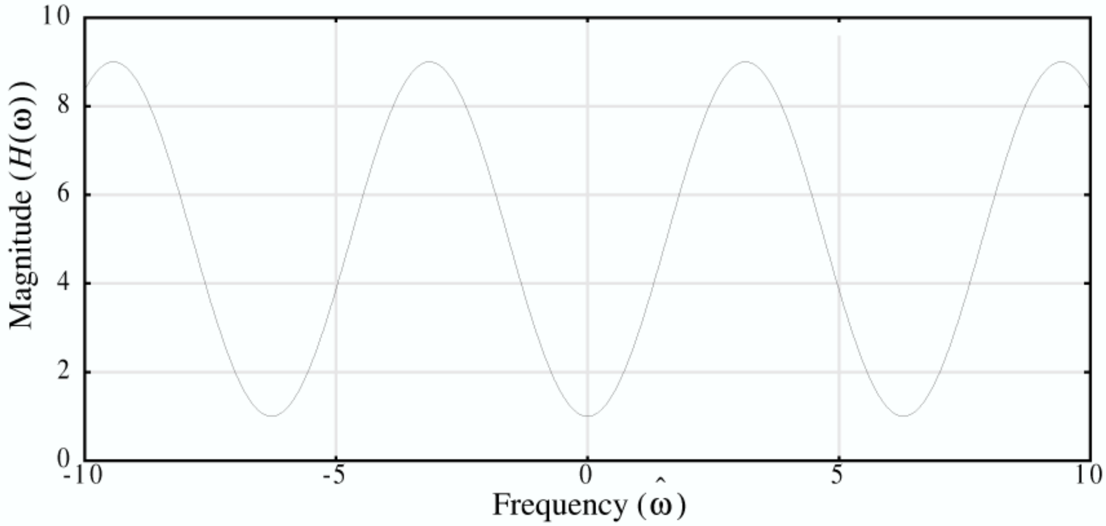
$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

For $y[n]$, $M = 2$ and $b_k = \{2, -5, 2\}$. Therefore,

$$H(\hat{\omega}) = 2e^{-j\hat{\omega}0} - 5e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} = (2e^{j\hat{\omega}} - 5 + 2e^{-j\hat{\omega}})e^{-j\hat{\omega}} = (4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}}$$

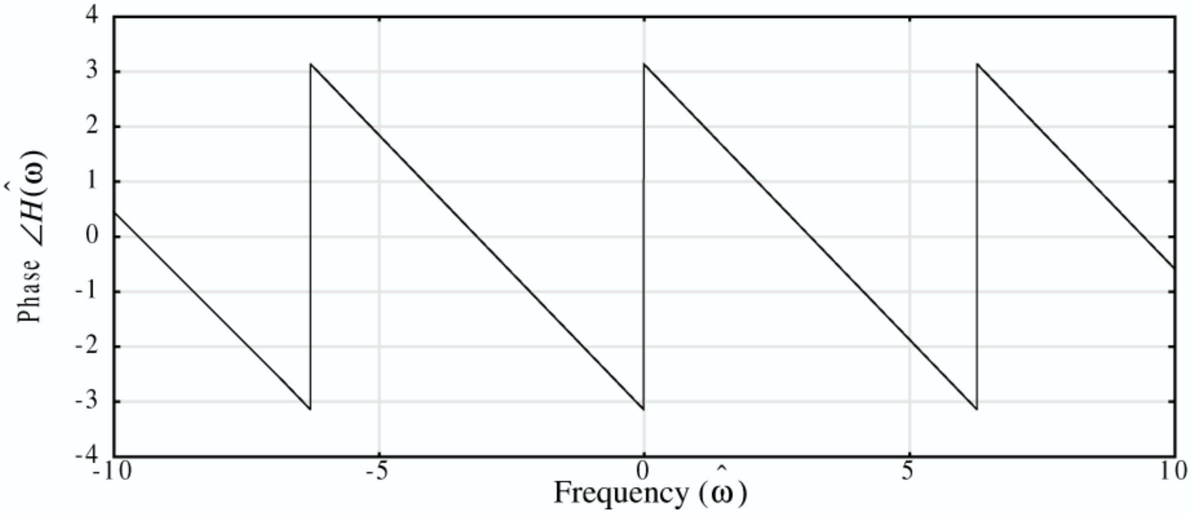
c) The figure below shows a plot for the magnitude of the frequency response

$$|H(\hat{\omega})| = |(4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}}| = |4\cos(\hat{\omega}) - 5|$$



The figure below shows a plot of the phase in the range of $-\pi < \theta < \pi$ for the frequency response (note the $+\pi$ because $4\cos(\hat{\omega}) - 5$ is always negative)

$$\angle H(\hat{\omega}) = \angle(4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}} = -(\hat{\omega} + \pi)$$



Chap6: Frequency response and sinusoidal output of LTI system

stusol/f_00/pset06_3.pdf

d) Given the input $x_2[n] = 4 + 4\cos(0.5\pi(n-1))$, $y_1[n]$ can be calculated as follows. First calculate $H(\hat{\omega})$ for the two frequencies in $x_2[n]$.

$$H(0) = (4\cos(0) - 5)e^{-j0} = -1$$

$$H(\pi/2) = (4\cos(\pi/2) - 5)e^{-j(\pi/2)} = -5e^{-j(\pi/2)} = 5e^{j(\pi/2)}$$

With these two frequency response values, $y_1[n]$ can be determined as follows

$$y_1[n] = (H(0))(4) + |H(\pi/2)||4|\cos(0.5\pi(n-1) + \arg(H(\pi/2)))$$

$$y_1[n] = -4 + 20\cos(0.5\pi(n-1) + \pi/2) = -4 + 20\cos(0.5\pi n)$$

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PROBLEM:

Let $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ be the impulse response of an LTI system and let

$$x[n] = 2e^{j(\pi/2)n}, \qquad -\infty < n < \infty$$

be the input to that system.

(a) Determine the frequency response $\mathcal{H}(\hat{\omega})$ of $h[n]$.
Note: We have also used the notation $H(e^{j\hat{\omega}})$ for the frequency response; i.e. $\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}})$.

$\mathcal{H}(\hat{\omega}) =$

(b) If $y[n] = h[n] * x[n]$, the output is a complex exponential of the form $Ae^{j(\omega_o n+\phi)}$, where A is a real positive number. Determine A, ϕ and ω_o.

$A =$

$\phi =$

$\omega_o =$

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SOLUTION

Let $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ be the impulse response of an LTI system and let

$$x[n] = 2e^{j\pi/2n}, \qquad -\infty < n < \infty$$

be the input to that system.

(a) Determine the frequency response $\mathcal{H}(\hat{\omega})$ of $h[n]$.[1]

$\mathcal{H}(\hat{\omega}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$

$$\mathcal{H}(\hat{\omega}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) = e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

(b) If $y[n] = h[n] * x[n]$, the output is a complex exponential of the form $Ae^{j(\omega_o n + \phi)}$, where A is a real positive number. Determine A, ϕ and ω_o.

$A = 4$

$\phi = -\pi/2$

$\omega_o = \pi/2$

Use $\mathcal{H}(\hat{\omega})$ at $\hat{\omega} = \frac{\pi}{2}$

$$\mathcal{H}(\tfrac{\pi}{2}) = (2 + 2\cos\tfrac{\pi}{2})e^{-j\pi/2}$$

$$= 2e^{-j\pi/2}$$

$$y[n] = \mathcal{H}(\tfrac{\pi}{2}) \cdot 2e^{j\frac{\pi}{2}n}$$

$$= 2e^{-j\pi/2} \cdot 2e^{j\frac{\pi}{2}n}$$

$$= 4e^{-j\pi/2}e^{j\frac{\pi}{2}n}$$

[1] We have also used the notation $H(e^{j\hat{\omega}})$ for the frequency response; i.e. $\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}})$.

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PROBLEM:

Consider the linear time-invariant system given by the difference equation

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6] + x[n-7] = \sum_{k=0}^{7} x[n-k]$$

(a) Find an expression for the frequency response $H(e^{j\hat{\omega}})$ of the system.

(b) Show that your answer in (a) can be expressed in the form

$$H(e^{j\hat{\omega}}) = \frac{\sin(8\hat{\omega}/2)}{\sin(\hat{\omega}/2)} e^{-j3.5\hat{\omega}}$$

(c) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above. You might want to check your plot by doing it in MATLAB with `freekz( )` or `freqz( )`.

(d) Suppose that the input is

$$x[n] = 3 + 3\cos(\hat{\omega}_0 n) \qquad \text{for} \quad -\infty < n < \infty$$

Find all possible non-zero frequencies $0 < \hat{\omega}_0 < \pi$ for which the output $y[n]$ is a constant for all n, i.e.,

$$y[n] = c \qquad \text{for} \quad -\infty < n < \infty$$

and find the value for c. (In other words, the sinusoid is removed by the filter.)

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PROBLEM:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

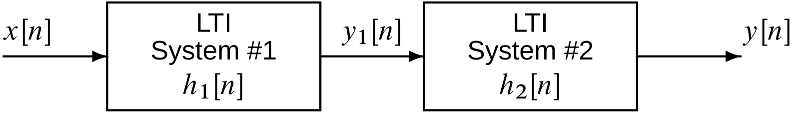

Figure 1: Cascade connection of two LTI systems.

Suppose that System #1 is a filter described by the difference equation

$$y_1[n] = -\tfrac{1}{2}x[n] + 3x[n-1] - \tfrac{1}{2}x[n-2]$$

and System #2 is described by the impulse response

$$h_2[n] = \delta[n-2],$$

(a) Determine the frequency response sequence, $H_1(e^{j\hat{\omega}})$, of the first system.

(b) Determine the frequency response, $H(e^{j\hat{\omega}})$, of the overall cascade system.

(c) Plot the magnitude and phase of the frequency response of the overall cascaded system.

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PROBLEM:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

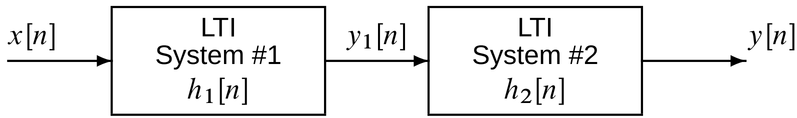

Figure 1: Cascade connection of two LTI systems.

Suppose that System #1 is a filter described by the impulse response

$$h_1[n] = \alpha\delta[n] + \delta[n-2]$$

and System #2 is described by the difference equation

$$y_2[n] = y_1[n-1] + \alpha y_1[n-3],$$

(a) Determine the frequency response, $H_2(e^{j\hat{\omega}})$, of the second system.

(b) Determine the frequency response, $H(e^{j\hat{\omega}})$, of the overall cascade system.

(c) For the case where $\alpha = \frac{1}{2}$, plot the magnitude of the overall frequency response of the cascaded system.

(d) When $\alpha = \frac{1}{2}$ and the input to this system is

$$x[n] = 10\cos(\tfrac{1}{2}\pi n + 0.25\pi)$$

Use the frequency response to compute the values of $y[n]$, over the range $-\infty \le n \le \infty$.

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PROBLEM:

Consider again the cascade system in Figure 1 with

$$h_1[n] = \delta[n] - \alpha\delta[n-1] \quad \text{and} \quad h_2[n] = \alpha^n(u[n] - u[n-6]).$$

(a) Determine $\mathcal{H}_1(\hat{\omega})$, the frequency response of the first system.

(b) Show that the frequency response of the second system is

$$\mathcal{H}_2(\hat{\omega}) = \frac{1-\alpha^6 e^{-j\hat{\omega}6}}{1-\alpha e^{-j\hat{\omega}}}.$$

(c) It is possible to show that $h[n] = h_1[n] * h_2[n] = \delta[n] - \alpha^6\delta[n-6]$. From $h[n]$ determine $\mathcal{H}(\hat{\omega})$ the frequency response of the overall system (from $x[n]$ to $y[n]$).

(d) Show that your result in part (c) is the product of the results in parts (a) and (b); i.e., $\mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega}) = \mathcal{H}(\hat{\omega})$.

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SOLUTION

Part A

$$\mathcal{H}_1(\hat{\omega}) = e^{-j0\hat{\omega}} - \alpha\, e^{-j1\hat{\omega}} = \boxed{1 - \alpha\, e^{-j\hat{\omega}}}$$

Part B

We first rewrite $h_2[n] = \alpha^n(u[n] - u[n-6])$ as follows

$$\begin{aligned} h_2[n] &= \alpha^n(\delta[n] + \delta[n-1] + \cdots + \delta[n-5]) \\ &= \alpha^0\delta[n] + \alpha^1\delta[n-1] + \cdots + \alpha^5\delta[n-5] \end{aligned}$$

from which we may see that the frequency response $\mathcal{H}_2(\hat{\omega})$ is given by

$$\begin{aligned} \mathcal{H}_2(\hat{\omega}) &= \alpha^0 e^{-j0\hat{\omega}} + \alpha^1 e^{-j1\hat{\omega}} + \cdots + \alpha^5 e^{-j5\hat{\omega}} \\ &= (\alpha e^{-j\hat{\omega}})^0 + (\alpha e^{-j\hat{\omega}})^1 + \cdots + (\alpha e^{-j\hat{\omega}})^5 \\ \alpha e^{-j\hat{\omega}}\mathcal{H}_2(\hat{\omega}) &= (\alpha e^{-j\hat{\omega}})^1 + (\alpha e^{-j\hat{\omega}})^2 + \cdots + (\alpha e^{-j\hat{\omega}})^6 \\ \mathcal{H}_2(\hat{\omega}) - \alpha e^{-j\hat{\omega}}\mathcal{H}_2(\hat{\omega}) &= (\alpha e^{-j\hat{\omega}})^0 - (\alpha e^{-j\hat{\omega}})^6 \\ \mathcal{H}_2(\hat{\omega}) &= \frac{1 - \alpha^6 e^{-j6\hat{\omega}}}{1 - \alpha e^{-j\hat{\omega}}} \end{aligned}$$

Part C

Given that $h[n] = \delta[n] - \alpha^6\delta[n-6]$, we may immediately write down

$$\mathcal{H}(\hat{\omega}) = 1e^{-j0\hat{\omega}} - \alpha^6 e^{-j6\hat{\omega}} = \boxed{1 - \alpha^6 e^{-j6\hat{\omega}}}$$

Part D

$$\mathcal{H}_1(\hat{\omega})\,\mathcal{H}_2(\hat{\omega}) = (1 - \alpha\, e^{-j\hat{\omega}})\frac{1 - \alpha^6 e^{-j6\hat{\omega}}}{1 - \alpha e^{-j\hat{\omega}}} = 1 - \alpha^6 e^{-j6\hat{\omega}} = \mathcal{H}(\hat{\omega})$$

PROBLEM:

Consider the linear time-invariant system given by the difference equation

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3] + x[n-4] - x[n-5] = \sum_{k=0}^{5} (-1)^k x[n-k]$$

(a) Find an expression for the frequency response $H(e^{j\hat{\omega}})$ of the system.

(b) Show that your answer in (a) can be simplified and expressed in the form

$$H(e^{j\hat{\omega}}) = \frac{\sin(3(\hat{\omega}-\pi))}{\sin(\frac{1}{2}(\hat{\omega}-\pi))}\, e^{-j2.5(\hat{\omega}-\pi)}$$

Hint: use the fact that $(-1)^k = e^{\pm j\pi k}$.

(c) Sketch the frequency response (magnitude and phase) versus $\hat{\omega}$ from the formula above. Notice that you get a *Dirichlet* shape, but its peak is no longer centered at $\hat{\omega} = 0$. You might want to check your plot by doing it in MATLAB with `freekz( )` or `freqz( )`.

(d) Suppose that the input signal is

$$x[n] = 7 + 9\cos(\hat{\omega}_0 n) \qquad \text{for} \quad -\infty < n < \infty$$

Find all possible non-zero frequencies $0 < \hat{\omega}_0 < \pi$ for which the output $y[n]$ is zero for all n. In other words, the sinusoid and DC are removed by the filter.

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PROBLEM:

For the following TRUE/FALSE questions, give an explanation and circle the correct answer:

(a) **TRUE** or **FALSE**: A signal $x(t)$ contains a maximum frequency component at $f = 200$ Hz. If $x(t)$ is sampled at twice its Nyquist rate, then $f_s = 800$ Hz.

(b) **TRUE** or **FALSE**: The frequency response $\mathcal{H}(\hat{\omega})$ is always periodic with a period of 2π.

(c) **TRUE** or **FALSE**: If a filter has an impulse response $h[n]$ that is $h[n] = \delta[n] - \delta[n-1]$, the the response of this filter to the DC input $x[n] = 7$ will be $y[n] = 7\delta[n] - 7\delta[n-1]$

(d) **TRUE** or **FALSE**: If the impulse response is $h[n] = \delta[n-1]$ then the magnitude of the frequency response $\mathcal{H}(\hat{\omega})$ is equal to one for all frequencies.

(e) **TRUE** or **FALSE**: A 2-point running average filter, $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-2]$, has a frequency response $\mathcal{H}(\hat{\omega})$ that is zero at $\hat{\omega} = 0$.

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SOLUTION

For the following TRUE/FALSE questions, circle the correct answer:

(a) **TRUE** or **FALSE**: A signal $x(t)$ contains a maximum frequency component at $f = 200$ Hz. If $x(t)$ is sampled at twice its Nyquist rate, then $f_s = 800$ Hz.

$f_s \geq 2f_{MAX} = 2(200) = 400 \leftarrow$ Nyquist

Twice Nyquist $= 2(400) = 800$

(b) **TRUE** or **FALSE**: The frequency response $\mathcal{H}(\hat{\omega})$ is always periodic with a period of 2π.

We also plot from $-\pi$ to $+\pi$

(c) **TRUE** or **FALSE**: If a filter has an impulse response $h[n]$ that is $h[n] = \delta[n] - \delta[n-1]$, the the response of this filter to the DC input $x[n] = 7$ will be $y[n] = 7\delta[n] - 7\delta[n-1]$

DC in will give DC out. $y[n]$ will be a constant

(d) **TRUE** or **FALSE**: If the impulse response is $h[n] = \delta[n-1]$ then the magnitude of the frequency response $\mathcal{H}(\hat{\omega})$ is equal to one for all frequencies.

$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}$ $\quad |\mathcal{H}(\hat{\omega})| = 1$

(e) **TRUE** or **FALSE**: A 2-point running average filter, $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-2]$, has a frequency response $\mathcal{H}(\hat{\omega})$ that is zero at $\hat{\omega} = 0$.

$\mathcal{H}(\hat{\omega}) = \frac{1}{2} + \frac{1}{2}e^{-j2\hat{\omega}} \Rightarrow \mathcal{H}(0) = \frac{1}{2} + \frac{1}{2} = 1$

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}}). \tag{1}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. *Hint: Multiply out the factors to obtain a sum of powers of* $e^{-j\hat{\omega}}$.

(b) What is the impulse response of this system?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \leq \hat{\omega} \leq \pi$ will $y[n] = 0$ for all n? *Hint: In this part, the answer is most obvious in the factored form of Eq. (1).*

(d) Use superposition to determine the output of this system when the input is

$$x[n] = 3 + \delta[n-2] + \cos(0.5\pi n + \pi/4) \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results.

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SOLUTION

Chap6: Impulse response from factored frequency response
stusol/sp_02/pset07_8.pdf

Part A

$$\begin{aligned}
\mathcal{H}(\hat\omega) &= (1+e^{-j\hat\omega})(1-e^{-j\pi/3}e^{-j\hat\omega})(1-e^{j\pi/3}e^{-j\hat\omega}) \\
&= (1+e^{-j\hat\omega})(1-e^{-j\hat\omega}+e^{-j2\hat\omega}) \\
&= 1-e^{-j\hat\omega}+e^{-j2\hat\omega}+e^{-j\hat\omega}-e^{-j2\hat\omega}+e^{-j3\hat\omega} \\
&= 1+e^{-j3\hat\omega}
\end{aligned}$$

and so, from the structure of $\mathcal{H}(\hat\omega)$ we can infer that $\boxed{y[n] = x[n] + x[n-3]}$

Part B

Plugging $x[n] = \delta[n]$ into the difference equation yields

$$\boxed{h[n] = \delta[n] + \delta[n-3]}$$

Part C

If $x[n] = Ae^{j\phi}e^{j\hat\omega n}$ then $y[n] = \mathcal{H}(\hat\omega)x[n]$ in which case $y[n] = 0$ when $\hat\omega$ is chosen so that $\mathcal{H}(\hat\omega) = 0$. From the structure of (3), we can see that this occurs in three different cases (modulo 2π):

1. $1 + e^{-j\hat\omega} = 0$ $\quad\boxed{\hat\omega = \pi}$
2. $1 - e^{-j\pi/3}e^{-j\hat\omega} = 0$ $\quad\boxed{\hat\omega = -\pi/3}$
3. $1 - e^{j\pi/3}e^{-j\hat\omega} = 0$ $\quad\boxed{\hat\omega = \pi/3}$

Part D

First note that $\mathcal{H}(0) = 2$ and that $\mathcal{H}(0.5\pi) = 1 + e^{-j1.5\pi} = 1 + j = \sqrt{2}e^{j\pi/4}$:

$$\begin{aligned}
y[n] &= 3\mathcal{H}(0) + h[n-2] + |\mathcal{H}(0.5\pi)|\cos\left(0.5\pi n + \pi/4 + \angle\mathcal{H}(0.5\pi)\right) \\
&= 3(2) + (\delta[n-2] + \delta[n-5]) + \sqrt{2}\cos\left(0.5\pi n + \pi/4 + \pi/4\right) \\
&= \boxed{6 + \delta[n-2] + \delta[n-5] + \sqrt{2}\cos\left(0.5\pi n + \pi/2\right)}
\end{aligned}$$

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PROBLEM:

Suppose that $\mathcal{S}$ is a linear, time-invariant system whose exact form is unknown. It needs to be tested by running some inputs into the system, and then observing the output signals. Suppose that the following input/output pairs are the result of the tests:

$$
\begin{aligned}
x[n] = \delta[n] \quad &\longrightarrow \quad y[n] = \delta[n] - \delta[n-3] \\
x[n] = \cos(2\pi n/3) \quad &\longrightarrow \quad y[n] = 0 \\
x[n] = \cos(\pi n/3 + \pi/2) \quad &\longrightarrow \quad y[n] = 2\cos(\pi n/3 + \pi/2)
\end{aligned}
$$

(a) Make a plot of the signal: $x[n] = 3\delta[n] - 2\delta[n-2] + \delta[n-3]$.

(b) What is the output of the system when the input is $x[n] = 3\delta[n] - 2\delta[n-2] + \delta[n-3]$.

(c) Determine the output when the input is $x[n] = \cos(\pi(n-3)/3)$.

(d) Is the following statement true or false: "$H(\pi/2) = 0$." EXPLAIN

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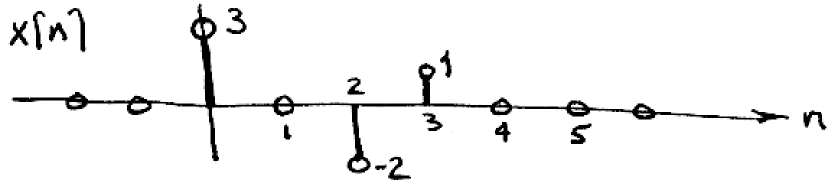
SOLUTION

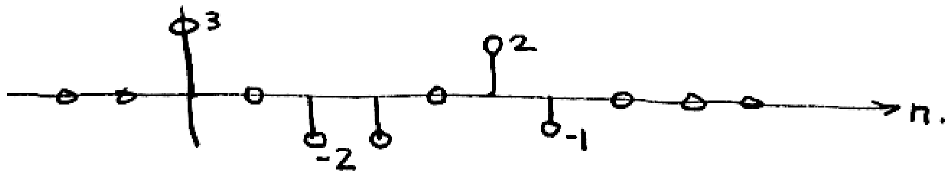
(a)



(b) Use LINEARITY & TIME-INVARIANCE

$$\left.\begin{aligned} 3\delta[n] &\longrightarrow 3\delta[n] - 3\delta[n-3] \\ -2\delta[n-2] &\longrightarrow -2\delta[n-2] + 2\delta[n-5] \\ \delta[n-3] &\longrightarrow \underline{\delta[n-3] - \delta[n-6]} \end{aligned}\right\} \text{Add these together}$$

$$\text{OUTPUT} = 3\delta[n] - 2\delta[n-2] - 2\delta[n-3] + 2\delta[n-5] - \delta[n-6].$$



(c) Use the third input/output pair:

$$\mathcal{H}(\pi/3) = 2 \qquad \text{(no phase).}$$

$$\therefore \cos(\pi(n-3)/3) \longrightarrow 2\cos(\pi(n-3)/3).$$

(d) There is no <u>direct</u> evidence about $\mathcal{H}(\pi/2)$.

<u>BUT</u> use impulse response to get $\{b_k\}$.

$$h[n] = \delta[n] - \delta[n-3].$$

$$\Rightarrow \{b_k\} = \{1, 0, 0, -1\}$$

$$\Rightarrow H(e^{j\hat{\omega}}) = \mathcal{H}(\hat{\omega}) = 1 - e^{-j3\hat{\omega}}$$

$$\mathcal{H}(\pi/2) = 1 - e^{-j3\pi/2} = 1 + j$$

$$\therefore \mathcal{H}(\pi/2) \neq 0$$

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PROBLEM:

Suppose that $\mathcal{S}$ is a linear, time-invariant system whose exact form is unknown. It needs to be tested by running some inputs into the system, and then observing the output signals. Suppose that the following input/output pairs are the result of the tests:

$$
\begin{aligned}
x[n] = \delta[n] - \delta[n-1] &\quad\longrightarrow\quad y[n] = 4\delta[n] - 4\delta[n-4] \\
x[n] = \cos(\pi n/2) &\quad\longrightarrow\quad y[n] = 0 \\
x[n] = \cos(\pi n/3) &\quad\longrightarrow\quad y[n] = 6.93\cos(\pi n/3 - \pi/2)
\end{aligned}
$$

(a) Make a plot of the signal: $x[n] = 4\delta[n] - 4\delta[n-4]$.

(b) Use linearity and time-invariance to find the output of the system when the input is

$$x[n] = 3\delta[n] - 3\delta[n-3]$$

(c) Determine the output when the input is $x[n] = 7\cos(\pi(n-2)/3)$.

(d) Determine the output when the input is $x[n] = 9\sin(\pi n/2)$

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PROBLEM:

For each of the following frequency response, pick one of the representations below that defines *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, or S_6, in the box next to each frequency response. In addition, evaluate the frequency response at $\hat{\omega} = 0, \ \pm\pi$ and $\hat{\omega} = \pm\frac{1}{2}\pi$ as requested for each case;*simplify* the answer to **polar form** and write it in the space provided.

ANS =

$$\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$$

$\mathcal{H}(\frac{1}{2}\pi) =$

ANS =

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$$

$\mathcal{H}(\pi) =$

ANS =

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(2\cos(\hat{\omega}))$$

$\mathcal{H}(0) =$

ANS =

$$\mathcal{H}(\hat{\omega}) = -e^{-j2\hat{\omega}}$$

$\mathcal{H}(0) =$

POSSIBLE ANSWERS: (impulse response, filter coefficients or difference equation)

$S_1: \quad b_k = \{0, \ 1, \ 0, \ -1\}$

$S_2: \quad y[n] = x[n] + x[n-2]$

$S_3: \quad h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$S_4: \quad h[n] = -\delta[n-2]$

$S_5: \quad y[n] = x[n-1] + x[n-3]$

$S_6: \quad b_k = \{1, \ 0, \ -1\}$

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SOLUTION

For each of the following frequency response, pick one of the representations below that defines *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, or S_6, in the box next to each frequency response. In addition, evaluate the frequency response at $\hat{\omega} = 0, \pi$ and $\hat{\omega} = \pm\frac{1}{2}\pi$ as requested for each case; *simplify* the answer to **polar form** and write it in the space provided.

$\boxed{\text{ANS} = 6}$ $\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$

$\boxed{\mathcal{H}(\tfrac{1}{2}\pi) = 2}$

$$\mathcal{H}(\pi/2) = 1 - e^{-j2(\pi/2)} = 1 - e^{-j\pi} = 1-(-1) = 2$$

$\boxed{\text{ANS} = 1}$ $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$

$\boxed{\mathcal{H}(\pi) = 0}$

$$\mathcal{H}(\pi) = e^{-j\pi} - e^{-j3\pi} = -1-(-1) = 0$$

$\boxed{\text{ANS} = 2}$ $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(2\cos(\hat{\omega}))$

$\boxed{\mathcal{H}(0) = 2}$

$$\mathcal{H}(0) = e^{-j0}(2\cos(0)) = 2$$

$\boxed{\text{ANS} = 4}$ $\mathcal{H}(\hat{\omega}) = -e^{-j2\hat{\omega}}$

$\boxed{\mathcal{H}(0) = 1e^{j\pi}}$

$$\mathcal{H}(0) = -e^{-j0} = -1 = e^{j\pi}$$

POSSIBLE ANSWERS: (impulse response, filter coefficients or difference equation)

$S_1:$ $b_k = \{0, 1, 0, -1\}$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{3} b_k e^{-j\hat{\omega}k} = e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$$

$S_2:$ $y[n] = x[n] + x[n-2]$

$b_k = \{1, 0, 1\}$

$$\mathcal{H}(\hat{\omega}) = 1 + e^{-j2\hat{\omega}} = e^{-j\hat{\omega}}(e^{j\hat{\omega}} + e^{-j\hat{\omega}})$$
$$= e^{-j\hat{\omega}}(2\cos\hat{\omega})$$

$S_3:$ $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$b_k = \{1, 1, 1\}$

$$\mathcal{H}(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$S_4:$ $h[n] = -\delta[n-2]$

$$\mathcal{H}(\hat{\omega}) = \sum_{n=0}^{2} h[n] e^{-jn\hat{\omega}} = -e^{-j2\hat{\omega}}$$

$S_5:$ $y[n] = x[n-1] + x[n-3]$

$$b_k = \{0, 1, 0, 1\} \Rightarrow \mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j3\hat{\omega}}$$

$S_6:$ $b_k = \{1, 0, -1\}$

$$\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$$

PROBLEM:

Pick the correct frequency response and enter the number in the answer box:

Difference Equation or Impulse Response	Frequency Response
(a) $h[n] = \delta[n-1]$ **ANS =**	1. $\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$
	2. $\mathcal{H}(\hat{\omega}) = 2je^{-j2\hat{\omega}}\sin(\hat{\omega})$
(b) $y[n] = x[n-1] - x[n-3]$ **ANS =**	3. $\mathcal{H}(\hat{\omega}) = 2e^{-j2\hat{\omega}}\cos(\hat{\omega})$
	4. $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos(\hat{\omega}))$
(c) $h[n] = \delta[n] - \delta[n-2]$ **ANS =**	5. $\mathcal{H}(\hat{\omega}) = \dfrac{\sin\hat{\omega}}{\sin(\frac{1}{2}\hat{\omega})}$
(d) $y[n] = x[n] + x[n-1] + x[n-2]$ **ANS =**	6. $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}$

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SOLUTION

Pick the correct frequency response and enter the number in the answer box:

Difference Equation or Impulse Response

(a) $h[n] = \delta[n-1]$ $\quad \sum \delta[n-1] e^{-j\hat{\omega}n} = e^{-j\hat{\omega}}$

ANS = 6

(b) $y[n] = x[n-1] - x[n-3]$ $\quad e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$

ANS = 2

(c) $h[n] = \delta[n] - \delta[n-2]$ $\quad 1 - e^{-j2\hat{\omega}}$

ANS = 1

Frequency Response

1. $\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$
2. $\mathcal{H}(\hat{\omega}) = 2je^{-j2\hat{\omega}}\sin(\hat{\omega})$
3. $\mathcal{H}(\hat{\omega}) = 2e^{-j2\hat{\omega}}\cos(\hat{\omega})$
4. $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos(\hat{\omega}))$
5. $\mathcal{H}(\hat{\omega}) = \dfrac{\sin\hat{\omega}}{\sin(\frac{1}{2}\hat{\omega})}$
6. $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}$

(c) $e^{-j\hat{\omega}} - e^{-j3\hat{\omega}} = e^{-j2\hat{\omega}}(e^{j\hat{\omega}} - e^{-j\hat{\omega}})$

$$= 2j \sin\hat{\omega}\, e^{-j2\hat{\omega}}$$

(d) $y[n] = x[n] + x[n-1] + x[n-2]$

ANS = 4

$1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} = e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}})$

$$= e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$$

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PROBLEM:

For each of the following frequency responses, pick one of the representations below that defines *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, or S_6, in the box next to each frequency response. In addition, evaluate the frequency response at $\hat{\omega} = 0,\ \pm\pi$ and $\hat{\omega} = \pm\frac{1}{2}\pi$ as requested for each case;*simplify* the answer to **polar form** and write it in the space provided.

ANS = $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j3\hat{\omega}}$

$\mathcal{H}(-\pi) =$

ANS = $\mathcal{H}(\hat{\omega}) = e^{-j2\hat{\omega}}(2j\sin(\hat{\omega}))$

$\mathcal{H}(\frac{1}{2}\pi) =$

ANS = $\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$

$\mathcal{H}(-\frac{1}{2}\pi) =$

ANS = $\mathcal{H}(\hat{\omega}) = e^{-j3\hat{\omega}/2}(2j\sin(3\hat{\omega}/2))$

$\mathcal{H}(0) =$

POSSIBLE ANSWERS: (impulse response, filter coefficients or difference equation)

$S_1: \quad b_k = \{1,\ 0,\ -1\}$

$S_2: \quad h[n] = \delta[n] + \delta[n-3]$

$S_3: \quad h[n] = \delta[n-1] + \delta[n-3]$

$S_4: \quad y[n] = x[n] - x[n-3]$

$S_5: \quad b_k = \{0,\ 1,\ 0,\ -1\}$

$S_6: \quad y[n] = x[n] + x[n-1] + x[n-2]$

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SOLUTION

For each of the following frequency responses, **pick one of the representations below that defines** *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, or S_6, in the box next to each frequency response. In addition, evaluate the frequency response at $\hat{\omega} = 0$, $\pm\pi$ and $\hat{\omega} = \pm\frac{1}{2}\pi$ as requested for each case; *simplify* the answer to **polar form** and write it in the space provided.

$\boxed{\text{ANS} = S_3}$ $\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j3\hat{\omega}} = \delta[n-1] + \delta[n-3]$

$\boxed{\mathcal{H}(-\pi) = -2}$

$\boxed{\text{ANS} = S_5}$ $\mathcal{H}(\hat{\omega}) = e^{-j2\hat{\omega}}(2j\sin(\hat{\omega})) = e^{-j2\hat{\omega}}(e^{j\hat{\omega}} - e^{-j\hat{\omega}}) = e^{-j\hat{\omega}} - e^{-j3\hat{\omega}}$

$\boxed{\mathcal{H}(\tfrac{1}{2}\pi) = 2e^{-j\frac{\pi}{2}}}$

$\boxed{\text{ANS} = S_1}$ $\mathcal{H}(\hat{\omega}) = 1 - e^{-j2\hat{\omega}} = \delta[n] - \delta[n-2] = \{1, 0, -1\}$

$\boxed{\mathcal{H}(-\tfrac{1}{2}\pi) = 2}$

$\boxed{\text{ANS} = S_4}$ $\mathcal{H}(\hat{\omega}) = e^{-j3\hat{\omega}/2}(2j\sin(3\hat{\omega}/2)) = e^{-j\frac{3\hat{\omega}}{2}}(e^{j\frac{3\hat{\omega}}{2}} - e^{-j\frac{3\hat{\omega}}{2}}) = 1 - e^{-j3\hat{\omega}}$

$\boxed{\mathcal{H}(0) = 0}$

POSSIBLE ANSWERS: (impulse response, filter coefficients or difference equation)

$S_1: \quad b_k = \{1,\ 0,\ -1\}$

$S_2: \quad h[n] = \delta[n] + \delta[n-3]$

$S_3: \quad h[n] = \delta[n-1] + \delta[n-3]$

$S_4: \quad y[n] = x[n] - x[n-3]$

$S_5: \quad b_k = \{0,\ 1,\ 0,\ -1\}$

$S_6: \quad y[n] = x[n] + x[n-1] + x[n-2]$

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PROBLEM:

Pick the correct frequency response (from the list on the right) and enter the number in the answer box:

Time-Domain Description

(a) $y[n] = x[n] + x[n-1] + x[n-2]$

ANS =

(b) $y[n] = x[n] + x[n-1]$

ANS =

(c) $h[n] = \delta[n-1] + \delta[n-3]$

ANS =

(d) $h[n] = \delta[n-1] - \delta[n-3]$

ANS =

(e) $\{b_k\} = \{\,1,\ 0,\ -1\,\}$

ANS =

(f) Select **all** systems (from the list on the right) that **null out** DC. Enter all numbers that apply.

ANS =

Frequency Response

1 $H(e^{j\hat{\omega}}) = 1 - e^{-j2\hat{\omega}}$

2 $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\cos(\hat{\omega})$

3 $H(e^{j\hat{\omega}}) = 2je^{-j2\hat{\omega}}\sin(\hat{\omega})$

4 $H(e^{j\hat{\omega}}) = \dfrac{\sin(\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}\, e^{-j\hat{\omega}/2}$

5 $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos(\hat{\omega}))$

6 $H(e^{j\hat{\omega}}) = \dfrac{\sin(2\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}\, e^{-j3\hat{\omega}/2}$

7 $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}$

8 None of the above

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PROBLEM:

For each of the following frequency responses on the left, pick one of the representations, S_1 through S_8 on the right, that defines *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, S_6, S_7, or S_8, in the box next to each frequency response.

ANS =	(a)	$1 + e^{j\hat{\omega}}$	S_1	$b_k = \{1, 0, 1\}$
ANS =	(b)	$2e^{-3j\hat{\omega}}$	S_2	$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$
ANS =	(c)	$\dfrac{\sin(2\hat{\omega})}{\sin(\hat{\omega}/2)} e^{-3j\hat{\omega}/2}$	S_3	$h[n] = 0.5\delta[n] + 0.5\delta[n-2]$
ANS =	(d)	$e^{-j\hat{\omega}} \cos(\hat{\omega})$	S_4	$b_k = \{1, 1, 1, 1\}$
			S_5	$y[n] = x[n] + x[n-1]$
			S_6	$h[n] = \delta[n] - \delta[n-1]$
			S_7	$y[n] = x[n] + 2x[n-3]$
			S_8	$h[n] = 2\delta[n-3]$

SOLUTION

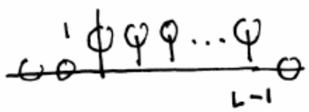
For each of the following frequency responses on the left, pick one of the representations, S_1 through S_8 on the right, that defines *exactly* the same LTI system. Write your answer S_1, S_2, S_3, S_4, S_5, S_6, S_7, or S_8, in the box next to each frequency response.

S_5 ANS =	(a)	$1+e^{-j\hat{\omega}}$	S_1	$b_k = \{1,0,1\}$
S_8 ANS =	(b)	$2e^{-3j\hat{\omega}}$	S_2	$y[n] = \frac{1}{3}\{x[n]+x[n-1]+x[n-2]\}$
S_4 ANS =	(c)	$\dfrac{\sin(2\hat{\omega})}{\sin(\hat{\omega}/2)}e^{-3j\hat{\omega}/2}$	S_3	$h[n] = 0.5\delta[n]+0.5\delta[n-2]$
S_3 ANS =	(d)	$e^{-j\hat{\omega}}\cos(\hat{\omega})$	S_4	$b_k = \{1,1,1,1\}$
			S_5	$y[n] = x[n]+x[n-1]$
			S_6	$h[n] = \delta[n]-\delta[n-1]$
			S_7	$y[n] = x[n]+2x[n-3]$
			S_8	$h[n] = 2\delta[n-3]$

a) $H(\hat{\omega}) = 1 + e^{-j\hat{\omega}} \Rightarrow h[n] = \delta[n] + \delta[n-1]$

$y[n] = x[n] + x[n-1] \Rightarrow S_5$

b) $H(\hat{\omega}) = 2e^{-3j\hat{\omega}} \Rightarrow h[n] = 2\delta[n-3] \Rightarrow S_8$

c) 

$\Rightarrow H(\hat{\omega}) = e^{-(\frac{L-1}{2})\hat{\omega}} \dfrac{\sin(L\hat{\omega}/2)}{\sin(\hat{\omega}/2)}$

$\Rightarrow L = 4 \Rightarrow S_4$

d) $e^{-j\hat{\omega}}\cos\hat{\omega} = \frac{1}{2} + \frac{1}{2}e^{-2j\hat{\omega}} \Rightarrow h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-2]$

$\Rightarrow S_3$

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PROBLEM:

A few quick questions:

(a) Consider the following MATLAB program:

```
nn = 0:16000;
yy = 11*cos(1.2*pi*nn+pi/3);
soundsc(yy,8000)
```

Neglecting the end effects in the convolution, the frequency ***in hertz*** for the output signal produced by the `soundsc( )` statement, i.e., the frequency that you hear.

Frequency = $\boxed{}$ Hz

(b) Evaluate $\left|H(e^{j\hat{\omega}})\right|^2$, where $H(e^{j\hat{\omega}}) = 2je^{-j(3\hat{\omega}^2+\hat{\omega})}\sin(2\hat{\omega})$.

(c) Determine the ***minimum period*** (in seconds) of the following signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k/20)}{k} e^{j20\pi kt}$$

Period = $\boxed{}$ secs.

(d) Solve the following relationship for A and ϕ

$$12\cos(3t) + A\cos(3t+\phi) = 16\cos(3t-\pi/2)$$

$A = \boxed{}$

$\phi = \boxed{}$

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PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = x[n+1] + x[n] + x[n-1]. \tag{1}$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}.$$

(c) For the system of Equation (1), determine the output $y_2[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.75\pi(n-1)).$$

A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n])^2. \tag{2}$$

Note: in parts (b), (c), (e), and (f), express your answer in terms of cosine functions. Do not leave any square powers of cosine functions in your answers.

(d) Determine whether or not the system defined by Equation (2) is (i) linear; (ii) time-invariant; (iii) causal.

(e) For the system of Equation (2), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}.$$

(f) For the system of Equation (2), determine the output $y_2[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.75\pi(n-1)).$$

(g) For which system does superposition hold?

(h) For which system does the output contain frequencies that are not present in the input signal?

(i) Which system can cause aliasing of sinusoidal components of the input?

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SOLUTION

A discrete-time system is defined by the input/output relation

$$y[n] = x[n+1] + x[n] + x[n-1]. \tag{1}$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal.

Linear: yes

Definition: A system is linear if $x_1[n] \rightarrow y_1[n]$ and $x_2[n] \rightarrow y_2[n]$, then

$$x[n] = \alpha x_1[n] + \beta x_2[n] \rightarrow y[n] = \alpha y_1[n] + \beta y_2[n]$$

For this system,

$$\begin{aligned} y[n] &= (\alpha x_1[n+1] + \beta x_2[n+1]) + (\alpha x_1[n] + \beta x_2[n]) + (\alpha x_1[n-1] + \beta x_2[n-1]) \\ &= \alpha(x_1[n+1] + x_1[n] + x_1[n-1]) + \beta(x_2[n+1] + x_2[n] + x_2[n-1]) \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

and thus the system is linear.

Time invariant: yes

Definition: A system is time invariant if $x[n-n_0] \rightarrow y[n-n_0]$.

For this system,

$$\begin{aligned} y[n-n_0] &= x[(n+1)-n_0] + x[n-n_0] + x[(n-1)-n_0] \\ &= x[(n-n_0)+1] + x[n-n_0] + x[(n-n_0)-1] \end{aligned}$$

where the latter statement is the result of the system when the input is $x[n-n_0]$, and thus the system is time invariant.

Causal: no

This system is not causal because it requires future input information in the term $x[n+1]$.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}.$$

First, we can compute the frequency response of the LTI system in general as

$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}} \\ &= 1 + 2\cos\hat{\omega} \end{aligned}$$

and for the specific frequencies in this problem as

$$\begin{aligned} \mathcal{H}(0.75\pi) &= 1 - \sqrt{2} = (\sqrt{2}-1)e^{j\pi} \approx 0.414e^{j\pi} \\ \mathcal{H}(-0.75\pi) &= 1 - \sqrt{2} = (\sqrt{2}-1)e^{-j\pi} \approx 0.414e^{-j\pi} \end{aligned}$$

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This results in an output as follows:

$$\begin{aligned} y_1[n] &= \mathcal{H}(0.75\pi)e^{j0.75\pi n} + \mathcal{H}(-0.75\pi)e^{-j0.75\pi n} \\ &= (\sqrt{2}-1)\left(e^{j\pi}e^{j0.75\pi n} + e^{-j\pi}e^{-j0.75\pi n}\right) \\ &= (\sqrt{2}-1)\cdot 2\cos(0.75\pi n + \pi) \\ &\approx 0.828\cos(0.75\pi n + \pi) \end{aligned}$$

(c) For the system of Equation (1), determine the output $y_2[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.75\pi(n-1)).$$

First, we convert the input into a sum of complex exponentials:

$$\begin{aligned} x_2[n] &= 4 + 4\cos(0.75\pi n - 0.75\pi) \\ &= 4 + 2e^{-j0.75\pi}e^{j0.75\pi n} + 2e^{j0.75\pi}e^{-j0.75\pi n} \end{aligned}$$

For this problem, we must add the additional frequency of 0, which has the frequency response of $\mathcal{H}(0) = 3$. The resulting output is

$$\begin{aligned} y_2[n] &= 3\cdot 4 + (\sqrt{2}-1)e^{j\pi}\cdot 2e^{-j0.75\pi}e^{j0.75\pi n} + (\sqrt{2}-1)e^{-j\pi}\cdot 2e^{j0.75\pi}e^{-j0.75\pi n} \\ &= 12 + 2(\sqrt{2}-1)\left(e^{j0.25\pi}e^{j0.75\pi n} + e^{-j0.25\pi}e^{-j0.75\pi n}\right) \\ &= 12 + 4(\sqrt{2}-1)\cos(0.75\pi n + 0.25\pi) \\ &\approx 12 + 1.6569\cos(0.75\pi n + \pi/4) \end{aligned}$$

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A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n])^2. \tag{2}$$

(d) Determine whether or not the system defined by Equation (2) is (i) linear; (ii) time-invariant; (iii) causal.

Linear: no

For this system,

$$\begin{aligned} y[n] &= (\alpha x_1[n] + \beta x_2[n])^2 \;=\; \alpha^2 (x_1[n])^2 + 2\alpha\beta(x_1[n]x_2[n]) + \beta^2 (x_2[n])^2 \\ &\neq \alpha(x_1[n])^2 + \beta(x_2[n])^2 \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Because $y[n] \neq \alpha y_1[n] + \beta y_2[n]$, the system is *not* linear.

Time invariant: yes

If we use the delayed input, $x[n-n_0]$, as the input to the system, the output is equal to $(x[n-n_0])^2$. If $x[n]$ is the input, then $y[n] = (x[n])^2$, and delaying the output gives $y[n-n_0] = (x[n-n_0])^2$. Since delaying the input gives the same result as delaying the output, the system is time invariant.

Causal: yes

This system is causal because it does not require future input information (it only requires present input information).

(e) For the system of Equation (2), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}$$

The output can be computed as follows:

$$\begin{aligned} y_1[n] &= (x_1[n])^2 \\ &= \left(e^{j0.75\pi n} + e^{-j0.75\pi n}\right)^2 \\ &= e^{j1.5\pi n} + 2 + e^{-j1.5\pi n} \end{aligned}$$

Note that the frequencies $\hat{\omega} = 1.5\pi$ and $\hat{\omega} = -1.5\pi$ do not fall in the range $-\pi < \hat{\omega} \leq \pi$, so they have aliases within that range at $\hat{\omega} = -0.5\pi$ and $\hat{\omega} = 0.5\pi$, and the equation can be rewritten as follows:

$$\begin{aligned} y_1[n] &= e^{-j0.5\pi n} + 2 + e^{j0.5\pi n} \\ &= 2 + 2\cos(\pi n/2) \end{aligned}$$

(f) For the system of Equation (2), determine the output $y_2[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.75\pi(n-1)).$$

As in Part (c), $x_2[n]$ can be rewritten as follows:

$$x_2[n] = 4 + 2e^{-j0.75\pi}e^{j0.75\pi n} + 2e^{j0.75\pi}e^{-j0.75\pi n}$$

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From this, we can compute $y_2[n]$ as follows:

$$\begin{aligned}
y_2[n] &= \left(4 + 2e^{-j0.75\pi}e^{j0.75\pi n} + 2e^{j0.75\pi}e^{-j0.75\pi n}\right)^2 \\
&= 24 + 16e^{-j0.75\pi}e^{j0.75\pi n} + 16e^{j0.75\pi}e^{-j0.75\pi n} + 4e^{-j1.5\pi}e^{j1.5\pi n} + 4e^{j1.5\pi}e^{-j1.5\pi n} \\
&= 24 + 16e^{-j0.75\pi}e^{j0.75\pi n} + 16e^{j0.75\pi}e^{-j0.75\pi n} + 4e^{j0.5\pi}e^{-j0.5\pi n} + 4e^{-j0.5\pi}e^{j0.5\pi n} \\
&= 24 + 32\cos(0.75\pi n - 0.75\pi) + 8\cos(0.5\pi n - 0.5\pi)
\end{aligned}$$

(g) For which system does superposition hold?

Superposition is equivalent to linearity. Thus, it holds in the first system and not in the second.

(h) For which system does the output contain frequencies that are not present in the input signal?

The output of the second system includes a frequency of $\hat{\omega} = 0.5\pi$, which was not found in the input.

(i) Which system can cause aliasing of sinusoidal components of the input?

The second system had an alias of $\hat{\omega} = 1.5\pi$ to $\hat{\omega} = -0.5\pi$.

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PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = \begin{cases} 1 & \text{if } |x[n]| \geq 0.5 \\ 0 & \text{if } |x[n]| < 0.5 \end{cases}$$

(a) For the system above, determine the output $y_1[n]$ when the input is

$$x_1[n] = \cos(0.5\pi n)$$

(b) Explain why the result from part (a) proves that the system is not an LTI system.

(c) Is the system linear? or time-invariant? or neither?

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula:

$$H(e^{j\hat{\omega}}) = \left(1 - je^{-j\hat{\omega}}\right)\left(1 + je^{-j\hat{\omega}}\right)\left(1 - e^{j3\pi/4}e^{-j\hat{\omega}}\right)\left(1 - e^{-j3\pi/4}e^{-j\hat{\omega}}\right)$$

(a) If the input is a complex exponential of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for which values of $-\pi \leq \hat{\omega} \leq \pi$ will $y[n] = 0$ for all n?
Hint: In this part, the answer is easy to obtain if you use the factored form above.

(b) Use superposition to determine the output of this system when the input is

$$x[n] = 5 + 9\delta[n-4] + 7\cos(0.5\pi n - 3\pi/4) \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results. This is what it means to apply the principle of *Superposition*.

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PROBLEM:

The following MATLAB code will compute a time response and the frequency response of a digital filter:

```
bb = [ 1  0  -1 ];
xn = [ 1, -1, -1, -1, 1, zeros(1,3) ];
yn = firfilt( bb, xn );
subplot(2,1,1), stem( [0:9], yn(1:10) );   %--- TIME RESPONSE
w = -pi : (pi/100) : pi;
H = freqz( bb, 1, w );
subplot(2,1,2), plot( w, abs(H) )      %--- FREQUENCY RESPONSE
```

(a) Make the plot of `yn` that will be done by the MATLAB `stem` function (in line #4).

(b) Again referring to the MATLAB code above, make the plot of the magnitude response vs. $\hat{\omega}$ over the range $-\pi \leq \hat{\omega} \leq \pi$. Justify by giving a $\boxed{\text{simple}}$ formula for the frequency response $H(e^{j\hat{\omega}})$. Remember that the magnitude should never be negative.

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SOLUTION

The following MATLAB code will compute a time response and the frequency response of a digital filter:

```
bb = [ 1  0  -1 ];
xn = [ 1, -1, -1, -1, 1, zeros(1,3) ];
yn = firfilt( bb, xn );
subplot(2,1,1), stem( [0:9], yn(1:10) );   %--- TIME RESPONSE
%
w = -pi : (pi/100) : pi;
H = freqz( bb, 1, w );
subplot(2,1,2), plot( w, abs(H) )      %--- FREQUENCY RESPONSE
```

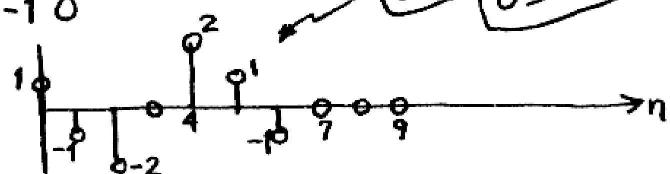
(a) Make the plot of yn that will be done by the MATLAB stem function (in line #4).

$\{b_k\} = \{1, 0, -1\} \Rightarrow y[n] = x[n] - x[n-2]$

Think of shifting xn by 2 & then subtracting

$$\begin{array}{rcccccccc} & [1 & -1 & -1 & -1 & 1 & 0 & 0 & 0] \\ - & [0 & 0 & 1 & -1 & -1 & -1 & 1 & 0] \\ \hline & 1 & -1 & -2 & 0 & 2 & 1 & -1 & 0 \end{array}$$

y[n] over $0 \le n \le 9$

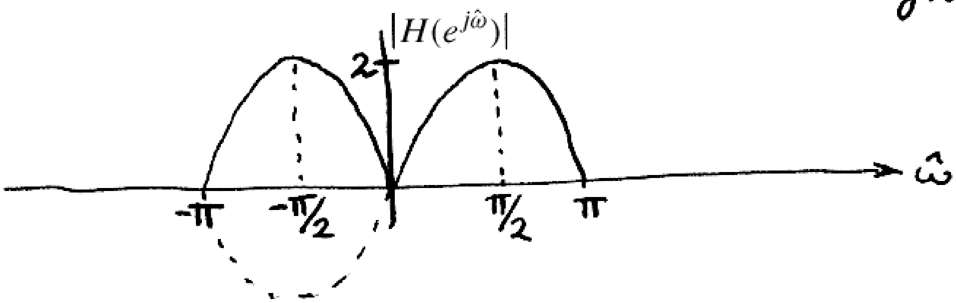


(b) Again referring to the MATLAB code above, make the plot of the magnitude response vs. $\hat{\omega}$ over the range $-\pi \le \hat{\omega} \le \pi$. Justify by giving a $\boxed{\text{simple}}$ formula for the frequency response $H(e^{j\hat{\omega}})$. Remember that the magnitude should never be negative.

$\{b_k\} = \{1, 0, -1\} \Rightarrow H(e^{j\hat{\omega}}) = 1 - e^{-j2\hat{\omega}}$

$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(e^{j\hat{\omega}} - e^{-j\hat{\omega}}) = 2je^{-j\hat{\omega}} \sin\hat{\omega}$

MAGNITUDE IS $2\sin\hat{\omega}$ if you ignore the minus sign.



NOTE: $H(e^{j\hat{\omega}})$ is periodic with period $= 2\pi$

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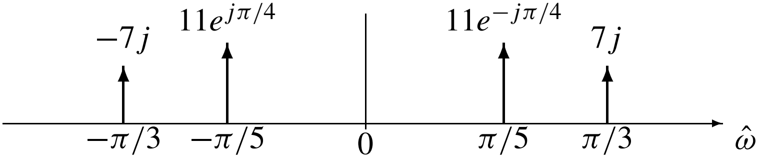
PROBLEM:

An FIR filter is characterized by the following frequency response:

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

(a) If the input to the filter is a signal with the following spectrum, determine a formula for the input signal, $x[n]$ for $-\infty < n < \infty$.



(b) Using the input signal from part (a), determine the output, $y[n]$ for $-\infty < n < \infty$.

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SOLUTION

(a) $x[n] = 22\cos\left(\frac{\pi}{5}n - \frac{\pi}{4}\right) + 14\cos\left(\frac{\pi}{3}n + \frac{\pi}{2}\right)$

by reading values off the spectrum.

(b) Evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \frac{\pi}{5}$ and $\hat{\omega} = \frac{\pi}{3}$

$$H(e^{j\pi/5}) = \frac{\sin(5 \cdot \pi/5)}{\sin(\frac{1}{2} \cdot \pi/5)} e^{-j5 \cdot \pi/5} = 0$$

$-1 = e^{-j\pi}$

$$H(e^{j\pi/3}) = \frac{\sin(5\pi/3)}{\sin(\pi/6)} e^{-j5\pi/3} = \frac{-\frac{1}{2}\sqrt{3}}{1/2} e^{j\pi/3} = \sqrt{3}\, e^{-j2\pi/3}$$

Thus $y[n]$ has only the $\frac{\pi}{3}$ component

$$y[n] = 14\sqrt{3}\cos\left(\frac{\pi}{3}n + \frac{\pi}{2} - \frac{2\pi}{3}\right)$$

$$= 14\sqrt{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{6}\right)$$

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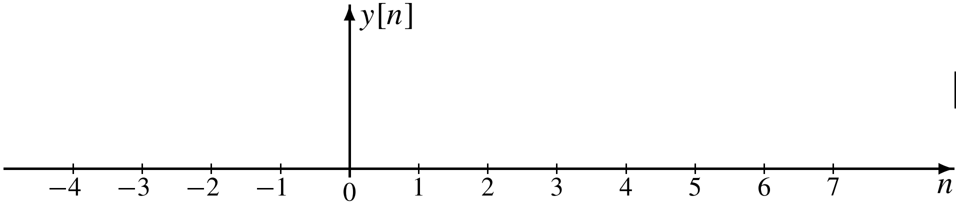
PROBLEM:

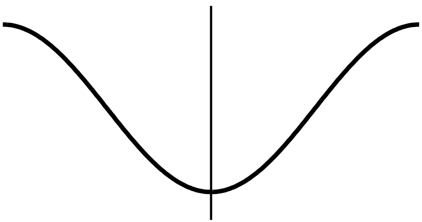
(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{9, -19, 9\}$, make a $\boxed{\text{plot}}$ of the output when the input is the signal: $x[n] = \delta[n-2] + \delta[n-3]$



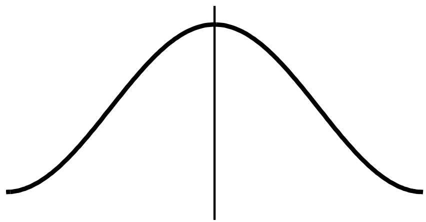
Plot zero values also

(b) The magnitude of the frequency response, $\mathcal{H}(\hat{\omega})$, for the filter in part (a) has one of the shapes below. The vertical line in each plot is located at $\hat{\omega} = 0$. Choose the correct one and then draw the horizontal axis with correct labels. **In addition, label the important features such as the locations of peaks and valleys and the values at those frequencies.**

HIGH-PASS



LOW-PASS



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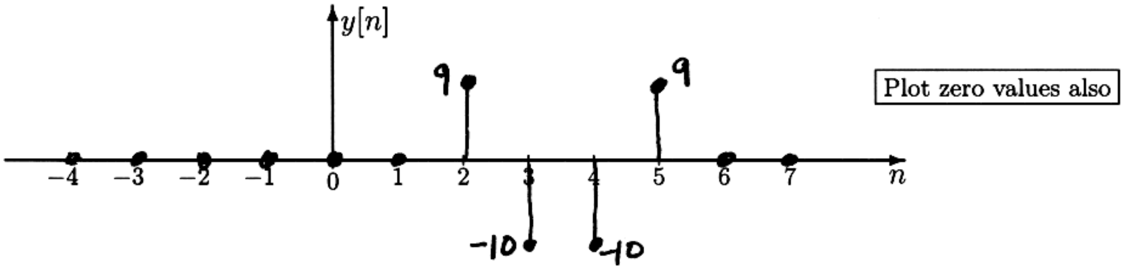
SOLUTION

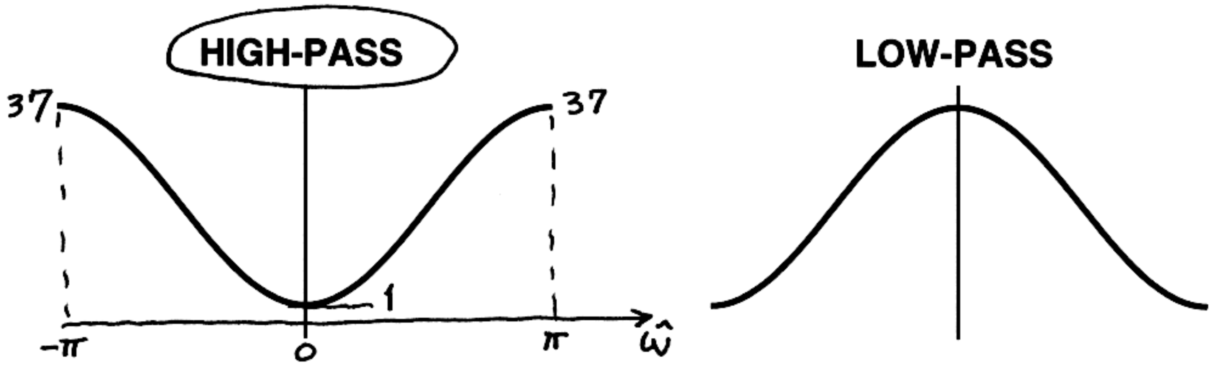
(a) If the filter coefficients of an FIR filter are $\{b_k\} = \{9, -19, 9\}$, make a $\boxed{\text{plot}}$ of the output when the input is the signal: $x[n] = \delta[n-2] + \delta[n-3]$



Use convolution:

```
9   -19   9
0    0    1    1
-------------------------
0    0    9   -19    9
              9   -19    9
-------------------------
0    0    9   -10  -10    9
```

(b) The magnitude of the frequency response, $\mathcal{H}(\hat{\omega})$, for the filter in part (a) has one of the shapes below. The vertical line in each plot is located at $\hat{\omega} = 0$. Choose the correct one and then draw the horizontal axis with correct labels. **In addition, label the important features such as the locations of peaks and valleys and the values at those frequencies.**



$$\mathcal{H}(\hat{\omega}) = 9 - 19e^{-j\hat{\omega}} + 9e^{-j2\hat{\omega}}$$
$$= e^{-j\hat{\omega}}(9e^{j\hat{\omega}} - 19 + 9e^{-j\hat{\omega}})$$
$$= e^{-j\hat{\omega}}(-19 + 18\cos\hat{\omega})$$
$$|\mathcal{H}(\hat{\omega})| = 19 - 18\cos\hat{\omega}$$

At $\hat{\omega} = 0$

$\mathcal{H}(0) = -1 \quad |\mathcal{H}(0)| = 1$

At $\hat{\omega} = \pi$

$\mathcal{H}(\pi) = 9 + 19 + 9 = 37$

PROBLEM:

For a 4-point moving average filter, find the output of the filter when the input signal is a sinusoid:

$$x[n] = 3\sin(0.2\pi n)$$

Express your answer as a formula that is real-valued.

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SOLUTION

$$H(e^{j\hat{\omega}}) = \frac{1}{4} e^{-j3\hat{\omega}/2} \frac{\sin 2\hat{\omega}}{\sin \hat{\omega}/2}$$ FREQ RESPONSE OF 4-PT AVG.

$$x[n] = 3\sin(0.2\pi n) = \frac{3}{2j} e^{j0.2\pi n} - \frac{3}{2j} e^{-j0.2\pi n}$$

MUST PROCESS EACH PART THRU FILTER!

$$e^{j\hat{\omega}n} \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow H(e^{j\hat{\omega}})e^{j\hat{\omega}n}.$$

$$\therefore \quad y[n] = \frac{3}{2j} H(e^{j0.2\pi}) e^{j0.2\pi n} - \frac{3}{2j} H(e^{-j0.2\pi}) e^{-j0.2\pi n}$$

$$= \frac{1}{4} e^{-j3(0.2\pi)/2} \frac{\sin 0.4\pi}{\sin 0.1\pi} = 0.769\, e^{-j0.3\pi}$$

$$H(e^{-j0.2\pi}) = 0.769\, e^{+j0.3\pi}$$

$$\therefore \quad y[n] = (1.5e^{-j0.5\pi})(0.769e^{-j0.3\pi})e^{j0.2\pi n} + (1.5e^{+j0.5\pi})(0.769e^{+j0.3\pi})e^{-j0.2\pi n}$$

$$y[n] = 1.154\, e^{j(0.2\pi n - 0.8\pi)} + 1.154\, e^{-j(0.2\pi n - 0.8\pi)}$$

$$= 2.308 \cos(0.2\pi n - 0.8\pi).$$

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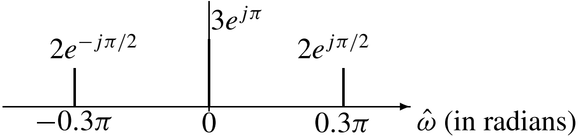
PROBLEM:

A discrete-time signal $x[n]$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x[n]$. Make sure to express $x[n]$ as a real-valued signal.

(b) Determine the formula for the output signal $y[n]$.

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PROBLEM:

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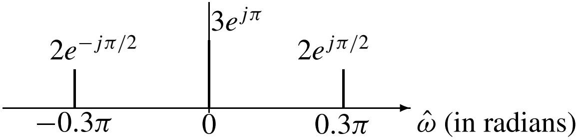

(a) Write an equation for $x[n]$. Make sure to express $x[n]$ as a real-valued signal.

(b) Use the MATLAB GUI called `ltidemo.m` to determine the output $y[n]$ when the FIR filter is a 9-point averaging filter. Include a screen shot of the result from `ltidemo`.

(c) Determine the formula for the output signal $y[n]$. Do this calculation by hand.

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PROBLEM:

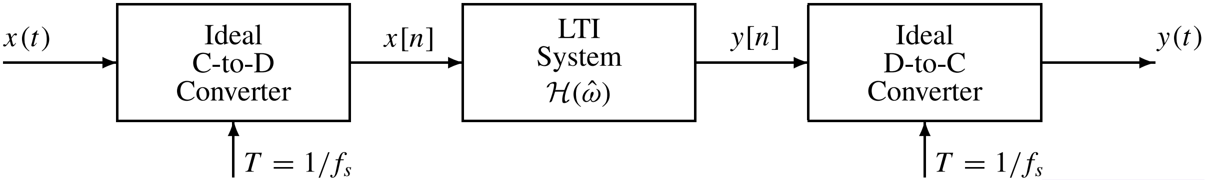
The input to the C-to-D converter in the figure below is

$$x(t) = 3 + 4\cos(2000\pi t) + 5\cos(4000\pi t - 2\pi/3)$$

The frequency response for the digital filter (LTI system) is

$$\mathcal{H}(\hat{\omega}) = \frac{\sin(2.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j2\hat{\omega}}$$

If $f_s = 10000$ samples/second, determine an expression for $y(t)$, the output of the D-to-C converter.



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SOLUTION

The input to the C-to-D converter in the figure below is

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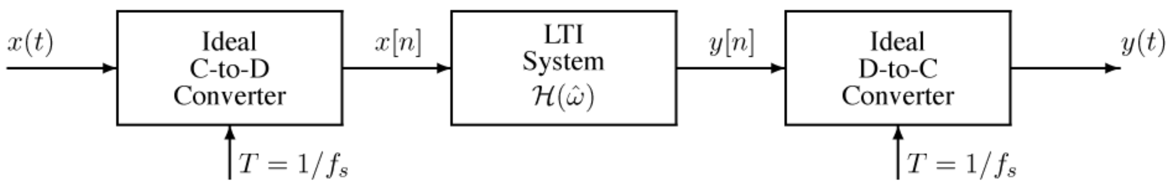

For $f_s = 10000$ samples/second, the output of the ideal C-to-D converter is

$$\begin{aligned} x[n] &= x(n/f_s) = 3 + 4\cos(2000\pi(n/10000)) + 5\cos(4000\pi(n/10000) - 2\pi/3) \\ &= 3 + 4\cos(0.2\pi n) + 5\cos(0.4\pi n - 2\pi/3) \\ &= 3 + 2e^{j0.2\pi n} + 2e^{-j0.2\pi n} + 2.5e^{-j2\pi/3}e^{j0.4\pi n} + 2.5e^{j2\pi/3}e^{-j0.4\pi n} \end{aligned}$$

The frequency response at each of these frequencies is

$$\begin{aligned} \mathcal{H}(0) &= 5 \\ \mathcal{H}(0.2\pi) &= \frac{\sin(0.5\pi)}{\sin(0.1\pi)} e^{-j0.4\pi} \approx 3.24e^{-j0.4\pi} \\ \mathcal{H}(-0.2\pi) &\approx 3.24e^{j0.4\pi} \\ \mathcal{H}(0.4\pi) &= \frac{\sin(\pi)}{\sin(0.2\pi)} e^{-j0.8\pi} = 0 \\ \mathcal{H}(-0.4\pi) &= 0 \end{aligned}$$

The output of the LTI system is thus

$$\begin{aligned} y[n] &= \mathcal{H}(0)3 + \mathcal{H}(0.2\pi)2e^{j0.2\pi n} + \mathcal{H}(-0.2\pi)2e^{-j0.2\pi n} + \\ &\qquad \mathcal{H}(0.4\pi)2.5e^{-j2\pi/3}e^{j0.4\pi n} + \mathcal{H}(-0.4\pi)2.5e^{j2\pi/3}e^{-j0.4\pi n} \\ &\approx 15 + 6.48e^{-j0.4\pi}e^{j0.2\pi n} + 6.48e^{j0.4\pi}e^{-j0.2\pi n} \\ &= 15 + 12.96\cos(0.2\pi n - 0.4\pi) \end{aligned}$$

Finally, if the output sampling rate is $f_s = 10000$ samples/second, the output of the ideal D-to-C converter is obtained by replacing n in $y[n]$ with $f_s t$

$$\begin{aligned} y(t) = y[f_s t] &\approx 15 + 12.96\cos(0.2\pi[10000t] - 2\pi/5) \\ &= 15 + 12.96\cos(2000\pi t - 2\pi/5) \end{aligned}$$

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(d) The frequency response in Equation (3) is written as a product of factors suggesting that it could be implemented as a cascade of several systems. By suitably grouping the factors and multiplying them together, obtain a representation as the cascade of *two* systems each of which has only *real* filter coefficients. Give the frequency responses and impulse responses of the two systems and draw a block diagram of the cascade system.

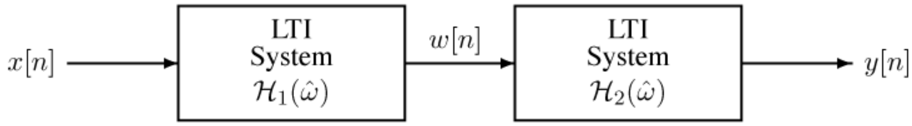
In the derivation in Part (a), we saw that

$$\mathcal{H}(\hat{\omega}) = (1 - e^{-j\hat{\omega}})(1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}})$$

This system can be implemented by the cascade of two systems with frequency responses and impulse responses as follows:

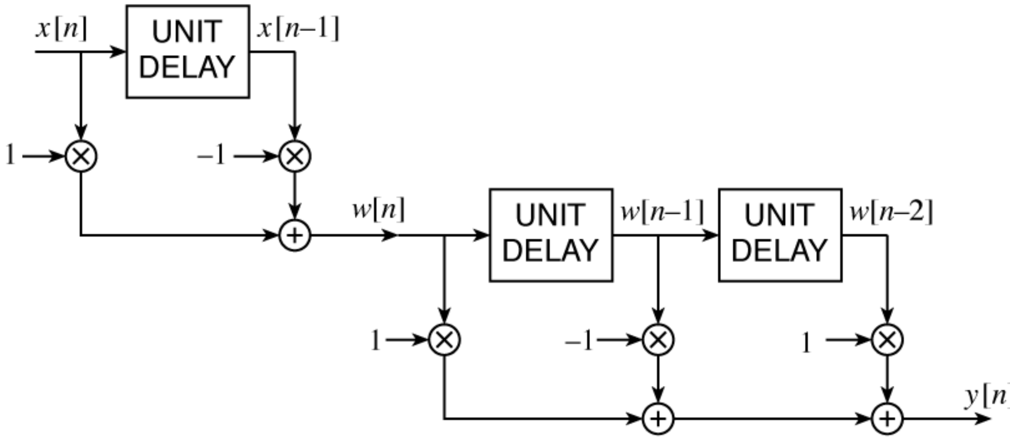
$$\begin{aligned}
\mathcal{H}_1(\hat{\omega}) &= 1 - e^{-j\hat{\omega}} \\
h_1[n] &= \delta[n] - \delta[n-1] \\
\mathcal{H}_2(\hat{\omega}) &= 1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \\
h_2[n] &= \delta[n] - \delta[n-1] + \delta[n-2]
\end{aligned}$$

because the filter coefficients of the first system are $\{b_k\} = \{1,\ -1\}$, and the filter coefficients of the second system are $\{b_k\} = \{1,\ -1,\ 1\}$.



Here is the detailed FIR filter structure with all the multipliers, adders and delays to implement the cascade of the two difference equations:

$$\begin{aligned}
w[n] &= x[n] - x[n-1] \\
y[n] &= w[n] - w[n-1] + w[n-2]
\end{aligned}$$



PROBLEM:

A linear time-invariant filter is described by the difference equation

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3]$$

(a) *Impulse Response:* Determine the impulse response of this system. Plot $h[n]$ as a function of n.

(b) When the input to the system is $x[n] = \exp(j\pi n/4)$ determine the functional form for the output signal $y[n]$. Find numerical values for the mangitude and phase.

(c) What is the output if the input is

$$x[n] = 4 + \cos[0.5\pi(n-1)] - 3\cos[0.25\pi n]$$

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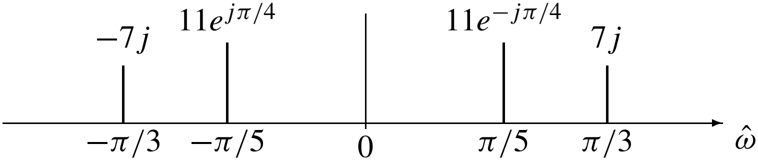
PROBLEM:

An FIR filter is characterized by the following frequency response:

$$\mathcal{H}(\hat{\omega}) = \frac{\sin(5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

(a) If the input to the filter is a signal with the following spectrum, determine a formula for the input signal, $x[n]$ for $-\infty < n < \infty$.



(b) Using the input signal from part (a), determine the output, $y[n]$ for $-\infty < n < \infty$.

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 + \tfrac{1}{2}e^{-j\hat{\omega}})(1 - e^{j\pi/4}e^{-j\hat{\omega}})(1 - e^{-j\pi/4}e^{-j\hat{\omega}})$$

(a) Write the difference equation for the FIR filter that gives the relation between the input $x[n]$ and the output $y[n]$. Give numerical values for the filter coefficients.

(b) What is the output of this FIR filter if the input is $x[n] = \delta[n]$?

(c) Evaluate the frequency response $\mathcal{H}(\hat{\omega})$ at the frequencies $\hat{\omega} = 0$, $\hat{\omega} = \pi$ and $\hat{\omega} = \pi/4$. Do the calculations by hand by manipulating the complex number formulas.

(d) Use MATLAB to make plots of the magnitude and phase of the frequency response. Include these plots in your homework solution. Mark the values that you determined in part (b).

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SOLUTION

$$H(\hat\omega) = (1+\tfrac{1}{2}e^{-j\hat\omega})(1-e^{j\pi/4}e^{-j\hat\omega})(1-e^{-j\pi/4}e^{-j\hat\omega})$$

$$= (1+\tfrac{1}{2}e^{-j\hat\omega})\left[1-\overbrace{(e^{j\pi/4}+e^{-j\pi/4})}^{2\cos\pi/4=\sqrt{2}}e^{-j\hat\omega}+e^{-j\hat\omega 2}\right]$$

$$= 1-\sqrt{2}\,e^{-j\hat\omega}+e^{-j\hat\omega 2} + \tfrac{1}{2}e^{-j\hat\omega} - \tfrac{\sqrt{2}}{2}e^{-j\hat\omega 2} + \tfrac{1}{2}e^{-j\hat\omega 3}$$

(a) $H(\hat\omega) = 1 - 0.9142\,e^{-j\hat\omega} + 0.2929\,e^{-j\hat\omega 2} + 0.5\,e^{-j\hat\omega 3}$

$$y[n] = x[n] - 0.9142\,x[n-1] + 0.2929\,x[n-2] + 0.5\,x[n-3]$$

(b) $h[n] = \delta[n] - 0.9142\,\delta[n-1] + 0.2929\,\delta[n-2] + 0.5\,\delta[n-3]$

(c) from (a):

$$H(0) = 1 - 0.9142 + 0.2929 + 0.5 = 0.8787$$
$$H(\pi) = 1 + 0.9142 + 0.2929 - 0.5 = 1.7071$$
$$H(\tfrac{\pi}{4}) = 1 - 0.9142\,e^{-j\pi/4} + 0.2929\,e^{j\pi/2} + 0.5\,e^{-j3\pi/4} = 0$$

from the origional $H(\hat\omega)$

$$H(0) = (1+\tfrac{1}{2})(1-e^{j\pi/4})(1-e^{-j\pi/4}) = 0.8787$$
$$H(\pi) = (1-\tfrac{1}{2})(1+e^{-j3\pi/4})(1-e^{-j5\pi/4}) = 1.7071$$
$$H(\pi/4) = (1+e^{-j\pi/4})(\underbrace{1-\cancel{e^{j(\pi/4-\pi/4)}}}_{=0})(1-e^{-j\pi/2}) = 0$$

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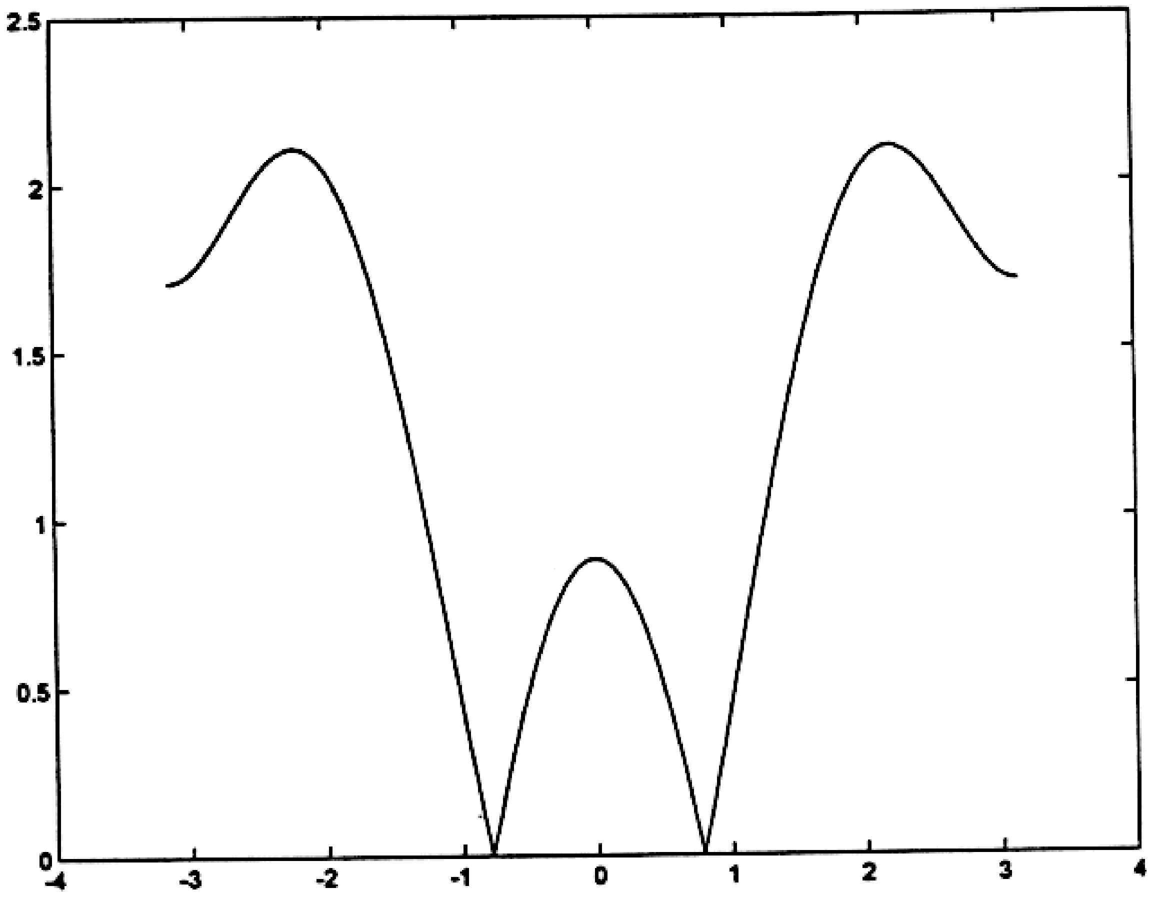
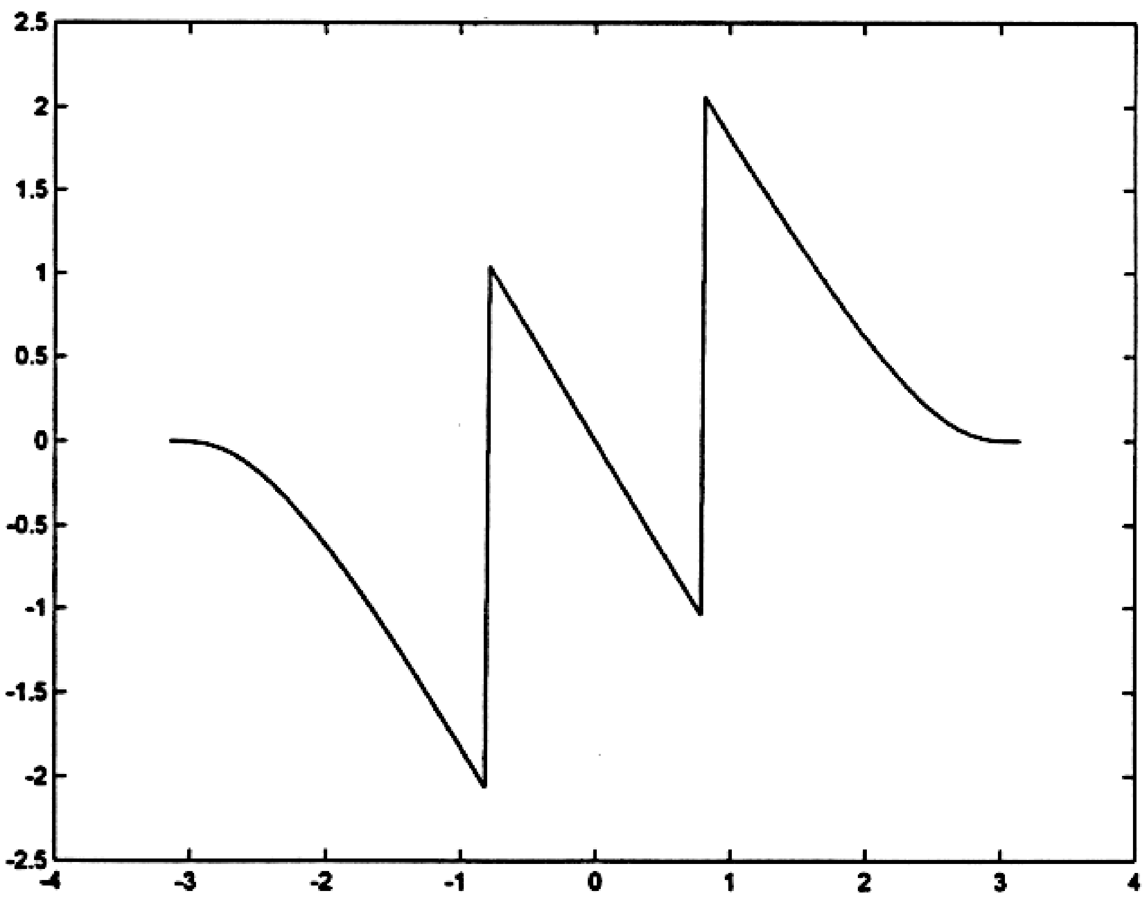
```
%-- problem
%
bb = [ 1, -0.9142, 0.2929, 0.5];
ww = -pi:pi/100:pi;
H = freekz(bb,1,ww);
plot(ww,abs(H));
pause
plot(ww,angle(H))
```





PROBLEM:

A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n+1])^2. \tag{1}$$

(a) Determine whether or not the system defined by (1) is (i) linear; (ii) time-invariant; (iii) causal.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}.$$

Express your answer in terms of cosine functions. Do not leave any squared powers of cosine functions in your answers.

(c) For the system of Equation (1), determine the output $y_2[n]$ when the input is

$$x_2[n] = 4 + 4\cos(0.75\pi(n-1)).$$

(d) This system produces output contain frequencies that are not present in the input signal. Explain how this system might cause "aliasing" of sinusoidal components of the input.

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SOLUTION

DEFINITIONS

(i) LINEAR IF: $a f(x_1) + b f(x_2) = f(a x_1 + b x_2)$

(ii) TIME INVARIANT IF:

$y[n-n_0] = f(x[n-n_0])$ -or- delayed output = f(delayed input)

(iii) CAUSAL IF OUTPUT DOES NOT DEPEND ON FUTURE VALUES

(i) LINEAR?

a i

$$\alpha x_1[n+1]^2 + \beta x_2[n+2]^2 \stackrel{?}{=} \left(\alpha x_1[n+1] + \beta x_2[n+1]\right)^2$$

$$= \alpha^2 x_1[n+1]^2 + \underbrace{\alpha\beta\, x_1[n+1]\, x_2[n+1]}_{\text{New CROSS-PRODUCT TERM}} + \beta^2 x_2[n+1]$$

Not Identical

NOT IDENTICAL

(CAN CAUSE NEW FREQUENCIES...)

✗ LINEAR
✓ NON-LINEAR

(ii) TIME INVARIANT?

SUBSTITUTE for n: m-m₀

a ii

$$y[m-m_0] = (x[m-m_0])^2$$

$$= f(x[m-m_0])$$

TRUE ✓

(iii) CAUSAL?

a iii

$$y[\text{now}] = (x[\text{now}+1])^2$$

now future

✗ CAUSAL
✓ NON-CAUSAL - Depends on future

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b) FIND $y_1[n]$ FOR $x_1[n] = 2\cos\left(\frac{3}{4}\pi n\right)$

$$y_1[n] = \left(x[n+1]\right)^2$$

$$y_1[n] = 2\cos\left(\tfrac{3}{4}\pi(n+1)\right)$$

$$= e^{-j\frac{3}{4}\pi(n+1)} + e^{+j\frac{3}{4}\pi(n+1)}$$

(CROSS MULTIPLY exp() TERMS
COLLECT TERMS
INVERSE EULER → COSINE)

or

(TRIG IDENTITY
$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$)

$$y_1[n] = 2\cos\left(\tfrac{3}{4}\pi(n+1) + \tfrac{3}{4}\pi(n+1)\right) + 2\cos(0)$$

$$= 2\cos\left(\tfrac{3}{2}\pi(n+1)\right) + 2$$

$$= 2\cos\left(\tfrac{1}{2}\pi(n+1)\right) + 2$$

0 rad/sec = "DC"

$\frac{\pi}{2}$ rad/sample

GONE ← $\frac{3\pi}{4}$ rad/sample

INITIAL FREQ

NEW FREQUENCIES: RESULT OF non-Linear SYSTEM

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c

FIND $y_2[n]$ WHEN $x_2[n] = 4 + 4\cos\left(\frac{3}{4}\pi(n-1)\right)$

$$y_2[n] = \left[4 + 4\cos\left(\tfrac{3}{4}\pi(n-1+1=n)\right)\right]^2$$

NON-LINEAR - SO WE CANNOT USE SCALING OR SUPERPOSITION

SO, PLUG + CHUG

$$= 4^2\left[1 + \cos\left(\tfrac{3}{4}\pi n\right)\right]^2$$

$$= 16\left[1 + 2\cos\left(\tfrac{3}{4}\pi n\right) + \cos\left(\tfrac{3}{4}\pi n\right)^2\right]$$

$$= 8\left[2 + 4\cos\left(\tfrac{3}{4}\pi n\right) + \underbrace{\cos\left(\tfrac{3}{2}\pi n\right) + 1}_{\text{FROM TRIG IDENTITY}}\right]$$

$$= 24 + 32\cos\left(\tfrac{3}{4}\pi n\right) + 8\cos\left(\tfrac{3}{2}\pi n\right)$$

$$= \underbrace{24 + 32\cos\left(\tfrac{3}{4}\pi n\right)}_{\text{FREQUENCIES IN INPUT SIGNAL}} + \underbrace{8\cos\left(\tfrac{1}{2}\pi n\right)}_{\text{A NEW FREQUENCY}}$$

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PROBLEM:

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 - e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}}). \tag{1}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.

(b) What is the output if the input is $x[n] = \delta[n]$?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?

(d) The frequency response in Equation (1) is written as a product of factors suggesting that it could be implemented as a cascade of several systems. By suitably grouping the factors and multiplying them together, obtain a representation as the cascade of *two* systems each of which has only *real* filter coefficients. Give the frequency responses and impulse responses of the two systems and draw a block diagram of the cascade system.

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SOLUTION

The frequency response of a linear time-invariant filter is given by the formula

$$\mathcal{H}(\hat{\omega}) = (1 - e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}}) \tag{3}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$.

First rearrange the frequency response as follows:

$$\begin{aligned}
\mathcal{H}(\hat{\omega}) &= (1 - e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}}) \\
&= (1 - e^{-j\hat{\omega}})\left[1 - (e^{j\pi/3} + e^{-j\pi/3})e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right] \\
&= (1 - e^{-j\hat{\omega}})\left[1 - (1)e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right] \\
&= 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}
\end{aligned}$$

From this equation, we can derive the filter coefficients: $\{b_k\} = \{1, -2, 2, -1\}$. Thus, the output of the filter is given by the following difference equation:

$$y[n] = x[n] - 2x[n-1] + 2x[n-2] - x[n-3]$$

(b) What is the output if the input is $x[n] = \delta[n]$?

When the input to the filter is the unit impulse sequence, the output is unit impulse response:

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3]$$

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?

For inputs of this form, the output of the filter is zero for all n when the frequency response of zero, i.e., when $\mathcal{H}(\hat{\omega}) = 0$ at a particular frequency. From Equation (3), the frequency response is zero when one of the factors is zero, i.e., when any ne of the following conditions is true:

$$\begin{aligned}
(1 - e^{-j\hat{\omega}}) &= 0 \\
(1 - e^{j\pi/3}e^{-j\hat{\omega}}) &= 0 \\
(1 - e^{-j\pi/3}e^{-j\hat{\omega}}) &= 0
\end{aligned}$$

These conditions are true when $\hat{\omega} = 0$, $\hat{\omega} = \pi/3$, and $\hat{\omega} = \pi/3$, respectively. For example, we can solve the middle one:

$$\begin{aligned}
(1 - e^{j\pi/3}e^{-j\hat{\omega}}) &= 0 \\
e^{j\hat{\omega}} - e^{j\pi/3} &= 0 \\
e^{j\hat{\omega}} &= e^{j\pi/3} \qquad \Rightarrow \qquad \hat{\omega} = \pi/3
\end{aligned}$$

PROBLEM:

For the *aliased sinc* function:

$$\text{asinc}(\hat{\omega}, 11) = \frac{\sin(5\frac{1}{2}\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

(a) Make a plot of $\text{asinc}(\hat{\omega}, 11)$ over the range $-4\pi \leq \hat{\omega} \leq +4\pi$. Label all the zero crossings.

(b) Determine the period of $\text{asinc}(\hat{\omega}, 11)$. Is it equal to 2π; why, or why not?

(c) Find the maximum value of the function.

NOTE: the *aliased sinc* function is defined via: $\text{asinc}(\hat{\omega}, L) = \dfrac{\sin(L\hat{\omega}/2)}{\sin(\frac{1}{2}\hat{\omega})}$

In MATLAB consult help on `diric` for more information.

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PROBLEM:

For the *modified Dirichlet* function:

$$\tilde{\mathcal{D}}(\hat{\omega}, 5) = \frac{\sin(2.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

(a) Make a plot of $\tilde{\mathcal{D}}(\hat{\omega}, 5)$ over the range $-2\pi \le \hat{\omega} \le +2\pi$. Label all the zero crossings.

(b) Determine the period of $\tilde{\mathcal{D}}(\hat{\omega}, 5)$. Is it equal to 2π; why, or why not?

(c) Find the maximum value of the function.

Note: the unmodified *Dirichlet* function is defined via: $\mathcal{D}(\hat{\omega}, L) = \dfrac{\sin(L\hat{\omega}/2)}{L\sin(\frac{1}{2}\hat{\omega})}$, so $\tilde{\mathcal{D}}(\hat{\omega}, 5) = 5\mathcal{D}(\hat{\omega}, 5)$.

In MATLAB consult help on `diric` for more information.

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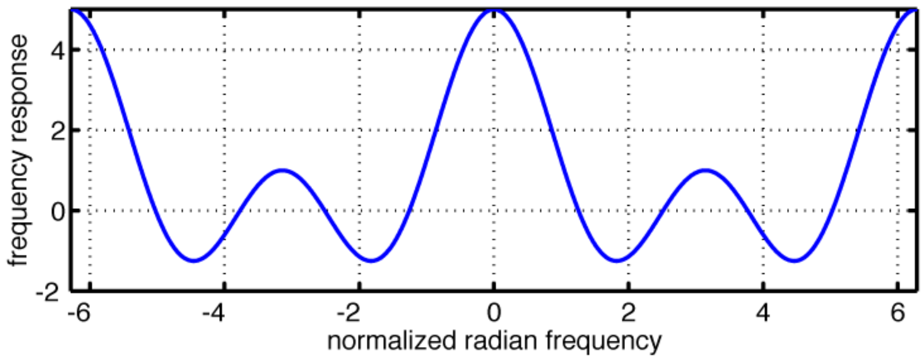
SOLUTION

For the *modified Dirichlet* function:

$$\tilde{\mathcal{D}}(\hat{\omega},5) = \frac{\sin(2.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

(a) Make a plot of $\tilde{\mathcal{D}}(\hat{\omega},5)$ over the range $-2\pi \le \hat{\omega} \le +2\pi$. Label all the zero crossings.

The zero crossings occur when $\sin(2.5\hat{\omega}) = 0$ and $\sin(\frac{1}{2}\hat{\omega}) \neq 0$. In the range, $-2\pi \le \hat{\omega} \le +2\pi$, these conditions are true at $\hat{\omega} = \pm 2\pi/5, \pm 4\pi/5, \pm 6\pi/5, \pm 8\pi/5$.



(b) Determine the period of $\tilde{\mathcal{D}}(\hat{\omega},5)$. Is it equal to 2π; why, or why not?

For both the numerator, $\sin(2.5\hat{\omega})$, and the denominator, $\sin(\frac{1}{2}\hat{\omega})$, of the function, the following properties hold true:

$$f(\hat{\omega}+2\pi k) = \begin{cases} -f(\hat{\omega}) & k \text{ odd} \\ f(\hat{\omega}) & k \text{ even} \end{cases}$$

Thus, for all k, the signs cancel,

$$\frac{\sin(2.5(\hat{\omega}+2\pi k))}{\sin(\frac{1}{2}(\hat{\omega}+2\pi k))} = \frac{\sin(2.5\hat{\omega}+\pi k)}{\sin(\frac{1}{2}\hat{\omega}+\pi k)} = \frac{(-1)^k \sin(2.5\hat{\omega})}{(-1)^k \sin(\frac{1}{2}\hat{\omega})} = \frac{\sin(2.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

and the function is periodic with a period of 2π.

(c) Find the maximum value of the function.

From the figure above, we see that the maximum value of the function occurs at $\hat{\omega} = 2\pi k$, where k is an integer. At these frequencies, the numerator and denominator of the function are both equal to 0. Thus, we can use L'Hopital's Rule to determine the value:

$$\lim_{\hat{\omega}\to 2\pi k} \tilde{\mathcal{D}}(\hat{\omega},5) = \lim_{\hat{\omega}\to 2\pi k} \frac{\sin(2.5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} = \left.\frac{2.5\cos(2.5\hat{\omega})}{\frac{1}{2}\cos(\frac{1}{2}\hat{\omega})}\right|_{\hat{\omega}=2\pi k} = \frac{\pm 2.5}{\pm 0.5} = 5$$

PROBLEM:

A linear time-invariant filter is described by the difference equation

$$y[n] = -x[n] + 2x[n-1] - x[n-2]$$

(a) Obtain an expression for the frequency response of this system.

(b) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(c) What is the output if the input is $x[n] = 5 + 5\cos(0.5\pi n + \pi/2)$?

(d) What is the output if the input is the *unit impulse sequence* $x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$

(e) What is the output if the input is the *unit step sequence* $x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0. \end{cases}$

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SOLUTION

A linear time-invariant filter is described by the difference equation

$$y[n] = -x[n] + 2x[n-1] - x[n-2]$$

(a) Obtain an expression for the frequency response of this system.

For this problem $\{b_k\} = \{-1, 2, -1\}$ and $M = 2$. The frequency response is computed as follows:

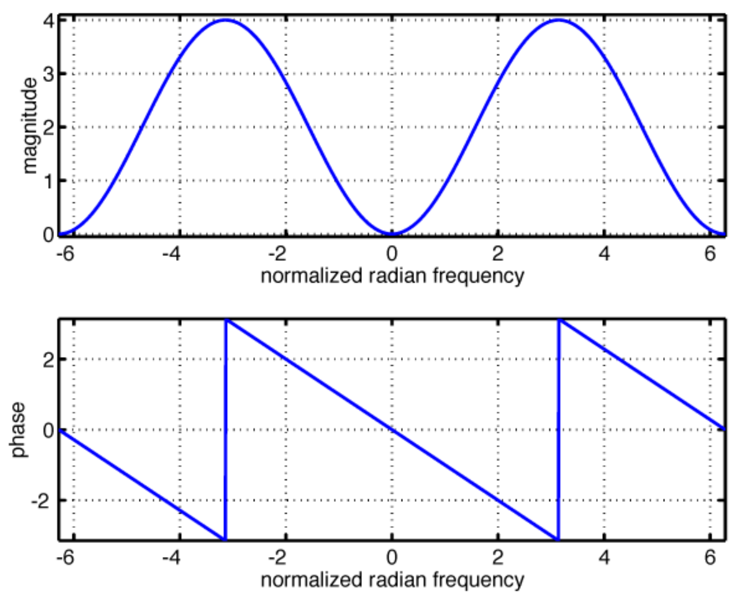
$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \\ &= -1 + 2e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}\end{aligned}$$

(b) Make a sketch of the frequency response (magnitude and phase) as a function of frequency.

Using symmetry to simplify the expression for $\mathcal{H}(\hat{\omega})$:

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= -1 + 2e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}}\left(-e^{j\hat{\omega}} + 2 - e^{-j\hat{\omega}}\right) \\ &= e^{-j\hat{\omega}}\left(2 - 2\cos\hat{\omega}\right)\end{aligned}$$

Because $(2 - 2\cos\hat{\omega}) \geq 0$ for all frequencies, the magnitude of the frequency response is $|\mathcal{H}(\hat{\omega})| = (2 - 2\cos\hat{\omega})$, and the phase is $\angle\mathcal{H}(\hat{\omega}) = -\hat{\omega}$ (in the range $-\pi < \hat{\omega} \leq \pi$).



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(c) What is the output if the input is $x[n] = 5 + 5\cos(0.5\pi n + \pi/2)$?

First convert $x[n]$ to a sum of complex exponentials as follows:

$$\begin{aligned} x[n] &= 5 + 5\cos\left(\frac{\pi}{2}n + \frac{\pi}{2}\right) \\ &= 5e^{j0n} + \frac{5}{2}e^{j\pi/2}e^{j(\pi/2)n} + \frac{5}{2}e^{-j\pi/2}e^{-j(\pi/2)n} \end{aligned}$$

Thus, there are three input frequencies: $\hat{\omega} = 0, \pi/2, -\pi/2$. We can evaluate $\mathcal{H}(\hat{\omega})$ at each of these frequencies to get

$$\begin{aligned} \mathcal{H}(0) &= e^{-j0}\,(2 - 2\cos(0)) = 0 \\ \mathcal{H}(\pi/2) &= e^{-j\pi/2}\,(2 - 2\cos(\pi/2)) = 2e^{-j\pi/2} \\ \mathcal{H}(-\pi/2) &= e^{j\pi/2}\,(2 - 2\cos(-\pi/2)) = 2e^{j\pi/2} \end{aligned}$$

Thus, the resulting output is

$$\begin{aligned} y[n] &= \mathcal{H}(0)\left(5e^{j0n}\right) + \mathcal{H}(\pi/2)\left(\frac{5}{2}e^{j\pi/2}e^{j(\pi/2)n}\right) + \mathcal{H}(-\pi/2)\left(\frac{5}{2}e^{-j\pi/2}e^{-j(\pi/2)n}\right) \\ &= 5e^{j(\pi/2)n} + 5e^{-j(\pi/2)n} \\ &= 10\cos\left(\frac{\pi}{2}n\right) \end{aligned}$$

(d) What is the output if the input is the *unit impulse sequence* $x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$

$$y[n] = \delta[n] * h[n] = h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$$

(e) What is the output if the input is the *unit step sequence* $x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0. \end{cases}$

$$y[n] = u[n] * h[n] = \begin{cases} 0 & n < 0 \\ -1 & n = 0 \\ 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

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PROBLEM:

Circle the correct answer to each of these short answer questions, and give a brief explanation:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 99\cos(0.4\pi n - 0.8\pi)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 10,000 samples/second.

   (a) $f = 8000$ Hz
   (b) $f = 4000$ Hz
   (c) $f = 2000$ Hz
   (d) $f = 1000$ Hz
   (e) $f = 0.2$ Hz

2. A continuous-time signal $x(t)$ is defined by the Fourier Series sum: $x(t) = \sum_{k=-10}^{10} jke^{j16\pi kt}$.
   The Nyquist Rate for sampling $x(t)$ is

   (a) 20 Hz
   (b) 40 Hz
   (c) 80 Hz
   (d) 160 Hz
   (e) 320 Hz

3. A rotating disk with one spot is spinning *clockwise* at the rate of 10 revolutions per second. If the disk is illuminated with a strobe light that flashes once every 0.2 seconds, determine the movement of the spot that you will see.

   (a) The spot appears to *stand still.*
   (b) The spot appears to rotate *counter-clockwise* at a rate of 1 revolutions per second.
   (c) The spot appears to rotate *counter-clockwise* at a rate of 2 revolutions per second.
   (d) The spot appears to rotate *clockwise* at a rate of 1 revolutions per second.
   (e) The spot appears to rotate *clockwise* at a rate of 2 revolutions per second.

4. Suppose that the discrete-time signal $x[n] = \cos(0.8\pi n)$ is the input to an FIR filter whose frequency response is shown on the next page. Determine the output signal, $y[n]$.

   (a) $y[n] = 3\cos(\hat{\omega})\, e^{-j\hat{\omega}} \cos(0.8\pi n)$
   (b) $y[n] = 0.62\cos(0.4\pi n + 0.2\pi)$
   (c) $y[n] = 0.62\cos(0.8\pi n - 0.2\pi)$
   (d) $y[n] = 0.62\cos(0.8\pi n + 0.2\pi)$
   (e) $y[n] = 0.5\cos(0.8\pi n + 0.2\pi)$
   (f) $y[n] = 0$

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Circle the correct answer to each of these short answer questions, and give a brief explanation:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 99\cos(0.4\pi n - 0.8\pi)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 10,000 samples/second.

   (a) $f = 8000$ Hz
   (b) $f = 4000$ Hz
   (c) $f = 2000$ Hz
   (d) $f = 1000$ Hz
   (e) $f = 0.2$ Hz

$$\frac{2\pi f}{f_s} = \hat{\omega} \Rightarrow f = \frac{\hat{\omega}}{2\pi} \cdot f_s = \frac{0.4\pi}{2\pi} \cdot 10{,}000$$

$$f = (0.2)(10{,}000) = 2{,}000 \text{ Hz}$$

2. A continuous-time signal $x(t)$ is defined by the Fourier Series sum: $x(t) = \sum_{k=-10}^{10} jke^{j16\pi kt}$. The Nyquist Rate for sampling $x(t)$ is

   (a) 20 Hz
   (b) 40 Hz
   (c) 80 Hz
   (d) 160 Hz
   (e) 320 Hz

Highest frequency is when $k=10$

$e^{j160\pi t} \Rightarrow \omega = 160\pi$ rad/sec

$f_{MAX} = 80$ Hz

$\Rightarrow$ Nyquist $f_s = 160$ Hz

3. A rotating disk with one spot is spinning *clockwise* at the rate of 10 revolutions per second. If the disk is illuminated with a strobe light that flashes once every 0.2 seconds, determine the movement of the spot that you will see.

   (a) The spot appears to *stand still.*
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   (c) The spot appears to rotate *counter-clockwise* at a rate of 2 revolutions per second.
   (d) The spot appears to rotate *clockwise* at a rate of 1 revolutions per second.
   (e) The spot appears to rotate *clockwise* at a rate of 2 revolutions per second.

10 rev/s $\Rightarrow$ 1 rev every 0.1 secs
$\therefore$ flashing once per 2 revs.

4. Suppose that the discrete-time signal $x[n] = \cos(0.8\pi n)$ is the input to an FIR filter whose frequency response is shown on the next page. Determine the output signal, $y[n]$.

   (a) $y[n] = 3\cos(\hat{\omega})\, e^{-j\hat{\omega}} \cos(0.8\pi n)$
   (b) $y[n] = 0.62\cos(0.4\pi n + 0.2\pi)$
   (c) $y[n] = 0.62\cos(0.8\pi n - 0.2\pi)$
   (d) $y[n] = 0.62\cos(0.8\pi n + 0.2\pi)$
   (e) $y[n] = 0.5\cos(0.8\pi n + 0.2\pi)$
   (f) $y[n] = 0$

Evaluate $\mathcal{H}(\hat{\omega})$ at $\hat{\omega} = 0.8\pi$

$\hat{\omega} = 2\pi(0.4)$

Reading from plot

$\mathcal{H}(\hat{\omega}) \approx 0.6\, e^{j0.6}$

0.6 rads $\approx 0.2\pi$ rads

$\therefore y[n] \approx 0.6\cos(0.8\pi n + 0.2\pi)$

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PROBLEM:

Consider the linear time-invariant system described by the difference equation

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] = \sum_{k=0}^{4} x[n-k]$$

(a) Find an expression for the frequency response $H(e^{j\hat{\omega}})$ of the system.

(b) Show that your answer in (a) can be expressed in the form

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega}/2)}{\sin(\hat{\omega}/2)} e^{-j\hat{\omega}2}.$$

(c) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above (or plot it using `freqz( )`).

(d) Suppose that the input is

$$x[n] = 10 + 10\cos(\hat{\omega}_0 n) \qquad \text{for } -\infty < n < \infty$$

Find a non-zero frequency $0 < \hat{\omega}_0 < \pi$ for which the output $y[n]$ is a constant for all n, i.e.,

$$y[n] = c \qquad \text{for } -\infty < n < \infty$$

and find the value for c. (In other words, the sinusoid is removed by the filter.)

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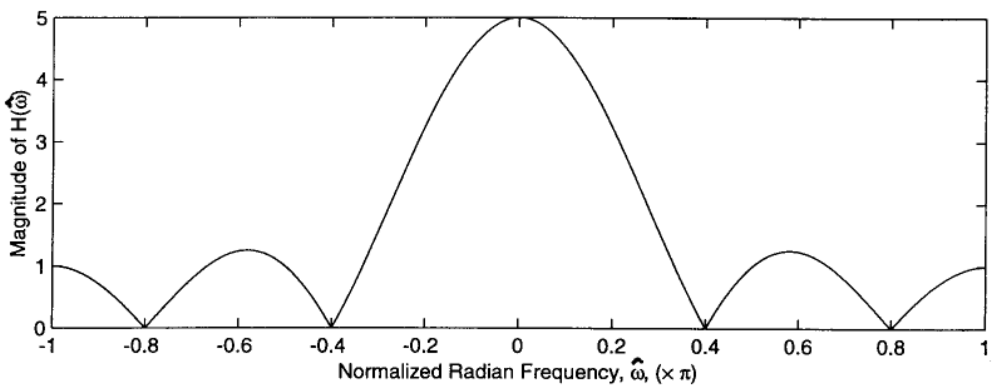
SOLUTION

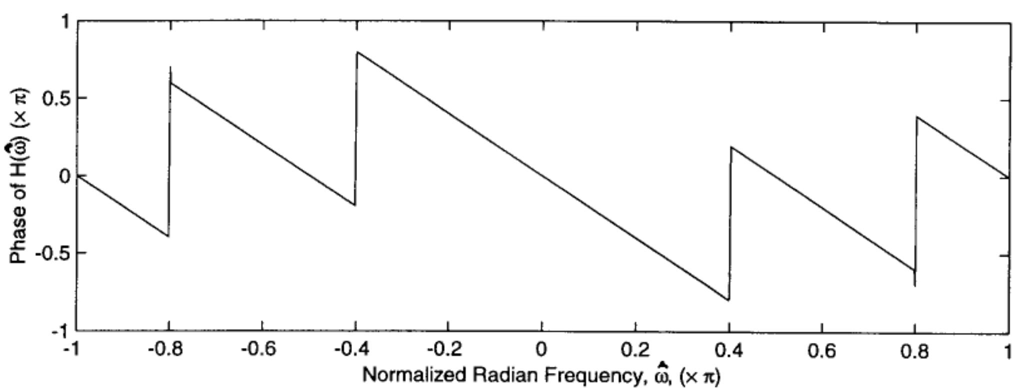
(a) $$\mathcal{H}(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}} = \frac{1-e^{-j5\hat{\omega}}}{1-e^{-j\hat{\omega}}}$$

(b) $$\mathcal{H}(\hat{\omega}) = \frac{1-e^{-j5\hat{\omega}}}{1-e^{-j\hat{\omega}}} = \frac{e^{-j\frac{5}{2}\hat{\omega}}\left[e^{j\frac{5}{2}\hat{\omega}} - e^{-j\frac{5}{2}\hat{\omega}}\right]}{e^{-j\hat{\omega}/2}\left[e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}\right]} =$$

$$= e^{-j2\hat{\omega}} \frac{\cancel{2j}\sin(\frac{5}{2}\hat{\omega})}{\cancel{2j}\sin(\hat{\omega}/2)} = e^{-j2\hat{\omega}} \frac{\sin(\frac{5}{2}\hat{\omega})}{\sin(\frac{\hat{\omega}}{2})}$$

(c)





Using FREQZ from MATLAB we can obtain $\mathcal{H}(\hat{\omega})$ in both magnitude & phase. These are plotted above.

(d) $x(n) = 10 + 10\cos(\hat{\omega}_o n) \qquad -\infty < n < +\infty$

Since $y(n) = c \sim \mathcal{H}(\hat{\omega}_o) = 0$ so $\hat{\omega}_o$ must correspond

to zeros of $\mathcal{H}(z)$ since $\mathcal{H}(\hat{\omega}_o) = H(z)\big|_{z=e^{j\hat{\omega}_o}} = 0$

If $z_1 = e^{j2\pi/5}$ or $z_2 = e^{j(2\pi/5)2}$

Therefore $\omega_o = 2\pi/5$ or $4\pi/5$ for $0 < \omega_o < \pi$

The value of c is given by:

$c = 10\,\mathcal{H}(\hat{\omega} = 0) = 50$

PROBLEM:

A discrete-time system is defined by the input/output relation

$$y[n] = (-1)^n x[n]$$

One characteristic of a LTI system is that a sinusoidal input at frequency $\hat{\omega}_0$ will give a sinusoidal output at the same frequency—no new frequency components will appear in the output. If frequencies other than $\hat{\omega}_0$ are contained in the output, then we can conclude that the system is not LTI.

(a) For the system above, determine the output $y_1[n]$ when the input is

$$x_1[n] = \cos(0.5\pi n)$$

Does this input-output pair $\{x_1[n],\ y_1[n]\}$ allow us to conclude that the system is not LTI?
Hint: Which frequencies are present in the output signal, $y_1[n]$?

(b) Exhibit one sinusoidal input signal, $A\cos(\hat{\omega}_0 n + \phi)$, for which the output contains frequency components not contained in the input signal.

(c) Is the system linear? or time-invariant? or neither? Explain.

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PROBLEM:

The frequency response of the filter above is

$$\mathcal{H}(\hat{\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 7 + 2\cos(0.5\pi n + \pi)$ for $-\infty < n < \infty$,
determine a simple mathematical expression for the output signal $y[n]$.

$y[n] =$

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SOLUTION

The frequency response of the filter above is

$$\mathcal{H}(e^{j\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 7 + 2\cos(0.5\pi n + \pi)$ for $-\infty < n < \infty$,
determine a simple mathematical expression for the output signal $y[n]$.

$$\boxed{y[n] = 7 + \sqrt{2}\cos(0.5\pi n + \pi/2)}$$

Recall that

$$x[n] = \cos(n\hat{\omega}) \Rightarrow y[n] = |H(\hat{\omega})| \cos(n\hat{\omega} + \angle H(\hat{\omega}))$$

Therefore, with

- $H(\hat{\omega})\big|_{\omega=0} = 1$
- $H(\hat{\omega})\big|_{\hat{\omega}=\pi/2} = \cos(\pi/4)\, e^{-j\pi/2} = \frac{1}{\sqrt{2}} e^{-j\pi/2}$

then

$$y[n] = 7 + \frac{1}{\sqrt{2}}\, 2\cos(0.5\pi n + \pi - \pi/2)$$

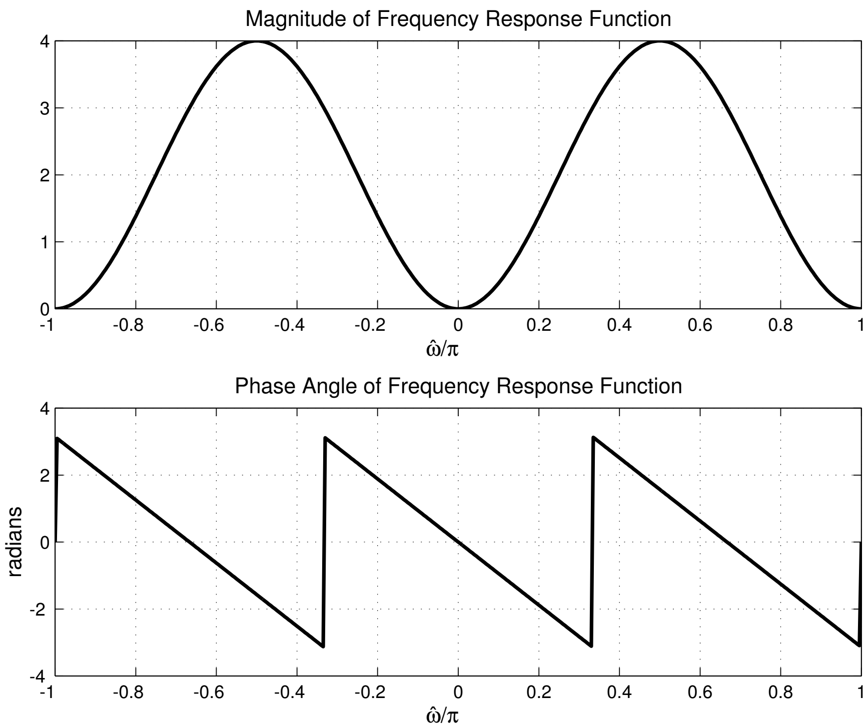
$$= 7 + \sqrt{2}\cos(0.5\pi n + \pi/2)$$

PROBLEM:

A linear time-invariant system is defined by the system function

$$H(z) = -z^{-1} + 2z^{-3} - z^{-5}$$

The magnitude and phase of the frequency response of this system are plotted in the following figure. **Note that the frequency scale is $\hat{\omega}/\pi$.**



(a) This filter is a *lowpass* *bandpass* *highpass* filter. (Circle one.)

(b) Use the above graph to determine (as accurately as you can) the output $y[n]$ of this system when the input is

$$x[n] = 10 + 10\cos(0.5\pi n).$$

Mark the points on the graph that you used in your solution.

(c) Determine an expression for the frequency response, $H(e^{j\hat{\omega}})$. Write your answer in the form $H(e^{j\hat{\omega}}) = A(\hat{\omega})e^{-j\hat{\omega}n_0}$, where $A(\hat{\omega})$ is real and n_0 is an integer.

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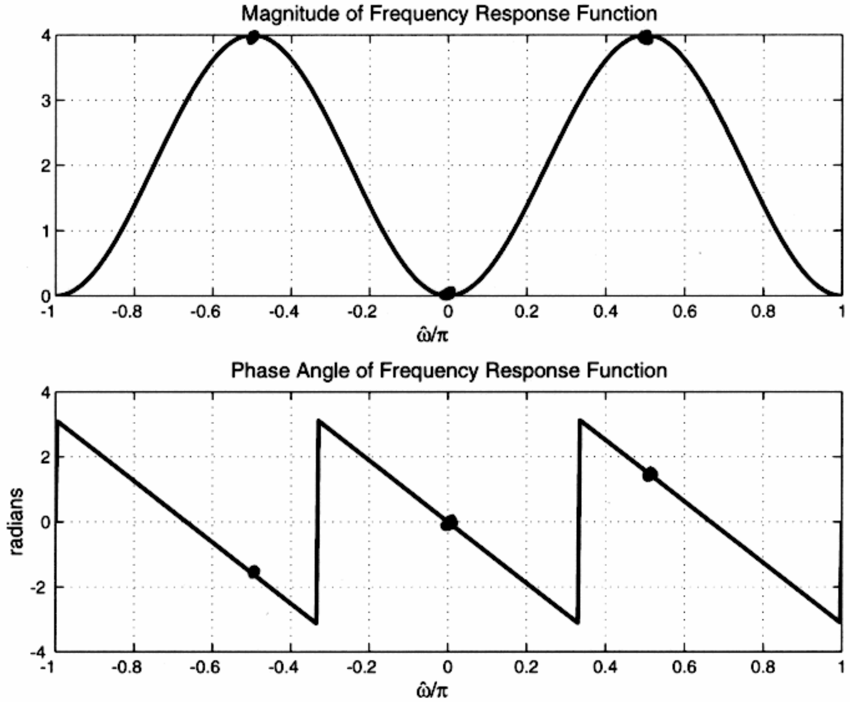
SOLUTION

A linear time-invariant system is defined by the system function

$$H(z) = -z^{-1} + 2z^{-3} - z^{-5}$$

The magnitude and phase of the frequency response of this system are plotted in the following figure. **Note that the frequency scale is $\hat{\omega}/\pi$.**



(a) This filter is a *lowpass* *bandpass* *highpass* filter. (Circle one.)

(b) Use the above graph to determine (as accurately as you can) the output $y[n]$ of this system when the input is

$$x[n] = 10 + 10\cos(0.5\pi n). = 10 + 5e^{j.5\pi n} + 5e^{-j.5\pi n}$$

Mark the points on the graph that you used in your solution.

$$y[n] = 10H(e^{j0}) + 5H(e^{j.5\pi})e^{j.5\pi n} + 5H(e^{-j.5\pi})e^{-j.5\pi n}$$

$$= 0 + 5.4\,e^{j1.6}e^{j.5\pi n} + 5.4e^{-j1.6}e^{-j.5\pi n}$$

$$= 40\cos(.5\pi n + 1.6) \qquad \left(1.6 \underset{\text{see (b)}}{\text{actually}} = \pi/2\right)$$

(c) Determine an expression for the frequency response, $H(e^{j\hat{\omega}})$. To receive full credit write your answer in the form $H(e^{j\hat{\omega}}) = A(\hat{\omega})e^{-j\hat{\omega}n_0}$, where $A(\hat{\omega})$ is real and n_0 is an integer.

$$H(e^{j\hat{\omega}}) = -e^{-j\hat{\omega}} + 2e^{-j\hat{\omega}3} - e^{-j\hat{\omega}5}$$

$$= \left(2 - e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}\right)e^{-j\hat{\omega}3}$$

$$= \left(2 - 2\cos 2\hat{\omega}\right)e^{-j\hat{\omega}3}$$

PROBLEM:

A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n-1])^2. \tag{1}$$

(a) Determine whether or not the system defined by (1) is (i) linear; (ii) time-invariant; (iii) causal.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}.$$

Express your answer in terms of cosine functions. Do not leave any squared powers of cosine functions in your answers. Note that this system produces output contain frequencies that are not present in the input signal. Explain how this system might cause "aliasing" of sinusoidal components of the input.

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SOLUTION

$$y[n] = (x[n-1])^2$$

a) Linearity test:

Let $x[n] = a_1 x_1[n] + a_2 x_2[n]$, then

$$y[n] = (a_1 x_1[n-1] + a_2 x_2[n-1])^2 \neq (a_1 x_1[n-1])^2 + (a_2 x_2[n-1])^2$$

so $y[n]$ is not linear.

Time-invariance test:

We have to show that a delay in $x[n]$ by n_0, $x[n-n_0]$, results in a delay in $y[n]$ to $y[n-n_0]$.

To do this, we start by letting $w[n]$ be the output when the input $x[n]$ is delayed by n_0 giving

$$w[n] = (x[n-1-n_0])^2$$

Then if we delay $y[n]$ by n_0, we obtain

$$y[n-n_0] = (x[n-n_0-1])^2$$

We can see that

$$y[n-n_0] = w[n]$$

so $y[n]$ is a time-invariant system.

Causal: $y[n]$ is causal since it depends only on $x[n-1]$ which is in the past.

b) $y_1[n] = (e^{j0.75\pi(n-1)} + e^{-j0.75\pi(n-1)})^2 = e^{j1.5\pi(n-1)} + e^{-j1.5\pi(n-1)} + 2e^{j0}$

$\qquad = 2\cos(1.5\pi(n-1)) + 2$

Notice that this sampled sinusoid has a frequency outside of the range $[-\pi, \pi]$ that a D-to-C converter assumes for signal reconstruction. This signal will therefore result in the folded alias as follows

$$y_1[n] = 2\cos((-0.5)\pi(n-1)) + 2 = 2\cos(0.5\pi(n-1)) + 2$$

instead of the desired signal.

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PROBLEM:

Circle the correct answer to each of these short answer questions:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 8\cos(0.3\pi n - \pi/4)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 20 samples/second.

   (a) $f = 3$ Hz

   (b) $f = 6$ Hz

   (c) $f = 17$ Hz

   (d) $f = 34$ Hz

   (e) $f = 0.3$ Hz

   (f) $f = 0.15$ Hz

2. A signal $x(t)$ is defined by: $x(t) = \sum_{k=-50}^{50} k^2 e^{j2\pi kt}$. The Nyquist Rate for sampling $x(t)$ is

   (a) 1 Hz

   (b) 2 Hz

   (c) 25 Hz

   (d) 50 Hz

   (e) 100 Hz

3. For the following MATLAB code: `yy = firfilt( [0,1,2,0,-5], xx )`
   pick the correct difference equation for the filter being implemented.

   (a) $y[n] = \delta[n]$

   (b) $y[n] = x[n] + 2x[n-1] - 5x[n-2]$

   (c) $y[n] = x[n] + 2x[n-1] - 5x[n-3]$

   (d) $y[n] = x[n-1] + 2x[n-2] - 5x[n-3]$

   (e) $y[n] = x[n-1] + 2x[n-2] - 5x[n-4]$

4. If $\mathcal{H}(\hat{\omega})$ is the frequency response of a digital filter, and the input is $x[n] = 5 + 7\cos(0.3\pi n)$, then a concise way to define the output is:

   (a) $y[n] = \mathcal{H}(0.3\pi)(5 + 7\cos(0.3\pi n))$

   (b) $y[n] = \Re e\{5 + 7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$

   (c) $y[n] = \Re e\{7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$

   (d) $y[n] = \Re e\{5\mathcal{H}(0) + 7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$

   (e) $y[n] = 5\mathcal{H}(0) + 7\mathcal{H}(0.3\pi)\cos(0.3\pi n)$

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SOLUTION

Circle the correct answer to each of these short answer questions:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 8\cos(0.3\pi n - \pi/4)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 20 samples/second.

   (a) $f = 3$ Hz
   (b) $f = 6$ Hz
   (c) $f = 17$ Hz
   (d) $f = 34$ Hz
   (e) $f = 0.3$ Hz
   (f) $f = 0.15$ Hz

$$2\pi f_a/f_s = \hat{\omega} \Rightarrow f_a = \frac{\hat{\omega}}{2\pi} f_s$$
$$= \frac{0.3\pi}{2\pi}(20) = 3$$

2. A signal $x(t)$ is defined by: $x(t) = \sum_{k=-50}^{50} k^2 e^{j2\pi kt}$. The Nyquist Rate for sampling $x(t)$ is

   (a) 1 Hz
   (b) 2 Hz
   (c) 25 Hz
   (d) 50 Hz
   (e) 100 Hz

$$k_{max} = 50 \Rightarrow f_{MAX} = \frac{2\pi(50)}{2\pi} = 50\text{ Hz}$$
$$f_s \geq 2f_{MAX} \Rightarrow f_s = 100 \text{ samples/sec}$$

3. For the following MATLAB code: `yy = firfilt( [0,1,2,0,-5], xx )` pick the correct difference equation for the filter being implemented.

   (a) $y[n] = \delta[n]$
   (b) $y[n] = x[n] + 2x[n-1] - 5x[n-2]$
   (c) $y[n] = x[n] + 2x[n-1] - 5x[n-3]$
   (d) $y[n] = x[n-1] + 2x[n-2] - 5x[n-3]$
   (e) $y[n] = x[n-1] + 2x[n-2] - 5x[n-4]$

$$b_k = \{0, \underset{b_1}{1}, \underset{b_2}{2}, 0, \underset{b_4}{-5}\}$$

4. If $\mathcal{H}(\hat{\omega})$ is the frequency response of a digital filter, and the input is $x[n] = 5 + 7\cos(0.3\pi n)$, then a concise way to define the output is:

   (a) $y[n] = \mathcal{H}(0.3\pi)(5 + 7\cos(0.3\pi n))$
   (b) $y[n] = \Re e\{5 + 7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$
   (c) $y[n] = \Re e\{7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$
   (d) $y[n] = \Re e\{5\mathcal{H}(0) + 7\mathcal{H}(0.3\pi)e^{j0.3\pi n}\}$
   (e) $y[n] = 5\mathcal{H}(0) + 7\mathcal{H}(0.3\pi)\cos(0.3\pi n)$

Mult by $\mathcal{H}(0)$

Mult by $\mathcal{H}(0.3\pi)$

$\mathcal{H}(0.3\pi)$ could be complex

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PROBLEM:

Circle the correct answer to each of these short answer questions:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 10\cos(0.2\pi n - \pi/3)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 2000 samples/second.

   (a) $f = 3200$ Hz

   (b) $f = 1600$ Hz

   (c) $f = 400$ Hz

   (d) $f = 200$ Hz

   (e) $f = 0.3$ Hz

   (f) $f = 0.15$ Hz

2. A signal $x(t)$ is defined by: $x(t) = \cos(2\pi t)\cos(10\pi t)$. The Nyquist Rate for sampling $x(t)$ is

   (a) 1 Hz

   (b) 2 Hz

   (c) 10 Hz

   (d) 11 Hz

   (e) 12 Hz

3. If $\mathcal{H}(\hat{\omega})$ is the frequency response of a digital filter, and the input is $x[n] = 16 + 4\cos(0.2\pi n)$, then a concise way to define the output is:

   (a) $y[n] = 16 + 4\mathcal{H}(0.2\pi)\cos(0.2\pi n)$

   (b) $y[n] = \Re e\{16\mathcal{H}(0) + 4\mathcal{H}(0.2\pi)e^{j0.2\pi n}\}$

   (c) $y[n] = 16\mathcal{H}(0) + 4\mathcal{H}(0.2\pi)\cos(0.2\pi n)$

   (d) $y[n] = \mathcal{H}(0.2\pi)(16 + 4\cos(0.2\pi n))$

   (e) $y[n] = \Re e\{16 + 4\mathcal{H}(0.2\pi)e^{j0.2\pi n}\}$

4. For the following MATLAB code: `yy = firfilt( [1,-5,0,3], [1,0,0,0,0,0] )` pick the correct mathematical formula for the output signal.

   (a) $y[n] = \delta[n]$

   (b) $y[n] = \delta[n] - 5\delta[n-1] + 3\delta[n-2]$

   (c) $y[n] = \delta[n] - 5\delta[n-1] + 3\delta[n-3]$

   (d) $y[n] = \delta[n-1] - 5\delta[n-2] + 3\delta[n-3]$

   (e) $y[n] = \delta[n-1] - 5\delta[n-2] + 3\delta[n-4]$

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SOLUTION

Circle the correct answer to each of these short answer questions:

1. Suppose that the discrete-time signal $x[n]$ is $x[n] = 10\cos(0.2\pi n - \pi/3)$ determine the frequency (in Hz) of the analog signal $y(t)$ that will be reconstructed by the ideal D-to-C converter operating at a sampling rate of 2000 samples/second.

   (a) $f = 3200$ Hz
   (b) $f = 1600$ Hz
   (c) $f = 400$ Hz
   (d) $f = 200$ Hz
   (e) $f = 0.3$ Hz
   (f) $f = 0.15$ Hz

   $\hat{\omega} = \omega / f_s \Rightarrow f_a = \frac{\omega}{2\pi} = \frac{\hat{\omega}}{2\pi} f_s$

   $f_a = \frac{0.2\pi}{2\pi}(2000) = 200$

2. A signal $x(t)$ is defined by: $x(t) = \cos(2\pi t)\cos(10\pi t)$. The Nyquist Rate for sampling $x(t)$ is

   (a) 1 Hz
   (b) 2 Hz
   (c) 10 Hz
   (d) 11 Hz
   (e) 12 Hz

   Maximum frequency is $10\pi + 2\pi$
   $= 2\pi(6)$

   $f_s \geq 2 f_{MAX} = 2 \times 6 = 12$

3. If $\mathcal{H}(\hat{\omega})$ is the frequency response of a digital filter, and the input is $x[n] = 16 + 4\cos(0.2\pi n)$, then a concise way to define the output is:

   $\mathcal{H}(0)$ $\mathcal{H}(0.2\pi)$

   (a) $y[n] = 16 + 4\mathcal{H}(0.2\pi)\cos(0.2\pi n)$
   (b) $y[n] = \Re e\{16\mathcal{H}(0) + 4\mathcal{H}(0.2\pi)e^{j0.2\pi n}\}$
   (c) $y[n] = 16\mathcal{H}(0) + 4\mathcal{H}(0.2\pi)\cos(0.2\pi n)$ ← Doesn't work if $\mathcal{H}(0.2\pi)$ is complex
   (d) $y[n] = \mathcal{H}(0.2\pi)(16 + 4\cos(0.2\pi n))$
   (e) $y[n] = \Re e\{16 + 4\mathcal{H}(0.2\pi)e^{j0.2\pi n}\}$

4. For the following MATLAB code:

```
yy = firfilt( [1,-5,0,3], [1,0,0,0] )
```

   $[1,0,0,0] \to \delta[n]$

   pick the correct mathmatical formula for the output signal.

   (a) $y[n] = \delta[n]$
   (b) $y[n] = \delta[n] - 5\delta[n-1] + 3\delta[n-2]$
   (c) $y[n] = \delta[n] - 5\delta[n-1] + 3\delta[n-3]$
   (d) $y[n] = \delta[n-1] - 5\delta[n-2] + 3\delta[n-3]$
   (e) $y[n] = \delta[n-1] - 5\delta[n-2] + 3\delta[n-4]$

PROBLEM:

A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n+1])^3. \tag{1}$$

(a) Determine whether or not the system defined by (1) is (i) linear; (ii) time-invariant; (iii) causal.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.6\pi n) = e^{j0.6\pi n} + e^{-j0.6\pi n}.$$

Express your answer in terms of cosine functions. Do not leave any powers of cosine functions in your answers. Note that this system produces output contain frequencies that are not present in the input signal. Explain how this system might cause "aliasing" of sinusoidal components of the input.

See Problem 6.4 of Problem Set #6, Fall 2000 for a problem like this.

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SOLUTION

a.i) The system is <u>not</u> linear because:

$$x[n] \rightarrow y[n] = (x[n+1])^3$$

$$c\,x[n] \rightarrow y_1[n] = (c\,x[n+1])^3 =$$

$$= c^3 (x[n+1])^3 = c^3 y[n] \neq c\,y[n].$$

ii) The system is time-invariant because:

$$x[n-n_0] \rightarrow y_1[n] = (x[n-n_0+1])^3 =$$

$$= y[n-n_0].$$

iii) The system is <u>not</u> causal because $y[n]$ depends on $x[n+1]$.

b) $$y_1[n] = \left(e^{j0.6\pi n} + e^{-j0.6\pi n}\right)^3 =$$

$$= e^{j1.8\pi n} + 3e^{j1.2\pi n}e^{-j0.6\pi n} +$$

$$+ 3e^{j0.6\pi n}e^{-j1.2\pi n} + e^{-j1.8\pi n}$$

Note that $1.8\pi > \pi$, so:

$$e^{j1.8\pi n} = e^{j(1.8\pi - 2\pi)n} = e^{-j0.2\pi n}$$

Similarly: $e^{-j1.8\pi n} = e^{j0.2\pi n}$

So: $y_1[n] = 2\cos(0.2\pi n) + 6\cos(0.6\pi n)$

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PROBLEM:

Consider the linear time-invariant system described by the difference equation

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] = \sum_{k=0}^{3} x[n-k]$$

(a) Find an expression for the frequency response $H(\hat{\omega})$ of the system.

(b) Show that your answer in (a) can be expressed in the form

$$H(e^{j\hat{\omega}}) = \frac{\sin(2\hat{\omega})}{\sin(\hat{\omega}/2)} e^{-j3\hat{\omega}/2}.$$

(c) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above (or plot it using `freqz( )`).

(d) Suppose that the input is

$$x[n] = 1 + 2\cos(n\hat{\omega}_0) \text{ for } -\infty < n < \infty$$

Find a non-zero frequency $0 < \hat{\omega}_0 < \pi$ for which the ouput $y[n]$ is a constant for all n, i.e.,

$$y[n] = c \qquad \text{for } -\infty < n < \infty$$

and find the value for c. (In other words, the sinusoid is removed by the filter.)

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SOLUTION

a) $\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{3} e^{-jk\hat{\omega}} = \dfrac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}}$

(using again $\sum_{k=0}^{N} x^k = \dfrac{1-x^{N+1}}{1-x}$, or simply

set $z = e^{j\hat{\omega}}$ in the expression for $H(z)$)

b) $\mathcal{H}(\hat{\omega}) = \dfrac{e^{-j2\hat{\omega}}}{e^{-j\frac{\hat{\omega}}{2}}} \dfrac{e^{j2\hat{\omega}}}{e^{j\frac{\hat{\omega}}{2}}} \dfrac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}} =$

$= e^{-j\frac{3\hat{\omega}}{2}} \dfrac{e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}}{e^{j\frac{\hat{\omega}}{2}} - e^{-j\frac{\hat{\omega}}{2}}} =$

$= e^{-j\frac{3\hat{\omega}}{2}} \dfrac{2j\sin 2\hat{\omega}}{2j\sin\frac{\hat{\omega}}{2}} = \dfrac{\sin(2\hat{\omega})}{\sin\frac{\hat{\omega}}{2}} e^{-j\frac{3\hat{\omega}}{2}}$

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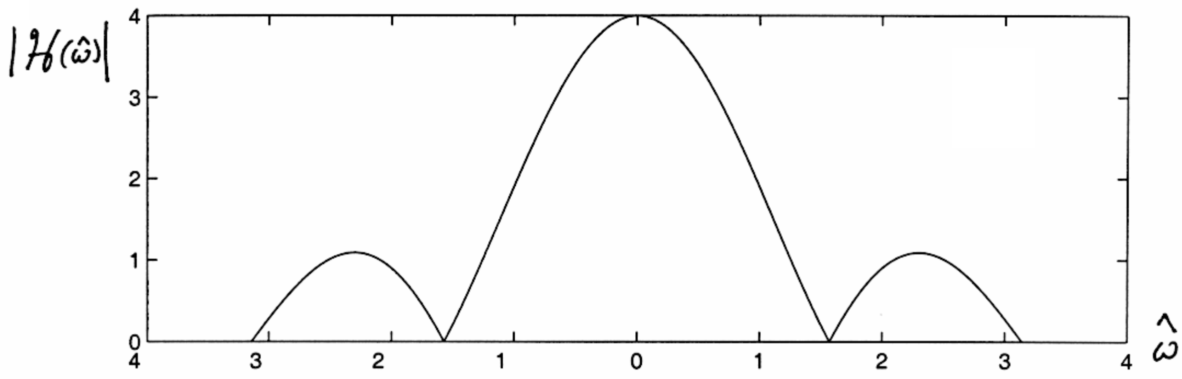
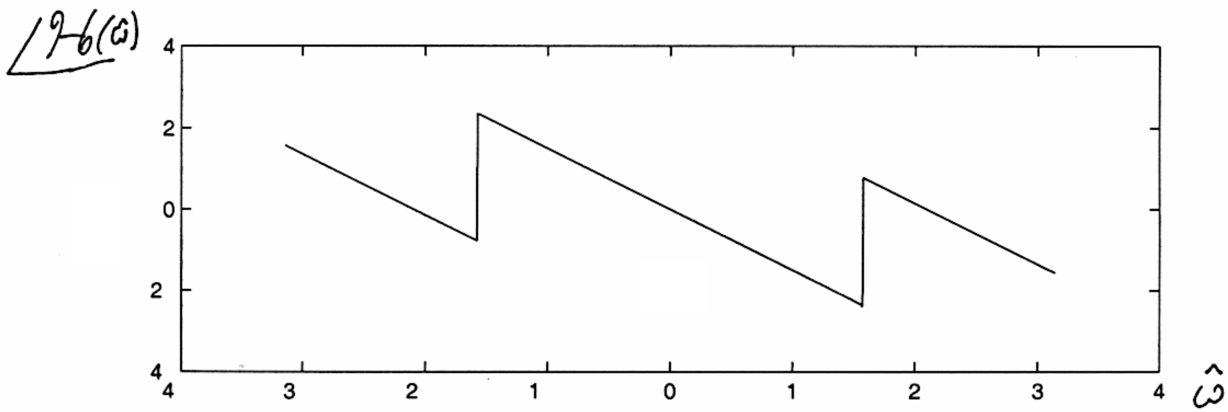
c)





d) The sinusoid is removed by the filter if $\mathcal{H}(\hat{\omega}_0) = 0$. This means we have to find those values of $\hat{\omega}_0$, $0 < \hat{\omega}_0 \leq \pi$ such that $\sin(2\hat{\omega}_0) = 0$ ($\hat{\omega}_0 = 0$ is excluded because $\mathcal{H}(0) = 4$).

Since $\sin\theta = 0 \iff \theta = m\pi, \quad m = 0, \pm 1, \pm 2, \ldots$

we must have $2\hat{\omega}_0 = m\pi$, i.e. $\hat{\omega}_0 = \frac{\pi}{2}$ or

$\hat{\omega}_0 = \pi$ (other values for m give values for $\hat{\omega}_0$ that are outside the specified interval).

For those values of $\hat{\omega}_0$, the output is:

$$y[n] = \mathcal{H}(0) \cdot 1 = 4.$$

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