Homework 5

STA 131B | A. Farris | Winter 2025

A pdf copy of your homework solutions is due at 5pm on Friday, February 21. Submission of the pdf will be through Gradescope. You must write up and turn in your own solutions, but if you do work on them with some of your classmates, list their names as collaborators on your submission. You must explain your answers for full credit.

Problems

1. Suppose that X has the MGF

$$m_X(t) = 0.5e^t + 0.25e^{2t} + 0.125e^{3t} + 0.0625e^{4t} + 0.03125e^{5t} + \dots$$

   What is the PMF of X? If possible, identify this distribution by name.

2. If $X_1 \sim \text{NB}(\ell, p_1)$ and $X_2 \sim \text{NB}(\ell, p_2)$ are independent,[1] is it accurate to say $X_1 + X_2 \sim \text{NB}(\ell, p_1 + p_2)$?

3. Suppose that $X_1 \sim \text{Poisson}(4)$, $X_2 \sim \text{Poisson}(4)$, and $X_3 \sim \text{Poisson}(2)$ are mutually independent.[2]
   (a) Is it accurate to say that $2X \sim \text{Poisson}(8)$?
   (b) Is it accurate to say that $X_1 + X_2 \sim \text{Poisson}(8)$?
   (c) Is it accurate to say that $X_1 + X_3 \sim \text{Poisson}(6)$?

4. Suppose that you have an IID sample $X_1, \dots, X_n$ from a distribution with probability density function

$$f(x) = \begin{cases} \theta(1-x)^{\theta-1} & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

   for some unknown $0 < \theta < 1$.
   (a) For each $i = 1, \dots, n$, what is the distribution of $Y_i = -\log(1 - X_i)$? If possible, identify this distribution by name.
   (b) Find the MLE of θ on the basis of the sample $X_1, \dots, X_n$.
   (c) What is the PDF of the MLE that you computed in part (c)?
   (d) Compute the bias and the MSE for $\hat{\theta}_{mle}$, for fixed sample size n.

[1] *Hint:* use the MGF obtained in discussion section this week!
[2] *Hint:* Note that, whenever $X \sim \text{Poisson}(\lambda)$, $m_X(t) = \exp(\lambda(e^t - 1))$.

5. Suppose that you have an IID sample $X_1, \ldots, X_n$ from a distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

for some unknown $\lambda > 0$.

(a) A possible estimator for λ here is $\hat{\lambda} = n \min\{X_1, \ldots, X_n\}$. Find the distribution of $\hat{\lambda}$ by first computing its CDF. What is the name of this distribution?

(b) Compute the bias and the MSE for $\hat{\lambda}$. Would you say that this estimator is better than the MLE or the MOM estimator?