Linear Systems

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1 Determininistic Linear Systems

In the most general case we consider

$$\dot{x} = A(t)x(t) + b(t)$$

Sometimes, we consider, redundantly, the factorization

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

Table 1: caption

	deterministic		stochastic	
	homogeneous	inhomogenenous	homogeneous	inhomogenenous
constant-coeff.	$\dot{x} = Ax$	$\dot{x} = Ax + b$	$\dot{x} = Ax + \nu$	$\dot{x} = Ax + b + \nu$
time-invariant	$\dot{x} = Ax$	$\dot{x} = Ax + b(t)$	$\dot{x} = Ax + \nu$	$\dot{x} = Ax + b(t) + \nu$
time-variant	$\dot{x} = A(t)x$	$\dot{x} = A(t)x + b(t)$	$\dot{x} = A(t)x + \nu$	$\dot{x} = A(t)x + b(t) + \nu$

Solutions

Table 2

	deterministic	
	homogeneous	inhomogenenous
constant-coeff.	$x(t+\Delta t) = e^{A\Delta t}x(t)$	$x(t+\Delta t) = e^{A\Delta t}x(t) + \varphi_1(A\Delta t)b$
time-invariant	$x(t+\Delta t) = e^{A\Delta t}x(t)$	$x(t+\Delta t) = e^{A\Delta t}x(t) + \int_0^{\Delta t} e^{A(\Delta t - \Delta\tau)} b(t+\Delta\tau)\,\mathrm{d}\Delta\tau$
time-variant (commutative)	$\dot{x}(t+\Delta t) = e^{\int_t^{t+\Delta t} A(\tau)\mathrm{d}\tau} x(t)$	$\dot{x}(t+\Delta t) = e^{\int_t^{t+\Delta t} A(\tau)\mathrm{d}\tau} x(t) + \int_t^{t+\Delta t} e^{-\int_\tau^{t+\Delta t} A(s)\mathrm{d}s} b(\tau)\,\mathrm{d}\tau$
time-variant	no closed form	no closed form

1.1 Linear-Gaussian Systems

We assume $x_t \sim \mathcal{N}(\mu_t, \Sigma_t)$. Then what is the distribution of $x(t+\Delta t)$?

Lemma 1 (Expectation/Variance under linear/affine transformations)**.**

$$\mathbf{E}[Ax+b] = A\mathbf{E}[x] + b \qquad \mathbf{V}[Ax+b] = A\mathbf{V}[x]A^\top$$

Lemma 2 (Normal under linear/affine transformation)**.**

$$x \sim \mathcal{N}(\mu, \Sigma) \implies Ax + b \sim \mathcal{N}(A\mu + b, A\Sigma A^\top)$$

Proof. Via characteristic functions. Note that $\varphi_{AX+b}(\mathbf{t}) = e^{ib^\top \mathbf{t}}\varphi_X(A^\top \mathbf{t})$. then

$$\begin{aligned}\varphi_y(\mathbf{t}) = \mathbf{E}_y[e^{i\mathbf{y}^\top \mathbf{t}}] = \mathbf{E}_x[e^{i(A\mathbf{x}+b)^\top \mathbf{t}}] = \mathbf{E}_x[e^{i(A\mathbf{x}+b)^\top \mathbf{t}}] &= e^{ib^\top \mathbf{t}}\mathbf{E}_x[e^{i\mathbf{x}^\top A^\top \mathbf{t}}] \\ &= e^{ib^\top \mathbf{t}} e^{i\mu^\top A^\top \mathbf{t} - \frac{1}{2}\mathbf{t}A\Sigma A^\top \mathbf{t}} = e^{i(A\mu+b)^\top \mathbf{t} - \frac{1}{2}\mathbf{t}A\Sigma A^\top \mathbf{t}}\end{aligned}$$

□

Lemma 3 (characteristic function of gaussian)**.**

$$x \sim \mathcal{N}(\mu, \Sigma) \implies \varphi(\mathbf{t}) = \mathbf{E}_x[e^{ix^\top \mathbf{t}}] = e^{i\mu^\top \mathbf{t} - \frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}}$$

2 Stochastic Linear Systems

First consider the homogeneous case:

$$\dot{x}_t \;= A_t x_t + \nu_t$$

where ν_t is a zero-mean white noise process, i.e. $\mathbf{E}[\nu_t] = 0$ and $\mathbf{E}[\nu_t \nu_s] = \delta_{ts} Q_t$.
Then

$$\dot{\Sigma} = A_t \Sigma_t + \Sigma_t A_t^\top + Q_t$$