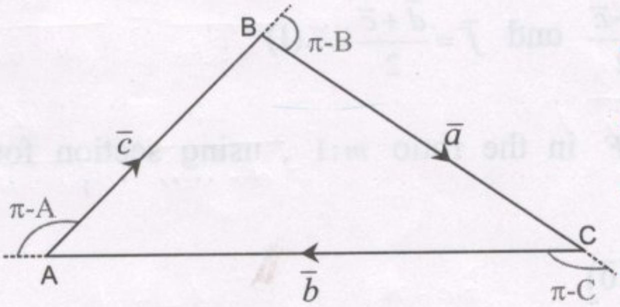
6. Vectors and Three Dimensional Geometry

Ex. (1) Using vectors prove the Projection rule.



Solution : We have to prove Projection rule,

$a = b\cos C + c\cos B$

Let $\overline{BC} = \bar{a}, \overline{CA} = \bar{b}, \overline{AB} = \bar{c}$

By triangle law of addition of vectors, we have

$\overline{AB} + \overline{BC} + \overline{CA} = \bar{0}$

$\bar{c} + \bar{a} + \bar{b} = \bar{0}$

Taking a dot product with $\bar{a}$ on both sides, we get

$\bar{a}\cdot\left(\bar{a} + \bar{b} + \bar{c}\right) = \bar{a}\cdot\bar{0}$

$\bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c} = \bar{a}\cdot\bar{0}$

If $\bar{p}$ and $\bar{q}$ are any two vectors, then $\bar{p}\cdot\bar{q} = pq\cos\theta$, where θ is angle between $\bar{p}$ and $\bar{q}$.

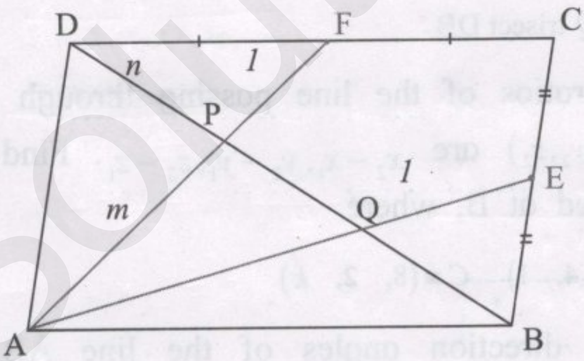
$(a)(a)\cos 0 + (a)(b)\cos(\pi - C) + (a)(c)\cos(\pi - B) = 0$

Divide throughout by a, we get

$a\cos 0 + b(-\cos C) + c(-\cos B) = 0$

$a = b\cos C + c\cos B$.

Ex. (2) ABCD is a parallelogram. E and F are mid points of BC and CD respectively. AE and AF meet diagonal BD in Q and P respectively. Show that P and Q trisect BD.



Solution : Without loss of generality let $A(\bar{0})$ be origin. $B(\bar{b}), C(\bar{c}), D(\bar{d})$ are the other three vertices of parallelogram.

$E(\bar{e})$ and $F(\bar{f})$ are the midpoints of BC and DC.

By midpoint formula, $\bar{e} = \dfrac{\bar{b}+\bar{c}}{2}$ and $\bar{f} = \dfrac{\bar{d}+\bar{c}}{2}$ ....(I)

Also $\bar{c} = \bar{b} + \bar{d}$ ...(II)

Now let point P divides AF in the ratio $m:1$, using section formula, we get

$$\bar{p} = \frac{m(\bar{f}) + 1(\bar{0})}{m+1} = \frac{m\left(\dfrac{\bar{d}+\bar{c}}{2}\right) + 1(\bar{0})}{m+1}$$

From (II), we get

$$\bar{p} = \frac{m}{2(m+1)}(\bar{d} + \bar{b} + \bar{d}) = \frac{m}{2(m+1)}(\bar{b} + 2\bar{d})$$

$$= \frac{m}{2(m+1)}\bar{b} + \frac{m}{(m+1)}\bar{d} \text{...(III)}$$

Also, let point P divides DB in the ratio $n:1$, using section formula, we get

$$\bar{p} = \frac{n(\bar{b}) + 1(\bar{d})}{n+1} = \frac{n}{n+1}\bar{b} + \frac{1}{n+1}\bar{d} \text{...(IV)}$$

From (III) and (IV), we get

$$\frac{m}{2(m+1)} = \frac{n}{n+1} \text{...(V)}$$

$$\frac{m}{(m+1)} = \frac{1}{n+1} \text{...(VI)}$$

Divide (V) by (VI), we get $\dfrac{1}{2} = n$,

$\therefore\ DP : PB = n:1 = 1:2$ ...(VII)

By symmetry, we get $BQ:QD = 1:2$ ...(VIII)

From (VII) and (VIII), P and Q trisect DB.

Ex. (3) Show that direction ratios of the line passing through points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Find k' if Δ ABC is right angled at B, where

$A \equiv (5, 6, 4)$, $B \equiv (4, 4, 1)$, $C \equiv (8, 2, k')$.

Solution: Let α, β, γ be the direction angles of the line AB, and

$\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the line AB .

Also $\bar{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ *and* $\bar{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, so we have

$\overline{AB} = \bar{b} - \bar{a} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Now , $\overline{AB}\cdot\hat{i} = \left[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right]\cdot\hat{i} = x_2 - x_1$ ...(I)

And also $\overline{AB}\cdot\hat{i} = |\overline{AB}||\hat{i}|\cos\alpha = AB\cos\alpha$ (II)

From (I) and (II), we have

$x_2 - x_1 = AB\cos\alpha$, similarly $y_2 - y_1 = AB\cos\beta$ $z_2 - z_1 = AB\cos\gamma$

As $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are proportional to $\cos\alpha, \cos\beta, \cos\gamma$.

Therefore, direction ratios of line AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

As $A \equiv (5,\ 6,\ 4), B \equiv (4,\ 4,\ 1),\ C \equiv (8,\ 2,\ k')$ then $\bar{a} \equiv 5\hat{i} + 6\hat{j} + 4\hat{k}$,

$\bar{b} = 4\hat{i} + 4\hat{j} + \ldots\hat{k}$ and $\bar{c} = 8\hat{i} + 2\hat{j} + k'\hat{k}$

Also $\overline{AB} = \bar{b} - \bar{a} = (4\hat{i} + 4\hat{j} + \hat{k}) - (5\hat{i} + 6\hat{j} + 4\hat{k}) = -\hat{i} - 2\hat{j} - 3\hat{k}$ and

$\overline{BC} = \bar{c} - \bar{b} = (8\hat{i} + 2\hat{j} + k'\hat{k}) - (4\hat{i} + 4\hat{j} + \hat{k}) = 4\hat{i} - 2\hat{j} + (k' - 1)\hat{k}$

Δ ABC is right angled at B, we have $\overline{AB}\cdot\overline{BC} = 0$,

$(-\hat{i} - 2\hat{j} - 3\hat{k}).[4\hat{i} - 2\hat{j} + (k' - 1)\hat{k}] = 0$

$-4 + 4 - 3(k' - 1) = 0$

$k' = 1$

Ex. (4) Prove that $(\bar{a} + 2\bar{b} - \bar{c}).\left[(\bar{a} - \bar{b})\times(\bar{a} - \bar{b} - \bar{c})\right] = 3\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$.

Solution: Consider

$(\bar{a} + 2\bar{b} - \bar{c})\cdot\left[(\bar{a} - \bar{b})\times(\bar{a} - \bar{b} - \bar{c})\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[(\bar{a} - \bar{b})\times\bar{a} - (\bar{a} - \bar{b})\times\bar{b} - (\bar{a} - \bar{b})\times\bar{c}\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[\bar{a}\times\bar{a} - \bar{b}\times\bar{a} - \bar{a}\times\bar{b} + \bar{b}\times\bar{b} - \bar{a}\times\bar{c} + \bar{b}\times\bar{c}\right]$

As $\bar{a}\times\bar{a} = \bar{0}$, $\bar{b}\times\bar{b} = \bar{0}$ and $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[\bar{0} + \bar{a}\times\bar{b} - \bar{a}\times\bar{b} + \bar{0} - \bar{a}\times\bar{c} + \bar{b}\times\bar{c}\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[-\bar{a}\times\bar{c} + \bar{b}\times\bar{c}\right]$

$= \bar{a}\cdot(-\bar{a}\times\bar{c} + \bar{b}\times\bar{c}) + 2\bar{b}\cdot(-\bar{a}\times\bar{c} + \bar{b}\times\bar{c}) - \bar{c}\cdot(-\bar{a}\times\bar{c} + \bar{b}\times\bar{c})$

As $\bar{a}\cdot(\bar{b}\times\bar{c}) = [\bar{a}\ \ \bar{b}\ \ \bar{c}]$

$= -[\bar{a}\ \ \bar{a}\ \ \bar{c}] + [\bar{a}\ \bar{b}\ \bar{c}] - 2[\bar{b}\ \ \bar{a}\ \ \bar{c}] + 2[\bar{b}\ \bar{b}\ \bar{c}] + [\bar{a}\ \ \bar{c}\ \ \bar{a}] - [\bar{c}\ \bar{b}\ \bar{c}]$

As $[\bar{a}\ \ \bar{a}\ \ \bar{c}] = 0$

$=-0+\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]-2\left[\bar{b}\quad\bar{a}\quad\bar{c}\right]+2(0)+0-0$

$=\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]-2\left[\bar{b}\quad\bar{a}\quad\bar{c}\right]$

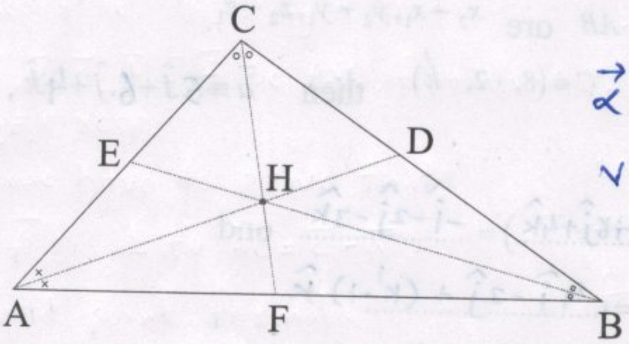
As $[\bar{a}\quad\bar{a}\quad\bar{c}]=0$

$[\bar{b}\ \bar{a}\ \bar{c}]=-\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]$

$=\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]+2\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]$

$=3\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]$

Ex. (5) Using vectors prove that bisectors of angles of a triangle are concurrent.



Let ABC be a triangle and $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be the p.v.s of the vertices A, B and C resp.

Solution :
Let AD, BE and CF be the internal bisectors of $\angle A$, $\angle B$ and $\angle C$ resp.

We know that D divides BC in the ratio of AB:AC i.e. c:b

p.v of D is $\dfrac{c\vec{\gamma}+b\vec{\beta}}{c+a}$, p.v. of E is $\dfrac{c\vec{\gamma}+a\vec{\alpha}}{c+a}$

p.v of F is $\dfrac{a\vec{\alpha}+b\vec{\beta}}{a+b}$

The point dividing AD in ratio b+c : a is $\dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

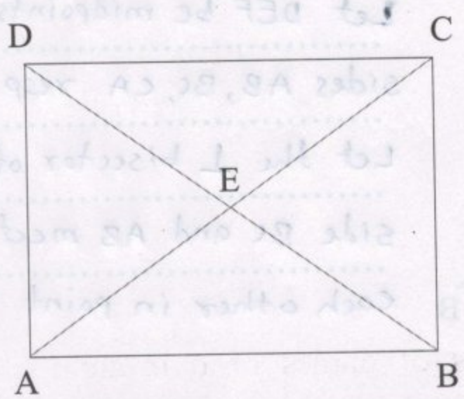
— " — BE — " — a+c : b $\dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

— " — CF — " — a+b : c is $\dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

Since the point $\dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$ lies on all the three internal bisectors AD, BE and CF.

Hence the internal bisectors are concurrent.

Ex. (6) Using vectors prove that a quadrilateral is a rectangle if and only if its diagonals are congruent and bisect each other.



Let ABCD be a rectangle

Let $\bar{a}\ \bar{b}\ \bar{c}\ \bar{d}\ \bar{e}$ be P.V. of points A,B,C,D,E

Since ABCD is a rectangle

$\bar{AB} = \bar{DC}$ (opp. sides of rectangle)

$\bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{b} + \bar{d} = \bar{c} + \bar{a}$

Solution :

$\therefore \dfrac{\bar{b}+\bar{d}}{2} = \dfrac{\bar{c}+\bar{a}}{2} = \bar{e}$ (say)

is mid point of BD and AC

Diagonals BD bisect AC at $E(\bar{e})$ — (I)

Now

$\bar{AC} = \bar{AB} + \bar{BC} = \bar{BC} + \bar{AB}$

$\bar{BD} = \bar{BC} + \bar{CD} = \bar{BC} + \bar{BA}$

$\bar{BD} = \bar{BC} - \bar{AB}$ $\quad \because \bar{CD} = \bar{BA}$

$|\bar{AC}|^2 = \bar{AC} \cdot \bar{AC}$

$= (\bar{BC} + \bar{AB}) \cdot (\bar{BC} + \bar{AB})$

$= \bar{BC} \cdot \bar{BC} + \bar{BC} \cdot \bar{AB} + \bar{AB} \cdot \bar{BC} + \bar{AB} \cdot \bar{AB}$

$= |\bar{BC}|^2 + 0 + 0 + |\bar{AB}|^2$

$(\because \bar{AB} \perp \bar{BC})$

$\therefore |\bar{AC}|^2 = |\bar{BC}|^2 + |\bar{AB}|^2$

$|\bar{BD}|^2 = \bar{BD} \cdot \bar{BD}$

|||y

$\therefore |\bar{BD}|^2 = |\bar{BC}|^2 + |\bar{AB}|^2$

$\therefore |\bar{AC}|^2 = |\bar{BD}|^2$

$\therefore AC = BD$ — (II)

from I & II the diagonals of a rectangle are congu. and bisect each other.

Conversely :-

Let diagonals AC and BD of □ ABCD are congu. and bisect each other at right angle (w.l.o.g)

$\therefore$ □ ABCD is ∥ grm

Now $AC \perp BD$

$\therefore \bar{AC} \cdot \bar{BD} = 0$

$\therefore (\bar{BC} + \bar{AB}) \cdot (\bar{BC} - \bar{AB}) = 0$

$\therefore \bar{BC} \cdot \bar{BC} - \bar{BC} \cdot \bar{AB} + \bar{AB} \cdot \bar{BC} - \bar{AB} \cdot \bar{AB} = 0$

$\therefore |\bar{BC}|^2 - |\bar{AB}|^2 = 0$

$\therefore |\bar{BC}|^2 = |\bar{AB}|^2$

$\therefore BC = AB$

i.e. adjacent sides AB & BC of ∥gm ABCD are equal.

$\therefore$ ABCD is a rhombus

AC = BD (given)

$\therefore |\bar{AC}|^2 = |\bar{BD}|^2$

$\therefore \bar{AC} \cdot \bar{AC} = \bar{BD} \cdot \bar{BD}$

$\therefore (\bar{BC} + \bar{AB}) \cdot (\bar{BC} + \bar{AB}) = (\bar{BC} - \bar{AB}) \cdot (\bar{BC} - \bar{AB})$

After simplifying we get

$2(\bar{BC} \cdot \bar{AB}) = -2(\bar{AC} \cdot \bar{AB})$

$\therefore 4(\bar{BC} \cdot \bar{AB}) = 0$

$\therefore \bar{BC} \cdot \bar{AB} = 0$

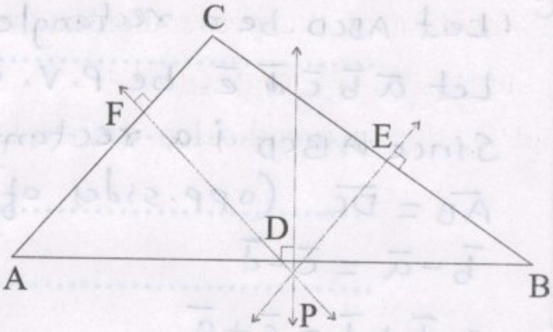
$\therefore BC \perp AB$

the adjacent sides of a rhombus ABCD are $\perp$ to each other

Hence ABCD is a Square

$\therefore$ □ ABCD is a rectangle

Ex. (7) Using vectors prove that the perpendicular bisectors of the sides of a triangle are concurrent.



Let DEF be midpoints of sides AB, BC, CA resp

Let the $\perp$ bisector of side BC and AB meet each other in point P

Solution :

Choose P as the origin and

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be P.V. of points A, B, C, D, E, F resp.

Here we have to prove that

$\overline{PF} = \bar{f}$ is $\perp$ to $\overline{AC} = \bar{c} - \bar{a}$

by mid point formula

$\bar{d} = \dfrac{\bar{a}+\bar{b}}{2}, \bar{e} = \dfrac{\bar{b}+\bar{c}}{2}, \bar{f} = \dfrac{\bar{a}+\bar{c}}{2}$

Now $\overline{PD} = \bar{d} \perp \overline{AB} = \bar{b} - \bar{a}$

$\therefore \bar{d} \cdot (\bar{b} - \bar{a}) = 0$

$\therefore \dfrac{(\bar{a}+\bar{b})}{2} \cdot (\bar{b} - \bar{a}) = 0$

$\therefore (\bar{b}+\bar{a}) \cdot (\bar{b} - \bar{a}) = 0$

$\bar{b}\cdot\bar{b} - \bar{b}\cdot\bar{a} + \bar{a}\cdot\bar{b} - \bar{a}\cdot\bar{a} = 0$

$\therefore |\bar{b}|^2 - |\bar{a}|^2 = 0$

$\therefore b^2 = a^2$ ——— (1)

Also $\overline{PE} = \bar{e} \perp \overline{BC} = \bar{c} - \bar{d}$

$\therefore \bar{e} \cdot (\bar{c} - \bar{b}) = 0$

$\therefore \left(\dfrac{\bar{b}+\bar{c}}{2}\right) \cdot (\bar{c} - \bar{b}) = 0$

$\therefore (\bar{c}+\bar{b})(\bar{c}-\bar{b}) = 0$

$\therefore \bar{c}\cdot\bar{c} - \bar{c}\cdot\bar{b} + \bar{b}\cdot\bar{c} - \bar{b}\cdot\bar{b} = 0$

$\therefore |\bar{c}|^2 - |\bar{b}|^2 = 0$

$\therefore c^2 - b^2 = 0$

$\therefore b^2 = c^2$ ——— (2)

from (1) and (2)

$a^2 = c^2$

$\therefore a^2 - c^2 = 0$

$\therefore |\bar{a}|^2 - |\bar{c}|^2 = 0$

$\therefore \bar{a}\cdot\bar{a} - \bar{c}\cdot\bar{c} = 0$

$\therefore \bar{a}\cdot\bar{a} - \bar{a}\cdot\bar{c} + \bar{a}\cdot\bar{c} - \bar{c}\cdot\bar{c} = 0$

$\therefore \bar{a}(\bar{a} - \bar{c}) + \bar{c}\cdot(\bar{a} - \bar{c}) = 0$

$\therefore (\bar{a}+\bar{c}) \cdot (\bar{a} - \bar{c}) = 0$

$\therefore \left(\dfrac{\bar{a}+\bar{c}}{2}\right) \cdot (\bar{a} - \bar{c}) = 0$

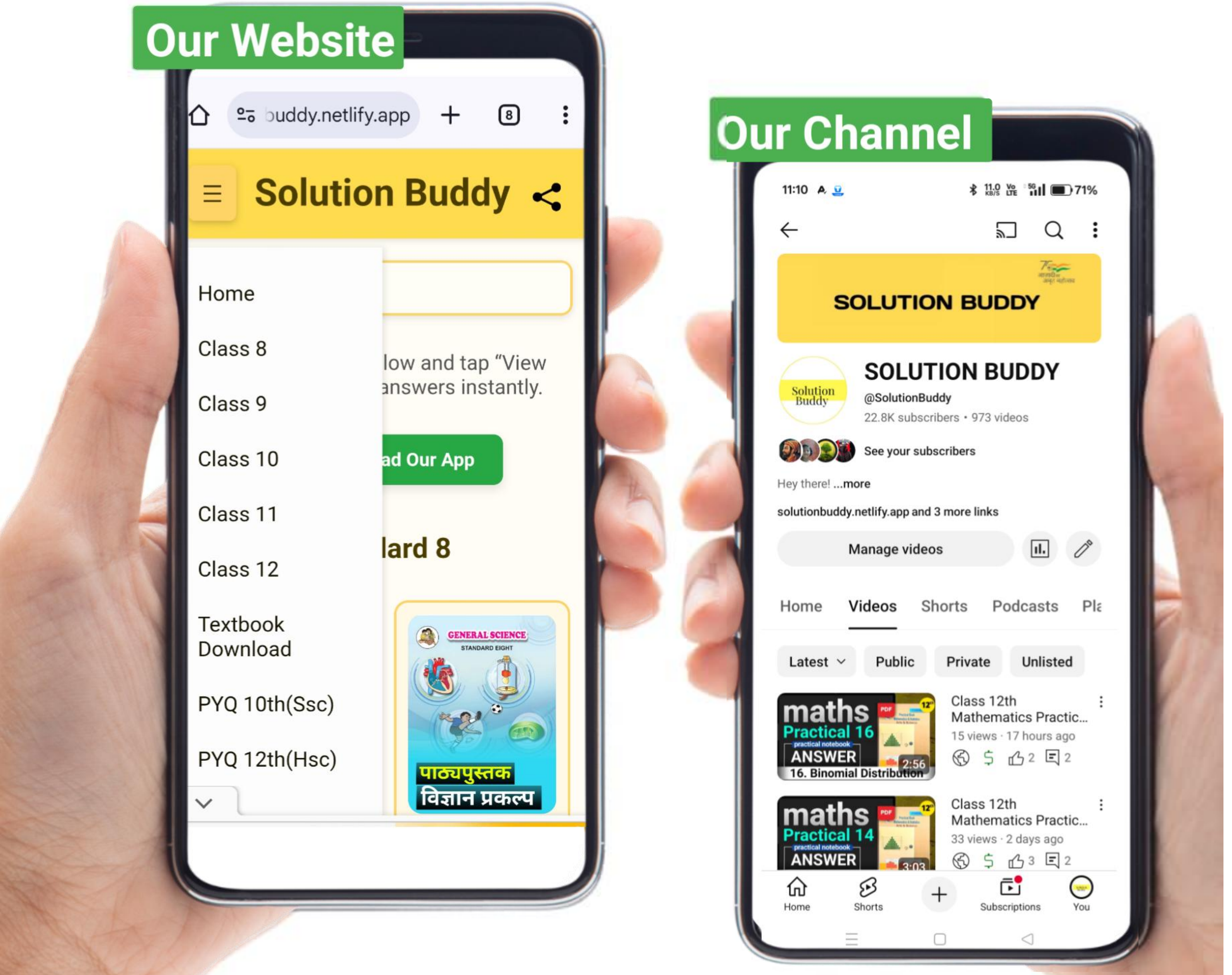
$\bar{f} \cdot (\bar{a} - \bar{c}) = 0$

$\therefore \overline{PF} \cdot \overline{CA} = 0$

$\therefore \overline{PF} \perp \overline{CA}$

$\therefore$ the $\perp$ bisectors of sides of ΔABC are concurrent.

Sign of Teacher :



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