

The solution based on a Theorem published in a book named “combinatorial optimization polyhedra and efficiency” (page111–page112).
<http://www.springer.com/mathematics/applications/book/978-3-540-44389-6>

Let $D = (V, A)$ be a directed graph (with n vertices) and let $l : A \rightarrow \mathbb{R}$. The *mean length* of a directed cycle (directed closed walk) $C = (v_0, a_1, v_1, \dots, a_t, v_t)$ with $v_t = v_0$ and $t > 0$ is $l(C)/t$. Karp [1978] gave the following polynomial-time method for finding a directed cycle of minimum mean length. For each $v \in V$ and each $k = 0, 1, 2, \dots$, let $d_k(v)$ be the minimum length of a walk with exactly k arcs, ending at v . So for each v one has

$$(8.7) \quad d_0(v) = 0 \text{ and } d_{k+1}(v) = \min\{d_k(u) + l(a) \mid a = (u, v) \in \delta^{\text{in}}(v)\}.$$

Now Karp [1978] showed:

Theorem 8.10. *The minimum mean length of a directed cycle in D is equal to*

$$(8.8) \quad \min_{v \in V} \max_{0 \leq k \leq n-1} \frac{d_n(v) - d_k(v)}{n - k}.$$

Proof. We may assume that the minimum mean length is 0, since adding ε to the length of each arc increases both minima in the theorem by ε . So we must show that (8.8) equals 0.

First, let minimum (8.8) be attained by v . Let P_n be a walk with n arcs ending at v , of length $d_n(v)$. So P_n can be decomposed into a path P_k , say, with k arcs ending at v , and a directed cycle C with $n - k$ arcs (for

some $k < n$). Hence $d_n(v) = l(P_n) = l(P_k) + l(C) \geq l(P_k) \geq d_k(v)$ and so $d_n(v) - d_k(v) \geq 0$. Therefore, (8.8) is nonnegative.

To see that it is 0, let $C = (v_0, a_1, v_1, \dots, a_t, v_t)$ be a directed cycle of length 0. Then $\min_r d_r(v_0)$ is attained by some r with $n - t \leq r < n$ (as it is attained by some $r < n$ (since each circuit has nonnegative length), and as we can add C to the shortest walk ending at v_0). Fix this r .

Let $v := v_{n-r}$, and split C into walks

$$(8.9) \quad \begin{aligned} P &:= (v_0, a_1, v_1, \dots, a_{n-r}, v_{n-r}) \text{ and} \\ Q &:= (v_{n-r}, a_{n-r+1}, v_{n-r+1}, \dots, a_t, v_t). \end{aligned}$$

Then $d_n(v) \leq d_r(v_0) + l(P)$, and therefore for each k :

$$(8.10) \quad d_k(v) + l(Q) \geq d_{k+(t-(n-r))}(v_0) \geq d_r(v_0) \geq d_n(v) - l(P).$$

This implies $d_n(v) - d_k(v) \leq l(C) = 0$. So the minimum (8.8) is at most 0. ■

We can find an algorithm of time complexity $O(n^3)$.