Ph219C/CS219C

Exercises
Due: Thursday 18 May 2023

2.1 Distingushing two nonorthogonal pure states

a) Consider an ensemble in which the two pure states of a single qubit

$$|\varphi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \tag{1}$$

occur equiprobably, where $\theta \in [0, 2\pi)$. The two states both lie in the xz plane of the Bloch sphere, and the angle between them is θ. Suppose that a state is drawn from this ensemble and then the Pauli observable

$$\boldsymbol{Z} = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{2}$$

is measured. Compute the information gain achieved by this measurement.

b) Now suppose that, instead of $\boldsymbol{Z}$, the rotated observable

$$\boldsymbol{Z}' = |\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp| \tag{3}$$

is measured, where

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/4 - \pi/4) \\ \sin(\theta/4 - \pi/4) \end{pmatrix}, \quad |\psi^\perp\rangle = \begin{pmatrix} \cos(\theta/4 + \pi/4) \\ \sin(\theta/4 + \pi/4) \end{pmatrix}. \tag{4}$$

Compute the information gain in this case. Make a plot showing the two functions of θ computed in parts (a) and (b). Which has the higher information gain?

c) For $\theta = \pi/2$ and for $\theta = 2\pi/3$, find the numerical value of the information gain, in bits, achieved by measuring $\boldsymbol{Z}$ and $\boldsymbol{Z}'$.

d) For the three pure states of a single qubit described in Section 10.6.4 of the notes,

$$|\varphi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_1\rangle = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad |\varphi_2\rangle = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \tag{5}$$

consider the ensemble in which the three two-qubit states

$$|\Phi_a\rangle = |\varphi_a\rangle \otimes |\varphi_a\rangle, \quad a = 1,2,3 \tag{6}$$

occur equiprobably. A state is drawn from this ensemble, and we are to measure the two-qubit state with the goal of gaining information about the value of a. Instead of doing the collective measurement described in the notes, consider an adaptive strategy in which the two qubits are measured separately. On the first qubit we perform the POVM with the three outcomes

$$\boldsymbol{E}_a = \frac{2}{3}\left(\boldsymbol{I} - |\varphi_a\rangle\langle\varphi_a|\right), \quad a = 1,2,3; \tag{7}$$

this measurement excludes one of the three states, but provides no information to distinguish the other two states. Then we perform a measurement on the second qubit that is conditioned on the outcome of the first measurement. Guided by the results of (a), (b), (c), choose the most informative measurement to perform on the second qubit, and then compute the information gain achieved by this adaptive procedure. Compare with the information gain achieved by the collective measurement described in Section 10.6.4.

2.2 An ensemble of four pure states

a) Consider an ensemble in which the four pure states of a single qubit

$$|\varphi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\varphi_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad |\varphi_2\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \quad |\varphi_3\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}, \tag{8}$$

occur equiprobably. Compute the information gain achieved by measuring $\boldsymbol{Z}$ for a state drawn from this ensemble.

b) Now compute the information gain achieved by measuring $\boldsymbol{Z}'$ for $\theta = \pi/2$. Compare with the result of (a).

2.3 An infinite ensemble

a) We now consider an ensemble with an infinite number of possible pure states, namely the uniform distribution on $\theta \in [0, 2\pi)$ where

$$|\varphi(\theta)\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}. \tag{9}$$

That is, when a state is sampled from this ensemble, a value in the interval $[\theta, \theta + d\theta]$ occurs with probability $d\theta/2\pi$. How much information do we gain about the value of θ by measuring $\boldsymbol{Z}$? To avoid having to take a difference of infinite quantities, answer this question by computing the conditional entropy $H(Y|\theta)$ and using

$$I(\theta; Y) = H(Y) - H(Y|\theta), \tag{10}$$

where Y denotes the probability distribution of measurement outcomes.

b) Hoping to gain more information, let's try a POVM with many outcomes rather than a two-outcome orthogonal measurement. Choose the POVM elements to be

$$\boldsymbol{E}_k = \frac{2}{n}|\phi_k\rangle\langle\phi_k|, \quad k = 0, 1, 2, \ldots, n{-}1, \tag{11}$$

where

$$\phi_k = \frac{2\pi k}{n}. \tag{12}$$

Check the normalization condition

$$\sum_k \boldsymbol{E}_k = \boldsymbol{I}, \tag{13}$$

and compute the information gain achieved by this measurement. Compare with the result of (a).

2.4 A quantum version of Fano's inequality

a) In a d-dimensional system, suppose a density operator $\boldsymbol{\rho}$ approximates the pure state $|\psi\rangle$ with fidelity

$$F = \langle\psi|\boldsymbol{\rho}|\psi\rangle = 1 - \varepsilon. \tag{14}$$

Show that

$$H(\boldsymbol{\rho}) \le H_2(\varepsilon) + \varepsilon \log_2(d-1). \tag{15}$$

Hint: Recall that if a complete orthogonal measurement performed on the state $\boldsymbol{\rho}$ has distribution of outcomes X, then $H(\boldsymbol{\rho}) \le H(X)$, where $H(X)$ is the Shannon entropy of X.

b) As in §10.7.2, suppose that the noisy channel $\mathcal{N}^{A\to B}$ acts on the pure state ψ_{RA}, and is followed by the decoding map $\mathcal{D}^{B\to C}$. Show that

$$H(R)_{\boldsymbol{\rho}} - I_c(R\,\rangle B)_{\boldsymbol{\rho}} \le 2H(RC)_{\boldsymbol{\sigma}}, \tag{16}$$

where

$$\boldsymbol{\rho}_{RB} = \mathcal{N}(\psi_{RA}), \quad \boldsymbol{\sigma}_{RC} = \mathcal{D}\circ\mathcal{N}(\psi_{RA}). \tag{17}$$

Therefore, if the decoder's output (the state of RC) is almost pure, then the coherent information of the channel $\mathcal{N}$ comes close to matching its input entropy. **Hint**: Use the data processing inequality $I_c(R\,\rangle C)_{\boldsymbol{\sigma}} \le I_c(R\,\rangle B)_{\boldsymbol{\rho}}$ and the subadditivity of von Neumann entropy. It is convenient to consider the joint pure state of the reference system, the output, and environments of the dilations of $\mathcal{N}$ and $\mathcal{D}$.

c) Suppose that the decoding map recovers the channel input with high fidelity,

$$F(\mathcal{D}\circ\mathcal{N}(\psi_{RA}), \psi_{RC}) = 1-\varepsilon. \tag{18}$$

Show that

$$H(R)_{\boldsymbol{\rho}} - I_c(R\,\rangle B)_{\boldsymbol{\rho}} \le 2H_2(\varepsilon) + 2\varepsilon\log_2(d^2-1), \tag{19}$$

assuming that R and C are d-dimensional. This is a quantum version of Fano's inequality, which we may use to derive an upper bound on the quantum channel capacity of $\mathcal{N}$.