1

The MGF can be written as:

$$m_X(t) = \sum_{k=1}^{\infty} p_k e^{kt}$$

From the given MGF:

$$P(X = 1) = 0.5$$
$$P(X = 2) = 0.25$$
$$P(X = 3) = 0.125$$
$$...$$

From the pattern:

$$P(X = k) = \frac{1}{2^k},\ k \in \mathbb{N}$$

The distribution is:

$$X \sim Geometric(0.5),\ X \in \mathbb{N}$$

2

From discussion section, the MGF is:

$$m_{X_1}(t) = \left(\frac{p_1}{1-(1-p_1)e^t}\right)^l$$

$$m_{X_2}(t) = \left(\frac{p_2}{1-(1-p_2)e^t}\right)^l$$

Since X_1, X_2 are independent:

$$m_{X_1+X_2}(t) = (m_{X_1}(t))(m_{X_2}(t))$$

$$m_{X_1+X_2}(t) = \left(\frac{p_1}{1-(1-p_1)e^t}\right)^l \left(\frac{p_2}{1-(1-p_2)e^t}\right)^l$$

The MGF of $NB(l, p_1 + p_2)$ is:

$$\left(\frac{p_1+p_2}{1-(1-p_1-p_2)e^t}\right)^l$$

Since a sum is not equal to a product when raised to a power, we can conclude:

$$X_1 + X_2 \nsim NB(l, p_1 + p_2)$$

3

(a) Given:

$$m_{X_1}(t) = exp(4(e^t - 1))$$

Let:

$$Y = 2X_1$$

Then the MGF is:

$$M_Y(t) = E[e^{t(2X)}]$$
$$M_Y(t) = E[(e^{2t})^X]$$
$$M_Y(t) = M_X(2t)$$

Plugging in the given MGF:

$$m_{Y_1}(t) = exp(4(e^{2t} - 1)) \neq exp(8(e^t - 1))$$

Therefore:

$$2X_1 \nsim Poisson(8)$$

(b) Given:

$$M_{X_1}(t) = exp(4(e^t - 1))$$
$$M_{X_2}(t) = exp(4(e^t - 1))$$

Let:

$$Y = X_1 + X_2$$

Then:

$$M_Y(t) = E[e^{t(X_1+X_2)}]$$
$$M_Y(t) = E[(e^t)^{X_1+X_2}]$$
$$M_Y(t) = E[(e^t)^{X_1}(e^t)^{X_2}]$$
$$M_Y(t) = E[(e^{tX_1})(e^{tX_2})]$$

Since X_1 and X_2 are independent:

$$M_Y(t) = E[e^{tX_1}]E[e^{tX_2}]$$
$$M_Y(t) = M_{X_1}(t)M_{X_2}(t)$$

Plugging in the given MGF:

$$M_Y(t) = exp(4(e^t - 1))exp(4(e^t - 1))$$
$$M_Y(t) = exp(8(e^t - 1))$$

Therefore:

$$X_1 + X_2 \sim Poisson(8)$$

(c) Using the same proof from (b), we can conclude:

$$X_1 + X_3 \sim Poisson(6)$$

4

(a)

$$F_{Y_i}(y) = P(Y_i \leq y)$$
$$F_{Y_i}(y) = P(-log(1 - X_i) \leq y)$$
$$F_{Y_i}(y) = P(log(1 - X_i) > -y)$$
$$F_{Y_i}(y) = P(1 - X_i > e^{-y})$$
$$F_{Y_i}(y) = P(1 - e^{-y} > X_i)$$
$$F_{Y_i}(y) = P(X_i \leq 1 - e^{-y})$$
$$F_{Y_i}(y) = F_{X_i}(1 - e^{-y})$$
$$f_{Y_i}(y) = e^{-y}\ f_{X_i}(1 - e^{-y})$$
$$f_{Y_i}(y) = e^{-y}\theta(1 - (1 - e^{-y}))^{\theta-1}$$
$$f_{Y_i}(y) = e^{-y}\theta(e^{-y})^{\theta-1}$$
$$f_{Y_i}(y) = \theta(e^{-y})^{\theta}$$
$$Y_i \sim Exponential(\theta)$$

(b)

$$L(\theta) = \prod_{i=1}^{n} \theta(1 - X_i)^{\theta-1}$$

$$l(\theta) = n\ log(\theta) + (\theta - 1)\sum_{i=1}^{n} log(1 - X_i)$$

$$\frac{\delta l(\theta)}{\delta\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log(1 - X_i) = 0$$

$$\sum_{i=1}^{n} log(1 - X_i) = -\frac{n}{\theta}$$

$$\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} log(1 - X_i)}$$

(c) We can use the transformation from (a):

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} Y_i}$$

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} Y_i}$$

Let $Y = \sum_{i=1}^{n} Y_i$:

$$\hat{\theta}_{MLE} = \frac{n}{Y}$$

Since Y is the sum of n independent exponential variables:

$$Y \sim Gamma(n, \theta)$$

Now find the CDF:

$$F_{\hat{\theta}}(x) = P(\hat{\theta} < x)$$

$$F_{\hat{\theta}}(x) = P\left(\frac{n}{Y} < x\right)$$
$$F_{\hat{\theta}}(x) = P\left(\frac{n}{x} < Y\right)$$
$$F_{\hat{\theta}}(x) = 1 - P\left(Y < \frac{n}{x}\right)$$
$$F_{\hat{\theta}}(x) = 1 - F_Y\left(\frac{n}{x}\right)$$
$$f_{\hat{\theta}}(x) = -f_Y\left(\frac{n}{x}\right) * \frac{-n}{x^2}$$
$$f_{\hat{\theta}}(x) = f_Y\left(\frac{n}{x}\right) * \frac{n}{x^2}$$

Since Y is a Gamma distribution:

$$f_Y(y) = \frac{\theta^n}{(n-1)!} * y^{n-1}e^{-\theta y}$$

Plug in the PDF:

$$f_{\hat{\theta}}(x) = \frac{\theta^n}{(n-1)!} * \left(\frac{n}{x}\right)^{n-1} e^{-\theta\frac{n}{x}} * \frac{n}{x^2}$$

(d) Since Y follows a gamma distribution and $\hat{\theta} = \frac{n}{Y}$:

$$\hat{\theta} \sim Inv - Gamma(n, n\theta)$$

$$E[\hat{\theta}] = \frac{n\theta}{n-1}$$

Bias:

$$Bias(\hat{\theta}) = E[\hat{\theta}_{MLE}] - \theta$$
$$Bias(\hat{\theta}) = \frac{n\theta}{n-1} - \theta$$

MSE:

$$MSE(\hat{\theta}) = Var[\hat{\theta}] + Bias[\hat{\theta}]^2$$

Since $\hat{\theta}$ follows an inverse gamma distribution:

$$Var[\hat{\theta}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

Plug into MSE:

$$MSE(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)^2(n-2)} + \left[\frac{n\theta}{n-1} - \theta\right]^2$$

5

(a)

$$F_{\hat{\lambda}}(x) = P(\hat{\lambda} < x)$$
$$F_{\hat{\lambda}}(x) = 1 - P(\hat{\lambda} \geq x)$$
$$F_{\hat{\lambda}}(x) = 1 - P(\{nX_1 \geq x \cap ... \cap nX_n \geq x\})$$

Since $X_1, ..., X_n$ are independent:

$$F_{\hat{\lambda}}(x) = 1 - P(nX_1 \geq x)...P(nX_n \geq x)$$

Since $X_1, ..., X_n$ follow the same distribution:

$$F_{\hat{\lambda}}(x) = 1 - [P(nX_1 \geq x)]^n$$
$$F_{\hat{\lambda}}(x) = 1 - \left[1 - P\left(X_1 < \frac{x}{n}\right)\right]^n$$
$$F_{\hat{\lambda}}(x) = 1 - \left[1 - F_{X_1}\left(\frac{x}{n}\right)\right]^n$$

Since X_1 is an exponential distribution, the CDF is:

$$F_{X_1}(x) = 1 - e^{-\lambda x}$$

Now plug in the CDF:

$$F_{\hat{\lambda}}(x) = 1 - \left[1 - [1 - e^{-\lambda \frac{x}{n}}]\right]^n$$
$$F_{\hat{\lambda}}(x) = 1 - \left[e^{-\lambda \frac{x}{n}}\right]^n$$
$$F_{\hat{\lambda}}(x) = 1 - e^{-\lambda x}$$

From the CDF:

$$\hat{\lambda} \sim Exponential(\lambda)$$
$$f_{\hat{\lambda}}(x) = \lambda e^{-\lambda x}$$

(b) Bias:

$$Bias(\hat{\lambda}) = E[\hat{\lambda}] - \lambda$$

Since $\hat{\lambda} \sim Exponential(\lambda)$:

$$E[\hat{\lambda}] = \frac{1}{\lambda}$$
$$Bias(\hat{\lambda}) = \frac{1}{\lambda} - \lambda$$

MSE:

$$MSE[\hat{\lambda}] = Var[\hat{\lambda}] + Bias[\hat{\lambda}]^2$$

Since $\hat{\lambda} \sim Exponential(\lambda)$:

$$Var[\hat{\lambda}] = \frac{1}{\lambda^2}$$
$$MSE[\hat{\lambda}] = \frac{1}{\lambda^2} + \left[\frac{1}{\lambda} - \lambda\right]^2$$