3. INDEFINITE INTEGRATION

Let us Study

- Definition and Properties
- Different Techniques : **1.** by substitution **2.** by parts **3.** by partial fraction

Introduction :

In differential calculus, we studied differentiation or derivatives of some functions. We saw that derivatives are used for finding the slopes of tangents, maximum or minimum values of the function.

Now we will try to find the function whose derivative is known, or given $f(x)$. We will find $g(x)$ such that $g'(x) = f(x)$. Here the integration of $f(x)$ with respect to x is $g(x)$ or $g(x)$ is called the primitive of $f(x)$. For example, we know that the derivative of x^3 *w. r. t.* x is $3x^2$. So $\frac{d}{dx}x^3 = 3x^2$; and integral of $3x^2$ *w. r. t.* x is x^3. This is shown with the sign of integration namely '$\int$'. We write $\int 3x^2\,dx = x^3$.

In this chapter we restrict ourselves only to study the methods of integration. The theory of integration is developed by Sir Isaac Newton and Gottfried Leibnitz.

$\int f(x)\,dx = g(x)$, read as an integral of $f(x)$ with respect to x, is $g(x)$. Since the derivative of constant function with respect to x is zero (0), we can also write

$\int f(x)\,dx = g(x) + c$, where c is an arbitary constant and c can take infinitely many values.

For example :

$f(x) = x^2 + c$ represents familly of curves for different values of c.

$f'(x) = 2x$ gives the slope of the tangent to $f(x) = x^2 + c$.

In the figure we have shown the curves

$y = x^2$, $y = x^2 + 4$, $y = x^2 - 5$.

Note that at the points (2, 4), (2, 8) (2, −1) respectivelly on those curves, the slopes of tangents are 2 (2) = 4.

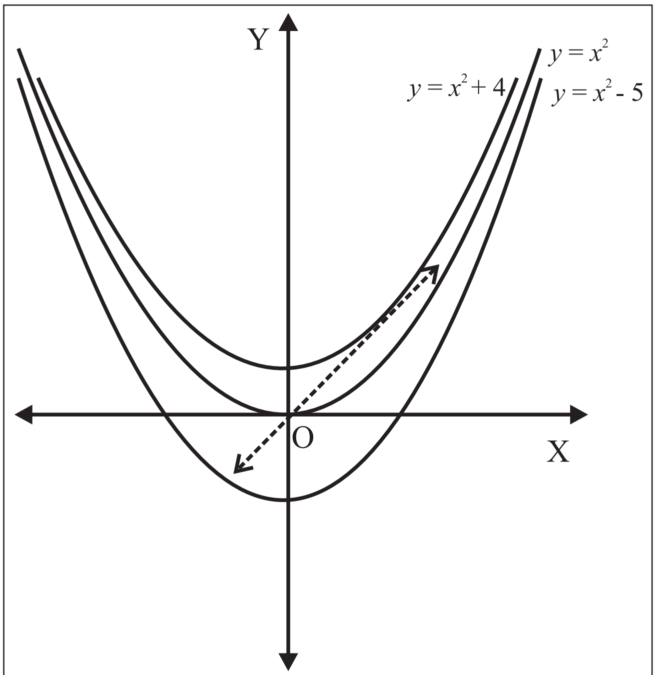

Fig. 3.1.1

3.1.1 Elementary Integration Formulae

(i) $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)\,x^n}{(n+1)}, \quad n \neq -1$$

$$= x^n \quad \Rightarrow \quad \therefore \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

$$\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{(n+1)\cdot a}\right) = \frac{(n+1)\,(ax+b)^n}{(n+1)}$$

$$= (ax+b)^n \quad \Rightarrow \quad \therefore \quad \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{n+1}\cdot\frac{1}{a} + c$$

This result can be extended for n replaced by any rational $\frac{p}{q}$.

(ii) $$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x,\ a > 0 \quad \Rightarrow \quad \therefore \quad \int a^x\,dx = \frac{a^x}{\log a} + c$$

$$\int A^{ax+b}\,dx = \frac{A^{ax+b}}{\log A}\cdot\frac{1}{a} + c,\ A > 0$$

(iii) $$\frac{d}{dx}e^x = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x + c$$

$$\int e^{ax+b}\,dx = e^{ax+b}\cdot\frac{1}{a} + c$$

(iv) $$\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + c$$

$$\int \cos(ax+b)\,dx = \sin(ax+b)\cdot\frac{1}{a} + c$$

(v) $$\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + c$$

$$\int \sin(ax+b)\,dx = -\cos(ax+b)\cdot\frac{1}{a} + c$$

(vi) $$\frac{d}{dx}\tan x = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\,dx = \tan x + c$$

$$\int \sec^2(ax+b)\,dx = \tan(ax+b)\cdot\frac{1}{a} + c$$

(vii) $$\frac{d}{dx}\sec x = \sec x \cdot \tan x \quad \Rightarrow \quad \int \sec x \cdot \tan x\,dx = \sec x + c$$

$$\int \sec(ax+b)\cdot\tan(ax+b)\,dx = \sec(ax+b)\cdot\frac{1}{a} + c$$

(viii) $$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \operatorname{cosec}(ax+b)\cdot\cot(ax+b)\,dx = -\operatorname{cosec}(ax+b)\cdot\frac{1}{a} + c$$

(ix) $$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\cot(ax+b)\cdot\frac{1}{a} + c$$

(x) $$\frac{d}{dx}\log x = \frac{1}{x},\ x > 0 \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \log x + c,\ x \neq 0.$$

$$\therefore \text{ also} \int \frac{1}{(ax+b)}\,dx = \log(ax+b)\ \frac{1}{a} + c$$

We assume that the trigonometric functions and logarithmic functions are defined on the respective domains.

3.1.2

Theorem 1 : If f and g are real valued integrable functions of x, then

$$\int [f(x) + g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

Theorem 2 : If f and g are real valued integrable functions of x, then

$$\int [f(x) - g(x)]\ dx = \int f(x)\ dx - \int g(x)\ dx$$

Theorem 3 : If f and g are real valued integrable functions of x, and k is constant, then

$$\int k\,[f(x)]\ dx = k \int f(x)\ dx$$

Proof : 1. Let $\int f(x)\ dx = g_1(x) + c_1$ and $\int g(x)\ dx = g_2(x) + c_2$ then

$\frac{d}{dx}[(g_1(x) + c_1)] = f(x)$ and $\frac{d}{dx}[(g_2(x) + c_2)] = g(x)$

$$\therefore \quad \frac{d}{dx}[(g_1(x) + c_1) + (g_2(x) + c_2)]$$

$$= \frac{d}{dx}[(g_1(x) + c_1)] + \frac{d}{dx}[(g_2(x) + c_2)]$$

$$= f(x) + g(x)$$

By definition of integration.

$$\int f(x) + g(x)\, dx = (g_1(x) + c_1) + (g_2(x) + c_2)$$

$$= \int f(x)\ dx + \int g(x)\ dx$$

Note : Students can construct the proofs of the other two theorems (Theorem 2 and Theorem 3).

SOLVED EXAMPLES

Ex. : Evaluate the following :

1. $\int (x^3 + 3^x)\ dx$

Solution : $\int (x^3 + 3^x)\ dx$

$$= \int x^3\, dx + \int 3^x\, dx$$

$$= \frac{x^4}{4} + \frac{3^x}{\log 3} + c$$

2. $\displaystyle\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)dx$

Solution : $\displaystyle\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)dx$

$$= \int \sin x \; dx + \int \frac{1}{x}\, dx + \int \frac{1}{\sqrt[3]{x}}\, dx$$

$$= \int \sin x \; dx + \int \frac{1}{x}\, dx + \int x^{-\frac{1}{3}}\, dx$$

$$= -\cos x + \log x + \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c$$

$$= -\cos x + \log x + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c$$

3. $\displaystyle\int (\tan x + \cot x)^2 \; dx$

Solution : $\displaystyle\int (\tan x + \cot x)^2 \; dx$

$$= \int (\tan^2 x + 2\tan x \cdot \cot x + \cot^2 x)\; dx$$

$$= \int (\tan^2 x + 2 + \cot^2 x)\; dx$$

$$= \int (\sec^2 x - 1 + 2 + \mathrm{cosec}^2 x - 1)\; dx$$

$$= \int (\sec^2 x + \mathrm{cosec}^2 x)\; dx$$

$$= \int \sec^2 x \; dx + \int \mathrm{cosec}^2 x \; dx$$

$$= \tan x + (-\cot x) + c$$

$$= \tan x - \cot x + c$$

4. $\displaystyle\int \frac{\sqrt{x}+1}{x+\sqrt{x}}\, dx$

Solution : $\displaystyle\int \frac{\sqrt{x}+1}{x+\sqrt{x}}\, dx$

$$= \int \frac{\sqrt{x}+1}{\sqrt{x}(\sqrt{x}+1)}\, dx$$

$$= \int \frac{1}{\sqrt{x}}\, dx$$

$$= 2\cdot\int \frac{1}{2\sqrt{x}}\, dx$$

$$= 2\sqrt{x} + c$$

5. $\displaystyle\int \frac{e^{4\log x} - e^{5\log x}}{x^5}\, dx$

Solution : $\displaystyle\int \frac{e^{4\log x} - e^{5\log x}}{x^5}\, dx$

$$= \int \frac{e^{\log x^4} - e^{\log x^5}}{x^5}\, dx, \qquad \because a^{\log_a f(x)} = f(x)$$

$$= \int \left(\frac{x^4 - x^5}{x^5}\right) dx$$

$$= \int \left(\frac{1}{x} - 1\right) dx$$

$$= \log(x) - x + c$$

6. $\displaystyle\int \frac{2x+3}{5x-1}\, dx$

Solution : $\dfrac{\mathrm{N}}{\mathrm{D}} = \mathrm{Q} + \dfrac{\mathrm{R}}{\mathrm{D}}$

$$\begin{array}{r|l} & \quad \frac{2}{5} \\ (5x-1) & \quad 2x+3 \\ & -\; 2x - \frac{2}{5} \\ & -\qquad + \\ \hline & \qquad 3+\frac{2}{5} = \frac{17}{5} \end{array}$$

$$\therefore \quad 2x+3 = \frac{2}{5}(5x-1) + 3 + \frac{2}{5}$$

$$\mathrm{I} = \int \left[\frac{2}{5} + \frac{\frac{17}{5}}{5x-1}\right] dx$$

$$= \frac{2}{5}x + \frac{17}{5}\log(5x-1)\cdot\frac{1}{5} + c$$

$$= \frac{2}{5}x + \frac{17}{25}\log(5x-1) + c$$

7. $$\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\,dx$$

Solution : $$\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\,dx$$

$$= \int \left(\frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\right)\cdot\left(\frac{\sqrt{3x+1}+\sqrt{3x-5}}{\sqrt{3x+1}+\sqrt{3x-5}}\right) dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{3x+1-3x+5}\,dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{6}\,dx$$

$$= \frac{1}{6}\cdot\int\left((3x+1)^{\frac{1}{2}}+(3x-5)^{\frac{1}{2}}\right) dx$$

$$= \frac{1}{6}\cdot\left\{\int (3x+1)^{\frac{1}{2}}\,dx + \int (3x-5)^{\frac{1}{2}}\,dx\right\}$$

$$= \frac{1}{6}\cdot\left\{\frac{(3x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3}+\frac{(3x-5)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3}\right\}+c$$

$$= \frac{1}{18}\cdot\left\{\frac{2}{3}(3x+1)^{\frac{3}{2}}+\frac{2}{3}(3x-5)^{\frac{3}{2}}\right\}+c$$

$$= \frac{1}{27}\cdot\left\{(3x+1)^{\frac{3}{2}}+(3x-5)^{\frac{3}{2}}\right\}+c$$

8. $$\int \frac{2x-7}{\sqrt{3x-2}}\,dx$$

Solution : Express $(2x-7)$ in terms of $(3x-2)$

$$2x-7 = \frac{2}{3}(3x-2)+\frac{4}{3}-7$$

$$= \frac{2}{3}(3x-2)-\frac{17}{3}$$

$$I = \int\left[\frac{\frac{2}{3}(3x-2)-\frac{17}{3}}{\sqrt{3x-2}}\right] dx$$

$$= \int\left[\frac{\frac{2}{3}(3x-2)}{\sqrt{3x-2}}-\frac{\frac{17}{3}}{\sqrt{3x-2}}\right] dx$$

$$= \frac{2}{3}\int \sqrt{3x-2}\,dx - \frac{17}{3}\int\frac{1}{\sqrt{3x-2}}\,dx$$

$$= \frac{2}{3}\int (3x-2)^{\frac{1}{2}}\,dx - \frac{17}{3}\int\frac{1}{\sqrt{3x-2}}\,dx$$

$$= \frac{2}{3}\cdot\frac{(3x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\cdot\frac{1}{3}-\frac{17}{3}\cdot 2\cdot\left(\sqrt{3x-2}\right)\cdot\frac{1}{3}+c$$

$$= \frac{4}{27}\cdot(3x-2)^{\frac{3}{2}}-\frac{34}{9}\cdot(3x-2)^{\frac{1}{2}}+c$$

9. $$\int \frac{x^3}{x-1}\,dx$$

Solution :

$$I = \int \frac{x^3-1+1}{x-1}\,dx$$

$$= \int\left(\frac{x^3-1}{x-1}+\frac{1}{x-1}\right) dx$$

$$= \int\left(\frac{(x-1)(x^2+x+1)}{(x-1)}+\frac{1}{x-1}\right) dx$$

$$= \int\left(x^2+x+1+\frac{1}{x-1}\right) dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+x+\log(x-1)+c$$

10. $$\int \frac{3^x-4^x}{5^x}\,dx$$

Solution :

$$I = \int\left(\frac{3^x}{5^x}-\frac{4^x}{5^x}\right) dx$$

$$= \int\left[\left(\frac{3}{5}\right)^x-\left(\frac{4}{5}\right)^x\right] dx$$

$$= \frac{\left(\frac{3}{5}\right)^x}{\log\frac{3}{5}}-\frac{\left(\frac{4}{5}\right)^x}{\log\frac{4}{5}}+c$$

11. $\int \cos^3 x \; dx$

Solution : $\cos 3A = 4\cos^3 A - 3\cos A$

$$I = \int \frac{1}{4}(\cos 3x + 3\cos x) \; dx$$
$$= \frac{1}{4}\left(\sin 3x \cdot \frac{1}{3} + 3\cdot\sin x\right) + c$$
$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$$

12. $\int \sqrt{1 + \sin 3x} \; dx$

Solution :

$$I = \int \sqrt{\cos^2\frac{3x}{2} + \sin^2\frac{3x}{2} + 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2}} \; dx$$
$$= \int \sqrt{\left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right)^2} \; dx$$
$$= \int \left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right) \; dx$$
$$= \sin\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} - \cos\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} + c$$
$$= \frac{2}{3}\left(\sin\frac{3x}{2} - \cos\frac{3x}{2}\right) + c$$

13. $\int \sin^4 x \; dx$

Solution :

$$I = \int (\sin^2 x)^2 \; dx$$
$$= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \; dx$$
$$= \frac{1}{4}\cdot\int (1 - 2\cos 2x + \cos^2 2x) \; dx$$
$$= \frac{1}{4}\cdot\int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] \; dx$$
$$= \frac{1}{4}\cdot\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) \; dx$$
$$= \frac{1}{4}\cdot\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) \; dx$$
$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - 2\sin 2x\cdot\frac{1}{2} + \frac{1}{2}\sin 4x\cdot\frac{1}{4}\right] + c$$
$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + c$$

14. $\int \sin 5x \cdot \cos 7x \; dx$

Solution : We know that

$$2\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)$$

$$I = \frac{1}{2}\int 2\sin 5x\cdot\cos 7x \; dx$$
$$= \frac{1}{2}\int [\sin(5x+7x) + \sin(5x-7x)] \; dx$$
$$= \frac{1}{2}\int [\sin(12x) + \sin(-2x)] \; dx$$
$$= \frac{1}{2}\int (\sin 12x - \sin 2x) \; dx$$
$$= \frac{1}{2}\cdot\left[-\cos 12x\cdot\frac{1}{12} + \cos 2x\cdot\frac{1}{2}\right] + c$$
$$I = -\frac{1}{24}\cos 12x + \frac{1}{4}\cos 2x + c$$

15. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x\cdot\cos^2 x} \; dx$

Solution : $I = \int \left(\frac{\sin^3 x}{\sin^2 x\cdot\cos^2 x} - \frac{\cos^3 x}{\sin^2 x\cdot\cos^2 x}\right) \; dx$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right) \; dx$$
$$= \int \left(\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\right) \; dx$$
$$= \int (\sec x\cdot\tan x - \operatorname{cosec} x\cdot\cot x) \; dx$$
$$= \sec x - (-\operatorname{cosec} x) + c$$
$$I = \sec x + \operatorname{cosec} x + c$$

16. $\displaystyle\int \frac{1}{1 - \sin x}\ dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \left(\frac{1}{1 - \sin x}\right)\left(\frac{1 + \sin x}{1 + \sin x}\right) dx \\
&= \int \frac{1 + \sin x}{1 - \sin^2 x}\ dx \\
&= \int \frac{1 + \sin x}{\cos^2 x}\ dx \\
&= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx \\
&= \int (\sec^2 x + \sec x \cdot \tan x)\ dx \\
&= \tan x + \sec x + c
\end{aligned}$$

17. $\displaystyle\int \left(\frac{\cos x}{1 - \cos x}\right) dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \left(\frac{\cos x}{1 - \cos x}\right)\left(\frac{1 + \cos x}{1 + \cos x}\right) dx \\
&= \int \frac{\cos x\,(1 + \cos x)}{1 - \cos^2 x}\ dx \\
&= \int \left(\frac{\cos x + \cos^2 x)}{\sin^2 x}\right) dx \\
&= \int \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) dx \\
&= \int (\operatorname{cosec} x \cdot \cot x + \cot^2 x)\ dx \\
&= \int (\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x - 1)\ dx \\
&= (-\operatorname{cosec} x) + (-\cot x) - x + c \\
&= -\operatorname{cosec} x - \cot x - x + c
\end{aligned}$$

Activity :

18. $\displaystyle\int \frac{\cos x - \cos 2x}{1 - \cos x}\ dx$

Solution :

$$\begin{aligned}
&\int \frac{\cos x - \cos 2x}{1 - \cos x}\ dx \\
&= \int \frac{\cos x - (\ldots\ldots\ldots\ldots)}{1 - \cos x}\ dx \\
&= \int \frac{\cos x - \ \ldots\ldots\ldots\ldots}{1 - \cos x}\ dx \\
&= \int \frac{\cos x\,(\,1 - \cos x) + \ \ldots\ldots\ldots\ldots}{1 - \cos x}\ dx \\
&= \int \left[\cos x + \frac{\ldots\ldots\ldots\ldots}{1 - \cos x}\right] dx \\
&= \int [\cos x + (1 + \cos x)]\ dx \\
&= \int (1 + 2\cos x)\ dx \\
&= x + 2\sin x + c
\end{aligned}$$

19. $\int \sin^{-1}(\cos 3x) \; dx$

Solution :

$$I = \int \sin^{-1}\left(\sin \frac{\pi}{2} - 3x\right) dx$$

$$= \int \left(\frac{\pi}{2} - 3x\right) dx$$

$$= \frac{\pi}{2} x - 3\frac{x^2}{2} + c$$

20. $\int \tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right) dx$

Solution :

$$I = \int \cot^{-1}\left(\frac{1 + \cos 2x}{\sin 2x}\right) dx$$

$$= \int \cot^{-1}\left(\frac{2\cos^2 x}{2\sin x \cdot \cos x}\right) dx$$

$$= \int \cot^{-1}(\cot x) \; dx$$

$$= \int x \; dx = \frac{x^2}{2} + c$$

21. $\int \tan^{-1}\sqrt{\frac{1 - \sin x}{1 + \sin x}} \; dx$

Solution :

$$I = \int \tan^{-1}\sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}} \; dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}} \; dx$$

$$= \int \tan^{-1}\sqrt{\tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} \; dx$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] dx$$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{\pi}{4} x - \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$= \frac{\pi}{4} x - \frac{x^2}{4} + c$$

EXERCISE 3.1

I. Integrate the following functions *w. r. t. x* :

(i) $x^3 + x^2 - x + 1$ (ii) $x^2\left(1 - \frac{2}{x}\right)^2$

(iii) $3\sec^2 x - \frac{4}{x} + \frac{1}{x\sqrt{x}} - 7$

(iv) $2x^3 - 5x + \frac{3}{x} + \frac{4}{x^5}$ (v) $\frac{3x^3 - 2x + 5}{x\sqrt{x}}$

II. Evaluate :

(i) $\int \tan^2 x \; dx$ (ii) $\int \frac{\sin 2x}{\cos x} dx$

(iii) $\int \frac{\sin x}{\cos^2 x} dx$ (iv) $\int \frac{\cos 2x}{\sin^2 x} dx$

(v) $\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx$ (vi) $\int \frac{\sin x}{1 + \sin x} dx$

(vii) $\int \frac{\tan x}{\sec x + \tan x} dx$ (viii) $\int \sqrt{1 + \sin 2x} \; dx$

(ix) $\int \sqrt{1 - \cos 2x} \; dx$ (x) $\int \sin 4x \cdot \cos 3x \; dx$

III. Evaluate :

(i) $\int \frac{x}{x + 2} dx$ (ii) $\int \frac{4x + 3}{2x + 1} dx$

(iii) $\int \frac{5x + 2}{3x - 4} dx$ (iv) $\int \frac{x - 2}{\sqrt{x + 5}} dx$

(v) $\int \frac{2x - 7}{\sqrt{4x - 1}} dx$ (vi) $\int \frac{\sin 4x}{\cos 2x} dx$

(vii) $\int \sqrt{1 + \sin 5x} \; dx$ (viii) $\int \cos^2 x \; dx$

(ix) $\int \frac{2}{\sqrt{x} - \sqrt{x + 3}} dx$

(x) $\int \frac{3}{\sqrt{7x - 2} - \sqrt{7x - 5}} dx$

IV. $f'(x) = x - \frac{3}{x^3}$, $f(1) = \frac{11}{2}$ then find $f(x)$.

3.2 Methods of integration :

We have evaluated the integrals which can be reduced to standard forms by algebric or trigonometric simplifications. This year we are going to study three special methods of reducing an integral to a standard form, namely –

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction

3.2.1 Integration by substitution :

Theorem 1 : If $x = \phi(t)$ is a differentiable function of t, then $\int f(x)\ dx = \int f[\phi(t)]\ \phi'(t)\, dt$.

Proof : $x = \phi(t)$ is a differentiable function of t.

$$\therefore \quad \frac{dx}{dt} = \phi'(t)$$

Let $\int f(x)\, dx = g(x) \Rightarrow \frac{d}{dx}[g(x)] = f(x)$

By Chain rule,

$$\frac{d}{dt}[g(x)] = \frac{d}{dx}[g(x)]\ \frac{dx}{dt}$$

$$= f(x)\ \frac{dx}{dt}$$

$$= f[\phi(t)]\ \ \phi'(t)$$

By definition of integration,

$$g(x) = \int f[\phi(t)]\ \ \phi'(t)\ \ dt$$

$$\therefore \quad \int f(x)\ \ dx = \int f[\phi(t)]\ \ \phi'(t)\ \ dt$$

For example 1 : $\int 3x^2 \sin(x^3)\ \ dx$

Let $x^3 = t$

$\therefore \quad 3x^2\, dx = dt$

$= \int \sin t\ \ dt$

$= -\cos t + c$

$= -\cos(x^3) + c$

Corollary I :

If $\int f(x) \, dx = g(x) + c$

then $\int f(ax+b) \, dx = g(ax+b) \dfrac{1}{a} + c$

Proof : Let $I = \int f(ax+b) \, dx$

put $ax + b = t$

Differentiating both the sides

$$a \, dx = 1 \, dt \Rightarrow dx = \frac{1}{a} dt$$

$$\begin{aligned} I &= \int f(t) \, \frac{1}{a} \, dt \\ &= \frac{1}{a} \cdot \int f(t) \, dt \\ &= \frac{1}{a} \cdot g(t) + c \\ &= \frac{1}{a} \cdot g(ax+b) + c \end{aligned}$$

$$\therefore \quad \int f(ax+b) \, dx = g(ax+b) \frac{1}{a} + c$$

For example : $\int \sec^2 (5x - 4) \, dx$

$$= \frac{1}{5} \tan (5x - 4) + c$$

Corollary II :

$$\int [f(x)]^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c, \; n \neq -1$$

Proof : Let $I = \int [f(x)]^{n+1} \cdot f'(x) \, dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \, dx = dt$

$$\begin{aligned} I &= \int [t]^n \, dt \\ &= \frac{t^{n+1}}{n+1} + c \;, \qquad n \neq -1 \\ &= \frac{[f(x)]^{n+1}}{n+1} + c \end{aligned}$$

$$\therefore \quad \int [f(x)]^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

For example : $\displaystyle\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} \, dx$

$$= \int [(\sin^{-1} x)^3] \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{(\sin^{-1} x)^4}{4} + c$$

Corollary III :

$$\int \frac{f'(x)}{f(x)} dx = \log (f(x)) + c$$

Proof : Consider $\displaystyle\int \frac{f'(x)}{f(x)} dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \, dx = dt$

$$\begin{aligned} I &= \int \frac{1}{t} \, dt \\ &= \log (t) + c \\ &= \log \big(f(x) \big) + c \end{aligned}$$

$$\therefore \quad \int \frac{f'(x)}{f(x)} dx = \log \big(f(x) \big) + c$$

For example : $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} \, dx$$

$$\frac{d}{dx} \sin x = \cos x$$

$$= \int \left(\frac{\frac{d}{dx} \sin x}{\sin x} \, dx \right)$$

$$= \log (\sin x) + c$$

Corollary IV :

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

Proof : Consider $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx$

put $f(x) = t$

Differentiating both the sides

$f'(x)\;dx = dt$

$$I = \int \frac{1}{\sqrt{t}}\;dt$$

$$= 2\cdot\int \frac{1}{2\sqrt{t}}\;dt$$

$$= 2\sqrt{t} + c$$

$$= 2\sqrt{f(x)} + c$$

$$\therefore \quad \int \frac{f'(x)}{\sqrt{f(x)}}\;dx = 2\sqrt{f(x)} + c$$

For example : $\int \frac{1}{x\sqrt{\log x}}\;dx$

$$= \int\left(\frac{\frac{1}{x}}{\sqrt{\log x}}\right) dx$$

$$= \int\left(\frac{\frac{d}{dx}\log x}{\sqrt{\log x}}\;dx\right)$$

$$= 2\sqrt{\log x} + c$$

Using corollary III, $\int \frac{f'(x)}{f(x)}\;dx = \log\left(f(x)\right) + c$ we find the integrals of some trigonometric functions.

3.2.2 Integrals of trignometric functions :

1. $\int \tan x\;\;dx$

Solution :

$$I = \int \tan x\;\;dx$$

$$= \int \frac{\sin x}{\cos x}\;dx$$

$$= -\int \frac{-\sin x}{\cos x}\;dx$$

$$= -\log(\cos x) + c$$

$$= \log(\sec x) + c$$

Activity :

2. $\int \cot(5x - 4)\;\;dx$

Solution :

$$I = \int \frac{\ldots\ldots\ldots\ldots}{\sin(5x-4)}\;dx$$

$$= \frac{1}{\ldots\ldots}\int \frac{5\cos(5x-4)}{\ldots\ldots\ldots\ldots}\;dx$$

$$\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\right) = \ldots\ldots\ldots\ldots$$

$$= \frac{1}{5}\log[\sin(5x-4)] + c$$

3. $\int \sec x \; dx = \log(\sec x + \tan x) + c$

Solution : Let $I = \int \sec x \; dx$

$$= \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x} \; dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \; dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} \; dx$$

$\because \; \frac{d}{dx}(\sec x + \tan x) = \sec x \cdot \tan x + \sec^2 x$

$\therefore \; \int \sec x \; dx = \log(\sec x + \tan x) + c$

Also,

$$\int \sec x \; dx = \log\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right] + c$$

Activity :

4. $\int \operatorname{cosec} x \; dx = \log(\operatorname{cosec} x - \cot x) + c$

Solution : Let $I = \int \operatorname{cosec} x \; dx$

$$= \int \frac{(\operatorname{cosec} x)(\ldots\ldots\ldots\ldots)}{(\ldots\ldots\ldots\ldots)} \; dx$$

$$= \int \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots} \; dx$$

$$= \int \frac{-\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x}{\ldots\ldots\ldots} \; dx$$

$\frac{d}{dx}(\operatorname{cosec} x - \cot x)$

$= \; \ldots\ldots\ldots\ldots\ldots\ldots$

$= \; \log(\operatorname{cosec} x - \cot x) + c$

$\therefore \; \int \operatorname{cosec} x \; dx = \log(\operatorname{cosec} x - \cot x) + c$

Also,

$$\int \operatorname{cosec} x \; dx = \log\left(\tan\frac{x}{2}\right) + c$$

SOLVED EXAMPLES

Ex. : Evaluate the following functions :

1. $\int \frac{\cot(\log x)}{x} \; dx$

Solution : Let $I = \int \frac{\cot(\log x)}{x} \; dx$

put $\log x = t$

$\therefore \; \frac{1}{x} \; dx = 1 \; dt$

$= \int \cot t \; dt$

$= \log(\sin t) + c$

$= \log(\sin \log x) + c$

2. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \; dx$

Solution : Let $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}} \; dx$

put $\sqrt{x} = t$

$\therefore \; \frac{1}{2\sqrt{x}} \; dx = 1 \; dt$

$\therefore \; \frac{1}{\sqrt{x}} \; dx = 2 \; dt$

$= 2 \cdot \int \cos t \; dt$

$= 2 \cdot \sin t + c$

$= 2 \cdot \sin\sqrt{x} + c$

3. $\displaystyle\int \frac{\sec^8 x}{\operatorname{cosec} x}\ dx$

Solution : $\displaystyle I = \int \sec^7 x \cdot \sec x \cdot \frac{1}{\operatorname{cosec} x}\ dx$

$\displaystyle = \int \sec^7 x \cdot \frac{1}{\cos x} \cdot \sin x\ \ dx$

$= \int \sec^7 x \cdot \tan x\ \ dx$

$= \int \sec^6 x \cdot \sec x \cdot \tan x\ \ dx$

put $\sec x = t$

$\therefore\ \ \sec x \cdot \tan x\ \ dx = dt$

$= \int t^6\ \ dt$

$\displaystyle = \frac{t^7}{7} + c$

$\displaystyle = \frac{\sec^7 x}{7} + c$

4. $\displaystyle\int \frac{1}{x + \sqrt{x}}\ dx$

Solution : $\displaystyle I = \int \frac{1}{x + \sqrt{x}}\ dx$

$\displaystyle = \int \frac{1}{\sqrt{x}\,(\sqrt{x} + 1)}\ dx$

put $\sqrt{x} + 1 = t$

$\displaystyle \therefore\ \ \frac{1}{2\sqrt{x}}\ dx = 1\ \ dt$

$\displaystyle \therefore\ \ \frac{1}{\sqrt{x}}\ dx = 2\ \ dt$

$\displaystyle = \int \frac{1}{t} \cdot 2\ \ dt$

$\displaystyle = 2 \cdot \int \frac{1}{t}\ \ dt$

$= 2 \cdot \log(t) + c$

$= 2 \cdot \log(\sqrt{x} + 1) + c$

5. $\int 5^{5^x} \cdot 5^x\ \ dx$

Solution : $I = \int 5^{5^x} \cdot 5^x\ \ dx$

put $5^x = t$

$\therefore\ \ 5^x \cdot \log 5\ \ dx = 1\ dt$

$\displaystyle 5^x\ \ dx = \frac{1}{\log 5}\ \ dt$

$\displaystyle = \int 5^t \cdot \frac{1}{\log 5}\ \ dt$

$\displaystyle I = \frac{1}{\log 5} \cdot \int 5^t\ \ dt$

$\displaystyle = \frac{1}{\log 5} \cdot 5^t \cdot \frac{1}{\log 5} + c$

$\displaystyle = \left(\frac{1}{\log 5}\right)^2 \cdot 5^{5^x} + c$

6. $\displaystyle\int \frac{1}{1 + e^{-x}}\ dx$

Solution : $\displaystyle I = \int \frac{1}{1 + e^{-x}}\ dx$

$\displaystyle = \int \frac{1}{1 + \frac{1}{e^x}}\ dx$

$\displaystyle = \int \frac{1}{\frac{e^x + 1}{e^x}}\ dx$

$\displaystyle = \int \frac{e^x}{e^x + 1}\ dx$

$\displaystyle \because\ \ \frac{d}{dx}\left(e^x + 1\right) = e^x$

$= \log[\,e^x + 1\,] + c$

7. $\displaystyle\int \frac{e^x(1 + x)}{\cos(x \cdot e^x)}\ dx$

Solution : put $x \cdot e^x = t$

Differentiating both sides

$(x \cdot e^x + e^x \cdot 1)\ \ dx = 1\ dt$

$e^x(1 + x)\ \ dx = 1\ dt$

$\displaystyle I = \int \frac{1}{\cos t}\ dt$

$= \int \sec t\ \ dt$

$= \log(\sec t + \tan t) + c$

$= \log(\sec(xe^x) + \tan(xe^x)) + c$

8. $\displaystyle\int \frac{1}{3x + 7x^{-n}}\, dx$

Solution : Consider $\displaystyle\int \frac{1}{3x + 7x^{-n}}\, dx$

$$= \int \frac{1}{3x + \frac{7}{x^n}}\, dx = \int \frac{1}{\frac{3x^{n+1} + 7}{x^n}}\, dx$$

$$= \int \frac{x^n}{3x^{n+1} + 7}\, dx$$

put $3x^{n+1} + 7 = t$

Differentiate *w. r. t. x*

$3(n + 1)\, x^n \quad dx = dt$

$$\therefore \quad x^n \quad dx = \frac{1}{3(n + 1)}\, dt$$

$$= \int \frac{\frac{1}{3(n+1)}}{t}\, dt$$

$$= \frac{1}{3(n + 1)} \cdot \log(t) + c$$

$$= \frac{1}{3(n + 1)} \cdot \log\left(3x^{n+1} + 7\right) + c$$

9. $\displaystyle\int (3x + 2)\sqrt{x - 4}\quad dx$

Solution : put $x - 4 = t$

$\therefore \quad x = 4 + t$

Differentiate

$1 \quad dx = 1 \quad dt$

$$= \int [\, 3(4 + t) + 2\,] \cdot \sqrt{t}\ dt$$

$$= \int (14 + 3t)\ t^{\frac{1}{2}}\quad dt$$

$$= \int \left(14t^{\frac{1}{2}} + 3t^{\frac{3}{2}}\right)\quad dt$$

$$= 14\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 3\,\frac{t^{\frac{5}{2}}}{\frac{5}{2}}$$

$$= \frac{28}{3}(x - 4)^{\frac{3}{2}} + \frac{6}{5}(x - 4)^{\frac{5}{2}} + c$$

10. $\displaystyle\int \frac{\sin(x + a)}{\cos(x - b)}\quad dx$

Solution :

$$= \int \frac{\sin[(x - b) + (a + b)]}{\cos(x - b)}\quad dx$$

$$= \int \frac{\sin(x - b)\cdot\cos(a + b) + \cos(x - b)\cdot\sin(a + b)}{\cos(x - b)}\quad dx$$

$$= \int \left[\frac{\sin(x - b)\cdot\cos(a + b)}{\cos(x - b)} + \frac{\cos(x - b)\cdot\sin(a + b)}{\cos(x - b)}\right] dx$$

$$= \int [\cos(a + b)\cdot\tan(x - b) + \sin(a + b)]\quad dx$$

$$= \cos(a + b)\cdot\log\left(\sec(x - b)\right) + x\cdot\sin(a + b) + c$$

11. $\displaystyle\int \frac{e^x + 1}{e^x - 1}\, dx$

Solution :

$$I = \int \frac{e^x - 1 + 2}{e^x - 1}\, dx$$

$$= \int \left(\frac{e^x - 1}{e^x - 1} + \frac{2}{e^x - 1} \right) dx$$

$$= \int \left(1 + \frac{2}{e^x - 1} \right) dx$$

$$= \int dx + \int \frac{2}{e^x (1 - e^{-x})}\, dx$$

$$= \int 1 dx + 2 \int \frac{e^{-x}}{1 - e^{-x}}\, dx$$

put $(1 - e^{-x}) = t$

Differentiate $w.\, r.\, t.\, x$

$-(e^{-x})\,(-1) \quad dx = 1\, dt$

$e^{-x} \quad dx = 1\, dt$

$$I = \int 1 dx + 2 \int \frac{1}{t}\, dt$$

$$= x + 2 \cdot \log(t) + c$$

$$= x + 2 \log(1 - e^{-x}) + c$$

$$\therefore \int \frac{e^x + 1}{e^x - 1}\, dx = x + 2 \log(1 - e^{-x}) + c$$

12. $\displaystyle\int \frac{1}{1 - \tan x}\, dx$

Solution :

$$I = \int \frac{1}{1 - \frac{\sin x}{\cos x}}\, dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\, dx$$

$$= \int \frac{\cos x}{\sqrt{2}\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \cos x \right)}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos x}{\cos \frac{\pi}{4} \cos x - \sin \frac{\pi}{4} \sin x}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos x}{\cos\left(x + \frac{\pi}{4} \right)}\, dx$$

put $x + \frac{\pi}{4} = t \quad \therefore x = t - \frac{\pi}{4}$

Differentiating both sides

$1\, dx = 1\, dt$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos\left(t - \frac{\pi}{4} \right)}{\cos t}\, dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos t \cdot \cos \frac{\pi}{4} + \sin t \cdot \sin \frac{\pi}{4}}{\cos t}\, dt$$

$$= \frac{1}{\sqrt{2}} \int \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \tan t \right] dt$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left[t + \log(\sec t) \right] + c$$

$$= \frac{1}{2} \left[x + \frac{\pi}{4} + \log \sec\left(x + \frac{\pi}{4} \right) \right] + c$$

To evaluate the integrals of type $\displaystyle\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x}\, dx$, express the Numerator as N $= \lambda$ (D) + μ (D)' , find the constants λ & μ by comparing the coefficients of like terms and then integrate the function.

EXERCISE 3.2 (A)

I. Integrate the following functions w. r. t. x :

1. $\dfrac{(\log x)^n}{x}$
2. $\dfrac{(\sin^{-1} x)^{\frac{3}{2}}}{\sqrt{1-x^2}}$
3. $\dfrac{1+x}{x \cdot \sin(x+\log x)}$
4. $\dfrac{x \cdot \sec^2(x^2)}{\sqrt{\tan^3(x^2)}}$
5. $\dfrac{e^{3x}}{e^{3x}+1}$
6. $\dfrac{(x^2+2)}{(x^2+1)} \cdot a^{x+\tan^{-1} x}$
7. $\dfrac{e^x \cdot \log(\sin e^x)}{\tan(e^x)}$
8. $\dfrac{e^{2x}+1}{e^{2x}-1}$
9. $\sin^4 x \cdot \cos^3 x$
10. $\dfrac{1}{4x+5x^{-11}}$
11. $x^9 \cdot \sec^2(x^{10})$
12. $e^{3\log x} \cdot (x^4+1)^{-1}$
13. $\dfrac{\sqrt{\tan x}}{\sin x \cdot \cos x}$
14. $\dfrac{(x-1)^2}{(x^2+1)^2}$
15. $\dfrac{2\sin x \cdot \cos x}{3\cos^2 x + 4\sin^2 x}$
16. $\dfrac{1}{\sqrt{x}+\sqrt{x^3}}$
17. $\dfrac{10x^9 + 10^x \cdot \log 10}{10^x + x^{10}}$
18. $\dfrac{x^{n-1}}{\sqrt{1+4x^n}}$
19. $(2x+1)\sqrt{x+2}$
20. $x^5 \cdot \sqrt{a^2+x^2}$
21. $(5-3x)(2-3x)^{-\frac{1}{2}}$
22. $\dfrac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}$
23. $\dfrac{x^2}{\sqrt{9-x^6}}$
24. $\dfrac{1}{x(x^3-1)}$
25. $\dfrac{1}{x \cdot \log x \cdot \log(\log x)}$

II. Integrate the following functions w. r. t. x :

1. $\dfrac{\cos 3x - \cos 4x}{\sin 3x + \sin 4x}$
2. $\dfrac{\cos x}{\sin(x-a)}$
3. $\dfrac{\sin(x-a)}{\cos(x+b)}$
4. $\dfrac{1}{\sin x \cdot \cos x + 2\cos^2 x}$
5. $\dfrac{\sin x + 2\cos x}{3\sin x + 4\cos x}$
6. $\dfrac{1}{2+3\tan x}$
7. $\dfrac{4e^x - 25}{2e^x - 5}$
8. $\dfrac{20+12e^x}{3e^x+4}$
9. $\dfrac{3e^{2x}+5}{4e^{2x}-5}$
10. $\cos^8 x \cdot \cot x$
11. $\tan^5 x$
12. $\cos^7 x$
13. $\tan 3x \cdot \tan 2x \cdot \tan x$
14. $\sin^5 x \cdot \cos^8 x$
15. $3^{\cos^2 x} \cdot \sin 2x$
16. $\dfrac{\sin 6x}{\sin 10x \cdot \sin 4x}$
17. $\dfrac{\sin x \cdot \cos^3 x}{1+\cos^2 x}$

3.2.3 Some Special Integrals

1. $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
2. $\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$
3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$
4. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
5. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right) + c$
6. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left(x+\sqrt{x^2+a^2}\right) + c$
7. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

While evaluating an integral there is no unique substitution, we can use some standard substitutions and try.

No.	Function	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ ($x = a\cos\theta$ can also be used.)
2.	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$
4.	$\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos 2\theta$

1. $$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Proof :

Let $\quad I = \displaystyle\int \frac{1}{x^2+a^2}\, dx$

put $x = a\cdot\tan\theta \Rightarrow \tan\theta = \dfrac{x}{a}$

i.e. $\theta = \tan^{-1}\left(\dfrac{x}{a}\right)$

$\therefore dx = a\cdot\sec^2\theta \; d\theta$

$$I = \int \frac{1}{a^2\cdot\tan^2\theta + a^2}\cdot a\cdot\sec^2\theta \; d\theta$$

$$= \int \frac{a\cdot\sec^2\theta}{a^2(\tan^2\theta + 1)}\, d\theta$$

$$= \int \frac{\sec^2\theta}{a\cdot\sec^2\theta}\, d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $$\int \frac{1}{x^2+5^2}\, dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$$

Alternatively

Cosider,

$$\frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right] + \frac{d}{dx}c$$

$$= \frac{1}{a}\,\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{a}\right) + 0$$

$$= \frac{1}{a}\,\frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{1}{a^2}\,\frac{1}{\frac{a^2+x^2}{a^2}}$$

$$= \frac{1}{x^2+a^2}$$

Therefore,

by definition of integration

$$\therefore \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

2. $\displaystyle\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{x^2 - a^2}\, dx$

$\displaystyle = \int \frac{1}{(x+a)(x-a)}\, dx$

$\displaystyle = \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$

$\displaystyle = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$

$\displaystyle = \frac{1}{2a}\,[\log(x-a) - \log(x+a)] + c$

$\displaystyle = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

e.g. $\displaystyle \int \frac{1}{x^2 - 9}\, dx = \frac{1}{2(3)}\log\left(\frac{x-3}{x+3}\right) + c$

Activity :

3. $\displaystyle\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

Proof : Consider,

$\displaystyle I = \int \frac{1}{a^2 - x^2}\, dx$

$\displaystyle = \int \frac{1}{(\ldots.)(\ldots.)}\, dx$

$\displaystyle = \int \frac{1}{2a}\left[\frac{1}{\ldots.} + \frac{1}{\ldots.}\right] dx$

$\displaystyle = \frac{1}{2a}\int\left[\frac{\ldots.}{\ldots.} + \frac{1}{a+x}\right] dx$

$\displaystyle = \frac{1}{2a}\,[\log(a+x) - \log(a-x)] + c$

$\displaystyle = \frac{1}{2a}\log\left(\frac{\ldots.}{\ldots.}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

e.g. $\displaystyle \int \frac{1}{16 - x^2}\, dx = \frac{1}{2(4)}\log\left(\frac{4+x}{4-x}\right) + c$

4. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{\sqrt{a^2 - x^2}}\, dx$

put $x = a\sin\theta \Rightarrow \sin\theta = \dfrac{x}{a}$

$\displaystyle \therefore \quad \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a\cos\theta\, d\theta$

$\displaystyle I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$

$\displaystyle I = \int \frac{a\cdot\cos\theta}{a\sqrt{1-\sin^2\theta}}\, d\theta$

$\displaystyle = \int \frac{\cos\theta}{\cos\theta}\, d\theta$

$\displaystyle = \int 1\, d\theta$

$= \theta + c$

$\displaystyle \therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

e.g. $\displaystyle \int \frac{1}{\sqrt{81 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{9}\right) + c$

5. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

Proof : Let $\displaystyle I = \int \frac{1}{\sqrt{x^2-a^2}}\,dx$

put $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore$ $dx = a\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}}\cdot a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}\, d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2\cdot\tan^2\theta}}\, d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{a\cdot\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \log(\sec\theta+\tan\theta)+c$$

$$= \log(\sec\theta+\sqrt{\sec^2\theta-1})+c$$

$$= \log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)+c_1$$

$$= \log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)+c_1$$

$$= \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+c_1$$

$$= \log\left(x+\sqrt{x^2-a^2}\right)-\log a+c_1$$

$$= \log\left(x+\sqrt{x^2-a^2}\right)+c$$

where $c = c_1 - \log a$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$$

e.g. $\displaystyle\int \frac{1}{\sqrt{x^2-16}}\,dx = \log\left(x+\sqrt{x^2-16}\right)+c$

Activity :

6. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left(x+\sqrt{a^2+x^2}\right)+c$

Proof : use substitution $x = a \tan\theta$

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Activity :

7. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$

Proof : Let $\displaystyle I = \int \frac{1}{x\sqrt{x^2-a^2}}\,dx$

put $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore$ $dx = a\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{a\sec\theta\tan\theta\, dx}{a\sec\theta\sqrt{\ldots\ldots - a^2}} \quad \ldots\ldots\ldots$$

$$= \int \frac{\tan\theta}{\sqrt{a^2(\ldots\ldots\ldots)}}\, d\theta$$

$$= \frac{1}{a}\int 1\, d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$$

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$$

e.g. $\displaystyle\int \frac{1}{x\sqrt{x^2-64}}\,dx = \frac{1}{8}\sec^{-1}\left(\frac{x}{8}\right)+c$

3.2.4

In order to evaluate the integrals of type $\int \frac{1}{ax^2+bx+c}\,dx$ and $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

we can use the following steps.

(1) Write ax^2+bx+c as, $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, $a>0$ and take a or $\sqrt{a}$ out of the integral sign.

(2) $\left(x^2+\frac{b}{a}x\right)$ or $\left(\frac{b}{a}x-x^2\right)$ is expressed by the method of completing square by adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$.

(3) Express the quadractic expression as a sum or difference of two squares

i.e. $\left((x+\beta)^2 \pm \alpha^2\right)$ or $\left(\alpha^2-(x+\beta)^2\right)$

(4) We know that $\int f(x)\,dx = g(x)+c \quad \Rightarrow \quad \int f(x+\beta)\,dx = g(x+\beta)+c$

$$\int f(\alpha x+\beta)\,dx = \frac{1}{\alpha}g(\alpha x+\beta)+c$$

(5) Use the standard integral formula and express the result in terms of x.

3.2.5

In order to evaluate the integral of type $\int \frac{1}{a\sin^2 x + b\cos^2 x + c}\,dx$

we can use the following steps.

(1) Divide the numerator and denominator by $\cos^2 x$ or $\sin^2 x$.

(2) In denominator replace $\sec^2 x$ by $1+\tan^2 x$ and /or $\operatorname{cosec}^2 x$ by $1+\cot^2 x$, if exists.

(3) Put $\tan x = t$ or $\cot x = t$ so that the integral reduces to the form $\int \frac{1}{at^2+bt+c}\,dt$

(4) Use the standard integral formula and express the result in terms of x.

3.2.6

To evaluate the integral of the form $\int \frac{1}{a\sin x + b\cos x + c}\,dx$, we use the standard substitution $\tan\frac{x}{2} = t$.

If $\tan\frac{x}{2} = t$ then (i) $\sec^2\frac{x}{2}\,\frac{1}{2}\,dx = 1\,dt$

$$\text{i.e. } dx = \frac{2}{\sec^2\frac{x}{2}}\,dt = \frac{2}{1+\tan^2\frac{x}{2}}\,dt = \frac{2\,dt}{1+t^2}$$

(ii) $$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

(iii) $\cos x = \dfrac{1 - \tan^2 \dfrac{x}{2}}{1 + \tan^2 \dfrac{x}{2}} = \dfrac{1 - t^2}{1 + t^2}$

We put $\tan x = t$ for the integral of the type $\displaystyle\int \frac{1}{a \sin 2x + b \cos 2x + c}\, dx$

therefore $dx = \dfrac{1}{1 + t^2}\, dt$

$\sin 2x = \dfrac{2t}{1 + t^2}$

and $\cos 2x = \dfrac{1 - t^2}{1 + t^2}$

With this substitution the integral reduces to the form $\displaystyle\int \frac{1}{ax^2 + bx + c}\, dx$. Now use the standard integral formula and express the result in terms of x.

SOLVED EXAMPLES

Ex. : Evaluate :

1. $\displaystyle\int \frac{1}{4x^2 + 11}\, dx$

Solution : $\displaystyle I = \int \frac{1}{4\left(x^2 + \frac{11}{4}\right)}\, dx$

$$= \frac{1}{4} \int \frac{1}{x^2 + \left(\frac{\sqrt{11}}{2}\right)^2}\, dx$$

$\because \displaystyle\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$I = \frac{1}{4}\left[\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\right] \tan^{-1}\left[\frac{x}{\left(\frac{\sqrt{11}}{2}\right)}\right] + c$$

$$= \frac{1}{2\sqrt{11}} \tan^{-1}\left(\frac{2x}{\sqrt{11}}\right) + c$$

2. $\displaystyle\int \frac{1}{a^2 - b^2 x^2}\, dx$

Solution : $\displaystyle I = \int \frac{1}{b^2\left(\frac{a^2}{b^2} - x^2\right)}\, dx$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2}\, dx$$

$\because \displaystyle\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left(\frac{a + x}{a - x}\right) + c$

$$I = \frac{1}{b^2} \frac{1}{2\left(\frac{a}{b}\right)} \log\left(\frac{\frac{a}{b} + x}{\frac{a}{b} - x}\right) + c$$

$$= \frac{1}{b^2} \frac{1}{2\left(\frac{a}{b}\right)} \log\left(\frac{\frac{a}{b} + x}{\frac{a}{b} - x}\right) + c$$

$$= \frac{1}{2ab} \log\left(\frac{a + bx}{a - bx}\right) + c$$

3. $\displaystyle\int \frac{1}{\sqrt{3x^2-7}}\,dx$

Solution : $\displaystyle I = \int \frac{1}{\sqrt{3\left(x^2-\frac{7}{3}\right)}}\,dx$

$$= \int \frac{1}{\sqrt{3}\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}}\,dx$$

$$\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}}\log\left(x+\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}\right)+c$$

$$= \frac{1}{\sqrt{3}}\log\left(x+\sqrt{x^2-\frac{7}{3}}\right)+c$$

4. $\displaystyle\int \frac{1}{x^2+8x+12}\,dx$

Solution : $\displaystyle I = \int \frac{1}{x^2+8x+16-4}\,dx$

$$= \int \frac{1}{(x+4)^2-(2)^2}\,dx$$

$$\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$$

$$I = \frac{1}{2(2)}\log\left(\frac{(x+4)-2}{(x+4)+2}\right)+c$$

$$= \frac{1}{4}\log\left(\frac{x+2}{x+6}\right)+c$$

$$\therefore \int \frac{1}{x^2+8x+12}\,dx = \frac{1}{4}\log\left(\frac{x+2}{x+6}\right)+c$$

5. $\displaystyle\int \frac{1}{\sqrt{3x^2-4x+2}}\,dx$

Solution : $\displaystyle = \int \frac{1}{\sqrt{3\left(x^2-\frac{4}{3}x+\frac{2}{3}\right)}}\,dx$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$= \int \frac{1}{\sqrt{3}\sqrt{x^2-\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{2}{3}}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+\left(\frac{2}{3}-\frac{4}{9}\right)}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}}\,dx$$

$$\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+c$$

$$= \frac{1}{\sqrt{3}}\log\left(\left(x-\frac{2}{3}\right)+\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right)+c$$

$$= \frac{1}{\sqrt{3}}\log\left(\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4}{3}x+\frac{2}{3}}\right)+c$$

6. $$\int \frac{1}{3-10x-25x^2}\,dx$$

Solution :

$$I = \int \frac{1}{25\left(\frac{3}{25}-\frac{10}{25}x-x^2\right)}\,dx$$

$$= \int \frac{1}{25\left[\frac{3}{25}-\left(x^2+\frac{2}{5}x\right)\right]}\,dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of }x\right)^2\right.$$

$$\left.= \left(\frac{1}{2}\left(\frac{2}{5}\right)\right)^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}\right\}$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}-\frac{1}{25}\right)}\,dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)+\frac{1}{25}}\,dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{4}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)}\,dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\left(\frac{2}{5}\right)^2-\left(x-\frac{1}{5}\right)^2}\,dx$$

$$\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$$

$$I = \frac{1}{25}\cdot\frac{1}{2\left(\frac{2}{5}\right)}\cdot\log\left(\frac{\frac{2}{5}+\left(x-\frac{1}{5}\right)}{\frac{2}{5}-\left(x-\frac{1}{5}\right)}\right)+c$$

$$= \frac{1}{5}\cdot\log\left(\frac{1+5x}{3-5x}\right)+c$$

Activity :

7. $$\int \frac{1}{\sqrt{1+x-x^2}}\,dx$$

Solution : $$I = \int \frac{1}{\sqrt{1-(\ldots\ldots\ldots\ldots)}}\,dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of }x\right)^2\right.$$

$$\left.= \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \int \frac{1}{\sqrt{1-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-(\ldots\ldots\ldots\ldots\ldots\ldots)}}\,dx$$

$$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$I = \ldots\ldots\ldots\ldots\ldots\ldots$

$= \ldots\ldots\ldots\ldots\ldots\ldots$

8. $$\int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1}\,dx$$

Solution : $$I = \int \frac{\sin 2x}{3(\sin^2 x)^2 - 4(\sin^2 x) + 1}\,dx$$

put $\sin^2 x = t$ $\therefore 2\sin x\cos x\,dx = 1\,dt$

$\therefore \sin 2x\,dx = 1\,dt$

$$= \int \frac{1}{3t^2 - 4t + 1}\,dt$$

$$= \int \frac{1}{3\left(t^2 - \frac{4}{3}t + \frac{1}{3}\right)}\,dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2\right.$$

$$\left.= \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$I = \frac{1}{3}\int \frac{1}{t^2 - \frac{4}{3}t + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}}\,dt$$

$$= \frac{1}{3}\int \frac{1}{\left(t^2 - \frac{4}{3}t + \frac{4}{9}\right) - \frac{1}{9}}\,dt$$

$$= \frac{1}{3}\int \frac{1}{\left(t - \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2}\,dt$$

$$= \frac{1}{3}\,\frac{1}{2\left(\frac{1}{3}\right)}\log\left(\frac{\left(t - \frac{2}{3}\right) - \frac{1}{3}}{\left(t - \frac{2}{3}\right) + \frac{1}{3}}\right) + c$$

$$= \frac{1}{2}\log\left(\frac{3t - 3}{3t - 1}\right) + c$$

$$= \frac{1}{2}\log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

$$\therefore \int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1}\,dx$$

$$= \frac{1}{2}\log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

9. $$\int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}}\,dx$$

Solution :

$$I = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1}{e^x} - e^x}}\,dx$$

$$= \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^x)^2}{e^x}}}\,dx$$

$$= \int \frac{\sqrt{e^x}}{\frac{\sqrt{1 - (e^x)^2}}{\sqrt{e^x}}}\,dx$$

$$= \int \frac{\sqrt{e^x}\cdot\sqrt{e^x}}{\sqrt{1 - (e^x)^2}}\,dx$$

$$= \int \frac{e^x}{\sqrt{1 - (e^x)^2}}\,dx$$

put $e^x = t$

$\therefore e^x\,dx = 1\,dt$

$$I = \int \frac{1}{\sqrt{1 - t^2}}\,dt$$

$$= \sin^{-1}(t) + c$$

$$= \sin^{-1}(e^x) + c$$

$$\therefore \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}}\,dx = \sin^{-1}(e^x) + c$$

10. $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)\,dx$

Solution : $I = \int\left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right)dx$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}}\,dx$$

put $\sqrt{\tan x} = t \quad \therefore \tan x = t^2 \ \therefore\ x = \tan^{-1} t^2$

$\therefore \quad 1\,dx = \frac{1}{1+(t^2)^2}\,2t\,dt$

$\because \quad \sec^2 x\,dx = 2t\,dt$

$$\therefore \quad dx = \frac{2t}{\sec^2 x}\,dx = \frac{2t}{1+\tan^2 x}\,dx = \frac{2t}{1+t^4}\,dt$$

$$= \int \frac{t^2+1}{t}\,\frac{2t}{1+t^4}\,dt = 2\int\frac{t^2+1}{t^4+1}\,dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt = 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

put $t-\frac{1}{t} = u \qquad \because \left[\frac{d}{dt}\left(t-\frac{1}{t}\right) = 1+\frac{1}{t^2}\right]$

$$\therefore \left(t-\left(-\frac{1}{t^2}\right)\right)dt = 1\,du$$

$$\therefore \left(1+\frac{1}{t^2}\right)dt = 1\,du$$

$$I = 2\int\frac{1}{u^2+2}\,du$$

$$= 2\int\frac{1}{u^2+(\sqrt{2})^2}\,du$$

$$= 2\,\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$= \sqrt{2}\,\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+c$$

$$= \sqrt{2}\,\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right)+c$$

$$= \sqrt{2}\,\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\cdot\sqrt{\tan x}}\right)+c$$

11. $\int\frac{1}{5-4\cos x}\,dx$

Solution : put $\tan\frac{x}{2} = t$

$\therefore \quad dx = \frac{2}{1+t^2}\,dt$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int\frac{1\left(\frac{2}{1+t^2}\right)}{5-4\left(\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int\frac{\frac{2}{1+t^2}}{\frac{5(1+t^2)-4(1-t^2)}{1+t^2}}\,dt$$

$$= \int\frac{2}{5-5t^2-4-4t^2}\,dt$$

$$= \int\frac{2}{9t^2+1}\,dt$$

$$= \int\frac{1}{9\left(t^2+\frac{1}{9}\right)}\,dt$$

$$= \frac{2}{9}\int\frac{1}{t^2+\left(\frac{1}{3}\right)^2}\,dt$$

$$= \frac{2}{9}\,\frac{1}{\left(\frac{1}{3}\right)}\tan^{-1}\left(\frac{t}{\left(\frac{1}{3}\right)}\right)+c$$

$$= \frac{2}{3}\tan^{-1}(2t)+c$$

$$= \frac{2}{3}\tan^{-1}\left(2\tan\frac{x}{2}\right)+c$$

$$\therefore \int\frac{1}{5-4\cos x}\,dx = \frac{2}{3}\tan^{-1}\left(2\tan\frac{x}{2}\right)+c$$

12. $\displaystyle\int \frac{1}{2-3\sin 2x}\,dx$

Solution : put $\tan x = t$

$\therefore \quad dx = \dfrac{1}{1+t^2}\,dt$ and $\sin 2x = \dfrac{2}{1+t^2}$

$$I = \int \frac{1\left(\frac{1}{1+t^2}\right)}{2-3\left(\frac{2}{1+t^2}\right)}\,dt$$

$$= \int \frac{\frac{1}{1+t^2}}{\frac{2(1+t^2)-3(2t)}{1+t^2}}\,dt$$

$$= \int \frac{1}{2+2t^2-6t}\,dt = \int \frac{1}{2(t^2-3t+1)}\,dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}(-3)\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}\right\}$$

$$= \frac{1}{2}\int \frac{1}{t^2-3t+\frac{9}{4}-\frac{9}{4}+1}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t^2-3t+\frac{9}{4}\right)-\frac{5}{4}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\,\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left(\frac{\left(t-\frac{3}{2}\right)-\frac{\sqrt{5}}{2}}{\left(t-\frac{3}{2}\right)+\frac{\sqrt{5}}{2}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{2t-3-\sqrt{5}}{2t-3+\sqrt{5}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{2\tan x-3-\sqrt{5}}{2\tan x-3+\sqrt{5}}\right)+c$$

13. $\displaystyle\int \frac{1}{3-2\sin x+5\cos x}\,dx$

Solution : put $\tan \dfrac{x}{2} = t$

$\therefore \quad dx = \dfrac{2}{1+t^2}$

$\therefore \quad \sin x = \dfrac{2}{1+t^2}\,dt$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$I = \int \frac{1\left(\frac{2}{1+t^2}\right)}{3-2\left(\frac{2}{1+t^2}\right)+5\left(\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{3(1+t^2)-2(2t)+5(1-t^2)}{1+t^2}}\,dt$$

$$= \int \frac{2}{3+3t^2-4t+5-5t^2}\,dt$$

$$= \int \frac{2}{8-4t-2t^2}\,dt$$

$$= \int \frac{1}{4-2t-t^2}\,dt$$

$$= \int \frac{1}{4-(t^2+2t)}\,dt$$

$$= \int \frac{1}{4-(t^2+2t+1-1)}\,dt$$

$$= \int \frac{1}{5-(t^2+2t+1)}\,dt$$

$$= \int \frac{1}{(\sqrt{5})^2-(t+1)^2}\,dt$$

$$= \frac{1}{2(\sqrt{5})}\log\left(\frac{\sqrt{5}+(t+1)}{\sqrt{5}-(t+1)}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}+1+\tan\frac{x}{2}}{\sqrt{5}-1-\tan\frac{x}{2}}\right)+c$$

Activity : 14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\, dx$

Solution : put $\tan\frac{x}{2} = t$ $\quad \therefore dx = \ldots\ldots\ldots$

$\therefore \quad \sin x = \ldots\ldots\ldots$ and $\cos x = \ldots\ldots\ldots$

$$I = \int \frac{1\left(\frac{2}{1+t^2}\right)}{\ldots\ldots\ldots + \sqrt{3}\ \ldots\ldots\ldots}\, dt$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{\ldots\ldots}{1+t^2}}\, dt$$

$$= \int \frac{2}{\ldots\ldots\ldots}\, dt$$

$$= \int \frac{2}{\sqrt{3}\left(1 - \left(t^2 - \frac{2}{\sqrt{3}}t\right)\right)}\, dt$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{1 - \left(t^2 - \frac{2}{\sqrt{3}}t + \frac{1}{3} - \frac{1}{3}\right)}\, dt$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{1 - \left(t^2 - \frac{2}{\sqrt{3}}t + \frac{1}{3}\right) + \frac{1}{3}}\, dt$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{\left(\ldots\ldots\right) - \left(t - \frac{1}{\sqrt{3}}\right)^2}\, dt$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{(\ldots)^2 - (\ldots)^2}\, dt$$

$$= \frac{2}{\sqrt{3}}\,\frac{1}{2\,(\ldots)}\cdot\log\left(\frac{\qquad}{\qquad}\right) + c$$

$$= \frac{\quad}{\quad}\log\left(\frac{\qquad}{\qquad}\right) + c$$

$$= \frac{1}{2}\log\left(\frac{1 + \sqrt{3}\tan\frac{x}{2}}{3 - \sqrt{3}\tan\frac{x}{2}}\right) + c$$

Alternative method :

14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\, dx$

Solution : For any two positive numbers a and b, we can find an angle θ, such that

$\therefore \quad \sin\theta = \dfrac{a}{\sqrt{a^2 - b^2}}$ and $\cos\theta = \dfrac{b}{\sqrt{a^2 - b^2}}$

Using this we express $\sin x - \sqrt{3}\cos x$

$= \sqrt{1+3}\,(\cos\theta\cdot\sin x - \sin\theta\cdot\cos x)$

$= 2\sin(x - \theta)$

$= 2\sin\left(x - \frac{\pi}{3}\right)$

$$\therefore I = \int \frac{1}{2\cdot\sin\left(x - \frac{\pi}{3}\right)}\, dx$$

$$= \frac{1}{2}\int \operatorname{cosec}\left(x - \frac{\pi}{3}\right) dx$$

$$= \frac{1}{2}\log\left(\operatorname{cosec}\left(x - \frac{\pi}{3}\right) - \cot\left(x - \frac{\pi}{3}\right)\right) + c$$

$$= \frac{1}{2}\log\left(\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)\right) + c$$

15. $\displaystyle\int \frac{1}{3+2\sin^2 x+5\cos^2 x}\,dx$

Solution : Divide Numerator and Denominator by $\cos^2 x$

$$I = \int \frac{\frac{1}{\cos^2 x}}{\frac{3+2\sin^2 x+5\cos^2 x}{\cos^2 x}}\,dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x+2\tan^2 x+5}\,dx$$

$$= \int \frac{\sec^2 x}{3(1+\tan^2 x)+2\tan^2 x+5}\,dx$$

$$= \int \frac{\sec^2 x}{5\tan^2 x+8}\,dx$$

$$= \frac{1}{5}\cdot\int \frac{\sec^2 x}{\tan^2 x+\frac{8}{5}}\,dx$$

put $\tan x = t \;\therefore\; \sec^2 x\,dx = 1\,dt$

$$I = \frac{1}{5}\int \frac{1}{t^2+\frac{8}{5}}\,dt$$

$$= \frac{1}{5}\int \frac{1}{t^2+\left(\frac{\sqrt{8}}{\sqrt{5}}\right)^2}\,dt$$

$$= \frac{1}{5}\,\frac{1}{\frac{\sqrt{8}}{\sqrt{5}}}\tan^{-1}\left(\frac{t}{\frac{\sqrt{8}}{\sqrt{5}}}\right)+c$$

$$= \frac{1}{\sqrt{5}}\,\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{5}\,t}{2\sqrt{2}}\right)+c$$

$$= \frac{1}{2\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$$

$$\therefore \int \frac{1}{3+2\sin^2 x+5\cos^2 x}\,dx = \frac{1}{2\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$$

16. $\displaystyle\int \frac{\cos\theta}{\cos 3\theta}\,d\theta$

Solution : $I = \displaystyle\int \frac{\cos\theta}{4\cos^3\theta-3\cos\theta}\,d\theta$

$$= \int \frac{1}{4\cos^2\theta-3}\,d\theta$$

Divide Numerator and Denominator by $\cos^2\theta$

$$I = \int \frac{\frac{1}{\cos^2\theta}}{\frac{4\cos^2\theta-3}{\cos^2\theta}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{4-3\sec^2\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta}{4-3(1+\tan^2\theta)}\,d\theta$$

$$= \int \frac{\sec^2\theta}{1-3\tan^2\theta}\,d\theta$$

put $\tan\theta = t \;\therefore\; \sec^2\theta\;\; d\theta = 1\,dt$

$$I = \int \frac{1}{1-3t^2}\,dt$$

$$= \frac{1}{3}\cdot\int \frac{1}{\frac{1}{3}-t^2}\,dt$$

$$= \frac{1}{3}\cdot\int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2-t^2}\,dt$$

$$= \frac{1}{3}\,\frac{1}{2\left(\frac{1}{\sqrt{3}}\right)}\log\left(\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right)+c$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{1+\sqrt{3}\,t}{1-\sqrt{3}\,t}\right)+c$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{1+\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta}\right)+c$$

$$\therefore \int \frac{\cos\theta}{\cos 3\theta}\,d\theta = \frac{1}{2\sqrt{3}}\log\left(\frac{1+\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta}\right)+c$$

EXERCISE 3.2 (B)

I. Evaluate the following :

1. $\int \frac{1}{4x^2 - 3}\, dx$
2. $\int \frac{1}{25 - 9x^2}\, dx$
3. $\int \frac{1}{7 + 2x^2}\, dx$
4. $\int \frac{1}{\sqrt{3x^2 + 8}}\, dx$
5. $\int \frac{1}{\sqrt{11 - 4x^2}}\, dx$
6. $\int \frac{1}{\sqrt{2x^2 - 5}}\, dx$
7. $\int \sqrt{\frac{9 + x}{9 - x}}\, dx$
8. $\int \sqrt{\frac{2 + x}{2 - x}}\, dx$
9. $\int \sqrt{\frac{10 + x}{10 - x}}\, dx$
10. $\int \frac{1}{x^2 + 8x + 12}\, dx$
11. $\int \frac{1}{1 + x - x^2}\, dx$
12. $\int \frac{1}{4x^2 - 20x + 17}\, dx$
13. $\int \frac{1}{5 - 4x - 3x^2}\, dx$
14. $\int \frac{1}{\sqrt{3x^2 + 5x + 7}}\, dx$
15. $\int \frac{1}{\sqrt{x^2 + 8x - 20}}\, dx$
16. $\int \frac{1}{\sqrt{8 - 3x + 2x^2}}\, dx$
17. $\int \frac{1}{\sqrt{(x - 3)(x + 2)}}\, dx$
18. $\int \frac{1}{4 + 3\cos^2 x}\, dx$
19. $\int \frac{1}{\cos 2x + 3\sin^2 x}\, dx$
20. $\int \frac{\sin x}{\sin 3x}\, dx$

II. Integrate the following functions *w. r. t. x* :

1. $\int \frac{1}{3 + 2\sin x}\, dx$
2. $\int \frac{1}{4 - 5\cos x}\, dx$
3. $\int \frac{1}{2 + \cos x - \sin x}\, dx$
4. $\int \frac{1}{3 + 2\sin x - \cos x}\, dx$
5. $\int \frac{1}{3 - 2\cos 2x}\, dx$
6. $\int \frac{1}{2\sin 2x - 3}\, dx$
7. $\int \frac{1}{3 + 2\sin 2x + 4\cos 2x}\, dx$
8. $\int \frac{1}{\cos x - \sin x}\, dx$
9. $\int \frac{1}{\cos x - \sqrt{3}\sin x}\, dx$

3.2.6 Integral of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

The integral of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c}\,dx + \int \frac{B}{ax^2+bx+c}\,dx \text{ for some constants } A \text{ and } B.$$

The numerator, $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

i.e. $\quad \text{N} = A\frac{d}{dx}\text{D}+B$

The first integral is evaluated by putting $ax^2+bx+c=t$

The Second integral is evaluated by expressing the integrand in the form either

$\frac{1}{A^2+t^2}$ or $\frac{1}{t^2-A^2}$ or $\frac{1}{A^2-t^2}$ and applying the methods discussed previously.

The integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ is evaluated by expressing the integral in the form

$$\int \frac{A\frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}\,dx + \int \frac{B}{\sqrt{ax^2+bx+c}}\,dx \quad \text{for constants } A \text{ and } B.$$

The numerator, $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

The first integral is evaluated by putting $ax^2+bx+c=t$

The second integral is evaluated by expressing the integrand in the form either

$\frac{1}{\sqrt{A^2+t^2}}$ or $\frac{1}{\sqrt{t^2-A^2}}$ or $\frac{1}{\sqrt{A^2-t^2}}$ and applying the methods which discussed previously.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{2x-3}{3x^2+4x+5}\,dx$

Solution : $2x-3 = A\dfrac{d}{dx}\left(3x^2+4x+5\right)+B$

$2x-3 = A(6x+4)+B$

$= (6A)x+(4A+B)$

compairing the sides/ the co–efficients of like variables and constants

$6A = 2$ and $4A+B = -3$

$\Rightarrow A = \dfrac{1}{3}$ and $B = -\dfrac{13}{3}$

$$= \int \frac{\frac{1}{3}\cdot\frac{d}{dx}(3x^2+4x+5)+\left(-\frac{13}{3}\right)}{3x^2+4x+5}\,dx$$

$$= \frac{1}{3}\cdot\int \frac{\frac{d}{dx}(3x^2+4x+5)}{3x^2+4x+5}\,dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\,dx$$

$$= \frac{1}{3}\cdot\int \frac{6x+4}{3x^2+4x+5}\,dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\,dx$$

$= I_1 - I_2$ . . . . . (i)

$\therefore\ I_1 = \dfrac{1}{3}\cdot\displaystyle\int \frac{6x+4}{3x^2+4x+5}\,dx$

put $3x^2+4x+5 = t$

$\therefore\ (6x+4)\,dx = 1\,dt$

$I_1 = \dfrac{1}{3}\displaystyle\int \frac{1}{t}\,dt$

$= \dfrac{1}{3}\log(t)+c_1$

$= \dfrac{1}{3}\log\left(3x^2+4x+5\right)+c_1$ . . . . . (ii)

$\therefore\ I_2 = \dfrac{13}{3}\displaystyle\int \frac{1}{3x^2+4x+5}\,dx$

$$= \frac{13}{3}\,\frac{1}{3}\int \frac{1}{x^2+\frac{4}{3}x+\frac{5}{3}}\,dx$$

$\because\ \left\{\left(\dfrac{1}{2}\text{ coefficient of } t\right)^2\right.$

$$= \left(\frac{1}{2}\left(\frac{4}{3}\right)\right)^2 = \left(\frac{2}{3}\right)^2 = \left.\frac{4}{9}\right\}$$

$$I_2 = \frac{13}{9}\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{5}{3}}\,dx$$

$$= \frac{13}{9}\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}+\frac{11}{9}}\,dx$$

$$= \frac{13}{9}\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}\,dx$$

$\because\ \displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$= \frac{13}{9}\,\frac{1}{\frac{\sqrt{11}}{3}}\tan^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right)+c_1$$

$I_2 = \dfrac{13}{3\sqrt{11}}\tan^{-1}\left(\dfrac{3x+2}{\sqrt{11}}\right)+c_2$ . . . . . (iii)

thus, from (i), (ii) and (iii)

$\therefore\ \displaystyle\int \frac{2x-3}{3x^2+4x+5}\,dx$

$$= \frac{1}{3}\log\left(3x^2+4x+5\right) - \frac{13}{3\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)+c$$

$(\because c_1+c_2 = c)$

2. $\int \sqrt{\frac{x-5}{x-7}}\, dx$

Solution : $I = \int \sqrt{\frac{(x-5)\ (x-5)}{(x-7)\ (x-5)}}\, dx = \int \sqrt{\frac{(x-5)^2}{x^2-12x+35}}\, dx$

$\because \quad x-5 = A\frac{d}{dx}\left(x^2-12x+35\right)+B$

$x-5 = A(2x-12)+B$

$= (2A)x+(-12A+B)$

compairing, the co–efficients of like variables and constants

$2A = 1$ and $-12A+B = -5$

$\Rightarrow A = \frac{1}{2}$ and $B = 1$

$$I = \int \frac{\frac{1}{2}\frac{d}{dx}(x^2-12x+35)+(1)}{\sqrt{x^2-12x+35}}\, dx$$

$$= \frac{1}{2}\cdot\int \frac{\frac{d}{dx}(x^2-12x+35)}{\sqrt{x^2-12x+35}}\, dx + \int \frac{1}{\sqrt{x^2-12x+35}}\, dx$$

$= I_1 + I_2 \qquad \ldots\ldots (i)$

$\therefore \quad I_1 = \frac{1}{2}\int \frac{2x-12}{\sqrt{x^2-12x+35}}\, dx$

put $x^2-12x+35 = t$

$\therefore \quad (2x-12)\, dx = 1\, dt$

$I_1 = \frac{1}{2}\int \frac{1}{\sqrt{t}}\, dt$

$= \int \frac{1}{2\sqrt{t}}\, dt$

$= \sqrt{t} + c_1$

$= \sqrt{x^2-12x+35} + c_1 \qquad \ldots\ldots (ii)$

$\therefore \quad I_2 = \int \frac{1}{\sqrt{x^2-12x+35}}\, dx$

$= \int \frac{1}{\sqrt{x^2-12x+36-1}}\, dx$

$= \int \frac{1}{\sqrt{(x-6)^2-(1)^2}}\, dx$

$\because \quad \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

$I_2 = \log\left((x-6)+\sqrt{(x-6)^2-1}\right)+c_2$

$= \log\left((x-6)+\sqrt{x^2-12x+35}\right)+c_2$

$\ldots\ldots (iii)$

Thus, from (i), (ii) and (iii)

$\int \sqrt{\frac{x-5}{x-7}}\, dx$

$= \sqrt{x^2-12x+35}+\log\left((x-6)+\sqrt{x^2-12x+35}\right)+c$

$\left(c_1+c_2 = c\right)$

Activity :

3. $\int \sqrt{\frac{8-x}{x}}\, dx$

Solution : $= \int \sqrt{\frac{(8-x)}{x} \times \frac{(\quad)}{(\quad)}}\, dx \quad = \int \sqrt{\frac{(\qquad)^2}{8x-x^2}}\, dx \quad = \int \frac{(8-x)}{\sqrt{8x-x^2}}\, dx$

$\because \quad 8-x = A\,\frac{d}{dx}\left(8x-x^2\right) + B$

$8-x = A\,(\ldots\ldots\ldots\ldots) + B$

$= (8A+B) - 2Ax$

compairing, the co–efficients of like variables and constants

$8A + B = \ldots\ldots \qquad$ and $\;-2A = -1$

$\Rightarrow \; A = \frac{\cdots}{\cdots}$ and $\; B = \ldots\ldots$

$$= \int \frac{\frac{1}{2}\frac{d}{dx}(8x-x^2) + (4)}{\sqrt{8x-x^2}}\, dx$$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(8x-x^2)}{\sqrt{8x-x^2}}\, dx + 4\int \frac{1}{\sqrt{8x-x^2}}\, dx$$

$$= \frac{1}{2}\int \frac{8-2x}{\sqrt{8x-x^2}}\, dx + 4\int \frac{1}{\sqrt{8x-x^2}}\, dx$$

$= I_1 + I_2 \qquad \ldots\ldots \text{(i)}$

$\therefore \quad I_1 = \frac{1}{2}\int \frac{8-2x}{\sqrt{8x-x^2}}\, dx$

put $\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots = t$

$\therefore \quad (\ldots\ldots\ldots\ldots)\, dx = 1\, dt$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t}}\, dt$$

$$= \int \frac{1}{2\sqrt{t}}\, dt$$

$= \sqrt{t} + c_1$

$= \sqrt{8x-x^2} + c_1 \qquad \ldots\ldots \text{(ii)}$

$$\therefore \quad I_2 = 4\int \frac{1}{\sqrt{8x-x^2}}\, dx$$

$$= 4\int \frac{1}{\sqrt{-(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)}}\, dx$$

$$= 4\int \frac{1}{\sqrt{\ldots\ldots\ldots - (\ldots\ldots\ldots)}}\, dx$$

$$= 4\int \frac{1}{\sqrt{\ldots\ldots\ldots - (x-4)^2}}\, dx$$

$$\because \quad \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$I_2 = 4\sin^{-1}\left(\frac{x-4}{4}\right) + c_2 \qquad \ldots\ldots \text{(iii)}$

thus, from (i), (ii) and (iii)

$$\therefore \quad \int \sqrt{\frac{8-x}{x}}\, dx$$

$$= \sqrt{8x-x^2} + 4\sin^{-1}\left(\frac{x-4}{4}\right) + c$$

$(\because c_1 + c_2 = c)$

EXERCISE 3.2 (C)

I. Evaluate :

1. $\int \frac{3x+4}{x^2+6x+5}\, dx$
2. $\int \frac{2x+1}{x^2+4x-5}\, dx$
3. $\int \frac{2x+3}{2x^2+3x-1}\, dx$
4. $\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\, dx$
5. $\int \frac{7x+3}{\sqrt{3+2x-x^2}}\, dx$
6. $\int \sqrt{\frac{x-7}{x-9}}\, dx$
7. $\int \sqrt{\frac{9-x}{x}}\, dx$
8. $\int \frac{3\cos x}{4\sin^2 x+4\sin x-1}\, dx$
9. $\int \sqrt{\frac{e^{3x}-e^{2x}}{e^x+1}}\, dx$

3.3 Integration by parts :

This method is useful when the integrand is expressed as a product of two different types of functions; one of which can be differentiated and the other can be integrated conveniently.

The following theorem gives the rule of integration by parts.

3.3.1 Theorem : If u and v are two differentiable functions of x then

$$\int u\, v\, dx = u \int v\, dx - \int \left(\frac{d}{dx} u\right)\left(\int v\, dx\right) dx$$

Proof : Let $\int v\, dx = w$ ... (i) $\Rightarrow$ $v = \frac{dw}{dx}$ ... (ii)

Consider, $\frac{d}{dx}(u\, w) = u \frac{d}{dx} w + w \frac{d}{dx} u$

$$= u\, v + w \frac{du}{dx}$$

By definition of integration

$$u\, w = \int \left[u\, v + w \frac{du}{dx}\right] dx$$

$$= \int u\, v\, dx + \int w \frac{du}{dx}\, dx$$

$$= \int u\, v\, dx + \int \frac{du}{dx}\, w\, dx$$

$$\therefore \quad u \int v\, dx = \int u\, v\, dx + \int \frac{du}{dx} \int v\, dx\, dx$$

$$\therefore \quad \int u\, v\, dx = u \int v\, dx - \int \left(\frac{d}{dx} u\right)\left(\int v\, dx\right) dx$$

For example : $\int x \cdot e^x \, dx = x\int e^x \, dx - \int \left(\frac{d(x)}{dx} \int e^x \, dx \right) dx$

$$= x\,e^x - \int (1)\, e^x \, dx$$

$$= x\,e^x - \int e^x \, dx$$

$$= x\,e^x - e^x + c$$

now let us reverse the choice of u and v

$$\therefore \quad \int e^x \cdot x \, dx = e^x \int x^1 \, dx - \int \frac{d}{dx} e^x \left(\int x \, dx \right) dx$$

$$= e^x \, \frac{x^2}{2} - \int e^x \, \frac{x^2}{2} \, dx$$

$$= \frac{1}{2} e^x x^2 - \frac{1}{2} \int e^x x^2 \, dx$$

We arrive at an integral $\int e^x x^2 \, dx$ which is more difficult, but it helps to get $\int e^x x^2 \, dx$

Thus it is essential to make a proper choise of the first function and the second function. The first function to be selected will be the one, which comes first in the order of **L I A T E**.

L	**Logarithmic function.**
I	**Inverse trigonometric function.**
A	**Algebric function.**
T	**Trigonometric function.**
E	**Exponential function.**

For example : $\int \sin x \; x \, dx$

$= \int x \sin x \, dx$ . . . . . by LIATE

$= x \int \sin x \, dx - \int \frac{d}{dx} x \left(\int \sin x \, dx \right) dx$

$= x\,(-\cos x) - \int (1)\,(-\cos x)\, dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c$

SOLVED EXAMPLES

1. $\int x^2\, 5^x\, dx$

Solution :

$$I = x^2 \int 5^x\, dx - \int \frac{d}{dx} x^2 \left(\int 5^x\, dx\right) dx$$

$$= x^2\, 5^x \frac{1}{\log 5} - \int 2x\, 5^x \frac{1}{\log 5}\, dx$$

$$= \frac{1}{\log 5} x^2\, 5^x - \frac{2}{\log 5}\left\{x \int 5^x\, dx - \int \frac{d}{dx} x \left(\int 5^x\, dx\right) dx\right\}$$

$$= \frac{1}{\log 5} x^2\, 5^x - \frac{2}{\log 5}\left\{x\, 5^x \frac{1}{\log 5} - \int (1)\left(5^x \frac{1}{\log 5}\right) dx\right\}$$

$$= \frac{1}{\log 5} x^2\, 5^x - \frac{2}{\log 5}\left\{\frac{1}{\log 5} x\, 5^x - \int \frac{1}{\log 5} 5^x\, dx\right\}$$

$$= \frac{1}{\log 5} x^2\, 5^x - \frac{2}{\log 5}\left\{\frac{1}{\log 5} x\, 5^x - \frac{1}{\log 5} 5^x \frac{1}{\log 5}\right\} + c$$

$$= \frac{1}{\log 5} x^2\, 5^x - \frac{2}{(\log 5)^2} x\, 5^x + \frac{2}{(\log 5)^3} 5^x + c$$

$$\therefore \int x^2\, 5^x\, dx = \frac{5^x}{\log 5}\left\{x^2 - \frac{2x}{\log 5} + \frac{2}{(\log 5)^2}\right\} + c$$

2. $\int x \tan^{-1} x\, dx$

Solution :

$$I = \int (\tan^{-1} x) x\, dx \qquad \text{..... by LIATE}$$

$$= \tan^{-1} x \int x\, dx - \int \frac{d}{dx} \tan^{-1} x \left(\int x\, dx\right) dx$$

$$= \tan^{-1} x \frac{x^2}{2} - \int \frac{1}{1+x^2} \frac{x^2}{2}\, dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2}\, dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2}\right] dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + c$$

$$\therefore \int x \tan^{-1} x\, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

3. $\displaystyle\int \frac{x}{1-\sin x}\, dx$

Solution : $\displaystyle I = \int \frac{x}{1-\sin x}\,\frac{(1+\sin x)}{(1+\sin x)}\, dx$

$$= \int \frac{x\,(1+\sin x)}{1-\sin^2 x}\, dx = \int \frac{x\,(1+\sin x)}{\cos^2 x}\, dx = \int x\left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int x\,(\sec^2 x + \sec x \tan x)\, dx$$

$$= \int x \sec^2 x\, dx + \int x \sec x \tan x\, dx$$

$$= \left(x \int \sec^2 x\, dx - \int \frac{d}{dx} x \left(\int \sec^2 x\, dx\right) dx\right) + \left(x \int \sec x \tan x\, dx - \int \frac{d}{dx} x \left(\int \sec x \tan x\, dx\right) dx\right)$$

$$= x \tan x - \int (1) \tan x\, dx + x \sec x - \int (1) \sec x\, dx$$

$$= x \tan x - \log(\sec x) + x \sec x - \log(\sec x + \tan x) + c$$

$$= x(\sec x + \tan x) - \log(\sec x) - \log(\sec x + \tan x) + c$$

$$\therefore \int \frac{x}{1-\sin x}\, dx = x(\sec x + \tan x) - \log[(\sec x)(\sec x + \tan x)] + c$$

4. $\int e^{2x} \sin 3x\, dx$

Solution : $I = \int e^{2x} \sin 3x\, dx$

Here we use repeated integration by parts.

To evaluate $\int e^{ax} \sin(bx + c)\, dx$; $\int e^{ax} \cos(bx+c)\, dx$ any function can be taken as a first function.

$$I = e^{2x} \int \sin 3x\, dx - \int \frac{d}{dx} e^{2x} \left(\int \sin 3x\, dx\right) dx$$

$$= e^{2x}\left(-\cos 3x \frac{1}{3}\right) - \int e^{2x}\, 2 \left(-\cos 3x \frac{1}{3}\right) dx$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x\, dx$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3}\left(e^{2x} \int \cos 3x \cdot dx - \int \frac{d}{dx} e^{2x} \left(\int \cos 3x\, dx\right) dx\right)$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3}\left[e^{2x}\left(\sin 3x \frac{1}{3}\right) - \int e^{2x}\, 2 \left(\sin 3x \frac{1}{3}\right) dx\right]$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x\, dx$$

$$I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} I$$

$$I + \frac{4}{9} I = \frac{e^{2x}}{9}[-3\cos 3x + 2 \sin 3x] + c$$

$$\frac{13}{9} I = \frac{e^{2x}}{9}[2 \sin 3x - 3 \cos 3x] + c$$

$$= \frac{e^{2x}}{13}[2 \sin 3x - 3 \cos 3x] + c$$

$$\therefore \int e^{2x} \sin 3x\, dx = \frac{e^{2x}}{13}[2 \sin 3x - 3 \cos 3x] + c$$

Activity :

Prove the following results.

(i) $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right] + c$

(ii) $\int e^{ax}\cos(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) + b\cos(bx+c)\right] + c$

5. $\displaystyle\int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

Solution :

$$
\begin{aligned}
\text{I} &= \int \log(\log x)\,1\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\int 1\,dx - \int \frac{d}{dx}\log(\log x)\int 1\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - \int \frac{1}{\log x}\,\frac{1}{x}\,(x)\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - \int \frac{1}{\log x}\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - \int (\log x)^{-1}\,1\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - \left\{(\log x)^{-1}\int 1\,dx + \int \frac{d}{dx}(\log x)^{-1}\int 1\,dx\,dx\right\} + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - \left\{(\log x)^{-1}\,x - \int -1(\log x)^{-2}\,\frac{1}{x}\,x\,dx\right\} + \int \frac{1}{(\log x)^2}\,dx \\
&= \log(\log x)\,x - (\log x)^{-1}\,x - \int (\log x)^{-2}\,dx + \int \frac{1}{(\log x)^2}\,dx \\
&= x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\,dx + \int \frac{1}{(\log x)^2}\,dx
\end{aligned}
$$

$$\therefore \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx = x\log(\log x) - \frac{x}{\log x} + c$$

Note that :

To evaluate the integrals of type $\int \sin^{-1}x\,dx$; $\int \tan^{-1}x\,dx$; $\int \sec^{-1}x\,dx$; $\int \log x\,dx$, take the second function (v) to be 1 and then apply integration by parts.

$\int\sqrt{a^2-x^2}\,dx$; $\int\sqrt{a^2+x^2}\,dx$; $\int\sqrt{x^2-a^2}\,dx$

6. $\int \sqrt{a^2 - x^2}\ dx$

Solution : Let $I = \int \sqrt{a^2 - x^2}\ 1\, dx$

$$= \sqrt{a^2 - x^2} \int 1\, dx - \int \frac{d}{dx} \sqrt{a^2 - x^2} \left(\int 1\, dx \right) dx$$

$$= \sqrt{a^2 - x^2}\ x - \int \frac{1}{2\sqrt{a^2 - x^2}} (-2x)\,(x)\, dx$$

$$= \sqrt{a^2 - x^2}\ x + \int \frac{x^2}{\sqrt{a^2 - x^2}}\ dx$$

$$= \sqrt{a^2 - x^2}\ x + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\ dx$$

$$= \sqrt{a^2 - x^2}\ x + \int \left[\frac{a^2}{\sqrt{a^2 - x^2}} - \frac{(a^2 - x^2)}{\sqrt{a^2 - x^2}} \right] dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\ dx - \int \sqrt{a^2 - x^2}\, dx$$

$$I = x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\ dx - I$$

$$\therefore \quad I + I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{c}{2}$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c_1 \quad \textit{where } c_1 = \frac{c}{2}$$

e.g. $\int \sqrt{9 - x^2}\ dx = \frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c$

with reference to the above example solve these :

7. $\int \sqrt{a^2 + x^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + c$

8. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$

9. $\int x \sin^{-1} x \, dx$

Solution : $I = \int \sin^{-1} x \; x \, dx$ . . . . . by LIATE

$$= \sin^{-1} x \int x \, dx - \int \frac{d}{dx} \sin^{-1} x \left(\int x \, dx \right) dx$$

$$= \sin^{-1} x \, \frac{x^2}{2} - \int \frac{1}{\sqrt{1 - x^2}} \, \frac{x^2}{2} \, dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{1 - (1 - x^2)}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \left[\frac{1}{\sqrt{1 - x^2}} - \frac{(1 - x^2)}{\sqrt{1 - x^2}} \right] dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{dx}{\sqrt{1 - x^2}} + \frac{1}{2} \int \sqrt{1 - x^2} \, dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} (x) \right] + c$$

$$= \frac{1}{2} x^2 \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} - \frac{1}{4} \sin^{-1} x + c$$

$$\therefore \int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} - \frac{1}{4} \sin^{-1} x + c$$

Activity :

10. $\int \cos^{-1} \sqrt{x} \, dx$

Solution : put $\sqrt{x} = t$

$\therefore \quad x = t^2$

differentiating *w.r.t.* x

$\therefore \quad 1 \, dx = 2t \, dt$

$I = \int \cos^{-1} t \; 2t \, dt$

refer previous (example no. 9) example and solve it.

11. $\int \sqrt{4 + 3x - 2x^2}\; dx$

Solution : $I = \int \sqrt{4 - 2x^2 + 3x}\; dx$

$$= \int \sqrt{4 - 2\left(x^2 - \frac{3}{2}x\right)}\; dx$$

$$= \int \sqrt{2}\sqrt{2 - \left(x^2 - \frac{3}{2}x\right)}\; dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left[\frac{1}{2}\left(-\frac{3}{2}\right)\right]^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}\right\}$$

$$I = \sqrt{2}\int \sqrt{2 - \left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)}\; dx$$

$$= \sqrt{2}\int \sqrt{2 - \left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + \frac{9}{16}}\; dx$$

$$= \sqrt{2}\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2}\; dx$$

$$\because \int \sqrt{a^2 - x^2}\; dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= \sqrt{2}\left[\frac{\left(x - \frac{3}{4}\right)}{2}\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2} + \frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\sin^{-1}\left(\frac{x - \frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)\right] + c$$

$$= \sqrt{2}\left[\frac{4x - 3}{8}\sqrt{2 + \frac{3}{2}x - x^2} + \frac{41}{32}\sin^{-1}\left(\frac{4x - 3}{\sqrt{41}}\right)\right] + c$$

$$\therefore \int \sqrt{4 + 3x - 2x^2}\; dx = \frac{4x - 3}{8}\sqrt{4 + 3x - 2x^2} + \frac{41}{16\sqrt{2}}\sin^{-1}\left(\frac{4x - 3}{\sqrt{41}}\right) + c$$

Note that :

3.3.2 :

To evaluate the integral of type $\int (px + q)\sqrt{ax^2 + bx + c}\;\; dx$

we express the term $px + q = A\,\frac{d}{dx}(ax^2 + bx + c) + B$ . . . for constants A, B.

Then the integral will be evaluated by the useual known methods.

3.3.3 Integral of the type $\int e^x [f(x) + f'(x)] \; dx = e^x \cdot f(x) + c$

Let $e^x \; f(x) = t$

Differentiating *w. r. t. x*

$$\left[e^x [f'(x) + f(x)]\right] = \frac{dt}{dx}$$

$$e^x [f(x) + f'(x)] = \frac{dt}{dx}$$

By definition of integration,

$$\therefore \int e^x [f(x) + f'(x)] \; dx = t + c$$

$$\therefore \int e^x [f(x) + f'(x)] \; dx = e^x \cdot f(x) + c$$

e.g. $\int e^x [\tan x + \sec^2 x] \; dx = e^x \cdot \tan x + c$

$$\left(\because \frac{d}{dx} \tan x = \sec^2 x\right)$$

SOLVED EXAMPLES

1. $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) dx$

Solution :

$$I = \int e^x \left(\frac{2 + 2\sin x \cdot \cos x}{2 \cdot \cos^2 x}\right) dx$$

$$= \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}\right) dx$$

$$= \int e^x [\sec^2 x + \tan x] \, dx$$

$$= \int e^x [\tan x + \sec^2 x] \, dx$$

$$\therefore f(x) = \tan x \Rightarrow \quad f'(x) = \sec^2 x$$

$$\therefore \int e^x [f(x) + f'(x)] \; dx = e^x \cdot f(x) + c$$

$$I = e^x \cdot \tan x + c$$

$$\therefore \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) dx = e^x \cdot \tan x + c$$

2. $\int e^x \left[\frac{x + 2}{(x + 3)^2}\right] dx$

Solution :

$$I = \int e^x \left[\frac{x + 3 - 1}{(x + 3)^2}\right] dx$$

$$= \int e^x \left[\frac{x + 3}{(x + 3)^2} + \frac{-1}{(x + 3)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{x + 3} + \frac{-1}{(x + 3)^2}\right] dx$$

$$\therefore f(x) = \frac{1}{x + 3} \Rightarrow \quad f'(x) = \frac{-1}{(x + 3)^2}$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

$$= e^x \cdot \left(\frac{1}{x + 3}\right) + c$$

$$= \frac{e^x}{x + 3} + c$$

$$\therefore \int e^x \left[\frac{x + 2}{(x + 3)^2}\right] dx = \frac{e^x}{x + 3} + c$$

3. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Solution : put $\tan^{-1}x = t$

$\therefore \quad x = \tan t$

differentiating *w. r. t. x*

$\therefore \quad \frac{1}{1+x^2}\,dx = 1\,dt$

$$I = \int e^t\,[\,1 + \tan t + \tan^2 t\,]\,dt$$

$$= \int e^t\,[\,\tan t + (1 + \tan^2 t)]\,dt$$

$$= \int e^t\,[\,\tan t + \sec^2 t\,]\,dt$$

Here $\quad f(t) = \tan t$

$\Rightarrow \quad f'(t) = \sec^2 t$

$$I = e^t \cdot f(t) + c$$

$$= e^t \cdot \tan t + c$$

$$= e^{\tan^{-1}x} \cdot x + c$$

$$\therefore \int e^{\tan^{-1}x}\cdot\left(\frac{1+x+x^2}{1+x^2}\right)dx = e^{\tan^{-1}x}\; x + c$$

4. $\int \frac{(x^2+1)\cdot e^x}{(x+1)^2}\,dx$

Solution :

$$I = \int e^x\left[\frac{x^2+1}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{x^2-1+2}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right]dx$$

Here $\quad f(x) = \frac{x-1}{x+1}$

$$\Rightarrow \quad f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\because \quad \int [\,f(x) + f'(x)]\;dx = e^x\; f(x) + c$$

$$I = e^x\left(\frac{x-1}{x+1}\right) + c$$

$$\therefore \int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx = e^x\left(\frac{x-1}{x+1}\right) + c$$

EXERCISE 3.3

I. Evaluate the following :

1. $\int x^2 \cdot \log x\,dx$
2. $\int x^2 \cdot \sin 3x\,dx$
3. $\int x \cdot \tan^{-1} x\,dx$
4. $\int x^2 \cdot \tan^{-1} x\,dx$
5. $\int x^3 \cdot \tan^{-1} x\,dx$
6. $\int (\log x)^2\;dx$
7. $\int \sec^3 x\,dx$
8. $\int x \cdot \sin^2 x\,dx$
9. $\int x^3 \cdot \log x\,dx$
10. $\int e^{2x} \cdot \cos 3x\,dx$
11. $\int x \cdot \sin^{-1} x\,dx$
12. $\int x^2 \cdot \cos^{-1} x\,dx$
13. $\int \frac{\log(\log x)}{x}\,dx$
14. $\int \frac{t\,\sin^{-1} t}{\sqrt{1-t^2}}\,dt$
15. $\int \cos\sqrt{x}\,dx$
16. $\int \sin\theta \cdot \log(\cos\theta)\,d\theta$
17. $\int x \cdot \cos^3 x\,dx$
18. $\int \frac{\sin(\log x)^2}{x} \cdot \log x\,dx$
19. $\int \frac{\log x}{x}\,dx$
20. $\int x \cdot \sin 2x \cdot \cos 5x\,dx$
21. $\int \cos\left(\sqrt[3]{x}\right)dx$

II. Integrate the following functions *w. r. t. x* :

1. $e^{2x} \cdot \sin 3x$
2. $e^{-x} \cdot \cos 2x$
3. $\sin(\log x)$
4. $\sqrt{5x^2 + 3}$
5. $x^2 \cdot \sqrt{a^2 - x^6}$
6. $\sqrt{(x-3)(7-x)}$
7. $\sqrt{4^x(4^x + 4)}$
8. $(x+1)\sqrt{2x^2+3}$
9. $x\sqrt{5 - 4x - x^2}$
10. $\sec^2 x \cdot \sqrt{\tan^2 x + \tan x - 7}$
11. $\sqrt{x^2 + 2x + 5}$
12. $\sqrt{2x^2 + 3x + 4}$

III. Integrate the following functions *w. r. t. x* :

1. $(2 + \cot x - \operatorname{cosec}^2 x) \cdot e^x$
2. $\left(\dfrac{1 + \sin x}{1 + \cos x}\right) \cdot e^x$
3. $e^x \cdot \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$
4. $\left(\dfrac{x}{(x+1)^2}\right) \cdot e^x$
5. $\dfrac{e^x}{x}\,[x(\log x)^2 + 2(\log x)]$
6. $e^{5x} \cdot \left(\dfrac{5x \cdot \log x + 1}{x}\right)$
7. $e^{\sin^{-1} x} \cdot \left(\dfrac{x + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}\right)$
8. $\log(1 + x)^{(1+x)}$
9. $\operatorname{cosec}(\log x)\,[1 - \cot(\log x)]$

3.4 Integration by partial fraction :

If $f(x)$ and $g(x)$ are two polynomials then $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ is called a rational algebraic function. $\dfrac{f(x)}{g(x)}$ is called a proper rational function provided degree of $f(x)$ < degree of $g(x)$; otherwise it is called **improper rational function**.

If degree of $f(x) \geq$ degree of $g(x)$ i.e. $\dfrac{f(x)}{g(x)}$ is an improper rational function then express it as in the form Quotient + $\dfrac{\text{Remainder}}{g(x)}$, $g(x) \neq 0$ where $\dfrac{\text{Remainder}}{g(x)}$ is proper rational function.

Lets see the three different types of the proper rational function $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ where the denominator $g(x)$ is expressed as

(i) a non-repeated linear factors

(ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

No.	Rational form	Partial form
(i)	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x-c)}$
(ii)	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-c)}$
(iii)	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

Type (i) : $\displaystyle\int \frac{px^2+qx+r}{(x-a)(x-b)(x-c)}\,dx$ i.e. denominator is expressed as non-repeated Linear factors.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{3x^2+4x-5}{(x^2-1)(x+2)}\,dx$

Solution : $\displaystyle I = \int \frac{3x^2+4x-5}{(x-1)(x+1)(x+2)}\,dx$

Consider, $$\frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{C}{(x+2)}$$

$$= \frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$\therefore\quad 3x^2+4x-5 = A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)$

at $x=1$, $\quad 3(1)^2+4(1)-5 = A(2)(3)+B(0)+C(0)$

$2=6A \quad\Rightarrow\quad A=\dfrac{1}{3}$

at $x=-1$, $\quad 3(-1)^2+4(-1)-5 = A(0)+B(-2)(1)+C(0)$

$-6=-2B \quad\Rightarrow\quad B=3$

at $x=-2$, $\quad 3(-2)^2+4(-2)-5 = A(0)+B(0)+C(-3)(-1)$

$-1=3C \quad\Rightarrow\quad C=-\dfrac{1}{3}$

Thus, $$\frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}$$

$$\therefore\ I = \int\left[\frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}\right]dx \quad = \frac{1}{3}\log(x-1)+3\log(x+1)-\frac{1}{3}\log(x+2)+c$$

$$= \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c \qquad \therefore \int\frac{3x^2+4x-5}{(x^2-1)(x+2)}\,dx = \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c$$

2. $\displaystyle\int \frac{2x^2-3}{(x^2-5)(x^2+4)}\,dx$

Solution : Consider, $\dfrac{2x^2-3}{(x^2-5)(x^2+4)}$

Let $x^2 = m$

$\therefore \quad = \dfrac{2m-3}{(m-5)(m+4)}$ . . . proper rational function.

Now, $\dfrac{2m-3}{(m-5)(m+4)} = \dfrac{A}{(m-5)} + \dfrac{B}{(m+4)} = \dfrac{A(m+4)+B(m-5)}{(m-5)(m+4)}$

$\therefore \quad 2m-3 = A(m+4) + B(m-5)$

at $m = 5$, $\quad 2(5) - 3 = A(9) + B(0)$

$7 = 9A \quad \Rightarrow \quad A = \dfrac{7}{9}$

at $m = -4$, $\quad 2(-4) - 3 = A(0) + B(-9)$

$-11 = -9B \quad \Rightarrow \quad B = \dfrac{11}{9}$

Thus, $\dfrac{2m-3}{(m-5)(m+4)} = \dfrac{\left(\frac{7}{9}\right)}{(m-5)} + \dfrac{\left(\frac{11}{9}\right)}{(m+4)}$ i.e. $\dfrac{2x^2-3}{(x^2-5)(x^2+4)} = \dfrac{\left(\frac{7}{9}\right)}{x^2-5} + \dfrac{\left(\frac{11}{9}\right)}{x^2+4}$

$$\therefore \text{I} = \int \left[\frac{\left(\frac{7}{9}\right)}{x^2-5} + \frac{\left(\frac{11}{9}\right)}{x^2+4}\right] dx$$

$$= \frac{7}{9}\cdot\int \frac{1}{x^2-(\sqrt{5})^2}\,dx + \frac{11}{9}\cdot\int \frac{1}{x^2+(2)^2}\,dx$$

$$= \frac{7}{9}\cdot\frac{1}{2(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{9}\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \text{I} = \frac{7}{18(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \int \frac{2x^2-3}{(x^2-5)(x^2+4)}\,dx = \frac{7}{18(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

3. $\displaystyle\int \frac{1}{(\sin\theta)(3+2\cos\theta)}\,d\theta$

Solution : $\displaystyle I = \int \frac{1}{(\sin\theta)(3+2\cos\theta)}\,d\theta = \int \frac{\sin\theta}{(1-\cos^2\theta)(3+2\cos\theta)}\,d\theta$

$$= \int \frac{\sin\theta}{(1-\cos\theta)(1+\cos\theta)(3+2\cos\theta)}\,d\theta$$

put $\cos\theta = t$ $\quad\therefore\quad -\sin\theta\cdot d\theta = 1\,dt$

$\therefore\quad \sin\theta\cdot d\theta = -1\,dt$

Consider, $\displaystyle \frac{-1}{(1-t)(1+t)(3+2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(3+2t)}$

$$= \frac{A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)}{(1-t)(1+t)(3+2t)}$$

$\therefore\quad -1 = A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)$

at $t = 1$, $\quad -1 = A(2)(5) + B(0) + C(0)$

$\displaystyle -1 = 10A \quad\Rightarrow\quad A = -\frac{1}{10}$

at $t = -1$, $\quad -1 = A(0) + B(2)(1) + C(0)$

$\displaystyle -1 = 2B \quad\Rightarrow\quad B = -\frac{1}{2}$

at $\displaystyle t = -\frac{3}{2}$, $\quad\displaystyle -1 = A(0) + B(0) + C\left(+\frac{5}{2}\right)\left(-\frac{1}{2}\right)$

$\displaystyle -1 = -\frac{5}{4}C \quad\Rightarrow\quad C = \frac{4}{5}$

Thus, $\displaystyle \frac{-1}{(1-t)(1+t)(3+2t)} = \frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}$

$$\therefore I = \int \left[\frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}\right] dt$$

$$= -\frac{1}{10}\log(1-t)\cdot\frac{1}{(-1)} - \frac{1}{2}\log(1+t) + \frac{4}{5}\log(3+2t)\cdot\frac{1}{2} + c$$

$$= \frac{1}{10}\log(1-\cos\theta) - \frac{1}{2}\log(1+\cos\theta) + \frac{4}{10}\log(3+2\cos\theta) + c$$

$$= \frac{1}{10}\left(\log\frac{(1-\cos\theta)(3+2\cos\theta)^4}{(1+\cos\theta)^5}\right) + c \qquad \because \log a^m = m\cdot\log a$$

4. $\displaystyle\int \frac{1}{2\cos x + \sin 2x}\,dx$

Solution : $\displaystyle I = \int \frac{1}{2\cos x + \sin 2x}\,dx \quad = \int \frac{1}{2\cos x + 2\sin x \cdot \cos x}\,dx \quad = \int \frac{1}{2(\cos x)(1+\sin x)}\,dx$

$$= \frac{1}{2}\cdot\int \frac{\cos x}{\cos^2 x\,(1+\sin x)}\,dx = \frac{1}{2}\int \frac{\cos x}{(1-\sin^2 x)(1+\sin x)}\,dx$$

put $\sin x = t$ $\quad\therefore\quad \cos x\,dx = 1\,dt$

$$= \frac{1}{2}\cdot\int \frac{1}{(1-t^2)(1+t)}\,dt \quad = \frac{1}{2}\cdot\int \frac{1}{(1-t)(1+t)(1+t)}\,dt = \frac{1}{2}\cdot\int \frac{1}{(1-t)(1+t)^2}\,dt$$

Consider, $\displaystyle \frac{1}{(1-t)(1+t)^2} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2} = \frac{A(1+t)^2 + B(1-t)(1+t) + C(1-t)}{(1-t)(1+t)^2}$

$$\therefore \quad 1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$$

at $t = 1$, $\quad 1 = A(2)^2 + B(0) + C(0)$

$1 = 4A \quad \Rightarrow \quad A = \dfrac{1}{4}$

at $t = -1$, $\quad 1 = A(0) + B(0) + C(2)$

$1 = 2C \quad \Rightarrow \quad C = \dfrac{1}{2}$

at $t = 0$, $\quad 1 = A(1)^2 + B(1)(1) + C(1)$

$1 = A + B + C$

$1 = \dfrac{1}{4} + B + \dfrac{1}{2} \quad \Rightarrow \quad B = \dfrac{1}{4}$

Thus, $\displaystyle \frac{1}{(1-t)(1+t)^2} = \frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}$

$$\therefore\ I = \int\left[\frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}\right]dt = \frac{1}{2}\left[\frac{1}{4}\log(1-t)\cdot\frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2}\cdot\frac{(-1)}{(1+t)}\right] + c$$

$$= \frac{1}{2}\left[\frac{1}{4}\log(1-t)\cdot\frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2}\cdot\frac{-1}{1+t}\right] + c$$

$$= \frac{1}{8}\left[-\log(1-\sin x) + \log(1+\sin x) - \frac{2}{1+\sin x}\right] + c \quad = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x}\right] + c$$

$$\therefore \quad \int \frac{1}{2\cos x + \sin 2x}\cdot dx = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x}\right] + c$$

5. $\displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\, d\theta$

Solution : $\displaystyle I = \int \frac{(\tan\theta)(1+\tan^2\theta)}{1+\tan^3\theta}\, d\theta \quad = \int \frac{(\tan\theta)(1+\tan^2\theta)}{1+\tan^3\theta}\, d\theta = \int \frac{\tan\theta \quad \sec^2\theta}{1+\tan^3\theta}\, d\theta$

put $\tan\theta = x \qquad \therefore \quad \sec^2\theta\, d\theta = 1\, dx$

$$= \int \frac{x}{1+x^3}\, dx = \int \frac{x}{(1+x)(1-x+x^2)}\, dx$$

Consider, $\displaystyle \frac{x}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{(1-x+x^2)}$

$$= \frac{A(1-x+x^2) + Bx + C(1+x)}{(1+x)(1-x+x^2)}$$

$$\therefore x \;= A(1-x+x^2) + (Bx+C)(1+x) = A - Ax + Ax^2 + Bx + Bx^2 + C + Cx$$

$$0\,x^2 + 1\cdot x + 0 = (A+B)\,x^2 + (-A+B+C)\,x + (A+C)$$

compairing the co-efficients of like powers of variables.

$0 = A + B \qquad$ **. . . (I)**

$1 = -A + B + C \qquad$ **. . . (II)** $\qquad$ and

$0 = A + C \qquad$ **. . . (III)**

Solving these equations, we get $A = -\dfrac{1}{3}$; $B = \dfrac{1}{3}$ and $C = \dfrac{1}{3}$

Thus, $\displaystyle \frac{x}{(1+x)(1-x+x^2)} = \frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1-x+x^2)}$

$$\therefore I = \int \left[\frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1-x+x^2)} \right] dx \quad = -\frac{1}{3}\cdot\int \frac{1}{1+x}\, dx + \frac{1}{3}\cdot\int \frac{x+1}{1-x+x^2}\, dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\, dx + \frac{1}{3}\cdot\frac{1}{(2)}\int \frac{2x-1+3}{x^2-x+1}\, dx \qquad \because \quad \frac{d}{dx}\, x^2 - x + 1 = 2x - 1$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\, dx + \frac{1}{3}\cdot\frac{1}{2}\cdot\int \frac{2x-1+3}{x^2-x+1}\, dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\, dx + \frac{1}{6}\cdot\int \frac{2x-1}{x^2-x+1}\, dx + \frac{1}{6}\cdot\int \frac{3}{x^2-x+1}\, dx$$

$= \quad I_1 + I_2 + I_3 \qquad$ **. . . (IV)**

$$\therefore \quad I_1 = -\frac{1}{3}\int \frac{1}{1+x}\, dx \quad = -\frac{1}{3}[\log(1+x)]$$

$$= -\frac{1}{3}\log\left(1+\tan\theta\right) + c_1 \qquad \ldots \textbf{(V)}$$

$$\therefore \quad I_2 = \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\, dx \quad = \frac{1}{6}[\log(x^2-x+1)] + c_2$$

$$= \frac{1}{6}\log\left(\tan^2\theta - \tan\theta + 1\right) + c_2 \qquad \ldots \textbf{(VI)}$$

$$\therefore \quad I_3 = \frac{1}{6}\int \frac{3}{x^2-x+1}\, dx$$

$$= \frac{1}{2}\int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1}\, dx \qquad \because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\right]\tan^{-1}\left[\frac{x-\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\right] + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$\therefore \quad I_3 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + c \qquad \ldots \textbf{(VII)}$$

$$\therefore \quad \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\, d\theta = -\frac{1}{3}\log\left(1+\tan\theta\right) + \frac{1}{6}\log\left(\tan^2\theta - \tan\theta + 1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right) + c$$

EXERCISE 3.4

I. Integrate the following *w. r. t. x* :

1. $\dfrac{x^2+2}{(x-1)(x+2)(x+3)}$
2. $\dfrac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$
3. $\dfrac{12x+3}{6x^2+13x-63}$

4. $\dfrac{2x}{4-3x-x^2}$

5. $\dfrac{x^2+x-1}{x^2+x-6}$

6. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

7. $\dfrac{12x^2-2x-9}{(4x^2-1)(x+3)}$

8. $\dfrac{1}{x(x^5+1)}$

9. $\dfrac{2x^2-1}{x^4+9x^2+20}$

10. $\dfrac{x^2+3}{(x^2-1)(x^2-2)}$

11. $\dfrac{2x}{(2+x^2)(3+x^2)}$

12. $\dfrac{2^x}{4^x-3\cdot 2^x-4}$

13. $\dfrac{3x-2}{(x+1)^2(x+3)}$

14. $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

15. $\dfrac{1}{x(1+4x^3+3x^6)}$

16. $\dfrac{1}{x^3-1}$

17. $\dfrac{(3\sin x-2)\cdot\cos x}{5-4\sin x-\cos^2 x}$

18. $\dfrac{1}{\sin x+\sin 2x}$

19. $\dfrac{1}{2\sin x+\sin 2x}$

20. $\dfrac{1}{\sin 2x+\cos x}$

21. $\dfrac{1}{\sin x\cdot(3+2\cos x)}$

22. $\dfrac{5\cdot e^x}{(e^x+1)(e^{2x}+9)}$

23. $\dfrac{2\log x+3}{x(3\log x+2)[(\log x)^2+1]}$

3.5 Something Interesting :

Students/ now familier with the integration by parts.

The result is $\int u\,v\,dx = u\int v\,dx - \int\left(\dfrac{d}{dx}u\right)\left(\int v\,dx\right)dx$,

u and v are differentiable functions of x and $u\cdot v$ follows L I A T E order.

This result can be extended to the generalisation as -

$\int u\,v\,dx = u\,v_1 - u'\,v_2 + u''\,v_3 - u'''\,v_4 + \dots$

(′) dash indicates the derivative.

($_1$) subscript indicates the integration.

This result is more useful where the first function (u) is a polynomial, because $\dfrac{d^n u}{dx^n} = 0$ for some n.

For example : $\int x^2\cos 3x\,dx$

$$= x^2\left(\sin 3x\,\frac{1}{3}\right) - (2x)\left(-\cos 3x\,\frac{1}{3}\,\frac{1}{3}\right) + (2)\left(-\sin 3x\,\frac{1}{3}\,\frac{1}{9}\right) - (0)$$

$$= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + c$$

verify this example with usual rule of integration by parts.

Let us Remember

- We can always add arbitarary constant c to the integration obtained :

(I) i.e. $\dfrac{d}{dx}\, g(x) = f(x) \quad \Rightarrow \quad \int f(x)\; dx = g(x) + c$

$f(x)$ is integrand, $g(x)$ is integral of $f(x)$ with respect to x, c is arbitarary constant.

(II) $\int f(ax+b)\; dx = g(ax+b)\; \dfrac{1}{a} + c$

(III) (1) $\int [f(x)]^n \;\; f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$ (2) $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \log\left(f(x)\right) + c$

(3) $\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$

(IV) (1) $\int x^n \;\; dx = \dfrac{x^{n+1}}{n+1} + c$ (2) $\displaystyle\int \frac{1}{\sqrt{x}}\;\; dx = 2\sqrt{x} + c$

(3) $\int (k)\, dx = kx + c$ (4) $\int a^x \;\; dx = \dfrac{a^x}{\log a} + c$

(5) $\int e^x \;\; dx = e^x + c$ (6) $\displaystyle\int \frac{1}{x}\;\; dx = \log(x) + c$

(7) $\int \sin x \;\; dx = -\cos x + c$ (8) $\int \cos x \;\; dx = \sin x + c$

(9) $\int \tan x \;\; dx = \log(\sec x) + c$ (10) $\int \cot x \;\; dx = \log(\sin x) + c$

(11) $\int \sec x \;\; dx = \log(\sec x + \tan x) + c$ (12) $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + c$

$= \log\left[\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right] + c \qquad\qquad = \log\left[\tan\left(\dfrac{x}{2}\right)\right] + c$

(13) $\int \sec^2 x \;\; dx = \tan x + c$ (14) $\int \operatorname{cosec}^2 x \;\; dx = -\cot x + c$

(15) $\int \sec x \;\; \tan x \;\; dx = \sec x + c$ (16) $\int \operatorname{cosec} x \;\; \cot x \;\; dx = -\operatorname{cosec} x + c$

(17) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\;\; dx = \sin^{-1} x + c$ (18) $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}\;\; dx = \cos^{-1} x + c$

(19) $\displaystyle\int \frac{1}{1+x^2}\;\; dx = \tan^{-1} x + c$ (20) $\displaystyle\int \frac{-1}{1+x^2}\;\; dx = \cot^{-1} x + c$

(21) $\displaystyle\int \frac{1}{x\;\sqrt{x^2-1}}\;\; dx = \sec^{-1} x + c$ (22) $\displaystyle\int \frac{-1}{x\;\sqrt{x^2-1}}\;\; dx = \operatorname{cosec}^{-1} x + c$

(23) $\displaystyle\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

(24) $\displaystyle\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left(\frac{x - a}{x + a}\right) + c$

(25) $\displaystyle\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left(\frac{a + x}{a - x}\right) + c$

(26) $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(27) $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left| x + \sqrt{x^2 - a^2}\right| + c$

(28) $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log\left| x + \sqrt{x^2 + a^2}\right| + c$

(29) $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

(30) $\displaystyle\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

(31) $\displaystyle\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right) + c$

(32) $\displaystyle\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right) + c$

(V) If u and v are differentiable functions of x then $\displaystyle\int u\,v\, dx = u\int v\, dx - \int\left(\frac{d}{dx}u\right)\left(\int v\, dx\right) dx$ where $u\ v$ follows the L I A T E order.

(VI) $\displaystyle\int e^x\,[\,f(x) + f'(x)]\ dx = e^x\ f(x) + c$

(VII) For the integration of type $\displaystyle\int \frac{f(x)}{g(x)}\, dx$, $g(x) \neq 0$ where $\dfrac{f(x)}{g(x)}$ proper rational function.

(i) non-repeated linear factors (ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

(VIII) $\displaystyle\int \frac{1}{x^2 + a^2}\, dx$, $\displaystyle\int \frac{1}{x^2 - a^2}\, dx$, $\displaystyle\int \frac{1}{a^2 - x^2}\, dx$

$\displaystyle\int \frac{1}{ax^2 + bx + c}\, dx$
Method of completing square

$\displaystyle\int \frac{1}{a\sin x + b\cos x + c}\, dx$
put $\tan\left(\dfrac{x}{2}\right) = t$

$\displaystyle\int \frac{px + q}{ax^2 + bx + c}\, dx$
$px + q = A\dfrac{d}{dx}(ax^2 + bx + c) + B$

$\displaystyle\int \frac{1}{a\sin^2 x + b\cos^2 x + c}\, dx$
Divide Nr and Dr by $\cos^2 x$

(VIII) $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\, dx$, $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$

$\displaystyle\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$

$\displaystyle\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx$

MISCELLANEOUS EXERCISE 3

(I) Choose the correct option from the given alternatives :

(1) $\displaystyle\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}\,dx =$

(A) $\dfrac{1}{2}\sqrt{x+1}+c$ (B) $\dfrac{2}{3}(x+1)^{\frac{3}{2}}+c$ (C) $\sqrt{x+1}+c$ (D) $2\,(x+1)^{\frac{3}{2}}+c$

(2) $\displaystyle\int \frac{1}{x+x^5}\,dx = f(x)+c$, then $\displaystyle\int \frac{x^4}{x+x^5}\,dx =$

(A) $\log x - f(x)+c$ (B) $f(x)+\log x+c$ (C) $f(x)-\log x+c$ (D) $\dfrac{1}{5}x^5 f(x)+c$

(3) $\displaystyle\int \frac{\log(3x)}{x\log(9x)}\,dx =$

(A) $\log(3x)-\log(9x)+c$ (B) $\log(x)-(\log 3)\cdot\log(\log 9x)+c$

(C) $\log 9-(\log x)\cdot\log(\log 3x)+c$ (D) $\log(x)+(\log 3)\cdot\log(\log 9x)+c$

(4) $\displaystyle\int \frac{\sin^m x}{\cos^{m+2}x}\,dx =$

(A) $\dfrac{\tan^{m+1}x}{m+1}+c$ (B) $(m+2)\tan^{m+1}x+c$ (C) $\dfrac{\tan^m x}{m}+c$ (D) $(m+1)\tan^{m+1}x+c$

(5) $\displaystyle\int \tan(\sin^{-1}x)\,dx =$

(A) $(1-x^2)^{-\frac{1}{2}}+c$ (B) $(1-x^2)^{\frac{1}{2}}+c$ (C) $\dfrac{\tan^m x}{\sqrt{1-x^2}}+c$ (D) $-\sqrt{1-x^2}+c$

(6) $\displaystyle\int \frac{x-\sin x}{1-\cos x}\,dx =$

(A) $x\cot\left(\dfrac{x}{2}\right)+c$ (B) $-x\cot\left(\dfrac{x}{2}\right)+c$ (C) $\cot\left(\dfrac{x}{2}\right)+c$ (D) $x\tan\left(\dfrac{x}{2}\right)+c$

(7) If $f(x)=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $g(x)=e^{\sin^{-1}x}$, then $\int f(x)\cdot g(x)\,dx =$

(A) $e^{\sin^{-1}x}\,(\sin^{-1}x-1)+c$ (B) $e^{\sin^{-1}x}\,(1-\sin^{-1}x)+c$

(C) $e^{\sin^{-1}x}\,\overline{(\sin^{-1}x+1)}+c$ (D) $e^{\sin^{-1}x}\,(\sin^{-1}x-1)+c$

(8) If $\int \tan^3 x\sec^3 x\,dx = \left(\dfrac{1}{m}\right)\sec^m x-\left(\dfrac{1}{n}\right)\sec^n x+c$, then $(m, n) =$

(A) $(5, 3)$ (B) $(3, 5)$ (C) $\left(\dfrac{1}{5}, \dfrac{1}{3}\right)$ (D) $(4, 4)$

(9) $\displaystyle\int \frac{1}{\cos x - \cos^2 x}\, dx =$

(A) $\log(\operatorname{cosec} x - \cot x) + \tan\left(\frac{x}{2}\right) + c$ (B) $\sin 2x - \cos x + c$

(C) $\log(\sec x + \tan x) - \cot\left(\frac{x}{2}\right) + c$ (D) $\cos 2x - \sin x + c$

(10) $\displaystyle\int \frac{\sqrt{\cot x}}{\sin x \cos x}\, dx =$

(A) $2\sqrt{\cot x} + c$ (B) $-2\sqrt{\cot x} + c$ (C) $\frac{1}{2}\sqrt{\cot x} + c$ (D) $\sqrt{\cot x} + c$

(11) $\displaystyle\int \frac{e^x(x-1)}{x^2}\, dx =$

(A) $\frac{e^x}{x} + c$ (B) $\frac{e^x}{x^2} + c$ (C) $\left(x - \frac{1}{x}\right)e^x + c$ (D) $xe^{-x} + c$

(12) $\int \sin(\log x)\, dx =$

(A) $\frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$ (B) $\frac{x}{2}[\sin(\log x) + \cos(\log x)] + c$

(C) $\frac{x}{2}[\cos(\log x) - \sin(\log x)] + c$ (D) $\frac{x}{4}[\cos(\log x) - \sin(\log x)] + c$

(13) $\int x^x(1 + \log x)\, dx =$

(A) $\frac{1}{2}(1 + \log x)^2 + c$ (B) $x^{2x} + c$ (C) $x^x \log x + c$ (D) $x^x + c$

(14) $\int \cos^{-\frac{3}{7}} x \sin^{-\frac{11}{7}} x\, dx =$

(A) $\log\left(\sin^{-\frac{4}{7}} x\right) + c$ (B) $\frac{4}{7}\tan^{\frac{4}{7}} x + c$

(C) $-\frac{7}{4}\tan^{-\frac{4}{7}} x + c$ (D) $\log\left(\cos^{\frac{3}{7}} x\right) + c$

(15) $\displaystyle 2\int \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}\, dx =$

(A) $\sin 2x + c$ (B) $\cos 2x + c$ (C) $\tan 2x + c$ (D) $2\sin 2x + c$

(16) $\displaystyle\int \frac{dx}{\cos x\sqrt{\sin^2 x - \cos^2 x}}\, dx =$

(A) $\log\left(\tan x - \sqrt{\tan^2 x - 1}\right) + c$ (B) $\sin^{-1}(\tan x) + c$

(C) $1 + \sin^{-1}(\cot x) + c$ (D) $\log\left(\tan x + \sqrt{\tan^2 x - 1}\right) + c$

(17) $\int \frac{\log x}{(\log ex)^2}\, dx =$

(A) $\frac{x}{1+\log x}+c$ (B) $x(1+\log x)+c$ (C) $\frac{1}{1+\log x}+c$ (D) $\frac{1}{1-\log x}+c$

(18) $\int [\sin(\log x)+\cos(\log x)]\, dx =$

(A) $x\cos(\log x)+c$ (B) $\sin(\log x)+c$ (C) $\cos(\log x)+c$ (D) $x\sin(\log x)+c$

(19) $\int \frac{\cos 2x-1}{\cos 2x+1}\, dx =$

(A) $\tan x - x + c$ (B) $x+\tan x + c$ (C) $x-\tan x+c$ (D) $-x-\cot x+c$

(20) $\int \frac{e^{2x}+e^{-2x}}{e^x}\, dx =$

(A) $e^x-\frac{1}{3e^{3x}}+c$ (B) $e^x+\frac{1}{3e^{3x}}+c$ (C) $e^{-x}+\frac{1}{3e^{3x}}+c$ (D) $e^{-x}-\frac{1}{3e^{3x}}+c$

(II) Integrate the following with respect to the respective variable :

(1) $(x-2)^2\sqrt{x}$ (2) $\frac{x^7}{x+1}$ (3) $(6x+5)^{\frac{3}{2}}$

(4) $\frac{t^3}{(t+1)^2}$ (5) $\frac{3-2\sin x}{\cos^2 x}$ (6) $\frac{\sin^6\theta+\cos^6\theta}{\sin^2\theta\cdot\cos^2\theta}$

(7) $\cos 3x\cdot\cos 2x\cdot\cos x$ (8) $\frac{\cos 7x-\cos 8x}{1+2\cos 5x}$ (9) $\cot^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

(III) Integrate the following :

(1) $\frac{(1+\log x)^3}{x}$ (2) $\cot^{-1}(1-x+x^2)$ (3) $\frac{1}{x\cdot\sin^2(\log x)}$

(4) $\sqrt{x}\sec\left(x^{\frac{3}{2}}\right)\cdot\tan\left(x^{\frac{3}{2}}\right)$ (5) $\log(1+\cos x)-x\cdot\tan\left(\frac{x}{2}\right)$ (6) $\frac{x^2}{\sqrt{1-x^6}}$

(7) $\frac{1}{(1-\cos 4x)(3-\cot 2x)}$ (8) $\log(\log x)+(\log x)^{-2}$ (9) $\frac{1}{2\cos x+3\sin x}$

(10) $\frac{1}{x^3\cdot\sqrt{x^2-1}}$ (11) $\frac{3x+1}{\sqrt{-2x^2+x+3}}$ (12) $\log(x^2+1)$

(13) $e^{2x}\cdot\sin x\cdot\cos x$ (14) $\frac{x^2}{(x-1)(3x-1)(3x-2)}$ (15) $\frac{1}{\sin x+\sin 2x}$

(16) $\sec^2 x\cdot\sqrt{7+2\tan x-\tan^2 x}$ (17) $\frac{x+5}{x^3+3x^2-x-3}$ (18) $\frac{1}{x\cdot(x^5+1)}$

(19) $\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}$ (20) $\sec^4 x\cdot\operatorname{cosec}^2 x$

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