

**Consider an experiment where you roll a 6-sided die**

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{x \in \Omega | x >= 3\} = \{3, 4, 5, 6\}$$

$$B = \{x \in \Omega | x <= 2\} = \{1, 2\}$$

Note that A and B are disjoint,  $A \cup B = \emptyset$

**Def: A probability measure is a function that takes any event  $A \subset \Omega$  as input and returns a real number  $P(A)$  such that the following hold**

1.  $P(\Omega) = 1$
2.  $P(A) \geq 0$  for any event  $A \subset \Omega$
3. Let  $A_1$  and  $A_2$  be disjoint events. Then  $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ .
  - More generally, let  $A_1, A_2, \dots$  be disjoint events, ie  $A_i \cap A_j = \emptyset$  for  $i \neq j$ . Then  $P(\cap_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ .

**Some simple consequences of Axioms**

1.  $P(A^c) = 1 - P(A)$ 
  - Proof:  $\Omega = A \cup A^c$
  - $\rightarrow 1 = P(\Omega) = P(A) + P(A^c) - P(A)$  by 3rd axiom
  - $\rightarrow P(A^c) = 1 - P(A)$
2. Suppose  $A \subset B$ . Then  $P(A) \leq P(B)$ 
  - Proof:  $B = A \cup (B \setminus A)$  by disjoint union
  - $\rightarrow P(B) = P(A) + P(B \setminus A)$  by 3rd axiom
  - $\rightarrow P(A) \leq P(B)$

**Example: Consider the experiment of flipping a coin twice**

- $\Omega = \{HH, HT, TH, TT\}$
- Suppose  $A_1 = \{HH\}, A_2 = \{HT\}, A_3 = \{TH\}, A_4 = \{TT\}$ 
  - $\rightarrow P(A_1) = P(A_2) = P(A_3) = P(A_4)$
- Let's use the axioms to check  $P(A_i) = .25$  for all i
  - $P(\Omega) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$  by 3rd axiom
  - $\rightarrow 1 = P(A_1) + P(A_2) + P(A_3) + P(A_4)$  by 1st axiom
  - $\rightarrow 1 = 4 * P(A_1)$  by assumption
  - $\rightarrow P(A_1) = .25$
- Let  $C = \{1\text{st or } 2\text{nd toss is head}\}$ , ie  $C = \{HH, HT, TH\}$ 
  - $C = A_1 \cup A_2 \cup A_3$  disjoint union
  - $P(C) = P(A_1) + P(A_2) + P(A_3)$
  - For any event B in this experiment, we would have  $P(B) = \frac{|B|}{|\Omega|}$
- In general, let  $\Omega = \{w_1, \dots, w_N\}$  have N outcomes
  - Assume  $P(\{w_1\}) = P(\{w_2\}) = P(\{w_3\})$
  - Then for any event  $A \subset \Omega$ , we have  $P(A) = \frac{|A|}{N} = \frac{|A|}{|\Omega|}$
  - Caution: The assumption of equitable outcomes is key

## Multiplication Principle

- Suppose there are 2 experiments to be performed
- There are  $n_1$  outcomes in the first experiment
- For every outcome in the first experiment, there are  $n_2$  outcomes in the second experiment
- Then there are  $n_1 * n_2$  total possible outcomes

**Example:** Suppose a hat can be made in 8 sizes and 12 colors

- Then there are  $8 * 12$  possible hats

## Extended Multiplication Principle

- Suppose there are  $k$  experiments to be performed
- There are  $n_1$  outcomes in the first experiment
- For every outcome in the first experiment, there are  $n_2$  outcomes in the second experiment
- ...
- For every outcome of the first  $k - 1$  experiments, there are  $n_k$  outcomes in the  $k$ th experiment
- Then there are  $n_1 * n_2 * ... * n_k$  total possible outcomes

**Example:** How many possible DNA sequences of length  $10^6$  are there?

- A DNA sequence has the form  $x_1, x_2, ..., x_{10^6}$  where  $x_i \in \{A, T, C, G\}$
- Answer: Think  $k = 10^6, n_1 = 4, n_2 = 4, ..., n_k = 4$
- So, the number of possible sequences is  $4^{10^6}$

**Def:** A permutation is just an ordered arrangement of things

- $(1, 2, 3) \neq (3, 2, 1)$
- Q: How many possible permutations of  $n$  items are there?
- A: Think  $k=n$  experiments,  $n_1 = n, n_2 = n - 1, n_3 = n - 2, ..., n_k = 1$ 
  - $\rightarrow n * (n - 1) * ... * 1 = n!$
  - Caution:  $0! = 1$

**Example:** How many possible passwords can be formed from the letters QWERTY?

- Ans:  $6!$
- Each letter can only be used once.

## Sampling without replacement

- Let  $1 \leq r \leq n$
- Q: How many ways can we sample  $r$  of  $n$  items without replacement?
- A: Think  $k = r, n_1 = n, n_2 = n - 1, n_r = n - (r - 1)$ 
  - $\rightarrow n * (n - 1) * ... * (n - (r - 1)) = \frac{n!}{(n-r)!}$