

5.5

$$X = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix}$$

$$Y'Y = \begin{bmatrix} 16 & 5 & 10 & 15 & 13 & 22 \end{bmatrix} \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 1259 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 17 \\ 17 & 55 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 81 \\ 261 \end{bmatrix}$$

5.5 Idempotent Proof

Using R:

$$H^2 = \begin{bmatrix} 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \\ -0.1463 & 0.6585 & 0.3902 & 0.122 & 0.122 & -0.1463 \\ 0.0244 & 0.3902 & 0.2682 & 0.1463 & 0.1463 & 0.0244 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \\ -0.1463 & 0.6585 & 0.3902 & 0.122 & 0.122 & -0.1463 \\ 0.0244 & 0.3902 & 0.2683 & 0.1463 & 0.1463 & 0.0244 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \end{bmatrix}$$

$$\therefore H^2 = H$$

By Hand:

$$\begin{aligned}
H^2 &= (X(X'X)^{-1}X')(X(X'X)^{-1}X') \\
\rightarrow H^2 &= (X(X^{-1}X'^{-1})X')(X(X'X)^{-1}X') \\
\rightarrow H^2 &= (XX^{-1}X'^{-1}X')(X(X'X)^{-1}X') \\
\rightarrow H^2 &= ((XX^{-1})(X'^{-1}X'))(X(X'X)^{-1}X') \\
\rightarrow H^2 &= (II)(X(X'X)^{-1}X') \\
\rightarrow H^2 &= I(X(X'X)^{-1}X') \\
\rightarrow H^2 &= X(X'X)^{-1}X' \\
\rightarrow H^2 &= H
\end{aligned}$$

$\therefore H$ is idempotent

5.17

$$W_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Y$$

$$W_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} Y$$

$$W_3 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} Y$$

$$W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_1 - Y_2 \\ Y_1 - Y_2 - Y_3 \end{bmatrix}$$

$$E\{W\} = \begin{bmatrix} \mu_1 + \mu_2 + \mu_3 \\ \mu_1 - \mu_2 \\ \mu_1 - \mu_2 - \mu_3 \end{bmatrix}$$

5.24.a

$$1. \hat{\beta} = \begin{bmatrix} 0.439 \\ 4.6098 \end{bmatrix}$$

$$2. e = \begin{bmatrix} -2.878 \\ -0.0488 \\ 0.3415 \\ 0.7317 \\ -1.2683 \\ 3.122 \end{bmatrix}$$

$$3. SSR = 145.2073$$

$$4. SSE = 20.2927$$

$$5. s^2\{b\} = \begin{bmatrix} 6.8055 & -2.1035 \\ -2.1035 & 0.7424 \end{bmatrix}$$

$$6. E\{Y_h\} = 18.878$$

$$7. s^2\{pred\} = 6.9292$$

5.24.b

$$1. s\{b_0, b_1\} = -2.1035$$

$$2. s\{b_0\} = 6.8055$$

$$3. s\{b_1\} = 0.7424$$

6.1.a

$$Y = \beta^T X + \epsilon$$

$$X = \begin{bmatrix} 1 & x_{11} & x_{11}x_{12} \\ 1 & x_{21} & x_{21}x_{22} \\ 1 & x_{31} & x_{31}x_{32} \\ 1 & x_{41} & x_{41}x_{42} \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

6.1.b

$$\log(Y) = \beta^T X + \epsilon$$

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

6.22.a

This is not a linear regression model. We can apply the following transformations:

$$X'_{i1} = X_{i1}$$

$$X'_{i2} = 10^{X_{i2}}$$

$$X'_{i3} = X_{i1}^2$$

Our linear regression model after the transformation:

$$Y_i = \beta_0 + \beta_1 X'_{i1} + \beta_2 X'_{i2} + \beta_3 X'_{i3} + \epsilon_i$$

6.22.b

This is not a linear regression model. It is not possible to transform this.

Example 6.9

b_0 : The intercept in the linear regression model.

b_1 : The slope of x_1 in the linear regression model.

b_2 : The slope of x_2 in the linear regression model.

Appendix

```
# Function
r_matrix_to_mathjax <- function(mat) {
  nrow <- dim(mat)[1]

  result <- "\\begin{bmatrix}"
  for (i in 1:nrow) {
    result <- paste(result, "", paste(mat[i, ], collapse = "&"), "\\\\" )
  }
  result <- paste(result, "\\end{bmatrix}")

  return(result)
}
```

```
# 5.5
x1 <- c(4, 1, 2, 3, 3, 4)
Y <- as.matrix(c(16, 5, 10, 15, 13, 22))
X <- cbind(1, x1)

Xt <- t(X)
Yt <- t(Y)
Yt.Y <- Yt %*% Y
Xt.X <- Xt %*% X
Xt.Y <- Xt %*% Y

H <- round(X %*% solve(Xt %*% X) %*% Xt, 4)
H.2 <- round(H %*% H, 4)
```

```
# 5.24
beta_hat <- solve(Xt.X) %*% Xt %*% Y

Y_hat <- X %*% beta_hat
residuals <- Y - Y_hat

Y_mean <- mean(Y)
SSR <- sum((Y_hat - Y_mean)^2)

SSE <- sum(residuals ** 2)

n <- length(x1)
dfSSE <- n - 2
MSE <- SSE / dfSSE
Var.b <- solve(Xt.X) * as.numeric(MSE)

X1h <- 4
Yh <- beta_hat[1] + beta_hat[2] * X1h

X1h <- 4
x_h <- c(1, X1h)
s2_pred <- MSE * (1 + t(x_h) %*% solve(Xt.X) %*% x_h)
```