Ph/CS 219a, Fall 2022
Quantum Computation
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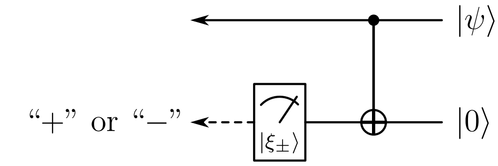
Problem set #2
Due Thursday, October 13, 2022

1. Computation by measurement. (10 points) Suppose we can measure an arbitrary state with respect to this basis:

$$|\xi_+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\varphi}|1\rangle\Big), \qquad |\xi_-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - e^{i\varphi}|1\rangle\Big). \tag{1}$$

Consider the following circuit acting on a pure state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$:



a) Find the quantum state $|\eta\rangle$ of the two qubits just before the measurement. Then calculate the unnormalized output state $|\widetilde{\psi}_\pm\rangle = \big(I \otimes \langle\xi_\pm|\big)|\eta\rangle$ for each measurement outcome (i.e. "+" or "−").

b) Show that $p_\pm = \langle\widetilde{\psi}_\pm|\widetilde{\psi}_\pm\rangle = 1/2$ (i.e. each outcome occurs with probability 1/2 regardless of the input state). Now, define the normalized *conditional states* $|\psi_\pm\rangle = p_\pm^{-1/2}|\widetilde{\psi}_\pm\rangle$ and interpret them as the result of application of some unitaries U_+, U_- to the initial state $|\psi\rangle$. (Note that such an interpretation is not always possible since normalizing a state is generally a nonlinear operation. But in this case, we are lucky.)

c) Explain how to use this circuit together with a classically controlled σ^z to implement the unitary gate

$$\Lambda\big(e^{-i\varphi}\big) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{2}$$

(*Classically controlled* means that maintaining coherence between the control bit states is not necessary. In our case, we use the measurement outcome as the control.)

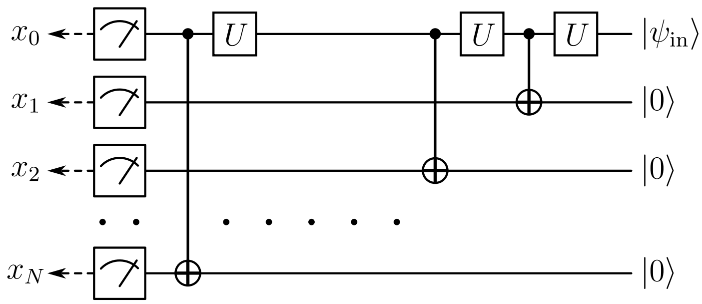
2. (10 points) Recall the *quantum Zeno effect* discussed in class: if we measure the quantum state of a system frequently, its Schrödinger evolution is suppressed. For a simple model, let

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{where} \quad \theta = \frac{\pi}{2N}. \tag{3}$$

In one case, we start with the initial state $|\psi_{\text{in}}\rangle = |0\rangle$, apply the operator U N times, and measure the qubit in the standard basis. Since

$$U^N = \begin{pmatrix} \cos(N\theta) & -\sin(N\theta) \\ \sin(N\theta) & \cos(N\theta) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

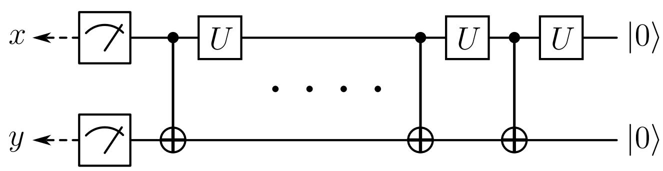
the qubit state before the measurement is $U^N|0\rangle = |1\rangle$, and the measured value is always 1. On the other hand, if we measure the qubit after each application of the operator U, it behaves classically and flips with probability $\sin^2\theta$ at each step. To conform to the standard model of quantum computation, we can mimic the measurements by copying the qubit in the standard basis and delaying the readout until the very end. The quantum circuit looks like this:



One can show formally that the probability distribution of the output bits $x_0, x_1, \ldots, x_N$ is given by a Markov chain:

$$p(x_0, x_1, \ldots, x_N) = q(0, x_1) \prod_{j=2}^{N} q(x_{j-1}, x_j)\, \delta_{x_0, x_N}, \quad \text{where} \quad q(x,y) = \begin{cases} \cos^2\theta, & \text{if } x = y, \\ \sin^2\theta, & \text{if } x \neq y. \end{cases} \tag{4}$$

Let us, however, do something different. Instead of copying the main qubit into a fresh ancillary qubit each time, we will try to reuse the ancilla. Specifically, consider the operator $\big(\mathrm{CNOT}(U \otimes I)\big)^N$ followed by measurement in the standard basis:



Calculate the probability to get a measurement outcome (x, y).