5.5

$$X = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix}$$

$$Y'Y = \begin{bmatrix} 16 & 5 & 10 & 15 & 13 & 22 \end{bmatrix} \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 1259 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 17 \\ 17 & 55 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 16 \\ 5 \\ 10 \\ 15 \\ 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 81 \\ 261 \end{bmatrix}$$

5.5 Idempotent Proof

Using R:

$$H^2 = \begin{bmatrix} 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \\ -0.1463 & 0.6585 & 0.3902 & 0.122 & 0.122 & -0.1463 \\ 0.0244 & 0.3902 & 0.2682 & 0.1463 & 0.1463 & 0.0244 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \\ -0.1463 & 0.6585 & 0.3902 & 0.122 & 0.122 & -0.1463 \\ 0.0244 & 0.3902 & 0.2683 & 0.1463 & 0.1463 & 0.0244 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.1951 & 0.122 & 0.1463 & 0.1707 & 0.1707 & 0.1951 \\ 0.3659 & -0.1463 & 0.0244 & 0.1951 & 0.1951 & 0.3659 \end{bmatrix}$$

$\therefore H^2 = H$

By Hand:

$H^2 = (X(X'X)^{-1}X')(X(X'X)^{-1}X')$

$\rightarrow H^2 = (X(X^{-1}X'^{-1})X')(X(X'X)^{-1}X')$

$\rightarrow H^2 = (XX^{-1}X'^{-1}X')(X(X'X)^{-1}X')$

$\rightarrow H^2 = ((XX^{-1})(X'^{-1}X'))(X(X'X)^{-1}X')$

$\rightarrow H^2 = (II)(X(X'X)^{-1}X')$

$\rightarrow H^2 = I(X(X'X)^{-1}X')$

$\rightarrow H^2 = X(X'X)^{-1}X'$

$\rightarrow H^2 = H$

$\therefore H$ is idempotent

5.17

$W_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix} Y$

$W_2 = \begin{bmatrix}1 & -1 & 0\end{bmatrix} Y$

$W_3 = \begin{bmatrix}1 & -1 & -1\end{bmatrix} Y$

$W = \begin{bmatrix}W_1 \\ W_2 \\ W_3\end{bmatrix} = \begin{bmatrix}Y_1 + Y_2 + Y_3 \\ Y_1 - Y_2 \\ Y_1 - Y_2 - Y_3\end{bmatrix}$

$E\{W\} = \begin{bmatrix}\mu_1 + \mu_2 + \mu_3 \\ \mu_1 - \mu_2 \\ \mu_1 - \mu_2 - \mu_3\end{bmatrix}$

5.24.a

1. $\hat{\beta} = \begin{bmatrix}0.439 \\ 4.6098\end{bmatrix}$
2. $e = \begin{bmatrix}-2.878 \\ -0.0488 \\ 0.3415 \\ 0.7317 \\ -1.2683 \\ 3.122\end{bmatrix}$
3. $SSR = 145.2073$
4. $SSE = 20.2927$
5. $s^2\{b\} = \begin{bmatrix}6.8055 & -2.1035 \\ -2.1035 & 0.7424\end{bmatrix}$
6. $E\{Y_h\} = 18.878$
7. $s^2\{pred\} = 6.9292$

5.24.b

1. $s^2\{b_0, b_1\} = -2.1035$

   $s\{b_0, b_1\} = 1.4503i = NaN$

2. $s^2\{b_0\} = 6.8055$
3. $s\{b_1\} = 0.8616$

6.1.a

$Y = \beta^T X + \epsilon$

$X = \begin{bmatrix} 1 & x_{11} & x_{11}x_{12} \\ 1 & x_{21} & x_{21}x_{22} \\ 1 & x_{31} & x_{31}x_{32} \\ 1 & x_{41} & x_{41}x_{42} \end{bmatrix}$

$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$

6.1.b

$log(Y) = \beta^T X + \epsilon$

$X = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \end{bmatrix}$

$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$

6.22.a

This is not a linear regression model. We can apply the following transformations:

$X'_{i1} = X_{i1}$

$X'_{i2} = 10^{X_{i2}}$

$X'_{i3} = X_{i1}^2$

Our linear regression model after the transformation:

$Y_i = \beta_0 + \beta_1 X'_{i1} + \beta_2 X'_{i2} + \beta_3 X'_{i3} + \epsilon_i$

6.22.b

This is not a linear regression model. It is not possible to transform this.

Example 6.9

b_0 : The intercept in the linear regression model.

b_1 : The slope of x_1 in the linear regression model.

b_2 : The slope of x_2 in the linear regression model.

Appendix

```r
# Function
r_matrix_to_mathjax <- function(mat) {
  nrow <- dim(mat)[1]

  result <- "\\begin{bmatrix}"
  for (i in 1:nrow) {
    result <- paste(result, "", paste(mat[i, ], collapse = "&"), "\\\\")
  }
  result <- paste(result, "\\end{bmatrix}")

  return(result)
}
```

```r
# 5.5
x1 <- c(4, 1, 2, 3, 3, 4)
Y <- as.matrix(c(16, 5, 10, 15, 13, 22))
X <- cbind(1, x1)

Xt <- t(X)
Yt <- t(Y)
Yt.Y <- Yt %*% Y
Xt.X <- Xt %*% X
Xt.Y <- Xt %*% Y

H <- round(X %*% solve(Xt %*% X) %*% Xt, 4)
H.2 <- round(H %*% H, 4)
```

```r
# 5.24
beta_hat <- solve(Xt.X) %*% Xt %*% Y

Y_hat <- X %*% beta_hat
residuals <- Y - Y_hat

Y_mean <- mean(Y)
SSR <- sum((Y_hat - Y_mean)^2)

SSE <- sum(residuals ** 2)

n <- length(x1)
dfSSE <- n - 2
MSE <- SSE / dfSSE
Var.b <- solve(Xt.X) * as.numeric(MSE)

X1h <- 4
Yh <- beta_hat[1] + beta_hat[2] * X1h

X1h <- 4
x_h <- c(1, X1h)
s2_pred <- MSE * (1 + t(x_h) %*% solve(Xt.X) %*% x_h)
```