Pair of Straight Lines

EXERCISE 4.1 [PAGES 119 - 120]

Exercise 4.1 | Q 1.1 | Page 119

Find the combined equation of the following pair of line:

$2x + y = 0$ and $3x - y = 0$

Solution: The combined equation of the lines $2x + y = 0$ and $3x - y = 0$ is

$(2x + y)(3x - y) = 0$

$\therefore 6x^2 - 2xy + 3xy - y^2 = 0$

$\therefore 6x^2 + xy - y^2 = 0$

Exercise 4.1 | Q 1.2 | Page 119

Find the combined equation of the following pair of line:

$x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

Solution: The combined equation of the lines $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ is

$(x + 2y - 1)(x - 3y + 2) = 0$

$\therefore x^2 - 3xy + 2x + 2xy - 6y^2 + 4y - x + 3y - 2 = 0$

$\therefore x^2 - xy - 6y^2 + x + 7y - 2 = 0$

Exercise 4.1 | Q 1.3 | Page 119

Find the combined equation of the following pair of line:

passing through (2, 3) and parallel to the coordinate axes.

Solution: Equations of the coordinate axes are $x = 0$ and $y = 0$

$\therefore$ The equations of the lines passing through (2, 3) and parallel to the coordinate axes are $x = 2$ and $y = 3$.

i.e. $x - 2 = 0$ and $y - 3 = 0$

$\therefore$ their combined equation is

$(x - 2)(y - 3) = 0$

$\therefore xy - 3x - 2y + 6 = 0$

Exercise 4.1 | Q 1.4 | Page 119

Find the combined equation of the following pair of line:

passing through (2, 3) and perpendicular to the lines 3x + 2y - 1 = 0 and x - 3y + 2 = 0

Solution: Let L_1 and L_2 be the lines passing through the point (2, 3) and perpendicular to the lines 3x + 2y - 1 = 0 and x - 3y + 2 = 0 respectively.

Slopes of the lines 3x + 2y - 1 = 0 and x - 3y + 2 = 0 are $\frac{-3}{2}$ and $\frac{-1}{-3} = \frac{1}{3}$ respectively.

∴ slopes of the lines L_1 and L_2 pass through the point (2, 3), their equations are

y - 3 = 2/3(x - 2) and y - 3 = -3(x - 2)

∴ 3y - 9 = 2x - 4 and y - 3 = - 3x + 6

∴ 2x - 3y + 5 = 0 and 3x + y - 9 = 0

their combined equation is

(2x - 3y + 5)(3x + y - 9) = 0

∴ $6x^2 + 2xy - 18x - 9xy - 3y^2 + 27y + 15x + 5y - 45 = 0$

∴ $6x^2 - 7xy - 3y^2 - 3x + 32y - 45 = 0$

Exercise 4.1 | Q 1.5 | Page 119

Find the combined equation of the following pair of line:

passing through (-1, 2), one is parallel to x + 3y - 1 = 0 and other is perpendicular to 2x - 3y - 1 = 0

Solution: Let L_1 be the line passing through the point (-1, 2) and parallel to the line x + 3y - 1 = 0 whose slope is – 1/3.

∴ slope of the line L_1 is – 1/3

∴ equation of the line L_1 is

y - 2 = – 1/3 (x + 1)

∴ 3y - 6 = - x - 1

∴ x + 3y - 5 = 0

Let L_1 be the line passing through (-1, 2) and perpendicular to the line 2x - 3y - 1 = 0 whose slope is

$$\frac{-2}{-3} = \frac{2}{3}$$

∴ slope of the line L_2 is – 1/3

∴ equation of the line L_2 is

$y - 2 = -1/3 (x + 1)$

$\therefore 2y - 4 = - 3x - 3$

$\therefore 3x + 2y - 1 = 0$

Hence, the equations of the required lines are

$x + 3y - 5 = 0$ and $3x + 2y - 1 = 0$

$\therefore$ their combined equation is

$(x + 3y - 5)(3x + 2y - 1) = 0$

$\therefore 3x^2 + 2xy - x + 9xy + 6y^2 - 3y - 15x - 10y + 5 = 0$

$\therefore 3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$

Exercise 4.1 | Q 2.1 | Page 119

Find the separate equation of the line represented by the following equation:

$3y^2 + 7xy = 0$

Solution: $3y^2 + 7xy = 0$

$\therefore y(3y + 7x) = 0$

$\therefore$ the separate equations of the lines are $y = 0$ and $7x + 3y = 0$

Exercise 4.1 | Q 2.2 | Page 119

Find the separate equation of the line represented by the following equation:

$5y^2 + 9y^2 = 0$

Solution:

$5y^2 + 9y^2 = 0$

$\therefore \left(\sqrt{5\mathrm{x}}\right)^2 - \left(\sqrt{3\mathrm{y}}\right)^2 = 0$

$\therefore \left(\sqrt{5\mathrm{x}} + 3\mathrm{y}\right)\left(\sqrt{5x} - 3\mathrm{y}\right) = 0$

the separate equations of the lines are $\left(\sqrt{5\mathrm{x}} + 3\mathrm{y}\right) = 0$ and $\left(\sqrt{5\mathrm{x}} - 3\mathrm{y}\right) = 0$

Exercise 4.1 | Q 2.3 | Page 119

Find the separate equation of the line represented by the following equation:

$x^2 - 4xy = 0$

Solution: $x^2 - 4xy = 0$

$\therefore x(x - 4y) = 0$

$\therefore$ the separate equations of the lines are $x = 0$ and $x - 4y = 0$

Exercise 4.1 | Q 2.4 | Page 119

Find the separate equations of the lines represented by the equation $3x^2-10xy-8y^2=0$

Solution:

Given pairs of lines $3x^2 - 10xy - 8y^2 = 0$

$$3x^2 - 12xy + 2xy - 8y^2 = 0$$

$$3x(x - 4y) + 2y(x - 4y) = 0$$

$$(x - 4y)(3x + 2y) = 0$$

Separated equations

$$3x + 2y = 0 \quad \text{and} \quad x - 4y = 0$$

Exercise 4.1 | Q 2.5 | Page 119

Find the separate equation of the line represented by the following equation:

$$3x^2 - 2\sqrt{3}xy - 3y^2 = 0$$

Solution:

$$3x^2 - 2\sqrt{3}xy - 3y^2 = 0$$

$$\therefore 3x^2 - 3\sqrt{3}xy + \sqrt{3}xy - 3y^2 = 0$$

$$\therefore 3x\left(x - \sqrt{3}y\right) + \sqrt{3}y\left(x - \sqrt{3}y\right) = 0$$

$$\therefore \left(x - \sqrt{3}y\right)\left(3x + \sqrt{3}y\right) = 0$$

The separate equations of the lines are

$x - \sqrt{3}y = 0$ and $3x + \sqrt{3}y = 0$

Exercise 4.1 | Q 2.6 | Page 119

Find the separate equation of the line represented by the following equation:

$x^2 + 2(\text{cosec } \alpha)xy + y^2 = 0$

Solution: $x^2 + 2(\text{cosec } \alpha)xy + y^2 = 0$

i.e. $y^2 + 2(\text{cosec } \alpha)xy + x^2 = 0$

Dividing by x^2, we get,

$$\left(\frac{y}{x}\right)^2 + 2\text{cosec}\alpha . \left(\frac{y}{x}\right) + 1 = 0$$

$$\therefore \frac{y}{x} = \frac{-2\text{cosec}\alpha \ \pm\sqrt{4\text{cosec}^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-2\text{cosec}\alpha \ \pm 2\sqrt{\text{cosec}^2\alpha - 1}}{2}$$

$= - \text{cosec } \alpha \pm \cot \alpha$

$\therefore \frac{y}{x} = (\cot\alpha - \text{cosec}\alpha)$ and

$\frac{y}{x} = -(\text{cosec}\alpha + \cot\alpha)$

The separate equations of the lines are

$(\text{cosec } \alpha - \cot \alpha)\ x + y = 0$ and $(\text{cosec } \alpha - \cot \alpha)\ x + y = 0$

Exercise 4.1 | Q 2.7 | Page 119

Find the separate equation of the line represented by the following equation:

$x^2 + 2xy \tan \alpha - y^2 = 0$

Solution:

$x^2 + 2xy \tan \alpha - y^2 = 0$

Dividing by y^2

$$\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right)\tan\alpha - 1 = 0$$

$$\therefore \frac{x}{y} = \frac{-2\tan\alpha \pm \sqrt{4\tan^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-2\tan\alpha \pm 2\sqrt{\tan^2\alpha - 1}}{2}$$

= - tan α ± sec α

$\left(\frac{x}{y}\right) = (\sec\alpha - \tan\alpha)$ and

$$\left(\frac{x}{y}\right) = -(\tan\alpha + \sec\alpha)$$

The separate equations of the lines are

(sec α - tan α) x + y = 0 and (sec α + tan α)x - y = 0

Exercise 4.1 | Q 3.1 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by following equation:

$5x^2 - 8xy + 3y^2 = 0$

Solution: Comparing the equation $5x^2 - 8xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 5, 2h = - 8, b= 3

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 - 8xy + 3y^2 = 0$

$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{8}{3}$ and $m_1 m_2 = \frac{a}{b} = \frac{5}{3}$ ....(1)

Now required lines are perpendicular to these lines

$\therefore$ their slopes are $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

i.e. $m_1 y = - x$ and $m_2 y = - x$

i.e. $x + m_1 y = 0$ and $x + m_2 y = 0$

$\therefore$ their combined equation is

$(x + m_1 y)(x + m_2 y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

$\therefore \mathbf{x^2 + \frac{8}{3}xy + \frac{5}{3}y^2 = 0}$ ...[By (1)]

$\therefore 3x^2 + 8xy + 5y^2 = 0$

Exercise 4.1 | Q 3.2 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$5x^2 + 2xy - 3y^2 = 0$

Solution: Comparing the equation $5x^2 + 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 5, 2h = 2, b = - 3$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 2xy - 3y^2 = 0$

$$\therefore \mathrm{m_1 + m_2 = \frac{-2h}{b} = \frac{-2}{-3} = \frac{2}{3}} \text{ and } \mathrm{m_1 m_2 = \frac{a}{b} = \frac{5}{-3}} \quad ....(1)$$

Now required lines are perpendicular to these lines

$\therefore$ their slopes are $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

$\therefore$ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

$\therefore \mathbf{x^2 + \frac{2}{3}xy - \frac{5}{3}y^2 = 0}$ ...[By (1)]

$\therefore 3x^2 + 2xy - 5y^2 = 0$

Exercise 4.1 | Q 3.3 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$xy + y^2 = 0$

Solution: Comparing the equation $xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 0$, $2h = 1$, $b = 1$

Let m_1 and m_2 be the slopes of the lines represented by $xy + y^2 = 0$

$\therefore \mathbf{m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{1} = -1}$ and $\mathbf{m_1m_2 = \frac{a}{b} = \frac{0}{1} = 0}$ ....(1)

Now required lines are perpendicular to these lines

$\therefore$ their slopes are $\mathbf{\frac{-1}{m_1}}$ and $\mathbf{\frac{-1}{m_2}}$

Since these lines are passing through the origin, their separate equations are

$y = \mathbf{\frac{-1}{m_1}x}$ and $y = \mathbf{\frac{-1}{m_2}x}$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

$\therefore$ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

$\therefore \mathbf{x^2 - xy + 0.y^2 = 0}$ ...[By (1)]

$\therefore x^2 - xy = 0$

[**Note: : Answer in the textbook is incorrect.**]

Exercise 4.1 | Q 3.4 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$3x^2 - 4xy = 0$

Solution: Consider $3x^2 - 4xy = 0$

$\therefore x(3x - 4y) = 0$

$\therefore$ separate equations of the lines are $x = 0$ and $3x - 4y = 0$

Let m_1 and m_2 be the slopes of these lines.

Then m_1 does not exist and $m_2 = \frac{3}{4}$

Now, required lines are perpendicular to these lines.

$\therefore$ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since m_1 does not exist, $-\frac{1}{m_1} = 0$

Also, $m_2 = \frac{3}{4}, -\frac{1}{m_2} = -\frac{4}{3}$

Since these lines are passing through the origin, their

separate equations are $y = 0$ and $y = -\frac{4}{3}x$, i.e. $4x + 3y = 0$

$\therefore$ their combined equation is

$y(4x + 3y) = 0$

$\therefore 4xy + 3y^2 = 0$

Exercise 4.1 | Q 4.1 | Page 119

Find k, if the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.

Solution: Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, a = 1, 2h = k, b = - 3.

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{k}{-3} = \frac{k}{3}$$

$$\text{and } m_1m_2 = \frac{a}{b} = \frac{1}{-3} = -\frac{1}{3}$$

Now, $m_1 + m_2 = 2(m_1m_2)$ ...(given)

$$\therefore \frac{k}{3} = 2\left(-\frac{1}{3}\right)$$

$\therefore k = - 2.$

Exercise 4.1 | Q 4.2 | Page 119

Find k, the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$ differ by 4.

Solution: Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, a = 3, 2h = k, b = -1.

Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{k}{-1} = k$$

$$\text{and } m_1m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$= k^2 - 4 (- 3)$

$= k^2 + 12$ ....(1)

But $|m_1 - m_2| = 4$

$\therefore (m_1 - m_2)^2 = 16$ ....(2)

$\therefore$ from (1) and (2), $k^2 + 12 = 16$

$\therefore k^2 = 4$

$\therefore k = \pm 2$

Exercise 4.1 | Q 4.3 | Page 119

Find k, the slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.

Solution: Comparing the equation $kx^2 + 4xy - y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, a = k, 2h = 4, b = -1.

Let m_1 and m_2 be the slopes of the lines represented by $kx^2 + 4xy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{4}{-1} = 4$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{k}{-1} = -k$$

We are given that $m_2 = m_1 + 8$

$\therefore m_1 + m_1 + 8 = 4$

$\therefore 2m_1 = - 4 \quad \therefore m_1 = - 2$ ...(1)

Also, $m_1(m_1 + 8) = - k$

$(-2)(- 2 + 8) = - k$ ...[By (1)]

$\therefore (-2)(6) = - k$

$\therefore - 12 = - k$

$\therefore k = 12$

Exercise 4.1 | Q 5.1 | Page 120

Find the condition that the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$

Given that $4x + 5y = 0$ is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$

The slope of the line 4x + 5y = 0 is $-\frac{4}{5}$

$\therefore$ m = $-\frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$

$$\therefore b\left(-\frac{4}{5}\right)^2 + 2h\left(-\frac{4}{5}\right) + a = 0$$

$$\therefore \frac{16b}{25} - \frac{8h}{5} + a = 0$$

$\therefore$ 16b - 40h + 25a = 0

$\therefore$ 25a + 16b = 40h

This is the required condition.

Exercise 4.1 | Q 5.2 | Page 120

Find the condition that the line 3x + y = 0 may be perpendicular to one of the lines given by $ax^2 + 2hxy + by^2 = 0$

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$

Since one line is perpendicular to the line 3x + y = 0 whose slope is $-\frac{3}{1} = -3$

$\therefore$ slope of that line = m = $\frac{1}{3}$

$\therefore$ m = $\frac{1}{3}$ is the root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$$\therefore b\left(\frac{1}{3}\right)^2 + 2h\left(\frac{1}{3}\right) + a = 0$$

$$\therefore \frac{b}{9} + \frac{2h}{3} + a = 0$$

$\therefore$ b + 6h + 9a = 0

$\therefore$ 9a + b + 6h = 0

This is the required condition.

Exercise 4.1 | Q 6 | Page 120

If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$, show that $ap^2 + 2hpq + bq^2 = 0$.

Solution: To prove: $ap^2 + 2hpq + bq^2 = 0$.

Let the slope of the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ be m_1 and m_2

Then, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

Slope of the line $px + qy = 0$ is $\frac{-p}{q}$

But one of the lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$

$$\Rightarrow m_1 = \frac{q}{p}$$

Now, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } \left(\frac{q}{p}\right) m_2 = \frac{a}{b}$$

$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } m_2 = \frac{ap}{bq}$$

$$\Rightarrow \frac{q}{p} + \frac{ap}{bq} = \frac{-2h}{b}$$

$$\Rightarrow \frac{bq^2 + ap^2}{pq} = -2h$$

$$\Rightarrow bq^2 + ap^2 = -2h\,pq$$

$$\Rightarrow ap^2 + 2hpq + bq^2 = 0$$

Exercise 4.1 | Q 7 | Page 120

Find the combined equation of the pair of lines through the origin and making an equilateral triangle with the line $y = 3$.

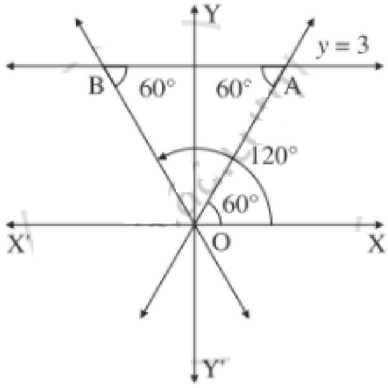
Solution: Let OA and OB be the lines through the origin making an angle of 60° with the line $y = 3$.

∴ OA and OB make an angle of 60° and 120° with the positive direction of the X-axis.

$\therefore$ slope of OA = tan 60° = $\sqrt{3}$

$\therefore$ equation of the line OA is

y = $\sqrt{3}$x i.e. $\sqrt{3}$x - y = 0



Slope of OB = tan 120° = tan (180° - 60°)

= - tan 60° = - $\sqrt{3}$

$\therefore$ equation of the line OB is

y = - $\sqrt{3}$x, i.e. $\sqrt{3}$x + y = 0

$\therefore$ required joint equation of the lines is

$$\left(\sqrt{3}x - y\right)\left(\sqrt{3}x + y\right) = 0$$

i.e. $3x^2 - y^2 = 0$

Exercise 4.1 | Q 8 | Page 120

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other, show that $16h^2 = 25ab$.

Solution: Let m_1 and m_2 be the slopes of the lines given by $ax^2 + 2hxy + by^2 = 0$

$\therefore m_1 + m_2 = -\frac{2h}{b}$

and $m_1 m_2 = \frac{a}{b}$

We are given that $m_2 = 4m_1$

$\therefore m_1 + 4m_1 = -\frac{2h}{b}$

$\therefore 5m_1 = \frac{-2h}{b}$

$\therefore m_1 = -\frac{2h}{5b}$ .....(1)

Also, $m_1(4m_1) = \frac{a}{b}$

$\therefore 4m_1^2 = \frac{a}{b}$

$\therefore m_1^2 = \frac{a}{4b}$

$\therefore \left(\frac{-2h}{5b}\right)^2 = \frac{a}{4b}$ ...[By(1)]

$\therefore \frac{4h^2}{25b^2} = \frac{a}{4b}$

$\therefore \frac{4h^2}{25b} = \frac{a}{4}$, as $b \neq 0$

$\therefore 16h^2 = 25ab$

This is the required condition.

Exercise 4.1 | Q 9 | Page 120

If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisect an angle between the coordinate axes, then show that $(a + b)^2 = 4h^2$.

Solution: The auxiliary equation of the lines given by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.

Since one of the lines bisects an angle between the coordinate axes, that line makes an angle of 45° or 135° with the positive direction of X-axis.

$\therefore$ the slope of that line = tan 45° or tan 135°

$\therefore m = \tan 45° = 1$

or $m = \tan 135° = \tan (180° - 45°)$

$= - \tan 45° = - 1$

$\therefore m = \pm 1$ are the roots of the auxiliary equation $bm^2 + 2hm + a = 0$.

$\therefore b(\pm 1)^2 + 2h(\pm 1) + a = 0$

$\therefore b \pm 2h + a = 0$

$\therefore a + b = \pm 2h$

$\therefore (a + b)^2 = 4h^2$

This is the required condition.

EXERCISE 4.2 [PAGE 124]

Exercise 4.2 | Q 1 | Page 124

. Show that the lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Solution: Comparing the equation $3x^2 - 4xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = 3, 2h = -4, b = -3$.

Since $a + b = 3 + (- 3) = 0$, the lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Exercise 4.2 | Q 2 | Page 124

Show that the lines represented by $x^2 + 6xy + 9y^2 = 0$ are coincident.

Solution: Comparing the equation $x^2 + 6xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = 1, 2h = 6$ i.e. $h = 3$ and $b = 9$.

Since $h^2 - ab = (3)^2 - 1(9)$
$= 9 - 9 = 0$,
the lines represented by $x^2 + 6xy + 9y^2 = 0$ are coincident.

Exercise 4.2 | Q 3 | Page 124

Find the value of k if lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other.

Solution: Comparing the equation $kx^2 + 4xy - 4y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = k, 2h = 4 and b = - 4.

Since lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other,

a + b = 0

∴ k - 4 = 0

∴ k = 4

Exercise 4.2 | Q 4.1 | Page 124

Find the measure of the acute angle between the line represented by:

$$3x^2 - 4\sqrt{3}xy + 3y^2 = 0$$

Solution:

Comparing the equation

$3x^2 - 4\sqrt{3}xy + 3y^2 = 0$ with

$ax^2 + 2hxy + by^2 = 0$, we get,

a = 3, 2h = - $4\sqrt{3}$ i.e. h = $-2\sqrt{3}$ and b = 3

Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{\left(-2\sqrt{3}\right)^2 - 3(3)}}{3 + 3}\right|$$

$$= \left| \frac{2\sqrt{12 - 9}}{6} \right|$$

$$= \left| \frac{2\sqrt{3}}{6} \right|$$

$$\therefore \tan\theta = \frac{1}{\sqrt{3}} = \tan 30°$$

$$\therefore \theta = 30°$$

Exercise 4.2 | Q 4.2 | Page 124

Find the measure of the acute angle between the line represented by:

$4x^2 + 5xy + y^2 = 0$

Solution: Comparing the equation

$4x^2 + 5xy + y^2 = 0$ with

$ax^2 + 2hxy + by^2 = 0$, we get,

a = 4, 2h = 5 i.e. h = 5/2 and b = 1

Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 4(1)}}{4 + 1} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{25}{4}\right) - 4}}{5} \right|$$

$$= \left| \frac{2 \times \frac{3}{2}}{5} \right|$$

$$\therefore \tan\theta = \frac{3}{5}$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Exercise 4.2 | Q 4.3 | Page 124

Find the measure of the acute angle between the line represented by:

$2x^2 + 7xy + 3y^2 = 0$

Solution: Comparing the equation

$2x^2 + 7xy + 3y^2 = 0$ with

$ax^2 + 2hxy + by^2 = 0$, we get,

a = 2, 2h = 7 i.e. h = 7/2 and b = 3

Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 2(3)}}{2 + 3} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{49}{4}\right) - 6}}{5} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{49-24}{4}\right)}}{5} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{25}{4}\right)}}{5} \right|$$

$$= \frac{2 \times \left(\frac{5}{2}\right)}{5}$$

$$= \frac{5}{5}$$

$\tan \theta = 1$

$\therefore \theta = \tan 1 = 45°$

$\therefore \theta = 45°$

Exercise 4.2 | Q 4.4 | Page 124

Find the measure of the acute angle between the line represented by:

$(a^2 + 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$

Solution: Comparing the equation

$(a^2 + 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with

$Ax^2 + 2Hxy + By^2 = 0$, we have,

$A = a^2 + 3b^2$, $H = 4ab$ and $B = b^2 - 3a^2$

$\therefore H^2 - AB = 16a^2b^2 - (a^2 - 3b^2)(b^2 - 3a^2)$

$= 16a^2b^2 + (a^2 - 3b^2)(3a^2 - b^2)$

$= 16a^2b^2 + 3a^4 - 10a^2b^2 + 3b^4$

$= 3a^4 + 6a^2b^2 + 3b^4$

$= 3(a^4 + 2a^2b^2 + b^4)$

$= 3(a^2 + b^2)^2$

$\therefore\ \sqrt{H^2 - AB} = \sqrt{3}\left(a^2 + b^2\right)$

Also, $A + B = (a^2 - 3b^2) + (b^2 - 3a^2)$

$= -2(a^2 + b^2)$

Let θ be the acute angle between the lines, then

$\therefore \tan\theta = \left|\dfrac{2\sqrt{H^2 - AB}}{A + B}\right|$

$= \left|\dfrac{2\sqrt{3}\left(a^2 + b^2\right)}{-2\left(a^2 + b^2\right)}\right|$

$= \sqrt{3} = \tan 60°$

$\therefore \theta = 60°$

Exercise 4.2 | Q 5 | Page 124

Find the combined equation of lines passing through the origin each of which making an angle of 30° with the line $3x + 2y - 11 = 0$

Solution: The slope of the line $3x + 2y - 11 = 0$ is $m_1 = -3/2$

Let m be the slope of one of the lines making an angle of 30° with the line $3x + 2y - 11 = 0$

The angle between the lines having slopes m and m_1 is 30°.

$\therefore \tan 30° = \left|\dfrac{m - m_1}{1 + m.m_1}\right|$, where $\tan 30° = \dfrac{1}{\sqrt{30}}$

$$\therefore \frac{1}{\sqrt{30}} = \left|\frac{m - \left(-\frac{3}{2}\right)}{1 + m\left(-\frac{3}{2}\right)}\right|$$

$$\therefore \frac{1}{\sqrt{30}} = \left|\frac{2m + 3}{2 - 3m}\right|$$

On squaring both sides, we get,

$$\frac{1}{3} = \frac{(2m + 3)^2}{(2 - 3m)^2}$$

∴ $(2 - 3m)^2 = 3(2m + 2)^2$

∴ $4 - 12m + 9m^2 = 3(4m^2 + 12m + 9)$

∴ $4 - 12m + 9m^2 = 12m^2 + 36m + 27$

∴ $3m^2 + 48m + 23 = 0$

This is the auxiliary equation of the two lines and their joint equation is obtained by putting m = y/x

∴ the combined equation of the two lines is

$$3\left(\frac{y}{x}\right)^2 + 48\left(\frac{y}{x}\right) + 23 = 0$$

$$\therefore \frac{3y^2}{x^2} + \frac{48y}{x} + 23 = 0$$

∴ $3y^2 + 48xy + 23x^2 = 0$

∴ $23x^2 + 48xy + 3y^2 = 0$

Exercise 4.2 | Q 6 | Page 124

If the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is equal to the angle between the lines $2x^2 - 5xy + 3y^2 = 0$, then show that $100 (h^2 - ab) = (a + b)^2$.

Solution:

The acute angle θ between the lines $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| \quad ....(1)$$

Comparing the equation $2x^2 - 5xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get.

a = 2, 2h = -5, i.e. $h = -\frac{5}{2}$ and b = 3

Let α be the acute angle between the lines $2x^2 - 5xy + 3y^2 = 0$

$$\therefore \tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 2(3)}}{2 + 3}\right|$$

$$= \left|\frac{\left(2\frac{\sqrt{25}}{4} - 6\right)}{5}\right|$$

$$= \left|\frac{2 \times \frac{1}{2}}{5}\right|$$

$$\therefore \tan\alpha = \frac{1}{5} \quad \text{....(2)}$$

If θ = α, then tan θ = tan α

$$\therefore \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \frac{1}{5} \quad \text{....[By (1) and (2)]}$$

$$\therefore \frac{4\left(h^2 - ab\right)}{(a + b)^2} = \frac{1}{25}$$

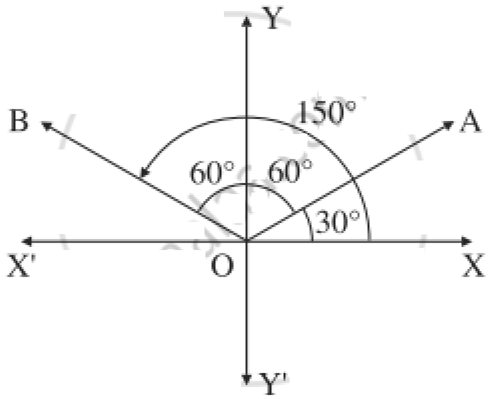
$\therefore 100\ (h^2 - ab) = (a + b)^2$

This is the required condition.

Exercise 4.2 | Q 7 | Page 124

Find the combined equation of lines passing through the origin and each of which making an angle of 60° with the Y-axis.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.

Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

∴ slope of OA = tan 30° = $\frac{1}{\sqrt{3}}$

∴ equation of the line OA is

y = $\frac{1}{\sqrt{3}}$ x i.e. $\mathbf{x} - \sqrt{3}\mathbf{y} = 0$

Slope of OB = tan 150° = tan (180° - 30°)

= - tan 30° = $-\frac{1}{\sqrt{3}}$

∴ equation of the line OB is

y = $-\frac{1}{\sqrt{3}}\mathrm{x}$ i.e. x + $\sqrt{3}$y = 0

∴ required combined equation is

$$\left(\mathrm{x} - \sqrt{3}\mathrm{y}\right)\left(\mathrm{x} + \sqrt{3}\mathrm{y}\right) = 0$$

i.e. $x^2 - 3y^2 = 0$

EXERCISE 4.3 [PAGES 127 - 128]

Exercise 4.3 | Q 1.1 | Page 127

Find the joint equation of the pair of the line through the point (2, -1) and parallel to the lines represented by $2x^2 + 3xy - 9y^2 = 0$.

Solution: The combined equation of the given lines is

$2x^2 + 3xy - 9y^2 = 0$

i.e. $2x^2 + 6xy - 3xy - 9y^2 = 0$

i.e. $2x(x + 3y) - 3y(x + 3y) = 0$

i.e. $(x + 3y)(2x - 3y) = 0$

∴ their separate equations are

$x + 3y = 0$ and $2x - 3y = 0$

∴ their slopes are $m_1 = \frac{-1}{3}$ and $m_2 = \frac{-2}{-3} = \frac{2}{3}$

The slopes of the lines parallel to these lines are m_1 and m_2 i.e. $-\frac{1}{3}$ and $\frac{2}{3}$.

∴ the equations of the lines with these slopes and through the point (2, -1) are

$y + 1 = -\frac{1}{3}(x - 2)$ and $y + 1 = \frac{2}{3}(x - 2)$

i.e. $3y + 3 = - x + 2$ and $3y + 3 = 2x - 4$

i.e. $x + 3y + 1 = 0$ and $2x - 3y - 7 = 0$

∴ the joint equation of these lines is

$(x + 3y + 1)(2x - 3y - 7) = 0$

∴ $2x^2 - 3xy - 7x + 6xy - 9y^2 - 21y + 2x - 3y - 7 = 0$

∴ $2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0$

Exercise 4.3 | Q 1.2 | Page 127

Find the joint equation of the pair of the line through the point (2, -3) and parallel to the lines represented by $x^2 + xy - y^2 = 0$.

Solution: The combined equation of the given lines is

$x^2 + xy - y^2 = 0$ ....(1)

with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1, 2h = 1, b = - 1$

Let m_1 and m_2 be the slopes of the lines represented by (1).

Then $m_1 + m_2 = -\frac{2h}{b} = \frac{-1}{-1} = 1$ and $m_1 m_2 = \frac{a}{b} = \frac{1}{-1} = -1$ .....(2)

The slopes of the lines parallel to these lines are m_1 and m_2 .

∴ the equations of the lines with these slopes and through the point (2, - 3) are

$y + 3 = m_1(x - 2)$ and $y + 3 = m_2 (x - 2)$

i.e. $m_1(x - 2) - (y + 3) = 0$ and $m_2(x - 2) - (y + 3) = 0$

∴ the joint equation of these lines is

$[m_1 (x - 2) - (y + 3)][m_2(x - 2)- (y + 3)] = 0$

$\therefore m_1m_2 (x - 2)^2 - m_1 (x - 2)(y + 3) - m_2(x - 2)(y+3) + (y + 3)^2 = 0$

$\therefore m_1m_2 (x - 2)^2 - (m_1 + m_2)(x - 2)(y + 3) + (y + 3)^3 = 0$

$$\therefore -(x-2)^2 - (x-2)(y+3) + (y+3)^2 = 0 \quad \text{....[By (2)]}$$

$$\therefore (x-2)^2 + (x-2)(y+3) - (y+3)^2 = 0$$

$$\therefore \left(x^2 - 4x + 4\right) + (xy + 3x - 2y - 6) - \left(y^2 + 6y + 9\right) = 0$$

$$\therefore x^2 - 4x + 4 + xy + 3x - 2y - 6 - y^2 - 6y - 9 = 0$$

$$\therefore x^2 + xy - y^2 - x - 8y - 11 = 0$$

Exercise 4.3 | Q 2 | Page 127

Show that the equation $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.

Solution: Comparing the equation

$x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

a = 1, h = 1, b = 2, g = 1, f = 1, c = 1.

The given equation represents a pair of lines, if

$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 - ab \geq 0$

Now, $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(2 - 1) - 1(1 - 1) + 1(1 - 2)$

$= 1 - 0 - 1 = 0$

and $h^2 - ab = (1)^2 - 1(2) = -1 < 0$

$\therefore$ given equation does not represent a pair of lines.

Exercise 4.3 | Q 3 | Page 127

Show that the equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of lines.

Solution: Comparing the equation

$2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$

with ax2 + 2hxy + by2 + 2gx + 2fy + c = 0, we get,

$a = 2$, $h = -\frac{1}{2}$, $b = -3$, $f = \frac{19}{2}$ and $c = -20$

$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$= \begin{vmatrix} 2 & -\frac{1}{2} & -3 \\ -\frac{1}{2} & -3 & \frac{19}{2} \\ -3 & \frac{19}{2} & -20 \end{vmatrix}$

Taking $\frac{1}{2}$ common from each row, we get, D = $\frac{1}{8}\begin{vmatrix} 4 & -1 & -6 \\ -1 & -6 & 19 \\ -6 & 19 & -40 \end{vmatrix}$

$$= \frac{1}{8}[4(240 - 361) + 1(40 + 114) - 6(-19 - 36)]$$

$$= \frac{1}{8}[4(-121) + 154 - 6(-55)]$$

$$= \frac{11}{8}[4(-11) + 14 - 6(-5)]$$

$$= \frac{11}{8}(-44 + 14 + 30) = 0$$

Also, h^2 - ab = $\left(-\frac{1}{2}\right)^2 - 2(-3) = \frac{1}{4} + 6 = \frac{25}{4} > 0$

∴ the given equation represents a pair of lines.

Exercise 4.3 | Q 4 | Page 127

Show that the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also, find the acute angle between them.

Solution: Comparing the equation

$2x^2 + xy - y^2 + x + 4y - 3 = 0$ with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get,

a = 2, h = $\frac{1}{2}$, b = - 1, g = $\frac{1}{2}$, f = 2, c = -3.

∴ D = $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$

Taking $\frac{1}{2}$ common from each row, we get,

$$D = \frac{1}{8}\begin{vmatrix} 4 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -6 \end{vmatrix}$$

$$= \frac{1}{8}[4(12 - 16) - 1(-6 - 4) + 1(4 + 2)]$$

$$= \frac{1}{8}[4(-4) - 1(-10) + 1(6)]$$

$$= \frac{1}{8}(-16 + 10 + 6) = 0$$

Also, $h^2 - ab = \left(\frac{1}{2}\right)^2 - 2(-1) = \frac{1}{4} + 2 = \frac{9}{4} > 0$

$\therefore$ the given equation represents a pair of lines.

Let θ be the acute angle between the lines

$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

$$= \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 2(-1)}}{2 - 1}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4} + 2}}{1}\right|$$

$$= 2 \times \frac{3}{2} = 3$$

$\therefore \theta = \tan^{-1}(3)$

Exercise 4.3 | Q 5.1 | Page 127

Find the separate equation of the line represented by the following equation:

$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$

Solution: $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$

$\therefore (x - 2)^2 - 2(x - 2)(y + 1) - (x - 2)(y + 1) + 2(y + 1)^2 = 0$

$\therefore (x - 2)[(x - 2) - 2(y + 1)]- (y + 1)[(x - 2) - 2(y + 1)] = 0$

$\therefore (x - 2)(x - 2 - 2y - 2) - (y + 1)(x - 2 - 2y - 2) = 0$

$\therefore (x - 2)(x - 2y - 4) - (y + 1)(x - 2y - 4) = 0$

$\therefore (x - 2y - 4)(x - 2 - y - 1) = 0$

$\therefore (x - 2y - 4)(x - y - 3) = 0$

$\therefore$ the separate equations of the lines are

$x - 2y - 4 = 0$ and $x - y - 3 = 0$

Alternative Method:

$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$ ...(1)

Put $x - 2 = X$ and $y + 2 = Y$

$\therefore$ (1) becomes,

$X^2 - 3XY + 2Y^2 = 0$

$\therefore X^2 - 2XY - XY + 2Y^2 = 0$

$\therefore X(X - 2Y) - Y(X - 2Y) = 0$

$\therefore (X - 2Y)(X - Y) = 0$

$\therefore$ the separate equations of the lines are

$X - 2Y = 0$ and $X - Y = 0$

$\therefore (x - 2) - 2(y + 1) = 0$ and $(x - 2) - (y + 1) = 0$

$\therefore x - 2y - 4 = 0$ and $x - y - 3 = 0$

Exercise 4.3 | Q 5.2 | Page 127

Find the separate equation of the line represented by the following equation:

$10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$

Solution: $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$ ....(1)

Put $x + 1 = X$ and $y - 2 = Y$

$\therefore$ (1) becomes

$10X^2 + XY - 3Y^2 = 0$

$10X^2 + 6XY - 5XY - 3Y^2 = 0$

$2X(5X + 3Y) - Y(5X + 3Y) = 0$

$(2X - Y)(5X + 3Y) = 0$

$5X + 3Y = 0$ and $2X - Y = 0$

$5X + 3Y = 0$

$5(x + 1) + 3(y - 2) = 0$

$5x + 5 + 3y - 6 = 0$

$\therefore 5x + 3y - 1 = 0$

$2X - Y = 0$

$2(x + 1) - (y - 2) = 0$

$2x + 2 - y + 2 = 0$

$\therefore 2x - y + 4 = 0$

Exercise 4.3 | Q 6.1 | Page 127

Find the value of k, if the following equations represent a pair of line:

$3x^2 + 10xy + 3y^2 + 16y + k = 0$

Solution: Comparing the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get, $a = 3$, $h = 5$, $b = 3$, $g = 0$, $f = 8$, $c = k$.

Now, given equation represents a pair of lines.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore (3)(3)(k) + 2(8)(0)(5) - 3(8)^2 - 3(0)^2 - k(5)^2 = 0$

$\therefore 9k + 0 - 192 - 0 - 25k = 0$

$\therefore -16k - 192 = 0$

$\therefore -16k = 192$

$\therefore k = -12$

Exercise 4.3 | Q 6.2 | Page 127

Find the value of k, if the following equations represent a pair of line:

$kxy + 10x + 6y + 4 = 0$

Solution: Comparing the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get, a = 0, h = k/2, b = 0, g = 5, f = 3, c = 4

Now, given equation represents a pair of lines.

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\therefore\ (0)(0)(4) + 2(3)(5)\left(\frac{k}{2}\right) - 0(3)^2 - 0(5)^2 - 4\left(\frac{k}{2}\right)^2 = 0$$

$\therefore$ $0 + 15k - 0 - 0 - k^2 = 0$

$\therefore$ $15k - k^2 = 0$

$\therefore$ $- k(k - 15) = 0$

$\therefore$ k = 0 or k = 15

If k = 0, then the given equation becomes

10x + 6y + 4 = 0 which does not represent a pair of lines.

$\therefore$ $k \neq 0$

Hence, k = 15.

Exercise 4.3 | Q 6.3 | Page 127

Find the value of k, if the following equations represent a pair of line:

$x^2 + 3xy + 2y^2 + x - y + k = 0$

Solution: Comparing the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get, a = 1, h = $\frac{3}{2}$, b = 2, g = $\frac{1}{2}$, f = $-\frac{1}{2}$, c = k.

Now, given equation represents a pair of lines.

$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

i.e. $\begin{vmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & k \end{vmatrix} = 0$

Taking out $\frac{1}{2}$ common from each row, we get,

$\frac{1}{8}\begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$

$\therefore \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$

$\therefore$ 2(8k - 1) - 3(6k + 1) + 1(- 3 - 4) = 0

$\therefore$ 16k - 2 - 18k - 3 - 7 = 0

$\therefore$ - 2k - 12 = 0

$\therefore$ - 2k = 12 $\quad\therefore$ k = - 6

Exercise 4.3 | Q 7 | Page 128

Find p and q, if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines.

Solution: The given equation represents a pair of lines perpendicular to each other

$\therefore$ (coefficient of x^2) + (coefficient of y^2) = 0

$\therefore$ p + 3 = 0 $\quad\therefore$ p = -3

With this value of p, the given equation is

$-3x^2 - 8xy + 3y^2 + 14x + 2y + q = 0$

Comparing this equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

a = -3, h = - 4, b = 3, g = 7, f = 1 and c = q

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & g \end{vmatrix}$$

= - 3(3q - 1) + 4(- 4q - 7) + 7(- 4 - 21)

= - 9q + 3 - 16q - 28 - 175

= - 25q - 200

= - 25 (q + 8)

Since the given equation represents a pair of lines, D = 0

∴ - 25(q + 8) = 0

∴ q = - 8

Hence, p = - 3 and q = - 8.

Exercise 4.3 | Q 8 | Page 128

Find p and q, if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

Solution: The given equation is $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

a = 2, h = 4, b = p, g = q/2, f = 1, c = - 15

Since the lines are parallel, $h^2 = ab$

∴ $(4)^2 = 2p$

∴ p = 8

Since the given equation represents a pair of lines

D = $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ where b = p = 8

i.e. $\begin{vmatrix} 2 & 4 & \frac{q}{2} \\ 4 & 8 & 1 \\ \frac{q}{2} & 1 & -15 \end{vmatrix} = 0$

i.e. $2(-120 - 1) - 4\left(-60 - \frac{q}{2}\right) + \frac{q}{2}(4 - 4q) = 0$

i.e. - 242 + 240 + 2q + 2q - $2q^2$ = 0

i.e. - $2q^2$ + 4q - 2 = 0

i.e. q^2 - 2q + 1 = 0

i.e. $(q - 1)^2$ = 0

∴ q - 1 = 0

∴ q = 1

Hence, p = 8 and q = 1

Exercise 4.3 | Q 9 | Page 128

Equations of pairs of opposite sides of a parallelogram are x^2 - 7x + 6 = 0 and y^2 - 14y + 40 = 0. Find the joint equation of its diagonals.

Solution: Let ABCD be the parallelogram such that the combined equation of sides AB and CD is x^2 - 7x + 6 = 0 and the combined equation of sides BC and AD y^2 - 14y + 40 = 0

The separate equations of the lines represented by x^2 - 7x + 6 = 0, i.e. (x - 1)(x - 6) = 0 are x - 1 = 0 and x - 6 = 0

Let equation of the side AB be x - 10 and equation of side CD be x - 6 = 0

The separate equations of the lines represented by y^2 - 14y + 40 = 0, i.e. (y - 4)(y - 10) = 0 are y - 4 = 0 and y - 10 = 0

Let equation of the side BC be y - 4 = 0 and equation of side AD be y - 10 = 0

Coordinates of the vertices of the parallelogram are A(1, 10), B(1, 4), C(6, 4) and D(6, 10)

∴ equation of the diagonal AC is

$$\frac{\mathrm{y - 10}}{\mathrm{x} - 1} = \frac{10 - 4}{1 - 6} = \frac{6}{-5}$$

∴ 5y + 50 = 6x - 6

∴ 6x + 5y - 56 = 0

and equation of the diagonal BD is

$$\frac{\mathrm{y} - 4}{\mathrm{x} - 1} = \frac{4 - 10}{1 - 6} = \frac{-6}{-5} = \frac{6}{5}$$

∴ 5y - 20 = 6x - 6

∴ 6x - 5y + 14 = 0

Hence, the equations of the diagonals are 6x + 5y - 56 = 0 and 6x - 5y + 14 = 0

∴ the joint equation of the diagonals is

(6x + 5y - 56)(6x - 5y + 14) = 0

$$\therefore 36\mathrm{x}^2 - 30\mathrm{xy} + 84\mathrm{x} + 30\mathrm{xy} - 25\mathrm{y}^2 + 70\mathrm{y} - 336\mathrm{x} + 280\mathrm{y} - 784 = 0$$

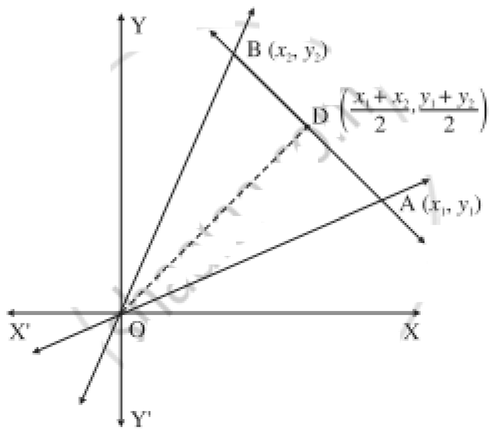
$$\therefore 36\mathrm{x}^2 - 25\mathrm{y}^2 - 252\mathrm{x} + 350\mathrm{y} - 784 = 0$$

[Note: Answer in the textbook is incorrect]

Exercise 4.3 | Q 10 | Page 128

ΔOAB is formed by the lines $x^2 - 4xy + y^2 = 0$ and the line AB. The equation of line AB is $2x + 3y - 1 = 0$. Find the equation of the median of the triangle drawn from O.

Solution:



Let D be the midpoint of seg AB where A is (x_1, y_1) and B is (x_2, y_2).

Then D has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The joint (combined) equation of the lines OA and OB is $x^2 - 4xy + y^2 = 0$ and the equation of the line AB is $2x + 3y - 1 = 0$

$\therefore$ points A and B satisfy the equations $2x + 3y - 1 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

We eliminate x from the above equations, i.e., put $x = \frac{1 - 3y}{2}$ in the equation $x^2 - 4xy + y^2 = 0$, we get,

$\therefore \left(\frac{1-3y}{2}\right)^2 - 4\left(\frac{1-3y}{2}\right)y + y^2 = 0$

$\therefore (1-3y)^2 - 8(1-3y)y + 4y^2 = 0$

$\therefore 1 - 6y + 9y^2 - 8y + 24y^2 + 4y^2 = 0$

$\therefore 37y^2 - 14y + 1 = 0$

The roots y_1 and y_2 of the above quadratic equation are the y-coordinates of the points A and B.

$\therefore y_1 + y_2 = -\frac{b}{a} = \frac{14}{37}$

$\therefore$ y-coordinate of D = $\frac{y_1 + y_2}{2} = \frac{7}{37}$

Since D lies on the line AB, we can find the x-coordinate of D as

$2x + 3\left(\frac{7}{37}\right) - 1 = 0$

$\therefore 2x = 1 - \frac{21}{37} = \frac{16}{37}$

$\therefore x = \frac{8}{37}$

$\therefore$ D is $\left(\frac{8}{37}, \frac{7}{37}\right)$

$\therefore$ equation of the median OD is $\frac{x}{8/37} = \frac{y}{7/37}$

i.e. $7x - 8y = 0$

Exercise 4.3 | Q 11 | Page 128

Find the coordinates of the points of intersection of the lines represented by $x^2 - y^2 - 2x + 1 = 0$

Solution: Consider, $x^2 - y^2 - 2x + 1 = 0$

$\therefore (x^2 - 2x + 1) - y^2 = 0$

$\therefore (x - 1)^2 - y^2 = 0$

$\therefore (x - 1 + y)(x - 1 - y) = 0$

$\therefore (x + y - 1)(x - y - 1) = 0$

∴ separate equations of the lines are

$x + y - 1 = 0$ and $x - y + 1 = 0$

To find the point of intersection of the lines, we have to solve

$x + y - 1 = 0$ ...(1)

and $x - y + 1 = 0$ ...(2)

Adding (1) and (2), we get,

$2x = 0$

$\therefore x = 0$

Substituting $x = 0$ in (1), we get,

$0 + y - 1 = 0$

$\therefore y = 1$

∴ coordinates of the point of intersection of the lines are (0, 1).

[**Note:** Answer in the textbook is incorrect.]

MISCELLANEOUS EXERCISE 4 [PAGES 129 - 130]

Miscellaneous Exercise 4 | Q 1.01 | Page 129

Choose correct alternatives:

If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then h = ________

1. ± 2
2. ± 3
3. **± 4**
4. ± 5

Solution: If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then h = **± 4.**

Miscellaneous Exercise 4 | Q 1.02 | Page 129

Choose correct alternatives:

If the lines represented by $kx^2 - 3xy + 6y^2 = 0$ are perpendicular to each other, then

1. k = 6
2. **k = - 6**
3. k = 3
4. k = - 3

Solution: k = - 6

Miscellaneous Exercise 4 | Q 1.03 | Page 129

Choose correct alternatives:

Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is

1. $2m^2 + 3m - 9 = 0$
2. **$9m^2 - 3m - 2 = 0$**
3. $2m^2 - 3m + 9 = 0$
4. $- 9m^2 - 3m + 2 = 0$

Solution: Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is **$9m^2 - 3m - 2 = 0$.**

Miscellaneous Exercise 4 | Q 1.04 | Page 129

Choose correct alternatives:

The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is 2

1. **2**
2. 1
3. 3
4. 4

Solution: The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is **2.**

Miscellaneous Exercise 4 | Q 1.05 | Page 129

Choose correct alternatives:

If two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then tan (α + β) = ______.

1. h/a+b
2. h/a-b
3. 2h/a+b

4. 2h/a-b

Solution:

If two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis,

then $\tan(\alpha + \beta) = \frac{2h}{a - b}$

Explanation:

$m_1 = \tan\alpha$, $m_2 = \tan\beta$

$$\therefore \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$= \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{-2h/b}{1 - (a/b)} = \frac{2h}{a - b}$$

Miscellaneous Exercise 4 | Q 1.06 | Page 129

Choose correct alternatives:

If the slope of one of the two lines given by $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$ is twice that of the other, then $ab : h^2$ = _____.

1. 1 : 2
2. 2 : 1
3. 8 : 9
4. 9 : 8

Solution:

If the slope of one of the two lines given by $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$ is twice that of the other, then $ab : h^2$ = **9 : 8.**

Explanation:

$m_1 + m_2 = \frac{-2b}{h}$ and $m_1 m_2 = \frac{b}{a}$

where $m_1 = 2m_2$

$\therefore 2m_2 + m_2 = -\frac{2b}{h}$ and $2m_2 \times m_2 = \frac{b}{a}$

$\therefore m2 = \frac{-2b}{3h}$ and $m_2^2 = \frac{b}{2a}$

$\therefore \left(\frac{-2b}{3h}\right)^2 = \frac{b}{2a}$

$\therefore \frac{4b^2}{9h^2} = \frac{b}{2a}$

$\therefore \frac{ab}{h^2} = \frac{9}{8}$

Miscellaneous Exercise 4 | Q 1.07 | Page 130

Choose correct alternatives:

The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is _______.

1. **$5x^2 + 4xy - 3y^2 = 0$**
2. $3x^2 + 4xy - 5y^2 = 0$
3. $3x^2 - 4xy + 5y^2 = 0$
4. $5x^2 + 4xy + 3y^2 = 0$

Solution: The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is **$\underline{5x^2 + 4xy - 3y^2 = 0}$.**

Miscellaneous Exercise 4 | Q 1.08 | Page 130

Choose correct alternatives:

If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, π/4, then $4h^2$ = _______.

1. $a^2 + 4ab + b^2$
2. **$a^2 + 6ab + b^2$**
3. $(a + 2b)(a + 3b)$
4. $(a - 2b)(2a + b)$

Solution:

If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{\pi}{4}$, then $4h^2$ = **$a^2 + 6ab + b^2$.**

Miscellaneous Exercise 4 | Q 1.09 | Page 130

Choose correct alternatives:

If the equation $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines, then the values of p and q are respectively.

1. - 3 and - 7
2. **- 7 and - 3**
3. 3 and 7
4. - 7 and 3

Solution: - 7 and - 3

Miscellaneous Exercise 4 | Q 1.1 | Page 130

Choose correct alternatives:

The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is

1. $4/\sqrt{3}$ sq units
2. **$8/\sqrt{3}$ sq units**
3. $16/\sqrt{3}$ sq units
4. $15/\sqrt{3}$ sq units

Solution: The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is **$8/\sqrt{3}$ sq units**

Miscellaneous Exercise 4 | Q 1.11 | Page 130

Choose correct alternatives:

The combined equation of the coordinate axes is

1. x + y = 0
2. xy = k
3. **xy = 0**
4. x - y = k

Solution: The combined equation of the coordinate axes is **xy = 0.**

Miscellaneous Exercise 4 | Q 1.12 | Page 130

Choose correct alternatives:

If $h^2 = ab$, then slopes of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio

1. 1:2
2. 2:1
3. 2:3
4. **1:1**

Solution: If $h^2 = ab$, then slopes of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio **1:1.**

Hint: If $h^2 = ab$, then lines are coincident. Therefore slopes of the lines are equal.

Miscellaneous Exercise 4 | Q 1.13 | Page 130

Choose correct alternatives:

If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2$ = _______

1. ab
2. 2ab
3. 7ab
4. **9ab**

Solution: If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2$ = **9ab.**

Miscellaneous Exercise 4 | Q 1.14 | Page 130

Choose correct alternatives:

If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then k = _______.

1. ± 3
2. $\pm 5\sqrt{5}$
3. 0
4. **$\pm 3\sqrt{5}$**

Solution: If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then $k = \pm 3\sqrt{5}$

Explanation:

$(x - 2y)^2 + k(x - 2y) = 0$

$\therefore (x - 2y)(x - 2y + k) = 0$

$\therefore$ equations of the lines are $x - 2y = 0$ and $x - 2y + k = 0$ which are parallel to each other.

$$\therefore \left|\frac{k-0}{\sqrt{1+4}}\right| = 3$$

$\therefore$ k = $\pm 3\sqrt{5}$

MISCELLANEOUS EXERCISE 4 [PAGES 130 - 132]

Miscellaneous Exercise 4 | Q 1.01 | Page 130

Find the joint equation of the line:

x - y = 0 and x + y = 0

Solution: Find the joint equation of the line x - y = 0 and x + y = 0 is

(x - y)(x + y) = 0

$\therefore$ $x^2 - y^2 = 0$

Miscellaneous Exercise 4 | Q 1.02 | Page 130

Find the joint equation of the line:

x + y - 3 = 0 and 2x + y - 1 = 0

Solution: Find the joint equation of the line x + y - 3 = 0 and 2x + y - 1 = 0

(x + y - 3)(2x + y - 1) = 0

$\therefore$ $2x^2 + xy - x + 2xy + y^2 - y - 6x - 3y + 3 = 0$

$\therefore$ $2x^2 + 3xy + y^2 - 7x - 4y + 3 = 0$

Miscellaneous Exercise 4 | Q 1.03 | Page 130

Find the joint equation of the line passing through the origin having slopes 2 and 3.

Solution: We know that the equation of the line passing through the origin and having slope m is y = mx. Equations of the lines passing through the origin and having slopes 2 and 3 are y = 2x and y = 3x respectively. i.e. their equations are

2x - y = 0 and 3x - y = 0 respectively.

$\therefore$ their joint equation is

(2x - y)(3x - y) = 0

$\therefore$ $6x^2 - 2xy - 3xy + y^2 = 0$

∴ $6x^2 - 5xy + y^2 = 0$

Miscellaneous Exercise 4 | Q 1.04 | Page 130

Find the joint equation of the line passing through the origin and having inclinations 60° and 120°.

Solution: Slope of the line having inclination θ is tan θ.

Inclinations of the given lines are 60° and 120°

∴ their slopes are $m_1 = \tan 60° = \sqrt{3}$ and

$m_2 = \tan 120° = \tan (180° - 60°)$

$= - \tan 60° = - \sqrt{3}$.

Since the lines pass through the origin, their equations are

$y = \sqrt{3}x$ and $y = -\sqrt{3}x$

i.e. $\sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$

∴ the joint equation of these lines is

$$\left(\sqrt{3}x - y\right)\left(\sqrt{3}x + y\right) = 0$$

∴ $3x^2 - y^2 = 0$

Miscellaneous Exercise 4 | Q 1.05 | Page 130

Find the joint equation of the line passing through (1, 2) and parallel to the coordinate axes

Solution: Equations of the coordinate axes are x = 0 and y = 0

∴ the equations of the lines passing through (1, 2) and parallel to the coordinate axes are x = 1 and y = 2.

i.e. x - 1 = 0 and y - 2 = 0

∴ their combined equation is

(x - 1)(y - 2) = 0

∴ x(y - 2) - 1(y - 2) = 0

∴ xy - 2x - y + 2 = 0

Miscellaneous Exercise 4 | Q 1.06 | Page 130

Find the joint equation of the line passing through (3, 2) and parallel to the lines x = 2 and y = 3.

Solution: Equations of the lines passing through (3, 2) and parallel to the lines x = 2 and y = 3 are x = 3 and y = 2.

i.e. x - 3 = 0 and y - 2 = 0

∴ their joint equation is

(x - 3)(y - 2) = 0

∴ xy - 2x - 3y + 6 = 0

Miscellaneous Exercise 4 | Q 1.07 | Page 131

Find the joint equation of the line passing through (-1, 2) and perpendicular to x + 2y + 3 = 0 and 3x - 4y - 5 = 0

Solution: Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines x + 2y + 3 = 0 and 3x - 4y - 5 = 0 respectively.

Slopes of the lines x + 2y + 3 = 0 and 3x - 4y - 5 = 0 are $-\frac{1}{2}$ and $-\frac{3}{-4} = \frac{3}{4}$ respectively.

∴ slopes of the lines L_1 and L_2 are 2 and $\frac{-4}{3}$ respectively.

Since the lines L_1 and L_2 pass through the point (-1, 2), their equations are

∴ $(y - y_1) = m(x - x_1)$

∴ (y - 2)= 2(x + 1)

⇒ y - 1 = 2x + 2

$\Rightarrow$ **2x - y + 4 = 0** and

$\therefore$ (y - 2) = $\left(\frac{-4}{3}\right)$(x + 1)

$\Rightarrow$ 3y - 6 = (- 4)(x + 1)

$\Rightarrow$ 3y - 6 = - 4x - 4

$\Rightarrow$ 4x + 3y - 6 + 4 = 0

$\Rightarrow$ **4x + 3y - 2 = 0**

their combined equation is

$\therefore$ (2x - y + 4)(4x + 3y - 2) = 0

$\therefore$ $8x^2 + 6xy - 4x - 4xy - 3y^2 + 2y + 16x + 12y - 8 = 0$

$\therefore$ $8x^2 + 2xy + 12x - 3y^2 + 14y - 8 = 0$

Miscellaneous Exercise 4 | Q 1.08 | Page 131

Find the joint equation of the line passing through the origin and having slopes 1 + √3 and 1 - √3

Solution:

Let l_1 and l_2 be the two lines. Slopes of l_1 is 1 + $\sqrt{3}$ and that of l_2 is 1 - $\sqrt{3}$

Therefore the equation of a line (l_1) passing through the origin and having slope is

y = $\left(1+\sqrt{3}\right)\mathrm{x}$

$\therefore$ $\left(1+\sqrt{3}\right)\mathrm{x}-\mathrm{y}=0$ ...(1)

Similarly, the equation of the line (l_2) passing through the origin and having slope is

y = $\left(1-\sqrt{3}\right)\mathrm{x}$

$\therefore \left(1-\sqrt{3}\right)x - y = 0$ ...(2)

From (1) and (2) the required combined equation is

$$\left[\left(1+\sqrt{3}\right)x - y\right]\left[\left(1-\sqrt{3}\right)x - y\right] = 0$$

$$\therefore \left(1+\sqrt{3}\right)x\left[\left(1-\sqrt{3}\right)x - y\right] - y\left[\left(1-\sqrt{3}\right)x - y\right] = 0$$

$$\therefore \left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)x^2 - \left(1+\sqrt{3}\right)xy - \left(1-\sqrt{3}\right)xy + y^2 = 0$$

$$\therefore \left((1)^2 - \left(\sqrt{3}\right)^2\right)x^2 - \left[\left(1+\sqrt{3}\right) + \left(1-\sqrt{3}\right)\right]xy + y^2 = 0$$

$$\therefore (1-3)x^2 - 2xy + y^2 = 0$$

$$\therefore -2x^2 - 2\ xy + y^2 = 0$$

$\therefore 2x^2 + 2xy - y^2 = 0$

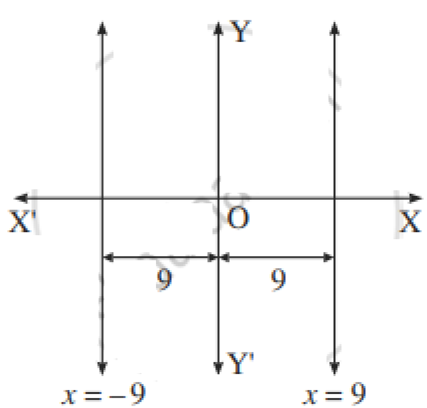
This is the required combined equation.

Miscellaneous Exercise 4 | Q 1.09 | Page 131

Find the joint equation of the line which are at a distance of 9 units from the Y-axis.

Solution: Equations of the lines, which are parallel to the Y-axis and at a distance of 9 units from it, are x = 9 and x = - 9

i.e. x - 9 = 0 and x + 9 = 0



∴ their combined equation is

$(x - 9)(x + 9) = 0$

$\therefore x^2 - 81 = 0$

Miscellaneous Exercise 4 | Q 1.1 | Page 131

Find the joint equation of the line passing through the point (3, 2), one of which is parallel to the line x - 2y = 2, and other is perpendicular to the line y = 3.

Solution:

Let L_1 be the line passes through (3, 2) and parallel to the line x - 2y = 2 whose slope is $\dfrac{-1}{-2} = \dfrac{1}{2}$

$\therefore$ slope of the line L_1 is $\dfrac{1}{2}$

$\therefore$ equation of the line L_2 is

$$y - 2 = \frac{1}{2}(x - 3)$$

$\therefore 2y - 4 = x - 3$

$\therefore x - 2y + 1 = 0$

Let L_2 be the line passes through (3, 2) and perpendicular to the line y = 3.

$\therefore$ equation of the line L_2 is of the form x = a. Since L_2 passes through (3, 2), 3 = a.

$\therefore$ equation of the line L_2 is x = 3, i.e. x - 3 = 0

Hence, the equations of the required lines are

x - 2y + 1 = 0 and x - 3 = 0

$\therefore$ their joint equation is

$(x - 2y + 1)(x - 3) = 0$

$\therefore x^2 - 2xy + x - 3x + 6y - 3 = 0$

$\therefore x^2 - 2xy - 2x + 6y - 3 = 0$

Miscellaneous Exercise 4 | Q 1.11 | Page 131

Find the joint equation of the line passing through the origin and perpendicular to the lines x + 2y = 19 and 3x + y = 18

Solution: Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines $x + 2y = 19$ and $3x + y = 18$ respectively.

Slopes of the lines $x + 2y = 19$ and $3x + y = 18$ are $-1/2$ and $-3/1 = -3$ respectively.

$\therefore$ slopes of the lines L_1 and L_2 are 2 and 1/3 respectively.

Since the lines L_1 and L_2 pass through the origin, their equations are

$y = 2x$ and $y = 1/3\ x$

i.e. $2x - y = 0$ and $x - 3y = 0$

$\therefore$ their combined equation is

$(2x - y)(x - 3y) = 0$

$\therefore 2x^2 - 6xy - xy + 3y^2 = 0$

$\therefore 2x^2 - 7xy + 3y^2 = 0$

Miscellaneous Exercise 4 | Q 2.1 | Page 131

Show that the following equations represents a pair of line:

$x^2 + 2xy - y^2 = 0$

Solution: Comparing the equation $x^2 + 2xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $2h = 2$ i,e, $h = 1$, and $b = - 1$

$\therefore h^2 - ab = (1)^2 - 1(- 1) = 1 + 2 = 2 > 0$

Since the equation $x^2 + 2xy - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.2 | Page 131

Show that the following equations represents a pair of line:

$4x^2 + 4xy + y^2 = 0$

Solution: Comparing the equation $4x^2 + 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 4$, $2h = 4$ i,e, $h = 2$, and $b = 1$

$\therefore h^2 - ab = (2)^2 - 4(1) = 4 - 4 = 0$

Since the equation $4x^2 + 4xy + y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab = 0$, the given equation represents a pair of lines which are real and coincident.

Miscellaneous Exercise 4 | Q 2.3 | Page 131

Show that the following equations represent a pair of line:

$x^2 - y^2 = 0$

Solution: Comparing the equation $x^2 - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 1, 2h = 0 i,e, h = 0, and b = - 1

$\therefore h^2 - ab = (0)^2 - 1(-1) = 0 + 1 = 1 > 0$

Since the equation $x^2 - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.4 | Page 131

Show that the following equations represent a pair of line:

$x^2 + 7xy - 2y^2 = 0$

Solution: Comparing the equation $x^2 + 7xy - 2y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 1, 2h = 7 i,e, $h = \frac{7}{2}$, and b = - 2

$$\therefore h^2 - ab = \left(\frac{7}{2}\right)^2 - 1(-2)$$

$$= \frac{49}{4} + 2$$

$$= \frac{57}{4} \text{ i.e. } 14.25 = 14 > 0$$

Since the equation $x^2 + 7xy - 2y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.5 | Page 131

Show that the following equations represent a pair of line:

$$x^2 - 2\sqrt{3}xy - y^2 = 0$$

Solution:

Comparing the equation $\mathbf{x^2 - 2\sqrt{3}xy - y^2 = 0}$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 1, 2h = $-2\sqrt{3}$ i,e, h = $\sqrt{3}$, and b = 1

$\therefore h^2 - ab = \left(\sqrt{3}\right)^2 - 1$ (1)

$= 3 - 1 = 2 > 0$

Since the equation $\mathbf{x^2 - 2\sqrt{3}xy - y^2 = 0}$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 3.1 | Page 131

Find the separate equation of the line represented by the following equation:

$6x^2 - 5xy - 6y^2 = 0$

Solution: $6x^2 - 5xy - 6y^2 = 0$

$\therefore 6x^2 - 9xy + 4xy - 6y^2 = 0$

$\therefore 3x(2x - 3y) + 2y(2x - 3y) = 0$

$\therefore (2x - 3y)(3x + 2y) = 0$

the separate equations of the lines are

$2x - 3y = 0$ and $3x + 2y = 0$.

Miscellaneous Exercise 4 | Q 3.2 | Page 131

Find the separate equation of the line represented by the following equation:

$x^2 - 4y^2 = 0$

Solution: $x^2 - 4y^2 = 0$

$\therefore x^2 - (2y)^2 = 0$

$\therefore (x - 2y)(x + 2y) = 0$

the separate equations of the lines are

$x - 2y = 0$ and $x + 2y = 0$

Miscellaneous Exercise 4 | Q 3.3 | Page 131

Find the separate equation of the line represented by the following equation:

$3x^2 - y^2 = 0$

Solution:

$3x^2 - y^2 = 0$

$$\therefore \left(\sqrt{3}x\right)^2 - y^2 = 0$$

$$\therefore \left(\sqrt{3}x - y\right)\left(\sqrt{3}x + y\right) = 0$$

the separate equations of the lines are

$\sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$

Miscellaneous Exercise 4 | Q 3.4 | Page 131

Find the separate equation of the line represented by the following equation:

$2x^2 + 2xy - y^2 = 0$

Solution: $2x^2 + 2xy - y^2 = 0$

The auxiliary equation is $- m^2 + 2m + 2 = 0$

$\therefore m^2 - 2m - 2 = 0$

$$\therefore m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4 + 8}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2}$$

$$= 1 \pm \sqrt{3}$$

$\therefore m_1 = 1 + \sqrt{3}$ and $m_2 = 1 - \sqrt{3}$ are the slopes of the lines.

$\therefore$ their separate equations are

$y = m_1x$ and $y = m_2x$

i.e. $y = \left(1+\sqrt{3}\right)x$ and $y = \left(1-\sqrt{3}\right)x$

i.e. $\left(\sqrt{3}+1\right)x - y = 0$ and $\left(\sqrt{3}-1\right)x + y = 0$

Miscellaneous Exercise 4 | Q 4.1 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$x^2 + 4xy - 5y^2 = 0$

Solution: Comparing the equation $x^2 + 4xy - 5y^2 = 0$ with $ax_2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $2h = 4$, $b = -5$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + 4xy - 5y^2 = 0$

$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{4}{5}$ and $m_1m_2 = \frac{a}{b} = \frac{-1}{5}$ ...(1)

Now, required lines are perpendicular to these lines

$\therefore$ their slopes are $\frac{-1}{m_1}$ and $\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

$\therefore$ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

$\therefore x^2 + \frac{4}{5}xy - \frac{1}{5}y^2 = 0$ ....[By(1)]

$\therefore 5x^2 + 4xy - y^2 = 0$

Miscellaneous Exercise 4 | Q 4.2 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$2x^2 - 3xy - 9y^2 = 0$

Solution: Comparing the equation $2x^2 - 3xy - 9y^2 = 0$ with $ax_2 + 2hxy + by^2 = 0$, we get,

$a = 2, 2h = -3, b = -9$

Let m_1 and m_2 be the slopes of the lines represented by $2x^2 - 3xy - 9y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{3}{9} \text{ and } m_1m_2 = \frac{a}{b} = -\frac{2}{9} \quad ...(1)$$

Now, required lines are perpendicular to these lines

$\therefore$ their slopes are $\frac{-1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

$\therefore$ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

$$\therefore x^2 + \left(-\frac{3}{9}\right)xy + \left(-\frac{2}{9}\right)y^2 = 0 \quad ....[By(1)]$$

$$\therefore 9x^2 - 3xy - 2y^2 = 0$$

Miscellaneous Exercise 4 | Q 4.3 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$x^2 + xy - y^2 = 0$

Solution: Comparing the equation $x^2 + xy - y^2 = 0$ with $ax_2 + 2hxy + by^2 = 0$, we get,

$a = 1, 2h = 1, b = -1$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + xy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{-1} = 1 \text{ and } m_1 m_2 = \frac{a}{b} = \frac{1}{-1} = -1 \quad ...(1)$$

Now, required lines are perpendicular to these lines

$$\therefore \text{ their slopes are } \frac{-1}{m_1} \text{ and } -\frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

i.e. $m_1 y = - x$ and $m_2 y = - x$

i.e. $x + m_1 y = 0$ and $x + m_2 y = 0$

$\therefore$ their combined equation is

$(x + m_1 y)(x + m_2 y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

$$\therefore x^2 + 1xy + (-1)y^2 = 0 \quad ....[\text{By}(1)]$$

$$\therefore x^2 + xy - y^2 = 0$$

Miscellaneous Exercise 4 | Q 5.1 | Page 131

Find k, if the sum of the slopes of the lines given by $3x^2 + kxy - y^2 = 0$ is zero.

Solution: Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 3$, $2h = k$, $b = -1$

Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-1} = k$$

Now, $m_1 + m_2 = 0$ ...(Given)

$\therefore k = 0$

Miscellaneous Exercise 4 | Q 5.2 | Page 131

Find k, if the sum of the slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is equal to their product.

Solution: Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 1, 2h = k, b = - 3

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-3} = \frac{k}{3}$$

$$m_1 m_2 = \frac{a}{b} = \frac{1}{-3} = \frac{-1}{3}$$

Now, $m_1 + m_2 = m_1 m_2$ ...(Given)

$$\therefore \frac{k}{3} = \frac{-1}{3}$$

$\therefore k = - 1.$

Miscellaneous Exercise 4 | Q 5.3 | Page 131

Find k, if the slope of one of the lines given by $3x^2 - 4xy + ky^2 = 0$ is 1.

Solution: The auxiliary equation of the lines given by $3x^2 - 4xy + ky^2 = 0$ is $km^2 - 4m + 3 = 0$

Given, slope of one of the lines is 1.

$\therefore$ m = 1 is the root of the auxiliary equation $km^2 - 4m + 3 = 0$

$\therefore k(1)^2 - 4(1) + 3 = 0$

$\therefore k - 4 + 3 = 0$

$\therefore k = 1$

Miscellaneous Exercise 4 | Q 5.4 | Page 131

Find k, if one of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the line $5x + 3y = 0$.

Solution:

The auxiliary equation of the lines given by $3x^2$ - kxy + $5y^2$ = 0 is $5m^2$ - km + 3 = 0

Now, one line is perpendicular to the line 5x + 3y = 0, whose slope is $-\frac{5}{3}$

∴ slope of that line = m = $\frac{3}{5}$

∴ m = $\frac{3}{5}$ is the root of the auxiliary equation $5m^2$ - km + 3 = 0

$$\therefore 5\left(\frac{3}{5}\right)^2 - k\left(\frac{3}{5}\right) + 3 = 0$$

$$\therefore \frac{9}{5} - \frac{3k}{5} + 3 = 0$$

∴ 9 - 3k + 15 = 0

∴ 3k = 24

∴ k = 8

Miscellaneous Exercise 4 | Q 5.5 | Page 131

Find k if the slope of one of the lines given by $3x^2 + 4xy + ky^2 = 0$ is three times the other.

Solution: $3x^2 + 4xy + ky^2 = 0$

∴ divide by x^2

$$\frac{3x^2}{x^2} + \frac{4xy}{x^2} + \frac{ky^2}{x^2} = 0$$

$$3 + \frac{4y}{x} + \frac{ky^2}{x^2} = 0 \quad ....(1)$$

∴ y = mx

$$\therefore \frac{y}{x} = m$$

Put $\frac{y}{x} = m$ in equation (1)

Comparing the equation $km^2 + 4m + 3 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, a = k , 2h = 4, b = 3

$m_1 = 3m_2$ ...(given condition)

$m_1 + m_2 = \frac{-2h}{k} = -\frac{4}{k}$

$m_1 m_2 = \frac{a}{b} = \frac{3}{k}$

$m_1 + m_2 = -\frac{4}{k}$

$4m_2 = -\frac{4}{k}$ .....$(m_1 = 3m_2)$

$m_2 = -\frac{1}{k}$

$m_1 m_2 = \frac{3}{k}$

$3m_2^2 = \frac{3}{k}$ ....$(m_1 = 3m_2)$

$3\left(-\frac{1}{k}\right)^2 = \frac{3}{k}$ ......$(m_2 = -\frac{1}{k})$

$\frac{1}{k^2} = \frac{1}{k}$

$k^2 = k$

k = 1 or k = 0

Miscellaneous Exercise 4 | Q 5.6 | Page 131

Find k, if the slopes of lines given by $kx^2 + 5xy + y^2 = 0$ differ by 1.

Solution: Comparing the equation $kx^2 + 5xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = k, 2h = 5 \text{ i.e. } h = \frac{5}{2}$

$m_1 + m_2 = \frac{-2h}{b} = -\frac{5}{1} = -5$

and $m_1 m_2 = \frac{a}{b} = \frac{k}{1} = k$

the slope of the line differ by $(m_1 - m_2) = 1$ ....(1)

$\therefore (m_1 - m_2)^2 = (m_1 + m2)^2 - 4m_1m_2$

$(m_1 - m_2)^2 = (-5)^2 - 4(k)$

$(m_1 - m_2)^2 = 25 - 4k$

$1 = 25 - 4k$ .....[By(1)]

$4k = 24$

$k = 6$

Miscellaneous Exercise 4 | Q 5.7 | Page 131

Find k, if one of the lines given by $6x^2 + kxy + y^2 = 0$ is $2x + y = 0$.

Solution: The auxiliary equation of the lines represented by $6x^2 + kxy + y^2 = 0$ is $m^2 + km + 6 = 0$

Since one of the line is $2x + y = 0$ whose slope is $m = -2$.

$\therefore m = -2$ is the root of the auxiliary equation $m^2 + km + 6 = 0$.

$\therefore (-2)^2 + k(-2) + 6 = 0$

$\therefore 4 - 2k + 6 = 0$

$\therefore 2k = 10$

$\therefore k = 5$

Miscellaneous Exercise 4 | Q 6 | Page 131

Find the joint equation of the pair of lines which bisect angles between the lines given by $x^2 + 3xy + 2y^2 = 0$

Solution: $x^2 + 3xy + 2y^2 = 0$

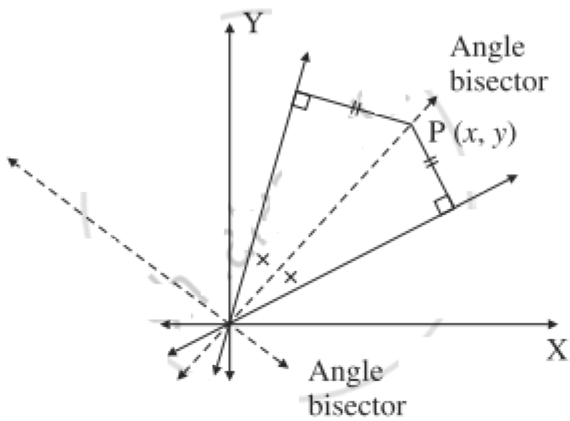
$\therefore x^2 + 2xy + xy + 2y^2 = 0$

$\therefore x(x + 2y) + y(x + 2y) = 0$

$\therefore$ (x + 2y)(x + y) = 0

$\therefore$ separate equations of the lines represented by $x^2 + 3xy + 2y^2 = 0$ are x + 2y = 0 and x + y = 0

Let P (x, y) be any point on one of the angle bisector. Since the points on the angle bisectors are equidistant from both the lines,



the distance of P(x, y) from the line x + 2y = 0

= the distance of P(x, y) from the line x + y = 0

$$\therefore \left|\frac{x + 2y}{\sqrt{1+4}}\right| = \left|\frac{x + y}{\sqrt{1+1}}\right|$$

$$\therefore \frac{(x + 2y)^2}{5} = \frac{(x + y)^2}{2}$$

$\therefore 2(x + 2y)^2 = 5(x + y)^2$

$\therefore 2(x^2 + 4xy + 4y^2) = 5(x^2 + 2xy + y^2)$

$\therefore 2x^2 + 8xy + 8y^2 = 5x^2 + 10xy + 5y^2$

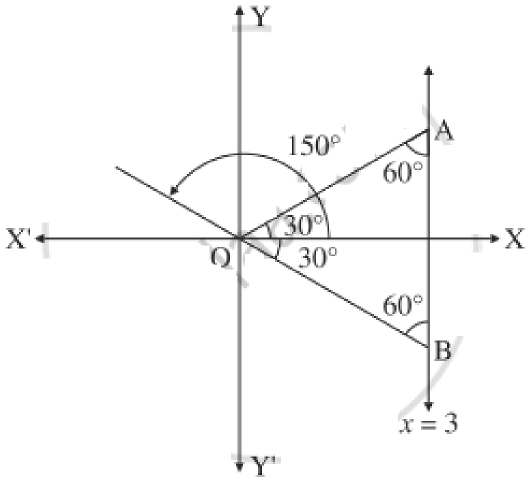
$\therefore 3x^2 + 2xy - 3y^2 = 0$

This is the required joint equation of the lines which bisect the angles between the lines represented by $x^2 + 3xy + 2y^2 = 0$

Miscellaneous Exercise 4 | Q 7 | Page 131

Find the joint equation of the pair of lines through the origin and making an equilateral triangle with the line x = 3.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the line x = 3.

∴ OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

∴ slope of OA = tan 30° = $\frac{1}{\sqrt{3}}$

∴ equation of the line OA is y = $\frac{1}{\sqrt{3}}$x

∴ $\sqrt{3}y = x$

∴ x - $\sqrt{y} = 0$

Slope of OB = tan 150° = tan (180° - 30°)

= - tan 30° = - $\frac{1}{\sqrt{3}}$

∴ equation of the line OB is y = $\frac{-1}{\sqrt{3}}$x

∴ $\sqrt{3}y = -x$

∴ x + $\sqrt{3}y = 0$

∴ required combined equation of the lines is

$\left(x - \sqrt{3}y\right)\left(x + \sqrt{3}y\right) = 0$

i.e. $x^2 - 3y^2 = 0$

Miscellaneous Exercise 4 | Q 8 | Page 131

Show that the lines x^2 - 4xy + y^2 = 0 and x + y = 10 contain the sides of an equilateral triangle. Find the area of the triangle.

Solution: We find the joint equation of the pair of lines OA and OB through origin, each making an angle of 60° with x + y = 10 whose slope is - 1.

Let OA(or OB) has slope m.

∴ its equation is y - mx ....(1)

$$\text{Also, } \tan 60° = \left|\frac{m-(-1)}{1+m(-1)}\right|$$

$$\therefore \sqrt{3} = \left|\frac{m+1}{1-m}\right|$$

Squaring both sides, we get,

$$3 = \frac{(m+1)^2}{(1-m)^2}$$

∴ 3(1 - 2m + m^2) = m^2 + 2m + 1

∴ 3 - 6m + 3m^2 = m^2 + 2m + 1

∴ 2m^2 - 8m + 2 = 0

∴ m^2 - 4m + 1 = 0

$$\therefore \left(\frac{y}{x}\right) - 4\left(\frac{y}{x}\right) + 1 = 0 \quad \text{...[By(1)]}$$

∴ y^2 - 4xy + x^2 = 0

∴ x^2 - 4xy + y^2 = 0 is the joint equation of the two lines through the origin each making an angle of 60° with x + y = 10

∴ x^2 - 4xy + y^2 = 0 and x + y = 10 form a triangle OAB which is equilateral.

Let seg OM perpendicular line AB whose question is x + y = 10

$\therefore$ OM = $\left|\frac{-10}{\sqrt{1+1}}\right| = 5\sqrt{2}$

$\therefore$ area of equilateral Δ OAB $= \frac{(\text{OM})^2}{\sqrt{3}} = \frac{\left(5\sqrt{2}\right)^2}{\sqrt{3}}$

$= \frac{50}{\sqrt{3}}$ sq units.

Miscellaneous Exercise 4 | Q 9 | Page 131

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is three times the other, prove that $3h^2 = 4ab$.

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$

We are given that $m_2 = 3m_1$

$$\therefore m_1 + 3m_1 = -\frac{2h}{b}$$

$$\therefore 4m_1 = -\frac{2h}{b}$$

$$\therefore m_1 = -\frac{h}{2b}$$

Also, $m_1(3m_1) = \frac{a}{b}$

$$\therefore 3m_1^2 = \frac{a}{b}$$

$$\therefore 3\left(-\frac{h}{2b}\right)^2 = \frac{a}{b} \quad \text{....[By (1)]}$$

$$\therefore \frac{3h^2}{4b^2} = \frac{a}{b}$$

$\therefore 3h^2 = 4ab$, as $b \neq 0$

Miscellaneous Exercise 4 | Q 10 | Page 132

Find the combined equation of bisectors of angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

Solution: Comparing the equation $5x^2 + 6xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 5$, $2h = 6$, $b = -1$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 6xy - y^2 = 0$.

$$\therefore m1 + m2 = \frac{-2h}{b} = \frac{-6}{-1} = 6 \text{ and } m_1 m_2 = \frac{a}{b} = \frac{5}{-1} = -5 \quad ...(1)$$

The separate equations of the lines are

$y = m_1x$ and $y = m_2x$, where $m_1 \neq m_2$

i.e. $m_1x - y = 0$ and $m_2x - y = 0$.

Let P(x, y) be any point on one of the bisector of the angles between the lines.

$\therefore$ the distance of P from the line $m_1x - y = 0$ is equal to the distance of P from the line $m_2x - y = 0$.

$$\therefore \left|\frac{m_1x - y}{\sqrt{m_1^2 + 1}}\right| = \left|\frac{m_2x - y}{\sqrt{m_2^2 + 1}}\right|$$

Squaring both sides, we get,

$$\frac{(m_1x - y)^2}{m_1^2 + 1} = \frac{m_2x - y}{m_2^2 + 1}$$

$$\therefore \left(m_2^2 + 1\right)(m_1x - y)^2 = \left(m_1^2 + 1\right)(m_2x - y)$$

$$\therefore \left(m_2^2 + 1\right)\left(m_1^2x^2 - 2m_1xy + y^2\right) = \left(m_1^2 + 1\right)\left(m_2^2x^2 - 2m_2xy + y^2\right)$$

$$\therefore m_1^2m_2^2x^2 - 2m_1m_2^2y^2xy + m_2^2y^2 + m_1^2x^2 - 2m_1xy + y^2 = m_1^2m_2^2x^2 - 2m_1^2m_2xy + m_1^2y^2 + m_2^2x^2 - 2m_2xy + y^2$$

$$\therefore (m_1^2 - m_2^2)x^2 + 2m_1m_2(m_1 - m_2)xy - 2(m_1 - m_2)xy - (m_1^2 - m_2^2)y^2 = 0$$

Dividing throughout by $m_1 - m_2$ ($\neq 0$), we get,

$$(m_1 + m_2)x^2 + 2m_1m_2xy - 2xy - (m_1 + m_2)y^2 = 0$$

$\therefore 6x^2 - 10xy - 2xy - 6y^2 = 0$ ...[By (1)]

$\therefore 6x^2 - 12xy - 6y^2 = 0$

$\therefore x^2 - 2xy - y^2 = 0$

This is the joint equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

Miscellaneous Exercise 4 | Q 11 | Page 132

Find an if the sum of the slope of lines represented by $ax^2 + 8xy + 5y^2 = 0$ is twice their product.

Solution: Comparing the equation $ax^2 + 8xy + 5y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, a = a, 2h = 8, b = 5

Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 8xy + 5y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{8}{5}$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{a}{5}$$

Now, $(m_1 + m_2) = 2(m_1 m_2)$

$$-\frac{8}{5} = 2\left(\frac{a}{5}\right)$$

a = 4

Miscellaneous Exercise 4 | Q 12 | Page 132

If the line 4x - 5y = 0 coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$, then show that 25a + 40h + 16b = 0

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.

Given that 4x - 5y = 0 is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

The slope of the line 4x - 5y = 0 is $\frac{-4}{-5} = \frac{4}{5}$

$\therefore$ m = $\frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$$\therefore \mathrm{b}\left(\frac{4}{5}\right)^2 + 2\mathrm{h}\left(\frac{4}{5}\right) + \mathrm{a} = 0$$

$$\therefore \frac{16\mathrm{b}}{25} + \frac{8\mathrm{h}}{5} + \mathrm{a} = 0$$

$\therefore$ 16b + 40h + 25a = 0 i.e.

$\therefore$ 25a + 40h + 16b = 0

Miscellaneous Exercise 4 | Q 13.1 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

Solution: Comparing this equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

a = 9, h = -3, b = 1, g = 9, f = - 3 and c = 8

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 9 & -3 & 9 \\ -3 & 1 & -3 \\ 9 & -3 & 8 \end{vmatrix}$$

= 9(8 - 9) + 3(- 24 + 27) + 9(9 - 9)

= 9(-1) + 3(3) + 9(0)

= - 9 + 9 + 0 = 0

and $h^2 - ab = (-3)^2 - 9(1) = 9 - 9 = 0$

∴ the given equation represents a pair of lines.

Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{(-3)^2 - 9(1)}}{10}\right|$$

$$= \left|\frac{2\sqrt{9 - 9}}{10}\right| = 0$$

$\therefore \tan\theta = \tan 0°$

$\therefore \theta = 0°$.

Miscellaneous Exercise 4 | Q 13.2 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$2x^2 + xy - y^2 + x + 4y - 3 = 0$

Solution: Comparing this equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$a = 2,\ h = \frac{1}{2},\ b = -1,\ g = \frac{1}{2},\ f = 2$ and $c = -3$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$$

$= 2(3-4) - \frac{1}{2}\left(-\frac{3}{2} - 1\right) + \frac{1}{2}\left(1 + \frac{1}{2}\right)$

$= -2 + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$

$= -2 + 1 + 1$

= - 2 + 2 = 0

∴ the given equation represents a pair of lines.

Let θ be the acute angle between the lines.

∴ $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$

$= \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - (2)(-1)}}{2-1}\right|$

$= \left|\frac{2\sqrt{\frac{1}{4} + 2}}{1}\right|$

$= 2\sqrt{\frac{9}{4}} = 3$

∴ tan θ = tan 3

∴ $\theta = \tan^{-1}(3)$

Miscellaneous Exercise 4 | Q 13.3 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$(x - 3)^2 + (x - 3)(y - 4) - 2(y - 4)^2 = 0$

Solution: Put x - 3 = X and y - 4 = Y in the given equation, we get,

$X^2 + XY - 2Y^2 = 0$

Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $h = 1/2$, $b = -2$

This is the homogeneous equation of second degree

and $h^2 - ab = \left(\frac{1}{2}\right)^2 - 1(-2)$

$$= \frac{1}{4} + 2 = \frac{9}{4} > 0$$

Hence, it represents a pair of lines passing through the new origin (3, 4).

Let θ be the acute angle between the lines.

$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

here $a = 1$, $2h = 1$, i.e. $h = \frac{1}{2}$ and $b = -2$

$\therefore \tan\theta = \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 1(-2)}}{1 - 2}\right|$

$$= \left|\frac{2\left(\sqrt{\frac{1}{4} + 2}\right)}{-1}\right|$$

$$= \left|\frac{2 \times \frac{3}{2}}{-1}\right|$$

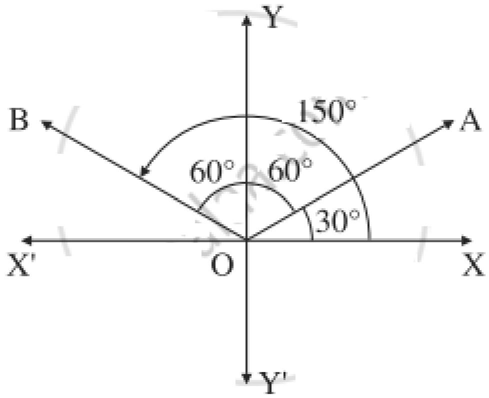
$\therefore \tan\theta = 3$

$\therefore \theta = \tan^{-1}(3)$

Miscellaneous Exercise 4 | Q 14 | Page 132

Find the combined equation of lines passing through the origin and each of which making an angle of 60° with the Y-axis.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.

Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

$\therefore$ slope of OA = tan 30° = $\frac{1}{\sqrt{3}}$

$\therefore$ equation of the line OA is

y = $\frac{1}{\sqrt{3}}$ x i.e. $x - \sqrt{3}y = 0$

Slope of OB = tan 150° = tan (180° - 30°)

= - tan 30° = $-\frac{1}{\sqrt{3}}$

$\therefore$ equation of the line OB is

y = $-\frac{1}{\sqrt{3}}x$ i.e. x + $\sqrt{3}$y = 0

$\therefore$ required combined equation is

$$\left(x - \sqrt{3}y\right)\left(x + \sqrt{3}y\right) = 0$$

i.e. $x^2 - 3y^2 = 0$

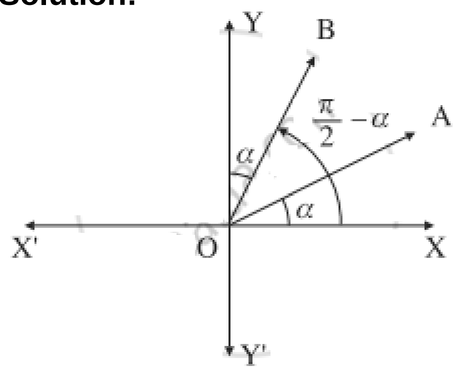
Miscellaneous Exercise 4 | Q 15 | Page 132

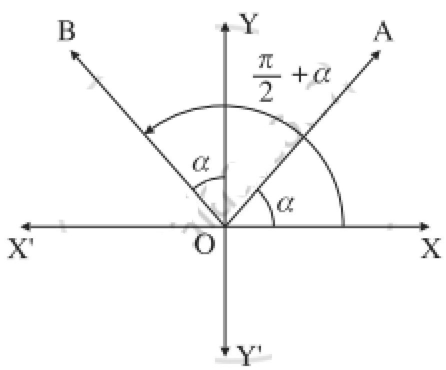
If the lines represented by $ax^2 + 2hxy + by^2 = 0$ make angles of equal measure with the coordinate axes, then show that a ± b.

OR

Show that, one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ will make an angle of the same measure with the X-axis as the other makes with the Y-axis, if $a = \pm b$.

Solution:





Let OA and OB be the two lines through the origin represented by $ax^2 + 2hxy + by^2 = 0$.

Since these lines make angles of equal measure with the coordinate axes, they make angles α and $\pi/2 - \alpha$ with the positive direction of X-axis or α and $\pi/2 + \alpha$ with the positive direction of X-axis.

∴ slope of the line OA = $m_1 = \tan \alpha$

and slope of the line OB = m_2

$$= \tan\left(\frac{\pi}{2} - \alpha\right) \text{or} \tan\left(\frac{\pi}{2} + \alpha\right)$$

i.e. $m_2 = \cot \alpha$ or $m_2 = -\cot \alpha$

∴ $m_1 m_2 = \tan \alpha \times \cot \alpha = 1$

OR $m_1m_2 = \tan\alpha\,(-\cot\alpha) = -1$

i.e. $m_1m_2 = \pm 1$

But $m_1m_2 = ab$

$\therefore a/b = \pm 1$

$\therefore a = \pm b$

This is the required condition.

Miscellaneous Exercise 4 | Q 16 | Page 132

Show that the combined equation of the pair of lines passing through the origin and each making an angle α with the line $x + y = 0$ is $x^2 + 2(\sec 2\alpha)xy + y^2 = 0$

Solution: Let OA and OB be the required lines.

Let OA (or OB) has slope m.

$\therefore$ its equation is $y = mx$ ...(1)

It makes an angle α with $x + y = 0$ whose slope is -1.

$$\therefore \tan\alpha = \left|\frac{m+1}{1+m(-1)}\right|$$

Squaring both sides, we get,

$$\tan^2\alpha = \frac{(m+1)^2}{(1-m)^2}$$

$\therefore \tan^2\alpha\,(1 - 2m + m^2) = m^2 + 2m + 1$

$\therefore \tan^2\alpha - 2m\tan^2\alpha + m^2\tan^2\alpha = m^2 + 2m + 1$

$\therefore (\tan^2\alpha - 1)m^2 - 2(1 + \tan^2\alpha)m + (\tan^2\alpha - 1) = 0$

$$\therefore m^2 - 2\left(\frac{1+\tan^2\alpha}{\tan^2\alpha - 1}\right)m + 1 = 0$$

$$\therefore m^2 + 2\left(\frac{1+\tan^2\alpha}{1 - \tan^2\alpha}\right)m + 1 = 0$$

$$\therefore m^2 + 2(\sec 2\alpha)m + 1 = 0 \ldots \left[\because \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$$

$$\therefore \frac{y^2}{x^2} + 2(\sec 2\alpha)\frac{y}{x} + 1 = 0$$

$$\therefore y^2\ 2xy\ \sec 2\alpha + x^2 = 0 \quad \text{...[By (1)]}$$

$$\therefore y^2 + 2xy\sec 2\alpha + x^2 = 0$$

$\therefore x^2 + 2(\sec 2\alpha)xy + y^2 = 0$ is the required equation.

Miscellaneous Exercise 4 | Q 17 | Page 132

Show that the line 3x + 4y + 5 = 0 and the lines $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$ form the sides of an equilateral triangle.

Solution: The slope of the line 3x + 4y + 5 = 0 is $m_1 = -3/4$

Let m be the slope of one of the line making an angle of 60° with the line 3x + 4y + 5 = 0. The angle between the lines having slope m and m_1 is 60°.

$\therefore$ tan 60° = $\left|\frac{m - m_1}{1 + m.m_1}\right|$, where tan 60° = $\sqrt{3}$

$$\therefore \sqrt{3} = \left|\frac{m - \left(-\frac{3}{4}\right)}{1 + m\left(-\frac{3}{4}\right)}\right|$$

$$\therefore \sqrt{3} = \left|\frac{4m + 3}{4 - 3m}\right|$$

On squaring both sides, we get,

$$3 = \frac{(4m + 3)^2}{(4 - 3m)^2}$$

∴ $3(4 - 3m)^2 = (4m + 3)^2$

∴ $3(16 - 24m + 9m^2) = 16m^2 + 24m + 9$

∴ $48 - 72m + 27m^2 = 16m^2 + 24m + 9$

∴ $11m^2 - 96m + 39 = 0$

This is the auxiliary equation of the two lines and their joint equation is obtained by putting m = y/x.

∴ the combined equation of the two lines is

$$11\left(\frac{y}{x}\right)^2 - 96(y'/x) + 39 = 0$$

$$\therefore \frac{11y^2}{x^2} - \frac{96y}{x} + 39 = 0$$

∴ $11y^2 - 96xy + 39x^2 = 0$

∴ $39x^2 - 96xy + 11y^2 = 0$

∴ $39x^2 - 96xy + 11y^2 = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with the line 3x + 4y + 5 = 0

The equation $39x^2 - 96xy + 11y^2 = 0$ can be written as: $- 39x^2 + 96xy - 11y^2 = 0$

$$\text{i.e. } \left(9x^2 - 48x^2\right) + (24xy + 72xy) + \left(16y^2 - 27y^2\right) = 0$$

$$\text{i.e. } \left(9x^2 + 24xy + 16y^2\right) - 3\left(16x^2 - 24xy + 9y^2\right) = 0$$

i.e. $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$

Hence, the line 3x + 4y + 5 = 0 and the lines $(3x + 4y)^2 - 3(4x - 3y)^2$ form the sides of an equilateral triangle.

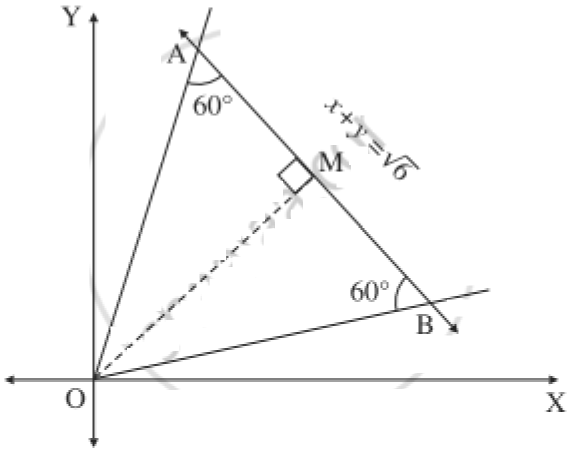
Miscellaneous Exercise 4 | Q 18 | Page 132

Show that the lines $x^2 - 4xy + y^2 = 0$ and the line x + y = √6 form an equilateral triangle.

Find its area and perimeter.

Solution: $x^2 - 4xy + y^2 = 0$ and x + y = √6 form a triangle OAB which is equilateral.

Let OM be the perpendicular from the origin O to AB whose equation is x + y = √6



$$\therefore \text{OM} = \left|\frac{-\sqrt{6}}{\sqrt{1+1}}\right| = \sqrt{3}$$

$$\therefore \text{ area of } \Delta \text{ OAB} = \frac{\text{OM}^2}{\sqrt{3}}$$

$$= \frac{\left(\sqrt{3}\right)^2}{\sqrt{3}} = \sqrt{3} \text{ sq.units.}$$

In right-angled triangle OAM,

$$\sin 60° = \frac{\text{OM}}{\text{OA}}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{\text{OA}}$$

∴ length of the each side of the equilateral triangle OAB = 2 units.

∴ perimeter of Δ OAB = 3 × length of each side

= 3 × 2 = 6 units

Miscellaneous Exercise 4 | Q 19 | Page 132

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is square of the slope of the other line, show that $a^2b + ab^2 + 8h^3 = 6abh$.

Solution: Let m be the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

Then the other line has slope m^2

$\therefore m + m^2 = \frac{-2h}{b}$ ....(1) and

$(m)(m^2) = \frac{a}{b}$

i.e. $m^3 = \frac{a}{b}$ ....(2)

$\therefore (m + m^2)^3 = m^3 + (m^2)^3 + 3(m)(m^2)(m + m^2)$ .....[$\because (p + q)^3 = p^3 + q^3 + 3pq(p + q)$]

$$\therefore \left(\frac{-2h}{b}\right)^3 = \frac{a}{b} + \frac{a^2}{b^2} + 3\frac{a}{b}\left(\frac{-2h}{b}\right)$$

$$\therefore \frac{-8h^3}{b^3} = \frac{a}{b} + \frac{a^2}{b^2} - \frac{6ah}{b^2}$$

Multiplying by b^3, we get,

$-8h^3 = ab^2 + a^2b - 6abh$

$\therefore a^2b + ab^2 + 8h^3 = 6abh$

This is the required condition.

Miscellaneous Exercise 4 | Q 20 | Page 132

Prove that the product of length of perpendiculars drawn from $P(x_1, y_1)$ to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$$\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{\overline{a - b}^2 + 4h^2}}\right|$$

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$

$\therefore$ $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$ ...(1)

The separate equations of the lines represented by $ax^2 + 2hxy + by^2 = 0$ are

$y = m_1x$ and $y = m_2x$

i.e. $m_1x - y = 0$ and $m_2x - y = 0$

Length of perpendicular from $P(x_1, y_1)$ on

$m_1x - y = 0$ is $\left|\frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}}\right|$

Length of perpendicular form $P(x_1, y_1)$ on

$m_2x - y = 0$ is $\left|\frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}}\right|$

$\therefore$ product of lengths of perpendiculars

$$= \left|\frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}}\right| \times \left|\frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}}\right|$$

$$= \left|\frac{m_1m_2x_1^2 - (m_1 + m_2)x_1y_1 + y_1^2}{\sqrt{m_1^2m_2^2 + m_1^2 + m_2^2 + 1}}\right|$$

$$= \frac{m_1 m_2 x_1^2 - (m_1 + m_2)x_1 y_1 + y_1^2}{\sqrt{m_1^2 m_2^2 + (m_1 + m_2)^2 - 2m_1 m_2 + 1}}$$

$$= \left| \frac{\frac{a}{b} \cdot x_1^2 - \frac{-2h}{b} x_1 y_1 + y_1^2}{\sqrt{\frac{a^2}{b^2} + \frac{-2h}{b} - \frac{2a}{b} + 1}} \right| \quad \text{...(By (1))}$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{a^2 + 4h^2 - 2ab + b^2}} \right|$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{(a^2 - 2ab + b^2) + 4h^2}} \right|$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}} \right|$$

Miscellaneous Exercise 4 | Q 21 | Page 132

Show that the difference between the slopes of the lines given by $(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan\theta + (\sin^2\theta)y^2 = 0$ is two.

Solution: Comparing the equation

$(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan\theta + (\sin^2\theta)y^2 = 0$

with $ax^2 + 2hxy + by^2 = 0$, wew get,

$a = \tan^2\theta + \cos^2\theta$,
$2h = -2\tan\theta$
$b = \sin^2\theta$

Let m_1 and m_2 be the slopes of the lines represented by the given equation.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\left[\frac{-\,2\tan\theta}{\sin^2\theta}\right] = \frac{2\tan\theta}{\sin^2\theta} \quad ....(1)$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{\tan^2\theta + \cos^2\theta}{\sin^2}\theta \quad .....(2)$$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$= \left(\frac{2\tan\theta}{\sin^2\theta}\right)^2 - 4\left(\frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{4\tan^2\theta}{\sin^4\theta} - 4\left(\frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{4\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{\sin^4\theta} - 4\left[\frac{\left(\frac{\sin^2\theta}{\cos^2\theta} + \cos^2\theta\right)}{\sin^2\theta}\right]$$

$$= \frac{4}{\sin^2\theta.\cos^2\theta} - \frac{4\left(\sin^2\theta + \cos^4\theta\right)}{\sin^2\theta.\cos^2\theta}$$

$$= 4\left[\frac{1 - \sin^2\theta - \cos^4\theta}{\sin^2\theta.\cos^2\theta}\right]$$

$$= 4\left[\frac{\cos^2\theta - \cos^4\theta}{\sin^2\theta.\cos^2\theta}\right]$$

$$= 4\left[\frac{\cos^2\theta(1-\cos^2\theta)}{\sin^2\theta.\cos^2\theta}\right] = 4$$

∴ $|m_1 - m_2| = 2$

∴ the slopes differ by 2.

Miscellaneous Exercise 4 | Q 22 | Page 132

Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.

Solution: Comparing the equation $ay^2 + bxy + ex + dy = 0$ with $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, we get,

$A = 0$, $H = \frac{b}{2}$, $B = a$, $G = \frac{e}{2}$, $F = \frac{d}{2}$, $C = 0$

The given equation represents a pair of lines,

$$\text{if} \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\text{i.e. if} \begin{vmatrix} 0 & \frac{b}{2} & \frac{e}{2} \\ \frac{b}{2} & a & \frac{d}{2} \\ \frac{e}{2} & \frac{d}{2} & 0 \end{vmatrix} = 0$$

$$\text{i.e. if } 0 - \frac{b}{2}\left(0 - \frac{ed}{4}\right) + \frac{e}{2}\left(\frac{bd}{4} - \frac{ae}{2}\right) = 0$$

$$\text{i.e. if } \frac{bed}{8} + \frac{bed}{8} - \frac{ae^2}{4} = 0$$

i.e. if $bed - ae^2 = 0$

i.e. if $e(bd - ae) = 0$

i.e. if $e = 0$ or $bd - ae = 0$

i.e. if $e = 0$ or $bd = ae$

This is the required condition.

Miscellaneous Exercise 4 | Q 23 | Page 132

If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, show that $(3a + b)(a + 3b) = 4h^2$.

Solution: Since the lines $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, the angle between the lines

$ax^2 + 2hxy + by^2 = 0$ is 60°.

$$\therefore \tan 60° = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$\therefore \sqrt{3} = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$\therefore 3(a + b)^2 = 4(h^2 - ab)$

$\therefore 3(a^2 + 2ab + b^2) = 4h^2 - 4ab$

$\therefore 3a^2 + 6ab + 3b^2 + 4ab = 4h^2$

$\therefore 3a^2 + 10ab + 3b^2 = 4h^2$

$\therefore 3a^2 + 9ab + ab + 3b^2 = 4h^2$

$\therefore 3a(a + 3b) + b(a + 3b) = 4h^2$

$\therefore (3a + b)(a + 3b) = 4h^2$

This is the required condition.

Miscellaneous Exercise 4 | Q 24 | Page 132

If the line $x + 2 = 0$ coincides with one of the lines represented by the equation $x^2 + 2xy + 4y + k = 0$, then prove that $k = -4$.

Solution: One of the lines represented by $x^2 + 2xy + 4y + k = 0$ is $x + 2 = 0$. .....(1)

Let the other line represented by (1) be $ax + by + c = 0$

$\therefore$ their combined equation is $(x + 2)(ax + by + c) = 0$

$\therefore ax^2 + bxy + cx + 2ax + 2by + 2c = 0$

$\therefore ax^2 + bxy + (2a + c)x + 2by + 2c = 0$ ...(2)

As the equations (1) and (2) are the combined equations of the same two lines, they are identical.

∴ by comparing their corresponding coefficients, we get,

$$\frac{a}{1} = \frac{b}{2} = \frac{2b}{4} = \frac{2c}{k} \text{ and } 2a + c = 0$$

$$\therefore a = \frac{2c}{k} \text{ and } c = -2a$$

$$\therefore a = \frac{2(-2a)}{k}$$

$$\therefore 1 = \frac{-4}{k}$$

$$\therefore k = -4$$

Miscellaneous Exercise 4 | Q 25 | Page 132

Prove that the combined of the pair of lines passing through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b} \quad ....(1)$$

Now, the required lines are perpendicular to these lines.

$$\therefore \text{their slopes are } -\frac{1}{m_1} \text{ and } -\frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = -\frac{1}{m_1}x \quad \text{and} \quad y = -\frac{1}{m_2}x$$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

∴ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

$$\therefore x^2 \frac{-2h}{b} x + \frac{a}{b} y^2 = 0 \quad ...[By(1)]$$

$$\therefore bx^2 - 2hxy + ay^2 = 0$$

Miscellaneous Exercise 4 | Q 26 | Page 132

If equation $ax^2 - y^2 + 2y + c = 1$ represents a pair of perpendicular lines, then find a and c.

Solution: The given equation represents a pair of lines perpendicular to each other.

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore a - 1 = 0$

$\therefore a = 1$

With this value of a, the given equation is

$x^2 - y^2 + 2y + c - 1 = 0$

Comparing this equation with

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, we get,

$A = 1, H = 0, B = -1, G = 0, F = 1, C = c - 1$

Since the given equation represents a pair of lines,

$$D = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & c - 1 \end{vmatrix} = 0$$

$$\therefore 1(- c + 1 - 1) - 0 + 0 = 0$$

$$\therefore - c = 0$$

$$\therefore c = 0$$

Hence, a = 1, c = 0.