Homework 6

STA 131B | A. Farris | Winter 2025

A pdf copy of your homework solutions is due at 5pm on Tuesday, March 18. Submission of the pdf will be through Gradescope. You must write up and turn in your own solutions, but if you do work on them with some of your classmates, list their names as collaborators on your submission. You must explain your answers for full credit.

Problems

1. For this exercise, we will assume that every user of a web browser uses either Firefox, Chrome, Safari, or Edge.

   We will independently sample n of these users; let the respective numbers of users of each browser in the sample be Y_1, Y_2, Y_3, Y_4.

   Assume that the proportion of users of Chrome is known to be two times larger than the proportion of Firefox users. Let π represent the proportion of Firefox users in the population.

   (a) Find a constant c_1 which is such that $\hat{\pi}_1 = c_1 Y_1$ is an unbiased estimator for π_1.

   (b) Find a constant c_2 which is such that $\hat{\pi}_2 = c_2 Y_2$ is an unbiased estimator for π_2.

   (c) Which of the estimators obtained in the previous two parts is the better estimator? To answer this, compute the MSE for each one.

   (d) Let's use the Rao-Blackwell approach to improve upon these estimators.

      i. Define a new estimator by $\hat{\pi}_3 = \mathrm{E}(\hat{\pi}_2 | Y_3 + Y_4)$. Is this an unbiased estimator for π?

      ii. Compute the MSE of $\hat{\pi}_3$. Is this a better estimator than $\hat{\pi}_2$?

2. Assume that $X_1, X_2 \overset{iid}{\sim} \mathrm{Poisson}(\lambda)$, where $\lambda > 0$; and, let $\hat{\theta}_1 = \frac{2}{3}X_1 + \frac{1}{3}X_2$, and $T = X_1 + X_2$.

   (a) Compute $\mathrm{E}(\hat{\theta}_1)$. Is this an unbiased estimator for θ?

   (b) Compute $\mathrm{MSE}(\hat{\theta}_1)$.

   (c) Compute the conditional probability mass function for X_i given T, where $i = 1, 2$. What is the name of this conditional distribution?

   (d) As we have seen in lecture, T is a sufficient statistic for λ. Using the Rao-Blackwell approach, together with T, find a new estimator, $\hat{\theta}_2$, that is an improvement on $\hat{\theta}_1$.

   (e) Compute $\mathrm{MSE}(\hat{\theta}_2)$. Is it possible to find a better unbiased estimator for θ than $\mathrm{MSE}(\hat{\theta}_2)$?

3. Suppose that you have an IID sample $X_1, \ldots, X_n$ from a distribution with PDF

$$f(x; \theta) = \begin{cases} x^{-\theta} & \text{if} \quad x \geq 1 \ , \\ 0 & \text{o.w.} \end{cases}$$

   where $\theta > 0$ is an unknown parameter. Find a sufficient statistic for θ.

4. Assume that you have an IID sample $X_1, \ldots, X_n$ from a distribution with PDF

$$f(x; \theta) = \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{(x-\mu)^2}{6}\right)$$

for any x, where μ is an unknown parameter. Find a sufficient statistic for μ.

5. Suppose that you have an IID sample $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$, where λ is assumed to be random, with $\lambda \sim \text{Gamma}(1.5, 3)$.

   (a) What is your prediction for the value of λ, based only on the prior distribution, under squared error loss (i.e. what is the prior mean)?

   (b) Compute the variance of the prior distribution.

   (c) What is the Bayes estimator for the value of λ given the data, under squared error loss (i.e. what is the posterior mean)?

   (d) Compute the variance of the posterior distribution.

6. Suppose that you have an IID sample $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$, where λ is assumed to be random, with $\lambda \sim \text{Gamma}(13.5, 27)$.

   (a) What is your prediction for the value of λ, based only on the prior distribution, under squared error loss (i.e. what is the prior mean)?

   (b) Compute the variance of the prior distribution. If you were say that you are k times more confident in the prediction based on this prior than you would have been in the previous problem, what is k?

   (c) What is the Bayes estimator for the value of λ given the data, under squared error loss (i.e. what is the posterior mean)? How does this compare to the estimator obtained in the previous problem?

   (d) Compute the variance of the posterior distribution.

7. Assume that you have an IID sample $X_1, \ldots, X_n \sim \text{Geometric}(\pi)$, where π is assumed to be random, with $\pi \sim \text{Beta}(2, 2)$.

   Compute the Bayes estimator for π from the data, under squared error loss.