π Day [Key]

Rancho San Joaquin Middle School ASB

3.14.23

Instructions

All problems below have answers in the set $\{ n \in \mathbb{N} \cup \{0\} \mid 0 \leq n \leq 100 \}$, and are arranged in random order (but all classes still have the same problems). Between Friday, March 10th, and Tuesday, March 14th, you may submit your solutions at this link. For both seventh and eighth grade, the advisement class who submits the most correct answers will receive a pie party. Good luck and happy π Day!

Problems

1. Mrs. Snowden makes pie. She now has so many apple and pumpkin pies, that she puts several of both in a bag. She then pulls out pies, until either the last apple or the last pumpkin pie is chosen. If the current probability for the last pie chosen being apple is $\frac{3}{5}$, but after adding 10 pumpkin pies, the new probability is $\frac{2}{5}$. How many pumpkin pies need to be added to move the probability from $\frac{2}{5}$ to $\frac{1}{5}$?

2. Mrs. Snowden makes pie. She lays out all her pies in rows. If she groups them 20 to a row, she will have n left over. If she groups them n to a row, she will have $\frac{n}{2}$ left over. Furthermore, if she groups them $n-4$ to a row, she will also have $\frac{n}{2}$ left over. Find the smallest possible number of pies she has.

3. Mrs. Snowden makes pie. She makes three types of pies: blueberry, cherry, and watermelon. She sells blueberry pies for $\$7$, cherry pies for $\$4$, and watermelon also for $\$4$. If she sold 28 pies in total and made $\$178$, how many blueberry pies did she sell?

4. Mrs. Snowden makes pie. You have a finite number of nickels, while infinite pennies—but no other denominations of coin. If you can buy a $\$5$ pie with a minimum of 128 coins, how many coins are necessary to buy a $\$10$ pie? Express your answer as the sum of the digits of the number of coins.

5. Mrs. Snowden makes pie. Each apple pie weighs the same as four apples, but by weight, only 12.5% of a pie is actually apple. How many pies can be made with 10 apples and sufficient other ingredients?

6. Mrs. Snowden makes pie. A pie is a cylinder, with a volume equal to three-fifths the surface area. If the radius is two units, and the surface area may be expressed as $a\sqrt[b]{c}\pi$, where $a, b, c \in \mathbb{N}$ find the minimum possible value of $a + b + c$.

7. Mrs. Snowden makes pie. Her pies (two-dimensional disk) each contain one white and one black chocolate chip (points distributed uniformly at random within the pie). In each pie, denote the diameter of the pie through the black chocolate chip by d. Let p be the probability that the distance from the white chip to d is less than the distance to the edge of the pie. Find the sum of the numerator and denominator of p when expressed in simplest form.

8. Mrs. Snowden makes pie. Three types of pie, pineapple, pear, and papaya pie, are put in a bag. Pies are then randomly removed, without replacement. For all three pies, the probability that the pie will be the second type of pie completely removed, is equal. If there are 10 pineapple pies, what is the sum of all possible number of total pies in the bag?

9. Mrs. Snowden makes pie. Two pies of radius 1 are externally tangent to each other, while a third pie is also externally tangent to both. If the sum of the areas of the pies is a hundred times the area of the triangle connecting the three circle centers, how many solutions are there for the radius of the third pie?

10. Mrs. Snowden makes pie. She also delivers pie; today, Mrs. Snowden begins at the origin and makes deliveries to the points $(1, 2)$ and $(2, 1)$, in any order, before returning back to the origin. Mrs. Snowden can only travel vertically or horizontally, one unit at a time, and can only move toward her next stop. How many paths are possible?

11. Mrs. Snowden makes pie. She has numbered each of the dozen slices of a circular pie from 1 to 12, clockwise, in that order. She first eats the slice numbered 12, so that there are now 11 slices and a gap. At each step, the gap can be filled by an adjacent slice (each of the two adjacent slices have an equal likelihood of being that slice), but a new gap appears. For example, in the first step, with probability $\frac{1}{2}$, the slice numbered 11 could move to occupy the gap, thus moving the gap counterclockwise. When the gap first reaches the slice numbered 6, what is the expected value of the sum of the number of times that the slices numbered 5 or 7 were reached? Round your answer to the nearest integer.

12. Mrs. Snowden makes pie. A person is considered fed when they receive a nonzero portion of pie. If she is to feed 280 total people, and makes four planar cuts per solid right circular cylindrical pie, how many pies are necessary?

13. Mrs. Snowden makes pie. Her pies are so tasty that she even participates in live pie cook-offs. Each of her 15 pies is judged against each of her her opponent's 15 pies. A pie is considered *delicious* if it wins against at least 10 opposing pies. What is the maximum possible number of *delicious* pies?

14. Mrs. Snowden makes pie. People enjoy her pie so much, that you, the last person in line, see 2023 people before you, all waiting to buy pie. Pie can contain apples or no apples, and each person in line has a 50% chance of buying a pie with apples. However, Mrs. Snowden only has enough apple pies to sell 1012 of them. If you really want to buy apple pie, and will do so if they are not already sold out, what is $a + b$, where $\frac{a}{b}$ is the probability that you will buy apple pie, and $a, b \in \mathbb{N}, \gcd(a, b) = 1$?

15. Mrs. Snowden makes pie. Her first pie has a volume of π. However, with each successive pie, the volume decreases by a factor of 10. What is the total volume of all her pies? Round your answer to the nearest integer.

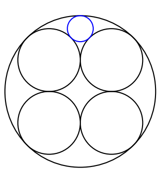
16. Mrs. Snowden makes pie. She bakes one per day. If she starts baking on Thursday, $\sqrt{5}$ Day (February 24th), how many will she have after Tuesday, π Day, March 14th? The year is not necessarily 2023.

17. Mrs. Snowden makes pie. The amount of pie baked each day may be modelled by a cubic polynomial. If she bakes one on Monday, two on Tuesday, four on Wednesday, and eight on Thursday, how many are baked on Friday?

18. Mrs. Snowden makes pie. A pie is composed of both a plate, which has a constant mass, and the edible portion, which can decrease. Currently, no slices have been eaten. If she were to eat five sixth of the edible portion of the pie, or a thousand grams, the mass of the entire pie would decrease by a factor of 5. Furthermore, eating all the remaining slices would again decrease the mass by a factor of 5. Now with just the pan left, subtracting the mass of three more slices, would increase the mass by a factor of -5 (the mass would go negative). How many slices are there?

19. Mrs. Snowden makes pie. Three types of pie cost different amounts, in an arithmetic progression. Once tip is applied (a constant proportional increase), the cost of the pies now form a geometric progression. However, once tax is applied after tip (another constant proportional increase), the costs are again in an arithmetic progression. How many distinct pairs of tax and tip exist?

20. Mrs. Snowden makes pie. A pie is a disk. Mrs. Snowden only has a single pie with a diameter of 12 inches, but wants to create more pies by cutting, to make it look like she did more work. She first creates four pies, each internally tangent to the large pie, and externally tangent to two others, all of equal radii. She then cuts out a sixth pie, externally tangent to the first pie and two of the second through fifth pies that themselves are tangent to each other. If the diameter of the first pie is 12 inches, and given that the diameter of the sixth pie may be expressed in the form $a + b\sqrt{p}$, where a and b are integers, and p is a prime, what is the value of $a + b + p$?



21. Mrs. Snowden makes pie. To prevent cooling, she places them under a heat lamp, which will heat any pie sufficiently if completely within a distance of 6.5 units. However, the lamp is placed 6 units above the ground. How many full pies (disks of one unit in diameter) on the ground can be heated at one time?

22. Mrs. Snowden makes pie. If she starts the day with 60 pies, but then sells three-quarters of them, how many more did she sell than keep?

23. Mrs. Snowden makes pie. She currently has 36 apple pies, 48 pineapple pies, 20 strawberry pies, 16 cherry pie, and 24 pumpkin pies. If she chooses 80 pies to serve at a feast, with each pie being equally likely, how many are expected to be apple?

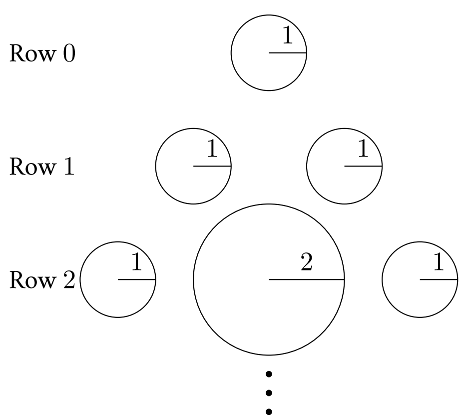
24. Mrs. Snowden makes pie. When selling the pie, the total price includes tax and tip (both multiplicative). Tip is 3.75 times the percentage of tax (e.g., if tax were 10%, then tip would be 37.5%). If only tax is added to the subtotal, the price would be $\$16$, but if only tip were added, the price would be $\$18.75$. What is the price, in dollars, when both tax and tip are added?

25. Mrs. Snowden makes pie. She has 3 pumpkin pies and 19 pineapple pies, and pies of the same type are indistinguishable. If she randomly arranges the pie in a line, what is the expected number of distinct locations where a fourth pumpkin pie could be placed? Placing a pumpkin pie in front of a pumpkin pie is not distinct from placing it after.

26. Mrs. Snowden makes pie. One of her pies has a radius of 10 and an area A. When she expresses A in scientific notation with a base of b, she gets $a \times b^n$, where $n \in \mathbb{Z}$ and $1 \le a < b$. There is a number $a_{\text{max}} \in \mathbb{R}$ such that for all a, where $b \in \mathbb{Z}$ and $b < A$, the inequality $a < a_{\text{max}}$ holds. If a_{max} can be expressed as $a\pi^{\frac{1}{b}}$, where a is an integer, find $a + b$.

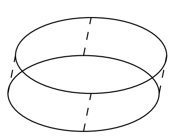
27. Mrs. Snowden makes pie. A pie contains 10 congruent slices, each a 36° sector of a circle, arranged in that circle. If someone randomly eats a subset of these pies, what is the probability that no remaining slice shares an edge with any other remaining slice? If your answer can be expressed as $\frac{a}{b}$, where a and b are integers with no common factor other than 1, find the sum of the digits of $a + b$.

28. Mrs. Snowden makes pie. She arranges them in Pie-scal's Triangle, which has the structure of Pascal's Triangle, except each value n of the triangle is replaced with a circular pie of radius n units, instead. What is the total perimeter of the pies in Row Five of Pie-scal's Triangle? Given that the answer is of the form $a\pi$, where a is an integer, find a.

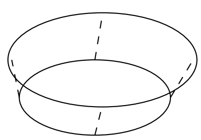


29. Mrs. Snowden makes pie. The pie is the region bounded between two horizontal planes of an inverted right-circular cone. Two cross-sections are created, one vertical and one horizontal, both passing through the point which is the intersection of the vertical center and the axis of the cone. The ratio of the vertical cross-section area to the horizontal cross-section area is $1 : 20\pi$. What is the ratio of the height of the pie to the average horizontal radius of the pie?

30. Mrs. Snowden makes pie. To optimize the process, she first creates a huge cone of pie, and then makes 24 cuts perpendicular to the axis, for equally thick pies. How many of those pies are actually pie-shaped (a ratio of greater to lesser base areas between 100% and 144%, inclusive)?



100%



144%

31. Mrs. Snowden makes pie. A pie is made with two ounces of flour. If she has five pounds of flour, and sufficient other ingredients, how many pies can she make?

32. Mrs. Snowden makes pie. When approximating the pie to be a polyhedron, it turns out that the pie has 182 faces and 200 edges. How many vertices does it have?

33. Mrs. Snowden makes pie. A pie can contain apples or no apples (two equally likely options), and cinammon or no cinammon (also both equally likely). However, a pie with cinammon can not have no apples (she first determines whether to add cinammon, and based on that result, determines whether there are apples). If adding apples contributes 4 grams of sugar, adding cinammon contributes 8 grams of sugar, and everything else is already 6 grams of sugar, what is the expected amount of sugar in grams in a pie?

34. Mrs. Snowden makes pie. She makes 2 cylindrical pies. One pie has a diameter of 3 feet and a height of $\frac{1}{2}$ foot, while the other has a radius of 4 inches and a height of 2 inches. What is the maximum number of small pies she can fit in the large pie, if she can reshape pies?

35. Mrs. Snowden makes pie. The number of pies of certain, not necessarily distinct types of her pies, forms the set $\{n+1, n+2, 8, n+4, n+5\}$, where n is an integer. Given that she has no other pies, and that the median number of each type of pie is 7, find the sum of all possible values of n.

36. Mrs. Snowden makes pie. She bakes P pies. If $\pi^P < P^\pi$, how many pies did she bake?

37. Mrs. Snowden makes pie. A pie is circular, and slices are formed by creating lines through the center. When Mrs. Snowden serves eight people, four cuts are necessary, and each will be chosen by randomly selecting a point on the edge, uniformly at random, and drawing the diameter through that point. Given that the probability that no single person eats more than a fourth of the pie can be expressed in lowest terms as $\frac{a}{b}$, find $a+b$.

38. Mrs. Snowden makes pie. A pie costs $\$10$ per pound. I have $\$800$, and want to buy as many one-pound pies as possible. However, I feel scammed, and only want to pay for the 80% that is actually edible. How many more pies could I get this way?

39. Mrs. Snowden makes pie. However, her oven is special, and she can only increase or decrease the temperature by 1 degree Fahrenheit or double the temperature in degrees Fahrenheit. Given that the current temperature is 70 degrees Fahrenheit, there is a temperature between 360 and 555 degrees Fahrenheit that requires the fewest number of operations to reach. Find the sum of the digits of that temperature in degrees Fahrenheit.

40. Mrs. Snowden makes pie. We also happen to be playing chess. I receive a pie for every win, of probability $\frac{2}{3}$, but lose a pie otherwise. If we play until one person reaches 40 wins, how many pies am I expected to have? Round your answer to the nearest whole number of pies.

Answer Key

1. 30
2. 56
3. 22
4. 16
5. 20
6. 22
7. 7
8. 90
9. 2
10. 36
11. 2
12. 20
13. 22
14. 3
15. 3
16. 20
17. 15
18. 12
19. 1
20. 14
21. 19
22. 30
23. 20
24. 20
25. 20
26. 12
27. 13
28. 64
29. 20
30. 20
31. 40
32. 20
33. 13
34. 60

35. 12
36. 3
37. 7
38. 20
39. 12
40. 20

Solutions

1. Let there be a apple pies and p pumpkin pies. We note that the probability of choosing the last apple pie in the bag, instead of the last pumpkin pie in the bag, is equivalent to $\frac{a}{a+p}$. In other words, we have $\frac{a}{a+p} = \frac{3}{5}$, and $\frac{a}{a+p+10} = \frac{2}{5}$. Cross-multiplying both equations, we have $5a = 3a + 3p$ and $5a = 2a + 2p + 20$. Therefore, $a + p = 20$, $a = 12$, and $p = 8$, which solves the equation. However, we want to know how many pumpkin pies would have to be added to move the probability to $\frac{1}{5}$. In other words, we want $a + p = 60$, so we need 30 pumpkin pies added.

2. Let us first allow the number of pies to be N. Our three sentences correspond to the following statements.

$$\begin{aligned} N &\equiv n && \pmod{20} \\ N &\equiv \frac{n}{2} && \pmod{n} \\ N &\equiv \frac{n}{2} && \pmod{n-4} \end{aligned}$$

   We now have cases on n. First of all, we can immediately ignore all odd cases, because we will have an integer number of pies remaining, when grouping in rows of $n-4$, which itself is an integer. Furthermore, $n > 4$, because it is stated that we can group the pies in rows of $n-4$, and we can not group a negative or 0 number of pies together. Now, for $n \equiv 2 \pmod{4}$, we see that $\frac{n}{2}$ is odd, but both N and n are even, so clearly the second equation is not satisfied. Therefore, $n \equiv 0 \pmod{4}$. Next, for $n = 8$, we have $\frac{n}{2} = n - 4$, so clearly the third equation can not be satisfied. For $n = 12$, it happens that N is a multiple of 4, by the first equation, so then we have $4k = 6 \pmod{8}$, which can not possibly be satisfied. Therefore, $n = 16$. Checking the first few cases, $N = 16, 36, 56, \ldots$, we see that the lowest possibility is 56.

3. We simply solve by a system of equations. We can represent watermelon and cherry pies with the same variable, because they cost the same amount, and we don't care about the precise number for either.

4. It would take 100 nickels all by themselves to pay for the \$5 pie, but we require more, which means we've already run out of nickels. Thus, the other \$5 dollars is composed of pennies entirely, so we add exactly 500 pennies, bringing the total count up to 628.

5. A single pie contains $4 \times 12.5\% = \frac{1}{2}$ of an apple. So, 10 apples generates $10 \cdot \frac{1}{\frac{1}{2}} = 20$ pies.

6. We simply apply the formulae for cylindrical surface area and volume. Our answer is 20π. Clearly, we want to express our answer as $20\sqrt[1]{1}\pi$, to minimize the sum.

7. Note that p is simply the area of the satisfying region divided by the area of the disk. In other words, given a diameter d already placed, we find all satisfactory locations for the white chocolate chip. This area motivates integration. We will find the length of the segment that contains all the satisfactory points for each slice x units to the left of the center (we consider only a fourth of the pie, above and to the left of the center of the diameter, as the area is symmetric otherwise). This involves finding the maximum height h above the diameter. Clearly, the distance from the white chip to the circle lies on the radius, in order to be perpendicular. Now, if we assume that the radius is 1, then the length to use is $1 - \sqrt{x^2 + h^2}$, by the Pythagorean Theorem. If we set this equal to h, subtract one from both sides, and then after simplifying, we get $h = \frac{1-x^2}{2}$. Note that when $x = 1$, or the point is all the way to the left of the circle, then clearly no height works, and $h = 0$. But if $x = 0$, or the point is above the center, then $h = \frac{1}{2}$, which is also expected. Now, the last step is to integrate. We simply integrate the distance from 0 to 1, as we consider only the one quadrant. This comes out to be $\int_0^1 \frac{1-x^2}{2} \mathrm{d}x$, although dividing by the area of the quadrant, or $\frac{\pi}{4}$, we get $\frac{4}{\pi} \int_0^1 \frac{1-x^2}{2} \mathrm{d}x$. Our integral is fairly simple to evaluate, and our final answer of p turns out to be $\frac{4}{3\pi}$.

8. We will first prove a simple lemma, that we could equivalently ask for the second type of pie to be chosen. Note that this can easily be seen as equivalent, when we reverse the order of the pies mentioned in the explanation for pies running out. Therefore, we only want to find the second pie chosen. First, let us assign variables. We have b and c, the number of pear and papaya pies, respectively. We will now split into cases based on the first pie chosen. Let the first pie chosen be pineapple. This has a probability $\frac{20}{20+b+c}$. Then, we can either choose a pineapple pie again, or we can choose either a pear or papaya pie. Therefore, we ignore the pineapple pie probability, and note that the probability now of choosing a pear pie second, is $\frac{b}{b+c}$. Repeating for other combinations of pie for first and second, we have a system of three equations for three variables. Solving, we get the solutions three possible total numbers of pies, either $40, 60$, or 80. Their sum is 180. (We double-check with Wolfram|Alpha, and this is indeed the correct answer.)

9. We find an equation for the radius by connecting the centers of circles and Pythagorean's Theorem, and then by Descartes' Rule of Signs, there are either 2 or 0 positive roots, so we assume there are 2. (Well actually, we can use the intermediate value theorem to prove that there is at least one root. Clearly, when all the circles are equal in radius, the total pie area is much less than a hundred times the area of the triangle. However, when the radius is very tiny, or the radius is huge, the pie area is normally sized and the triangle is tiny, or the pie area is huge and the triangle is normally sized, so the ratio grows without bound.)

10. There are two orderings of the three points, which cannot intersect. Each ordering allows for $\binom{3}{1}^2\binom{2}{1} = 18$ paths, so there are a total of $18 \cdot 2 = 36$ paths.

11. Without loss of generality, let us first assume that after a long while, the gap has made it to position 7, without yet having made it to position 5 (the gap could also have made it to position 5 first, but the two situations are equivalent). Now, there are two options of equal probability, either that the hole moves to position 6, and our answer is 1, or the hole moves away to position 8. Now, the hole will eventually have to move back to position 7, without affecting the count yet (the probability of moving all the way around the pie to position 5 is negligible; recall that the problem asks for the expected value rounded to the nearest integer). Then, there are again two possibilities, either that the count ends with 2 (which happens now with a probability of $\frac{1}{4}$, or the hole moves away again). Now, we are essentially taking an infinite sum, $\sum_{n=1}^{\infty} \frac{n}{2^n}$, which is our answer, which is 2. (Also, simulations confirm that the answer is close to 2—a million trials were done, with a mean of 2.000949 and a standard deviation of 1.4164357025265941, a sigma excess of around 352.)

12. It is a well-known result that a disk can be split into a maximum of 7 regions by 3 lines. Thus, as the fourth cut can cut through the existing regions at most once, the number of regions at the end is at most $7 \times 2 = 14$. Note that this is achievable by simply having the first three pies cut perpendicular to the horizontal, for 7 pies, followed by a horizontal cut, cutting each one of those in two. Now, since there are 280 mouths to fill, there are $\frac{280}{14} = 20$ pies necessary.

13. There are $15 \cdot 15 = 225$ pairs. Each pie requires 10 favorable battles, so we have a maximum of $\lfloor \frac{225}{10} \rfloor = 22$ pies that win at least 10 battles. Note that we can achieve this when each pie on one side wins 10 battles, while the losses on the other side are split so that there are eight pies with 15 losses, two with 10, four with 5, and one with none.

14. You can only buy apple pie if at most 1011 of the 2023 people before him buys apple pie. For each n, the number of people before you who bought apple pie, there are $\binom{2023}{n}$ ways for this to happen, and a probability $\left(\frac{1}{2}\right)^n \cdot \left(1 - \frac{1}{2}\right)^{2023-n} = \frac{1}{2^{2023}}$. However, the two probabilities are symmetric, e.g., there are just as many ways for at most 1011 people to buy apple pies, as there are for at least 1012 people to buy apple pies ahead of you. Additionally, all outcomes are equally weighted, so we see that there is a $\frac{1}{2}$ probability that there are enough apple pies for you to buy an apple pie.

15. We apply the formula for an infinite geometric series. Our sum is clearly $S = \frac{\pi}{\frac{9}{10}}$. Note that if $S < 3.5$, we would round to 3, but if $S > 3.5$, we would round to 4. Since $3.5 \cdot \frac{9}{10} > \pi$, we have $S < 3.5$, so we round S to 3.

16. Since we have one pie per day, we need only count the number of days between February 24th and March 14th, inclusive. Let us first assume this year is not a leap year. Then, February would have 28 days, so there are 5 days in February to count, along with 14 days in March, for a total of 19 days. However, the offset in the day of the week should be given by the number of days minus one (because we included the 24th in our count, which would increase the number of days, but not the offset) modulo 7. On the other hand, $19 - 1 \pmod 7$ is 4, not the 5 which would be expected for Tuesday minus Thursday. But if the year was a leap year, then we would increase the number of days, and the offset of 5 would be satisfied. In this case, the number of days, and thus the number of pies, is 20.

17. Use n^{th} differences.

18. First of all, we look at the second situation. If the mass of the edible portion of the pie is m, then $\frac{m}{6}$ of the pie is four times the mass of the pan, so in all, the edible portion is 24 times as heavy as the pan. However, if $\frac{5}{6}$ of the pie is 1000 grams, then the entire pie is 1200 grams in mass, while the pan is 50 grams in mass. Additionally, after eating 3 slices, the mass can turn from 50 grams to -250 grams, so clearly each slice is 100 grams in mass. Since there are 1200 grams of slices, we have 12 slices.

19. If you multiply each term of a geometric proportion by a constant amount, then the result is still a geometric proportion. Therefore, the only possibility for it also being an arithmetic progression is for a constant progression.

20. We use the Pythagorean Theorem to get a radius of $3 - 2\sqrt{2}$ if the radius were 1. However, we then scale by 12 (6 because we initially divided by 6 and 2 because we calculated the radius not the diameter) to get $36 - 24\sqrt{2}$.

21. The criteria for sufficient heating is equivalent to any pie within a distance of 2.5 units from the foot of the heat lamp. The optimal packing is hexagonal, with one pie's center at that heat lamp. Then, we can fit nineteen pies, in accordance with the centered hexagonal formula, although the proof is left up to the reader.

22. 45 pies are sold, and 15 are kept, so the difference is 30 pies.

23. If Mrs. Lockliear chooses 80 to serve at a feast, each pie has a $\frac{80}{36+48+20+16+24} = \frac{5}{9}$ of being selected. There are 36 apple pies, and $36 \cdot \frac{5}{9} = 20$ pies.

24. We simply set up a system of equations and solve.

25. The locations of the pumpkin pies do not affect the number of positions that the next pumpkin pie can be placed.

26. First, we see that $A = 100\pi \approx 314.15$. Clearly, we have $n = \lfloor \log_b A \rfloor$. Now, when b is a perfect root of A, we have $a = 1$, which is clearly not maximized (and indeed, this is a local minima, because $a = 1$ has very few solutions). Therefore, we want b to be slightly more than a perfect root of A. More rigorously, we note that in general, increasing b decreases a, because n stays constant, except for when n itself decreases. Clearly, then, we want to choose the largest perfect root of A, so we choose the square root (the first root does not satisfy the less than A requirement). The square root of 100π is $10\sqrt{\pi}$. This "point," $(10\sqrt{\pi}, 10\sqrt{\pi})$ would be a hole in the function, but we do have $\lim_{b \to 10\sqrt{\pi}} b \times b^2 = A$, so this is the valid $a_{\max}$.

27. Let us consider the simpler case of a arranging slices in a linear fashion. Then, this is equivalent to the function $C(n)$, the number of ways to choose a subset of the set $\{1, 2, \cdots, n\}$ such that no two consecutive elements are both in the subset. Clearly, $C(0) = 1, C(1) = 2, C(2) = 3, C(3) = 5$, which looks like a shifted Fibonacci sequence. Indeed, we see that when we increment n by one, we can modify existing subsets by either including the new element, which forces the previous element not to be included, or by not including the new element, allowing the previous element to either be included or not. In other words, our recursion is $C(n+1) = C(n-1) + C(n)$, which is the same recursion for the Fibonacci sequence. Now, we want to count the number of ways to choose the subsets for the $n = 10$ case. Let us first number the slices from 1 to 10. Then, there are three cases for the inclusivity of the first and tenth slices. If they were both eaten, then we have $C(8) = 55$ options. If one was eaten and the other not, then we only have $C(7) = 34$ options, but there are two of these, so in total, we have $55 + 2 \cdot 34 = 123$ options. Note that at least one must be eaten. However, there are clearly $2^1 0$ ways to eat a subset of the pies, so we have a final answer of $\frac{123}{1024}$. Now, $123 + 1024 = 1147$, and the sum of the digits here is 13.

28. It is a well-known fact that the sum of the values in Row n of Pascal's Triangle is 2^n. Applying this here, and then multiplying by 2π, to find the perimeters, we get 64π.

29. Let the height of the pie be h, and the average radius of the two bases be r. The vertical cross-section is a trapezoid, with area rh, and the horizontal cross-section has an area $r^2\pi$. The ratio is then $\frac{r}{h}\pi$, so clearly, the ratio is 20.

30. For the n^{th} pie, where the first pie is simply the cone, the ratio of base radii is $\frac{n+1}{n}$, because radius scales linearly with height from the top. Taking the square root on all sides of the given inequality, we have that $1 < \frac{n+1}{n} < 1.2$. Clearly, the left inequality will always be satisfied, but the right inequality will only be for $n > 5$. There are 20 pies between the 6^{th} and 25^{th}, inclusive.

31. 5 pounds is equal to 80 ounces, but as each pie requires two ounces of flour, 40 pies can be made.

32. We use Euler's formula for planar graphs.

33. Not including the default sugar in a pie, the expected value of choosing cinnamon leads to $8 + 4 = 12$ grams of sugar, the expected value of not is $\frac{4}{2} = 2$ grams of sugar, so taking the average, and then adding the default sugar, we get 13.

34. We divide volumes.

35. $3, 4,$ and 5 all work (if there are duplicates, we can choose whether or not to count them both).

36. This is the only positive integer P that satisfies the equation. We can prove that no more exist by establishing that exponentials grow faster than polynomials.

37. Let us define the first cut to be located at an angle of $0°$, the second cut to be at $\alpha°$ from the first (where, without loss of generality, $0 < \alpha < 90°$) and the third cut to be at $\beta°$, also from the first (where, without loss of generality, $0 < \beta < 180°$. Now, for a given α, we have four cases to consider. When $0 < \beta < \alpha, \alpha < \beta < 90°, 90° < \beta < \alpha + 90°$, and

$\alpha + 90\circ < \beta < 180°$. The probabilities that the final cut will satisfy are $\frac{\alpha}{180°}$, $\frac{\beta}{180°}$, 100%, and $\frac{\alpha+180°-\beta}{180°}$, respectively. Integrating, we get $\frac{1}{4}$. However, we take the complement, because we want the probability that no one eats more than a fourth, and we found the probability that someone does eat more than a fourth. This is $\frac{3}{4}$.

38. Only paying for the 80% that is edible means paying $\$8$ per pie. Therefore, 100 pies may be bought in this method. However, with $\$10$ per pie, from before, only 80 pies could be bought. The difference is 20 pies.

39. Clearly, as $\log_2\left(\frac{360}{70}\right) > 2$, at least 3 operations are necessary. However, three operations are not actually sufficient, as the values possible are $67, 69, 71, 73, 136, 137, 138, 139, 140, 141, 142, 143, 144, 276, 278, 279, 281, 282, 284, 560$. The only one achievable by adding an operation somewhere, is 552.

40. When the first person reaches 40 pies, it is overwhelmingly more likely that that person is me. Therefore, we may neglect the expected value for the second case when Mrs. Lockliear wins. Now, when I win, I have 40 pies, and Mrs. Lockliear, winning only half as much as me, has 20 pies. The difference is also 20 pies.