# Fault-tolerant quantum memory

**Path to fault-tolerant quantum computation based on stabilizer codes**

1) Fault-tolerant error correction

2) Logical Clifford + classical operations

-- If a self-dual CSS code is used, these can be implemented transversally

-- With other codes, the implementation is more complex

This lecture is focused on tasks 1 and 2 assuming sparse faults (at most one per gadget and not in adjacent gadgets). To deal with faults occurring at a constant rate and to prove the threshold theorem, one can use either surface codes or concatenated codes. In the later case, the two tasks are interdependent.

3) Add some non-Clifford gates to achieve universality

-- Use "magic" ancillas, e.g. $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$

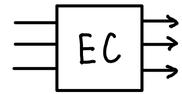
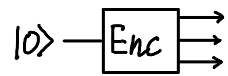
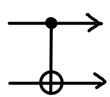
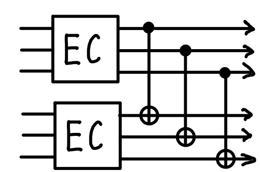
-- Other methods

**Alternative paths**

-- Non-Abelian anyons

-- Continuous variables codes

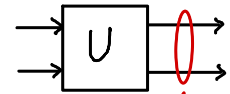
## Elementary physical operations and their logical counterparts

| Physical operation | Logical counterpart |
|---|---|
| 0) Idling or leakage cleanup<br><br>Physical qubits can be out of the computational subspace (in states $\|2\rangle, \|3\rangle, \dots$). We include an elementary operation that replaces any leaked state by $\|0\rangle$ or $\|1\rangle$ but does not alter any superposition of the latter. |  (fault-tolerant error correction circuit)<br><br>At the logical level, "leaked" (or uncorrectable) states are the result of bad errors. They have to be considered if one wants to add another error correction layer on top of an existing one. |
| 1) Qubit initialization in state $\|0\rangle$ |  The circuit Enc can be realized as the encoding for the trivial code, $\|\xi\rangle \mapsto \|0^{n-1}\rangle \otimes \|\xi\rangle$, followed by a sequence of Clifford gates<br><br>Fault-tolerant realization:<br>Use the error-correcting circuit for the code $\mathcal{M}_0$ obtained from $\mathcal{M}$ by adding $Z_L$ as an extra stabilizer operator. |
| 2) Measurements in the $\|0\rangle, \|1\rangle$ basis | The eigenvalue of $Z_L$ can be measured as part of the syndrome for $\mathcal{M}_0$. |
| 3) Classically controlled Clifford gates<br><br>$H, K,$  | $H^{\otimes n} \cdot EC, \quad K^{\otimes n} \cdot EC,$  |

4) Operations with classical bits (assumed to be absolutely reliable)

## Faults in a unitary gate $U$

A general fault involves interaction of the qubits with the environment




The error operator $E_j$ effectively acts after the gate

Actual gate: $\tilde{U} \approx \underbrace{U}_{\text{unitary acting on the qubits}} \otimes \underbrace{Y_0}_{\text{unitary acting on the environment}}$,

$$\tilde{U} = \Big( I \otimes Y_0 + \sum_{j\neq 0} \underbrace{E_j}_{\text{error operators (nontrivial Pauli matrices)}} \otimes \underbrace{Y_j}_{\text{not unitary: } \|Y_j\| \le \delta} \Big) (U \otimes I)$$

In the simplest case, faults are incoherent (i.e. the corresponding operators $Y_j$ leave the environment in mutually orthogonal states) and occur with total probability $p = \sum_{j\neq 0} \|Y_j\|^2 = O(\delta^2)$

In general, the total error is quantified by the operator norm: $\|\tilde{U} - U \otimes Y_0\| \le \sum_{j\neq 0} \|Y_j\| = O(\delta)$

The second case is worse due to the possibility of quantum interference between a fault and faultless operation:

$$\mathrm{Tr}_{env.}\big(\tilde{U}(\rho \otimes \rho_{env})\tilde{U}^{+}\big) - U\rho U^{+} = \underbrace{\sum_{j\neq 0} E_j U \rho U^{+} \cdot \mathrm{Tr}\big(Y_j \rho_{env.} Y_0^{+}\big)}_{\text{interference: norm } = O(\delta)} + \text{h.c.} + \underbrace{\text{quadratic terms}}_{\text{norm } = O(\delta^2)}$$

use the $E_j \otimes Y_j$ term    use $I \otimes Y_0$

**Mitigation of error interference** (in the case where $U$ is a Clifford gate)

Choose a Pauli operator $P$ at random. Implement $U$ as a product of three gates: $P \cdot U \cdot \underbrace{(U^{-1} P\, U)}_{\text{Pauli}}$

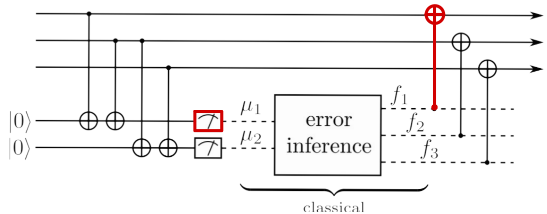
The error $E_j$ in the middle $U$ becomes $P E_j P$ and averages to 0 because $P$ is random

(This trick is commonly used in NMR to cancel unwanted interactions between nuclear spins.)

**Fault patterns in a gadget** (i.e. a circuit operating on a code blocks)

We decompose each physical gate $\widetilde{U}_k$ into the "correct" term $U_k \otimes Y_{k,0}$ and fault terms $\sum_{j\neq 0} E_{k,j} U_k \otimes Y_{k,j}$

For example:




The location of fault terms define a *fault pattern*. Fault patterns are subdivided into "admissible" and "bad". For example, we may call a fault pattern admissible if it contains at most one fault.

$$\boxed{\widetilde{U} = \underbrace{\widetilde{U}_{\text{admissible}}} + \widetilde{U}_{\text{bad}}, \qquad \|\widetilde{U}_{\text{bad}}\| \leq O(n\delta)}$$

should represent a logical gate *W* in a pair of subsystem codes.
(A *subsystem code* is a combination of a usual code and a choice of admissible errors that may be left uncorrected.)

**Subsystem encoding (recap)**

$$\widetilde{V} : \mathcal{L} \otimes \underbrace{\mathcal{E}'} \to \mathcal{N},$$

reduced error space: $\mathcal{E}' = \mathcal{E}/\mathcal{E}_0$, where $\mathcal{E}_0$ is the space of errors annihilating all code vectors

$$\widetilde{V}\left(\underbrace{|\psi\rangle}_{\text{logical state}} \otimes \underbrace{|E'\rangle}_{\text{reduced error}}\right) = \underbrace{E}_{\text{a corresponding error operator}} V|\psi\rangle$$

encoding for the usual code

## Realization of logical gates using subsystem encodings

Logical gate: $W : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$

Subsystem encodings: $\underbrace{\widetilde{V}_1 : \mathcal{L}_1 \otimes \mathcal{E}_1' \longrightarrow \mathcal{N}_1}_{\text{input}}, \quad \underbrace{\widetilde{V}_2 : \mathcal{L}_2 \otimes \mathcal{E}_2' \longrightarrow \mathcal{N}_2}_{\text{output}}$

(Part of) the physical operator: $\widetilde{U}_{\text{admissible}} : \mathcal{N}_1 \otimes \mathcal{F}_1 \longrightarrow \mathcal{N}_2 \otimes \mathcal{F}_2$, $\quad \widetilde{U}_{\text{admissible}} = \sum_j A_j \otimes Y_j$

environment before and after

(sum over combinations of error terms)

### The realization condition

$\exists \widetilde{C} : \mathcal{E}_1 \otimes \mathcal{F}_1 \rightarrow \mathcal{E}_2 \otimes \mathcal{F}_2$ (not necessarily unitary) that makes this diagram commute:

$$\begin{tikzcd}
\mathcal{N}_1 \otimes \mathcal{F}_1 \arrow[r, "\widetilde{U}_{\text{admissible}}"] & \mathcal{N}_2 \otimes \mathcal{F}_2 \\
\mathcal{L}_1 \otimes \mathcal{E}_1' \otimes \mathcal{F}_1 \arrow[u, "\widetilde{V}_1 \otimes I_{\mathcal{F}_1}"'] \arrow[r, "W \otimes \widetilde{C}"] & \mathcal{L}_2 \otimes \mathcal{E}_2' \otimes \mathcal{F} \arrow[u, "\widetilde{V}_2 \otimes I_{\mathcal{F}_2}"']
\end{tikzcd}$$

This condition can be satisfied term by term, neglecting the environment.

$\exists\, C_j : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ such that

$$\begin{tikzcd}
\mathcal{N}_1 \arrow[r, "A_j"] & \mathcal{N}_2 \\
\mathcal{L}_1 \otimes \mathcal{E}_1' \arrow[u, "\widetilde{V}_1"'] \arrow[r, "W \otimes C_j"] & \mathcal{L}_2 \otimes \mathcal{E}_2' \arrow[u, "\widetilde{V}_2"']
\end{tikzcd}$$

commutes

Then $\widetilde{C} = \sum_j C_j \otimes Y_j$

The previous formalism can be extended to measurements. However, we will treat them less rigorously...

## Fault-tolerant error correction (basic scheme)

We use stabilizer code $\mathcal{M}$ of type *[[n,1]]* that protects from one error.

In certain cases, one residual error is allowed.

Usual encoding (no residual errors): $V : \mathcal{B} \to \mathcal{B}^{\otimes n}$

Subsystem encoding (up to 1 residual error): $\widetilde{V} : \mathcal{B} \otimes \mathcal{E}' \to \mathcal{B}^{\otimes n}$, where $\mathcal{E} = \mathcal{E}(n,1)$

At most one fault in the error correction circuit is allowed

### Requirements for the EC circuit

If there are no faults:

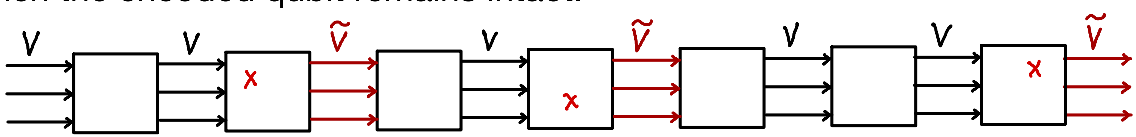
A single error is corrected, i.e. the circuit realizes the identity operator from encoding $\widetilde{V}$ to encoding $V$. (Since $V$ is a special case of $\widetilde{V}$, the circuit will also accept a qubit encoded by $V$.

In general (assuming up to 1 fault):

A single error may be introduced, i.e. the circuit realizes the identity operator from $V$ to $\widetilde{V}$.

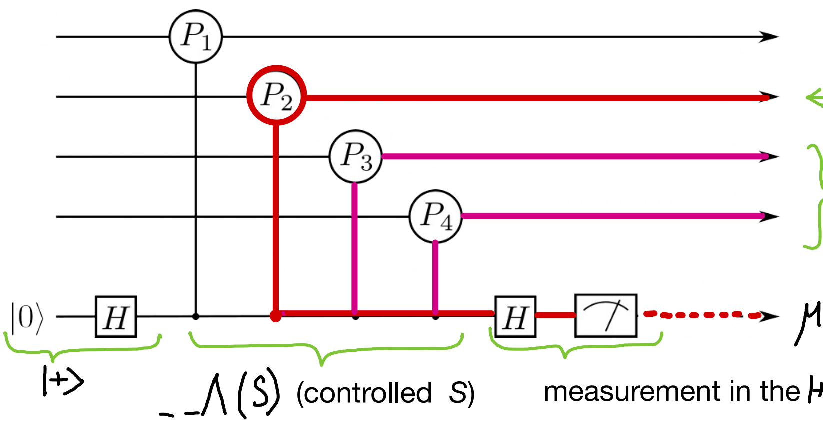
### How it works

If we run the EC circuit multiple times and there is at most one fault in any pair of adjacent runs, then the encoded qubit remains intact.



## Problem with syndrome measurement (recap)

Consider the eigenvalue measurement for $S = P_1 \cdots P_m$ $\quad e.v.(S) = (-1)^{\mu}$




A fault in the controlled $P_2$ has multiple effects:

-- This code qubit is affected directly

-- An $X$ error in the ancilla propagates, effecting the operator $P_3 P_4$

-- A $Z$ error in the ancilla does not propagate but results in the incorrect measurement of $\mu$

### Comments

-- An incorrectly measured $\mu$ can be fixed by repeated measurements

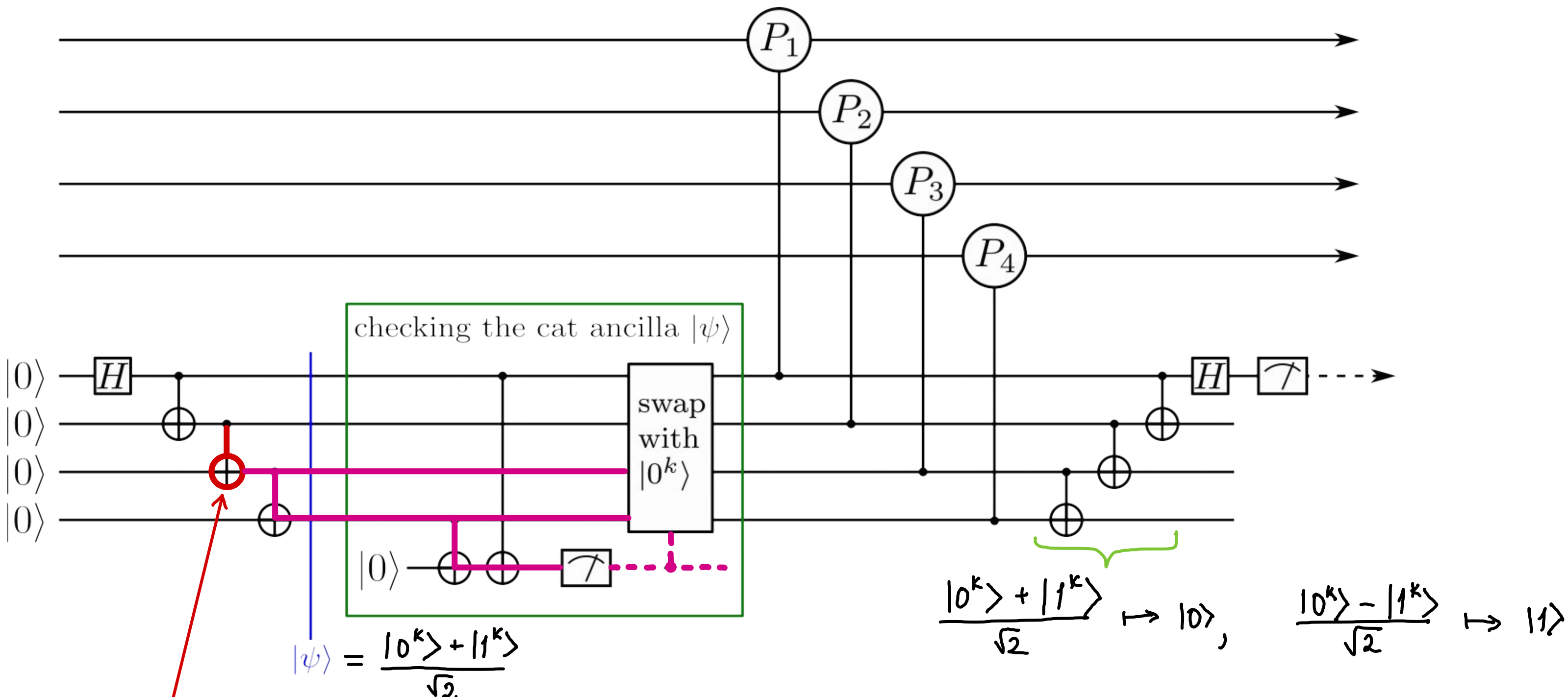
-- Errors in the code qubits may be considered up to the stabilizer $S$. For example,

$$P_2 P_3 P_4 = P_1 S \equiv P_1$$

The worst case is when the error hits in the middle, e.g. $E = P_3 P_4 \equiv P_1 P_2$

**Fault-tolerant stabilizer measurement based on a cat ancilla** (used by Shor in his original paper)

To avoid error propagation, the ancillary qubit is replaced by its encoding using the repetition code. An $X$ error can be detected by an additional measurement. If the test fails, the cat ancilla is replaced by a dummy state, $|0^k\rangle$




A fault in this gate propagates, but is then stopped

If there is at most 1 fault, no more than 1 code qubit is affected. But the measurement result may still be incorrect.

## Fixing measurement errors

Since all syndrome bits are used, we must ensure the consistency of the whole syndrome.

### The simplest procedure

Measure the syndrome twice, get two vectors $\mu^{(1)}, \mu^{(2)}$. If $\mu^{(1)} = \mu^{(2)}$, infer and correct the error. Otherwise stop.

If $\mu^{(1)} \neq \mu^{(2)}$, there must be a fault in the current circuit, and therefore, we are not required to correct any pre-existing errors.

### A more advanced procedure

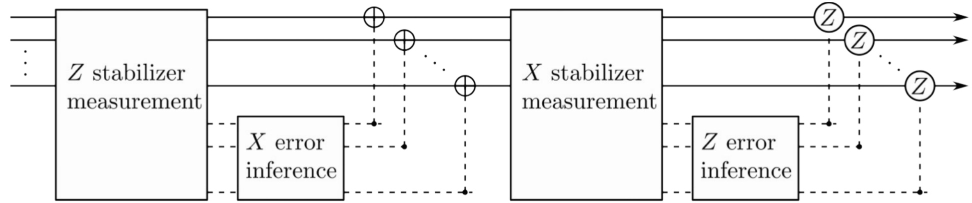
Measure the whole syndrome 3 times, and choose a consistent result:

If $\mu^{(1)} = \mu^{(2)}$, use $\mu^{(1)}$, otherwise use $\mu^{(3)}$

Possible outcomes by fault location: $(\cancel{\mu^{(1)}}, \mu^{(2)} = \mu^{(3)})$, $(\mu^{(1)}, \cancel{\mu^{(2)}}, \mu^{(3)})$, $(\mu^{(1)} = \mu^{(2)}, \cancel{\mu^{(3)}})$

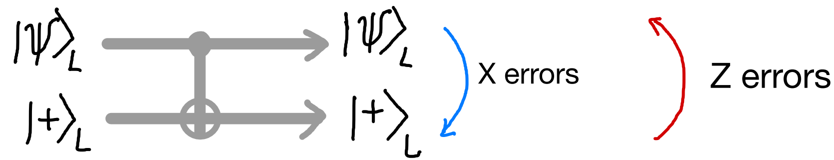
## Fault-tolerant error correction circuit based on encoded $|0\rangle$ and $|+\rangle$ ancillas

(Due to Steane; works for CSS codes)




**Z stabilizer measurement** (= X error detection): general idea

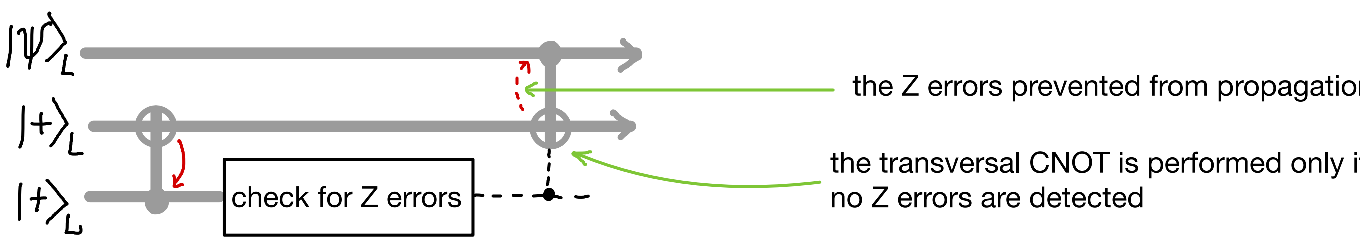
If we perform a logical CNOT like this,




then the logical state is unchanged, but any $X$ errors in the physical qubits are propagated from the first code block (representing the logical qubit) to the second block (a $|+\rangle$ ancilla), where they can be measured <u>destructively</u>, by measuring every qubit.

### Problem and its conceptual solution

Also, Z errors propagate from the $|+\rangle$ ancilla to the code qubits. If there was a fault during the preparation of the ancilla, there may be many such errors! But we can detect them before too late.



**Destructive measurement of X or Z errors in a $|+\rangle_L$ ancilla**

Let the code stabilizers be $S_j^{(z)} = \sigma^z(f_j^{(z)})$, $\quad S_k^{(x)} = \sigma^x(f_k^{(x)})$

The state $|+\rangle_L$ is determined by those stabilizers together with $X_L = \sigma^x(h^{(x)})$.

$$\boxed{\tilde{D}_z = D_z = \text{linear span } \{f_j^{(z)}\}, \qquad \tilde{D}_x = D_z^{\perp} = \text{linear span } \{f_j^{(x)}, h^{(x)}\}}$$

**To detect X errors** (up to $\tilde{D}_x$) and find the $\tilde{D}_z$ syndrome:

Measure each qubit in the $|0\rangle, |1\rangle$ basis. Let the outcome be $g^{(x)}$.

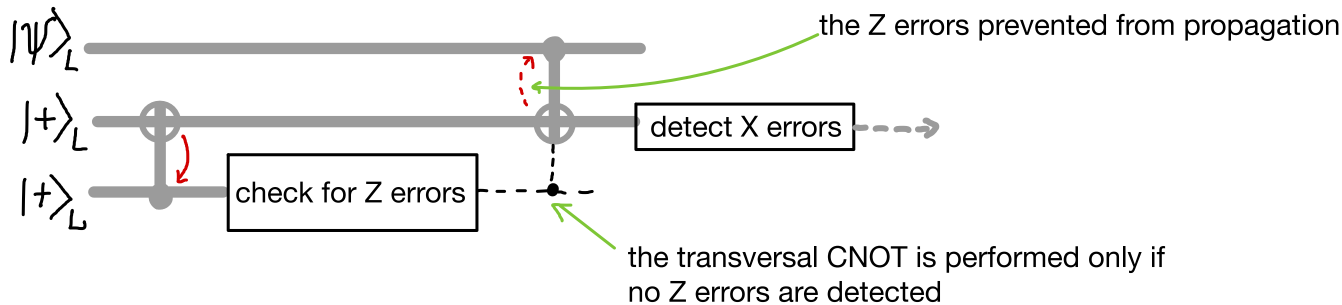
Then the $Z$-syndrome is given by $\mu_j = (g^{(x)}, f_j^{(z)})$.

**To detect Z errors** (up to $\tilde{D}_z$) and find the $\tilde{D}_x$ syndrome:

Measure each qubit in the $|+\rangle, |-\rangle$ basis. Let the outcome be $g^{(z)}$.

Then the $Z$-syndrome is given by $\mu_j = (g^{(z)}, f_j^{(x)})$ and $\mu = (g^{(z)}, h^{(x)})$.

## *Z* stabilizer measurement: schematic




## Detailed circuit