1. Rotational Dynamics

Can you recall?

1. What is circular motion?
2. What is the concept of centre of mass?
3. What are kinematical equations of motion?
4. Do you know real and pseudo forces, their origin and applications?

1.1 Introduction:

Circular motion is an essential part of our daily life. Every day we come across several revolving or rotating (rigid) objects. During revolution, the object (every particle in the object) undergoes circular motion about some point outside the object or about some other object, while during rotation the motion is about an axis of rotation passing through the object.

1.2 Characteristics of Circular Motion:

1) It is an accelerated motion: As the direction of velocity changes at every instant, it is an accelerated motion.
2) It is a periodic motion: During the motion, the particle repeats its path along the same trajectory. Thus, the motion is periodic in space.

1.2.1 Kinematics of Circular Motion:

As seen in XIth Std, in order to describe a circular motion, we use the quantities angular displacement $\vec{\theta}$, angular velocity $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ and angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ which are analogous to displacement $\vec{s}$, velocity $\vec{v} = \frac{d\vec{s}}{dt}$ and acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ used in translational motion.

Also, the tangential velocity is given by $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is the angular velocity.

Here, the position vector $\vec{r}$ is the radius vector from the centre of the circular motion. The magnitude of $\vec{v}$ is $v = \omega r$.

Direction of $\vec{\omega}$ is *always* along the axis of rotation and is given by the right-hand thumb rule. To know the direction of $\vec{\omega}$, curl the fingers of the right hand along the sense of rotation, with the thumb outstretched. The outstretched thumb then gives the direction of $\vec{\omega}$.

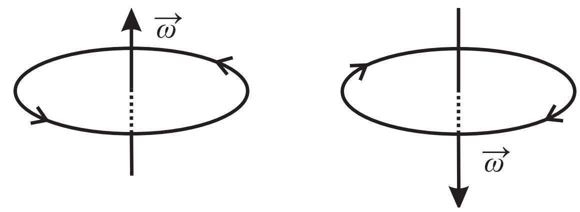


Fig. 1.1: Directions of angular velocity.

If T is period of circular motion or periodic time and n is the frequency, $\omega = 2\pi n = \frac{2\pi}{T}$

Uniform circular motion: During circular motion if the speed of the particle remains constant, it is called Uniform Circular Motion (UCM). In this case, only the direction of its velocity changes at every instant in such a way that the velocity is always tangential to the path. The acceleration responsible for this is the centripetal or radial acceleration $\vec{a}_r = -\omega^2\vec{r}$

For UCM, its magnitude is constant and it is $a = \omega^2 r = \frac{v^2}{r} = v\omega$. It is always directed towards the centre of the circular motion (along $-\vec{r}$), hence called *centripetal*.

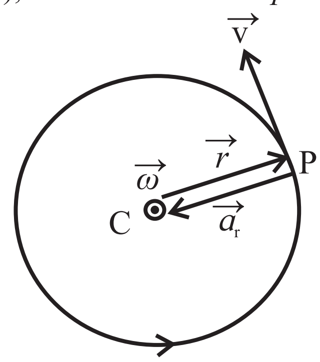


Fig. 1.2: Directions of linear velocity and acceleration.

Illustration: Circular motion of any particle of a fan rotating uniformly.

Non-uniform circular motion: When a fan is switched ON or OFF, the speeds of particles of the fan go on increasing or decreasing for some time, however their directions are always tangential to their circular trajectories.

During this time, it is a non-uniform circular motion. As the velocity is still tangential, the centripetal or radial acceleration $\vec{a}_r$ is still there. However, for non-uniform circular motion, the magnitude of $\vec{a}_r$ is *not* constant.

The acceleration responsible for changing the magnitude of velocity is directed along or opposite to the velocity, hence always tangential and is called as tangential acceleration $\vec{a}_T$.

As magnitude of tangential velocity $\vec{v}$ is changing during a non-uniform circular motion, the corresponding angular velocity $\vec{\omega}$ is also changing at every instant. This is due to the angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Though the motion is non-uniform, the particles are still in the same plane. Hence, the direction of $\vec{\alpha}$ is still along the axis of rotation. For increasing speed, it is along the direction of $\vec{\omega}$ while during decreasing speed, it is opposite to that of $\vec{\omega}$.

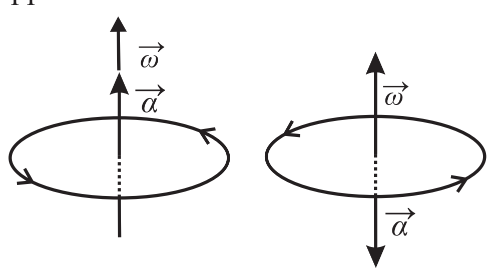
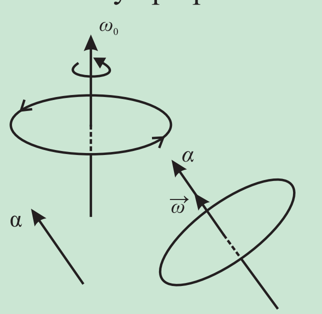


Fig. 1.3: Direction of angular acceleration.

Do you know?

If the angular acceleration $\vec{\alpha}$ is along any direction other than axial, it will have a component perpendicular to the axis. Thus, it will change the direction of $\vec{\omega}$ also, which will change the plane of rotation as $\vec{\omega}$ is always perpendicular to the plane of rotation.



If $\vec{\alpha}$ is constant in magnitude, but always perpendicular to $\vec{\omega}$, it will always change only the direction of $\vec{\omega}$ and never its magnitude thereby continuously changing the plane of rotation. (This is similar to an acceleration $\vec{a}$ perpendicular to velocity $\vec{v}$ changing only its direction).

If the angular acceleration $\vec{\alpha}$ is **constant and along the axis of rotation**, all $\vec{\theta}$, $\vec{\omega}$ and $\vec{\alpha}$ will be directed along the axis. This makes it possible to use scalar notation and write the kinematical equations of motion analogous to those for translational motion as given in the Table 1.1 at the end of the Chapter.

Example 1.1 : A fan is rotating at 90 rpm. It is then switched OFF. It stops after 21 rotations. Calculate the time taken by it to stop assuming that the frictional torque is constant.

Solution:

$n_0 = 90\,\text{rpm} = 1.5\,\text{rps} \quad \therefore \omega_0 = 2\pi n_0 = 3\pi \frac{\text{rad}}{\text{s}}$

The angle through which the blades of the fan move while stopping is $\theta = 2\pi N = 2\pi\,(21) = 42\,\pi$ rad, $\omega = 0$ (fan stops). Using equations analogous to kinematical equations of motion

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{\omega^2 - \omega_0^2}{2\theta}$$

$$\therefore \frac{0 - 3\pi}{t} = \frac{0 - (3\pi)^2}{2(42\pi)} \quad \therefore t = 28\,\text{s}$$

Remark: One can also use the unit 'revolution' for angle and get rid of π throughout (for such data). In this case, $\omega_0 = 1.5$ rps and $\theta = 21$ rev.

1.2.2 Dynamics of Circular Motion (Centripetal Force and Centrifugal Force):

i) Centripetal force (CPF): As seen above, the acceleration responsible for circular motion is the centripetal or radial acceleration $\vec{a}_r = -\omega^2 \vec{r}$. The force providing this acceleration is the centripetal or radial force, $\text{CPF} = -m\omega^2 \vec{r}$

Remember this

(i) The word *centripetal* is NOT the name or type of that force (like *gravitational force, nuclear force, etc*). It is the adjective or property of that force saying that the direction of this force is along the radius and towards centre (centre seeking).

(ii) While performing circular or rotational motion, the resultant of all the *real* forces acting upon the body is (or, must be) towards the centre, *hence* we call this *resultant* force to be centripetal force. Under the action of this *resultant* force, the direction of the velocity is always maintained tangential to the circular track.

The vice versa need not be true, i.e., the resultant force directed towards the centre may not always result into a circular motion. (In the Chapter 7 you will know that during an S.H.M. also the force is always directed to the centre of the motion). For a motion to be circular, correspondingly matching tangential velocity is also essential.

(iii) Obviously, this discussion is in an inertial frame of reference in which we are *observing* that the body is performing a circular motion.

(iv) In magnitude, centripetal force

$$= mr\omega^2 = \frac{mv^2}{r} = mv\omega$$

ii) Centrifugal force (CFF):

Visualize yourself on a merry-go-round rotating uniformly. If you close your eyes, you will not know that you are performing a circular motion but you will feel that you are at rest. In order to explain that you are at rest, you need to consider a force equal in magnitude to the resultant real force, but directed opposite, i.e., away from the centre. This force, $\left(+m\omega^2\vec{r}\right)$ is the centrifugal (away from the centre) force. It is a pseudo force arising due to the centripetal acceleration of the frame of reference.

It must be understood that centrifugal force is a non-real force, but NOT an imaginary force. Remember, before the merry-go-round reaches its uniform speed, you were *really* experiencing an outward pull (because, centrifugal force is greater than the resultant force towards the centre). A force measuring instrument can record it as well.

On reaching the uniform speed, in the frame of reference of merry-go-round, this centrifugal force exactly balances the resultant of all the *real* forces. The resultant force in that frame of reference is thus zero. Thus, only in such a frame of reference we can say that the centrifugal force balances the centripetal force. It must be remembered that in this case, centrifugal force means the '*net pseudo force*' and centripetal force means the '*resultant of all the real forces*'.

There are two ways of writing force equation for a circular motion:

Resultant force $= -m\omega^2\vec{r}$ or

$$m\omega^2\vec{r} + \sum\left(\text{real forces}\right) = 0$$

Activity

Attach a suitable mass to spring balance so that it stretches by about half is capacity. Now whirl the spring balance so that the mass performs a horizontal motion. You will notice that the balance now reads more mass for the same mass. Can you explain this?

1.3 Applications of Uniform Circular Motion:

1.3.1 Vehicle Along a Horizontal Circular Track:

Figure 1.4 shows vertical section of a car on a horizontal circular track of radius *r*. Plane of figure is a vertical plane, perpendicular to the track but includes only centre C of the track. Forces acting on the car (considered to be a particle) are (i) weight *mg*, vertically downwards, (ii) normal reaction *N*, vertically upwards that balances the weight *mg* and (iii) force of static friction f_s between road and the tyres. This is static friction because it prevents the vehicle from outward slipping or skidding. This is the resultant force which is centripetal.

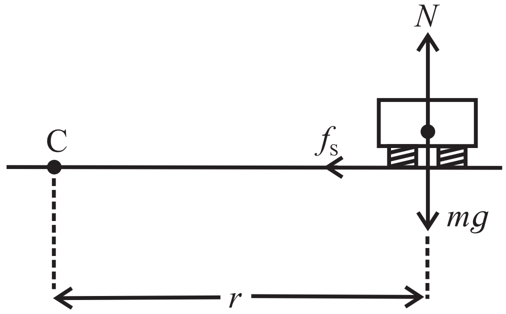


Fig. 1.4: Vehicle on a horizontal road.

While working in the frame of reference attached to the vehicle, it balances the centrifugal force.

$$\therefore mg = N \text{ and } f_s = mr\omega^2 = \frac{mv^2}{r}$$

$$\therefore \frac{f_s}{N} = \frac{r\omega^2}{g} = \frac{v^2}{rg}$$

For a given track, radius r is constant. For given vehicle, $mg = N$ is constant. Thus, as the speed v increases, the force of static friction f_s also increases. However, f_s has an upper limit $(f_s)_{max} = \mu_s . N$, where μ_s is the coefficient of static friction between road and tyres of the vehicle. This imposes an upper limit to the speed v.

At the maximum possible speed v_s, we can write

$$\frac{(f_s)_{max}}{N} = \mu_s = \frac{v^2_{max}}{rg} \therefore v_{max} = \sqrt{\mu_s rg}$$

Do you know?

(i) In the discussion till now, we had assumed the vehicle to be a point. In reality, if it is a four wheeler, the resultant normal reaction is due to all the four tyres. Normal reactions at all the four tyres are never equal while undergoing circular motion. Also, the centrifugal force acts through the centre of mass, which is not at the ground level, but above it. Thus, the frictional force and the centrifugal force result into a torque which may topple the vehicle (even a two wheeler).

(ii) For a two wheeler, it is a must for the rider to incline with respect to the vertical to prevent toppling.

Use your brain power

(I) Obtain the condition for not toppling for a four-wheeler. On what factors does it depend, and in what way? Think about the normal reactions – where are those and how much are those! What is the recommendation on loading the vehicle for not toppling easily? If a vehicle topples while turning, which wheels leave the contact? Why? How does it affect the tyres? What is the recommendation for this?

(II) Determine the angle to be made with the vertical by a two wheeler rider while turning on a horizontal track.

Hint: For both (I) and (II) above, find the torque that balances the torque due to centrifugal force and torque due to static friction force.

(III) We have mentioned about static friction between road and the tyres. Why is it static? What about the kinetic friction between road and the tyres?

(IV) What do you do if your vehicle is trapped on a slippery or a sandy road? What is the physics involved?

1.3.2 Well (or Wall) of Death: (मौत का कुआँ):

This is a vertical cylindrical wall of radius r inside which a vehicle is driven in horizontal circles. This can be seen while performing stunts.

As shown in the Fig. 1.5, the forces acting on the vehicle (assumed to be a point) are (i) Normal reaction N acting horizontally and

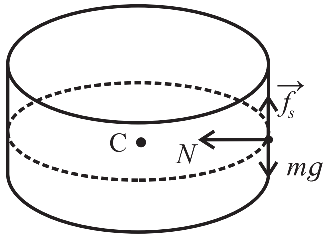


Fig. 1.5: Well of death.

towards the centre, (ii) Weight mg acting vertically downwards, and (iii) Force of static friction f_s acting vertically upwards between vertical wall and the tyres. It is static friction because it has to prevent the downward slipping. Its magnitude is equal to mg, as this is the only upward force.

Normal reaction N is thus the resultant centripetal force (or the only force that can balance the centrifugal force). Thus, in magnitude,

$$N = mr\omega^2 = \frac{m\text{v}^2}{r} \text{ and } mg = f_s$$

Force of static friction f_s is always less than or equal to $\mu_s N$.

$$\therefore f_s \le \mu_s N \therefore mg \le \mu_s\left(\frac{m\text{v}^2}{r}\right)$$

$$\therefore g \le \frac{\mu_s \text{v}^2}{r} \therefore \text{v}^2 \ge \frac{rg}{\mu_s}$$

$$\therefore \text{v}_{min} = \sqrt{\frac{rg}{\mu_s}}$$

Remember this

(i) N should always be equal to $\frac{m\text{v}^2}{r}$

$$\therefore N_{min} = \frac{m\text{v}^2_{min}}{r} = \frac{mg}{\mu_s}$$

(ii) In this case, $f_s = \mu_s N$ is valid only for the minimum speed as f_s should always be equal to mg.

(iii) During the derivation, the vehicle is assumed to be a particle. In reality, it is not so. During revolutions in such a well, a two-wheeler rider is *never* horizontal, else, the torque due to her/his weight will topple her/him. Think of the torque that balances the torque due to the weight. What about a four-wheeler?

(iv) In this case, the angle made by the road surface with the horizontal is 90°, i.e., if the road is banked at 90°, it imposes a lower limit on the turning speed. In the previous sub-section we saw that for an unbanked (banking angle 0) road there is an upper limit for the turning speed. *It means that for any other banking angle* ($0 < \theta < 90°$), *the turning speed will have the upper as well as the lower limit.*

Example 1.2: A motor cyclist (to be treated as a point mass) is to undertake horizontal circles inside the cylindrical wall of a well of inner radius 4 m. Coefficient of static friction between the tyres and the wall is 0.4. Calculate the minimum speed and frequency necessary to perform this stunt. (Use $g = 10 \text{ m/s}^2$)

Solution:

$$\text{v}_{\text{min}} = \sqrt{\frac{rg}{\mu_s}} = \sqrt{\frac{4\times 10}{0.4}} = 10 \text{ m s}^{-1} \quad \text{and}$$

$$n_{\text{min}} = \frac{\text{v}_{\text{min}}}{2\pi r} = \frac{10}{2\times\pi\times 4} \cong 0.4 \text{ rev s}^{-1}$$

1.3.3 Vehicle on a Banked Road:

As seen earlier, while taking a turn on a horizontal road, the force of static friction between the tyres of the vehicle and the road provides the necessary centripetal force (or balances the centrifugal force). However, the frictional force is having an upper limit. Also, its value is usually not constant as the road surface is not uniform. Thus, in real life, we should not depend upon it, as far as possible. For this purpose, the surfaces of curved roads are tilted with the horizontal with some angle θ. This is called banking of a road or the road is said to be banked.

Figure 1.6 Shows the vertical section of a vehicle on a curved road of radius r banked

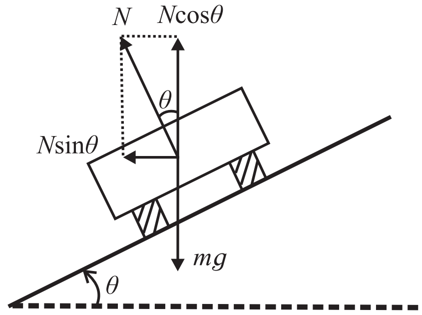


Fig 1.6: Vehicle on a banked road.

at an angle θ with the horizontal. Considering the vehicle to be a point and ignoring friction (not eliminating) and other non-conservative forces like air resistance, there are two forces acting on the vehicle, (i) weight *mg*, vertically downwards and (ii) normal reaction *N*, perpendicular to the surface of the road. As the motion of the vehicle is along a horizontal circle, the resultant force must be horizontal and directed towards the centre of the track. It means, the vertical force *mg* must be balanced. Thus, we have to resolve the normal reaction *N* along the vertical and along the horizontal. Its vertical component $N\cos\theta$ balances weight *mg*. Horizontal component $N\sin\theta$ being the resultant force, must be the necessary centripetal force (or balance the centrifugal force). Thus, in magnitude,

$N\cos\theta = mg$ and

$$N\sin\theta = mr\omega^2 = \frac{m\text{v}^2}{r} \therefore \tan\theta = \frac{\text{v}^2}{rg} \quad \text{--- (1.1)}$$

(a) Most safe speed: For a particular road, *r* and θ are fixed. Thus, this expression gives us the expression for the *most safe* speed (*not* a minimum or a maximum speed) on this road as $\text{v}_s = \sqrt{rg\tan\theta}$

(b) Banking angle: While designing a road, this expression helps us in knowing the angle of banking as

$$\theta = \tan^{-1}\left(\frac{\text{v}^2}{rg}\right) \quad \text{--- (1.2)}$$

(c) Speed limits: Figure 1.7 and 1.8 show vertical section of a vehicle on a *rough* curved road of radius *r*, banked at an angle θ. If the vehicle is running exactly at the speed $\text{v}_s = \sqrt{rg\tan\theta}$, the forces acting on the vehicle

Use your brain power

As a civil engineer, you are given contract to construct a curved road in a ghat. In order to obtain the banking angle θ, you need to decide the speed limit. How will you decide the values of speed v and radius *r*?

are (i) weight *mg* acting vertically downwards and (ii) normal reaction *N* acting perpendicular to the road. As seen above, only at this speed, the resultant of these two forces (which is $N\sin\theta$) is the necessary centripetal force (or balances the centrifugal force). In practice, vehicles never travel exactly with this speed. For speeds other than this, the component of force of static friction between road and the tyres helps us, up to a certain limit.

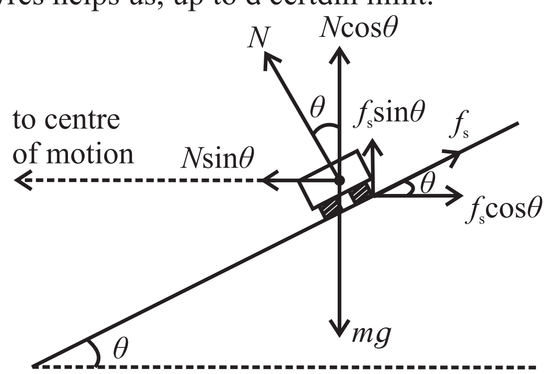


Fig 1.7: Banked road : lower speed limit.

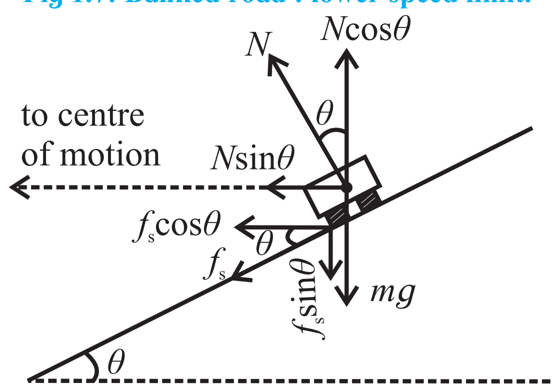


Fig 1.8: Banked road : upper speed limit.

For speeds $\text{v}_1 < \sqrt{rg\tan\theta}$, $\frac{m\text{v}_1^2}{r} < N\sin\theta$ (or $N\sin\theta$ is greater than the centrifugal force $\frac{m\text{v}_1^2}{r}$). In this case, the direction of force of static friction f_s between road and the tyres is directed along the inclination of the road, upwards (Fig. 1.7). Its horizontal component is parallel and opposite to $N\sin\theta$. These two forces take care of the necessary centripetal force (or balance the centrifugal force).

$\therefore mg = f_s \sin\theta + N\cos\theta$ and

$\frac{mv_1^2}{r} = N\sin\theta - f_s\cos\theta$

For minimum possible speed, f_s is maximum and equal to $\mu_s N$. Using this in the equations above and solving for minimum possible speed, we get

$$(v_1)_{min} = v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)} \quad \text{--- (1.3)}$$

For $\mu_s \geq \tan\theta$, v_{min} = 0. This is true for most of the rough roads, banked at smaller angles.

(d) For speeds $v_2 > \sqrt{rg\tan\theta}$, $\frac{mv_2^2}{r} > N\sin\theta$ (or N sinθ is less than the centrifugal force $\frac{mv_2^2}{r}$). In this case, the direction of force of static friction f_s between road and the tyres is directed along the inclination of the road, downwards (Fig. 1.8). Its horizontal component is parallel to $N\sin\theta$. These two forces take care of the necessary centripetal force (or balance the centrifugal force).

$\therefore mg = N\cos\theta - f_s\sin\theta$ and

$\frac{mv_2^2}{r} = N\sin\theta + f_s\cos\theta$

For maximum possible speed, f_s is maximum and equal to $\mu_s N$. Using this in the equations above, and solving for maximum possible speed, we get

$$(v_2)_{max} = v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1-\mu_s\tan\theta}\right)} \quad \text{--- (1.4)}$$

If μ_s = cot θ, $v_{max} = \infty$. But $(\mu_s)_{max} = 1$. Thus, for $\theta \geq 45°$, $v_{max} = \infty$. However, for heavily banked road, minimum limit may be important. *Try to relate the concepts used while explaining the well of death.*

(e) For μ_s = 0, both the equations 1.3 and 1.4 give us $v = \sqrt{rg\tan\theta}$ which is the *safest* speed on a banked road as we don't take the help of friction.

Example 1.3: A racing track of radius of curvature 9.9 m is banked at $\tan^{-1}0.5$. Coefficient of static friction between the track and the tyres of a vehicle is 0.2. Determine the speed limits with 10 % margin. (Take g = 10 m/s²)

Solution:

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)}$$

$$= \sqrt{9.9\times10\left(\frac{0.5-0.2}{1+(0.2\times0.5)}\right)}$$

$$= \sqrt{27} = 5.196\ m/s$$

Allowed v_{min} should be 10% higher than this.

$$\therefore (v_{min})_{allowed} = 5.196\times\frac{110}{100}$$

$$= 5.716\ \frac{m}{s}$$

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1-\mu_s\tan\theta}\right)}$$

$$= \sqrt{9.9\times10\left(\frac{0.5+0.2}{1-(0.2\times0.5)}\right)}$$

$$= \sqrt{77} = 8.775\ m/s$$

Allowed v_{max} should be 10% lower than this.

$$\therefore (v_{max})_{allowed} = 8.775\times\frac{90}{100} = 7.896\ m/s$$

Use your brain power

- If friction is zero, can a vehicle move on the road? Why are we not considering the friction in deriving the expression for the banking angle?
- What about the kinetic friction between the road and the tyres?

1.3.4 Conical Pendulum:

A tiny mass (assumed to be a point object and called a bob) connected to a long, flexible, massless, inextensible string, and suspended to a rigid support is called a pendulum. If the string is made to oscillate in a single vertical plane, we call it a *simple* pendulum (to be studied in the Chapter 5).

We can also revolve the string in such a way that the string moves along the surface of a right circular cone of vertical axis and the point object performs a (practically) uniform horizontal circular motion. In such a case the system is called a *conical* pendulum.

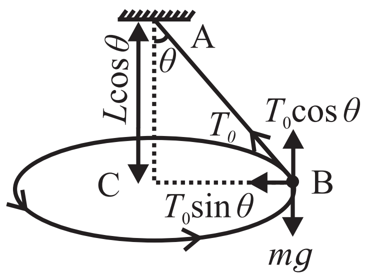


Fig. 1.9 (a): In an inertial frame

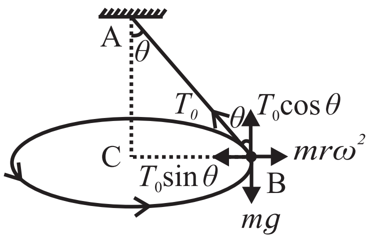


Fig. 1.9 (b): In a non- inertial frame

Figure 1.9 shows the vertical section of a conical pendulum having bob (point mass) of mass m and string of length L. In a given position B, the forces acting on the bob are (i) its weight mg directed vertically downwards and (ii) the force T_0 due to the tension in the string, directed along the string, towards the support A. As the motion of the bob is a horizontal circular motion, the resultant force must be horizontal and directed towards the centre C of the circular motion. For this, all the vertical forces must cancel. Hence, we shall resolve the force T_0 due to the tension. If θ is the angle made by the string with the vertical, at any position (semi-vertical angle of the cone), the vertical component $T_0 \cos\theta$ balances the weight mg. The horizontal component $T_0 \sin\theta$ then becomes the resultant force which is *centripetal*.

$\therefore T_0 \sin\theta =$ centripetal force $= mr\omega^2$ --- (1.5)

Also, $T_0 \cos\theta = mg$ --- (1.6)

Dividing eq (1.5) by Eq. (1.6), we get,

$$\omega^2 = \frac{g\sin\theta}{r\cos\theta}$$

Radius r of the circular motion is $r = L\sin\theta$.

If T is the period of revolution of the bob,

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L\cos\theta}}$$

$$\therefore \text{Period } T = 2\pi\sqrt{\frac{L\cos\theta}{g}} \quad \text{--- (1.7)}$$

Frequency of revolution,

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L\cos\theta}} \quad \text{--- (1.8)}$$

In the frame of reference attached to the bob, the centrifugal force should balance the resultant of all the real forces (which we call CPF) for the bob to be at rest.

$\therefore T_0 \sin\theta = mr\omega^2$ --- (in magnitude). This is the same as the Eq. (1.5)

Remember this

(i) For a given set up, L and g are constant. Thus, both period and frequency depend upon θ.

(ii) During revolutions, the string can NEVER become horizontal. This can be explained in two different ways.

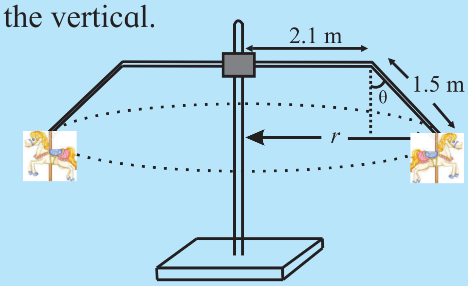
(a) If the string becomes horizontal, the force due to tension will also be horizontal. Its vertical component will then be zero. In this case, nothing will be there to balance *mg*.

(b) For horizontal string, $\theta = 90°$. This will indicate the frequency to be infinite and the period to be zero, which are impossible. Also, in this case, the tension $T_0 = \frac{mg}{\cos\theta}$ in the string and the kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$ of the bob will be infinite.

Activity

A stone is tied to a string and whirled such that the stone performs horizontal circular motion. It can be seen that the string is NEVER horizontal.

Example 1.4: A merry-go-round usually consists of a central vertical pillar. At the top of it there are horizontal rods which can rotate about vertical axis. At the end of this horizontal rod there is a vertical rod fitted like an elbow joint. At the lower end of each vertical rod, there is a horse on which the rider can sit. As the merry-go-round is set into rotation, these vertical rods move away from the axle by making some angle with the vertical.



The figure above shows vertical section of a merry-go-round in which the '*initially vertical*' rods are inclined with the vertical at $\theta = 37^0$, during rotation. Calculate the frequency of revolution of the merry-go-round. (Use $g = \pi^2$ m/s^2 and sin 37° = 0.6)

Solution: Length of the horizontal rod, $H = 2.1$ m

Length of the 'initially vertical' rod, $V = 1.5$ m, $\theta = 37°$

$\therefore$ Radius of the horizontal circular motion of the rider $= H + V \sin 37° = 3.0$ m

If T is the tension along the inclined rod,

$T\cos\theta = mg$ and $T\sin\theta = mr\omega^2 = 4\pi^2 mrn^2$

$$\therefore \tan\theta = \frac{4\pi^2 rn^2}{g}$$

$$\therefore n = \sqrt{\frac{\tan\theta}{4r}} = \frac{1}{4}\text{rev s}^{-1} \quad \ldots\text{as } g = \pi^2$$

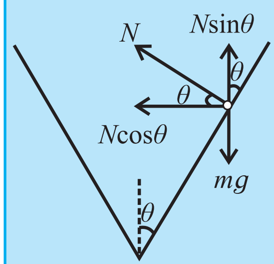
Example 1.5: Semi-vertical angle of the conical section of a funnel is 37^0. There is a small ball kept inside the funnel. On rotating the funnel, the maximum speed that the ball can have in order to remain in the funnel is 2 m/s. Calculate inner radius of the brim of the funnel. Is there any limit upon the frequency of rotation? How much is it? Is it lower or upper limit? Give a logical reasoning. (Use $g = 10$ m/s^2 and sin $37^0 = 0.6$)

Solution: $N\sin\theta = mg$ and $N\cos\theta = \dfrac{mv^2}{r}$

$$\therefore \tan\theta = \frac{rg}{v^2} \therefore r = \frac{v^2\tan\theta}{g}$$

$$\therefore r_{max} = \frac{v^2_{max}\tan\theta}{g} = 0.3\,\text{m}$$

$$v = r\omega = 2\pi rn$$



If we go for the lower limit of the speed (while rotating), $v \to 0 \therefore r \to 0$, but the frequency n increases. Hence a specific upper limit is not possible in the case of frequency. Thus, the practical limit on the frequency of rotation is its lower limit. It will be possible for $r = r_{max}$

$$\therefore n_{min} = \frac{v_{max}}{2\pi r_{max}} = \frac{1}{0.3\pi} \cong 1\,\text{rev / s}$$

Activity

Using a funnel and a marble or a ball bearing try to work out the situation in the above question. Try to realize that as the marble goes towards the brim, its linear speed increases but its angular speed decreases. When nearing the base, it is the other way.

1.4 Vertical Circular Motion:

Two types of vertical circular motions are commonly observed in practice:

(a) A controlled vertical circular motion such as a giant wheel or similar games. In this case the speed is either kept constant or NOT totally controlled by gravity.

(b) Vertical circular motion controlled *only* by gravity. In this case, we initially supply the necessary energy (mostly) at the lowest point. Then onwards, the entire kinetics is governed by the gravitational force. During the motion, there is interconversion of kinetic energy and gravitational potential energy.

1.4.1 Point Mass Undergoing Vertical Circular Motion Under Gravity:

Case I: Mass tied to a string:

The figure 1.10 shows a bob (treated as a point mass) tied to a (practically) massless, inextensible and flexible string. It is whirled along a vertical circle so that the bob performs a vertical circular motion and the string rotates in a vertical plane. At any position of the bob, there are only two forces acting on the bob:

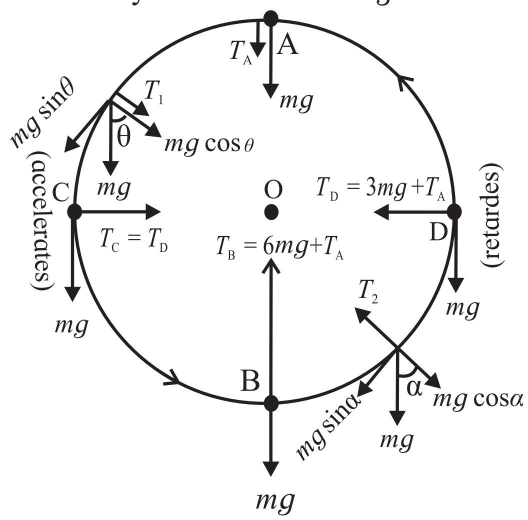


Fig 1.10: Vertical circular motion.

(a) its weight *mg*, vertically downwards, which is constant and (b) the force due to the tension along the string, directed along the string and towards the centre. Its magnitude changes periodically with time and location.

As the motion is non uniform, the resultant of these two forces is *not* directed towards the center *except* at the uppermost and the lowermost positions of the bob. At all the other positions, part of the resultant is tangential and is used to change the speed.

***Uppermost position* (A):** Both, weight *mg* and force due to tension T_A are downwards, i.e., towards the centre. In this case, their resultant is used only as the centripetal force. Thus, if v_A is the speed at the uppermost point, we get,

$$mg + T_A = \frac{mv_A^2}{r} \qquad \text{--- (1.9)}$$

Radius *r* of the circular motion is the length of the string. For minimum possible speed at this point (or if the motion is to be realized with minimum possible energy),

$$T_A = 0 \;\therefore (v_A)_{min} = \sqrt{rg} \qquad \text{--- (1.10)}$$

***Lowermost position* (B):** Force due to the tension, T_B is vertically upwards, i.e., towards the centre, and opposite to *mg*. In this case also their resultant is the centripetal force. If v_B is the speed at the lowermost point, we get,

$$T_B - mg = \frac{mv_B^2}{r} \qquad \text{--- (1.11)}$$

While coming down from the uppermost to the lowermost point, the vertical displacement is $2r$ and the motion is governed only by gravity. Hence the corresponding decrease in the gravitational potential energy is converted into the kinetic energy.

$$\therefore mg(2r) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$$\therefore v_B^2 - v_A^2 = 4rg \qquad \text{--- (1.12)}$$

Using this in the eq (1.11), and using (v_A) min from Eq. (1.10) we get,

$$(v_B)_{min} = \sqrt{5rg} \qquad \text{--- (1.13)}$$

Subtracting eq (1.9) from eq (1.11) , we can write,

$$T_B - T_A - 2mg = \frac{m}{r}\left(v_B^2 - v_A^2\right) \qquad \text{--- (1.14)}$$

Using eq (1.12) and rearranging, we get,

$$T_B - T_A = 6mg \qquad \text{--- (1.15)}$$

***Positions when the string is horizontal* (C and D):** Force due to the tension is the only force towards the centre as weight *mg* is perpendicular to the tension. Thus, force due to the tension is the centripetal force used to change the direction of the velocity and weight *mg* is used only to change the speed.

Using similar mathematics, it can be shown that

$$T_C - T_A = T_D - T_A = 3mg \quad \text{and}$$

$$(v_C)_{min} = (v_D)_{min} = \sqrt{3rg}$$

***Arbitrary positions*:** Force due to the tension and weight are neither along the same line, nor perpendicular. Tangential component of weight is used to change the speed. It decreases the speed while going up and increases it while coming down.

Remember this

1. Equation (1.15) is independent of v and r.
2. T_A can never be exactly equal to zero in the case of a string, else, the string will slack. $\therefore$ $T_B > 6\,mg$.
3. None of the parameters (including the linear and angular accelerations) are constant during such a motion. Obviously, kinematical equations given in the table1 are not applicable.
4. We can determine the position vector or velocity at any instant using the energy conservation. But as the function of the radius vector is not integrable (definite integration is not possible), *theoretically* it is *not* possible to determine the period or frequency. However, experimentally the period can be measured.
5. Equations (1.10) and (1.13) give only the respective *minimum* speeds at the uppermost and the lowermost points. Any higher speeds obeying the equation (1.14) are allowed.
6. In reality, we have to continuously supply some energy to overcome the air resistance.

Case II: Mass tied to a rod: Consider a bob (point mass) tied to a (practically massless and rigid) rod and whirled along a vertical circle. *The basic difference between the rod and the string is that the string needs some tension at all the points, including the uppermost point.* Thus, a certain minimum speed, Eq. (1.10), is necessary at the uppermost point in the case of a string. In the case of a rod, as the rod is rigid, such a condition is not necessary. Thus (practically) zero speed is possible at the uppermost point.

Using similar mathematics, it is left to the readers to show that

$$\left(v_{lowermost}\right)_{min} = \sqrt{4rg} = 2\sqrt{rg}$$

$$v_{min} \text{ at the } \textit{rod horizontal} \text{ position} = \sqrt{2rg}$$

$$T_{lowermost} - T_{uppermost} = 6mg$$

1.4.2 Sphere of Death (मृत्यु गोल):

This is a popular show in a circus. During this, two-wheeler rider (or riders) undergo rounds inside a hollow sphere. Starting with small horizontal circles, they eventually perform revolutions along vertical circles. The dynamics of this vertical circular motion is the same as that of the point mass tied to the string, except that the force due to tension T is replaced by the normal reaction force N.

If you have seen this show, try to visualize that initially there are nearly horizontal circles. The linear speed is more for larger circles but angular speed (frequency) is more for smaller circles (while starting or stopping). This is as per the theory of conical pendulum.

1.4.3 Vehicle at the Top of a Convex Over-Bridge:

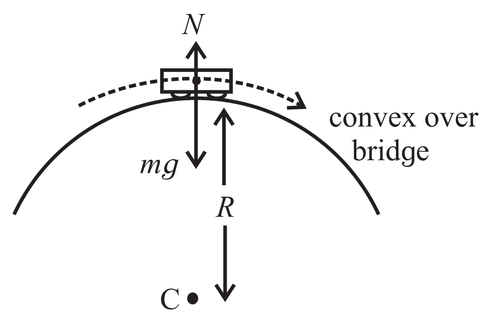


Fig. 1.11: Vehicle on a convex over-bridge.

Figure shows a vehicle at the top of a convex over bridge, during its motion (part of vertical circular motion). Forces acting on the vehicle are (a) Weight mg and (b) Normal reaction force N, both along the vertical line (topmost position). The resultant of these two must provide the necessary centripetal force (vertically downwards) if the vehicle is at the uppermost position. Thus, if v is the speed at the uppermost point,

$$mg - N = \frac{mv^2}{r}$$

As the speed is increased, N goes on decreasing. Normal reaction is an indication of contact. Thus, for just maintaining contact, $N = 0$. This imposes an upper limit on the speed as $v_{max} = \sqrt{rg}$

Do you know?

Roller coaster is a common event in the amusement parks. During this ride, all the parts of the vertical circular motion described above can be experienced. The major force that we experience during this is the normal reaction force. Those who have experienced this, should try to recall the changes in the normal reaction experienced by us during various parts of the track.

Use your brain power

- What is expected to happen if one travels fast over a speed breaker? Why?
- How does the normal force on a concave suspension bridge change when a vehicle is travelling on it with constant speed?

Example 1.6: A tiny stone of mass 20 g is tied to a practically massless, inextensible, flexible string and whirled along vertical circles. Speed of the stone is 8 m/s when the centripetal force is exactly equal to the force due to the tension.
Calculate minimum and maximum kinetic energies of the stone during the entire circle.

Let $\theta = 0$ be the angular position of the string, when the stone is at the lowermost position. Determine the angular position of the string when the force due to tension is numerically equal to weight of the stone.
Use g = 10 m/s^2 and length of the string = 1.8 m

Solution: When the string is horizontal, the force due to the tension is the centripetal force. Thus, vertical displacements of the bob for minimum and maximum energy positions are radius r each.

If K.E.$_{max}$ and K.E.$_{min}$ are the respective kinetic energies at the uppermost and the lowermost points,

$$\text{K.E.}_{max} - \frac{1}{2}m(8)^2 = mgr \quad \text{and}$$

$$\frac{1}{2}m(8)^2 - \text{K.E.}_{min} = mgr$$

$$\therefore \frac{1}{2}(0.02)(8)^2 - \text{K.E.}_{min} = (0.02)(10)(1.8)$$

$$\therefore \text{K.E.}_{min} = 0.28\,\text{J}$$

$$\text{K.E.}_{max} - \frac{1}{2}(0.02)(8)^2 = (0.02)(10)(1.8)$$

$$\therefore \text{K.E.}_{max} = \frac{1}{2}mv_{max}^2 = 1\,\text{J}$$

$$\therefore v_{max} = \sqrt{\frac{2(\text{K.E.})_{max}}{m}} = 10\ \text{m s}^{-1}$$

at the lowermost position, for which $\theta = 0$.

$$T - mg\cos\theta = \frac{mv^2}{r} \quad \text{--- at any angle } \theta,$$

where the speed is v.
Thus, if $T = mg$, we get,

$$mg - mg\cos\theta = \frac{mv^2}{r}$$

$$\therefore rg(1-\cos\theta) = v^2 \qquad \text{--- (A)}$$

Vertical displacement at the angular position θ is $r(1-\cos\theta)$. Thus, the energy equation at this position can be written as

$$\frac{1}{2}m(10)^2 - \frac{1}{2}mv^2 = mg\left[r(1-\cos\theta)\right]$$

By using Eq. A, we get

$$50 - \frac{1}{2}rg(1-\cos\theta) = rg(1-\cos\theta)$$

$$\therefore 50 = \frac{3}{2}rg(1-\cos\theta)$$

$$\therefore \cos\theta = \frac{-23}{27} \therefore \theta = 148^0 25'$$

1.5 Moment of Inertia as an Analogous Quantity for Mass:

In XIth Std. we saw that angular displacement, angular velocity and angular acceleration respectively replace displacement, velocity and acceleration for various kinematical equations. Also, torque is an analogous quantity for force. Expressions of linear momentum, force (for a fixed mass) and kinetic energy include mass as a common term. In order to have their rotational analogues, we need a replacement for mass.

If we open a door (with hinges), we give a certain angular displacement to it. The efforts needed for this depend not only upon the mass of the door, but also upon the (perpendicular) distance from the axis of rotation, where we apply the force. Thus, the quantity analogous to mass includes not only the mass, but also takes care of the distance wise distribution of the mass around the axis of rotation. To know the exact relation, let us derive an expression for the rotational kinetic energy which is the sum of the translational kinetic energies of all the individual particles.

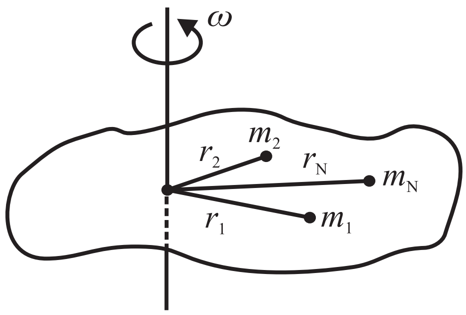


Fig. 1.12: A body of N particles.

Figure 1.12 shows a rigid object rotating with a constant angular speed ω about an axis perpendicular to the plane of paper. For theoretical simplification let us consider the object to be consisting of N particles of masses m_1, m_2, ...m_N at respective perpendicular distances r_1, r_2, ...r_N from the axis of rotation. As the object rotates, all these particles perform UCM with the same angular speed ω, but with different linear speeds $v_1 = r_1\omega,\ v_2 = r_2\omega,\ \ldots\ldots\ v_N = r_N\omega$.

Translational K.E. of the first particle is

$$\text{K.E.}_1 = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 r_1^2\omega^2$$

Similar will be the case of all the other particles. Rotational K.E. of the object, is the sum of individual translational kinetic energies. Thus, rotational K.E.

$$= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 \ldots\ldots + \frac{1}{2}m_N r_N^2\omega^2$$

$\therefore$ Rotational K.E.

$$= \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2 \ldots + m_N r_N^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

Where $I = m_1 r_1^2 + m_2 r_2^2 \ldots\ldots + m_N r_N^2 = \sum_{i=1}^{N} m_i r_i^2$

If $I = \sum m_i r_i^2$ replaces mass m and angular speed ω replaces linear speed v, rotational K.E. $= \frac{1}{2}I\omega^2$ is analogous to translational K.E. $= \frac{1}{2}mv^2$. Thus, I is defined to be the rotational inertia or moment of inertia (M.I.) *of the object about the given axis* of rotation. It is clear that the moment of inertia of an object depends upon (i) individual masses and (ii) the distribution of these masses about the *given* axis of rotation. For a different axis, it will again depend upon the mass distribution around that axis and *will* be different if there is no symmetry.

During this discussion, for simplicity, we assumed the object to be consisting of a finite number of particles. In practice, usually, it is not so. For a homogeneous rigid object of mathematically integrable mass distribution, the moment of inertia is to be obtained by integration as $I = \int r^2 dm$. If integrable mass distribution is not known, it is not possible to obtain the moment of inertia theoretically, but it can be determined experimentally.

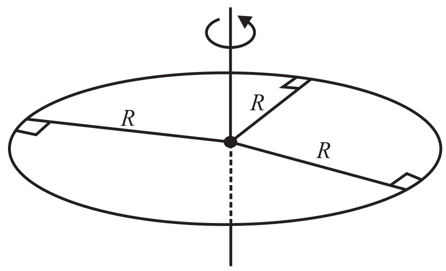


Fig. 1.13: Moment of Inertia of a ring.

1.5.1 Moment of Inertia of a Uniform Ring:

An object is called a uniform ring if its mass is (practically) situated uniformly on the circumference of a circle (Fig 1.13). Obviously, it is a two dimensional object of negligible thickness. If it is rotating about its own axis (line perpendicular to its plane and passing through its centre), its entire mass M is practically at a distance equal to its radius R form the axis. Hence, the expression for the moment of inertia of a uniform ring of mass M and radius R is $I = MR^2$.

1.5.2 Moment of Inertia of a Uniform Disc:

Disc is a two dimensional circular object of negligible thickness. It is said to be uniform if its mass per unit area and its composition is the same throughout. The ratio $\sigma = \frac{m}{A} = \frac{\text{mass}}{\text{area}}$ is called the surface density.

Consider a uniform disc of mass M and radius R rotating about its own axis, which is the line perpendicular to its plane and passing through its centre $\therefore \sigma = \frac{M}{\pi R^2}$.

As it is a uniform circular object, it can be considered to be consisting of a number of concentric rings of radii increasing from (practically) zero to R. One of such rings of mass dm is shown by shaded portion in the Fig. 1.14.

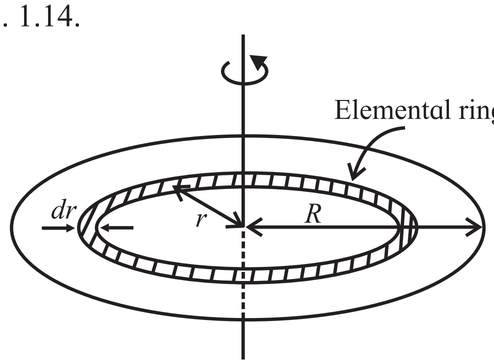


Fig .1.14: Moment of Inertia of a disk.

Width of this ring is dr, which is so small that the entire ring can be considered to be of average radius r. (In practical sense, dr is less than the least count of the instrument that measures r, so that r is constant for that ring).

Area of this ring is $A = 2\pi r.\text{dr}$ $\therefore \sigma = \frac{dm}{2\pi r.dr}$

$\therefore$ d$m = 2\pi\sigma r.dr$.

As it is a ring, this entire mass is at a distance r from the axis of rotation. Thus, the moment of inertia of this ring is $I_r = dm\ (r^2)$

Moment of inertia (I) of the disc can now be obtained by integrating I_r from $r = 0$ to $r = R$.

$$\therefore I = \int_0^R I_r = \int_0^R dm \cdot r^2 = \int_0^R 2\pi\sigma r \cdot dr \cdot r^2 = 2\pi\sigma \int_0^R r^3 \cdot dr$$

$$\therefore I = 2\pi\sigma \left(\frac{R^4}{4}\right) = 2\pi \left(\frac{M}{\pi R^2}\right)\left(\frac{R^4}{4}\right) = \frac{1}{2} MR^2$$

Using similar method, expressions for moment of inertias of objects of several integrable geometrical shapes can be derived. Some of those are given in the Table 3 at the end of the topic.

1.6 Radius of Gyration:

As stated earlier, theoretical calculation of moment of inertia is possible only for mathematically integrable geometrical shapes. However, experimentally we can determine the moment of inertia of any object. It depends upon mass of that object and how that mass is distributed from or around the given axis of rotation. If we are interested in knowing only the mass distribution around the axis of rotation, we can express moment of inertia of any object as $I = MK^2$, where M is mass of that object. It means that the mass of that object is effectively at a distance K from the given axis of rotation. In this case, K is defined as the *radius of gyration* of the object about the given axis of rotation. In other words, if K is radius of gyration for an object, $I = MK^2$ is the moment of inertia of that object. Larger the value of K, *farther* is the mass from the axis.

Consider a uniform ring and a uniform disc, both of the same mass M and same radius R. Let I_r and I_d be their respective moment of inertias.

If K_r and K_d are their respective radii of gyration, we can write,

$I_r = MR^2 = MK_r^2$ $\therefore K_r = R$ and

$I_d = \frac{1}{2} MR^2 = MK_d^2$ $\therefore K_d = \frac{R}{\sqrt{2}}$ $\therefore K_d < K_r$

It shows mathematically that K is decided by the distribution of mass. In a ring the entire mass is distributed at the distance R, while for a disc, its mass is distributed between 0 and R. Among any objects of same mass and radius, ring has the largest radius of gyration and hence maximum M.I.

1.7 Theorem of Parallel Axes and Theorem of Perpendicular Axes:

Expressions of moment of inertias of

regular geometrical shapes given in the table 3 are about their axes of symmetry. These are derived by integration. However, every time the axis need not be the axis of symmetry. In simple transformations it may be parallel or perpendicular to the symmetrical axis. For example, if a rod is rotated about one of its ends, the axis is parallel to its axis of symmetry. If a disc or a ring is rotated about its diameter, the axis is perpendicular to the central axis. In such cases, simple transformations are possible in the expressions of moment of inertias. These are called theorem of parallel axes and theorem of perpendicular axes.

1.7.1 Theorem of Parallel Axes:

In order to apply this theorem to *any* object, we need two axes parallel to each other with one of them passing through the centre of mass of the object.

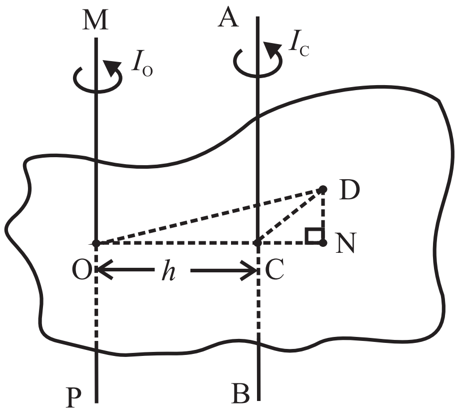


Fig. 1.15: Theorem of parallel axes.

Figure 1.15 shows an object of mass *M*. Axis MOP is any axis passing through point O. Axis ACB is passing through the centre of mass C of the object, parallel to the axis MOP, and at a distance *h* from it ($\therefore\ h = \text{CO}$). Consider a mass element *dm* located at point D. Perpendicular on OC (produced) from point D is DN. Moment of inertia of the object about the axis ACB is $I_C = \int (\text{DC})^2\, dm$, and about the axis MOP it is $I_O = \int (\text{DO})^2\, dm$.

$$\therefore I_O = \int (\text{DO})^2\, dm = \int \left([\text{DN}]^2 + [\text{NO}]^2 \right) dm$$

$$= \int \left([\text{DN}]^2 + [\text{NC}]^2 + 2.\text{NC.CO} + [\text{CO}]^2 \right) dm$$

$$= \int \left([\text{DC}]^2 + 2\text{NC}.h + h^2 \right) dm$$

$$= \int (\text{DC})^2\, dm + 2h \int \text{NC}.dm + h^2 \int dm$$

Now, $\int (\text{DC})^2\, dm = I_C$ and $\int dm = M$.

NC is the distance of a point from the centre of mass. Any mass distribution is symmetric about the centre of mass. Thus, from the definition of the centre of mass, $\int \text{NC}.dm = 0$.

$$\therefore I_O = I_C + M.h^2$$

This is the mathematical form of the theorem of parallel axes.

It states that, "The moment of inertia (I_O) of an object about any axis is the sum of its moment of inertia (I_C) about an axis parallel to the given axis, and passing through the centre of mass *and* the product of the mass of the object and the square of the distance between the two axes (Mh^2)."

Use your brain power

In Fig. 1.15, the point D is chosen such that we have to extend OC for the perpendicular DN to fall on it. What will happen to the final expression of I_0, if point D is so chosen that the perpendicular DN falls directly on OC?

1.7.2 Theorem of Perpendicular Axes:

This theorem relates the moment of inertias of a *laminar* object about three mutually perpendicular and concurrent axes, two of them in the plane of the object and the third perpendicular to the object. *Laminar* object is like a leaf, or any two dimensional object, e.g., a ring, a disc, any plane sheet, etc.

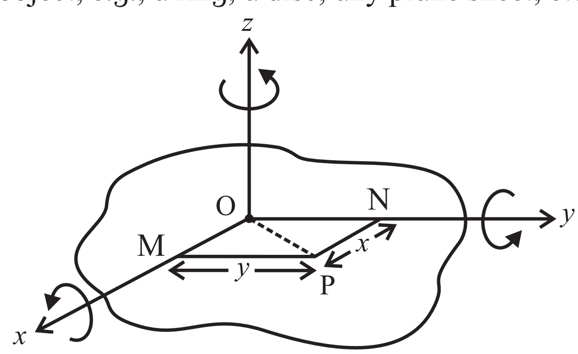


Fig. 1.16: Theorem of perpendicular axes.

Figure 1.16 shows a rigid laminar object able to rotate about three mutually perpendicular axes x, y and z. Axes x and y are in the plane of the object while the z axis is perpendicular to it, and all are concurrent at O. Consider a mass element dm located at any point P. PM $= y$ and PN $= x$ are the perpendiculars drawn from P respectively on the x and y axes. The respective perpendicular distances of point P from x, y and z axes will then be y, x and $\sqrt{y^2+x^2}$. If I_x, I_y and I_z are the respective moments of inertia of the body about x, y and z axes, we can write,

$$\therefore I_x = \int y^2 \mathrm{d}m,\ I_y = \int x^2 \mathrm{d}m \text{ and}$$

$$I_z = \int \left(y^2 + x^2\right) \mathrm{d}m$$

$$\therefore I_z = \int y^2 \mathrm{d}m + \int x^2 \mathrm{d}m = I_x + I_y$$

This is the mathematical form of the theorem of perpendicular axes.

It states that, "The moment of inertia (I_z) of a laminar object about an axis (z) perpendicular to its plane is the sum of its moments of inertia about two mutually perpendicular axes (x and y) in its plane, all the three axes being concurrent".

Example 1.7: A *flywheel* is a mechanical device specifically designed to efficiently store rotational energy. For a particular machine it is in the form of a uniform 20 kg disc of diameter 50 cm, able to rotate about its own axis. Calculate its kinetic energy when rotating at 1200 rpm. Use $\pi^2 = 10$. Calculate its moment of inertia, in case it is rotated about a tangent in its plane.

Solution: (I) As the flywheel is in the form of a uniform disc rotating about its own axis, $I_z = \frac{1}{2}MR^2$

$\therefore$ Rotational kinetic energy

$$= \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)4\pi^2 n^2$$

$\therefore$ Rotational kinetic energy

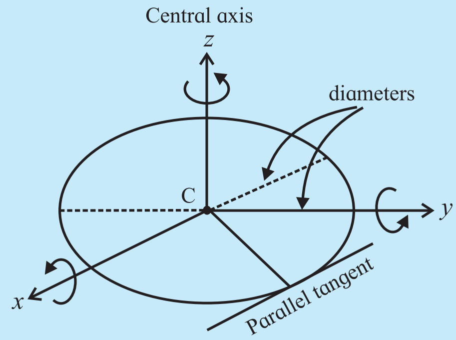
$$= \mathrm{M}\pi^2 (Rn)^2 = 20 \times 10 \times (0.25 \times 20)^2 = 5000\ \mathrm{J}$$

(II) Consider any two mutually perpendicular diameters x and y of the flywheel. If the flywheel rotates about these diameters, these three axes (own axis and two diameters) will be mutually perpendicular and concurrent. Thus, perpendicular axes theorem is applicable. Let I_d be the moment of inertia of the flywheel, when rotating about its diameter. $\therefore I_d = I_x = I_y$

Thus, according to the theorem of perpendicular axes,

$$I_z = \frac{1}{2}MR^2 = I_x + I_y = 2I_d$$

$$\therefore I_d = \frac{1}{4}MR^2$$



As the diameter passes through the centre of mass of the (uniform) disc, $I_d = I_C$

Consider a tangent in the plane of the disc and parallel to this diameter. It is at the distance $h = R$ from the diameter. Thus, parallel axes theorem is applicable about these two axes.

$$\therefore I_{T,\ \text{parallel}} = I_o = I_c + Mh^2 = I_d + MR^2$$

$$= \frac{1}{4}MR^2 + MR^2 = \frac{5}{4}MR^2$$

$$\therefore I_{T,\ \text{parallel}} = \frac{5}{4}MR^2 = \frac{5}{4}\ 20 \times 0.25^2$$

$$= 1.5625\ \mathrm{kg\ m^2}$$

1.8 Angular Momentum or Moment of Linear Momentum:

The quantity in rotational mechanics, analogous to linear momentum is *angular momentum* or moment of linear momentum. It is similar to the torque being moment of a force. If $\vec{p}$ is the instantaneous linear momentum of a particle undertaking a circular motion, its angular momentum at that instance is given by $\vec{L} = \vec{r} \times \vec{p}$, were $\vec{r}$ is the position vector from the axis of rotation.

In magnitude, it is the product of linear momentum and its perpendicular distance from the axis of rotation. $\therefore L = P \times r \sin\theta$, where θ is the smaller angle between the directions of $\vec{P}$ and $\vec{r}$.

1.8.1 Expression for Angular Momentum in Terms of Moment of Inertia:

Figure 1.12 in the section 1.5 shows a rigid object rotating with a constant angular speed ω about an axis perpendicular to the plane of paper. For theoretical simplification let us consider the object to be consisting of N number of particles of masses m_1, m_2, ….. m_N at respective perpendicular distances r_1, r_2, …..r_N from the axis of rotation. As the object rotates, all these particles perform UCM with same angular speed ω, but with different linear speeds $v_1 = r_1\ \omega$, $v_2 = r_2\ \omega$, ….. $v_N = r_N\ \omega$.

Directions of individual velocities $\vec{v}_1$, $\vec{v}_2$, etc., are along the tangents to their respective tracks. Linear momentum of the first particle is of magnitude $p_1 = m_1 v_1 = m_1 r_1\ \omega$. Its direction is along that of $\vec{v}_1$.

Its angular momentum is thus of magnitude $L_1 = p_1 r_1 = m_1 r_1^2 \omega$

Similarly, $L_2 = m_2 r_2^2 \omega$, $L_3 = m_3 r_3^2 \omega$, ……. $L_N = m_N r_N^2 \omega$

For a rigid body with a fixed axis of rotation, all these angular momenta are directed along the axis of rotation, and this direction can be obtained by using right hand thumb rule. As all of them have the same direction, their magnitudes can be algebraically added. Thus, magnitude of angular momentum of the body is given by

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots + m_N r_N^2 \omega$$
$$= \left(m_1 r_1^2 + m_2 r_2^2 + \ldots + m_N r_N^2\right)\omega = I\omega$$

Where, $I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_N r_N^2$ is the moment of inertia of the body about the given axis of rotation. The expression for angular momentum $L = I\omega$ is analogous to the expression $p = mv$ of linear momentum, if the moment of inertia I replaces mass, which is its physical significance.

1.9 Expression for Torque in Terms of Moment of Inertia:

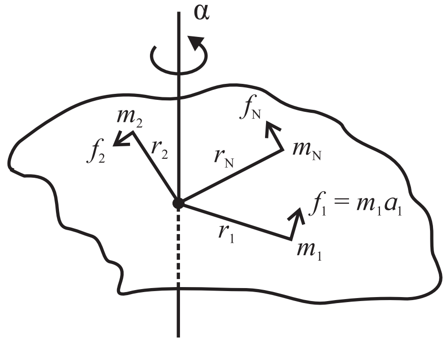


Fig 1.17: Rigid rotating object.

Figure 1.17 shows a rigid object rotating with a constant angular acceleration α about an axis perpendicular to the plane of paper. For theoretical simplification let us consider the object to be consisting of N number of particles of masses m_1, m_2, ….. m_N at respective perpendicular distances r_1, r_2, …..r_N from the axis of rotation. As the object rotates, all these particles perform circular motion with same angular acceleration α, but with different linear (tangential) accelerations $a_1 = r_1\alpha, a_2 = r_2\alpha, \ldots\ldots, a_N = r_N\alpha,$ etc.

Force experienced by the first particle is $f_1 = m_1 a_1 = m_1 r_1 \alpha$

As these forces are tangential, their respective perpendicular distances from the axis are r_1, r_2, …..r_N.

Thus, the torque experienced by the first particle is of magnitude $\tau_1 = f_1 r_1 = m_1 r_1^2 \alpha$

Similarly, $\tau_2 = m_2 r_2^2 \alpha$, $\tau_3 = m_3 r_3^2 \alpha$ …….. $\tau_N = m_N r_N^2 \alpha$

If the rotation is restricted to a single plane, directions of all these torques are the same, and along the axis. Magnitude of the resultant torque is then given by

$$\tau = \tau_1 + \tau_2 + \ldots + \tau_N$$
$$= \left(m_1 r_1^2 + m_2 r_2^2 \ldots + m_N r_N^2\right)\alpha = I\alpha$$

where, $I = m_1 r_1^2 + m_2 r_2^2 \ldots + m_N r_N^2$ is the moment of inertia of the object about the given axis of rotation.

The relation $\tau = I\alpha$ is analogous to $f = ma$ for the translational motion if the moment of inertia I replaces mass, which is its physical significance.

1.10 Conservation of Angular Momentum:

In the article 4.7 of XI[th] Std. we have seen the conservation of linear momentum which says that linear momentum of an isolated system is conserved in the absence of an external unbalanced force. As seen earlier, torque and angular momentum are the respective analogous quantities to force and linear momentum in rotational dynamics. With suitable changes this can be transformed into the conservation of angular momentum.

As seen in the section 1.8, angular momentum or the moment of linear momentum of a system is given by $\vec{L} = \vec{r} \times \vec{p}$

where $\vec{r}$ is the position vector from the axis of rotation and $\vec{p}$ is the linear momentum.

Differentiating with respect to time, we get,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

Now, $\frac{d\vec{r}}{dt} = \vec{v}$ and $\frac{d\vec{p}}{dt} = \vec{F}$.

$$\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m\left(\vec{v} \times \vec{v}\right)$$

Now $\left(\vec{v} \times \vec{v}\right) = 0$

$$\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

But $\vec{r} \times \vec{F}$ is the moment of force or torque $\vec{\tau}$.

$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt}$$

Thus, if $\vec{\tau} = 0$, $\frac{d\vec{L}}{dt} = 0$ or $\vec{L} =$ constant.

Hence, angular momentum $\vec{L}$ is conserved in the absence of external unbalanced torque $\vec{\tau}$. This is the principle of conservation of angular momentum, analogous to the conservation of linear momentum.

Examples of conservation of angular momentum: During some shows of ballet dance, acrobat in a circus, sports like ice skating, diving in a swimming pool, etc., the principle of conservation of angular momentum is realized. In all these applications the product $L = I\omega = I\left(2\pi n\right)$ is constant (once the players acquire a certain speed). Thus, if the moment of inertia I is increased, the angular speed and hence the frequency of revolution n decreases. Also, if the moment of inertia is decreased, the frequency increases.

(i) Ballet dancers: During ice ballet, the dancers have to undertake rounds of smaller and larger radii. The dancers come together while taking the rounds of smaller radius (near the centre). In this case, the moment of inertia of their system becomes minimum and the frequency increases, to make it thrilling. While outer rounds, the dancers outstretch their legs and arms. This increases their moment of inertia that reduces the angular speed and hence the linear speed. This is essential to prevent slipping.

(ii) Diving in a swimming pool (during competition): While on the diving board, the divers stretch their body so as to increase the moment of inertia. Immediately after leaving the board, they fold their body. This reduces the moment inertia considerably. As a result, the frequency increases and they can complete more rounds in air to make the show attractive. Again, while entering into water they stretch their body into a streamline shape. This allows them a smooth entry into the water.

Example 1.8: A spherical water balloon is rotating at 60 rpm. In the course of time, 48.8 % of its water leaks out. With what frequency will the remaining balloon rotate now? Neglect all non-conservative forces.

Solution: $\frac{m_1}{m_2} = \frac{V_1}{V_2} = \left(\frac{R_1}{R_2}\right)^3 \quad \therefore \frac{R_1}{R_2} = \left(\frac{m_1}{m_2}\right)^{\frac{1}{3}}$

Also, $\frac{m_1}{m_2} = \frac{100}{100-48.8} = \frac{100}{51.2} = \frac{1}{0.512}$

$$\therefore \left(\frac{m_1}{m_2}\right)^{\frac{1}{3}} = \frac{1}{0.8} = 1.25$$

$n_1 = 60\,\text{rpm} = 1\,\text{rps}, \qquad n_2 = ?$

Being sphere, moment of inertia

$I = \frac{2}{5}mR^2 \therefore \frac{I_1}{I_2} = \left(\frac{m_1}{m_2}\right)\left(\frac{R_1}{R_2}\right)^2 = \left(\frac{m_1}{m_2}\right)^{\frac{5}{3}}$

According to principle of conservation of angular momentum, $I_1\omega_1 = I_2\omega_2$

$\therefore I_1 2\pi n_1 = I_2 2\pi n_2 \therefore n_2 = \left(\frac{I_1}{I_2}\right)n_1 = \left(\frac{m_1}{m_2}\right)^{\frac{5}{3}}$

$$n_1 = (1.25)^5 \times 1 = 3.052\,\text{rps}$$

Example 1.9: A ceiling fan having moment of inertia 2 kg-m² attains its maximum frequency of 60 rpm in '2π' seconds. Calculate its power rating.

Solution:

$\omega_0 = 0,\ \omega = 2\pi n = 2\pi \times 2 = 4\pi \text{ rad / s}$

$\alpha = \frac{\omega - \omega_0}{t} = \frac{4\pi - 0}{2\pi} = 2 \text{ rad/s}^2$

$\therefore P = \tau \cdot \omega = I\alpha \cdot \omega = 2 \times 2 \times 4\pi$

$\quad = 16\pi \text{ watt} \cong 50 \text{ watt}$

1.11 Rolling Motion:

The objects like a cylinder, sphere, wheels, etc. are quite often seen to perform rolling motion. In the case of pure rolling, two motions are undertaking simultaneously; circular motion and linear motion. Individual motion of the particles (except the one at the centre of mass) is too difficult to describe. However, for theory considerations we can consider the actual motion to be the result of

(i) rotational motion of the body as a whole, about its own symmetric axis and

(ii) linear motion of the body assuming it to be concentrated at its centre of mass. In other words, the centre of mass performs purely translational motion.

Accordingly, the object possesses two types of kinetic energies, rotational and translational. Sum of these two is its total kinetic energy.

Consider an object of moment of inertia I, rolling uniformly. Following quantities can be related.

v = Linear speed of the centre of mass

R = Radius of the body

ω = Angular speed of rotation of the body

$\therefore \omega = \frac{v}{R}$ for any particle

M = Mass of the body

K = Radius of gyration of the body $\therefore I = MK^2$

Total kinetic energy of rolling = Translational K.E. + Rotational K.E.

$$\therefore E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\left(MK^2\right)\left(\frac{v}{R}\right)^2$$

$$= \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right) \qquad \text{--- (1.18)}$$

It must be remembered that static friction is essential for a purely rolling motion. In this case, it prevents the sliding motion. You might have noticed that many a times while rolling down, the motion is initially a purely rolling motion that later on turns out to be a sliding motion. Similarly, if you push a sphere-like object along a horizontal surface, initially it slips for some distance and then starts rolling.

1.11.1 Linear Acceleration and Speed While Pure Rolling Down an Inclined Plane:

Figure 1.18 shows a rigid object of mass M and radius R, rolling down an inclined plane, without slipping. Inclination of the plane with the horizontal is θ.

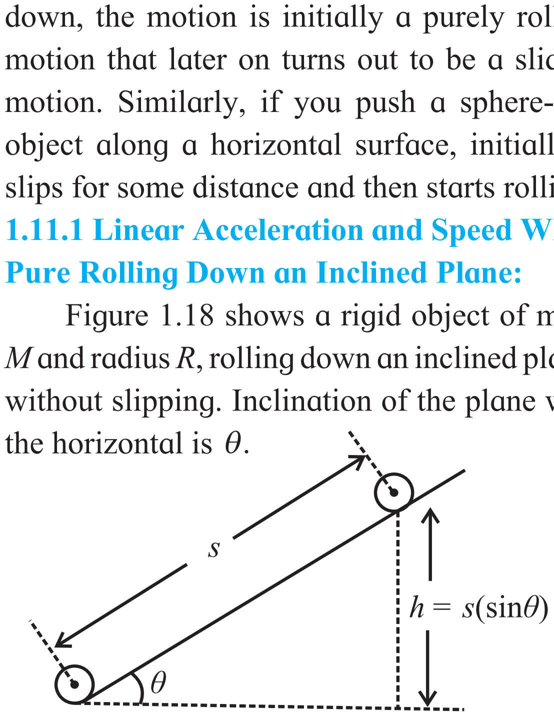


Fig. 1.18: Rolling along an incline.

As the objects starts rolling down, its gravitational P.E. is converted into K.E. of rolling. Starting from rest, let v be the speed of the centre of mass as the object comes down through a vertical distance h.

From Eq. (1.18),

$$E = \frac{1}{2}M\mathrm{v}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}M\mathrm{v}^2\left(1+\frac{K^2}{R^2}\right)$$

$$\therefore E = Mgh = \frac{1}{2}M\mathrm{v}^2\left(1+\frac{K^2}{R^2}\right)$$

$$\therefore \mathrm{v} = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}} \quad \text{--- (1.19)}$$

Linear distance travelled along the plane is $s = \frac{h}{\sin\theta}$

During this distance, the linear velocity has increased from zero to v. If a is the linear acceleration along the plane,

$$2as = \mathrm{v}^2 - u^2 \therefore 2a\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)} - 0$$

$$\therefore a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} \quad \text{--- (1.20)}$$

For pure sliding, without friction, the acceleration is $g\sin\theta$ and final velocity is $\sqrt{2gh}$. Thus, during pure rolling, the factor $\left(1+\frac{K^2}{R^2}\right)$ is effective for both the expressions.

Remarks :

I) For a rolling object, if the expression for moment of inertia is of the form n (MR^2), the numerical factor n gives the value of $\frac{K^2}{R^2}$ for that object.

For example, for a uniform solid sphere, $I = \frac{2}{5}MR^2 = MK^2 \therefore \frac{K^2}{R^2} = \frac{2}{5}$

Similarly,

$\frac{K^2}{R^2} = 1$, for a ring or a hollow cylinder

$\frac{K^2}{R^2} = \frac{1}{2}$ for a uniform disc or a solid cylinder

$\frac{K^2}{R^2} = \frac{2}{3}$ for a thin walled hollow sphere

(II) When a rod rolls, it is actually a cylinder that is rolling.

(III) While rolling, the ratio 'Translational K.E.: Rotational K.E.: Total K.E.' is

$$1 : \frac{K^2}{R^2} : \left(1+\frac{K^2}{R^2}\right)$$

For example, for a hollow sphere, $\frac{K^2}{R^2} = \frac{2}{3}$

Thus, for a rolling hollow sphere,

Translational K.E.: Rotational K.E.: Total K.E. $= 1 : \frac{2}{3} : \left(1+\frac{2}{3}\right) = 3 : 2 : 5$

Percentage wise, 60% of its kinetic energy is translational and 40% is rotational.

Table 1.1 : Analogous kinematical equations (ω_0 is the initial angular velocity)

Equation for translational motion	Analogous equation for rotational motion
$\mathrm{v}_{av} = \frac{u+\mathrm{v}}{2}$	$\omega_{av} = \frac{\omega_0+\omega}{2}$
$a = \frac{d\mathrm{v}}{dt} = \frac{\mathrm{v}-u}{t}$ $\therefore \mathrm{v} = u + at$	$\alpha = \frac{d\omega}{dt} = \frac{\omega-\omega_0}{t}$ $\therefore \omega = \omega_0 + \alpha t$
$s = \mathrm{v}_{av} \cdot t$ $= \left(\frac{u+\mathrm{v}}{2}\right)t$ $= ut + \frac{1}{2}at^2$	$\theta = \omega_{av} \cdot t$ $= \left(\frac{\omega_0+\omega}{2}\right)t$ $= \omega_0 t + \frac{1}{2}\alpha t^2$
$\mathrm{v}^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Internet my friend

http://hyperphysics.phy-astr.gsu.edu/hbase/mi.html

https://issacphysics.org

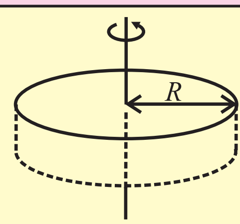
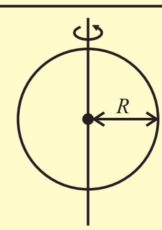
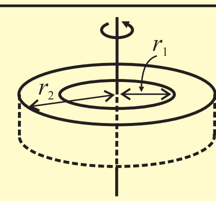
https://www.engineeringtoolbox.com

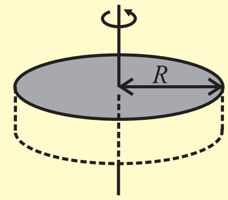
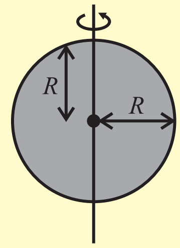
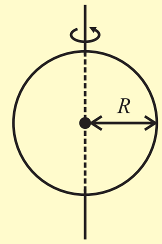
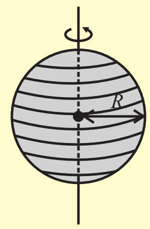
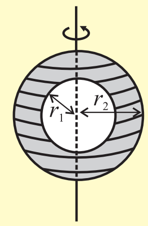
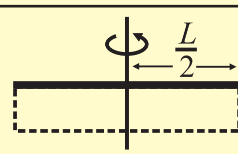
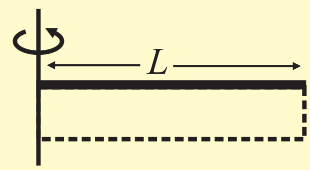
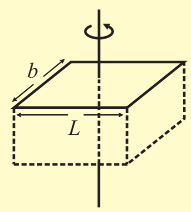
https://opentextbc.ca/physicstextbook

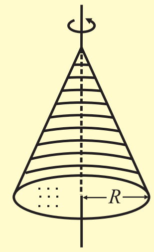
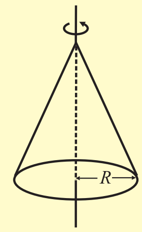
Table 2: Analogous quantities between translational motion and rotational motion:

Translational motion		Rotational motion		
Quantity	Symbol/ expression	Quantity	Symbol/ expression	Inter-relation, if possible
Linear displacement	$\vec{s}$	Angular displacement	$\vec{\theta}$	$\vec{s} = \vec{\theta} \times \vec{r}$
Linear velocity	$\vec{v} = \frac{d\vec{s}}{dt}$	Angular velocity	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$	$\vec{v} = \vec{\omega} \times \vec{r}$
Linear acceleration	$\vec{a} = \frac{d\vec{v}}{dt}$	Angular acceleration	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$	$\vec{\alpha} = \vec{a} \times \vec{r}$
Inertia or mass	m	Rotational inertia or moment of inertia	I	$I = \int r^2 dm = \sum m_i r_i^2$
Linear momentum	$\vec{p} = m\vec{v}$	Angular momentum	$\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{p}$
Force	$\vec{f} = \frac{d\vec{p}}{dt}$	Torque	$\vec{\tau} = \frac{d\vec{L}}{dt}$	$\vec{\tau} = \vec{r} \times \vec{f}$
Work	$W = \vec{f} \cdot \vec{s}$	Work	$W = \vec{\tau} \cdot \vec{\theta}$	------
Power	$P = \frac{dW}{dt} = \vec{f} \cdot \vec{v}$	Power	$P = \frac{dW}{dt} = \vec{\tau} \cdot \vec{\omega}$	------

Table 3: Expressions for moment of inertias for some symmetric objects:

Object	Axis	Expression of moment of inertia	Figure
Thin ring or hollow cylinder	Central	$I = MR^2$	
Thin ring	Diameter	$I = \frac{1}{2} MR^2$	
Annular ring or thick walled hollow cylinder	Central	$I = \frac{1}{2} M \left(r_2^2 + r_1^2\right)$	

Uniform disc or solid cylinder	Central	$I=\frac{1}{2}MR^2$	
Uniform disc	Diameter	$I=\frac{1}{4}MR^2$	
Thin walled hollow sphere	Central	$I=\frac{2}{3}MR^2$	
Solid sphere	Central	$I=\frac{2}{5}MR^2$	
Uniform symmetric spherical shell	Central	$I=\frac{2}{5}M\frac{\left(r_2^5-r_1^5\right)}{\left(r_2^3-r_1^3\right)}$	
Thin uniform rod or rectangular plate	Perpendicular to length and passing through centre	$I=\frac{1}{12}ML^2$	
Thin uniform rod or rectangular plate	Perpendicular to length and about one end	$I=\frac{1}{3}MR^2$	
Uniform plate or rectangular parallelepiped	Central	$I=\frac{1}{12}M(L^2+b^2)$	

Uniform solid right circular cone	Central	$I = \frac{3}{10} MR^2$	
Uniform hollow right circular cone	Central	$I = \frac{1}{2} MR^2$	

Exercises

Use g = 10 m/s^2, unless, otherwise stated.

1. Choose the correct option.

i) When seen from below, the blades of a ceiling fan are seen to be revolving anticlockwise and their speed is decreasing. Select correct statement about the directions of its angular velocity and angular acceleration.

(A) Angular velocity upwards, angular acceleration downwards.

(B) Angular velocity downwards, angular acceleration upwards.

(C) Both, angular velocity and angular acceleration, upwards.

(D) Both, angular velocity and angular acceleration, downwards.

ii) A particle of mass 1 kg, tied to a 1.2 m long string is whirled to perform vertical circular motion, under gravity. Minimum speed of a particle is 5 m/s. Consider following statements.

P) Maximum speed must be $5\sqrt{5}$ m/s.

Q) Difference between maximum and minimum tensions along the string is 60 N.

Select correct option.

(A) Only the statement P is correct.

(B) Only the statement Q is correct.

(C) Both the statements are correct.

(D) Both the statements are incorrect.

iii) Select correct statement about the formula (expression) of moment of inertia (M.I.) in terms of mass M of the object and some of its distance parameter/s, such as R, L, etc.

(A) Different objects must have different expressions for their M.I.

(B) When rotating about their central axis, a hollow right circular cone and a disc have the same expression for the M.I.

(C) Expression for the M.I. for a parallelepiped rotating about the transverse axis passing through its centre includes its depth.

(D) Expression for M.I. of a rod and that of a plane sheet is the same about a transverse axis.

iv) In a certain unit, the radius of gyration of a uniform disc about its central and transverse axis is $\sqrt{2.5}$. Its radius of gyration about a tangent in its plane (in the same unit) must be

(A) $\sqrt{5}$ (B) 2.5

(C) $2\sqrt{2.5}$ (D) $\sqrt{12.5}$

v) Consider following cases:

(P) A planet revolving in an elliptical orbit.

(Q) A planet revolving in a circular orbit. Principle of conservation of angular momentum comes in force in which of these?

(A) Only for (P)
(B) Only for (Q)
(C) For both, (P) and (Q)
(D) Neither for (P), nor for (Q)

X) A thin walled hollow cylinder is rolling down an incline, without slipping. At any instant, the ratio "Rotational K.E.: Translational K.E.: Total K.E." is

(A) 1:1:2 (B) 1:2:3
(C) 1:1:1 (D) 2:1:3

2. Answer in brief.

i) Why are curved roads banked?

ii) Do we need a banked road for a two-wheeler? Explain.

iii) On what factors does the frequency of a conical pendulum depend? Is it independent of some factors?

iv) Why is it useful to define radius of gyration?

v) A uniform disc and a hollow right circular cone have the same formula for their M.I., when rotating about their central axes. Why is it so?

3. While driving along an unbanked circular road, a two-wheeler rider has to lean with the vertical. Why is it so? With what angle the rider has to lean? Derive the relevant expression. Why such a leaning is *not* necessary for a four wheeler?

4. Using the energy conservation, derive the expressions for the minimum speeds at different locations along a vertical circular motion controlled by gravity. Is zero speed possible at the uppermost point? Under what condition/s? Also prove that the difference between the extreme tensions (or normal forces) depends only upon the weight of the object.

5. Discuss the necessity of radius of gyration. Define it. On what factors does it depend and it does not depend? Can you locate some similarity between the centre of mass and radius of gyration? What can you infer if a uniform ring and a uniform disc have the same radius of gyration?

6. State the conditions under which the theorems of parallel axes and perpendicular axes are applicable. State the respective mathematical expressions.

7. Derive an expression that relates angular momentum with the angular velocity of a rigid body.

8. Obtain an expression relating the torque with angular acceleration for a rigid body.

9. State and explain the principle of conservation of angular momentum. Use a suitable illustration. Do we use it in our daily life? When?

10. Discuss the interlink between translational, rotational and total kinetic energies of a rigid object that rolls without slipping.

11. A rigid object is rolling down an inclined plane. Derive expressions for the acceleration along the track and the speed after falling through a certain vertical distance.

12. Somehow, an ant is stuck to the rim of a bicycle wheel of diameter 1 m. While the bicycle is on a central stand, the wheel is set into rotation and it attains the frequency of 2 rev/s in 10 seconds, with uniform angular acceleration. Calculate (i) Number of revolutions completed by the ant in these 10 seconds. (ii) Time taken by it for first complete revolution and the last complete revolution.

[Ans:10 rev., $t_{\text{first}} = \sqrt{10}\,\text{s}$, $t_{\text{last}} = 0.5132\,\text{s}$]

13. Coefficient of static friction between a coin and a gramophone disc is 0.5. Radius of the disc is 8 cm. Initially the centre of the coin is 2 cm away from the centre of the disc. At what minimum frequency will it start slipping from

there? By what factor will the answer change if the coin is almost at the rim? (use $g = \pi^2$ m/s^2)

[Ans: 2.5 rev/s, $n_2 = \frac{1}{2} n_1$]

14. Part of a racing track is to be designed for a radius of curvature of 72 m. We are not recommending the vehicles to drive faster than 216 kmph. With what angle should the road be tilted? At what height will its outer edge be, with respect to the inner edge if the track is 10 m wide?

[Ans: $\theta = \tan^{-1}(5) = 78.69^{\circ}, h = 9.8\,\text{m}$]

15. The road in the example 14 above is constructed as per the requirements. The coefficient of static friction between the tyres of a vehicle on this road is 0.8, will there be any lower speed limit? By how much can the upper speed limit exceed in this case?

[Ans: $v_{\min} \cong 88\,\text{kmph}$, no upper limit as the road is banked for $\theta > 45^{\circ}$]

16. During a stunt, a cyclist (considered to be a particle) is undertaking horizontal circles inside a cylindrical well of radius 6.05 m. If the necessary friction coefficient is 0.5, how much minimum speed should the stunt artist maintain? Mass of the artist is 50 kg. If she/he increases the speed by 20%, how much will the force of friction be?

[Ans: $v_{\min} = 11$ m/s, $f_s = mg = 500\,\text{N}$]

17. A pendulum consisting of a massless string of length 20 cm and a tiny bob of mass 100 g is set up as a conical pendulum. Its bob now performs 75 rpm. Calculate kinetic energy and increase in the gravitational potential energy of the bob. (Use $\pi^2 = 10$)

[Ans: cos $\theta =$ 0.8, K.E. = 0.45 J, $\Delta(\text{P.E}) = 0.04$ J]

18. A motorcyclist (as a particle) is undergoing vertical circles inside a sphere of death. The speed of the motorcycle varies between 6 m/s and 10 m/s. Calculate diameter of the sphere of death. What are the minimum values are possible for these two speeds?

[Ans: Diameter = 3.2 m, $(v_1)_{\min} = 4$ m/s, $(v_2)_{\min} = 4\sqrt{5}\,\text{m/s}$]

19. A metallic ring of mass 1 kg has moment of inertia 1 kg m^2 when rotating about one of its diameters. It is molten and remoulded into a thin uniform disc of the same radius. How much will its moment of inertia be, when rotated about its own axis.

[Ans: 1 kg m^2]

20. A big dumb-bell is prepared by using a uniform rod of mass 60 g and length 20 cm. Two identical solid thermocol spheres of mass 25 g and radius 10 cm each are at the two ends of the rod. Calculate moment of inertia of the dumb-bell when rotated about an axis passing through its centre and perpendicular to the length.

[Ans: 24000 g cm^{-2}]

21. A flywheel used to prepare earthenware pots is set into rotation at 100 rpm. It is in the form of a disc of mass 10 kg and radius 0.4 m. A lump of clay (to be taken equivalent to a particle) of mass 1.6 kg falls on it and adheres to it at a certain distance x from the centre. Calculate x if the wheel now rotates at 80 rpm.

[Ans: $x = \frac{1}{\sqrt{8}}\,\text{m} = 0.35$ m]

22. Starting from rest, an object rolls down along an incline that rises by 3 units in every 5 units (along it). The object gains a speed of $\sqrt{10}$ m/s as it travels a distance of $\frac{5}{3}$ m along the incline. What can be the possible shape/s of the object?

[Ans: $\frac{K^2}{R^2} = 1$. Thus, a ring or a hollow cylinder]

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