1.6

(a)

$$E[x_t] = \beta_1 + \beta_2 t$$

The mean is not constant, so this is not stationary.

(b)

$$y_t = x_t - x_{t-1}$$
$$y_t = \beta_1 + \beta_2 t + w_t - \beta_1 - \beta_2(t-1) - w_{t-1}$$
$$y_t = \beta_2 + w_t - w_{t-1}$$

$$E[y_t] = E[\beta_2] + E[w_t] - E[w_{t-1}]$$
$$E[y_t] = \beta_2$$

The mean is constant.

$$Var[y_t] = Var[\beta_2 + w_t - w_{t-1}]$$
$$Var[y_t] = Var[w_t - w_{t-1}]$$
$$Var[y_t] = Var[w_t] + Var[w_{t-1}] - 2Cov[w_t, w_{t-1}]$$
$$Var[y_t] = Var[w_t] + Var[w_{t-1}]$$
$$Var[y_t] = 2\sigma_w^2$$

The variance is constant.

Now solve for autocovariance:

$$Cov(y_t, y_{t+h})$$

When $h = 0$:

$$Cov(y_t, y_{t+h}) = Var(y_t)$$
$$Cov(y_t, y_{t+h}) = 2\sigma_w^2$$

When $h = 1$:

$$Cov(y_t, y_{t+h}) = Cov(\beta_2 + w_t - w_{t-1}, \beta_2 + w_{t+1} - w_t)$$
$$Cov(y_t, y_{t+h}) = Cov(w_t - w_{t-1}, w_{t+1} - w_t)$$
$$Cov(y_t, y_{t+h}) = Cov(w_t - w_{t-1}, w_{t+1} - w_t)$$
$$Cov(y_t, y_{t+h}) = -Var(w_t)$$
$$Cov(y_t, y_{t+h}) = -\sigma_w^2$$

Otherwise:

$$Cov(y_t, y_{t+h}) = 0$$

The process is stationary because the mean and variance are constant, and the autocovariance only depends on h.

(c)

Show that the $E[v_t] = \beta_1 + \beta_2 t$:

$$E[v_t] = E[\frac{1}{2q+1}\sum_{j=-q}^{q} x_{t-j}]$$

$$E[v_t] = E[\frac{1}{2q+1}\sum_{j=-q}^{q} (\beta_1 + \beta_2(t-j) + w_{t-j})]$$

$$E[v_t] = \frac{1}{2q+1}\sum_{j=-q}^{q} (\beta_1 + \beta_2(t-j))$$

$$E[v_t] = \frac{1}{2q+1}\left[(2q+1)(\beta_1 + \beta_2 t) - \beta_2 \sum_{j=-q}^{q} j\right]$$

Note that the following is a symmetric sum and cancels out:

$$\sum_{j=-q}^{q} j$$

Then:

$$E[v_t] = \frac{1}{2q+1}(2q+1)(\beta_1 + \beta_2 t)$$

$$E[v_t] = \beta_1 + \beta_2 t$$

Now solve for autocovariance:

$$Cov(v_t, v_{t+h}) = Cov(\frac{1}{2q+1}\sum_{j=-q}^{q} x_{t-j}, \frac{1}{2q+1}\sum_{j=-q}^{q} x_{t+h-j})$$

$$Cov(v_t, v_{t+h}) = Cov(\frac{1}{2q+1}\sum_{j=-q}^{q} w_{t-j}, \frac{1}{2q+1}\sum_{j=-q}^{q} w_{t+h-j})$$

$$Cov(v_t, v_{t+h}) = \frac{1}{(2q+1)^2}\sum_{j=-q}^{q}\sum_{k=-q}^{q} Cov(w_{t-j}, w_{t+h-k})$$

When $j + h = k$:

$$Cov(w_{t-j}, w_{t+h-k}) = 1$$

Else:

$$Cov(w_{t-j}, w_{t+h-k}) = 0$$

Then:

$$Cov(v_t, v_{t+h}) = \frac{1}{(2q+1)^2}\sum_{j=-q}^{q} \sigma_w^2 * 1_{\{-q \le j+h \le q\}}$$

We know that the number of valid $j + h$ is:

$$2q + 1 - |h|$$

Therefore:

$$Cov(v_t, v_{t+h}) = \frac{2q+1-|h|}{(2q+1)^2}\sigma_w^2$$

1.7

$$cov(v_s, v_t) = cov\left\{w_{s-1} + 2w_s + w_{s+1}, w_{t-1} + 2w_t + w_{t+1}\right\}$$

When $s = t$:

$$cov(v_s, v_t) = cov\left\{w_{t-1} + 2w_t + w_{t+1}, w_{t-1} + 2w_t + w_{t+1}\right\}$$
$$cov(v_s, v_t) = var(w_{t-1}) + var(2w_t) + var(w_{t+1})$$
$$cov(v_s, v_t) = var(w_{t-1}) + 4var(w_t) + var(w_{t+1})$$
$$cov(v_s, v_t) = 6\sigma_w^2$$

When $|s - t| = 1$:

$$cov(v_s, v_t) = cov\left\{w_t + 2w_{t+1} + w_{t+2}, w_{t-1} + 2w_t + w_{t+1}\right\}$$
$$cov(v_s, v_t) = cov(w_t, 2w_t) + cov(2w_{t+1}, w_{t+1})$$
$$cov(v_s, v_t) = 2cov(w_t, w_t) + 2cov(w_{t+1}, w_{t+1})$$
$$cov(v_s, v_t) = 4\sigma_w^2$$

When $|s - t| = 2$:

$$cov(v_s, v_t) = cov\left\{w_{t+1} + 2w_{t+2} + w_{t+3}, w_{t-1} + 2w_t + w_{t+1}\right\}$$
$$cov(v_s, v_t) = cov\left\{w_{t+1}, w_{t+1}\right\}$$
$$cov(v_s, v_t) = \sigma_w^2$$

Else:

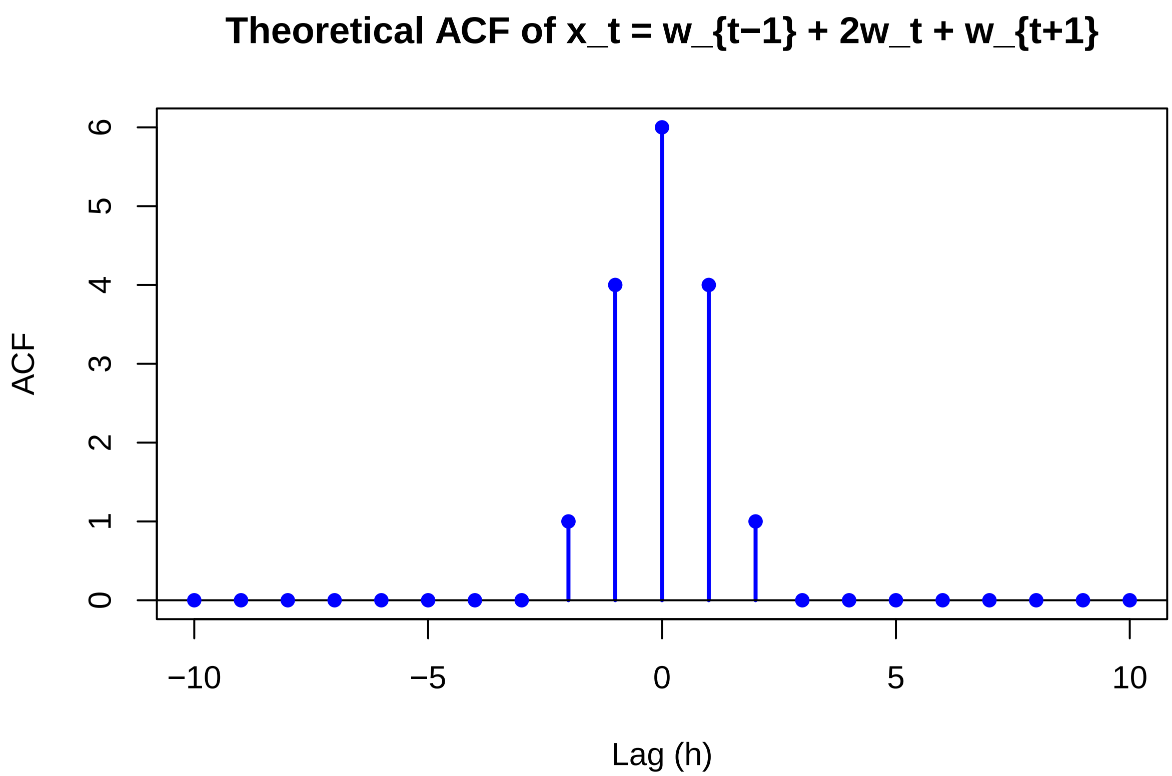
$$cov(v_s, v_t) = 0$$

Now plot it:

```r
lags <- -10:10

# Compute theoretical ACF values
acf_values <- sapply(lags, function(h) {
  if (h == 0) {
    return(6)
  } else if (abs(h) == 1) {
    return(4)
  } else if (abs(h) == 2) {
    return(1)
  } else {
    return(0)
  }
})

# Plot
plot(lags, acf_values, type="h", lwd=2, col="blue", ylim=c(0, 6),
     main="Theoretical ACF of x_t = w_{t-1} + 2w_t + w_{t+1}",
     xlab="Lag (h)", ylab="ACF")
abline(h=0, col="black")
points(lags, acf_values, pch=16, col="blue")
```



1.15

$$E[x_t] = E[w_t w_{t-1}]$$

Since white noise is independent:

$$E[x_t] = E[w_t]E[w_{t-1}]$$

Given the expectation of white noise:

$$E[x_t] = 0$$

Now we want to find:

$$Cov(x_t, x_{t+h})$$

When $h = 0$:

$$Cov(x_t, x_{t+h}) = Cov(w_t w_{t-1}, w_t w_{t-1})$$

$$Cov(x_t, x_{t+h}) = Var(w_t w_{t-1})$$

$$Cov(x_t, x_{t+h}) = E[(w_t w_{t-1})^2] - E[w_t w_{t-1}]^2$$

$$Cov(x_t, x_{t+h}) = E[w_t^2 w_{t-1}^2] - (E[w_t]E[w_{t-1}])^2$$

$$Cov(x_t, x_{t+h}) = E[w_t^2]E[w_{t-1}^2]$$

$$Cov(x_t, x_{t+h}) = (E[w_t^2] - E[w_t]^2)(E[w_{t-1}^2] - E[w_{t-1}]^2)$$

$$Cov(x_t, x_{t+h}) = Var[w_t]Var[w_{t-1}]$$

$$Cov(x_t, x_{t+h}) = \sigma_w^4$$

When $h = \pm 1$:

$$Cov(x_t, x_{t+h}) = Cov(w_t w_{t-1}, w_t w_{t+1})$$

$$Cov(x_t, x_{t+h}) = E[w_t w_{t-1} w_t w_{t+1}] - E[w_t w_{t-1}]E[w_t w_{t+1}]$$
$$Cov(x_t, x_{t+h}) = E[w_t^2]E[w_{t-1}]E[w_{t+1}] - E[w_t]^2 E[w_{t-1}]E[w_{t+1}]$$
$$Cov(x_t, x_{t+h}) = 0$$

Else:

$$Cov(x_t, x_{t+h}) = 0$$

The process is stationary because the mean and variance are constant, and the autocovariance only depends on h.