Toric code and anyons

Surface codes (including the toric tode we will discuss shortly):

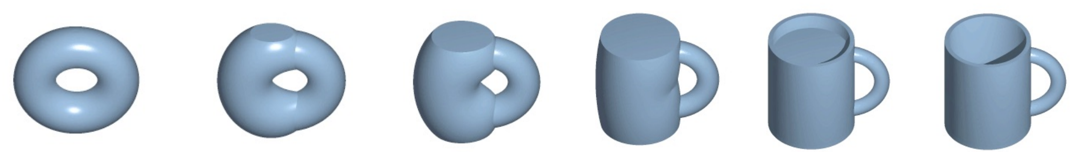
-- are quantum LDPC codes:

- each stabilizer operator acts on *O(1)* qubits,
- each qubit enters *O(1)* stabilizer operators;

-- protect from random errors occurring at a constant rate;
-- are good for fault-tolerant quantum computation;
-- can be realized by quantum Hamiltonians such that error correction occurs at the physical level;
-- are related to *anyons* -- quasiparticles with more complex statistics than bosons or fermions.

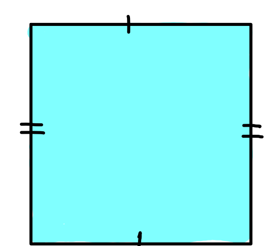
Some simple topology (surface codes and anyons involve topology too)

Solid torus (a 3D shape):



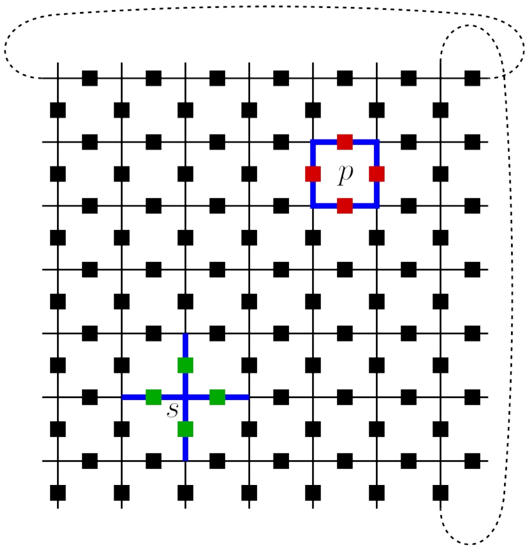
The 2D torus T^2 is the surface of the solid torus (a 2D manifold)

It is obtained by glueing the two sides of a square (this gives a cylinder),
and also glueing the top and the bottom



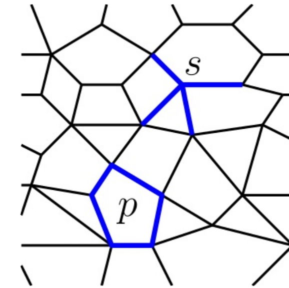
Definition of the toric code

The qubits are associated with the edges of a lattice (or a graph) on the torus



$m \times m$ lattice

$n = 2m^2$ qubits



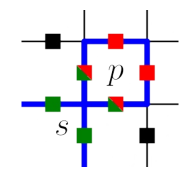
The stabilizer operators are associated with vertices s and plaquettes p:

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x \,, \qquad B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z$$

(a CSS code)

$$\boxed{A_s B_p = B_p A_s}$$

because star(s) and boundary(p) share 0 or 2 qubits



Redundancy in the definition: $\prod_s A_s = 1 = \prod_p B_p$

Number of independent stabilizer operators: $\ell = (m^2-1)\cdot 2 = n-2$

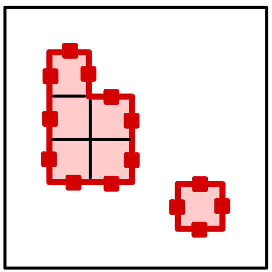
Number of logical qubits: $K = n - \ell = 2$

The groups $D_Z \subseteq D_X^{\perp} \subseteq \mathbb{Z}_2^n$ (The fact that $\mathbb{F}_2=\{0,1\}$ is a field is not important for the moment, so we denote it by $\mathbb{Z}_2$)

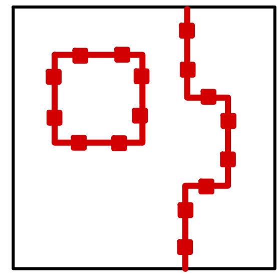
It is convenient to think of elements of $\mathbb{Z}_2$ as sets of qubits (or edges)

$D_Z =$ boundaries of various regions (i.e. unions of plaquettes)

$D_X^{\perp} =$ $\mathbb{Z}_2$- cycles (each vertex has an even # of incident edges)



When we add adjacent plaquettes modulo 2, the shared edges cancel, and we are left with the outline



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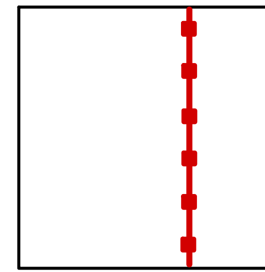
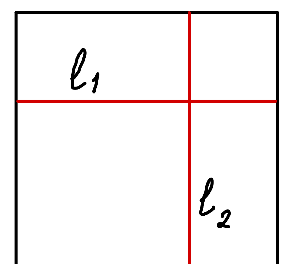


Logical Z operators: $D_X^{\perp} / D_Z = H_1(T^2, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ ($\mathbb{Z}_2$ homology group of the torus)

There are two basis cycles ℓ_1, ℓ_2 (modulo boundaries): one going horizontally and the other vertically



The groups $D_x \subseteq D_z^{\perp} \subseteq \mathbb{Z}_2^n$

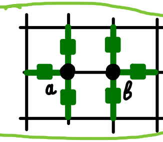
In this case, we think of group elements as $\mathbb{Z}_2$-valued functions

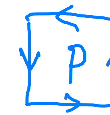
D_x is generated by vertex stars

$$D_x = \left\{ \text{functions } w: \text{edges} \to \mathbb{Z}_2 \text{ of the form } w(s,s') = \underbrace{v}_{\text{characteristic function of a set of vertices}}(s) - v(s') \right\} \quad (\mathbb{Z}_2 \text{ gradients})$$

For example, let $v(a)=v(b)=1$, 0 for other vertices.

Then $w(s,s') = \begin{cases} 1 \text{ on green edges,} \\ 0 \text{ otherwise} \end{cases}$





$$D_z^{\perp} = \left\{ \text{functions } w: \text{edges} \to \mathbb{Z}_2 \text{ such that for all plaquettes } p, \sum_{j \in \text{boundary}(p)} w(j) = 0 \right\}$$

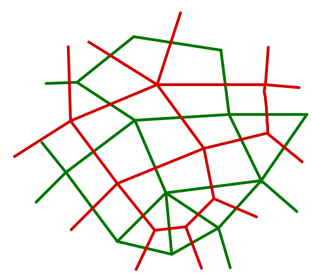
($\mathbb{Z}_2$ cocycles, or curl-free functions)

> Logical X operators: $D_z^{\perp} / D_x = H^1(T^2, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ ($\mathbb{Z}_2$ cohomology group of the torus)

Poincare duality

$$H^1(M, \mathbb{Z}_2) \cong H_1(M, \mathbb{Z}_2)$$

for any closed surface (2D-manifold) M



$$D_x(\text{graph}) = D_z(\text{dual graph})$$

Code vectors (using the general theory of CSS codes)

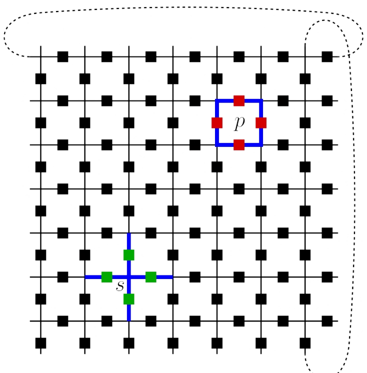
$$|\psi_K\rangle = \frac{1}{\sqrt{D_x}} \sum_{w \in C} |w\rangle$$ -- basis vectors of $\mathcal{M}$

$$C \in D_z^\perp / D_x = \tilde{C} + D_x = \{\tilde{C} + v : v \in D_x\} \cong \mathbb{Z}_2^{2}$$ # of logical qubits

The meaning of two logical bits: a cocycle $\tilde{C} \in D_z^\perp$ is characterized modulo D_x by two invariants:

$$C_1 = \sum_{j \in \text{horizontal line}} w(j), \qquad C_2 = \sum_{j \in \text{vertical line}} w(j) \qquad (C_1, C_2 \in \mathbb{F}_2)$$

Error syndrome = set of particles



$B_p = \prod \sigma_j^z, \quad A_s = \prod \sigma_j^x$

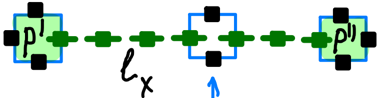
Let $|\xi\rangle \in \mathcal{M}$, $\quad |\psi\rangle = \underbrace{\sigma(g)}_{\text{error}} |\xi\rangle$

The syndrome of g is defined by the eigenvalues of the stabilizer operators:

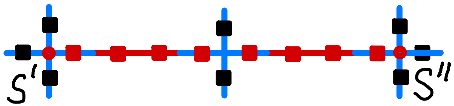
$B_p \sigma(g) = -\sigma(g) B_p \iff B_p|\psi\rangle = -|\psi\rangle$ -- $\mathbb{Z}_2$- vortex at plaquette p

$A_s \sigma(g) = -\sigma(g) A_s \iff A_s|\psi\rangle = -|\psi\rangle$ -- $\mathbb{Z}_2$- charge at vertex s

Detectable errors (which produce particles)



these stabilizer operators commute with the errors because they share 2 qubits with them



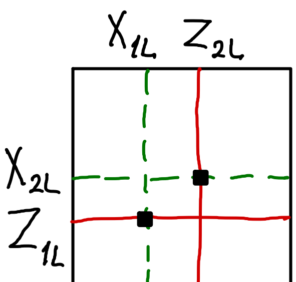
$$\sigma^x(\ell_x) = \prod_{j\in\ell} \sigma^x_j$$ creates vortices at p', p''

$$B_p\, \sigma^x(\ell_x) = -\sigma^x(\ell_x)\, B_p$$ for $p = p', p''$

$$\sigma^z(\ell_z) = \prod_{j\in\ell_z} \sigma^z_j$$ creates charges at s', s''

$$A_s\, \sigma^z(\ell_z) = -\sigma^z(\ell_z)\, A_s$$ for $s = s', s''$

Logical operators (undetectable by the code)



$$X_{1L} = \sigma^x(\,\vdots\,), \qquad Z_{1L} = \sigma^z(\,—\,)$$

$$X_{2L} = \sigma^x(---), \qquad Z_{2L} = \sigma^z(\,|\,)$$

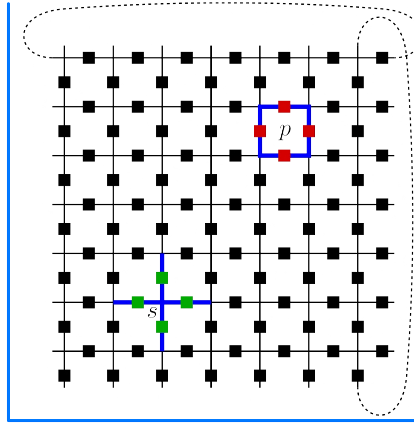
$$\boxed{d = m}$$

-- the smallest weight of a nontrivial cycle or cocycle (represented by a straight line going across the torus)

Toric code: summary

$$\underbrace{D_z}_{\text{boundaries}} \subseteq \underbrace{D_x^{\perp}}_{\mathbb{Z}_2 \text{ cycles (unions of loops)}}$$

$$\underbrace{D_x \subseteq D_z^{\perp}}_{\text{boundaries and cycles on the dual lattice}}$$



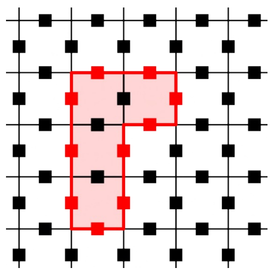
$m \times m$ lattice, $n = 2m^2$

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x$$

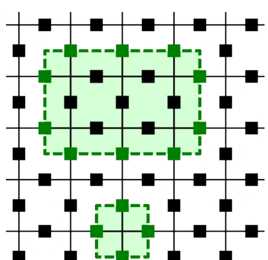
$$B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z$$

Trivial errors:

$\sigma^z(g)$, $g \in D_z$

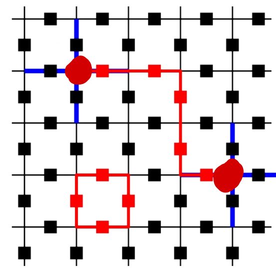


$\sigma^x(g)$, $g \in D_x$



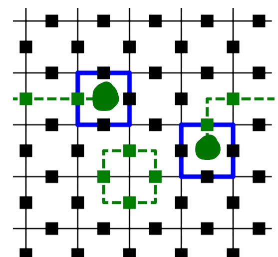
Detectable errors:

$g \notin D_x^{\perp}$



"electric charges" (e)

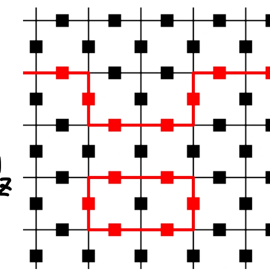
$g \notin D_z^{\perp}$



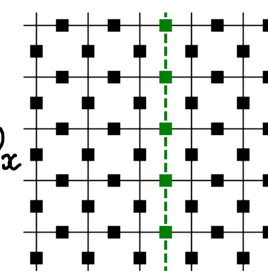
"magnetic vortices" (m)

Bad errors:

$g \in D_x^{\perp} \setminus D_z$



$g \in D_z^{\perp} \setminus D_x$



Toric code as a physical system

Hamiltonian:

$$\boxed{H_{TC} = -\sum_s A_s - \sum_p B_p}$$

Ground states:

$$A_s|\xi\rangle = |\xi\rangle, \qquad B_p|\xi\rangle = |\xi\rangle \quad \text{for all } s, p$$

$$H_{TC}|\xi\rangle = E_g|\xi\rangle \qquad E_g = -2m^2 = -n$$

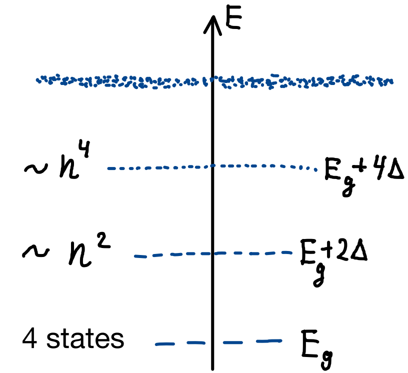
Excited states:

$$A_s|\psi\rangle = -|\psi\rangle, \qquad B_p|\psi\rangle = -|\psi\rangle$$

for an even number of sites and plaquettes

$$E_{ex} = E_g + 2\ell\Delta \qquad\qquad \Delta = 2$$

Energy spectrum



We will later examine stability to *time-independent* perturbations (different from instantaneous errors)

For example,

$$H = H_{TC} \underbrace{- h_x \sum_j \sigma_j^x - h_z \sum_j \sigma_j^z}_{\text{perturbation}}$$

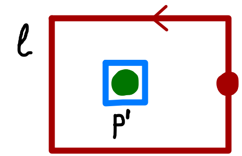
Quasiparticle braiding and statistics

Moving an e-particle (charge) around an m-particle (vortex):

The m-particle is defined by the condition $B_{p'}|\psi\rangle = -|\psi\rangle$

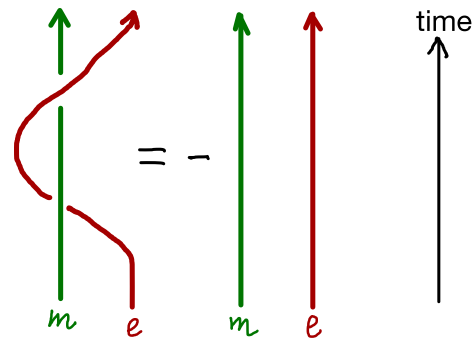
Moving the e-particle is described by the operator

$$W = \sigma^z(\ell) = \prod_{p \text{ inside } \ell} B_p$$

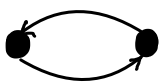


We conclude that $W|\psi\rangle = -|\psi\rangle$

Space-time (i.e. 2+1 dimensional) picture



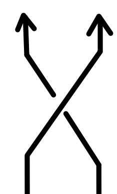
Compare this result with usual particles:



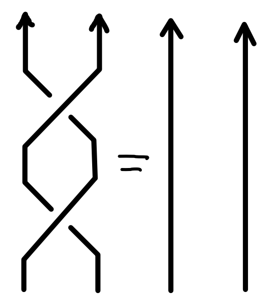
Bosons: $R|\psi\rangle = |\psi\rangle$

Fermions: $R|\psi\rangle = -|\psi\rangle$

R is the counterclockwise exchange (braiding) operator

$R =$ 

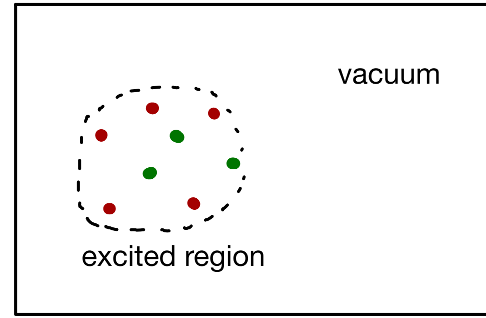
In both cases,

$R^2 =$  $= I$

e-particles and m-particles are bosons when considered separately. Taken together, they are *anyons* -- particles with more complex statistics than bosons or fermions.

Superselection sectors

Superselection sectors are equivalence classes of arbitrary excitations (simple or composite). They describe what will remain of an excited region when most particles annihilate each other.



Superselection sectors for the toric code anyons:

$$1, \quad e, \quad m, \quad \varepsilon = em$$

$$e \times e \times e \times e \times e \times m \times m \times m = em$$

Fusion rules

Superselection sectors form an Abelian group under fusion:

$$e \times e = m \times m = \varepsilon \times \varepsilon = 1$$

$$L = \{1, e, m, \varepsilon\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$e \times m = \varepsilon, \quad e \times \varepsilon = m, \quad m \times \varepsilon = e$$

There are also more complex, *non-Abelian* anyons, for example, the so-called *Ising anyons*:

$$L = \{1, \sigma, \varepsilon\}$$

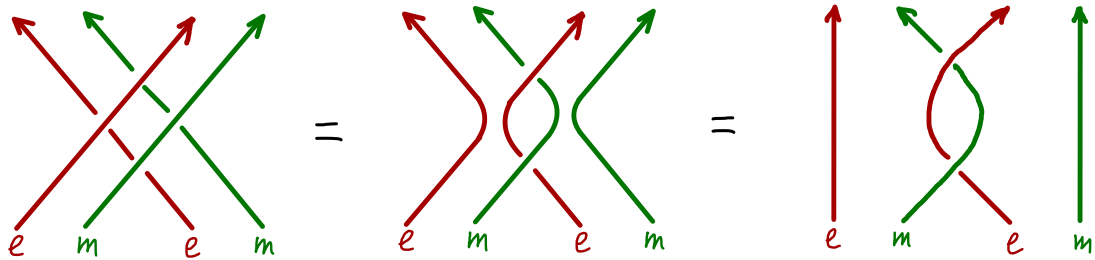
$$\sigma \times \sigma = 1 + \varepsilon$$

Before the fusion, there are two practically indistinguishable states with two σ-particles. The fusion is a measurement that tells them apart. A pair of σ-particles can be used as a logical qubit!

$$\sigma \times \varepsilon = \sigma$$

$$\varepsilon \times \varepsilon = 1$$

Braiding of composite particles



$= - I_{em,em}$

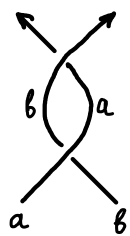
Conclusion: ε- particles are fermions!

Algebraic data for Abelian anyons

(This is a preview: we will discuss this topic on the next lecture)

1) L is an Abelian group under fusion $(\times)$

2) Double braiding:



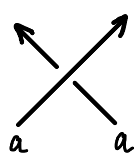
$= R_{ba} R_{ab} = W_{a,b} \, I_{a,b}$

$W : L \times L \to U(1)$

$W_{a \times b, c} = W_{a,c} W_{b,c}$

$W_{a, b \times c} = W_{a,b} W_{a,c}$

3) Braiding of identical particles



$= R_{aa} = \theta_a \, I_{a,a}$

$\theta : L \to U(1)$

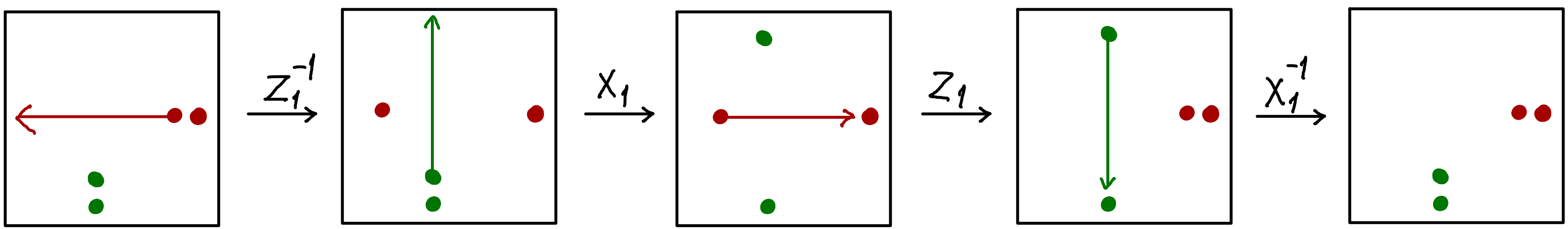
$\theta_a^2 = W_{a,a}$

$\theta_{a \times b} = \theta_a \theta_b W_{a,b}$

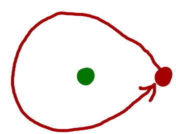
Braiding rules imply ground state degeneracy on the torus

We compare two processes. One of them occurs on the torus. We begin with a ground state, create a pair of particles, move one of them around the torus, and annihilate with the second particle. This effects some logical operator on the ground state, and we can proceed with a similar move. The other process occurs on the plane, but is *topologically equivalent* to a certain sequence of moves on the torus. The difference is that all particle pairs are created in advance, and we never annihilate them.

In the following picture, the particles go not all the way around the torus, but far enough so that they move *relative to each other* in the same way. Thus, the actual process happens on the plane but is equivalent to a sequence of logical operators on the torus.



The above sequence is topologically equivalent to braiding:



Thus, $\boxed{X_1^{-1} Z_1 X_1 Z_1^{-1} = -1}$

Now, the ground space (= code space) can be determined as a representation of the operator algebra generated by X_1, Z_1, X_2, Z_2