Introduction to Optimization
University of California Davis
Fall 2022
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Homework 1 Solution

Due Oct. 5

1 Problem Set

1: Determine if the following problems are bounded or unbounded and then determine if they are feasible or infeasible (5pts each):

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq 8 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \geq 50 \\ & x_1 + 3x_2 + x_3 + 2x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq -1 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solutions:

- The first LP problem

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq 8 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

  is feasible since $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ is a feasible solution (you may take whatever feasible solutions you like). It's also bounded since all the coefficients of variables in the constraints are positive, so we cannot increase any of the variables to infinity.

- The second LP problem

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \geq 50 \\ & x_1 + 3x_2 + x_3 + 2x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

is feasible since $(x_1, x_2, x_3, x_4) = (25, 0, 0, 0)$ is a feasible solution (you may take whatever feasible solutions you like), but it's unbounded since we can increase x_1 to infinity without violating any of the constraints.

- The third LP problem

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq -1 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

is infeasible since $2x_1 + x_2 + x_3 + 3x_4 \leq -1$ can never hold (left hand side is nonnegative).

2.a: Solve following **by hand** using the simplex method (18 pts each):

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} max \quad & x_1 + 3x_2 \\ s.t. & - x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solutions:

- For LP

$$\begin{aligned} max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ s.t. \quad & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

by adding slack variables we get:

$$\begin{aligned} & \zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ & w_1 = 5 - 2x_1 - x_2 - x_3 - 3x_4 \\ & w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4 \\ & x_1, x_2, x_3, x_4, w_1, w_2 \geq 0 \end{aligned}$$

this initial table with $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ is feasible, so we can start our simplex method from this point. Now we increase x_1, then w_1 will be the leaving variable, we get the next table:

$$\begin{aligned} & \zeta = 15 - 3w_1 + 5x_2 + 2x_3 \\ & x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 \\ & w_2 = \frac{1}{2} + \frac{1}{2}w_1 - \frac{5}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ & x_1, x_2, x_3, x_4, w_1, w_2 \geq 0 \end{aligned}$$

now increase x_3 and w_2 will be leaving, we have

$$\begin{aligned} & \zeta = 17 - w_1 - 4w_2 - 5x_2 - 2x_4 \\ & x_1 = 2 - w_1 + \frac{1}{2}w_2 + 2x_2 - x_4 \\ & x_3 = 1 + w_1 - w_2 - 5x_2 - x_4 \\ & x_1, x_2, x_3, x_4, w_1, w_2 \geq 0 \end{aligned}$$

which is already optimal. The optimal solution is $(x_1, x_2, x_3, x_4) = (2, 0, 1, 0)$.

- For LP

$$\begin{aligned} max \quad & x_1 + 3x_2 \\ s.t. -& x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The point $(x_1, x_2) = (0, 0)$ is not feasible, thus we use the Phase I method: consider

$$\begin{aligned} max \quad & -x_0 \\ s.t. -& x_1 - x_2 - x_0 \leq -3 \\ & -x_1 + x_2 - x_0 \leq -1 \\ & x_1 + 2x_2 - x_0 \leq 4 \\ & x_1, x_2, x_0 \geq 0 \end{aligned}$$

Now we have the initial table:

$$\begin{aligned} &\zeta = -x_0 \\ &w_1 = -3 + x_1 + x_2 + x_0 \\ &w_2 = -1 + x_1 - x_2 + x_0 \\ &w_3 = 4 - x_1 - 2x_2 + x_0 \\ &x_0, x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

this is still infeasible. To make it feasible we increase x_0 and w_1 will be leaving, we have

$$\begin{aligned} &\zeta = -3 - w_1 + x_1 + x_2 \\ &x_0 = 3 + w_1 - x_1 - x_2 \\ &w_2 = 2 - w_1 - 2x_2 \\ &w_3 = 7 + w_1 - 2x_1 - 3x_2 \\ &x_0, x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

with initial feasible solution $(x_0, x_1, x_2, w_1, w_2, w_3) = (3, 0, 0, 0, 2, 7)$. Now we increase x_1 and x_0 will be leaving, we get

$$\begin{aligned} &\zeta = -x_0 \\ &x_1 = 3 + w_1 - x_0 - x_2 \\ &w_2 = 2 - w_1 - 2x_2 \\ &w_3 = 1 - w_1 + 2x_0 - x_2 \\ &x_0, x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

Now that we are done with the Phase I, we move to Phase II, i.e. the original problem

$$\begin{aligned} &\zeta = 3 + w_1 + 2x_2 \\ &x_1 = 3 + w_1 - x_2 \\ &w_2 = 2 - w_1 - 2x_2 \\ &w_3 = 1 - w_1 - x_2 \\ &x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

Now we enter x_2 and w_3 will be leaving (or you can choose w_2, it's the same), we get

$$\begin{aligned}
\zeta &= 5 - w_1 - 2w_3 \\
x_1 &= 2 + 2w_1 + w_3 \\
x_2 &= 1 - w_1 - w_3 \\
w_2 &= w_1 + 2w_3 \\
&x_1, x_2, w_1, w_2, w_3 \geq 0
\end{aligned}$$

this is already optimal with optimal solution $(x_1, x_2) = (2, 1)$.

2.b: The first problem has 4 variables and only 2 linear constraints (in addition to the non-negativity condition). The second has 2 variables, but 4 constraints. Which do you think is more difficult to solve and why (4 pts)?

Solutions:

- The first one is easier, since we do not need to solve the auxiliary problem and the simplex method only takes a few steps to find the final solution. To solve the second one we need to solve the auxiliary problem and then do simplex method.
- The second one is easier, since we can directly use a graph to visualize the feasible region and find the optimal solution via analytical geometry, but the dimension of the first one is 4 so we can not visualize the feasible region.

3: Solve the Klee-Minty problem, as described in the notes/text, for n=3 (15 pts). *Solution*: when $n = 3$, the Klee-Minty problem has the following form:

$$\begin{aligned} max \quad 4x_1 + 2x_2 + x_3 \\ s.t. \quad x_1 \leq 1 \\ 4x_1 + x_2 \leq 100 \\ 8x_1 + 4x_2 + x_3 \leq 10000 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

we have the initial table

$$\begin{aligned} &\zeta = 4x_1 + 2x_2 + x_3 \\ &w_1 = 1 - x_1 \\ &w_2 = 100 - 4x_1 - x_2 \\ &w_3 = 10000 - 8x_1 - 4x_2 - x_3 \\ &x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

Now we enter x_3 and w_3 will be leaving,

$$\begin{aligned} &\zeta = 10000 - 4x_1 - 2x_2 - w_3 \\ &w_1 = 1 - x_1 \\ &w_2 = 100 - 4x_1 - x_2 \\ &x_3 = 10000 - 8x_1 - 4x_2 - w_3 \\ &x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

Notice: if we pivot according to the maximum coefficient rule, i.e. we pivot x_1 first, then x_2 and x_3, we will have to pivot 3 times. Now if n is really large, we will have to pivot n times, which will cost a very long time for the computer to implement.

4: Suppose that you have been put in charge of managing the clothing supply for a small, newly established space colony. The colony has n colonist each of whom need at least two pairs of shoes, three jump suits and a hat. You can make each of these products from some combination of cotton, synthetic threads and glue, all of which you need to import. You can also import already made clothing for a fixed cost. For each item that you produce in the factory you also incur a cost of d per item (a pair of shoes, jumpsuit and hat each count as an 'item') for the use of electricity in your factory. Write down (but do not solve) a linear program to minimize the cost of acquiring all of the clothing which your colony requires. You may assume that you are allowed to make and import fractional quantities of each of these goods. (20 pts)
Hint: Let $\mathbf{x} = \{x_i\}_{i=1}^6$ = amount imported of cotton, thread, glue, shoes, jumpsuits, hats, and $\mathbf{c} = \{c_i\}_{i=1}^6$ be the cost of importing each of these raw materials and completed items
Bonus:(Up to 5 pts): What can you say about this problem as n changes?

Solution: For convenience we split $\mathbf{x}$ into two parts, the first three forms a vector of raw materials $\mathbf{x}_{\mathrm{M}}$ and last three will be $\mathbf{x}_{\mathrm{I}}$, the vector for product items. Similarly we have $\mathbf{c}_{\mathrm{M}}$ and $\mathbf{c}_{\mathrm{I}}$, when splitting the vector $\mathbf{c}$. Denote y_1, y_2 and y_3 be the number of shoes, suits and hats produced, correspondingly. then in total we will have $x_4 + y_1$ shoes, $x_5 + y_2$ suits and $x_6 + y_3$ hats. We immediately have three constraints:

$$\begin{aligned} x_4 + y_1 &\geq 2n \\ x_5 + y_2 &\geq 3n \\ x_6 + y_3 &\geq n \\ x_i, y_j &\geq 0 \end{aligned}$$

Assume matrix $A \in \mathbb{R}^3$ where $a_{i,j}$ represents the unit of material i we need in order to produce one item j. For example $a_{1,1}$ is the unit of cotton we need to produce one shoe. Then in total we will need $a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3$ cottons, and threads and glues similarly. Now written in a matrix form we will have that

$$A\mathbf{y} = \mathbf{x}_{\mathrm{M}} \text{ or } A\mathbf{y} \leq \mathbf{x}_{\mathrm{M}}$$

Now the objective will be three parts. The first part is the cost for all materials, $\mathbf{c}_{\mathrm{M}}^\top \mathbf{x}_{\mathrm{M}} = \mathbf{c}_{\mathrm{M}}^\top A\mathbf{y}$; the second part is the cost for importing all products $\mathbf{c}_{\mathrm{I}}^\top \mathbf{x}_{\mathrm{I}}$; the third part is the electricity cost for producing in the factory, which is $d(y_1 + y_2 + y_3)$. Eventually our optimization problem will be

$$\begin{aligned} \min \quad & \mathbf{c}_{\mathrm{M}}^\top A\mathbf{y} + \mathbf{c}_{\mathrm{I}}^\top \mathbf{x}_{\mathrm{I}} + d(y_1 + y_2 + y_3) \\ s.t. \quad & x_4 + y_1 \geq 2n \\ & x_5 + y_2 \geq 3n \\ & x_6 + y_3 \geq n \\ & x_i, y_j \geq 0 \end{aligned}$$

or equivalently

$$
\begin{aligned}
\min \quad & \mathbf{c}_{\mathrm{M}}^{\top}\mathbf{x}_{\mathrm{M}} + \mathbf{c}_{\mathrm{I}}^{\top}\mathbf{x}_{\mathrm{I}} + d(y_1 + y_2 + y_3) \\
s.t. \quad A\mathbf{y} &= \mathbf{x}_{\mathrm{M}} \\
x_4 + y_1 &\geq 2n \\
x_5 + y_2 &\geq 3n \\
x_6 + y_3 &\geq n \\
x_i, y_j &\geq 0
\end{aligned}
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