3.1

For $MA(1)$, the autocorrelation function is given as:

$$\gamma(h) = Cov[x_{t+h}, x_t]$$
$$\gamma(h) = Cov[w_{t+h} + \theta w_{t+h-1}, w_t + \theta w_{t-1}]$$

When $h = 0$:

$$\gamma(0) = Cov[w_t + \theta w_{t-1}, w_t + \theta w_{t-1}]$$
$$\gamma(0) = Var[w_t + \theta w_{t-1}]$$
$$\gamma(0) = (1 + \theta^2)\sigma_w^2$$

When $h = 1$:

$$\gamma(1) = Cov[w_{t+1} + \theta w_t, w_t + \theta w_{t-1}]$$
$$\gamma(1) = E[(w_{t+1} + \theta w_t)(w_t + \theta w_{t-1})] - E[w_{t+1} + \theta w_t]E[w_t + \theta w_{t-1}]$$
$$\gamma(1) = E[(w_{t+1} + \theta w_t)(w_t + \theta w_{t-1})] - 0$$
$$\gamma(1) = E[\theta w_t w_t]$$
$$\gamma(1) = \theta\sigma_w^2$$

Solve for $\rho_x(1)$:

$$\rho_x(1) = \frac{\gamma(1)}{\gamma(0)}$$
$$\rho_x(1) = \frac{\theta\sigma_w^2}{(1 + \theta^2)\sigma_w^2}$$
$$\rho_x(1) = \frac{\theta}{1 + \theta^2}$$

Find which θ results in minimum and maximum values of $\rho_x(1)$:

$$\frac{\delta\rho_x(1)}{\delta\theta} = \frac{\theta}{1 + \theta^2}$$
$$\frac{\delta\rho_x(1)}{\delta\theta} = \frac{1 - \theta^2}{(1 + \theta^2)^2}$$

Set $\frac{\delta\rho_x(1)}{\delta\theta} = 0$, then the maximum and minimum values are at:

$$\theta = \pm 1$$

At $\theta = 1$:

$$\rho_x(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

At $\theta = -1$:

$$\rho_x(1) = \frac{-1}{1 + 1} = -\frac{1}{2}$$

We know that $\theta = \pm 1$ are global maximums and minimums since both limits to infinity are within the range of the minimum and maximum:

$$\lim_{\theta \to -\infty} \frac{\theta}{1 + \theta^2} = \lim_{\theta \to -\infty} \frac{1}{2\theta} = 0$$
$$\lim_{\theta \to \infty} \frac{\theta}{1 + \theta^2} = \lim_{\theta \to -\infty} \frac{1}{2\theta} = 0$$

Since $-\frac{1}{2} \leq \rho_x(1) \leq \frac{1}{2}$:

$$|\rho_x(1)| \leq \frac{1}{2}$$

3.4

(a)

Rewriting the model with backshift operator:

$$x_t = .8Bx_t - .15B^2x_t + w_t - .3Bw_t$$

$$x_t - .8Bx_t + .15B^2x_t = w_t - .3Bw_t$$

$$(1 - .3B)(1 - .5B)x_t = (1 - .3B)w_t$$

We can reduce the model to:

$$(1 - .5B)x_t = w_t$$

This is an $AR(1)$ model.

AR polynomial:

$$\phi(B) = 1 - .5B$$

MA polynomial:

$$\theta(B) = 1$$

Check the roots of AR polynomial:

$$1 - .5z = 0$$

$$z = 2$$

Since the root is outside of the unit circle, this process is causal.

There is no MA component, so this process is trivially invertible.

The causal representation is:

$$x_t = .5x_{t-1} + w_t$$

$$x_t = .5(.5x_{t-2} + w_{t-1}) + w_t$$

$$\dots$$

$$x_t = \sum_{j=0}^{\infty} .5^j w_{t-j}$$

The invertible representation is:

$$w_t = x_t - .5x_{t-1}$$

(b)

Rewriting with backshift operator:

$$x_t = Bx_t - .5B^2x_t + w_t - Bw_t$$

$$x_t - Bx_t + .5B^2x_t = w_t - Bw_t$$

$$(1 - B + .5B^2)x_t = (1 - B)w_t$$

Since we can't factor $1 - B + .5B^2$, our model is already in simplest form with no redundant parameters.

This is an $ARMA(2, 1)$ model.

AR polynomial:

$$\phi(B) = 1 - B + .5B^2$$

MA polynomial:

$$\theta(B) = 1 - B$$

Check the roots of AR polynomial:

$$1 - z + .5z^2 = 0$$

Using the quadratic formula:

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4 * .5 * 1}}{2 * .5}$$

$$z = 1 \pm \sqrt{-1}$$

$$z = 1 \pm i$$

Since the complex roots are outside of the unit circle, this process is causal.

Check the roots of MA polynomial:

$$1 - z = 0$$

$$z = 1$$

Since the root is on the unit circle, this process is NOT invertible.

Since this process is causal, this can be written as:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

To find the causal representation, derive the Coefficients. Let:

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$$

Since:

$$x_t = \frac{\theta(B)}{\phi(B)} w_t$$

We can use the identity:

$$\phi(B)\psi(B) = \theta(B)$$

In time-domain convolution, this gives the recursion:

$$\sum_{k=0}^{2} \phi_k \psi_{j-k} = \theta_j$$

Where:

- $\phi_0 = 1$, $\phi_1 = -1$, $\phi_2 = 0.5$
- $\theta_0 = 1$, $\theta_1 = -1$, $\theta_j = 0$ for $j \geq 2$

We initialize:

- $\psi_{-1} = 0$
- $\psi_0 = 1$

Then recursively compute:

- $\psi_1 = \psi_0 - 0.5\psi_{-1} + \theta_1 = 1 - 0 - 1 = 0$
- $\psi_2 = \psi_1 - 0.5\psi_0 + 0 = 0 - 0.5 = -0.5$
- $\psi_3 = \psi_2 - 0.5\psi_1 = -0.5 - 0 = -0.5$
- $\psi_4 = \psi_3 - 0.5\psi_2 = -0.5 + 0.25 = -0.25$
- etc.

The coefficients follow the second-order linear recurrence:

$$\begin{cases} \psi_0 = 1 \\ \psi_1 = 0 \\ \psi_j = \psi_{j-1} - 0.5\psi_{j-2}, \quad j \geq 2 \end{cases}$$

The causal representation of the process is:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

Where:

$$\begin{cases} \psi_0 = 1 \\ \psi_1 = 0 \\ \psi_j = \psi_{j-1} - 0.5\psi_{j-2}, \quad j \geq 2 \end{cases}$$