Equiprobable Outcomes

- Let $\Omega = \{w_1, ..., w_N\}$ where $P\{w_1\} = ... = P\{w_N\}$. Let $A \subset \Omega$.
- Then $P(A) = \frac{|A|}{|\Omega|}$

Extended Multiple Principle

- Suppose there is a sequence of k experiments to be performed
- Suppose there are n_1 potential outcomes in the first experiment
- And for every outcome in the first experiment, there are n_2 outcomes in the second experiment
- ...
- And for every outcome in the first k-1 experiments, there are n_k outcomes in the kth experiment
- then, there are $n_1 * n_2 * ... * n_k$ total outcomes.

Permutations

- How many permutations of n items are there? n!

Sampling when order matters

- How many samples of size r can be formed without replacement from n items? $\frac{n!}{(n-r)!}$
- How many samples of size r can be formed with replacement from n items?
  - $n_1 = n$
  - $n_2 = n$
  - ...
  - $n_r = n$
  - Answer: n^r

Definition: A combination is an unordered arrangement of things, ie {1, 2, 3} = {3, 2, 1}

- How many combinations of size r can be formed from n items?
- Ex: n = 4, r = 3
  - n items -> {a, b, c, d} | Combinations | Permutations | | — | — | | {a, b, c} | (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a) | | {a, b, d} | ... | | {b, c, d} | ... | | {a, c, d} | ... | | all combinations of size 3 | all samples of size 3 taken without replacement | | Let x = number of things on left | y = number of things on the right |
  - Note: $y = r! * x$
  - Answer: $x = \frac{y}{r!} = \frac{n!}{r!(n-r)!}$
  - This is the number of combinations of size r that can be formed n times
- Binomial coefficient: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$; "n-choose-r"

Another perspective of binomials

- Think of partitioning n items into 2 buckets such that the first bucket has r items and the second bucket has n - r
- How many such partitions are there? $\binom{n}{r}$

Example

- You run a business with 9 employees, and you want to partition them into three shifts (morning, afternoon, night).
- How many ways can this be done? We must allocate 3 to morning, 4 to afternoon, 2 to night
- $n_1 = \binom{9}{3}$
- $n_2 = \binom{6}{4}$
- $n_3 = \binom{2}{2}$
- The number of such partitions is $\binom{9}{3}\binom{6}{4}\binom{2}{2} = \frac{9!}{3!4!2!}$

Multinomial Coefficient

- In general, suppose there are n items to partition into m buckets, with sizes $n_1, n_2, ..., n_m$ where $\sum n_i = n$
- The number of ways to do this is the following: $\frac{n!}{n_1!n_2!...n_m!} = \binom{n}{n_1,...,n_m}$

Example

- How many passwords can be formed from the letters S, T, A, T, I, S, T, I, C, S?
- Think of filling in 10 spaces with the different letters
- How many ways can we arrange the 2 S's in different buckets?
- For each arrangement, how can we arrange the rest of the letters?
- the multinomial coefficient says the number of passwords is $\binom{10}{3,3,1,2,1} = \frac{10!}{3!3!2!}$

Example: Birthday Paradox

- There are n people in a room
- Assume all lists of n birthdays are equally likely
- Ω = {all lists of n birthdays}
- A = {at least two people have the same birthday}
- P(A) = ?
- The question is only interesting when n <= 365
- $P(A) = \frac{|A|}{|\Omega|}$
- It's actually easier to use
- $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|}$
- $|\Omega| = 365^n$
- $|A^c| = 365 * 365 * 363 * ... * (365 - (n-1))$
- $P(A) = 1 - \frac{365}{365} * \frac{364}{365} * \frac{363}{365} * ... * \frac{365-(n-1)}{365}$
- Note: n = 23, then P(A) is roughly 0.5

Conditional Probability

- How to account for side information?
- Example: you roll a 6-sided die twice
- A = {sum of the 2 rolls is equal to 8}; |A| = 5
- If all outcomes are equally likely, $P(A) = \frac{|A|}{|\Omega|} = \frac{5}{36}$
- B = {first roll is a 3}
  - Lets calculate P(A) given B
  - Heuristic answer: $\frac{1}{6} = \frac{6}{36}$