homework 2

question 5.2

5.2 *Binary hypothesis test and SNR*
Consider the binary choice in Gaussian noise, as shown in Example 5.2. With the threshold of $k/2$, the SNR is also maximized at the decision point. Since the possible signal values are known, the maximization of SNR means that the hypothesized noise variance is minimized when the decision boundaries are optimally chosen. Prove that SNR is maximized when the threshold is $k/2$.

answer 5.2

threshold = `k/2`

The variance of this Gaussian noise as a function of the threshold γ is

$$\mathrm{Var}(\gamma) = \frac{1}{2}\left(\int_{-\infty}^{\gamma} x^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-x^2}{2\sigma^2}}\,dx + \int_{-\infty}^{\gamma} x^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{\frac{-(x-k)^2}{2\sigma^2}}\,dx\right)$$
$$+ \frac{1}{2}\left(\int_{\gamma}^{\infty} (x-k)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-k)^2}{2\sigma^2}}\,dx + \int_{\gamma}^{\infty} (x-k)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-x^2}{2\sigma^2}}\,dx\right)$$

We find the maximum of this is we take the derivative of the variance w.r.t γ then set it equal to 0 and solve for γ.

$$\frac{d}{d\gamma} Var(\gamma) = \gamma^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-\gamma^2}{2\sigma^2}} + \gamma^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(\gamma-k)^2}{2\sigma^2}}$$

$$- (\gamma - k)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(\gamma-k)^2}{2\sigma^2}} - (\gamma - k)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-\gamma^2}{2\sigma^2}} = 0$$

$$\gamma = k/2$$

∴ SNR is maximized when the threshold is = `k/2`.

question 5.4

5.4 *MAP and the LRT*

Show that the MAP decision rule is equivalent to the likelihood ratio test.

answer 5.4

MAP(maximum a posteriori) decision rule

choose H_1 if $P(s_1|z) > P(s_0|z)$, and H_0 otherwise.

Bayes theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

$$\frac{P(z|S_1)P(S_1)}{\cancel{P(z)}} = \frac{P(z|S_0)P(S_0)}{\cancel{P(z)}}$$

This simplifies to the **likelihood ratio test**

$$\frac{P(z|S_1)}{P(z|S_0)} > \frac{P(S_0)}{P(S_1)}$$

question 5.7

5.7 *Thresholds with costs*

Analytically compute the threshold in Example 5.5.

Example 5.5 Binary choice in Gaussian noise with cost functions

Return to Example 5.2, where now the costs are $c_{00}=c_{11}=0$, but $c_{01}=10$ and $c_{10}=1$. Let $k=5$ and $\sigma=1$. Compute the threshold value that minimizes the average cost.

Solution: The prior probabilities are equal so that the cost is $.5(Q(\gamma)+10Q(5-\gamma))$. That is

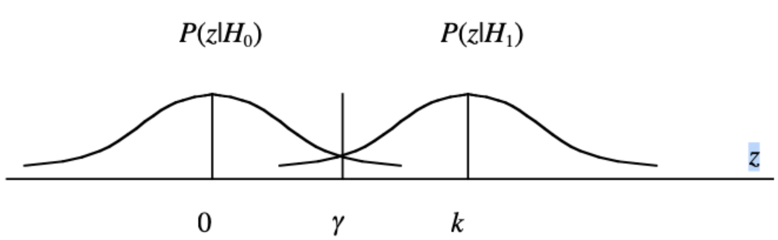
$$\bar{c} = \frac{0.5}{\sqrt{2\pi}}\left(\int_{\gamma}^{\infty} e^{-z^2/2}dz + 10\int_{5-\gamma}^{\infty} e^{-z^2/2}dz\right)$$

To solve, differentiate with respect to γ and set the result equal to zero. Alternatively, it may be observed that the minimum will occur with γ just a little bit less than the optimal value without cost, owing to the exponential dependence of the Q function with its argument. $\gamma=2.5$, 2.0, and 1.5 result in average costs of 0.034, .0179, and .0346 respectively. Thus $\gamma=2$ is close to being optimal.

Example 5.2 Binary choice in Gaussian noise

A signal voltage z can be zero (H_0) or k (H_1), each hypothesis with probability 1/2. The voltage measurement is perturbed by AWGN of variance σ^2. Compute the decision threshold for the MAP criterion, and the error probabilities $P(D_1|H_0)$ and $P(D_0|H_1)$, where D_1 means that H_1 was decided, and D_0 means H_0 was decided.

Solution: The situation is depicted in Figure 5.2



answer 5.5

The threshold value that minimizes the average cost will be found when the derivative of the cost w.r.t $\gamma = 0$.

$$\frac{d}{d\gamma}\bar{C} = \frac{d}{d\gamma}\left[\frac{1/2}{\sqrt{2\pi}}\left(\int_{\gamma}^{\infty} e^{-\frac{z^2}{2}}\,dz + 10\int_{5-\gamma}^{\infty} e^{-z^2/2}\,dz\right)\right]$$

$$\frac{d}{d\gamma}\bar{C} = \frac{1/2}{\sqrt{2\pi}}\left[e^{-\frac{z^2}{2}}\Big|_{\gamma}^{\infty} + 10\, e^{-z^2/2}\Big|_{5-\gamma}^{\infty}\right]$$

$$\frac{d}{d\gamma}\bar{C} = \frac{1/2}{\sqrt{2\pi}}\left[\left(0 - e^{-\gamma^2/2}\right) + 10\left(0 - e^{-\frac{(5-\gamma)^2}{2}}\right)\right]$$

$$\frac{d}{d\gamma}\bar{C} = \cancel{\frac{1/2}{\sqrt{2\pi}}}\left(-e^{-\gamma^2/2} - 10\, e^{-\frac{(5-\gamma)^2}{2}}\right) = 0$$

neg err?

$$\ln\left[e^{-\gamma^2/2} = 10\, e^{-\frac{(5-\gamma)^2}{2}}\right]$$

$$-\gamma^2/2 = \ln\left(10 \cdot e^{-\frac{(5-\gamma)^2}{2}}\right) = \ln(10) \cdot \frac{-(5-\gamma)^2}{2}$$

$$\cancel{-\gamma^2} = 2\ln(10) \cdot -(5-\gamma)^2 = (2\ln(10) - 25) + 10\gamma - \cancel{\gamma^2}$$

$$\gamma = \frac{25 - 2\ln(10)}{10} \approx 2.039$$

question 5.27

5.27 *LMS algorithm*

Show that the gradient $\boldsymbol{G}_0$ in the LMS algorithm is equal to $-E[e_0\mathbf{u}_0^*]$.

$$G_o = \frac{1}{2} \frac{\partial J}{\partial w}$$

$$J = \frac{1}{N} \sum_{i=1}^{N} \| f(u_i) - d_i \|^2 = E\left[\| f(u) - d \|^2 \right]$$

$$J = E\left[\| w^T u - d \|^2 \right] = E\left[(w^T u - d)(w^T u - d)^* \right]$$

$$J = E\left[w^T u u^* (w^T)^* - w^T u d^* - d u^* (w^T)^* + d^2 \right]$$

$$\frac{\partial J}{\partial w} = 2 w^T u u^* - 2 d u^* = 2 \left(w^T u u^* - d u^* \right)$$

Let $\boxed{e_i = d_i - (w^T u)_i}$

$$\frac{\partial J}{\partial w} = -2 u^* \left(d - w^T u \right) = -2 u^* e$$

$$G_o = \frac{1}{2} E\left[-2 u_o^* \underbrace{\left(d_o - (w^T u)_o \right)}_{e_o} \right] = \frac{1}{2} E\left[-2 e_o u_o^* \right]$$

$$G_o = - E\left[e_o u_o^* \right]$$