Conditional Probabililty

- Let A and B be events P(B) > 0
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Multiplication Laws

1. $P(A \cap B) = P(A|B)P(B)$
2. Let A_1 through A_n be a sequence of events

- Then $P(\cap A_i) = P(A_1)P(A_2|A_1)...P(A_n|A_1 \cap ...A_{n-1})$

Law of Total Probability

- Let B_1 through B_n be a partition of Ω
- Let $\cup B_i = \Omega$, $B_i \cap B_j = \emptyset$, $i \neq j$
- Then for any event A, $P(A) = \sum P(A_i|B_i)P(B_i)$

Example

- A = {Randomly drawn person will vote in the US}
- B_i = {Randomly drawn person resides in state i}, i = 1 through 50
- $P(A) = \sum P(A|B_i)P(B_i)$
- $P(B_i)$ = number of people in state i / number of people in the US

Example: Occupational Mobility

- Occupations are categorized into 3 groups, upper, middle, and lower
- u_1 = {Randomly drawn father's occupation is in u}
- u_2 = {Randomly drawn son's occupation is in u}
- M_1 = {Randomly drawn father's occupation is in M}
- M_2 = {Randomly drawn son's occupation is in M}
- $L_1 = \ldots$
- $L_2 = \ldots$
- We are given

	U_2	M_2	L_2
U_1	.45	.48	.07
M_1	.05	.7	.25
L_1	.01	.5	.49

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- $P(U_1) = .1, P(M_1) = .4, P(L_1) = .5, ...$
- Law of total probability = $P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1)$
- Partition is U_1, M_1, L_1
- Note: $P(U_2) + P(M_2) + P(L_2) = 1$

How can we flip conditional probabilities?

- eg $P(U_1|U_2)$
- $P(U_1|U_2) = \frac{P(U_1 \cap U_2)}{P(U_2)}$
- Notice $P(U_1 \cap U_2) = P(U_2|U_1)P(U_1)$
- $P(U_1|U_2) = \frac{P(U_2|U_1)P(U_1)}{P(U_2)} = \frac{.45 * .1}{.07}$
- This is an illustration of Bayes' rule
- Two formulations
  1. For any events A and B, $P(A), P(B) > 0$
     - We have $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$
     - Note $P(B|A)P(A) = P(A|B)P(B)$
  2. Let B_1 to B_n be a partition of Ω
     - Then for my fixed $i \in 1 \to n$
     - $P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$
     - $P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum P(A|B_l)P(B_l)}$

Example: Lie Detector Test

- $L = \{\text{Person taking the test is lying}\}$
- $T = \{\text{Person taking the test is telling the truth}\}$
- $D_+ = \{\text{Positive test}\}$
- $D_- = \{\text{Negative alarm}\}$
- According to manufacturers data, we have $P(D_+|L) = .88$, $P(D_-|L) = .12$, $P(D_+|T) = .14$, $P(D_-|T) = .86$
- Let's try to calculate the probability that the person is telling the truth given that there was a false alarm
- $P(T|D_+) =?$
- Let's assume $P(T) = .99$, $P(L) = .01$
- $P(T|D_+) = \frac{P(D_+|T)P(T)}{P(D_+)} = \frac{P(D_+|T)P(T)}{P(D_+|T)P(T) + P(D_+|L)P(L)}$
- Probability of a false positive: $P(T|D_+) = .94$
- This is an issue when screening cuz false positives are too high, therefore these lie detector tests are controversial

Example: Gambler's ruin

- A gambler has k dollars
- They want to win N dollars
- $0 < k < N$
- The gambler plays the following game as many times as necessary until he goes bankrupt or reaching N dollars
- The game is flipping a fair coin where heads implies win $1 and tails implies lose $1
- A = {gambler goes bankrupt}
- Want to calculate $P(A) =?$
- Use $P_k(A)$ for $P(A)$
- Want to derive a formula for $P_k(A)$ in terms of k and N
- B = {1st toss is heads}

Techniques "first step analysis"

- $P_k(A) = P_k(A|B)P_k(B) + P_k(A|B^C)P(B^C)$
- Law of total probability with partition B, B^C

- $= .5P_k(A|B) + .5P_k(A|B^C)$
- $= .5P_{k+1}(A) + .5P_{k-1}(A)$
- Let $P_k = P_k(A)$, ie $P_k = .5P_{k+1} + .5P_{k-1}$ for any k
  - This is a recurrence relation
  - $\rightarrow 2P_k = P_{k+1} + P_{k-1}$
  - $\rightarrow P_k - P_{k-1} = P_{k+1} - P_k$ for all k
  - Let $b_k = P_k - P_{k-1}$
  - Key observation: $b_1 = b_2 = ... = c$ where c is a constant