Bayes' Theorem

- Conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Bayes' Rule: $P(B|A) = \frac{P(A|B) * P(B)}{P(A)}$
- Let $B_1 \to B_n$ be a partition of Ω
  $\to P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\Sigma_{l=1}^{n} P(A|B_l)P(B_l)}$

Gambler's Ruin

- Gambler has k dollars and wants to have N dollars, 0 < k < N
- Gambler plays following game until they reach N dollars or goes bankrupt
- Each involves tossing a fair coin where head wins 1 dollar and tails lose 1 dollar
- Let $A = \{\text{the gambler goes bankrupt}\}$

Q: What is $P(A)$?

- A: $P_k = 1 - \frac{k}{N}$
- Let $B = \{\text{1st toss is heads}\}$, $k =$ current dollars the gambler has
  $\to P_k(A) = P_k(A|B)P(B) + P_k(A|B^c)P(B^c)$
  $\to P_k(A) = P_k(A|B) * .5 + P_k(A|B^c) * .5$
  $\to P_k(A) = P_{k+1}(A) * .5 + P_{k-1}(A|B^c) * .5$
  $\to 2P_k = P_{k+1} + P_{k-1}$
  $\to P_k - P_{k-1} = P_{k+1} - P_k$
- Let $b_k = P_k - P_{k-1}$
  $\to b_k = b_{k+1}$
- Note: all b_k's are the same
  $\to P_k = P_{k-1} + (P_k - P_{k-1})$
  $\to P_k = P_{k-1} + b$
  $\to P_k = P_{k-2} + (P_{k-1} - P_{k-2}) + b$
  $\to P_k = P_{k-2} + 2b$
  ...
  $\to P_k = P_0 + kb$
  $\to P_k = 1 + kb$
- Need to find b
  $\to 0 - 1 = P_N - P_0 = (P_N - P_{N-1}) + (P_{N-1} - P_{N-2}) + ... + (P_1 - P_0)$
- Note: Telescoping sum, can replace all terms in between with b
  $\to -1 = N * b$
  $\to b = \frac{-1}{N}$
  $\to P_k = 1 - \frac{k}{N}$

Independent events

- Roughly speaking, we say 2 events are independent if knowing that one has occured doesn't provide any info about the other
- Def: Events A and B are independent if $P(A|B) = P(A), P(B|A) = P(B)$
- Rewrite:
  $\to \frac{P(A \cap B)}{P(B)} = P(A)$
  $\to P(A \cap B) = P(B)P(A)$

Example: You have a well shuffled deck of 52 cards and deal one

- $A = \{\text{dealt card is an ace}\}$

- $D = \{\text{dealt card is a diamond}\}$

Q: Are A and D independent?

- A: Yes, lets check
- Need to check $P(A \cap D) = P(A)P(D)$
- Equiprobable assumption: $P(A \cap D) = \frac{|A \cap D|}{|\Omega|} = \frac{1}{52}$
- $P(A) = \frac{4}{52}$
- $P(D) = \frac{13}{52}$
- $P(A)P(D) = \frac{4}{52} * \frac{13}{52} = \frac{1}{52}$

What about independent among more than 2 events?

Definition: We can say events $A_1 \to A_m$ are independent if:

- $P(\cap_j A_{ij}) = P(A_{i1})P(A_{i2})...P(A_{im})$
- for any $i_1 \to i_m, m \in [1, n]$

Note: It's possible to have events A, B, C such that

- A, B are independent
- A, C are independent
- B, C are independent
- BUT, it's possible to get info from C from $A \cap B$

Example: We toss a fair coin twice

- $A = \{\text{1st toss is head}\}$
- $B = \{\text{2st toss is head}\}$
- $A = \{\text{exactly one toss is head}\}$
- Claim $P(C|A \cap B) \neq P(C)$
  1. $P(C|A \cap B) = 0$
  2. $P(C) = \frac{2}{4} = \frac{1}{2}$

Example: Suppose everytime someone comes into contact with someone who has a cold, there is a $\frac{1}{8}$ probability you catch it

- Let $C_1 \to C_{10}$ be defined by $C_i = \{\text{you don't catch cold on } i\text{-th contact}\}$
- Let $B = \{\text{you catch cold after 10 contacts}\}$
  $\to B = \cup_{i=1}^{10} C_i^c$
- DeMorgan: $B^c = \cap C_i$
  $\to P(B) = 1 - P(B^c)$
  $\to P(B) = 1 - P(C_1)P(C_2)...P(C_{10})$
  $\to P(B) = 1 - (\frac{7}{8})^{10}$

Example: Circuit

x|–3–|x

-|-1-2-|-

- The components 1, 2, 3 behave randomly and can work or fail
- $A_i = \{i\text{-th component works}\}$
- $W = \{\text{circuit works}\}$
- $P(W) = ?$

- Assume $P(A_1) = P(A_2) = P(A_3) = p$
  $\rightarrow W = A_3 \cup (A_1 \cap A_2)$
  $\rightarrow P(W) = P(A_3) + (A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$
  $\rightarrow P(W) = p + p^2 - p^3$