# 11. Definite Integration - I

**Ex.** (1) Evaluate $\int_0^1 x^2\,dx$

**Solution :** $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$

$$= \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \frac{1}{3}$$

Evaluation of integral as a limit of sum $\int_0^1 x^2\,dx$

$f(x) = x^2$ a = 0 and b = 1

$x = a + rh$ and $h = \frac{b - a}{n}$

$h = \frac{1 - 0}{n}$

$nh = 1$

$f(x) = f(a + rh)$

$= f(a + rh)$

$= (rh)^2$

$= r^2h^2$

We know,

$$\int_a^b f(x).dx = \lim_{n\to\infty} \sum_{r=1}^{n} h.f\ (a + rh)$$

$$\therefore \int_0^1 x^2.dx = \lim_{n\to\infty} \sum_{r=1}^{n} h.r^2h^2$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h^3.r^2$$

$$= \lim_{n\to\infty} h^3.\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{h^3.n^3(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

$$= \frac{(1)^3(1)(1+0)(2+0)}{6} = \frac{1}{3}$$

**Ex.** (2) Evaluate $\int_1^3 (x^2+1)\, dx$

**Solution :** $f(x) = x^2 + 1, \quad a = 1, b = 3$

$$x = a + rh \qquad \text{and} \qquad h = \frac{b - a}{n}$$

$$x = 1 + rh \qquad \text{and} \qquad h = \frac{3-1}{n}$$

$$\therefore \ nh = 2$$

$$\begin{aligned} f(x) &= f(a + rh) \\ &= f(1 + rh) \\ &= (1 + rh)^2 + 1 \\ &= 1 + 2rh + r^2h^2 + 1 \\ &= 2 + 2rh + r^2h^2 \end{aligned}$$

We know,

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\, f(a + rh)$$

$$\begin{aligned} \int_1^3 (x^2+1)\, dx &= \lim_{n\to\infty} \sum_{r=1}^{n} h\, (2 + 2rh + r^2h^2) \\ &= \lim_{n\to\infty} \sum_{r=1}^{n} 2h + 2rh^2 + rh^3 \\ &= \lim_{n\to\infty} \left[ 2h \sum_{r=1}^{n} 1 + 2h^2 \sum_{r=1}^{n} r + h^3 \sum_{r=1}^{n} r^2 \right] \\ &= \lim_{n\to\infty} \left[ 2h\,(n) + 2h^2\, \frac{n\,(n+1)}{2} + h^3\, \frac{n\,(n+1)\,(2n+1)}{6} \right] \\ &= \lim_{n\to\infty} \left[ 2hn + h^2n^2(1)\,(1+\frac{1}{n}) + \frac{h^3n^3(1)\,(1+\frac{1}{n})\,(2+\frac{1}{n})}{6} \right] \\ &= 2\,(2) + (2)^2(1)\,(1+0) + \frac{(2)^3(1)\,(1+0)\,(2+0)}{6} \\ &= 8 + \frac{8}{3} \\ &= \frac{32}{3} \end{aligned}$$

$$\therefore \int_1^3 (x^2+1)\, dx = \frac{32}{3}$$

**Ex.** (3) Evaluate $\int_0^3 (4x+3)\,dx$

**Solution :** $f(x) = 4x + 3$, $a = 0$, $b = 3$

$x = a + rh$ and $h = \frac{b - a}{n}$

$\therefore x = rh$ and $h = \frac{3 - 0}{n}$

$\therefore nh = 3$

$f(x) = f(a + rh) = f(rh)$

$= 4(rh) + 3$

$= 4(rh) + 3$

We know,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\ f(a + rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h\,[(4rh + 3)]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} 4rh^2 + 3h$$

$$= \lim_{n\to\infty} (4h^2 \sum_{r=1}^{n} r + 3h \sum_{r=1}^{n} 1)$$

$$= \lim_{n\to\infty} \left[4h^2\ \frac{n(n+1)}{2} + 3h(n)\right]$$

$$= \lim_{n\to\infty} \left[2\,h^2n^2(1)\,(1 + \frac{1}{n}) + 3nh\right]$$

$$= 2(3)^2(1)(1+0) + 3(3)$$

$$= 18 + 9$$

$$= 27$$

$$\therefore \int_0^3 (4x + 3)\,dx = 27$$

**Ex.** (4) Evaluate $\int_0^4 (2x - 1)\,dx$

Let $I = \int_0^4 (2x - 1)\,dx$

$\int_a^b f(x)\,dx = \lim_{h\to 0} \left[h \sum_{r=1}^{n} f(a + rh)\right]$

$a=0,\ b=4\ ;\ h=\frac{b-a}{n}=\frac{4-0}{n}$

$\therefore nh=4$

$f(x)=2x-1$

$f(a+rh)=2(a+rh)-1$

$f(a+rh)=2(0+rh)-1$

$f(a+rh)=2rh-1$

$\therefore I=\lim_{h\to 0}\left[h\sum_{r=1}^{n}(2rh-1)\right]$

$=\lim_{h\to 0}h\left[\sum_{r=1}^{n}2rh-\sum_{r=1}^{n}1\right]$

$I=\lim_{h\to 0}h\left[2h\frac{n(n+1)}{2}-n\right]$

$=\lim_{h\to 0}h\left[nh\times n+nh-nh\right]$

$=\lim_{h\to 0}\left[nh\times nh\right]$

$=\lim_{h\to 0}4\times 4$

$=\lim_{h\to 0}16$

$\therefore I=16$

B. Evaluate the following definite integrals.

**Ex.** (1) $\int_0^{\pi} x\sin^2 x\, dx$

Let $I=\int_0^{\pi} x\sin^2 x\,dx$ — I

using

$\int_0^a f(x)dx=\int_0^a f(a-x)\,dx$

changing $x\to \pi-x$

$\therefore I=\int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx$

$=\int_0^{\pi}(\pi-x)\sin^2 x\,dx$

$(\because \sin(\pi-\theta)=\sin\theta)$

$=\int_0^{\pi}(\pi\sin^2 x-x\sin^2 x)\,dx$

$=\int_0^{\pi}\pi\sin^2 x\,dx-\int_0^{\pi}x\sin^2 x\,dx$

$=\int_0^{\pi}\pi\sin^2 x\,dx-I$ from I

$\therefore I+I=\frac{\pi}{2}\int_0^{\pi}2\sin^2 x\,dx$

$\cos 2\theta=1-\sin^2\theta$

$\Rightarrow 2\sin^2\theta=1-\cos 2\theta$

$\therefore 2I=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx$

$\therefore 2I=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}$

$\therefore 2I=\frac{\pi}{2}\left[\pi-\frac{\sin 2\pi}{2}-\left(0-\frac{\sin 0}{2}\right)\right]$

$\therefore 2I=\frac{\pi}{2}\left[\pi-\frac{0}{2}-\left(0-\frac{0}{2}\right)\right]$

$2I=\frac{\pi^2}{2}$

$I=\frac{\pi^2}{4}$

**Ex.** (2) $\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$

$I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}\,dx$

$= \int_{-a}^{a}\sqrt{\frac{(a-x)^2}{a^2-x^2}}\,dx$

$= \int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}\,dx$

$= \int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}\,dx - \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}\,dx$

$f_1(x) = \frac{a}{\sqrt{a^2-x^2}}$ $f_2(x) = \frac{x}{\sqrt{a^2-x^2}}$

$f_1(-x) = \frac{a}{\sqrt{a^2-(-x)^2}}$, $f_2(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}}$

$f_1(-x) = \frac{a}{\sqrt{a^2-x^2}}$, $f_2(-x) = \frac{-x}{\sqrt{a^2-x^2}}$

$f_1(x)$ is even and $f_2(x)$ is odd

$\therefore$ using def integration Property

$I = 2\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx - 0$

$= 2a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$

$= 2a \sin^{-1}\left(\frac{x}{a}\right)\Big]_0^a$

$= 2a\left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)\right]$

$= 2a\left(\sin^{-1}(1) - \sin^{-1}(0)\right)$

$= 2a \times \frac{\pi}{2}$

$I = \pi a$

**Ex.** (3) $\int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$

$I = \int_0^{\frac{\pi}{2}} \log\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$ — using Property and changing $x \to \frac{\pi}{2}-x$

$= \int_0^{\frac{\pi}{2}} \log \sin x\,dx$ — I

$= \int_0^{\frac{\pi}{2}} \log\left(2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)dx$

$= \int_0^{\frac{\pi}{2}} \log 2 + \log\sin\frac{x}{2} + \log\cos\frac{x}{2}\,dx$

Put $\frac{x}{2} = \theta \Rightarrow x = 2\theta \quad \therefore dx = 2d\theta$

when $x = 0, \theta = 0$, when $x = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

$I = \log 2\int_0^{\pi/2} 1\,dx + \int_0^{\frac{\pi}{4}} \log\sin\theta(2)d\theta + \int_0^{\pi/4} \log\cos\theta\, 2d\theta$

$= \log 2\,[\theta]_0^{\frac{\pi}{2}} + 2\int_0^{\pi/4} \log\sin\theta\,d\theta + 2\int_0^{\pi/4}\log\cos\theta\,d\theta$

put $\theta = \frac{\pi}{2} - \alpha \Rightarrow \alpha = \frac{\pi}{2} - \theta$

$\therefore d\theta = -d\alpha$, when $\theta = 0$ $\alpha = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

when $\theta = \frac{\pi}{4}$, $\alpha = \frac{\pi}{4}$

$\therefore = \log 2\left[\frac{\pi}{2} - 0\right] + 2\int_0^{\pi/4}\log\sin\theta\,d\theta +$

$2\int_{\pi/2}^{\pi/4} \log\cos\left(\frac{\pi}{2}-\alpha\right)(-1)\,d\alpha$

$I = \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4}\log\sin x\,dx + 2\int_{\pi/4}^{\frac{\pi}{2}}\log\sin x\,dx$

$= \frac{\pi}{2}\log 2 + 2\left[\int_0^{\frac{\pi}{4}}\log\sin x\,dx + \int_{\pi/4}^{\frac{\pi}{2}}\log\sin x\,dx\right]$

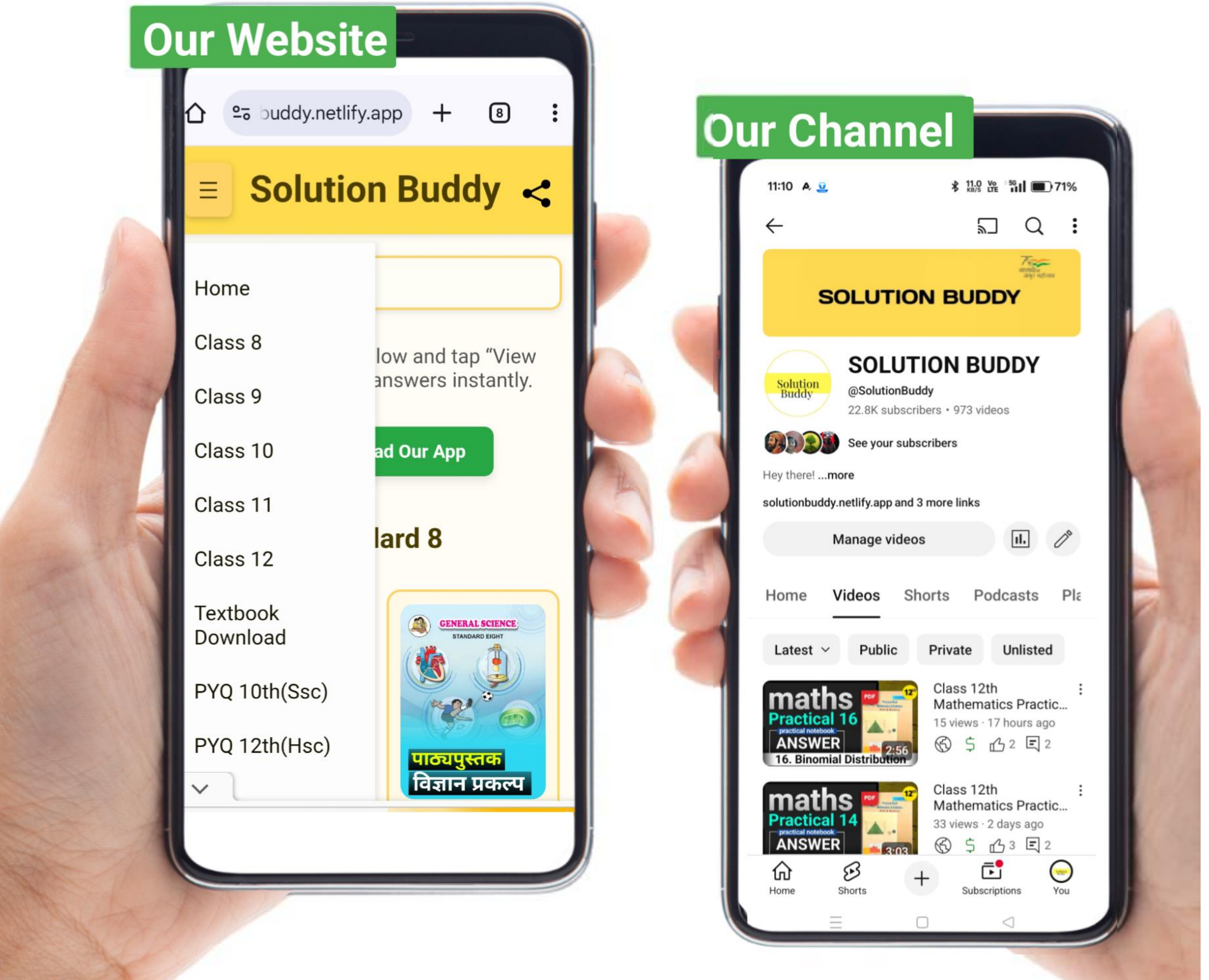
$= \frac{\pi}{2}\log 2 + 2\int_0^{\frac{\pi}{2}}\log\sin x\,dx = \frac{\pi}{2} + 2I$

$\Rightarrow I - 2I = \frac{\pi}{2}\log 2$

$-I = \frac{\pi}{2}\log 2$

$I = -\frac{\pi}{2}\log 2$

Sign of Teacher :

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