3 Trigonometry - II

Let's : Study

- Trigonometric functions of sum and difference of angles.
- Trigonometric functions of allied angles.
- Trigonometric functions of multiple angles.
- Factorization formulae.
- Trigonometric functions of angles of a triangle.

Let's Recall

In the previous chapter we have studied trigonometric functions in different quadrants.

3.1 Compound angle : Compound angles are sum or difference of given angles.

Following are theorems about trigonometric functions of sum and difference of two angles.

Let's Derive

Theorem : 1) For any two angles A and B, cos (A–B) = cos A cosB + sinA sin B

Proof :

Draw a unit standard circle. Take points P and Q on the circle so that OP makes an angle A with positive X-axis and OQ makes an angle B with positive X–axis.

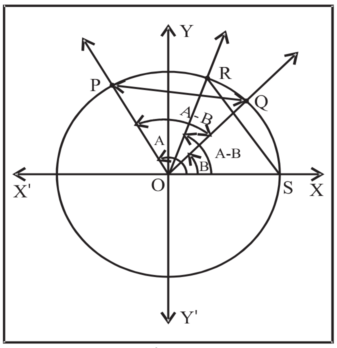

Fig. 3.1

$\therefore$ $m\angle XOP = A$, $m\angle XOQ = B$.

From figure OP = OQ = 1

$\therefore$ Co-ordinates of P and Q are (cosA,sinA) and (cos$B$, sin$B$) respectively .

$\therefore$ $d(PQ)$

$= \sqrt{(cosA-cosB)^2+(sinA-sinB)^2}$

$= \sqrt{cos^2A-2cosAcosB+cos^2B+sin^2A-2sinAsinB+sin^2B}$

$= \sqrt{(cos^2A+sin^2A)+\left(cos^2B+sin^2B\right)-2(cosAcosB+sinAsinB)}$

$= \sqrt{1+1-2(cosAcosB+sinA\ sinB)}$

$= \sqrt{2-2\left(cosAcosB+sinAsinB\right)}$

$[d(PQ)]^2 = 2-2\left(cosAcosB+sinAsinB\right)$ ...(1)

Now consider OQ as new X-axis.

$\therefore$ $m\angle$QOP = A–B

$\therefore$ Co-ordinates of P and Q are

(cos (A–B), sin(A–B)) and (1,0) respectively.

P ≡ (cos (A–B), sin (A–B)),Q ≡ (1,0)

$\therefore$ $d(PQ)$

$= \sqrt{[cos(A-B)-1]^2+[\sin(A-B)-0]^2}$

$= \sqrt{cos^2(A-B)-2\cos(A-B)+1+sin^2(A-B)}$

$= \sqrt{cos^2(A-B)+sin^2(A-B)+1-2\cos(A-B)}$

$= \sqrt{1+1-2\cos(A-B)}$

$= \sqrt{2-2\cos(A-B)}$

$[d(PQ)]^2 = 2-2cos(A-B)$.....(2)

From equation (1) and (2) we get

$2-2\cos(A-B) = 2-2(\cos A\cos B+\sin A\sin B)$

$\therefore -2\cos(A-B) = -2(\cos A\cos B+\sin A\sin B)$

$\therefore\ \cos(A-B) = \cos A\cos B + \sin A\sin B$

Theorem : 2) For any two angles A and B ,

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

Proof : We know that

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

Put $x = A, y = -B$ *we get*

$\cos(A+B) = \cos A\cos B+\sin A(-\sin B)$

$\because \cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta$

$$\boxed{\therefore \cos(A+B) = \cos A\cos B - \sin A\sin B}$$

Results :

1) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

Proof : We know that

$\cos(x-y) = \cos x\cos y+\sin x\sin y$

Put $x = \frac{\pi}{2},\ y = \theta$ *we get*

$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$

$= 0.\cos\theta + 1.\sin\theta$

$= \sin\theta$

$\therefore\ \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

Simillarly

2) $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

3) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

4) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

5) $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{\sin(\frac{\pi}{2}-\theta)}{\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$

6) $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

Theorem : 3) For any two angles A and B,

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

Proof : We know that $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

Putting $\theta = A - B$, we get

$\sin(A-B) = \cos\left[\frac{\pi}{2}-(A-B)\right]$

$= \cos\left((\frac{\pi}{2}-A)+B\right)$

$= \cos\left(\frac{\pi}{2}-A\right)\cos B - \sin\left(\frac{\pi}{2}-A\right)\sin B$

$\therefore\ \sin(A-B) = \sin A\cos B - \cos A\sin B$

$\left[\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta,\ \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right]$

Theorem : 4) For any two angles A and B,

$\sin(A+B) = \sin A\cos B+\cos A\sin B$ [verify]

Theorem : 5) For any two angles A and B,

$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$

Proof : Consider $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}}$ (dividing numerator

and demonator by $\cos A\cos B$)

$= \dfrac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Theorem : 6) For any two angles A and B,

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ (Activity)

Results :

1) If none of the angles A,B and (A+B) is a multiple of π

then, $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

2) If none of the angles A,B and (A–B) is a multiple of π

then, $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

SOLVED EXAMPLES

Ex. 1) Find the value of cos 15°

Solution : $\cos 15° = \cos(45° - 30°)$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Ex. 2) Find the value of $\tan\dfrac{13\pi}{12}$

Solution : $\tan\dfrac{13\pi}{12}$ $\tan\left(\pi + \dfrac{\pi}{12}\right)$

$$= \frac{\tan\pi + \tan\frac{\pi}{12}}{1 - \tan\pi \tan\frac{\pi}{12}}$$

$$= \frac{0 + \tan\frac{\pi}{12}}{1 + 0 \times \tan\frac{\pi}{12}}$$

$$= \tan\frac{\pi}{12}$$

$$= \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}}$$

$$= = 2 - \sqrt{3}$$

Ex. 3) Show that $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

Solution : L.H.S. $= \dfrac{\sin(x+y)}{\sin(x-y)}$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

(dividing numerator and denominator by $\cos x \cos y$)

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y}$$

= R.H.S.

Ex. 4) Show that :

tan3x tan2x tanx = tan3x – tan2x – tanx

Solution : tan (3x) = tan (2x+x)

$$\therefore \tan(3x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$\therefore$ tan3x [1–tan 2x tanx] = tan2x + tanx

$\therefore$ tan3x – tan3x tan2x tanx = tan2x + tanx

$\therefore$ tan3x – tan 2x – tanx = tan3x tan2x tanx

$\therefore$ tan3x tan2x tanx = tan3x – tan2x – tanx

Ex. 5) Show that

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$$

Solution : L.H.S. = $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$$= \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x$$

$$= \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\cos x$$

$$= \frac{2}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R. H.S.}$$

Ex. 6) If tan A – tan B = x and cot B – cot A = y

then show that cot (A–B) = $\frac{1}{x}+\frac{1}{y}$

Solution : cot B – cot A = y

$$\therefore \frac{1}{\tan B} - \frac{1}{\tan A} = y$$

$$\therefore \frac{\tan A - \tan B}{\tan A \tan B} = y$$

$$\therefore \frac{x}{\tan A \tan B} = y$$

$$\therefore \tan A \tan B = \frac{x}{y}$$

Now cot (A–B) = $\frac{1}{\tan(A-B)}$

$$= \frac{1+\tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{y}+\frac{1}{x}$$

$$\therefore \cot(A-B) = \frac{1}{y}+\frac{1}{x}$$

Ex. 7) If

$$\tan\alpha = \frac{1}{\sqrt{x(x^2+x+1)}}, \tan\beta = \frac{\sqrt{x}}{\sqrt{x^2+x+1}} \text{ and } \tan\gamma = \sqrt{x^{-3}+x^{-2}+x^{-1}}$$

then show that

$\alpha + \beta = \gamma$

Solution : We know that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$\therefore$ tan (α +β) =

$$= \left[\frac{\frac{1}{\sqrt{x(x^2+x+1)}}+\frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1-\frac{1}{\sqrt{x(x^2+x+1)}}\frac{\sqrt{x}}{\sqrt{x^2+x+1}}}\right]$$

$$= \frac{(x+1)\sqrt{x^2+x+1}}{\sqrt{x}.x(x+1)}$$

$$= \sqrt{\frac{x^2}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}$$

$$= \sqrt{x^{-1}+x^{-2}+x^{-3}}$$

= tan γ

$\therefore$ α +β = γ

Ex. 8) If $\sin A + \sin B = x$ and $\cos A + \cos B = y$ then show that $\sin(A+B) = \dfrac{2xy}{x^2+y^2}$

Solution :

$y^2+x^2 = (\cos A+\cos B)^2 + (\sin A+\sin B)^2$

$= \cos^2 A+2\cos A\cos B+\cos^2 B+\sin^2 A+\sin^2 B +2\sin A\sin B$

$y^2+x^2 = (\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+ 2(\cos A\cos B+\sin A\sin B)$

$= 1+1+2\cos(A-B)$

$\therefore x^2+y^2 = 2 + 2\cos(A-B)$ .... (I)

$y^2 - x^2 = (\cos A+\cos B)^2-(\sin A+\sin B)^2$

$= (\cos^2 A - \sin^2 A)+(\cos^2 B-\sin^2 B) +2[\cos A\cos B-\sin A\sin B]$

$= \cos 2A+\cos 2B+2\cos(A+B)$

$= 2\cos\left[\dfrac{2A+2B}{2}\right].\cos\left[\dfrac{2A-2B}{2}\right] + 2\cos(A+B)$

$= 2\cos(A+B)\cos(A-B) +2\cos(A+B)$

$= \cos(A+B)[2\cos(A-B)+2]$ [from (I)]

$y^2-x^2 = \cos(A+B)(x^2+y^2)$

$\therefore \dfrac{y^2-x^2}{x^2+y^2} = \cos(A+B)$

$\therefore \sin(A+B) =$

$\therefore \sin(A+B) = \sqrt{1-\left(\dfrac{y^2-x^2}{y^2+x^2}\right)^2}$

$= \sqrt{\dfrac{(y^2+x^2)^2-(y^2-x^2)^2}{(y^2+x^2)^2}}$

$= \sqrt{\dfrac{y^4+2x^2y^2+x^4-y^4+2x^2y^2-x^4}{(x^2+y^2)^2}}$

$= \sqrt{\dfrac{4x^2y^2}{(x^2+y^2)^2}}$

$= \dfrac{2xy}{x^2+y^2}$

EXERCISE 3.1

1) Find the values of

i) $\sin 15°$ ii) $\cos 75°$ iii) $\tan 105°$

iv) $\cot 225°$

2) Prove the following.

i) $\cos(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-y) - \sin(\frac{\pi}{2}-x)\sin(\frac{\pi}{2}-y) = -\cos(x+y)$

ii) $\tan\left(\frac{\pi}{4}+\theta\right) = \dfrac{1-\tan\theta}{1+\tan\theta}$

iii) $\left(\dfrac{1+\tan x}{1-\tan x}\right)^2 = \dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}$

iv) $\sin[(n+1)A].\sin[(n+2)A]+\cos[(n+1)A].\cos[(n+2)A]= \cos A$

v) $\sqrt{2}\cos\left(\frac{\pi}{4}-A\right) = \cos A +\sin A$

vi) $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{\cot x\cot y+1}{\cot x\cot y-1}$

vii) $\cos(x+y).\cos(x-y) = \cos^2 y - \sin^2 x$

viii) $\dfrac{\tan 5A-\tan 3A}{\tan 5A+\tan 3A} = \dfrac{\sin 2A}{\sin 8A}$

ix) $\tan 8\theta - \tan 5\theta - \tan 3\theta = \tan 8\theta\tan 5\theta\tan 3\theta$

x) $\tan 50° = \tan 40° + 2\tan 10°$

xi) $\dfrac{\cos 27°+\sin 27°}{\cos 27°-\sin 27°} = \tan 72°$

xii) $\tan 10° + \tan 35° + \tan 10°.\tan 35° = 1$

xiii) $\dfrac{\cot A\cot 4A+1}{\cot A\cot 4A-1} = \dfrac{\cos 3A}{\cos 5A}$

xiv) $\dfrac{\cos 15°-\sin 15°}{\cos 15°+\sin 15°} = \dfrac{1}{\sqrt{3}}$

3) If $\sin A = \frac{-5}{13}$, $\pi < A < \frac{3\pi}{2}$ and

$\cos B = \frac{3}{5}$, $\frac{3\pi}{2} < B < 2\pi$ then

find i) $\sin (A+B)$ ii) $\cos (A-B)$

iii) $\tan (A+B)$

4) If $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$, prove that $A+B = \frac{\pi}{4}$

Let's Learn

3.2 Trigonometric functions of allied angels.

Allied angles : If the sum or difference of the measures of two angles is an integral multiple of $\frac{\pi}{2}$ then these angles are said to be allied angles.

If θ is the measure of an angle the

$-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, \frac{3\pi}{2} \pm \theta, 2\pi - \theta$ are its allied angles.

We have already proved the following results :

1) $\sin (\frac{\pi}{2} - \theta) = \cos \theta$, $\cos (\frac{\pi}{2} - \theta) = \sin\theta$,
$\tan (\frac{\pi}{2} - \theta) = \cot \theta$

2) $\sin (\frac{\pi}{2} + \theta) = \cos \theta$, $\cos (\frac{\pi}{2} + \theta) = -\sin\theta$,
$\tan (\frac{\pi}{2} + \theta) = -\cot \theta$

Similarly we can also prove the following results :

1) $\sin (\pi - \theta) = \sin \theta$, $\cos (\pi - \theta) = -\cos\theta$,
$\tan (\pi - \theta) = -\tan \theta$

2) $\sin (\pi + \theta) = -\sin \theta$, $\cos (\pi + \theta) = -\cos\theta$,
$\tan (\pi + \theta) = \tan \theta$

3) $\sin (\frac{3\pi}{2} - \theta) = -\cos \theta$, $\cos (\frac{3\pi}{2} - \theta) = -\sin\theta$,
$\tan (\frac{3\pi}{2} - \theta) = \cot \theta$

4) $\sin (\frac{3\pi}{2} + \theta) = -\cos \theta$, $\cos (\frac{3\pi}{2} + \theta) = \sin\theta$,
$\tan (\frac{3\pi}{2} + \theta) = -\cot \theta$

5) $\sin (2\pi - \theta) = -\sin \theta$, $\cos (2\pi - \theta) = \cos\theta$, $\tan (2\pi - \theta) = -\tan \theta$

Above results are tabulated in following table .

allied angles/ Trigonometric functions	$-\theta$	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\pi - \theta$	$\pi + \theta$	$2\pi - \theta$	$2\pi + \theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos \theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan \theta$	$-\tan\theta$	$-\tan\theta$	$\tan\theta$

SOLVED EXAMPLES

Ex. 1) Find the values of

i) $(\sin 495°)$ ii) $\cos 930°$ iii) $\tan 840°$

Solution :

i) $\sin (495°) = \sin 495°$
$= \sin (360° + 135°)$
$= \sin 135°$
$= \sin (\frac{\pi}{2} + 45^0)$
$= \cos 45° = \frac{1}{\sqrt{2}}$

ii) $\cos 930°$
$= \cos (2 \times 360° + 210°)$
$= \cos 210°$
$= \cos (\pi + 30°)$
$= -\cos 30^0 = -\frac{\sqrt{3}}{2}$

iii) $\tan 840° = \tan (2 \times 360° + 120°) = \tan 120° = \tan (\frac{\pi}{2} + 30°) = -\cot 30° = -\sqrt{3}$

Ex. 2) Show that :

i) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

Solution :

L.H.S. $= \cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300°$

$= \cos 24° + \cos 55° + \cos (180° - 55°) + \cos (180° + 24^0) + \cos (360^0 - 60^0)$

$= \cos 24° + \cos 55° - \cos55° - \cos24° + \cos 60°$

$= \cos 60° = \dfrac{1}{2} = R.H.S.$

ii) $\sec 840°.\cot (-945°) + \sin 600°.\tan (-690°) = \dfrac{3}{2}$

Solution :

$\sec 840° = \sec (2 \times 360° + 120°) = \sec 120°$

$= \sec (90° + 30°) = -\operatorname{cossec} 30°$

$= -2$

$\cot (-945°) = -\cot 945° = -\cot (2 \times 360° + 225°)$

$= -\cot 225°$

$= -\cot (180° + 45°) = -\cot 45° = -1$

$\sin 600° = \sin (360° + 240°) = \sin 240°$

$= \sin (180° + 60°)$

$= -\sin 60° = -\dfrac{\sqrt{3}}{2}$

$\tan (-690°) = -\tan690° = -\tan (2 \times 360 - 30°)$

$= -(-\tan 30°)$

$= \tan 30 = \dfrac{1}{\sqrt{3}}$

L.H.S.

$= \sec840°.\cot (-945°) + \sin 600° \tan (-690°)$

$$= -2 \times -1 + \left(-\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}\right)$$

$$= 2 - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2} = R.\ H.\ S$$

iii)

$$\frac{\cos ec\left(90^o - \theta\right).\sin\left(180^o - \theta\right)\cot\left(360^o - \theta\right)}{\sec\left(180^o + \theta\right)\tan\left(90^o + \theta\right)\sin\left(-\theta\right)} = 1$$

L.H.S.

$$= \frac{\operatorname{coses}\left(90^o - \theta\right).\sin\left(180^o - \theta\right)\cot\left(360^o - \theta\right)}{\sec\left(180^o + \theta\right)\tan\left(90^o + \theta\right)\sin\left(-\theta\right)}$$

$$= \frac{\sec\theta\ \sin\theta\ (-\cot\theta)}{(-\sec\theta)(-\cot\theta)(-\sin\theta)}$$

$$= \frac{-\sec\theta\sin\theta\cot\theta}{-\sec\theta\cot\theta\sin\theta} = 1 = \text{R.H.S.}$$

iv) $$\frac{\cot\left(\frac{\pi}{2} + \theta\right)\sin(-\theta)\cot(\pi - \theta)}{\cos(2\pi - \theta)\sin(\pi + \theta)\tan(2\pi - \theta)} = -\operatorname{cosec}\theta$$

L.H.S.

$$= \frac{\cot\left(\frac{\pi}{2} + \theta\right)\sin(-\theta)\cot(\pi - \theta)}{\cos(2\pi - \theta)\sin(\pi + \theta)\tan(2\pi - \theta)}$$

$$= \frac{(-\tan\theta)(-\sin\theta)(-\cot\theta)}{\cos\theta(-\sin\theta)(-\tan\theta)}$$

$$= \frac{-\cot\theta}{+\cos\theta}$$

$$= -\frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta} = \frac{1}{\sin\theta} = -\operatorname{cosec}\theta = \text{R.H.S.}$$

Ex. 3) Prove the following :

i) $\sin\dfrac{\pi}{15} + \sin\dfrac{4\pi}{15} - \sin\dfrac{14\pi}{15} - \sin\dfrac{11\pi}{15} = 0$

Solution : L.H.S

$$= \sin\frac{\pi}{15} + \sin\frac{4\pi}{15} - \sin\frac{14\pi}{15} - \sin\frac{11\pi}{15}$$

$$= \sin\frac{\pi}{15} + \sin\frac{4\pi}{15} - \sin\left(\pi - \frac{\pi}{15}\right) - \sin\left(\pi - \frac{4\pi}{15}\right)$$

$$= \sin\frac{\pi}{15} + \sin\frac{4\pi}{15} - \sin\frac{\pi}{15} - \sin\frac{4\pi}{15}$$

$= 0$

$=$ R.H.S.

ii) $\sin^2\left(\frac{\pi}{4}-x\right)+\sin^2\left(\frac{\pi}{4}+x\right)=1$

Solution : consider $\frac{\pi}{4}-x=y \quad \therefore x=\frac{\pi}{4}-y$

$$\text{L.H.S.} = \sin^2\left(\frac{\pi}{4}-x\right)+\sin^2\left(\frac{\pi}{4}+x\right)$$

$$= \sin^2 y+\sin^2\left(\frac{\pi}{4}+\frac{\pi}{4}-y\right)$$

$$= \sin^2 y+\cos^2 y$$

$$= 1 = \text{R.H.S.}$$

iii) $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=2$

Solution : L.H.S.

$$= \sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$$

$$= \sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\left(\frac{4\pi+\pi}{8}\right)+\sin^2\left(\frac{4\pi+3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\left(\frac{\pi}{2}+\frac{\pi}{8}\right)+\sin^2\left(\frac{\pi}{2}+\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}$$

$= 1+1$

$= 2 = \text{R.H.S.}$

iv)

$$\cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{9\pi}{10}\right)=2$$

Solution : L.H.S

$$= \cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{9\pi}{10}\right)$$

$$= \cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{10}\right)+\cos^2\left(\frac{\pi}{2}+\frac{\pi}{10}\right)+\cos^2\left(\pi-\frac{\pi}{10}\right)$$

$$= \cos^2\left(\frac{\pi}{10}\right)+\sin^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{\pi}{10}\right)+\sin^2\left(\frac{\pi}{10}\right)$$

$= 1+1$

$= 2 = \text{R.H.S.}$

EXERCISE 3.2

1) Find the value of :

i) sin690°   ii) sin (495°)

iii) cos 315°   iv) cos (600°)

v) tan 225°   vi) tan (– 690°)

vii) sec 240°   viii) sec (– 855°)

ix) cosec 780°   x) cot (–1110°)

2) Prove the following:

i) $$\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}=\cot^2 x$$

ii) $$\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$$

iii) sec 840°.cot (– 945°) + sin 600° tan (– 690°) $= \frac{3}{2}$

iv) $$\frac{\operatorname{cosec}(90^\circ-x)\sin(180^\circ-x)\cot(360^\circ-x)}{\sec(180^\circ+x)\tan(90^\circ+x)\sin(-x)}=1$$

v) $$\frac{\sin^3(\pi+x)\sec^2(\pi-x)\tan(2\pi-x)}{\cos^2\left(\frac{\pi}{2}+x\right)\sin(\pi-x)\operatorname{cosec}^2-x}=\tan^3 x$$

vi) $\cos\theta+\sin(270^\circ+\theta)-\sin(270^\circ-\theta)+\cos(180^\circ+\theta)=0$

Let's Learn

3.3 Trigonometric functions of multiple angles.

Angles of the form 2θ, 3θ, 4θ etc. are integral multiple of θ these angles are called multiple angles and angles of the form $\frac{\theta}{2}, \frac{3\theta}{2}$ etc. are called submultiple angles of θ.

3.3.1 Trigonometric functions of double angles (2θ)

Theorem : For any angle θ,

1) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

2) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

3) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Proof: 1) $= \sin 2\theta = \sin(\theta+\theta)$

$= \sin\theta\cos\theta + \cos\theta\sin\theta$

$= 2\sin\theta\cos\theta \;......(1)$

$= \dfrac{2\sin\theta\cos\theta}{1}$

$= \dfrac{2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}$

$= \dfrac{2\sin\theta\cos\theta/\cos^2\theta}{\sin^2\theta+\cos^2\theta/\cos^2\theta}$

$= \dfrac{2\sin\theta/\cos\theta}{\dfrac{\sin^2\theta}{\cos^2\theta}+1}$

$= \dfrac{2\tan\theta}{1+\tan^2\theta} \;.....(2)$

From (1) and (2)

$\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

2) $\cos 2\theta = \cos(\theta+\theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta \;.......(1)$

$= \cos^2\theta - (1-\cos^2\theta)$

$= \cos^2\theta - 1 + \cos^2\theta$

$= 2\cos^2\theta - 1 \;.......(2)$

$= 2(1-\sin^2\theta) - 1$

$= 2 - 2\sin^2\theta - 1$

$= 1 - 2\sin^2\theta \;........(3)$

$= \cos^2\theta - \sin^2\theta$

$= \dfrac{\cos^2\theta-\sin^2\theta}{1}$

$= \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$

$= \dfrac{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta} \;.......(4)$

From (1), (2), (3) and (4) we get

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

3) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Note that the substitution $2\theta = t$ transforms $\sin 2\theta = 2\sin\theta.\cos\theta$ into $\sin t = 2\sin\dfrac{t}{2}.\cos\dfrac{t}{2}$.

Simillarly,

$\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\quad \cos t = \cos^2\dfrac{t}{2} - \sin^2\dfrac{t}{2}$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, $\quad \tan t = \dfrac{2\tan\dfrac{t}{2}}{1-\tan^2\dfrac{t}{2}}$

Also if $\tan\frac{\theta}{2} = t$ then $\sin\theta = \frac{2t}{1-t^2}$

and $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\tan\theta = \frac{2t}{1-t^2}$

3.3.2 Trigonometric functions of triple angle (3θ)

Theorem : 1) For any angle θ

1) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

2) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

3) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Proof:

1) $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos\theta.\cos\theta + (1 - 2\sin^2\theta)\ \sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta\ (1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\therefore\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(Activity)

2) $\cos 3\theta = \cos(2\theta + \theta)$

3) $\tan 3\theta = \tan(2\theta + \theta)$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$$

$$= \frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan\theta}{\left(1 - \frac{2\tan\theta}{1-\tan^2\theta}\right)\tan\theta}$$

$$= \frac{\frac{2\tan\theta + \tan\theta\left(1-\tan^2\theta\right)}{1-\tan^2\theta}}{\frac{\left(1-\tan^2\theta\right) - 2\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$\therefore\ \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

SOLVED EXAMPLES

Ex. 1) Prove that $1 + \tan\theta\tan\frac{\theta}{2} = \sec\theta$

Solution :

L.H.S $= 1 + \tan\theta\tan\left(\frac{\theta}{2}\right)$

$$= 1 + \frac{\sin\theta}{\cos\theta}.\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= 1 + \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}}$$

$$= 1 + \frac{2\sin^2\frac{\theta}{2}}{\cos\theta}$$

$$= 1 + \frac{1-\cos\theta}{\cos\theta}$$

$$= \frac{\cos\theta + 1 - \cos\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} = \sec\theta = R.H.S$$

Ex. 2) Prove that

$\tan 20° \tan 40° \tan 60° \tan 80° = 3$

Solution :

L.H.S.

$= \tan 20° \tan 40° \tan 60° \tan 80°$

$= \tan 20° \tan 40°.\sqrt{3}\ \tan 80°$

$= \sqrt{3}\ \tan 20°\ \tan\ (60° - 20°)\tan\ (60° + 20°)$

$$= \sqrt{3}\ \tan 20°. \frac{\tan 60^\circ - \tan 20^\circ}{1+\tan 60^\circ \tan 20^\circ} . \frac{\tan 60^\circ + \tan 20^\circ}{1-\tan 60^\circ \tan 20^\circ}$$

$$= \sqrt{3}\ \tan 20°. \frac{\sqrt{3} - \tan 20^\circ}{1+\sqrt{3}\tan 20^\circ} . \frac{\sqrt{3} + \tan 20^\circ}{1-\sqrt{3}\tan 20^\circ}$$

$$= \sqrt{3}\ \tan 20° . \frac{3 - \tan^2 20^\circ}{1-3\tan^2 20^\circ}$$

$$= \sqrt{3}\ \frac{3\tan 20^\circ - \tan^3 20^\circ}{1-3\tan^2 20^\circ}$$

$= \sqrt{3}$ tan [3 (20)]°

$= \sqrt{3}$ tan 60°

$= \sqrt{3}\ .\ \sqrt{3}$ = 3 = *R. H. S*

Ex. 3) Prove that 2cosec2x + cosec$x$ = sec$x$. cot (x/2)

Solution : L.H.S. = 2cosec2x + cosec$x$

$$= \frac{2}{\sin 2x} + \frac{1}{\sin x}$$

$$= \frac{2}{2\sin x \cos x} + \frac{1}{\sin x}$$

$$= \frac{1+\cos x}{\sin x \cos x}$$

$$= \frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)\cos x}$$

$$= \frac{\cos(x/2)}{\sin(x/2)} . \frac{1}{\cos x}$$

= cot (x/2) . sec$x$ = R. H. S.

Ex. 4) Prove that

$$\frac{\cos^3\theta - \cos 3\theta}{\cos\theta} + \frac{\sin^3\theta + \sin 3\theta}{\sin\theta} = 3$$

Solution : L.H.S.

$$= \frac{\cos^3\theta - \cos 3\theta}{\cos\theta} + \frac{\sin^3\theta + \sin 3\theta}{\sin\theta}$$

$$= \frac{\cos^3\theta - \left[4\cos^3\theta - 3\cos\theta\right]}{\cos\theta} + \frac{\sin^3\theta + \left[3\sin\theta - 4\sin^3\theta\right]}{\sin\theta}$$

$$= \frac{-3\cos^3\theta + 3\cos\theta}{\cos\theta} + \frac{3\sin\theta - 3\sin^3\theta}{\sin\theta}$$

$$= \frac{3\cos\theta\left(1-\cos^2\theta\right)}{\cos\theta} + \frac{3\sin\theta\left(1-\sin^2\theta\right)}{\sin\theta}$$

= 3 $\sin^2\theta$ + 3 $\cos^2\theta$

= 3($\sin^2\theta$ + $\cos^2\theta$)

= 3(1) = 3 = R.H.S.

Ex. 5) Prove that $\frac{\tan 5A + \tan 3A}{\tan 5A - \tan 3A}$ =4$\cos^2 A$.cos 4A

Soln. : L.H.S. $= \frac{\tan 5A + \tan 3A}{\tan 5A - \tan 3A}$

$$= \frac{\frac{\sin 5A}{\cos 5A} + \frac{\sin 3A}{\cos 3A}}{\frac{\sin 5A}{\cos 5A} - \frac{\sin 3A}{\cos 3A}}$$

$$= \frac{\frac{\sin 5A\cos 3A + \cos 5A\sin 3A}{\cos 5A\cos 3A}}{\frac{\sin 5A\cos 3A - \cos 5A\sin 3A}{\cos 5A\cos 3A}}$$

$$= \frac{\sin 5A\cos 3A + \cos 5A\sin 3A}{\sin 5A\cos 3A - \cos 5A\sin 3A}$$

$$= \frac{\sin 8A}{\sin 2A} = \frac{2\sin 4A\cos 4A}{\sin 2A}$$

$$= \frac{2.2\sin 2A\cos 2A\cos 4A}{\sin 2A}$$

= 4 cos2 A cos 4A = R. H. S.

Ex. 6) Show that $(\cos\theta + i\sin\theta)^3$
= cos 3θ + i sin 3θ, where $i^2 = -1$.

Solution : L.H.S.= $[\cos\theta + i\sin\theta]^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i(1 - \sin^2\theta)\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta + 3i\sin\theta - 3i\sin^3\theta - 3\cos\theta(1 - \cos^2\theta) - i\sin^3\theta$

$= \cos^3\theta + 3i\sin\theta - 3i\sin^3\theta - 3\cos\theta + 3\cos^3\theta - i\sin^3\theta$

$= [4\cos^3\theta - 3\cos\theta] + i\,[3\sin\theta - 4\sin^3\theta]$

$= \cos 3\theta + i\sin 3\theta$

= R.H.S.

Ex. 7) Show that

$$4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta = \sin 4\theta$$

Solution:

L.H.S $= 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

$= 4\sin\theta\cos\theta\,[\cos^2 - \sin^2\theta]$

$= 2.\,(2\sin\theta\cos\theta)\,(\cos^2\theta - \sin^2\theta)$

$= 2.\,\sin 2\theta\,.\,\cos 2\theta$

$= \sin 4\theta$

= R.H.S.

Ex. 8) Show that $\sqrt{\dfrac{1+\sin 2A}{1-\sin 2A}} = \tan\left(\dfrac{\pi}{4}+A\right)$

Solution :

$$\text{L.H.S.} = \sqrt{\frac{1+\sin 2A}{1-\sin 2A}}$$

$$= \sqrt{\frac{\sin^2 A+\cos^2 A+2\sin A\cos A}{\sin^2 A+\cos^2 A-2\sin A\cos A}}$$

$$= \sqrt{\frac{(\sin A+\cos A)^2}{(\cos A-\sin A)^2}}$$

$$= \frac{\sin A+\cos A}{\cos A-\sin A}$$

$$= \frac{\dfrac{\cos A+\sin A}{\cos A}}{\dfrac{\cos A-\sin A}{\cos A}}$$

$$= \frac{1+\dfrac{\sin A}{\cos A}}{1-\dfrac{\sin A}{\cos A}}$$

$$= \frac{1+\tan A}{1-\tan A}$$

$$= \frac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\tan A} \qquad [\therefore\ 1 = \tan\frac{\pi}{4}]$$

$$= \tan\left(\frac{\pi}{4}+A\right)$$

= R.H.S.

Ex. 9) Find $\sin\dfrac{x}{2}$, $\cos\dfrac{x}{2}$, $\tan\dfrac{x}{2}$

if $\tan x = \dfrac{4}{3}$, x lies in II quadrant.

Solution : we know that $1 + \tan^2\theta = \sec^2\theta$

$$\sec^2 x = 1 + \left(-\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{9+16}{9} = \frac{25}{9}$$

$$\sec x = \pm\frac{5}{3}$$

But x lies in II quadrant.

$\therefore$ $\sec x$ is negative.

$\therefore\ \sec x = -\dfrac{5}{3} \qquad \therefore\ \cos x = -\dfrac{3}{5}$

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$$

$\therefore\ \sin x = \dfrac{4}{5} \qquad [\therefore\ x$ lies in II quadrant$]$

$$\text{But } \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$$

$$= \sqrt{\frac{5+3}{2\times 5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$$

$$= \sqrt{\frac{5-3}{2\times 5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \sqrt{\frac{\frac{4}{5}}{\frac{1}{5}}} = \sqrt{\frac{4}{5}\times\frac{5}{1}} = \sqrt{4} = 2$$

$$\therefore\ \sin\frac{x}{2} = \frac{2}{\sqrt{5}},\ \cos\frac{x}{2} = \frac{1}{\sqrt{5}},\ \tan\frac{x}{2} = 2$$

Ex. 10) Find the value of $\tan\frac{\pi}{8}$

Solution : let $x = \frac{\pi}{8}$ $\therefore$ $2x = \frac{\pi}{4}$

we have $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$$\therefore \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

let $y = \tan\frac{\pi}{8}$ $\therefore$ $1 = \frac{2y}{1-y^2}$

$\therefore$ $1 - y^2 = 2y$

$\therefore$ $y^2 + 2y - 1 = 0$

$\therefore$ $y = \frac{-2+2\sqrt{2}}{2} = -1 \pm\sqrt{2}$

Since $\frac{\pi}{8}$ lies in I quadrant $y = \tan\frac{\pi}{8}$ positive

$\therefore \tan\frac{\pi}{8} = \sqrt{2} - 1$

Ex. 11) Prove that

$$\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$$

Solution : L.H.S.

$= \cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right)$

$= \frac{1+\cos 2x}{2} + \frac{1+\cos 2\left(x+\frac{\pi}{3}\right)}{2} + \frac{1+\cos 2\left(x-\frac{\pi}{3}\right)}{2}$

$= \frac{1}{2}$ [3 + cos 2x + cos $\left(2x+\frac{2\pi}{3}\right)$ + cos $\left(2x-\frac{2\pi}{3}\right)$

$= \frac{1}{2}$ [3 + cos 2x + 2cos2x cos $\frac{2\pi}{3}$]

$= \frac{1}{2}$ [3 + cos 2x + 2cos2x cos $\left(\pi-\frac{\pi}{3}\right)$]

$= \frac{1}{2}$ [3 + cos 2x – 2cos2x cos $\frac{\pi}{3}$]

$= \frac{1}{2}$ [3 + cos 2x – cos2x]

$= \frac{3}{2}$ = R. H. S.

Ex. 12) Find $\sin\frac{\pi}{10}$

Solution : $\frac{\pi^c}{10} = 18°$

Let, $\theta = 18°$, $2\theta = 36°$, $3\theta = 54°$

We have $2\theta + 3\theta = 90°$

$2\theta = 90° - 3\theta$

$\therefore$ $\sin 2\theta = \sin(90° - 3\theta)$

$2\sin\theta.\cos\theta = \cos 3\theta$

$\therefore$ $2\sin\theta.\cos\theta = 4.\cos^3\theta - 3\cos\theta$

$2\sin\theta = 4.\cos^2\theta - 3$

$2\sin\theta = 4(1 - \sin^2\theta) - 3$

$2\sin\theta = 4 - 4\sin^2\theta - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\therefore$ $\sin\theta = \frac{-2\pm\sqrt{4+(4)(4)(1)}}{2(4)}$

$= \frac{-2\pm 2\sqrt{5}}{2(4)}$

$\therefore$ $\sin\theta = \frac{-1\pm\sqrt{5}}{4}$

$\therefore$ $\sin\theta = \frac{-1+\sqrt{5}}{4}$

[$\because$ θ is an acute angle]

$\therefore$ $\sin 18° = \frac{-1+\sqrt{5}}{4}$

$\therefore$ $\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}$

EXERCISE 3.3

1) Find values of : i) sin $\frac{\pi}{8}$ ii) cos $\frac{\pi}{8}$

2) Find sin $2x$, cos $2x$, tan $2x$ if sec$x$ = $\frac{-13}{5}$, $\frac{\pi}{2} < x < \pi$

3) Prove the following:

i) $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$

ii) $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

iii) $(\cos x+\cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\left(\frac{x+y}{2}\right)$

iv) $(\cos x-\cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\left(\frac{x-y}{2}\right)$

v) $\tan x + \cot x = 2\operatorname{cosec} 2x$

vi) $\frac{\cos x+\sin x}{\cos x-\sin x} - \frac{\cos x-\sin x}{\cos x+\sin x} = 2\tan 2x$

vii) $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8x}}} = 2\cos x$

viii) $16\sin\theta\cos\theta\cos 2\theta\cos 4\theta\cos 8\theta = \sin 16\theta$

ix) $\frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x} = 2\cot 2x$

x) $\frac{\cos x}{1+\sin x} = \frac{\cot\left(\frac{x}{2}\right)-1}{\cot\left(\frac{x}{2}\right)+1}$

xi) $\frac{\tan\left(\frac{\theta}{2}\right)+\cot\left(\frac{\theta}{2}\right)}{\cot\left(\frac{\theta}{2}\right)-\tan\left(\frac{\theta}{2}\right)} = \sec\theta$

xii) $\frac{1}{\tan 3A-\tan A} - \frac{1}{\cot 3A-\cot A} = \cot 2A$

xiii) $\cos 7^\circ \cos 14^\circ \cos 28^\circ \cos 56^\circ = \frac{\sin 68^0}{16\cos 83^0}$

xiv) $\frac{\sin^2(-160^\circ)}{\sin^2 70^\circ} + \frac{\sin(180^\circ-\theta)}{\sin\theta} = \sec^2 20^\circ$

xv) $\frac{2\cos 4x+1}{2\cos x+1} = (2\cos x-1)(2\cos 2x-1)$

xvi) $\cos^2 x + \cos^2(x+120^\circ) + \cos^2(x-120^\circ) = \frac{3}{2}$

xvii) $2\operatorname{cosec} 2x + \operatorname{cosec} x = \sec x \cot\left(\frac{x}{2}\right)$

xviii) $4\cos x\cos\left(x+\frac{\pi}{3}\right) + \cos^2\left(\pi-\frac{\pi}{3}\right) = \cos 3x$

xix) $\sin x\tan\left(\frac{x}{2}\right) + 2\cos x = \frac{2}{1+\tan^2\left(\frac{x}{2}\right)}$

Let's :Learn

3.4 Factorization formulae:

Formulae for expressing sums and differences of trigonometric functions as products of sine and cosine functions are called factorization formulae. Formulae to express products in terms of sums and differences are called defactorization formulae.

3.4.1 Formulae for conversion of sum or difference into product.

Theorem: 9) For any angles C and D,

1) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

2) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

3) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

4) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Proof :

Let, $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

$\therefore$ A+B = C and A–B = D

using these values in equations

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$

we get

$$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Simillarly the equations,

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \text{ ...... (3)}$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B \text{ ...... (4)}$$

gives,

$$\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\therefore \cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\therefore \sin(-\theta)=\sin\theta$$

$$\therefore -\sin\left(\frac{C-D}{2}\right)=\sin\left(-\left(\frac{C-D}{2}\right)\right)$$

$$=\sin\left(\frac{D-C}{2}\right)$$

$$\therefore \cos C-\cos D=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

3.4.2 Formulae for conversion of product in to sum or difference :

For any angles A and B

1) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

2) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

3) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

4) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

SOLVED EXAMPLES

Ex. 1) Prove the following :

i) $\sin 40^\circ-\cos 70^\circ=\sqrt{3}\cos 80^\circ$

ii) $\cos 40^\circ+\cos 50^\circ+\cos 70^\circ+\cos 80^\circ$

$=\cos 20^\circ+\cos 10^\circ$

Solution :

i) L.H.S. $=\sin 40^\circ-\cos 70^\circ$

$=\sin(90^\circ-50^\circ)-\cos 70^\circ$

$=\cos 50^\circ-\cos 70^\circ$

$=-2\sin 60^\circ\sin(-10^\circ)$

$=2\sin 60^\circ\sin 10^\circ$

$=2\times\frac{\sqrt{3}}{2}\cos 80=\sqrt{3}\cos 80^\circ$

= R.H.S

ii) L.H.S.

$=\cos 40^\circ+\cos 50^\circ+\cos 70^\circ+\cos 80^\circ$

$=(\cos 80^\circ+\cos 40^\circ)+(\cos 70^\circ+\cos 50^\circ)$

$=2\cos\left(\frac{80+40}{2}\right)\cos\left(\frac{80-40}{2}\right)+2\cos\left(\frac{70+50}{2}\right)\cos\left(\frac{70-50}{2}\right)$

$=2\cos 60^\circ\cos 20^\circ+2\cos 60^\circ\cos 10^\circ$

$=2\cos 60^\circ(\cos 20^\circ+\cos 10^\circ)$

$=2\,\frac{1}{2}(\cos 20^\circ+\cos 10^\circ)$

$=\cos 20^\circ+\cos 10^\circ$ = R. H. S.

Ex. 2) Express the following as sum or difference of two trigonometric function:

i) $2\sin 4\theta\cos 2\theta$

Solution : $=2\sin 4\theta\cos 2\theta$

$=\sin(4\theta+2\theta)+\sin(4\theta-2\theta)$

$=\sin 6\theta+\sin 2\theta$

ii) 4 sin $\left(\frac{A+B}{2}\right)$ sin $\left(\frac{A-B}{2}\right)$

$= 2 \times [\cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right)$

$= 2[\cos B - \cos A]$

$= 2\cos B - 2\cos A$

Ex. 3) Show that

i) $\frac{sin8x + sin2x}{cos2x - cos8x} = \cos 3x$

Solution : L.H.S

$= \frac{sin8x + sin2x}{cos2x - cos8x}$

$= \frac{2\sin\left(\frac{8x+2x}{2}\right) cos\left(\frac{8x-2x}{2}\right)}{2sin\left(\frac{2x+8x}{2}\right) sin\left(\frac{8x-2x}{2}\right)}$

$= \frac{2\sin 5x \cos 3x}{2\sin 5x \sin 3x}$

$= \cot 3x$

= R. H. S.

ii) $\frac{sin2\propto + sin2\beta}{sin2\propto - sin2\beta} = \frac{\tan(\propto+\beta)}{\tan(\propto-\beta)}$

Solution :

L.H.S. $= \frac{sin2\propto + sin2\beta}{sin2\propto - sin2\beta}$

$= \frac{2\sin\left(\frac{2\propto+2\beta}{2}\right) cos\left(\frac{2\propto-2\beta}{2}\right)}{2cos\left(\frac{2\propto+2\beta}{2}\right) sin\left(\frac{2\propto-2\beta}{2}\right)}$

$= \frac{sin(\propto+\beta)}{cos(\propto+\beta)} \cdot \frac{cos(\propto-\beta)}{sin(\propto-\beta)}$

$= \tan(\alpha + \beta) \cdot \cot(\alpha - \beta)$

$= \frac{\tan(\propto+\beta)}{tan(\propto-\beta)}$

= *R. H.S.*

Ex. 4) Prove that following.

i) $\frac{\cos(7x-5y)+\cos(7y-5x)}{\sin(7x-5y)+\sin(7y-5x)} = \cot(x+y)$

Solution : L.H.S.

$= \frac{\cos(7x-5y)+\cos(7y-5x)}{\sin(7x-5y)+\sin(7y-5x)}$

$= \frac{2\cos\left(\frac{7x-5y+7y-5x}{2}\right)\cos\left(\frac{7x-5y-7y+5x}{2}\right)}{2sin\left(\frac{7x-5y+7y-5x}{2}\right)\cos\left(\frac{7x-5y-7y+5x}{2}\right)}$

$= \frac{\cos(x+y)\cos(6x-6y)}{\sin(x+y)\cos(6x-6y)}$

$= \frac{\cos(x+y)}{\sin(x+y}$ $= \cot(x+y) = R.\ H.\ S.$

ii) $\sin 6\theta + \sin 4\theta - \sin 2\theta = 4\cos\theta \sin 2\theta \cos 3\theta$

Solution : L.H.S.

$= \sin 6\theta + \sin 4\theta - \sin 2\theta$

$= 2\sin\left(\frac{6\theta+4\theta}{2}\right)\cos\left(\frac{6\theta-4\theta}{2}\right) - 2\sin\theta\cos\theta$

$= 2\sin 5\theta\cos\theta - 2\sin\theta\cos\theta$

$= 2\cos\theta\,[\sin 5\theta - \sin\theta]$

$= 2\cos\theta\left[2\cos\left(\frac{5\theta+\theta}{2}\right) sin\left(\frac{5\theta-\theta}{2}\right)\right]$

$= 2\cos\theta . 2\cos 3\theta \sin 2\theta$

$= 4\cos\theta\sin 2\theta\cos 3\theta$

= R.H.S.

iii) $\frac{cos3xsin9x - sinxcos5x}{cosxcos5x - sin3xsin9x} = \tan 8x$

Solution : L.H.S.

$= \frac{cos3xsin9x - sinxcos5x}{cosxcos5x - sin3xsin9x}$

$= \frac{2cos3xsin9x - 2sinxcos5x}{2cosxcos5x - 2sin3xsin9x}$

$$= \frac{[\sin(3x+9x)-\sin(3x-9x)]-[\sin(x+5x)+\sin(x-5x)]}{[\cos(x+5x)+\cos(x-5x)]-[\cos(9x-3x)-\cos(3x+9x)]}$$

$$= \frac{\sin 12x-\sin(-6x)-\sin 6x-\sin(-4x)}{\cos 6x+\cos(-4x)-\cos 6x+\cos 12x}$$

$$= \frac{\sin 12x+\sin 6x-\sin 6x+\sin 4x}{\cos 6x+\cos 4x-\cos 6x+\cos 12x}$$

$$= \frac{\sin 12x+\sin 4x}{\cos 12x+\cos 4x}$$

$$= \frac{2\sin\left(\frac{12x+4x}{2}\right)\cos\left(\frac{12-4x}{2}\right)}{2\cos\left(\frac{12x+4x}{2}\right)\cos\left(\frac{12x-4x}{2}\right)}$$

$$= \frac{\sin 8x}{\cos 8x} = \tan 8x = \text{R.H.S.}$$

iv) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

Solution : L.H.S.

$= \cos 20° \cos 40° \cos 60° \cos 80°$

$= \cos 20° \,.\, \cos 40° \,.\, \frac{1}{2} \,.\, \cos 80°$

$= \frac{1}{2} [\cos 20° \cos 40° \cos 80°]$

$= \frac{1}{4} [\cos(20°+40°) + \cos(20°-40°)] \cos 80°$

$= \frac{1}{4} [\cos 60°) + \cos(-20°)] \cos 80°$

$= \frac{1}{4} [\frac{1}{2} \cos 80° + \cos 20° \cos 80°]$

$= \frac{1}{4} [\frac{1}{2} \cos 80° + \cos 20° \cos 80°]$

$= \frac{1}{8} \cos 80° + \frac{1}{4} \,.\, \frac{1}{2} \,.\, 2\cos 20° \cos 80°$

$= \frac{1}{8} \cos 80° + \frac{1}{8} [\cos(20+80) + \cos(20-80)]$

$= \frac{1}{8} \cos 80° + \frac{1}{8} [\cos 100° + \cos(-60)°]$

$= \frac{1}{8} [\cos 80° + [\cos 180° - 80°)] + \frac{1}{8} \times \frac{1}{2}$

$= \frac{1}{8} [\cos 80° - \cos 80°] + \frac{1}{16} = \frac{1}{16}$

= *R. H.S.*

EXERCISE 3.4

1) Express the following as a sum or difference of two trigonometric function.

i) $2\sin 4x \cos 2x$

ii) $2\sin \frac{2\pi}{3} \cos \frac{\pi}{2}$

iii) $2\cos 4\theta \cos 2\theta$

iv) $2\cos 35° \cos 75°$

2) Prove the following :

i) $\frac{\sin 2x+\sin 2y}{\sin 2x-\sin 2y} = \frac{\tan(x+y)}{\tan(x-y)}$

ii) $\sin 6x + \sin 4x - \sin 2x = 4\cos x \sin 2x \cos 3x$

iii) $\frac{\sin x-\sin 3x+\sin 5x-\sin 7x}{\cos x-\cos 3x-\cos 5x+\cos 7x} = \cot 2x$

iv) $\sin 18° \cos 39° + \sin 6° \cos 15° = \sin 24° \cos 33°$

v) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

vi) $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

Let's :Learn

3.5 Trigonometric functions of angles of a triangle

Notation: In ΔABC; $m\angle BAC = A$,

$m\angle ABC = B$, $m\angle ACB = C$

$\therefore A + B + C = \pi$

Result 1) In ΔABC, $A + B + C = \pi$

$\therefore A + B = \pi - C$

$\therefore \sin(A + B) = \sin(\pi - C)$

$\therefore \sin(A + B) = \sin C$

Simillarly:

$\sin(B + C) = \sin A$ and

$\sin(C + A) = \sin B$

Result 2) In ΔABC, $A+B+C = \pi$

$\therefore B + C = \pi - C$

$\therefore \cos(B + C) = \cos(\pi - A)$

$\therefore \cos(B + C) = -\cos A$

Simillarly:

$\cos(A + B) = -\cos C$ and

$\cos(C + A) = -\cos B$

Result 3) for any ΔABC

i) $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$

$\sin\left(\frac{C+A}{2}\right) = \cos\frac{B}{2}$

ii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

$\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$

$\cos\left(\frac{C+A}{2}\right) = \sin\frac{B}{2}$

Proof.

i) In ΔABC, $A+B+C = \pi$ $\therefore A + B = \pi - C$

$\therefore \left(\frac{A+B}{2}\right) = \frac{\pi - C}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

$\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

Verify.

1) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$

2) $\cos\left(\frac{B+C}{2}\right) = \sin\frac{B}{2}$

ii) In ΔABC, $A+B+C = \pi$ $\therefore B + C = \pi - B$

$\therefore \cos\left(\frac{A+C}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) = \sin\frac{B}{2}$

$\therefore \cos\left(\frac{A+C}{2}\right) = \sin\frac{B}{2}$

Verify.

1) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

2) $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$

SOLVED EXAMPLES

Ex. 1) In ΔABC prove that

i) $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$

Solution : L.H.S. $= \sin 2A + \sin 2B - \sin 2C$

$= 2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) - \sin 2C$

$= 2\sin(A + B)\cos(A - B) - 2\sin C\cos C$

$= 2\sin(\pi - C)\cos(A - B) - 2\sin C\cos[\pi - (A+B)]$

$= 2\sin C\cos(A - B) + 2\sin C\cos(A + B)$

$= 2\sin C[\cos(A - B) + \cos(A + B)]$

$= 2\sin C . 2\cos\left(\frac{A-B+A+B}{2}\right)\cos\left(\frac{A-B-A-B}{2}\right)$

$= 4\sin C\cos A\cos B$

$= 4\cos A\cos B\sin C$

= R.H.S.

ii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution : L.H.S $= \cos A + \cos B + \cos C$

$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$

$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$

$= 1 + 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin^2\frac{C}{2}$

$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$

$= 1 + 2\sin\frac{C}{2}\ [\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)]$

$= 1 + 2\sin\frac{C}{2}\ [\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)]$

$= 1 + 2\sin\frac{C}{2}\ .2\sin\left(\frac{A-B+A+B}{4}\right)\sin\left(\frac{A+B-A+B}{4}\right)$

$= 1 + 4\sin\frac{C}{2}\ \sin\frac{A}{2}\ \sin\frac{B}{2}$

$= 1 + 4\sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$

= R.H.S.

iii) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

Solution : L.H.S. $= \sin^2 A + \sin^2 B - \sin^2 C$

$= \frac{1-cos2A}{2} + \frac{1-cos2B}{2} - sin^2 C$

$= \frac{1}{2}\ [2 - \cos 2A - \cos 2B] - \sin^2 C$

$= 1 - \frac{1}{2}\ [\cos 2A + \cos 2B] - \sin^2 c$

$= 1 - \frac{1}{2}.\ 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) - \sin^2 C$

$= 1 - \sin^2 C - \cos(A+B) + \cos(A-B)$

$= \cos^2 C - \cos[\pi - C]\cos(A-B)$

$= \cos^2 C + \cos C\cos(A-B)$

$= \cos C\ [\cos C + \cos(A-B)]$

$= \cos C\ [\cos[\pi - (A+B)] + \cos(A-B)]$

$= \cos C\ [-\cos(A+B) + \cos(A-B)]$

$= \cos C\ [\cos(A-B) - \cos(A+B)]$

$= \cos C.2\sin\left(\frac{A-B+A+B}{2}\right)\sin\left(\frac{A+B-A+B}{2}\right)$

$= 2\cos C\ \sin A\sin B$

$= 2\sin A\sin B\cos C$

= R. H.S.

iv) $\cot A\cot B + \cot B\cot + \cot C\ \cot A = 1$

Solution : In ΔABC, $A + B + C = \pi$

$\therefore\ A + B = \pi - C$

$\therefore\ \tan(A + B) = \tan(\pi - C)$

$\therefore\ \frac{tanA + tanB}{1 - tanA\ tanB} = \tan(\pi - C)$

$\therefore\ \tan A + \tan B = -\tan C + \tan A\tan B\tan C$

$\therefore\ \tan A + \tan B + \tan C = \tan A\tan B\tan C$

$\therefore\ \frac{1}{\cot A} + \frac{1}{\cot B} + \frac{1}{\cot C} = \frac{1}{\cot A}.\frac{1}{\cot B}.\frac{1}{\cot C}$

$\therefore\ \cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

v) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Solution : In ΔABC, $A + B + C = \pi$

$\therefore\ A + B = \pi - C\ \ \therefore\ \frac{A+B}{2} = \frac{\pi - C}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\therefore\ \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \cot\frac{C}{2}$

$\therefore\ \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$

$\therefore\ [\tan\frac{A}{2} + \tan\frac{B}{2}]\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$

$\therefore\ \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$

$\therefore\ \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1$

vi) $\dfrac{\cos A-\cos B+\cos C+1}{\cos A+\cos B+\cos C-1}=\cot\dfrac{A}{2}\cot\dfrac{C}{2}$

Solution : L.H.S. $=\dfrac{\cos A-\cos B+\cos C+1}{\cos A+\cos B+\cos C-1}$

$$=\frac{[\cos A-\cos B]+[1+\cos C]}{[\cos A+\cos B]-[1-\cos C]}$$

$$=\frac{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)+2\cos^2\frac{C}{2}}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\left(-2\sin^2\frac{C}{2}\right)}$$

$$=\frac{2\sin\left(\frac{\pi}{2}-\frac{c}{2}\right)\sin\left(\frac{B-A}{2}\right)+2cos^2\frac{C}{2}}{2\cos\left(\frac{\pi}{2}-\frac{c}{2}\right)\cos\left(\frac{A-B}{2}\right)+\left(-2sin^2\frac{C}{2}\right)}$$

$$=\frac{2\cos\frac{C}{2}\sin\left(\frac{B-A}{2}\right)+2cos^2\frac{C}{2}}{2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-2sin^2\frac{C}{2}}$$

$$=\frac{\cos\frac{C}{2}\,[\sin\left(\frac{B-A}{2}\right)+cos\frac{C}{2}]}{\sin\frac{C}{2}\,[\cos\left(\frac{A-B}{2}\right)-sin\frac{C}{2}]}$$

$$=\cot\frac{C}{2}\;\frac{\left[\sin\left(\frac{A+B}{2}\right)+\sin\left(\frac{B-A}{2}\right)\right]}{[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)]}$$

$$=\cot\frac{C}{2}\;\frac{\sin\left(\frac{A+B}{2}\right)+\sin\left(\frac{B-A}{2}\right)}{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)}$$

$$=\cot\frac{C}{2}\cdot\frac{2\sin\frac{\left(\frac{A+B}{2}+\frac{B-A}{2}\right)}{2}\cos\frac{\left(\frac{A+B}{2}-\frac{B-A}{2}\right)}{2}}{2\sin\frac{\left(\frac{A-B}{2}+\frac{A+B}{2}\right)}{2}\sin\frac{\left(\frac{A+B}{2}-\frac{A-B}{2}\right)}{2}}$$

$$=\cot\frac{C}{2}\cdot\frac{2\sin\frac{B}{2}\cos\frac{A}{2}}{2\sin\frac{A}{2}\sin\frac{B}{2}}$$

$$=\cot\frac{C}{2}\;\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$=\cot\frac{C}{2}\cot\frac{A}{2}$$

= R.H.S.

EXERCISE 3.5

In ΔABC, $A+B+C=\pi$ show that

1) $\cos 2A+\cos 2B+\cos 2C$
$=-1-4\cos A\cos B\cos C$

2) $\sin A+\sin B+\sin C$
$=4\cos\frac{A}{2}\,cos\frac{B}{2}\,cos\frac{C}{2}$

3) $\cos A+\cos B-\cos C$
$=4\cos\frac{A}{2}\,cos\frac{B}{2}\,sin\frac{C}{2}-1$

4) $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

5) $\sin^2\frac{A}{2}+sin^2\frac{B}{2}-sin^2\frac{C}{2}$
$=1-2\cos\frac{A}{2}\,cos\frac{B}{2}\,sin\frac{C}{2}$

6) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

7) $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

8) $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$

Let's Remember

1) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

2) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

3) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

4) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

5) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, $\cos\left(\frac{\pi}{2}-\theta\right)$

$= \sin\theta$, $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

6) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2}+\theta\right)$

$= -\sin\theta$, $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$,

7) $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

8) $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

9) $\sin(\pi-\theta) = \sin\theta$, $\cos(\pi-\theta)$

$= -\cos\theta$, $\tan(\pi-\theta) = -\tan\theta$

10) $\sin(\pi+\theta) = -\sin\theta$, $\cos(\pi+\theta)$

$= -\cos\theta$ $\tan(\pi+\theta) = \tan\theta$

11) $\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}-\theta\right)$

$= \sin\theta$, $\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta$

12) $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}+\theta\right)$

$= \sin\theta$, $\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$

13) $\sin(2\pi-\theta) = -\sin\theta$, $\cos(2\pi-\theta)$

$= \cos\theta$, $\tan(2\pi-\theta) = -\tan\theta$

14) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

15) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

16) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$

$= 1 - 2\sin^2\frac{\theta}{2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

17) $1+\cos\theta = 2\cos^2\frac{\theta}{2}$, $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$1+\cos 2\theta = 2\cos^2\theta$, $1-\cos 2\theta = 2\sin^2\theta$

18) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

19) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

20) For ΔABC,

$\sin(A+B) = \sin C$, $\sin(B+C) = \sin A$
$\sin(A+C) = \sin B$

$\cos(A+B) = -\cos C$, $\cos(B+C) = -\cos A$
$\cos(A+C) = -\cos B$

$\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$, $\sin\left(\frac{B+C}{2}\right)$

$= \cos\frac{A}{2}$, $\sin\left(\frac{A+C}{2}\right) = \cos\frac{B}{2}$

$\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$, $\cos\left(\frac{B+C}{2}\right)$

$= \sin\frac{A}{2}$, $\cos\left(\frac{A+C}{2}\right) = \sin\frac{B}{2}$

Activity :

Verify the following.

i) $\sin 18° = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72° = \cos\frac{2\pi}{5}$

ii) $\cos 36° = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54° = \sin\left(\frac{3\pi}{10}\right)$

iii) $\sin 72° = \sin\frac{2\pi}{5} = \sqrt{\frac{10+2\sqrt{5}}{4}} = \cos 18° = \cos\frac{\pi}{10}$

iv) $\sin 36° = \sin\frac{2\pi}{5} = \sqrt{\frac{10-2\sqrt{5}}{4}} = \cos 54° = \cos\frac{3\pi}{10}$

v) $\sin 15° = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos\frac{5\pi}{12}$

vi) $\cos 15° = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin\frac{5\pi}{12}$

vii) $\tan 15° = \tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

viii) $\tan 75° = \tan\frac{\pi}{12} = 2 + \sqrt{3}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

ix) $\tan(22.5°) = \tan\frac{\pi}{8}$

$= \sqrt{2} - 1 = \cot 67.5 = \cot\frac{3\pi}{8}$

x) $\tan(67.5°) = \tan\frac{3\pi}{8}$

$= \sqrt{2} + 1 = \cot(22..5°) = \cot\frac{\pi}{8}$

MISCELLANEOUS EXERCISE - 3

I) Select correct option from the given alternatives.

1) The value of $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A$ is equal to

A) $\sin A$ B) $\cos A$ C) $-\cos A$ D) $\sin 2A$

2) If $\tan A - \tan B = x$ and $\cot B - \cot A = y$ *then* $\cot(A-B) = \ldots$

A) $\frac{1}{y} - \frac{1}{x}$ B) $\frac{1}{x} - \frac{1}{y}$

C) $\frac{1}{x} + \frac{1}{y}$ D) $\frac{xy}{x-y}$

3) If $\sin\theta = n\sin(\theta + 2\alpha)$ *then* $\tan(\theta+\alpha)$ is equal to

A) $\frac{1+n}{2-n}\tan\alpha$ B) $\frac{1-n}{1+n}\tan\alpha$

C) $\tan\alpha$ D) $\frac{1+n}{1-n}\tan\alpha$

4) The value of $\frac{\cos\theta}{1+\sin\theta}$ is equal to……

A) $\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ B) $\tan\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)$

C) $\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ D) $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

5) The value of $\cos A \cos(60° - A)\cos(60° + A)$ is equal to…..

A) $\frac{1}{2}\cos 3A$ B) $\cos 3A$

C) $\frac{1}{4}\cos 3A$ D) $4\cos 3A$

6) The value of

$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is ....

A) $\frac{1}{16}$ B) $\frac{1}{64}$ C) $\frac{1}{128}$ D) $\frac{1}{256}$

7) If $\alpha+\beta+\varkappa=\pi$ then the value of $\sin^2\alpha+\sin^2\beta-\sin^2\varkappa$ is equal to…..

A) $2\sin\alpha$ B) $2\sin\alpha\cos\beta\sin\varkappa$

C) $2\sin\alpha\sin\beta\cos\varkappa$ D) $2\sin\alpha\sin\beta\sin\varkappa$

8) Let $0 < A, B < \frac{\pi}{2}$ satisfying the equation $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$ then $A+2B$ is equal to....

A) π B) $\frac{\pi}{2}$ C) $\frac{\pi}{4}$ D) 2π

9) In ΔABC if $\cot A \cot B \cot C > 0$ then the triangle is….

A) Acute angled B) right angled

C) obtuse angled

D) isosceles right angled

10) The numerical value of $\tan 20^0 \tan 80^0 \cot 50^0$ is equal to…..

A) $\sqrt{3}$ B) $\frac{1}{\sqrt{3}}$ C) $2\sqrt{3}$ D) $\frac{1}{2\sqrt{3}}$

II) Prove the following.

1) $\tan 20° \tan 80° \cot 50° = \sqrt{3}$

2) If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$ then prove $\cot\alpha\tan\beta = -1$

3) $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$

4) $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$

5) $\cos 12° + \cos 84° + \cos 156° + \cos 132° = -\frac{1}{2}$

6) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

7) $\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

8) $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

9) $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

10) $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$

11) $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$

12) If $\sin 2A = \lambda \sin 2B$ then prove that $\frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$

13) $\frac{2\cos 2A+1}{2\cos 2A-1} = \tan(60°+A)\tan(60°-A)$

14) $\tan A + \tan(60°+A) + \tan(120°+A) = 3\tan 3A$

15) $3\tan^6 10° - 27\tan^4 10° + 33\tan^2 10° = 1$

16) $\text{cosec } 48° + \text{cosec } 96° + \text{cosec } 192° + \text{cosec } 384° = 0$

17) $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

18) $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = \cot A$

19) If $A + B + C = \frac{3\pi}{2}$ then $\cos 2A + \cos 2B + \cos 2C = 1 - 4\sin A \sin B \sin C$

20) In any triangle ABC, $\sin A - \cos B = \cos C$ then $\angle B = \pi/2$

21) $\frac{\tan^3 x}{1+\tan^2 x} + \frac{\cot^3 x}{1+\cot^2 x} = \sec x \operatorname{cosec} x - 2\sin x \cos x$

22) $\sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$

23) $\sin 18° = \frac{\sqrt{5}-1}{4}$

24) $\cos 36° = \frac{\sqrt{5}+1}{4}$

25) $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}$

26) $\sin \frac{\pi^c}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

27) $\tan \frac{\pi}{8} = \sqrt{2} - 1$

28) $\tan 6° \tan 42° \tan 66° \tan 78° = 1$

29) $\sin 47° + \sin 61° - \sin 11° - \sin 25° = \cos 7°$

30) $\sqrt{3} \operatorname{cosec} 20° - \sec 20° = 4$

31) In ΔABC, $\angle C = \frac{2\pi}{3}$ then prove that $\cos^2 A + \cos^2 B - \cos A \cos B = \frac{3}{4}$