1. A quantum magic square. Alice and Bob claim they have constructed an impossible magic square, a 3×3 table with entries $m_{jk} = \pm 1$ such that

$$\prod_{k=1}^{3} m_{jk} = -1 \quad \text{for all } j, \qquad \prod_{j=1}^{3} m_{jk} = 1 \quad \text{for all } k. \tag{1}$$

However, they are unwilling to share their discovery. Charlie is skeptical and challenges them to prove this unlikely claim without having to reveal their secret. After some negotiation, Alice and Bob agree to take a test. They are placed in separate rooms (so that they cannot communicate with each other) and asked independent questions. Alice is asked to show a single row of the magic square, (m_{j1}, m_{j2}, m_{j3}), and Bob is asked to produce one column, (m_{1k}, m_{2k}, m_{3k}). Then Charlie can verify that the row and the column satisfy the magic square conditions and that Alice's and Bob's answers for m_{jk} agree. He hopes that the only way to consistently pass this test is for Alice and Bob to actually have such a square, though they may use different squares in repeated trials.

Let us define the test formally, not supposing that a magic square exists. Alice and Bob are given two independent random number $j, k \in \{1, 2, 3\}$ (each value occurs with probability $\frac{1}{3}$). They must reply with three numbers, $a_1, a_2, a_3 \in \{+1, -1\}$ for Alice and $b_1, b_2, b_3 \in \{+1, -1\}$ for Bob, such that

$$a_1 a_2 a_3 = -1, \qquad b_1 b_2 b_3 = 1. \tag{2}$$

They pass the test if

$$a_k = b_j. \tag{3}$$

a) [4 points] Show that in the classical world, it is impossible to pass with probability greater than $\frac{8}{9}$.

b) [3 points] Show that the following nine operators M_{jk},

$$\begin{array}{ccc} \sigma^x \otimes \sigma^x, & \sigma^y \otimes \sigma^y, & \sigma^z \otimes \sigma^z, \\ \sigma^y \otimes \sigma^z, & \sigma^z \otimes \sigma^x, & \sigma^x \otimes \sigma^y, \\ \sigma^z \otimes \sigma^y, & \sigma^x \otimes \sigma^z, & \sigma^y \otimes \sigma^x, \end{array} \tag{4}$$

form the kind of magic square that is impossible with numbers. Specifically, these operators are Hermitian, $M_{jk}^2 = 1$, any two operators in the same row or column commute, and

$$\prod_{k=1}^{3} M_{jk} = -1 \quad \text{for all } j, \qquad \prod_{j=1}^{3} M_{jk} = 1 \quad \text{for all } k. \tag{5}$$

(To minimize the writing, just show that $\sigma^x \otimes \sigma^x$ and $\sigma^y \otimes \sigma^y$ commute; then multiply the operators in the first row and in the first column.)

c) [8 points] Let Alice and Bob share two copies of the spin singlet $|\Phi\rangle = \frac{1}{\sqrt{2}}\bigl(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\bigr)$. Describe a strategy that would allow them to pass the test with probability 1. (Use all the properties from the previous question and check whatever else that is needed for a correctness proof.)

Remark. The difference between classical and quantum strategies is even more drastic if the test is repeated. The multiple trials can be done in parallel so that Alice and Bob receive, say, k numbers and respond with $3k$ numbers. This situation can be contrasted with the case where the two parties may be asked arbitrary long questions, but can only say "yes" or "no", and the outcome (for each given pair of questions) depends on whether the answers are the same or different. In this case, a theorem of Tsirelson asserts that the use of entangled states cannot increase the success probability more than by factor $K \approx 1.782$.

2. Classical and quantum correlations. [10 points] As shown in class, quantum correlations can violate the CHSH inequality by factor $\sqrt{2}$ at most. Tsirelson also proved[1] that the maximum violation of a general Bell inequality for measurements with ± 1 outcomes is determined by a so-called *Grothendieck constant* K. Its exact value is unknown. Grothendieck in his original paper[2] gave the upper bound $K \leq \sinh(\pi/2) \approx 2.301$. An elegant proof of a stronger bound is due to Krivine[3]:

$$K \leq \frac{\pi}{2\ln\bigl(1+\sqrt{2}\bigr)} \approx 1.782. \tag{6}$$

There are also lower bounds, for example, $K \geq \sqrt{2} \approx 1.414$ because such violation of Bell inequalities is indeed realized. In this and the next problem, you are asked to follow the logic and reconstruct certain parts of Tsirelson's and Krivine's proofs.

In this game, Alice and Bob are given the following challenge. After some prior meeting (where they can discuss their strategy and generate shared EPR pairs) they are placed in separate rooms and asked one question each, to which they may respond "yes" or "no". More formally, Alice receives number $j \in \{1, \dots, n\}$, and Bob receives $k \in \{1, \dots, m\}$, where the constants n and m are fixed throughout this problem. Given this information only, Alice should produce $a = \pm 1$, and Bob should produce $b = \pm 1$. The payoff (shared by both players) is $H_{jk}ab$, where H_{jk} are some real numbers known to Alice and Bob. For example, the left-hand side of the CHSH inequality, $\langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle \leq 2$ corresponds to the payoff matrix $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Of course, Alice and Bob want to maximize their expected payoff. But it will be more convenient for us to analyze the problem without explicit use of the payoff matrix. Instead, we will discuss possible correlations between Alice's and Bob's answers they can achieve without communication. Let us consider the correlation matrix C corresponding to a particular playing strategy. It is an $n \times m$ table with entries

$$C_{jk} = \langle a_j b_k\rangle \stackrel{\text{def}}{=} \text{expectation value of } ab \text{ given } j \text{ and } k. \tag{7}$$

[1] B. S. Tsirelson, "Quantum analogs of the Bell inequalities. The case of two spatially separated domains", *J. Math. Sci.* **36**(3), 557–570 (1987). Translated from Russian: *Zapiski LOMI* **142**, 174-194 (1985).

[2] A. Grothendieck, "Résumé de la théorie métrique des produits tensoriels topologiques" (French), *Bol. Soc. Mat. Sao Paulo* **8**, 1–79 (1953).

[3] J. L. Krivine, "Constantes de Grothendieck et fonctions de type positif sur les sphéres" (French), *Advances in Mathematics* **31**, 16–30 (1979).

We call C a *classical correlation matrix* if it can be realized by some classical strategy. Without loss of generality, we may assume that Alice and Bob have agreed what they will say in each particular case (perhaps by flipping some coins during the decision process) and act deterministically in the game itself. Their answers to questions j and k are determined by functions $a = a_j(r)$ and $b = b_j(r)$, where r is a shared random number. For each value of r, the correlation coefficients $C_{jk} = C_{jk}(r)$ have the form

$$C_{jk} = a_j b_k, \qquad a_1, \ldots, a_n,\ b_1, \ldots, b_m = \pm 1 \qquad \text{(classical deterministic)}. \tag{8}$$

The complete correlation matrix is a convex linear combination of such matrices:

$$C = \sum_r p_r\, C(r), \qquad p_r \geq 0, \quad \sum_r p_r = 1 \qquad \text{(classical)}. \tag{9}$$

Similarly, we call C a *quantum correlation matrix* if it can be realized by doing some measurements on a shared entangled state $|\Psi\rangle \in \mathcal{A} \otimes \mathcal{B}$. Alice's and Bob's measurements are described by some Hermitian operators with ± 1 eigenvalues, $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. For example, when Alice receives question j, she uses the projectors $\frac{1}{2}(1 + A_j)$ and $\frac{1}{2}(1 - A_j)$ to determine her answer. Thus, the correlation matrix is

$$C_{jk} = \big\langle \Psi \big| (A_j \otimes B_k) \big| \Psi \big\rangle \qquad \text{(quantum)}. \tag{10}$$

In addition, we call any matrix with entries $|C_{jk}| \leq 1$ a *conceivable correlation matrix*. For example, of these three matrices,

$$C^{(1)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad C^{(3)} = 0.9 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$C^{(1)}$ is classical, $C^{(2)}$ is quantum (it corresponds to the maximum violation of the CHSH inequality), and $C^{(3)}$ is not even quantum (though conceivable). Let us also introduce some notation for the set of all correlation matrices of each type:

$$\Lambda_{\mathrm{c}} \ \text{(classical)}, \qquad \Lambda_{\mathrm{q}} \ \text{(quantum)}, \qquad \Lambda_{\infty} \ \text{(conceivable)}. \tag{11}$$

Theorem [Tsirelson 1985]. C is a quantum correlation matrix if an only if there are some vectors $\vec{u}_1, \ldots, \vec{u}_n$, $\vec{v}_1, \ldots, \vec{v}_m$ in a real Euclidean space such that

$$C_{jk} = \vec{u}_j \cdot \vec{v}_k, \qquad |\vec{u}_j| = |\vec{v}_k| = 1. \tag{12}$$

(The Euclidean inner product is denoted by $\vec{u} \cdot \vec{v}$ to distinguish it from the Hermitian inner product in quantum mechanics. The dimension of the Euclidean space is not restricted, but one may set it to $n + m$ without loss of generality.)

The representation (12) of quantum correlation matrices is very convenient. In particular, the relation to the classical case follows from the Grothendieck inequality: if H is an arbitrary real $n \times m$ matrix (representing the payoff), then

$$\max_{C \in \Lambda_{\mathrm{q}}} \sum_{j,k} H_{jk} C_{jk} \leq K \max_{C \in \Lambda_{\mathrm{c}}} \sum_{j,k} H_{jk} C_{jk}. \tag{13}$$

The *Grothendieck constant* is the smallest number K such that this inequality holds for all n and m. One can dispense with the payoff matrix and simply write

$$\Lambda_{\mathrm{q}} \leq K\Lambda_{\mathrm{c}}, \tag{14}$$

where $K\Lambda_{\mathrm{c}}$ means "set Λ_{c} scaled by factor K around the origin". But this inequality is the subject of the next problem; this one is concerned with Tsirelson's theorem.

Questions:

a) Let C be a quantum correlation matrix (as defined by Eq. (10)). Prove that it can be represented in the form (12). **Hint:** Let $|\xi_j\rangle = (A_j \otimes I)|\Psi\rangle$ and $|\eta_k\rangle = (I \otimes B_k)|\Psi\rangle$. Turn $|\xi_j\rangle$, $|\eta_k\rangle$ into real vectors by doubling the space dimension.

b) Let Alice's and Bob's parts of the Hilbert space have dimension N, and let

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{s=1}^{N} |s, s\rangle. \tag{15}$$

Show that

$$\langle\Psi|A \otimes B|\Psi\rangle = \frac{1}{N} \operatorname{Tr}(AB^T). \tag{16}$$

c) Define the following Hermitian operators acting on l qubits:

$$\gamma_{2s-1} = \left(\prod_{p=1}^{s-1} \sigma_p^z\right) \sigma_s^x, \qquad \gamma_{2s} = \left(\prod_{p=1}^{s-1} \sigma_p^z\right) \sigma_s^y \qquad (s = 1, \ldots, l). \tag{17}$$

(Here, σ_j^α stands for the tensor product of the Pauli operator σ^α on the j-th qubit and the identity operators on the other qubits.) Show that $\gamma_1, \ldots, \gamma_{2l}$ satisfy these anticommutation relations:

$$\gamma_r\gamma_q + \gamma_q\gamma_r = 2\delta_{rq}. \tag{18}$$

d) Let C be defined by Eq. (12), where $\vec{u}_j, \vec{v}_k \in \mathbb{R}^{2l}$, and let Alice and Bob share the entangled state (15), where $N = 2^l$. Construct some Hermitian operators A_j, B_k with ± 1 eigenvalues such that C is represented by Eq. (10).

3. The Grothendieck inequality. [Extra credit 10 points] Like in the previous problem, we fix some positive integers n, m and consider real $n \times m$ matrices. Let us summarize the definitions of the sets of classical, quantum, and conceivable correlation matrices:

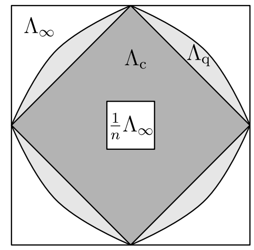
$$\Lambda_{\mathrm{c}} = \text{convex hull}\big\{C :\, C_{jk} = a_j b_k \text{ for some } a_j, b_k \in \{+1, -1\}\big\} \qquad \text{(classical)}, \tag{19}$$

$$\Lambda_{\mathrm{q}} = \big\{C :\, C_{jk} = \vec{u}_j \cdot \vec{v}_k \text{ for some } \vec{u}_j, \vec{v}_k \in \mathbb{R}^l \text{ s.t. } |\vec{u}_j| = |\vec{v}_k| = 1\big\} \qquad \text{(quantum)}, \tag{20}$$

$$\Lambda_\infty = \big\{C :\, |C_{jk}| \leq 1\big\} \qquad \text{(conceivable)}. \tag{21}$$

(However, it is sometimes easier to think about classical matrices in terms of probabilistic strategies.) These sets are compact, i.e. bounded and closed[4]. The three sets are related as follows:

$$\frac{1}{n}\Lambda_\infty \subseteq \Lambda_{\rm c} \subseteq \Lambda_{\rm q} \subseteq \Lambda_\infty. \tag{22}$$



The first inclusion is not obvious, but it follows from the statement below.

Questions:

a) Let C be a matrix such that $|C_{jk}| \le \frac{1}{n}$. Show that C is a classical correlation matrix.

b) Each of the sets $\Lambda_{\rm c}$, $\Lambda_{\rm q}$, Λ_∞ satisfies the following conditions:

   1. Let C be an arbitrary matrix; then $tC \in \Lambda$ for sufficiently small positive t.
   2. If $C \in \Lambda$, then $-C \in \Lambda$ *(symmetry).*
   3. If $C', C'' \in \Lambda$ and $t \in [0,1]$, then $(1-t)C' + tC'' \in \Lambda$ *(convexity).*
   4. If $C', C'' \in \Lambda$, then $C'C'' \in \Lambda$ *(multiplicativity).*

   Here XY denotes the Hadamard (entry-wise) product of two matrices: $(XY)_{jk} = X_{jk}Y_{jk}$. Some of these properties are obvious, and the nontrivial cases of property 1 follow from the previous result. You only need to prove 3 and 4 for quantum correlation matrices. **Hint:** Choose the new vectors $\vec{u}_j, \vec{v}_k$ from a larger space.

c) Given a compact set of matrices, Λ that satisfies the above conditions, one can define the corresponding norm $\|X\|$ for any matrix X:

$$\|X\| = \min\bigl\{\alpha \ge 0 :\, X \in \alpha\Lambda\bigr\} = \begin{cases} 0 & \text{if } X = 0, \\ \bigl(\max\{t :\, tX \in \Lambda\}\bigr)^{-1} & \text{if } X \ne 0. \end{cases} \tag{23}$$

   Thus, there are three norms, $\|X\|_{\rm c}$, $\|X\|_{\rm q}$, and $\|X\|_\infty$, but we are now concerned with their common properties:

   1. $\|X\| \ge 0$. Furthermore, $\|X\| = 0$ if and only if $X = 0$.
   2. $\|aX\| = |a|\,\|X\|$.
   3. $\|X+Y\| \le \|X\| + \|Y\|$.
   4. $\|XY\| \le \|X\|\,\|Y\|$.

[4]For example, $\Lambda_{\rm q}$ is compact because it is a continuous image of a compact set, namely, the product of $n+m$ unit spheres. (If you don't understand this, don't worry: it's just a technicality.)

Together with the completeness property, this conditions constitute the axioms of a *Banach norm*. (The completeness holds automatically since the space of matrices is finite-dimensional.) These things are standard and can be found in analysis textbooks. But please write a proof of property 3 to make sure you understand it.

d) Let

$$C \in \Lambda_\infty, \qquad Q = \sin\left(\frac{\pi}{2} C\right), \tag{24}$$

where the sine function is applied entry-wise, i.e. $Q_{jk} = \sin\left(\frac{\pi}{2} C_{jk}\right)$. Show that if Q is a quantum correlation matrix, then C is a classical correlation matrix. **Hint:** Sample a random vector $\vec{r} \in \mathbb{R}^l$ from a rotationally-symmetric Gaussian distribution, $p(\vec{r}) \sim e^{-\vec{r}\cdot\vec{r}}$, and define Alice's and Bob's classical strategies as follows:

$$a_j(\vec{r}) = \mathrm{sgn}(\vec{u}_j \cdot \vec{r}), \qquad b_k(\vec{r}) = \mathrm{sgn}(\vec{v}_k \cdot \vec{r}). \tag{25}$$

The product $a_j(\vec{r}) b_k(\vec{r})$ depends only on the projection of $\vec{r}$ onto the plane spanned by vectors $\vec{u}_j$, $\vec{v}_k$.

e) For an arbitrary Banach norm, show that

$$\|\sin x\| \leq \sinh\big(\|x\|\big). \tag{26}$$

Hint: Use the Taylor expansion.

f) Prove the following version of the Grothendieck inequality:

$$\Lambda_{\mathrm{q}} \subseteq \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} \Lambda_{\mathrm{c}}. \tag{27}$$

Hint: Assuming that $\|C\|_{\mathrm{q}} \leq a$ for a suitable constant a, show that $C \in \Lambda_{\mathrm{c}}$.