Chapter 4: Definite Integration

EXERCISE 4.1 [PAGE 156]

Exercise 4.1 | Q 1 | Page 156

Evaluate the following integrals as limit of a sum : $\int_1^3 (3x - 4) \cdot dx$

SOLUTION

Let f(x) = 3x – 4, for 1 ≤ x ≤ 3
Divide the closed interval [1, 3] into n subintervals each of length h at the points

1, 1 + h, 1 + 2h, 1 + rh, ..., 1 + nh = 3

∴ nh = 2

∴ h = $\frac{2}{n}$ and $as \to \infty, h \to 0$

Here, a = 1

∴ f(a + rh) = f(1 + rh) = 3(1 + rh) – 4 = 3rh – 1

$$\because \int_a^b f(x) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a + rh) \cdot h$$

$$\therefore \int_1^3 (3x - 4) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(3rh - 1) \cdot h$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (3r \cdot \frac{2}{n} - 1) \cdot \frac{2}{n} \quad ...[\because h = \frac{2}{n}]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (\frac{12}{n^2} - \frac{2}{n})$$

$$= \lim_{n\to\infty} [\frac{12}{n^2} \sum_{r=1}^{n} r - \frac{2}{n} \sum_{r=1}^{n} 1]$$

$= \lim_{n\to\infty} [\frac{12}{n^2} \frac{n(n+1)}{2} \frac{2}{n} \cdot n]$

$= \lim_{n\to\infty} [6(\frac{n+1}{n}) - 2]$

$= \lim_{n\to\infty} [6(1 + \frac{1}{n}) - 2]$

= 6(1 + 0) – 2 ...$[\because \lim_{n\to\infty} \frac{1}{n} = 0]$

= 4.

Exercise 4.1 | Q 2 | Page 156

Evaluate the following integrals as limit of a sum : $\int_0^4 x^2 \cdot dx$

SOLUTION

Let f(x) = x^2, for 0 ≤ x ≤ 4
Divide the closed interval [0, 4] into n subintervals each of length h at the points
0, 0 + h, 0 + 2h, ..., 0 + rh, ..., 0 + nh = 4
i.e. 0, h, 2h, ..., rh, ..., nh = 4

∴ h = $\frac{4}{n}$ as $n \to \infty, h \to \infty$

Here , a = 0

∴ f(a + rh) = f(0 + rh) = f(rh) = r^2h^2

$\because \int_a^b f(x) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a+rh) \cdot h$

$\therefore \int_0^4 x^2 \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} r^2 h^2 \cdot h$

$= \lim_{n\to\infty} \sum_{r=1}^{n} r^2 \frac{+64}{n^3} \ldots [\because h = \frac{4}{n}]$

$= \lim_{n\to\infty}[\frac{64}{n^3}\sum_{r=1}^{n} r^2]$

$= \lim_{n\to\infty}[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}]$

$= \lim_{n\to\infty}[\frac{64}{6}(\frac{n+1}{n})(\frac{2n+1}{n})]$

$= \lim_{n\to\infty}[\frac{64}{6}(1+\frac{1}{n})(2+\frac{1}{n})]$

$= \frac{64}{6}(1+0)(2+0)\ldots[\because \lim_{n\to\infty}\frac{1}{n} = 0]$

$= \frac{64}{3}$.

Exercise 4.1 | Q 3 | Page 156

Evaluate the following integrals as limit of a sum : $\int_0^2 e^x \cdot dx$

SOLUTION

Let f(x) = ex, for 0 ≤ x ≤ 2
DIvide the closed interval [0, 2] into n equal subntervals each of length h at the points
0, 0 + h, 0 + 2h, ..., 0 + rh, ... 0 + nh = 2
i.e. 0,h, 2h, ..., rh, ..., nh = 2

∴ h = $\frac{2}{n}$ and $n \to \infty, h \to 0$

Here, a = 0

∴ f(a + rh) = f(0 + rh) = f(rh) = erh

$\therefore \int_a^b f(x)\cdot dx = \lim_{n\to\infty}\sum_{r=1}^{n} f(a+rh)\cdot h$

$\therefore \int_0^2 e^x\cdot dx = \lim_{n\to\infty}\sum_{r=1}^{n} e^{rh}\cdot h$

$= \lim_{h \to 0} [h \sum_{r=1}^{n} e^{rh}]$ ...[as$n \to \infty, h \to 0$]

Now, $\sum_{r=1}^{n} e^{rh} = e^h + e^{2h} + \ldots + e^{rh}$

$= \frac{e^h \left[(e^h)^n - 1 \right]}{e^h - 1}$

$= \frac{e^h \left[e^{rh} - 1 \right]}{e^h - 1}$

$= \frac{e^h \cdot (e^2 - 1)}{e^h - 1}$ ... $\left[\because h = \frac{2}{n} \therefore nh = 2 \right]$

$= (e^2 - 1) \frac{e^h}{e^b - 1}$

$\therefore \int_0^2 e^x \cdot dx = \lim_{h \to 0} \frac{h(e^2 - 1)e^h}{e^h - 1}$

$= (e^2 - 1) \lim_{h \to 0} \frac{e^h}{\left(\frac{e^h - 1}{h} \right)}$

$= (e^2 - 1) \frac{\lim_{h \to 0} e^h}{\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right)}$

$= (e^2 - 1) \cdot \frac{e^0}{1} \ldots [\because \lim_{h \to 0} \frac{e^{h-1}}{h} = 1]$

= $(e^2 - 1)$.

Exercise 4.1 | Q 4 | Page 156

Evaluate the following integrals as limit of a sum : $\int_0^2 (3x^2 - 1) \cdot dx$

SOLUTION

Let $f(x) = 3x^2 – 1$, for $0 \le x \le 2$.
DIvide the closed interval [0, 2] int n subintervals each of length h at the points.
0, 0 + h, 0 + 2h, ..., 0 + rh, .., 0 + nh = 2
i.e. 0, h, 2h, ..., rh, ..., nh = 2

$\therefore$ h = $\frac{2}{n}$ and as $n \to \infty, h \to 0$

Here, a = 0

$\therefore f(a + rh) = f(0 + rh) = f(rh) = 3(rh)^2 – 1 = 3r^2h^2 – 1$

$$\because \int_a^b f(x) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a + rh) \cdot h$$

$$= \int_0^2 (3x^2 - 1) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} (3r^2h^2 - 1) \cdot h$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (3r^2 \times \frac{4}{n^2} - 1) \cdot \frac{2}{n} \ldots [\because h = \frac{2}{n}]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (\frac{24r^2}{n^3} - \frac{2}{n})$$

$$= \lim_{n\to\infty} [\frac{24}{n^3} \sum_{r=1}^{n} r^2 - \frac{2}{n} \sum_{r=1}^{n} 1]$$

$$= \lim_{n\to\infty} [\frac{24}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{2}{n} \cdot n]$$

$$= \lim_{n\to\infty} [4 \cdot (\frac{n+1}{n})(\frac{2n+1}{n}) - 2]$$

$$= \lim_{n\to\infty} [4(1 + \frac{1}{n})(2 + \frac{1}{n}) - 2]$$

$$= 4(1 + 0)(2 + 0) - 2 \ldots [\because \lim_{n\to\infty} \frac{1}{n} = 0]$$

$= 8 – 2 = 6.$

Exercise 4.1 | Q 5 | Page 156

Evaluate the following integrals as limit of a sum : $\int_1^3 x^3 \cdot dx$

SOLUTION

Let f(x) = x^3, for 1 ≤ x ≤ 3.
DIvide the closed interval [1, 3] into n equal subintervals each of length h at the points 1, 1 + h, 1 + 2h, ..., 1 + rh, ..., 1 + nh = 3

∴ nh = 2

∴ h = $\frac{2}{n}$ and as $n \to \infty, h \to 0.$

Here , a = 1.

∴ f(a + rh) = f(1 + rh) =)1 + rh$)^3$

= 1 + 3rh + $3r^2h^2$ + r^3h^3

$$\because \int_a^b f(x) \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a+rh) \cdot h$$

$$\therefore \int_1^3 x^3 \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} (1+3rh+3r^2h^2+r3h^3) \cdot h$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (h+3rh^2+3r^2h^3+r^3h^4)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} [\frac{2}{n}+3r(\frac{2}{n})^2+3r^2(\frac{2}{n})^3+r^3(\frac{2}{n})^4]\ldots[\because h=\frac{2}{n}]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} [\frac{2}{n}+\frac{12r}{n^2}+\frac{24r^2}{n^3}+\frac{16r^3}{n^4}]$$

$= \lim_{n\to\infty} [\frac{2}{n} \sum_{r=1}^{n} 1 + \frac{12}{n^2} \sum_{r=1}^{n} r + \frac{24}{n^3} \sum_{r=1}^{n} r^2 + \frac{16}{n^4} \sum_{r=1}^{n} r^3]$

$= \lim_{n\to\infty} [\frac{2}{n} \cdot n + \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + \frac{24}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4}]$

$= \lim_{n\to\infty} [2 + 6(\frac{n+1}{n}) + 4(\frac{n+1}{n})(\frac{2n+1}{n}) + 4(\frac{n+1}{n})^2]$

$= \lim_{n\to\infty} [2 + 6(1 + \frac{1}{n}) + 4(1 + \frac{1}{n})(2 + \frac{1}{n}) + 4(1 + \frac{1}{n})^2]$

$= [2 + 6(1+0) + 4(1+0)(2+0) + 4(1+0)^2] \ldots [\because \lim_{n\to\infty} \frac{1}{n} = 0]$

= 2 + 6 + 8 + 4

= 20.

EXERCISE 4.2 [PAGES 171 - 172]

Exercise 4.2 | Q 1.01 | Page 171

Evaluate : $\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx$

SOLUTION

$$\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx = \int_1^9 \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) \cdot dx$$

$$= \int_1^9 x^{\frac{1}{2}} \cdot dx + \int_1^9 x^{-\frac{1}{2}} \cdot dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9 + \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9$$

$$= \frac{2}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] + 2\left[9^{\frac{1}{2}} - 1^{\frac{1}{2}}\right]$$

$$= \frac{2}{3}\left[\left(3^2\right)^{\frac{3}{2}} - 1\right] + 2[3-1]\Big]$$

$$= \frac{2}{3}[27 - 1] + 4$$

$$= \frac{52}{3} + 4$$

$$= \frac{64}{3}.$$

Exercise 4.2 | Q 1.02 | Page 171

Evaluate : $\int_2^3 \frac{1}{x^2 + 5x + 6} \cdot dx$

SOLUTION

$$\int_2^3 \frac{1}{x^2 + 5x + 6} \cdot dx$$

$$= \int_2^3 \frac{1}{(x+2)(x+3)} \cdot dx$$

$$= \int_2^3 \frac{(x+3) - (x+2)}{(x+2)(x+3)} \cdot dx$$

$$= \int_2^3 \left[\frac{1}{x+2} - \frac{1}{x+3}\right] \cdot dx$$

$$= [\log(x+2) - \log(x+3)]_2^3$$

$$= \left[\log\left|\frac{x+2}{x+3}\right|\right]_2^3$$

$$= \log\left(\frac{3+2}{3+3}\right) - \log\left(\frac{2+2}{2+3}\right)$$

$$= \log\ \frac{5}{6} - \log\ \frac{4}{5}$$

$$= \log\left(\frac{5}{6} \times \frac{5}{4}\right)$$

$= \log\left(\frac{25}{24}\right).$

Exercise 4.2 | Q 1.03 | Page 171

Evaluate : $\int_0^{\frac{\pi}{4}} \cot^2 x \cdot dx$

SOLUTION

$$\int_0^{\frac{\pi}{4}} \cot^2 x \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\text{cosec}^2 x - 1\right) \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \text{cosec}^2 x \cdot dx - \int_0^{\frac{\pi}{4}} \cdot dx$$

$$= [-\cot x]_0^{\frac{\pi}{4}} - [x]_0^{\frac{\pi}{4}}$$

$$= \left[\left(-\frac{\cot\pi}{4}\right) - (-\cot 0)\right] - \left[\frac{\pi}{4} - 0\right]$$

$$= -1 + \cot 0 - \frac{\pi}{4}.$$

The integral does not exist since cot 0 is not defined.

Exercise 4.2 | Q 1.04 | Page 171

Evaluate : $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot dx$

SOLUTION

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot dx$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} \cdot dx$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1+\sin x}{1+\sin x} \cdot dx$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x} \cdot dx$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{\cos 2x} + \frac{six}{\cos^2 x}\right) \cdot dx$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left(\sec^2 x + \sec x \tan x\right) \cdot dx$$

$$= \left[\tan x + \sec x\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left(\tan \frac{\pi}{4} + \sec \frac{\pi}{4}\right) - \left[\tan\left(-\frac{\pi}{4}\right) + \sec\left(-\frac{\pi}{4}\right)\right]$$

$$= \left(1+\sqrt{2}\right) - \left(-\tan \frac{\pi}{4} + \sec \frac{\pi}{4}\right)$$

$$= \left(1+\sqrt{2}\right) - \left(-1+\sqrt{2}\right)$$

$$= 1+\sqrt{2}+1-\sqrt{2}$$

= 2.

Exercise 4.2 | Q 1.05 | Page 171

Evaluate : $\int_{3}^{5} \frac{1}{\sqrt{2x+3}-\sqrt{2x-3}} \cdot dx$

SOLUTION

$$\int_3^5 \frac{1}{\sqrt{2x+3}-\sqrt{2x-3}} \cdot dx$$

$$= \int_3^5 \frac{1}{\sqrt{2x+3}-\sqrt{2x-3}} \times \frac{\sqrt{2x+3}+\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}} \cdot dx$$

$$= \int_3^5 \frac{\sqrt{2x+3}+\sqrt{2x-3}}{(2x+3)-(2x-3)} \cdot dx$$

$$= \frac{1}{6}\int_3^5 (2+3)^{\frac{1}{2}} \cdot dx + \frac{1}{6}\int_3^5 (2x-3)^{\frac{1}{2}} \cdot dx$$

$$= \frac{1}{6}\left[\frac{2x+3^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}\right]_3^5 + \frac{1}{6}\left[\frac{(2x-3)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}\right]_3^5$$

$$= \frac{1}{18}\left[(10+3)^{\frac{3}{2}} - (6+3)^{\frac{3}{2}}\right] + \frac{1}{18}\left[(10-3)^{\frac{3}{2}} - (6-3)^{\frac{3}{2}}\right]$$

$$= \frac{1}{18}\left[13\sqrt{13} - 9\sqrt{9}\right] + \frac{1}{18}\left[7\sqrt{7} - 3\sqrt{3}\right]$$

$$= \frac{1}{18}\left(13\sqrt{13} - 27 + 7\sqrt{7} - 3\sqrt{3}\right)$$

$$= \frac{1}{18}\left(13\sqrt{13} + 7\sqrt{7} - 3\sqrt{3} - 27\right).$$

Exercise 4.2 | Q 1.06 | Page 171

Evaluate : $\int_0^1 \frac{x^2-2}{x^2+1} \cdot dx$

SOLUTION

$$\int_0^1 \frac{x^2-2}{x^2+1} \cdot dx$$

$$= \int_0^1 \frac{\left(x^2+1\right)-3}{x^2+1} \cdot dx$$

$$= \int_0^1 \left(1 - \frac{3}{x^2+1}\right) \cdot dx$$

$= [x - 3 \tan^{-1} x]^1$

$= (1 - 3\tan^{-1}1) - (0 - 3\tan^{-1} 0)$

$$= 1 - 3\left(\frac{\pi}{4}\right) - 0$$

$$= 1 - \frac{3\pi}{4}.$$

Exercise 4.2 | Q 1.07 | Page 171

Evaluate : $\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \cdot dx$

SOLUTION

$$\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \cdot dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} 2 \sin 4x \sin 3x \cdot dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} [\cos(4x-3x) - \cos(4x+3x)] \cdot dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos x \cdot dx - \frac{1}{2} \int^{\frac{\pi}{4}} \cos 7x \cdot dx$$

$$= \frac{1}{2}[\sin x]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\frac{\sin 7x}{7}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sin\ \frac{\pi}{4} - \sin 0\right] - \frac{1}{14}\left[\sin\ \frac{7\pi}{4} - \sin 0\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - 0\right] - \frac{1}{14}\left[\sin\left(2\pi - \frac{\pi}{4}\right) - 0\right]$$

$$= \frac{1}{2\sqrt{2}} - \frac{1}{14}\left(- \sin\ \frac{\pi}{4}\right)$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{14\sqrt{2}}$$

$$= \frac{7+1}{14\sqrt{2}}$$

$$= \frac{4}{7\sqrt{2}}.$$

Exercise 4.2 | Q 1.08 | Page 171

Evaluate : $\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin 2x} \cdot dx$

SOLUTION

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin 2x} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{(\sin x + \cos x)^2} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin x + \cos x) \cdot dx$$

$= \int_0^{\frac{\pi}{4}} \sin x \cdot dx + \int_0^{\frac{\pi}{4}} \cos x \cdot dx$

$= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_0^{\frac{\pi}{4}}$

$= \left[-\cos \frac{\pi}{4} - (-\cos 0)\right] + \left[\sin \frac{\pi}{4} - \sin 0\right]$

$= -\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} - 0$

= 1.

Exercise 4.2 | Q 1.09 | Page 171

Evaluate : $\int_0^{\frac{\pi}{4}} \sin^4 x \cdot dx$

SOLUTION

Consider $\sin^4 x = \left(\sin^2 x\right)^2$

$= \left(\frac{1 - \cos 2x}{2}\right)^2$

$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$

$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right]$

$= \frac{1}{4}\left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right]$

$\therefore \int_0^{\frac{\pi}{4}} \sin^4 x \cdot dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left[\frac{3}{2} - 2\cos 2x \frac{1}{2}\cos 4x\right] \cdot dx$

$$= \frac{3}{8}\int_0^{\frac{\pi}{4}} 1 \cdot dx - \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x \cdot dx + \frac{1}{8}\int_0^{\frac{\pi}{4}} \cos 4x \cdot dx$$

$$= \frac{3}{8}[x]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} + \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{3}{8}\left[\frac{\pi}{4} - 0\right] - \frac{1}{4}\left[\sin\ \frac{\pi}{2} - \sin 0\right] + \frac{1}{32}[\sin\pi - \sin 0]$$

$$= \frac{3\pi}{32} - \frac{1}{4}[1 - 0] + \frac{1}{32}(0 - 0)$$

$$= \frac{3\pi}{32} - \frac{1}{4}$$

$$= \frac{3\pi - 8}{32}.$$

Exercise 4.2 | Q 1.1 | Page 171

Evaluate : $\int_{-4}^{2} \frac{1}{x^2 + 4x + 13} \cdot dx$

SOLUTION

$$\int_{-4}^{2} \frac{1}{x^2 + 4x + 13} \cdot dx$$

$$= \int_{-4}^{2} \frac{1}{x^2 + 4x + 4 + 9} \cdot dx$$

$$= \int_{-4}^{2} \frac{1}{(x+2)^2 + 3^2} \cdot dx$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\right]_{-4}^{2}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{2+2}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{-4+2}{3}\right)$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) - \frac{1}{3}\tan^{-1}\left(-\frac{2}{3}\right)$$

$$= \frac{1}{3}\left[\tan^{-1}\ \frac{4}{3} + \tan^{-1}\ \frac{2}{3}\right].$$ ...[∵ $\tan^{-1}(-x) = -\tan^{-1} x$]

Exercise 4.2 | Q 1.11 | Page 171

Evaluate : $\int_0^4 \frac{1}{\sqrt{4x - x^2}} \cdot dx$

SOLUTION

$$\int_0^4 \frac{1}{\sqrt{4x - x^2}} \cdot dx$$

$$= \int_0^4 \frac{1}{\sqrt{4 - (x^2 - 4x + 4)}} \cdot dx$$

$$= \int_0^4 \frac{1}{\sqrt{2^2 - (x-2)^2}} \cdot dx$$

$$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^4$$

$$= \sin^{-1}\left(\frac{4-2}{2}\right) - \sin^{-1}\left(\frac{0-2}{2}\right)$$

$= \sin^{-1} 1 - \sin^{-1}(-1)$

$= 2 \sin^{-1} 1$ ...[∵ $\sin^{-1}(-x) = -\sin^{-1} x$]

$$= 2\left(\frac{\pi}{2}\right)$$

$= \pi$.

Exercise 4.2 | Q 1.12 | Page 171

Evaluate : $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx$

SOLUTION

$$\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{3-(x^2-2x+1)+1}} \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{(2)^2-(x-1)^2}} \cdot dx$$

$$= \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^1$$

$$= \sin^{-1}(0) - \sin^{1}\left(-\frac{1}{2}\right)$$

$$= 0 - \sin^{-1}\left(-\sin\ \frac{\pi}{6}\right)$$

$$= -\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= -\left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}.$$

Exercise 4.2 | Q 1.13 | Page 171

Evaluate : $\int_0^{\frac{\pi}{2}} x \sin x \cdot dx$

SOLUTION

$$\int_0^{\frac{\pi}{2}} x\sin x \cdot dx$$

$$= \left[x\int \sin x \cdot dx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left[\frac{d}{dx}(x)\int \sin x \cdot dx\right] \cdot dx$$

$$= [x(-\cos x)]_0^{\frac{\pi}{2}} - \int^{\frac{\pi}{2}} 1 \cdot (-\cos x) \cdot dx$$

$$= -[x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot dx$$

$$= -\left[\frac{\pi}{2}\cos\ \frac{\pi}{2} - 0\right] + [\sin x]_0^{\frac{\pi}{2}}$$

$$= 0 + \left(\sin\ \frac{\pi}{2} - \sin 0\right)$$

= 1.

Exercise 4.2 | Q 1.14 | Page 171

Evaluate : $\int_0^1 x\tan^{-1} x \cdot dx$

SOLUTION

Let I = $\int_0^1 x\tan^{-1} x \cdot dx$

$$= \int_0^1 (\tan^{-1} x)(x) \cdot dx$$

$$= \left[(\tan^{-1} x)\int x \cdot dx\right]_0^1 - \int_0^1\left[\frac{d}{dx}(\tan^{-1} x) \cdot \int x \cdot dx\right] \cdot dx$$

$$= \left[\frac{x^2\tan^{-1} x}{2}\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx$$

$$= \left(\frac{1^2 \tan^{-1} 1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{1 + x^2 - 1}{1 + x^2} \cdot dx$$

$$= \frac{\frac{\pi}{4}}{2} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1 + x^2}\right) \cdot dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}(x)\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\left(1 - \tan^{-1} 1\right) - 0\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

Exercise 4.2 | Q 1.15 | Page 171

Evaluate : $\int_0^\infty xe^{-x} \cdot dx$

SOLUTION

$$\int_0^\infty xe^{-x} \cdot dx$$

$$= \left[x \int e^{-x} \cdot dx\right]_0^\infty - \int_0^\infty \left[\frac{d}{dx}(x) \int e^{-x} \cdot dx\right] \cdot dx$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty - \int^\infty 1. \frac{e^{-x}}{(-1)} \cdot dx$$

$$= \left[-\frac{x}{e^x}\right]_0^\infty + \int_0^\infty e^{-x} \cdot dx$$

$$= \left[-\frac{x}{e^x}\right]_0^\infty + \left[-e^x\right]_0^\infty$$

= [0 – (–0)] + [0 – (–1)]

= 1. ...[$\because e^0 = 1, e^{-x} = 0$, when $x = \infty$]

Exercise 4.2 | Q 2.01 | Page 172

Evaluate : $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}} \cdot dx$

= $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1 - x^2)\sqrt{1 - x^2}} \cdot dx$

Put $\sin^{-1} x = t$

$\therefore \frac{1}{\sqrt{1 - x^2}} \cdot dx = dt$

Also, $x = \sin t$

When $x = \frac{1}{\sqrt{2}}, t = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

When $x = 0$, $t = \sin^{-1} 0 = 0$

$\therefore$ I = $\int_0^{\frac{\pi}{4}} \frac{t}{1 - \sin^2 t} \cdot dt$

= $\int_0^{\frac{\pi}{4}} \frac{t}{\cos^2 t} \cdot dt$

$= \int_0^{\frac{\pi}{4}} t \sec^2 t \cdot dt$

$= \left[t \int \sec^2 t \cdot dt\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[\frac{d}{dt}(t) \int \sec^2 t \cdot dt\right] \cdot dt$

$= [t \tan t]_0^{\frac{p}{4}} - \int_0^{\frac{\pi}{4}} 1 \cdot \tan t \cdot dt$

$= \left[\frac{\pi}{4} \tan \frac{\pi}{4} - 0\right] - [\log|\sec t|]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \left[\log\left(\sec \frac{\pi}{4}\right) - \log(\sec 0)\right]$

$= \frac{\pi}{4} - \left[\log \sqrt{2} - \log 1\right]$

$= \frac{\pi}{4} - \frac{1}{2} \log 2.$ ...[$\because$ log 1 = 0]

Exercise 4.2 | Q 2.02 | Page 172

Evaluate : $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 \tan^2 x + 4 \tan x + 1} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 \tan^2 x + 4 \tan x + 1} \cdot dx$

Put tan x = t

$\therefore$ $\sec^2 x \cdot dx = dt$

When x = 0, t = tan 0 = 0

When x = $\frac{\pi}{4}, t = \tan \frac{\pi}{4}$ = 1

$\therefore$ I = $\int_0^1 \frac{dt}{3t^2 + 4t + 1}$

$$= \frac{1}{3}\int_0^1 \frac{dt}{t^2 + \frac{4}{3}t + \frac{1}{3}}$$

$$= \frac{1}{3}\int_0^1 \frac{dt}{t^2 + \frac{4t}{3} + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}}$$

$$= \frac{dt}{(t + \frac{2}{3})2 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{1}{3}\frac{1}{2\left(\frac{1}{3}\right)}\left[\log\left|\frac{t + \frac{2}{3} - \frac{1}{3}}{t + \frac{2}{3} + \frac{1}{3}}\right|\right]_0^1$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + \frac{1}{3}}{1 + 1}\right) - \log\left(\frac{0 + \frac{1}{3}}{0 + 1}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{2}{3}\right) - \log\left(\frac{1}{3}\right)\right.$$

$$= \frac{1}{2}\log 2.$$

Exercise 4.2 | Q 2.03 | Page 172

Evaluate : $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} \cdot dx$$

Dividing each term by $\cos^4 x$, we get

I = $\int_0^{\frac{\pi}{4}} \frac{2\frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x}}{\frac{\sin^4 x}{\cos^4 x} + 1} \cdot dx$

= $\int_0^{\frac{\pi}{4}} \frac{2 \tan x \sec^2 x}{\left(\tan^2\right)^2 + 1} \cdot dx$

Put $\tan^2 x = t$

$\therefore$ $2\tan x \sec^2 x \cdot dx = dt$

When x = 0, t = $\tan^2 0 = 0$

When x = $\frac{\pi}{4}, t = \tan^2 \frac{\pi}{4}$ = 1

$\therefore$ I = $\int_0^1 \frac{dt}{1 + t^2}$

= $\left[\tan^{-1} t\right]_0^1$

= $\tan^{-1} 1 - \tan^{-1} 0$

= $\frac{\pi}{4} - 0$

= $\frac{\pi}{4}$.

Exercise 4.2 | Q 2.04 | Page 172

Evaluate : $\int_0^{2\pi} \sqrt{\cos x} \sin^3 x \cdot dx$

SOLUTION

Let I = $\int_0^{2\pi} \sqrt{\cos x}\sin^3 x \cdot dx$

= $\int_0^{2\pi} \sqrt{\cos x}\sin^2 x \sin x \cdot dx$

= $\int_0^{2\pi} \sqrt{\cos x}\left(1 - \cos^2 x\right)\sin x \cdot dx$

Put cos x = t

∴ − sin x·dx = dt

∴ sin x·dx = − dt

When x = 0, t = cos 0 = 1

When x = 2π, t = cos 2π = 1

∴ I = $\int_1^1 \sqrt{t}\left(1 - t^2\right)(-dt)$ = 0. ...$\left[\because \int_a^a f(x) \cdot dx = 0\right]$

Exercise 4.2 | Q 2.05 | Page 172

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x} \cdot dx$

Put $\tan\left(\frac{x}{2}\right)$ = t

∴ x = 2 $\tan^{-1}$ t

∴ dx = $\frac{2dt}{1 + t}$

and

cos x = $\dfrac{1-t^2}{1+t^2}$

When x = $\dfrac{\pi}{2}, t = \tan\left(\dfrac{\pi}{2}\right)$ = 1

When x = 0, t = tan 0 = 0

∴ I = $\dfrac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$

= $\displaystyle\int_0^1 \frac{2dt}{5(1+t^2)+4(1-t^2)}$

= 2 $\displaystyle\int_0^1 \frac{1}{t^2+9}\cdot dt$

= 2 $\left[\dfrac{1}{3}\tan^{-1}\ \dfrac{t}{3}\right]_0^1$

= 2 $\left[\dfrac{1}{3}\tan^{-1}\ \dfrac{1}{3}-\dfrac{1}{3}\tan^{-1}0\right]$

= $\dfrac{2}{3}\tan^{-1}\ \dfrac{1}{3}-\dfrac{2}{3}\times 0$

= $\dfrac{2}{3}\tan^{-1}\ \left(\dfrac{1}{3}\right).$

Exercise 4.2 | Q 2.06 | Page 172

Evaluate : $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{4-\sin^2 x}\cdot dx$

SOLUTION

Let I = $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{4-\sin^2 x}\cdot dx$

Put sin x = t

∴ cos x·dx = dt

When x = $\frac{\pi}{4}, t = \sin\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

When x = 0, t = sin 0 = 0.

∴ I = $\int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{2^2 - t^2}$

= $\left[\frac{1}{2(2)}\log\left|\frac{2+t}{2-t}\right|\right]_0^{\frac{1}{\sqrt{2}}}$

= $\frac{1}{4}\left[\log\left(\frac{2+\frac{1}{\sqrt{2}}}{2-\frac{1}{\sqrt{2}}}\right) - \log\left(\frac{2+0}{2\ -0}\right)\right]$

= $\frac{1}{4}\left[\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-1}\right) - \log 1\right]$

= $\frac{1}{4}\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-1}\right)$. ...[∵ log 1 = 0]

Exercise 4.2 | Q 2.07 | Page 172

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)} \cdot dx$

Put sin x = t

∴ cos x·dx = dt

When x = $\frac{\pi}{2}, t = \sin\ \frac{\pi}{2}$ = 1

When x = 0, t = sin 0 = 0

∴ I = $\int_0^1 \frac{dt}{(1+t)(2+t)}$

= $\int_0^1 \frac{(2+t)-(1+t)}{(1+t)(2+t)} \cdot dt$

= $\int_0^1 \left[\frac{1}{1+t} - \frac{1}{2+t}\right] \cdot dt$

= $\int_0^1 \frac{1}{1+t} \cdot dt - \int_0^1 \frac{1}{2+t} \cdot dt$

= $[\log|1+t|]_0^1 - [\log|2+t|]_0^1$

= [log(1 + 1) – log(1 + 0)] – [log(2 + 1) – log(2 + 0)]

= log 2 – log 3 + log 2 ...[∵ log 1 = 0]

= $\log\left(\frac{2\times 2}{3}\right)$

= $\log\left(\frac{4}{3}\right)$.

Exercise 4.2 | Q 2.08 | Page 172

Evaluate : $\int_{-1}^1 \frac{1}{a^2e^x + b^2e^{-x}} \cdot dx$

SOLUTION

Let I = $\int_{-1}^{1} \frac{e^x}{a^2(e^x)^2 + b^2} \cdot dx$

Put $e^x = t$

$\therefore e^x \cdot dx = dt$

When x = 1, t = e

When x = – 1, t = e^{-1} = $\frac{1}{e}$

$\therefore$ I = $\int_{\frac{1}{e}}^{e} \frac{dt}{a^2t^2 + b^2}$

= $\int_{\frac{1}{e}}^{e} \frac{dt}{(at)^2 + b^2}$

= $\left[\frac{1}{a} \cdot \frac{1}{b}\tan^{-1}\left(\frac{\text{at}}{b}\right)\right]_{\frac{1}{e}}^{e}$

= $\frac{1}{\text{ab}}\tan^{-1}\left(\frac{\text{ae}}{b}\right) - \frac{1}{\text{ab}}\tan^{-1}\left(\frac{a}{\text{be}}\right)$

= $(1)\text{ab}\left[\tan^{-1}\left(\frac{\text{ae}}{b}\right) - \tan^{-1}\left(\frac{a}{\text{be}}\right)\right]$.

Exercise 4.2 | Q 2.09 | Page 172

Evaluate : $\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} \cdot dx$

SOLUTION

Let I = $\int_0^{\pi} \frac{1}{3+2\sin x+\cos x} \cdot dx$

Put $\tan \frac{x}{2}$ = t

∴ x = 2 $\tan^{-1}$ t

∴ dx = $\frac{2dt}{1+t^2}$

and

sinx = $\frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$

When x = 0, t = tan0 = 0

When x = $\pi, t = \tan \frac{\pi}{2} = \infty$

∴ I = $\int_0^{\infty} \frac{1}{3+2\left(\frac{2t}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$

= $\int_0^{\infty} \frac{1}{2t^2+4t+4} \cdot dt$

= $\frac{2}{2} \int_0^{\infty} \frac{1}{t^2+2t+2} \cdot dt$

= $\int_0^{\infty} \frac{1}{(t^2+2t+1)+1} \cdot dt$

= $\int_0^{\infty} \frac{1}{(t^2+2t+1+1) \cdot dt}$

= $\int_0^{\infty} \frac{1}{(t+1)^2+(1)^2} \cdot dt$

= $\frac{1}{1}\left[\tan^{-1}\left(\frac{t+1}{1}\right)\right]_0^{\infty}$

= $\left[\tan^{-1}(t+1)\right]_0^{\infty}$

= $\tan^{-1}\infty - \tan^{1} 1$

= $\frac{\pi}{2} - \frac{\pi}{2}$

= $\frac{\pi}{4}$.

Exercise 4.2 | Q 2.1 | Page 172

Evaluate : $\int_0^{\frac{\pi}{4}} \sec^4 x \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{4}} \sec^4 x \cdot dx$

= $\int_0^{\frac{\pi}{4}} \sec^2 x \cdot \sec^2 x \cdot dx$

= $\int_0^{\frac{\pi}{4}} \left(1+\tan^2 x\right) \sec^2 x \cdot dx$

Put tan x = t

∴ $\sec^2 x \cdot dx = dt$

When x = 0, t = tan 0 = 0

When x = $\frac{\pi}{4}, t = \tan\ \frac{\pi}{4}$ = 1

∴ I = $\int_0^1 \left(1+t^2\right) \cdot dt$

= $\left[t+\frac{t^3}{3}\right]_0^1$

= $1+\frac{1}{3}-0$

= $\dfrac{4}{3}$.

Exercise 4.2 | Q 2.11 | Page 172

Evaluate : $\int_0^1 \sqrt{\dfrac{1-x}{1+x}} \cdot dx$

SOLUTION

Let I = $\int_0^1 \sqrt{\dfrac{1-x}{1+x}} \cdot dx$

Put x = cos θ

dx = - sinθ dθ

When x = 0, cos θ = 0 = cos $\dfrac{\pi}{2}$ ∴ θ = $\dfrac{\pi}{2}$

When x = 1, cos θ = 1 = cos 0 ∴ θ = 0

∴ I = $\int_{\frac{\pi}{2}}^0 \sqrt{\dfrac{-\cos\theta}{1+\cos\theta}} \cdot (-\sin\theta) d\theta$

= $\int_{\frac{\pi}{2}}^0 \sqrt{\dfrac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}} \left(-2\dfrac{\sin\theta}{2}\cos\ \dfrac{\theta}{2}\right) \cdot d\theta$

= $\int_{\frac{\pi}{2}}^0 \left(\dfrac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right) \left[-2\sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right)\right] \cdot d\theta$

= $\int_{\frac{\pi}{2}}^0 -2\sin^2\left(\dfrac{\theta}{2}\right) \cdot d\theta$

= $-\int_{\frac{\pi}{2}}^0 (1-\cos\theta) \cdot d\theta$

$= -[\theta - \sin\theta]_{\frac{\pi}{2}}^{0}$

$= -\left[(0 - \sin 0) - \left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right)\right]$

$= -\left[0 - \frac{\pi}{2} + 1\right]$

$= \frac{\pi}{2} - 1.$

Exercise 4.2 | Q 2.12 | Page 172

Evaluate : $\int_0^{\pi} \sin^3 x(1 + 2\cos x)(1 + \cos x)^2 \cdot dx$

SOLUTION

Let I = $\int_0^{\pi} \sin^3 x(1 + 2\cos x)(1 + \cos x)^2 \cdot dx$

$= \int_0^{\pi} \sin^2 x(1 + 2\cos x)(1 + \cos x)^2 \cdot \sin x \cdot dx$

$= \int_0^{\pi} (1 - \cos^2 x)(1 + 2\cos x)(1 + \cos x)^2 \cdot \sin x \cdot dx$

Put cos x = t

∴ – sin x·dx = dt.

∴ sin x·dx = – dt

When x = 0, t = cos 0 = 1

When x = π, t = cos π = – 1

∴ I = $\int_1^{-1} (1 - t^2)(1 + 2t)(1 + t)^2(-dt)$

$= -\int_1^{-1} (1 + 2t - t^2 - 2t^3)(1 + 2t + t^2) \cdot dt$

$$= -\int_1^{-1}\left(1 + 2t - t^2 - 2t^3 + 2t + 4t^2 - 2t^3 - 4t^4 + t^2 + 2t^3 - t^4 - 2t^5\right) \cdot dt$$

$$= \int_1^{-1}\left(1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5\right) \cdot dt$$

$$= \int_1^{-1}\left(1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5\right) \cdot dt$$

$$= -\left[t + 4\left(\frac{t^2}{2}\right) + 4\left(\frac{t^3}{3}\right) - 2\left(\frac{t^4}{4}\right) - 5\left(\frac{t^5}{5}\right) - 2\left(\frac{t^6}{6}\right)\right]_1^{-1}$$

$$= -\left[t + 2t^2\ \frac{4}{3}t^3 - \frac{1}{2}t^4 - t^5 - \frac{1}{3}t^6\right]_1^{-1}$$

$$= -\left[\left(-1 + 2 - \frac{4}{3} - \frac{1}{2} + 1 - \frac{1}{3}\right) - \left(1 + 2 + \frac{4}{3} - \frac{1}{2} - 1 - \frac{1}{3}\right)\right]$$

$$= -\left[-1 + 2 - \frac{4}{3} - \frac{1}{2} + 1 - \frac{1}{3} - 1 - 2 - \frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{3}\right]$$

$$= -\left[-\frac{8}{3}\right]$$

$$= \frac{8}{3}.$$

Exercise 4.2 | Q 2.13 | Page 172

Evaluate : $\int_0^{\frac{\pi}{2}} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

= $\int_0^{\frac{\pi}{2}} \tan^{-1}(\sin x) \cdot (2 \sin x \cos x) \cdot dx$

Put sinx = t

∴ cos x·dx= dt

When x = 0, t = sin 0 = 0.

When x = $\frac{\pi}{2}, t = \sin\ \frac{\pi}{2}$ = 1

∴ I = $\int_0^1 (\tan^1 t)(2t) \cdot dt$

= $\left[\tan^{-1} t \int 2t\ dt\right]_0^1 - \int_0^1 \left(\frac{d}{dt}(\tan^{-1} t) \int 2t\ dt\right) \cdot dt$

= $\left[\tan^{-1} t)\left(t^2\right)\right]_0^1 - \int_0^1 \frac{1}{1+t^2} \cdot t^2 \cdot dt$

= $t^2 \tan^{-1}\ t\Big]_0^1 - \int_0^1 \frac{\left(1+t^2\right)-1}{1+t^2} \cdot dt$

= $t^2 \tan^{-1}\ t\Big]_0^1 - \int_0^1 \frac{\left(1+t^2\right)-1}{1+t^2} \cdot dt$

= $\left[1 \cdot \tan^{-1} - 0\right] - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) \cdot dt$

= $\frac{\pi}{4} - \left[t - \tan^{-1} t\right]_0^1$

= $\dfrac{p}{4} - \left[\left(1 - \tan^{-1} 1\right) - 0\right]$

= $\dfrac{\pi}{4} - 1 + \dfrac{\pi}{4}$

= $\dfrac{\pi}{2} - 1.$

Exercise 4.2 | Q 2.14 | Page 172

Evaluate : $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{1} \frac{e^{\cos^{-1} x} \sin^{-1} x}{\sqrt{1 - x^2}} \cdot dx$

SOLUTION

Let I = $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{1} \frac{e^{\cos^{-1} x} \sin^{-1} x}{\sqrt{1 - x^2}} \cdot dx$

Put $\sin^{-1}$ x = t

$\therefore \dfrac{1}{\sqrt{1 - x^2}} \cdot dx$ = dt

When x = 1, t = $\sin^{-1} 1 = \dfrac{\pi}{2}$

When x = $\dfrac{1}{\sqrt{2}}, t = \dfrac{\sin^{-1} 1}{\sqrt{2}} = \dfrac{\pi}{4}$

Also, $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x = \dfrac{\pi}{2} - t$

$\therefore$ I = $\displaystyle\int_{\frac{i}{4}}^{\frac{\pi}{2}} e^{\frac{\pi}{2} - t} \cdot t\, dt$

= $e^{\frac{\pi}{2}} \displaystyle\int_{\frac{i}{4}}^{\frac{\pi}{2}} te^{-t} dt$

= $e^{\frac{\pi}{2}} \left\{ \left[t \int e^{-t} dt\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{i}{4}}^{\frac{\pi}{2}} \left[\frac{d}{dt}(t) \int e^{-t} dt\right] \cdot dt \right\}$

$$= e^{\frac{\pi}{2}}\left\{\left[-te^{-t}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{i}{4}}^{\frac{\pi}{2}}(1)\left(-e^{-t}\right)\cdot dt\right\}$$

$$= e^{\frac{\pi}{2}}\left\{\frac{-\pi}{2}e^{-\frac{\pi}{2}} + \frac{\pi}{4}e^{-\frac{\pi}{4}} + \int_{\frac{i}{4}}^{\frac{\pi}{2}} e^{-t}\cdot dt\right\}$$

$$= -\frac{\pi}{2}e^{o} + \frac{\pi}{4}e^{\frac{\pi}{2}-\frac{\pi}{4}} + e^{\frac{\pi}{2}}\left[-e^{-t}\right]^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2} + \frac{\pi}{4}e^{\frac{\pi}{4}} + e^{\frac{\pi}{2}}\left[-e^{-\frac{\pi}{2}} + e^{\frac{-\pi}{4}}\right]$$

$$= -\frac{\pi}{2} + e^{\frac{\pi}{4}}\frac{\pi}{4} - e^{o}+^{\frac{\pi}{2}-\frac{\pi}{4}}$$

$$= -\frac{\pi}{2} + e^{\frac{\pi}{4}}\frac{\pi}{4} - 1 + e^{\frac{\pi}{4}}$$

$$= e^{\frac{\pi}{4}}\left(\frac{\pi}{4}+1\right) - \left(\frac{\pi}{2}+1\right).$$

Exercise 4.2 | Q 2.15 | Page 172

Evaluate : $\int_1^3 \frac{\cos(\log x)}{x}\cdot dx$

SOLUTION

Let I = $\int_1^3 \frac{\cos(\log x)}{x}\cdot dx$

$= \int_1^3 \cos(\log x)\cdot\frac{1}{x}\cdot dx$

Put log x = t

$\therefore \frac{1}{x}\cdot dx$ = dt

When x = 1, t = log 1 = 0

When x = 3, t = log 3

$\therefore$ I = $\int_0^{\log 3} \cos t \cdot dt = [\sin t]_0^{\log 3}$

= sin (log 3) - sin 0

= sin (log 3).

Exercise 4.2 | Q 3.01 | Page 172

Evaluate the following: $\int_0^{a} \frac{1}{x+\sqrt{a^2-x^2}} \cdot dx$

SOLUTION

Let I = $\int_0^{a} \frac{1}{x+\sqrt{a^2-x^2}} \cdot dx$

Put x = a sin θ

$\therefore$ dx = a cos θ dθ

and

$\sqrt{a^2-x^2}$

= $\sqrt{a^1-a^2\sin^2\theta}$

= $\sqrt{a^2\left(1-\sin^2\theta\right)}$

= $\sqrt{a^2\cos^2\theta}$

= a cos θ

When x = 0, a sin θ = 0 $\therefore$ θ = 0

When x - a, a sin θ = a $\therefore$ θ = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\frac{\pi}{2}} \frac{a\cos\theta d\theta}{a\sin\theta + a\cos\ \theta}$

$\therefore$ I = $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\ \theta} \cdot d\theta$ ...(1)

We use the property, $\int_0^a f(a-x)\cdot dx$.

Hence in I, we change θ by $\left(\frac{\pi}{2}\right)-\theta$.

$\therefore$ I = $\int_0^{\frac{\pi}{2}} \frac{\cos\left[\left(\frac{\pi}{2}\right)-\theta\right]}{\sin\left[\left(\frac{\pi}{2}\right)-\theta\right]+\cos\left[\left(\frac{\pi}{2}\right)-\theta\right]}\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\cdot d\theta$ ...(2)

Adding (1) and (2), we get

2I = $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\cdot d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} \frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} 1\cdot d\theta = [\theta]_0^{\frac{\pi}{2}}$

= $\left(\frac{\pi}{2}\right)-0$

= $\frac{\pi}{2}$

$\therefore$ I = $\frac{\pi}{4}$.

Exercise 4.2 | Q 3.02 | Page 172

Evaluate the following : $\int_0^{\frac{\pi}{2}} \log(\tan x)\cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \log(\tan x) \cdot dx$

We use the property, $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$.

Here, $a = \frac{\pi}{2}$.

Hence changing x by $\frac{\pi}{2} - x$, we get

I = $\int_0^{\frac{\pi}{2}} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] \cdot dx$

= $\int_0^{\frac{\pi}{2}} \log(\cot x) \cdot dx$

= $\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\tan x}\right) \cdot dx$

= $\int_0^{\frac{\pi}{2}} \log(\tan x)^{-1} \cdot dx$

= $\int_0^{\frac{\pi}{2}} - \log(\tan x) \cdot dx$

= $- \int_0^{\frac{\pi}{2}} \log(\tan x) \cdot dx$

= – I

∴ 2I = 0

∴ I = 0.

Exercise 4.2 | Q 3.03 | Page 172

Evaluate the following : $\int_0^1 \log\left(\frac{1}{x} - 1\right) \cdot dx$

SOLUTION

Let I = $\int_0^1 \log\left(\frac{1}{x} - 1\right) \cdot dx$

= $\int_0^1 \log\left(\frac{1 - x}{x}\right) \cdot dx$

= $\int_0^1 [\log(1 - x) - \log x] \cdot dx$ ...(1)

We use the property $\int_0^a f(x) \cdot dx = \int_0^a f(a - x) \cdot dx$

Here, a = 1

Hence in I, changing x to 1 – x, we get

I = $\int_0^1 [\log|1 - (1 - x)| - \log(1 - x)] \cdot dx$

= $\int_0^1 [\log x - \log(1 - x)] \cdot dx$

= $-\int_0^1 [\log(1 - x) - \log x] \cdot dx$

= – 1 ...[By (1)]

∴ 2I = 0

∴ I = 0.

Exercise 4.2 | Q 3.04 | Page 172

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$

We use the property, $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$.

Here $a = \frac{\pi}{2}$.

Hence In I, we change x by $\frac{\pi}{2} - x$.

∴ I = $\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}$

= $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} \cdot dx$

= $-\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$

= – I

∴ 2I = 0

∴ I = 0.

Exercise 4.2 | Q 3.05 | Page 172

Evaluate the following : $\int_0^3 x^2 (3-x)^{\frac{5}{2}} \cdot dx$

SOLUTION

Let I = $\int_0^3 x^2 (3-x)^{\frac{5}{2}} \cdot dx$

We use the property $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

Here, a = 3

Hence in I, changing x to 3 – x, we get

I = $\int_0^3 (3-x)^2[3-(3-x)]^{\frac{5}{2}} \cdot dx$

= $\int_0^3 (9-6x+x^2)x^{\frac{5}{2}} \cdot dx$

= $\int_0^3 \left[9x^{\frac{5}{2}} - 6x^{\frac{7}{2}} + x^{\frac{9}{2}}\right] \cdot dx$

= $9\int_0^3 x^{\frac{5}{2}} \cdot dx - 6\int_0^3 x^{\frac{7}{2}} \cdot dx + \int_0^3 x^{\frac{9}{2}} \cdot dx$

= $9\left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^3 - 6\left[\frac{x^{\frac{9}{2}}}{\frac{9}{2}}\right]_0^3 + 9\left[\frac{x^{\frac{11}{2}}}{\frac{11}{2}}\right]_0^3$

= $9\left[\frac{2.3^{\frac{7}{2}}}{7} - 0\right] - 6\left[\frac{2.3^{\frac{9}{2}}}{9} - 0\right] + \left[\frac{2}{11} \cdot 3^{\frac{11}{2}} - 0\right]$

= $\frac{18}{7}3^{\frac{7}{2}} - \frac{2.6}{9} \cdot 3^{\frac{7}{2}} \cdot 3 + \frac{2}{11} \cdot 3^{\frac{7}{2}} \cdot 3^2$

= $2(3)^{\frac{7}{2}}\left[\frac{9}{7} - 2 + \frac{9}{11}\right]$

= $2(3)^{\frac{7}{2}}\left[\frac{99-154+63}{77}\right]$

= $2(3)^{\frac{7}{2}} \times \frac{8}{77}$

= $\frac{16}{77}(3)^{\frac{7}{2}}$.

Exercise 4.2 | Q 3.06 | Page 172

Evaluate the following : $\int_{-3}^3 \frac{x^3}{9-x^2} \cdot dx$

SOLUTION

Let I = $\int_{-3}^{3} \frac{x^3}{9-x^2} \cdot dx$

Let f(x) = $\frac{x^3}{9-x^2}$

$\therefore$ f(−x) = $\frac{(-x)^3}{9-(-x)^2}$

= $\frac{-x^3}{9-x^2}$

= $-f(x)$

$\therefore$ f is an odd function.

$\therefore \int_{-3}^{3} f(x) \cdot dx = 0, \text{i.e.} \int_{-3}^{3} \frac{x^3}{9-x^2} \cdot dx$ = 0.

Exercise 4.2 | Q 3.07 | Page 172

Evaluate the following : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2+\sin x}{2-\sin x}\right) \cdot dx$

SOLUTION

Let I = $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2-\sin x}{2+\sin x}\right) \cdot dx$

Let f(x) = $\log\left(\frac{2-\sin x}{2+\sin x}\right)$

$\therefore$ f(− x)= $\log\left[\frac{2-\sin(-x)}{2+\sin(-x)}\right]$

$= \log\left(\frac{2+\sin x}{2-\sin x}\right)$

$= -\log\left(\frac{2-\sin x}{2+\sin x}\right)$

= – f(x)

∴ f is an odd function.

$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\cdot dx = 0$

$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2-\sin x}{2+\sin x}\right)\cdot dx = 0.$

Exercise 4.2 | Q 3.08 | Page 172

Evaluate the following : $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\cdot dx$

SOLUTION

Let I = $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\cdot dx$

$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left[\frac{x}{2-\cos 2x}+\frac{\frac{\pi}{4}}{2-\cos 2x}\right]$

$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2-\cos 2x}\cdot dx + \frac{\pi}{4}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x}\cdot dx$

$= \mathrm{I}_1 + \frac{\pi}{4}\mathrm{I}_2$ ...(1)

Let f(x) = $\frac{x}{2-\cos 2x}$

$\therefore$ f(– x) = $\dfrac{-x}{2-\cos[2(-x)]}$

= $\dfrac{-x}{2-\cos 2x}$

= – f(x)

$\therefore$ f is an odd function

$\therefore \displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} f(x)\cdot dx$ = 0

i.e. $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2-\cos 2x}\cdot dx$ = 0, i.e. I_1 = 0 ...(2)

In I_2, put tan x = t

$\therefore$ x = $\tan^{-1}$t

$\therefore$ dx = $\dfrac{1}{1+t^2}\cdot dt$

and

cos 2x = $\dfrac{1-t^2}{1+t^2}$

When x = $-\dfrac{\pi}{4}, t = \tan\left(-\dfrac{\pi}{4}\right)$ = – 1

When x = $\dfrac{\pi}{4}, t = \dfrac{\tan\pi}{4}$ = 1.

$\therefore I_2$ = $\displaystyle\int_{-1}^{1} \frac{1}{2-\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{1}{1+t^2}\cdot dt$

= $\displaystyle\int_{-1}^{1} \frac{1}{2(1+t^2)-(1-t^2)}\cdot dt$

$= \int_{-1}^{1} \frac{1}{3t^2 + 1} \cdot dt$

$= \int_{-1}^{1} \frac{1}{\left(\sqrt{3}t\right)^2 + 1}$

$= \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}t}{1}\right)\right]_{-1}^{1}$

$= \frac{1}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1}\left(-\sqrt{3}\right)\right]$

$= \frac{1}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} + \tan^{-1} \sqrt{3}\right]$

$= \frac{1}{\sqrt{3}} \left[\frac{\pi}{3} + \frac{\pi}{3}\right]$

$= \frac{2\pi}{3\sqrt{3}}$ ...(3)

From (1), (2) and (3), we get

$I = 0 + \frac{\pi}{4} \left[\frac{2\pi}{3\sqrt{3}}\right]$

$= \frac{\pi^2}{6\sqrt{3}}$.

Exercise 4.2 | Q 3.09 | Page 172

Evaluate the following : $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$

SOLUTION

Let I = $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$

Let f(x) = $x^3 \sin^4 x$

∴ f(−x) = $(-x)^3 \sin^4(-x)$

= $-x^3 \sin^4 x$

= $-f(x)$

∴ f is an odd function.

∴ $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} f(x) \cdot dx = 0, \text{i.e.} \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$ = 0.

Exercise 4.2 | Q 3.1 | Page 172

Evaluate the following : $\int_0^1 \frac{\log(x+1)}{x^2+1} \cdot dx$

SOLUTION

Let I = $\int_0^1 \frac{\log(x+1)}{x^2+1} \cdot dx$

Put x = tan θ.

∴ dx = $\sec^2\theta \cdot d\theta$

and

$x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$

When x = 0, tan θ = 0 ∴ θ = 0

When x = 1, tan θ = 1 ∴ θ = $\frac{\pi}{4}$

$\therefore$ I = $\int_0^{\frac{\pi}{4}} \frac{\log(\tan\theta + 1)}{\sec^2\theta} \cdot \sec 2\theta \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) \cdot d\theta$ ...(1)

We use the property, $\int_0^a f(x) \cdot dx = \int_0^a f(a - x) \cdot dx$.

Here, a = pi/(4).

Hence changing θ by $\frac{\pi}{4} - \theta$, we have,

I = $\int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log\left(\frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\right) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + \tan\theta}\right) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} [\log 2 - \log(1 + \tan\theta)] \cdot d\theta$

= $\log 2 \int_0^{\frac{\pi}{4}} 1 \cdot d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) \cdot d\theta$

= $(\log 2)[\theta]_0^{\frac{\pi}{4}} - \mathrm{I}$

= $\frac{\pi}{4} \log 2 - \mathrm{I}$

$\therefore$ 2I = $\frac{\pi}{4} \log 2$

$\therefore$ I = $\frac{\pi}{8} \log 2$.

Exercise 4.2 | Q 3.11 | Page 172

Evaluate the following : $\int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx$

SOLUTION

Let I = $\int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx$

= $\int_{-1}^{1} \left[\frac{x^3}{\sqrt{x^2 + 4}} + \frac{2}{\sqrt{x^2 + 4}} \right] \cdot dx$

= $\int_{-1}^{1} \frac{x^3}{\sqrt{x^2 + 4}} \cdot dx + 2 \int \frac{1}{\sqrt{x^2 + 4}} \cdot dx$

= $I_1 + 2I_2$ ...(1)

Let f(x) = $\frac{x^3}{\sqrt{x^2 + 4}}$

∴ f(− x) = $\frac{(-x)^3}{\sqrt{(-x)^2 + 4}}$

= $\frac{x^3}{\sqrt{x^2 + 4}}$

= − f(x)

∴ f is an odd function.

$\therefore \int_{-1}^{1} \cdot dx$ = 0, i.e.

$I_1 = \int_{-1}^{1} = \frac{x^3}{\sqrt{x^2+4}} \cdot dx = 0$ ...(2)

$\because (-x)^2 = x^2$

$\therefore \frac{1}{\sqrt{x^2+4}}$ is an even function.

$\therefore \int_{-1}^{1} f(x) \cdot dx = 2\int_{0}^{1} f(x) \cdot dx$

$\therefore I_2 = 2\int_{0}^{1} \frac{1}{\sqrt{x^2+4}} \cdot dx$

$= 2\left[\log\left(x+\sqrt{x^2+4}\right)\right]_0^1$

$= 2g\left(1+\sqrt{1+4}\right) - \log\left(0+\sqrt{0+4}\right)\Big]$

$= 2\left[\log\left(\sqrt{5}+1\right) - \log 2\right]$

$= 2\log\left(\frac{\sqrt{5}+1}{2}\right)$ ...(3

From (1), (2) and (3, we get

$I = 0 + 2\left[2\log\left(\frac{\sqrt{5}+1}{2}\right)\right]$

$= 4\log\left(\frac{\sqrt{5}+1}{2}\right)$.

Exercise 4.2 | Q 3.12 | Page 172

Evaluate the following : $\int_{-a}^{a} \frac{x+x^3}{16-x^2} \cdot dx$

SOLUTION

Let I = $\int_{-a}^{a} \frac{x+x^3}{16-x^2} \cdot dx$

Let f(x) = $\frac{x+x^3}{16-x^2}$

∴ f(−x) = $\frac{(-x)+(-x)^3}{16-(-x)^2}$

= $\frac{-(x+x^3)}{16-x^2}$

= $-f(x)$

∴ f is an odd function.

∴ $\int_{-a}^{a} f(x) \cdot dx = 0, \text{i.e.} \int_{a}^{a} \frac{x+x^3}{16-x^2} \cdot dx$ = 0.

Exercise 4.2 | Q 3.13 | Page 172

Evaluate the following : $\int_{0}^{1} t^2\sqrt{1-t} \cdot dt$

SOLUTION

We use the property,

$\int_{0}^{a} f(t) \cdot dt = \int_{0}^{a} f(a-t) \cdot dt$

∴ $\int_{0}^{1} t^2\sqrt{t}(1-t) \cdot dt = \int_{0}^{1} (1-t)^2\sqrt{1-1+t} \cdot dt$

= $\int_{0}^{1} \left(1-2t+t^2\right)\sqrt{t} \cdot dt$

$$= \int_0^1 \left(t^{\frac{1}{2}} - 2t^{\frac{3}{2}} + t^{\frac{5}{2}}\right) \cdot dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2 \cdot \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3}(1)^{\frac{3}{2}} - \frac{4}{5}(1)^{\frac{5}{2}} + \frac{2}{7}(1)^{\frac{7}{2}} - 0$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} - 0$$

$$= \frac{70 - 84 + 30}{105}$$

$$= \frac{16}{105}.$$

Exercise 4.2 | Q 3.14 | Page 172

Evaluate the following : $\int_0^\pi x \sin x \cos^2 x \cdot dx$

SOLUTION

Let I = $\int_0^\pi x \sin x \cos^2 x \cdot dx$

$$= \frac{1}{2}\int_0^a x(2\sin x \cos x)\cos x \cdot dx$$

$$= \frac{1}{2}\int_0^\pi x(\sin 2x \cos x) \cdot dx$$

$$= \frac{1}{4}\int_0^\pi x(2\sin 2x \cos x) \cdot dx$$

$$= \frac{1}{4}\int_0^\pi [\sin(2x + x) + \sin(2x - x)] \cdot dx$$

$$= \frac{1}{4}\left[\int_0^\pi x \sin 3x \cdot dx + \int_0^\pi x \sin x \cdot dx\right]$$

$$= \frac{1}{4}[\mathrm{I}_1 + \mathrm{I}_2] \qquad ...(1)$$

$$\mathrm{I}_1 = \int_0^{\pi} x \sin 3x \cdot dx$$

$$= \left[x \int \sin 3x \cdot dx\right]_0^{\pi} - \int \left[\left\{\frac{d}{dx}(x) \int \sin 3x \cdot dx\right\}\right] \cdot dx$$

$$= \left[x\left(\frac{-\cos 3x}{3}\right)\right]_0^{\pi} - \int_0^{\pi} 1\left(\frac{-\cos 3x}{3}\right) \cdot dx$$

$$= \left[-\frac{\pi \cos 3\pi}{3} + 0\right] + \frac{1}{3}\int_0^{\pi} \cos 3x \cdot dx$$

$$= -\frac{\pi}{3}(-1) + \frac{1}{3}\left[\frac{\sin 3x}{3}\right]_0^{\pi}$$

$$= \frac{\pi}{3} + \frac{1}{3}[0 - 0]$$

$$= \frac{\pi}{3} \qquad ...(2)$$

$$\mathrm{I}_2 = \int_0^{\pi} x \sin x \cdot dx$$

$$= \left[x \int \sin x \cdot dx\right]_0^{\pi} - \int_0^{\pi} \left[\left\{\frac{d}{dx}(x) \int \sin x \cdot dx\right\}\right] \cdot dx$$

$$= [x(-\cos x)]_0^{\pi} - \int_0^{\pi} 1 \cdot (-\cos x) \cdot dx$$

$$= [-\pi \cos \pi + 0] + \int_0^{\pi} \cos x \cdot dx$$

= $-\pi(-1)+[\sin x]_0^\pi$

= $\pi+[\sin\pi-\sin 0]$

= $\pi+(0-0)$

= π ...(3)

From (1), (2) and (3), we get

I = $\frac{1}{4}\left[\frac{\pi}{3}+\pi\right]$

= $\frac{1}{4}\left(\frac{4\pi}{3}\right)$

= $\frac{\pi}{3}$.

Exercise 4.2 | Q 3.15 | Page 172

Evaluate the following : $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\cdot dx$

SOLUTION

Let I = $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\cdot dx$

Put x = sin θ

∴ dx = cos θ dθ

and

$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}$ = cos θ

When x = 0, sin θ = 0 ∴ θ = 0

When x = 1, sin θ = 1 ∴ θ = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\frac{\pi}{2}} \log \sin \theta \cdot d\theta$

Using the property, $\int_0^{2a} f(x) \cdot dx = \int_0^{a} [f(x) + f(2a - x)] \cdot dx$, we get

I = $\int_0^{\frac{\pi}{4}} \left[\log \sin \theta + \log \sin\left(\frac{\pi}{2} - \theta\right)\right] \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} (\log \sin \theta + \log \cos \theta) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log \sin \theta \cos \theta \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log\left(\frac{2 \sin \theta \cos \theta}{2}\right) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} (\log \sin 2\theta - \log 2) \cdot d\theta$

= $\int_0^{\frac{\pi}{4}} \log \sin 2\theta \cdot d\theta - \int_0^{\frac{\pi}{4}} \log 2 \cdot d\theta$

= $I_1 - I_2$ ...(Say)

I_2 = $\int_0^{\frac{\pi}{4}} \log 2 \cdot d\theta$

= $\log 2 \int_0^{\frac{\pi}{4}} 1 \cdot d\theta$

= $\log 2 [\theta]_0^{\frac{\pi}{4}}$

= $(\log 2)\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi}{4} \log 2$

$I_1 = \int_0^{\frac{\pi}{4}} \log \sin 2\theta \cdot d\theta$

Put $2\theta = t$.

Then $d\theta = \frac{dt}{2}$

When $\theta = 0$, $t = 0$

When $\theta = \frac{\pi}{4}, t = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\therefore I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \times \frac{dt}{2}$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin \theta \cdot d\theta$

$= \frac{1}{2} \mathrm{I} \quad \ldots \left[\because \int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt\right]$

$\therefore I = \frac{1}{2} \mathrm{I} - \frac{\pi}{4} \log 2$

$\therefore I = \frac{1}{2} \mathrm{I} - \frac{\pi}{4} \log 2$

$\therefore \frac{1}{2} \mathrm{I} = -\frac{\pi}{4} \log 2$

$\therefore I = -\frac{\pi}{2} \log 2$

$= \frac{\pi}{2} \log\left(\frac{1}{2}\right)$.

MISCELLANEOUS EXERCISE 4 [PAGES 175 - 177]

Miscellaneous Exercise 4 | Q 1.01 | Page 175

Choose the correct option from the given alternatives :

$$\int_2^3 \frac{dx}{x(x^3-1)} =$$

$\frac{1}{3}\log\left(\frac{208}{189}\right)$

$\frac{1}{3}\log\left(\frac{189}{208}\right)$

$\log\left(\frac{208}{189}\right)$

$\log\left(\frac{189}{208}\right)$

SOLUTION

$\frac{1}{3}\log\left(\frac{208}{189}\right)$

Miscellaneous Exercise 4 | Q 1.02 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cdot dx}{(1+\cos x)^2} =$$

$\frac{4-\pi}{2}$

$\frac{\pi-4}{2}$

$4-\frac{\pi}{2}$

$\frac{4+\pi}{2}$

SOLUTION

$$\frac{4-\pi}{2}$$

Miscellaneous Exercise 4 | Q 1.03 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3} \cdot dx =$$

- 3 + 2π
- 2 + π
- **4 – π**
- 4 + π

SOLUTION

4 – π

Miscellaneous Exercise 4 | Q 1.04 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\frac{\pi}{2}} sn^6 x \cos^2 x \cdot dx =$$

- $\frac{7\pi}{256}$
- $\frac{3\pi}{256}$
- $\frac{5\pi}{256}$
- $\frac{-5\pi}{256}$

SOLUTION

$$\frac{5\pi}{256}$$

Miscellaneous Exercise 4 | Q 1.05 | Page 175

Choose the correct option from the given alternatives :

If $\dfrac{dx}{\sqrt{1+x}-\sqrt{x}} = \dfrac{k}{3}$, then k is equal to

$\sqrt{2}\left(2\sqrt{2}-2\right)$

$\dfrac{\sqrt{2}}{3}\left(2-2\sqrt{2}\right)$

$\dfrac{2\sqrt{2}-2}{3}$

$4\sqrt{2}$

SOLUTION

$4\sqrt{2}$

Miscellaneous Exercise 4 | Q 1.06 | Page 175

Choose the correct option from the given alternatives :

$\int_1^2 \dfrac{1}{x^2} e^{\frac{1}{x}} \cdot dx =$

$\sqrt{e}+1$

$\sqrt{e}-1$

$\sqrt{e}\left(\sqrt{e}-1\right)$

$\dfrac{\sqrt{e}-1}{e}$

SOLUTION

$\sqrt{e}\left(\sqrt{e}-1\right)$

Miscellaneous Exercise 4 | Q 1.07 | Page 175

Choose the correct option from the given alternatives :

If $\left[\dfrac{1}{\log x} - \dfrac{1}{(\log x)^2}\right] \cdot dx = a + \dfrac{b}{\log 2}$, then

a = e, b = − 2

a = e, b = 2

a = − e, b = 2

a = − e, b = − 2

SOLUTION

a = e, b = − 2

Miscellaneous Exercise 4 | Q 1.08 | Page 175

Choose the correct option from the given alternatives :

Let $I_1 = \int_e^{e^2} \dfrac{dx}{\log x}$ and $I_2 = \int_1^2 \dfrac{e^x}{x} \cdot dx$, then

$I_1 = \dfrac{1}{3} I_2$

$I_1 + I_2 = 0$

$I_1 = 2I_2$

$\mathbf{I_1 = I_2}$

SOLUTION

$I_1 = I_2$

Miscellaneous Exercise 4 | Q 1.09 | Page 176

Choose the correct option from the given alternatives :

$$\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx =$$

- 9
- $\frac{9}{2}$
- 0
- 1

SOLUTION

9/2

Miscellaneous Exercise 4 | Q 1.1 | Page 176

Choose the correct option from the given alternatives :

The value of $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \log\left(\frac{2+\sin\theta}{2-\sin\theta}\right) \cdot d\theta$ is

- **0**
- 1
- 2
- π

SOLUTION

0

Miscellaneous Exercise 4 | Q 2.01 | Page 176

Evaluate the following : $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3\cos x+\sin x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3\cos x + \sin x} \cdot dx$

Put Numerator = $\mathrm{A}(\text{Denominator}) + \mathrm{B}\left[\frac{d}{dx}(\text{Denominator})\right]$

∴ cos x = $\mathrm{A}(3\cos x + \sin x) + \mathrm{B}\left[\frac{d}{dx}(3\cos x + \sin x)\right]$

= A(3 cos x + sin x) + B(– 3 sin x + cos x)

∴ cos x + 0· sin x = (3A ++ B)cos x (A – 3B) sin x

Comapring the coefficient od sin x and cos x on both the sides, we get

3A + B = 1 ...(1)

A – 3B = 0 ...(2)

Multiplying equation (1) by 3, we get

9A + 3B = 3 ...(3)

Adding (2) and (3), we get

10A = 3

∴ A = $\frac{3}{10}$

∴ from (1), $3\left(\frac{3}{10}\right) \mathrm{B} = 1$

∴ B = $1 - \frac{9}{10} = \frac{1}{10}$

∴ cos x = $\frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)$

∴ I = $\int_0^{\frac{\pi}{2}} \left[\frac{\frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)}{3\cos x + \sin x}\right] \cdot dx$

= $\int_0^{\frac{\pi}{2}} \left[\frac{3}{10} + \frac{\frac{1}{10}(-3\sin x + \cos x)}{3\cos x + \sin x}\right] \cdot dx$

$$= \frac{3}{10}\int_0^{\frac{\pi}{2}} 1 \cdot dx + \frac{1}{10}\int_0^{\frac{\pi}{2}} \frac{-3\sin x + \cos x}{3\cos x + \sin x} \cdot dx$$

$$= \frac{3}{10}\int_0^{\frac{\pi}{2}} + \frac{1}{10}[\log|3\cos x + \sin x|]_0^{\frac{\pi}{2}} \quad ...\left[\because \int \frac{f\prime(x)}{f(x)} \cdot dx = \log \int |f(x)| + c\right]$$

$$= \frac{3}{10}\left[\frac{\pi}{2} - 0\right] + \frac{1}{10}\left[\log\left|3\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right| - \log|3\cos 0 + \sin 0|\right]$$

$$= \frac{3\pi}{20} + \frac{1}{10}[\log|3 \times 0 + 1| - \log|3 \times 1 + 0|]$$

$$= \frac{3\pi}{20} + \frac{1}{10}[\log 1 - \log 3]$$

$$= \frac{3\pi}{20} - \frac{1}{10}\log 3. \quad ...[\because \log 1 = 0]$$

Miscellaneous Exercise 4 | Q 2.02 | Page 176

Evaluate the following : $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^3} \cdot d\theta$

SOLUTION

Let I = $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^3} \cdot d\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^3} \cdot d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^3} \cdot d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^2} \cdot d\theta$$

Put $\cos\ \frac{\theta}{2} - \sin\ \frac{\theta}{2}$ = t

$\therefore \left(-\frac{1}{2}\sin\ \frac{\theta}{2} + \frac{1}{2}\cos\ \frac{\theta}{2}\right) \cdot d\theta$ = dt

$\therefore \left(\cos\ \frac{\theta}{2} - \sin\ \frac{\theta}{2}\right) \cdot d\theta = 2 \cdot dt$

When θ = $\frac{\pi}{4}, t = \cos\ \frac{\pi}{8} + \sin\ \frac{\pi}{8}$

When θ = $\frac{\pi}{2}, t = \cos\ \frac{\pi}{4} + \sin\ \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore$ I = $\int_{\cos\ \frac{\pi}{8}+\sin\ \frac{\pi}{8}}^{\sqrt{2}} \frac{1}{t^2} \cdot 2dt$

= $2\int_{\cos\ \frac{\pi}{8}+\sin\ \frac{\pi}{8}}^{\sqrt{2}} t^{-2} \cdot dt$

= $2\left[\frac{t^{-1}}{-1}\right]_{\cos\ \frac{\pi}{8}+\sin\ \frac{\pi}{8}}^{\sqrt{2}}$

= $\left[\frac{-2}{t}\right]_{\cos\ \frac{\pi}{8}+\sin\ \frac{\pi}{8}}^{\sqrt{2}}$

= $-\frac{2}{\sqrt{2}} + \frac{2}{\cos\ \frac{\pi}{8} + \sin\ \frac{\pi}{8}}$

= $\frac{2}{\cos\ \frac{\pi}{8} + \sin\ \frac{\pi}{8}} - \sqrt{2}.$

Miscellaneous Exercise 4 | Q 2.03 | Page 176

Evaluate the following : $\int_0^1 \frac{1}{1+\sqrt{x}} \cdot dx$

SOLUTION

Let I = $\displaystyle\int_0^1 \frac{1}{1+\sqrt{x}} \cdot dx$

Put $\sqrt{x}$ = t

∴ x = t^2 and dx = 2t·dt

When x = 0, t = 0

When x = , t = 1

∴ I = $\displaystyle\int_0^1 \frac{1}{1+t} 2t \cdot dt$

= $\displaystyle 2\int_0^1 \frac{t}{1+t} \cdot dt$

= $\displaystyle 2\int_0^1 \frac{(1+t)-1}{1+t} \cdot dt$

= $\displaystyle 2\int_0^1 \left(1-\frac{1}{1+t}\right) \cdot dt$

= $2[t - \log|1+t|]_0^1$

= $2[1 - \log 2 - 0 + \log 1]$

= 2(1 - log 2) ...[∵ log 1 = 0]

= 2 – 2log 2

= 2 – log 4.

Miscellaneous Exercise 4 | Q 2.04 | Page 176

Evaluate the following : $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1+\cos 2x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} \cdot dx$

= $\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{2\cos^2 x} \cdot dx$

= $\frac{1}{2} \int_0^{\frac{\pi}{4}} \tan^3 x \cdot \sec^2 x \cdot dx$

Put tan x = t

∴ $\sec^2 x \cdot dx = dt$

When x = 0, t = tan 0 = 0

When x = $\frac{\pi}{4}, t = \tan \frac{\pi}{4}$ = 1

∴ I = $\frac{1}{2} \int_0^1 t^3 \cdot dt$

= $\frac{1}{2} \cdot \left[\frac{t^4}{4}\right]_0^1$

= $\frac{1}{8} [t^4]_0^1$

= $\frac{1}{8} [1 - 0]$

= $\frac{1}{8}$.

Miscellaneous Exercise 4 | Q 2.05 | Page 176

Evaluate the following : $\int_0^1 t^5 \sqrt{1 - t^2} \cdot dt$

SOLUTION

Let I = $\int_0^1 t^5\sqrt{1-t^2}\cdot dt$

Put t = sin θ

∴ dt = cos θ dθ

When t = 1, θ = $\sin^{-1}1 = \frac{\pi}{2}$

When t = 0, θ = $\sin^{-1}0 = 0$

∴ I = $\int_0^{\frac{\pi}{2}} \sin^5\theta\sqrt{1-\sin^2\theta}\cos\theta\cdot d\theta$

I = $\int_0^{\frac{\pi}{2}} \sin^5\theta\cdot\cos\theta\cdot\cos\theta\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} \sin^5\theta\left(1-\sin^2\theta\right)\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} \left(\sin^5\theta-\sin^7\theta\right)\cdot d\theta$

= $\int_0^{\frac{\pi}{2}} \sin^5\theta\cdot d\theta - \int_0^{\frac{\pi}{2}} \sin^7\theta d\theta.$

Using Reduction formula, we get

I = $\frac{4}{5}\cdot\frac{2}{3}-\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$

= $\frac{8}{15}\left[1-\frac{6}{7}\right]$

= $\frac{8}{15}\times\frac{1}{7}$

= $\frac{8}{105}.$

Miscellaneous Exercise 4 | Q 2.06 | Page 176

Evaluate the following : $\int_0^1 \left(\cos^{-1} x^2\right) \cdot dx$

SOLUTION

Let I = $\int_0^1 \left(\cos^{-1} x^2\right) \cdot dx$

Put $\cos^{-1}x = t$

$\therefore x = \cos t$

$\therefore dx = -\sin t \cdot dt$

When x = 0, t = $\cos^{-1}0 = \frac{\pi}{2}$

When x = 1, t = $\cos^{-1}1 = 0$

$\therefore$ I = $\int_{\frac{\pi}{2}}^0 t^2 \cdot (-\sin t) \cdot dt$

$= -\int_{\frac{\pi}{2}}^0 t^2 \sin t \cdot dt$

$= \int_0^{\frac{\pi}{2}} t^2 \sin t \cdot dt \quad \ldots\left[\because \int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx\right]$

$= \left[t^2 \int \sin t \cdot dt\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[\frac{d}{dx}(t^2) \int \sin t \cdot dt\right] \cdot dt$

$= \left[t^2(\cos t)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2t \cdot (-\cos t) \cdot dt$

$= \left[-t^2 \cos t\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} t \cdot \cos t \cdot dt$

$= \left[-\frac{\pi}{4}\cos \ \frac{\pi}{2} + 0\right] + 2\left\{\left[t \int \cos t \cdot dt\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[\frac{d}{dt}(t) \int \cos t \cdot dt\right] \cdot dt\right\}$

$= 0 + 2\left\{[t \sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1 \cdot \sin t \cdot dt\right\}$ ...$\left[\because \cos \ \frac{\pi}{2} = 0\right]$

$= 2[t \sin t]_0^{\frac{\pi}{2}} - 2[(-\cos t)]_0^{\frac{\pi}{2}}$

$= 2\left[\frac{\pi}{2} \sin \ \frac{\pi}{2} - 0\right] - 2\left[-\cos \ \ \frac{\pi}{2} + \cos 0\right]$

$= 2\left[\frac{\pi}{2} \times 1\right] - 2[-0 + 1]$

$= \pi - 2.$

Miscellaneous Exercise 4 | Q 2.07 | Page 176

Evaluate the following : $\int_{-1}^{1} \frac{1 + x^3}{9 - x^2} \cdot dx$

SOLUTION

Let I = $\int_{-1}^{1} \frac{1 + x^3}{9 - x^2} \cdot dx$

$= \int_{-1}^{1} \left[\frac{1}{9 - x^2} + \frac{x^3}{9 - x^2}\right] \cdot dx$

$= \int_{-1}^{1} \frac{1}{9 - x^2} \cdot dx + \int_{-1}^{1} \frac{x^3}{9 - x^2} \cdot dx$

$\therefore$ I = $I_1 + I_2$ ....(1)

$I_1 = \int_{-1}^{1} \frac{1}{3^2 - x^2} \cdot dx$

$= \frac{1}{2 \times 3}\left[\log\left|\frac{3 + x}{3 - x}\right|\right]_{-1}^{1}$

$= \frac{1}{6}\left[\log\left(\frac{4}{2}\right) - \log\left(\frac{2}{4}\right)\right]$

$= \frac{1}{6}\left[\log\left(\frac{2}{\frac{1}{2}}\right)\right]$

$= \frac{1}{6}\log 4$

$= \frac{1}{6}\log 2^2$

$= \frac{1}{6} \times 2\log 2$

$= \frac{1}{3}\log 2$ ...(2)

$I_2 = \int_{-1}^{1} \frac{x^3}{9 - x^2} \cdot dx$

Let $f(x) = \frac{x^3}{9 - x^2}$

$\therefore f(-x) = \frac{(-x)^3}{9 - (-x)^2}$

$= \frac{(-x)^3}{9 - x^2}$

$= -f(x)$

$\therefore$ f is an odd function.

$\therefore \int_{-1}^{1} f(x) \cdot dx = 0$

$\therefore$ $I_2 = \int_{-1}^{1} \frac{x^3}{9 - x^2} \cdot dx = 0$ ...(3)

From (1),(2) and (3), we get

$I = \frac{1}{3} \log 2 + 0$

$= \frac{1}{3} \log 2.$

Miscellaneous Exercise 4 | Q 2.08 | Page 176

Evaluate the following : $\int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx$

SOLUTION

Let $I = \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx$ ...(1)

We use the property, $\int_0^a f(x) \cdot dx = \int_0^a f(a - x) \cdot dx$

Here a = π.

Hence changing x by π – x, we get

$I = \int_0^{\pi} (\pi - x) \cdot \sin(\pi - x) \cdot [\cos(\pi - x)]^4 \cdot dx$

$= \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx$ ...(2)

Adding(1) and (2), we get

$2I = \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx + \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx$

$= \int_0^{\pi} (x + \pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx$

= $\pi \int_0^{\pi} \sin x \cdot \cos^4 x \cdot dx$

$\therefore$ I = $\frac{\pi}{2} \int_0^{\pi} \cos^4 x \cdot \sin x \cdot dx$

Put cos = t

∴ – sinx · dx = dt

∴ sinx · dx = – dt

When x 0, t = cos 0 = 1

When x = π cos π = – 1

$\therefore$ I = $\frac{\pi}{2} \int_1^{-1} t^4(-dt)$

= $-\frac{\pi}{2} \int_1^{-1} t^4 \cdot dt$

= $-\frac{\pi}{2} \left[\frac{t^5}{5}\right]_1^{-1}$

= $-\frac{\pi}{10} \left[t^5\right]_1^{-1}$

= $-\frac{\pi}{10} \left[(-1)^5 - (1)^5\right]$

= $-\frac{\pi}{10}(-1 - 1)$

= $\frac{2\pi}{10}$

= $\frac{\pi}{5}$.

Miscellaneous Exercise 4 | Q 2.09 | Page 176

Evaluate the following : $\int_0^{\pi} \frac{x}{1 + \sin^2 x} \cdot dx$

SOLUTION

Let I = $\int_0^\pi \frac{x}{1+\sin^2 x} \cdot dx$ ...(1)

We use the property, $\int a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

Here a = π.

Hence in I, changing x to π – x, we get

I = $\int_0^\pi \frac{\pi - x}{1+\sin^2(\pi - x)} \cdot dx$

= $\int_0^\pi \frac{\pi - x}{1+\sin^2 x} \cdot dx$

= $\int_0^\pi \frac{\pi}{1+\sin^2 x} \cdot dx$

= $-\int_0^\pi \frac{x}{1+\sin^2 x} \cdot dx$

= $\int_0^\pi \frac{\pi}{1 \sin^2 x} \cdot dx - \mathrm{I}$ ...[By (1)]

∴ 2I = $\pi \int_0^\pi \frac{1}{1+\sin^2 x} \cdot dx$

Dividing numerator and denominator by $\cos^2 x$, we get

2I = $\pi \int_0^\pi \frac{\sec^2 x}{\sec^2 x + \tan^2 x} \cdot dx$

= $\pi \int_0^\pi \frac{\sec^2 x}{1 + 2\tan^2 x} \cdot dx$

Put tan x = t

∴ $\sec^2 x \cdot dx = dt$

When x = π, t = tan π = 0

When x = 0, t = tan 0 = 0

∴ 2I = $\pi \int_0^{\pi} \frac{dt}{1+2^2}$ = 0

∴ I = 0. ...$[\because \int_a^a f(x) \cdot dx = 0]$

Miscellaneous Exercise 4 | Q 3.01 | Page 176

Evaluate the following : $\int_0^1 \left(\frac{1}{1+x^2}\right) \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

SOLUTION

Let I = $\int_0^1 \left(\frac{1}{1+x^2}\right) \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

Put x = tan t, i.e. t = $\tan^{-1}$x

∴ dx = $\sec^2$t dt

When x = 1, t = $\tan^{-1}$1 = $\frac{\pi}{4}$

When x = 0, t = tan–1 0 = 0

∴ I = $\int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\tan^2 t}\right) \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right) \sec^2 t \cdot dt$

= $\int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 t} \sin^{-1}(\sin 2t) \sec^2 t \cdot dt$

= $\int_0^{\frac{\pi}{4}} 2t \cdot dt$

$= 2\int_0^{\frac{\pi}{4}} t \cdot dt$

$= 2\left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}}$

$= 2\left[\frac{\pi}{32} - 0\right]$

$= \frac{\pi^2}{16}.$

Miscellaneous Exercise 4 | Q 3.02 | Page 176

Evaluate the following : $\int_0^{\frac{\pi}{2}} \frac{1}{6 - \cos x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} \frac{1}{6 - \cos x} \cdot dx$

Put $\tan\left(\frac{x}{2}\right)$ = t

∴ x = 2 $\tan^{-1}$ t

∴ dx = $\frac{2dt}{1 + t}$

and

cos x = $\frac{1 - t^2}{1 + t^2}$

When x = $\frac{\pi}{2}, t = \tan\left(\frac{\pi}{2}\right)$ = 1

When x = 0, t = tan 0 = 0

$\therefore$ I = $\dfrac{\frac{2dt}{1+t^2}}{6-\cos\left(\frac{1-t^2}{1+t^2}\right)}$

= $\displaystyle\int_0^1 \frac{2dt}{6(1+t^2)+1(1-t^2)}$

= 2 $\displaystyle\int_0^1 \frac{1}{t^2+7}\cdot dt$

= 2 $\left[\dfrac{1}{35}\tan^{-1}\ \dfrac{t}{5}\right]_0^1$

= 2 $\left[\dfrac{1}{35}\tan^{-1}\ \dfrac{1}{3}-\dfrac{1}{5}\tan^{-1}0\right]$

= $\dfrac{2}{35}\tan^{-1}\ \dfrac{1}{3}-\dfrac{7}{5}\times 0$

= $\dfrac{2}{\sqrt{35}}\tan^{-1}\ \sqrt{\dfrac{7}{5}}.$

Miscellaneous Exercise 4 | Q 3.03 | Page 176

Evaluate the following : $\displaystyle\int_0^a \frac{1}{a^2+ax-x^2}\cdot dx$

SOLUTION

Let I = $\int_0^a \frac{1}{a^2 + ax - x^2} \cdot dx$

a^2 + ax – x^2 = $a^2 - \left(x^2 - ax + \frac{a^2}{4}\right) + \frac{a^2}{4}$

= $\frac{5a^2}{4} - \left(x - \frac{a}{2}\right)^2$

= $\left(\frac{\sqrt{5a}}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2$

∴ I = $\int_0^a \frac{dx}{\left(\frac{\sqrt{5}a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2}$

= $\frac{1}{\frac{2\times\sqrt{5}a}{2}} \cdot \left[\log\left|\frac{\frac{\sqrt{5}a}{2} + x - \frac{a}{2}}{\frac{\sqrt{5}a}{2} - x + \frac{a}{2}}\right|\right]_0^a$

= $\frac{1}{\sqrt{5}a}\left[\log\left|\frac{\frac{\sqrt{5}a}{2} + a - \frac{a}{2}}{\frac{\sqrt{5}a}{2} - a + \frac{a}{2}}\right| - \log\left|\frac{\frac{\sqrt{5}a}{2} - \frac{a}{2}}{\frac{\sqrt{5}a}{2} + \frac{a}{2}}\right|\right]$

= $\frac{1}{\sqrt{5}a}\left[\log\left|\frac{\frac{\sqrt{5}}{2} + \frac{1}{2}}{\frac{\sqrt{5}}{2} - \frac{1}{2}}\right| - \log\left|\frac{\frac{\sqrt{5}}{2} - \frac{1}{2}}{\frac{\sqrt{5}}{2} + \frac{1}{2}}\right|\right]$

= $\frac{1}{\sqrt{5}a}\left[\log\left|\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\right| - \log\left|\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right|\right]$

= $\frac{1}{\sqrt{5}a}\log\left|\frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}-1}\right|$

$$= \frac{1}{\sqrt{5}a} \log \left[\left(\frac{\sqrt{5}+1}{\sqrt{5}-1} \right)^2 \right]$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{5+1+2\sqrt{5}}{5+1\ -2\sqrt{5}} \right|$$

$$= \frac{1}{\sqrt{5}a} \log\ \frac{6+2\sqrt{5}}{6-2\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{6+2\sqrt{5}}{6-2\sqrt{5}} \times \frac{6+2\sqrt{5}}{6+2\sqrt{5}} \right|$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{36+20+24\sqrt{5}}{36-20} \right|$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{56+24\sqrt{5}}{16} \right|$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{7+3\sqrt{5}}{2} \right|.$$

Miscellaneous Exercise 4 | Q 3.04 | Page 176

Evaluate the following : $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx$

SOLUTION

Let I = $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx$ ...(1)

We use the property, $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$.

Here $a = \frac{\pi}{5}, b = \frac{3\pi}{10}$.

Hence changing x by $\frac{\pi}{5} + \frac{3\pi}{10} - x$, we get,

I = $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)}{\sin\left(\frac{\pi}{5} + \frac{\pi}{10} - x\right) + \cos\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)} \cdot dx$

= $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} \cdot dx$

= $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\cos x}{\cos x + \sin x} \cdot dx$ ...(2)

Adding (1) and (2), we get,

2I = $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx + \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\cos x}{\cos x + \sin x} \cdot dx$

= $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x + \cos x}{\sin x + \cos x} \cdot dx$

= $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} 1 \cdot dx = [x]_{\frac{\pi}{5}}^{\frac{3\pi}{10}}$

= $\frac{3\pi}{10} - \frac{\pi}{5}$

$= \dfrac{\pi}{10}$

$\therefore$ I = $\dfrac{\pi}{20}$.

Miscellaneous Exercise 4 | Q 3.05 | Page 176

Evaluate the following : $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

SOLUTION

Let I = $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

Put x = tan t, i.e. t = $\tan^{-1}$x

$\therefore$ dx = $\sec^2$t·dt

When x = 0, t = tan−1 0 = 0

When x = 1, t = $\tan^{-1}$ = $\dfrac{\pi}{4}$

$\therefore$ I = $\displaystyle\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right) \sec^2 t \cdot dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2t) \sec^2 t \cdot dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} 2t \sec^2 t \cdot dt$

$= \displaystyle\left[2t \int \sec^2 t \cdot dt\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[\frac{d}{dx}(2t) \int \sec^2 t \cdot dt\right]$

$= \displaystyle[2t \tan t]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\tan t \cdot dt$

$$= \left[2 \cdot \frac{\pi}{4} \tan \frac{\pi}{4} - 0\right] - 2 \log(\sec t)\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - 2\left[\log\left(\sec \frac{\pi}{4}\right) - \log(\sec 0)\right]$$

$$= \frac{\pi}{2} - 2\left[\log \sqrt{2} - \log 1\right]$$

$$= \frac{\pi}{2} - 2\left[\frac{1}{2} \log 2 - 0\right]$$

$$= \frac{\pi}{2} - \log 2.$$

Miscellaneous Exercise 4 | Q 3.06 | Page 176

Evaluate the following : $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1 + \cos 2x + \sin 2x} \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1 + \cos 2x + \sin 2x} \cdot dx$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{2 \cos^2 x + 2 \sin x \cos x} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)(\cos x + \sin x)}{2 \cos x(\cos x + \sin x)} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{2 \cos x} \cdot dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}\right] \cdot dx$$

$$= \frac{1}{2} \left[\int_0^{\frac{\pi}{4}} 1 \cdot dx - \int_0^{\frac{\pi}{4}} \tan x \cdot dx\right]$$

$= \frac{1}{2}\left\{[x]_0^{\frac{\pi}{4}} - [\log(\sec x)]_0^{\frac{\pi}{4}}\right\}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right) - \left(\log\sec\ \frac{\pi}{4} - \log\sec 0\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \log\sqrt{2} + \log 1\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \log\sqrt{2}\right]$. ...[∵ log 1 = 0]

Miscellaneous Exercise 4 | Q 3.07 | Page 176

Evaluate the following : $\int_0^{\frac{\pi}{2}} [2\log(\sin x) - \log(\sin 2x)] \cdot dx$

SOLUTION

Let I = $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x) \cdot dx$

$= \int_0^{\frac{\pi}{2}} [2\log\sin x - \log(2\sin x\cos x)] \cdot dx$

$= \int_0^{\frac{\pi}{2}} [2\log\sin x - (\log 2 + \log\sin x + \log\cos x)] \cdot dx$

$= \int_0^{\frac{\pi}{2}} (2\log\sin x - \log 2 - \log\sin x - \log\cos x) \cdot dx$

$= \int_0^{\frac{\pi}{2}} (\log\sin x - \log\cos x - \log 2) \cdot dx$

$= \int_0^{\frac{\pi}{2}} \log\sin x \cdot dx - \int_0^{\frac{\pi}{2}} \log\cos x \cdot dx - \log 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= \int_0^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] \cdot dx - \int_0^{\frac{\pi}{2}} \log\cos x \cdot dx - \log 2[x]_0^{\frac{\pi}{2}}$ $\ldots\left[\because \int_0^a f(x) \cdot dx = i\int_0^a f(a-x) \cdot dx\right]$

$= \int_0^{\frac{\pi}{2}} \log\cos x \cdot dx - \int_0^{\frac{\pi}{2}} \log\cos x \cdot dx - \log 2\left[\frac{\pi}{2} - 0\right]$

$= -\frac{\pi}{2}\log 2.$

Miscellaneous Exercise 4 | Q 3.08 | Page 176

Evaluate the following : $\int_0^{\pi} \left(\sin^{-1} x + \cos^{-1} x\right)^3 \sin^3 x \cdot dx$

SOLUTION

Let I = $\int_0^{\pi} \left(\sin^{-1} x + \cos^{-1} x\right)^3 \sin^3 x \cdot dx$

We know that, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

and

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\therefore 4\sin^3 x = 3 \sin x - \sin 3x$

$\therefore \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$\therefore$ I = $\int_0^{\pi} \left(\frac{\pi}{2}\right)^3 \left[\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right] \cdot dx$

$= \frac{\pi^3}{8} \times \frac{3}{4} \int_0^{\pi} \sin x \cdot dx - \frac{\pi^2}{8} \times \frac{1}{4} \int_0^{\pi} \sin 3x$

$= \frac{3\pi^3}{32} [-\cos \pi - (-\cos 0)] - \frac{\pi^3}{32} \left[-\frac{\cos 3\pi}{3} - \left(\frac{-\cos 0}{3}\right)\right]$

$= \frac{3\pi^3}{32} [1 + 1] - \frac{\pi^3}{32} \left[\frac{1}{3} + \frac{1}{3}\right]$

$= \frac{6\pi^3}{32} - \frac{2\pi^3}{96}$

$= \frac{18\pi^3 - 2\pi^3}{96}$

$= \frac{16\pi^3}{96}$

$= \dfrac{\pi^3}{6}$.

Miscellaneous Exercise 4 | Q 3.09 | Page 176

Evaluate the following : $\int_0^4 \left[\sqrt{x^2 + 2x + 3}\right]^{-1} \cdot dx$

SOLUTION

Let I = $\int_0^4 \left[\sqrt{x^2 + 2x + 3}\right]^{-1} \cdot dx$

$= \int_0^4 \dfrac{1}{\sqrt{x^2 + 2x + 1 + 2}} \cdot dx$

$= \int_0^4 \dfrac{1}{\sqrt{(x+1)^2 + 2}} \cdot dx$

$= \left[\log\left[x + 1 + \sqrt{(x+1)^2 + 2}\right]\right]_0^4$

$= \log\left[4 + 1 + \sqrt{5^2 + 2}\right] - \log\left[0 + 1 + \sqrt{1^2 + 2}\right]$

$= \log\left(5 + 3\sqrt{3}\right) - \log\left(1 + \sqrt{3}\right)$

$= \log\left(\dfrac{5 + 3\sqrt{3}}{1 + \sqrt{3}}\right)$.

Miscellaneous Exercise 4 | Q 3.1 | Page 176

Evaluate the following : $\int_{-2}^3 |x - 2| \cdot dx$

SOLUTION

|x – 2| = 2 – x, if x < 2

= x – 2, if x ≥ 2

$\therefore \int_{-2}^{3} |x-2| \cdot dx = \int_{-2}^{3} |x-2| \cdot dx + \int_{2}^{3} |x-2| \cdot dx$

= $\int_{-2}^{3} (2-x) \cdot dx + \int_{2}^{3} (x-2) \cdot dx$

= $\left[\frac{(2-x)^2}{(-2)}\right]_{-2}^{2} + \left[\frac{(x-2)^2}{2}\right]_{3}^{2}$

= $\left[0 - \frac{(4)^2}{(-2)^2}\right] + \left[\frac{1^2}{2} - \frac{0^2}{2}\right]$

= $8 + \frac{1}{2}$

= $\frac{17}{2}$.

Miscellaneous Exercise 4 | Q 4.1 | Page 177

Evaluate the following : if $\int_{a}^{a} \sqrt{x} \cdot dx = 2a \int_{0}^{\frac{\pi}{2}} \sin^3 x \cdot dx$, find the value of $\int_{a}^{a+1} x \cdot dx$

SOLUTION

It is given that

$\int_{a}^{a} \sqrt{x} \cdot dx = 2a \int_{a}^{\frac{\pi}{2}} \sin^3 x \cdot dx$

$\therefore \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0}^{a} = 2a \cdot \frac{2}{3}$ ...[Using Reduction Formula]

$\therefore \left[\frac{2a^{\frac{3}{2}}}{3} - 0\right] = \frac{4a}{3}$

$\therefore \frac{2a\sqrt{a}}{3} = \frac{4a}{3}$

$\therefore 2a\left(\sqrt{a} - 2\right)$ = 0

$\therefore$ a = 0 or $\sqrt{a}$ = 2

i.e. a = 0 or a = 4

When a = 0, $\int_a^{a+1} x \cdot dx = \int_0^1 x \cdot dx$

= $\left[\frac{x^2}{2}\right]_0^1$

= $\frac{1}{2} - 0$

= $\frac{1}{2}$

When a = 4, $\int_a^{a+1} d \cdot dx = \int_4^5 x \cdot dx$

= $\left[\frac{x^2}{2}\right]_4^5$

= $\frac{25}{2} - \frac{16}{2}$

= $\frac{9}{2}$.

Miscellaneous Exercise 4 | Q 4.2 | Page 177

Evaluate the following : If $\int_0^k \frac{1}{2 + 8x^2} \cdot dx = \frac{\pi}{16}$, find k

SOLUTION

Let I = $\int_0^k \frac{1}{2+8x^2} \cdot dx$

= $\frac{1}{8}\int_0^k \frac{1}{x^2+\left(\frac{1}{2}\right)^2} \cdot dx$

= $\frac{1}{8} \times \frac{1}{\left(\frac{1}{2}\right)}\left[\tan^{-1}\left(\frac{x}{\left(\frac{1}{2}\right)}\right)\right]_0^k$

= $\frac{1}{4}\left[\tan^{-1} 2x\right]_0^k$

= $\frac{1}{4}\left[\tan^{-1} 2k - \tan^{-1} 0\right]$

= $\frac{1}{4}\tan^{-1} 2k$

∴ I = $\frac{\pi}{16}$ $\text{gives} \frac{1}{4}\tan^{-1} 2k = \frac{\pi}{16}$

∴ $\tan^{-1} 2k = \frac{\pi}{4}$

∴ 2k = $\tan \frac{\pi}{4}$ = 1

∴ k = $\frac{1}{2}$.

Miscellaneous Exercise 4 | Q 4.3 | Page 177

Evaluate the following : If f(x) = a + bx + cx^2, show that $\int_0^1 f(x) \cdot dx = \left(\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]\right.$

SOLUTION

$$\int_0^1 f(x) \cdot dx = \int_0^1 (a + bx + cx^2) \cdot dx$$

$$= a \int_0^1 1 \cdot dx + b \int_0^1 x \cdot dx + c \int_0^1 x^2 \cdot dx$$

$$= \left[ax + \frac{\mathrm{bx}^2}{2} + \frac{\mathrm{cx}^3}{3}\right]_0^1$$

$$= a + \frac{b}{2} + \frac{c}{3} \qquad \text{...(1)}$$

Now, $f(0) = a + b(0) + c(0)^2$ = a

$$f\left(\frac{1}{2}\right) = a + b\left(\frac{1}{2}\right) + c\left(\frac{1}{2}\right)^2 = a + \frac{b}{2} + \frac{c}{4}$$

and

$f(1) = a + b(1) + c(1)^2$ = a + b + c

$$\therefore \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

$$= \frac{1}{6}\left[a + 4\left(a + \frac{b}{2} + \frac{c}{4}\right) + (a + b + c)\right]$$

$$= \frac{1}{6}[a + 4a + 2b + c + a + b + c]$$

$$= \frac{1}{6}[6a + 3b + 2c]$$

$$= a + \frac{b}{2} + \frac{c}{3} \qquad \text{...(2)}$$

$\therefore$ from (1) and (2),

$$\int_0^1 f(x) \cdot dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$$