Example: Poker

- $\Omega = \{w_1, ..., w_n\}$
- Assume: $P(\{w_1\}) = ... = P(\{w_n\})$
- A hand is just a set of five cards drawn from deck
- A = { full house }
- 3 cards of one rank
- 2 cards of another rank
- $P(A) = \frac{|A|}{|\Omega|}$
- $|\Omega| = \binom{52}{5}$
- Choose one rank for the triple (13 ways)
- Choose one rank for the pair (12 ways)
- Choose three of four suits for triple ($\binom{4}{3}$)
- Choose two of four suits for the pair ($\binom{4}{3}$)
- $P(A) = \frac{13*12*\binom{4}{3}*\binom{4}{3}}{\binom{52}{5}}$
- 2 of one rank and 3 other cards all of different ranks
- Choose one rank for pair 13 (13)
- Choose 3 of 12 remaining ranks for single cards ($\binom{12}{3}$)
- Choose 2 suits for pair ($\binom{4}{2}$)
- Choose 1 suit for each single card (4^3)
- $P(B) = \frac{13*\binom{12}{3}*\binom{4}{2}*4^3}{\binom{52}{5}}$

Example: Ecology "Capture Recapture"

- There is a large population of n animals
- You catch 10 animals as a sample without replacement
- You tag the animals and release them back
- You catch a new sample of size 20 without replacement
- Let $m \in \{0, ..., 10\}$
- Let $A_m = \{$there are m tagged animals in sample size of 20$\}$
- $P(A_m) = ?$
- Assume all samples of size 20 without replacement are equally likely
- $|\Omega| = \binom{n}{20}$
- Imagine you are filling up a tray of 20 items, part of it contains the m tagged animals, and 20 - m untagged
- There are $\binom{10}{m}$ ways of choosing tagged animals
- There are $\binom{n-10}{20-m}$ ways of choosing untagged animals
- If you plug the observed m into this formula, it becomes only a function of n, "likelihood function"
- $\hat{n}$ maximizes estimate of population size

Conditional Probability

- We roll a 6 sided die twice

	1	2	3	4	5	6
1						
2						*
3					*	
4				*		
5			*			
6		*				

-
- A = {sum of two rolls is 8}
- $P(A) = \frac{5}{36}$
- B = {we roll a 3 on the first die}
- $P(A|B) = \frac{1}{6}$

Def

- Let A and B be events in some general sample space Ω. Assume $P(B) > 0$
- Probability of A given B is $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- If $B = \Omega$, then $P(A|\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)} = P(A)$
- Consider the previous example, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Comment: Conditional probability is a new probability measure that satisfies all the axioms

- Let's fix an event B and define a new probability measure $\tilde{P}(A) = P(A|B)$
- $\tilde{P}(\Omega) = 1$
  - i.e. $\frac{P(\Omega \cap B)}{P(B)} = 1$
- $\tilde{P}(A) >= 0$
- $\tilde{P}(\cup A_i) = \sum \tilde{P}(A_i)$
  - A_i are disjoint

Multiplication laws for conditional probability

1. $P(A \cap B) = P(A|B)P(B)$
2. Let $A_1 ... A_n$ be events (possibly not disjoint)

- $P(\cap A_i) = P(A_1)P(A_2|A_1)...P(A_n|A_1 \cap A_2 \cap ... \cap A_{n-1})$

Example

- Jar with 4 marbles, 3 red, 1 blue
- You draw 2 marbles without replacement
- $R_1 = \{\text{first draw is red}\}$
- $R_2 = \{\text{second draw is red}\}$
- $P(R_1 \cap R_2) = P(R_1)P(R_2|R_1)$

Law of Total Probability

- Let B_1 through B_n be a partition of Ω
- i.e, all B_i are disjoint and $\cup B_i = \Omega$
- Let $A \subset \Omega$ be any event
  - Then $P(A) = \sum P(A|B_i)P(B_i)$
- $B_i = \{\text{randomly drawn customer comes from zipcode i}\}$
- A = {randomly drawn customer purchases the product}
- Divide and conquer strategy

Why is Law of Total Probability True?

- $\sum P(A|B_i)P(B_i)$
- $\rightarrow \sum P(A \cap B_i)$
- $\rightarrow \sum P(C_i)$
- Note that C_i are disjoint
- $\rightarrow P(\cup C_i)$ third axiom
- $\rightarrow P(\cup(A \cap B_i))$
- $\rightarrow P(A \cap (\cup B_i))$ distribution law