2.1

a) Set Y as measurement outcomes.

Measure operator $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ with two possible outcomes 1 (Set $y=0$) and -1 (set $y=1$).

Here:

$$\rho = \frac{1}{2}|\varphi_0\rangle\langle\varphi_0| + \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| .$$

$$p(x_0) = p(x_1) = 1/2 .$$

$$\begin{cases} p(0|x_0) = 1, & p(1|x_0) = 0 \\ p(0|x_1) = \cos^2(\theta/2), & p(1|x_1) = \sin^2(\theta/2) \end{cases} .$$

Entropy:

$$H(X) = \log_2 2 = 1 \text{ bit} .$$

$$H(X|Y=0) = \frac{1}{1+\cos^2\theta/2} \log(1+\cos^2\theta/2) + \frac{\cos^2\theta/2}{2+\cos^2\theta/2} \log\left(\frac{1+\cos^2\theta/2}{\cos^2\theta/2}\right)$$

$$H(X|Y=1) = 0 .$$

Since $P_Y(y=0) = \frac{1}{2}(1+\cos^2\theta/2)$, $P_Y(y=1) = \frac{1}{2}\sin^2\theta/2$.

we have :

$$H(X|Y) = \sum_y P_Y(y)\, H(X|Y=y)$$

$$= \frac{1}{2}(1+\sin^2\theta/2)\log(1+\sin^2\theta/2) - \frac{1}{2}\cos^2\theta/2 \log\cos^2\theta/2 .$$

$$I_1(X;Y) = H(X) - H(X|Y)$$
$$= 1 - \frac{1}{2}(1+\sin^2\theta/2)\log(1+\sin^2\theta/2)$$
$$+ \frac{1}{2}\cos^2\theta/2 \log(\sin^2\theta/2).$$

b) Similarly, for $Z' = |\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp|$, we have :

$$p(x_0) = p(x_1) = \frac{1}{2}$$
$$p(y=0|x=x_0) = tr(|\psi\rangle\langle\psi|\varphi_0\rangle\langle\varphi_0|) = \cos^2(\frac{\theta}{4} - \frac{\pi}{4})$$
$$p(y=1|x=x_0) = tr(|\psi^\perp\rangle\langle\psi^\perp|\varphi_0\rangle\langle\varphi_0|) = \cos^2(\frac{\theta}{4} + \frac{\pi}{4}).$$

$$p(y=0|x=x_1) = [\cos(\theta/4 - \pi/4)\cos(\theta/2) + \sin(\theta/4 - \pi/4)\sin(\theta/2)]^2$$
$$= \cos^2(\frac{\theta}{4} + \frac{\pi}{4})$$

$$p(y=1|x=x_1) = [\cos(\theta/4 + \pi/4)\cos(\theta/2) + \sin(\theta/4 + \pi/4)\sin(\theta/2)]^2$$
$$= \cos^2(\frac{\theta}{4} - \frac{\pi}{4}) .$$

Thus $P_Y(y=0) = \frac{1}{2}\left(\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\right) = \frac{1}{2}$

$P_Y(y=1) = \frac{1}{2}\left(\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\right) = \frac{1}{2}$

$$\Rightarrow \quad H(X|Y=0) = -\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right) - \cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right).$$

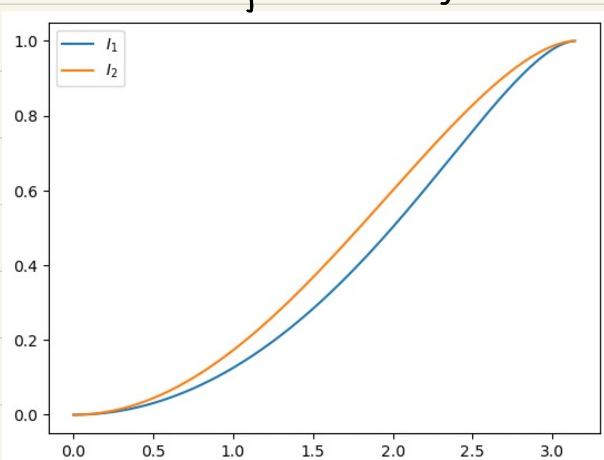
$$H(X|Y=1) = H(X|Y=0).$$

Then $H(X|Y) = H(X|Y=0)$ since $P(Y=0) = P(Y=1) = \frac{1}{2}$

$$I_2(X;Y) = 1 + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right).$$

Plot of I_1 and I_2 as function of θ:

$I(X;Y)$



Measurement along Z' has higher info. gain.

θ

c) $\theta = \frac{\pi}{2}$:

$I_1(X;Y) \simeq 0.3113$ bits

$I_2(X;Y) \simeq 0.3991$ bits

$\theta = \frac{2\pi}{3}$:

$I_1(X;Y) \simeq 0.5488$ bits

$I_2(X;Y) \simeq 0.6454$ bits

d) Choose to measure along Z' with respect to one of the remaining two state vectors.

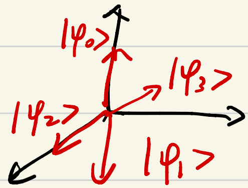
Denote measurement results as Y (on the 1st qubit) and Z (on the second qubit).

Then information gain:

$$
\begin{aligned}
I(X;YZ) &= H(X) - H(X|YZ) \\
&= H(X) - H(X|Y) \\
&\quad + H(X|Y) - H(X|YZ) \\
&= \log_2 3 - 1 + 0.6454 \quad (\theta = \tfrac{2}{3}\pi) \\
&\simeq 1.23036 \text{ bits} \\
&< 1.369068 \text{ bits}.
\end{aligned}
$$

The info. gain is still less than the collective measurement in lecture note.

2.2 $\theta_1 = \pi/2$, $\theta_2 = -\pi/2$.



a) $P_X(x) = \frac{1}{4}$ for $x = 0, 1, 2, 3$.

$P_{Y|X=0}(y=0) = 1$, $P_{Y|X=0}(y=1) = 0$.

$P_{Y|X=1}(y=0) = 0$. $P_{Y|X=1}(y=1) = 1$

$P_{Y|X=2}(y=0) = P_{Y|X=2}(y=1) = \frac{1}{2}$

$P_{Y|X=3}(y=0) = P_{Y|X=3}(y=1) = \frac{1}{2}$.

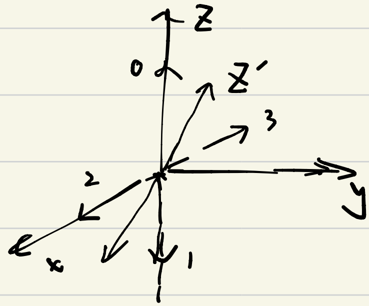
Then $H(X) = \log_2 4 = 2.$ bits.

$$H(X|Y) = \langle H(X|Y=y) \rangle_y$$
$$= \frac{1}{2}\log_2 2 + \frac{1}{4} \times 2 \log_2 4 = \frac{1}{2} + 1 = \frac{3}{2}.$$

$\Rightarrow$ $I(X;Y) = H(X) - H(X|Y) = \frac{1}{2}$ bits.

b) $Z' = |\psi\rangle\langle\psi| - |\psi^{\perp}\rangle\langle\psi^{\perp}|$

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\cdot\frac{1}{2} - \frac{\pi}{4}\right) \\ \sin\left(\frac{\theta}{2}\cdot\frac{1}{2} - \frac{\pi}{4}\right) \end{pmatrix}$$



$$|\psi^{\perp}\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\cdot\frac{1}{2} + \frac{\pi}{4}\right) \\ \sin\left(\frac{\theta}{2}\cdot\frac{1}{2} + \frac{\pi}{4}\right) \end{pmatrix}$$

$\theta = \pi/2$.

$$|\psi\rangle = \begin{pmatrix} \cos\frac{-\pi}{8} \\ \sin\frac{-\pi}{8} \end{pmatrix} \qquad |\psi^{\perp}\rangle = \begin{pmatrix} \cos\frac{3}{8}\pi \\ \sin\frac{3}{8}\pi \end{pmatrix}.$$

$$P_{Y|x=0}(y=0) = \cos^2\frac{\pi}{8}$$
$$P_{Y|x=0}(y=1) = \cos^2\frac{3}{8}\pi$$

$$P_{Y|x=1}(y=0) = \sin^2\frac{\pi}{8} = \cos^2\frac{3}{8}\pi$$
$$P_{Y|x=1}(y=1) = \sin^2\frac{3}{8}\pi = \cos^2\frac{\pi}{8}$$

$$P_{Y|x=2}(y=0) = \cos^2\frac{3}{8}\pi$$
$$P_{Y|x=2}(y=1) = \cos^2\frac{\pi}{8}$$
$$P_{Y|x=3}(y=0) = \cos^2\frac{\pi}{8} = \sin^2\frac{3}{8}\pi$$
$$P_{Y|x=3}(y=1) = \cos^2\frac{3}{8}\pi = \sin^2\frac{\pi}{8}$$

$$\Rightarrow H(X|Y) = -\cos^2\frac{\pi}{8}\log_2\frac{\cos^2\frac{\pi}{8}}{2} - \cos^2\frac{3}{8}\pi\log_2\frac{\cos^2\frac{3\pi}{8}}{2}$$

$$= 1 - \cos^2\frac{\pi}{8}\log_2\cos^2\frac{\pi}{8} - \cos^2\frac{3}{8}\pi\log_2\cos^2\frac{3}{8}\pi.$$

Then:

$$I'(X;Y) = H(X) - H(X|Y)$$

$$= 1 - \left(-\cos^2\frac{\pi}{8}\log_2\cos^2\frac{\pi}{8} - \cos^2\frac{3\pi}{8}\log_2\cos^2\frac{3}{8}\pi\right).$$

$$= 0.39912$$

$$< I(X;Y).$$

2.3

a) Measure in Z, then:

$$P_{Y|\Theta=\theta}(y=0) = \cos^2(\theta/2)$$
$$P_{Y|\Theta=\theta}(y=1) = \sin^2(\theta/2)$$

Then $H(Y|\Theta=\theta) = H_2(\cos^2\theta/2)$

$$H(Y|\Theta) = \int \frac{d\theta}{2\pi} H_2(\cos^2\theta/2)$$

$$= \int_\theta H_2(\cos^2\theta/2)$$

$$= -2\int_\theta \cos^2\theta/2 \log_2 \cos^2\theta/2 \quad .$$

By symmetry: $H(Y) = \log_2 2 = 1$ bit .

Thus:

$$I(\theta;Y) = H(Y) - H(Y|\Theta) = 1 + 2\int_\theta \cos^2\theta/2 \log_2 \cos^2\theta/2$$

b) $E_k = \frac{2}{n} |\phi_k\rangle\langle\phi_k|$, $\phi_k = 2\pi \frac{k}{n}$.

Then $|\phi_k\rangle = \begin{pmatrix} \cos(\pi \frac{k}{n}) \\ \sin(\pi \frac{k}{n}) \end{pmatrix}$.

$$|\phi_k\rangle\langle\phi_k| = \frac{1}{2} \begin{pmatrix} 1+\cos\phi_k & \sin\phi_k \\ \sin\phi_k & 1-\cos\phi_k \end{pmatrix}.$$

Completeness of E_k:

$$\Rightarrow \quad \frac{2}{n} \sum_k |\phi_k\rangle\langle\phi_k| = \frac{2}{n} \begin{pmatrix} \frac{n}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix} = I.$$

$$\Rightarrow \quad \sum_k E_k = I.$$

Info. gain:

$$I(\theta;Y) = H(Y) - H(Y|\theta)$$

$$P_Y(y=a) = tr(\rho E_a) = tr(\rho E_b) = \frac{1}{n}, \quad a \neq b.$$

$$\Rightarrow \quad H(Y) = \log_2 n.$$

$$P_{Y|\theta=\theta}(y=k) = tr(\rho_\theta E_k)$$

$$= \frac{2}{n} tr(|\varphi(\theta)\rangle\langle\varphi(\theta)|\phi_k\rangle\langle\phi_k|)$$

$$= \frac{2}{n} |\langle\varphi(\theta)|\phi_k\rangle|^2$$

$$= \frac{2}{n} \cos^2\left(\frac{\theta-\theta_k}{2}\right)$$

Then $H(Y|\theta=\theta) = -\sum_k P_{Y|\theta=\theta}(y=k) \log_2 P_{Y|\theta=\theta}(y=k)$

$$= -\frac{2}{n}\sum_k \cos^2\left(\frac{\theta-\phi_k}{2}\right) \log_2 \cos^2\left(\frac{\theta-\phi_k}{2}\right)$$
$$- \log_2 \frac{2}{n}$$

Integration:

$$H(Y|\theta) = -\frac{2}{n}\sum_k \int_\theta \cos^2\left(\frac{\theta-\phi_k}{2}\right) \log_2 \cos^2\left(\frac{\theta-\phi_k}{2}\right)$$

$$- \log_2 \frac{2}{n}$$

$$= -\frac{2}{n}\cdot n \int_\theta \cos^2\left(\frac{\theta}{2}\right) \log_2 \cos^2\left(\frac{\theta}{2}\right) - 1 + \log_2 n$$

Thus $I(\theta; Y) = H(Y) - H(Y|\theta)$
$$= 1 + 2\int_\theta \cos^2\frac{\theta}{2} \log_2 \cos^2\frac{\theta}{2}$$

We see the info. gain in b) is the same as a).

2.4

a) Define $|\psi^\perp\rangle$ that $\langle\psi|\psi^\perp\rangle = 0$.

$|\psi^\perp\rangle$ is in $(d-1)$-dimensional space.

$\text{Prob.}(|\psi^\perp\rangle) = \varepsilon$, $\text{Prob.}(|\psi\rangle) = 1-\varepsilon$.

Span $|\psi^\perp\rangle$ in basis $\{|i\rangle\}$, $i = 0, 1, \cdots, d-2$.

Then $\text{Prob.}(|i\rangle) = \text{Prob.}(|\psi^\perp\rangle) \cdot \underbrace{\text{Prob.}(|i\rangle | |\psi^\perp\rangle)}_{p_i}$

$$= \varepsilon \cdot p_i$$

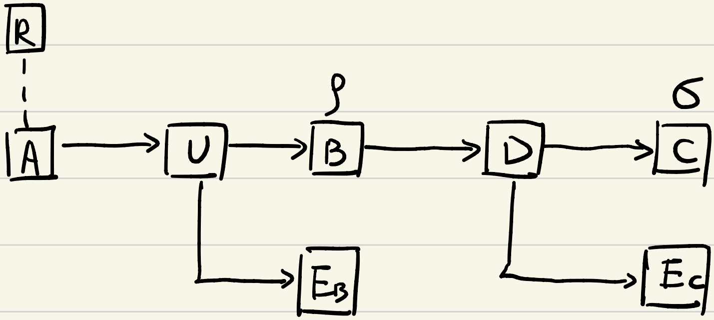
$$\Rightarrow H(\rho) \leq H(X) = -(1-\varepsilon)\log_2(1-\varepsilon) - \sum_i \varepsilon p_i \log_2(\varepsilon p_i)$$

$$= H_2(\varepsilon) + \varepsilon H(X^\perp)$$

$$\leq H_2(\varepsilon) + \varepsilon \log_2(d-1)$$

maximal Shannon entropy of uniform distribution in $(d-1)$-dimensional space.

b)



We need to prove:

$$I_c(R\rangle B)_\rho + 2H(RC)_\sigma - H(R)_\rho \geq 0 .$$

From quantum data processing inequality:

$$\begin{aligned}
\text{LHS} &\geq I_c(R\rangle C)_\sigma + 2H(RC)_\sigma - H(R)_\rho \\
&= H(C)_\sigma - H(RC)_\sigma + 2H(RC)_\sigma - H(R)_\rho \\
&= H(C)_\sigma + H(RC)_\sigma - H(R)_\rho \\
&= H(C)_\sigma + H(RC)_\sigma - H(CE_BE_C)_\sigma \\
&\geq H(C)_\sigma + H(RC)_\sigma - H(C)_\sigma - H(E_BE_C)_\sigma \\
&= H(RC)_\sigma - H(E_BE_C)_\sigma \\
&= 0 .
\end{aligned}$$

$\square$.

c) Since R and C are d-dimensional, we have:

$$\dim(RC) = d^2$$

From b):

$$H(R)_\rho - I_c(R\rangle B)_\rho \leq 2H(RC)_\sigma .$$

From a):

$$F = \langle \psi_{RA} | \rho_{AC} | \psi_{RA} \rangle = 1 - \varepsilon .$$

$$\Rightarrow \quad H(RC)_\sigma \leq H_2(\varepsilon) + \varepsilon \log_2(d^2 - 1) .$$

Therefore:

$$H(R)_\rho - I_c(R\rangle B)_\rho \leq 2H_2(\varepsilon) + 2\varepsilon \log_2(d^2 - 1) .$$

$\square$.