MAT168 HW1

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Problem Set

Question 1

Part 1

$\zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4$
$w_1 = 8 - 2x_1 - x_2 - x_3 - 3x_4$
$w_2 = 2 - x_1 - 3x_2 - x_3 - 2x_4$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

This problem is feasible since all constraints are met when:
$x_1, x_2, x_3, x_4 = 0$

This problem is bounded because for any x_i, it is bounded when all other $x_j = 0$ such that $i \neq j$.

Part 2

This problem is feasible since $x_1, x_2, x_3, x_4 = 50$ satisfies all constraints.

This problem is unbounded since increasing x_1 to infinity will still satisfy all constraints.

Part 3

This problem is infeasible because x_1, x_2, x_3, x_4 cannot be less than zero due to a constraint. But, when $x_1, x_2, x_3, x_4 = 0$, the constraint $2x_1 + x_2 + x_3 + 3x_4 \leq -1$ isn't satisfied.

This problem is bounded because for any x_i, it is bounded when all other $x_j = 0$ such that $i \neq j$.

Question 2a

Part 1

$\zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4$
$w_1 = 5 - 2x_1 - x_2 - x_3 - 3x_4$
$w_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

Attempt to replace non-basic variable x_4.
$w_1 = 5 - 3x_4 \rightarrow x_4 = \frac{5}{3}$
$w_2 = 3 - 2x_4 \rightarrow x_4 = \frac{3}{2}$

x_4 enters and w_2 leaves.

Rewrite constraints.
$w_1 = 5 - 2x_1 - x_2 - x_3 - 3(\frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2)$
$w_1 = 5 - 2x_1 - x_2 - x_3 - \frac{9}{2} + \frac{3}{2}x_1 + \frac{9}{2}x_2 + \frac{3}{2}x_3 + \frac{3}{2}w_2$
$w_1 = \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}x_2 + \frac{1}{2}x_3 + \frac{3}{2}w_2$

$2x_4 = 3 - x_1 - 3x_2 - x_3 - w_2$
$x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$

Rewrite objective function.
$\zeta = 6x_1 + 8x_2 + 5x_3 + 9(\frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2)$
$\zeta = 6x_1 + 8x_2 + 5x_3 + \frac{27}{2} - \frac{9}{2}x_1 - \frac{27}{2}x_2 - \frac{9}{2}x_3 - \frac{9}{2}w_2$
$\zeta = \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}w_2$

New objective function and constraints.
$\zeta = \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}w_2$
$w_1 = \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}x_2 + \frac{1}{2}x_3 + \frac{3}{2}w_2$
$x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$
$x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

Attempt to replace non-basic variable x_1.
$w_1 = \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}x_2 + \frac{1}{2}x_3 + \frac{3}{2}w_2$
$x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$

x_1 enters, w_1 leaves.
$\frac{1}{2}x_1 = \frac{1}{2} + \frac{7}{2}x_2 + \frac{1}{2}x_3 + \frac{3}{2}w_2 - w_1$
$x_1 = 1 + 7x_2 + x_3 + 3w_2 - 2w_1$

$x_4 = \frac{3}{2} - \frac{1}{2} - \frac{7}{2}x_2 - \frac{1}{2}x_3 - \frac{3}{2}w_2 + w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}w_2$
$x_4 = 1 - 5x_2 - x_3 + w_1 - 2w_2$

Rewrite objective function:
$\zeta = \frac{27}{2} + \frac{3}{2} + \frac{21}{2}x_2 + \frac{3}{2}x_3 + \frac{9}{2}w_2 - \frac{6}{2}w_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}w_2$
$\zeta = 15 + 5x_2 + 2x_3 - 3w_1$

x_2 enters, x_4 leaves.
$x_1 = 1 + \frac{7}{2} - \frac{7}{2}x_3 + \frac{7}{2}w_1 - 7w_2 - \frac{7}{2}x_4 + x_3 + 3w_2 - 2w_1$
$x_1 = \frac{9}{2} - \frac{5}{2}x_3 + \frac{3}{2}w_1 - 4w_2 - \frac{7}{2}x_4$

$5x_2 = 1 - x_3 + w_1 - 2w_2 - x_4$
$x_2 = \frac{1}{5} - \frac{1}{5}x_3 + \frac{1}{5}w_1 - \frac{2}{5}w_2 - \frac{1}{5}x_4$

Rewrite objective function:
$\zeta = 15 + 1 - x_3 + w_1 - 2w_2 - x_4 + 2x_3 - 3w_1$
$\zeta = 16 - 2w_2 - x_4 + x_3 - 2w_1$

x_3 enters and x_2 leaves.
$x_1 = \frac{9}{2} - \frac{5}{2}(1 + w_1 - 2w_2 - x_4 - 5x_2) + \frac{3}{2}w_1 - 4w_2 - \frac{7}{2}x_4$

$x_3 = 1 + w_1 - 2w_2 - x_4 - 5x_2$

Rewrite objective function:
$\zeta = 17 - 4w_2 - 2x_4 - 5x_2 - w_1$

The optimal solution is at:
$x_1 = 2$
$x_3 = 1$
$\zeta = 17$

Part 2

Auxiliary problem.
$\zeta = -x_0$
$- x_1 - x_2 - x_0 \leq -3$
$- x_1 + x_2 - x_0 \leq -1$
$x_1 + 2x_2 - x_0 \leq 4$
$x_1, x_2, x_0 \geq 0$

Rewrite constraints.
$x_3 = -3 + x_1 + x_2 + x_0$
$x_4 = -1 + x_1 - x_2 + x_0$
$x_5 = 4 - x_1 - 2x_2 + x_0$
$x_3, x_4, x_5 \geq 0$

x_0 enters and x_3 leaves, since that equation in the dictionary is the most infeasible.

$\zeta = -(x_3 + 3 - x_1 - x_2)$
$\zeta = -x_3 - 3 + x_1 + x_2$

$x_0 = 3 + x_3 - x_1 - x_2$

$x_4 = -1 + x_1 - x_2 + (3 + x_3 - x_1 - x_2)$
$x_4 = 2 - 2x_2 + x_3$

$x_5 = 4 - x_1 - 2x_2 + (3 + x_3 - x_1 - x_2)$

$x_5 = 7 - 2x_1 - 3x_2 + x_3$

New objective function and constraints.
$\zeta = -x_3 - 3 + x_1 + x_2$
$x_0 = 3 + x_3 - x_1 - x_2$
$x_4 = 2 - 2x_2 + x_3$
$x_5 = 7 - 2x_1 - 3x_2 + x_3$
$x_1, x_2, x_0, x_3, x_4, x_5 \geq 0$

Attempt to move x_2.
$x_2 = 3$
$x_2 = 1$
$x_2 = \frac{7}{3}$

x_2 enters and x_4 leaves.
$\zeta = -x_3 - 3 + x_1 + (1 + \frac{1}{2}x_3 - \frac{1}{2}x_4)$
$\zeta = -2 + x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4$

$x_0 = 3 + x_3 - x_1 - (1 + \frac{1}{2}x_3 - \frac{1}{2}x_4)$
$x_0 = 3 + x_3 - x_1 - 1 - \frac{1}{2}x_3 + \frac{1}{2}x_4$
$x_0 = 2 - x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4$

$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$

$x_5 = 7 - 2x_1 - 3(1 + \frac{1}{2}x_3 - \frac{1}{2}x_4) + x_3$
$x_5 = 7 - 2x_1 - 3 - \frac{3}{2}x_3 + \frac{3}{2}x_4 + x_3$
$x_5 = 4 - 2x_1 - \frac{1}{2}x_3 + \frac{3}{2}x_4$

New objective function and constraints.
$\zeta = -2 + x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4$
$x_0 = 2 - x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4$
$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$

$x_5 = 4 - 2x_1 - \frac{1}{2}x_3 + \frac{3}{2}x_4$
$x_1, x_2, x_0, x_3, x_4, x_5 \geq 0$

Attempt to move x_1.
$x_1 = 2$

x_1 enters and x_0 leaves.
$\zeta = -2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - x_0 - \frac{1}{2}x_3 - \frac{1}{2}x_4$
$\zeta = -x_0$

$x_1 = 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - x_0$

$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$

$x_5 = 4 - 2(2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - x_0) - \frac{1}{2}x_3 + \frac{3}{2}x_4$
$x_5 = 4 - 4 - \frac{2}{2}x_3 - \frac{2}{2}x_4 + 2x_0 - \frac{1}{2}x_3 + \frac{3}{2}x_4$
$x_5 = -\frac{3}{2}x_3 + 2x_0 + \frac{1}{2}x_4$

New objective function and constraints:
$\zeta = -x_0$
$x_1 = 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - x_0$
$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$
$x_5 = -\frac{3}{2}x_3 + 2x_0 + \frac{1}{2}x_4$
$x_1, x_2, x_0, x_3, x_4, x_5 \geq 0$

Do simplex. Replace $-x_0$ in objective function with $x_1 + 3x_2$.
$\zeta = x_1 + 3x_2$
$\zeta = 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + 1 + \frac{3}{2}x_3 - \frac{3}{2}x_4$
$\zeta = 3 + 2x_3 - x_4$

New objective function and constraints.
$\zeta = 3 + x_3$
$x_1 = 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4$

$x_2 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$
$x_5 = -\frac{3}{2}x_3 + \frac{1}{2}x_4$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

Attempt to replace x_3.
$x_3 = 0$

x_3 enters and x_5 exits.

Rewrite constraints:
$x_1 = 2 + \frac{1}{2}(\frac{1}{3}x_4 - \frac{2}{3}x_5) + \frac{1}{2}x_4$

$x_2 = 1 + \frac{1}{2}(\frac{1}{3}x_4 - \frac{2}{3}x_5) - \frac{1}{2}x_4$

$\frac{3}{2}x_3 = \frac{1}{2}x_4 - x_5$
$x_3 = \frac{1}{3}x_4 - \frac{2}{3}x_5$

Rewrite objective function:
$\zeta = 3 + 2(\frac{1}{3}x_4 - \frac{2}{3}x_5) - x_4$
$\zeta = 3 - \frac{4}{3}x_5 - \frac{1}{3}x_4$

The optimal solution is:
$x_1 = 2$
$x_2 = 1$
$\zeta = x_1 + 3x_2 = 5$

Question 2b

I think the problem with more constraints was more difficult to solve just because we had to do the auxiliary problem before doing the simplex problem.

Question 3

we want to solve:
$\zeta = 100x_1 + 10x_2 + x_3$
$x_1 \leq 1$
$20x_1 + x_2 \leq 100$
$200x_1 + 20x_2 + x_3 \leq 10000$
$x_1, x_2, x_3 \geq 0$

Rewrite the constraints:
$w_1 = 1 - x_1$
$w_2 = 100 - 20x_1 - x_2$
$w_3 = 10000 - 200x_1 - 20x_2 - x_3$
$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

Attempt to replace basic variable x_3.
$x_3 = 10000$

x_3 enters and w_3 exits.

Rewrite constraints:
$w_1 = 1 - x_1$
$w_2 = 100 - 20x_1 - x_2$
$x_3 = 10000 - 200x_1 - 20x_2 - w_3$
$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

Rewrite objective function:
$\zeta = 100x_1 + 10x_2 + (10000 - 200x_1 - 20x_2 - w_3)$
$\rightarrow \zeta = 10000 - 100x_1 - 10x_2 - w_3$

Increasing any non-basic variables will decrease the objective function. Therefore, the optimal solution is at:
$x_3 = 10000$
$x_1, x_2 = 0 \ \zeta = 10000$

Question 4

Let $\mathbf{x} = \{x\}_{i=1}^{6}$ be the amount imported of cotton, thread, glue, shoes, jumpsuits, hats, and $\mathbf{c} = \{c\}_{i=1}^{6}$ be the cost of importing each of these raw materials and completed items.

We are producing some items in our factory. Let $\mathbf{q} = \{q\}_{i=4}^{6}$ be the amount of shoes, jumpsuits, hats produced.

Manufacturing consumes materials. Let matrix $A = [a_{i,j}]$ where $\{j\}_1^3$ be the cotton, thread, glue needed for each product and $\{i\}_1^3$ be the recipe for shoes, jumpsuits, and hats.

We want to minimize our total cost (import and production costs). Our objective is to minimize $\mathbf{c} + \mathbf{q} * d$ or to maximize $-\mathbf{c} - \mathbf{q} * d$. We can write our objective function as:
$\zeta = -\mathbf{c} - \mathbf{q} * d$

Let the colonists' requirements (the minimum amount of clothing needed) be some constraints. The constraints are as shown:
$x_4 + q_4 \geq 2n$
$x_5 + q_5 \geq 3n$
$x_6 + q_6 \geq n$

Manufacturing is limited to the materials imported. The additional constraints are as shown:
$q_4 * a_{1,1} + q_5 * a_{1,2} + q_6 * a_{1,3} \leq x_1$
$q_4 * a_{2,1} + q_5 * a_{2,2} + q_6 * a_{2,3} \leq x_2$
$q_4 * a_{3,1} + q_5 * a_{3,2} + q_6 * a_{3,3} \leq x_3$

In general, whenever n increases, both $\mathbf{x}$ and $\mathbf{c}$ will increase too. Whenever n decreases, both $\mathbf{x}$ and $\mathbf{c}$ will decrease too. This makes sense because more clothing is needed when more people are in the colony, thus increasing the cost. On the other side, less clothing is needed when less people are in the colony, thus decreasing the cost.

Collaboration

All collaborators are listed (in alphabetical order) below:

- Anne
- Dhruv
- Fengqin
- Sterling
- Zhongning

Academic Integrity

On my personal integrity as a student and member of the UCD community, I have not given, nor received and unauthorized assistance on this assignment.
Signature: Andrew Jowe