HW1

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Problem 1

First we input the data.

```
mu = c(3, 1, 4)
sigma = matrix(c(6, 1, -2, 1, 13, 4, -2, 4, 4),3,3)
```

Here we use the fact that if $X \sim \mathcal{N}_d(\mu, \Sigma)$, for $A \in \mathbb{R}^{k \times d}$, $AX \sim \mathcal{N}_k(A\mu, A\Sigma A^T)$.

(a)

```
A = t(c(2, -1, 3))
A%*%mu
```

```
##      [,1]
## [1,]   17
```

```
A%*%sigma%*%t(A)
```

```
##      [,1]
## [1,]   21
```

(b)

```
A = rbind(c(1, 0, 0),c(0, 0, 1))
A%*%mu
```

```
##      [,1]
## [1,]    3
## [2,]    4
```

```
A%*%sigma%*%t(A)
```

```
##      [,1] [,2]
## [1,]    6   -2
## [2,]   -2    4
```

(c)

```
A = rbind(c(1, 0, 0),c(0, 0, 1),c(0.5, 0, 0.5))
A%*%mu
```

```
##      [,1]
## [1,]  3.0
## [2,]  4.0
```

```
## [3,]  3.5
```

```
A%*%sigma%*%t(A)
```

```
##      [,1] [,2] [,3]
## [1,]    6   -2  2.0
## [2,]   -2    4  1.0
## [3,]    2    1  1.5
```

Problem 2

In this problem, we utilize the fact that if $X \in \mathbb{R}^m, Y \in \mathbb{R}^n$ and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_{m+n}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx}, \Sigma_{xy} \\ \Sigma_{yx}, \Sigma_{yy} \end{pmatrix}\right)$$

we therefore have

$$Y|X \sim \mathcal{N}_n(\mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}(X - \mu_x), \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy})$$

This formula is complicated. Please write it in your cheat sheet in exams!

Some rule of thumb to memorize. Since you are actually get the distribution of Y, the conditional mean and covariance should be the same dimension as that of Y. Look at $\Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$, you can think of x's here are cancelled out and remember that our conditional covariance will be reduced so the sign is negative. For the mean part, similarly, in $\Sigma_{yx}\Sigma_{xx}^{-1}(X - \mu_x)$, x's are cancelled out, there is one x in Σ_{yx}, two reciprocal of x in Σ_{xx}^{-1} and one x in $(X - \mu_x)$.

Then we input the data

```
mu_y = c(2, -1)
mu_x = c(3, 1)
sigma_yy = rbind(c(7, 3),c(3, 6))
sigma_yx = matrix(c(-3, 2, 0, 4),2,2)
sigma_xy = t(sigma_yx)
sigma_xx = cbind(c(5, -2),c(-2, 4))

sigma_yy
```

```
##      [,1] [,2]
## [1,]    7    3
## [2,]    3    6
```

```
sigma_yx
```

```
##      [,1] [,2]
## [1,]   -3    0
## [2,]    2    4
```

```
sigma_xx
```

```
##      [,1] [,2]
## [1,]    5   -2
## [2,]   -2    4
```

Now do the math

```
step_1 <- sigma_yx%*%solve(sigma_xx)
step_1
```

```
##        [,1]   [,2]
## [1,] -0.75 -0.375
## [2,]  1.00  1.500
```

```
step_2 <-mu_y-sigma_yx%*%solve(sigma_xx)%*%mu_x
step_2
```

```
##        [,1]
## [1,]  4.625
## [2,] -5.500
```

```
step_3 <- sigma_yy-sigma_yx%*%solve(sigma_xx)%*%sigma_xy
step_3
```

```
##      [,1] [,2]
## [1,] 4.75    6
## [2,] 6.00   -2
```

So we have

$$\mathbb{E}(Y|X) = \begin{bmatrix} 4.625 \\ -5.5 \end{bmatrix} + \begin{bmatrix} -0.75 & -0.375 \\ 1 & 1.5 \end{bmatrix} X$$

$$\mathbf{COV}(Y|X) = \begin{bmatrix} 4.75 & 6 \\ 6 & -2 \end{bmatrix}$$

Our distribution is:

$$y|x \sim \mathcal{N}(\mathbb{E}(Y|X), \mathbf{COV}(Y|X))$$
$$y|x \sim \mathcal{N}(\begin{bmatrix} 4.625 \\ -5.5 \end{bmatrix} + \begin{bmatrix} -0.75 & -0.375 \\ 1 & 1.5 \end{bmatrix} X, \begin{bmatrix} 4.75 & 6 \\ 6 & -2 \end{bmatrix})$$