10. Applications of Derivatives – II

Ex. (1). Check the validity of the Rolle's theorem for the function $f(x) = x^3 - 4x + 1$. Find all values of c in the interval (-2, 2) such that $f'(c) = 0$.

Solution : Given that $f(x) = x^3 - 4x + 1$ … (I)

$f(x)$ is a polynomial which is continuous on [-2, 2] and it is differentiable on (-2, 2)

Let $a = -2$ and $b = 2$

For $x = a = -2$ from (I) we get,

$f(a) = f(-2) = (-2)^3 - 4(-2) + 1 = -8 + 8 + 1 = 1$

For $x = b = 2$ from (I) we get,

$f(b) = f(2) = (2)^3 - 4(2) + 1 = 8 - 8 + 1 = 1$

So, here $f(a) = f(b)$ i.e. $f(-2) = f(2) = 1$

Hence conditions of Rolle's theorem are satisfied. So, there exists $c \in (-2, 2)$ such that $f'(c) = 0$.

Differentiating (I) *w. r. t. x.*

$f'(x) = 3x^2 - 4 \therefore f'(c) = 3c^2 - 4$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 4 = 0 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$

$\therefore c_1 = -\frac{2}{\sqrt{3}}$ and $c_2 = \frac{2}{\sqrt{3}}$ both belong to (-2, 2).

Ex. (2). Determine the local extrema of the function $f(x) = \sin x - \cos x$ in $[0, 2\pi]$.

Solution : Given that $f(x) = \sin x - \cos x$ … (I)

Differentiate w. r. t. x.

$f'(x) = \cos x + \sin x$ … (II)

$f'(x) = 0$, for extreme values of $f(x)$

i.e. $\cos x + \sin x = 0 \Rightarrow \tan x = -1$

we have $\tan\left(\frac{3\pi}{4}\right) = -1$ and $\tan\left(\frac{7\pi}{4}\right) = -1$

$\therefore x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are the values at which $f'(x) = 0$ and

$f(x)$ has its extreme values. Also, both $\frac{3\pi}{4}, \frac{7\pi}{4} \in [0, 2\pi]$

Differentiate (II) w. r. t. x.

$f''(x) = -\sin x + \cos x$ .... (III)

For $x = \frac{3\pi}{4}$, from (III) we get [angle $\frac{3\pi}{4}$ is in II quadrant]

$$f''\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$$

$$f''\left(\frac{3\pi}{4}\right) = -\sqrt{2} < 0$$

$\therefore$ For $x = \frac{3\pi}{4}$, $f(x)$ has a maxima.

$$f\max = \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$\therefore f\max = \sqrt{2}$

For $x = \frac{7\pi}{4}$, from (III) [$\frac{7\pi}{4}$ is in IV quadrant]

$$f''\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$f''\left(\frac{7\pi}{4}\right) = \sqrt{2} > 0$$

$\therefore$ For $x = \frac{7\pi}{4}$, $f(x)$ has a minima.

$$f\min = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$\therefore f\min = -\sqrt{2}$

Ex. (3). Given the function $f(x) = x^3 - 2x^2 - x + 1$. Find all points c satisfying the conditions of the Lagrange's Mean Value Theorem for the function on the interval [-2, 2].

Solution : Given that $f(x) = x^3 - 2x^2 - x + 1$ ... (I)

f (x) is a polynomial which is continuous on [-2, 2]and it is differentiable on [-2, 2]. So, f (x) satisfies the conditions of LMVT.

$\therefore$ There exists a $c \in (-2, 2)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $a = -2$ and $b = 2$

For $x = a = -2$ from (I) we get,

$f(a) = f(-2) = (-2)^3 - 2(-2)^2 - (-2) + 1 = -13$

For $x = b = 2$ from (I) we get,

$f(b) = f(2) = (2)^3 - 2 (2)^2 - 2 + 1 = -1$

Differentiate (I) w. r. t. x.

$f'(x) = 3x^2 - 4x - 1 \quad \therefore f'(c) = 3c^2 - 4c - 1$

Now, $f'(c) = \dfrac{-1-(-13)}{2-(-2)} = 3$

Thus, $3c^2 - 4c - 1 = 3$ i.e. $3c^2 - 4c - 1 = 3$

$3c^2 - 4c - 4 = 0 \Rightarrow 3c^2 - (6c) + (2c) - 4 = 0$

$\Rightarrow 3c(c-2) + 2(c-2) = 0$ i.e. $(c-2)(3c+2) = 0$

$\Rightarrow c-2 = 0$ or $3c+2 = 0. \Rightarrow c = 2$ or $c = -\frac{2}{3}$

But $c = 2 \notin (-2, 2)$ and $c = -\frac{2}{3} \in (-2, 2)$

Hence LMVT is verified.

Ex. (4). The sum of two positive numbers is 24. Find the numbers so that the sum of their squares is minimum.

Solution : Let one of the numbers be x so the other number is $24-x$

Let S be the sum of the squares of the numbers.

$\therefore S = (24-x)^2 + (x)^2 = x^2 + 576 - 48 x + x^2$

$S = 2x^2 - 48x + 576$ .... (I)

Differentiate (I) w. r. t. x.

$\dfrac{dS}{dx} = 4x - 48$ .... (II)

For extreme values of S, we have $\dfrac{dS}{dx} = 0$

$\therefore 4x - 48 = 0 \qquad \therefore x = 12$

Therefore at x = 12 either there is a maxima or minima.

Differentiate (II) w. r. t. x.

$\dfrac{d^2S}{dx^2} = 4$ .... (III)

Substituting x = 12 in (III), we get,

$\left(\dfrac{d^2S}{dx^2}\right)_{x=12} = 4 > 0$

Therefore S has a minima at x = 12

Therefore the required numbers are 12 and 24 – 12 = 12

Ex. (5). Find the volume of the largest box that can be made by cutting equal squares out of the corners of a piece of cardboard of dimensions 15 cm by 24 cm, and then turning up the sides.

Solution : ....

Let the side of square be x cm

$\therefore$ Length of box $= (24-20x)$ cm

breadth of box $= (15-2x)$ cm

volume $= v = l \times b \times h$

$v = (24-2x)\,x\,(15-2x)$

$v = (24x-2x^2)(15-2x)$

$= 360x - 48x^2$

$= 30x^2 + 4x^3$

$\therefore v = 4x^3 - 78x^2 + 360x$

diff w.r.to x

$\frac{dv}{dx} = 12x^2 - 156x + 360$

diff w.r.to x

$\frac{d^2v}{dx^2} = 24x - 156$

For max volume $\frac{dv}{dx} = 0$ & $\frac{d^2v}{dx^2} < 0$

Let $\frac{dv}{dx} = 0$

$\therefore 12x^2 - 156x + 360 = 0$

$x^2 - 13x + 30 = 0$ solving

$\therefore x = 10$ or $x = 3$

$\frac{d^2v}{dx^2} = 24 \times 10 - 156 = 84 > 0$

$\frac{d^2v}{dx^2} = 24 \times 3 - 156 = -84 < 0$

Volume is max for $x = 3$

Put $x = 3$ in V

$V = 4(3)^3 - 78(3)^2 + 360 \times 3$

$= 108 - 702 + 1080$

$V = 486$ cu.cm.

Ex. (6). Examine the function $f(x) = x^3 - 5x^2 + 8x - 4$ for maxima and minima.

Solution :

$f(x) = x^3 - 5x^2 + 8x - 4$ — I

diff w.r.to x

$f'(x) = 3x^2 - 10x + 8$ — II

diff w.r.to x

$f''(x) = 6x - 10$ — III

Let $f'(x) = 0$

$\therefore 3x^2 - 10x + 8 = 0$ from II

on solving

$x = 2$, or $x = \frac{4}{3}$

Put $x = 2$ in III

$f''(2) = 6 \times 2 - 10 = 2 > 0$

function is min. at $x = 2$

min value of function

$f(2) = 2^3 - 5(2)^2 + 8 \times - 4 = \underline{0}$

Now put $x = \frac{4}{3}$ in (III)

$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = -2 < 0$

$\therefore f(x)$ is max at $x = \frac{4}{3}$

$\therefore$ max value of function is

$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 5\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) - 4$

$= \frac{64}{27} - \frac{80}{9} + \frac{32}{3} - 4$

$= \frac{4}{27}$

$\therefore$ maxima $= \frac{4}{27}$ and minima $= 0$

Ex. (7). Find two positive numbers x and y, such that x + y = 60 and xy^3 is maximum.

Solution:

$x + y = 60$

$x = 60 - y$

$xy^3 = (60-y)y^3$

$\therefore xy^3 = 60y^3 - y^4$

Let $f(y) = 60y^3 - y^4$ —(I)

diff w.r.to y

$f'(y) = 180y^2 - 4y^3$ —(II)

diff w.r.to. y

$f''(y) = 360y - 12y^2$ — (III)

f(y) is Maximum if

$f'(y) = 0$ and $f''(y) < 0$

Let $f'(y) = 0$

$\therefore 180y^2 - 4y^3 = 0$

$\therefore 4y^2(45-y) = 0$

$\therefore 4y^2 = 0$ or $45 - y = 0$

$\therefore y = 0$ or $y = 45$

since y is a positive

$\therefore y = 0$ is not possible

put $y = 45$ in (III)

$f''(45) = 360 \times 45 - 4(45)^3$

$f''(45) = 16200 - 24300$

$\therefore f'(45) = -8100 < 0$

For $y = 45$, $f'(y) = 0$, $f''(y) < 0$

$\therefore f(y) = (60-y)y^3$

$= xy^3$ is max

for $y = 45$ and

$x = 60 - y$

$= 60 - 45$

$= 15$

$\therefore x = 15$ and $y = 45$ are two required positive numbers such that

$x + y = 6$

and xy^3 is maximum.

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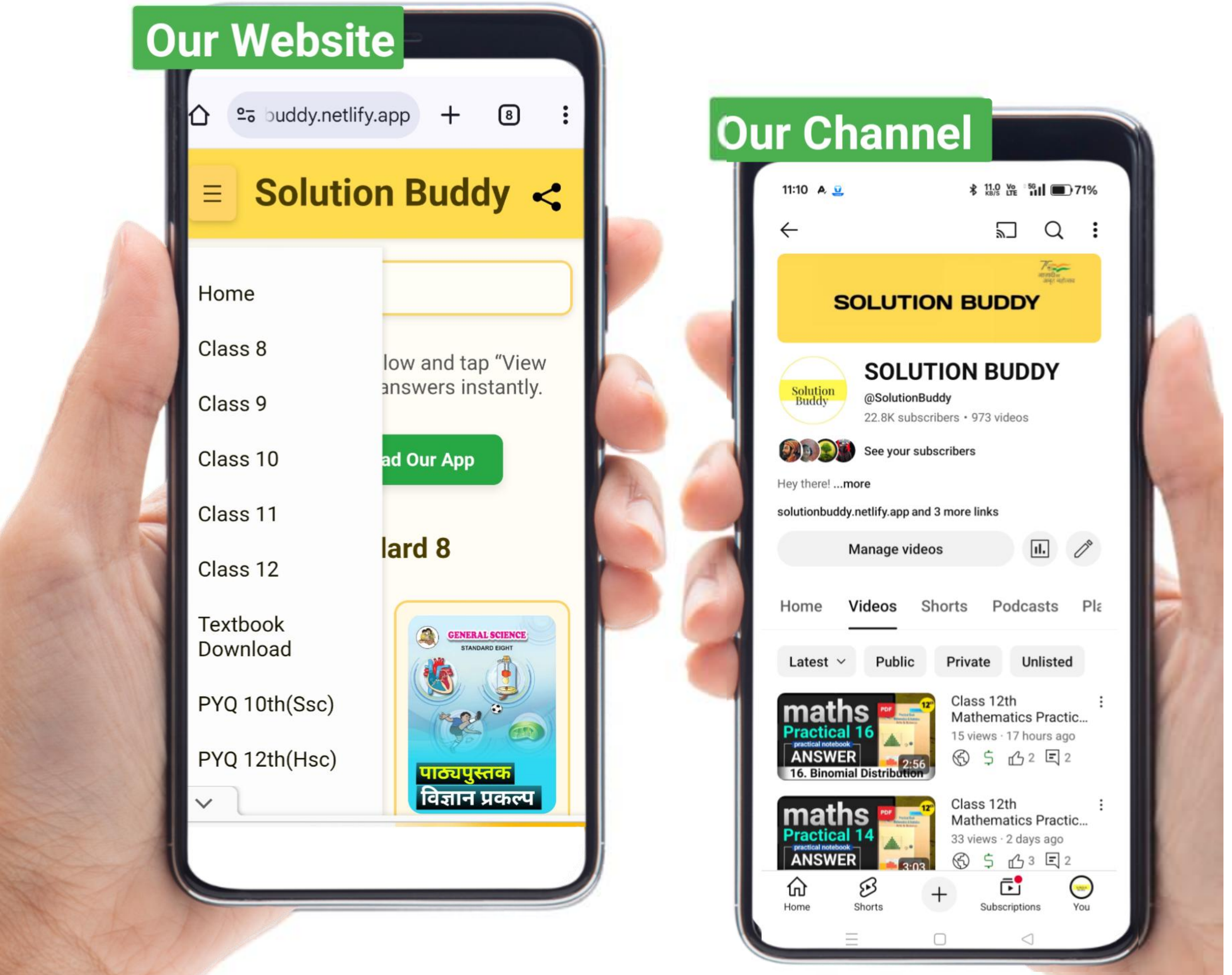

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