FREE-FALL TIME IN QUANTUM THEORY

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Abstract

This project aims to understand the concept of free fall and discuss this motion in the quantum universe. We, as a team of three, study "Classical/Quantum Dynamics in a Uniform Gravitational Field: A. Unobstructed Free Fall" by Nicholas Wheeler from Reed College Physics Department (August, 2002). This report summarizes the idea of free fall action and discusses several methods such as Hamiltonian and Lagrangian to solve complex problems in a simpler way. Since our project is based on discussing and studying this specific article in our weekly meeting with Professor Durmuş Ali Demir, we try to summarize what we understand while working on this topic and aim to explain the topic in a compact way by giving important calculations and plots from the article.

Keywords: quantum, modern physics, free fall, mechanics.

1. Introduction

Science proves that macroscopic objects fall to the ground at the same time when released from the same height with zero initial velocity in a frictionless system. Their duration of falling, what we call as the free-fall time is universal; we see this universality from Galileo's Pisa Tower experiment. In his experiment Galileo proved that the acceleration of a body near the Earth's surface is independent of its mass, he disproved the Aristotelian theory which claims that the speed of a falling object is proportional to its mass (Adler & Coulter, 1978). Furthermore, this universality is what underlies the geometric theory of gravitation, the General Relativity. But the interesting point here is, that this universality above does not hold for quantum particles. The reason for that can be counted as the appearance of the Planck constant as a parameter of the mass-velocity-distance dimension.

In this project, as a team of three, we will analyze and dive into the Schrodinger equation for a subatomic particle in the concept of free-fall and try to guess the exact cause of the break in universality. In addition, we will use Fisher's information on quantum probability distribution to determine under what conditions the universality is broken.

While studying about free-fall, our professor leads our team with one main single source which is "Classical/Quantum Dynamics in a Uniform Gravitational Field: A. Unobstructed Free Fall" by Nicholas Wheeler from Reed College Physics Department (August, 2002). Wheeler's article specifically discussed our project topic and explain all the steps through to the solution with detailed calculations. He started his discussion with one-dimensional dynamical systems and goes on with the classical dynamics of free fall from Galileo and Newton. After discussing the classical theory, the article dives into the concept of quantum motion in a uniform gravitational field (Wheeler, 2002). Lagrangian and Hamiltonian approaches to this topic play the main roles while calculating and proving the actions as Wheeler does in his article (Wheeler, 2002).

Since it is a need to solve problems with step-by-step proofs in science it is a must to use plots while showing and comparing the results. That is why in our project we focused on graphs and use MATLAB or Wolfram Alpha to use the benefits of technology in science. Since two of us come from engineering departments we used our engineering background while plotting and discussing the results. Due to the limited time, we had, we could not be able to come to the end of the article but with the help of our Professor Durmuş Ali Demir, we highlighted the main concepts and used our previous basic knowledge about quantum to understand the free fall from Schrodinger's view. The detailed information on our discussions and plots can be found in the following sections of the report.

2. Classical Dynamics of Free Fall: What Galileo and Newton say about the problem?

To be familiar with the dynamics of free fall in quantum mechanics, we first started with the classical dynamics of free fall and discussed what were the scientists' general ideas about free fall in the classical era. Working in one dimension we started with the 2 nd law of Newton.

$$F(x) = m\ddot{x}$$

In this equation F(x) is the force applied on the particle, m is the mass of the particle and a is the acceleration of the article For a particle in a gravitational field this equation becomes;

$$m_g g = ma$$

Here m_g symbolizes a so called "gravitational charge" (charge used in reference to the coulomb interaction) As it is the law of our universe, m_g is equal to the inertial mass of our particle. Using this our equation becomes,

$$g = \ddot{x}$$

In order to obtain x as a function of time we take the integral of both sides twice. Also using the fact that g is a downward facing vector in 1-d space to convert the vectorial equation to an equation of magnitudes, we obtain,

$$-gt^2/2 + bt + a = x(t)$$

Here a and b are constants of integration. b is the initial velocity of the particle and a is the initial position of the particle. We got the primitive solution into spacetime translation using the general solution where a and b are equal to 0 .

$$x = \frac{-1}{2} g t^2$$

if we change $x - x_0$ for $x, t - t_0$ for t we obtain,

$$(x - x_0) = \frac{-1}{2} g (t - t_0)^2$$

which is equal to the formula that we are familiar with;

$$x = \left(x_0 - \frac{1}{2} g t_0^2\right) + (g t_0) t - \frac{1}{2} g t^2$$

Which shows us that we can transform the free fall equations on space and time.

3. Lagrangian, Hamiltonian and 2-Point Action

Hamilton's principle of least action (used as stationary action in some cases) states that a particle will always follow the path where action is stationary. Although action is a functional, in order to get an intuitive understanding, we can imagine that it is a function of a single variable. Using this analogy, particle would take the path where the action function's first derivative is zero. But in our case this process requires variational principles. To start,

$$S = \int_{t_0}^{t_1} L(q, \dot{q}, t)$$

Where S is action, q_i and $\dot{q}_i$ are generalized position and velocity respectively. In this paper we will do a 1-dimensional analysis so q_i and $\dot{q}_i$ are just q and $\dot{q}$.

As we discussed we need to minimize the action. In order to do so we must take its variation. This is analogous to taking the differential of a function in single variable calculus. This process is as follows.

We vary the path by some vdel (q) so we replace q by vdel $(q) + q$. $\delta(q)$ is an arbitrary small function.

$$S[q + \delta q] = L[q + \delta q, \dot{q} + \delta \dot{q}, t]$$

Subtracting S from both sides results in, $\delta S = \delta \int_{t_0}^{t_1} L(q, \dot{q}, t) dt$

Taking the variation of the Lagrangian gives,

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\right) dt$$

And using integration by parts to integrate 2nd term,

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\right) dt + \left(\frac{\partial L}{\partial \dot{q}} \delta q\right)\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}) dt}\right.$$

2nd term is zero since start and end points are fixed. (we do not vary the end points with respect to time since those are fixed by the problem.)

$$S = \int_{t_0}^{t_1} \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) dt$$

Since variation of q is an arbitrary small function in order for variation of action to be zero the rest of the integrand must be equal to zero. This provides,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

This is the Euler Lagrange equation. For a given Lagrangian a particle must always have an equation of motion that satisfies the equation above.

Lagrangian for a particle in a gravitational field is,

$$\frac{1}{2} m\ddot{x} - mgx = L$$

And the Hamiltonian is defined as,

$$\frac{1}{2} m\ddot{x} + mgx = H$$

Hamiltonian can be obtained via legendre transformation from the Lagrangian. On 1-d case this transformation becomes,

$$p\dot{x} - L = H$$

p is the momentum of the particle and it is defined as,

$$p = \frac{\partial L}{\partial \dot{x}}$$

Coming back to action and substituting $x(t)$ and L obtained above in to the action equation we obtain,

$$S[x(t)] = \int_{t_0}^{t_1} L(\dot{x}(t), x(t)) dt$$

$$S\left[a + bt - \frac{1}{2}gt^2\right] = \int_{t_0}^{t_1} L\left(b - gt, a + bt - \frac{1}{2}gt^2\right) dt = m\left[\begin{array}{r}\left(\frac{1}{2}b^2 - ag\right)(t_1 - t_0) - bg(t_1^2 - t_0^2) \\ +\frac{1}{3}g^2(t_1^3 - t_0^3)\end{array}\right]$$

With appropriate transformations that were used above we finally obtain the free fall action as,

$$S(x_1, t_1; x_0, t_0) = \frac{1}{2} m\left[\frac{(x_1 - x_0)^2}{t_1 - t_0} - g(x_0 + x_1)(t_1 - t_0) - \frac{1}{12} g^2 (t_1 - t_0)^3\right]$$

In order to solve Schrodinger's equation, we will use a technique called separation of variables. This technique is based on the assumption that the unknown function of the partial differential equation can be written in the following form (for 1-d wave function with a time independent potential),

$$\psi(x,t) = X(x)T(t)$$

This assumption helps us to reduce the partial differential equation to two ordinary differential equation. Substituting our separated wave function into the Schrodinger equation we obtain,

$$i\hbar\left(\frac{dT}{dt}\right)X = \left(\frac{-\hbar^2}{2m}\right)\left(\frac{d^2X}{dx^2}\right)T + mgxX(x)T(t)$$

Dividing both sides by X(x)T(t) we obtain,

$$i\hbar\left(\frac{dT}{dt}\right)/T = \left(\frac{-\hbar^2}{2m}\right)\left(\frac{d^2X}{dx^2}\right)/X + mgx$$

Looking at LHS we see only t dependent functions. Looking at the RHS we only see x dependent functions this means that both sides are equal to a constant independent of both t and x. In our case this constant is the energy. We will denote the energy of the particle as E.

Right hand side provides,

$$\frac{-\hbar^2}{2m}\frac{d^2X}{dx^2} + mgxX = EX$$

Left hand side provides,

$$i\hbar\frac{dT}{dt} = ET$$

Using $\frac{d}{dt}\ln\left(f(t)\right) = \frac{df(t)}{dt}/f(t)$ we obtain,

$$T(t) = e^{\frac{-i}{\hbar}Et}$$

Using this time dependence we obtain,

$$\Psi(x,t) = \psi(x)e^{\frac{-i}{\hbar}Et}$$

4. Quantum Mechanical Free Fall According to Schrodinger

This leaves us with the time independent Schrodinger equation which is,

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + mgx\right)\psi(x) = E\psi(x)$$

Using

$$y = \left(\frac{2m^2 g^{1/3}}{\hbar^2}\right)\left(x - \frac{E}{mg}\right) = k(x-a) \quad \text{and} \quad k = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}$$

we get,

$$\left(\frac{d}{dy}\right)^2 \psi(y) = y\psi(y)$$

where,

$$a = \frac{E}{mg} \quad \text{and} \quad y = z - \alpha$$

Variable a in this equation represents a particle's classical "turning point" of the trajectory which shows the dimensionless state of the turning point with the $\alpha =$ ka formula.

Furthermore, we have used Airy's differential equation on the time-independent equation of Schrödinger to get solutions in the form of linear Airy functions Ai (y) and Bi (y) for a particle in a one dimensional constant force field ("Airy function - Wikipedia", 2022).

$$Ai(y) = \frac{1}{\pi}\int_0^\infty \cos\left(yu + \frac{1}{3}u^3\right) du$$

To have a deeper observation of Airy's construction, we can say

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (g(u)e^{iyu}) du$$

with the above equation we reach: $f'' - yf = 0$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-u^2 g(u) + ig(u)\frac{d}{du}\right] e^{iyu} du = 0$$

Integration by parts gives

$$\frac{1}{2\pi} ig(u)e^{iyu}\,\Big|_{-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} [u^2 g(u) + ig'(u)] e^{iyu} du = 0$$

Leading term disappears if we take $g(\pm\infty) = 0$. Then it is only left with a first-order differential equation $u^2 g(u) + ig'(u) = 0$ of which the general solution is $g(u) = A\,.\,e^{i\frac{1}{3}u^3}$. Thus, we have

$$f(y) = A\,.\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(yu+\frac{1}{3}u^3)} = A\,.\frac{1}{\pi}\int_0^\infty \cos(yu + \frac{1}{3}u^3)\, du$$

We achieve $\int_{-\infty}^{+\infty} Ai(y)dy = 1$ that Airy assigned the value $A = 1$

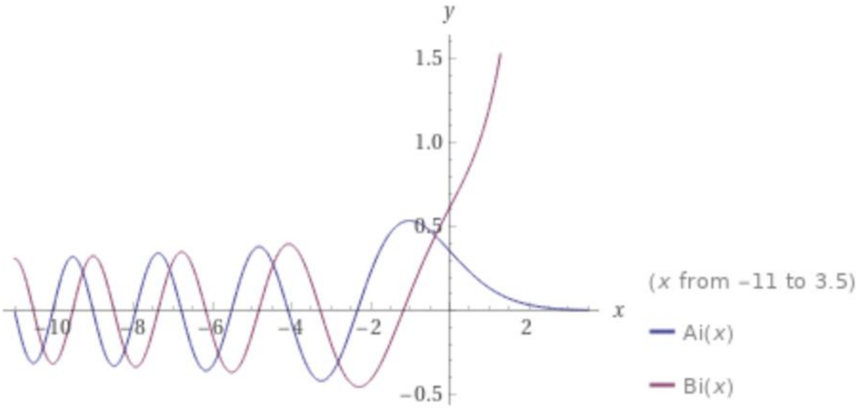

Figure 1: Airy functions Ai(x) Bi(x)

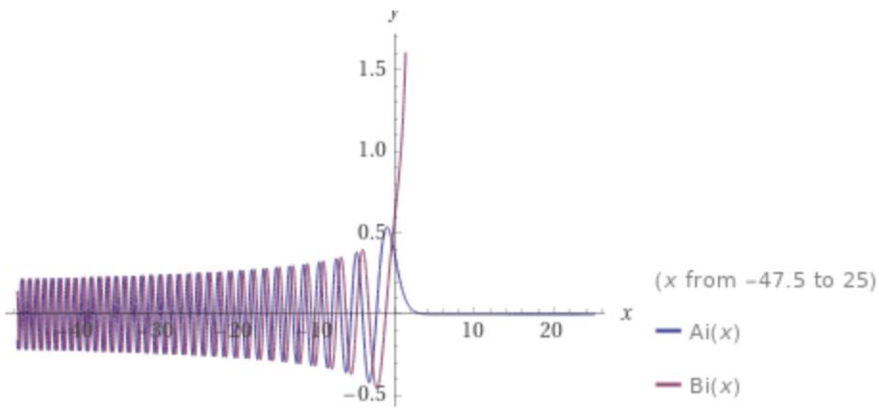

Figure 2: Airy functions Ai(x) Bi(x)

We have displayed the Schrodinger equation solutions in physical form using the $\mathrm{a} = \mathrm{E}/\mathrm{mg}$ we derived,

$$\psi_E(x) = (\text{ normalization factor }) \cdot Ai\big(k(x - a_E)\big)$$

where
$$a_E = \frac{E}{mg}$$

which reduces to,

$$\psi_E(z) = N \cdot Ai(z - \alpha_\epsilon)$$

where N is the normalization factor

Using the eigenfunctions ψ_E, we have realized that all have the same graphical shape and are translated to each other. No ground state can be observed since our potential does not have a point where its first derivative is zero.

$$U(x) = \mathrm{mg}\ x, -\infty < x < \infty$$

The eigenfunctions E(x) are similar with the free particle function in the sense that they need to assemble into wave packets because they are not individually normalizable. The eigenfunctions includes complete orthonormal sets. We can prove this by letting

$$f(z,\alpha) = \mathrm{Ai}\ (z-\alpha)$$

and checking if,

$$\int_{-\infty}^{\infty} f(z,\alpha) f(z,\beta) dz = \delta(\alpha-\beta)$$

and

$$\int_{-\infty}^{\infty} f(y,\alpha) f(z,\alpha) d\alpha = \delta(y-z)$$

In order to do so we will prove the first one and with no loss of generality we will have proved the second one.

In order to do so we will prove the equation below,

$$\int_{-\infty}^{+\infty} Ai(z-\alpha)Ai(z-\beta)dz = \delta(\alpha-\beta)$$

Starting with,

$$\int_{-\infty}^{+\infty} Ai(z-\alpha)Ai(z-\beta)dz$$

we plug in the Airy's function formulas and obtain,

$$= \left(\frac{1}{2\pi}\right)^2 \iiint \ e^{i\left[(z-\alpha)u+\frac{1}{3}u^3\right]} e^{i\left[(z-\beta)v+\frac{1}{3}v^3\right]}$$

collecting the terms with z in one integral we get,

$$\frac{1}{2\pi}\iint \ e^{i\frac{1}{3}(u^3+v^3)} e^{-i(\alpha u+\beta v)} \left[\frac{1}{2\pi}\int \ e^{iz(u+v)} dz\right] dudv$$

After using dirac delta representation of the z integral and setting

$$u = -v$$

we get

$$\frac{1}{2\pi}\int \ e^{i\frac{1}{3}(v^3-v^3)} e^{iv(\alpha-\beta)} dv = \delta(\alpha-\beta)$$

which means that our wave functions are orthogonal.

$$\int_{-\infty}^{+\infty} \psi_{\epsilon}^{*}(z)\psi_{\epsilon''}(z)dz = N^2 \int_{-\infty}^{+\infty} Ai(z-\alpha_{\epsilon'})Ai(z-\alpha\epsilon'')dz$$
$$= N^2\delta(\epsilon' - \epsilon'') = \delta(\epsilon' - \epsilon'')$$

Which also means that they have to obey the following,

$$\int_{-\infty}^{+\infty} \psi_{\epsilon}^{*}(z')\psi_{\epsilon}(z'')d\epsilon = \delta(z' - z'')$$

As the final topic of our work, we will derive the expression for the quantum propagator of our free fall particle.

5. Construction and Structure of the Free Fall Propagator

The general form of the propagator is,

$$K(x_!, t_1; x_0, t_0) = \sum \psi_n(x_1)\psi_n^{*}(x_0)e^{-\frac{i}{n}E_n(t_1-t_0)}$$

Where ψ_n is the eigenfunction for the E_n eigenvalue. This propagator describes how a particle's position evolves in time using some given wavefunction at some initial time and position via the following equation;

$$\psi(x, t_0) \rightarrow \psi(x, t) = \int K(x, t; x_0, t_0)\psi(x_0, t_0)dx_0$$

Substituting our propagator and using dimensionless variables equation above provides the equation below in our case.

$$K(z, t; z_0, 0) = \int_{-\infty}^{+\infty} \psi_{\epsilon}(z)\psi_{\epsilon}^{*}(z_0)e^{i\epsilon\theta}d\epsilon$$

These dimensionless variables are given as such,

$$E = \epsilon_g \cdot \epsilon = \left(mgl_g\right) \cdot \epsilon$$
$$t = \tau_g \cdot \theta = \left(\frac{\hbar}{mgl_g}\right) \cdot \epsilon$$

where,

$$l_g = \left(\frac{1}{k}\right)$$

These provide,

$$\frac{1}{\hbar}Et = \theta$$

Using these new variables in our formula for the time independent wave function we obtain,

$$\psi_\epsilon(x) = Ai(z-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}u^3+[z-\alpha]u\right)}du$$

Now, we substitute this wavefunction into propagator equation which we obtained before,

$$\text{К} \ = \ \left(\frac{1}{2\pi}\right)^2 \iiint e^{\ i\left(\frac{1}{3}u^3+[z-a]u\right)}\, e^{i\left(\frac{1}{3}v^3+[z_0-a]v\right)} e^{-ia\theta}\ dudvda$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiint e^{\ i\left(u^3+v^3\right)} e^{i(zu+z_0v)} e^{-i(u+v+\theta)a} dadudv$$

$$= \left(\frac{1}{2\pi}\right) \iint e^{i\,\frac{1}{3}\left(u^3+v^3\right)}\ e^{i(zu+z_0v)} \delta(u+v+\ \theta) dudv$$

$$= \left(\frac{1}{2\pi}\right) \int e^{i\,\frac{1}{3}\left[v^3-(v+\theta)^3\right]}\ e^{i[z_0v-z(v+\theta)]}dv$$

But $v^3-(v+\theta)^3 = -3v^2\theta-3v\theta^2-\theta^3$ so

$$= \tfrac{1}{2\pi} e^{-i\left(\frac{1}{3}\theta^3+z\theta\right)} \cdot \int e^{-i\theta v^2 - i\left[\theta^2+(z-z_0)\right]}dv$$

Formally the integral is Gaussian, and its elementary evaluation provides

$$\text{К} = \frac{1}{2\pi} e^{-i\left(\frac{1}{3}\theta^3+z\theta\right)} \sqrt{\frac{2\pi}{2i\theta}}\, e^{-\frac{1}{4i\theta}\left[\theta^2+(z-z_0)\right]^2}$$

$$= \sqrt{\frac{1}{4\pi i\theta}}\, exp\left\{i\left[\frac{(z-z_0)^2}{40} + \left[\frac{1}{2}(z-z_0)-z\right]\theta + \left[\frac{1}{4}-\frac{1}{3}\right]\theta^3\right]\right\}$$

This integral is a gaussian. But it requires some work to turn it into that form.

$$\text{К} \ = \ \frac{1}{2\pi} e^{-i(\frac{1}{3}\theta^3+z\theta)} \int e^{\left(-i\theta v^2 - i\left[\theta^2+(z-z_0)\right]v\right)}dv$$

$$\text{К} \ = \ \frac{1}{2\pi} e^{-i(\frac{1}{3}\theta^3+z\theta)} \int e^{\left(-i\theta(v^2-v\left(\theta+\left(\frac{z-z_0}{\theta}\right)\right)\right)}dv$$

$$(v+A)^2 - A^2 = v^2 + v(\theta + \frac{z-z_0}{\theta})$$

$$v^2 + 2Av = v^2 + \theta(\theta + \frac{z-z_0}{\theta})$$

$$A = \frac{1}{2}(\theta + \frac{z-z_0}{\theta})$$

$$К = \frac{1}{2\pi} e^{-i(\frac{1}{3}\theta^3 + z\theta)} . e^{i\frac{\theta}{4}(\theta + \frac{z-z_0}{\theta})^2} \int e^{-i\theta(v + \frac{1}{2}(\theta + \frac{z-z_0}{\theta}))^2} dv$$

$$v + \frac{1}{2}\left(\theta + \frac{z-z_0}{\theta}\right) = w \qquad\qquad dv = dw$$

$$К = \frac{1}{2\pi} e^{-i(\frac{\theta^3}{3} + z\theta)} . e^{i\frac{\theta}{4}(\theta + \frac{z-z_0}{\theta})^2} . \int e^{-i\theta w^2} dw$$

$$\int e^{-i\theta w^2 dw} = \sqrt{\frac{\pi}{i\theta}}$$

$$К = \frac{1}{\sqrt{4\pi i\theta}} e^{-i(\frac{\theta^3}{3} + z\theta)} . e^{i\frac{\theta}{4}(\theta + \frac{z-z_0}{\theta})^2}$$

$$К = \frac{1}{\sqrt{4\pi i\theta}} e^{-i(\frac{\theta^3}{12} + \frac{\theta(z-z_0)}{2} - \frac{(z-z_0)^2}{4\theta})}$$

Substituting $z = kx$ and $\theta = \frac{t}{k\hbar} mg$, then we obtain,

$$К(x_1 t_1; x_0, t_0) = \sqrt{\frac{m}{2\pi\hbar(t_1 - t_0)}} e^{\frac{i}{\hbar} S(x_1 t_1; x_0, t_0)}$$

After obtaining the equation above we realize that the exponent is the free fall action.

6. Discussion and Conclusion

In this report we deeply observed and discussed the Wheeler's article about free fall in quantum. We start analyzing this concept from classical approach and dive into quantum mechanics after using several methods such as Hamiltonian, Lagrangian, Gaussian, Airy functions and Fourier. Since we have a limited time to discuss whole material in a single semester, we could not reach the end of the paper but highlighted the main parts of the article which ends up by reaching the action of the free fall. Through these calculations we showed each part step by step to have a better understanding of the approaches used in quantum. Moreover, to visualize the functions we use Wolfram Alpha and put some plots between the equations. As one last thing, with this research-based project we strengthened our ability to combine causality with our abstract intelligence and widen our perspective of modern physics, thanks to Durmuş Ali Demir Hoca.

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