More about surface codes and anyons

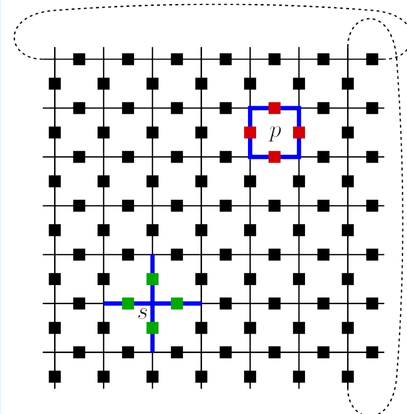
Perturbation stability of the toric code Hamiltonian

$$H_{TC} = -\sum_s A_s - \sum_p B_p$$

$$\downarrow$$

$$H = H_{TC} - h_x \sum_j \sigma_j^x - h_z \sum_j \sigma_j^z$$

$$(h_x,\ h_z \ll 1)$$

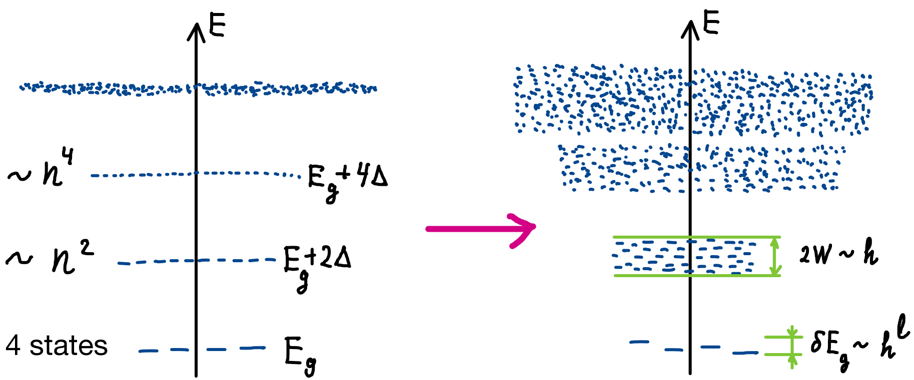


$\ell \times \ell$ lattice, $n = 2\ell^2$

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x$$

$$B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z$$

Effect on the energy spectrum (summary)



-- Quasiparticles become mobile; their eigenstates are no longer localized and their previously degenerate spectrum becomes a band of width $w \sim h$

-- Each k-particle level has width kw

-- The ground states split by $\delta E_g \sim e^{-\alpha \ell}$

Quasiparticle hopping and kinetic energy

Simpler model: Heisenberg ferromagnet (# of particles is preserved)

$$H = -J \sum_s \underbrace{\vec{\sigma}_s \cdot \vec{\sigma}_{s+1}}_{\sigma_s^x \sigma_s^x + \sigma_s^y \sigma_s^y + \sigma_s^z \sigma_s^z}$$

Restriction to two spins:

$$\vec{\sigma}_s \cdot \vec{\sigma}_{s+1} = \begin{array}{cc} \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array} \end{array}$$

Ground state: $|\xi\rangle = |\ldots\uparrow\uparrow\uparrow\ldots\rangle, \qquad E_g = -2nJ$

Single-particle states (form a subspace preserved by the Hamiltonian)

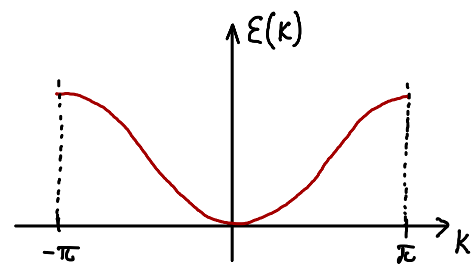
Position basis: $|\psi_s\rangle = |\ldots \uparrow \underbrace{\downarrow}_{s} \uparrow \ldots\rangle$

$$H - E_g = J \begin{pmatrix} \ddots & \ddots & & \\ \ddots & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{array}{c} \vdots \\ |\psi_{s-1}\rangle \\ |\psi_s\rangle \\ |\psi_{s+1}\rangle \\ \vdots \end{array}$$

Eigenstates of the Hamiltonian: $|\tilde{\psi}_{\underbrace{k}_{\text{momentum}}}\rangle \sim \sum_s e^{iks} |\psi_s\rangle$

Equation on the eigenvalues: $(H - E_g)|\tilde{\psi}_k\rangle = \varepsilon(k)|\tilde{\psi}_k\rangle$

Kinetic energy: $\varepsilon(k) = J\left(4 - 2e^{ik} - 2e^{-ik}\right) = 4J(1 - \cos k)$



Another relatively simple Hamiltonian: Transverse field Ising model (TFIM)

$$H = \underbrace{-J \sum_s \sigma^z_s \sigma^z_{s+1}}_{H_{Ising}} \underbrace{- h \sum_s \sigma^x_s}_{V}$$

In this lecture, we assume that $h \ll J$.

Properties of H_{Ising}:

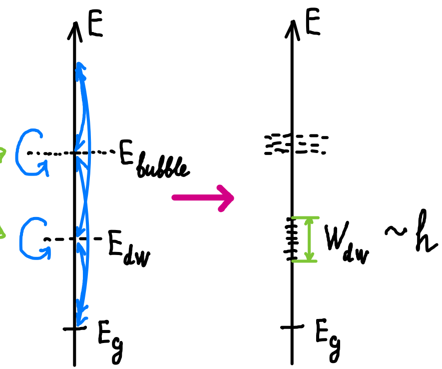
Ground states: $|\xi_\uparrow\rangle = |\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle$, $|\xi_\downarrow\rangle = |\ldots\downarrow\downarrow\downarrow\downarrow\ldots\rangle$

Single-particle states (domain walls): $|\psi_{\uparrow,s}\rangle = |\ldots\uparrow\uparrow\underset{s}{\uparrow}\underset{s+1}{\downarrow}\downarrow\downarrow\ldots\rangle$, $E_{dw} = E_g + 2J$

Two-particle states (bubbles): $|\psi_{\uparrow,s',s''}\rangle = |\ldots\uparrow\underset{s'}{\uparrow}\downarrow\downarrow\underset{s''}{\downarrow}\uparrow\uparrow\ldots\rangle$, $E_{bubble} = E_g + 4J$

First-order perturbation effects

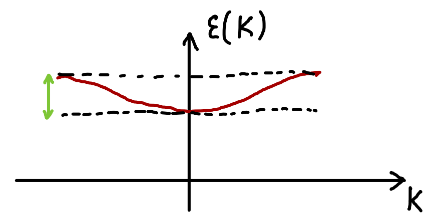
We consider matrix elements of V between unperturbed states of equal energy



Energy of a domain wall: $E_{dw} = E_g + \varepsilon(k)$

$$\varepsilon(k) \approx 2J - 2h\cos k$$

(to the first order in h/J)



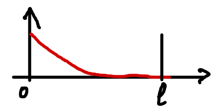
$$\varepsilon(k) = 2\sqrt{(J - h e^{ik})(J - h e^{-ik})}$$

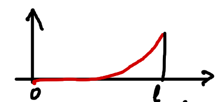
(exact result)

Higher-order perturbation effects in TFIM

Admixture of domain wall and bubble states to the ground state

$$|\Psi_\uparrow\rangle = f_0|\xi_\uparrow\rangle + \sum_s f_{\mathrm{l,dw}}(s)\underbrace{|\Psi_{\downarrow,s}\rangle}_{|\underset{s}{\downarrow}\underset{s+1}{\downarrow}\uparrow\uparrow\uparrow\dots\rangle} + \sum_s f_{\mathrm{r,dw}}(s)\underbrace{|\Psi_{\uparrow,s}\rangle}_{|\dots\uparrow\uparrow\uparrow\underset{s}{\downarrow}\underset{s+1}{\downarrow}\rangle} + \sum_{s'<s''} f_{\mathrm{bubble}}(s',s'')\underbrace{|\Psi_{\uparrow,s',s''}\rangle}_{|\dots\uparrow\underset{s'}{\uparrow}\downarrow\downarrow\underset{s''}{\downarrow}\uparrow\uparrow\dots\rangle}$$



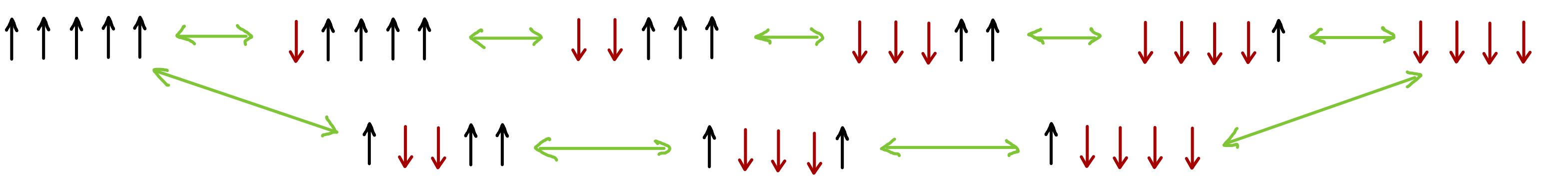


Wavefunctions: $f_{\mathrm{l,dw}}(s) \sim e^{-\alpha s}$ $\quad f_{\mathrm{r,dw}}(s) \sim e^{-\alpha(\ell - s)}$ $\quad f_{\mathrm{bubble}}(s',s'') \sim e^{-\alpha(s''-s')}$

(evanescent waves with momentum $k = \pm i\alpha$)

$$\boxed{\varepsilon(\pm i\alpha) = 0 \Rightarrow e^{-\alpha} = \frac{h}{J}}$$

Tunneling between the $\uparrow$ and $\downarrow$ ground states



Effective Hamiltonian: $$\boxed{H_{eff} = -u\left(|\Psi_\downarrow\rangle\langle\Psi_\uparrow| + |\Psi_\uparrow\rangle\langle\Psi_\downarrow|\right), \quad u \sim e^{-\alpha\ell} = \left(\frac{h}{J}\right)^\ell}$$

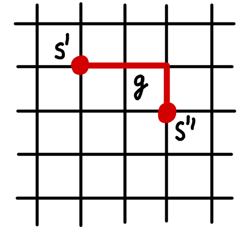
Back to the toric code Hamiltonian

$$H = -J_e \sum_s A_s - J_m \sum_p B_p - \underline{h_x \sum_j \sigma_j^x} - \underline{h_z \sum_j \sigma_j^z}$$

Let $h_z \ll J_e$, $h_x \ll J_m$

Admixture of two-charge states to the ground state

$$|\xi\rangle \mapsto |\xi\rangle + \frac{h_z}{2\Delta_e}|\Psi_{2e}\rangle, \qquad \Delta_e = 2J_e$$



The "strings of Z-errors", g, plays the role of the bubble. However, it is unphysical. Only its boundary, i.e. the pair of sites s', s'', has a physical meaning.

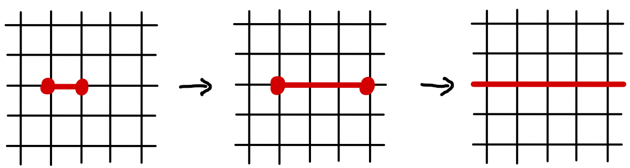
$$|\Psi_{2e}\rangle = \sum_{s',s''} f(s',s'')\cdot\sigma^z(g)|\xi\rangle,$$

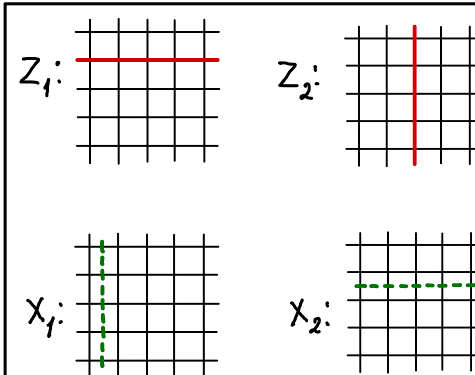
shortest string

$$f(s',s'') \sim e^{-\alpha|g|}, \qquad e^{-\alpha} \approx \frac{h_z}{\Delta_e}$$

Tunneling across the torus effects a logical operator on the code space

Logical Z_1:





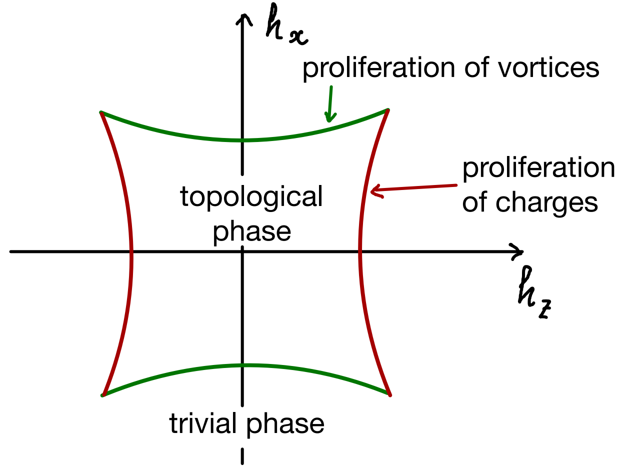
Effective Hamiltonian:

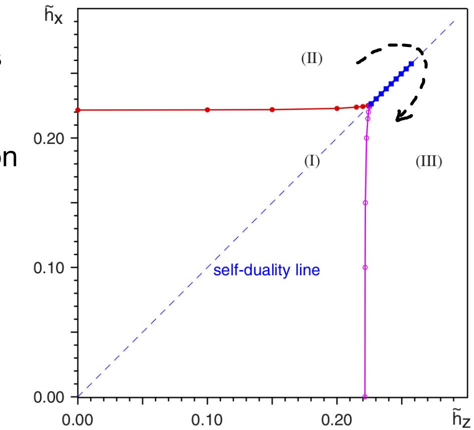
$$\boxed{H_{eff} = -u_{e1}Z_1 - u_{e2}Z_2 - u_{m1}X_1 - u_{m2}X_2}$$

$$u_{e1} = u_{e2} \sim \left(\begin{matrix}\text{\# of places to nucleate}\\ \text{a pair of charges}\end{matrix}\right)\cdot\left(\begin{matrix}\text{tunneling}\\ \text{amplitude}\end{matrix}\right) \sim n\cdot\left(\frac{h_z}{\Delta}\right)^{\ell}$$

Phase diagram

$$H = -\sum_s A_s - \sum_p B_p - h_x \sum_j \sigma_j^x - h_z \sum_j \sigma^z$$

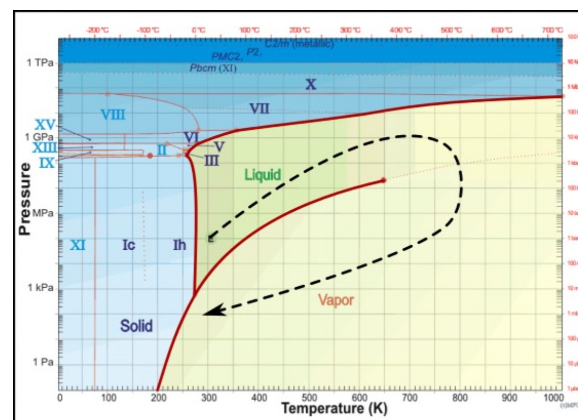




Tupitsyn, Kitaev, Prokof'ev, Stamp (2010)

Related to the gauge Higgs model, a 3D statistical mechanics model studied by Fradkin and Shenker in 1979. The latter can be simulated on a classical computer.

This phase diagram is similar to the phase diagram of water:



Thermal excitations

The previous discussion was about ground state properties. At finite temperature, there is, on average, a finite density of quasiparticles. Pairs or charges or vortices occasionally appear due to interaction with the thermal environment. Before they annihilate, a quasiparticle can travel across the torus, causing a logical error.

Average # of ee pairs in the system:

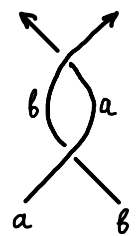
$$\langle N_e \rangle \sim n^2 \cdot e^{-\frac{2\Delta e}{T}} \ll 1$$

(at small *T*)

Some general properties of Abelian anyons

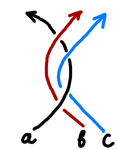
1) The set of superselection sectors L is an Abelian group under fusion $(\times)$

2) Double braiding

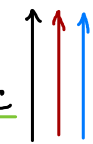


$$= R_{b,a} R_{a,b} = W_{a,b}\, I_{a,b}$$

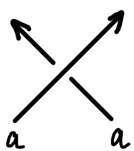
$$\boxed{\begin{aligned} &W : L \times L \to U(1) \\ &W_{a\times b,\, c} = W_{a,c}\, W_{b,c} \\ &\underline{W_{a,\, b\times c}} = W_{a,b}\, W_{a,c} \end{aligned}}$$



$$= \underline{W_{a,\, b\times c}}$$

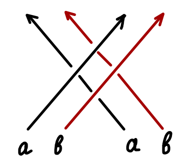


3) Braiding of identical particles

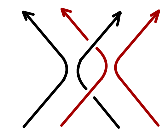


$$= R_{a,a} = \theta_a\, I_{a,a}$$

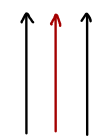
$$\boxed{\begin{aligned} &\theta : L \to U(1) \\ &\theta_a^2 = W_{a,a} \\ &\underline{\theta_{a\times b} = \theta_a\, \theta_b\, W_{a,b}} \end{aligned}}$$



$$= \theta_a\, \theta_b$$



$$= \underline{\theta_a\, \theta_b\, W_{a,b}}$$

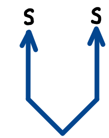


Example: semions ("half-fermions")

$$L = \{1, S\} \cong \mathbb{Z}_2, \qquad S \times S = 1, \qquad W_{S,S} = -1, \qquad \theta_S = i$$

Complication: associativity relations

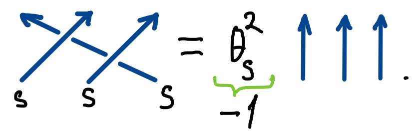
We have $S \times S = 1$. Hence, we can create two s particles from the vacuum by some operator:



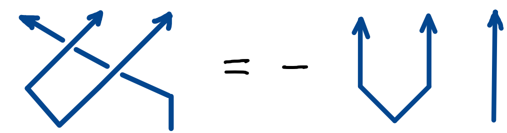
(Caution: This operator is not invariant under $180°$ rotation or braiding)

This operator is unique up to an overall phase, and we fix the phase.

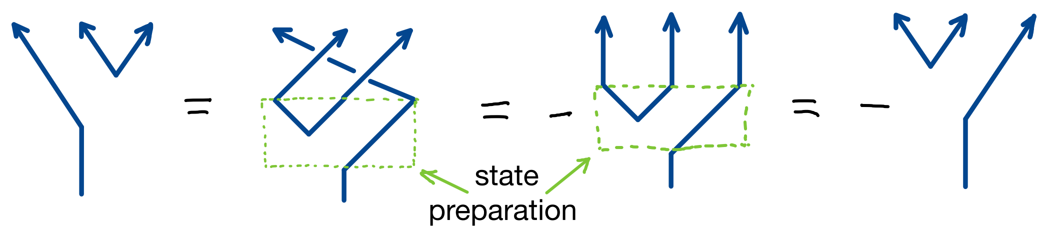
We also know that

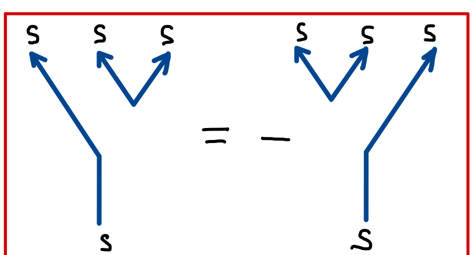


Therefore,

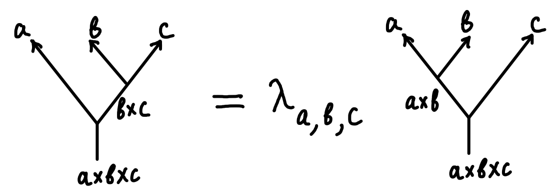


Now,





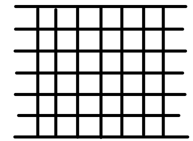
In general (but still considering Abelian anyons), the associativity relations are given by a 3-cocycle $\lambda: L \times L \times L \to U(1)$

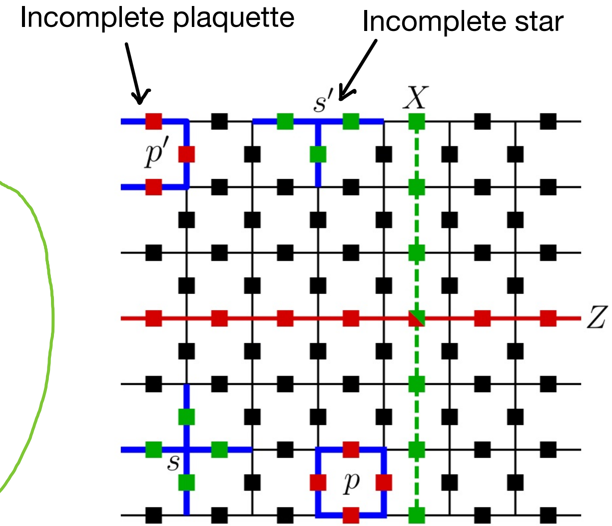


(The vertices are particle splitting operators)

Surface codes with boundaries

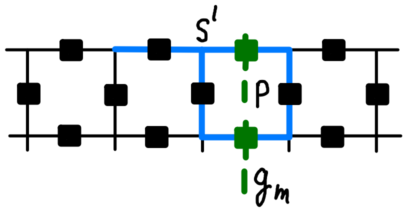
Motivation: Planar layout with local interaction between the physical qubits.





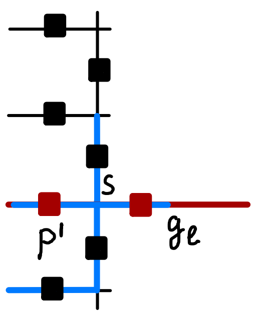
Two types of boundary

Smooth boundary (absorbs m-particles)



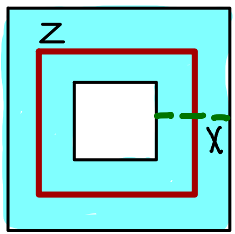
The operator $\sigma^x(g_m)$, which transports an m-particle, commutes with $A_{s'}, B_p$

Rough boundary (absorbs e-particles):

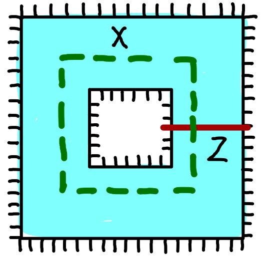


The operator $\sigma^z(g_e)$, transporting an e-particle, commutes with A_s and $B_{p'}$

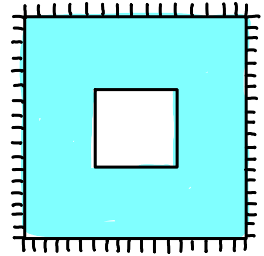
Other topologies



1 qubit

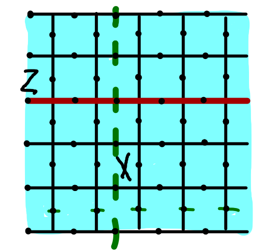


1 qubit



0 qubits

cylinder:
the two sides are
glued together



1 qubit

this cycle is trivial because it is equal to the product of incomplete stars at the bottom

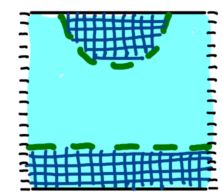
General rules

The code is defined on a manifold with two complementary boundaries: B_m (smooth) and B_e (rough)

$$D_z^\perp = \text{Cycles}(M, B_m; \mathbb{Z}_2), \qquad D_x = \text{Boundaries}(M, B_m; \mathbb{Z}_2)$$

cycles on M *relative to* B_m
(can end on B_m)

relative boundaries of regions: the part of the boundary that lies in B_m is ignored

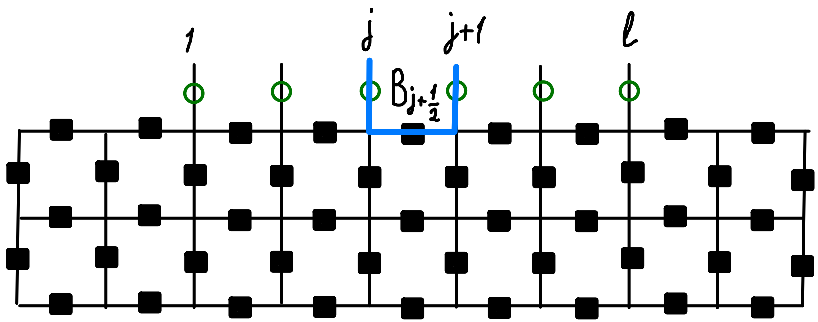


All logical operators modulo trivial ones

X-type: $\sigma^X(g_m), \quad g_m \in D_z^\perp / D_x = H_1(M, B_m; \mathbb{Z}_2)$

Z-type: $\sigma^Z(g_e), \quad g_e \in D_x^\perp / D_z = H_1(M, B_e; \mathbb{Z}_2) \cong H^1(M, B_m; \mathbb{Z}_2)$

Transition between rough and smooth boundary in terms of TFIM



operators σ_j^x act on the "active" spins, denoted by hollow circles

$$H = -J\sum_s A_s - J\sum_p B_p - \underbrace{h\sum_{j=1}^{l}\sigma_j^x}$$

commutes with all other terms except the incomplete plaquettes

$B_{j+\frac{1}{2}}$ $(j=1,..,l-1)$

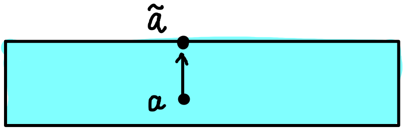
$h=0$: rough boundary is undisturbed

$h\to\infty$: active spins freeze in the $|+\rangle$ state

The operator σ_j^x, $B_{j+\frac{1}{2}}$ have the same commutation relations as σ_j^x, $\sigma_j^z\sigma_{j+1}^z$ in TFIM

The transverse field Ising model has a phase transition at *h*=*J*. Therefore, the rough boundary in our model has the same topological properties as in the surface code for *h*<*J*. In the opposite case, *h*>*J*, the boundary becomes equivalent to smooth boundary.

Boundary excitations and domain walls

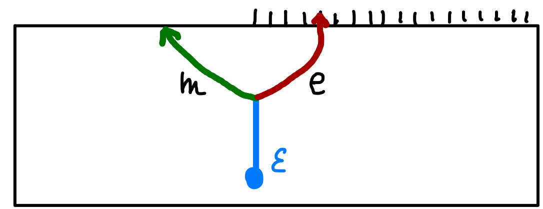


$a \mapsto \tilde{a}$: bulk particle of type a becomes a boundary particle of type $\tilde{a}$

Smooth boundary: $\begin{cases} 1, m \mapsto 1 \\ e, \varepsilon \mapsto \varepsilon \end{cases}$

Rough boundary: $\begin{cases} 1, e \mapsto 1 \\ m, \varepsilon \mapsto \varepsilon \end{cases}$

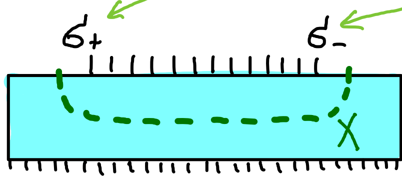
An interesting process, in which a bulk ε-particle splits into m and e, which disappear at suitable boundaries. Thus, one can fuse an ε-particle with a domain wall not even touching it!



Fusion rules

smooth on the left, rough on the right

the other way around



$\sigma_+ \times \sigma_- = 1 + \varepsilon$

$$\boxed{\begin{aligned} \sigma_+ \times \sigma_- &= \sigma_- \times \sigma_+ = 1 + \varepsilon \\ \varepsilon \times \sigma_+ &= \sigma_+ \times \varepsilon = \sigma_+ \\ \varepsilon \times \sigma_- &= \sigma_- \times \varepsilon = \sigma_- \\ \varepsilon \times \varepsilon &= 1 \end{aligned}}$$

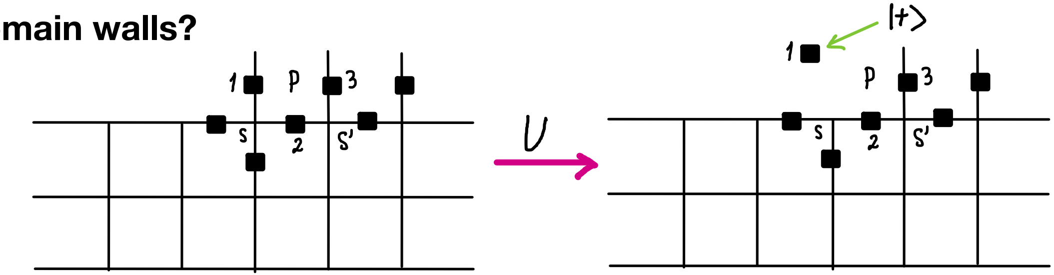
The fusion outcome depends on the state of the logical qubit:

if $X|\psi\rangle = |\psi\rangle$, then the domain walls fuse into 1;

if $X|\psi\rangle = -|\psi\rangle$, then they fuse into ε

How do we actually move and fuse domain walls?

Moving σ_+ one step to the right



$$U = CNOT[3,1] \cdot CNOT[2,1]$$

$$\sigma_j^\alpha \mapsto U \sigma_j^\alpha U^{-1}$$

$$\sigma_1^z \mapsto \sigma_1^z \sigma_2^z \sigma_3^z \qquad \sigma_1^x \mapsto \sigma_1^x$$

$$\sigma_2^z \mapsto \sigma_2^z \qquad \sigma_2^x \mapsto \sigma_1^x \sigma_2^x$$

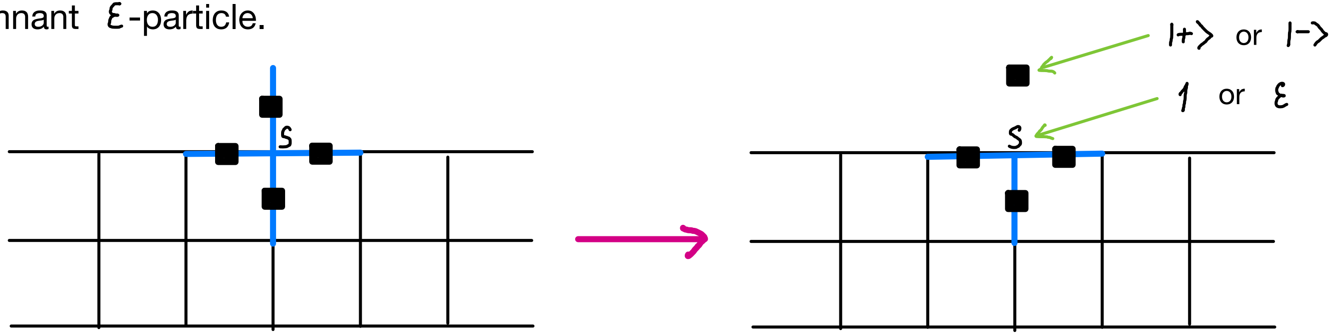
$$\sigma_3^z \mapsto \sigma_3 \qquad \sigma_3^x \mapsto \sigma_1^x \sigma_3^x$$

$A_s \mapsto \tilde{A}_s \Rightarrow$ if $|\psi\rangle$ is a codeword, then $U|\psi\rangle$ is a codeword for the new code

$B_p \mapsto \sigma_1^z \Rightarrow U|\psi\rangle$ has $|+\rangle$ in qubit 1

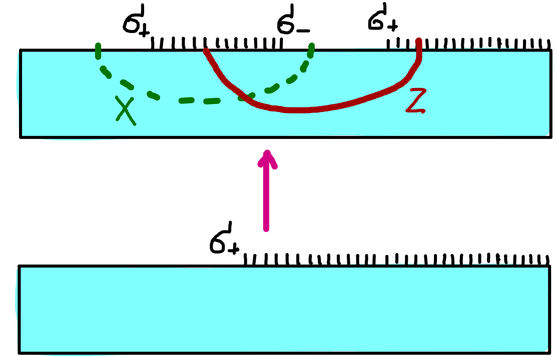
Fusing

At the last step, we remove a qubit and replace the full star operator A_s with the incomplete star $\tilde{A}_s$. If the qubit is in the $|+\rangle$ state, then $\tilde{A}_s|\psi\rangle = |\psi\rangle$. Otherwise, $\tilde{A}_s|\psi\rangle = -|\psi\rangle$, and hence, there is a remnant ε-particle.

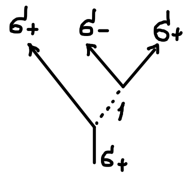


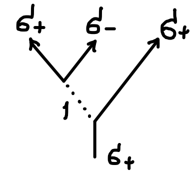
Some associativity relations

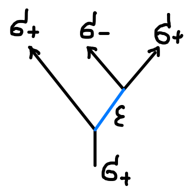
Splitting σ_+ into $\sigma_+, \sigma_-, \sigma_+$ is equivalent to adding a logical qubit

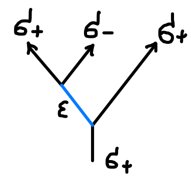


Its initial state depends on the exact splitting process:

$|0\rangle =$ 

$|+\rangle =$ 

$|1\rangle =$ 

$|-\rangle =$ 

Thus,