Jul 2 180

Asymptotic boundaries

A function $f(n)$ is an **asymptotic upper bound** of a function $g(n)$ if

If there are a constant $c > 0$ and a natural number m such that

$$g(n) \leq cf(n)$$

for all $n > m$

Formal definition

$$\exists c > 0\ \exists m \in N\ \forall n > m(\ g(n) \leq cf(n)\ )$$

It is denoted by $g(n) = O(f(n))$.

$$g = O(f)$$

$$g(n) \in O(f(n))$$

Big O notation

Ex $10x^2 + 100x + 1000 = O(x^2)$

$10x^2 + 100x + 1000 \leq 3000x^2$

$c = 3000$

$10n^2 + 100n + 1000$

$1000n^2 + 1000n^2 + 1000n^2 = 3000n^2$

$2^x + 1000000x = O(2^x)$

$2^x + 1{,}000{,}000x = O(2^x)$ $n > m$

$c = 2{,}000{,}000$

$2^n + 1{,}000{,}000n$

$1{,}000{,}000 \cdot 2^n + 1{,}000{,}000 \cdot 2^n$

$2^n > n$ $n = 1$ $2 > 1$

$n = 2$ $4 > 2$

$c = 2$

$2 \cdot 2^n = 2^n + 2^n$

$2^n > 1{,}000{,}000n$

A function $f(n)$ is an **asymptotic lower bound** of a function $g(n)$ if

If there are a constant $k > 0$ and a natural number m such that

$$g(n) \geq kf(n)$$

for all $n > m$

Formal definition

$$\exists k > 0 \ \exists m \in N \ \forall n > m \ (g(n) \geq kf(n))$$

It is denoted by $g(n) = \Omega(f(n))$.

$$g = \Omega(f)$$

$$g(n) \in \Omega(f(n))$$

Examples:

$$10x^2 + 100x + 1000 = \Omega(x^2)$$

$$2^x = \Omega(2^x + 1000000x)$$

$$2^x = \Omega(10x^{100})$$

Pr 1 If $g = O(f)$, then $f = \Omega(g)$ $\qquad \frac{1}{c} = c^{-1}$

proof $\exists c > 0 \ \exists m \in N \ \forall n > m \ (g(n) \leq c \cdot f(n))$

$(c^{-1} \cdot g(n) \leq f(n))$

$k = c^{-1}$

$f = \Omega(g)$

Corrilary 1 If $f = O(g)$ & $g = O(f)$ then $g = \Theta(f)$

Proposition 2 Θ is a symmetric relation

$g = \Theta(f)$ iff $f = \Theta(g)$

Proposition 3 O is a transitive relation

If $g = O(f)$ & $f = O(h)$, then $O(h)$

proof $\exists c > 0 \ \exists m \in N \ \forall n > m \ (g(n) \leq c \cdot f(n))$

$\exists k > 0 \ \exists p \in N \ \forall n > p \ (f(n) \leq k \cdot h(n))$

$\exists d > 0 \ \exists q \in N \ \forall n > q \ (g(n) \leq \underbrace{c \cdot k}_{d} \cdot h(n))$

$q = \max(m, p)$

$g = O(h)$

Proposition 4

a) $g = O(f)$ & $h = O(f) \Rightarrow g+h = O(f)$

b) $g = \Omega(f)$ & $h = \Omega(f) \Rightarrow g+h = \Omega(f)$

c) $g = \Theta(f)$ & $h = \Theta(f) \Rightarrow g+h = \Theta(f)$

Proof (a)

$\exists c > 0 \ \exists m \in N \ \forall n > m \ (g(n) \leq c \cdot f(n))$

$\exists k > 0 \ \exists p \in N \ \forall n > p \ (h(n) \leq k \cdot f(n))$

$g(n) + h(n) \leq c \cdot f(n) + k \cdot f(n)$

$\exists d > 0 \ \exists q \in N \ \forall n > q \ \left((g+h)(n) \leq \underbrace{(c+k)}_{d} \cdot f(n) \right)$

$q = \max(m, p)$

Propisition 5

a) $g = O(f)$ & $a > 0 \Rightarrow a \cdot g = O(f)$

b) $g = \Omega(f)$ & $a > 0 \Rightarrow a \cdot g = \Omega(f)$

c) $g = \Theta(f)$ & $a > 0 \Rightarrow a \cdot g = \Theta(f)$

proof (b)

$\exists k > 0 \ \exists m \in N \ \forall n > m \ (g(n) \geq k \cdot f(n))$

$\exists d > 0 \ \exists m \in N \ \forall n > m \ (a \cdot g(n) \geq \underbrace{a \cdot k}_{d} \cdot f(n))$

$g(n), f(n) \ldots > 0$

1000

$f(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ if $n < 1000$

$f(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ if $1000 < m < 2000$

$n = 2000 - m$

$f(t) = 2000$ if $t > 2000$

Prop 6 If $g = O(f)$ & $f \le h$, then $g = O(h)$

Proof If $f \le h$, then $f = O(h)$

By Pr 3, $g = O(h)$

By Pr 7, $h \le g$ & $g = O(f)$, then $h = O(f)$

Order $\le$

1) $f \le f$

2) $f \le g$ & $g \le h \Rightarrow f \le h$

3) $f \le g$ & $g \le f \Rightarrow f = g$

Strict order $<$

either $f < g$ or $g < f$

Order relations

$g \le f$ if $g = O(f)$

$g < f$ if $g = O(f)$ but $f \ne O(g)$

same order of growth

$10x \le 10^{10}x$, $10x \ge 10^{10}x$

$10^{10}x < 2^x$

$x^{100} < 2^x$

$\ln x^{100} < x$

A function $f(n)$ is a **diminishing upper bound** of a function $g(n)$ if

If for any constant $c > 0$, there is a natural number m such that

$$g(n) \le cf(n)$$

for all $n > m$

Formal definition

$$\forall c > 0 \ \exists m \in N \ \forall n > m(g(n) \le cf(n))$$

It is denoted by $g(n) = o(f(n))$.

Examples:

$10x^2 + 100x + 1000 = o(x^3)$

$1000x^{100} = o(2^x)$

$10x^2 + 100x + 1000 = o(x^3)$

$c > 0 \quad m \quad c = 3$

$10x^2 + 100x + 1000 < 3x^2$

$x^3 + x^3 + x^3 = 3x^3 \qquad x > 10$

$x \cdot x^2 \qquad x^2 \cdot x$

$c > 0$

$10x^2 + 100x + 1000 \qquad cx^3$

$10c^{-1}x^2 + 100c^{-1}x + 1000c^{-1} \qquad x^3$

$\frac{1}{3}x^3 + \frac{1}{3}x^3 + \frac{1}{3}x^3 \qquad x > 30c^{-1}$

$10c^{-3}x^2 \quad 100c^{-3}x \quad 1000c^{-3}$

$g(n) \leq f(n) \leq 9 \cdot f(n)$

$n > 1$

A function $f(n)$ is a **growing lower bound** of a function $g(n)$ if

If for any constant $c > 0$, there is a natural number m such that

$g(n) \geq cf(n)$

for all $n > m$

Formal definition

$\forall c > 0 \; \exists m \in N \forall n > m (\, g(n) \geq cf(n) \,)$

It is denoted by $g(n) = \omega(f(n))$.

Examples:

$x^3 = \omega(10x^2 + 100x + 1000)$

$2^x = \omega(1000x^{100})$

Prop 8 a) if $g = o(f)$, then $g = O(f)$

b) if $g = \omega(f)$, then $g = \Omega(f)$

Prop 9 if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c > 0$, then $f = \Theta(g)$

Proof $\exists m \; \forall n > m \left(\frac{c}{2} < \frac{f(n)}{g(n)} < \frac{3}{2}c \right)$

$\underbrace{\frac{1}{2}c\, g(n) < f(n)} < \frac{3}{2} < g(n)$

$g(n) < 2c^{-1} \cdot f(n) \Rightarrow g(n) = O(f)$

$f(n) < \frac{3}{2}c \cdot g(n) \Rightarrow f(n) = O(g)$

$\Rightarrow f = \Theta(g)$