Sources: Alexei Kitaev's Characterization of Quantum Channels notes, John Preskill's chapter 3 notes, Robert Griffiths' Quantum Channels, Kraus Operators, POVMs notes.

GOAL: Prove that a superoperator T is completely positive iff $\breve{T}$ is Hermitian and positive (i.e. positive semi-definite).

1. Prove that T is completely positive if $\breve{T}$ is Hermitian and positive.

   Recall that the matrix $\breve{T}$ is related to the superoperator T in the following way:

   $$\breve{T}_{JJ'} = \breve{T}_{jk;j'k'} = T_{jj';kk'}, \tag{1}$$

   where $T_{jj';kk'}$ constitutes the elements of the four-dimensional tensor T.

   Because $\breve{T}$ is Hermitian, we can write a spectral decomposition for it, $\breve{T} = MAM^{\dagger}$, where the columns (rows) of M ($M^{\dagger}$) correspond to the eigenvectors of $\breve{T}$, and the real, non-negative values in the diagonal matrix A are its eigenvalues (non-negative since $\breve{T}$ is positive).

   Let's rewrite $\breve{T}$ as $\Sigma_a \breve{M}_a \breve{M}_a^{\dagger}$, where $\breve{M}_a$ ($\breve{M}_a^{\dagger}$) corresponds to a column (row) vector of M ($M^{\dagger}$) multiplied by the square root of its eigenvalue, which is also non-negative and real. $\Sigma_a \breve{M}_a \breve{M}_a^{\dagger}$ is also of course Hermitian and positive, which we can show in a different way:

   $$Hermitian : (\Sigma_a \breve{M}_a \breve{M}_a^{\dagger})^{\dagger} = \Sigma_a \breve{M}_a \breve{M}_a^{\dagger} \tag{2}$$

   $$Positive : \Sigma_a \langle\psi| \breve{M}_a \breve{M}_a^{\dagger} |\psi\rangle = \Sigma_a || \breve{M}_a^{\dagger} |\psi\rangle ||^2 \geq 0 \rightarrow \Sigma_a \breve{M}_a \breve{M}_a^{\dagger} \geq 0 \tag{3}$$

   In index notation,

   $$\breve{T}_{jk;j'k'} = \breve{T}_{JJ'} = \Sigma_a (\breve{M}_a)_J (\breve{M}_a^{*})_{J'} = \Sigma_a (\breve{M}_a)_{jk} (\breve{M}_a^{*})_{j'k'}. \tag{4}$$

   We can now reconstruct $T_{jj';kk'}$ by swapping the two middle indices of $\breve{T}_{jk;j'k'}$:

   $$T_{jj';kk'} = \Sigma_a (M_a)_{j;k} (M_a^{*})_{j';k'}. \tag{5}$$

   Recall that for an arbitrary operator $X \in \mathbf{L}(\mathcal{L})$:

   $$(TX)_{jj'} = \Sigma_{kk'} T_{jj';kk'} X_{kk'}. \tag{6}$$

   Combining the two previous equations, we get:

$$(TX)_{jj'} = \Sigma_{akk'}(M_a)_{j;k}X_{kk'}(M_a^*)_{j';k'}. \tag{7}$$

Let's rewrite this without the indices:

$$TX = \Sigma_a M_a X M_a^\dagger. \tag{8}$$

This is very reminiscent of the equivalence between quantum channels in the Kraus representation and quantum channels as CPTP linear superoperators, except that we have not implemented the trace-preserving condition yet. Regardless, we can check whether or not T is a completely positive map by checking if TX and $(T \otimes I)X'$ preserve Hermiticity and positivity.

First, we let $X = X^\dagger$ and $X \geq 0$. We can expand X so that $X = \Sigma_i p_i |\phi_i\rangle \langle\phi_i|$, for $p_i \geq 0$. We verify that TX also holds these properties:

$$(TX)^\dagger = (\Sigma_a M_a X M_a^\dagger)^\dagger = \Sigma_a M_a X M_a^\dagger = TX \tag{9}$$

$$\begin{aligned}\langle\psi| TX |\psi\rangle &= \Sigma_{ai} p_i \langle\psi| M_a |\phi_i\rangle \langle\phi_i| M_a^\dagger |\psi\rangle \\ &= \Sigma_a p_i || \langle\phi_i| M_a^\dagger |\psi\rangle ||^2 \geq 0 \rightarrow TX \geq 0\end{aligned} \tag{10}$$

Consequently, T is a positive map.

Next, we consider X', which is an operator that exists on an augmented Hilbert space. Let $X' = X'^\dagger$ and $X' \geq 0$. We can expand X' so that $X' = \Sigma_i p'_i |\phi'_i\rangle \langle\phi'_i|$, for $p'_i \geq 0$. We verify that $(T \otimes I)X'$ also holds these properties:

$$\begin{aligned}((T \otimes I)X')^\dagger &= (\Sigma_a (M_a \otimes I)X'(M_a^\dagger \otimes I))^\dagger \\ &= \Sigma_a (M_a \otimes I)X'(M_a^\dagger \otimes I) = (T \otimes I)X'\end{aligned} \tag{11}$$

$$\begin{aligned}\langle\psi| (T \otimes I)X' |\psi\rangle &= \Sigma_{ai} p'_i \langle\psi| (M_a \otimes I) |\phi'_i\rangle \langle\phi'_i| (M_a^\dagger \otimes I) |\psi\rangle \\ &= \Sigma_a p'_i || \langle\phi'_i| (M_a^\dagger \otimes I) |\psi\rangle ||^2 \geq 0 \rightarrow (T \otimes I)X' \geq 0\end{aligned} \tag{12}$$

Consequently, T is a completely positive map. T is completely positive if $\breve{T}$ is Hermitian and positive.

2. Brief aside regarding the equivalence between different definitions of quantum channels.

   Let's now implement the trace-preserving property on our superoperator T:

   $$\delta_{kk'} = \Sigma_j T_{jj,kk'} = \Sigma_{aj} (M_a)_{j;k} (M_a^*)_{j;k'} = (\Sigma_a M_a^\dagger M_a)_{kk'}. \tag{13}$$

   In other words, $\Sigma_a M_a^\dagger M_a = I$. This is the exact condition which is necessary to ensure that the channel $\mathcal{E}(X) = \Sigma_a M_a X M_a^\dagger$ is trace-preserving.

   We have therefore shown that the quantum channel that is represented by a CPTP linear superoperator is equivalent to the quantum channel $\mathcal{E}(X)$ in the Kraus representation.

3. Prove that if T is completely positive, $\breve{T}$ is Hermitian and positive.

   If T is completely positive, the result of the map $(T \otimes I)X'$ will be a positive semi-definite operator for any positive semi-definite operator X', where X' lies on the augmented Hilbert space $\mathcal{L} \otimes \mathcal{N}$.

   Since we are allowed to choose X', let $X' = \Sigma_{jkj'k'} |jk\rangle \langle j'k'| \delta_{jk}\delta_{j'k'}$, or in index notation, $(X')_{jkj'k'} = \delta_{jk}\delta_{j'k'}$. X' is the unnormalized maximally entangled state between both of our Hilbert spaces. We have constrained our Hilbert spaces $\mathcal{L}$ and $\mathcal{N}$ to be of the same dimension, which is a choice that we are allowed to make, since it does not impose any restrictions on T itself.

   Let us compute $(T \otimes I)X'$:

   $$\begin{aligned}((T \otimes I)X')_{jkj'k'} &= \Sigma_{ll'} T_{jj';ll'} X'_{lkl'k'} \\ &= \Sigma_{ll'} T_{jj';ll'} \delta_{lk}\delta_{l'k'} = T_{jj';kk'} = \breve{T}_{jk;j'k'} = \breve{T}_{JJ'} = \breve{T}\end{aligned} \tag{14}$$

   We can immediately conclude that $\breve{T}$ is positive semi-definite (Hermitian and positive), since T is completely positive. If T is completely positive, $\breve{T}$ is Hermitian and positive.

To conclude, a superoperator T is completely positive iff $\breve{T}$ is Hermitian and positive.

Additional note: this proof can also be done using the channel state duality.