4 METHOD OF INDUCTION AND BINOMIAL THEOREM

Let's Study

- Mathematical Induction
- Binomial Theorem
- General term of expansion
- Expansion for negative and fractional index
- Binomial coefficients

Let's Learn

Introduction :

The earliest implicit proof by induction was given by Al Karaji around 100 AD. The first explicit formulation of the principle was given by Pascal in 1665. The Mathematical Induction is a powerful method, easy to use for proving many theorems.

4.1 Principle of Mathematical Induction :

Principle of Mathematical Induction consists of the following four 4 steps:

Step 1 : (Foundation) To prove $P(n)$ is true for $n = 1$

(It is advisable to check if $P(n)$ is true for $n = 2, 3$ also if P(1) is trivial).

Step 2 : (Assumption) To assume $P(n)$ is true for $n = k$.

Step 3 : (Succession) To prove that $P(n)$ is true for $n = k + 1$.

Step 4 : (Induction) To conclude that $P(n)$ is true for all $n \in N$

Row of dominos standing close to each other gives us the idea of how the Principle of Mathematical Induction works.

Step 1 : (Foundation) The 1st domino falls down.

(followed by it 2^{nd} also falls down. Then 3^{rd}, 4^{th} and so on.)

Step 2 : (Assumption) Assume if k^{th} domino falls down.

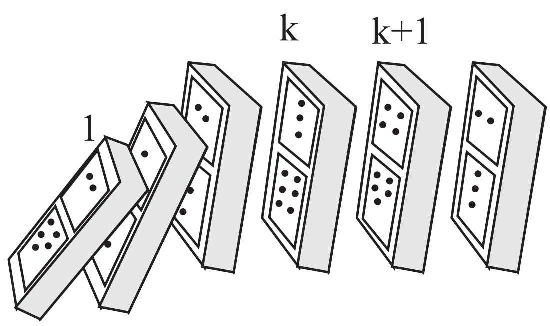

Fig. 4.1

Step 3 : (Succession) Followed by k^{th} domino, $(k + 1)^{th}$ domino will also fall down.

Step 4 : (Induction) It is true that all the dominos will fall down.

We will see how to use the principle of mathematical induction to prove statements.

Stepwise Explanation :

Step 1. (Foundation) Formulate the statement of the theorem as P(n) say, for any positive integer n and verify it for integer $n = 1$. In fact, it is often instructive, though not necessary, to verify the statement for $n = 2$ and 3. This gives better insight into the theorem.

Step 2. (Assumption) Assume that the statement P(n) is true for a positive integer k.

Step 3. (Succession) Prove the statement for $n = k + 1$.

Step 4. (Induction) Now invoke the principle of Mathematical induction. Conclude that the theorem is true for any positive integer n.

Illustration :

Let us prove a theorem with this method. The theorem gives the sum of the first n positive integers.

It is stated as P(n) : $1 + 2 + 3 \ldots + n = n(n+1)/2$.

Step 1 : (Foundation)

To prove *P(n)* is true for n = 1

L.H.S = 1 R.H.S = $\frac{1(1+1)}{2} = 1$ which is trivially true.

Check that $1 + 2 = \frac{2\times(2+1)}{2}$ and

$1 + 2 + 3 = \frac{3\times(3+1)}{2}$, so P(2) and P(3) are also true.

Step 2 : (Assumption) Assume that P(*n*) is true for n = k and in particular,

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

Step 3 : (Succession) To prove P(*n*) is true for n = k +1 that is

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Here L.H.S. = $1 + 2 + 3 + \ldots + k + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ (by step 2)

$= (k + 1) (\frac{k}{2} + 1)$

$= \frac{(k+1)(k+2)}{2}$ = R.H.S.

Thus, P(k + 1) is proved.

Step 4 : (Induction) Now by the Principle of Mathematical induction, the statement P(n) is proved for all positive integers n.

SOLVED EXAMPLES

Ex.1 By method of induction, prove that.

$1.3 + 2.5 + 3.7 + \ldots. + n(2n+1) = \frac{n}{6}(n+1)(4n+5)$ for all $n \in N$

Solution :

Let $P(n) \equiv 1.3 + 2.5 + 3.7 + \ldots. + n(2n+1)$,

for all $n \in N$

$= \frac{n}{6}(n+1)(4n+5)$

Step (I) : (Foundation) To prove P(1) is true

Let n = 1

L. H. S. = 1.3 = 3

R. H. S. $= \frac{1}{6}(1 + 1)(4.1 + 5)$

$= \frac{1}{6}(2)(9) = 3$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P(1) is true

Step (II) : (Assumption) Assume that let P(k) is true

$\therefore 1.3 + 2.5 + 3.7 + \ldots. + k(2k+1)$

$= \frac{k}{6}(k+1)(4k+5)$ ...(i)

Step (III) : (Succession) To prove that P(k+1) is true.

i.e. $1.3 + 2.5 + 3.7 + \ldots. + (k+1)[2(k+1)+1]$

$= \frac{(k+1)}{6}(k+1+1)[4(k+1)+5]$

i.e. $1.3 + 2.5 + 3.7 + \ldots. + (k+1)(2k+3)$

$= \frac{(k+1)}{6}(k+2)(4k+9)$

Now

$$\text{L.H.S.} = 1.3 + 2.5 + 3.7 + ... + (k+1)(2k+3)$$

$$= 1.3+2.5+...+k(2k+1)+(k+1)(2k+3)$$

$$= \frac{k}{6}(k+1)(4k+5)+(k+1)(2k+3)$$

... from (i)

$$= (k+1)\left[\frac{k(4k+5)}{6}+2k+3\right]$$

$$= (k+1)\frac{\left[4k^2+5k+12k+18\right]}{6}$$

$$= \frac{(k+1)(k+2)(4k+9)}{6}$$

= R.H.S.

$\therefore$ P (k+1) is true.

Step (IV) : (Induction) From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore$ $1.3 + 2.5 + 3.7 + ... + n(2n+1)$

$$= \frac{n}{6}(n+1)(4n+5), \text{ for all } n \in N.$$

Ex.2 By method of induction, prove that.

$$\sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Solution : Let $P(n) \equiv \sum_{r=1}^{n} ax^{r-1}$

$$= a + ax + ax^2 + ... + ax^{n-1} = a\left(\frac{1-x^n}{1-x}\right)$$

Step (I) : To prove that P(1) is true

Let n = 1

$\therefore$ L. H. S. = a

$$\text{R. H. S.} = a\left(\frac{1-x}{1-x}\right) = a$$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true.

$$\sum_{r=1}^{k} ax^{r-1} = a + ax + ax^2 + ... + ax^{k-1}$$

$$= a\left[\frac{1-x^k}{1-x}\right] \quad ....(i)$$

Step (III) : To prove that P(k+1) is true

i.e. $a + ax + ax^2 + ... + ax^k = a\left[\frac{1-x^{k+1}}{1-x}\right]$

Now, L.H.S. $= a + ax + ax^2 + ... + ax^{k-1} + ax^k$

$$= a\left[\frac{1-x^k}{1-x}\right] + ax^k \quad \text{[by (i)]}$$

$$= \frac{a(1-x^k)+ax^k(1-x)}{(1-x)}$$

$$= \frac{a[1-x^k+x^k-x^{k+1}]}{(1-x)}$$

$$= a\left[\frac{1-x^{k+1}}{1-x}\right]$$

= R. H. S.

$\therefore$ P(k+1) is true.

Step (IV) : From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore \sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Ex.3 By method of induction, prove that.

$5^{2n} - 1$ is divisible by 6, for all $n \in N$.

Solution : $5^{2n} - 1$ is divisible by 6, if and only if

$5^{2n} - 1$ is a multiple of 6

Let P(n) be $5^{2n} - 1 = 6m$, $m \in N$.

Step (I) : To prove that P(1) is true, Let n = 1

$\therefore$ $5^{2n} - 1 = 25 - 1 = 24 = 6 \cdot 4$

$\therefore\ 5^{2n} - 1$ is a multiple of 6

$\therefore$ P(1) is true.

Step (II) : Assume that P(k) is true.

$\therefore\ 5^{2k} - 1 = 6a,$ where $a \in N$

$\therefore\ 5^{2k} = 6a + 1$ ...(i)

Step (III) : To prove that $P(k+1)$ is true

i.e. to prove that $5^{2(k+1)} - 1$ is a multiple of 6

i.e. $5^{2k+2} - 1 = 6b,$ $b \in N$

Now $5^{2k+2} - 1 = 5^{2k}.5^2 - 1$

$= (6a + 1)\ 25 - 1$ by (i)

$= 150a + 24 = 6(25a + 4)$

$= 6b$

Step (IV) : From all the steps above

$P(n) = 5^{2n} - 1$ is divisible by 6,

for all $n \in N$

Note :

1) $5<5$ is not a true statement, whereas $5 \leq 5$, $5 \geq 5$ are true statements.

2) $2 = 3$, $2 > 3$, $2 \geq 3$ are not true statements, whereas $2 < 3$, $2 \leq 3$are true statements,

Ex. 4) By method of induction, prove that

$n\,! \geq 2^n\,;\ \forall\ n \in N,\ n \geq 4.$

Solution : Step I : (Foundation) Since $P(n)$ is stated for $n \geq 4$. Take $n = 4$

L.H.S. = 4! = 24, R.H.S. = 2^4 = 16.

Since 24≥16, $P(n)$ is true for $n = 4$

[$P(n)$ is not true for $n = 1, 2, 3$ (Verify!)]

Step (II) : (Assumption) Assume that let P(k) is true.

i.e. $k\,! \geq 2^k\,;\ k \in N,\ k \geq 4.$

Step (III) : (Succession) To prove that $P(k+1)$ is true.

i.e. to prove that $(k + 1)! \geq 2^{k+1},\ k + 1 \geq 4.$

L.H.S. = $(k+1)! = (k + 1)k!$

Since $k \geq 4$, $k + 1 > 4+1$, i.e. $k + 1 \geq 5$,
also $k + 1 \geq 2$ (why?)

and from Step II, $k\,! \geq 2^k\,;\ k \geq 4.$

Therefore, L.H.S. = $(k+1)k! \geq 2.2^k = 2^{k+1}$ =R.H.S.

i.e. $(k+1)! \geq 2^{k+1},\ k + 1 \geq 4$

Therefore $P(k+1)$ is true.

Step (IV) : (Induction) From all steps above, P(n) is true for $\forall\ n \in N,\ n \geq 4.$

Ex. 5) Given that (recurrence relation) $t_{n+1} = 3t_n + 4$, $t_1 = 1$, prove by induction that (general statement) $t_n = 3^n - 2$.

Solution : The statement $P(n)$ has L.H.S. a recurrence relation $t_{n+1} = 3t_n + 4$, $t_1 = 1$ and R.H.S. a general statement $t_n = 3^n - 2$.

Step I : (Foundation) To prove $P(1)$ is true.

L.H.S. = 1, R.H.S = $3^1 - 2 = 3 - 2 = 1$
So $P(1)$ is true.

For n = 2, L.H.S. = $t_2 = 3t_1 + 4 = 3(1) + 4$
= 7

Now R.H.S. = $t_2 = 3^2 - 2 = 9 - 2 = 7$. P(2) is also true.

Step II : (Assumption) Assume that $P(k)$ is true.

i.e. for $t_{k+1} = 3t_k + 4$, $t_1 = 1$, then $t_k = 3^k - 2$

Step III : (Sucession) To prove that $P(k + 1)$ is ture.

i.e. to prove $t_{k+1} = 3^{k+1} - 2$

Since $t_{k+1} = 3t_k + 4$, and $t_k = 3^k - 2$ (From Step II) $t_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 6 + 4$

$= 3^{k+1} - 2.$

Therefore $P(k + 1)$ is true.

Step IV: (Induction) From all the steps above $P(n)$, $t_n = 3^n - 2$ is true for $\forall$ n $\in$ N, where $t_{n+1} = 3t_n + 4$, $t_1 = 1$.

Ex.6 By method of induction, prove that.

$2^n > n$, for all $n \in N$.

Solution : Let $P(n) = 2^n > n$

Step (I) : To prove that P(1) is true, Let n = 1

L.H.S. $= 2^1 = 2$

R.H.S. = 1

2 > 1 Which is true

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true, $k \in N$

$\therefore 2^k > k$ ...(i)

Step (III) : To prove that P(k+1) is true

i.e. $2^{k+1} > k + 1$

Now $2^{k+1} = 2^k.2^1 > k.2$ ...by (i)

$\therefore 2^{k+1} > 2k$

$\therefore 2^{k+1} > k + k$

$\therefore 2^{k+1} > k + 1$ ($\because k \geq 1$)

$\therefore$ P(k + 1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore 2^n > n$, for all $n \in N$.

Remarks : (1) In the proof of P(n) by method of induction, both the conditions viz. (i) P(1) is true and (ii) P(k+1) is true when P(k) is true, must be satisfied. (2) In some problems, second step is satisfied but the first step is not satisfied. Hence the result is not valid for all $n \in N$.

for example,

let $P(n) \equiv 1.6+2.9+3.12+....+n(3n + 3) = n^3 + 3n^2 + 2n + 3$

Let us assume that P(k) is true.

$\therefore 1{\cdot}6+2{\cdot}9+3{\cdot}12+....+k(3k + 3) = k^3 + 3k^2 + 2k + 3$ ...(i)

We have to prove that P(k+1) is true,

i.e. to prove that

$1{\cdot}6 + 2{\cdot}9 + 3{\cdot}12 +....+ (k + 1)(3k + 6) = (k+1)^3+3(k+1)^2+2(k+1)+3$

L.H.S. $= 1{\cdot}6 + 2{\cdot}9 + 3{\cdot}12 +....+ (k + 1)(3k + 6)$

$= 1{\cdot}6+2{\cdot}9+3{\cdot}12 +....+ k(3k+3) + (k+1)(3k+6)$

$= k^3 + 3k^2 + 2k + 3 + (k + 1)(3k + 6)$ by (i)

$= k^3 + 3k^2 + 2k + 3 + 3k^2 + 6k + 3k + 6$

$= k^3 + 3k^2 + 3k + 1 + 3k^2+6k+3+2k+2+3$

$= (k + 1)^3 + 3(k^2 + 2k + 1) + 2(k + 1) + 3$

$= (k + 1)^3 + 3(k + 1)^2 + 2(k + 1) + 3$

= R. H. S.

$\therefore$ P(k + 1) is true.

If P(k) is true then P (k+1) is true.

Now we examine the result for n = 1

L.H.S. $= 1{\cdot}6 = 6$

R. H. S. $= 1^3 + 3(1)^2 + 2(1) + 3$

$= 9$

$\therefore$ L. H. S. $\neq$ R. H. S.

$\therefore$ P(1) is not true

$\therefore$ P (n) is not true for all $n \in N$.

EXERCISE 4.1

Prove by method of induction, for all $n \in N$.

(1) $2 + 4 + 6 + ... + 2n = n(n+1)$

(2) $3 + 7 + 11 + ...$ + to n terms $= n(2n+1)$

(3) $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(4) $1^2+3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n}{3}(2n-1)(2n+1)$

(5) $1^3+3^3 + 5^3 + ...$ to n terms $= n^2(2n^2-1)$

(6) $1{\cdot}2+2{\cdot}3+3{\cdot}4+.....+n(n+1) = \dfrac{n}{3}(n+1)(n+2)$

(7) $1{\cdot}3+3{\cdot}5+5{\cdot}7+.....$ to n terms $= \dfrac{n}{3}(4n^2+6n-1)$

(8) $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} +....+ \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

(9) $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots \text{ to n terms} = \frac{n}{3(2n+3)}$

(10) $(2^{3n}-1)$ is divisible by 7.

(11) $(2^{4n}-1)$ is divisible by 15.

(12) $3^n - 2n - 1$ is divisible by 4.

(13) $5 + 5^2 + 5^3 + \ldots. + 5^n = \frac{5}{4}(5^n - 1)$

(14) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(15) Given that $t_{n+1} = 5t_n + 4$, $t_1=4$, prove by method of induction that $t_n = 5^n - 1$

(16) Prove by method of induction

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 2n\\ 0 & 1\end{pmatrix} \quad \forall\, n \in N$$

4.2 Binomial Theorem for positive integral index :

We know that

$(a+b)^0 =$ 1

$(a+b)^1 =$ 1a + 1b

$(a+b)^2 =$ $1a^2 + 2ab + 1b^2$

$(a+b)^3 =$ $1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 =$ $1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

The coefficients of these expressions are arranged by Pascal's triangle as follows and are expressed in the form of ^nC_r

Index	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Index	
0	$^0C_0 = 1$
1	$^1C_0 = 1$ $^1C_1 = 1$
2	$^2C_0 = 1$ $^2C_1 = 2$ $^2C_2 = 1$
3	$^3C_0 = 1$ $^3C_1 = 3$ $^3C_2 = 3$ $^3C_3 = 1$
4	$^4C_0=1$ $^4C_1=4$ $^4C_2= 6$ $^4C_3= 4$ $^4C_4=1$

Now, we will study how to expand binomials of higher powers.

Theorem : If a, b $\in$ R and n $\in$ N, then

$(a+b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0 b^n$

Proof : We prove this theorem by method of induction.

Let P(n) be $(a+b)^n =$

${}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0 b^n$

Step (I) : Let n = 1

$\therefore$ L. H. S. = $(a+b)^1 = a + b$

R. H. S. = ${}^1C_0\, a^1b^0 + {}^1C_1\, a^0b^1 = a + b$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P (1) is true.

Step (II) : Let P(k) be true.

$\therefore (a+b)^k = {}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1} b^1 + {}^kC_2 a^{k-2}b^2 + .. + {}^kC_k a^0 b^k$ ....(i)

Step (III) : We have to prove that P(k+1) is true.

i.e. to prove that

$(a+b)^{k+1} =$

${}^{k+1}C_0\, a^{k+1}b^0 + {}^{k+1}C_1 a^kb^1 + {}^{k+1}C_2 a^{k-1}b^2 + \ldots$

$+ {}^{k+1}C_{k+1} a^0\, b^{k+1}$

Now L. H. S. = $(a+b)^{k+1}$

$= (a+b)\,(a+b)^k$

$= (a+b)\,[{}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1} b^1 + {}^kC_2 a^{k-2}b^2 + ..$

$+ {}^kC_k a^0\, b^k]$ by (i)

$= a[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1}b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0b^k]$

$+ b[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1}\, b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0\, b^k]$

$= [{}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_2a^{k-1}b^2+...+{}^kC_kab^k]+$
$[{}^kC_0a^kb+{}^kC_1a^{k-1}\ b^2+{}^kC_2a^{k-2}b^3+...+{}^kC_ka^0\ b^{k+1}]$
$= {}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_0a^kb^1+{}^kC_2a^{k-1}b^2+$
${}^kC_1a^{k-1}b^2+...+{}^kC_ka^1\ b^k+ {}^kC_{k-1}a^1b^k + {}^kC_ka^0\ b^{k+1}$
$= {}^kC_0a^{k+1}b^0 + ({}^kC_1+{}^kC_0)a^kb^1+({}^kC_2+{}^kC_1)a^{k-1}b^2+$
$... + ({}^kC_k+ {}^kC_{k-1})\ a^1b^k + {}^kC_ka^0\ b^{k+1}$

But we know that

${}^kC_0 = 1 = {}^{k+1}C_{k+1},\ {}^kC_1+ {}^kC_0 = {}^{k+1}C_1.$

${}^kC_2+{}^kC_1 = {}^{k+1}C_2,....\ {}^kC_k + {}^kC_{k-1} = {}^{k+1}C_k,....$

$${}^kC_k = 1 = {}^{k+1}C_{k+1}$$

$\therefore$ L.H.S. $= {}^{k+1}C_0a^{k+1}b^0+{}^{k+1}C_1a^kb^1+{}^{k+1}C_2a^{k-1}b^2 + ... + {}^{k+1}C_{k+1}a^0b^{k+1}$

= R.H.S.

$\therefore$ P (k+1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore (a+b)^n={}^nC_0\ a^nb^0+{}^nC_1a^{n-1}\ b^1+{}^nC_2a^{n-2}b^2...+{}^nC_na^0\ b^n$, for all $n \in N$.

Remarks :

(1) The expansion of $(a+b)^n$ contains n+1 terms.

(2) First term is a^n and last term is b^n.

(3) In each term, the sum of indices of a and b is always n.

(4) In successive terms, the index of a decreases by 1 and index of b increases by 1.

(5) Coefficients of the terms in binomial expansion equidistant from both the ends are equal. i.e. coefficients are symmetric.

(6) $(a-b)^n = {}^nC_0a^nb^0 - {}^nC_1a^{n-1}\ b^1+{}^nC_2a^{n-2}b^2 - \cdots + (-1)^n{}^nC_na^0\ b^n$.

In the RHS, the first term is positive and consequent terms are alternately negative and positive.

SOLVED EXAMPLES

Ex. 1 : Expand $(x^2 + 3y)^5$

Solution : Here $a = x^2$, $b = 3y$ and $n = 5$ using binomial theorem,

$$(x^2+3y)^5 = {}^5C_0(x^2)^5(3y)^0 + {}^5C_1(x^2)^4(3y)^1 + {}^5C_2(x^2)^3(3y)^2 + {}^5C_3(x^2)^2(3y)^3 + {}^5C_4(x^2)^1(3y)^4 + {}^5C_5(x^2)^0(3y)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore (x^2+3y)^5 = 1(x^{10})(1)+5(x^8)(3y)+10(x^6)(9y^2)+10(x^4)(27y^3)+5(x^2)(81y^4)+1(1)(243y^5)$$

$$\therefore (x^2+3y)^5 = x^{10}+15x^8y+90x^6y^2+270x^4y^3+405x^2y^4+243y^5$$

Ex. 2 : Expand $\left(2x - \frac{y}{2}\right)^5$

Solution : Here $a = 2x$, $b = \frac{y}{2}$ and $n = 5$

Using binomial theorem,

$$\left(2x - \frac{y}{2}\right)^5 = {}^5C_0(2x)^5\left(\frac{y}{2}\right)^0 - {}^5C_1(2x)^4\left(\frac{y}{2}\right)^1 + {}^5C_2(2x)^3\left(\frac{y}{2}\right)^2 - {}^5C_3(2x)^2\left(\frac{y}{2}\right)^3 + {}^5C_4(2x)^1\left(\frac{y}{2}\right)^4 - {}^5C_5(2x)^0\left(\frac{y}{2}\right)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore \left(2x - \frac{y}{2}\right)^5 = 1(32x^5)(1) - 5(16x^4)\left(\frac{y}{2}\right) + 10(8x^3)\left(\frac{y^2}{4}\right) - 10(4x^2)\left(\frac{y^3}{8}\right) + 5(2x)\left(\frac{y^4}{16}\right) - 1(1)\left(\frac{y^5}{32}\right)$$

$$\therefore \left(2x - \frac{y}{2}\right)^5 = 32x^5 - 40x^4y + 20x^3y^2 - 5x^2y^3 + \frac{5}{8}xy^4 - \frac{y^5}{32}$$

Ex.3 : Expand $\left(\sqrt{5}+\sqrt{3}\right)^4$

Solution : Here a = $\sqrt{5}$, b = $\sqrt{3}$ and n = 4

Using binomial theorem,

$$\left(\sqrt{5}+\sqrt{3}\right)^4 = {}^4C_0\left(\sqrt{5}\right)^4\left(\sqrt{3}\right)^0 + {}^4C_1\left(\sqrt{5}\right)^3\left(\sqrt{3}\right)^1 + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_3\left(\sqrt{5}\right)^1\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{5}\right)^0\left(\sqrt{3}\right)^4$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$,

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 1(25)(1) + 4\left(5\sqrt{5}\right)\left(3\sqrt{3}\right) + 6(5)(3) + 4\left(\sqrt{5}\right)\left(3\sqrt{3}\right) + 1(1)(9)$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 25 + \left(20\sqrt{15}\right) + 90 + \left(12\sqrt{15}\right) + 9$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 124 + \left(32\sqrt{15}\right)$$

Ex.4 : Evaluate $\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$

Solution : $\left(\sqrt{2}+1\right)^5 = {}^5C_0\left(\sqrt{2}\right)^5 + {}^5C_1\left(\sqrt{2}\right)^4 + {}^5C_2\left(\sqrt{2}\right)^3 + {}^5C_3\left(\sqrt{2}\right)^2 + {}^5C_4\left(\sqrt{2}\right)^1 + {}^5C_5\left(\sqrt{2}\right)^0$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_4 = {}^5C_1 = 5$, ${}^5C_2 = {}^5C_3 = \frac{5.4}{2.1} = 10$

$$\therefore \left(\sqrt{2}+1\right)^5 = 1\left(4\sqrt{2}\right) + 5(4) + 10\left(2\sqrt{2}\right) + 10(2) + 5\left(\sqrt{2}\right) + 1$$

$$\therefore \left(\sqrt{2}+1\right)^5 = \left(4\sqrt{2}\right) + 20 + \left(20\sqrt{2}\right) + 20 + \left(5\sqrt{2}\right) + 1 \quad \text{... (i)}$$

Similarly,

$$\left(\sqrt{2}-1\right)^5 = \left(4\sqrt{2}\right) - 20 + \left(20\sqrt{2}\right) - 20 + \left(5\sqrt{2}\right) - 1 \quad \text{... (ii)}$$

Subtracting (ii) from (i) we get,

$$\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$$

$$= \left(4\sqrt{2}+20+20\sqrt{2}+20+5\sqrt{2}+1\right) - \left(4\sqrt{2}-20+20\sqrt{2}-20+5\sqrt{2}-1\right)$$

$$= 2(20 + 20 + 1)$$

$$= 82$$

$$\therefore \left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5 = 82$$

Ex. 5 (Activity) : Using binomial theorem, find the value of $(99)^4$

Solution : We have $(99)^4 = (\square - 1)^4$

$$\therefore (99)^4 = {}^4C_0(\square)^4 - {}^4C_1(\square)^3 + {}^4C_2(\square)^2 - {}^4C_3(\square)^1 + {}^4C_4(\square)^0$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$

$$\therefore (99)^4 = 1(10)^{\square} - 4(10)^{\square} + 6(10)^{\square} - 4(10)^{\square} + 1(1)$$

$$= \square - \square + \square - \square + \square = \square$$

Ex. 6 : Find the value of $(2.02)^5$ correct upto 4 decimal places.

Solution : $(2.02)^5 = [2+0.02]^5$

$$= {}^5C_0(2)^5(0.02)^0 + {}^5C_1(2)^4(0.02)^1 + {}^5C_2(2)^3(0.02)^2 + {}^5C_3(2)^2(0.02)^3 + {}^5C_4(2)^1(0.02)^4 + {}^5C_5(2)^0(0.02)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$, ${}^5C_2 = {}^5C_3 = 10$

$$\therefore (2.02)^5 = 1(32)(1) + 5(16)(0.02) + 10(8)(0.0004) + 10(4)(0.000008) + 5(2)(0.00000016) + 1(0.0000000032)$$

Ignore last two terms for four decimal places

$\therefore \quad (2.02)^5 = 32 + 1.60 + 0.0320 + 0.0003$

$\therefore \quad (2.02)^5 = 33.6323.$

Ex. 7 : Without expanding, find the value of

$(2x-1)^5 + 5(2x-1)^4(1-x) + 10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5$

Solution : We notice that 1, 5, 10, 10, 5, 1 are the values of ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$ and ${}^5C_5$ respectively.

Hence, given expression can be written as

$$\begin{aligned}
&{}^5C_0(2x-1)^5 + {}^5C_1(2x-1)^4(1-x) \\
&+ {}^5C_2(2x-1)^3(1-x)^2 + {}^5C_3(2x-1)^2(1-x)^3 \\
&+ {}^5C_4(2x-1)(1-x)^4 + {}^5C_5(1-x)^5 \\
&= [(2x-1) + (1-x)]^5 \\
&= (2x - 1 + 1 - x)^5 \\
&= x^5
\end{aligned}$$

$\therefore \quad (2x-1)^5 + 5(2x-1)^4(1-x) + 10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5 = x^5$

EXERCISE 4.2

(1) Expand (i) $\left(\sqrt{3}+\sqrt{2}\right)^4$ (ii) $\left(\sqrt{5}-\sqrt{2}\right)^5$

(2) Expand (i) $(2x^2 + 3)^4$ (ii) $\left(2x-\dfrac{1}{x}\right)^6$

(3) Find the value of

(i) $\left(\sqrt{3}+1\right)^4 - \left(\sqrt{3}-1\right)^4$

(ii) $\left(2+\sqrt{5}\right)^5 + \left(2-\sqrt{5}\right)^5$

(4) Prove that

(i) $\left(\sqrt{3}+\sqrt{2}\right)^6 + \left(\sqrt{3}-\sqrt{2}\right)^6 = 970$

(ii) $\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5 = 352$

(5) Using binomial theorem, find the value of

(i) $(102)^4$ (ii) $(1.1)^5$

(6) Using binomial theorem, find the value of

(i) $(9.9)^3$ (ii) $(0.9)^4$

(7) Without expanding, find the value of

(i) $(x+1)^4 - 4(x+1)^3(x-1) + 6(x+1)^2(x-1)^2 - 4(x+1)(x-1)^3 + (x-1)^4$

(ii) $(2x-1)^4 + 4(2x-1)^3(3-2x) + 6(2x-1)^2(3-2x)^2 + 4(2x-1)^1(3-2x)^3 + (3-2x)^4$

(8) Find the value of $(1.02)^6$, correct upto four places of decimals.

(9) Find the value of $(1.01)^5$, correct upto three places of decimals.

(10) Find the value of $(0.9)^6$, correct upto four places of decimals.

4.3 General term in expansion of $(a+b)^n$

In the expansion of $(a+b)^n$, we denote the terms by $t_1, t_2, t_3, \ldots, t_r, t_{r+1}, \ldots t_n \ldots$ then

$t_1 = {}^nC_0\, a^n b^0$

$t_2 = {}^nC_1\, a^{n-1} b^1$

$t_3 = {}^nC_2\, a^{n-2} b^2$

. . .

. . .

. . .

$t_r = {}^nC_{r-1}\, a^{n-r+1} b^{r-1}$

$t_{r+1} = {}^nC_r\, a^{n-r} b^r$

t_{r+1} is called a general term for all $r \in N$ and $0 \le r \le n$. Using this formula, we can find any term of the expansion.

4.3 Middle term (s) in the expansion of $(a+b)^n$:

(i) In $(a+b)^n$ if n is even then the number of terms in the expansion is odd. So the only middle term is $\left(\dfrac{n+2}{2}\right)^{th}$ term.

(ii) In $(a+b)^n$ if n is odd then the number of terms in the expansion is even. So the two middle terms are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

SOLVED EXAMPLES

Ex. 1 : Find the fifth term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$

Solution : Here $a = 2x^2$, $b = \frac{3}{2x}$, $n = 8$

For t_5, $r = 4$

Since, $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (2x^2)^{8-4}\left(\frac{3}{2x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(2x^2)^4\left(\frac{3}{2x}\right)^4$$

$$= 70(16x^8)\left(\frac{81}{16x^4}\right)$$

$$= 5670x^4$$

$\therefore$ The fifth term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$ is $5670x^4$

Ex. 2 : Find the middle term(s) in the expansion of $\left(x^2+\frac{2}{x}\right)^8$

Solution : Here $a = x^2$, $b = \frac{2}{x}$, $n = 8$

Now n is even, hence $\left(\frac{n+2}{2}\right) = \left(\frac{8+2}{2}\right) = 5$

$\therefore$ Fifth term is the only middle term.

For t_5, $r = 4$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (x^2)^{8-4}\left(\frac{2}{x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(x^2)^4\left(\frac{2}{x}\right)^4$$

$$= 70\,(x^8)\,\frac{16}{x^4}$$

$$= 1120x^4$$

Ex. 3 : Find the middle terms in the expansion of $\left(2x-\frac{1}{4x}\right)^9$

Solution : Here $a = 2x$, $b = -\frac{1}{4x}$, $n = 9$

Now n is odd $\left(\frac{n+1}{2}\right) = 5$ $\left(\frac{n+3}{2}\right) = 6$

$\therefore$ Fifth and sixth terms are the middle terms.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

For t_5, $r = 4$

$$\therefore\quad t_5 = {}^9C_4\,(2x)^{9-4}\left(-\frac{1}{4x}\right)^4$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(32x)^5\left(-\frac{1}{4x}\right)^4$$

$$= 126\,(2x^5)\left(\frac{1}{256x^4}\right)$$

$$= \left(\frac{63x}{4}\right)$$

For t_6, $r = 5$

$$\therefore\quad t_6 = {}^9C_5\,(2x)^{9-5}\left(\frac{-1}{4x}\right)^5$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(2x)^4\left(\frac{-1}{4x}\right)^5$$

$$= 126\,(16x^4)\left(\frac{-1}{1024x^5}\right)$$

$$= -\frac{63}{32x}$$

$\therefore$ The middle terms are $\left(\frac{63x}{4}\right)$ and $-\frac{63}{32x}$

Ex. 4 : Find the coefficient of x^7 in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$

Solution : $a = x^2$, $b = \frac{1}{x}$, $n = 11$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_{r+1} = {}^{11}C_r (x^2)^{11-r}\left(\frac{1}{x}\right)^r$$

$$= {}^{11}C_r x^{22-2r} x^{-r}$$

$$= {}^{11}C_r x^{22-3r}$$

To get coefficient of x^7, we must have

$$x^{22-3r} = x^7$$

$\therefore \quad 22 - 3r = 7$

$\therefore \quad r = 5$

$\therefore \quad {}^{11}C_5 = \frac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2\cdot1} = 462$

$\therefore$ Coefficient of x^7 is 462.

Ex. 5 : Find the coefficient of x^{-2} in the expansion of $\left(2x-\frac{1}{\sqrt{3}x^2}\right)^{10}$

Solution : Here $a = 2x$, $b = \frac{-1}{\sqrt{3}x^2}$, $n = 10$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$= {}^{10}C_r (2x^2)^{10-r}\left(\frac{-1}{\sqrt{3}x^2}\right)^r$$

$$= {}^{10}C_r (2)^{10-r} x^{10-r}\left(\frac{-1}{\sqrt{3}}\right)^r x^{-2r}$$

$$= {}^{10}C_r (2)^{10-r}\left(\frac{-1}{\sqrt{3}}\right)^r x^{10-3r}$$

To get coefficient of x^{-2}, we must have

$$x^{10-3r} = x^{-2}$$

$\therefore\ 10 - 3r = -2$

$\therefore\ -3r = -12$

$\therefore\ r = 4$

$\therefore$ coefficient of x^{-2}

$$= {}^{10}C_4 (2)^{10-4}\left(\frac{-1}{\sqrt{3}}\right)^4 = \frac{10.9.8.7}{4.3.2.1}(2)^6\left(\frac{-1}{\sqrt{3}}\right)^4$$

$$= 210\ (64)\left(\frac{1}{9}\right)$$

$$= \frac{4480}{3}$$

$\therefore$ Coefficient of x^{-2} is $\frac{4480}{3}$

Ex. 6 : Find the term independent of x, in the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Solution : Here $a = \sqrt{x}$, $b = \frac{-2}{x^2}$, $n = 10$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^{10}C_r \left(\sqrt{x}\right)^{10-r}\left(\frac{-2}{x^2}\right)^r$$

$$= {}^{10}C_r\, x^{\left(\frac{10-r}{2}\right)}(-2)^r x^{-2r}$$

$$= {}^{10}C_r (-2)^r\, x^{\frac{10-5r}{2}}$$

To get the term independent of x, we must have

$$x^{\frac{10-5r}{2}} = x^0$$

$\therefore\ \frac{10-5r}{2} = 0$

$\therefore\ 10 - 5r = 0$

$\therefore\ r = 2$

$\therefore$ the term independent of x is

$${}^{10}C_2 (-2)^2 = \frac{10.9}{2.1}(-2)^2 = 45(4) = 180$$

$\therefore$ the term independent of x is 180.

EXERCISE 4.3

(1) In the following expansions, find the indicated term.

(i) $\left(2x^2+\dfrac{3}{2x}\right)^8$, 3^{rd} term

(ii) $\left(x^2-\dfrac{4}{x^3}\right)^{11}$, 5^{th} term

(iii) $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$, 7^{th} term

(iv) In $\left(\dfrac{1}{3}+a^2\right)^{12}$, 9^{th} term

(v) In $\left(3a+\dfrac{4}{a}\right)^{13}$, 10^{th} term

(2) In the following expansions, find the indicated coefficients.

(i) x^3 in $\left(x^2+\dfrac{3\sqrt{2}}{x}\right)^9$ (ii) x^8 in $\left(2x^5-\dfrac{5}{x^3}\right)^8$

(iii) x^9 in $\left(\dfrac{1}{x}+x^2\right)^{18}$ (iv) x^{-3} in $\left(x-\dfrac{1}{2x}\right)^5$

(v) x^{-20} in $\left(x^3-\dfrac{1}{2x^2}\right)^{15}$

(3) Find the constant term (term independent of x) in the expansion of

(i) $\left(2x+\dfrac{1}{3x^2}\right)^9$ (ii) $\left(x-\dfrac{2}{x^2}\right)^{15}$

(iii) $\left(\sqrt{x}-\dfrac{3}{x^2}\right)^{10}$ (iv) $\left(x^2-\dfrac{1}{x}\right)^9$

(v) $\left(2x^2-\dfrac{5}{x}\right)^9$

(4) Find the middle terms in the expansion of

(i) $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^{12}$ (ii) $\left(x^2+\dfrac{1}{x}\right)^7$

(iii) $\left(x^2-\dfrac{2}{x}\right)^8$ (iv) $\left(\dfrac{x}{a}-\dfrac{a}{x}\right)^{10}$

(v) $\left(x^4-\dfrac{1}{x^3}\right)^{11}$

(5) In the expansion of $(k+x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.

(6) Find the term containing x^6 in the expansion of $(2-x)(3x+1)^9$

(7) The coefficient of x^2 in the expansion of $(1+2x)^m$ is 112. Find m.

4.4 Binomial Theorem for Negative Index or Fraction.

If n is negative then n! is not defined.

We state binomial theorem in another form.

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots..$$

$$\frac{n(n-1)\ldots.(n-r+1)}{r!}a^{n-r}b^r + \ldots..+ b^n$$

Here $t_{r+1} = \dfrac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}a^{n-r}b^r$

Consider the binomial theorem

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots. + x^n$

$= 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots.+ x^n$

This is a finite sum.

The theorem has an extension to the case where 'n' is negative or faction. We state it here without proof.

For $|x|<1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \frac{n(n-1)....(n-r+1)}{r!}x^r + \dots.$$

Here n is not an integer and the terms on the RHS are infinite, the series does not terminate.

Here there are infinite number of terms in the expansion. The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)x^r}{r!}, \ r \geq 0$$

Remarks : (1) If $|x|<1$ and n is any real number, not a positive integer, then

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

The general term is given by

$$t_{r+1} = \frac{(-1)^r n(n-1)(n-2)...(n-r+1)}{r!}x^r$$

(ii) If n is any real number and $|b|<|a|$, then

$$(a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n = a^n\left(1+\frac{b}{a}\right)^n$$

Note : While expanding $(a+b)^n$ where n is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index, fraction. $|x|<1$

(1) $$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

(2) $$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

(3) $$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

(4) $$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(5) $$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

(6) $$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \dots$$

SOLVED EXAMPLES

Ex.1 : State first four terms in the expansion of $\frac{1}{(a-b)^4}$ where $|b|<|a|$

Solution : We have $\frac{1}{(a-b)^4} = (a-b)^{-4}$

$$= \left[a\left(1-\frac{b}{a}\right)\right]^{-4}$$

$$= a^{-4}\left[1+(-4)\left(-\frac{b}{a}\right)+\frac{(-4)(-5)}{2!}\left(-\frac{b}{a}\right)^2+\frac{(-4)(-5)(-6)}{3!}\left(-\frac{b}{a}\right)^3+\dots\right]$$

$$= a^{-4}\left[1+4\frac{b}{a}+\frac{20}{2}\frac{b^2}{a^2}+\frac{120}{6}\frac{b^3}{a^3}+\dots\right]$$

$$= a^{-4}\left[1+\frac{4b}{a}+\frac{10b^2}{a^2}+\frac{20b^3}{a^3}+\dots\right]$$

Ex. 2 : State first four terms in the expansion of $\frac{1}{(a+b)}$, $|b|<|a|$

Solution : $= \frac{1}{(a+b)} = (a+b)^{-1}$

$$= a^{-1}\left(1+\frac{b}{a}\right)^{-1}$$

$$a^{-1}\left[1+(-1)\left(\frac{b}{a}\right)+\frac{(-1)(-2)}{2!}\left(\frac{b}{a}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{b}{a}\right)^3+\dots\right]$$

$$= a^{-1}\left[1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\dots\right]$$

Ex. 3 : State first four terms in the expansion of

$(2-3x)^{-1/2}$ if $|x| < \frac{2}{3}$

Solution : $|x| < \frac{2}{3}$

$$\therefore \left|\frac{3x}{2}\right| < 1$$

We have $(2-3x)^{-1/2}$

$$= 2^{-1/2}\left(1-\frac{3x}{2}\right)^{-\frac{1}{2}}$$

$$= 2^{-\frac{1}{2}}\left[1+\left(-\frac{1}{2}\right)\left(\frac{-3x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{-3x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{-3x}{2}\right)^3+\ldots\right]$$

$$= 2^{-\frac{1}{2}}\left[1+\frac{3x}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{9x^2}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(\frac{-27x^3}{8}\right)+\ldots.\right]$$

$$= 2^{-\frac{1}{2}}\left[1+\frac{3x}{4}+\frac{27x^2}{32}+\frac{135x^3}{128}+\ldots\right]$$

Ex. 4 : Find the value of $\sqrt{30}$ upto 4 decimal places.

Solution:

$$\sqrt{30} = (25+5)^{1/2}$$

$$= (25)^{1/2}\left(1+\frac{5}{25}\right)^{\frac{1}{2}}$$

$$= 5\left(1+\frac{1}{5}\right)^{\frac{1}{2}}$$

$$= 5\left[1+\frac{1}{10}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{25}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{6}\left(\frac{1}{125}\right)+\ldots\right]$$

$$= 5\left[1+\frac{1}{10}-\frac{1}{200}+\frac{1}{2000}-\ldots\right]$$

$= 5\,[\,1 + 0.1 - 0.005 + 0.0005]$

(upto 4 decimal places)

$= 5[1.0955]$

$= 5.4775$

EXERCISE 4.4

(1) State, by writing first four terms, the expansion of the following, where $|x|<1$

(i) $(1+x)^{-4}$ (ii) $(1-x)^{1/3}$

(iii) $(1-x^2)^{-3}$ (iv) $(1+x)^{-1/5}$

(v) $(1+x^2)^{-1}$

(2) State, by writing first four terms, the expansion of the following, where $|b|<|a|$

(i) $(a-b)^{-3}$ (ii) $(a+b)^{-4}$

(iii) $(a+b)^{1/4}$ (iv) $(a-b)^{-1/4}$

(v) $(a+b)^{-1/3}$

(3) Simplify first three terms in the expansion of the following

(i) $(1+2x)^{-4}$ (ii) $(1+3x)^{-1/2}$

(iii) $(2-3x)^{1/3}$ (iv) $(5+4x)^{-1/2}$

(v) $(5-3x)^{-1/3}$

(4) Use binomial theorem to evaluate the following upto four places of decimals.

(i) $\sqrt{99}$ (ii) $\sqrt[3]{126}$

(iii) $\sqrt[4]{16.08}$ (iv) $(1.02)^{-5}$

(v) $(0.98)^{-3}$

4.5 Binomial Coefficients :

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_n$ in the expansion of $(a+b)^n$ are called the binomial coefficients and denoted by $C_0, C_1, C_2, \ldots C_n$ respectively

Now $(1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + \ldots + {}^nC_nx^n \ldots$(i)

Put $x = 1$ we get

$(1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

$\therefore 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n$

$\therefore {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$

$\therefore C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n$

$\therefore$ The sum of all binomial coefficients is 2^n

(ii) Put $x = -1$, in equation (i) we get

$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\, {}^nC_n$

$\therefore 0 \quad = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots + (-1)^n\, {}^nC_n$

$\therefore {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots + (-1)^n\, {}^nC_n = 0$

$\therefore {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots$

$\therefore C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots$

$C_0, C_2, C_4, \ldots\ldots$ are called as even coefficients.

$C_1, C_3, C_5, \ldots\ldots$ are called as odd coefficients.

Let $C_0 + C_2 + C_4 + \ldots. = C_1 + C_3 + C_5 + \ldots. = k$

Now $C_0 + C_1 + C_2 \div C_3 + \ldots. + C_n = 2^n$

$\therefore (C_0 + C_2 + C_4 + \ldots.) + (C_1 + C_3 + C_5 + \ldots.) = 2^n$

$\therefore k + k = 2^n$

$\therefore 2k = 2^n$

$\therefore k = \dfrac{2^n}{2}, \quad k = 2^{n-1}$

$\therefore C_0 + C_2 + C_4 + \ldots. = C_1 + C_3 + C_5 + \ldots. = 2^{n-1}$

$\therefore$ The sum of even coefficients = The sum of odd coefficients = 2^{n-1}

SOLVED EXAMPLES

Ex.1 : Show that $C_0 + C_1 + C_2 + \ldots. + C_{10} = 1024$

Solution : We have $C_0 + C_1 + C_2 + \ldots. \; C_n = 2^n$

Put $n = 10$, we get

$C_0 + C_1 + C_2 + \ldots. C_{10} = 2^{10}$

$\therefore C_0 + C_1 + C_2 + \ldots. + C_{10} = 1024$

Ex. 2 : Show that

$C_0 + C_2 + C_4 + \ldots. + C_{12} = C_1 + C_3 + C_5 + \ldots. + C_{11} = 2048$

Solution : We have

$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + \ldots. + C_{n-1} + C_n = 2^n$

Put $n = 12$, we get

$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + \ldots + C_{11} + C_{12} = 2^{12} = 4096$ (i)

We know that

The sum of even coefficients = The sum of odd coefficients.

$\therefore \quad C_0 + C_2 + C_4 + \ldots. + C_{12} = C_1 + C_3 + C_5 + \ldots. + C_{11} = k$ ...(ii)

Now from (i)

$(C_0 + C_2 + C_4 + \ldots. + C_{12}) + (C_1 + C_3 + C_5 + \ldots. + C_{11}) = 4096$

$\therefore k + k = 4096$

$\therefore 2k = 4096$

$\therefore k = 2048$

$\therefore C_0 + C_2 + C_4 + \ldots. + C_{12} = C_1 + C_3 + C_5 + \ldots + C_{11} = 2048$

Ex. 3 : Prove that

$C_1 + 2C_2 + 3C_3 + 4C_4 + \ldots + nC_n = n.2^{n-1}$

Solution :

L.H.S. = $C_1 + 2C_2 + 3C_3 + 4C_4 + \ldots + nC_n$

$$= n + 2\frac{n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + \ldots. + n.1$$

$$= n\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+....+1\right]$$

$$= n\,[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + ...+ {}^{n-1}C_{n-1}]$$

$$= n\,[C_0 + C_1 + C_2 + ...+ C_{n-1}]$$

$$= n.2^{n-1}$$

$$= \text{R.H.S.}$$

$$\therefore\ C_1 + 2C_2 + 3C_3 + ... + nC_n = n.2^{n-1}$$

Ex. 4 : Prove that

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+....+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$

Solution : L.H.S.

$$=\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\frac{C_3}{4}....+\frac{C_n}{n+1}$$

$$=1+\frac{n}{2}+\frac{1}{3}\frac{n(n-1)}{2!}+\frac{1}{4}\frac{n(n-1)(n-2)}{3!}+...+\frac{1}{n+1}$$

$$=\frac{1}{(n+1)}\left[(n+1)+\frac{n(n+1)}{2!}+\frac{(n+1)n(n-1)}{3!}+...+1\right]$$

$$=\frac{1}{(n+1)}\,[{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + ...+ {}^{n+1}C_{n+1}]$$

$$=\frac{1}{(n+1)}\,[1+{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 +...+ {}^{n+1}C_{n+1}-1]$$

$$=\frac{1}{(n+1)}\,[{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + ...+ {}^{n+1}C_{n+1}-1]$$

$$=\frac{1}{(n+1)}\,[C_0 + C_1 + C_2 + ...+ C_{n+1}-1]$$

$$=\frac{1}{(n+1)}\,(2^{n+1}-1)$$

$$=\frac{(2^{n+1}-1)}{n+1} = \text{R.H.S.}$$

$$\therefore\ C_0+\frac{C_1}{2}+\frac{C_2}{3}+....+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$

EXERCISE 4.5

Show That

(1) $C_0 + C_1 + C_2 + ...\ C_8 = 256$

(2) $C_0 + C_1 + C_2 + ...\ C_9 = 512$

(3) $C_1 + C_2 + C_3 + ...\ C_7 = 127$

(4) $C_1 + C_2 + C_3 + ...\ C_6 = 63$

(5) $C_0+C_2+C_4+C_6+C_8=C_1+C_3+C_5+C_7= 128$

(6) $C_1 + C_2 + C_3 + ...\ C_n = 2^n - 1$

(7) $C_0+2C_1+3C_2+4C_3+...+(n+1)C_n=(n+2)2^{n-1}$

Let's Remember

- Step (I) Foundation : To prove P(1) is true

  Step (II) Assumption : To assume P(k) is true.

  Step (III) Succession : To prove that P(k+1) is true.

  Step (IV) Induction :P(n) is true for all $n \in N$.

- If $a, b \in R$ and $n \in N$, then

  $(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_n\, a^0 b^n$

  $(a-b)^n = {}^nC_0\, a^n b^0 - {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 .... + (-1)^n\, {}^nC_n\, a^0 b^n$

- General term in the expansion of $(a+b)^n$ is $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

- If $|x| < 1$ and n is any real number then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

$$(1+x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+...$$

- $C_0 + C_1 + C_2 + ... + C_n = 2^n$

- $C_0 + C_2 + C_4 + C_6 + ... = C_1 + C_3 + C_5 + C_7 + ... = 2^{n-1}$

MISCELLANEOUS EXERCISE - 4

(I) Select the correct anwsers from the given alternatives.

(1) The total number of terms in the expression of $(x+y)^{100} + (x-y)^{100}$ after simplification is :

A) 50 B) 51 C) 100 D) 202

(2) The middle term in the expansion of $(1+x)^{2n}$ will be :

A) $(n-1)^{th}$ B) n^{th} C) $(n+1)^{th}$ D) $(n+2)^{th}$

(3) In the expansion of $(x^2-2x)^{10}$, the coefficient of x^{16} is

A) −1680 B) 1680 C) 3360 D) 6720

(4) The term not containing x in expansion of $(1-x)^2 \left(x+\dfrac{1}{x}\right)^{10}$ is

A) ${}^{11}C_5$ B) ${}^{10}C_5$ C) ${}^{10}C_4$ D) ${}^{10}C_7$

(5) The number of terms in expansion of $(4y+x)^8-(4y-x)^8$

A) 4 B) 5 C) 8 D) 9

(6) The value ${}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots\ldots + {}^{14}C_{11}$ is

A) $2^{14}-1$ B) $2^{14}-14$ C) 2^{12} D) $2^{13}-14$

(7) The value ${}^{11}C_2 + {}^{11}C_4 + {}^{11}C_6 + {}^{11}C_8$ is equal to

A) $2^{10}-1$ B) $2^{10}-11$ C) $2^{10}+12$ (D) $2^{10}-12$

(8) In the expansion of $(3x+2)^4$, the coefficient of middle term is

A) 36 B) 54 C) 81 D) 216

(9) The coefficient of the 8^{th} term in the expansion of $(1+x)^{10}$ is :

A) 7 B) 120 C) ${}^{10}C_8$ D) 210

(10) If the coefficient of x^2 and x^3 in the expansion of $(3+ax)^9$ are the same, then the value of a is

A) $-\dfrac{7}{9}$ B) $-\dfrac{9}{7}$ C) $\dfrac{7}{9}$ D) $\dfrac{9}{7}$

(II) Answer the following.

(1) Prove, by method of induction, for all $n \in N$

(i) $8 + 17 + 26 + \ldots. + (9n-1) = \dfrac{n}{2}(9n+7)$

(ii) $1^2 + 4^2 + 7^2 + \ldots + (3n-2)^2 = \dfrac{n}{2}(6n^2-3n-1)$

(iii) $2 + 3.2 + 4.2^2 + \ldots + (n+1)2^{n-1} = n.\, 2^n$

(iv) $\dfrac{1}{3.4.5}+\dfrac{2}{4.5.6}+\dfrac{3}{5.6.7}+\ldots+\dfrac{n}{(n+2)(n+3)(n+4)} = \dfrac{n(n+1)}{6(n+3)(n+4)}$

(2) Given that $t_{n+1} = 5t_n - 8$, $t_1 = 3$, prove by method of induction that $t_n = 5^{n-1} + 2$

(3) Prove by method of induction

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & -2n+1 \end{pmatrix}, \ \forall\, n \in N$$

(4) Expand $(3x^2 + 2y)^5$

(5) Expand $\left(\dfrac{2x}{3}-\dfrac{3}{2x}\right)^4$

(6) Find third term in the expansion of $\left(9x^2-\dfrac{y^3}{6}\right)^4$

(7) Find tenth term in the expansion of $\left(2x^2+\dfrac{1}{x}\right)^{12}$

(8) Find the middle term (s) in the expansion of

(i) $\left(\dfrac{2a}{3}-\dfrac{3}{2a}\right)^6$ (ii) $\left(x-\dfrac{1}{2y}\right)^{10}$

(iii) $(x^2+2y^2)^7$ (iv) $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^9$

(9) Find the coefficients of

(i) x^6 in the expansion of $\left(3x^2-\frac{1}{3x}\right)^9$

(ii) x^{60} in the expansion of $\left(\frac{1}{x^2}+x^4\right)^{18}$

(10) Find the constant term in the expansion of

(i) $\left(\frac{4x^2}{3}+\frac{3}{2x}\right)^9$ (ii) $\left(2x^2-\frac{1}{x}\right)^{12}$

(11) Prove by method of induction

(i) $\log_a x^n = n \log_a x$, $x > 0$. $n \in N$

(ii) $15^{2n-1}+1$ is divisible by 16, for all $n \in N$.

(iii) $5^{2n} - 2^{2n}$ is divisible by 3, for all $n \in N$.

(12) If the coefficient of x^{16} in the expansion of $(x^2 + ax)^{10}$ is 3360, find a.

(13) If the middle term in the expansion of $\left(x+\frac{b}{x}\right)^6$ is 160, find b.

(14) If the coefficient of x^2 and x^3 in the expansion of $(3 + kx)^9$ are equal, find k.

(15) If the constant term in the expansion of $\left(x^3+\frac{k}{x^8}\right)^{11}$ is 1320, find k.

(16) Show that there is no term containing x^6 in the expansion of $\left(x^2-\frac{3}{x}\right)^{11}$.

(17) Show that there is no constant term in the expansion of $\left(2x-\frac{x^2}{4}\right)^9$

(18) State, first four terms in the expansion of $\left(1-\frac{2x}{3}\right)^{-1/2}$

(19) State, first four terms in the expansion of $(1-x)^{-1/4}$

(20) State, first three terms in the expansion of $(5 + 4x)^{-1/2}$

(21) Using binomial theorem, find the value of $\sqrt[3]{995}$ upto four places of decimals.

(22) Find approximate value of $\frac{1}{4.08}$ upto four places of decimals.

(23) Find the term independent of x in the in expansion of $(1 -x^2)\left(x+\frac{2}{x}\right)^6$

(24) $(a + bx)(1 - x)^6 = 3 -20x + cx^2 + \ldots$.. then find a, b, c.

(25) The 3rd term of $(1+x)^n$ is $36x^2$. Find 5th term.

(26) Suppose $(1+kx)^n = 1-12x + 60x^2 - \ldots$. find k and n.