11. Definite Integration - I

Ex. (1) Evaluate $\int_0^1 x^2\, dx$

Solution : $\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1$

$$= \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \frac{1}{3}$$

Evaluation of integral as a limit of sum $\int_0^1 x^2\, dx$

$f(x) = x^2 \qquad a = 0$ and $b = 1$

$x = a + rh \quad$ and $h = \frac{b - a}{n}$

$h = \frac{1 - 0}{n}$

$nh = 1$

$f(x) = f(a + rh)$

$= f(a + rh)$

$= (rh)^2$

$= r^2h^2$

We know,

$$\int_a^b f(x).dx = \lim_{n\to\infty} \sum_{r=1}^{n} h.f\ (a + rh)$$

$$\therefore \int_0^1 x^2.dx = \lim_{n\to\infty} \sum_{r=1}^{n} h.r^2h^2$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h^3.r^2$$

$$= \lim_{n\to\infty} h^3.\frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{h^3.n^3(1)\,(1+\frac{1}{n})\,(2+\frac{1}{n})}{6}$$

$$= \frac{(1)^3(1)\,(1+0)\,(2+0)}{6} = \frac{1}{3}$$

Ex. (2) Evaluate $\int_1^3 (x^2+1)\,dx$

Solution : $f(x) = x^2 + 1, \quad a = 1, b = 3$

$$x = a + rh \quad \text{and} \quad h = \frac{b - a}{n}$$

$$x = 1 + rh \quad \text{and} \quad h = \frac{3-1}{n}$$

$$\therefore nh = 2$$

$$\begin{aligned} f(x) &= f(a + rh) \\ &= f(1 + rh) \\ &= (1 + rh)^2 + 1 \\ &= 1 + 2rh + r^2h^2 + 1 \\ &= 2 + 2rh + r^2h^2 \end{aligned}$$

We know,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\, f(a + rh)$$

$$\int_1^3 (x^2+1)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\,(2 + 2rh + r^2h^2)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} 2h + 2rh^2 + rh^3$$

$$= \lim_{n\to\infty} \left[2h \sum_{r=1}^{n} 1 + 2h^2 \sum_{r=1}^{n} r + h^3 \sum_{r=1}^{n} r^2 \right]$$

$$= \lim_{n\to\infty} \left[2h\,(n) + 2h^2 \frac{n\,(n+1)}{2} + h^3 \frac{n\,(n+1)\,(2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \left[2hn + h^2n^2(1)\,(1+\frac{1}{n}) + \frac{h^3n^3(1)\,(1+\frac{1}{n})\,(2+\frac{1}{n})}{6} \right]$$

$$= 2\,(2) + (2)^2(1)\,(1+0) + \frac{(2)^3(1)\,(1+0)\,(2+0)}{6}$$

$$= 8 + \frac{8}{3}$$

$$= \frac{32}{3}$$

$$\therefore \int_1^3 (x^2+1)\,dx = \frac{32}{3}$$

Ex. (3) Evaluate $\int_0^3 (4x+3)\,dx$

Solution : $f(x) = 4x + 3$, $a = 0$, $b = 3$

$x = a + rh$ and $h = \dfrac{b - a}{n}$

$\therefore x = rh$ and $h = \dfrac{3 - 0}{n}$

$\therefore nh = 3$

$f(x) = f(a + rh) = f(rh)$

$= 4(rh) + 3$

$= 4(rh) + 3$

We know,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\, f(a + rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h\,[(4rh + 3)]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} 4rh^2 + 3h$$

$$= \lim_{n\to\infty} \left(4h^2 \sum_{r=1}^{n} r + 3h \sum_{r=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \left[4h^2\ \frac{n(n+1)}{2} + 3h(n)\right]$$

$$= \lim_{n\to\infty} \left[2\,h^2 n^2 (1)\left(1 + \frac{1}{n}\right) + 3nh\right]$$

$$= 2(3)^2(1)(1+0) + 3(3)$$

$$= 18 + 9$$

$$= 27$$

$$\therefore \int_0^3 (4x+3)\,dx = 27$$

Ex. (4) Evaluate $\int_0^4 (2x - 1)\,dx$

Let $I = \int_0^4 (2x-1)\,dx$

$\int_a^b f(x)\,dx = \lim_{h\to 0} \left[h \sum_{r=1}^{n} f(a+rh)\right]$

$a=0,\ b=4;\ h=\frac{b-a}{n}=\frac{4-0}{n}$

$\therefore nh=4$

$f(x)=2x-1$

$f(a+rh)=2(a+rh)-1$

$f(a+rh)=2(0+rh)-1$

$f(a+rh)=2rh-1$

$\therefore I=\lim_{h\to 0}\left[h\sum_{r=1}^{n}(2rh-1)\right]$

$=\lim_{h\to 0} h\left[\sum_{r=1}^{n}2rh-\sum_{r=1}^{n}1\right]$

$I=\lim_{h\to 0} h\left[2h\,\frac{n(n+1)}{2}-n\right]$

$=\lim_{h\to 0} h\left[nh\times n+nh-nh\right]$

$=\lim_{h\to 0}\left[nh\times nh\right]$

$=\lim_{h\to 0} 4\times 4$

$=\lim_{h\to 0} 16$

$\therefore I=16$

B. Evaluate the following definite integrals.

Ex. (1) $\int_0^{\pi} x\sin^2 x\, dx$

Let $I=\int_0^{\pi} x\sin^2 x\, dx$ — I

using

$\int_0^a f(x)dx=\int_0^a f(a-x)\,dx$

changing $x\to\pi-x$

$\therefore I=\int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx$

$=\int_0^{\pi}(\pi-x)\sin^2 x\,dx$

$(\because \sin(\pi-\theta)=\sin\theta)$

$=\int_0^{\pi}(\pi\sin^2 x-x\sin^2 x)\,dx$

$=\int_0^{\pi}\pi\sin^2 x\,dx-\int_0^{\pi}x\sin^2 x\,dx$

$=\int_0^{\pi}\pi\sin^2 x\,dx-I$ from I

$\therefore I+I=\frac{\pi}{2}\int_0^{\pi}2\sin^2 x\,dx$

$\cos 2\theta=1-\sin^2\theta$

$\Rightarrow 2\sin^2\theta=1-\cos 2\theta$

$\therefore 2I=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx$

$\therefore 2I=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}$

$\therefore 2I=\frac{\pi}{2}\left[\pi-\frac{\sin 2\pi}{2}-\left(0-\frac{\sin 0}{2}\right)\right]$

$\therefore 2I=\frac{\pi}{2}\left[\pi-\frac{0}{2}-\left(0-\frac{0}{2}\right)\right]$

$2I=\frac{\pi^2}{2}$

$I=\frac{\pi^2}{4}$

Ex. (2) $\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$

$$I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}\,dx$$

$$= \int_{-a}^{a}\sqrt{\frac{(a-x)^2}{a^2-x^2}}\,dx$$

$$= \int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= \int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}\,dx - \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$f_1(x) = \frac{a}{\sqrt{a^2-x^2}} \quad f_2(x) = \frac{x}{\sqrt{a^2-x^2}}$$

$$f_1(-x) = \frac{a}{\sqrt{a^2-(-x)^2}},\ f_2(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}}$$

$$f_1(-x) = \frac{a}{\sqrt{a^2-x^2}},\ f_2(-x) = \frac{-x}{\sqrt{a^2-x^2}}$$

$f_1(x)$ is even and $f_2(x)$ is odd

$\therefore$ using def integration Property

$$I = 2\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx - 0$$

$$= 2a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= 2a\sin^{-1}\left(\frac{x}{a}\right)\Big]_0^a$$

$$= 2a\left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)\right]$$

$$= 2a\left(\sin^{-1}(1) - \sin^{-1}(0)\right)$$

$$= 2a\times\frac{\pi}{2}$$

$$I = \pi a$$

Ex. (3) $\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx$

$$I = \int_0^{\frac{\pi}{2}}\log\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$$ – using Property and changing $x \to \frac{\pi}{2}-x$

$$= \int_0^{\frac{\pi}{2}}\log\sin x\,dx \quad — I$$

$$= \int_0^{\frac{\pi}{2}}\log\left(2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{2}}\log 2 + \log\sin\frac{x}{2} + \log\cos\frac{x}{2}\,dx$$

Put $\frac{x}{2} = \theta \Rightarrow x = 2\theta \quad \therefore dx = 2d\theta$

when $x = 0,\ \theta = 0$, when $x = \frac{\pi}{2},\ \theta = \frac{\pi}{4}$

$$I = \log 2\int_0^{\pi/2} 1\,dx + \int_0^{\frac{\pi}{4}}\log\sin\theta(2)d\theta + \int_0^{\pi/4}\log\cos\theta\,2\,d\theta$$

$$= \log 2\,[\theta]_0^{\frac{\pi}{2}} + 2\int_0^{\pi/4}\log\sin\theta\,d\theta + 2\int_0^{\pi/4}\log\cos\theta\,d\theta$$

put $\theta = \frac{\pi}{2} - \alpha \Rightarrow \alpha = \frac{\pi}{2} - \theta$

$\therefore d\theta = -d\alpha$, when $\theta = 0$ $\alpha = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

when $\theta = \frac{\pi}{4},\ \alpha = \frac{\pi}{4}$

$$\therefore = \log 2\left[\frac{\pi}{2} - 0\right] + 2\int_0^{\pi/4}\log\sin\theta\,d\theta +$$
$$2\int_{\pi/2}^{\pi/4}\log\cos\left(\frac{\pi}{2}-\alpha\right)(-1)\,d\alpha$$

$$I = \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4}\log\sin x\,dx + 2\int_{\pi/4}^{\frac{\pi}{2}}\log\sin x\,dx$$

$$= \frac{\pi}{2}\log 2 + 2\left[\int_0^{\frac{\pi}{4}}\log\sin x\,dx + \int_{\pi/4}^{\frac{\pi}{2}}\log\sin x\,dx\right]$$

$$= \frac{\pi}{2}\log 2 + 2\int_0^{\frac{\pi}{2}}\log\sin x\,dx = \frac{\pi}{2} + 2I$$

$$= I - 2I = \frac{\pi}{2}\log 2$$

$$-I = \frac{\pi}{2}\log 2$$

$$I = -\frac{\pi}{2}\log 2$$

Sign of Teacher :

Q. 26. A solenoid of length π m and 5 cm in diameter has a winding of 1000 turns and carries a current of 5A. Calculate the magnetic field at its centre along the axis.

SECTION – D

Attempt any THREE questions of the following : **[12]**

Q. 27. What is Ferromagnetism? Explain it on the basis of domain theory.

Q. 28. Obtain an expression for average power dissipated in a series LCR circuit.

Q. 29. Distinguish between interference and diffraction of light.

A double slit arrangement produces interference fringes for sodium light of wavelength 589 nm, that are 0.20 degree apart. What is the angular fringe separation if the entire arrangement is immersed in water?

(R.I. of water = 1.33)

Q. 30. State Einstein's photoelectric equation and mention physical significance of each term involved in it.

The wavelength of incident light is 4000Å. Calculate the energy of incident photon.

Q. 31. State any four uses of Van de Graaff generator.

In a parallel plate air capacitor, intensity of electric field is changing at the rate of 2×10^{11} V/ms. If area of each plate is 20 cm^2, calculate the displacement current.

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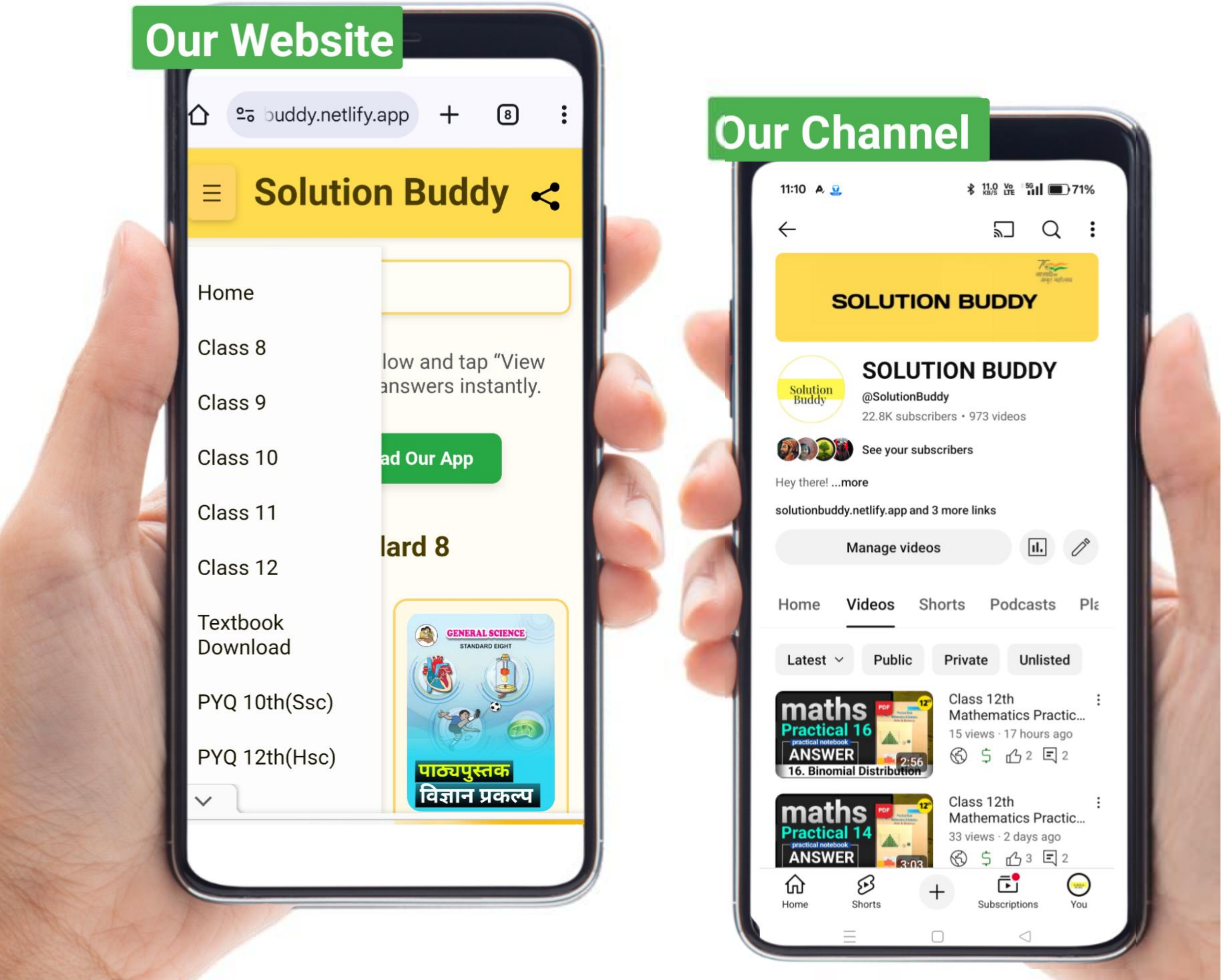

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