3 Trigonometric Functions

Let's Study

INTRODUCTION :

We are familiar with algebraic equations. In this chapter we will learn how to solve trigonometric equations, their principal and general solutions, their properties. Trigonometric functions play an important role in integral calculus.

Let's learn.

3.1 Trigonometric Equations and their solutions:

Trigonometric equation :

Definition : An equation involving trigonometric function (or functions) is called trigonometric equation.

For example : $\sin\theta = \dfrac{1}{2}$, $\tan\theta = 2$, $\cos3\theta = \cos5\theta$ are all trignometric equations, $x = a\sin(\omega t + \alpha)$ is also a trigonometric equation.

Solution of Trigonometric equation :

Definition : A value of a variable in a trigonometric equation which satisfies the equation is called a *solution of the trigonometric equation*.

A trigonometric equation can have more than one solutions.

For example, $\theta = \frac{\pi}{6}$ satisfies the equation, $\sin\theta = \frac{1}{2}$, Therefore $\frac{\pi}{6}$ is a solution of the trigonometric equation $\sin\theta = \frac{1}{2}$,

$\frac{\pi}{4}$ is a solution of the trigonometric equation $\cos\theta = \frac{1}{\sqrt{2}}$.

$\frac{7\pi}{4}$ is a solution of the trigonometric equation $\cos\theta = \frac{1}{\sqrt{2}}$.

Is π a solution of equation $\sin\theta - \cos\theta = 1$? Can you write one more solution of this equation? Equation $\sin\theta = 3$ has no solution. Can you justify it?

Because of periodicity of trigonometric functions, trigonometric equation may have infinite number of solutions. Our interest is in finding solutions in the interval $[0, 2\pi)$.

Principal Solutions :

Definition : A solution α of a trigonometric equation is called a principal solution if $0 \le \alpha < 2\pi$.

$\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the principal solutions of trigonometric equation $\sin\theta = \frac{1}{2}$.

Note that $\frac{13\pi}{6}$ is a solution but not principal solution of $\sin\theta = \frac{1}{2}$, $\left\{\because \frac{13\pi}{6} \notin [0, 2\pi)\right\}$

0 is the principal solution of equation $\sin\theta = 0$ but 2π is not a principal solution.

Trigonometric equation $\cos\theta = -1$ has only one principal solution. $\theta = \pi$ is the only principal solution of this equation.

Solved Examples

Ex. (1) Find the principal solutions of $\sin\theta = \frac{1}{\sqrt{2}}$.

Solution :

As $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ *and* $0 \le \frac{\pi}{4} < 2\pi$, $\frac{\pi}{4}$ is a principal solution.

By allied angle formula, $\sin\theta = \sin(\pi - \theta)$.

$\therefore \sin\frac{\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{3\pi}{4}$ *and* $0 \le \frac{3\pi}{4} < 2\pi$

$\therefore \frac{3\pi}{4}$ is also a principal solution.

$\therefore \frac{\pi}{4}$ *and* $\frac{3\pi}{4}$ are the principal solutions of $\sin\theta = \frac{1}{\sqrt{2}}$.

Ex.(2) Find the principal solutions of $\cos\theta = \frac{1}{2}$.

Solution : As $\cos\frac{\pi}{3} = \frac{1}{2}$ and $0 \le \frac{\pi}{3} < 2\pi$, $\frac{\pi}{3}$ is a principal solution.

By allied angle formula, $\cos\theta = \cos(2\pi - \theta)$.

$\therefore \cos\frac{\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\frac{5\pi}{3}$ *and* $0 \le \frac{5\pi}{3} < 2\pi$

$\therefore \frac{5\pi}{3}$ is also a principal solution.

$\therefore \frac{\pi}{3}$ *and* $\frac{5\pi}{3}$ are the principal solutions of $\cos\theta = \frac{1}{2}$.

Ex. (3) Find the principal solutions of $\cos\theta = -\frac{1}{2}$

Solution : We known that $\cos\frac{\pi}{3} = \frac{1}{2}$

As $\cos(\pi - \theta) = \cos(\pi + \theta) = -\cos\theta$,

$\cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$ *and* $\cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

$\therefore \cos\frac{2\pi}{3} = -\frac{1}{2}$ *and* $\cos\frac{4\pi}{3} = -\frac{1}{2}$

Also $0 \le \frac{2\pi}{3} \le 2\pi$ *and* $0 \le \frac{4\pi}{3} < 2\pi$. Therefore $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are principal solutions of $\cos\theta = -\frac{1}{2}$

Ex.(4) Find the principal solutions of cot $\theta = -\sqrt{3}$

Solution : We know that cot $\theta = -\sqrt{3}$ if and only if tan $\theta = -\frac{1}{\sqrt{3}}$

We know that tan $\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Using identities, $\tan(\pi - \theta) = -\tan\theta$ and $\tan(2\pi - \theta) = -\tan\theta$, we get

$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$ and $\tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

An $0 \le \frac{5\pi}{6} < 2\pi$ and $0 \le \frac{11\pi}{6} < 2\pi$

$\therefore \frac{5\pi}{6}$ and $\frac{11\pi}{6}$ are required principal solutions.

The General Solution :

Definition : The solution of a trigonometric equation which is generalized by using its periodicity is called the general solution

For example : All solutions of the equation $\sin\theta = \frac{1}{2}$, are $\left\{....., -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, ...\right\}$. We can generate all these solutions from the expression $n\pi + (-1)^n \frac{\pi}{6},\ n \in Z$. The solution $n\pi + (-1)^n \frac{\pi}{6}, n \in Z$ is called the general solution of $\sin\theta = \frac{1}{2}$.

Theorem 3.1 : The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

Proof : As $\sin\theta = \sin\alpha$, α is a solution.

As $\sin(\pi-\alpha) = \sin\alpha$, $\pi-\alpha$ is also a solution. Using periodically, we get

$\sin\theta = \sin\alpha = \sin(2\pi+\alpha) = \sin(4\pi+\alpha) = ...$ and

$\sin\theta = \sin(\pi-\alpha) = \sin(3\pi-\alpha) = \sin(5\pi-\alpha) = ...$

$\therefore$ $\sin\theta = \sin\alpha$ if and only if $\theta = \alpha, 2\pi+\alpha, 4\pi+\alpha, ...$ or $\theta = \pi-\alpha, 3\pi-\alpha, 5\pi-\alpha, ...$

$\therefore$ $\theta = ..., \alpha, \pi-\alpha, 2\pi+\alpha, 3\pi-\alpha, 4\pi+\alpha, 5\pi-\alpha, ...$

$\therefore$ The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

Theorem 3.2 : The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

Proof : As $\cos\theta = \cos\alpha$, α is a solution.

As $\cos(-\alpha) = \cos\alpha$, $-\alpha$ is also a solution.

Using periodically, we get

$\cos\theta = \cos\alpha = \cos(2\pi+\alpha) = \cos(4\pi+\alpha) = ...$ and

$\cos\theta = \cos(-\alpha) = \cos(2\pi-\alpha) = \cos(4\pi-\alpha) = ...$

$\therefore \cos\theta = \cos\alpha$ if and only if $\theta = \alpha, 2\pi+\alpha, 4\pi+\alpha, ...$or $\theta = -\alpha, 2\pi-\alpha, 4\pi-\alpha, 6\pi-\alpha, ...$

$\therefore$ The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.3 : The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

Proof : We know that $\tan\theta = \tan\alpha$ if and only if $\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

If and only if $\sin\theta\cos\alpha = \cos\theta\sin\alpha$

If and only if $\sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

If and only if $\sin(\theta-\alpha) = \sin 0$

If and only if $\theta-\alpha = n\pi + (-1)^n \times 0 = n\pi$, where $n \in Z$.

If and only if $\theta = n\pi + \alpha$, where $n \in Z$.

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

Remark : For $\theta \in R$, we have the following :

(i) $\sin\theta = 0$ if and only if $\theta = n\pi$, where $n \in Z$.

(ii) $\cos\theta = 0$ if and only if $\theta = (2n+1)\frac{\pi}{2}$, where $n \in Z$.

(iii) $\tan\theta = 0$ if and only if $\theta = n\pi$, where $n \in Z$.

Theorem 3.4 : The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\sin^2\theta = \sin^2\alpha$

$\therefore \quad \sin\theta = \pm \sin\alpha$

$\therefore \quad \sin\theta = \sin\alpha$ or $\sin\theta = -\sin\alpha$

$\therefore \quad \sin\theta = \sin\alpha$ or $\sin\theta = \sin(-\alpha)$

$\therefore \quad \theta = n\pi + (-1)^n\alpha$ or $\theta = n\pi + (-1)^n(-\alpha)$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Alternative Proof : $\sin^2\theta = \sin^2\alpha$

$$\therefore \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2\pi\pi \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.5 : The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\cos^2\theta = \cos^2\alpha$

$$\therefore \quad \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2n\pi = \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.6 : The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\tan^2\theta = \tan^2\alpha$

$$\therefore \quad \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+\tan^2\alpha}{1-\tan^2\alpha} \quad \text{by componendo and dividendo}$$

$$\therefore \quad \therefore \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} \quad \text{by invertendo}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2n\pi \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Solved Examples

Ex.(1) Find the general solution of

$(i)\ \sin\theta = \frac{\sqrt{3}}{2}$ $\quad (ii)\ \cos\theta = \frac{1}{\sqrt{2}}$ $\quad (iii)\ \tan\theta = \sqrt{3}$

Solution : (i) We have $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \sin\theta = \sin\frac{\pi}{3}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{\pi}{3}$ is $\theta = n\pi + (-1)^n \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \frac{\sqrt{3}}{2}$ is $\theta = n\pi + (-1)^n \frac{\pi}{3}$, where $n \in Z$.

(ii) We have $\cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\theta = \cos\frac{\pi}{4}$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{\pi}{4}$ is $\theta = 2n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \frac{1}{\sqrt{2}}$ is $\theta = 2n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

(iii) $\tan\theta = \sqrt{3}$

$\therefore \quad \tan\theta = \tan\frac{\pi}{3}$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \tan\frac{\pi}{3}$ is $\theta = n\pi + \frac{\pi}{3}$ where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \sqrt{3}$ is $\theta = n\pi + \frac{\pi}{3}$, where $n \in Z$.

Ex. (2) Find the general solution of

(i) $\sin\theta = -\frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{1}{2}$ (iii) $\cot\theta = -\sqrt{3}$

Solution : (i) $\sin\theta = -\frac{\sqrt{3}}{2}$

$\therefore \quad \sin\theta = \sin\frac{4\pi}{3}$ (As $\sin\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin(\pi+A) = -\sin A$)

The general solution of $\sin\theta = \sin\alpha$ is $\alpha = n\pi + (-1)^n\alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{4\pi}{3}$ is $\theta = n\pi + (-1)^n \frac{4\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = -\frac{\sqrt{3}}{2}$ is $\theta = n\pi + (-1)^n \frac{4\pi}{3}$, where $n \in Z$.

(ii) $\cos\theta = -\frac{1}{2}$

$\therefore$ $\cos\theta = \cos\frac{2\pi}{3}$ $\left(\text{As}\cos\frac{\pi}{3}=\frac{1}{2}\text{ and }\cos(\pi-A)=-\cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{2\pi}{3}$ is $\theta = 2n\pi \pm \frac{2\pi}{3}$ where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = -\frac{1}{2}$ is $\theta = 2n\pi \pm \frac{2\pi}{3}$, where $n \in Z$.

(iii) $\cot\theta = -\sqrt{3}$ $\therefore$ $\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore$ $\tan\theta = \tan\frac{5\pi}{6}$ $\left(\text{As}\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\text{ and }\tan(\pi-A)=-\tan A\right)$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi+\alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \tan\frac{5\pi}{6}$ is $\theta = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

$\therefore$ The general solution of $\cot\theta = -\sqrt{3}$ is $\theta = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

Ex. (3) Find the general solution of

(i) $\operatorname{cosec}\theta = 2$ (ii) $\sec\theta + \sqrt{2} = 0$

Solultion : (i) We have $\operatorname{cosec}\theta = 2$ $\therefore \sin\theta = \frac{1}{2}$

$\therefore$ $\sin\theta = \sin\frac{\pi}{6}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{\pi}{6}$ is $\theta = n\pi + (-1)^n\frac{\pi}{6}$, where $n \in Z$.

$\therefore$ The general solution of $\operatorname{cosec}\theta = 2$ is $\theta = n\pi + (-1)^n\frac{\pi}{6}$, where $n \in Z$.

(ii) We have $\sec\theta + \sqrt{2} = 0$ $\therefore$ $\cos\theta = -\frac{1}{\sqrt{2}}$

$\therefore$ $\cos\theta = \cos\frac{3\pi}{4}$ $\left(\text{As}\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}\text{ and }\cos(\pi-A)=-\cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{3\pi}{4}$ is $\theta = 2n\pi \pm \frac{3\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\sec\theta = \sqrt{2}$ is $\theta = 2n\pi \pm \frac{3\pi}{4}$, where $n \in Z$.

Ex. (4) Find the general solution of

(i) $\cos 2\theta = -\frac{1}{\sqrt{2}}$ (ii) $\tan 3\theta = -1$ (iii) $\sin 4\theta = \frac{\sqrt{3}}{2}$

Solution : (i) We have $\cos 2\theta = -\frac{1}{\sqrt{2}}$

$\therefore$ $\cos 2\theta = \cos\frac{3\pi}{4}\left(As \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos(\pi - A) = \cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos 2\theta = \cos\frac{3\pi}{4}$ is $2\theta = 2n\pi \pm \frac{3\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\cos 2\theta = -\frac{1}{\sqrt{2}}$ is $\theta = n\pi \pm \frac{3\pi}{8}$, where $n \in Z$.

(ii) We have $\tan 3\theta = -1$

$\therefore$ $\tan 3\theta = \tan\frac{3\pi}{4}\left(As \tan\frac{\pi}{4} = 1 \text{ and } \tan(\pi - A) = -\tan A\right)$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan 3\theta = \tan\frac{3\pi}{4}$ is $3\theta = n\pi + \frac{3\pi}{4}$ where $n \in Z$.

$\therefore$ The general solution of $\tan 3\theta = -1$ is $\theta = \frac{n\pi}{3} + \frac{\pi}{4}$, where $n \in Z$.

(iii) $\sin 4\theta = \frac{\sqrt{3}}{2}$

$\therefore$ $\sin 4\theta = \sin\frac{\pi}{3}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin 4\theta = \sin\frac{\pi}{3}$ is $4\theta = n\pi + (-1)^n\frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin 4\theta = \frac{\sqrt{3}}{2}$ is $\theta = \frac{n\pi}{4} + (-1)^n\frac{\pi}{12}$ where $n \in Z$.

Ex. (5) Find the general solution of

(i) $4\cos^2\theta = 1$ (ii) $4\sin^2\theta = 3$ (iii) $\tan^2\theta = 1$

Solution : (i) We have $4\cos^2\theta = 1$

$\therefore$ $\cos^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\therefore$ $\cos^2\theta = \cos^2\frac{\pi}{3}$

The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos^2\theta = \cos^2\frac{\pi}{3}$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $4\cos^2\theta = 1$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

(ii) We have $4\sin^2\theta = 3$

$\therefore$ $\sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore$ $\sin^2\theta = \sin^2\frac{\pi}{3}$

The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin^2\theta = \sin^2\frac{\pi}{3}$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $4\sin^2\theta = 3$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

(iii) We have $\tan^2\theta = 1$

$\therefore$ $\tan^2\theta = \tan^2\frac{\pi}{4}$

The general solution of $\tan^2\theta = \alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan^2\theta = \tan^2\frac{\pi}{4}$ is $\theta = n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\tan^2\theta = 1$ is $\theta = n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

Ex. (6) Find the general solution of $\cos 3\theta = \cos 2\theta$

Solution : We have $\cos 3\theta = \cos 2\theta$

$\therefore$ $\cos 3\theta - \cos 2\theta = 0$

$\therefore$ $-2\sin\frac{5\theta}{2}\sin\frac{\theta}{2} = 0$

$\therefore$ $\sin\frac{5\theta}{2} = 0$ or $\sin\frac{\theta}{2} = 0$

$\therefore$ $\frac{5\theta}{2} = n\pi$ or $\frac{\theta}{2} = n\pi$ where $n \in Z$.

$\therefore$ $\theta = \frac{2n\pi}{5}$, $n \in Z$.

$\therefore$ $\theta = \frac{2n\pi}{5}$ where $n \in Z$ is the required general solution.

Alternative Method : We know that the general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$,where $n \in Z$.

$\therefore$ The general solution of $\cos 3\theta = \cos 2\theta$ is $3\theta = 2n\pi \pm 2\theta$, where $n \in Z$.

$\therefore$ $3\theta = 2n\pi - 2\theta$ or $3\theta = 2n\pi + 2\theta$, where $n \in Z$.

$\therefore$ $5\theta = 2n\pi$ or $\theta = 2n\pi$,where $n \in Z$.

$\therefore$ $\theta = \dfrac{2n\pi}{5}$, $n \in Z$ where $n \in Z$ is the required general solution.

Ex. (7) Find the general solution of $\cos 5\theta = \sin 3\theta$

Solution : We have $\cos 5\theta = \sin 3\theta$

$\therefore$ $\cos 5\theta = \cos\left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $5\theta = 2n\pi \pm \left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $5\theta = 2n\pi - \left(\dfrac{\pi}{2} - 3\theta\right)$ or $5\theta = 2n\pi + \left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $\theta = n\pi - \dfrac{\pi}{4}$ or $\theta = \dfrac{n\pi}{4} + \dfrac{\pi}{16}$, where $n \in Z$ are the required general solutions.

Ex. (8) Find the general solution of $\sec^2 2\theta = 1 - \tan 2\theta$

Solution : Given equation is $\sec^2 2\theta = 1 - \tan 2\theta$

$\therefore$ $1 + \tan^2 2\theta = 1 - \tan 2\theta$

$\therefore$ $\tan^2 2\theta + \tan 2\theta = 0$

$\therefore$ $\tan 2\theta\ (\tan 2\theta + 1) = 0$

$\therefore$ $\tan 2\theta = 0$ or $\tan 2\theta + 1 = 0$

$\therefore$ $\tan 2\theta = \tan 0$ or $\tan 2\theta = \tan\dfrac{3\pi}{4}$

$\therefore$ $2\theta = n\pi$ or $2\theta = n\pi + \dfrac{3\pi}{4}$, where $n \in Z$.

$\therefore$ $\theta = \dfrac{n\pi}{2}$ or $\theta = \dfrac{n\pi}{2} + \dfrac{3\pi}{8}$, where $n \in Z$ is the required general solution.

Ex. (9) Find the general solution of $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution : We have $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$\therefore$ $(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$

$\therefore$ $2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$

$\therefore$ $(2\cos 2\theta + 1)\sin 3\theta = 0$

$\therefore$ $\sin 3\theta = 0$ or $\cos 2\theta = -\dfrac{1}{2}$

$\therefore$ $\sin 3\theta = 0$ or $\cos 2\theta = \cos\dfrac{2\pi}{3}$

$\therefore \quad 3\theta = n\pi$ or $2\theta = 2n\pi \pm \dfrac{2\pi}{3}$, where $n \in Z$.

$\therefore \quad \theta = \dfrac{n\pi}{3}$ or $\theta = n\pi \pm \dfrac{\pi}{3}$, where $n \in Z$ is the required general solution.

Ex. (10) Find the general solution of $\cos\theta - \sin\theta = 1$

Solution : We have $\cos\theta - \sin\theta = 1$

$$\therefore \quad \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \cos\left(\theta + \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\therefore \quad \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\therefore \quad \theta + \frac{\pi}{4} = 2n\pi - \frac{\pi}{4} \quad \text{or} \quad \theta + \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}$$

$\therefore \quad \theta = 2n\pi - \dfrac{\pi}{2}$ or $\theta = 2n\pi$, where $n \in Z$ is the required general solution.

Exercise 3.1

1) Find the principal solutions of the following equations :

(i) $\cos\theta = \dfrac{1}{2}$ (ii) $\sec\theta = \dfrac{2}{\sqrt{3}}$ (iii) $\cot\theta = \sqrt{3}$ (iv) $\cot\theta = 0$

2) Find the principal solutions of the following equations:

(i) $\sin\theta = -\dfrac{1}{2}$ (ii) $\tan\theta = -1$ (iii) $\sqrt{3}\operatorname{cosec}\theta + 2 = 0$

Find the general solutions of the following equations :

3) (i) $\sin\theta = \dfrac{1}{2}$ (ii) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (iii) $\tan\theta = \dfrac{1}{\sqrt{3}}$ (iv) $\cot\theta = 0$

4) (i) $\sec\theta = \sqrt{2}$ (ii) $\operatorname{cosec}\theta = -\sqrt{2}$ (iii) $\tan\theta = -1$

5) (i) $\sin 2\theta = \dfrac{1}{2}$ (ii) $\tan\dfrac{2\theta}{3} = \sqrt{3}$ (iii) $\cot 4\theta = -1$

6) (i) $4\cos^2\theta = 3$ (ii) $4\sin^2\theta = 1$ (iii) $\cos 4\theta = \cos 2\theta$

7) (i) $\sin\theta = \tan\theta$ (ii) $\tan^3\theta = 3\tan\theta$ (iii) $\cos\theta + \sin\theta = 1$

8) Which of the following equations have solutions ?

(i) $\cos 2\theta = -1$ (ii) $\cos^2\theta = -1$ (iii) $2\sin\theta = 3$ (iv) $3\tan\theta = 5$

3.2 Solution of triangle

3.2.1 Polar co-ordinates : Let O be a fixed point in a plane. Let OX be a fixed ray in the plane. O is called the pole and ray OX is called the polar axis. Let P be a point in the plane other than pole O.

Let OP = r and m $\angle$XOP = θ. The ordered pair (r, θ) determines the position of P in the plane. They are called the polar co-ordinates of P . 'r' is called the radius vector and θ is called the vectorial angle of point P.

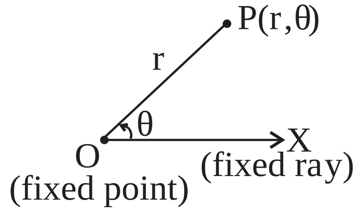


Fig 3.1

Remarks :

i) Vectorial angle θ is the smallest non-negative angle made by OP with the ray OX.

ii) $0 \leq \theta < 2\pi$

iii) Pole has no polar co-ordinates.

3.2.2 Relation between the Cartesian and the Polar co-ordinates: Let O be the pole and OX be the polar axis of polar co-ordinates system. We take line along OX as the X - axis and line perpendicular to OX through O as the Y - axis.

Let P be any point in the plane other than origin. Let (x , y) and (r , θ) be Cartsian and polar co-ordinates of P. To find the relation between them.

By definition of trigonometric functions, we have $\sin\theta = \dfrac{y}{r}$ and

$$\cos\theta = \frac{x}{r}$$

$\therefore$ $x = r\cos\theta$ and $y = r\sin\theta$

This is the relation between Cartesian and polar co-ordinates.

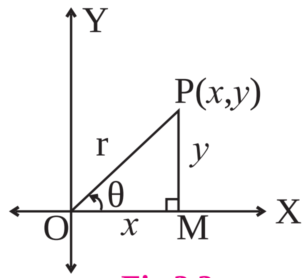


Fig 3.2

Ex. (1) Find the Cartesian co-ordinates of the point whose polar co-ordinates are $\left(2, \dfrac{\pi}{4}\right)$

Solution : Given $r = 2$ and $\theta = \dfrac{\pi}{4}$

Using $x = \text{r}\cos\theta$ and $y = \text{r}\sin\theta$, we get

$$x = 2\cos\frac{\pi}{4} = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$y = 2\sin\frac{\pi}{4} = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

The required Cartesian co-ordinates are $\left(\sqrt{2}, \sqrt{2}\right)$.

Ex. (2) Find the polar co-ordinates of point whose Cartesian co-ordinates are $\left(\dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\right)$

Solution : From the co-ordinates of the given point we observe that point lies in the fourth quadrant.

$$r^2 = x^2 + y^2$$

$$\therefore \quad r^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore \quad r = 1$

$x = r\cos\theta \;, y = r\sin\theta$

$\therefore \quad \frac{1}{\sqrt{2}} = 1 \times \cos\theta$ and $-\frac{1}{\sqrt{2}} = 1 \times \sin\theta$

$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$

$\therefore \quad \theta = \frac{7\pi}{4}$

$\therefore$ The required polar co-ordinates are $\left(1, \frac{7\pi}{4}\right)$.

3.2.3 Solving a Triangle :

Three sides and three angles of a triangle are called the elements of the triangle. If we have a certain set of three elements of a triangle, in which at least one element is a side, then we can determine other three elements of the triangle. To solve a triangle means to find unkown elements of the triangle. Using three angles of a triangle we can't solve it. At least one side should be known. In ΔABC, we use the following notations : $l(BC) = BC = a$, $l(CA) = AC = b$, $l(AB) = AB = c$. This notation is called as the usual notation. Following are some standard relations between elements of triangle.

3.2.4 The Sine Rule : In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of ΔABC.

Proof : Let AD be perpendicular to BC.

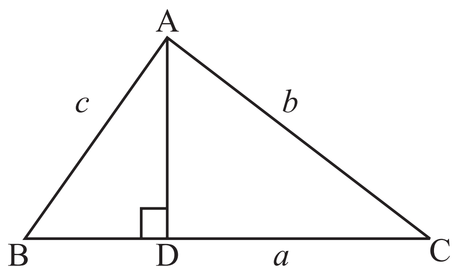


Fig 3.3

$AD = b\sin C$

$\therefore \quad A(\Delta ABC) = \frac{1}{2} BC \times AD$

$= \frac{1}{2} a \times b\sin C$

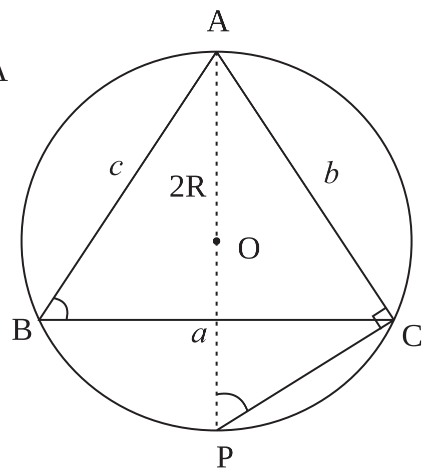
$\therefore \quad A(\Delta ABC) = \frac{1}{2} ab\sin C$

$\therefore \quad 2A(\Delta ABC) = ab\sin C$

Similarly $2A(\Delta ABC) = ac\sin B$ and $2A(\Delta ABC) = bc\sin A$

$\therefore \quad bc\sin A = ac\sin B = ab\sin C$

Divide by abc,



$\therefore \quad \frac{bc\sin A}{abc} = \frac{ac\sin B}{abc} = \frac{ab\sin C}{abc}$

$\therefore \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \ldots (1)$

To prove that each ratio is equal to 2R.

As the sum of three angles is 180^0, at least one of the angle of the triangle is not right angle. Suppose A is not right angle.

Draw diameter through A. Let it meet circle in P.

$\therefore$ AP = 2R and ΔACP is a right angled triangle. $\angle$ ABC and $\angle$ APC are inscribed in the same arc.

$\therefore$ m $\angle$ ABC = m $\angle$ APC

$$\therefore \sin B = \sin P = \frac{b}{AP} = \frac{b}{2R}$$

$$\therefore \sin B = \frac{b}{2R}$$

$$\therefore \frac{b}{\sin B} = 2R \qquad \dots (2)$$

From (1) and (2), we get

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Different forms of Sine rule : Following are the different forms of the Sine rule.
In ΔABC.

(i) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

(ii) a = 2R sin A, b = 2R sinB, c = 2R sin C

(iii) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$

(iv) b sin A = a sin B, c sin B = b sin C, c sin A = a sin C

(v) $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}, \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

Ex.(1) In ΔABC if A = 30º, B = 60º then find the ratio of sides.

Solution : To find a : b : c

Given A = 30º, B = 60º.

As A, B, C are angles of the triangle, A + B + C = 180º

$\therefore$ C = 90º

By Sine rule,

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin 30^0} = \frac{b}{\sin 60^0} = \frac{c}{\sin 90^0}$$

$$\therefore \frac{a}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{1}$$

$\therefore \quad a : b : c = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$\therefore \quad a : b : c = 1 : \sqrt{3} : 2$

Ex.(2) In ΔABC if a = 2, b = 3 and $\sin A = \frac{2}{3}$ then find B.

Solution : By sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \quad \frac{2}{\frac{2}{3}} = \frac{3}{\sin B}$

$\therefore \quad \sin B = 1$

$\therefore \quad B = 90^{\circ} = \frac{\pi}{2}$

Ex. (3) In ΔABC, prove that a(sin B – sin C) + b(sin C – sin A) + c(sin A – sin B) = 0

Solution : L.H.S. = a(sin B – sin C) + b(sin C – sin A) + c(sin A – sin B)

= a sin B – a sin C + b sin C – b sin A + c sin A – c sin B

= (a sin B – b sin A) + (bsin C – c sin B) + (c sin A – a sin C)

= 0 + 0 + 0

= 0 =R.H.S.

Ex.(4) In ΔABC, prove that (a – b) sin C + (b – c) sin A + (c – a) sin B = 0

Solution : L.H.S. = (a – b) sin C + (b – c) sin A + (c – a) sin B

= (a sin C – b sin C) + (b sin A – c sin A) + (c sin B – a sin B)

= (a sin C – c sin A) + (b sin A – a sin B) + (c sin B – b sin C)

= 0 + 0 + 0 = 0 = R.H.S.

3.3.5 The Cosine Rule : In ABC,

(i) $a^2 = b^2 + c^2 - 2bc \cos A$ (ii) $b^2 = c^2 + a^2 - 2ca \cos B$ (iii) $c^2 = a^2 + b^2 - 2ab \cos C$

Proof : Take A as the origin, X - axis along AB and the line perpendicular to AB through A as the Y - axis. The co-ordinates of A, B and C are (0,0). (c, 0) and (b cos A, b sin A) respectively.

To prove that $a^2 = b^2 + c^2 - 2bc \cos A$

L.H.S. = $a^2 = BC^2$

= $(c - b \cos A)^2 + (0 - b \sin A)^2$ (by distance formula)

= $c^2 + b^2 \cos^2 A - 2\,bc \cos A + b^2 \sin^2 A$

= $c^2 + b^2 \cos^2 A + b^2 \sin^2 A - 2bc \cos A$

= $c^2 + b^2 - 2\,bc \cos A$

= R.H.S.

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can prove that

$b^2 = c^2 + a^2 - 2ca \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

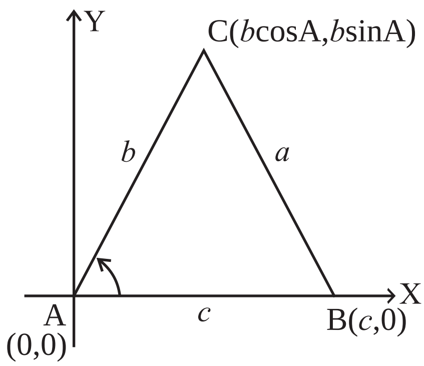


Fig 3.5

Remark : The cosine rule can be stated as : In ΔABC,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \ \cos B = \frac{c^2+a^2-b^2}{2ca}, \ \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Ex.(5) In ΔABC, if a = 2, b = 3, c = 4 then prove that the triangle is obtuse angled.

Solution : We know that the angle opposite to largest side of a triangle is the largest angle of the triangle.

Here side AB is the largest side. C is the largest angle of ΔABC. To show that C is obtuse angle.

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{2^2+3^2-4^2}{2(3)(4)} = -\frac{3}{24} = -\frac{1}{8}$$

As cos C is negative, C is obtuse angle.

$\therefore$ ΔABC is obtuse angled triangle.

Ex.(6) In ΔABC, if $A = 60^0$, b = 3 and c = 8 then find a. Also find the circumradius of the triangle.

Solution : By Cosine rule, $a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore$ $a^2 = 3^2 + 8^2 - 2(3)(8)\cos(60^0)$

$= 9 + 64 - 48 \times \frac{1}{2}$

$= 73 - 24 = 49$

$\therefore$ $a^2 = 49$

$\therefore$ $a = 7$

Now by sine rule $\frac{a}{\sin A} = 2R$

$\therefore$ $\frac{7}{\sin 60^\circ} = 2R$

$\therefore$ $\frac{7}{\frac{\sqrt{3}}{2}} = 2R$

$\therefore$ $R = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$

The circumradius of the ΔABC is $\frac{7\sqrt{3}}{3}$

Ex. (7) In ΔABC prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution :

L.H.S. = $a(b\cos C - c\cos B)$

$= ab\cos C - ac\cos B$

$= \frac{1}{2}(2ab\cos C - 2ac\cos B)$

$= \frac{1}{2}\{(a^2+b^2-c^2) - (c^2+a^2-b^2)\}$

$= \quad \frac{1}{2}\{ a^2 + b^2 - c^2 - c^2 - a^2 + b^2\}$

$= \quad \frac{1}{2}\{ 2b^2 - 2c^2\}$

$= \quad b^2 - c^2 = \text{R.H.S.}$

3.3.6 The projection Rule : In ΔABC,

(i) $a = b \cos C + c \cos B$

(ii) $b = c \cos A + a \cos C$

(iii) $c = a \cos B + b \cos A$

Proof : Here we give proof of one of these three statements, by considering all possible cases.

To prove that $a = b \cos C + c \cos B$

Let altitude drawn from A meets BC in D.

BD is called the peojection of AB on BC.

DC is called the projection of AC on BC.

$\therefore$ Projection of AB on BC = $c \cos B$

And projection of AC on BC = DC = $b \cos C$

Case (i) B and C are acute angles.

$\therefore$ Projection of AB on BC = BD = $c \cos B$

And projection of AC on BC = DC = $b \cos C$

From figure we have,

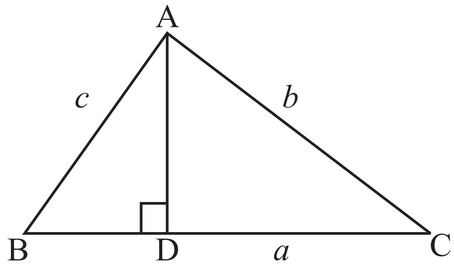


Fig 3.6

$a = BC = BD + DC$

$= c \cos B + b \cos C$

$= b \cos C + c \cos B$

$\therefore a = b \cos C + c \cos B$

Case (ii) B is obtuse angle.

$\therefore$ Projection AB on BC = BD = $c \cos(\pi - B) = -c \cos B$

And projection of AC on BC = DC = $b \cos C$

From figure we have,

$a = BC = DC - BD$

$= b \cos C - (-c \cos B)$

$= b \cos C + c \cos B$

$\therefore a = b \cos C + c \cos B$

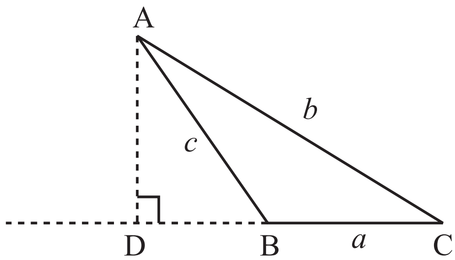


Fig 3.7

Case (iii) B is right angle. In this case D coincides with B.

R.H.S. = $b \cos C + c \cos B$

$= BC + 0$

$= a = \text{L.H.S.}$

$\therefore a = b \cos C + c \cos B$

Similarly we can prove the cases where C is obtuse angle and C right angle.

Therefore in all possible cases, $a = b \cos C + c \cos B$

Similarly we can prove other statements.

Ex.(8) In ΔABC, prove that $(a + b) \cos C + (b + c) \cos A + (c + a) \cos B = a + b + c$

Solution :

L.H.S. $= (a + b) \cos C + (b + c) \cos A + (c + a) \cos B$

$= (a \cos C + b \cos C) + (b \cos A + c \cos A) + (c \cos B + a \cos B)$

$= (a \cos C + c \cos A) + (b \cos A + a \cos B) + (c \cos B + b \cos C)$

$= a + b + c =$ R.H.S.

Ex.(9) In ΔABC, prove that $a(\cos C - \cos B) = 2(b - c) \cos^2 \left(\frac{A}{2}\right)$

Solution : By Projection rule, we have $a \cos C + c \cos A = b$ and $a \cos B + b \cos A = c$

$\therefore$ $a \cos C = b - c \cos A$ and $a \cos B = c - b \cos A$

L.H.S. $= a(\cos C - \cos B)$

$= a \cos C - a \cos B$

$= (b - c \cos A) - (c - b \cos A)$

$= b - c \cos A - c + b \cos A$

$= (b - c) + (b - c) \cos A$

$= (b - c)(1 + \cos A)$

$= (b - c) \times 2\cos^2 \frac{A}{2}$

$= 2(b - c) \cos^2 \frac{A}{2}$

$=$ R.H.S.

Ex.(10) Prove the Cosine rule using the Projection rule.

Solution : Given: In ΔABC, $a = b \cos C + c \cos B$

$b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

Mulitiply these equations by a,b,c respectively.

$a^2 = ab \cos C + ac \cos B$

$b^2 = bc \cos A + ab \cos C$

$c^2 = ac \cos B + bc \cos A$

$a^2 + b^2 - c^2 = (ab \cos C + ac \cos B) + (bc \cos A + ab \cos C) - (ac \cos B + bc \cos A)$

$= ab \cos C + ac \cos B + bc \cos A + ab \cos C - ac \cos B - bc \cos A$

$= 2ab \cos C$

$\therefore$ $a^2 + b^2 - c^2 = 2ab \cos C$ $\therefore$ $c^2 = a^2 + b^2 - 2ab \cos C$.

Similarly we can prove that

$a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = c^2 + a^2 - 2ca \cos B$.

3.3.7 Applications of Sine rule, Cosine rule and Projection rule:

(1) Half angle formulae : In ΔABC, if $a + b + c = 2s$ then

(i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Proof : (i) We have, $1 - \cos A = 2\sin^2 \frac{A}{2}$

$$\therefore \quad 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right) = 2\sin^2 \frac{A}{2} \qquad \text{[by cosine rule]}$$

$$\therefore \quad \frac{2bc - b^2 - c^2 + a^2}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{a^2 - (b - c)^2}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{a - (b - c)\}\{a + (b - c)\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{a + b + c - 2b\}\{a + b + c - 2c\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{2s - 2b\}\{2s - 2c\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{(s - b)(s - c)}{bc} = \sin^2 \frac{A}{2}$$

$$\therefore \quad \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}$$

(ii) We have, $1 + \cos A = 2\cos^2 \frac{A}{2}$

$$\therefore \quad 1 + \left(\frac{b^2 + c^2 - a^2}{2bc}\right) = 2\cos^2 \frac{A}{2} \qquad \text{by cosine rule}$$

$$\therefore \quad \frac{2bc + b^2 + c^2 - a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b^2 + c^2 + 2bc) - a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b + c)^2 - a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b + c + a)(b + c - a)}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b+c+a)(b+c+a-2a)}{2bc} = 2\cos^2\frac{A}{2}$$

$$\therefore \quad \frac{(2s)(2s-2a)}{2bc} = 2\cos^2\frac{A}{2}$$

$$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

(iii) $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly we can prove that

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ab}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

(2) Heron's Formula : If a,b,c are sides of ΔABC and a + b + c = 2s then

$$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

Proof : We know that $A(\Delta ABC) = \frac{1}{2}$ ab sin C

$$\therefore \quad A(\Delta ABC) = \frac{1}{2}\, ab\, 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= ab\sqrt{\frac{(s-b)(s-a)}{ab}}\sqrt{\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)}$$

(3) Napier's Analogy : In ΔABC, $\tan\left(\frac{B-C}{2}\right) = \frac{(b-c)}{(b+c)} \cot\frac{A}{2}$

Proof : By sine rule b = 2R sin B and c =2Rsin C

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \tan\left(\frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \tan\left(\frac{B-C}{2}\right) = \frac{(b-c)}{(b+c)}\cot\frac{A}{2}$$

Similarly we can prove that

$$\tan\left(\frac{C-A}{2}\right) = \frac{(c-a)}{(c+a)}\cot\frac{B}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{(a-b)}{(a+b)}\cot\frac{C}{2}$$

Solved Examples

Ex.(1) In ΔABC if a = 13, b = 14, c = 15 then find the values of

(i) cosA (ii) $\sin\frac{A}{2}$ (iii) $\cos\frac{A}{2}$ (iv) $\tan\frac{A}{2}$ (v) A(ΔABC) (vi) sinA

Solution :

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

(s - a) = 21 – 13 = 8
(s - b) = 21 – 14 = 7
(s - c) = 21 – 15 = 6

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{13^2 + 15^2 - 14^2}{2(13)(15)} = \frac{198}{390} = \frac{33}{65}$$

(ii) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b(s-c)}{bc}}$

$$= \sqrt{\frac{7 \times 6}{14 \times 15}} = \frac{1}{\sqrt{5}}$$

(iii) $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

$$= \sqrt{\frac{21 \times 8}{14 \times 15}} = \frac{2}{\sqrt{5}}$$

(iv) $\tan \dfrac{A}{2} = \dfrac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \dfrac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \dfrac{1}{2}$

(v) $A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{21 \times 8 \times 7 \times 6} = 84 \text{ sq. unit}$$

(vi) $\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$

$$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$$

Ex.(2) In ΔABC prove that $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \dfrac{a+b+c}{b+c-a} \cot \dfrac{A}{2}$

Solution : L.H.S. = $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2}$

$$= \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} + \frac{1}{\tan \frac{C}{2}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s(s-a)^2}{(s-b)(s-c)(s-a)}} + \sqrt{\frac{s(s-b)^2}{(s-a)(s-c)(s-b)}} + \sqrt{\frac{s(s-c)^2}{(s-b)(s-a)(s-c)}}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \left\{\sqrt{(s-a)^2} + \sqrt{(s-b)^2} + \sqrt{(s-c)^2}\right\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{(s-a)+(s-b)+(s-c)\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{3s-(a+b+c)\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{3s-2s\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \times \text{s}$$

$$= \sqrt{\frac{s}{(s-b)(s-c)}} \times \frac{s}{\sqrt{(s-a)}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \times \frac{s}{(s-a)}$$

$$= \frac{2s}{(2s-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{(a+b+c-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{b+c-a} \cot \frac{A}{2} = \text{R.H.S.}$$

Exerecise 3.2

1) Find the Cartesian co-ordinates of the point whose polar co-ordinates are :

(i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (ii) $\left(4, \frac{\pi}{2}\right)$ (iii) $\left(\frac{3}{4}, \frac{3\pi}{4}\right)$ (iv) $\left(\frac{1}{2}, \frac{7\pi}{3}\right)$

2) Find the of the polar co-ordinates point whose Cartesian co-ordinates are.

(i) $\left(\sqrt{2}, \sqrt{2}\right)$ (ii) $\left(0, \frac{1}{2}\right)$ (iii) $\left(1, -\sqrt{3}\right)$ (iv) $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

3) In ΔABC, if A = 45º, B = 60º then find the ratio of its sides.

4) In ΔABC, prove that $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right) \cos \frac{A}{2}$.

5) With usual notations prove that $2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\} = a - b + c$.

6) In ΔABC, prove that $a^3 \sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B) = 0$

7) In ΔABC, if cot A, cot B, cot C are in A.P. then show that a^2, b^2, c^2 are also in A.P

8) In ΔABC, if a cos A = b cos B then prove that the triangle is right angled or an isosceles traingle.

9) With usual notations prove that $2(bc \cos A + ac \cos B + ab \cos C) = a^2 + b^2 + c^2$

10) In ΔABC, if a = 18, b = 24, c = 30 then find the values of

(i) cosA (ii) $\sin \frac{A}{2}$ (iii) $\cos \frac{A}{2}$ (iv) $\tan \frac{A}{2}$ (v) $A(\Delta ABC)$ (vi) sinA

11) In ΔABC prove that $(b + c - a) \tan \frac{A}{2} = (c + a - b) \tan \frac{B}{2} = (a + b - c) \tan \frac{C}{2}$

12) In ΔABC prove that $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{[A(\Delta ABC)]^2}{abcs}$

3.3 Inverse Trigonometric Function :

We know that if a function $f : A \to B$ is one - one and onto then its inverse function, denoted by $f^{-1} : B \to A$, exists. For $x \in A$ and $y \in B$ if $y = f(x)$ then $x = f^{-1}(y)$.

Clearly, the domain of f^{-1} = the range of f and the range of f^{-1} = the domain of f. Trigonometric ratios defines functions, called trigonometric functions or circular functions. Their inverse functions are called inverse trigonometric functions or inverse circular functions. Before finding inverse of trigonometric (circular) function , let us revise domain , range and period of the trigonometric function. We summarise them in the following table .

No trigonometric fuction is one-one. An equation of the type $\sin\theta = k$, $(|k| \le 1)$ has infinitely many solutions given by $\theta = n\pi + (-1)^n \alpha$, where $\sin\alpha = k$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

Function	Domain	Range	Period
sin	R	[–1, 1]	2π
cos	R	[–1, 1]	2π
tan	$R-\left\{\frac{(2n+1)\pi}{2} : n\in Z\right\}$	R	π
cot	$R-\{n\pi : n\in Z\}$	R	π
sec	$R-\left\{\frac{(2n+1)\pi}{2} : n\in Z\right\}$	R–(–1, 1)	2π
cosec	$R-\{n\pi : n\in Z\}$	R–(–1, 1)	2π

There are infinitely many elements in the domain for which the sine function takes the same value. This is true for other trigonometric functions also.

We therefore arrive at the conclusion that inverse of trigonometric functions do not exist. However from the graphs of these functions we see that there are some intervals of their domain, on which they are one-one and onto. Therefore, on these intervals we can define their inverses.

3.3.1 Inverse sine function: Consider the function sin : $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\rightarrow[-1,1]$. It can be verifide from the graph that with this domain and range it is one-one and onto function. Therefore inverse sine function exists. It is denoted by

$\sin^{-1}: [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

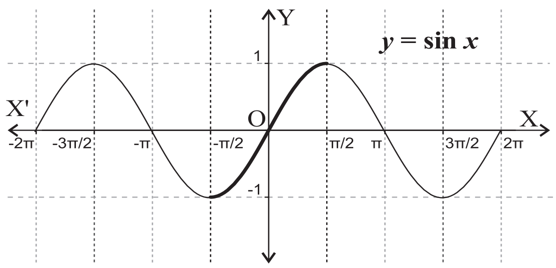


Fig 3.8(a)

For $x \in [-1,1]$ $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

we write $\sin^{-1}x = \theta$ if $\sin\theta = x$.

Here θ is known as the principal value of $\sin^{-1} x$.

For example:

1) $\sin\frac{\pi}{6}=\frac{1}{2}$, where $\frac{1}{2}\in[-1, 1]$ and $\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore\ \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$,

The principal value of $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$,

However, though $\sin\frac{5\pi}{6}=\frac{1}{2}$,

we cannot write $\sin^{-1}\frac{1}{2}=\frac{5\pi}{6}$ as $\frac{5\pi}{6}\notin\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

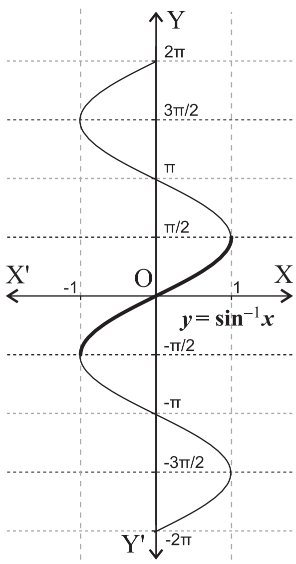


Fig 3.8(b)

2) $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, where $-\frac{1}{\sqrt{2}} \in [-1, 1]$ and $-\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

The principal value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $-\frac{\pi}{4}$.

Note:

1. $\sin(\sin^{-1}x) = x$, for $x \in [-1, 1]$
2. $\sin^{-1}(\sin y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

3.3.2 Inverse cosine function: Consider the function $\cos : [0, \pi] \to [-1, 1]$. It can be verified from the graph that it is a one-one and onto function. Therefore its inverse fuctions exsit. It is denoted by $\cos^{-1}$.

Thus, $\cos^{-1}: [-1, 1] \to [0, \pi]$.

For $x \in [-1, 1]$ and $\theta \in [0, \pi]$, we write $\cos^{-1} x = \theta$ is $\cos\theta = x$. Here θ is known as the principal value of $\cos^{-1}x$.

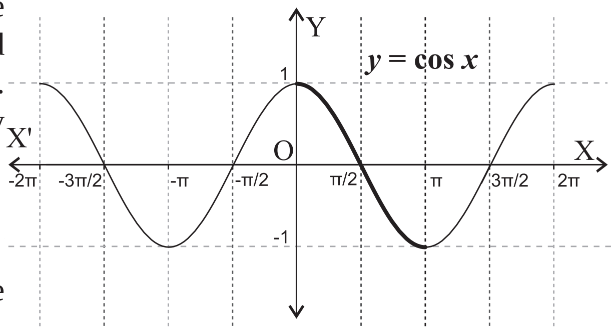


Fig 3.9(a)

For example : $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, where $\frac{1}{\sqrt{2}} \in [-1, 1]$ and $\frac{\pi}{4} \in [0, \pi]$

$\therefore \quad \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

The principal value of $\cos^{-1}\frac{1}{\sqrt{2}}$ is $\frac{\pi}{4}$

Though, $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,

We cannot write $\cos^{-1}\frac{1}{\sqrt{2}} = -\frac{\pi}{4}$ as $\frac{-\pi}{4} \notin [0, \pi]$

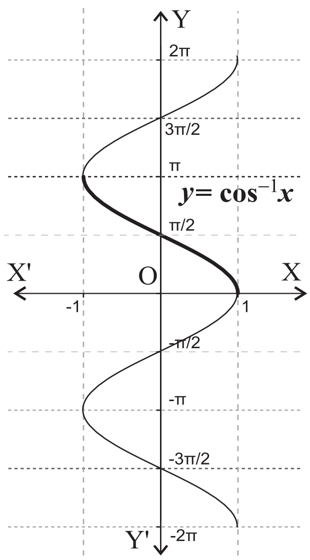


Fig 3.9(b)

Note: 1. $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$
2. $\cos^{-1}(\cos y) = y$, for $y \in [0, \pi]$

3.3.3 Inverse tangent function : Consider the function tan $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to$ R. It can be verified from the graph that it is a one-one and onto function .

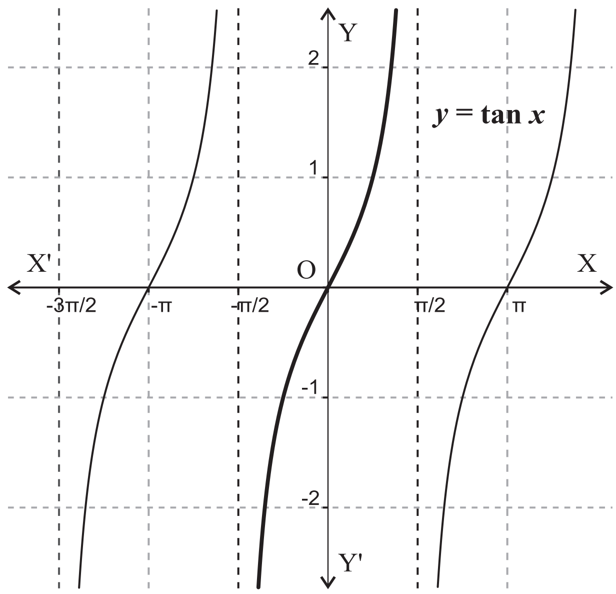


Fig 3.10(a)

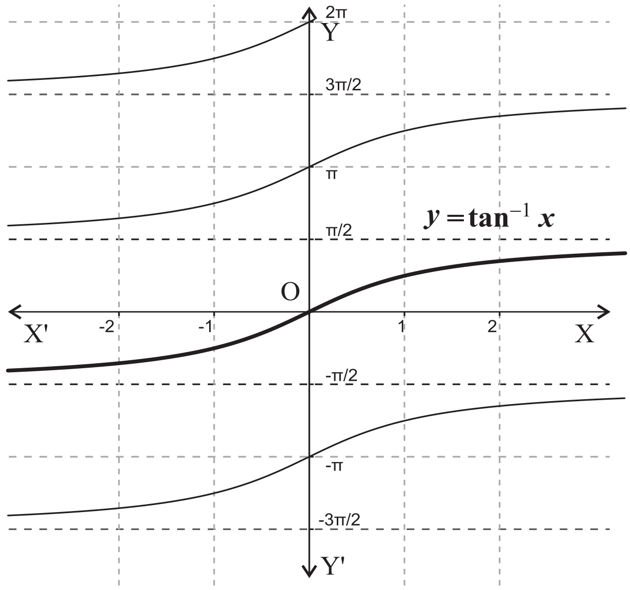


Fig 3.10(b)

Therefore, its inverse function exist. It is denoted by $\tan^{-1}$

Thus, $\tan^{-1}$: R $\to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

For $x \in$ R and $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we write $\tan^{-1} x = \theta$ if $\tan\theta = x$

For example $\tan\frac{\pi}{4} = 1$, where $1 \in$ R and $\frac{\pi}{4} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore$ $\tan^{-1}(1) = \frac{\pi}{4}$.

The principal value of $\tan^{-1} 1$ = is $\frac{\pi}{4}$.

Note: 1. $\tan(\tan^{-1} x) = x$, for $x \in$ R 2. $\tan^{-1}(\tan y) = y$, for $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

3.3.4 Inverse cosecant function : Consider the function cosec : $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \{0\} \to R - (-1, 1)$. It can be verified from the graph that it is a one-one and onto function. Therefore, its inverse function exists. It is denoted by cosec^{-1}

Thus, cosec^{-1}: $R - (-1, 1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

For $x \in R - (-1, 1)$ and

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$,

we write $\text{cosec}^{-1} x = \theta$ if $\text{cosec}\, \theta = x$

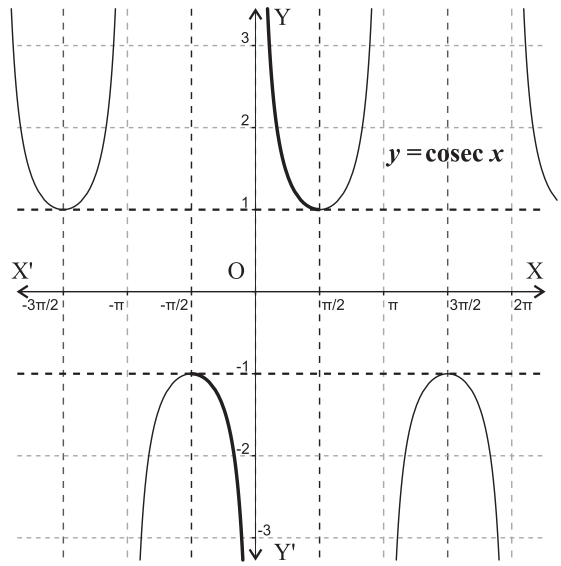


Fig 3.11(a)

For example $\text{cosec}\, \frac{\pi}{6} = 2$,

where $2 \in R - (-1, 1)$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\text{cosec}^{-1}(2) = \frac{\pi}{6}$

The principal value of $\text{cosec}^{-1} 2$ is $\frac{\pi}{6}$.

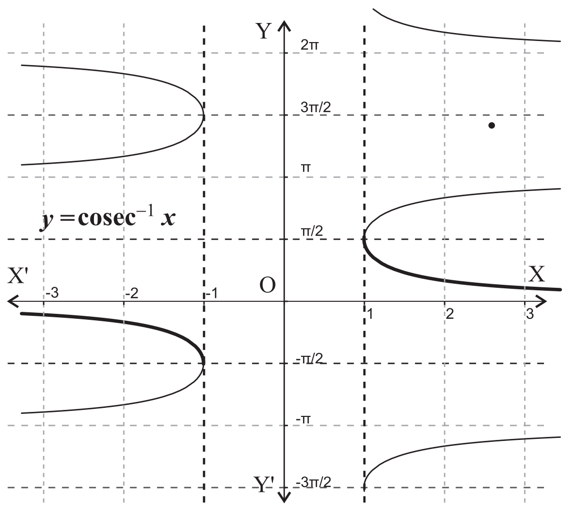


Fig 3.11(b)

Note: 1. $\text{cosec}(\text{cosec}^{-1}x) = x$, for $x \in R - (-1, 1)$

2. $\text{cosec}^{-1}(\text{cosec}\, y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

3.3.5 Inverse secant function : consider the function

$\sec : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to R - (-1, 1)$

It can be verified from the graph that it is a one-one and onto function. Therefore its inverse function exists. It is denoted by $\sec^{-1}$

Thus, $\sec^{-1} : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

For $x \in R - (-1, 1)$ and $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$,
we write $\sec^{-1} x = \theta$ if $\sec \theta = x$
For example $\sec \pi = -1$, where $-1 \in R - (-1, 1)$
and $\pi \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore$ $\sec^{-1}(-1) = \pi$

The principal value of $\sec^{-1}(-1)$ is π .

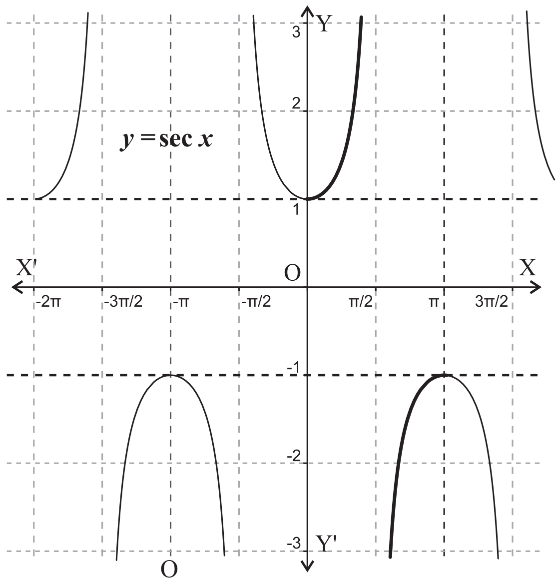


Fig 3.12(a)

Note: 1. $\sec(\sec^{-1}x) = x$, for $x \in R - (-1, 1)$

2. $\sec^{-1}(\sec^{-1} y) = y$, for $y \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

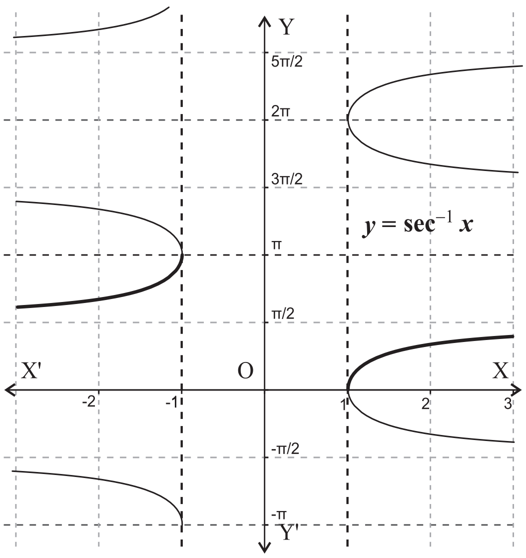


Fig 3.12(b)

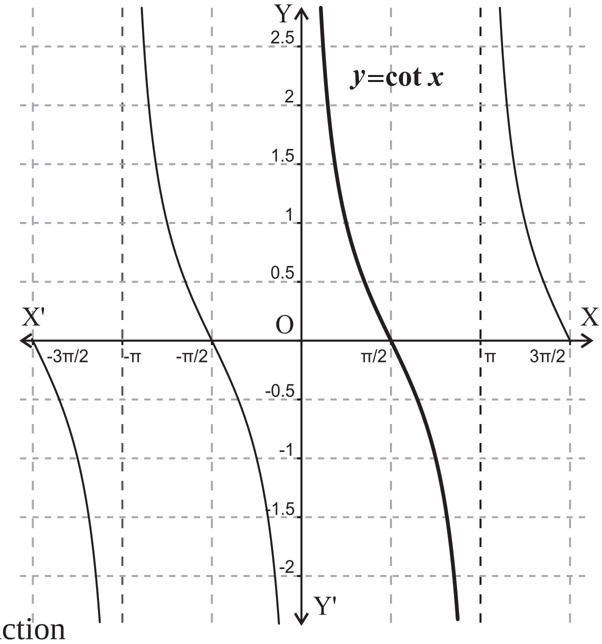


Fig 3.13(a)

3.4.6 Inverse cotangent function : Consider the function cot : $(0, \pi) \to R$. It can be verifide from the graph that it is a one-one and onto function. Therefore, its inverse exists. It is denoted by $\cot^{-1}$.

Thus, $\cot^{-1} : R \to (0, \pi)$

For $x \in R$ and $\theta \in (0, \pi)$, we write

$\cot^{-1} x = \theta$ if $\cot\theta = x$

For example: $\cot\dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}$, where $\dfrac{1}{\sqrt{3}} \in R$

and $\dfrac{\pi}{3} \in (0, \pi)$

$$\therefore \quad \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

The principal value of $\cot^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ is $\dfrac{\pi}{3}$.

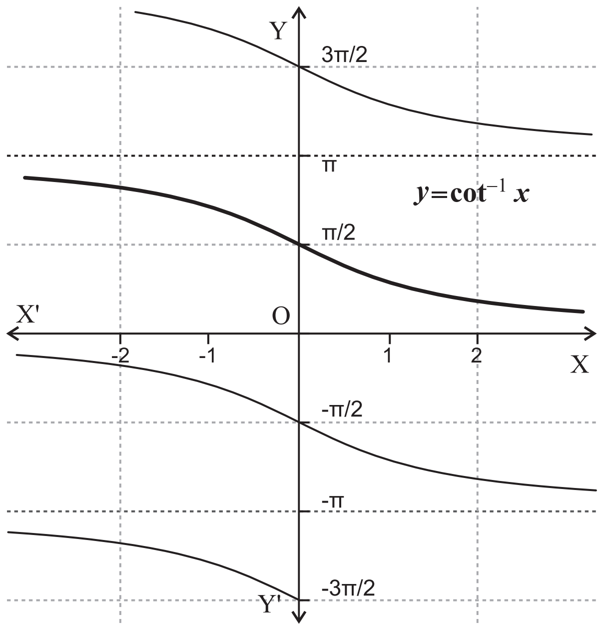


Fig 3.13(b)

Note: 1. $\cot(\cot^{-1}x) = x$, for $x \in R$

2. $\cot^{-1}(\cot y) = y$, for $y \in (0, \pi)$

3.4.7 Principal Values of Inverse Trigonometric Functions :

The following table shows domain and range of all inverse trigonometic functions. The value of function in the range is called the principal value of the function.

$\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} : [-1,1] \to [0, \pi]$
$\tan^{-1} : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} : R - (-1,1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} : R - (-1,1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$

3.4.8 Properties of inverse trigonometric functions :

i) If $-1 \leq x \leq 1$ and $x \neq 0$ then $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

Proof : By the conditions on x, $\sin^{-1} x$ and $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ are defined.

As $-1 \leq x \leq 1$ and $x \neq 0$, $\frac{1}{x} \in R - (-1, 1)$ ...(1)

Let $\sin^{-1} x = \theta$

$\therefore \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta \neq 0$ (as $x \neq 0$)

$\therefore \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ ...(2)

$\therefore \quad \sin\theta = x$

$\therefore \quad \operatorname{cosec}\theta = \frac{1}{x}$ ...(3)

From (1), (2) and (3) we get

$$\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \theta$$

$$\therefore \quad \theta = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$\therefore \quad \sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Similarly we can prove the following result.

(i) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ if $-1 \leq x \leq 1$ and $x \neq 0$

(ii) $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$ if $x > 0$

Proof : Let $\tan^{-1} x = \theta$

$\therefore \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. As $x > 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \tan\theta = x$

$\therefore \quad \cot\theta = \frac{1}{x}$, where $\frac{1}{x} \in R$ ....(1)

As $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right) \subset (0, \pi)$ ...(2)

From (1) and (2) we get $\cot^{-1}\left(\frac{1}{x}\right) = \theta$

$\therefore \quad \theta = \cot^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$

(iii) Similarly we can prove that : $\tan^{-1} x = -\pi + \cot^{-1}\left(\frac{1}{x}\right)$ if $x < 0$.

Activity : Verify the above result for $x = -\sqrt{3}$

iv) if $-1 \le x \le 1$ then $\sin^{-1}(-x) = -\sin^{-1}(x)$

Proof : As $-1 \le x \le 1$, $x \in [-1, 1]$

$\therefore \quad -x \in [-1,1]$ ...(1)

Let $\sin^{-1}(-x) = \theta$

$\therefore \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$

Now $\sin(-\theta) = -\sin\theta = -x$

$\therefore \quad \sin(-\theta) = -x$ ...(2)

Also from (1) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\therefore \quad \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}$

$\therefore \quad -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2}$

$\therefore \quad -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ...(3)

From (1), (2) and (3) we can write

$\therefore \quad \sin^{-1}(-x) = -\theta$

$\therefore \quad \sin^{-1}(-x) = -\sin^{-1}(x)$

Similarly we can prove the following results.

v) If $-1 \le x \le 1$ then $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

vi) For all $x \in R$, $\tan^{-1}(-x) = -\tan^{-1}(x)$

vii) If $|x| \ge 1$ then $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$

viii) If $|x| \ge 1$ then $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

ix) For all $x \in R$, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

x) If $-1 \le x \le 1$ then $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

Proof : Let $\sin^{-1} x = \theta$, where $x \in [-1, 1]$ and $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\therefore \quad -\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\therefore \quad \dfrac{\pi}{2} - \theta \in [0, \pi]$, the principal domain of the cosine fuction.

$\therefore \quad \cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = x$

$\therefore \quad \cos^{-1} x = \dfrac{\pi}{2} - \theta$

$\therefore \quad \theta + \cos^{-1} x = \dfrac{\pi}{2}$

$\therefore \quad \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

Similarly we can prove the following results.

xi) For $x \in R$, $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

xii) For $x \ge 1$, $\text{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$

xiii) If $x > 0$, $y > 0$ and $xy < 1$ then

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

Proof : Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$

$\therefore \quad \tan\theta = x$ and $\tan\phi = y$

As $x > 0$ and $y > 0$, we have $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$

$\therefore \quad 0 < \theta + \phi < \pi \qquad ...(1)$

Also $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \dfrac{x+y}{1-xy}$

As $x > 0$, $y > 0$ and $xy < 1$, x, y and $1-xy$ are all positive.

$\therefore$ $\dfrac{x+y}{1-xy}$ is positive.

$\tan(\theta+\phi)$ is positive. ....(2)

From (1) and (2) we get $(\theta+\phi) \in \left(0,\dfrac{\pi}{2}\right)$, the part of the principal domain of the tangent function.

$$\therefore \quad \theta+\phi = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\therefore \quad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Similarly we can prove the following results.

xiv) If $x > 0$, $y > 0$ and $xy > 1$ then

$$\tan^{-1}x + \tan^{-1}y = \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

xv) If $x > 0$, $y > 0$ and $xy = 1$ then

$$\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2}$$

xvi) If $x > 0$, $y > 0$ then

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Ex.(1) Find the principal values of the following :

(i) $\sin^{-1}\left(-\dfrac{1}{2}\right)$ (ii) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (iii) $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

Solution : (i) $\sin^{-1}\left(-\dfrac{1}{2}\right)$

We have, $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$, where $-\dfrac{1}{2} \in [-1,1]$ and $-\dfrac{\pi}{6} \in \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

$$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

The principal value of $\sin^{-1}\left(-\dfrac{1}{2}\right)$ is $-\dfrac{\pi}{6}$.

(ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

We have, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, where $\frac{\sqrt{3}}{2} \in [-1,1]$ and $\frac{\pi}{6} \in [0, \pi]$

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

(iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

We have, $\cot\frac{2\pi}{3} = \frac{-1}{\sqrt{3}}$, where $\frac{-1}{\sqrt{3}} \in R$ and $\frac{\pi}{6} \in (0, \pi)$

$\therefore$ The principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$

Ex.(2) Find the values of the following

(i) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

(iii) $\sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

(iv) $\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right)$

Solution : (i) $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(2\pi - \frac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$, as $-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{3}\right) = -\frac{\pi}{3}$

(ii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$, as $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) We have $\cos^{-1}(-x) = \pi - \cos^{-1} x$

$$\therefore \quad \sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \sin\left(\pi - \cos^{1}\frac{1}{\sqrt{2}}\right)$$

$$= \quad \sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}$$

(iv) Let $\cos^{-1}\dfrac{4}{5} = \theta$ and $\tan^{-1}\dfrac{5}{12} = \phi$

$$\therefore \quad \cos\theta = \frac{4}{5} \text{ and } \tan\phi = \frac{5}{12}$$

$$\therefore \quad \sin\theta = \frac{3}{5} \text{ and } \sin\phi = \frac{5}{13}, \ \cos\phi = \frac{12}{13}$$

$$\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right) = \sin(\theta + \phi)$$

$$= \quad \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= \quad \frac{3}{5}\ \frac{12}{13} + \frac{4}{5}\ \frac{5}{13} = \frac{56}{65}$$

$$\therefore \quad \sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right) = \frac{56}{65}$$

Ex.(3) Find the values of the following :

(i) $\sin\left[\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{3}{5}\right)\right]$

(ii) $\cos\left[\cos^{-1}\left(-\dfrac{1}{2}\right) + \tan^{-1}\sqrt{3}\right]$

Solution : (i) We have if $-1 \le x \le 1$ then

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Here $-1 < \dfrac{3}{5} < 1$

$$\therefore \quad \sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$$

$$\sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1.$$

(ii) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ *and* $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

$\therefore \quad \cos\left[\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\sqrt{3}\right]$

$= \quad \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$

$= \quad \cos \pi$

$= -1.$

Ex.(4) If $|x| \leq 1$, show that

$\sin(\cos^{-1}x) = \cos(\sin^{-1}x)$.

Solution :

We have for $|x| \leq 1$.

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\therefore \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x,$$

Now using $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

We have $\sin(\cos^{-1}x) = \sin\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \cos(\sin^{-1}x)$

$\therefore \quad \sin(\cos^{-1}x) = \cos(\sin^{-1}x)$

Ex.(5) Prove the following

(i) $2\tan^{-1}\left(-\frac{1}{3}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

(ii) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Solution :

(i) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$, as $xy = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} < 1$

$$= \quad \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\frac{1}{3}}\right) = \tan^{-1}\frac{3}{4} = \theta \text{, (say)}$$

$\therefore \quad \tan\theta = \frac{3}{4} \qquad \therefore \quad 0 < \theta < \frac{\pi}{2}$

$\therefore \quad \sin\theta = \frac{3}{5} \qquad \therefore \ \theta = \sin^{-1}\frac{3}{5}$

$\therefore \quad 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4} = \sin^{-1}\frac{3}{5}$

$\therefore \quad 2\tan^{-1}\frac{1}{3} + \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

(ii) $\quad 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$ as seen in (i)

$\therefore \quad 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$ and $xy = \frac{3}{4} \times \frac{1}{7} = \frac{3}{28} < 1$

$= \quad \tan^{-1}\left(\dfrac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\frac{1}{7}}\right) = \tan^{-1}1 = \frac{\pi}{4}$

Ex.(6) Prove that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

Solution: We use the result:

$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy > 1$

Here $xy = 1 \times 2 = 2 > 1$

$\therefore \quad \tan^{-1}1 + \tan^{-1} 2 = \pi + \tan^{-1}\left(\frac{1+2}{1-(1)(2)}\right)$

$= \quad \pi + \tan^{-1}\left(\frac{3}{1-2}\right)$

$= \quad \pi + \tan^{-1}(-3)$

$= \quad \pi - \tan^{-1}3$ (As, $\tan^{-1}(-x) = -\tan^{-1} x$)

$\therefore \quad \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

Ex.(7) Prove that $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Solution: Let $\cos^{-1}\frac{4}{5} = \theta$

Then $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{4}{5}$

$\therefore \quad \sin\theta = \dfrac{3}{5}$

Let $\cos^{-1}\dfrac{12}{13} = \phi$

Then $0 < \phi < \dfrac{\pi}{2}$ and $\cos\phi = \dfrac{12}{13}$

$\sin\phi = \dfrac{5}{13}$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$= \left(\dfrac{4}{2}\right)\left(\dfrac{12}{13}\right) - \left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)$

$= \dfrac{48}{65} - \dfrac{15}{65} = \dfrac{33}{65}$

$\therefore \quad \cos(\theta + \phi) = \dfrac{33}{65} \qquad .....(1)$

Also $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$

$\therefore \quad 0 < \theta + \phi < \pi.$

$\therefore$ from (1), $\theta + \phi = \cos^{-1}\dfrac{33}{65}$

$\therefore \quad \cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$

Exereise 3.3

1) Find the principal values of the following :

(i) $\sin^{-1}\left(\dfrac{1}{2}\right)$ (ii) $\operatorname{cosec}^{-1}(2)$ (iii) $\tan^{-1}(-1)$

(iv) $\tan^{-1}\left(-\sqrt{3}\right)$ (v) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ (vi) $\cos^{-1}\left(-\dfrac{1}{2}\right)$

2) Evaluate the following :

(i) $\tan^{-1}(1) + \cos^{-1}\left(\dfrac{1}{2}\right) + \sin^{-1}\left(\dfrac{1}{2}\right)$

(ii) $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$

(iii) $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

(iv) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) + \cot^{-1}\left(\sqrt{3}\right)$

3) Prove the following :

(i) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = -\dfrac{3\pi}{4}$

(ii) $\sin^{-1}\left(-\dfrac{1}{2}\right) + \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) = \cos^{-1}\left(-\dfrac{1}{2}\right)$

(iii) $\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{12}{13}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)$

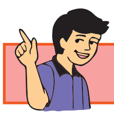
(iv) $\cos^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{4}{5}\right) = \dfrac{\pi}{2}$

(v) $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

(vi) $2\tan^{-1}\left(\dfrac{1}{3}\right) = \tan^{-1}\left(\dfrac{3}{4}\right)$

(vii) $\tan^{-1}\left[\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right] = \dfrac{\pi}{4} + \theta$ if $\theta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

(viii) $\tan^{-1}\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{\theta}{2}$, if $\theta \in (0, \pi)$



Let's remember!

* An equation involving trigonometric function (or functions) is called a trigonometric equation.
* A value of α variable in a trigonometric equation which satisfies the equation is called a solution of the trigonometric equation.
* A solution α of a trigonometric equation is called a principal solution if $0 \le \alpha < 2\pi$.
* The solution of a trigonometric equation which is generalized by using its periodicity is called the general solution.
* The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.
* The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.
* The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.
* The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
* The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
* The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

* The Sine Rule : In ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ = 2R, where R is the circumradius of ΔABC.

Following are the different forms of the Sine rule.

(i) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

(ii) a = 2R sinA, b = 2R sinB, c = 2R sinC

(iii) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$

(iv) b sinA = a sinB , c sinB = b sinC, c sinA = a sinC

(v) $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}, \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

* The Cosine Rule : In ΔABC,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = c^2 + a^2 - 2ca \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

* The Projection Rule : In ΔABC.

a = b cos C + c cos B

b = c cos A + a cos C

c = a cos B + b cos A

* Half angle formulae : In ΔABC , if a + b + c = 2s then

(i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}, \sin\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}}, \sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}, \cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}, \cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(iii) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}, \tan\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}, \tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

* Heron's Formula : If a,b,c are sides of ΔABC and a + b + c = 2s then

$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

* Napier's Analogy : In ΔABC , $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{(b-c)}{(b+c)}\cot\dfrac{A}{2}$

* Inverse Trigonometric functions :

(i) $\sin(\sin^{-1}x) = x$, for $x \in [-1,1]$

(ii) $\sin^{-1}(\sin y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\cos(\cos^{-1}x) = x$, for $x \in [-1, 1]$

(iv) $\cos^{-1}(\cos y) = y$, for $y \in [0, \pi]$

(v) $\tan(\tan^{-1}x) = x$, for $x \in R$

(vi) $\tan^{-1}(\tan y) = y$, for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, for $x \in R - (-1, 1)$

(viii) $\operatorname{cosec}^{-1}(\operatorname{cosec} y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(ix) $\sec(\sec^{-1} x) = x$, for $x \in R - (-1, 1)$

(x) $\sec^{-1}(\sec y) = y$, for $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(xi) $\cot(\cot^{-1} x) = x$, for $x \in R$

(xii) $\cot^{-1}(\cot y) = y$, for $y \in (0, \pi)$

∗ Properties of inverse trigonometric functions :

(i) $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ if $-1 \le x \le 1$ and $x \ne 0$

(ii) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ if $-1 \le x \le 1$ and $x \ne 0$

(iii) $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$ if $x > 0$

(iv) If $-1 \le x \le 1$ then $\sin^{-1}(-x) = -\sin^{-1}(x)$

(v) If $-1 \le x \le 1$ then $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(vi) For all $x \in R$, $\tan^{-1}(-x) = -\tan^{-1}x$

(vii) If $|x| \ge 1$ then $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$

(viii) If $|x| \ge 1$ then $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

(ix) For all $x \in R$, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

(x) If $-1 \le x \le 1$ then $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}$

(xi) For $x \in R$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(xii) For $x \ge 1$, $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

(xiii) If $x > 0$, $y > 0$ and $xy < 1$ then $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

(xiv) If $x > 0$, $y > 0$ and $xy > 1$ then $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\dfrac{x+y}{1-xy}$

(xv) If $x > 0$, $y > 0$ and $xy = 1$ then $\tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{2}$

(xvi) If $x > 0$, $y > 0$ then $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Miscellaneous Exercise 3

I) Select the correct option from the given alternatives.

1) The principal of solutions equation $\sin\theta = \dfrac{-1}{2}$ are ____________.

a) $\dfrac{5\pi}{6}, \dfrac{\pi}{6}$ b) $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ c) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ d) $\dfrac{7\pi}{6}, \dfrac{\pi}{6}$

2) The principal solution of equation $\cot\theta = \sqrt{3}$ ___________.

a) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ b) $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

c) $\dfrac{\pi}{6}, \dfrac{8\pi}{6}$ d) $\dfrac{7\pi}{6}, \dfrac{\pi}{6}$

3) The general solution of $\sec x = \sqrt{2}$ is __________.

a) $2n\pi \pm \dfrac{\pi}{4}$, $n \in Z$ b) $2n\pi \pm \dfrac{\pi}{2}$, $n \in Z$

c) $n\pi \pm \dfrac{\pi}{2}$, $n \in Z$, d) $2n\pi \pm \dfrac{\pi}{3}$, $n \in Z$

4) If $\cos p\theta = \cos q\theta$, $p \neq q$ then ________.

a) $\theta = \dfrac{2n\pi}{p \pm q}$ b) $\theta = 2n\pi$

c) $\theta = 2n\pi \pm p$ d) $np + q$

5) If polar co-ordinates of a point are $\left(2, \dfrac{\pi}{4}\right)$ then its cartesian co-ordinates are ______.

a) $(2, \sqrt{2})$ b) $(\sqrt{2}, 2)$

c) $(2, 2)$ d) $(\sqrt{2}, \sqrt{2})$

6) If $\sqrt{3}\cos x - \sin x = 1$, then general value of x is _________.

a) $2n\pi \pm \dfrac{\pi}{3}$ b) $2n\pi \pm \dfrac{\pi}{6}$

c) $2n\pi \pm \dfrac{\pi}{3} - \dfrac{\pi}{6}$ d) $n\pi + (-1)^n \dfrac{\pi}{3}$

7) In ΔABC if $\angle A = 45^0$, $\angle B = 30^0$ then the ratio,then a: b: c = __________.

a) $2 : \sqrt{2} : \sqrt{3}+1$ b) $\sqrt{2} : 2 : \sqrt{3}+1$

c) $2\sqrt{2} : \sqrt{2} : \sqrt{3}$ d) $2 : 2\sqrt{2} : \sqrt{3}+1$

8) In ΔABC, if $c^2 + a^2 - b^2 = ac$, then $\angle B$ = ___________.

$a)\ \frac{\pi}{4}$ $b)\ \frac{\pi}{3}$ $c)\ \frac{\pi}{2}$ $d)\ \frac{\pi}{6}$

9) In ΔABC, $ac \cos B - bc \cos A$ = _____________.

a) $a^2 - b^2$ b) $b^2 - c^2$

c) $c^2 - a^2$ d) $a^2 - b^2 - c^2$

10) If in a triangle, the angles are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$ then A is equal to ___________.

a) 30^0 b) 60^0

c) 75^0 d) 45^0

11) $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ = ____________.

a) $\frac{7\pi}{6}$ b) $\frac{5\pi}{6}$

c) $\frac{\pi}{6}$ d) $\frac{3\pi}{2}$

12) The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is _____________.

a) $\left(-\frac{2\pi}{3}\right)$ b) $\frac{4\pi}{3}$

c) $\frac{5\pi}{3}$ d) $-\frac{\pi}{3}$

13) If $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\alpha$, then α = ______________.

a) $\frac{63}{65}$ b) $\frac{62}{65}$

c) $\frac{61}{65}$ d) $\frac{60}{65}$

14) If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then x =

a) -1 b) $\frac{1}{6}$ c) $\frac{2}{6}$ d) $\frac{3}{2}$

15) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) =$ __________.

a) $\tan^{-1}\left(\frac{4}{5}\right)$ b) $\frac{\pi}{2}$

c) 1 d) $\frac{\pi}{4}$

16) $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) =$ __________.

$a)\ \frac{17}{7}$ $b)\ -\frac{17}{7}$

$c)\ \frac{7}{17}$ $d)\ -\frac{7}{17}$

17) The principal value branch of $\sec^{-1} x$ is __________.

$a)\ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ $b)\ [0, \pi] - \left[\frac{\pi}{2}\right]$

$c)\ (0, \pi)$ $d)\ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

18) $\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right] =$ __________.

$a)\ \frac{1}{\sqrt{2}}$ $b)\ \frac{\sqrt{3}}{2}$ $c)\ \frac{1}{2}$ $d)\ \frac{\pi}{4}$

19) If tan θ + tan 2θ + tan 3θ = tanθ tan 2θ tan3θ, then the general value of the θ is _______.

a) nπ b) $\frac{n\pi}{6}$

c) $n\pi \pm \frac{\pi}{4}$ d) $\frac{n\pi}{2}$

20) If any ΔABC, if a cos B = b cos A, then the triangle is ________.

a) Equilateral triangle b) Isosceles triangle

c) Scalene d) Right angled

II: Solve the following

1) Find the principal solutions of the following equations :

(i) $\sin 2\theta = -\frac{1}{2}$ (ii) $\tan 3\theta = -1$ (iii) $\cot\theta = 0$

2) Find the principal solutions of the following equations :

(i) $\sin 2\theta = -\frac{1}{\sqrt{2}}$ (ii) $\tan 5\theta = -1$ (iii) $\cot 2\theta = 0$

3) Which of the following equations have no solutions ?

(i) $\cos 2\theta = \frac{1}{3}$ (ii) $\cos^2\theta = -1$ (iii) $2\sin\theta = 3$ (iv) $3\sin\theta = 5$

4) Find the general solutions of the following equations :

i) $\tan\theta = -\sqrt{3}$ ii) $\tan^2\theta = 3$ iii) $\sin\theta - \cos\theta = 1$ iv) $\sin^2\theta - \cos^2\theta = 1$

5) In ΔABC prove that $\cos\left(\frac{A-B}{2}\right) = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$

6) With usual notations prove that $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$.

7) In ΔABC prove that $(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2$

8) In ΔABC if $\cos A = \sin B - \cos C$ then show that it is a right angled triangle.

9) If $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-c)}$ then show that a^2, b^2, c^2, are in A.P.

10) Solve the triangle in which $a = \sqrt{3}+1 \cdot b = \sqrt{3}-1$ and $C = 60^0$

11) In ΔABC prove the following :

(i) $a\sin A - b\sin B = c\sin(A-B)$

(ii) $\frac{c-b\cos A}{b-c\cos A} = \frac{\cos B}{\cos C}$

(iii) $a^2\sin(B-C) = (b^2-c^2)\sin A$

(iv) $ac\cos B - bc\cos A = (a^2-b^2)$

(v) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(vi) $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$

(vii) $\frac{b-c}{a} = \frac{\tan\frac{B}{2} - \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}$

12) In ΔABC if a^2, b^2, c^2, are in A.P. then $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are also in A.P.

13) In ΔABC if $C = 90^o$ then prove that $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

14) In ΔABC if $\frac{\cos A}{a}=\frac{\cos B}{b}$ then show that it is an isosceles triangle.

15) In ΔABC if sin2 A + sin2 B = sin2 C then prove that the triangle is a right angled triangle.

16) In ΔABC prove that $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

17) With usual notations show that $(c^2 - a^2 + b^2)\tan A = (a^2 - b^2 + c^2)\tan B = (b^2 - c^2 + a^2)\tan C$

18) In ΔABC if $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$ then prove that a , b ,c are in A.P.

19) Show that $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

20) Show that $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

21) Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) if\ x\in[0,1]$

22) Show that $\frac{9\pi}{5}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

23) Show that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ for -\frac{1}{\sqrt{2}}\le x\le 1$

24) If $\sin(\sin^{-1}\frac{1}{5}+\cos^{-1}x)=1$ then find the value of x.

25) If $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$ then find the value of x .

26) If 2 $\tan^{-1}(\cos x) = \tan^{-1}(\text{cosec } x)$ then find the value of x.

27) Solve: $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, \text{for } x>0$

28) If $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ then find the value of x .

29) If $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{2}$ then find the value of x.

30) Show that $\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{2}{9}$

31) Show that $\cot^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{3}=\cot^{-1}\frac{3}{4}$

32) Show that $\tan^{-1}\frac{1}{2}=\frac{1}{3}\tan^{-1}\frac{11}{2}$

33) Show that $\cos^{-1}\frac{\sqrt{3}}{2}+2\sin^{-1}\frac{\sqrt{3}}{2}=\frac{5\pi}{6}$

34) Show that $2\cot^{-1}\frac{3}{2}+\sec^{-1}\frac{13}{12}=\frac{\pi}{2}$

35) Prove the following :

(i) $\cos^{-1} x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $x < 0$.

(ii) $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $x < 0$.

36) If $|x| < 1$, then prove that $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

37) If x, y, z, are positive then prove that $\tan^{-1}\frac{x-y}{1+xy}+\tan^{-1}\frac{y-z}{1+yz}+\tan^{-1}\frac{z-x}{1+zx}=0$

38) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ then, show that $xy + yz + zx = 1$

39) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ then show that $x^2 + y^2 + z^2 + 2xyz = 1$.

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