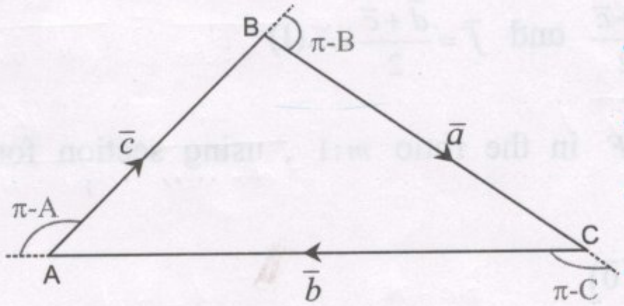
6. Vectors and Three Dimensional Geometry

Ex. (1) Using vectors prove the Projection rule.



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Solution : We have to prove Projection rule,

$a = b\cos C + c\cos B$

Let $\overline{BC} = \bar{a}, \overline{CA} = \bar{b}, \overline{AB} = \bar{c}$

By triangle law of addition of vectors, we have

$\overline{AB} + \overline{BC} + \overline{CA} = \bar{0}$

$\bar{c} + \bar{a} + \bar{b} = \bar{0}$

Taking a dot product with $\bar{a}$ on both sides, we get

$\bar{a}\cdot(\bar{a}+\bar{b}+\bar{c}) = \bar{a}\cdot\bar{0}$

$\bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c} = \bar{a}\cdot\bar{0}$

If $\bar{p}$ and $\bar{q}$ are any two vectors, then $\bar{p}\cdot\bar{q} = pq\cos\theta$, where θ is angle between $\bar{p}$ and $\bar{q}$.

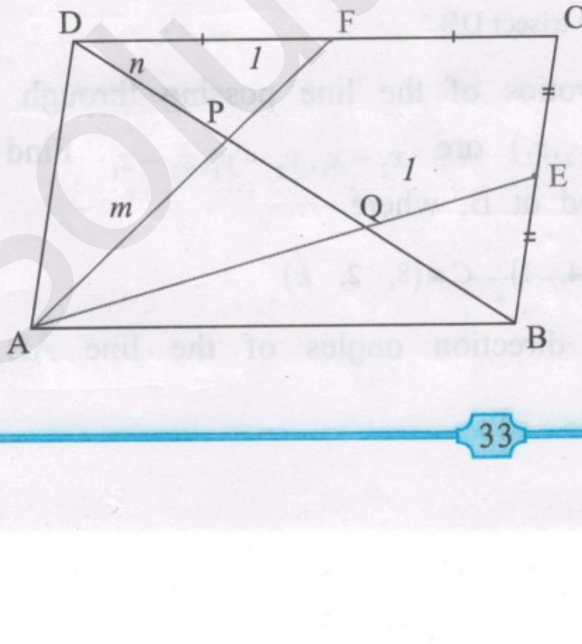
$(a)(a)\cos 0 + (a)(b)\cos(\pi - C) + (a)(c)\cos(\pi - B) = 0$

Divide throughout by a, we get

$a\cos 0 + b(-\cos C) + c(-\cos B) = 0$

$a = b\cos C + c\cos B$.

Ex. (2) ABCD is a parallelogram. E and F are mid points of BC and CD respectively. AE and AF meet diagonal BD in Q and P respectively. Show that P and Q trisect BD.



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Solution : Without loss of generality let $A(\bar{0})$ be origin. $B(\bar{b}), C(\bar{c}), D(\bar{d})$ are the other three vertices of parallelogram.

$E(\bar{e})$ and $F(\bar{f})$ are the midpoints of BC and DC.

By midpoint formula, $\bar{e} = \dfrac{\bar{b}+\bar{c}}{2}$ and $\bar{f} = \dfrac{\bar{d}+\bar{c}}{2}$ ....(I)

Also $\bar{c} = \bar{b} + \bar{d}$ ...(II)

Now let point P divides AF in the ratio $m:1$, using section formula, we get

$$\bar{p} = \frac{m(\bar{f})+1(\bar{0})}{m+1} = \frac{m\left(\frac{\bar{d}+\bar{c}}{2}\right)+1(\bar{0})}{m+1}$$

From (II), we get

$$\bar{p} = \frac{m}{2(m+1)}\left(\bar{d}+\bar{b}+\bar{d}\right) = \frac{m}{2(m+1)}\left(\bar{b}+2\bar{d}\right)$$

$$= \frac{m}{2(m+1)}\bar{b} + \frac{m}{(m+1)}\bar{d} \text{...(III)}$$

Also, let point P divides DB in the ratio $n:1$, using section formula, we get

$$\bar{p} = \frac{m(\bar{b})+1(\bar{d})}{n+1} = \frac{n}{n+1}\bar{b} + \frac{1}{n+1}\bar{d} \text{...(IV)}$$

From (III) and (IV), we get

$$\frac{m}{2(m+1)} = \frac{n}{n+1} \text{...(V)}$$

$$\frac{m}{(m+1)} = \frac{1}{n+1} \text{...(VI)}$$

Divide (V) by (VI), we get $\frac{1}{2} = n$,

$\therefore$ $DP:PB = n:1 = 1:2$ ...(VII)

By symmetry, we get $BQ:QD = 1:2$ ...(VIII)

From (VII) and (VIII), P and Q trisect DB.

Ex. (3) Show that direction ratios of the line passing through points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Find k' if Δ ABC is right angled at B, where

$A \equiv (5, \ 6, \ 4)$, $B \equiv (4, \ 4, \ 1)$, $C \equiv (8, \ 2, \ k')$.

Solution: Let α, β, γ be the direction angles of the line AB, and

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$\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the line AB .

Also $\bar{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ *and* $\bar{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, so we have

$\overline{AB} = \bar{b} - \bar{a} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Now , $\overline{AB}\cdot\hat{i} = \left[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right]\cdot\hat{i} = x_2 - x_1$ ...(I)

And also $\overline{AB}\cdot\hat{i} = |\overline{AB}||\hat{i}|\cos\alpha = AB\cos\alpha$ (II)

From (I) and (II), we have

$x_2 - x_1 = AB\cos\alpha$, similarly $y_2 - y_1 = AB\cos\beta$ $z_2 - z_1 = AB\cos\gamma$

As $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are proportional to $\cos\alpha, \cos\beta, \cos\gamma$.

Therefore, direction ratios of line AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

As $A \equiv (5,\ 6,\ 4), B \equiv (4,\ 4,\ 1)$, $C \equiv (8,\ 2,\ k')$ then $\bar{a} \equiv 5\hat{i} + 6\hat{j} + 4\hat{k}$,

$\bar{b} = 4\hat{i} + 4\hat{j} + \ldots\hat{k}$ and $\bar{c} = 8\hat{i} + 2\hat{j} + k'\hat{k}$

Also $\overline{AB} = \bar{b} - \bar{a} = (4\hat{i} + 4\hat{j} + \hat{k}) - (5\hat{i} + 6\hat{j} + 4\hat{k}) = -\hat{i} - 2\hat{j} - 3\hat{k}$ and

$\overline{BC} = \bar{c} - \bar{b} = (8\hat{i} + 2\hat{j} + k'\hat{k}) - (4\hat{i} + 4\hat{j} + \hat{k}) = 4\hat{i} - 2\hat{j} + (k' - 1)\hat{k}$

Δ ABC is right angled at B, we have $\overline{AB}\cdot\overline{BC} = 0$,

$(-\hat{i} - 2\hat{j} - 3\hat{k}).[4\hat{i} - 2\hat{j} + (k' - 1)\hat{k}] = 0$

$-4 + 4 - 3(k' - 1) = 0$

$k' = 1$

Ex. (4) Prove that $(\bar{a} + 2\bar{b} - \bar{c}).\left[(\bar{a} - \bar{b}) \times (\bar{a} - \bar{b} - \bar{c})\right] = 3\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$.

Solution: Consider

$(\bar{a} + 2\bar{b} - \bar{c})\cdot\left[(\bar{a} - \bar{b}) \times (\bar{a} - \bar{b} - \bar{c})\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[(\bar{a} - \bar{b}) \times \bar{a} - (\bar{a} - \bar{b}) \times \bar{b} - (\bar{a} - \bar{b}) \times \bar{c}\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[\bar{a} \times \bar{a} - \bar{b} \times \bar{a} - \bar{a} \times \bar{b} + \bar{b} \times \bar{b} - \bar{a} \times \bar{c} + \bar{b} \times \bar{c}\right]$

As $\bar{a} \times \bar{a} = \bar{0}$, $\bar{b} \times \bar{b} = \bar{0}$ and $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[\bar{0} + \bar{a} \times \bar{b} - \bar{a} \times \bar{b} + \bar{0} - \bar{a} \times \bar{c} + \bar{b} \times \bar{c}\right]$

$= (\bar{a} + 2\bar{b} - \bar{c})\cdot\left[-\bar{a} \times \bar{c} + \bar{b} \times \bar{c}\right]$

$= \bar{a}\cdot(-\bar{a} \times \bar{c} + \bar{b} \times \bar{c}) + 2\bar{b}\cdot(-\bar{a} \times \bar{c} + \bar{b} \times \bar{c}) - \bar{c}\cdot(-\bar{a} \times \bar{c} + \bar{b} \times \bar{c})$

As $\bar{a}\cdot(\bar{b} \times \bar{c}) = [\bar{a}\ \ \bar{b}\ \ \bar{c}]$

$= -[\bar{a}\ \ \bar{a}\ \ \bar{c}] + [\bar{a}\ \ \bar{b}\ \ \bar{c}] - 2[\bar{b}\ \ \bar{a}\ \ \bar{c}] + 2[\bar{b}\ \ \bar{b}\ \ \bar{c}] + [\bar{a}\ \ \bar{c}\ \ \bar{a}] - [\bar{c}\ \ \bar{b}\ \ \bar{c}]$

As $[\bar{a}\ \ \bar{a}\ \ \bar{c}] = 0$

$= -0 + \left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right] - 2\left[\bar{b} \ \ \bar{a} \ \ \bar{c}\right] + 2(0) + 0 - 0$

$= \left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right] - 2\left[\bar{b} \ \ \bar{a} \ \ \bar{c}\right]$

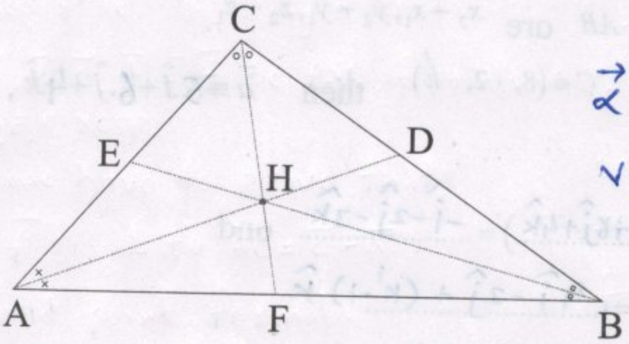
As $[\bar{a} \ \ \bar{a} \ \ \bar{c}] = 0$

$[\bar{b} \ \bar{a} \ \bar{c}] = -\left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right]$

$= \left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right] + 2\left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right]$

$= 3\left[\bar{a} \ \ \bar{b} \ \ \bar{c}\right]$

Ex. (5) Using vectors prove that bisectors of angles of a triangle are concurrent.



Let ABC be a triangle and $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be the p.v.s of the vertices A, B and C resp.

Solution :

Let AD, BE and CF be the internal bisectors of $\angle A, \angle B$ and $\angle C$ resp.

We know that D divides BC in the ratio of AB : AC i.e. c : b

p.v of D is $\dfrac{c\vec{\gamma} + b\vec{\beta}}{c+a}$, p.v. of E is $\dfrac{c\vec{\gamma} + a\vec{\alpha}}{c+a}$

p.v of F is $\dfrac{a\vec{\alpha} + b\vec{\beta}}{a+b}$

The point dividing AD in ratio b+c : a is $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$

— " — BE — " — a+c : b $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$

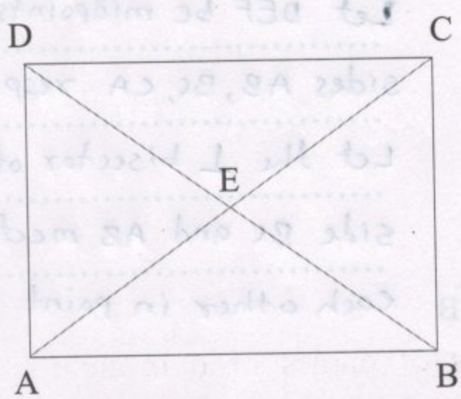
— " — CF — " — a+b : c is $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$

Since the point $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$ lies on all the three internal bisectors AD, BE and CF.

Hence the internal bisectors are concurrent.

Solution Study

Ex. (6) Using vectors prove that a quadrilateral is a rectangle if and only if its diagonals are congruent and bisect each other.



Let ABCD be a rectangle

Let $\bar{a}\ \bar{b}\ \bar{c}\ \bar{d}\ \bar{e}$ be P.V. of points A,B,C,D,E

Since ABCD is a rectangle

$\overline{AB} = \overline{DC}$ (opp. sides of rectangle)

$\bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{b} + \bar{d} = \bar{c} + \bar{a}$

Solution :

$\therefore \dfrac{\bar{b}+\bar{d}}{2} = \dfrac{\bar{c}+\bar{a}}{2} = \bar{e}$ (say)

is mid point of BD and AC

Diagonals BD bisect AC at $E(\bar{e})$ —— (I)

Now

$\overline{AC} = \overline{AB} + \overline{BC} = \overline{BC} + \overline{AB}$

$\overline{BD} = \overline{BC} + \overline{CD} = \overline{BC} + \overline{BA}$

$\overline{BD} = \overline{BC} - \overline{AB}$ $\because \overline{CD} = \overline{BA}$

$|\overline{AC}|^2 = \overline{AC} \cdot \overline{AC}$

$= (\overline{BC} + \overline{AB}) \cdot (\overline{BC} + \overline{AB})$

$= \overline{BC} \cdot \overline{BC} + \overline{BC} \cdot \overline{AB} + \overline{AB} \cdot \overline{BC} + \overline{AB} \cdot \overline{AB}$

$= |\overline{BC}|^2 + 0 + 0 + |\overline{AB}|^2$

$(\because \overline{AB} \perp \overline{BC})$

$\therefore |\overline{AC}|^2 = |\overline{BC}|^2 + |\overline{AB}|^2$

$|\overline{BD}|^2 = \overline{BD} \cdot \overline{BD}$

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$\therefore |\overline{BD}|^2 = |\overline{BC}|^2 + |\overline{AB}|^2$

$\therefore |\overline{AC}|^2 = |\overline{BD}|^2$

$\therefore AC = BD$ —— (II)

from I & II the diagonals of a rectangle are congu. and bisect each other.

Conversely :-

Let diagonals AC and BD of □ABCD are congu. and bisect each other at right angle (w.l.o.g)

$\therefore$ □ABCD is ∥gm

Now $AC \perp BD$

$\therefore \overline{AC} \cdot \overline{BD} = 0$

$\therefore (\overline{BC} + \overline{AB}) \cdot (\overline{BC} - \overline{AB}) = 0$

$\therefore \overline{BC} \cdot \overline{BC} - \overline{BC} \cdot \overline{AB} + \overline{AB} \cdot \overline{BC} - \overline{AB} \cdot \overline{AB} = 0$

$\therefore |\overline{BC}|^2 - |\overline{AB}|^2 = 0$

$\therefore |\overline{BC}|^2 = |\overline{AB}|^2$

$\therefore BC = AB$

i.e. adjacent sides AB & BC of ∥gm ABCD are equal.

$\therefore$ ABCD is a rhombus

$AC = BD$ (given)

$\therefore |\overline{AC}|^2 = |\overline{BD}|^2$

$\therefore \overline{AC} \cdot \overline{AC} = \overline{BD} \cdot \overline{BD}$

$\therefore (\overline{BC} + \overline{AB}) \cdot (\overline{BC} + \overline{AB}) = (\overline{BC} - \overline{AB}) \cdot (\overline{BC} - \overline{AB})$

After simplifying we get

$2(\overline{BC} \cdot \overline{AB}) = -2(\overline{AC} \cdot \overline{AB})$

$\therefore 4(\overline{BC} \cdot \overline{AB}) = 0$

$\therefore \overline{BC} \cdot \overline{AB} = 0$

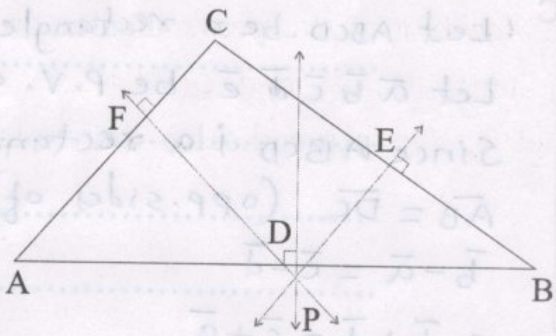
$\therefore BC \perp AB$

the adjacent sides of a rhombus ABCD are $\perp$ to each other

Hence ABCD is a Square

$\therefore$ □ABCD is a rectangle

Ex. (7) Using vectors prove that the perpendicular bisectors of the sides of a triangle are concurrent.



Let DEF be midpoints of sides AB, BC, CA resp

Let the $\perp$ bisector of side BC and AB meet each other in point P

Solution :

Choose P as the origin and

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be P.V. of points A, B, C, D, E, F resp.

Here we have to prove that

$\overline{PF} = \bar{f}$ is $\perp$ to $\overline{AC} = \bar{c} - \bar{a}$

by mid point formula

$\bar{d} = \dfrac{\bar{a}+\bar{b}}{2}, \bar{e} = \dfrac{\bar{b}+\bar{c}}{2}, \bar{f} = \dfrac{\bar{a}+\bar{c}}{2}$

Now $\overline{PD} = \bar{d} \perp \overline{AB} = \bar{b} - \bar{a}$

$\therefore \bar{d} \cdot (\bar{b} - \bar{a}) = 0$

$\therefore \dfrac{(\bar{a}+\bar{b})}{2} \cdot (\bar{b} - \bar{a}) = 0$

$\therefore (\bar{b}+\bar{a}) \cdot (\bar{b} - \bar{a}) = 0$

$\bar{b}\cdot\bar{b} - \bar{b}\cdot\bar{a} + \bar{a}\cdot\bar{b} - \bar{a}\cdot\bar{a} = 0$

$\therefore |\bar{b}|^2 - |\bar{a}|^2 = 0$

$\therefore b^2 = a^2$ —— (1)

Also $\overline{PE} = \bar{e} \perp \overline{BC} = \bar{c} - \bar{d}$

$\therefore \bar{e} \cdot (\bar{c} - \bar{b}) = 0$

$\therefore \left(\dfrac{\bar{b}+\bar{c}}{2}\right) \cdot (\bar{c} - \bar{b}) = 0$

$\therefore (\bar{c}+\bar{b})(\bar{c} - \bar{b}) = 0$

$\therefore \bar{c}\cdot\bar{c} - \bar{c}\cdot\bar{b} + \bar{b}\cdot\bar{c} - \bar{b}\cdot\bar{b} = 0$

$\therefore |\bar{c}|^2 - |\bar{b}|^2 = 0$

$\therefore c^2 - b^2 = 0$

$\therefore b^2 = c^2$ —— (2)

from (1) and (2)

$a^2 = c^2$

$\therefore a^2 - c^2 = 0$

$\therefore |\bar{a}|^2 - |\bar{c}|^2 = 0$

$\therefore \bar{a}\cdot\bar{a} - \bar{c}\cdot\bar{c} = 0$

$\therefore \bar{a}\cdot\bar{a} - \bar{a}\cdot\bar{c} + \bar{a}\cdot\bar{c} - \bar{c}\cdot\bar{c} = 0$

$\therefore \bar{a}(\bar{a} - \bar{c}) + \bar{c}\cdot(\bar{a} - \bar{c}) = 0$

$\therefore (\bar{a}+\bar{c})\cdot(\bar{a} - \bar{c}) = 0$

$\therefore \left(\dfrac{\bar{a}+\bar{c}}{2}\right)\cdot(\bar{a} - \bar{c}) = 0$

$\bar{f} \cdot (\bar{a} - \bar{c}) = 0$

$\therefore \overline{PF} \cdot \overline{CA} = 0$

$\therefore \overline{PF} \perp \overline{CA}$

$\therefore$ the $\perp$ bisectors of sides of ΔABC are concurrent.

Sign of Teacher :

Q. 26. A solenoid of length π m and 5 cm in diameter has a winding of 1000 turns and carries a current of 5A. Calculate the magnetic field at its centre along the axis.

SECTION – D

Attempt any THREE questions of the following : **[12]**

Q. 27. What is Ferromagnetism? Explain it on the basis of domain theory.

Q. 28. Obtain an expression for average power dissipated in a series LCR circuit.

Q. 29. Distinguish between interference and diffraction of light.

A double slit arrangement produces interference fringes for sodium light of wavelength 589 nm, that are 0.20 degree apart. What is the angular fringe separation if the entire arrangement is immersed in water?

(R.I. of water = 1.33)

Q. 30. State Einstein's photoelectric equation and mention physical significance of each term involved in it.

The wavelength of incident light is 4000Å. Calculate the energy of incident photon.

Q. 31. State any four uses of Van de Graaff generator.

In a parallel plate air capacitor, intensity of electric field is changing at the rate of 2×10^{11} V/ms. If area of each plate is 20 cm^2, calculate the displacement current.

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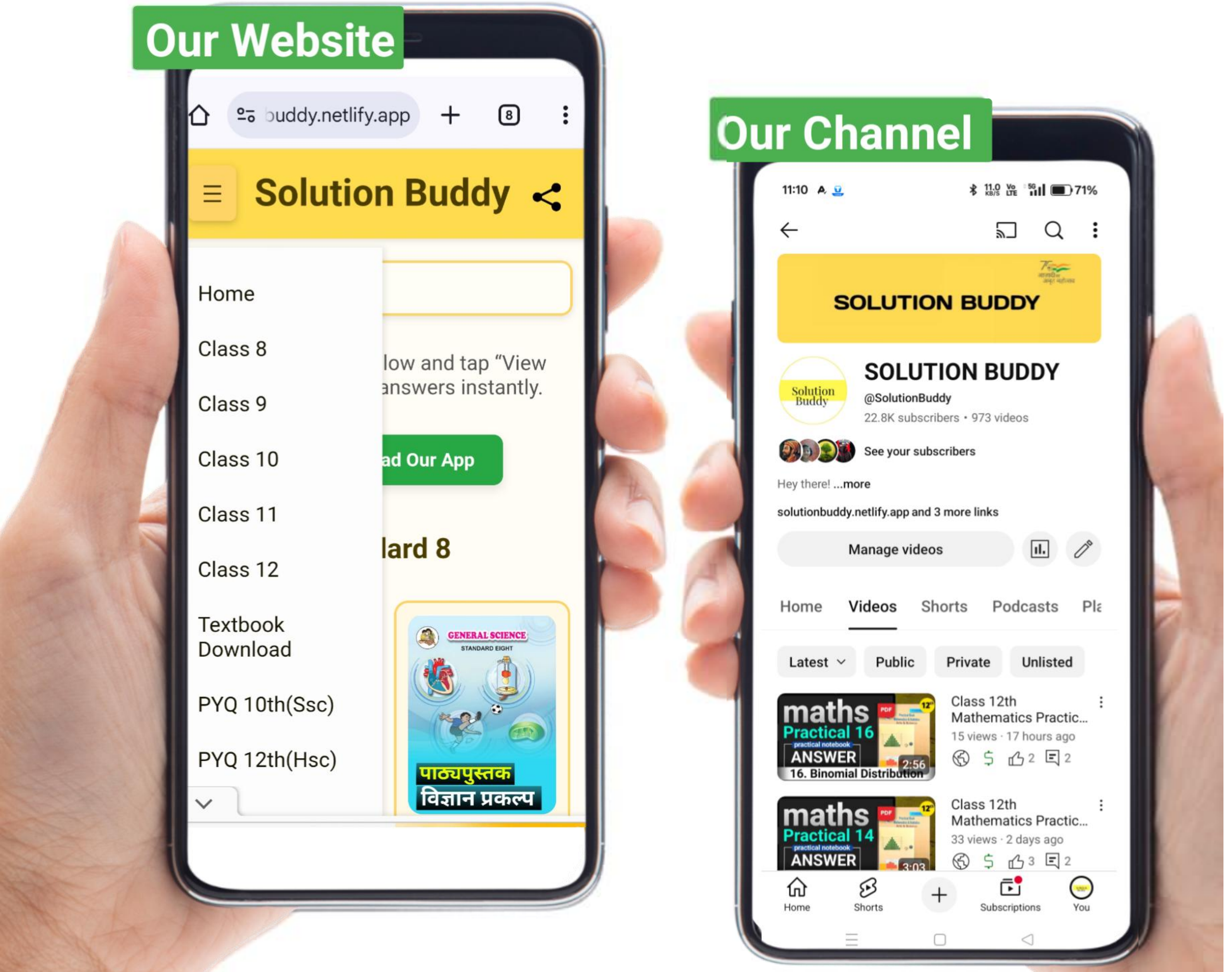

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