Network Flow

A flow network is a directed graph $g = (V, E)$

source node $s \in V$

sink node $t \in V$

Capacity $c : E \to \mathbb{R}^{++}$

Assumptions:

1) $(a,b) \in E \Rightarrow b \neq s$
2) $(a,b) \in E \Rightarrow a \neq t$
3) $\forall d \in V \ \exists (a,b) \in E \Rightarrow d = a \lor d = b$

f.n. does not have fluctuations

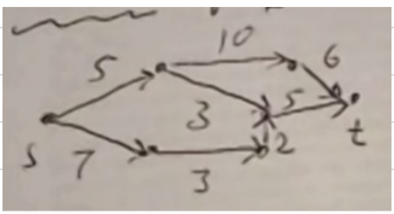
A flow is $f : E \to \mathbb{R}^+$ (s-t flow)

conditions 1) $0 \leq f(e) \leq c(e) \quad \forall e \in E$

2) $\forall v \in V$ s.t. $v \neq s, t$ $\left(\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out } v} f(e) \right)$

Consistency condition

The value of f $\quad v(f) = \sum_{e \text{ out } s} f(e) = f^{out}(s)$



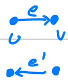
Find the flow with max value

The residual graph $G_f = (V_f, E_f)$

1) $V_f = V$

2) If $e = (u, v) \in E$ & $f(e) < c(e)$, then $e = (u, v) \in E_f$ & $c_f(e) = c(e) - f(e)$

3) If $e = (u, v) \in E$ & $f(e) > 0$



then $e' = (v, u) \in E_f$ & $c_f(e') = f(e)$

backward edge

Let P be a simple path from s to t in G_f

bottleneck $(P, f) = \min \{ c_f(e) \text{ s.t. } e \in P \}$

Augment (f, P)

Let $b = $ bottleneck (P, f)

For each $e = (u, v) \in P$

If $e = (u, v)$ is a forward edge

then increase $f(e)$ in G by b

$f'(e) = f(e) + b$

else [$e' = (u, v)$ is a backward edge]

decrease $f(e)$ in G by b

$f'(e) = f(e) - b$

Endif

End for

Return f'

Pr1 f' is a flow in g

proof 1. capacity cond

a) $e = (u, v)$ is forward edge $0 \le f(e) \le f'(e) = f(e) + b \le f(e) + (c(e) - f(e)) = c(e)$

b) $e = (v, u)$ backward edge

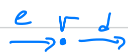
$c(e) \ge f(e) \ge f'(e) = f(e) - b \ge f(e) - f(e) = 0$

2) consistency cond

$e = (u, v)$ forw. e. $\quad f'(e) = f(e) + b$

$\Rightarrow$

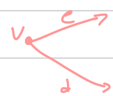
$d = (v, w)$ forw. e. $\quad f'(e) = f(d) + b$



$e = (u, v)$ is backw. e. $\quad f'(e) = f(e) - b \quad e = (v, u)$

$\Rightarrow$

$d = (v, w)$ is forw. e. $\quad f'(d) = f(d) + b$



c) $e = (u, v)$ is f. e.

$d = (v, w)$ is b. e.

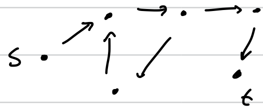
d) $e = (u, v)$ is b. e.

$d = (v, w)$ is b. e.

Proved

Ford - Fulkerson Algorithm (FFA) 1956

Max - Flow



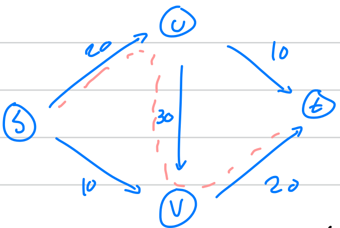
Put $f(e) = 0$ for $\forall\, e \in E$

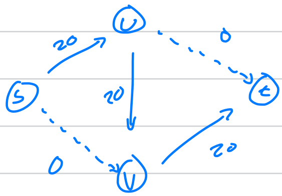
While $\exists$ an s-t path in g_f

- Let P be a simple s-t path in g_f
- Augment $(g, f) = f'$
- Update g_f to $g_{f'}$

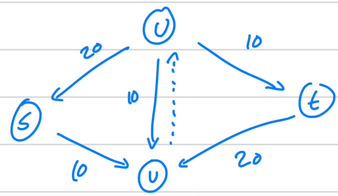
End While

Return f_{end}



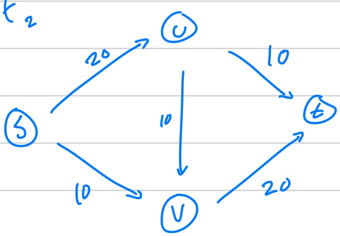


g_{f_1}

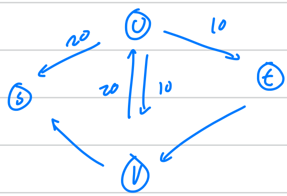


$V(f_0) = 0$ $V(f) = 20$

f_2



g_{f_2}



$V(f_2) = 30$

f_2 is max flow