Consider an experiment where you roll a 6-sided die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{x \in \Omega | x >= 3\} = \{3, 4, 5, 6\}$

$B = \{x \in \Omega | x <= 2\} = \{1, 2\}$

Note that A and B are disjoint, $A \cup B = \emptyset$

Def: A probability measure is a function that takes any event $A \subset \Omega$ as input and returns a real number P(A) such that the following hold

1. $P(\Omega) = 1$
2. $P(A) >= 0$ for any event $A \subset \Omega$
3. Let A_1 and A_2 be disjoint events. Then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
    - More generally, let $A_1, A_2, ...$ be disjoint events, ie $A_i \cap A_j = \emptyset$ for $i \neq j$. Then $P(\cap_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Some simple consequences of Axioms

1. $P(A^c) = 1 - P(A)$
    - Proof: $\Omega = A \cup A^c$
    - $\rightarrow 1 = P(\Omega) = P(A) + P(A^c) - P(A)$ by 3rd axiom
    - $\rightarrow P(A^c) = 1 - P(A)$
2. Suppose $A \subset B$. Then $P(A) <= P(B)$
    - Proof:$B = A \cup (B \backslash A)$ by disjoint union
    - $\rightarrow P(B) = P(A) + P(B \backslash A)$ by 3rd axiom
    - $\rightarrow P(A) <= P(B)$

Example: Consider the experiment of flipping a coin twice

- $\Omega = \{HH, HT, TH, TT\}$
- Suppose $A_1 = \{HH\}, A_2 = \{HT\}, A_3 = \{TH\}, A_4 = \{TT\}$
    - $\rightarrow P(A_1) = P(A_2) = P(A_3) = P(A_4)$
- Let's use the axioms to check $P(A_i) = .25$ for all i
    - $P(\Omega) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$ by 3rd axiom
    - $\rightarrow 1 = P(A_1) + P(A_2) + P(A_3) + P(A_4)$ by 1st axiom
    - $\rightarrow 1 = 4 * P(A_1)$ by assumption
    - $\rightarrow P(A_1) = .25$
- Let C = {1st or 2nd toss is head}, ie $C = \{HH, HT, TH\}$

- $C = A_1 \cup A_2 \cup A_3$ disjoint union
  - $P(C) = P(A_1) + P(A_2) + P(A_3)$
  - For any event B in this experiment, we would have $P(B) = \frac{|B|}{|\Omega|}$
- In general, let $\Omega = \{w_1, ..., w_N\}$ have N outcomes
  - Assume $P(\{w_1\}) = P(\{w_2\}) = P(\{w_3\})$
  - Then for any event $A \subset \Omega$, we have $P(A) = \frac{|A|}{N} = \frac{|A|}{|\Omega|}$
  - Caution: The assumption of equitable outcomes is key

Multiplication Principle

- Suppose there are 2 experiments to be performed
- There are n_1 outcomes in the first experiment
- For every outcome in the first experiment, there are n_2 outcomes in the second experiment
- Then there are $n_1 * n_2$ total possible outcomes

Example: Suppose a hat can be made in 8 sizes and 12 colors

- Then there are $8 * 12$ possible hats

Extended Multiplication Principle

- Suppose there are k experiments to be performed
- There are n_1 outcomes in the first experiment
- For every outcome in the first experiment, there are n_2 outcomes in the second experiment
- ...
- For every outcome of the first $k - 1$ experiemnts, there are n_k outcomes in the kth experiment
- Then there are $n_1 * n_2 * ... * n_k$ total possible outcomes

Example: How many possible DNA sequences of length 10^6 are there?

- A DNA sequence has the form $x_1, x_2, ..., x_{10^6}$ where $x_i \in \{A, T, C, G\}$
- Answer: Think $k = 10^6, n_1 = 4, n_2 = 4, ..., n_k = 4$
- So, the number of possible sequences is 4^{10^6}

Def: A permutation is just on ordered arrangement of things

- $(1, 2, 3) \neq (3, 2, 1)$
- Q: How many possible permutations of n items are there?
- A: Think k=n experiments, $n_1 = n, n_2 = n - 1, n_3 = n - 2, ..., n_k = 1$
  - $\rightarrow n * (n-1) * ... * 1 = n!$
  - Caution: $0! = 1$

Example: How many possible passwords can be formed from the letters QWERTY?

- Ans: 6!
- Each letter can only be used once.

Sampling without replacement

- Let $1 <= r <= n$
- Q: How many ways can we sample r of n items without replacement?
- A: Think k = r, $n_1 = n, n_2 = n - 1, n_r = n - (r - 1)$
  - $\rightarrow n * (n-1) * ... * (n - (r-1)) = \frac{n!}{(n-r)!}$