1

Step 1

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\ dx$$

$$E[X] = \lambda \int_0^\infty x e^{-\lambda x}\ dx$$

$$E[X] = \frac{1}{\lambda} = \bar{X}$$

$$\hat{\lambda}_{MOM} = \frac{1}{\bar{X}}$$

This result will match the result from MLE.

Step 2

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$l(\lambda) = n\ log\ \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{\delta l(\lambda)}{\delta \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\hat{\lambda}_{MLE} = \frac{1}{\bar{X}}$$

This result matches the MOM.

2

a - obtain MOM estimator

$$E[X] = \int_0^\infty \frac{x^2}{\theta^2} exp\left(-\frac{x^2}{2\theta^2}\right)\ dx$$

$$E[X] = \frac{\sqrt{\pi}\theta}{\sqrt{2}} = \bar{X}$$

$$\hat{\theta}_{MOM} = \frac{\sqrt{2}}{\sqrt{\pi}}\bar{X}$$

a - compute bias and MSE

$$Bias[\hat{\theta}_{MOM}] = E[\hat{\theta}_{MOM}] - \theta$$

$$Bias[\hat{\theta}_{MOM}] = \frac{\sqrt{2}}{\sqrt{\pi}}E[\bar{X}] - \theta$$

$$Bias[\hat{\theta}_{MOM}] = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}\theta}{\sqrt{2}} - \theta$$

$$Bias[\hat{\theta}_{MOM}] = 0$$

$$MSE[\hat{\theta}_{MOM}] = Var[\hat{\theta}_{MOM}] + Bias[\hat{\theta}_{MOM}]$$

$$MSE[\hat{\theta}_{MOM}] = Var[\hat{\theta}_{MOM}] + 0$$

$$MSE[\hat{\theta}_{MOM}] = \frac{2}{\pi}Var[\bar{X}]$$

$$MSE[\hat{\theta}_{MOM}] = \frac{2}{\pi}\frac{Var[X]}{n}$$

$$E[X^2] = \int_0^\infty \frac{x^3}{\theta^2} exp\left(-\frac{x^2}{2\theta^2}\right)\ dx$$

$$E[X^2] = 2\theta^2$$

$$Var[X] = E[X^2] - (E[X])^2 = 2\theta^2 - \frac{\pi}{2}\theta^2$$

$$Var[X] = \left(2 - \frac{\pi}{2}\right)\theta^2$$

$$MSE[\hat{\theta}_{MOM}] = \frac{2\theta^2(2 - \frac{2}{\pi})}{n\pi}$$

b - obtain MLE

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} exp\left(-\frac{x_i^2}{2\theta^2}\right)$$

$$l(\theta) = \sum_{i=1}^{n} log(x_i) - 2n\ log\ \theta - \sum_{i=1}^{n} \frac{x_i^2}{2\theta^2}$$

$$\frac{\delta l(\theta)}{\delta\theta} = -\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{x_i^2}{\theta^3} = 0$$

$$\sum_{i=1}^{n} \frac{x_i^2}{\theta^3} = \frac{2n}{\theta}$$

$$\sum_{i=1}^{n} x_i^2 = 2n\theta^2$$

$$\theta^2 = \frac{1}{2n}\sum_{i=1}^{n} x_i^2$$

$$\hat{\theta}_{MLE} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} x_i^2}$$

This result differs from our MOM.