10. Applications of Derivatives – II

Ex. (1). Check the validity of the Rolle's theorem for the function $f(x) = x^3 - 4x + 1$. Find all values of c in the interval (-2, 2) such that $f'(c) = 0$.

Solution : Given that $f(x) = x^3 - 4x + 1$ … (I)

$f(x)$ is a polynomial which is continuous on [-2, 2] and it is differentiable on (-2, 2)

Let $a = -2$ and $b = 2$

For $x = a = -2$ from (I) we get,

$f(a) = f(-2) = (-2)^3 - 4(-2) + 1 = -8 + 8 + 1 = 1$

For $x = b = 2$ from (I) we get,

$f(b) = f(2) = (2)^3 - 4(2) + 1 = 8 - 8 + 1 = 1$

So, here $f(a) = f(b)$ i.e. $f(-2) = f(2) = 1$

Hence conditions of Rolle's theorem are satisfied. So, there exists $c \in (-2, 2)$ such that $f'(c) = 0$.

Differentiating (I) *w. r. t. x.*

$f'(x) = 3x^2 - 4 \therefore f'(c) = 3c^2 - 4$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 4 = 0 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$

$\therefore c_1 = -\frac{2}{\sqrt{3}}$ and $c_2 = \frac{2}{\sqrt{3}}$ both belong to (-2, 2).

Ex. (2). Determine the local extrema of the function $f(x) = \sin x - \cos x$ in $[0, 2\pi]$.

Solution : Given that $f(x) = \sin x - \cos x$… (I)

Differentiate w. r. t. x.

$f'(x) = \cos x + \sin x$ … (II)

$f'(x) = 0$, for extreme values of $f(x)$

i.e. $\cos x + \sin x = 0 \Rightarrow \tan x = -1$

we have $\tan\left(\frac{3\pi}{4}\right) = -1$ and $\tan\left(\frac{7\pi}{4}\right) = -1$

$\therefore x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are the values at which $f'(x) = 0$ and

$f(x)$ has its extreme values. Also, both $\frac{3\pi}{4}, \frac{7\pi}{4} \in [0, 2\pi]$

Differentiate (II) w. r. t. x.

$f''(x) = -\sin x + \cos x$ .... (III)

For $x = \frac{3\pi}{4}$, from (III) we get [angle $\frac{3\pi}{4}$ is in II quadrant]

$$f''\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$$

$$f''\left(\frac{3\pi}{4}\right) = -\sqrt{2} < 0$$

$\therefore$ For $x = \frac{3\pi}{4}$, $f(x)$ has a maxima.

$$f\max = \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$\therefore f\max = \sqrt{2}$

For $x = \frac{7\pi}{4}$, from (III) [$\frac{7\pi}{4}$ is in IV quadrant]

$$f''\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$f''\left(\frac{7\pi}{4}\right) = \sqrt{2} > 0$$

$\therefore$ For $x = \frac{7\pi}{4}$, $f(x)$ has a minima.

$$f\min = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$\therefore f\min = -\sqrt{2}$

Ex. (3). Given the function $f(x) = x^3 - 2x^2 - x + 1$. Find all points c satisfying the conditions of the Lagrange's Mean Value Theorem for the function on the interval [-2, 2].

Solution : Given that $f(x) = x^3 - 2x^2 - x + 1$ … (I)

f (x) is a polynomial which is continuous on [-2, 2]and it is differentiable on [-2, 2]. So, f (x) satisfies the conditions of LMVT.

$\therefore$ There exists a $c \in (-2, 2)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $a = -2$ and $b = 2$

For $x = a =$ -2 from (I) we get,

$f(a) = f(-2) = (-2)^3 - 2(-2)^2 - (-2) + 1 = -13$

For x = b = 2 from (I) we get,

$f(b) = f(2) = (2)^3 - 2 (2)^2 - 2 + 1 =$ ...-1...

Differentiate (I) w. r. t. x.

$f'(x) =$ ...$3x^2 - 4x - 1$... $\therefore f'(c) =$ ...$3c^2 - 4c - 1$...

Now, $f'(c) = \dfrac{-1-(-13)}{2-(-2)} = 3$

Thus, $3c^2 - 4c - 1 = 3$ i.e. $3c^2 - 4c - 1 =$ ...3...

$3c^2 - 4c -$ ...4... $= 0 \Rightarrow 3c^2 - (6c) + (2c) - 4 = 0$

$\Rightarrow 3c\,(c-2) + 2\,(c-2) = 0$ i.e. $(c-2)\,(3c + 2) = 0$

$\Rightarrow$ ...$c-2$... $= 0$ or ...$3c+2$... $= 0. \Rightarrow c =$ ...2... or $c =$ ...$-\frac{2}{3}$...

But $c =$ ...2... $\notin (-2, 2)$ and $c =$ ...$-\frac{2}{3}$... $\in (-2, 2)$

Hence LMVT is verified.

Ex. (4). The sum of two positive numbers is 24. Find the numbers so that the sum of their squares is minimum.

Solution : Let one of the numbers be x so the other number is ...$24-x$

Let S be the sum of the squares of the numbers.

$\therefore S = (24-x)^2 + (x)^2 = x^2 +$ ...576... $-$ ...48... $x + x^2$

$S =$ ...$2x^2 - 48x + 576$... .... (I)

Differentiate (I) w. r. t. x.

$\dfrac{dS}{dx} =$ ...$4x - 48$... .... (II)

For extreme values of S, we have $\dfrac{dS}{dx} = 0$

$\therefore$ ...$4x$... $- 48 = 0$ $\qquad \therefore x =$ ...12...

Therefore at x = ...12... either there is a maxima or minima.

Differentiate (II) w. r. t. x.

$\dfrac{d^2S}{dx^2} =$ ...4... .... (III)

Substituting x = 12 in (III), we get,

$\left(\dfrac{d^2S}{dx^2}\right)_{x=12} =$ ...4... > 0

Therefore S has a minima at x = ...12

Therefore the required numbers are ...12... and $24 -$ ...12... $=$ ...12...

Ex. (5). Find the volume of the largest box that can be made by cutting equal squares out of the corners of a piece of cardboard of dimensions 15 cm by 24 cm, and then turning up the sides.

Solution : ....

Let the side of square be x cm

$\therefore$ Length of box $= (24-20x)$ cm

breadth of box $= (15-2x)$ cm

volume $= V = l \times b \times h$

$V = (24-2x)\,x\,(15-2x)$

$V = (24x-2x^2)(15-2x)$

$= 360x - 48x^2$

$= 30x^2 + 4x^3$

$\therefore V = 4x^3 - 78x^2 + 360x$

diff w.r.to x

$\frac{dV}{dx} = 12x^2 - 156x + 360$

diff w.r.to x

$\frac{d^2V}{dx^2} = 24x - 156$

For max volume $\frac{dV}{dx} = 0$ & $\frac{d^2V}{dx^2} < 0$

Let $\frac{dV}{dx} = 0$

$\therefore 12x^2 - 156x + 360 = 0$

$x^2 - 13x + 30 = 0$ solving

$\therefore x = 10$ or $x = 3$

$\frac{d^2V}{dx^2} = 24 \times 10 - 156 = 84 > 0$

$\frac{d^2V}{dx^2} = 24 \times 3 - 156 = -84 < 0$

Volume is max for $x = 3$

Put $x = 3$ in V

$V = 4(3)^3 - 78(3)^2 + 360 \times 3$

$= 108 - 702 + 1080$

$V = 486$ cu.cm.

Ex. (6). Examine the function $f(x) = x^3 - 5x^2 + 8x - 4$ for maxima and minima.

Solution :

$f(x) = x^3 - 5x^2 + 8x - 4$ — I

diff w.r.to x

$f'(x) = 3x^2 - 10x + 8$ — II

diff w.r.to x

$f''(x) = 6x - 10$ — III

Let $f'(x) = 0$

$\therefore 3x^2 - 10x + 8 = 0$ from II

on solving

$x = 2$, or $x = \frac{4}{3}$

Put $x = 2$ in III

$f''(2) = 6 \times 2 - 10 = 2 > 0$

function is min. at $x = 2$

min value of function

$f(2) = 2^3 - 5(2)^2 + 8 \times 2 - 4 = \underline{0}$

Now put $x = \frac{4}{3}$ in (III)

$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = -2 < 0$

$\therefore f(x)$ is max at $x = \frac{4}{3}$

$\therefore$ max value of function is

$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 5\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) - 4$

$= \frac{64}{27} - \frac{80}{9} + \frac{32}{3} - 4$

$= \frac{4}{27}$

$\therefore$ maxima $= \frac{4}{27}$ and minima $= 0$

Ex. (7). Find two positive numbers x and y, such that x + y = 60 and xy^3 is maximum.

Solution :

$x + y = 60$

$x = 60 - y$

$xy^3 = (60-y)y^3$

$\therefore xy^3 = 60y^3 - y^4$

Let $f(y) = 60y^3 - y^4$ — (I)

diff w.r.to y

$f'(y) = 180y^2 - 4y^3$ — (II)

diff w.r.to. y

$f''(y) = 360y - 12y^2$ — (III)

f(y) is Maximum if

$f'(y) = 0$ and $f''(y) < 0$

Let $f'(y) = 0$

$\therefore 180y^2 - 4y^3 = 0$

$\therefore 4y^2(45-y) = 0$

$\therefore 4y^2 = 0$ or $45 - y = 0$

$\therefore y = 0$ or $y = 45$

since y is a positive

$\therefore y = 0$ is not possible

put $y = 45$ in (III)

$f''(45) = 360 \times 45 - 4(45)^3$

$f''(45) = 16200 - 24300$

$\therefore f'(45) = -8100 < 0$

For $y = 45$, $f'(y) = 0$, $f''(y) < 0$

$\therefore f(y) = (60-y)y^3$

$= xy^3$ is max

for $y = 45$ and

$x = 60 - y$

$= 60 - 45$

$= 15$

$\therefore x = 15$ and $y = 45$ are two required positive numbers such that

$x + y = 6$

and xy^3 is maximum.

Sign of Teacher :

Q. 26. A solenoid of length π m and 5 cm in diameter has a winding of 1000 turns and carries a current of 5A. Calculate the magnetic field at its centre along the axis.

SECTION – D

Attempt any THREE questions of the following : **[12]**

Q. 27. What is Ferromagnetism? Explain it on the basis of domain theory.

Q. 28. Obtain an expression for average power dissipated in a series LCR circuit.

Q. 29. Distinguish between interference and diffraction of light.

A double slit arrangement produces interference fringes for sodium light of wavelength 589 nm, that are 0.20 degree apart. What is the angular fringe separation if the entire arrangement is immersed in water?

(R.I. of water = 1.33)

Q. 30. State Einstein's photoelectric equation and mention physical significance of each term involved in it.

The wavelength of incident light is 4000Å. Calculate the energy of incident photon.

Q. 31. State any four uses of Van de Graaff generator.

In a parallel plate air capacitor, intensity of electric field is changing at the rate of 2×10^{11} V/ms. If area of each plate is 20 cm^2, calculate the displacement current.

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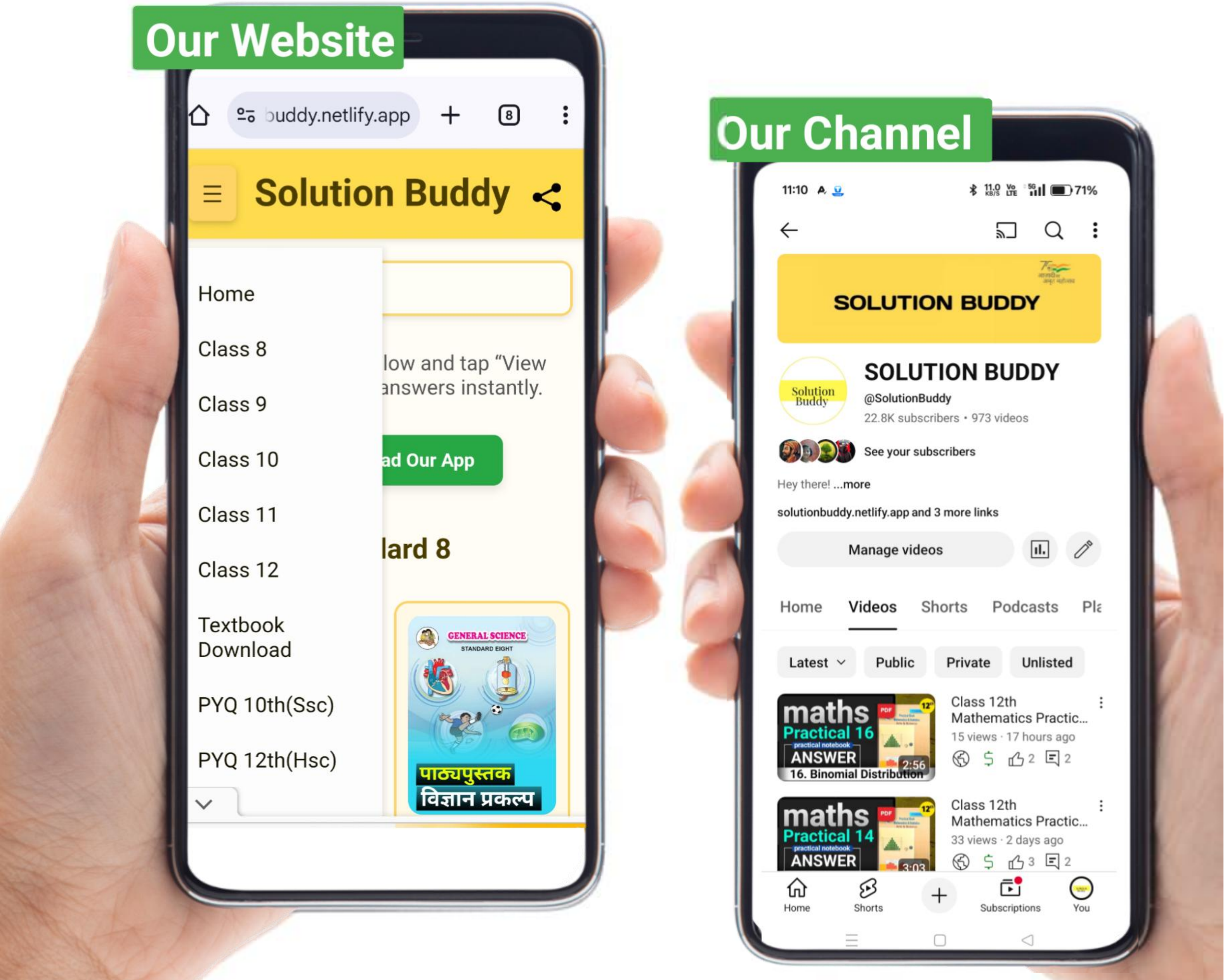

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