7 Conic sections

Let's Learn

- Conic sections : parabola, ellipse, hyperbola
- Standard equation of conics
- Equation of tangent to the conics
- Condition for tangency

Let's Recall

- Section formulae : Let A (x_1,y_1) and B(x_2,y_2) be two points in a plane. If P and Q divide seg AB in the ratio m:n internally and externally respectively then

$$P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) \text{ and}$$

$$Q = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

Introduction:

The Greek mathematician Archimedes and Apollonius studied the curves named conic sections. These curves are intersections of a plane with right circular cone. Conic sections have a wide range of applications such as planetary motions, in designs of telescopes and antennas, reflection in flash light, automobile headlights, construction of bridges, navigation, projectiles etc.

A straight line, a circle, parabola, ellipse and hyperbola are all conic sections. Of these we have studied circle and straight line.

Earlier we have studied different forms of equations of line, circle, and their properties. In this chapter we shall study some more curves, namely parabola, ellipse and hyperbola which are conic sections.

Let's Learn

7.1.1 Double cone:

Let l be a fixed line and m another line intersecting it at a fixed point V and inclined at an acute angle θ (fig 7.1.) Suppose we rotate the line m around the line l in such a way that the angle θ remains

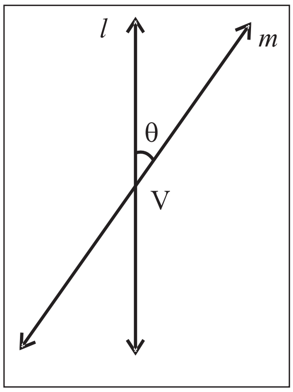


Fig.7.1

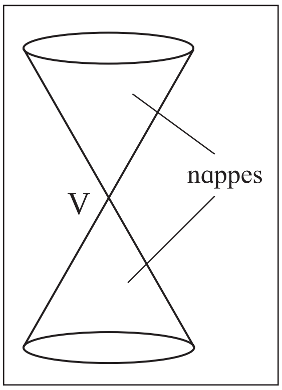


Fig.7.2

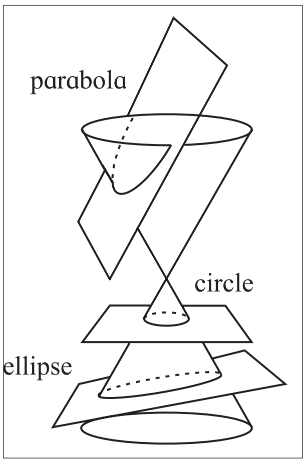


Fig.7.3(a)

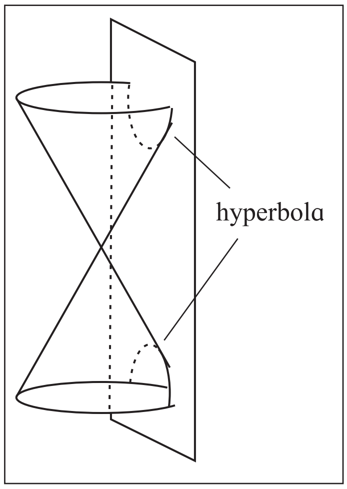


Fig.7.3(b)

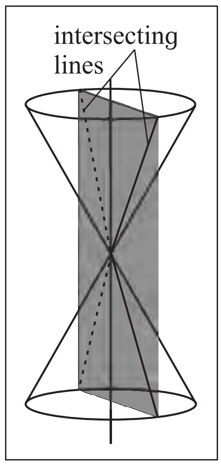


Fig.7.4

constant. Then the surface generated is a double-napped right circular cone.

The point V is called the vertex, the line *l* is called axis, and the rotating line m is called a generator of the cone. The vertex V separates the cone in to two parts called nappes (Fig.7.2).

7.1.2. Conic sections :

Let's construct

Take a carrot or a cone made of drawing paper and cut it with a plane satisfying the following conditions.

i) The plane is perpendicular to the axis and does not contain vertex, the intersection is a circle (studied earlier Fig. 7.3(a)).

ii) The plane is parallel to one position of the generator but does not pass through the vertex, we get a parabola (Fig. 7.3(a)).

iii) The plane is oblique to the axis and not parallel to the generator we get an ellipse (Fig. 7.3(a)).

iv) If a double cone is cut by a plane parallel to axis, we get parts of the curve at two ends called hyperbola (Fig. 7.3(b)).

v) A plane containing a generator and tangent to the cone, intersects the cone in that generator. We get pair to straight lines (Fig. 7.4).

7.1.3. Definition of a conic section and its equation:

A conic section or conic can be defined as the locus of the point P in a plane such that the ratio of the distance of P from a fixed point to its distance from a fixed line is constant.

The fixed point is called the focus of the conic section, denoted by S. The fixed straight line is called the directrix of conic section, denoted by *d*.

If S is the focus, P is any point on the conic section and segment PM is the length of perpendicular from P on the directrix, then by definition $\dfrac{SP}{PM}$ = constant. (fig. 7.5)

This constant ratio is called the eccentricity of the conic section, denoted by e.

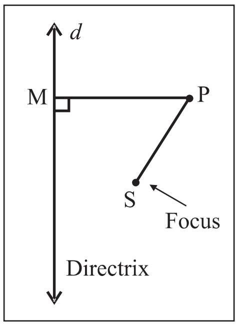


Fig. 7.5

Hence we write $\dfrac{SP}{PM} = e$

or SP = e PM. This is called Focus - Directrix property of the conic section. The nature of the conic section depends upon the value of e.

i) If e = 1, the conic section is called parabola.

ii) If $0 < e < 1$, the conic section is called an ellipse.

iii) If $e > 1$, the conic section is called hyperbola.

7.1.4. Some useful terms of conic sections:

1) **Axis:** A line about which a conic section is symmetric is called an axis of the conic section.

2) **Vertex :** The point of intersection of a conic section with its axis of symmetry is called a vertex.

3) **Focal Distance :** The distance of a point on a conic section from the focus is called the focal distance of the point.

4) **Focal chord :** A chord of a conic section passing through its focus is called a focal chord.

5) **Latus-Rectum:** A focal chord of a conic section which is perpendicular to the axis of symmetry is called the latus-rectum.

6) **Centre of a conic :** The point which bisects every chord of the conic passing through it, is called the centre of the conic.

7) **Double ordinate :** A chord passing through any point on the conic and perpendicular to the axis is called double ordinate.

7.1.5. Parabola

Definition: A parabola is the locus of the point in plane equidistant from a fixed point and a fixed line in that plane. The fixed point is called the focus and the fixed straight line is called the directrix.

Standard equation of the parabola:

Equation of the parabola in the standard form $y^2 = 4ax$.

Let S be the focus and d be the directrix of the parabola.

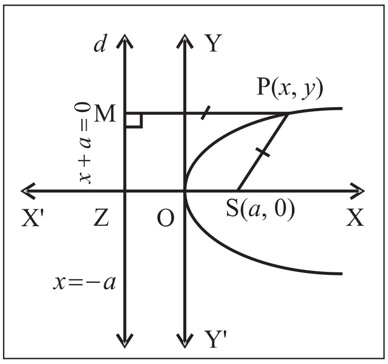


Fig. 7.6

Let SZ be perpendicular to the directrix. Bisect SZ at the point O. By the definition of parabola the midpoint O is on the parabola. Take O as the origin, line OS as the X - axis and the line through O perpendicular to OS as the Y - axis.

Let SZ = 2 a, $a > 0$.

Then the coordinates of the focus S are (a, 0) and the coordinates of Z are (–a, 0).

The equation of the directrix d is

$x = -a$, i.e. $x + a = 0$

Let P (x, y) be any point on the parabola. Draw segment PM perpendicular to the directrix d.

$\therefore$ M$= (-a, y)$

By using distance formula we have

$$SP = \sqrt{(x-a)^2+(y-0)^2},$$

$$PM = \sqrt{(x+a)^2+(y-y)^2}$$

By focus – directrix property of the parabola SP = PM

$$\sqrt{(x-a)^2+(y-0)^2} = \sqrt{(x+a)^2+(y-y)^2}$$

Squaring both sides $(x-a)^2 + y^2 = (x+a)^2$

that is $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

that is $y^2 = 4ax \ (a > 0)$

This is the equation of parabola in standard form.

Activity :

Trace the parabola using focus directrix property.

1) find the equation of parabola with focus at (2, 0) and directrix $x + 2 = 0$.

2) Find the equation of parabola with focus at (– 4 , 0) and directrix $x = 4$.

7.1.6. Tracing of the parabola $y^2 = 4ax$ ($a>0$)

1) **Symmetry :** Equation of the parabola can be written as $y = \pm 2\sqrt{ax}$ that is for every value of x, there are two values of y which are negatives of each other. Hence parabola is symmetric about X- axis.

2) **Region :** For every $x < 0$,the value of y is imaginary therefore entire part of the curve lies to the right of Y-axis.

3) **Intersection with the axes:** For $x = 0$ we have $y = 0$, therefore the curve meets the co ordinate axes at the origin O(0, 0)

4) **Shape of parabola:** As $x \to \infty$, $y \to \infty$. Therefore the curve extends to infinity as x grows large and opens in the right half plane Shape of the parabola $y^2 = 4$ ax (a > 0) is as shown in figure 7.6.

7.1.7 Some results

1) **Focal distance :** Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4\,ax$

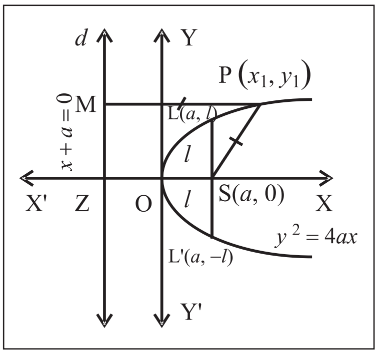


Fig.7.7

Let segment PM is perpendicular to the directrix d, then M is $(-a, y_1)$

$$SP = PM = \sqrt{(x_1 + a)^2 + (y_1 - y_1)^2} = x_1 + a$$

$\therefore$ focal distance $SP = x_1 + a$

$= a +$ abscissa of point P.

2) **Length of latus-Rectum:**

In figure 7.7 LSL' is the latus-rectum of the parabola $y^2 = 4\,ax$. By symmetry of the curve

$LS = L'S = l$ (say).

So the coordinates of L are (a, l)

Since L lies on $y^2 = 4\,ax$, $l^2 = 4a\,(a)$

$$l^2 = 4a^2$$

$$l = \pm 2a$$

As L is in the first quadrant, $\quad l > 0$

$$l = 2a$$

Length of latus rectum $LSL' = 2l = 2(2a) = 4a$

The co-ordinates of ends points of the latus rectum are L $(a, 2a)$ and L' $(a, -2a)$

Activity :

1) Find the length and end points of latus rectum of the parabola $x^2 = 8y$

2) Find the length and end points of latus rectum of the parabola $5y^2 = 16x$

7.1.8 Some other standard forms of parabola

$y^2 = -4ax$ $\qquad$ $x^2 = 4by$ $\qquad$ $x^2 = -4by$

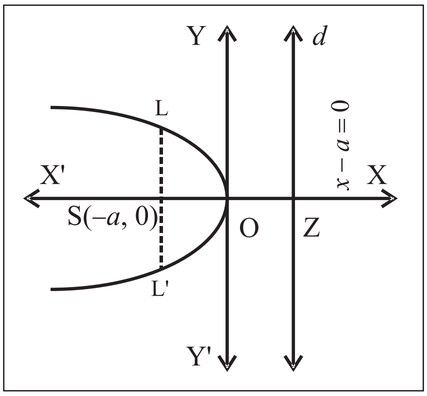


Fig.7.8

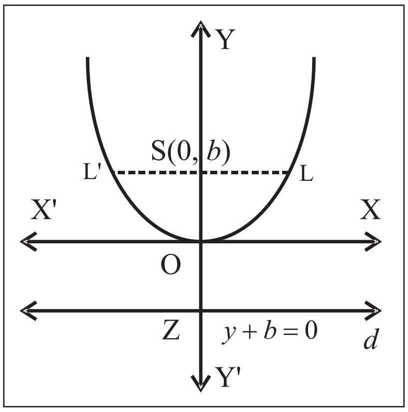


Fig.7.9

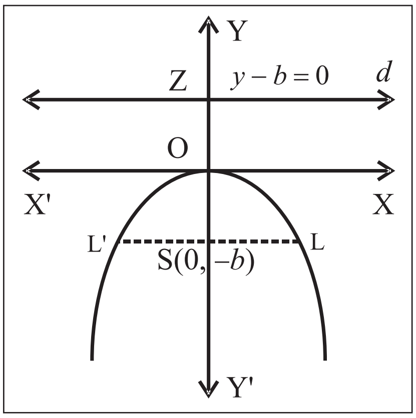


Fig.7.10

We summarize the properties of parabola in four standard forms

	Equation of the parabola / Terms	$y^2 = 4ax$	$x^2 = 4by$
1	Focus	(a, 0)	(0, b)
2	Equation of directrix	$x + a = 0$	$y + b = 0$
3	Vertex	0(0,0)	0(0,0)
4	End points of latus rectum	$(a, \pm 2a)$	$(\pm 2b, b)$
5	Length of latus rectum	$\|4a\|$	$\|4b\|$
6	Axis of symmetry	X-axis	Y-axis
7	Equation of axis	$y = 0$	$x = 0$
8	Tangent at vertex	Y-axis	X-axis
9	Focal distance of a point $P(x_1, y_1)$	$\|x_1 + a\|$	$\|y_1 + b\|$

Parameter : If the co-ordinates of a point on the curve are expressed as functions of a variable, that variable is called the parameter for the curve.

7.1.9 Parametric expressions of standard parabola $y^2 = 4ax$

$x = at^2$, $y = 2at$ are the expressions which satisfies given equation $y^2 = 4ax$ for any real value of t that is $y^2 = (2at)^2 = 4\ a^2\ t^2 = 4a\ (at^2) = 4ax$ where t is a parameter

$P(x_1, y) \equiv (at^2, 2at)$ describes the parabola $y^2 = 4ax$, where t is the parameter.

Activity :

1) For the parabola $y^2 = 12x$, find the parameter for the point a) (3, –6) b) (27,18)

2) Find the parameter for the point (9, –12) of the parabola $y^2 = 16x$

7.1.10 General forms of the equation of a parabola

If the vertex is shifted to the point (h, k) we get the following form

1) $(y - k)^2 = 4a\ (x - h)$

This represents a parabola whose axis of symmetry is $y - k = 0$ which is parallel to the X-axis, vertex is at (h, k) and focus is at $(h + a , k)$ and directrix is $x = h - a$.

It can be reduced to the form $x = Ay^2 + By + C$.

OR

$Y^2 = 4a\ X$, where $X = x - h$, $Y = y - k$

Activity :

1) Obtain the equation of the parabola with its axis parallel to Y– axis and vertex at (h, k)

2) Find the coordinates of the vertex, focus and equation of the directrix of the parabola $y^2 = 4x + 4y$

SOLVED EXAMPLES

Ex. 1) Find the coordinates of the focus, equation of the directrix, length of latus rectum and coordinates of end points of latus rectum of each of the following parabolas.

i) $y^2 = 28x$ ii) $3x^2 = 8y$

Solution:

i) $y^2 = 28x$

Equation of the parabola is $y^2 = 28x$

comparing this equation with $y^2 = 4ax$, we get $4a = 28 \therefore a = 7$

Coordinates of the focus are $S(a,0) = (7,0)$

Equation of the directrix is $x + a = 0$ that is $x + 7 = 0$

Length of latus rectum $= 4a = 4 \times 7 = 28$

End points of latus rectum are $(a,2a)$ and $(a,-2a)$. that is (7,14) and (7, –14)

ii) $3x^2 = 8y$

Equation of the parabola is $3x^2 = 8y$ that is

$x^2 = \frac{8}{3}y$

comparing this equation with $x^2 = 4by$, we get $4b = \frac{8}{3} \therefore b = \frac{2}{3}$

Co-ordinates of the focus are S $(0, b)$ = $(0, \frac{2}{3})$

Equation of the directrix is $y + b = 0$ that is

$y + \frac{2}{3} = 0$ that is $3y + 2 = 0$

Length of latus rectum $= 4b = 4 \times \left(\frac{2}{3}\right) = \frac{8}{3}$

Coordinates of end points of latus rectum are

(2b, b) and (–2b, b). that is $(\frac{4}{3}, \frac{2}{3})$ and $(-\frac{4}{3}, \frac{2}{3})$

Ex. 2) Find the equation of the parabola with vertex at the origin, axis along Y-axis and passing through the point (6,–3)

Solution:

The vertex of the parabola is at the origin, it's axis is along Y-axis. Hence equation of the parabola is of the form $x^2 = 4by$.

Now the point (6,–3) lies on this parabola. Hence the coordinates of the points satisfy the equation of the parabola.

$\therefore (6)^2 = 4\,b \times -3)$

$\therefore -12\,b = 36 \quad \therefore \; b = -3$

$\therefore$ equation of parabola is $x^2 = 4(-3)y$

$x^2 = -12y$ that is $x^2 + 12y = 0$.

Ex. 3) Find the equation of the parabola whose diretrix is $x + 3 = 0$

Solution:

Here equation of diretrixs is $x + a = 0$ that is $x + 3 = 0$ comparing we get $a = 3$.

$\therefore$ Equation of the parabola $y^2 = 4ax$ that is $y^2 = 12x$.

Ex. 4) Calculate the focal distance of point P on the parabola $y^2 = 20x$ whose ordinate is 10

Solution : Equation of parabola is $y^2 = 20x$

comparing this with $y^2 = 4ax$

we get $4a = 20 \therefore a = 5$

Here ordinate = y – coordinate = 10

$\therefore (10)^2 = 20\,x \qquad \therefore 20x = 100$

$\therefore x = \frac{100}{20} = 5$

Now focal distance $= a + x$

$= a$ + abscissa of point

$= 5 + 5 = 10$ units

Ex. 5) Find the equation of the parabola having (4,–8) as one of extremities of porabola.

Solution : Given that, one of the extrimities of the latus rectum of the parabola is (4, –8) therefore other must be (4,8). End-coordinates of latus - rectum $(a, \pm 2a)$ = $(4, \pm 8)$.

$\therefore a = 4$

Equation of parabola is $y^2 = 4ax$

$y^2 = 4(4)x \;\; \therefore \; y^2 = 16x$

Ex. 6) For the parabola $3y^2 = 16x$, find the parameter of the point $(3,-4)$

Solution : Equation of parabola is $3y^2 = 16x$

$\therefore y^2 = \frac{16}{3}x$ comparing this with $y^2 = 4ax$ we get

$4a = \frac{16}{3}$ $\therefore a = \frac{4}{3}$ Parametric equations of the parabola $y^2 = 4ax$ are $(at^2, 2at) = \left(\frac{4}{3}t^2, \frac{8}{3}t\right)$

$$\left(\frac{4}{3}t^2, \frac{8}{3}t\right) = (3,-4)$$

Equating second components we get $\frac{8}{3}t = -4$

$$\therefore t = -4 \times \frac{3}{8} = -\frac{3}{2}$$

$$\therefore \text{Parameter } t = -\frac{3}{2}$$

Ex. 7) Find the coordinates of the vertex and focus, the equation of the axis of symmetry, diretrix and tangent at the vertex of the parabola $x^2 + 4x + 4y + 16 = 0$

Solution : Equation of parabola is

$x^2 + 4x + 4y + 16 = 0$

$x^2 + 4x = -4y - 16$

$x^2 + 4x + 4 = -4y - 12$

$(x+2)^2 = -4(y+3)$

Comparing this equation with $X^2 = -4bY$

We get $X = x + 2$, $Y = y + 3$ and $4b = -4$

$\therefore b = -1$

Coordinates of the vertex are $X = 0$ and $Y = 0$ that is $x + 2 = 0$ and $y + 3 = 0$

$\therefore x = -2$ and $y = -3$

$\therefore$ Vertex $= (x, y) = (-2,-3)$

Coordinates of focus are given by $X = 0$ and $Y = +b$

that is $x + 2 = 0$ and $y + 3 = -1$

$\therefore x = -2$ and $y = -4$

$\therefore$ Focus $= (-2, -4)$

Equation of axis is $X = 0$ that is $x + 2 = 0$

Equation of diretrix is $Y + b = 0$ that is $y + 3 - 1 = 0$ that is $y + 2 = 0$

Equation of tangent at vertex is $Y = 0$ that is $y + 3 = 0$

7.1.11 Tangent :

A straight line which intersects the parabola in coinsident point is called a tangent of the parabola

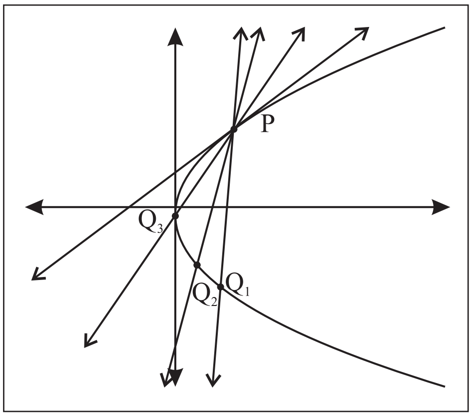


Fig. 7.11

Point Q moves along the curve to the point P. The limiting position of secant PQ is the tangent at P.

A tangent to the curve is the limiting position of a secant intersecting the curve in two points and moving so that those points of intersection come closer and finally coincide.

Tangent at a point on a parabola.

Let us find the equation of tangent to the parabola at a point on it in cartesion form and in parametrics form.

We find the equation of tangent to the parabola $y^2 = 4ax$ at the point $P(x_1, y_1)$ on it. Hence, obtain the equation of tangent at P(t).

Equation of the tangent to the curve $y = f(x)$ at point (x_1, y_1) on it is,

$$y - y_1 = [f'(x)]_{(x_1, y_1)} (x - x_1) \; [f'(x)]_{(x_1, y_1)}$$

We need to know the slope of the tangent at $P(x_1, y_1)$. From the theory of derivative of a function, the slope of the tangent is $\frac{dy}{dx}$ at (x_1, y_1)

and here $\left.\frac{dy}{dx}\right|_{(x1,\,y1)} = \frac{2a}{y_1}$

The equation of parabola is $y^2 = 4ax$, differentiate both sides with respect to x

We get $2y\frac{dy}{dx} = 4a$ (1)

$\therefore \frac{dy}{dx} = \frac{2a}{y}$

$\therefore \left.\frac{dy}{dx}\right|_{(x1,\,y1)} = \frac{2a}{y_1}$ = slope of the tangent at $P(x_1, y_1)$

$\therefore$ Equation of the tangent at $P(x_1, y_1)$ is

$y - y_1 = \frac{2a}{y_1}(x - x_1)$

$yy_1 - y_1^2 = 2a(x - x_1)$

$yy_1 - y_1^2 = 2ax - 2ax_1$

Now $P(x_1, y_1)$ lies on the parabola $\therefore y_1^2 = 4ax_1$

$\therefore yy_1 - 4ax_1 = 2ax - 2ax_1$

$\therefore yy_1 = 2ax + 2ax_1$

$\therefore yy_1 = 2a(x + x_1)$ ……………………(I)

This is the equation of the tangent at $P(x_1, y_1)$ on it

Now, t_1 is the parameter of point P

$\therefore P(x_1, y_1) = (at_1^2, 2at_1)$ that is $x_1 = at_1^2$, $y_1 = 2at_1$

Substituting these values in equation (1), we get $y(2at_1) = 2a(x + at_1^2)$

that is $y\,t_1 = x + a\,t_1^2$

This is the required equation of the tangent at P(t).

> Thus, the equation of tangent to the parabola $y^2 = 4ax$ at point (x, y) on it is $yy_1 = 2a(x + x_1)$ or $yt_1 = x + at_1^2$ where t_1 is the parameter.

7.1.12 Condition of tangency

To find the condition that the line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$. Also to find the point of contact.

Equation of the line is $y = mx + c$

$\therefore mx - y + c = 0$ ……………………(I)

equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

$\therefore 2ax - y_1 y + 2ax_1 = 0$ ……………..(II)

If the line given by equation (I) is a tangent to the parabola at (x_1, y_1). Equation (I) and equation (II) represents the same line.

Comparing the co-efficients of like terms in equations (I) and (II)

we get $\frac{2a}{m} = \frac{-y_1}{-1} = \frac{2ax_1}{c}$

$\therefore x_1 = \frac{c}{m}$ and $y_1 = \frac{2a}{m}$

But the point $P(x_1, y_1)$ lies on the parabola

$\therefore y_1^2 = 4ax_1$

$\therefore \left(\frac{2a}{m}\right)^2 = 4a\left(\frac{c}{m}\right)$

$\frac{4a^2}{m^2} = 4a\left(\frac{c}{m}\right)$

$\therefore c = \frac{a}{m}$

this is the required condition of tangency.

> Thus the line $y = mx + c$ is tangent the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the point of contact is $\left(\frac{c}{m}, \frac{2a}{m}\right)$ i.e. $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

> The equation of tangent in terms of slope is $y = mx + \frac{a}{m}$

7.1.13 Tangents from a point to a parabola

In general, two tangents can be drawn to a parabola $y^2 = 4ax$ from any point in its plane.

Let P (x_1, y_1) be any point in the plane of parabola. Equation of tangent to the parabola $y^2 = 4ax$ is

$y = mx + \frac{a}{m}$

Since the tangent passes through P (x_1, y_1), we have $y_1 = mx_1 + \frac{a}{m}$

$\therefore my_1 = m^2x_1 + a$

$m^2x_1 - my_1 + a = 0$ …………………………..(1)

$x_1m^2 - y_1 m + a = 0$

This is quadratic equation in m and in general it has two roots say, m_1 and m_2 which are the slopes of two tangents.

Thus, in general, two tangents can been drawn to a parabola from a given point in its plane.

If the tangent drawn from P are mutually perpendicular we have

$$m_1 m_2 = -1$$

From equation (1) $m_1 m_2 = \frac{a}{x_1}$ (product of roots)

$$\therefore \frac{a}{x_1} = -1$$

$$\therefore x_1 = -a$$

which is the equation of directrix.

Thus, the locus of the point, the tangents from which to the parabola are perpendicular to each other is the directrix of the parabola.

SOLVED EXAMPLES

Ex. 1) Find the equation of tangent to the parabola $y^2 = 9x$ at (1,–3).

Solution :

Equation of the parabola is $y^2 = 9x$;

comparing it with $y^2 = 4ax$

$$4a = 9 \quad \Rightarrow \quad a = \frac{9}{4}$$

Tangent is drawn to the parabola at $(1,-3) = (x_1,y_1)$

Equation of tangent to the parabola $y^2 = 4ax$ at (x_1,y_1) is $yy_1 = 2a(x + x_1)$

$\therefore$ Equation of tangent to the parabola $y^2 = 4x$ at (1,–3) is $y(-3) = 2\left(\frac{9}{4}\right)(x + 1)$

i.e. $-3y = \left(\frac{9}{2}\right)(x + 1)$

i.e. $-6y = 9x + 9$

i.e. $3x + 2y + 3 = 0$

Ex. 2) Find the equation to tangent to the parabola $y^2 = 12x$ from the point (2,5).

Solution :

Equation of the parabola is $y^2 = 12x$

comparing it with $y^2 = 4ax \Rightarrow 4a = 12$

$\therefore a = 3$

Tangents are drawn to the parabola from the point (2,5).

We know, equation of tangent to the parabola $y^2 = 4ax$ having slopes m is $y = mx + \frac{a}{m}$.

$$(5) = m(2) + \frac{(3)}{m}$$

$$5m = 2m^2 + 3$$

$$2m^2 - 5m + 3 = 0$$

$$2m^2 - 2m - 3m + 3 = 0$$

$$(2m - 3)(m - 1) = 0$$

$$m = \frac{3}{2} \quad \text{or} \quad m = 1$$

These are the slopes of tangents.

Therefore the equations of tangents by slope - point form are

$$(y - 5) = \frac{3}{2}(x - 2) \quad \text{and} \quad (y - 5) = 1(x - 2)$$

$$\therefore 2y - 10 = 3x - 6 \text{ and } y - 5 = x - 2$$

$$\therefore 3x - 2y + 4 = 0 \text{ and } x - y + 3 = 0$$

Ex. 3) Show that the tangents drawn from the point (–4,–9) to the parabola $y^2 = 16x$ are perpendicular to each other.

Solution :

Equation of the parabola is $y^2 = 16x$.

comparing it with $y^2 = 4ax \Rightarrow 4a = 16$

$\therefore a = 4$

Tangents are drawn to the parabola from point (–4,–9).

Equation of tangent to the parabola $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$

$$\therefore (-9) = m(-4) + \frac{4}{m}$$

$\therefore\ -9m = -4m^2 + 4$

$\therefore\ 4m^2 - 9m + 4 = 0$

m_1 and m_2 be the slopes (roots)

$$(m_1.m_2) = +\frac{\text{constant}}{\text{co-efficient of } m^2}$$

$$m_1.m_2 = -\frac{4}{4} \quad \therefore\ m_1.m_2 = -1$$

hence tangents are perpendicular to each other.

Activity :

1) Find the equation of tangent to the parabola $y^2 = 9x$ at the point (4,-6)

2) Find the equation of tangent to the parabola $y^2 = 24x$ having slope 3/2

3) Show that the line $y = x + 2$ touches the parabola $y^2 = 8x$. Find the coordinates of point of contact.

EXERCISE 7.1

1) Find co-ordinate of focus, equation of directrix, length of latus rectum and the co ordinate of end points of latus rectum of the parabola i) $5y^2 = 24x$ ii) $y^2 = -20x$ iii) $3x^2 = 8y$ iv) $x^2 = -8y$ v) $3y^2 = -16x$

2) Find the equation of the parabola with vertex at the origin, axis along Y-axis and passing through the point (–10,–5)

3) Find the equation of the parabola with vertex at the origin, axis along X-axis and passing through the point (3,4)

4) Find the equation of the parabola whose vertex is O (0,0) and focus at(–7,0).

5) Find the equation of the parabola with vertex at the origin, axis along X-axis and passing through the point i) (1,–6) ii) (2,3)

6) For the parabola $3y^2 = 16x$, find the parameter of the point a) (3,–4) b) (27,–12)

7) Find the focal distance of a point on the parabola $y^2 = 16x$ whose ordinate is 2 times the abscissa.

8) Find coordinate of the point on the parabola. Also find focal distance. i) $y^2 = 12x$ whose parameter is 1/3 ii) $2y^2 = 7x$ whose parameter is –2

9) For the parabola $y^2 = 4x$, find the coordinate of the point whose focal distance is 17.

10) Find length of latus rectum of the parabola $y^2 = 4ax$ passing through the point (2.–6).

11) Find the area of the triangle formed by the line joining the vertex of the parabola $x^2 = 12y$ to the end points of latus rectum.

12) If a parabolic reflector is 20cm in diameter and 5 cm deep, find its focus.

13) Find coordinate of focus, vertex and equation of directrix and the axis of the parabola $y = x^2 - 2x + 3$

14) Find the equation of tangent to the parabola
i) $y^2 = 12x$ from the point (2,5)
ii) $y^2 = 36x$ from the point (2,9)

15) If the tangent drawn from the point (–6,9) to the parabola $y^2 = kx$ are perpendicular to each other, find k.

16) Two tangents to the parabola $y^2 = 8x$ meet the tangents at the vertex in the point P and Q. If PQ = 4,prove that the equation of the locus of the point of intersection of two tangent is $y^2 = 8(x + 2)$.

17) Find the equation of common tangent to the parabola $y^2 = 4x$ and $x^2 = 32y$.

18) Find the equation of the locus of a point, the tangents from which to the parabola $y^2 = 18x$ are such that some of their slopes is -3

19) The tower of a bridge, hung in the form of a parabola have their tops 30 meters above the road way and are 200 meters apart. If the cable is 5meters above the road way at the centre of the bridge, find the length of the vertical supporting cable from the centre.

20) A circle whose centre is (4,–1) passes through the focus of the parabola $x^2 + 16y = 0$.

Show that the circle touches the diretrixs of the parabola.

7.2 Ellipse

Let's Study

- Standard equation of the ellipse.
- Equation of tangent to the ellipse.
- Condition for tangency.
- Auxilary circle and director circle of the ellipse

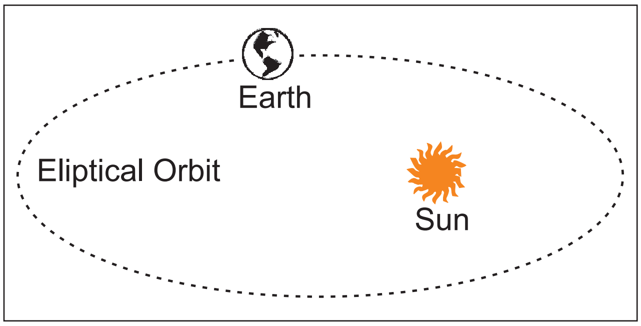


Fig. 7.12

The ellipse is the intersection of double napped cone with an oblique plane.

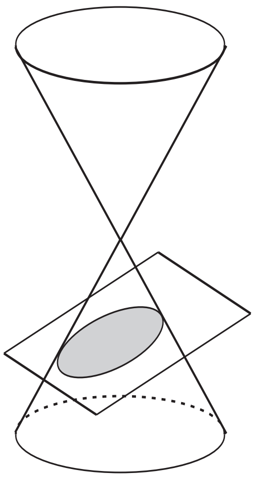


Fig. 7.14

> An ellipse is the locus of a point in a plane which moves so that its distance from a fixed point bears a constant ratio e ($0 < e < 1$) to its distance from a fixed line. The fixed point is called the focus S and the fixed line is called the directrix d.

If S is *a* fixed point is called focus and directrix d is *a* fixed line not containing the focus then by definition $\frac{PS}{PM}$ = e and PS = e PM , where PM is the perpendicular on the directrix and e is the real number with $0 < e < 1$ called eccentricity of the ellipse.

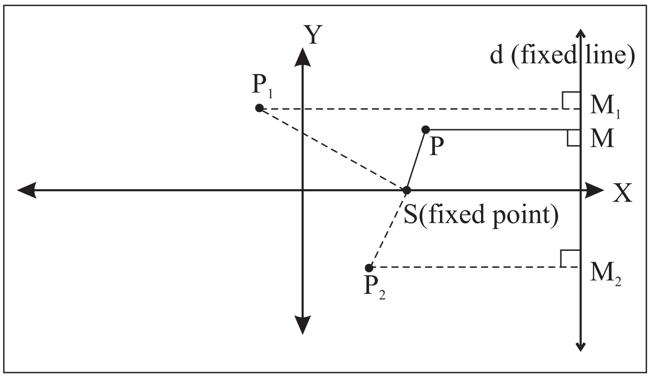
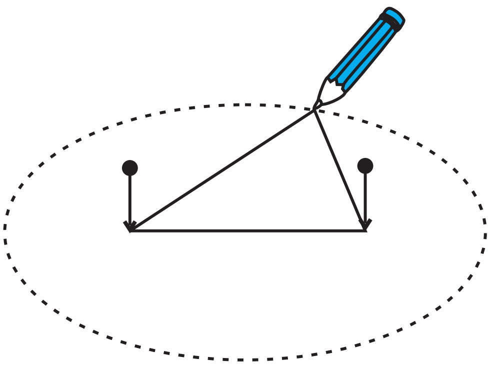


Fig. 7.13

7.2.1 Standard equation of ellipse

Let's derive the standard equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$$



How to draw an ellipse

Fig. 7.15

Let S be the focus, d be the directrix and e be the eccentricity of an ellipse.

Draw SZ perpendicular to directrix. let A and A' divide the segment SZ internally as well as externally in the ratio e : 1.

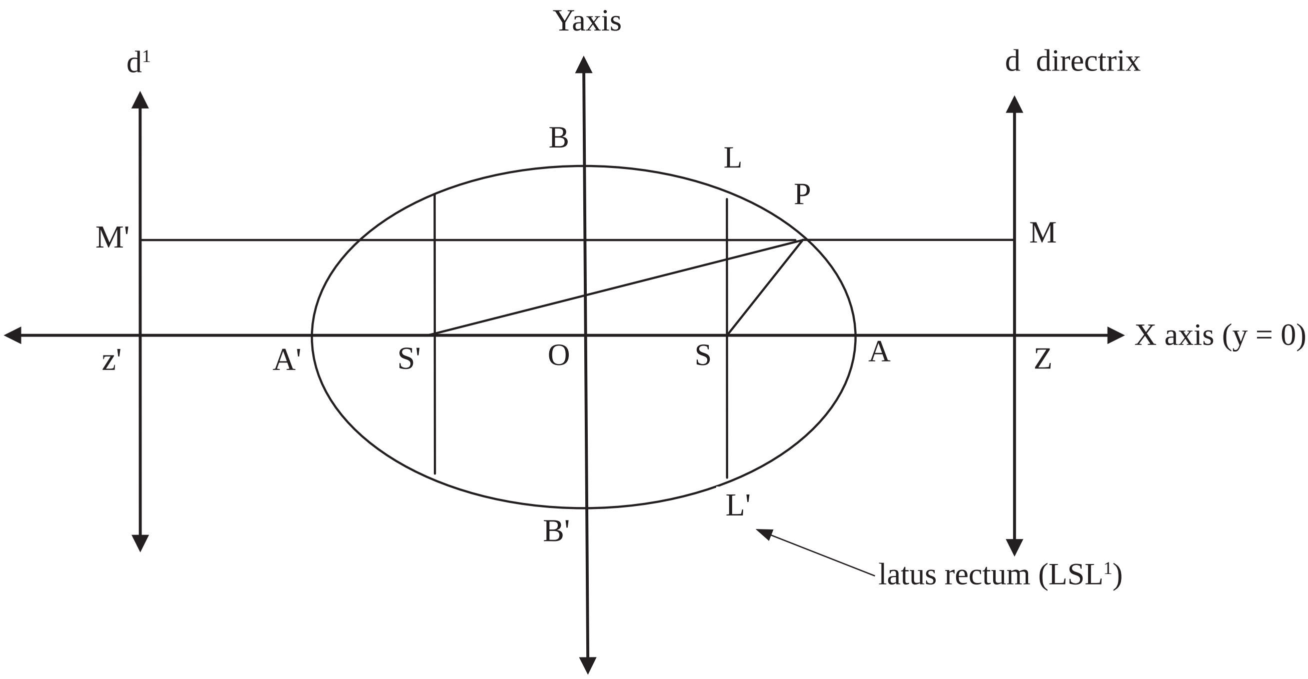


Fig. 7.16

let AA' = 2 a , midpoint O of segment AA' be the origin. Then O ≡ (0,0) , A ≡ (a, 0) and A' ≡ (– a ,0)

By definition of ellipse A and A' lie on ellipse.

$$\frac{SA}{AZ} = \frac{e}{1} \qquad \frac{SA'}{A'Z} = \frac{-e}{1}$$

Let P(x , y) be any point on the ellipse.

Since P is on the ellipse SP = e PM . . . (1)

therefore SA = e AZ .

Let Z ≡ (k,0) and S ≡ (h,0)

By section formula

$$a = \frac{ek+1h}{e+1} \qquad \text{also} -a = \frac{ek-1h}{e-1}$$

a e + a = e k + h . . . (2)

– a e + a = e k – h . . . (3)

Solving these equations , we get

k = a/e and h = a e

Focus S ≡ (ae , 0) and Z ≡ (a/e , 0)

Equation of the directrix is $x = \dfrac{a}{e}$

That is $x - \dfrac{a}{e} = 0$.

SP = focal distance

$$= \sqrt{(x-ae)^2 + (y-0)^2} \quad .....(4)$$

PM = distance of point P from directrix

$$= \left| \frac{x - \frac{a}{e}}{\sqrt{1^2 + 0^2}} \right| = \left| x - \frac{a}{e} \right| \quad ....(5)$$

From (1) , (4) and (5)

$$\sqrt{(x-ae)^2 + (y-0)^2} = e\left| x - \frac{a}{e} \right|$$

$$\sqrt{(x-ae)^2 + (y-0)^2} = |ex - a|$$

Squaring both sides

$(x - ae)^2 + (y - 0) = e^2x^2 - 2aex + a^2$

$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$

$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$

$(1 - e^2)\, x^2 + y^2 = a^2 (1 - e^2)$

Since $(1 - e^2) > 0$, Dividing both sides by $a^2 (1 - e^2)$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } b^2 = a^2 (1 - e^2) \text{ and } a > b$$

This is the standard equation of ellipse .

We also get for a point $P(x,y)$ on the locus PS' = e PM' . . . (6) where PM' is the perpendicular on directrix $x = -\frac{a}{e}$, from point P.

S'P = focal distance

$$= \sqrt{(x+ae)^2 + (y-0)^2} \quad .....(7)$$

PM' = distance of point P from directrix

$$= \left|\frac{x+\frac{a}{e}}{\sqrt{1^2+0^2}}\right| = \left|x+\frac{a}{e}\right| \quad ....(8)$$

From (6) , (7) and (8)

$$\sqrt{(x+ae)^2 + (y-0)^2} = e\left|x+\frac{a}{e}\right|$$

$$\sqrt{(x+ae)^2 + (y-0)^2} = |ex+a|$$

Squaring both sides

$(x + ae)^2 + (y - 0) = e^2x^2 + 2aex + a^2$

$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$

$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$

$(1 - e^2)\, x^2 + y^2 = a^2 (1 - e^2)$

Since $(1 - e^2) > 0$,

Dividing both sides by $a^2 (1 - e^2)$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } b^2 = a^2 (1 - e^2) \text{ and } a > b$$

Thus for the ellipse $(a e , 0)$ and $(- a e , 0)$ are two foci and $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ are corresponding two directrices.

$\therefore$ Standard equation of Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Note:

Equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$

i) The ellipse Intersects x-axis at $A(a, 0)$, $A'(-a, 0)$ and y-axis at $B(0, b)$, $B'(0, -b)$, these are the vertices of the ellipse.

ii) The line segment through the foci of the ellipse is called the major axis and the line segment through centre and perpendicular to major axis is the minor axis. The major axis and minor axis together are called principal axis of the ellipse. In the standard form X axis is the major axis and Y axis is the minor axis.

iii) The segment AA' of length $2a$ is called the major axis and the segment BB' of length 2b is called the minor axis. Ellipse is symmetric about both the axes.

iv) The origin O bisects every chord through it therefore origin O is called the centre of the ellipse.

v) latus rectum is the chord through focus which is perpendicular to major axis. It is bisected at the focus. There are two latera recta as there are two foci.

7.2.2 Some Results :

1) Distance between directrices

d (dd') is the same that of distance ZZ' ie. d (ZZ')

Z $(a/e, 0)$ and Z' $(-a/e, 0)$

$$\Rightarrow d(dd') = d(zz') = \left|\frac{a}{e} - \left(-\frac{a}{e}\right)\right|$$

$$= 2\,\frac{a}{e}$$

2) **End co-ordinates of latera recta**

Let LSL′ be the latus rectum of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \; a>b \qquad \text{(refer fig. 7.16)}$$

(SL and SL′ are the semi latus rectum)

Let : L ≡ (ae, l)

$$\frac{(ae)^2}{a^2}+\frac{(l^2)}{b^2}=1$$

$$e^2+\frac{l^2}{b^2}=1$$

$$\frac{l^2}{b^2}=1-e^2$$

$$l^2 = b^2 (1 - e^2)$$

$$l^2 = b^2\left(\frac{b^2}{a^2}\right) \qquad [\therefore\ b^2 = a^2 (1 - e^2)]$$

$$l^2=\frac{b^4}{a^2}$$

$$l=\pm\frac{b^2}{a}$$

$$L=\left(ae,\frac{b^2}{a}\right) \text{ and } L\equiv\left(ae,\frac{-b^2}{a}\right)$$

These are the co-ordinates of end points of latus rectum.

3) **Length of latus rectum**

$$l\,(LSL') = l\,(SL) + l\,(SL') = \frac{b^2}{a}+\frac{b^2}{a}=\frac{2b^2}{a}$$

4) SP and S'P are the focal distances of the point P on the ellipse. (ref. Fig.7.17)

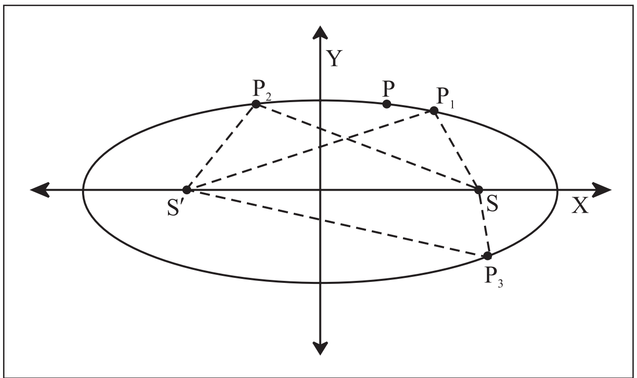


Fig. 7.17

SP = e PM and S'P = e PM'

Sum of focal distances of point P

SP + S'P = e PM + e PM'

= e (PM + PM')

= e (MM')

= e (distance between the directrix)

$$= e\left(2\times\frac{a}{e}\right)=2a$$

SP + S'P = $2a$ = constant

= length of major axis.

Sum of focal distances of point on the ellipse is the length of major axis which is a constant.

Using this property one can define and draw an ellipse. If S_1 and S_2 are two fixed points and a point P moves in the plane such that $PS_1 + PS_2$ is equal to constant K , where K > $d(S_1S_2)$, then the locus of P is an ellipse with S_1 and S_2 as foci. Here in the standard form K = $2a$.

5) A circle drawn with the major axis AA′ as a diameter is called an auxiliary circle of the ellipse.

6) **Parametric form of an ellipse**

P(x, y) be any point on the ellipse. Let Q be a point on the on the auxiliary circle such that QPN ⊥ to the major axis.

Let m ∠ XOQ = θ ∴ Q = (a cosθ, a sinθ)

Let P(x , y) ≡ (a cos θ, y).

$$\frac{(a\cos\theta)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\cos^2\theta+\frac{y^2}{b^2}=1$$

$$y^2 = b^2 (1 - \cos^2\theta)$$

$$y^2 = b^2 \sin^2\theta$$

$$y = \pm\, b\sin\theta$$

P(x, y) ≡ (a cosθ , b sinθ) ≡ P(θ)

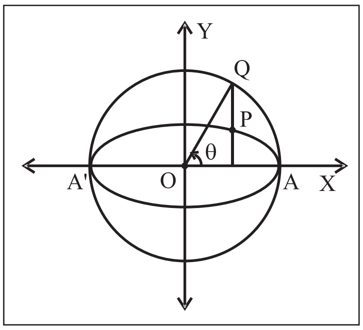


Fig. 7.18

Thus $x = a\cos\theta$ and $y = b\sin\theta$ is the parametric form of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$ where θ is the parameter which is called as an eccentric angle of the point P.

To find the eccentric angle of a point P(*x*,*y*) on the ellipse in terms of *x* and *y*.

If θ is the eccentric angle of P , we know that $x = a\cos\theta$, $y = b\sin\theta$ then $\tan\theta = \dfrac{\frac{y}{b}}{\frac{x}{a}} = \dfrac{ay}{bx}$, that is $\theta = \tan^{-1}\dfrac{ay}{bx}$

note that θ is not the angle made by OP with X axis.

7) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $(b > a)$ is other standard form of the ellipse.

It is called vertical ellipse. (Ref. figure 7.22)

1	Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ b>a$
2	Centre	0(0,0)	0(0,0)
3	Axes of symmetry	Both x axis and y axis	Both x axis and y axis
4	Vertices	A(a,0) A'($-a$,0) B(0,b) B'(0,$-b$)	A(a,0) A'($-a$,0) B(0,b) B'(0,$-b$)
5	Major axis and minor axis	X axis and Y axis	Y axis and X axis
6	Length of major axis	$2a$	$2b$
7	Length of minor axis	$2b$	$2a$
8	Relation between a b and c	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
9	Foci	S(ae, 0) S($-$ae,0)	S(0,be) S(0, $-$be)
10	Distance between foci	2 ae	2 be
11	Equation of directrix	$x=\frac{a}{e}$, and $x=-\frac{a}{e}$	$y=\frac{b}{e}$ and $y=-\frac{b}{e}$.
12	Distance between the directrix	$\frac{2a}{e}$	$\frac{2b}{e}$.
13	End points of latus rectum	$L=\left(ae,\frac{b^2}{a}\right)$ and $L'\equiv\left(ae,\frac{b^2}{a}\right)$	$L\left(\frac{a^2}{b},be\right)$ $L'\left(-\frac{a^2}{b},be\right)$
14	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
15	Parametric form	$x = a\cos\theta$ and $y = b\sin\theta$	$x = a\cos\theta$ and $y = b\sin\theta$
16	Equation of tangent at vertex	$x = a$, $x = -a$ and $y = b$, $y = -b$	$x = a$, $x = -a$ and $y = b$, $y = -b$
17	Sum of Focal distance of a point P(x_1, y_1) is the length of it's	2 a major axis	2 b major axis

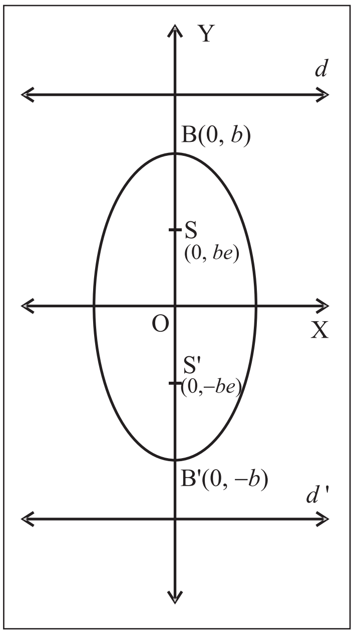


Fig. 7.19

SOLVED EXAMPLES

Ex. 1) Find the coordinates of the foci, the vertices, the length of major axis, the eccentricity and the length of the latus rectum of the ellipse

i) $\frac{x^2}{16}+\frac{y^2}{9}=1$

ii) $4x^2 + 3y^2 = 1$

iii) $3x^2 + 4y^2 = 1$

iv) $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

Solution :

i) Given equation of an ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing with standard equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a^2 = 16$; $b^2 = 9$

$a = 4$; $b = 3$ $(a > b)$

X-axis $(y = 0)$ is the major axis and y-axis $(x = 0)$ the minor axis.

Centre O (0, 0)

Vertices $A(\pm a, 0) \equiv (\pm 4, 0)$, $B(0, \pm b) \equiv (0, \pm 3)$

Length of major axis $(2a) = 2(4) = 9$

Length of minor axis $(2b) = 2(3) = 6$

By relation between a, b and e.

$b^2 = a^2(1 - e^2)$

$9 = 16(1 - e^2)$

$\frac{9}{16} = 1 - e^2$

$e^2 = 1 - \frac{9}{16}$

$e^2 = \frac{7}{16}$ that is $e = \pm\frac{\sqrt{7}}{4}$

but $0 < e < 1$ therefore $e = \frac{\sqrt{7}}{4}$

Foci $S(ae, 0) \equiv \left(4.\frac{\sqrt{7}}{4},0\right)=\left(\sqrt{7},0\right)$

$S'(-ae, 0) = \left(-4.\frac{\sqrt{7}}{4},0\right)=\left(-\sqrt{7},0\right)$

Distance between foc = 2 ae = $2\sqrt{7}$,

Equation of directrix $x = \pm\frac{a}{e}$

$x = \pm\frac{4}{\frac{\sqrt{7}}{4}}$ that is $x = \pm\frac{16}{\sqrt{7}}$

distance between directrix $= 2\frac{a}{e} = 2\left(\frac{16}{\sqrt{7}}\right) = \frac{32}{\sqrt{7}}$

End coordinates of latus rectum

$L\left(ae, \frac{b^2}{a}\right) = \left(\sqrt{7}, \frac{9}{4}\right)$

$L'\left(ae, \frac{-b^2}{a}\right) = \left(\sqrt{7}, -\frac{9}{4}\right)$

Length of latus rectum $= \frac{2b^2}{a} = 2\left(\frac{9}{4}\right) = \frac{9}{2}$

Parametric form $x = a\cos\theta$, $y = b\sin\theta$

That is $x = 4\cos\theta$, $y = 3\sin\theta$

ii) $3x^2 + 4y^2 = 1$

$$= \frac{x^2}{\frac{1}{3}} + \frac{y^2}{\frac{1}{4}} = 1$$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = \frac{1}{3}$, $b^2 = \frac{1}{4}$

$a = \frac{1}{\sqrt{3}}$ $b = \frac{1}{2}$ $(a > b)$

$\therefore$ Major axis is $\boxed{}$

By the relation between a, b & e

$\boxed{} = \boxed{}(1 - e^2)$

$(1 - e^2) = \frac{\boxed{}}{\boxed{}}$

$\therefore$ $e^2 = \boxed{}$ ⇨ $e = \frac{1}{2}$ $\because 0 < e < 1$

centre is at 0(0, 0)

vertices $(\pm a, 0) = (\pm \frac{1}{\sqrt{3}}, 0)$ and

$(0, \pm b) = (0, \pm \frac{1}{2})$

foci $\boxed{(\pm ae, 0)}$ = $(\pm \frac{1}{\sqrt{3}} \frac{1}{2}, 0)$

$$= \left(\pm \frac{1}{2\sqrt{3}}, 0\right)$$

distance between foci = $\boxed{} = \frac{1}{\sqrt{3}}$

Equation of $\boxed{}$ is $x = \pm \frac{a}{e}$

i.e. $x = \pm \frac{\boxed{}}{\boxed{}}$ ⇨ $\boxed{}$

Distance between directrices = $2\frac{a}{e}$

$= \boxed{}$

End points of Latera recta = $(ae; \pm \frac{b^2}{a})$

$= (\boxed{}, \pm \boxed{})$

Length of Latus rectum = $\boxed{} = \frac{\sqrt{3}}{2}$

Parametric form $\boxed{x = }$; $\boxed{y = }$

iii) $4x^2 + 3y^2 = 1$

$$\frac{x^2}{\left(\frac{1}{4}\right)} + \frac{y^2}{\left(\frac{1}{3}\right)} = 1$$

Comparing with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a^2 = \frac{1}{4}$, $b^2 = \frac{1}{3}$

$a = \frac{1}{2}$ b $= \frac{1}{\sqrt{3}}$ $b > a$

therefore y-axis is major axis

Y-axis (ie $x = 0$) is the major axis

x-axis (ie. $y = 0$) is the minor axis

Length of major axis $2b = 2\frac{1}{\sqrt{3}}$

length of minor axis $2a = 2\frac{1}{2} = 1$

Centre is at O(0 ,0)

Vertices $A(\pm a, 0) \equiv (\pm \frac{1}{2}, 0)$,

$B(0, \pm b) \equiv (0, \pm \frac{1}{\sqrt{3}})$

Relation between a, b, e

$a^2 = b^2(1 - e^2)$

$\frac{1}{4} = \frac{1}{3}(1 - e^2)$

$\therefore \frac{3}{4} = 1 - e^2$

$\therefore e^2 = 1 - \frac{3}{4}$ $e^2 = \frac{1}{4}$

$\therefore e = \frac{1}{2}$ $(\because 0 < e < 1)$

foci S (0, + be) and S′ (0, – be) ≡ (0, ± be) =

$$\left(0, \pm\frac{1}{\sqrt{3}}.\frac{1}{2}\right) = \left(0, \pm\frac{1}{2\sqrt{3}}\right)$$

distance between foci = 2 be = $2\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right) = \frac{1}{\sqrt{3}}$

equation of directrices is $y = \pm\frac{b}{e}$

$$y = \pm\frac{\left(\frac{1}{\sqrt{3}}\right)}{\left(\frac{1}{2}\right)}\ y = \pm\frac{2}{\sqrt{3}}$$

Distance between directrices $2\frac{b}{e} = \frac{4}{\sqrt{3}}$

and coordinates of latus rectum.

$$LL' \equiv \left(\pm\frac{a^2}{b}, be\right)$$

$$= \left(\pm\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{\sqrt{3}}\right)}, \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)\right)$$

$$= \left(\pm\frac{\sqrt{3}}{4}, \frac{1}{2\sqrt{3}}\right)$$

Length of latus rectum = $\frac{2a^2}{b} = \frac{\sqrt{2}}{3}$

Parametric form

$x = a\cos\theta,\ y = b\sin\theta$

$$x = \frac{1}{2}\cos\theta,\quad y = \frac{1}{\sqrt{3}}\sin\theta$$

iv) $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

By the method of completing square the above equation becomes $4(x-2)^2 + 9(y+3)^2 = 36$

That is $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$

Comparing with standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 9$; $b^2 = 4$

$a = 3$; $b = 2$ $(a > b)$

$Y = 0$ ie. $y + 3 = 0$ is the major axis and

$X = 0$ ie. $x - 2 = 0$ is the minor axis.

Centre $(X = 0, Y = 0) \equiv (x - 2 = 0,\ y + 3 = 0) \equiv (2, -3)$

Vertices $A(x - 2 = \pm 3,\ y + 3 = 0) \equiv (2 \pm 3, -3)$ i.e. $A(5,-3)$ and $A'(-1,-3)$

$B(x - 2 = 0,\ y + 3 = \pm 2) \equiv (2, -3 \pm 2)$

i.e. $B(2,-1)$ and $B'(2,-5)$

$A(5, -3)$, $A'(-1, -3)$. $B(2, -1)$ $B'(2, -5)$

Length of major axis (2a) = 2(3) = 6

Length of minor axis (2b) = 2(2) = 4

$b^2 = a^2(1 - e^2)$

$\therefore 4 = 9(1 - e^2)$ $\quad\therefore \frac{4}{9} = 1 - e^2$ $\quad\therefore e^2 = 1 - \frac{4}{9}$

$e^2 = \frac{5}{9}$ that is $e = \pm\frac{\sqrt{5}}{3}$

but $0 < e < 1$ therefore $e = \frac{\sqrt{5}}{3}$

Foci $\quad S\left(x - 2 = 3\frac{\sqrt{5}}{3},\ y + 3 = 0\right) \equiv (2 + \sqrt{5}, -3)$

$S'\left(x - 2 = -3\frac{\sqrt{5}}{3},\ y + 3 = 0\right) = (2 - \sqrt{5}, -3)$

Distance between foci = $2\sqrt{5}$

Equation of directrix $x - 2 = \pm\frac{3}{\left(\frac{\sqrt{5}}{3}\right)}$

that is $x = 2 \pm \frac{9}{\sqrt{5}}$

distance between directrix = $2\frac{a}{e} = 2\left(\frac{9}{\sqrt{5}}\right) = \frac{18}{\sqrt{5}}$

coordinates of end point of latera recta

$$L\left(ae, \frac{b^2}{a}\right) = \left(3\times\frac{\sqrt{5}}{3}, \frac{(2)^2}{3}\right) = \left(\sqrt{5}, \frac{4}{3}\right)$$

$$L'\left(ae, -\frac{b^2}{a}\right) = \left(\sqrt{5}, -\frac{4}{3}\right)$$

Length of latus rectum = $\frac{2b^2}{a} = 2\left(\frac{4}{3}\right) = \frac{9}{2}$

Parametric form X = 3 cosθ , Y = 2 sinθ

That is $x - 2 = 3\cos$, $y + 3 = 2\sin\theta$

i.e. $x = 2 + 3\cos\theta$, $y = -3 + 2\sin\theta$

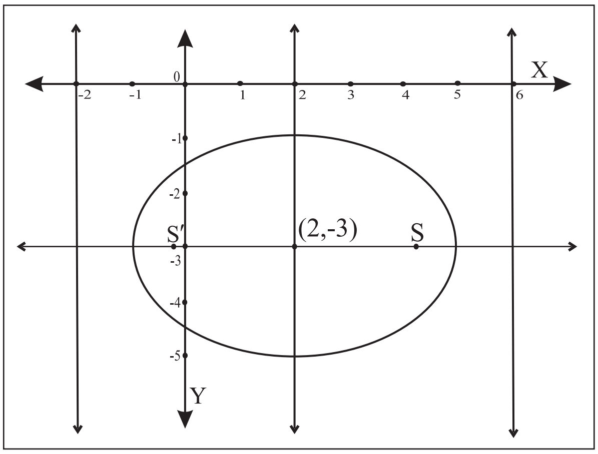


Fig. 7.20

Ex. 2) Find the equation of an ellipse having vertices (± 13, 0) and foci (± 5, 0)

Solution : Since vertices and foci are on the x-axis, the equation of an ellipse will be of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$

Vertices (± 13, 0) = (± a, 0) $\Rightarrow a = 13$

Foci (± 5, 0) = (±ae, 0) $\Rightarrow$ ae = 5

$$\therefore\ e = \frac{5}{13}$$

We know $b^2 = a^2(1 - e^2) = a^2 - a^2e^2$

$= (13)^2 - (5)^2 = 169 - 25 = 144$

Equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

Ex. 3) Find the eccentricity of an ellipse whose length of the latus rectum is one third of its minor axis.

Solution :

Length of latus rectum = $\frac{1}{3}$ (minor axis)

$\frac{2b^2}{a} = \frac{1}{3}(2b)$ that is $b = \frac{1}{3}a$

We know that $b^2 = a^2(1 - e^2)$

$$\frac{1}{9}a^2 = a^2(1 - e^2)$$

$\frac{1}{9} = 1 - e^2$ $\quad\therefore e^2 = 1 - \frac{1}{9}$

$e^2 = \frac{8}{9}$ that is $e = \pm\frac{2\sqrt{2}}{3}$

but $0 < e < 1$ $\quad\therefore\ e = \frac{2\sqrt{2}}{3}$

Activity :

Find the equation of an ellipse whose major axis is on the X-axis and passes through the points (4, 3) and (6, 2)

Solution :

Let equation an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$a > b$, since major axis is the X-axis.

Also ellipse passes through points (4, 3) and (6, 2)

$$\therefore \frac{(4)^2}{a^2} + \frac{(3)^2}{b^2} = 1 \text{ and } \frac{(6)^2}{a^2} + \frac{(2)^2}{b^2} = 1$$

Solve these equations simultaneous to set a^2 and b^2.

7.2.3 Special cases of an ellipse:

Consider the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2(1 - e^2)$ and $a > b$.

As $a \to b$ ($b > 0$) then observe that $e \to 0$ and shape of the ellipse is more rounded. Thus when a = b the ellipse reduces to a circle of radius a and two foci coincides with the centre .

7.2.4 Tangent to an ellipse :

A straight line which intersects the curve ellipse in two coincident point is called a tangent to the ellipse

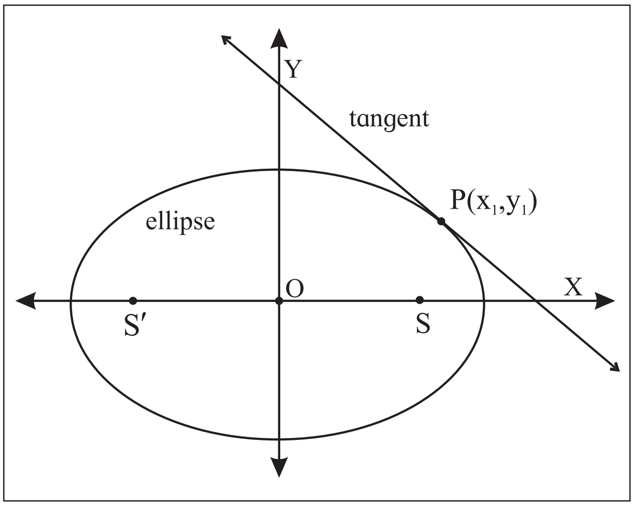


Fig. 7.21

To find the equation of tangent to the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ on it. Hence, to obtain the equation of tangent at $P(\theta_1)$.

We need to know the slope of the tangent at $P(x_1, y_1)$. From the theory of derivative of a function, the slope of the tangent is $\frac{dy}{dx}$ at (x_1, y_1)

The equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

differentiate both sides with respect to x

We get $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\therefore \left[\frac{dy}{dx}\right]_{(x_1,y_1)} = -\frac{b^2}{a^2}\frac{x_1}{y_1} = \text{slope of the tangent}$$

at $P(x_1,y_1)$.

$\therefore$ Equation of the tangent (by slope point form)

at $P(x_1, y_1)$ is $y - y_1 = -\frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1)$

$$a^2 y_1 (y - y_1) = - b^2 x_1 (x - x_1)$$

$$a^2 y_1\ y - a^2 y_1^2 = - b^2 x_1 x + b^2 x_1^2$$

$$b^2 x_1 x + a^2 y_1 y = b^2 x_1^2 + a^2 y_1^2$$

Dividing by $a^2 b^2$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

Now $P(x_1,y_1)$ lies on the ellipse $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

$$\therefore \frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This is the equation of the tangent at $P(x_1,y_1)$ on it

Now, θ_1 is the parameter of point P

$\therefore P(x_1,y_1) = (a\cos\theta_1, b\sin\theta_1)$ that is

$x_1 = a\cos\theta_1$, $y_1 = b\sin\theta_1$

Substituting these values in equation (1),

we get $\frac{x\, a\cos\theta_1}{a^2} + \frac{y\, b\sin\theta_1}{b^2} = 1$

$$\frac{x\cos\theta_1}{a} + \frac{y\sin\theta_1}{b} = 1$$

is the required equation of the tangent at $P(\theta_1)$.

7.2.5 Condition for tangency

To find the condition that the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also to find the point of contact.

Equation of the line is $y = mx + c$

that is $mx - y + c = 0\ .\ .\ .\ (1)$

equation of the tangent at $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

that is $\frac{x_1}{a^2}x+\frac{y_1}{b^2}y-1=0$ . . . (2)

If the line given by equation (1) is *a* tangent to the ellipse at $P(x_1,y_1)$.

Comparing coefficients of like terms in equation (1) and (2)

we get, $\frac{\left(\frac{x_1}{a^2}\right)}{m}=\frac{\left(\frac{y_1}{b^2}\right)}{1}=-\frac{1}{c}$

$\therefore \frac{\left(\frac{x_1}{a^2}\right)}{m}=\frac{-1}{c}$ and $\frac{\left(\frac{y_1}{b^2}\right)}{-1}=\frac{-1}{c}$

$\therefore \frac{x_1}{a^2m}=\frac{-1}{c}$ and $\frac{y_1}{b^2}=\frac{1}{c}$

$\therefore x_1=-\frac{a^2m}{c}$ and $y_1=\frac{b^2}{c}$

$P(x_1,y_1)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

$\therefore \frac{(-\frac{a^2m}{c})^2}{a^2}+\frac{(\frac{b^2}{c})^2}{b^2}=1$

$\therefore \frac{\frac{a^4m^2}{c^2}}{a^2}+\frac{(\frac{b^4}{c^2})}{b^2}=1$

$\therefore \frac{a^2m^2}{c^2}+\frac{b^2}{c^2}=1$

$\therefore a^2m^2+b^2=c^2$

i.e. $c^2=a^2m^2+b^2$

$\therefore c=\pm\sqrt{a^2m^2+b^2}$ is the condition for tangency.

The equation of tangent to the ellipse in terms of slope is

$$y=mx\pm\sqrt{a^2m^2+b^2}\,,\ P\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$$

Thus the line $y=mx+c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c=\pm\sqrt{a^2m^2+b^2}$ and the point of contact is $(-\frac{a^2m}{c},\frac{b^2}{c})$.

7.2.6 Tangents from a point to the ellipse

Two tangents can be drawn to the ellipse from any point outside the ellipse.

Let $P(x_1,y_1)$ be any point in plane of the ellipse.

The equation of tangent, with slope m to the ellipse is

$$y=mx\pm\sqrt{a^2m^2+b^2}.$$

This pass through (x_1,y_1)

$\therefore y_1=mx_1\pm\sqrt{a^2m^2+b^2}$

$\therefore y_1-mx_1=\pm\sqrt{a^2m^2+b^2}$, we solve it for m.

Squaring on both sides and simplifying we get the quadratic equation in *m*.

$$(x_1^2-a^2)m^2-2x_1y_1m+\left(y_1^2-b^2\right)=0$$

it has two roots say, m_1 and m_2 which are the slopes of two tangents.

Thus, in general, two tangents can be drawn to *a* ellipse from a given point in its plane.

Sum of the roots $=m_1+m_2=\frac{-(-2x_1y_1)}{(x_1^2-a^2)}$

$$=\frac{(2x_1y_1)}{(x_1^2-a^2)}$$

Product of roots = $m_1 m_2 = \dfrac{(y_1^2 - b^2)}{(x_1^2 - a^2)}$

7.2.7 Locus of point of intersection of perpendicular tangents

If the tangent drawn from P are mutually perpendicular then we have $m_1 m_2 = -1$

$$\therefore \frac{y_1^2 - b^2}{x_1^2 - a^2} = 1$$

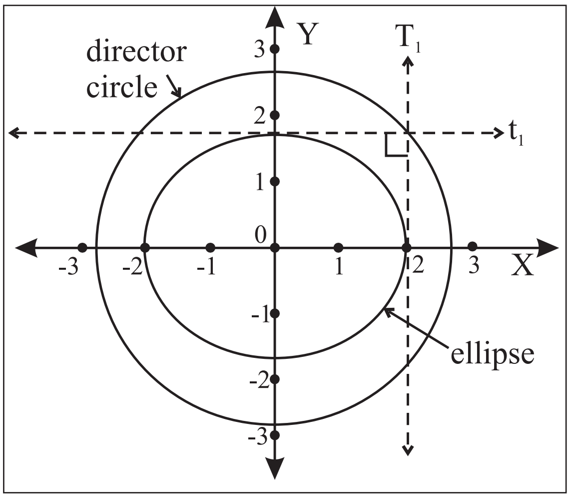


Fig. 7.22

$$\therefore (y_1^2 - b^2) = -(x_1^2 - a^2)$$

$$\therefore x_1^2 + y_1^2 = a^2 + b^2$$

This is the equation of standard circle with centre at origin and radius $\sqrt{a^2 + b^2}$ which is called the director circle of the ellipse.

7.2.8 Auxiliary circle, director circle of the ellipse

For the standard ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 (a > \text{b})$ the circle drawn with major axis as a diameter is called the auxillary circle of the ellipse and it's equation is $x^2 + y^2 = a^2$.

The locus of point of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is called the director circle of the ellipse and its equation is $x^2 + y^2 = a^2 + b^2$.

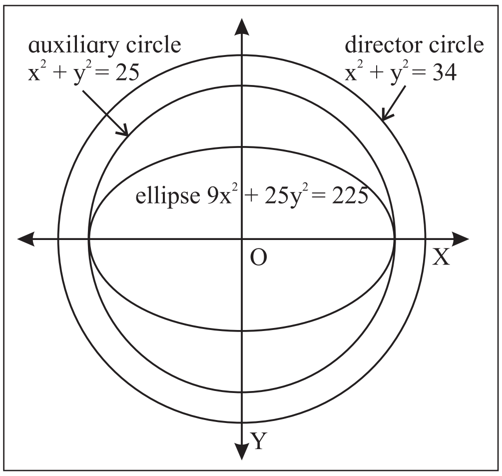


Fig. 7.23

SOLVED EXAMPLE

Ex. 1) Find the equation of tangent to the ellipse

i) $\dfrac{x^2}{8} + \dfrac{y^2}{6} = 1$ at the point $(2, \sqrt{3})$.

ii) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ at the point whose eccentric angle is $\pi/4$.

Solution :

i) Equation of the ellipse is $\dfrac{x^2}{8} + \dfrac{y^2}{6} = 1$

comparing it with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$a^2 = 8$ and $b^2 = 6$.

Tangent is drawn to the ellipse at point $(2, \sqrt{3})$ on it. Say $(x_1, y_1) \equiv (2, \sqrt{3})$.

We know that,

the equation of tangent to the ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at point $(x_1\, y_1)$ on it is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\therefore \frac{x(2)}{8} + \frac{y(\sqrt{3})}{6} = 1$$

$$\frac{x}{4} + \frac{\sqrt{3}\,y}{6} = 1$$

i.e. $6x + 4\sqrt{3}\,y = 24$

i.e. $3x + 2\sqrt{3}\,y = 12$

Thus required equation of tangent is

$3x + 2\sqrt{3}\,y = 12$.

ii) Equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

compairing it with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 25$ and $b^2 = 9$.

eccentric angle $\theta = \frac{\pi}{4}$.

By parametric form equation of tangent is

$$\frac{x \cdot \cos\theta}{a} + \frac{y \cdot \sin\theta}{b} = 1$$

i.e. $$\frac{x \cdot \cos\frac{\pi}{4}}{5} + \frac{y \cdot \sin\frac{\pi}{4}}{3} = 1$$

$$\frac{x \cdot \frac{1}{\sqrt{2}}}{5} + \frac{y \cdot \frac{1}{\sqrt{2}}}{3} = 1$$

$$\frac{x}{5\sqrt{2}} + \frac{y}{3\sqrt{2}} = 1$$

$3x + 5y = 15\sqrt{2}$

Ex. 2) Show that the line $2x + 3y = 12$ is tangent to the ellipse $4x^2 + 9y^2 = 72$.

Solution : Equation of the ellipse is $4x^2 + 9y^2 = 72$

i.e. $\frac{x^2}{18} + \frac{y^2}{8} = 1$

compairing it with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 18$ and $b^2 = 8$

Equation of line is $2x + 3y = 12$

i.e. $y = -\frac{2}{3}x + 4$

compairing it with $y = mx + c$

$m = -\frac{2}{3}$ and $c = 4$

We know that,

if the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then $c^2 = a^2m^2 + b^2$.

Here $c^2 = (4)^2 = 16$ and

$a^2m^2 + b^2 = (18)\left(-\frac{2}{3}\right)^2 + (8) = (18)\left(\frac{4}{9}\right) + 8$

$= (2)(4) + 8 = 16$

hence the given line is tangent to the given ellipse.

Ex. 3) Find the equations of tangents to the ellipse $4x^2 + 9y^2 = 36$ passing through the point $(2, -2)$.

Solution : Equation of the ellipse is $4x^2 + 9y^2 = 36$

i.e. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

compairing it with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 9$ and $b^2 = 4$

Equation of tangent in terms of slope m, to the ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

Point $(2, -2)$ lies on the tangent

$\therefore (-2) = m(2) \pm \sqrt{9m^2 + 4}$

$\therefore -2m - 2 = \pm\sqrt{9m^2 + 4}$

squaring both sides

$4m^2 + 8m + 4 = 9m^2 + 4$

$-5m^2 + 8m = 0$

$m(-5m + 8) = 0 \Rightarrow m = 0$ or $m = 8/5$

Equation of tangent line having slope m and passing through pt $(2, -2)$ is $y + 2 = m(x - 2)$

i.e. $y + 2 = 0(x - 2)$ or $y + 2 = \frac{8}{5}(x - 2)$

$y + 2 = 0$ $\qquad$ $5y + 10 = 8x - 16$

$8x - 5y - 26 = 0$

Thus equation of tangents are $y + 2 = 0$ and $8x - 5y - 26 = 0$

EXERCISE 7.2

1. Find the (i) lengths of the principal axes. (ii) co-ordinates of the focii (iii) equations of directrics (iv) length of the latus rectum (v) distance between focii (vi) distance between directrices of the ellipse:

   (a) $x^2/25 + y^2/9 = 1$ (b) $3x^2 + 4y^2 = 12$

   (c) $2x^2 + 6y^2 = 6$ (d) $3x^2 + 4y^2 = 1$.

2. Find the equation of the ellipse in standard form if

   i) eccentricity = 3/8 and distance between its focii = 6.

   ii) the length of major axis 10 and the distance between focii is 8.

   iii) distance between directrix is 18 and eccentricity is 1/3.

   iv) minor axis is 16 and eccentricity is 1/3.

   v) the distance between foci is 6 and the distance between directrix is 50/3.

   vi) The latus rectum has length 6 and foci are $(\pm 2, 0)$.

   vii) passing through the points $(-3, 1)$ and $(2, -2)$

   viii) the dist. between its directrix is 10 and which passes through $(-\sqrt{5}, 2)$

   ix) eccentricity is 2/3 and passes through $(2, -5/3)$.

3. Find the eccentricity of an ellipse, if the length of its latus rectum is one third of its minor axis.

4. Find the eccentricity of an ellipse if the distance between its directrix is three times the distance between its focii.

5. Show that the product of the lengths of ht perpendicular segments drawn from the foci to any tangent line to the ellipse $x^2/25 + y^2/16 = 1$ is equal to 16.

6. A tangent having slop $-1/2$ to the ellipse $3x^2 + 4y^2 = 12$ intersects the X and Y axes in the points A and B respectively. If O is the origin, find the area of the triangle.

7. Show that the line $x - y = 5$ is *a* tangent to the ellipse $9x^2 + 16y^2 = 144$. Find the point of contact.

8. Show that the line $8y + x = 17$ touches the ellipse $x^2 + 4y^2 = 17$. Find the point of contact.

9. Determine whether the line $x + 3y\sqrt{2} = 9$ is *a* tangent to the ellipse $x^2/9 + y^2/4 = 1$. If so, find the co-ordinates of the pt of contact.

10. Find k, if the line $3x + 4y + k = 0$ touches $9x^2 + 16y^2 = 144$.

11. Find the equation of the tangent to the ellipse (i) $x^2/5 + y^2/4 = 1$ passing through the point (2, -2).

    ii) $4x^2 + 7y^2 = 28$ from the pt (3, -2).

    iii) $2x^2 + y^2 = 6$ from the point (2, 1).

    iv) $x^2 + 4y^2 = 9$ which are parallel to the line $2x + 3y - 5 = 0$.

    v) $x^2/25 + y^2/4 = 1$ which are parallel to the line $x + y + 1= 0$.

    vi) $5x^2 + 9y^2 = 45$ which are $\perp$ to the line $3x + 2y + y = 0$.

    vii) $x^2 + 4y^2 = 20$, $\perp$ to the line $4x+3y = 7$.

12. Find the equation of the locus of *a* point the tangents form which to the ellipse $3x^2 + 5y^2 = 15$ are at right angles.

13. Tangents are drawn through *a* point P to the ellipse $4x^2 + 5y^2 = 20$ having inclinations θ_1 and θ_2 such that $\tan\theta_1 + \tan\theta_2 = 2$. Find the equation of the locus of P.

14. Show that the locus of the point of intersection of tangents at two points on an ellipse, whose eccentric angles differ by *a* constant, is an ellipse.

15. P and Q are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentric angles θ_1 and θ_2. Find the equation of the locus of the point of intersection of the tangents at P and Q if $\theta_1 + \theta_2 = \pi/2$.

16. The eccentric angles of two points P and Q the ellipse $4x^2 + y^2 = 4$ differ by $2\pi/3$. Show that the locus of the point of intersection of the tangents at P and Q is the ellipse $4x^2 + y^2 = 16$.

17. Find the equations of the tangents to the ellipse $x^2/16 + y^2/9 = 1$, making equal intercepts on co-ordinate axes.

18. A tangent having slope – ½ to the ellipse $3x^2 + 4y^2 = 12$ intersects the X and Y axes in the points A and B respectively. If O is the origin, find the area of the triangle.

7.3 Hyperbola

Let's Study

- Standard equation of the hyperbola.
- Equation of tangent to the hyperbola.
- condition for tangency.
- auxillary circle and director circle of the hyperbola.

The hyperbola is the intersection of double napped cone with plane parallel to the axis.

The hyperbola is the locus of *a* point in a plane which moves so that its distance from a fixed point bears *a* constant ratio *e* ($e > 1$) to its distance from a fixed line. The fixed point is called the focus S and the fixed line is called the directrix d.

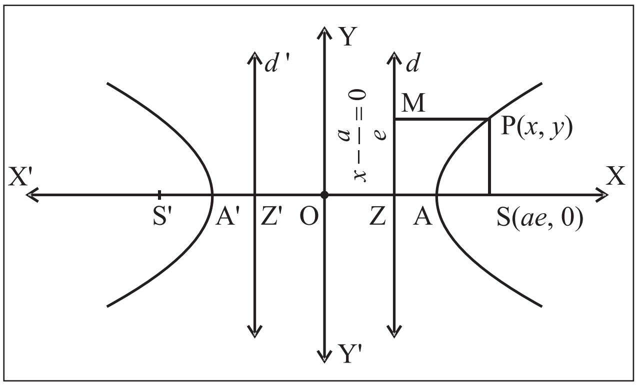


Fig. 7.24

If S is the focus and *d* is the directrix not containing the focus and P is the moving point, then $\frac{PS}{PM}$ = e, where PM is the perpendicular on the directrix. $e > 1$ called eccentricity of the hyperbola. (Fig. 7.24)

7.3.1 Standard equation of the hyperbola

Let S be the focus, *d* be the directrix and e be the eccentricity of a hyperbola.

Draw SZ perpendicular to directrix. Let A and A' divide the segment SZ internally and externally in the ratio e : 1. By definition of hyperbola A and A' lie on hyperbola.

Let AA' = 2 *a* , midpoint O of segment AA' be the origin. Then O ≡ (0,0) , A≡ (a,0) and A' ≡ (– a,0)

$$\frac{SA}{AZ} = \frac{e}{1} \qquad \left(\frac{SA'}{A'Z} = \frac{-e}{1}\right)$$

therefore SA = e AZ .

Let Z ≡ (k,0) and S ≡ (h,0)

By section formula for internal and external division.

$$a = \frac{ek + 1h}{e + 1} \qquad \text{also} \ -a = \frac{ek - 1h}{e - 1}$$

a e + a = e k + h . . . (2)

$-a$ e + a = e k – h . . . (3)

Solving these equations , we get

k = a/e and h = a e

Focus S ≡ (ae , 0) and Z ≡ (a/e , 0)

Equation of the directrix is $x = \frac{a}{e}$

That is $x - \frac{a}{e} = 0$

Let P(x,y) be a point on the hyperbola.

SP = focal distance

$$= \sqrt{(x-ae)^2 + (y-0)^2} \quad .....(4)$$

PM = distance of point P from the directrix

$$= \left| \frac{x - \frac{a}{e}}{\sqrt{1^2 + 0^2}} \right| = \left| x - \frac{a}{e} \right| \quad ....(5)$$

From (1) , (4) and (5)

$$\sqrt{(x-ae)^2 + (y-0)^2} = e\left| x - \frac{a}{e} \right|$$

$$= \sqrt{(x-ae)^2 + (y-0)^2} = |ex - a|$$

Squaring both sides

$(x - ae)^2 + (y - 0)^2 = e^2x^2 - 2\,aex + a^2$

$x^2 - 2\,aex + a^2e^2 + y^2 = e^2x^2 - 2\,aex + a^2$

$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$

$(1 - e^2)\,x^2 + y^2 = a^2\,(1 - e^2)$

Since e > 1

$(e^2 - 1)\,x^2 - y^2 = a^2\,(e^2 - 1)$

Dividing both sides by $a^2\,(e^2 - 1)$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2\,(e^2 - 1)$

This is the standard equation of hyperbola .

Let S' be ($-ae$,0) and d' be the line $x = -\frac{a}{e}$.

For any point P on the hyperbola, PM' is perpendicular on d' then it can be verified that PS '= e PM'.

Thus for the hyperbola (a e , 0) and ($-a$ e , 0) are two focii and $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ are corresponding two directrices.

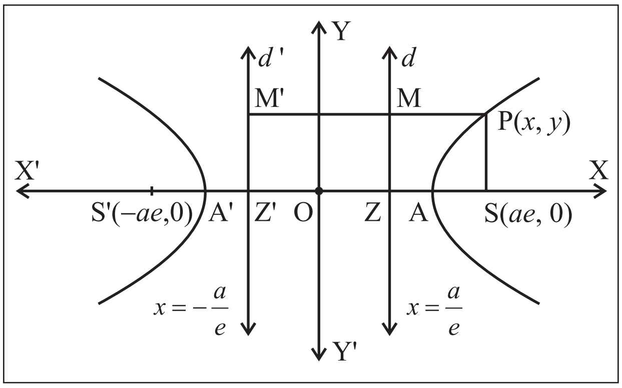


Fig. 7.25

7.3.2 Some useful terms of the hyperbola

Equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

i) The hyperbola intersects x-axis at A(a, 0), A′ ($-a$, 0).

ii) It does not intersects the y-axis.

iii) The segment AA′ of length 2a is called the transverse axis and the segment BB′ of length 2b is called the conjugate axis.

iii) The line segment through the foci of the hyperbola is called the transverse axis and the line segment through centre and perpendicular to transverse axis is the conjugate axis. The transverse axis and conjugate axis together are called principal axes of the hyperbola. In the standard form X axis is the transverse axis and Y axis is the conjugate axis.

iv) latus rectum is the chord passing through the focus which is perpendicular to transverse axis. It is bisected at the focus. There are two latera recta as there are two focii.

7.3.3 Some Results :

1) **Distance between directrices** i.e. d (dd') is the same that of distance ZZ′ ie. d (ZZ′)

Z ≡ (a/e, 0) and Z′ (– a/e , 0)

$$\Rightarrow \quad d\,(dd^1) = d\,(zz') = \left|\frac{a}{e} + \frac{a}{e}\right|$$

$$= 2\frac{a}{e}$$

2) **Let LSL′ be the latus rectum of the hyperbola.**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(SL and SL′ are the semi latus rectum)

Let L ≡ (ae, l)

$$\frac{(ae)^2}{a^2} + \frac{l^2}{b^2} = 1$$

$$e^2 + \frac{l^2}{b^2} = 1$$

$$\frac{l^2}{b^2} = 1 - e^2$$

$$l^2 = b^2\,(1 - e^2)$$

$$l^2 = b^2\left(\frac{b^2}{a^2}\right)$$

$$l^2 = \frac{b^4}{a^2}$$

$$l = \pm\frac{b^2}{a}$$

$$L = \left(ae, \frac{b^2}{a}\right) \text{ and } L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

These are the co-ordinates of end points of latus rectum.

3) **Length of latus rectum** = l (LL′)

$$= l\,(SL) + l\,(SL') = \frac{b^2}{a} + \frac{b^2}{a} = \frac{2b^2}{a}$$

4) **SP and S′P are the focal distances of the point P on the hyperbola.**

SP = e PM and S′P = e PM′

Difference between the focal distances of point P

SP – S′P = e PM – e PM′

= e (PM – PM′)

= e (MM′)

= e (distance between the directrices)

= e (2 a/e)

SP – S′P = 2a = constant

= length of major axis ie. transverse axis.

Difference between the focal distances of point on the hyperbola is the length of transverse axis which is a constant.

5) A circle drawn with the transverse axis AA′ as a diameter is called an auxiliary circle of the hyperbola and its equation is $x^2 + y^2 = a^2$.

- **Parametric Equation of the Hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **:**

Taking the transverse axis AA' as diameter. Draw a circle with centre at origin and radius 'a' so that its equation is $x^2+y^2 = a^2$. It is called the auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(I)

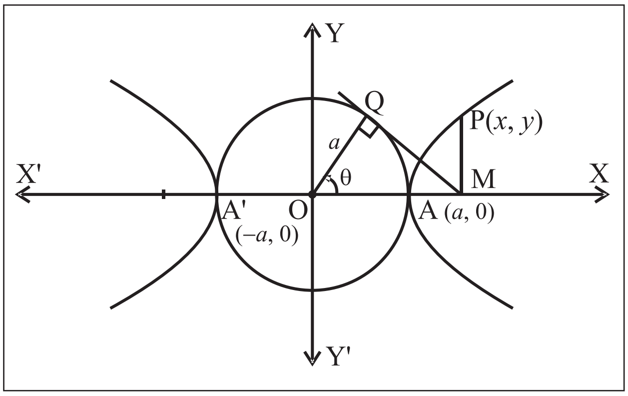


Fig. 7.26

Let P(x_1,y) be any point on the hyperbola. Draw PM perpendicular to OX. Draw the tangent MQ touching auxiliary circle at Q. Point Q is called the corresponding point of P on the auxiliary circle.

Let $m\angle XOQ = \theta$. Then by trigonometry co-ordinates of Q are (a cos θ, a sinθ).

Further,

$$x = \text{OM} = \frac{\text{OM}}{\text{OQ}} \,.\, \text{OQ} = \sec\theta .\, a = a \sec\theta$$

The point P(x,y) = P(a sec θ, y) ∴ P lies on the hyperbola-(I) therefore,

$$\frac{a^2\sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \frac{y^2}{b^2} = \sec^2\theta - 1 = \tan^2\theta$$

$$\therefore \frac{y}{b} = \pm \tan\theta$$

Since P lies in the first quadrant and angle θ < 90°, y and tanθ both are positive.

$\therefore y = b \tan\theta$

$\therefore P \equiv P\,(a \sec\theta, b \tan\theta)$

Substituting these co-ordinates in the LHS of equation of hyperbola (I), we get

$$\left(x-2=3\frac{\sqrt{5}}{3}, y+3=0\right) = \sec^2\theta - \tan^2\theta = 1$$

∴ for any value of θ, the point (a secθ, b tanθ) always lies on the hyperbola.

Let us denote this point by P(θ) = P (a secθ, b tanθ) where θ is called parameter also called the eccentric angle of point P.

The equations $x = a \sec\theta$, $y = b \tan\theta$ are called parametric equations of the hyperbola.

7) Other standard form of hyperbola.

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is called the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

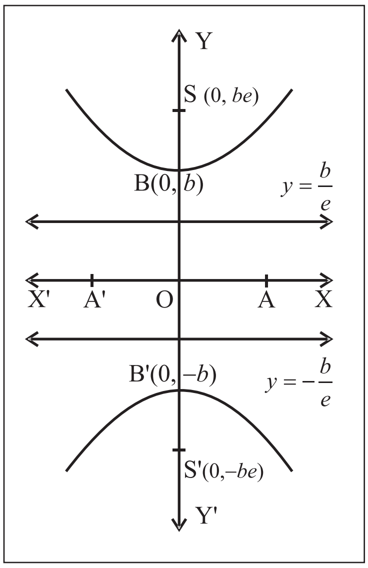


Fig. 2.27

1	Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
2	Centre	O(0,0)	O(0,0)
3	Axes of symmetry	Both x axis and y axis	Both x axis and y axis
4	Vertices	A(a,0) A′(–a,0) B(0,b) B′(0,–b)	A(a,0) A′(–a,0) B(0,b) B′(0,–b)
5	Transverse axis and Conjugate axis	X axis and Y axis	Y axis and X axis
6	Length of transverse axis	2a	2b
7	Length of conjugate axis	2b	2a

8	Foci	S(ae, 0) S(– ae,0)	S(0,be) S(0, – be)
9	Distance between foci	2 ae	2 be
10	Equation of directrix	$x = \frac{a}{e}$. and $x = -\frac{a}{e}$	$y = +\frac{b}{e}$ and $y = -\frac{b}{e}$.
11	Distance between the directrics	$\frac{2a}{e}$.	$\frac{2b}{e}$.
12	end points of latus rectum	$L = \left(ae, \frac{b^2}{a}\right)$ and $L' \equiv \left(ae, \frac{-b^2}{a}\right)$	$L \equiv (\frac{a^2}{b}$, be) and $L' \equiv (\frac{-a^2}{b}$, be)
13	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
14	Parametric form	$x = a$ secθ and $y = b$ tanθ	$x = a$ tanθ and y = b secθ
15	Equation of tangent at vertex	$x = a$, $x = -a$	y = b , y = – b
16	Sum of Focal distance of a point $P(x_1, y_1)$	2 a (length of major axis ie. transverse axis)	2 b (length of minor axis ie. conjugate axis)

SOLVED EXAMPLE

Ex. 1) Find the length of transverse axis, length of conjugate axis, the eccentricity, the co-ordinates of foci, equations of directrices and the length of latus rectum of the hyperbola

(i) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (ii) $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Solution :

(i) The equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

Comparing this with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $a^2 = 4$, $b^2 = 12$

$\therefore a = 2, b = 2\sqrt{3}$

Length of transverse axis = $2a = 2(2) = 4$

Length of conjugate axis = $2b$ = $2(2\sqrt{3})$

= $4\sqrt{3}$

Eccentricity $b^2 = a^2 (e^2 - 1)$

$$\therefore e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{4+12}}{2} = \frac{\sqrt{16}}{2} = \frac{4}{2} = 2$$

($\because$ e > 0)

$ae = 2(2) = 4$

$\therefore$ foci ($\pm ae$, 0) are ($\pm$ 4,0)

$\frac{a}{e} = \frac{2}{2} = 1$

$\therefore$ the equations of directrices $x = \pm \frac{a}{e}$ are $x = \pm 1$.

Length of latus rectum = $\frac{2b^2}{a} = \frac{2(12)}{2} = 12$

(ii) The equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Comparing this with the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

We have $a^2 = 16$, $b^2 = 9$

$\therefore\ a = 4, b = 3$

Length of transverse axis = $2b = 2(3) = 6$

Length of conjugate axis = $2a = 2(4) = 8$

Eccentricity $a^2 = b^2 (e^2 - 1)$

$$\therefore\ e = \frac{\sqrt{a^2+b^2}}{b} = \frac{\sqrt{16+9}}{3} = \frac{\sqrt{25}}{3} = \frac{5}{3}$$

$(\because\ e > 0)$

$be = 3\left(\frac{5}{3}\right) = 5$

$\therefore$ focii $(0, \pm be)$ are $(0,+5)$

$$\frac{b}{e} = \frac{3}{5/3} = \frac{9}{5}$$

$\therefore$ the equations of directrices $y = \pm\frac{b}{e}$ are

$y = \pm\frac{9}{5}$.

Length of latus rectum = $\frac{2a^2}{b} = \frac{2(16)}{3} = \frac{32}{3}$

Ex. 2) Find the equation of the hyperbola with the centre at the origin, transverse axis 12 and one of the foci at $(3\sqrt{5},0)$

Solution :

Let the equation of the required hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{..................(1)}$$

Length of transverse axis = $2a = 12$

$\therefore\ a = 6 \quad \therefore\ a^2 = 36$

Since focus $(ae, 0)$ is $(3\sqrt{5}, 0)$

$\therefore\ ae = 3\sqrt{5}$

$\therefore\ a^2e^2 = 45$

$\therefore\ a^2 + b^2 = 45$

$\therefore\ 36 + b^2 = 45$

$\therefore\ b^2 = 9$

Then from (1), the equation of the required hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

Ex. 3) Find the equation of the hyperbola referred to its principal axes whose distance between directrices is $\frac{18}{5}$ and eccentricity is $\frac{5}{3}$.

Solution :

The equation of the hyperbola referred to its principal axes be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{.................(1)}$$

Since eccentricity = $e = \frac{5}{3}$ and distance between directrices = $\frac{2a}{e} = \frac{18}{5}$

$\therefore\ \frac{a}{e} = \frac{9}{5}$

$\therefore\ a = \frac{9}{5}e = \frac{9}{5} \times \frac{5}{3}$, $a = 3$

$\therefore\ a^2 = 9$

Now $b^2 = a^2 (e^2 - 1) = 9\left(\frac{25}{9} - 1\right)$

$= 9 \times \frac{16}{9} = 16$

Then from (1), the equation of the required hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

7.3.4 Tangent to a hyperbola:

A straight line which intersects the curve hyperbola in two coincident points is called a tangent of the hyperbola

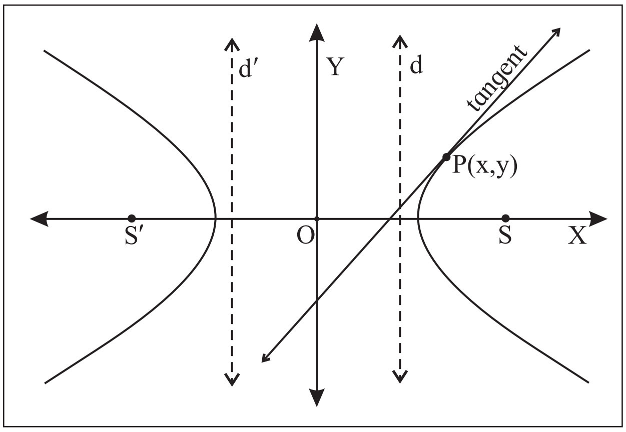


Fig. 7.28

Tangent at a point on a hyperbola.

To find the equation of tangent to the hyperbola.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ on it. Hence, to obtain the equation of tangent at $P(\theta_1)$.

We need to know the slope of the tangent at $P(x_1,y_1)$. From the theory of derivative of a function the slope of the tangent is $\frac{dy}{dx}$ at (x_1,y_1)

and here $\frac{dy}{dx}_{(x_1,y_1)}= m$

The equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

differentiate both sides with respect to x

We get $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\left(-\frac{2x}{a^2}\right)\left(-\frac{b^2}{2y}\right)=\frac{b^2x}{a^2y}$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}$ = slope of the tangent at $P(x_1,y_1)$

$\therefore$ Equation of the tangent to the hyperbola at $P(x_1,y_1)$ is $y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$

$$a^2y_1(y-y_1)=b^2x_1(x-x_1)$$

$$a^2y_1\,y-a^2y_1^2=b^2x_1\,x-b^2x_1^2$$

$$b^2x_1\,x + a^2y_1\,y = b^2x_1^2+a^2y_1^2$$

Dividing by a^2b^2

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

Now $P(x_1,y_1)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

$\therefore \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ....(1)

is the equation of the tangent at $P(x_1,y_1)$ on it

Now, θ_1 is the parameter of point P

$\therefore P(x_1,y_1)=(a\sec\theta_1, b\tan\theta_1)$ that is

$x_1=a\sec\theta_1$, $y_1=b\tan\theta_1$

Substituting these values in equation (1),

we get $\frac{x\,a\sec\theta_1}{a^2}-\frac{y\,b\tan\theta_1}{b^2}=1$

$$\frac{x\sec\theta_1}{a}-\frac{y\tan\theta_1}{b}=1$$

i.e. $\left(\frac{\sec\theta_1}{a}\right)x-\left(\frac{\tan\theta_1}{b}\right)y=1$

is the required equation of the tangent at $P(\theta_1)$.

7.3.5 Condition for tangency

To find the condition that the line $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Also to find the point of contact.

Equation of the line is $y = mx + c$

that is $mx-y+c=0$ ...(1)

equation of the tangent at $P(x_1,y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

that is $\frac{x_1}{a^2}x-\frac{y_1}{b^2}y-1=0$ . . . (2)

If the line given by equation (1) is a tangent to the hyperbola at (x_1,y_1).

Comparing similar terms in equation (1) and (2)

we get $\frac{\left(\frac{x_1}{a^2}\right)}{m}=\frac{\left(-\frac{y_1}{b^2}\right)}{-1}=\frac{-1}{c}$

$\therefore \frac{\left(\frac{x_1}{a^2}\right)}{m}=\frac{-1}{c}$ and $\frac{\left(-\frac{y_1}{b^2}\right)}{-1}=\frac{-1}{c}$

$$\therefore \frac{x_1}{a^2 m} = \frac{-1}{c} \quad \text{and} \quad \frac{y_1}{b^2} = -\frac{1}{c}$$

$$\therefore x_1 = -\frac{a^2 m}{c} \quad \text{and} \quad y_1 = \frac{-b^2}{c}$$

$P(x_1, y_1)$ lies on the hyperbola

$$\therefore \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\therefore \frac{\left(-\frac{a^2 m}{c}\right)^2}{a^2} - \frac{\left(-\frac{b^2}{c}\right)^2}{b^2} = 1$$

$$\therefore \frac{\left(\frac{a^4 m^2}{c^2}\right)}{a^2} - \frac{\left(\frac{b^4}{c^2}\right)}{b^2} = 1$$

$$\therefore \frac{a^2 m^2}{c^2} - \frac{b^2}{c^2} = 1$$

$\therefore c^2 = a^2m^2 - b^2$

$c = \pm \sqrt{a^2m^2 - b^2}$ is called the condition of tangency.

> Thus the line $y = mx + c$ is tangent to the hyperbola
>
> $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c = \pm\sqrt{a^2m^2 - b^2}$ and the point of contact is $\left(-\frac{a^2 m}{c}, -\frac{-b^2}{c}\right)$.
>
> The equation of tangent in terms slope is
>
> $y = mx \pm \sqrt{a^2m^2 - b^2}$,

7.3.6 Tangents from a point to the hyperbola

Two tangents can be drawn to a hyperbola from any point outside the hyperbola in its plane.

Let $P(x_1, y_1)$ be any point in plane of the hyperbola.

The equation of tangent to the hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}\ .$$

If the tangent passes through (x_1, y_1).

$$\therefore y_1 = m x_1 \pm \sqrt{a^2m^2 - b^2}\ .$$

$$\therefore y_1 - m x_1 = \pm \sqrt{a^2m^2 - b^2}\ .$$

Squaring on both sides and simplifying we get the quadratic equation in m which is found to be,

$$(x_1^2 - a^2)m^2 - 2x_1 y_1 m + (y_1^2 + b^2) = 0$$

it has two roots say, m_1 and m_2 which are the slopes of two tangents.

Thus, in general, two tangents can be drawn to a hyperbola from a given point in its plane.

$$\text{Sum of the roots} = m_1 + m_2 = \frac{-(-2x_1y_1)}{(x_1^2 - a^2)}$$

$$= \frac{(2x_1y_1)}{(x_1^2 - a^2)}$$

$$\text{Product of roots} = m_1 m_2 = \frac{\left(y_1^2 + b^2\right)}{\left(x_1^2 - a^2\right)}$$

7.3.7 Locus of point of intersection of perpendicular tangents :

If the tangent drawn from P are mutually perpendicular then we have $m_1 m_2 = -1$

$$\therefore (y_1^2 + b^2) = -(x_1^2 - a^2)$$

$$\therefore x_1^2 + y_1^2 = a^2 - b^2$$

Which is the equation of standard circle with Centre at origin and radius $\sqrt{a^2 - b^2}$ $(a > b)$. This is called the director circle of the hyperbola.

7.3.8 Auxiliary Circle, Director Circle

The director circle of the given hyperbola is the locus of a point, the tangents from which to the hyperbola are at right angles.

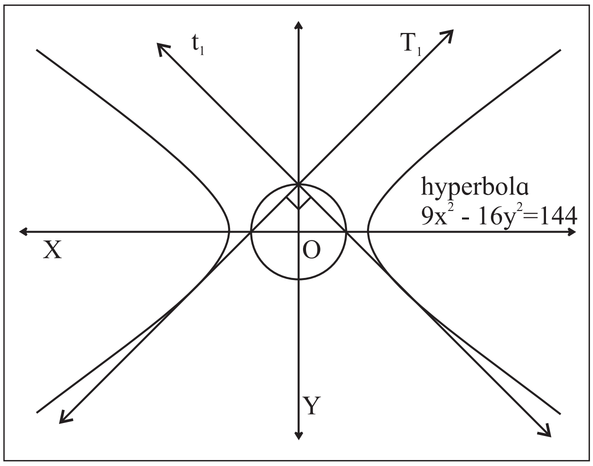


Fig. 7.29

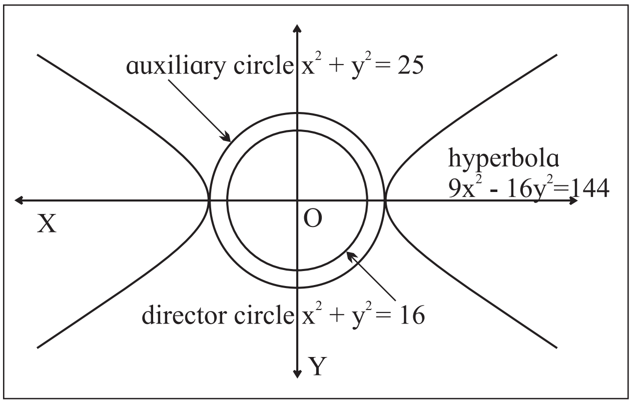


Fig. 7.30

For the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the circle drawn with transverse axis as a diameter is called the auxillary circle of the hyperbola and its equation is $x^2 + y^2 = a^2$.

The locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called the director circle of the hyperbola and its equation is $x^2 + y^2 = a^2 - b^2$. $(a > b)$.

auxillary circle $x^2 + y^2 = 25$;
director circle $x^2 + y^2 = 16$ of the
hyperbola $9x^2 - 16y^2 = 144$. (Refer fig 7.30)

7.3.9 Asymptote:

Consider the lines $\frac{x}{a} = \pm\frac{y}{b}$, they pass through origin O.

Consider the point P moving along the line so that distance OP goes on increasing, then the distance between P and hyperbola goes on decreasing but does not become zero. Here the distance between the point P and hyperbola is tending to zero, such a straight line is called an asymptote for the hyperbola

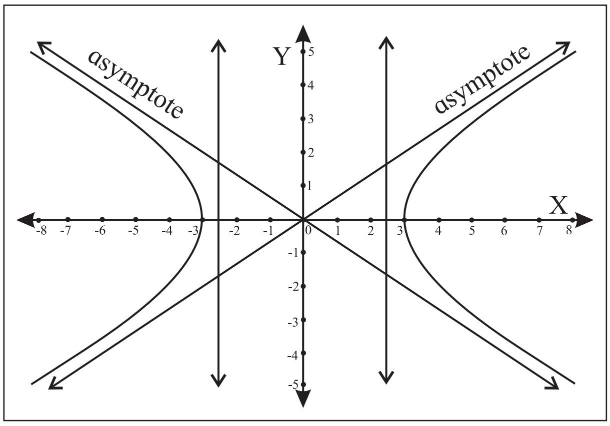


Fig. 7.31

SOLVED EXAMPLES

Ex.1 : Find the equation of the tangent to the hyperbola $2x^2 - 3y^2 = 5$ at a point in the third quadrant whose abscissa is -2.

Solution : Let $P(-2, y_1)$ be the point on the hyperbola

$\therefore\ 2(-2)^2 - 3y^2 = 5$

$\therefore\ 8 - 5 = 3y^2$

$\therefore\ 3 = 3y^2$

$\therefore\ y^2 = 1$

$\therefore\ y = \pm 1$

But P lies in the third quadrant.

$\therefore\ P \equiv (-2, -1)$

The equation of the hyperbola is

$\frac{x^2}{5/2} - \frac{y^2}{5/3} = 1$. Comparing this with the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have

$a^2 = \frac{5}{2}$, $b^2 = \frac{5}{3}$

The equation of tangent at

$P(x_1, y_1) \equiv P(-2. -1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

$\therefore\ 2xx_1 - 3yy_1 = 5$

$\therefore\ 2x(-2) - 3y(-1) = 5$

$\therefore\ -4x + 3y = 5$

$\therefore\ 4x - 3y + 5 = 0$

Ex. 2 : Show that the line $4x - 3y = 16$ touches the hyperbola $16x^2 - 25y^2 = 400$. Find the co-ordinates of the point of contact.

Solution : The equation of the hyperbola is $\frac{x^2}{25}-\frac{y^2}{16}=1$.

Comparing it with the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

we have $a^2 = 25$, $b^2 = 16$

The equation of the line is $3y = 4x - 16$

$\therefore\ y=\frac{4}{3}x-\frac{16}{3}$

$\therefore$ Comparing it with the equation

$y = mx + c$

We get $m = \frac{4}{3}$, $c = \frac{-16}{3}$

Now

$$a^2m^2 - b^2 = 25\left(\frac{16}{9}\right) - 16$$

$$=\frac{256}{9}=\left(\frac{-16}{3}\right)^2=c^2$$

Thus the condition of tangency is satisfied.

$\therefore$ the given line touches the hyperbola.

Let $P(x_1, y_1)$ be the point of contact.

$$\therefore\ x_1 = \frac{-a^2m}{c}=\frac{-25\left(\frac{4}{3}\right)}{\left(-\frac{16}{3}\right)}=\frac{25}{4}$$

and $y_1 = \frac{-b^2}{c} = \frac{-16}{-16/3} = 3$

$\therefore$ the point of contact is $P\left(\frac{25}{4}, 3\right)$

Ex. 3 : If the line $2x + y + k = 0$ is tangent to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ then find the value of k.

Solution : The equation of the hyperbola is $\frac{x^2}{6} - \frac{y^2}{8} = 1$.

The equation of the line is $2x + y + k = 0$

$\therefore\ y = -2x - k$

Putting this value of y in the equation of hyperbola, we get

$$\frac{x^2}{6} - \frac{(-2x-k)^2}{8} = 1$$

$\therefore\ 4x^2 - 3(4x^2 + 4kx + k^2) = 24$

$\therefore\ 4x^2 - 12(4x^2 - 12kx - 3k^2) = 24$

$\therefore\ 8x^2 + 12kx + (3k^2 + 24) = 0$ ....(1)

Since given line touches the hyperbola

$\therefore$ the quadratic equation (1) in x has equal roots.

$\therefore$ its discriminant = 0 i.e. $b^2 - 4ac = 0$

$\therefore\ (12k)^2 - 4(8)(3k^2 + 24) = 0$

$\therefore\ 144k^2 - 32(3k^2 + 24) = 0$

$\therefore\ 9k^2 - 2(3k^2 + 24) = 0$

$\therefore\ 9k^2 - 6k^2 - 48 = 0$

$\therefore\ 3k^2 = 48$

$\therefore\ k^2 = 16$

$\therefore\ k^2 = \pm 4$

Another Method :

Here $c = -k$, $m = -2$, $a^2 = b$, $b^2 = 8$

$\therefore c^2 = a^2m^2 - b^2$

$\therefore k^2 = [(-2)^2 , 6] = 8$

$\therefore k^2 = 16$

$\therefore k^2 = \pm 4$

Ex.4 : The line $x - y + 3 = 0$ touches the hyperbola whose foci are $(\pm\sqrt{41},\ 0)$. Find the equation of the hyperbola.

Solution : Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{.................(1)}$$

Its foci $(\pm ae, 0)$ are $(\pm\sqrt{41},\ 0)$

$\therefore ae = \sqrt{41}$

$\therefore a^2 e^2 = 41$

$\therefore a^2 + b^2 = 41$..............(2) $(a^2e^2 = a^2+b^2)$

The given line is $y = x + 3$

Comparing this with $y = mx + c$, we get

$m = 1$, $c = 3$

Since the given line touches the hyperbola

$\therefore$ It satisfies the condition of tangency.

$\therefore a^2m^2 - b^2 = c^2$

$\therefore a^2(1)^2 - b^2 = (3)^2$

$\therefore a^2 - b^2 = 9$...................(3)

By adding (2) and (3), we get

$2a^2 = 50 \quad \therefore a^2 = 25$

from (2), we get

$25 + b^2 = 41 \ \therefore b^2 = 16$

From (1), the equation of the required hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

EXERCISE 7.3

1) Find the length of transverse axis, length of conjugate axis, the eccentricity, the co-ordinates of foci, equations of directrices and the length of latus rectum of the hyperbola.

   i) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

   ii) $\frac{x^2}{25} - \frac{y^2}{16} = -1$

   iii) $16x^2 - 9y^2 = 144$

   iv) $21x^2 - 4y^2 = 84$

   v) $3x^2 - y^2 = 4$

   vi) $x^2 - y^2 = 16$

   vii) $\frac{y^2}{25} - \frac{x^2}{9} = 1$

   viii) $\frac{y^2}{25} - \frac{x^2}{144} = 1$

   ix) $\frac{x^2}{100} - \frac{y^2}{25} = +1$

   (x) $x = 2 \sec \theta$, $y = 2\sqrt{3} \tan \theta$

2) Find the equation of the hyperbola with centre at the origin, length of conjugate axis 10 and one of the foci (–7,0).

3) Find the eccentricity of the hyperbola, which is conjugate to the hyperbola $x^2 - 3y^2 = 3$.

4) If e and e' are the eccentricities of a hyperbola and its conjugate hyperbola respectively, prove that $\frac{1}{e^2} + \frac{1}{(e')^2} = 1$

5) Find the equation of the hyperbola referred to its principal axes.

   i) whose distance between foci is 10 and eccentricity $\frac{5}{2}$.

   ii) whose distance between foci is 10 and length of conjugate axis 6.

   iii) whose distance between directrices is $\frac{8}{3}$ and eccentricity is $\frac{3}{2}$.

iv) whose length of conjugate axis = 12 and passing through $(1_1 -2)$.

v) which passes through the points (6,9) and (3,0).

vi) whose vertices are (±7,0) and end points of conjugate axis are (0, ±3).

vii) whose foci are at (±2,0) and eccentricity $\frac{3}{2}$.

viii) whose length of transverse and conjugate axis are 6 and 9 respectively.

ix) whose length of transverse axis is 8 and distance between foci is 10.

6) Find the equation of the tangent to the hyperbola.

i) $3x^2 - y^2 = 4$ at the point $(2,2\sqrt{2})$.

ii) $3x^2 - 4y^2 = 12$ at the point (4,3).

iii) $\frac{x^2}{144} - \frac{y^2}{25} = 1$ at the point whose eccentric angle is $\frac{\pi}{3}$.

iv) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point in a first quadratures whose ordinate is 3.

v) $9x^2 - 16y^2 = 144$ at the point L of latus rectum in the first quadrant.

7) Show that the line $3x - 4y + 10 = 0$ is tangent till the hyperbola $x^2 - 4y^2 = 20$. Also find the point of contact.

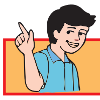
8) If the $3x - 4y = k$ touches the hyperbola $\frac{x^2}{5} - \frac{4y^2}{5} = 1$ then find the value of k.

9) Find the equations of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ making equal intercepts on the co-ordinate axes.

10) Find the equations of the tangents to the hyperbola $5x^2 - 4y^2 = 20$ which are parallel to the line $3x + 2y + 12 = 0$.

Conic section	Eccentricity	Equation of the curve $(x, f(x))$	Equation of tangent at point (x_1, y_1) on it	Point of contact of the target	Condition for tangency
circle	–	$x^2 + y^2 = a^2$	$xx_1 + yy_1 = a^2$	–	$c^2 = a^2m^2 + a^2$
parabola	e=1	$y^2 = 4ax$	$yy_1 = 2a(x_1 - y_1)$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$c = \frac{a}{m}$
ellipse	0<e<1	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\left(\frac{-a^2m}{c}, \frac{+b^2}{c}\right)$	$c^2 = a^2m^2 + b^2$
hyperbola	e>1	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$	$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$	$c^2 = a^2m^2 - b^2$

Curve	Equation of auxillary circle	Equation of director circle
$x^2 + y^2 = a^2$ (circle)	--	$x^2 + y^2 = a^2 + a^2$
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$	$x^2 + y^2 = a^2$	$x^2 + y^2 = a^2 + b^2$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$	$x^2 + y^2 = a^2$	$x^2 + y^2 = a^2 - b^2$



Let's Remember

- A conic section or a conic can be defined as the locus of the point P in a plane such that the ratio of the distance of P, from a fixed point to its distance from a fixed line is constant.

  The constant ratio is called the eccentricity of the conic section, denoted by 'e'.
- If e = 1 the conic section is called parabola if $0 < e < 1$ the conic section is called ellipse. if $e > 1$ the conic section is called hyperbola.
- eccentricity of rectangular hyperbola is $\sqrt{2}$.
- standard equations of curve.

  parabola $y^2 = 4ax$, $x^2 = 4by$

  ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (a>b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (b>a)

  hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \; a > b$,

  $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1 \; b > a$
- focal distance of a point P on the parabola $y^2 = 4ax$ is a abscissa of point P.
- sum of focal distances of point on the ellipse is the length of major axis.
- Difference between the focal distances of point on the hyperbola is the length of transverse axis.

MISCELLANEOUS EXERCISE - 7

(I) Select the correct option from the given alternatives.

1) The line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ if m is ....

   A) 1 B) 2 C) 3 D) 4

2) The length of latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is.........

   A) 4 B) 6 C) 8 D) 10

3) If the focus of the parabola is (0,–3) its directrix is $y = 3$ then its equation is

   A) $x^2 = -12y$ B) $x^2 = 12y$
   C) $y^2 = 12x$ D) $y^2 = -12x$

4) The coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4 are ....

   A) $(1/2, \pm 2)$ B) $(1, \pm 2\sqrt{2})$
   C) $(2, \pm 4)$ D) none of these

5) The end points of latus rectum of the parabola $y^2 = 24x$ are...........

   A) $(6, \pm 12)$ B) $(12, \pm 6)$
   C) $(6, \pm 6)$ D) none of these

6) Equation of the parabola with vertex at the origin and diretrix $x + 8 = 0$ is..............

   A) $y^2 = 8x$ B) $y^2 = 32x$
   C) $y^2 = 16x$ D) $x^2 = 32y$

7) The area of the triangle formed by the line joining the vertex of the parabola $x^2 = 12y$ to the end points of its latus rectum is...........

   A) 22 sq.units B) 20 sq.units
   C) 18 sq.units D) 14 sq.units

8) If $P\left(\dfrac{\pi}{4}\right)$ is any point on he ellipse $9x^2 + 25y^2 = 225$. S and S^1 are its foci then $SP.S^1P =$

   A) 13 B) 14 C) 17 D) 19

9) The equation of the parabola having (2, 4) and (2 , –4) as end points of its latus rectum is..............................

   A) $y^2 = 4x$ B) $y^2 = 8x$
   C) $y^2 = -16x$ D) $x^2 = 8y$

10) If the parabola $y^2 = 4ax$ passes through (3, 2) then the length of its latus rectum is…………………….…………

A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) $\frac{1}{3}$ D) 4

11) The eccentricity of rectangular hyperbola is

A) ½ B) 1 / (2 ½)

C) 2 ½ D) 1 / (3 ½)

12) The equation of the ellipse having foci (+ 4, 0) and eccentricity $\frac{1}{3}$ is,

A) $9x^2 + 16y^2 = 144$

B) $144x^2 + 9y^2 = 1296$

C) $128x^2 + 144y^2 = 18432$

D) $144x^2 + 128y^2 = 18432$

13) The equation of the ellipse having eccentricity $\frac{\sqrt{3}}{2}$ and passing through (− 8 , 3) is

A) $4x^2 + y^2 = 4$ B) $x^2 + 4y^2 = 100$

C) $4x^2 + y^2 = 100$ C) $x^2 + 4y^2 = 4$

14) If the line $4x - 3y + k = 0$ touches the ellipse $5x^2 + 9y^2 = 45$ then the value of k is

A) + 21 B) $\pm 3\sqrt{21}$ C) + 3 D) + 3 (21)

15) The equation of the ellipse is $16x^2 + 25y^2 = 400$. The equations of the tangents making an angle of 180^0 with the major axis are

A) $x = 4$ B) $y = \pm 4$ C) $x = -4$ D) $x = \pm 5$

16) The equation of the tangent to the ellipse $4x^2 + 9y^2 = 36$ which is perpendicular to the $3x + 4y = 17$ is,

A) $y = 4x + 6$ B) $3y + 4x = 6$

C) $3y = 4x + 6\sqrt{5}$ C) $3y = x + 25$

17) Eccentricity of the hyperbola $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ is

A) $\sqrt{\frac{17}{3}}$ B) $\sqrt{\frac{19}{3}}$ C) $\frac{\sqrt{19}}{3}$ D) $\frac{\sqrt{17}}{3}$

18) Centre of the ellipse $9x^2 + 5y^2 - 36x - 50y - 164 = 0$ is at

A) (2, 5) B) (1, −2) C) (−2, 1) D) (0, 0)

19) If the line $2x - y = 4$ touches the hyperbola $4x^2 - 3y^2 = 24$, the point of contact is

A) (1, 2) B) (2, 3) C) (3, 2) D)(−2, −3)

20) The foci of hyperbola $4x^2 - 9y^2 - 36 = 0$ are

A) $(\pm\sqrt{13}\,, 0)$ B) $(\pm\sqrt{11}\,, 0)$

C) $(\pm\sqrt{12}\,, 0)$ S) $(0\,, \pm\sqrt{12}\,)$

(II) Answer the following.

1) For each of the following parabolas, find focus, equation of the directrix, length of the latus rectum, and ends of the latus rectum. (i) $2y^2 = 17x$ (ii) $5x^2 = 24y$.

2) Find the Cartesian co-ordinates of the points on the parabola $y^2 = 12x$ whose parameters are (i) 2, (ii) −3.

3) Find the co-ordinates of a point of the parabola $y^2 = 8x$ having focal distance 10.

4) Find the equation of the tangent to the parabola $y^2 = 9x$ at the point (4, −6) on it.

5) Find the equation of the tangent to the parabola $y^2 = 8x$ at t = 1 on it.

6) Find the equations of the tangents to the parabola $y^2 = 9x$ through the point (4,10).

7) Show that the two tangents drawn to the parabola $y^2 = 24x$ from the point (−6,9) are at the right angle.

8) Find the equation of the tangent to the parabola $y^2 = 8x$ which is parallel to the line $2x + 2y + 5 = 0$. Find its point of contact.

9) A line touches the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$. Show that its equation is $y = \pm (x+2)$.

10) Two tangents to the parabola $y^2 = 8x$ meet the tangent at the vertex in P and Q. If PQ = 4, prove that the locus of the point of intersection of the two tangents is $y^2 = 8(x + 2)$.

11) The slopes of the tangents drawn from P to the parabola $y^2 = 4ax$ are m_1 and m_2, show that (i) $m_1 - m_2 = k$ (ii) $(m_1/m_2) = k$, where k is a constant.

12) The tangent at point P on the parabola $y^2 = 4ax$ meets the y − axis in Q. If S is the focus, show that SP subtends a right angle at Q.

13) Find the (i) lengths of the principal axes (ii) co-ordinates of the foci (iii) equations of directrices (iv) length of the latus rectum (v) Distance between foci (vi) distance between directrices of the curve (a) $x^2/25 + y^2/9 = 1$ (b) $16x^2 + 25y^2 = 400$ (c) $x^2/144 - y^2/25 = 1$ (d) $x^2 - y^2 = 16$

14) Find the equation of the ellipse in standard form if (i) eccentricity = 3/8 and distance between its foci=6. (ii) the length of major axis 10 and the distance between foci is 8. (iii) passing through the points (–3, 1) and (2, –2).

15) Find the eccentricity of an ellipse if the distance between its directrices is three times the distance between its foci.

16) For the hyperbola $x^2/100 - y^2/25 = 1$, prove that SA. S'A = 25, where S and S' are the foci and A is the vertex.

17) Find the equation of the tangent to the ellipse $x^2/5 - y^2/4 = 1$ passing through the point (2,–2).

18) Find the equation of the tangent to the ellipse $x^2 + 4y^2 = 100$ at (8,3).

19) Show that the line $8y + x = 17$ touches the ellipse $x^2 + 4y^2 = 17$. Find the point of contact.

20) Tangents are drawn through a point P to the ellipse $4x^2 + 5y^2 = 20$ having inclinations θ_1 and θ_2 such that $\tan\theta_1 + \tan\theta_2 = 2$. Find the equation of the locus of P.

21) Show that the product of the lengths of its perpendicular segments drawn from the foci to any tangent line to the ellipse $x^2/25 + y^2/16 = 1$ is equal to 16.

22) Find the equation of the hyperbola in the standard form if (i) Length of conjugate axis is 5 and distance between foci is 13. (ii) eccentricity is 3/2 and distance between foci is 12. (iii) length of the conjugate axis is 3 and distance between the foci is 5.

23) Find the equation of the tangent to the hyperbola, (i) $7x^2 - 3y^2 = 51$ at (–3, –2) (ii) $x = 3\sec\theta, y = 5\tan\theta$ at $\theta = \pi/3$ (iii) $x^2/25 - y^2/16 = 1$ at P(30°).

24) Show that the line $2x - y = 4$ touches the hyperbola $4x^2 - 3y^2 = 24$. Find the point of contact.

25) Find the equations of the tangents to the hyperbola $3x^2 - y^2 = 48$ which are perpendicular to the line $x + 2y - 7 = 0$

26) Two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make angles θ_1, θ_2, with the transverse axis. Find the locus of their point of intersection if $\tan\theta_1 + \tan\theta_2 = k$.