# General theory of quantum error correction

- Physical error models
- A nicer (but equivalent) formulation of the quantum error correction condition
- Subsystem encoding (of the logical qubit in the presence of a correctable error)
- Removing the error

**Probabilistic model:** each qubit is subjected to a Pauli error with probability *p*.

Depolarizing channel

$$T = (1-p)\, I \cdot I + \frac{p}{3}\sigma^x \cdot \sigma^x + \frac{p}{3}\sigma^y \cdot \sigma^y + \frac{p}{3}\sigma^z \cdot \sigma^z$$

$$T\rho = (1-p)\,\rho + \frac{p}{3}\sigma^x \rho\, \sigma^x + \cdots$$

Notation: $A \cdot B$

is the superoperator

$\rho \mapsto A \rho B$

Simple generalization

$$T = (1-p)\, I \cdot I + \sum_\alpha p_\alpha \sigma^\alpha \cdot \sigma^\alpha, \qquad p = p_x + p_y + p_z$$

This model allows for a classical probabilistic analysis ("likely" and "unlikely" errors)

However, it does not include all types of single-qubit errors.

## More general channels

Terminology:
A *channel* is a completely-positive trace-preserving superoperator

**Coherent errors:** $T = U \cdot U^\dagger, \quad \|U - I\| \le \delta$

**Amplitude-damping channel:** $T = M_0^\dagger \cdot M_0 + M_1^\dagger \cdot M_1$

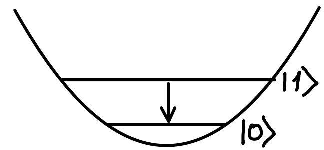
$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$|0\rangle \mapsto |0\rangle$

$|1\rangle \to |0\rangle$ with probability *p*

$|1\rangle \to |1\rangle$ with probability *1-p*




Error analysis involves the *diamond norm (see KSV 11.5)*

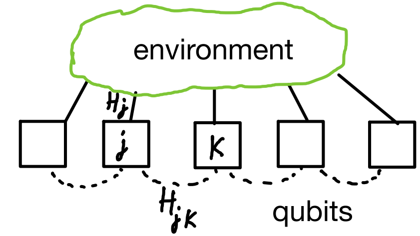
Error strength = $\|T - I\|_\diamond$, where

$$\|R\|_\diamond = \inf\Big\{\|A\|\,\|B\| \,:\, \mathrm{Tr}_{\mathcal{F}}(A \cdot B^\dagger) = T\Big\} = \sup_{X \neq 0} \frac{\|(T \otimes I_{\mathrm{L}(\mathcal{G})})X\|_1}{\|X\|_1}$$

$$\text{for} \quad T : \mathrm{L}(\mathcal{N}) \to \mathrm{L}(\mathcal{N}'), \quad \dim \mathcal{G} \geqslant \dim \mathcal{N}.$$

<u>Shortcoming</u>: the superoperator model does not cover errors that are correlated in space in time

**Interaction Hamiltonian model** (captures some types of correlated noise)




$$H = \underbrace{H_{qubits}(t) \otimes I + I \otimes H_{env}}_{H_0} + \underbrace{\sum_j H_j + \sum_{j<K} H_{jK}}_{V}$$

Interaction of the $j$-th qubit with environment: $H_j = \sum_\alpha \sigma_j^\alpha \otimes B_{j,\alpha}$

For simplicity, let $H_{qubits} = 0$ (quantum memory), and let $H_{jK} = 0$.

Thus, $$\boxed{H = H_0 + V, \quad H_0 = I \otimes H_{env}, \quad V = \sum_{j,\alpha} \sigma_j^\alpha \otimes B_{j,\alpha}}$$

We will consider the evolution operator $\boxed{U = e^{-iH\tau}}$ ($\tau$ is the wait time until error correction)

and show that
$$\boxed{U = U_{good} + U_{bad} \text{ such that } \begin{cases} U_{good} \in \mathcal{E}(n,r) \otimes \mathbb{L}(\mathcal{H}_{env}) \\ \|U_{bad}\| \text{ is small if } \|B_{j,\alpha}\| \text{ is small} \end{cases}}$$

## Expressing the evolution operator in the interaction representation

Evolution over time *t*: $$U(t) = e^{-i(H_0+V)t}$$

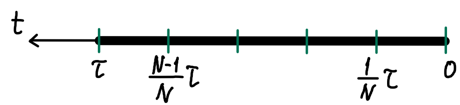
Difference between the evolutions with and without interaction: $$\breve{U}(t) = e^{iH_0 t} e^{-i(H_0+V)t}$$

Interaction representation of *V*: $$\breve{V}(t) = e^{iH_0 t} V e^{-iH_0 t}$$

$$\Rightarrow \quad \frac{dU}{dt} = -i(H_0+V)U$$

$$\frac{d\breve{U}}{dt} = e^{iH_0 t}(-iV)e^{-i(H_0+V)t} = -i\breve{V}\breve{U}$$

Discrete time approximation (Trotterization):




$$\breve{U}(t+\Delta t) \approx \left(1 - i\Delta t\, \breve{V}(t)\right)\breve{U}(t)$$

$$\breve{U} = \breve{U}(\tau) \approx \left(1 - i\frac{\tau}{N}\breve{V}\left(\tfrac{N-1}{N}\tau\right)\right)\cdots\left(1 - i\frac{\tau}{N}\breve{V}\left(\tfrac{1}{N}\tau\right)\right)\cdots\left(1 - i\frac{\tau}{N}\breve{V}(0)\right)$$

$$= 1 + \left(-i\frac{\tau}{N}\right)\sum_{N>j\geqslant 0}\breve{V}\left(\tfrac{j}{N}\tau\right) + \left(-i\frac{\tau}{N}\right)^2\sum_{N>j_2>j_1\geqslant 0}\breve{V}\left(\tfrac{j_2}{N}\tau\right)\breve{V}\left(\tfrac{j_1}{N}\tau\right) + \cdots$$

$t_1$ $\qquad$ $t_2$ $\qquad$ $t_1$

Back to the continuous time:

$$\breve{U} = 1 + (-i)\int_{\tau>t_1>0}\breve{V}(t_1)\,dt_1 + (-i)^2\iint_{\tau>t_2>t_1>0}\breve{V}(t_2)\,\breve{V}(t_1)\,dt_1\,dt_2 + \cdots$$

$$e^{iH_0\tau}e^{-iH\tau} = \breve{U} = \sum_{s=0}^{\infty} X_s, \qquad X_s = (-i)^s \int_{\tau>t_s>\cdots>t_1>0} \cdots \int \breve{V}(t_s)\cdots\breve{V}(t_1)\, dt_1\cdots dt_s$$

$X_s$ acts on at most $s$ qubits and the environment

where $\breve{V}(t) = e^{iH_0 t} V e^{-iH_0 t}$, $\quad V = \sum_{j,\alpha} \sigma_j^\alpha \otimes B_{j,\alpha}$

$$\boxed{\breve{U} = \breve{U}_{good} + \breve{U}_{bad}, \qquad \breve{U}_{good} = \sum_{s=0}^{r} X_s \in \mathcal{E}(n,r)\otimes \mathbb{L}(\mathcal{H}_{env}), \qquad \breve{U}_{bad} = \sum_{s=r+1}^{\infty} X_s}$$

$$U = e^{-iH_0\tau}\breve{U} = U_{good} + U_{bad}$$ ($H_0$ acts only on the environment)

Suppose $\boxed{\|B_{j,\alpha}\| \leq h}$ (for all $j,\alpha$)

Then $\|V\| \leq \sum_{j,\alpha} \underbrace{\|\sigma_j^\alpha\|}_{1} \cdot \underbrace{\|B_{j,\alpha}\|}_{\leq h} \leq 3nh$

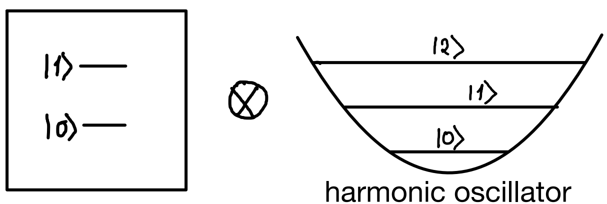
$$\|X_s\| \leq \frac{\|V\|^s \tau^s}{s!} \leq \frac{(3nh\tau)^s}{s!}$$

$$\boxed{\|U_{bad}\| \leq \sum_{s=r+1}^{\infty} \frac{(n\delta)^s}{s!} = O\left((n\delta)^{r+1}\right) \quad \text{where } \delta = 3h\tau}$$

## Beyond the bounded norm assumption

In reality, interaction with the environment is rarely bounded by the norm. However, errors may be suppressed due to energetic reasons:
If the Hamiltonian changes slowly, the total energy is conserved.

## Jaynes-Cumming model

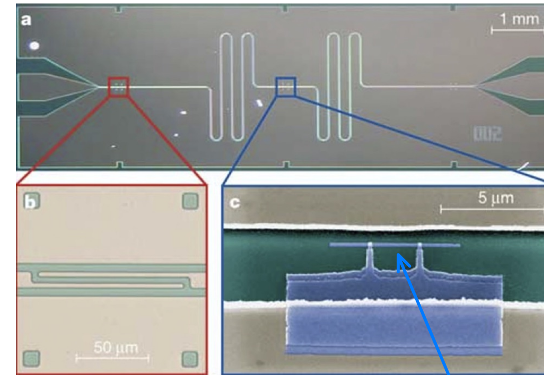



qubit

harmonic oscillator

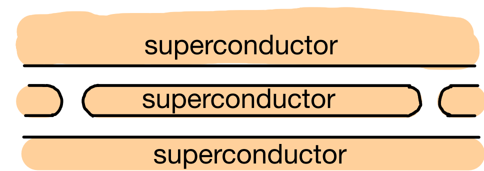
$$E_m = \left(m + \tfrac{1}{2}\right)\omega$$

Wallraff et al, Nature 431, 62 (2004)




Josephson junction-based qubit

Oscillator:




$$H = \underbrace{\left(E_0 |0\rangle\langle 0| + E_1 |1\rangle\langle 1|\right)}_{H_{qubit}} \otimes I + I \otimes \underbrace{\omega\left(a^{+}a + \tfrac{1}{2}\right)}_{H_{env}} + \underbrace{v\,|0\rangle\langle 1| \otimes a^{+} + v^{*}|1\rangle\langle 0| \otimes a}_{V},$$

(only resonant terms are included)

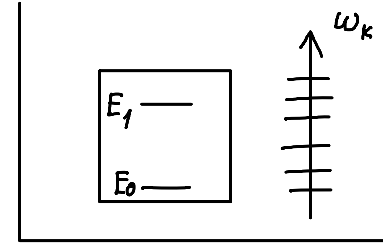
$$a^{+}|m\rangle = \sqrt{m+1}\,|m+1\rangle$$

$$\|a^{+}\| = \infty$$

Relevant states of the oscillator: $|0\rangle, |1\rangle$

If we restrict $a^{+}, a$ to this subspace, then the interaction is bounded

## Multiple oscillators




$$H = H_{qubit} \otimes I + I \otimes \left(\sum_k \omega_k \, a_k^+ a_k\right) + \underbrace{\sum_{k=1}^{N} \left(v_k |0\rangle\langle 1| \otimes a_k^+ + h.c.\right)}_{V}$$

Even after the restriction to the subspace with $\leqslant 1$ photons, $\|V\| \sim v\sqrt{N}$

$$C_k = \frac{v_k}{E_1 - (E_0 + \omega_k)}$$

Perturbation theory for the eigenvectors:

$$|\tilde{0}\rangle = |0\rangle \otimes |0\,0\,0\ldots\rangle$$

$$|\tilde{1}\rangle \approx |1\rangle \otimes |0\,0\,0\ldots\rangle + \sum_k C_k |0\rangle \otimes |0\cdots 0\,\underbrace{1}_{k\text{-th place}}\,0\cdots\rangle$$

If $\underline{\sum_k |C_k|^2 \ll 1}$, then the entanglement with the environment (for the eigenstates $|\tilde{1}\rangle$) is small;

otherwise the qubit will entangle with the environment and eventually decay.

**Decay rate:** $\underline{\Gamma = 2\pi\, |v|^2 \cdot \nu(E_1 - E_0)}$, where $\nu(\omega)$ is the density of states

The finite-time evolution can be modelled as the amplitude-damping channel with $1 - p = e^{-\Gamma\tau}$

**Quantum error correction condition**

$\boxed{\mathcal{N} \text{ -- physical Hilbert space}, \quad \mathcal{E} \subseteq \mathbf{L}(\mathcal{N}) \text{ -- error space}}$

**Original form.** A code $\mathcal{M} \subseteq \mathcal{N}$ *protects from errors in* $\mathcal{E} \subseteq \mathbf{L}(\mathcal{N})$ if

$$\forall\, |\xi_1\rangle, |\xi_2\rangle \in \mathcal{M} \quad \forall\, E_1, E_2 \in \mathcal{E}, \quad |\xi_1\rangle \perp |\xi_2\rangle \Rightarrow E_1|\xi_1\rangle \perp E_2|\xi_2\rangle. \tag{A}$$

**Nicer form.** A code $\mathcal{M}$ *protects from errors in* $\mathcal{E}$ if there is a function $c : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ such that

$$\forall\, |\xi_1\rangle, |\xi_2\rangle \in \mathcal{M} \quad \forall\, E_1, E_2 \in \mathcal{E}, \quad \boxed{\langle \xi_1 | E_1^\dagger E_2 | \xi_2 \rangle = c(E_1, E_2)\, \langle \xi_1 | \xi_2 \rangle.} \tag{B}$$

Note: $C(E_1,E_2)$ is a linear function of $E_1^+ E_2$

Obviously, (B) implies (A). We will prove the converse: $\underline{(A) \Rightarrow (B)}$

Let $E_1, E_2 \in \mathcal{E}$, and let $|\xi_1\rangle, \ldots |\xi_m\rangle$ be some orthonormal basis of $\mathcal{M}$.

If $j \neq k$, then $|\xi_j\rangle \perp |\xi_k\rangle \Rightarrow E_1|\xi_j\rangle \perp E_2|\xi_k\rangle \Rightarrow \langle \xi_j | E_1^+ E_2 | \xi_k \rangle = 0$

We now show that $\underline{C(E_1,E_2) := \langle \xi_j | E_1^+ E_2 | \xi_j \rangle \text{ does not depend on } j.}$

Let $|\eta_\pm\rangle = |\xi_j\rangle \pm |\xi_k\rangle$ (for some *j* and *k*)

$$|\eta_-\rangle \perp |\eta_+\rangle \implies 0 = \langle \eta_- | \underbrace{E_1^+ E_2}_{E} | \eta_+ \rangle = \langle \xi_j | E | \xi_j \rangle + \cancel{\langle \xi_j | E | \xi_k \rangle}$$
$$- \cancel{\langle \xi_k | E | \xi_j \rangle} - \langle \xi_k | E | \xi_k \rangle$$

**How is quantum information encoded after the action of error?**

Some errors are equivalent: $E_1 \equiv E_2$ meaning that $\underbrace{\forall |\xi\rangle \in \mathcal{M} \quad E_1|\xi\rangle = E_2|\xi\rangle}_{\Leftrightarrow \; E_1 - E_2 \equiv 0}$

Space of null errors: $\mathcal{E}_0 := \{ E \in \mathcal{E} : \quad E \equiv 0 \}$

Reduced error space: $\mathcal{E}' := \mathcal{E}/\mathcal{E}_0$ (quotient space)

Hermitian inner product on $\mathcal{E}'$: $\langle \xi_1 | E_1^\dagger E_2 | \xi_2 \rangle = \underbrace{C(E_1, E_2)}_{\langle E_1' | E_2' \rangle} \langle \xi_1 | \xi_2 \rangle$

Before the error, the logical qubits are encoded by an isometric embedding $V : \mathcal{L} \to \mathcal{N}$ s.t. $\mathrm{Im}\, V = \mathcal{M}$

After the error, we have a *subsystem encoding*: in addition to the logical qubits, the reduced error is also encoded

$$\boxed{\tilde{V} : \mathcal{L} \otimes \mathcal{E}' \to \mathcal{N}, \qquad \tilde{V}\left(|\psi\rangle \otimes |E'\rangle\right) = E\, V|\psi\rangle}$$

The quantum error correction condition says that $\tilde{V}^\dagger \tilde{V} = I$, i.e. that $\tilde{V}$ preserves the inner product:

$$\left(\langle \psi_1| \otimes \langle E_1'|\right) \tilde{V}^\dagger \tilde{V} \left(|\psi_2\rangle \otimes |E_2'\rangle\right) = \underbrace{\langle \psi_1 | V^\dagger}_{\xi_1 \in \mathcal{M}} E_1^\dagger E_2 \underbrace{V|\psi_2\rangle}_{\xi_2 \in \mathcal{M}} = \underbrace{C(E_1, E_2)}_{\langle E_1' | E_2' \rangle} \langle \psi_1 | \underbrace{V^\dagger V}_{I} | \psi_2 \rangle$$

## Removing the (reduced) error from the system

Recovery (error extraction) map: $R: \mathcal{N} \to \mathcal{N} \otimes \mathcal{E}'$, $\quad R^\dagger R = I_{\mathcal{N}}$

$$\boxed{R|\eta\rangle = \begin{cases} |\xi\rangle \otimes |E'\rangle & \text{if } |\eta\rangle = E|\xi\rangle \text{ for some } E \in \mathcal{E},\ |\xi\rangle \in \mathcal{M} \\ |\eta\rangle \otimes |I'\rangle & \text{if } |\eta\rangle \perp \mathcal{E}\mathcal{M} \quad \text{(this is rather arbitrary and not important)} \end{cases}}$$

## Finally, we can show rigorously how "good" errors are corrected (using our interaction model)

Let $|\xi\rangle = V|\psi\rangle \in \mathcal{M}$, and let $|\Psi_{env}\rangle$ be the initial state of the environment

$$|\xi\rangle \otimes |\Psi_{env}\rangle \xmapsto{\text{error}} U(|\xi\rangle \otimes |\Psi_{env}\rangle) \xmapsto{\text{recovery}} (R \otimes I_{env})U(|\xi\rangle \otimes |\Psi_{env}\rangle) \in \mathcal{N} \otimes \mathcal{E}' \otimes \mathcal{H}_{env}$$

We have: $U_{good} = \sum_j E_j \otimes Y_j$, $\quad E_j \in \mathcal{E} = \mathcal{E}(n,r)$, $\quad Y_j$ acts in $\mathcal{H}_{env}$.

$$(R \otimes I_{env}) U_{good}(\underbrace{|\xi\rangle}_{V|\psi\rangle} \otimes |\Psi_{env}\rangle) = \sum_j R E_j |\xi\rangle \otimes Y_j |\Psi_{env}\rangle = \underbrace{|\xi\rangle}_{V|\psi\rangle} \otimes \underbrace{\sum_j |E_j'\rangle \otimes Y_j}_{Q}|\Psi_{env}\rangle$$

$Q$ (captures the error and acts on the environment)

$$(R \otimes I_{env}) U_{good}(V \otimes I_{env}) = V \otimes Q \implies \boxed{\|(R \otimes I_{env}) U (V \otimes I_{env}) - V \otimes Q\| \leq O((n\delta)^{r+1})}$$