Characterization of quantum channels 10/9/2022

Many physical processes can be described as transformations of density matrices. We will see shortly that such transformations are linear. This motivates a very general definition: a *superoperator* is a linear map from operators to operators. For example, let the input operators act on a Hilbert space $\mathcal{L}$, and let us denote the space of all such operators by $\mathbf{L}(\mathcal{L})$. A superoperator T may map this space to $\mathbf{L}(\mathcal{M})$ for another Hilbert space $\mathcal{M}$. Given some bases of $\mathcal{L}$ and $\mathcal{M}$, we can write operators as matrices, and so T is represented by a tensor with four indices:

$$T:\,\mathbf{L}(\mathcal{L})\to\mathbf{L}(\mathcal{M}),\qquad (TX)_{jj'}=\sum_{k,k'}T_{jj';kk'}X_{kk'}. \tag{1}$$

However, not all superoperators are physical or even preserve the general properties of density matrices. The purpose of this note is to characterize physically realizable superoperators, also known as *quantum channels*. We will consider three different definitions:

1. Operational definition: A quantum channel corresponds to a process in which the input state is supplemented by an ancillary system in a fixed pure state, some unitary operation is performed, and then part of the system is discarded.

2. A matrix characterization (known as Choi's criterion), which imposes certain conditions on the coefficients $T_{jj';kk'}$.

3. An abstract mathematical definition: A quantum channel is a completely positive, trace-preserving superoperator T. (The preservation of the trace is almost obvious because if X is a density matrix and thus have unit trace, then TX should also be a density matrix. The complete positivity will be defined later.)

There is also a fourth characterization, the Kraus representation, which is described in detail in Preskill's lecture notes. We will now elaborate the above definitions and prove their equivalence.

Operational definition. First, it is legitimate to ask why add ancillas at the beginning and discard them (or some other part of the system) at the very end. One could do it the other way around or add and discard things multiple times. To answer this, all parts exist throughout the whole process. "Adding" and "discarding" them just imposes some constraints on the unitary operations: they cannot act nontrivially on a subsystem before it is "added" or after it has been "discarded". There are no constraints if we add all ancillas at once and keep them until the very end. One could also use a mixed ancillary state, but it is equivalent to a pure state with additional ancillas, which will never be touched. So our definition is the most general one.

Now, let us write some equations. Denoting the ancillary state by $|0\rangle$, the unitary operation by U, and the discarded part by $\mathcal{F}$, we get:

$$T\rho=\operatorname{Tr}_{\mathcal{F}}\Big(U\big(\rho\otimes|0\rangle\langle 0|\big)U^{\dagger}\Big). \tag{2}$$

Indeed, by simply adding the ancillas, we obtain the density matrix $\gamma = \rho \otimes |0\rangle\langle 0|$.[1] The unitary operation U acts on pure states as $|\psi\rangle \mapsto U|\psi\rangle$ or $|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger$ and this rule extends to arbitrary states by linearity; hence, γ is transformed into $U\gamma U^\dagger$. The partial trace over $\mathcal{F}$ corresponds to discarding $\mathcal{F}$.

There is some room for simplification. The addition of $|0\rangle$ followed by U can be described by a single linear map

$$V : \mathcal{L} \to \mathcal{M} \otimes \mathcal{F}, \qquad V|\xi\rangle = U\big(|\xi\rangle \otimes |0\rangle\big). \tag{3}$$

V is an *isometric embedding*, meaning that it preserves the length of vectors. This condition is expressed by the equation $V^\dagger V = I$.[2] Thus,

$$\boxed{T\rho = \operatorname{Tr}_{\mathcal{F}}(V\rho V^\dagger), \qquad \text{where} \quad V^\dagger V = I_{\mathcal{L}}.} \tag{4}$$

Matrix characterization. Let us introduce a variant of tensor T where the two middle indices are swapped:

$$\breve{T}_{jk;j'k'} = T_{jj;'kk'}. \tag{5}$$

This is convenient because we can now combine j and k into one large index J, and similarly combine j', k' into J'. We claim that equation (4) is equivalent to three conditions:

1. The matrix $\breve{T} = (\breve{T}_{JJ'})$ is Hermitian.
2. The matrix $\breve{T}$ is positive-semidefinite.
3. $\sum_j T_{jj';kk'} = \delta_{kk'}$.

To show this, we will first derive an explicit formula for T from equation (4). Writing the latter as

$$(T\rho)_{jj'} = \sum_{k,k',l} V_{jl;k}\rho_{kk'}V^*_{j'l;k'}, \tag{6}$$

we find that

$$T_{jj';kk'} = \sum_l V_{jl;k}V^*_{j'l;k'}, \tag{7}$$

or equivalently,

$$\breve{T}_{JJ'} = \sum_l \breve{V}_{Jl}\breve{V}^*_{Jl}, \qquad \text{where} \quad \breve{V}_{jk;l} = V_{jl;k}. \tag{8}$$

[1] We have omitted parentheses around $|0\rangle\langle 0|$, which is consistent with operator multiplication having higher precedence than tensor product.

[2] Unitarity is a combination of two conditions, $U^\dagger U = I$ and $UU^\dagger = I$. One follows from the other if the input and the output dimensions of U are finite and equal to each other. However, V may not satisfy the second condition because the output dimension, $\dim\mathcal{M} \cdot \dim\mathcal{F}$, is generally greater than $\dim\mathcal{L}$.)

That is, $\breve{T} = \breve{V}\breve{V}^\dagger$. It is clear that this $\breve{T}$ is Hermitian and positive-semidefinite. Conversely, any Hermitian positive-semidefinite matrix $\breve{T}$ admits a representation as $\breve{V}\breve{V}^\dagger$ for some $\breve{V}$. Finally, condition 3 above can easily be written in terms of V,

$$\sum_{j,l} V_{jl;k} V^*_{jl;k'} = \delta_{kk'}, \tag{9}$$

which is the same as $V^\dagger V = I$.

Abstract mathematical definition. A superoperator is called *positive* if it takes positive-semidefinite operators to positive-semidefinite operators. (Hermicity is assumed when we talk about positivity.) This property follows from the physical realizability as described by equation (4). However, we will need a stronger property called complete positivity. Namely, not only T is positive but it remains positive if we augment the input and the output with an arbitrary Hilbert space $\mathcal{N}$ on which the action is trivial; that is, $T \otimes I_{\mathbf{L}(\mathcal{N})}$ is positive. If T is physically realizable, then the augmentation can be achieved by replacing V with $V \otimes I_{\mathcal{N}}$ (which is also an isometric embedding). Therefore, physically realizable superoperators are completely positive. It seems surprising that for general superoperators, the augmentation may break positivity, but here is one example.

Let T be the transposition superoperator, that is,

$$(TX)_{jj'} = X_{j'j}. \tag{10}$$

The transposed of a positive-semidefinite matrix is also positive-semidefinite; hence T is positive. For simplicity, we assume that $\mathcal{L} = \mathcal{M} = \mathbb{C}^2$, i.e. T acts on a single qubit. Let us augment it with the trivial action on another qubit, $\mathcal{N} = \mathbb{C}^2$, to obtain $T \otimes I_{\mathbf{L}(\mathcal{N})}$, and let us apply it to the entangled state $|\psi\rangle\langle\psi|$, where $|\psi\rangle = |00\rangle + |11\rangle$. (The vector ψ could be normalized but that is not important for our purposes.) We have

$$(|\psi\rangle\langle\psi|)_{jkj'k'} = \delta_{jk}\delta_{j'k'}. \tag{11}$$

The superoperator $T \otimes I_{\mathcal{N}}$ swaps j and j' but not k and k'. Thus,

$$\big((T \otimes I_{\mathbf{L}(\mathcal{N})})(|\psi\rangle\langle\psi|)\big)_{jkj'k'} = \delta_{j'k}\delta_{jk'}, \tag{12}$$

or in matrix notation,

$$(T \otimes I_{\mathbf{L}(\mathcal{N})})(|\psi\rangle\langle\psi|) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

This matrix is *not* positive-semidefinite, as is evident from its middle 2×2 block, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Another surprising fact is that complete positivity is computationally simpler than just positivity. As we will see momentarily, checking the complete positivity is equivalent to checking conditions 1 and 2 on the the matrix $\breve{T}$, which is a standard task in linear algebra. But checking if T is positive is generally harder.

Finally, let us prove that any completely positive trace-preserving superoperator satisfies conditions 1–3 above. It is easy to see that condition 3 is equivalent to the trace preservation, so it is sufficient to prove that any completely positive superoperator T satisfies conditions 1 and 2. To this end, let $\mathcal{N}$ have the same dimension as the input space $\mathcal{L}$, and let us apply $T \otimes I_{\mathbf{L}(\mathcal{N})}$ to the operator $|\psi\rangle\langle\psi|$, where $|\psi\rangle = \sum_j |jj\rangle$. Thus we obtain the operator with the matrix elements

$$\big((T \otimes I_{\mathbf{L}(\mathcal{N})})(|\psi\rangle\langle\psi|)\big)_{jkj'k'} = \sum_{l,l'} T_{jj';ll'}(|\psi\rangle\langle\psi|)_{lkl'k'} = T_{jj';kk'} = \breve{T}_{jk;j'k'}. \tag{14}$$

Since T is completely positive, the first expression in the above equation is a positive-semidefinite Hermitian matrix (where we treat the first and second pairs of indices as single indices). But this is equivalent to conditions 1 and 2.

Thus, we have established the equivalence between the three definitions of a quantum channel. The statement that any completely positive trace-preserving superoperator admits representation (4) is often referred to as Steinespring's theorem. In fact, the classic Steinespring theorem is slightly different in two ways. First, it is more general and applicable to infinitely-dimensional Hilbert spaces. Second, the theorem is usually formulated in terms of the dual superoperator T^{dual}, which acts on observables rather than quantum states. It is defined by the equation

$$\mathrm{Tr}\big((T^{\text{dual}}Y)X\big) = \mathrm{Tr}\big(Y(TX)\big), \tag{15}$$

or equivalently,

$$T^{\text{dual}}_{jj';kk'} = T_{k'k;j'j}. \tag{16}$$

In this notation, the Steinespring theorem asserts that if $T^{\text{dual}} : \mathbf{L}(\mathcal{M}) \to \mathbf{L}(\mathcal{L})$ is completely positive, then it admits a representation of the form

$$T^{\text{dual}}Y = V^\dagger(Y \otimes I_{\mathcal{F}})V. \tag{17}$$

In addition, if T^{dual} is *unital*, i.e. takes $I_{\mathcal{M}}$ to $I_{\mathcal{L}}$, then $V^\dagger V = I_{\mathcal{L}}$.