Note: for any random variable, it must be $P(a < x \leq b) = F_X(b) - F_X(a)$, NOT $a \leq x$, it only not matters in continuous random variables

$X \sim Uniform[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = P(X \leq x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{if } x < a \\ 1 & \text{otherwise} \end{cases}$$

Quantile

Def: The pth quantile for $p \in (0, 1)$ for a continuous random variable X is defined as the solution x_p to the equation $F_X(x_p) = p$

Let's calculate x_p for $X \sim Uniform[a, b]$

$$F_X(x_p) = \frac{x_p - a}{b - a} = p$$

$$x_p = p(b - a) + a$$

$X \sim Exponential(\lambda), \ \lambda > 0$

Story: X represents the lifetime of a lightbulb

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's find $F_X(x) = \int_{-\infty}^{x} f_x(t) \ dt, \ x \geq 0$

$$= \int_0^x \lambda e^{-\lambda t} \ dt = 1 - e^{-\lambda x}, \ x \geq 0$$

Let's calculate the median

$$F_X(x) = \frac{1}{2}$$

$$1 - e^{-\lambda x} = \frac{1}{2}$$

$$x = \frac{ln(2)}{\lambda}$$

Memoryless property of exponential distribution

$$X \sim Exponential(\lambda) \rightarrow P(X > t + s | X > t) = P(X > s); \ t, s > 0$$

The probability that lightbulbs last more than $t + s$ seconds given that it has lasted more than t seconds is the same as the unconditional probability that it lasts more than s seconds.

Let's verify this equation

$$LHS = \frac{P(\{X > t+s\} \cap \{X > t\})}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)}$$

$$= \frac{1 - P(X \leq t+s)}{1 - P(x \leq t)} = \frac{1 - (1 - e^{-\lambda(t+s)})}{1 - (1 - e^{-\lambda t})} = e^{-\lambda s}$$

$$RHS = P(X > s) = 1 - P(X \leq s) = e^{-\lambda s}$$

Example: $X \sim N(\mu, \sigma^2)$; $\mu \in \mathbb{R}, \sigma^2 > 0$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

- μ is a shifting parameter (doesn't change shape)
- σ is a shape or scale parameter that doesn't affect the location

Story: The sum of a large number of small independent random variables approximately behave like a normal random variable

Consider a sample mean

$X_1 \to X_n$ independent samples (random variables)

$$\bar{X} = \frac{1}{n}(X_1 + ... + X_n)$$

Note: for normal distributions, $f_X = \phi$, $F_X = \Phi$

Example: $X \sim Gamma(a, \lambda)$; $a, \lambda > 0$

$$f_X(x) = \begin{cases} \frac{\lambda^a}{\Gamma(a)} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$Exponential(\lambda)$ **is a special case of** $Gamma(x, \lambda)$

When $a = 1$:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$