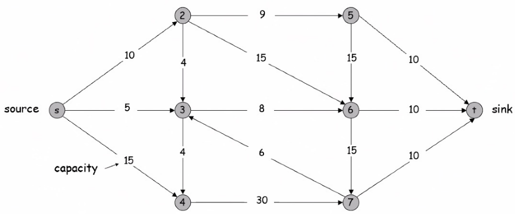
Aug 7

140

Minimum Cut Problem

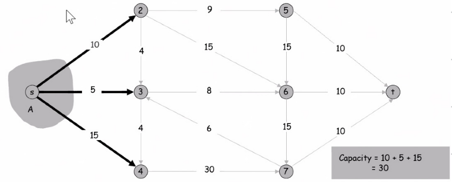
Flow network.
- Abstraction for material flowing through the edges.
- G = (V, E) = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- c(e) = capacity of edge e.



Cuts

Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



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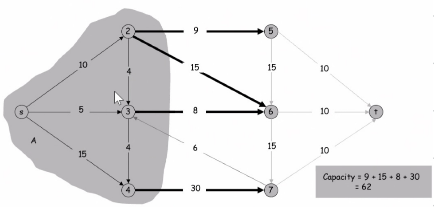
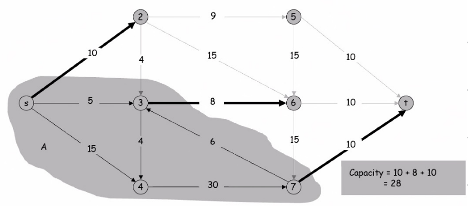


Minimum Cut Problem

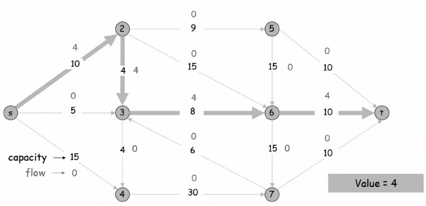
Min s-t cut problem. Find an s-t cut of minimum capacity.



Flows

Def. An s-t flow is a function that satisfies:
- For each e ∈ E: $0 \le f(e) \le c(e)$ [capacity]
- For each v ∈ V – {s, t}: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$



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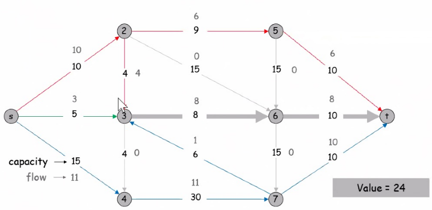
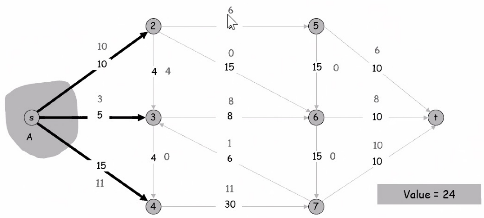


Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to } A} f(e) \; = \; v(f)$$



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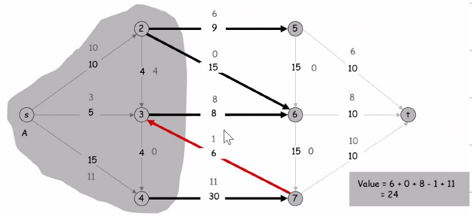

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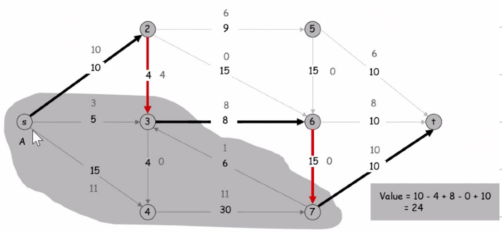


Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except v = s are 0 $\longrightarrow$

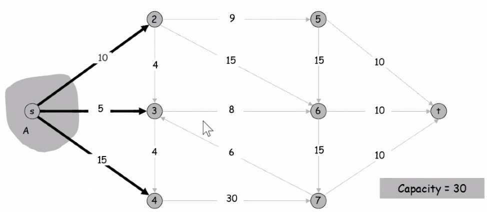
$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e).$$

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30

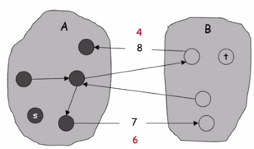


Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have v(f) ≤ cap(A, B).

Pf.

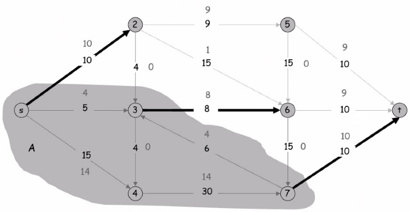
$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= \operatorname{cap}(A, B) \quad \blacksquare \end{aligned}$$



Certificate of Optimality

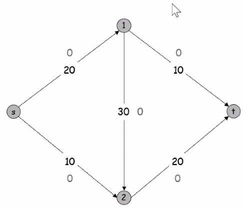
Corollary. Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 ⇒ Flow value ≤ 28



Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e ∈ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

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- Start with f(e) = 0 for all edge e ∈ E.
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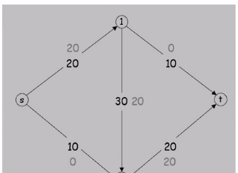
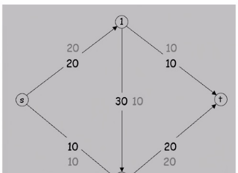
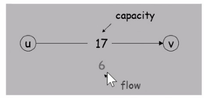
locally optimality ⇏ global optimality





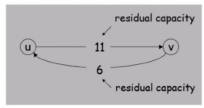
Residual Graph

Original edge: e = (u, v) ∈ E.
- Flow f(e), capacity c(e).



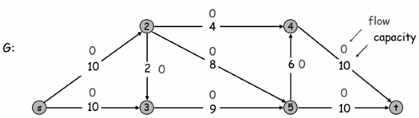
Residual edge.
- "Undo" flow sent.
- e = (u, v) and e^R = (v, u).
- Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$



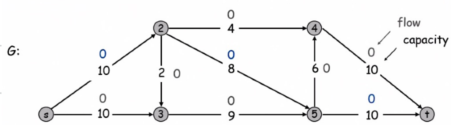
Residual graph: $G_f = (V, E_f)$.
- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

Ford-Fulkerson Algorithm



Flow value = 0

Ford-Fulkerson Algorithm



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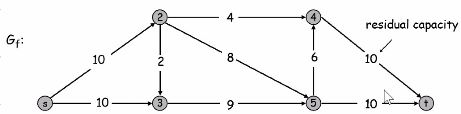

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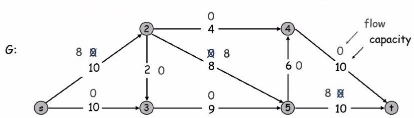

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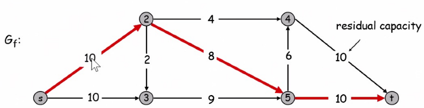

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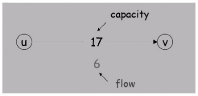

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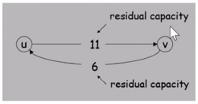

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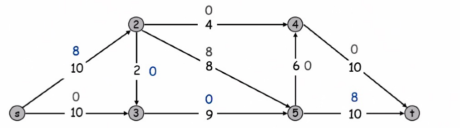
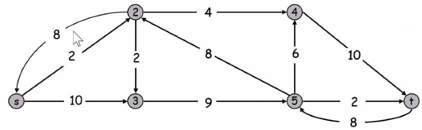
Ford-Fulkerson Algorithm

G:



Flow value = 8

G_f:



Ford-Fulkerson Algorithm

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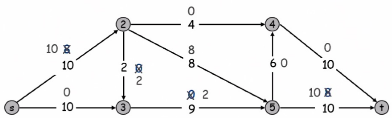

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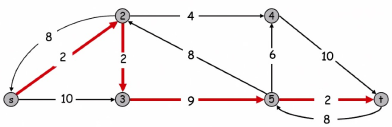
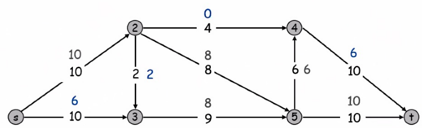
G_f:



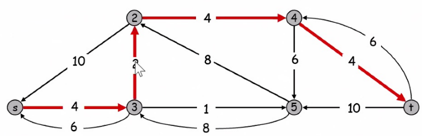
Ford-Fulkerson Algorithm

G:



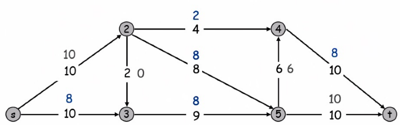
Flow value = 16

G_f:



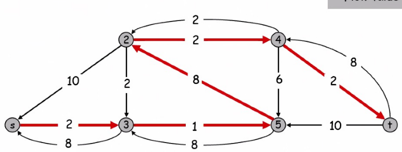
Ford-Fulkerson Algorithm

G:



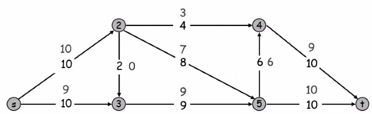
Flow value = 18

G_f:



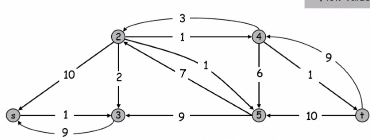
Ford-Fulkerson Algorithm

G:



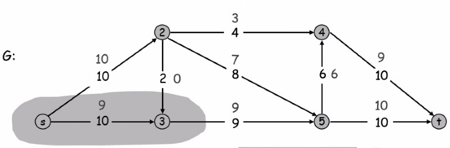
Flow value = 19

G_f:



Ford-Fulkerson Algorithm

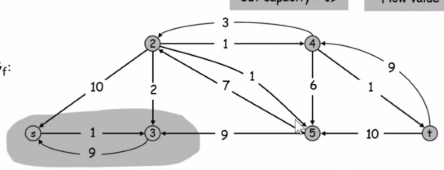
G:



Cut capacity = 19

Flow value = 19

G_f:



Ford-Fulkerson: A Greedy Max Flow Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b       forward edge
      else       f(eR) ← f(eR) - b     reverse edge
   }
   return f
}
```

A path with non-full forward edge
Or non-zero reverse edge

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Gf ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update Gf
   }
   return f
}
```

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forward edge
reverse edge

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   }
   return f
}
```

Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Pf. We prove both simultaneously by showing TFAE:
- (i) There exists a cut (A, B) such that v(f) = cap(A, B).
- (ii) Flow f is a max flow.
- (iii) There is no augmenting path relative to f.

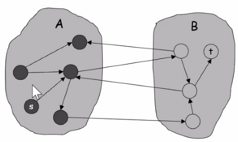
(i) ⇒ (ii) This was the corollary to weak duality lemma.

(ii) ⇒ (iii) We show contrapositive.
- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

(iii) ⇒ (i)
- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, s ∈ A.
- By definition of f, t ∉ A.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B) \quad \blacksquare
\end{aligned}
$$

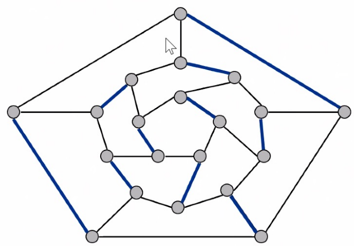


original network

Second equality holds since otherwise there will be backward edge.

Matching

Matching.
- Input: undirected graph G = (V, E).
- M ⊆ E is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value f(e) and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

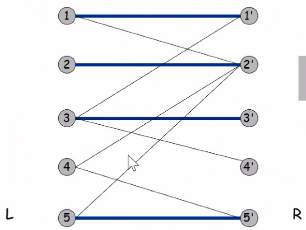
Theorem. The algorithm terminates in at most $v(f^*) \le nC$ iterations.
Pf. Each augmentation increase value by at least 1. ▪

Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.
Pf. Since algorithm terminates, theorem follows from invariant. ▪

Bipartite Matching

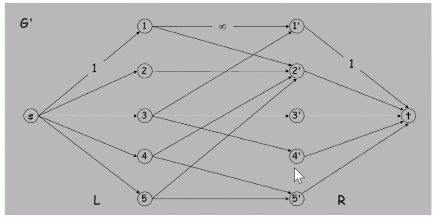
Bipartite matching.
- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



max matching
1-1', 2-2', 3-3' 4-4'

Bipartite Matching

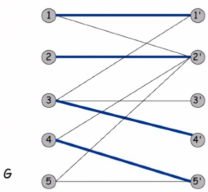
Max flow formulation.
- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.

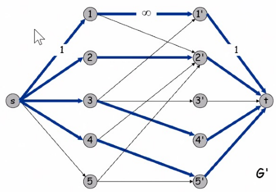


Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\leq$
- Given max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k. ▪





Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\geq$
- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ k is integral and can assume f is 0-1.
- Consider M = set of edges from L to R with $f(e) = 1$.
  - each node in L and R participates in at most one edge in M
  - $|M| = k$: consider cut $(L \cup s, R \cup t)$ ▪