181 extra credit

William Randall

March 2021

1 Extra Credit

1. **Let L_1 and L_2 be decidable languages. Prove that $L_1 \times L_2$ is decidable.**
   Let a TM M_1 decide L_1 and a TM M_2 decide L_2.
   TM M_{cross} takes as input $x \in (x_1, x_2) \in (\{0,1\}^*, \{0,1\}^*)$:

   (a) Simulate M_1 on x_1.
   If M_1 halts and **accepts** then **continue** to the next step.
   If M_1 halts and **rejects** then **reject**.

   (b) Simulate M_2 on x_2.
   If M_2 halts and **accepts** then **accept**.
   If M_2 halts and **rejects** then **reject**.

   Claim:
   TM M_{cross} decides $L_1 \times L_2$
   Proof:
   If $x = (x_1, x_2)$ belongs to $L_1 \times L_2$ then x_1 belongs to L_1 and x_2 belongs to L_2.

   If $x_1 \in L_1$ then simulating M_1 on x_1 will result in M_1 accepting x_1 because M_1 decides L_1.

   If $x_2 \in L_2$ then simulating M_2 on x_2 will result in M_2 accepting x_2 because M_2 decides L_2.

   By construction of M_{cross}, M_{cross} accepts if and only if x_1 is accepted by the simulation of M_1 on x_1 and x_2 is accepted by the simulation of M_2 on x_2 else M_{cross} will reject.

   If x is accepted by the machine M_{cross} then by construction both x_1 and x_2 were respectively accepted by M_1 and M_2. Since M_1 and M_2 are deciders of L_1 and L_2 respectively it must be that $x \in L_1 \times L_2$

   $\square$

2. **Let L1 and L2 be decidable languages. Prove that the following language is decidable**

$$\text{shuffle}(L_1, L_2) = \{x_1 y_1 x_2 y_2 \ldots x_n y_n \mid x_1 \ldots x_n \in L_1, y_1 \ldots y_n \in L_2\}$$

Let a TM M_1 decide L_1 and a TM M_2 decide L_2.

TM M_{shuffle} takes as input $x = x_1 x_2 x_3 \ldots x_n \in \{0,1\}^*$:

(a) If n is odd then **reject**.
Else move on to the next step.

(b) Simulate M_1 on $x_1 x_3 x_5 \ldots x_{n-1}$.
If M_1 halts and **accepts** then **continue** onto the next step.
If M_1 halts and **rejects** then **reject**.

(c) Simulate M_2 on $x_2 x_4 x_6 \ldots x_n$.
If M_2 halts and **accepts** then **accept**.
If M_2 halts and **rejects** then **reject**.

Claim

TM M_{shuffle} decides shuffle(L_1, L_2).

Proof

If $x = x_1 x_2 x_3 \ldots x_n$ belongs to shuffle(L_1, L_2) then $x_1 x_3 x_5 \ldots x_{n-1}$ belongs to L_1 and $x_2 x_4 x_6 \ldots x_n$ belongs to L_2.

If the length of the input n is odd then it cannot belong to shuffle(L_1, L_2) so it is automatically rejected.

If $x_1 x_3 x_5 \ldots x_{n-1} \in L_1$ then simulating M_1 on $x_1 x_3 x_5 \ldots x_{n-1}$ will result in M_1 accepting $x_1 x_3 x_5 \ldots x_{n-1}$ because M_1 decides L_1.

If $x_2 x_4 x_6 \ldots x_n \in L_2$ then simulating M_2 on $x_2 x_4 x_6 \ldots x_n$ will result in M_2 accepting $x_2 x_4 x_6 \ldots x_n$ because M_2 decides L_2.

By construction of M_{shuffle}, M_{shuffle} accepts if and only if x_1 is accepted by the simulation of M_1 on x_1 and x_2 is accepted by the simulation of M_2 on x_2 else M_{shuffle} will reject.

If x is accepted by the machine M_{shuffle} then by construction both $x_1 x_3 x_5 \ldots x_{n-1}$ and $x_2 x_4 x_6 \ldots x_n$ were respectively accepted by M_1 and M_2. Since M_1 and M_2 are deciders of L_1 and L_2 respectively it must be that $x \in \text{shuffle}(L_1, L_2)$

$\square$