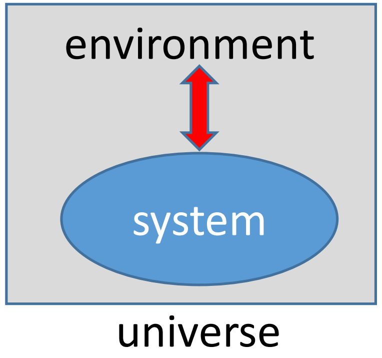
Ph/CS 219A
Quantum Computation

Lecture 3. Measurements



This is the second of several lectures in which we will develop the theory of *open quantum systems*.

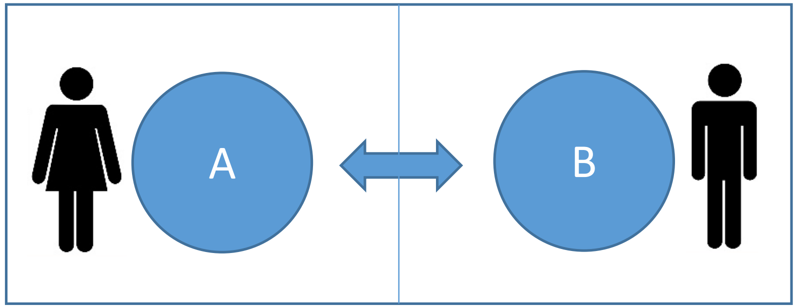
We say that a system is "open" if it can exchange energy and information with its environment.

Last time we talked about states of an open system, described by density operators: nonnegative Hermitian operators with unit trace. We also discussed the Schmidt decomposition of a bipartite pure state.

Today we will talk about measurements that are more general than the orthogonal measurements specified by our axioms for closed quantum systems. These are realized when a system interacts with an auxiliary system, followed by an orthogonal measurement on that auxiliary system.

Before we get to that, though, we'll discuss some further properties of density operators and bipartite pure states.

See Chapters 2 and 3 of the Lecture Notes. Note that Homework #1 has been posted, due Friday October 16



Properties of the density operator

$$\rho_A = \sum_{i,j,\mu} a_{i\mu}^* a_{j\mu} \, | i \rangle \langle j | \equiv \mathrm{tr}_B \left(| \psi \rangle \langle \psi | \right),$$

$$\text{where} \quad | \psi \rangle_{AB} = \sum a_{i\mu} \, | i, \mu \rangle_{AB}$$

The density operator is Hermitian. $\rho = \rho^\dagger$

The density operator is nonnegative. $\langle \phi | \rho | \phi \rangle = \sum_{i,j,\mu} a_{j\mu}^* a_{i\mu} \langle \phi | i \rangle \langle j | \phi \rangle = \sum_\mu \left| \sum_i a_{i\mu} \langle \phi | i \rangle \right|^2 \geq 0$

The density operator has unit trace. $\mathrm{tr} \rho = \sum |a_{i\mu}|^2 = \| \, | \psi \rangle_{AB} \|^2 = 1$

Hence there is an orthonormal basis in which the density operator is diagonal. The eigenvalues are nonnegative real numbers which sum to one. Interpret as an ensemble.

$$\rho = \sum_i p_i \, | i \rangle \langle i |, \quad p_i \geq 0, \quad \sum_i p_i = 1$$

$| \psi \rangle_{AB} = \sum_i \sqrt{p_i} \, | i \rangle_A \otimes | i' \rangle_B$ where $\{ | i \rangle_A \}$ and $\{ | i' \rangle_B \}$ are orthonormal bases.

(Schmidt decomposition). Density operators of A and B have the same nonzero eigenvalues:

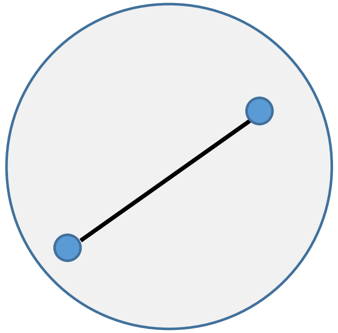
$$\mathrm{tr}_A \left(| \psi \rangle \langle \psi | \right) = \sum_i p_i \, | i' \rangle \langle i' |$$

The set of density operators is convex

Consider $\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2, \quad 0 \le \lambda \le 1.$

Is it a density operator? It is obviously Hermitian and has unit trace. But is it positive? Yes.

$$\langle\psi|\rho(\lambda)|\psi\rangle = \lambda\langle\psi|\rho_1|\psi\rangle + (1-\lambda)\langle\psi|\rho_2|\psi\rangle \ge 0$$

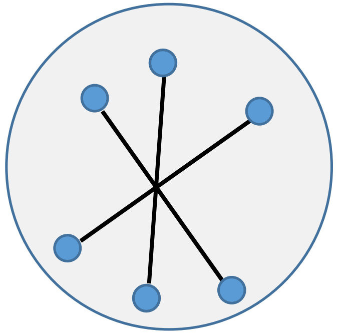


Interpretation: Charlie flips coins to sample from a distribution, then prepares either state 1 or state 2 with specified probabilities. What Charlie prepares this way is also a possible state.

$$\langle M\rangle = \lambda \mathrm{tr}\rho_1 M + (1-\lambda)\mathrm{tr}\rho_2 M = \mathrm{tr}\rho(\lambda) M$$

For whatever observable Alice chooses to measure, she can't tell the difference between measuring the state $\rho(\lambda)$ and measuring either state 1 or state 2 with specified probabilities.

For a given mixed state, there are many possible ways to prepare by sampling from an ensemble. But pure states are an exception: for them the preparation is unique. Suppose not …



$$\rho = |\psi\rangle\langle\psi| = \lambda\rho_1 + (1-\lambda)\rho_2 \quad \text{and} \quad \langle\psi^\perp|\psi\rangle = 0. \quad \text{Then}$$

$$0 = \langle\psi^\perp|\rho|\psi^\perp\rangle = \lambda\langle\psi^\perp|\rho_1|\psi^\perp\rangle + (1-\lambda)\langle\psi^\perp|\rho_2|\psi^\perp\rangle$$

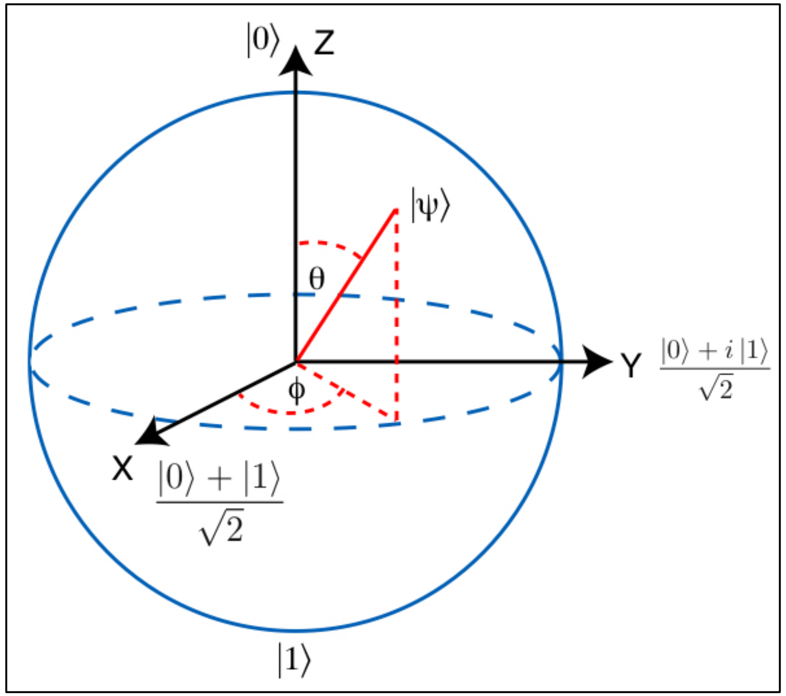
$$\Rightarrow \langle\psi^\perp|\rho_1|\psi^\perp\rangle = 0 \quad \text{and} \quad \langle\psi^\perp|\rho_2|\psi^\perp\rangle = 0.$$

Therefore: $\rho_1, \rho_2 \propto |\psi\rangle\langle\psi|$

A *pure* state is *extremal*. It cannot be obtained as a mixture of two other states.

Density operator of a qubit

$$\rho(\vec{P}) = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1+P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1-P_3 \end{pmatrix}$$



This is the most general Hermitian 2 X 2 matrix with unit trace.
But is it nonnegative? That is, are both eigenvalues ≥ 0?

$$\det \rho(\vec{P}) = \frac{1}{4}\left(1 - \vec{P}^2\right) \geq 0 \quad \Rightarrow \quad |\vec{P}| \leq 1.$$

The possible density operators correspond to points in a closed 3-dimensional ball with unit radius. This state space for a qubit is called the Bloch sphere (after Felix Bloch), even though it is really a ball not a sphere.

The boundary of the Bloch ball really is a sphere. For points on the boundary, the determinant is 0; that is the eigenvalues of the density operator are 0 and 1. These are the pure states of a qubit, previously discussed.

$$\rho(\hat{n}) = \frac{1}{2}\left(I + \hat{n}\cdot\vec{\sigma}\right).$$

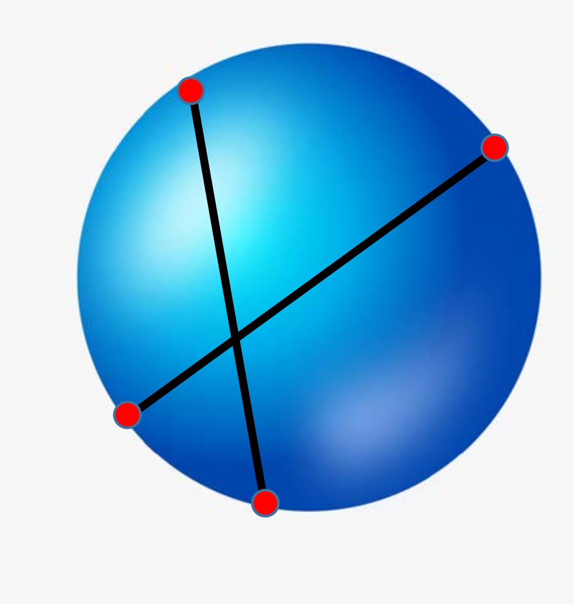
Notice that: $\left(\hat{n}\cdot\vec{\sigma}\right)^2 = I \Rightarrow \left(\hat{n}\cdot\vec{\sigma}\right)\rho(\hat{n}) = \rho(\hat{n}) = \rho(\hat{n})\left(\hat{n}\cdot\vec{\sigma}\right) \Rightarrow \rho(\hat{n}) = |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)|$

$$\mathrm{tr}\,\sigma_i\sigma_j = 2\delta_{ij} \Rightarrow \langle \hat{n}\cdot\vec{\sigma}\rangle = \mathrm{tr}\rho(\vec{P})(\hat{n}\cdot\sigma) = \frac{1}{2}\sum_{i,j}\mathrm{tr}\left(n_i\sigma_i\left(I + P_j\sigma_j\right)\right) = \sum_i n_i P_i = \hat{n}\cdot\vec{P}.$$

Polarization of the qubit.

Density operator of a qubit

$$\rho(\vec{P}) = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right) \Rightarrow \mathrm{tr}\rho(\vec{P})(\hat{n}\cdot\sigma) = \hat{n}\cdot\vec{P}.$$



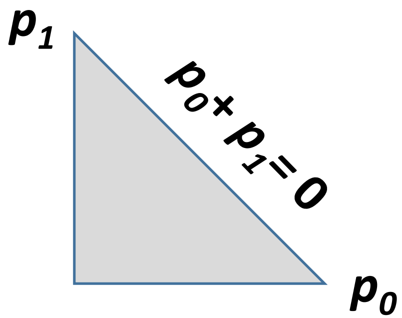
Every mixed state of a qubit can be expressed as the convex combination of two pure states, and can be so expressed in many ways, but there is a unique way to express (almost any) mixed state as a convex combination of two mutually orthogonal pure states (the antipodal points that determine a diameter which passes through the point). The exception is the maximally mixed state (the center of the ball), as all diameters pass through this point.

For the case of a qubit (two-dimensional Hilbert space), every state on the boundary of the ball is a pure state. That is not the case for higher dimensions. Consider d = 3:

$$\rho = p_0 \,|0\rangle\langle 0| + p_1 \,|1\rangle\langle 1| + p_2 \,|2\rangle\langle 2|.$$

States with p_0=0 are on the boundary, but these are not extremal unless p_1=0 or p_2=0.

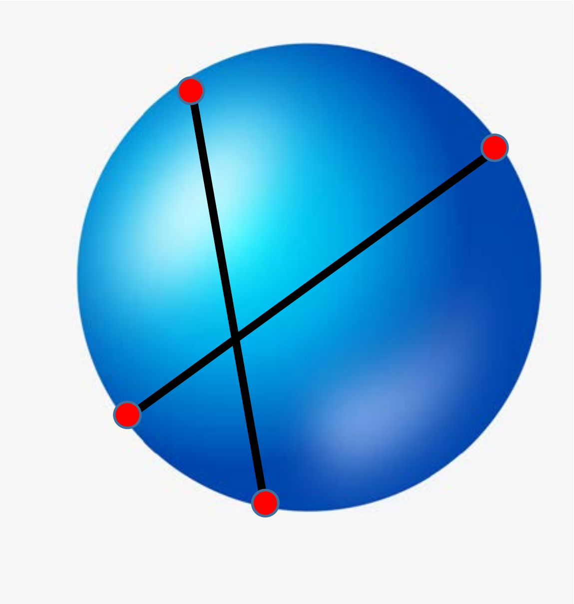
A classical probability distribution in d dimensions has d extremal points. The extremal distributions are those for which one outcome has probability 1, and all distributions are convex combinations of these. But the Bloch sphere has an infinite number of extremal points.



The ensemble interpretation is ambiguous.

Express the density operator as a convex combination of pure states which are normalized but not necessarily mutually orthogonal. This can be done in many ways.

$$\rho_A = \sum_a p_a |\phi_a\rangle\langle\phi_a| = \sum_\mu q_\mu |\psi_\mu\rangle\langle\psi_\mu|$$



Introducing a orthonormal bases on Bob's system B, we can *purify* these density operators; that is, find bipartite pure states of AB with the desired marginal density operator on A.

$$|\Phi_1\rangle_{AB} = \sum_a \sqrt{p_a}\, |\phi_a\rangle_A \otimes |\alpha_a\rangle_B$$

$$|\Phi_2\rangle_{AB} = \sum_\mu \sqrt{q_\mu}\, |\psi_\mu\rangle_A \otimes |\beta_\mu\rangle_B$$

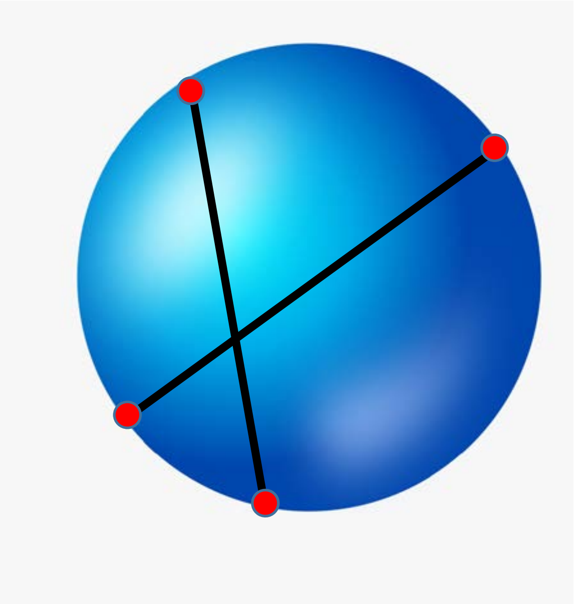
Now Bob can measure the first state in the alpha basis to realize the phi ensemble, or measure the second state in the beta basis to realize the psi ensemble.

How are these two purifications of Alice's density operator related? I claim that Bob can turn the first state into the second state by applying a unitary operation to system B alone. That means there is a single purification such that Bob can realize either ensemble by doing the appropriate measurement. This is called the *Hughston-Jozsa-Wootters (HJW) Theorem*, or the *Purification Theorem*.

The ensemble interpretation is ambiguous.

$$|\Phi_1\rangle_{AB} = \sum_a \sqrt{p_a}\,|\phi_a\rangle_A \otimes |\alpha_a\rangle_B$$

$$|\Phi_2\rangle_{AB} = \sum_\mu \sqrt{q_\mu}\,|\psi_\mu\rangle_A \otimes |\beta_\mu\rangle_B$$



Bob can turn the first state into the second state by applying a unitary operation to system B alone. This is the *Hughston-Jozsa-Wootters (HJW) Theorem.*

To see this, consider the basis in which Alice's density operator is diagonal, and the corresponding Schmidt decomposition of both states.

$$|\Phi_1\rangle_{AB} = \sum_k \sqrt{\lambda_k}\,|k\rangle_A \otimes |k_1'\rangle_B, \quad |\Phi_2\rangle_{AB} = \sum_k \sqrt{\lambda_k}\,|k\rangle_A \otimes |k_2'\rangle_B$$

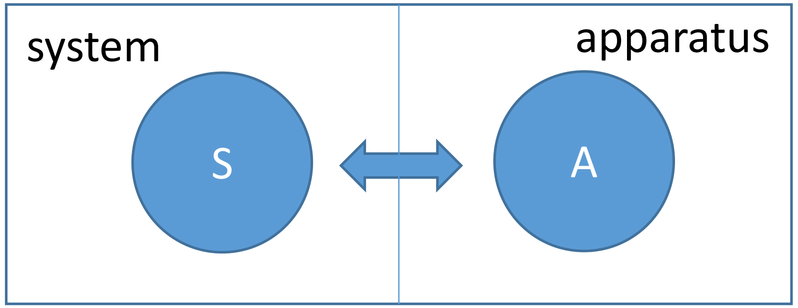
There is a unitary transformation *U* relating these two orthonormal bases for B. Therefore,

$$|\Phi_1\rangle_{AB} = \sum_a \sqrt{p_a}\,|\phi_a\rangle_A \otimes |\alpha_a\rangle_B = (I_A \otimes U_B)|\Phi_2\rangle_{AB} = \sum_\mu \sqrt{q_\mu}\,|\psi_\mu\rangle_A \otimes U_B|\beta_\mu\rangle_B = \sum_\mu \sqrt{q_\mu}\,|\psi_\mu\rangle_A \otimes |\gamma_\mu\rangle_B$$

And in fact there is another unitary *V* relating the alpha and gamma bases:

$$\sum_a \sqrt{p_a}\,|\phi_a\rangle_A \otimes |\alpha_a\rangle_B = \sum_\mu \sqrt{q_\mu}\,|\psi_\mu\rangle_A \otimes \sum_a |\alpha_a\rangle_B V_{a\mu} \Rightarrow \quad \sqrt{p_a}\,|\phi_a\rangle = \sum_\mu \sqrt{q_\mu}\,|\psi_\mu\rangle V_{a\mu}$$

Two different ensemble representations of the same density operator are always related this way for some unitary matrix *V*.



Quantum Measurement

To measure a quantum system, we first couple the (microscopic) system to a (macroscopic) apparatus, then observe the apparatus. This amplifies a microscopic property to a macroscopic scale accessible to us.

The interaction of system S and apparatus A is described by a unitary transformation acting jointly on S and A. Consider orthogonal projectors acting on the system S: $\{E_a\}, \quad E_a E_b = \delta_{ab} E_a, \quad E_a = E_a^\dagger, \quad \sum_a E_a = I$

There is a corresponding orthonormal basis $\{|a\rangle\}$ for the apparatus A. And the unitary acting on SA is

$$U_{SA} = \sum_{a,b} E_a \otimes |b+a\rangle\langle b|$$

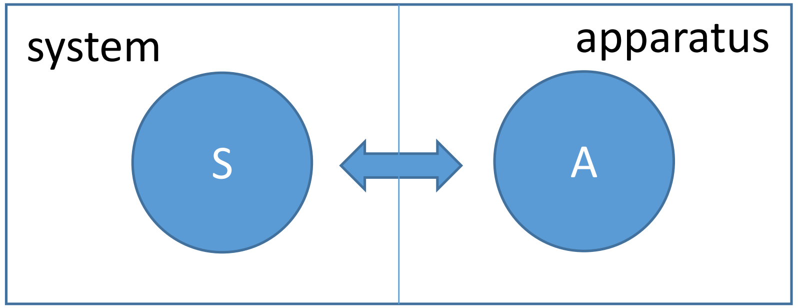
(The addition is modulo d, if the apparatus is d dimensional. When A interacts with S, the basis state of A is shifted by the amount a, if S is in the support of E_a.)

Let's check that this really is a unitary transformation:

$$UU^\dagger = \left(\sum_{a,b} E_a \otimes |b+a\rangle\langle b|\right)\left(\sum_{c,d} E_c \otimes |d\rangle\langle d+c|\right) = \left(\sum_{a,b} E_a \otimes |b+a\rangle\langle b+a| = I \otimes I\right)$$

Initialize the apparatus to a=0, and after the interaction, perform an orthogonal measurement on the apparatus.

$$U : |\psi\rangle \otimes |0\rangle \to \sum_a E_a |\psi\rangle \otimes |a\rangle \Rightarrow \mathrm{Prob}(a) = \langle I \otimes |a\rangle\langle a|\rangle = \langle \psi | E_a | \psi\rangle = \| E_a |\psi\rangle \|^2$$



Quantum Measurement

$$U : |\psi\rangle \otimes |0\rangle \to \sum_a E_a |\psi\rangle \otimes |a\rangle \Rightarrow$$

$$\mathrm{Prob}(a) = \langle I \otimes |a\rangle\langle a|\rangle = \langle \psi | E_a | \psi \rangle = \| E_a |\psi\rangle \|^2$$

By measuring the apparatus A in this "standard" orthogonal basis, we obtained the same result as if we had performed an orthogonal measurement on the system S. But … what if we measured the apparatus in a different orthogonal basis?

Example: A "Stern-Gerlach" interaction which correlates two qubits. $\left(\alpha|0\rangle + \beta|1\rangle\right)_S \otimes |0\rangle_A \to \alpha|00\rangle + \beta|11\rangle$

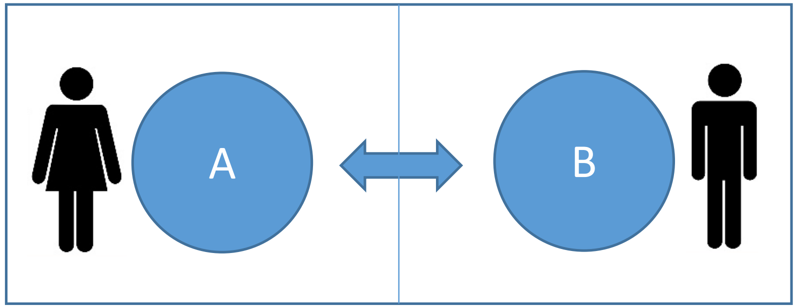
There is a preferred (Schmidt) basis in which the system and apparatus are perfectly correlated, but instead measure A in the basis:

$$|\pm\rangle_A = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)_A$$

Then the two outcomes each occur with probability ½, and the post-measurement states are not in general mutually orthogonal; they are

$$\sqrt{2}\langle \pm | (\alpha|00\rangle + \beta|11\rangle) = \alpha|0\rangle \pm \beta|1\rangle$$

If we repeat the measurement immediately, we won't necessarily get the same result again. The rules are different now.



Generalized Measurements

Let's be more general now. The unitary interaction of A and B creates an entangled state of A and B, then we do an orthogonal measurement on B. It is convenient to expand the state of AB in the B measurement basis.

$$U : |\psi\rangle_A \otimes |0\rangle_B \to |\psi'\rangle_{AB} = \sum_\mu M_\mu |\psi\rangle_A \otimes |\mu\rangle_B$$

If U is unitary, then the output state has norm 1 for any normalized input state. Therefore

$$1 = \| \sum_\mu M_\mu |\psi\rangle \otimes |\mu\rangle \|^2 = \sum_{\mu,\nu} \left(\langle \psi | M_\mu^\dagger \otimes \langle \mu | \right) \left(M_\nu |\psi\rangle \otimes |\nu\rangle \right) = \sum_\mu \langle \psi | M_\mu^\dagger M_\mu | \psi \rangle \Rightarrow \sum_\mu M_\mu^\dagger M_\mu = I_A$$

Now do the orthogonal measurement on B. The orthogonal projectors are $\{E_\mu = I_A \otimes |\mu\rangle\langle\mu|\}$

The probability of outcome μ is: $P(\mu) = \| E_\mu |\psi'\rangle_{AB} \|^2 = \| M_\mu |\psi\rangle_A \|^2$

And the post-measurement state is: $\dfrac{E_\mu |\psi'\rangle_{AB}}{\| E_\mu |\psi'\rangle_{AB} \|} = \dfrac{M_\mu |\psi\rangle_A}{\| M_\mu |\psi\rangle_A \|} \otimes |\mu\rangle_B$

Is the measurement "repeatable"?

$$P(\nu | \mu) = \frac{\| M_\nu M_\mu |\psi\rangle_A \|^2}{\| M_\mu |\psi\rangle_A \|^2}$$

To get the same outcome again for any input state, must be orthogonal measurement:

$$M_\nu M_\mu = \delta_{\nu\mu} M_\mu \times \text{phase}$$

Generalized Measurements

A *generalized measurement*, also called a *Positive Operator-Valued Measure (POVM)*, is a partition of unity by nonnegative Hermitian operators.

Change notation: $E_a = M_a^\dagger M_a$

Hermitian: $E_a = E_a^\dagger$

Nonnegative: $\langle \psi | E_a | \psi \rangle = \langle \psi | M_a^\dagger M_a | \psi \rangle \geq 0$

Complete: $\sum_a E_a = \sum_a M_a^\dagger M_a = I$

Therefore: $p_a = \langle \psi | E_a | \psi \rangle$

is a probability distribution.

For a mixed state (ensemble of pure states):

$$p_a = \text{tr } \rho E_a$$

Any POVM can be realized as a unitary interaction between system and apparatus followed by an orthogonal measurement performed on the apparatus. A positive operator has a square root.

In fact, apply polar decomposition to any measurement operator: $M_a = U_a \sqrt{M_a^\dagger M_a} = U_a \sqrt{E_a}$

Post measurement-state:

$$U_a \frac{\sqrt{E_a} | \psi \rangle}{\| \sqrt{E_a} | \psi \rangle \|}$$

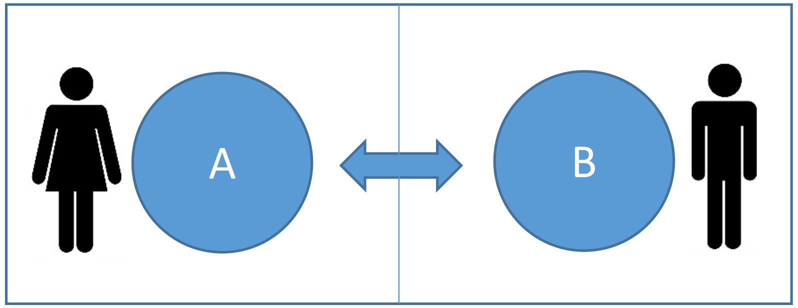
Freedom to apply any unitary after measuring.

If input state is mixed: $\rho' = \dfrac{M_a \rho M_a^\dagger}{\text{tr } M_a \rho M_a^\dagger}$

Quantum Channels

Suppose we make a generalized measurement, but we don't record the outcome.
Then we sum over all outcomes, each weighted by its probability, to find the output state.

$$\rho' = \mathcal{E}(\rho) = \sum_a p_a \rho_a' = \sum_a \operatorname{tr} M_a \rho M_a^\dagger \frac{M_a \rho M_a^\dagger}{\operatorname{tr} M_a \rho M_a^\dagger} = \sum_a M_a \rho M_a^\dagger, \quad \sum_a M_a^\dagger M_a = I.$$



This linear map of density operators to density operators is called a *quantum channel*, also known as a *superoperator*, or a *trace-preserving completely positive map* (TPCP map). (Why *completely* positive? We'll discuss that in the next lecture.)

Recall how we arrived at this map. Systems A and B were initially in a product state (not entangled). Then a unitary transformation was applied to the joint system AB, and finally we discarded system B.

This is how we would describe noise acting on quantum system A. System A interacts with its unobserved environment, B. We can imagine that the environment is measured in some orthogonal basis, but it does not matter whether it is really measured. Rather, we perform the partial trace of B to describe how A was affected by the interaction, with the understanding that B is unobserved (e.g., we measured but threw away the outcome).

We have found an *operator-sum representation*, also called a *Kraus representation*, of the map, in terms of the operation elements (Kraus operators) $\{M_a\}$. In the next lecture, we will study the properties of TPCP maps.