8. BINOMIAL DISTRIBUTION

Let us Study

- Bernoulli Trial
- Binomial distribution
- Mean and variance of Binomial Distribution.

Let us Recall

- Many experiments are dichotomous in nature. For example, a tossed coin shows a 'head' or 'tail', A result of student 'pass' or 'fail', a manufactured item can be 'defective' or 'non-defective', the response to a question might be 'yes' or 'no', an egg has 'hatched' or 'not hatched', the decision is 'yes' or 'no' etc. In such cases, it is customary to call one of the outcomes a 'success' and the other 'not success' or 'failure'. For example, in tossing a coin, if the occurrence of the head is considered a success, then occurrence of tail is a failure.

Let us Learn

8.1.1 Bernoulli Trial :

Each time we toss a coin or roll a die or perform any other experiment, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or failure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred to as 'success' or 'failure' are called Bernoulli trials.

Definition:

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) Each trial has exactly two outcomes : success or failure.

(ii) The probability of success remains the same in each trial.

Throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results in success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent trials. If the die is fair and has six numbers 1 to 6 written on six faces, then

$$p = \frac{1}{2} \text{ and } q = 1 - p \qquad \therefore\ q = \frac{1}{2}$$

For example :

Consider a die to be thrown 20 times. if the result is an even number, consider it a success, else it is a failure. Then $p = \frac{1}{2}$ as there are 3 even numbers in the possible outcomes.

If in the same experiment, we consider the result a success if it is a multiple of 3, then $p = \frac{1}{3}$ as there are 2 multiples of 3 among the six possible outcomes. Both above trials are Bernoulli trials.

SOLVED EXAMPLE

Ex. 1 : Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is

(i) replaced (ii) not replaced in the urn.

Solution :

(i) The number of trials is finite. When the drawing is done with replacement, the probability of success (say, red ball) is $p = \frac{7}{16}$ which is same for all six trials (draws). Hence, the drawing of balls with replacements are Bernoulli trials.

(ii) When the drawing is done without replacement, the probability of success (i.e. red ball) in first trial is $\frac{7}{16}$ in second trial is $\frac{6}{15}$ if first ball drawn is red and is $\frac{7}{15}$ if first ball drawn is black and so on. Clearly probability of success is not same for all trials, hence the trials are not Bernoulli trials.

8.2 Binomial distribution:

Consider the experiment of tossing a coin in which each trial results in success (say, heads) or failure (tails). Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways in which we have one success in six trials. Clearly, six different cases are there as listed below:

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS.

Similarly, two successes and four failures can have $\frac{6!}{4\,! \times 2\,!} = 15$ combinations.

But as n grows large, the calculation can be lengthy. To avoid this the number for certain probabilities can be obtained with Bernoullis formula.For this purpose, let us take the experiment made up of three Bernoulli trials with probabilities p and $q = 1 - p$ for success and failure respectively in each trial. The sample space of the experiment is the set

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The number of successes is a random variable X and can take values 0, 1, 2, or 3.The probability distribution of the number of successes is as below :

$$P(X=0) = P\text{ (no success)}$$

$$= P(\{\text{FFF}\}) = P(F)\cdot P(F)\cdot P(F)\text{, since trials are independent.}$$

$$= q\cdot q\cdot q = q^3$$

$$P(X=1) = P\text{ (one success)}$$

$$= P(\{\text{SFF, FSF, FFS}\})$$

$$= P(\{\text{SFF}\}) + P(\{\text{FSF}\}) + P(\{\text{FFS}\})$$

$$= P(S)\cdot P(F)\cdot P(F) + P(F)\cdot P(S)\cdot P(F) + P(F)\cdot P(F)\cdot P(S)$$

$$= p\cdot q\cdot q + q\cdot p\cdot q + q\cdot q\cdot p = 3pq^2$$

$$P(X=2) = P\text{ (two success)}$$

$$= P(\{\text{SSF, SFS, FSS}\})$$

$$= P(\{\text{SSF}\}) + P(\{\text{SFS}\}) + P(\{\text{FSS}\})$$

$$= P(S)\cdot P(S)\cdot P(F) + P(S)\cdot P(F)\cdot P(S) + P(F)\cdot P(S)\cdot P(S)$$

$$= p\cdot p\cdot q + p\cdot q\cdot p + q\cdot p\cdot p = 3p^2q$$

and $$P(X=3) = P\text{ (three successes)}$$

$$= P(\{\text{SSS}\})$$

$$= P(S)\cdot P(S)\cdot P(S)$$

$$= p^3$$

Thus, the probability distribution of X is

X	0	1	2	3
$P(X)$	q^3	$3q^2p$	$3qp^2$	p^3

Also, the binominal expansion of

$$(q+p)^3 \text{ is } q^3 + 3q^2p + 3qp^2 + p^3$$

Note that the probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q+p)^3$.

Also, since $q + p = 1$, it follows that the sum of these probabilities, as expected, is 1.Thus, we may conclude that in an experiment of n-Bernoulli trials, the probabilities of 0, 1, 2,..., n successes can be obtained as 1^{st}, 2^{nd}, 3^{rd} , . . . , $(n+1)^{th}$ terms in the expansion of $(q+p)^n$. To prove this assertion (result), let us find the probability of x successes in an experiment of n-Bernoulli trials.

Clearly, in case of x successes (S), there will be $(n-x)$ failures (F). Now x successes (S) and $(n-x)$ failures (F) can be obtained in $\dfrac{n\,!}{x\,!\,(n-x)\,!}$ ways.

In each of these ways the probability of x successes and $(n-x)$ failures

$$= P\,(x \text{ successes})\cdot P\,\big((n-x) \text{ failures}\big)$$

$$= \big(P\,(S)\cdot P\,(S). . . P\,(S)\ x \text{ times}\big)\cdot\big(P\,(F\,)\cdot P(F)\big)\cdot\ . . .\ \cdot\big(P(F)\cdot(n-x) \text{ times}\big)$$

$$= \big(\,p\cdot p\cdot p. . .\,p\ x \text{ times}\big)\ \big(q\cdot q\cdot q. . .\,q\ (n-x) \text{ times}\big)$$

$$= p^x\cdot q^{n-x}$$

Thus probability of getting x successes in n-Bernoulli trial is

$$P\,(x \text{ successes out of } n \text{ trials}) = \frac{n\,!}{x\,!\,(n-x)\,!}\times p^x\times q^{n-x} = {}^nC_x\, p^x\times q^{n-x}$$

Clearly, P (x successes), i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x+1)^{th}$ term in the binomial expansion of $(q+p)^n$.

Thus, the probability distribution of number of successes in an experiment consisting of n-Bernoulli trials may be obtained by the binomial expansion of $(q+p)^n$. Hence, this distribution of number of successes X can be written as

X	0	1	2	. . .	x	. . .	n
$P\,(X)$	${}^nC_0\, p^0\times q^n$	${}^nC_1 p^1\times q^{n-1}$	${}^nC_2\, p^2\times q^{n-2}$	. . .	${}^nC_x\, p^x\times q^{n-x}$	. . .	${}^nC_n\, p^n\times q^0$

The above probability distribution is known as binomial distribution with parameters n and p, because for given values of n and p, we can find the complete probability distribution. It is represented $X{\sim}B\,(n, p)$ as read as X follows binomial distribution with parameters n, p

The probability of x successes $P\,(X = x)$ is also denoted by $P\,(x)$ and is given by

$$P\,(x) = {}^nC_x\cdot q^{n-x}\times p^{\,x},\ x = 0, 1, . . . , n,\ \ (q = 1-p)$$

This $P\,(x)$ is called the **probability function** of the binomial distribution.

A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by $B\,(n, p)$ or $X{\sim}B\,(n, p)$.

Lets Note : (i) The number of trials should be fixed.

(ii) The trials should be independent.

SOLVED EXAMPLES

Ex. 1 : If a fair coin is tossed 10 times, find the probability of getting

(i) exactly six heads (ii) at least six heads (iii) at most six heads

Solution : The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, $X \sim B(n, p)$ with $n = 10$ and $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2}$ $\quad \therefore \quad q = \frac{1}{2}$

$$P(X = x) = {}^{n}C_{x}\, p^{x} \times q^{n-x}$$

$$= {}^{10}C_{x}\left(\frac{1}{2}\right)^{x} \times \left(\frac{1}{2}\right)^{n-x}$$

(i) Exactly six successes means $x = 6$

$$P(X = 6) = {}^{10}C_{6}\left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{10-6} = \frac{10\,!}{6\,!\,(10-6)\,!} \times \left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10}$$

$$= \frac{105}{512}$$

(ii) At least six successes means $x \geq 6$

$$P(X \geq 6) = \left[P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)\right]$$

$$= {}^{10}C_{6}\left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{4} + {}^{10}C_{7}\left(\frac{1}{2}\right)^{7} \times \left(\frac{1}{2}\right)^{3} + {}^{10}C_{8}\left(\frac{1}{2}\right)^{8} \times \left(\frac{1}{2}\right)^{2} + {}^{10}C_{9}\left(\frac{1}{2}\right)^{9} \times \left(\frac{1}{2}\right)^{1} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{0}$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9}{2 \times 1} \times \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10}$$

$$= \left(210 + 120 + 45 + 10 + 1\right) \times \frac{1}{1024}$$

$$= \frac{386}{1024} = \frac{193}{512}$$

(iii) At most six successes means $x \leq 6$

$$P(X \leq 6) = 1 - \left(\mathrm{P}(X > 6)\right.$$

$$= 1 - \left[P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)\right]$$

$$= 1 - \left[{}^{10}C_{7}\left(\frac{1}{2}\right)^{7} \times \left(\frac{1}{2}\right)^{3} + {}^{10}C_{8}\left(\frac{1}{2}\right)^{8} \times \left(\frac{1}{2}\right)^{2} + {}^{10}C_{9}\left(\frac{1}{2}\right)^{9} \times \left(\frac{1}{2}\right)^{1} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{0}\right]$$

$$= 1 - \left[\frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9}{2 \times 1} \times \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{10}\right]$$

$$= 1 - \left[\left(120 + 45 + 10 + 1\right) \times \frac{1}{1024}\right] = 1 - \frac{176}{1024} = 1 - \frac{88}{512} = \frac{512 - 88}{512} = \frac{424}{512} = \frac{53}{64}$$

Ex. 2 : Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution : Let X denote the number of defective eggs in the 10 eggs drawn.

Since the drawing is done with replacement, the trials are Bernoulli trials.

Probability of success $= \dfrac{1}{10}$

$$p = \frac{1}{10}, \qquad q = 1 - p = 1 - \frac{1}{10} \qquad \therefore \quad q = \frac{9}{10}$$

$$n = 10$$

$$X \sim B\left(10, \frac{1}{10}\right)$$

$$P(X = x) = {}^{10}C_x \left(\frac{1}{10}\right)^x \times \left(\frac{9}{10}\right)^{10-x}$$

Here $X \geq 1$

$$P(X \geq 1) = 1 - {}^{10}C_0 \left(\frac{1}{10}\right)^0 \times \left(\frac{9}{10}\right)^{10}$$

$$= 1 - 1 \times 1 \times \left(\frac{9}{10}\right)^{10}$$

$$= 1 - \left(\frac{9}{10}\right)^{10}$$

8.3 Mean and Variance of Binomial Distribution (Formulae without proof) :

Let $X \sim B(n, p)$ then mean or expected value of r.v. X is denoted by μ or $E(X)$ and given by $\mu = E(X) = np$.

The variance is denoted by $Var(X)$ and given by $Var(X) = npq$.

Standard deviation of X is denoted by $SD(X)$ or σ and given by $SD(X) = \sigma_x = \sqrt{Var(X)}$

For example : If $X \sim B(10, 0{\cdot}4)$ then find $E(X)$ and $Var(X)$.

Solution : Here $n = 10$, $p = 0{\cdot}4$, $\quad q = 1 - p$

$$q = 1 - 0{\cdot}4 = 0{\cdot}6$$

$$E(X) = np$$

$$= 10 \times 0{\cdot}4 = 4$$

$$Var(X) = npq$$

$$= 10 \times 0{\cdot}4 \times 0{\cdot}6$$

$$= 2.4$$

SOLVED EXAMPLES

Ex. 1 : Let the p.m.f. of r.v. X be

$$P(X = x) = {}^4C_x \left(\frac{5}{9}\right)^x \times \left(\frac{4}{9}\right)^{4-x}, \text{ for } x = 0, 1, 2, 3, 4.$$

then find $E(X)$ and $Var(X)$.

Solution : $P(X = x)$ is binomial distribution with $n = 4$, $p = \frac{5}{9}$ and $q = \frac{4}{9}$

$$E(X) = np$$

$$= 4 \times \left(\frac{5}{9}\right) = \frac{20}{9}$$

$$Var(X) = npq$$

$$= 4 \times \left(\frac{5}{9}\right) \times \left(\frac{4}{9}\right) = \frac{80}{81}$$

Ex. 2 : If $E(X) = 6$ and $Var(X) = 4{\cdot}2$, find n and p.

Solution : $E(X) = 6$ therefore $np = 6$ and $Var(X) = 4{\cdot}2$ therefore $npq = 4{\cdot}2$

$$\frac{npq}{np} = \frac{4{\cdot}2}{6} \qquad \therefore \quad q = 0{\cdot}7$$

$$\therefore p = 1 - q = 1 - 0{\cdot}7 \qquad \therefore \quad p = 0{\cdot}3$$

$$np = 6$$

$$\therefore n \times 0{\cdot}3 = 6 \qquad \therefore \quad n = \frac{6}{0{\cdot}3} = 20$$

EXERCISE 8.1

(1) A die is thrown 6 times. If 'getting an odd number' is a success, find the probability of

(i) 5 successes (ii) at least 5 successes (iii) at most 5 successes.

(2) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

(3) There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

(4) Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. find the probability that

(i) all the five cards are spades (ii) only 3 cards are spades (iii) none is a spade.

(5) The probability that a bulb produced by a factory will fuse after 150 days of use is 0·05. Find the probability that out of 5 such bulbs

(i) none (ii) not more than one (iii) more than one (iv) at least one

will fuse after 150 days of use.

(6) A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

(7) On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

(8) A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1/100. find the probability that he will win a prize

(i) at least once (ii) exactly once (iii) at least twice

(9) In a box of floppy discs it is known that 95% will work. A sample of three of the discs is selected at random. Find the probability that

(i) none (ii) 1 (iii) 2 (iv) all 3 of the sample will work.

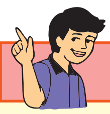
(10) Find the probability of throwing at most 2 sixes in 6 throws of a single die.

(11) It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

(12) Given that X~ B (n, p)

(i) If $n = 10$ and $p = 0{\cdot}4$, find $E(X)$ and $Var(X)$ (ii) If $p = 0{\cdot}6$ and $E(X) = 6$, find n and $Var(X)$.

(iii) If $n = 25$, $E(X) = 10$ find p and $SD(X)$. (iv) If $n = 10$, $E(X) = 8$, find $Var(X)$.



Let us Remember

- Trials of a random experiment are called **Bernoulli trials**, if they satisfy the following conditions :

  (i) Each trial has exactly two outcomes : success or failure.

  (ii) The probability of success remains the same in each trial.

  Thus probability of getting x successes in n-Bernoulli trial is

  $$P(x \text{ successes out of } n \text{ trials}) = \frac{n!}{x!\,(n-x)!} \times p^x \times q^{n-x} = {}^nC_x\, p^x \times q^{n-x}$$

  Clearly, $P(x \text{ successes})$, i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x+1)^{th}$ term in the binomial expansion of $(q+p)^n$.

- Let $X \sim B(n, p)$ then mean of expected value of r.v. X is denoted by μ.
  $E(X)$ and given by $\mu = E(X) = np$.

  The variance is denoted by $Var(X)$ and given by $Var(X) = npq$.

  Standard deviation of X is denoted by $SD(X)$ or σ and given by $SD(X) = \sigma_x = \sqrt{Var(X)}$

MISCELLANEOUS EXERCISE 8

(I) Choose the correct option from the given alternatives :

(1) A die is thrown 100 times. If getting an even number is considered a sucess, then the standard deviation of the number of successes is

(A) $\sqrt{50}$ (B) 5 (C) 25 (D) 10

(2) The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probablity of 2 successes is

(A) $\frac{128}{256}$ (B) $\frac{219}{256}$ (C) $\frac{37}{256}$ (D) $\frac{28}{256}$

(3) For a binomial distribution, $n = 5$. If $P(X = 4) = P(X = 3)$ then $p = \ldots$

(A) $\frac{1}{3}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{2}{3}$

(4) In a binomial distribution, $n = 4$. If $2\,P(X = 3) = 3\,P(X = 2)$ then $p = \ldots$

(A) $\frac{4}{13}$ (B) $\frac{5}{13}$ (C) $\frac{9}{13}$ (D) $\frac{6}{13}$

(5) If $X \sim B(4, p)$ and $P(X = 0) = \frac{16}{81}$, then $P(X = 4) = \ldots$

(A) $\frac{1}{16}$ (B) $\frac{1}{81}$ (C) $\frac{1}{27}$ (D) $\frac{1}{8}$

(6) The probability of a shooter hitting a target is $\frac{3}{4}$.

How many minimum number of times must he fire so that the probability of hitting the target at least once is more than 0·99 ?

(A) 2 (B) 3 (C) 4 (D) 5

(7) If the mean and variance of a binomial distribution are 18 and 12 respectively, then $n = \ldots$

(A) 36 (B) 54 (C) 18 (D) 27

(II) Solve the following :

(1) Let $X \sim B(10, 0{\cdot}2)$, Find (i) $P(X = 1)$ (ii) $P(X \geq 1)$ (iii) $P(X \leq 8)$.

(2) Let $X \sim B(n, p)$ (i) If $n = 10$, $E(X) = 5$, find p and $Var(X)$.

(ii) If $E(X) = 5$ and $Var(X) = 2{\cdot}5$, find n and p.

(3) If fair coin is tossed 10 times find the probability that it shows heads

(i) 5 times. (ii) in the first four tosses and tail in last six tosses.

(4) Probability that bomb will hit target is $0 \cdot 8$. Find the probability that out of 10 bombs dropped exactly 2 will miss the target.

(5) The probability that a mountain-bike rider travelling along a certain track will have a tyre burst is $0 \cdot 05$. Find the probability that among 17 riders : (i) exactly one has a burst tyre

(ii) at most three have a burst tyre (iii) two or more have burst tyres.

(6) Probability that a lamp in a classroom will burnt out will be $0 \cdot 3$. Six lamps are fitted in the classroom. If it is known that the classroom is unusable if the number of lamps burning in it is less than four, find the probability that classroom can not used at random occasion.

(7) Lot of 100 items contains 10 defective items. Five items are selected at random from the lot and sent to the retail store. What is the probability that the store will receive at most one defective item?

(8) A large chain retailer purchases certain kind of electric device from manufacturer. The manufacturer indicates that the defective rate of the device is 3% . The inspector of the retailer picks 20 items from a shipment. What is the probability that the store will receive at most one defective item?

(9) The probability that the certain kind of component will survive a check test is $0 \cdot 6$. Find the probability that exactly 2 of the next 4 components tested survive.

(10) An examination consists of 10 multiple-choice questions, in each of which a candidate has to deduce which one of five suggested answers is correct. A completely unprepared student guesses each answer completely randomly. What is the probability that this student gets 8 or more questions correct? Draw the appropriate moral !

(11) The probability that a machine will produce all bolts in a production run within specification is $0 \cdot 998$. A sample of 8 machines is taken at random. Calculate the probability that

(i) all 8 machines (ii) 7 or 8 machines

(iii) at least 6 machines will produce all bolts within specification

(12) The probability that a machine develops a fault within the first 3 years of use is $0 \cdot 003$. If 40 machines are selected at random, calculate the probability that 38 or more will not develop any faults within the first 3 years of use.

(13) A computer installation has 10 terminals. Independently, the probability that any one terminal will require attention during a week is $0 \cdot 1$. Find the probabilities that

(i) 0 (ii) 1 (iii) 2

(iv) 3 or more, terminals will require attention during the next week.

(14) In a large school, 80% of the pupils like mathematics. A visitor to the school asks each of 4 pupils, chosen at random, whether they like mathematics.

(i) Calculate the probabilities of obtaining an answer yes from 0, 1, 2, 3, 4 of the pupils

(ii) Find the probability that the visitor obtains the answer yes from at least 2 pupils:

(a) when the number of pupils questioned remains at 4

(b) when the number of pupils questioned is increased to 8.

(15) It is observed that, it rains on 12 days out of 30 days. Find the probability that

(i) it rains exactly 3 days of week. (ii) it will rain on at least 2 days of given week.

(16) If probability of success in a single trial is $0 \cdot 01$. How many trials are required in order to have probability greater than $0 \cdot 5$ of getting at least one success?

(17) In binomial distribution with five Bernoulli's trials, probability of one and two success are $0 \cdot 4096$ and $0 \cdot 2048$ respectively. Find probability of success.

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