Transverse field Ising model and Majorana chain

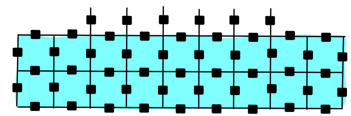
Different realizations of the transverse field Ising model (TFIM)

$$\boxed{H = -J\sum_{j=1}^{m-1} \underline{\sigma_j^z \sigma_{j+1}^z} - h\sum \underline{\sigma_j^x}}$$

$$\uparrow\ \uparrow\ \downarrow\ \uparrow\ \uparrow\ \uparrow\ \downarrow\ \downarrow$$

1) Spin chain: $|0\rangle_L = |\psi_\uparrow\rangle \approx |\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle$, $|1\rangle_L = |\psi_\downarrow\rangle \approx |\ldots\downarrow\downarrow\downarrow\downarrow\ldots\rangle$ unprotected

2) An interval of rough boundary surrounded by smooth boundary



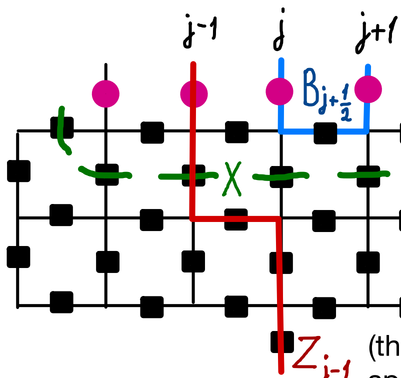
3) Majorana chain

(to be defined later)

protected; can be used as quantum memory elements

We will show that models 2 and 3 are *formally* equivalent to model 1, in the sense that the terms in the corresponding Hamiltonians are mapped to the operators $\underline{\sigma_j^z \sigma_{j+1}^z}$ and $\underline{\sigma_j^x}$ such that all algebraic relations are preserved. All three models have a two-fold degenerate ground state, which represents a logical qubit. However, different physical systems allow different *perturbations*. In particular, the logical Z is realized by a local operator (σ_j^z) in the spin chain and by nonlocal operators in the other two models. Nonlocal operators are unlikely to appear as perturbations to the Hamiltonian!

An interval of rough boundary surrounded by smooth boundary



(the product of σ^z on a path connecting active spin *j-1* with a "reference spin" on the bottom)

operators σ_j^x act on the "active" spins, denoted by circles

$$\boxed{H = -J\sum_s A_s - J\sum_p B_p - \underbrace{h\sum_{j=1}^{m}\sigma_j^x}}$$

commutes with all other terms except the incomplete plaquettes

$B_{j+\frac{1}{2}} \quad (j=1,..,l-1)$

We work in the subspace $\mathcal{H} = \{|\psi\rangle \in \mathcal{B}^{\otimes n} : A_s|\psi\rangle = |\psi\rangle,\ B_p|\psi\rangle = |\psi\rangle$ for all A_s, B_p except $B_{j+\frac{1}{2}}\}$

Operator mapping from the spin chain to our model

$$\sigma_j^x \longmapsto \sigma_j^x$$

$$\sigma_j^z \longmapsto Z_j$$

$$\sigma_j^z\sigma_{j+1}^z \longmapsto B_{j+\frac{1}{2}} \equiv Z_j Z_{j+1}$$

Logical X:

$$X \equiv \prod_{j=1}^{m} \sigma_j^x$$

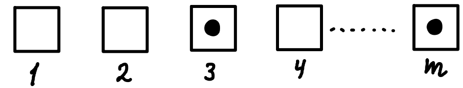
In the unperturbed case, each of these effects a logical *Z*

$\equiv$ means that the operators act in $\mathcal{H}$ in the same way

If $h \ll J$, both the spin chain and the surface code Hamiltonian protect from a logical *X* error. However, a logical *Z* error is likely to happen in the spin chain but not in the surface code.

Introducing fermions

m fermionic modes:



(a box can be empty or filled with a particle)

occupation numbers: $n_1 = 0$ $\quad n_3 = 1$ $\quad n_m = 1$

Fock states
(basis states of the Hilbert space): $|n_1, \dots, n_m\rangle\,, \qquad n_j = 0, 1$

(The Fock states may be identified with the basis states of m qubits, but the elementary operators are different)

Elementary operators

creation: $a_j^\dagger |n_1, \dots, n_{j-1}, 0, \dots\rangle = (-1)^{\sum_{s<j} n_s} |n_1, \dots, n_{j-1}, 1, \dots\rangle\,, \qquad a_j^\dagger |n_1, \dots, n_{j-1}, 1, \dots\rangle = 0$

annihilation: $a_j |n_1, \dots, n_{j-1}, 1, \dots\rangle = (-1)^{\sum_{s<j} n_s} |n_1, \dots, n_{j-1}, 0, \dots\rangle\,, \qquad a_j |n_1, \dots, n_{j-1}, 0, \dots\rangle = 0$

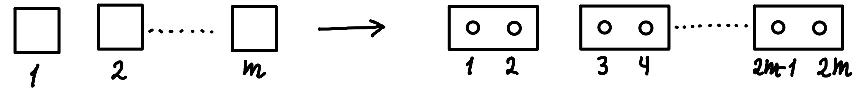
$$a_j a_k = -a_k a_j\,, \qquad a_j^\dagger a_k^\dagger = -a_k^\dagger a_j^\dagger\,, \qquad a_j^\dagger a_k + a_k a_j^\dagger = \delta_{jk}$$

Relation to qubit operators (Jordan-Wigner transformation)

$$a_j = Z \cdots Z \underbrace{\left(\frac{X + iY}{2}\right)}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} I \cdots I \qquad a_j^\dagger = Z \cdots Z \underbrace{\left(\frac{X - iY}{2}\right)}_{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} I \dots I$$

Majorana formalism

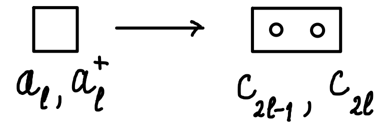
Let's split each fermionic mode in half:



Majorana operators

$$c_{2l-1} = a_l + a_l^+ = Z\cdots Z \underbrace{X}_{l\text{-th place}} I\cdots I$$

$$c_{2l} = \frac{a_l - a_l^+}{i} = Z\cdots Z \overbrace{Y} I\ldots I$$



$$\underbrace{c_j^2 = 1, \qquad c_j c_k = -c_k c_j \quad (j \neq k)}$$

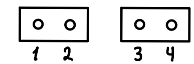
$$\boxed{c_j c_k + c_k c_j = 2\delta_{jk}}$$

$$c_{2l-1} c_{2l} = X_l Y_l = i Z_l$$

\# of fermions modulo 2

Fermionic parity: $P = (-1)^N = Z\cdots Z = \prod_{l=1}^{m} (-i c_{2l-1} c_{2l})$

Example: *m*=2



$c_1 = XI$, $c_3 = ZX$

$c_2 = YI$, $c_4 = ZY$

Toy Hamiltonian

$$H = \frac{i}{2}\left(A_{12} c_1 c_2 + A_{13} c_1 c_3 + A_{23} c_2 c_3\right)$$
$$= -\frac{1}{2}\left(A_{12} ZI - A_{13} YX + A_{23} XX\right)$$

Keeping the parity fixed: $n_2 = N - n_1 \pmod 2$

$$H \equiv -\frac{1}{2}\left(A_{12} Z - A_{13} Y + A_{23} X\right)$$

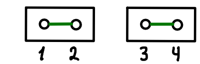
Reduction of the TFIM to a Majorana chain Hamiltonian

$$H = \underbrace{-J\sum_{l=1}^{m-1} X_l X_{l+1} - h\sum_{l=1}^{m} Z_l}$$

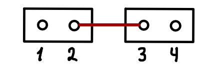
using dual basis: $X \leftrightarrow Z$, $|+\rangle = |\rightarrow\rangle$, $|-\rangle = |\leftarrow\rangle$

$$= h\sum_{l=1}^{m}\left(i\, c_{2l-1} c_{2l}\right) + J\sum_{l=1}^{m-1}\left(i\, c_{2l} c_{2l+1}\right)$$

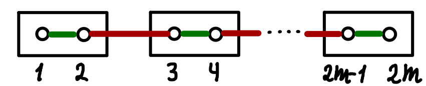
$$Z_l = -i\, c_{2l-1} c_{2l}$$



$$X_l X_{l+1} = -i\, c_{2l} c_{2l+1}$$



e.g. $X_1 X_2 = c_2 c_3$



Interpretation in terms of ordinary fermions (e.g. electrons)

$$i\, c_{2l-1} c_{2l} = -Z_l = \left\{\begin{matrix} -1 & \text{if } n_l = 0 \\ +1 & \text{if } n_l = 1 \end{matrix}\right\} 2 a_l^{\dagger} a_l - 1$$

$$i\, c_{2l} c_{2l+1} = \left(a_l - a_l^{\dagger}\right)\left(a_{l+1} + a_{l+1}^{\dagger}\right)$$

Terms like $a_j^{\dagger} a_k^{\dagger}$ are prohibited by the conservation of electric charge or some other quantum number (except perhaps for neutrinos). However, such terms appear in the mean-field description of superconductors.

In a superconductor, the total charge is conserved but these terms are allowed:

$$\hat{\Psi}\, a_j^{\dagger} a_k^{\dagger}, \qquad \hat{\Psi}^{\dagger} a_k a_j$$

borrowing/returning an electron pair from the condensate

Mean-field approximation: Ψ is treated as a c-number because there are many electron pairs in the condensate.

Quadratic fermionic Hamiltonians

$$\boxed{H(A) = \frac{i}{4}\sum_{j,k} A_{jk}\, c_j\, c_k}$$

A is a real skew-symmetric matrix

The normalization factor $\frac{i}{4}$ is chosen such that $\left[-iH(A), -iH(B)\right] = -i\,H\left([A,B]\right)$

Example:

$$H = \frac{i}{2}\left(A_{12}\, c_1 c_2 + A_{13}\, c_1 c_3 + A_{23}\, c_2 c_3\right)$$

$$= \frac{i}{4}\left(c_1, c_2, c_3, c_4\right)\begin{pmatrix} 0 & A_{12} & A_{13} & 0 \\ -A_{12} & 0 & A_{23} & 0 \\ -A_{13} & -A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

Reduction of a real skew-symmetric matrix to a standard form

$$A = Q\begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_m \\ & & & -\varepsilon_m & 0 \end{pmatrix} Q^T$$

$Q = (\vec{q}_1, \ldots, \vec{q}_{2m})$ is an orthogonal matrix, $\varepsilon_1, \ldots, \varepsilon_m \geq 0$

column vectors

Recipe: Find the eigenvalues and eigenvectors of the Hermitian matrix iA and organize them in pairs

$$\left((\varepsilon_1, \vec{u}_1), (-\varepsilon_1, \vec{u}_2)\right), \quad \left((\varepsilon_2, \vec{u}_3), (-\varepsilon_2, \vec{u}_4)\right):$$

If $\quad iA\,\vec{u}_{2\ell-1} = \varepsilon_\ell \vec{u}_{2\ell-1}, \quad$ then $\quad -iA\,\vec{u}^{\,*}_{2\ell-1} = \varepsilon_\ell\, \vec{u}^{\,*}_{2\ell-1}$ (by complex conjugation).

Let $\quad \vec{u}_{2\ell} := \vec{u}^{\,*}_{2\ell-1}, \quad \vec{q}_{2\ell-1} = \dfrac{\vec{u}_{2\ell-1} + \vec{u}_{2\ell}}{\sqrt{2}}, \quad \vec{q}_{2\ell} = i\,\dfrac{\vec{u}_{2\ell-1} - \vec{u}_{2\ell}}{\sqrt{2}} \quad \Rightarrow \quad \begin{cases} A\,\vec{q}_{2\ell-1} = -\varepsilon_\ell\, \vec{q}_{2\ell} \\ A\,\vec{q}_{2\ell} = \varepsilon_\ell\, \vec{q}_{2\ell-1} \end{cases}$

Diagonalization of the Hamiltonian

Let us define a new set of Majorana, annihilation, and creation operators called *normal modes*:

$$(\tilde{c}_1, .., \tilde{c}_{2m}) = (c_1, .., c_{2m})\, Q, \qquad \tilde{a}_\ell = \frac{\tilde{c}_{2\ell-1} + i\,\tilde{c}_{2\ell}}{2}, \qquad \tilde{a}_\ell^{+} = \frac{\tilde{c}_{2\ell-1} - i\,\tilde{c}_{2\ell}}{2}$$

Then

$$\boxed{H(A) = \frac{i}{2} \sum_{\ell=1}^{m} \varepsilon_\ell\, \tilde{c}_{2\ell-1} \tilde{c}_{2\ell} = \sum_{\ell=1}^{m} \underbrace{\varepsilon_\ell}_{\text{energies of elementary excitations}} \left(\tilde{a}_\ell^{+} \tilde{a}_\ell - \frac{1}{2} \right)}$$

($\pm\,\varepsilon$ are the eigenvalues of iA)

Ground state: $\boxed{\tilde{a}_\ell \,|\tilde{0}\rangle = 0 \quad \text{for } l=1,..,m}$

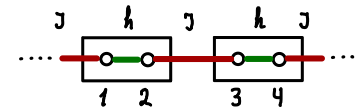
Eigenstates of the Hamiltonian: $|\tilde{n}_1, .., \tilde{n}_m\rangle = (\tilde{a}_1^{+})^{\tilde{n}_1} \cdots (\tilde{a}_m^{+})^{\tilde{n}_\ell}\, |\tilde{0}\rangle$

Many-body energy spectrum: $E_{\tilde{n}_1, .., \tilde{n}_m} = E_0 + \sum_{\ell=1}^{m} \varepsilon_\ell\, \tilde{n}_\ell\,,$ where $E_0 = -\frac{1}{2} \sum_{\ell=1}^{m} \varepsilon_\ell$

Excitation spectrum of the infinite Majorana chain

(We may keep it finite but ignore boundary conditions)

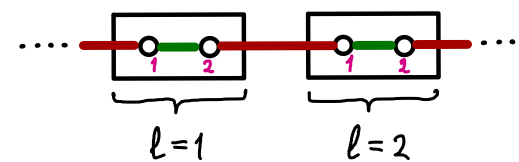
$$H = H(A), \qquad A = 2\begin{array}{c} \left(\begin{array}{cccccc} \ddots & \ddots & & & & \\ \ddots & & J & & & \\ -J & 0 & h & & & \\ & -h & 0 & J & & \\ & & -J & 0 & h & \\ & & & -h & 0 & J \ \ddots \\ & & & & -J & \ddots \end{array}\right) \\ \begin{array}{cccccc} \quad & 1 & 2 & 3 & 4 & \quad \end{array} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \\ \end{array}$$



Eigenvectors of *iA* are indexed by quasimomentum *k*

(momentum defined modulo 2π)

$$\vec{u}(k) = \begin{pmatrix} \vdots \\ f_1 \\ f_2 \\ f_1 e^{ik} \\ f_2 e^{ik} \\ \vdots \end{pmatrix} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \ell = 1 \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \ell = 2 \\ \\ \end{array} \qquad A\,\vec{u}(k) = \begin{pmatrix} \vdots \\ g_1 \\ g_2 \\ g_1 e^{ik} \\ g_2 e^{ik} \\ \vdots \end{pmatrix}$$



$\ell = \ldots, 0, 1, 2, 3, \ldots$ refers to a unit cell

$\alpha = 1, 2$ is the site index within a unit cell

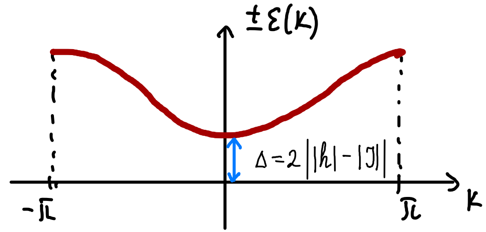
$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \tilde{A}(k) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad \tilde{A}(k) = 2\begin{pmatrix} 0 & h - Je^{-ik} \\ -h + Je^{ik} & 0 \end{pmatrix}$$

$$\boxed{\varepsilon(k) = 2\left|h - Je^{ik}\right|}$$

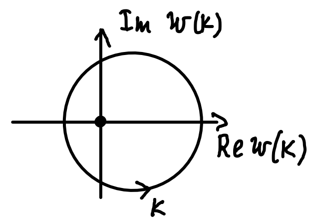
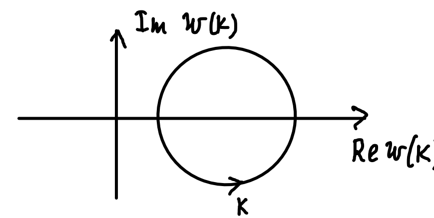
Phase transition (as reflected by the excitation spectrum)

$$H = -J\sum_{l=1}^{m-1} X_l X_{l+1} - h\sum_{l=1}^{m} Z_l$$

$$\mathcal{E}(k) = 2\,|\underbrace{h - J\, e^{ik}}_{w(k)}|$$



At $|h|=|J|$, the energy gap Δ vanishes. On opposite sides of this transition point, the complex-valued function $w(k)$ is topologically different:

$\vert h\vert < \vert J\vert$	$\vert h\vert > \vert J\vert$
	

We will also see that the $|J|<|h|$ phase does not have boundary modes, but the $|J|>|h|$ phase does

Many-body picture:

$$|\Psi_{\rightarrow}\rangle \approx |\cdots \rightarrow \rightarrow \rightarrow \cdots\rangle$$

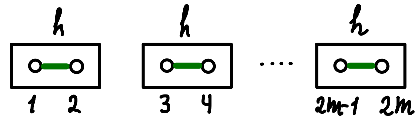
$$|\Psi_{\leftarrow}\rangle \approx |\cdots \leftarrow \leftarrow \leftarrow \cdots\rangle$$

$$|\Psi\rangle \approx |\cdots \uparrow \uparrow \uparrow \cdots\rangle$$

Boundary modes (a.k.a. "unpaired Majorana modes" or "Majorana zero modes")

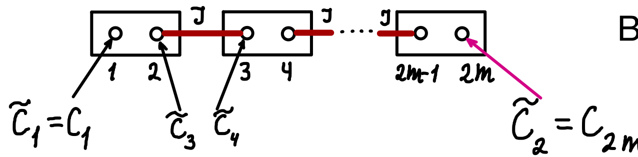
Extreme cases

$|J| < |h| \longrightarrow J = 0$



Energy gap $\Delta = 2|h|$

$|J| > |h| \longrightarrow h = 0$



Bulk gap $\Delta = 2|J|$

$$H = J\sum_{l=1}^{m-1} \underline{i\, c_{2l}\, c_{2l+1}} = \frac{i}{2}\Big(\underbrace{\varepsilon_1}_{0}\tilde{c}_1\tilde{c}_2 + \underbrace{\varepsilon_2}_{2J}\tilde{c}_3\tilde{c}_4 + \cdots\Big)$$

The boundary modes do not participate in the Hamiltonian, and therefore, the corresponding energy ε_1 is zero

Ground states are defined by the condition $\tilde{n}_2 = \cdots = \tilde{n}_m = 0 \Rightarrow \underline{-i\, c_{2l}\, c_{2l+1}\, |\xi\rangle = (\operatorname{sgn} J)\, |\xi\rangle}$

The boundary modes are unconstrained, i.e. the corresponding occupation number $\tilde{n}_1$ is arbitrary

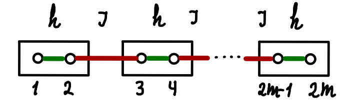
Fermionic parity of the ground states (corresponds to flipping all spins in TFIM, $\underline{Z \cdots Z |\psi_{\rightarrow}\rangle = |\psi_{\leftarrow}\rangle}$)

$$\underline{P = \prod_{l=1}^{m} \left(-i\, c_{2l-1} c_{2l}\right) \equiv -i\, c_1 c_{2m} \cdot (\operatorname{sgn} J)^{m-1}}$$

General case of boundary modes (for $|J|>|h|$)

$$H = \frac{i}{2}\varepsilon_1 \tilde{c}_1 \tilde{c}_2 + \cdots$$

If the chain is long, $\varepsilon_1 \approx 0$, and the boundary modes correspond to approximate null vectors of A: $A\vec{q}_1 \approx A\vec{q}_2 \approx 0$



$$A = 2\begin{array}{c} \begin{pmatrix} 0 & h & & & & & \\ -h & 0 & J & & & & \\ & -J & 0 & h & & & \\ & & -h & 0 & J & & \\ & & & -J & \ddots & \ddots & \\ & & & & \ddots & \ddots & J \\ & & & & & -J & 0 & h \\ & & & & & & -h & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ \\ 2m-1 \\ 2m \end{array}$$

$$\tilde{c}_s = \sum_{j=1}^{2m} c_j Q_{js} = (c_1, \dots, c_{2m})\,\vec{q}_s$$

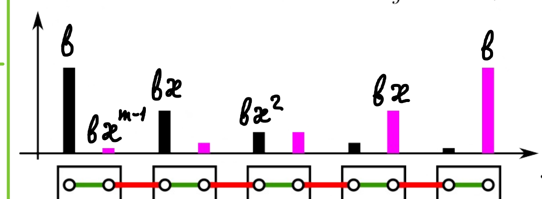
(we are interested in $s=1,2$)

$$\vec{q}_1 = b\begin{pmatrix} 1 \\ 0 \\ x \\ 0 \\ \vdots \\ x^{m-1} \\ 0 \end{pmatrix}, \quad \vec{q}_2 = b\begin{pmatrix} 0 \\ x^{m-1} \\ 0 \\ x^{m-2} \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$x = \frac{h}{J} \qquad (|x|<1)$$

$$b = \sqrt{1-x^2}$$

The j-th element of $\vec{q}_s$, i.e. the coefficient in front of c_j in $\tilde{c}_1, \tilde{c}_2$



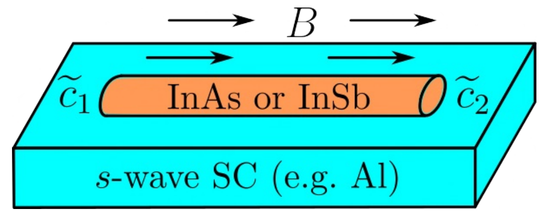
Boundary mode energy: $$\varepsilon_1 = \vec{q}_1^{\,T} \underbrace{A\,\vec{q}_2}_{2h\begin{pmatrix} bx^{m-1} \\ 0 \\ 0 \\ \vdots \end{pmatrix}} = 2h(1-x^2)\,x^{m-1}$$

Effective Hamiltonian and fermionic parity:

$$\boxed{H_{eff} = \frac{i}{2}\varepsilon_1 \tilde{c}_1 \tilde{c}_2, \quad \varepsilon_1 \sim \left(\frac{h}{J}\right)^m J}$$

$$\boxed{P = -i\,\tilde{c}_1 \tilde{c}_2 \cdot (\operatorname{sgn} J)^{m-1}}$$

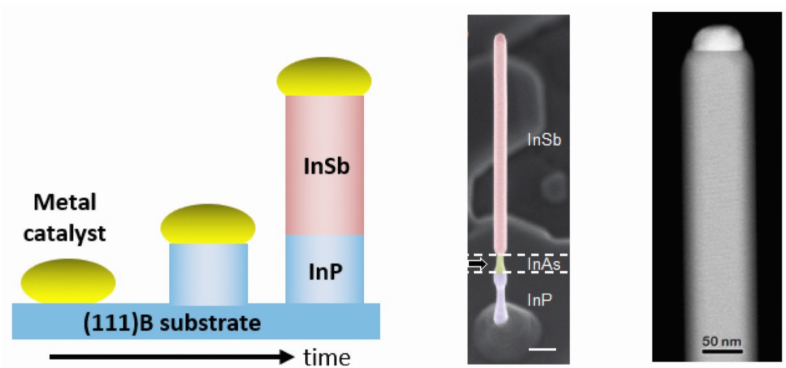
Physical realization of the Majorana chain (work in progress)



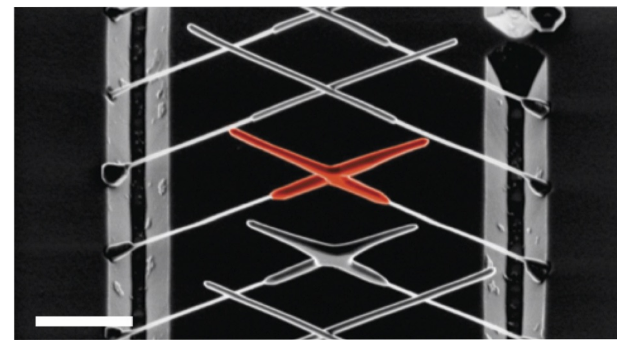
Device proposal (Lutchyn,Sau, Das Sarma 2010, Oreg, Refael, von Oppen 2010)

First experiment (Kouwenhoven's group 2012)

Nanowire growth

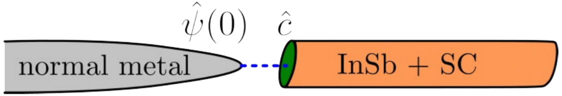


Lutchyn *at al,* arXiv:1707.04899



Gazibegovic *at al,* Nature 548, 434 (2017)

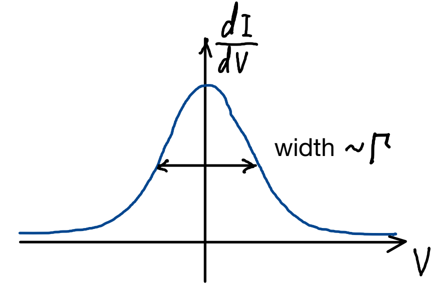
Tunneling (Law, Lee, Ng 2009)



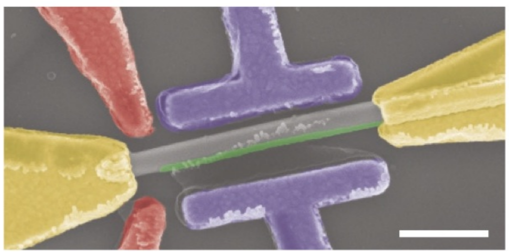
$$H = H_{metal} + t\,\frac{\hat{\psi}(0) + \hat{\psi}(0)^{+}}{2}\,\hat{c}$$

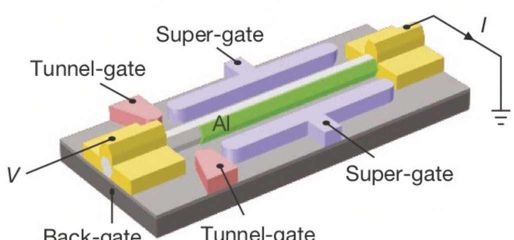
$T = 0$:

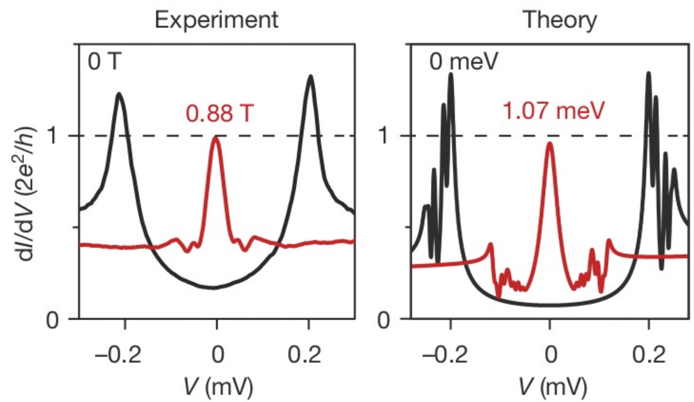
$$\frac{dI}{dV} = \underbrace{\frac{e^2}{2\pi\hbar}}_{\text{quantum conductance unit}} \cdot \frac{2}{1 + (2eV/\Gamma)^2}$$



Some (not quite successful) experiments







Zhang *et al*, doi:10.1038/nature26142 (2018)
(The paper was retracted due to problems with data)

Unfortunately, these results do not prove the existence of Majorana zero modes. Such plots are observed only in a fraction of samples, in a narrow parameter region, and might be a coincidence. So we have to wait until the device quality improves and more accurate measurements are done.