HW 2 notes.

2.1

$$H(\rho) = -\underbrace{p}_{\text{is the e.v. of } \rho} \log p - (1-p)\log(1-p)$$

a)

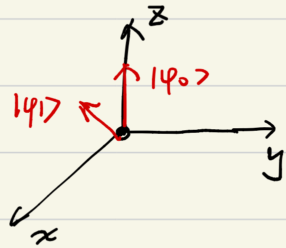
$H(\rho)$: Von-Neumann entropy

$H(X)$: Shannon entropy.

Information gain: (Classical mutual info.)

$$I(X;Y) = H(X) - \underbrace{H(X|Y)}_{H(XY) - H(Y)}$$

Y is measurement outcomes.



$p(x,y) = p(y|x)\,p(x)$

$p(x)$ known, $p(y|x) = tr(E_y\,\rho(x))$.

$p(y) = \sum_x p(x,y)$.

$$|\varphi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}.$$

$$\rho = \frac{1}{2}|\varphi_0\rangle\langle\varphi_0| + \frac{1}{2}|\varphi_1\rangle\langle\varphi_1|$$

$$p(x_0) = p(x_1) = \frac{1}{2}.$$

$$p(0|x_0) = 1, \quad p(1|x_0) = 0$$

$$p(0|x_1) = \cos^2(\theta/2), \quad p(1|x_1) = \sin^2(\theta/2).$$

$$p(0, x_0) = \frac{1}{2}, \quad p(1, x_0) = 0$$

$$p(0, x_1) = \frac{1}{2}\cos^2(\theta/2), \quad p(1, x_1) = \frac{1}{2}\sin^2(\theta/2)$$

Information gain is $I(X;Y) = H(X) - H(X|Y)$.

$$H(X) = -\log\frac{1}{2} = \log 2 = 1.$$

$$H(X|Y=0) = -p_0\log p_0 - p_1\log p_1$$

$$= \frac{1}{1+\cos^2\theta/2}\log(1+\cos^2\theta/2) + \frac{\cos^2\theta/2}{1+\cos^2\theta/2}\log\left(\frac{1+\cos^2\theta/2}{\cos^2\theta/2}\right)$$

$$H(X|Y=1) = 0.$$

$$P_Y(y=0) = \frac{1}{2}(1+\cos^2\theta/2). \quad P_Y(y=1) = \frac{1}{2}\sin^2\theta/2$$

Therefore, $H(X|Y) = \frac{1}{2}\log(1+\cos^2\theta/2) + \frac{1}{2}\cos^2\theta/2 \log \frac{1+\cos^2\theta/2}{\cos^2\theta/2}$

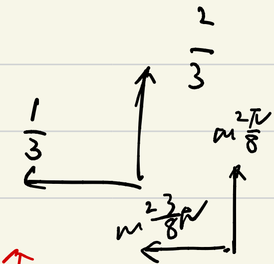
$= \frac{1}{2}(1+\cos^2\theta/2)\log(1+\cos^2\theta/2)$

$- \frac{1}{2}\cos^2\theta/2 \log \cos^2\theta/2$.

$I(X;Y) = H(X) - H(X|Y)$

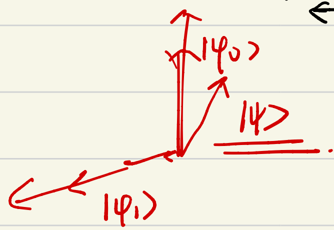
$= 1 - \frac{1}{2}(1+\cos^2\theta/2)\log(1+\cos^2\theta/2)$

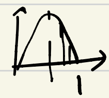
$+ \frac{1}{2}\cos^2\theta/2 \log(\cos^2\theta/2)$



$\theta = 0$, $I = 1 - 1 + 0 = 0$.

$\theta = \pi$, $I = 1 - 0 + 0 = 1$.





A.

b) $p(x_0) = p(x_1) = \frac{1}{2}$

$p(y=0|x=x_0) = tr(|\psi\rangle\langle\psi|\varphi_0\rangle\langle\varphi_0|) = \cos^2(\frac{\theta}{4} - \frac{\pi}{4})$

$p(y=1|x=x_0) = tr(|\psi^\perp\rangle\langle\psi^\perp|\varphi_0\rangle\langle\varphi_0|) = \cos^2(\frac{\theta}{4} + \frac{\pi}{4})$.

$P(y=0|x=x_1) = [\cos(\theta/4 - \pi/4)\cos(\theta/2) + \sin(\theta/4-\pi/4)\sin(\theta/2)]^2$

$= \cos^2(\frac{\theta}{4} + \frac{\pi}{4})$

$P(y=1|x=x_1) = [\cos(\theta/4+\pi/4)\cos(\theta/2) + \sin(\theta/4+\pi/4)\sin(\theta/2)]^2$

$= \cos^2(\frac{\theta}{4} - \frac{\pi}{4})$.

$P_{X|Y=0}(x_0|y=0) = \cos^2(\frac{\theta}{4} - \frac{\pi}{4})$ $P_{X|Y=0}(x_1|y=0) = \cos^2(\frac{\theta}{4} + \frac{\pi}{4})$.

... $y=1)$ .. $\cos^2(\frac{\theta}{4} + \frac{\pi}{4})$ ... $|y=0) = \cos^2(\frac{\theta}{4} - \frac{\pi}{4})$.

Thus $P_Y(y=0) = \frac{1}{2}\left(\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\right) = \frac{1}{2}$

$$P_Y(y=1) = \frac{1}{2}\left(\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\right) = \frac{1}{2}$$

$$\frac{1+\cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)}{2} + \frac{1+\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right)}{2}.$$

$$= 1 + \frac{1}{2}\Big(\underbrace{\cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)}_{-\sin\frac{\theta}{2}} + \underbrace{\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right)}_{+\sin\frac{\theta}{2}}\Big).$$

$$= 1.$$

$$\Rightarrow \quad H(X|Y=0) = -\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right) - \cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right).$$

$$H(X|Y=1) = H(X|Y=0).$$

Then $H(X|Y) = H(X|Y=0)$ since $P(Y=0)=P(Y=1)=\frac{1}{2}$

$$I(X;Y) = 1 + \cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}-\frac{\pi}{4}\right) + \cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right)\log\cos^2\left(\frac{\theta}{4}+\frac{\pi}{4}\right).$$

$$\theta = \pi \Rightarrow I(X;Y) = 1$$

$$\theta = 0 \Rightarrow I(X;Y) = 0. \qquad \square$$

c) $\theta = \frac{\pi}{2}$:

$I_1(X;Y) \simeq 0.3113$ bits.

$I_2(X;Y) \simeq 0.3991$ bits

$\theta = \frac{2\pi}{3}$

$I_1(X;Y) \simeq 0.5488$ bits

$I_2(X;Y) \simeq 0.6454$ bits

$$
\begin{array}{c|ccc}
 & a & b & c \\
\hline
\frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0
\end{array}
$$

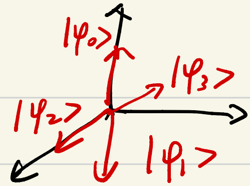
$\Rightarrow P(y=a) = \frac{1}{3}$.

d)

$$
\begin{aligned}
I(X;YZ) &= H(X) - H(X|YZ) \\
&= H(X) - H(X|Y) \\
&\quad + H(X|Y) - H(X|YZ). \\
&= \log_2 3 - 1 + 0.6454 \quad (\theta = \tfrac{2}{3}\pi) \\
&\simeq 1.23036. \text{ bits}
\end{aligned}
$$

2.2

$\theta_1 = \pi/2$, $\theta_2 = -\pi/2$.



a) $P_X(x) = \frac{1}{4}$ for $x = 0, 1, 2, 3$.

$P_{Y|x=0}(y=0) = 1$, $P_{Y|x=0}(y=1) = 0$.

$P_{Y|x=1}(y=0) = 0$, $P_{Y|x=1}(y=1) = 1$

$P_{Y|x=2}(y=0) = P_{Y|x=2}(y=1) = \frac{1}{2}$

$P_{Y|x=3}(y=0) = P_{Y|x=3}(y=1) = \frac{1}{2}$.

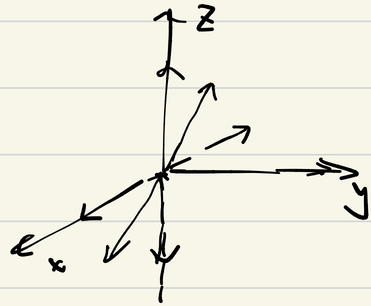
Then $H(X) = \log_2 4 = 2$. bits.

$$H(X|Y) = \langle H(X|Y=y)\rangle_y$$
$$= \frac{1}{2}\log_2 2 + \frac{1}{4}\times 2\log_2 4 = \frac{1}{2} + 1 = \frac{3}{2}.$$

$\Rightarrow$ $I(X;Y) = H(X) - H(X|Y) = \frac{1}{2}$ bits.

b) $Z' = |\psi\rangle\langle\psi| - |\psi^{\perp}\rangle\langle\psi^{\perp}|$

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\cdot\frac{1}{2} - \frac{\pi}{4}\right) \\ \sin\left(\frac{\theta}{2}\cdot\frac{1}{2} - \frac{\pi}{4}\right) \end{pmatrix}$$



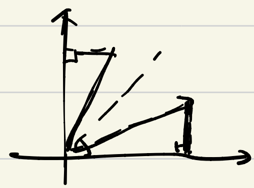
$$|\psi^{\perp}\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\cdot\frac{1}{2} + \frac{\pi}{4}\right) \\ \sin\left(\frac{\theta}{2}\cdot\frac{1}{2} + \frac{\pi}{4}\right) \end{pmatrix}$$

$\theta = \pi/2$.

$\frac{1}{2}\varphi = \left(\boxed{\frac{\theta}{2} - \frac{\pi}{2}}\right)\frac{1}{2}$

$\theta = \frac{\pi}{2} \Rightarrow \varphi = -\frac{\pi}{4}$

$$|\psi\rangle = \begin{pmatrix} \cos\frac{-\pi}{8} \\ \sin\frac{-\pi}{8} \end{pmatrix} \qquad |\psi^{\perp}\rangle = \begin{pmatrix} \cos\frac{3}{8}\pi \\ \sin\frac{3}{8}\pi \end{pmatrix}.$$



$P_{Y|X=0}(y=0) = \cos^2\frac{\pi}{8}$

$P_{Y|X=0}(y=1) = \cos^2\frac{3}{8}\pi$

$P_{Y|X=1}(y=0) = \sin^2\frac{\pi}{8} = \cos^2\frac{3}{8}\pi$

$P_{Y|X=1}(y=1) = \sin^2\frac{3}{8}\pi = \cos^2\frac{\pi}{8}$

$P_{Y|X=2}(y=0) = \cos^2\frac{3}{8}\pi$

$P_{Y|X=2}(y=1) = \cos^2\frac{\pi}{8}$

$P_{Y|X=3}(y=0) = \cos^2\frac{\pi}{8} = \sin^2\frac{3}{8}\pi$

$P_{Y|X=3}(y=1) = \cos^2\frac{3}{8}\pi = \sin^2\frac{\pi}{8}$

$$\Rightarrow H(X|Y) = -\cos^2\frac{\pi}{8}\log_2\frac{\cos^2\frac{\pi}{8}}{2} - \cos^2\frac{3}{8}\pi\log_2\frac{\cos^2\frac{3\pi}{8}}{2}$$

$$= 1 - \cos^2\frac{\pi}{8}\log_2\cos^2\frac{\pi}{8} - \cos^2\frac{3}{8}\pi\log_2\cos^2\frac{3}{8}\pi .$$

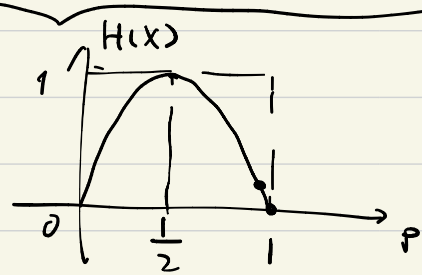
Then:

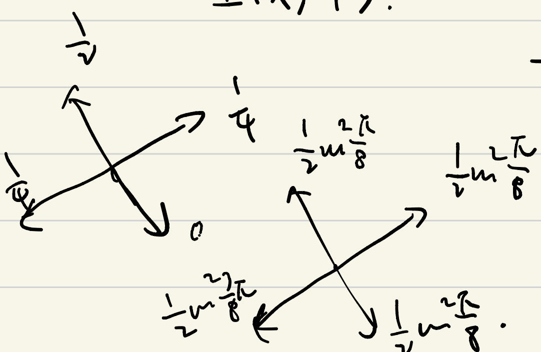
$$I'(X;Y) = H(X) - H(X|Y)$$

$$= 1 - \underbrace{\left(-\cos^2\frac{\pi}{8}\log_2\cos^2\frac{\pi}{8} - \cos^2\frac{3}{8}\pi\log_2\cos^2\frac{3}{8}\pi\right)}.$$

$$= 0.39912.$$

$$< I(X;Y).$$

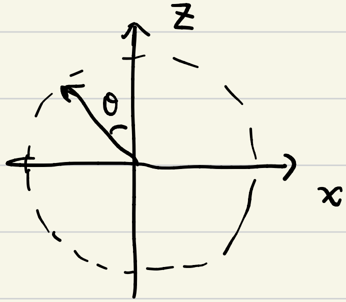




uncertainty is higher.

2.3

a)



$H(Y) = 1$ bit.
$= \log_2 2$.
by symmety.
(need to be checked).

Measure in Z, then:

$$P_{Y|\Theta=\theta}(y=0) = \cos^2(\theta/2)$$
$$P_{Y|\Theta=\theta}(y=1) = \sin^2(\theta/2).$$

Then $H(Y|\Theta=\theta) = H_2(\cos^2(\theta/2))$

$$H(Y|\Theta) = \int \frac{d\theta}{2\pi} \cdot H_2(\cos^2(\theta/2)).$$

$$\Rightarrow \quad I(\Theta; Y) = H(Y) - H(Y|\Theta)$$

$$H_2(\cos^2\theta/2) = -\cos^2\theta/2 \log_2 \cos^2\theta/2 - \sin^2\theta/2 \log_2 \sin^2\theta/2$$

$$H(Y|\Theta) = -2\int_{\theta} \cos^2\theta/2 \log_2 \cos^2\theta/2 .$$

$$\Rightarrow \quad I(\Theta; Y) = 1 + 2\int_{\theta} \cos^2\theta/2 \log_2 \cos^2\theta/2 .$$

b) $E_k = \frac{2}{n}|\phi_k\rangle\langle\phi_k|$, $\phi_k = 2\pi\frac{k}{n}$

Then $|\phi_k\rangle = \begin{pmatrix}\cos(2\pi\frac{k}{n}/2)\\ \sin(2\pi\frac{k}{n}/2)\end{pmatrix}$.

$$|\phi_k\rangle\langle\phi_k| = \begin{pmatrix}\cos^2(2\pi\frac{k}{n}/2) & \sin(2\pi\frac{k}{n}/2)\cos(2\pi\frac{k}{n}/2)\\ \sin(2\pi\frac{k}{n}/2)\cos(2\pi\frac{k}{n}/2) & \sin^2(2\pi\frac{k}{n}/2)\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}(1+\cos\phi_k) & \frac{1}{2}\sin(\phi_k)\\ \frac{1}{2}\sin(\phi_k) & \frac{1}{2}(1-\cos\phi_k)\end{pmatrix}$$

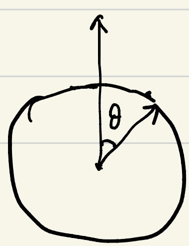
$$\frac{2}{n}\sum_k |\phi_k\rangle\langle\phi_k| \Rightarrow \frac{2}{n}\begin{pmatrix}\frac{n}{2} & 0\\ 0 & \frac{n}{2}\end{pmatrix} = I.$$

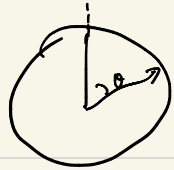
Thus $\sum_k E_k = I$.

$$I(\theta;Y) = H(Y) - H(Y|\theta).$$

$$P_Y(y=a) = tr(\rho E_a) = tr(\rho E_b) = \frac{1}{n}.$$

$$\Rightarrow H(Y) = \log_2 n.$$





$$P_{Y|\Theta=\theta}(y=a) = \mathrm{tr}(\rho_\theta E_a)$$

$$= \frac{2}{n}\,\mathrm{tr}\left(|\varphi(\theta)\rangle\langle\varphi(\theta)|\phi_k\rangle\langle\phi_k|\right)$$

$$= \frac{2}{n}\left|\langle\varphi(\theta)|\phi_k\rangle\right|^2$$

$$\Rightarrow \frac{2}{n}\left(\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi_k}{2}\right) + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi_k}{2}\right)\right)^2$$

$$\Rightarrow \underline{\frac{2}{n}\cdot\cos^2\left(\frac{\theta-\phi_k}{2}\right) = P_{Y|\Theta}(y=k)}\,.$$

Then $H(Y|\Theta=\theta) = -\sum_k P_{Y|\Theta}(y=k)\log_2 P_{Y|\Theta}(y=k)$

$$= -\frac{2}{n}\sum_k \cos^2\left(\frac{\theta-\phi_k}{2}\right)\log_2\cos^2\left(\frac{\theta-\phi_k}{2}\right)$$

$$- \sum_k P_{Y|\Theta}(y=k)\log_2\frac{2}{n}$$

$$= -\frac{2}{n}\sum_k \cos^2\left(\frac{\theta-\phi_k}{2}\right)\log_2\cos^2\left(\frac{\theta-\phi_k}{2}\right)$$

$$- \log_2\frac{2}{n}\,.$$

Integration:

$$H(Y|\Theta) = -\sum_k \frac{2}{n}\int_\theta \cos^2\left(\frac{\theta-\phi_k}{2}\right)\log_2\cos^2\left(\frac{\theta-\phi_k}{2}\right)$$

$$- \log_2\frac{2}{n}$$

$$= -2\int_\theta \cos^2\left(\frac{\theta}{2}\right)\log_2\cos^2\left(\frac{\theta}{2}\right)$$

$$- 1 + \log_2 n\,.$$

Thus $I(\theta;Y) = H(Y) - H(Y|\theta)$

$$= \log_2 n + 2\int_\theta \sin^2\left(\frac{\theta}{2}\right)\log_2 \sin^2\frac{\theta}{2}$$
$$+ 1 - \log_2 n$$
$$= 1 + 2\int_\theta \cos^2\left(\frac{\theta}{2}\right)\log_2 \cos^2\frac{\theta}{2}$$

Same as a).

2.4

a) $|\psi\rangle$ and $|\psi^\perp\rangle$ which is in $(d-1)$-dimensional space.

$\text{Prob.}(|\psi^\perp\rangle) = \varepsilon$. $\text{Prob}(|\psi\rangle) = 1-\varepsilon$.

$$\downarrow$$

$\text{Prob.}(|i\rangle) = \varepsilon \cdot p_i = \varepsilon \cdot P(|i\rangle \,|\, |\psi^\perp\rangle)$, $i = 0, 1, \cdots, d-2$

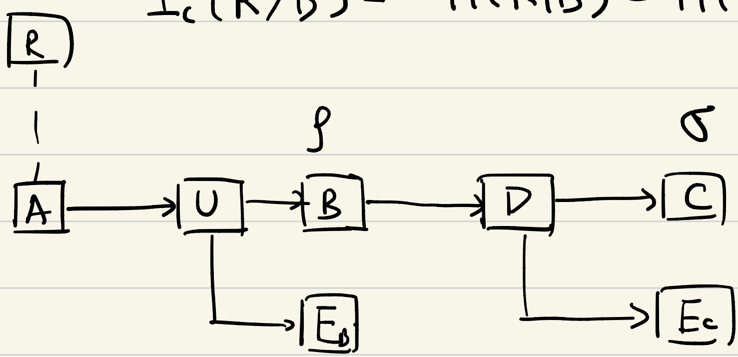
$$\Rightarrow H(\rho) \leq H(X) = -(1-\varepsilon)\log_2(1-\varepsilon)$$
$$- \sum_i \varepsilon p_i \log \varepsilon p_i$$
$$= H_2(\varepsilon) + \varepsilon H(X^\perp)$$

$$\leq H_2(\varepsilon) + \varepsilon \log_2(d-1).$$

uniform distribution
in the $(d-1)$-dimensional
space.

b)

$$I_c(R\rangle B) = -H(R|B) = H(B) - H(E)$$



$$|\psi_{RA}\rangle\langle\psi_{RA}|$$
$$\downarrow$$
$$\sum_a M_a |\psi_{RA}\rangle\langle\psi_{RA}| M_a^\dagger \otimes |a\rangle\langle a|_E$$
$$\Rightarrow |\psi_{RAE}\rangle = \sum_a M_a |\psi_{RA}\rangle \otimes |a\rangle.$$

$$I_c(R\rangle C)_\sigma \leq I_c(R\rangle B)_\rho$$

$$\boxed{H(R)_\rho - 2H(RC)_\sigma \leq I_c(R\rangle B)_\rho.}$$

$$I_c(R\rangle C)_\sigma = -H(R|C)_\sigma = H(C)_\sigma - H(RC)_\sigma$$

$$I_c(R\rangle B)_\rho - H(R)_\rho \geq -2H(RC)_\sigma$$

$$I_c(R\rangle B)_\rho = H(B)_\rho - H(RB)_\rho$$

RCE_BE_C.

$$I_c(R\rangle B)_\rho + 2H(RC)_\sigma - H(R)_\rho \geq 0$$

$$\begin{aligned}
LHS &\geq I_c(R\rangle C)_\rho + 2H(RC)_\sigma - H(R)_\rho \\
&= H(C)_\sigma - H(RC)_\sigma + 2H(RC)_\sigma - H(R)_\rho \\
&= H(C)_\sigma - H(RC)_\sigma - H(R)_\rho \\
&= \underbrace{H(RE_BE_C)}_{}{}_\sigma - H(E_BE_C)_\sigma - H(R)_\rho \\
&\geq H(R)_\sigma + H(E_BE_C)_\sigma - H(E_BE_C)_\sigma - H(R)_\rho \\
&= H(R)_\rho - H(R)_\rho \\
&= 0. \qquad \square.
\end{aligned}$$

c) $\dim(RC) = d^2$

From a) and b) we complete the proof.