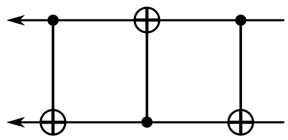
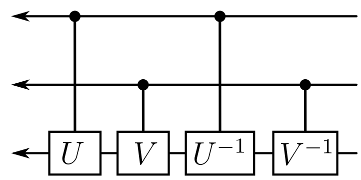
Ph/CS 219a, Fall 2022
Quantum Computation
Prof. Alexei Kitaev

Problem set #1
Due Thursday, October 6, 2022

1. (10 points) Find the operators represented by the following circuits:

a) $W_1 =$  ; b) $W_2 =$  .

Describe what these operators do and write them as matrices in the standard basis. (In the second case, U and V are arbitrary unitary operators. Write the result as a 4×4 matrix, where each element represents an operator acting on the third qubit.)

2. (10 points) Let the following gates be available (together with their inverses):

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \qquad \Lambda(\sigma^x) \quad \text{(controlled NOT)}. \tag{1}$$

Implement the controlled Hadamard operator $\Lambda(H)$ using these gates. **Hint:** First show how to implement $\Lambda(H)$ if we also have a single-qubit gate S such that $H = S\sigma^x S^{-1}$ (draw a circuit). Then construct S from H and R. The last part could be solved by trial-and-error, but the geometric picture of single-qubit gates will save you from tedious calculations or programming. Here is a brief summary of what we discussed in class.

Each single-qubit gate U defines a 3×3 rotation matrix $\Phi(U)$ with matrix elements $[\Phi(U)]_{\alpha\beta}$:

$$U\sigma^\beta U^{-1} = \sum_\alpha [\Phi(U)]_{\alpha\beta}\sigma^\alpha; \qquad U(\vec{n}\cdot\vec{\sigma})U^{-1} = \vec{n}'\cdot\vec{\sigma}, \text{ where } n'_\alpha = \sum_\beta [\Phi(U)]_{\alpha\beta} n_\beta. \tag{2}$$

Mathematically, $\Phi : \mathrm{U}(2) \to \mathrm{SO}(3)$ is a group homomorphism. For example,

$$\Phi(\sigma^x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} : \qquad \text{rotation about} \quad \vec{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{by angle } \pi; \tag{3}$$

$$\Phi(H) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} : \qquad \text{rotation about} \quad \vec{m} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{by angle } \pi; \tag{4}$$

$$\Phi(R) = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix} : \quad \text{rotation about} \quad \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{by angle } \frac{\pi}{4}. \tag{5}$$

Note that $\Phi(H)$ and $\Phi(R^2) = \Phi(R)^2$ generate the rotational symmetry group of a cube. As far as this problem is concerned, think how $\Phi(S)$ should act on $\vec{e}_x$.

3. (10 points) Suppose we can prepare qubits in the state $|0\rangle$ and act on them by the gates H, σ^x, σ^y, σ^z, and CNOT. It's clear that this set of operations is insufficient to create the state

$$|\eta_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + i|1\rangle\Big) \quad \text{or} \quad |\eta_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - i|1\rangle\Big), \tag{6}$$

even up to an overall phase factor. Indeed, the operations we use have real coefficients. (Well, almost: $\sigma^y = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ but we don't care about the overall factor.)

Show that it is, however, possible to copy an unknown state $|\psi\rangle$ with respect to the basis $\big\{|\eta_+\rangle, |\eta_-\rangle\big\}$. **Hint:** Using the above gate set, construct a circuit that performs $\Lambda(i\sigma^y)$, i.e., the controlled $i\sigma^y$. Prepare a suitable state in the first (controlling) qubit and send $|\psi\rangle$ to the second (controlled) qubit.