Ph/CS 219a, Fall 2022
Quantum Computation
Prof. Alexei Kitaev

Problem set #3
Due Thursday, October 20, 2022

1. Positivity vs. complete positivity. (10 points) Let $T : \mathbf{L}(\mathbb{C}^2) \to \mathbf{L}(\mathbb{C}^2)$ be a superoperator defined by the equations

$$TI = I, \qquad T\sigma^x = x\sigma^x, \qquad T\sigma^y = y\sigma^y, \qquad T\sigma^z = z\sigma^z, \tag{1}$$

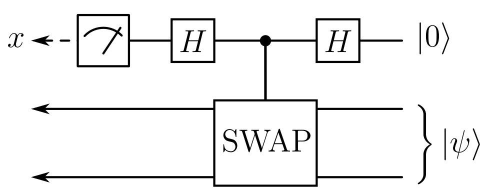
where x, y, z are some real numbers.

a) Find a necessary and sufficient condition for T being positive.

b) Find a necessary and sufficient condition for T being completely positive. **Hint:** Use the matrix representation $\breve{T}_{jkj'k'} = T_{jj'kk'}$ as discussed in class.

2. Swap test. [10 points] The swap test is a simple procedure that can be used to measure the overlap between two unknown but unentangled states $|\xi\rangle$, $|\eta\rangle$. The result is probabilistic, so the procedure has to be repeated multiple times with different copies of the given states. The test can also distinguish *some* entangled states from separable ones. (A bipartite state is called *separable* if it can be written as a probabilistic mixture of product states, see Eq. (2) below.)

Consider the following circuit, which includes a controlled swap operator:



, where $\quad \mathrm{SWAP}\,|a,b\rangle = |b,a\rangle \quad$ for all a, b.

a) Calculate the probability p_1 to get the measurement outcome $x = 1$ if $|\psi\rangle = |\xi\rangle \otimes |\eta\rangle$.

b) Now, let us replace the pure state $|\psi\rangle$ with the mixed separable state

$$\rho = \sum_j w_j |\xi_j\rangle\langle\xi_j| \otimes |\eta_j\rangle\langle\eta_j|, \qquad \text{where } \langle\xi_j|\xi_j\rangle = \langle\eta_j|\eta_j\rangle = 1, \quad w_j \geq 0, \quad \sum_j w_j = 1. \tag{2}$$

Calculate the probability p_1 in this case and show that $0 \leq p_1 \leq \frac{1}{2}$.

c) Calculate p_1 for $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$.

3. Half-way teleportation. [10 points] Alice and Bob share an unknown state $|\psi\rangle$ (Alice holds the first qubit and Bob the second). They want to perform the unitary operator

$$\Lambda^2(e^{i\varphi}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{pmatrix}. \tag{3}$$

This is easy to do if Alice and Bob also share two EPR pairs and have access to a classical communication channel: Alice teleports her qubit to Bob; he applies the gate and teleports the qubit back. How do they achieve their goal using only one EPR pair?

Hint: Alice does some partial measurement on her qubit and her part of the EPR pair, and tells the result to Bob. (Here "partial measurement" means the projection onto a pair of orthogonal two-dimensional subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2$; quantum coherence within each subspace is preserved.) Having received one classical bit from Alice, Bob creates a "copy" of Alice's qubit relative to the standard basis, $|x\rangle \mapsto |xx\rangle$. The required gate can now be realized in a straightforward way. The last part is to erase the extra copy of x. To achieve that, Bob does another partial measurement and communicates the result to Alice, who applies a suitable gate to her qubit. Thus, the problem splits into tasks: (i) create a remote copy (relative to the standard basis) using one EPR pair and one bit of communication; (ii) erase such a copy while maintaining quantum coherence, which involves sending one bit in the opposite direction.