1

Using the difference equation from lecture:

$$\gamma(h) - \phi_1\gamma(h-1) - ... - \phi_p\gamma(h-p) = 0$$

$$h \geq max\{p, q+1\}$$

Apply:

$$h \geq max\{1, 1+1\} = 2$$

$$\gamma(h) - \phi\gamma(h-1) = 0$$

$$\gamma(h) = c * \phi^h$$

When $h = 0$:

$$\gamma(0) - \phi\gamma(1) = \sigma^2 \sum_{j=0}^{1} \theta_j\psi_j$$

$$\gamma(0) - \phi\gamma(1) = \sigma^2(\psi_0 + \theta\psi_1)$$

When $h = 1$:

$$\gamma(1) - \phi\gamma(0) = \sigma^2 \sum_{j=1}^{1} \theta_j\psi_{j-1}$$

$$\gamma(1) - \phi\gamma(0) = \sigma^2\theta\psi_0$$

The causal representation is:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

Using the method of matching coefficients:

$$x_t = w_t + (\phi + \theta)\sum_{j=1}^{\infty} \phi^{j-1} w_{t-j}$$

Then:

$$\psi_0 = 1$$

$$\psi_1 = \phi + \theta$$

$$\gamma(0) - \phi\gamma(1) = \sigma^2(1 + \theta(\theta + \phi))$$

$$\gamma(1) - \phi\gamma(0) = \sigma^2\theta$$

Solve for $\gamma(0)$:

$$\gamma(1) = \sigma^2\theta + \phi\gamma(0)$$

$$\gamma(0) - \phi(\sigma^2\theta + \phi\gamma(0)) = \sigma^2(1 + \theta(\theta + \phi))$$

$$\gamma(0) - \phi\sigma^2\theta - \phi^2\gamma(0) = \sigma^2 + \sigma^2\theta(\theta + \phi)$$

$$\gamma(0) - \phi^2\gamma(0) = \sigma^2 + \sigma^2\theta^2 + \sigma^2\theta\phi + \phi\sigma^2\theta$$

$$(1 - \phi^2)\gamma(0) = \sigma^2 + \sigma^2\theta^2 + 2\phi\sigma^2\theta$$

$$(1 - \phi^2)\gamma(0) = \sigma^2(1 + \theta^2 + 2\phi\theta)$$

$$\gamma(0) = \sigma^2 \frac{1 + \theta^2 + 2\phi\theta}{(1 - \phi^2)}$$

$$\gamma(0) = \sigma^2 \frac{1 - \phi^2 + \phi^2 + \theta^2 + 2\phi\theta}{1 - \phi^2}$$
$$\gamma(0) = \sigma^2 \frac{1 - \phi^2 + (\phi + \theta)^2}{1 - \phi^2}$$
$$\gamma(0) = \sigma^2 \left(1 + \frac{(\phi + \theta)^2}{1 - \phi^2}\right)$$

Now solve for $\gamma(1)$:

$$\gamma(1) = \sigma^2\theta + \phi\gamma(0)$$
$$\gamma(1) = \sigma^2\theta + \phi\sigma^2 \left(1 + \frac{(\phi + \theta)^2}{1 - \phi^2}\right)$$
$$\gamma(1) = \sigma^2 \left(\theta + \phi \left(1 + \frac{(\phi + \theta)^2}{1 - \phi^2}\right)\right)$$
$$\gamma(1) = \sigma^2 \left(\theta + \phi + \phi \frac{(\phi + \theta)^2}{1 - \phi^2}\right)$$

Now solve for $\gamma(h)$ when $h \geq 2$:

$$\gamma(1) = c\phi$$
$$c = \frac{\gamma(1)}{\phi}$$
$$\gamma(h) = \phi^h * \frac{\gamma(1)}{\phi} = \gamma(1)\phi^{h-1}$$

Now solve for $\rho(h)$:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

When $h = 0$:

$$\rho(0) = 1$$

When $h \geq 1$:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$
$$\rho(h) = \frac{\gamma(1)\phi^{h-1}}{\gamma(0)}$$
$$\rho(h) = \phi^{h-1} \frac{\sigma^2 \left(\theta + \phi + \phi \frac{(\phi+\theta)^2}{1-\phi^2}\right)}{\sigma^2 \left(1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right)}$$
$$\rho(h) = \phi^{h-1} \frac{\theta + \phi + \phi \frac{(\phi+\theta)^2}{1-\phi^2}}{1 + \frac{(\phi+\theta)^2}{1-\phi^2}}$$
$$\rho(h) = \phi^{h-1} \frac{(\theta + \phi)(1 - \phi^2) + \phi(\phi + \theta)^2}{1 - \phi^2 + (\phi + \theta)^2}$$
$$\rho(h) = \phi^{h-1} \frac{\theta + \phi - \phi^2\theta - \phi^3 + \phi(\phi^2 + 2\phi\theta + \theta^2)}{1 - \phi^2 + \phi^2 + 2\phi\theta + \theta^2}$$
$$\rho(h) = \phi^{h-1} \frac{\theta + \phi - \phi^2\theta - \phi^3 + \phi^3 + 2\phi^2\theta + \phi\theta^2}{1 + 2\phi\theta + \theta^2}$$
$$\rho(h) = \phi^{h-1} \frac{\theta + \phi + \phi^2\theta + \phi\theta^2}{1 + 2\phi\theta + \theta^2}$$
$$\rho(h) = \phi^{h-1} \frac{(\theta + \phi)(1 + \phi\theta)}{1 + 2\phi\theta + \theta^2}$$

2

When $\theta = 0$:

$$\rho(0) = 1$$

$$\rho(h) = \phi^{h-1}\frac{\phi}{1};\ h \geq 1$$

$$\rho(h) = \phi^h;\ h \geq 1$$

This matches equation (3.8).

3

When $\phi = 0$:

$$\rho(0) = 1$$

$$\rho(1) = \frac{\theta}{1+\theta^2}$$

$$\rho(h) = 0;\ h > 1$$

This matches example (3.5).

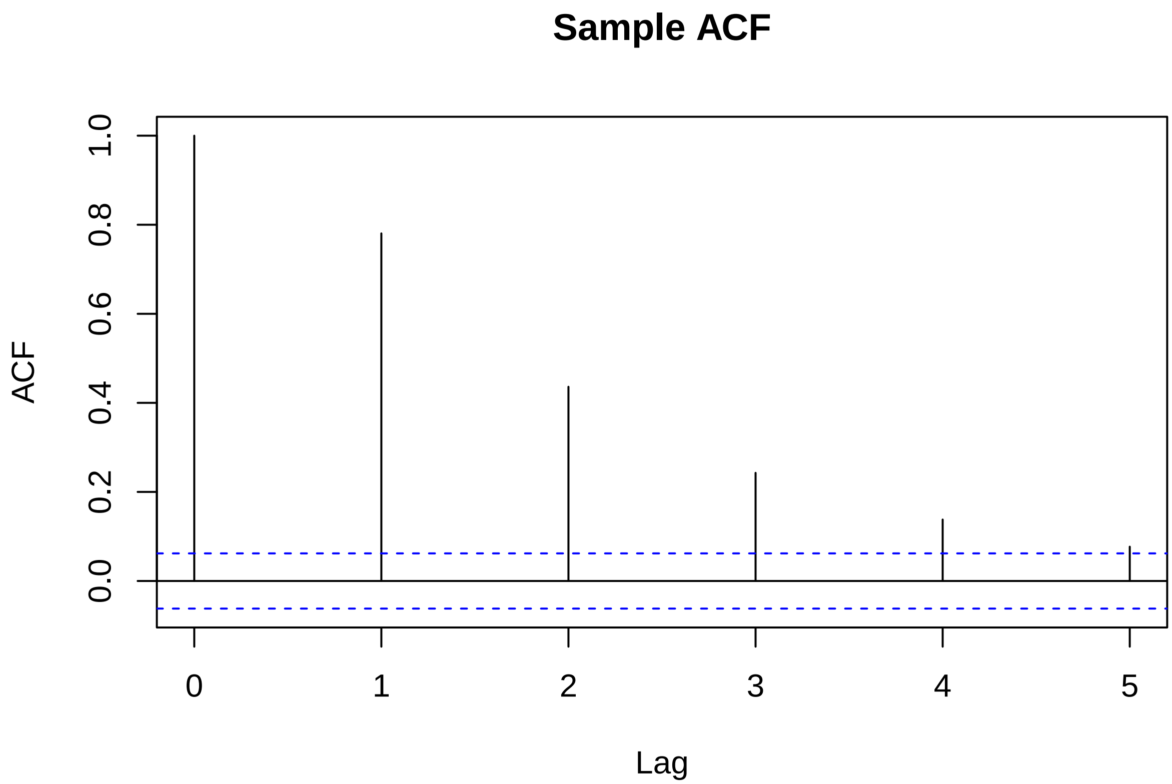
4

```r
# Load required library
library(forecast)

# Set parameters
phi <- 0.6
theta <- 0.9
n <- 1000  # sample size

# Simulate ARMA(1,1) process
set.seed(123)
x <- arima.sim(n = n, list(ar = phi, ma = theta))

# Plot sample ACF for lags 0-5
acf(x, lag.max = 5, main = "Sample ACF")
```




```r
# Compute and print theoretical ACF for comparison
sigma2 <- 1
gamma0 <- sigma2 * (1 + theta^2 + 2 * phi * theta) / (1 - phi^2)
gamma1 <- phi * gamma0 + theta * sigma2
rho1 <- gamma1 / gamma0

# Generate theoretical ACF values
rho <- numeric(6)
rho[1] <- 1
rho[2] <- rho1
for (h in 3:6) {
  rho[h] <- phi * rho[h - 1]
}

# Create data frame of theoretical ACF values
lags <- 0:5
theoretical_acf_values <- data.frame(
  Lag = lags,
  Theoretical_ACF = round(rho, 4)
)
```

Table 1: Theoretical ACF Values

Lag	Theoretical_ACF
0	1.0000
1	0.7993
2	0.4796
3	0.2878
4	0.1727

Lag	Theoretical_ACF
5	0.1036

There is not much difference between the sample ACF values and the theoretical ACF values, when n is large.