MAT168 HW1

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Problem Set

Question 1

Part 1

$\zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4$
$w_1 = 8 - 2x_1 - x_2 - x_3 - 3x_4$
$w_2 = 2 - x_1 - 3x_2 - x_3 - 2x_4$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$
This problem is feasible since all constraints are met when $x_1, x_2, x_3, x_4 = 0$.
When picking the highest bound, the ratio is $\frac{3}{2}$. Since the ratio is positive, this problem is bounded.

Part 2

The objective function is unbounded since x_j ($j \in N$ where N is the set of non-basic indexes) can infinitely increase without violating any constraints. Since the objective function is unbounded, it is also feasible.

Part 3

We have to solve the auxiliary problem.
$\zeta = -x_0$
$w_1 = -1 - 2x_1 - x_2 - x_3 - 3x_4 + x_0$
$w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4 + x_0$
$x_0, x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$
x_0 enters and w_1 leaves.
$\zeta = -(w_1 + 1 + 2x_1 + x_2 + x_3 + 3x_4)$
$\rightarrow \zeta = -w_1 - 1 - 2x_1 - x_2 - x_3 - 3x_4$
$x_0 = w_1 + 1 + 2x_1 + x_2 + x_3 + 3x_4$
$w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4 + x_0$
$x_0, x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$
This is a feasible dictionary, so we can move on to simplex.
A feasible solution is:
$x_0, x_1, x_2, x_3, x_4, w_1, w_2 = 0$
Therefore, this problem is feasible.
When picking the highest bound, the ratio is $\frac{3}{3}$. Since the ratio is positive, this problem is bounded.

Question 2a

Part 1

$\zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4$
$w_1 = 5 - 2x_1 - x_2 - x_3 - 3x_4$
$w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

Start with the feasible solution at:
$x_1, x_2, x_3, x_4, w_1, w_2 = 0$

Attempt to replace non-basic variable x_4.
$w_1 = 5 - 3x_4 \rightarrow x_4 = \frac{5}{3}$
$w_2 = 3 - 2x_4 \rightarrow x_4 = \frac{3}{2}$

New solution:
$x_1, x_2, x_3, w_1, w_2 = 0$
$x_4 = \frac{3}{2}$

x_4 enters and w_2 leaves.

Rewrite constraints.
$w_1 = 5 - 2x_1 - x_2 - x_3 - 3x_4$
$2x_4 = 3 - x_1 - 3x_2 - x_3 - w_2$
$\rightarrow x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

Rewrite objective function.
$\zeta = 6x_1 + 8x_2 + 5x_3 + 9(\frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2)$
$\rightarrow \zeta = 6x_1 + 8x_2 + 5x_3 + \frac{27}{2} - \frac{9}{2}x_1 - \frac{27}{2}x_2 - \frac{9}{2}x_3 - \frac{9}{2}w_2$
$\rightarrow \zeta = \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}w_2$

Attempt to replace non-basic variable x_1.
$2x_1 = 5 - 3 * \frac{3}{2} \rightarrow x_1 = \frac{1}{2} * \frac{1}{2}$
$\frac{1}{2}x_1 = \frac{3}{2} - \frac{3}{2} \rightarrow x_1 = 0$

Increasing x_1 will violate a constraint. Attempt to replace non-basic variable x_3.
$x_3 = 5 - 3 * \frac{3}{2}$
$\frac{1}{2}x_3 = \frac{3}{2} - \frac{3}{2} \rightarrow x_3 = 0$

Increasing x_1, x_3 will violate a constraint. Increasing x_2, w_2 decreases the objective function. Therefore, the optimal solution is:
$x_1, x_2, x_3 = 0$
$x_4 = \frac{3}{2}$

Part 2

Auxiliary problem.
$\zeta = -x_0$
$-x_1 - x_2 - x_0 \leq -3$
$-x_1 + x_2 - x_0 \leq -1$
$x_1 + 2x_2 - x_0 \leq 4$
$x_1, x_2, x_0 \geq 0$

Rewrite constraints.
$x_3 = -3 + x_1 + x_2 + x_0$
$x_4 = -1 + x_1 - x_2 + x_0$
$x_5 = 4 - x_1 - 2x_2 + x_0$
$x_3, x_4, x_5 \geq 0$

x_0 enters and x_3 leaves.
$\zeta = -(x_3 + 3 - x_1 - x_2)$
$\rightarrow \zeta = -x_3 - 3 + x_1 + x_2$
$x_0 = 3 + x_3 - x_1 - x_2$
$x_4 = -1 + x_1 - x_2 + (3 + x_3 - x_1 - x_2)$
$\rightarrow x_4 = 2 - 2x_2 + x_3$
$x_5 = 4 - x_1 - 2x_2 + (3 + x_3 - x_1 - x_2)$

$\rightarrow x_5 = 1 - 2x_1 - 3x_2 + x_3$
$x_1, x_2, x_0, x_3, x_4, x_5 \geq 0$

Start with the solution at:
$x_1, x_2, x_0, x_3, x_4, x_5 = 0$

Attempt to replace non-basic variable x_1.
$x_1 \leq 3$
$x_1 \leq \frac{1}{2}$

x_1 enters and x_5 leaves.

New solution is:
$x_1 = \frac{1}{2}$
$x_2, x_0, x_3, x_4, x_5 = 0$

Rewrite constraints.
$x_0 = 3 + x_3 - (\frac{1}{2} - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_5) - x_2$
$\rightarrow x_0 = \frac{5}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_5$
$x_4 = 2 - 2x_2 + x_3$
$x_1 = \frac{1}{2} - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_5$

Rewrite objective function.
$\zeta = -x_3 - 3 + (\frac{1}{2} - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_5) + x_2$
$\rightarrow \zeta = -\frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_5$

Dictionary is now feasible and $-x_0$ is maximized. Do simplex.
$\zeta = x_1 + 3x_2$
$\rightarrow \zeta = \frac{1}{2} + \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_5$
$x_0 = \frac{5}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_5$
$x_4 = 2 - 2x_2 + x_3$
$x_1 = \frac{1}{2} - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_5$

Attempt to replace non-basic variable x_2.
$x_0 = \frac{5}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_5 \rightarrow x_2$ is unbounded by this equation.

$2x_2 = 2 \rightarrow x_2 = 1$
$\frac{3}{2}x_2 = \frac{1}{2} - \frac{1}{2} \rightarrow x_2 = 0$

Increasing x_2 would violate a constraint.

Attempt to replace non-basic variable x_3.
$x_0 = \frac{5}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_5 \rightarrow x_3$ is unbounded by this equation.
$x_4 = 2 - 2x_2 + x_3$
$-\frac{1}{2}x_3 = \frac{1}{2} - \frac{1}{2} \rightarrow x_3 = 0$

Increasing x_2, x_3 would violate a constraint. Increasing x_5 decreases the objective function. Therefore, the optimal solution is:
$x_1 = \frac{1}{2}$
$x_2, x_0, x_3, x_4, x_5 = 0$

Question 2b

I think the problem with more constraints was more difficult to solve just because we had to do the auxiliary problem before doing the simplex problem.

Question 3

we want to solve:
$\zeta = 100x_1 + 10x_2 + x_3$
$x_1 \leq 1$
$20x_1 + x_2 \leq 100$
$200x_1 + 20x_2 + x_3 \leq 10000$
$x_1, x_2, x_3 \geq 0$

Rewrite the constraints:
$w_1 = 1 - x_1$
$w_2 = 100 - 20x_1 - x_2$
$w_3 = 10000 - 200x_1 - 20x_2 - x_3$
$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

Start with the feasible solution:
$x_1, x_2, x_3, w_1, w_2, w_3 = 0$

Attempt to replace basic variable x_3.
$x_3 = 10000$

x_3 enters and w_3 exits.

New solution:
$x_3 = 10000$
$x_1, x_2, w_1, w_2, w_3 = 0$

Rewrite constraints:
$w_1 = 1 - x_1$
$w_2 = 100 - 20x_1 - x_2$
$x_3 = 10000 - 200x_1 - 20x_2 - w_3$
$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

Rewrite objective function:
$\zeta = 100x_1 + 10x_2 + (10000 - 200x_1 - 20x_2 - w_3)$
$\rightarrow \zeta = 10000 - 100x_1 - 10x_2 - w_3$

Increasing any non-basic variables will decrease the objective function. Therefore, the optimal solution is at:
$x_3 = 10000$
$x_1, x_2 = 0$

Question 4

Let $\mathbf{x} = \{x\}_{i=1}^{6}$ be the amount imported of cotton, thread, glue, shoes, jumpsuits, hats, and $\mathbf{c} = \{c\}_{i=1}^{6}$ be the cost of importing each of these raw materials and completed items.

We are producing some items in our factory. Let $\mathbf{q} = \{q\}_{i=4}^{6}$ be the amount of shoes, jumpsuits, hats produced.

Manufacturing consumes materials. Let matrix $A = [a_{i,j}]$ where $\{j\}_1^3$ be the cotton, thread, glue needed for each product and $\{i\}_1^3$ be the recipe for shoes, jumpsuits, and hats.

We want to minimize our total cost (import and production costs). Our objective is to minimize $\mathbf{c} + \mathbf{q} * d$ or to maximize $-\mathbf{c} - \mathbf{q} * d$. We can write our objective function as:
$\zeta = -\mathbf{c} - \mathbf{q} * d$

Let the colonists' requirements (the minimum amount of clothing needed) be some constraints. The constraints are as shown:
$x_4 + q_4 \geq 2n$
$x_5 + q_5 \geq 3n$
$x_6 + q_6 \geq n$

Manufacturing is limited to the materials imported. The additional constraints are as shown:
$q_4 * a_{1,1} + q_5 * a_{1,2} + q_6 * a_{1,3} \leq x_1$
$q_4 * a_{2,1} + q_5 * a_{2,2} + q_6 * a_{2,3} \leq x_2$
$q_4 * a_{3,1} + q_5 * a_{3,2} + q_6 * a_{3,3} \leq x_3$

In general, whenever n increases, both $\mathbf{x}$ and $\mathbf{c}$ will increase too. Whenever n decreases, both $\mathbf{x}$ and $\mathbf{c}$ will decrease too. This makes sense because more clothing is needed when more people are in the colony, thus increasing the cost. On the other side, less clothing is needed when less people are in the colony, thus decreasing the cost.

Collaboration

Academic Integrity

On my personal integrity as a student and member of the UCD community, I have not given, nor received and unauthorized assistance on this assignment.
Signature: Andrew Jowe