Chapter 2: Applications of Derivatives

EXERCISE 2.1 [PAGE 72]

Exercise 2.1 | Q 1.1 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : y = x^2 + $2e^x$ 2 at (0, 4)

SOLUTION

y = x^2 + $2e^x$ 2

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 2e^x + 2\right)$$

= 2x + 2 x e^x +0

= 2x + $2e^x$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(0,4)} = 2(0) + 2e^0 = 2$$

= slope othe tangent at (0, 4)

∴ the equation of the tangent at (0, 4) is

y – 4 = 2(x –0)

∴ y – 4 = 2x

∴ 2x – y + 4 = 0

The slope of the normal at (0, 4)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(0,4)}} = -\frac{1}{2}$$

∴ the equaion oof the normal at (0, 4) is y – 4 = $-\frac{1}{2}(x-0)$

∴ 2y – 8 = – x

∴ x + 2y – 8 = 00

Hence, the equations of tangent and normal are

2x – y + 4 = 0 and x + 2y – 8 = 0 respectively.

Exercise 2.1 | Q 1.2 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : $x^3 + y^3 - 9xy = 0$ at (2, 4)

SOLUTION

$x^3 + y^3 - 9xy = 0$

Differentiating both sides w.r.t. x, we get

$3x^2 + 3y^2\frac{dy}{dx} - 9\left[x\frac{dy}{dx} + y.\frac{d}{dx}(x)\right] = 0$

$\therefore 3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9yx1 = 0$

$\therefore \left(3y^2 - 9x\right)\frac{dy}{dx} = 9y - 3x^2$

$\therefore \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(2,4)} = \frac{9(4) - 3(2)^2}{3(4)^2 - 9(2)}$

$= \frac{36 - 12}{48 - 18}$

$= \frac{24}{30}$

$= \frac{4}{5}$

= slope of the tangent at (2, 4)

∴ the equation of the tangent at (2, 4) is

$$y - 4 = \frac{4}{5}(x - 2)$$

∴ 5y – 20 = 4x – 8

∴ 4x – 5y + 12 = 0

The slope of normal at (2, 4)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(2,4)}}$$

$$= -\frac{5}{4}$$

∴ the equation of the normal at (2, 4) is

$$y - 4 = -\frac{5}{4}(x - 2)$$

∴ 4y – 16 = – 5x + 10

∴ 5x + 4y – 26 = 0

Hence, the equation of tangent and normal are

4x – 5y + 12 = 0 and 5x + 4y – 26 = 0 respectively.

Exercise 2.1 | Q 1.3 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them :

$$x^2 - \sqrt{3}xy + 2y^2 = 5\text{at}\left(\sqrt{3}, 2\right)$$

SOLUTION

$x^2 - \sqrt{3}xy + 2y^2$ = 5

Differentiating both sides w.r.t. x, we get

$2x - \sqrt{3}\left[x\frac{dy}{dx} + y.\frac{d}{dx}(x)\right] + 2 \times 2y\frac{dy}{dx}$ = 0

$\therefore$ $2x = \sqrt{3}x\frac{dy}{dx} = \sqrt{3}y \times 1 + 4y\frac{dy}{dx}$ = 0

$\therefore$ $\left(4y - \sqrt{3}x\right)\frac{dy}{dx} = \sqrt{3}y - 2x$

$\therefore$ $\frac{dy}{dx} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x}$

= $\frac{2x - \sqrt{3}x}{\sqrt{3}x - 4y}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{\text{at}\left(\sqrt{3},2\right)} = \frac{2\sqrt{3} - \sqrt{3}(2)}{\sqrt{3}\left(\sqrt{3}\right) - 4(2)}$ = 0

= slope of the tangent at $\left(\sqrt{3}, 2\right)$

$\therefore$ the equation of the tangent at $\left(\sqrt{3}, 2\right)$ is

y – 2 = $0\left(x - \sqrt{3}\right)$

$\therefore$ y – 2 = 0

$\therefore$ y = 2

The slope of normal at $\left(\sqrt{3}, 2\right)$

= $\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}\left(\sqrt{3},2\right)}}, \text{where}\left(\frac{dy}{dx}\right)_{\text{at}\left(\sqrt{3},2\right)}$ = 0

$\therefore$ the slope of normal at $\left(\sqrt{3}, 2\right)$ does not exist.

∴ equation of the normal is of the form x = k

Since, it pases through the point $\left(\sqrt{3}, 2\right), k = \sqrt{3}$

∴ equation of the normal is x = $\sqrt{3}$.

Hence, the equations of tangent and normal are
y = 2 and x = $\sqrt{3}$ respectively.

Exercise 2.1 | Q 1.4 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : 2xy + π sin y = 2π at $\left(1, \frac{\pi}{2}\right)$

SOLUTION

2xy + π sin y = 2π
Differentiating both sides w.r.t. x, we get

$$2\left[x\frac{dy}{dx} + y.\frac{d}{dx}(x)\right] + \pi\cos y\frac{dy}{dx} = 0$$

$$\therefore 2x\frac{dy}{dx} + 2y \times 1 + \pi\cos y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2y}{2x + \pi\cos y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}\left(1, \frac{\pi}{2}\right)} = \frac{-2\left(\frac{\pi}{2}\right)}{2(1) + \pi\cos\ \frac{\pi}{2}}$$

$$= \frac{-\pi}{2 + \pi(0)}$$

$$= -\frac{\pi}{2}$$

= slope of the tangent at $\left(1, \frac{\pi}{2}\right)$

$\therefore$ the equation of the tangent at $\left(1, \frac{\pi}{2}\right)$ is

$$y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)$$

$\therefore \pi y - \frac{\pi^2}{2} = 2x - 2$

$\therefore 2\pi y - \pi^2 = 4x - 4$

$\therefore 4x - 2\pi y + \pi^2 - 4 = 0$

Hence, the equations of tangent and normal are

$\pi x + 2y - 2\pi = 0$ and $4x - 2\pi y + \pi^2 - 4 = 0$ respectively.

Exercise 2.1 | Q 1.5 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : x sin 2y = y cos 2x at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

SOLUTION

x sin 2y = y cos 2x

Differentiating both sides w.r.t. x, we get

$$x\frac{d}{dx}(\sin 2y) + \sin 2y.\ \frac{d}{dx}(x) = y.\ \frac{d}{dx}(\cos 2x) + \cos 2x.\ \frac{dy}{dx}$$

$$= x.\cos 2y.\ \frac{d}{dx}(2y) + \sin 2y \times 1 = y.(-\sin 2x).\ \frac{d}{dx}(2x) + \cos 2x.\ \frac{dy}{dx}$$

$$\therefore x\cos 2y \times 2\frac{dy}{dx} + \sin 2y = -y\sin 2x \times 2 + \cos 2x.\ \frac{dy}{dx}$$

$$\therefore (2x\cos 2y - \cos 2x)\frac{dy}{dx} = -2y\sin 2x - \sin 2y$$

$$\therefore \frac{dy}{dx} = \frac{-2y\sin 2x - \sin 2y}{2x\cos 2y - \cos 2x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}\left(\frac{\pi}{4},\frac{\pi}{2}\right)} = \frac{-2\left(\frac{\pi}{2}\right)\sin\ \frac{\pi}{2} - \sin\pi}{2\left(\frac{\pi}{4}\right)\cos\pi - \cos\ \frac{\pi}{2}}$$

$$= \frac{-\pi(1) - 0}{\frac{\pi}{2}(-1) - 0}$$

$$= \frac{-\pi}{\left(\frac{-\pi}{2}\right)}$$

= 2

= slope of the tanget at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

∴ the equation of the tangent at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is

$$\therefore y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$$

$$\therefore y - \frac{\pi}{2} = 2x - \frac{\pi}{2}$$

∴ 2x – y = 0

The slope of normal at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}\left(\frac{\pi}{4},\frac{\pi}{2}\right)}}$$

$$= \frac{-1}{2}$$

∴ the equation of te noral at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is

$$y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$\therefore 2y - \pi = -x + \frac{\pi}{4}$$

$$\therefore 8y - 4\pi = -4x + \pi$$

∴ 4x + 8y – 5π = 0

Hence, the equation of the tangent and normal are

2 – y = 0 and 4x + 8y – 5π = 0 respectively.

Exercise 2.1 | Q 1.6 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : x = sin θ and y = cos 2θ at θ = π/6

SOLUTION

When $\theta = \frac{\pi}{6}, x = \sin \frac{\pi}{6}$ and $y = \cos \frac{\pi}{3}$

$$\therefore x = \frac{1}{2} \text{ and } y = \frac{1}{2}$$

Hence, the point at which we want to find the equations of tangent and normal is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Now, x = sinθ, y = cos 2θ

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\cos 2\theta) = -\sin 2\theta . \frac{d}{d\theta}(2\theta)$$

= – sin 2θ x 2

= – 2 sin 2θ

$$\therefore \frac{dx}{dy} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$= \dfrac{-2\sin 2\theta}{\cos\theta}$

$\therefore \left(\dfrac{dy}{dx}\right)_{at\theta=\frac{\pi}{6}} = \dfrac{-2\sin\ \frac{\pi}{3}}{\cos\ \frac{\pi}{6}}$

$= \dfrac{-2\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}$

= − 2

= slope of the tangent at θ = $\dfrac{\pi}{6}$

∴ the equation of the tangent at θ = $\dfrac{\pi}{6}$i.e. at$\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is

$y - \dfrac{1}{2} = -2\left(x - \dfrac{1}{2}\right)$

$\therefore y - \dfrac{1}{2} = -2x + 1$

∴ 2y − 1 = − 4x + 2

∴ 4x+ 2y − 3 = 0

The slope of normal at θ = $\dfrac{\pi}{6}$

$= -\dfrac{1}{\left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{6}}}$

$= \dfrac{-1}{-2} = \dfrac{1}{2}$

∴ equation of the normal at θ = $\dfrac{\pi}{6}$, i.e at$\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is

$y - \dfrac{1}{2} - \dfrac{1}{2}\left(x - \dfrac{1}{2}\right)$

$\therefore 2y - 1 = x - \frac{1}{2}$

$\therefore$ 4y – 2 = 2x – 1

$\therefore$ 2x – 4y + 1 = 0

Hence, equations of the tangent and normal are

4x + 2y – 3 = 0 and 2x – 4y + 1 = 0 respectively.

Exercise 2.1 | Q 1.7 | Page 72

Find the equations of tangents and normals to the following curves at the indicated points on them : $x = \sqrt{t}, y = t - \frac{1}{\sqrt{t}}$ at = 4.

SOLUTION

When t = 4, x = $\sqrt{4}$ and $y = 4 - \frac{1}{\sqrt{4}}$

$\therefore\ x = 2$ and $y = 4 - \frac{1}{2} = \frac{7}{2}$

Hence, the point at which we want to find the equations of tangent and normal is $\left(2, \frac{7}{2}\right)$.

Now, $x = \sqrt{t}, y = t - \frac{1}{\sqrt{t}}$

Differentiating x and y w.r.t. t, we get

$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{t}\right) = \frac{1}{2\sqrt{t}}$

and $\frac{dy}{dx} = \frac{d}{dt}\left(t - \frac{1}{\sqrt{t}}\right)$

= $\frac{1}{-\frac{1}{2}} t^{-\frac{3}{2}}$

= $1 + \frac{1}{2t^{\frac{3}{2}}}$

$= \dfrac{2t^{\frac{3}{2}} + 1}{2t^{\frac{3}{2}}}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

$= \dfrac{\left(\frac{2t^{\frac{3}{2}+1}}{2t^{\frac{3}{2}}}\right)}{\left(\frac{1}{2\sqrt{t}}\right)}$

$= \dfrac{2t^{\frac{3}{2}} + 1}{2t^{\frac{3}{2}}} \times 2\sqrt{t}$

$= \dfrac{2t^{\frac{3}{2}} + 1}{t}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\mathrm{at}t=4} = \dfrac{2(4)^{\frac{3}{2}} + 1}{4}$

$= \dfrac{2 \times 8 + 1}{4}$

$= \dfrac{17}{4}$

∴ the equation of the normal at t = 4, i.e. at $\left(2, \dfrac{7}{2}\right)$ is

$y - \dfrac{7}{2} = -\dfrac{4}{17}(x - 2)$

∴ 34y – 119 = – 8x + 16

∴ 8x + 34y – 135 = 0

Hence,, the equations of tangent and normal are

17x – 4y – 20 = 0 and 8x + 34y – 135 = 0 respectively.

Exercise 2.1 | Q 2 | Page 72

Find the point of the curve $y = \sqrt{x - 3}$ where the tangent is perpendicular to the line 6x + 3y – 5 = 0.

SOLUTION

Let the required point on the curve $y = \sqrt{x-3}$ be P(x_1, y_1).

Differentiating $y = \sqrt{x-3}$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x-3}\right)$$

$$= \frac{2}{2\sqrt{x-3}} \cdot \frac{d}{dx}(x-3)$$

$$= \frac{1}{2\sqrt{x-3} \times (1-0)}$$

$$= \frac{1}{2\sqrt{x-3}}$$

∴ slope of the tangent at (x_1, y_1)

$$= \left(\frac{dy}{dx}\right)_{\text{at}(x_1,y_1)}$$

$$= \frac{1}{2\sqrt{x_1-3}}$$

Since, this tangent is perndicular to 6x + 3y – 5 = 0

whose slope is $\frac{-6}{3}$ = – 2,

slope of the tangent = $\frac{-1}{-2} = \frac{1}{2}$

∴ $\frac{1}{2\sqrt{x_1-3}} = \frac{1}{2}$

∴ $\sqrt{x_1-3}$ = 1

∴ x_1 – 3 = 1

∴ x_1 = 4

$\therefore x_1 = 4$

Since, (x_1, y_1) lies on $y = \sqrt{x-3}, y_1 = \sqrt{x_1 - 3}$

When $x_1 = 4$, $y_1\sqrt{4-3} = \pm 1$

Hence, the quire points are (4, 1) and (4, –1).

Exercise 2.1 | Q 3 | Page 72

Find the points on the curve $y = x^3 - 2x^2 - x$ where the tangents are parallel to $3x - y + 1 = 0$.

SOLUTION

Let the required point on the curve

$y = x^3 - 2x^2 - x$ be $P(x_1, y_1)$.

Differentiating $y = x^3 - 2x^2 - x$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 2x^2 - x\right)$$

$= 3x^2 - 2x\ 2x - 1$

$= 3x^2 - 4x - 1$

$\therefore$ slope of the tangent at (x_1, y_1)

$$= \left(\frac{dy}{dx}\right)_{\text{at}(x_1, y_1)}$$

$= 3{x_1}^2 - 4x_1 - 1$

Since this tangent is parallelto $3x - y + 1 = 0$

where slope is $\frac{-3}{-1} = 3$,

slope of the tangent = 3

$\therefore\ 3{x_1}^2 - 4x_1 - 1 = 3$

$\therefore 3x_1^2 - 4x_1 - 4 = 0$

$\therefore 3x_1^2 - 6x_1 + 2x_1 - 4 = 0$

$\therefore 3x_1 (x_1 - 2) + 2(x_1 - 2) = 0$

$\therefore (x_1 - 2)(3x_1 + 2) = 0$

$\therefore x_1 - 2 \ 0$ or $3x_1 + 2 = 0$

$\therefore x_1 = 2 \ \ x_1 = -\frac{2}{3}$

Since, (x_1, y_1) lies on $y = x^3 - 2x^2 - x, y_1 = x_1^3 - 2x_1^2 - x_1$

When $x_1 = 2$, $y_1 = (2)3 - 2(2)2 - 2 = 8 - 8 - 2 = - 2$

When $x_1 = -\frac{2}{3}, y_1 = \left(\frac{-2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)^2 + \frac{2}{3}$

$= \frac{-8}{27} - \frac{8}{9} + \frac{2}{3}$

$= \frac{-14}{27}$

Hence the rquired points are $(2, -2)$ nd $\left(\frac{-2}{3}, \frac{-14}{27}\right)$.

Exercise 2.1 | Q 4 | Page 72

Find the equation of the tangents to the curve $x^2 + y^2 - 2x - 4y + 1 =0$ which a parallel to the X-axis.

SOLUTION

Let $P(x_1,y_1)$be the point on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ where the tangent is paralle to X-axis.

Differentiating $x^2 + y^2 - 2x - 4y + 1 = 0$ w.r.t. x, we get

$2x + 2y\dfrac{dy}{dx}\ 2 \times 1 - 4\dfrac{dy}{dx} + 0 = 0$

$\therefore (2y - 4)\dfrac{dy}{dx} = 2 - 2x$

$\therefore \dfrac{d}{dx} = \dfrac{2 - 2x}{2y - 4} = \dfrac{1 - x}{y - 2}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\text{at}(x_1, y_1)} = \dfrac{1 - x_1}{y_1 - 2}$

= slope of the tangent at (x_1, y_1)

Since,the tangent is parallel to X-axis,

slope of the tangent 0.

$\therefore \dfrac{1 - x_1}{y_1 - 2} = 0$

$\therefore 1 - x_1 = 0$

$\therefore x_1 = 1$

Since, (x_1, y_1) lies on $x^2 + y^2 - 2x - 4y + 1 = 0$,

$x_1^2 + y_1^2 - 2x_1 - 4y_1 + 1 = 0$

When $x_1 = 1$, $(1)^2 + y_1^2 - 2(1) - 4y_1 + 1 = 0$

$\therefore 1 + y_1^2 - 2 - 4y_1 + 1 = 0$
$\therefore y_1^2 - 4y_1 = 0$

$\therefore y_1(y_1 - 4) = 0$
$\therefore y_1 = 0$ or $y_1 = 4$

$\therefore$ the coordinates of the point are (1, 0) or (1, 4)

Since, the tangents aare parallel to X-axis their equations are of the form $y = k$

If it passes through the point (1, 0), $k = 0$ and if it passes through the point (1, 4), $k = 4$

Hence, te equations of the tanganes are $y = 0$ and $y = 4$.

Exercise 2.1 | Q 5 | Page 72

Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line $x + 3y = 4$.

SOLUTION

Let $P(x_1, y_1)$ be the foot of the required normal to the curve $3x^2 - y^2 = 8$.

Differentiating $3x^2 - y^2 = 8$ w.r.t. x, we get

$$3 \times 2x - 2y\frac{dy}{dx} \quad 0$$

$$\therefore -2y\frac{dy}{dx} = -6x$$

$$\therefore \frac{dy}{dx} = \frac{3x}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(x_1, y_1)}$$

$$= \frac{3x_1}{y_1}$$

= slope of the tangent at (x_1, y_1)

$\therefore$ slope of the normal at $P(x_1, y_1)$

$$= m_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(x_1, y1)}}$$

$$= -\frac{y_1}{3x_1}$$

The slope of the line $x + 3y = 4$ is $m_2 = \frac{-1}{3}$

Since, the normal at $P(x_1, y_1)$ is parallel to the line

$x + 3y = 4$, $m_1 = m_2$

$$\therefore -\frac{y_1}{3x_1} = \frac{1}{3}$$

$\therefore y_1 = x_1$

Since, (x_1, y_1) lies on the curve $3x^2 - y^2 = 8$,

$3x_1^2 - y_1^2 = 8$

$\therefore 3x_1^2 - x_1^2 = 8$ ...[$\because y_1 = x_1$]

$\therefore 2x_1^2 = 8$

$\therefore x_1 = \pm 2$

$\therefore x_1^2 = 4$

When $x_1 = 2$, $y_1 = 2$

When $x_1 = -2$, $y_1 = 2$

$\therefore$ the coordinate of the point P a$(2, 2)$ or$(-2, -2)$

and the slo of the normal is $m_1 = m_2 = \frac{1}{3}$

$\therefore$ the equation of the normaal at $(2, 2)$ is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$\therefore 3y - 6 = -x + 2$

$\therefore x + 3y - 8 = 0$

and the equation of the normal at $(-2, -2)$ is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$\therefore 3y - 6 = -x + 2$

$\therefore x + 3y + 8 = 0$

∴ the equation of the normaal at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

∴ 3y – 6 = – x + 2

∴ x + 3y – 8 = 0

and the equation of the normal at (– 2, – 2) is

$$y + 2 = -\frac{1}{3}(x + 2)$$

∴ 3y – 6 = – x + 2

∴ x + 3y + 8 = 0

Hence, the equations of the normals are

x + 3y – 8 = 0 and x + 3y + 8 = 0.

Exercise 2.1 | Q 6 | Page 72

If the line y = 4x – 5 touches the curves $y^2 = ax^3 + b$ at the point (2, 3), find a and b.

SOLUTION

$y^2 = ax^3 + b$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = a \times 3x^2 + 0$$

$$\therefore \frac{dy}{dx} == \frac{3ax^2}{2y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(2,3)}$$

$$= \frac{3a(2)^2}{2(3)}$$

= 2a

= slope of the tangent at (2, 3)

Since, the line y = 4x – 5 touches the curve at the point (2, 3), slope of the tangent at (2,

3) is 4.
$\therefore$ 2a = 4
$\therefore$ a = 2

Since (2, 3) lies on the curve $y^2 = ax^3 + b$,
$(3)^2 = a(2)^3 + b$
$\therefore$ 9 = 8a + b

$\therefore$ 9 = 8(2) + b ...[$\because$ a = 2]
$\therefore$ b = – 7
Hence, a = 2 and b = – 7.

Exercise 2.1 | Q 7 | Page 72
A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which y-coordinate is changing 8 times as fast as the x-coordinate.

SOLUTION

Let $P(x_1, y_1)$ be the point on the curve $6y = x^3 + 2$ whose y-coordinate is changing 8 times as fast as the coordinate.

Then $\left(\frac{dy}{dx}\right)_{\text{at}(x_1,y_1)} = 8\left(\frac{dx}{dt}\right)_{\text{at}(x_1,y_1)}$ ...(1)

Differentiating $6y = x^3 + 2$ w.r.t. t, we get

$$6\frac{dy}{dt} = \frac{d}{dt}\left(x^3 + 2\right) = 3x^2\frac{dx}{dt} + 0$$

$$\therefore 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

$$\therefore 2\left(\frac{dy}{dt}\right)_{\text{at}(x_1,y_2)} = x_1^2 . \left(\frac{dx}{dt}\right)_{\text{at}(x_1,y_1)}$$

$$\therefore 2 \times 8\left(\frac{dy}{dt}\right)_{\text{at}(x_1,y_2)} = x_1^2 . \left(\frac{dx}{dt}\right)_{\text{at}(x_1,y_1)} \quad \text{...[By (1)]}$$

$\therefore x_1^2 = 16$

$\therefore x_1 = \pm 4$

Now, (x_1, y_2) lies on the curve $6y = x^3 + 2$.

$\therefore 6y_1 = x^3 + 2$

When $x_1 = 4$, $6y_1 = (4)^3 + 2 = 66$

$\therefore y_1 = 11$

When $x_1 = 4$, $6y_1 = (-4)^3 + 2 = -62$

$\therefore y_1 = -\frac{31}{3}$

Hence, the required points on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

Exercise 2.1 | Q 8 | Page 72

A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area increasing, when its radius is 5 cm?

SOLUTION

Let r be the radius and S be the surface area of the soap bubble at any time t.

Then $S = 4\pi r^2$

Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt}$$

$$\therefore \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \qquad ...(1)$$

Now, $\frac{dr}{dt} = \frac{0.02cm}{sec}$ and r = 5 cm

$\therefore$ (1) gives, $\frac{dS}{dt} = 8\pi(5)(0.02)$

$= 0.8\pi$

Hence, the surface area of the soap bubble is increasing at the rate if $\frac{0.8\pi cm^2}{sec}$.

Exercise 2.1 | Q 9 | Page 72

The surface area of a spherical balloon is increasing at the rate of $2cm^2$/sec. At what rate is the volume of the balloon is increasing, when the radius of the balloon is 6 cm?

SOLUTION

Let r be the radius, S be the surface area and V be the volume of the spherical balloon at any time t.

Then S = $4\pi r^2$ and V = $\frac{4}{3}\pi r^3$

Differentiating w.r.t. t, we get

$$\frac{\mathrm{dS}}{\mathrm{dt}} = 4\pi \times 2r\frac{\mathrm{dr}}{\mathrm{dt}} = 8\pi r\frac{\mathrm{dr}}{dt} \quad ...(1)$$

and $\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{4}{3}\pi \times 3r^2\frac{\mathrm{dr}}{\mathrm{dt}} = 4\pi r^2\frac{\mathrm{dr}}{\mathrm{dt}}$

From (1), $\frac{\mathrm{dr}}{\mathrm{dt}} = \frac{1}{8\pi r}.\frac{\mathrm{dS}}{\mathrm{dt}}$

$\therefore \frac{\mathrm{dV}}{\mathrm{dt}} = 4\pi r^2 \times \frac{1}{8\pi r}\frac{\mathrm{dS}}{\mathrm{dt}}$

$\therefore \frac{\mathrm{dV}}{\mathrm{dt}} = \frac{\mathrm{r}}{2}.\frac{\mathrm{dS}}{\mathrm{dt}} \quad ...(2)$

Now, $\frac{\mathrm{dS}}{\mathrm{dt}} = \frac{2\mathrm{cm}^2}{\mathrm{sec}}$ and r = 6cm

$\therefore$ (2) gives, $\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{6}{2} \times 2 = 6$

Hence, the volume of the spherical balloon is increasing at the rate of $\frac{6\mathrm{cm}^3}{\mathrm{sec}}$.

Exercise 2.1 | Q 10 | Page 72

If each side of an equilateral triangle increases at the rate of √2cm/sec, find the rate of increase of its area when its side of length 3 cm.

SOLUTION

If x cm is the side of the equilateral triangle and A is its area, then

$$A = \frac{\sqrt{3}}{4}x^2$$

Differentiating w.r.t. t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x\frac{dx}{dt} = \frac{\sqrt{3}}{2}.x\frac{dx}{dt} \qquad ...(1)$$

Now, $\frac{dx}{dt} = \frac{\sqrt{2}cm}{sec}$ aand x = 3 cm

$\therefore$ (1) gives, $\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 3 \times \sqrt{2}$

$= \frac{3\sqrt{6}}{2}\frac{cm^2}{sec}$

Hence, rate of increase of the area of equilateral triangle

$= \frac{3\sqrt{6}}{2}\frac{cm^2}{sec}$.

Exercise 2.1 | Q 11 | Page 72

The volume of a sphere increases at the rate of 20cm^3/sec. Find the rate of change of its surface area, when its radius is 5 cm.

SOLUTION

Let r be the radius, S be the surface area and V be the volume of the sphere at any time t.

Then S = $4\pi r^2$ and V = $\frac{4}{3}\pi r^3$

Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}$$

and $\frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt} \qquad ...(1)$

From (1), $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$

$$\therefore \frac{\mathrm{dS}}{\mathrm{dt}} = 8\pi r \times \frac{1}{4\pi r^2}\frac{\mathrm{dV}}{\mathrm{dt}}$$

$$\therefore \frac{\mathrm{dS}}{\mathrm{dt}} = \frac{2}{r} . \frac{\mathrm{dV}}{\mathrm{dt}} \qquad ...(2)$$

Now, $\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{20\mathrm{cm}^3}{\mathrm{sec}}$ and r = 5 cm

$\therefore$ (2) gives, $\frac{\mathrm{dS}}{\mathrm{dt}} = \frac{2}{5} \times \mathbf{20} = 8$

Hence, the surface area of the sphere is changing at the rate $\frac{8\mathrm{cm}^2}{\mathrm{sec}}$.

Exercise 2.1 | Q 12 | Page 72

The edge of a cube is decreasing at the rate of 0.6cm/sec. Find the rate at which its volume is decreasing, when the edge of the cube is 2 cm.

SOLUTION

Let x be the edge of the cube and V be its volume at any time t.
Then V = x^3
Differentiating both sides w.r.t. t, we get

$$\frac{\mathrm{dV}}{\mathrm{dt}} = 3x^2\frac{\mathrm{dx}}{\mathrm{dt}}$$

Now, $\frac{dx}{dt} = \frac{0.6\mathrm{cm}}{\mathrm{sec}}$ and x = 2 cm

$$\therefore \frac{\mathrm{dV}}{dt} = 3(2)^2(0.6)$$

$$= 7.2$$

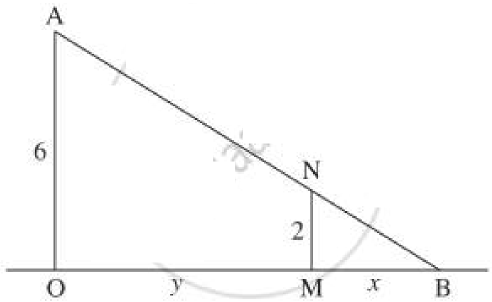
Hence, the volume of the cube is decreasing at the rate of $\frac{7.2\mathrm{cm}^3}{\mathrm{sec}}$.

Exercise 2.1 | Q 13 | Page 72

A man of height 2 metres walks at a uniform speed of 6km/hr away from a lamp post of 6 metres high. Find the rate at which the length of the shadow is increasing.

SOLUTION

Let OA be the lamp post, MN the man, MB = x, his shadow and OM = y, the distance of the man from lamp post at time t.



Then $\frac{dy}{dx} = \frac{6\text{km}}{\text{hr}}$ is the rate at which the man is moving away from the lamp post.

$\frac{dx}{dy}$ is the rate at which his shadow is increasing.

From the figure,

$\frac{x}{2} = \frac{x+y}{6}$

$\therefore 6x = 2x + 2y$

$\therefore 4x = 2y$

$\therefore x = \frac{1}{2}y$

$\therefore \frac{dx}{dt} = \frac{1}{2}\frac{dy}{dx}$

$= \frac{1}{2} \times 6 = \frac{3\text{km}}{\text{hr}}$

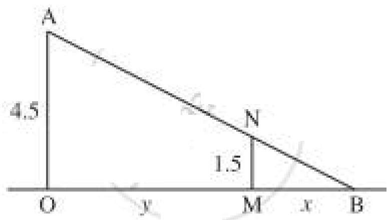
Hence, the length of shadow is increasing at the rate of $\frac{3\text{km}}{\text{hr}}$.

Exercise 2.1 | Q 14 | Page 72

A man of height 1.5 metres walks towards a lamp post of height 4.5 metres, at the rate of $\left(\frac{3}{4}\right)\frac{\text{metre}}{\text{sec.}}$

Find the rate at which
(i) his shadow is shortening
(ii) the tip of shadow is moving.

SOLUTION



Let OA be the lamp post, MN the man, MB = x his shadow and OM = y the distance of the man from lamp post at time t.

Then $\frac{dy}{dx} = \frac{3}{4}$ is the rate at which the man is moving towards the lamp post.

$\frac{dx}{dt}$ is the rate at which his shadow is shortening.

B is the tip of the shadow and it is at a distance of x + y from the post.

$\therefore \frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$ is the rate at which the tip of the shadow is moving.

From the figure,

$$\frac{x}{1.5} = \frac{x+y}{4.5}$$

$\therefore$ 45x = 15x + 15y

$\therefore$ 30x = 15y

$\therefore$ x = $\frac{1}{2}y$

$$\therefore \frac{dx}{dt} = \frac{1}{2} . \frac{dy}{dt} = \frac{1}{2}\left(\frac{3}{4}\right) = \left(\frac{3}{8}\right)\frac{\text{metre}}{\text{sec}}$$

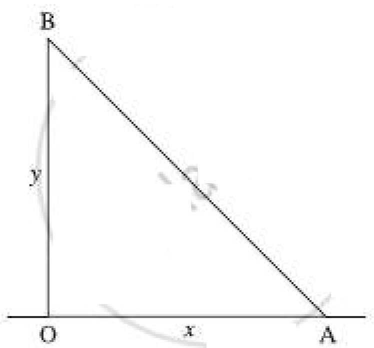
and

$$\frac{dx}{dt} + \frac{dy}{dt} = \frac{3}{8} + \frac{3}{4} = \left(\frac{9}{8}\right)\frac{\text{metres}}{\text{sec}}.$$

Exercise 2.1 | Q 15 | Page 72

A ladder 10 metres long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 metres per second, find how fast the top of the ladder is sliding down the wall, when the bottom is 6 metres away from the wall.

SOLUTION



Let AB be the ladder, where AB = 10 meters. Let at time t seconds, the end A of the ladder be x metres from the wall and the end B be y metres from the ground.

Since, OAB is a right angled triangle, by Pythagoras theorem

$x^2 + y^2 = 10^2$

i.e. $y^2 = 100 - x^2$

Differentiating w.r.t. t, we get

$$2y\frac{dy}{dt} = 0 - 2x\frac{dx}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{x}{y} . \frac{dx}{dt} \qquad ...(1)$$

Now, $\frac{dx}{dy} = \frac{12\text{metres}}{\text{sec}}$ is the rate at wh the bottom of the ladder s pulled horizontally and $\frac{dy}{dx}$ is the rate which the top of ladder B is sliding.

which the top of ladder B is sliding.

When x = 6, y^2 = 100 – 36 = 64

$\therefore$ y = 8

$\therefore$ (1) gives, $\frac{dy}{dx} = -\frac{6}{8}(1.2)$

$= \frac{6}{8} \times \frac{12}{10}$

$= -\frac{9}{10}$

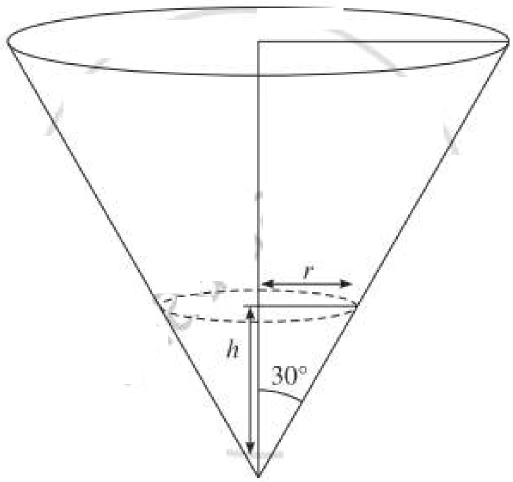
= – 0.9

Hence, the top of the ladder is sliding down the wall, at the rate of $\frac{0.9\text{metre}}{\text{sec}}$.

Exercise 2.1 | Q 16 | Page 72

If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, so that its depth (measured along the axis) increases at the rate of 1cm/sec. Find the rate at which the volume of water increasing when the depth is 2 cm.

SOLUTION



Let r be radius, h be the height, θ be the semi-vertical angle and V be the volume of the water at any time t.

Given : $\frac{\mathrm{dh}}{dt} = \frac{1\mathrm{cm}}{\sec}$, θ = 30°

Now, V = $\frac{1}{3}\pi r^2 h$

But, tan 30° = $\frac{r}{h}$

$\therefore \frac{1}{\sqrt{3}} = \frac{r}{h}$

$\therefore$ r = $\frac{h}{\sqrt{3}}$

$\therefore$ V = $\frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi}{9}h^3$

Differentiating w.r.t. t, we get,

$$\frac{dV}{dt} = \frac{\pi}{9} \times 3h^2 \frac{dh}{dt} = \frac{\pi}{3} h^2 \frac{dh}{dt}$$

When h = 2cm, then

$$\frac{dV}{dt} = \frac{\pi}{3} \times (2)^2 \times 1 = \frac{4\pi}{3}$$

Hence, the volume of water is increasing at the rate of $\left(\frac{4\pi}{3}\right)\frac{\text{cm}^3}{\text{sec}}$.

EXERCISE 2.2 [PAGE 75]

Exercise 2.2 | Q 1.1 | Page 75

Find the approximate values of : $\sqrt{8.95}$

SOLUTION

Let f(x) = $\sqrt{x}$.

Then f'(x) = $\frac{1}{2\sqrt{x}}$.

Take a = 9 and h = – 0.05.

Then f(a) = f(9) = $\sqrt{9}$ = 3 and

f'(a) = f'(9) = $\frac{1}{2\sqrt{9}} = \frac{1}{6}$.

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $\sqrt{8.95}$ = f(9 – 0.05)

≑ f(9) – (0.05)f'(9)

≑ $3 - 0.05 \times \frac{1}{6}$

≑ 3 – 0.0083 = 2.9917

∴ $\sqrt{8.95}$ ≑ 2.9917.

Exercise 2.2 | Q 1.2 | Page 75

Find the approximate values of : $\sqrt[3]{28}$

SOLUTION

Let f(x) = $\sqrt[3]{x}$

Then f'(x) = $\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

Take a = 27 and h = 1.

Then f(a) = f(27) = $\sqrt[3]{27}$ = 3

and f'(a) = f'(27) = $\frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{3 \times 9} = \frac{1}{27}$ = 0.03704

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $\sqrt[3]{28}$

= f(28) = f(27 + 1)

≑ f(27) – 1.f'(27)

≑ 3 + 1 x 0.03704

= 3.03704

∴ $\sqrt[3]{28}$ ≑ 3.03704.

Exercise 2.2 | Q 1.3 | Page 75

Find the approximate values of : $\sqrt[5]{31.98}$

SOLUTION

Let f(x) = $\sqrt[5]{x}$

Then f'(x) = $\frac{d}{dx}\left(x^{\frac{1}{5}}\right)$

$= \frac{1}{5}x^{-\frac{4}{5}}$

$= \frac{1}{5x^{\frac{4}{5}}}$

Take a = 2 and h = – 0.02.

Then f(a) = f(32) = $\sqrt[5]{32}$ = 2

f'(a) = f'(32) = $\frac{1}{5(32)^{\frac{4}{5}}}$

$= \frac{1}{5 \times 16}$

$= \frac{1}{80}$

= 0.0125

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $\sqrt[5]{31.98}$

= f(31.98)

= f(32 – 0.02)

≑ f(32 – 0.02.f'(32)

≑ 2 – 0.02 x 0.0125

≑ 2 – 0.000250

= 1.99975

∴ $\sqrt[5]{31.98}$ ≑ 1.99975.

Exercise 2.2 | Q 1.4 | Page 75

Find the approximate values of : $(3.97)^4$

SOLUTION

Let f(x) = x^4

Then f'(x) = $\frac{d}{dx}\left(x^4\right)$ = $4x^3$

Take a = 4 and h = – 0.03.

Then f(a) = f(4) = $(4)^4$ = 256 and

f'(a) = f'(4) = $4(4)^3$ = 256

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $(3.97)^4$ = f(3.97) = f(4 – 00.03)

≑ f(4) – (0.03)f'(4)

≑ 256 – 0.03 x 256

≑ 256 – 7.68

= 248.32

∴ $(3.97)^4$ ≑ 248.32.

Exercise 2.2 | Q 1.5 | Page 75

Find the approximate values of : $(4.01)^3$

SOLUTION

Let f(x) = x^3.
Then, f'(x) = $3x^2$
Take a = 4 and h = 0.01. Then
f(a) = f(4) = 4^3 = 64 and
f'(a) = f'(4) = 3 x 4^2 = 48.

The formula for approximation is
f(a + h) ≑ f(a) + h.f'(a)
∴ $(4.01)^3$ = f(4 + 0.01)
≑ f(4) + (0.01) f'(4)
≑ 64 + 0.01 x 48
≑ 64 + 0.48

= 64.48
∴ $(4.01)^3 \doteqdot 64.48$.

Exercise 2.2 | Q 2.1 | Page 75

Find the approximate values of : sin 61° , given that 1° = 0.0174°, √3 = 1.732

SOLUTION

Let f(x) = sin x

Then f'(x) = $\frac{d}{dx}(\sin x) = \cos x$

Take a = 60° = $\frac{\pi}{3}$ and $h = 1^{\circ}$ = 0.0174°

Then f(a) = $f\left(\frac{\pi}{3}\right)$

= $\sin \frac{\pi}{3}$

= $\frac{\sqrt{3}}{2}$

= $\frac{1.732}{2}$

= 0.866

and

f'(a) = $f\prime\left(\frac{\pi}{3}\right)$

= $\cos \frac{\pi}{3}$

= $\frac{1}{2}$

= 0.5

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ sin 61° = f(61°)

$= f\left(\frac{\pi}{3} + 0.0174\right)$

$\doteqdot f\left(\frac{\pi}{3}\right) + 0.0174.\, f\prime\left(\frac{\pi}{3}\right)$

≑ 0.866 + 0.0174 x 0.5

≑ 0.866 + 0.00870

= 0.8747

∴ sin 61° ≑ 0.8747.

Exercise 2.2 | Q 2.2 | Page 75

Find the approximate values of : sin (29° 30'), given that 1°= 0.0175°, √3 = 1.732

SOLUTION

Let f'(x) = sin x

Then f'(x) = $\frac{d}{dx}(\sin x) = \cos x$

Take a = 30° = $\frac{\pi}{6}$ and

h = – 30°

$= -\left(\frac{1}{2}\right)^{\circ}$

$= -\left(\frac{1}{2}\right) \times 0.0175$

= – 0.00875

Then f(a) = $f\left(\frac{\pi}{6}\right)$

$= \sin\ \frac{\pi}{6}$

$= \frac{1}{2}$

= 0.5

and

f'(a) = $f\prime\left(\frac{\pi}{6}\right)$

$= \cos\ \frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

$= \frac{1.732}{2}$

= 0.866

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ sin (29° 30°)

= f(29°30°)

$= f\left(\frac{\pi}{6} + 0.00875\right)$

$≑ f\left(\frac{\pi}{6}\right) - (0.00875).\ f\prime\left(\frac{\pi}{6}\right)$

≑ 0.5 – 0.00875 x 0.866

≑ 0.5 – 0.0075775

= 0.4924

π sin (29°30°) ≑ 0.4924.

Exercise 2.2 | Q 2.3 | Page 75

Find the approximate values of : cos(60° 30°), given that 1° = 0.0175°, √3 = 1.732

SOLUTION

Let f(x) = cos x

Then f'(x) = $\frac{d}{dx}(\cos x) = -\sin x$

Take a = 60° = $\frac{\pi}{3}$ and

h = 30°

= $\left(\frac{1}{2}\right)^{\circ}$

= $\left(\frac{1}{2} \times 00175\right)^{\circ}$

= 0.00875°

Then f(a) = $f\left(\frac{\pi}{3}\right)$

= $\cos \frac{\pi}{3}$

= $\frac{1}{2}$

= 0.5

f'(a) = $f\prime\left(\frac{\pi}{3}\right)$

≑ 0.5 + (0.00875) (– 0.8660)

≑ 0.5 – 0.0075775

= 0.4924225

π cos(60° 30°) ≑ 0.4924.

Exercise 2.2 | Q 2.4 | Page 75

Find the approximate values of : tan (45° 40°), given that 1° = 0.0175°.

SOLUTION

Let f(x) = tan x

Then f'(x) = $\dfrac{d}{dx}(\tan x) = \sec^2 x$

Take a = 45°

= $\dfrac{\pi}{4}$

and

h = 40°

= $\left(\dfrac{40}{60} \times 0.0175\right)^{\circ}$

= 0.01167°

Then f(a) = $f\left(\dfrac{\pi}{4}\right)$

= $\tan\ \dfrac{\pi}{4}$

= 1

and

f'(a) = $f\prime\left(\dfrac{\pi}{4}\right)$

= $\sec^2\ \dfrac{\pi}{4}$

= $\left(\sqrt{2}\right)^2$

= 2

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ tan(45° 40°)

= f(45° 40°)

$= f\left(\frac{\pi}{4} + 0.01167\right)$

$\doteqdot f\left(\frac{\pi}{4}\right) + (0.01167).\, f\prime\left(\frac{\pi}{4}\right)$

$\doteqdot$ 1 + 0.01167 x 2

= 1 + 0.02334

= 1.02334

$\therefore$ tan (45° 40°) $\doteqdot$ 1.02334.

Exercise 2.2 | Q 3.1 | Page 75

Find the approximate values of : $\tan^{-1}(0.999)$

SOLUTION

Let f(x) = $\tan^{-1}x$

Then f'(x) = $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Take a = 1 and h = – 0.001

Then f(a) = f(1) = $\tan^{-1}1$ = $\frac{\pi}{4}$

and f'(a) = f'(1) = $\frac{1}{1+1^2} = \frac{1}{2}$

The formula for approximation is

f(a + h) $\doteqdot$ f(a) + h.f'(a)

$\therefore$ $\tan^{-1}$ (0.999)

= f(0.999)

= f(1 – 0.001)

$\doteqdot$ f(1) – (0.001).f'(1)

$\doteqdot \frac{\pi}{4} - 0.001 \times \frac{1}{2}$

$= \frac{\pi}{4} - 0.0005$

$\therefore \tan^{-1}(0999) \doteqdot \frac{\pi}{4} - 0.0005.$

Remark: The answer can also be given as :

$\tan^{-1}(0.999) \doteqdot \frac{3.1416}{4} - 0.0005$

$\doteqdot 0.7854 - 0.0005$

$= 0.7849.$

Exercise 2.2 | Q 3.2 | Page 75

Find the approximate values of : $\cot^{-1}(0.999)$

SOLUTION

Let $f(x) = \cot^{-1} x$

$\therefore f'(x) = \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1 + x^2}$

Take a = and h = – 0.001

Then $f(a) = f(1) = \cot^{-1}1 = \frac{\pi}{4}$

and $f'(a) = f'(1) = \frac{-1}{1 + 1^2} = \frac{-1}{2}$

The formula for appromation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore \cot^{-1}(0.999)$

$= f(0.999)$

$= f(1 - 0.001)$

$\doteqdot f(1) - (0.001).f'(1)$

$\doteqdot \frac{\pi}{4} - (0001).\left(\frac{-1}{2}\right)$

$= \frac{\pi}{4} + 0.005$

$\therefore \cot^{-1}(0.999) \doteqdot \frac{\pi}{4} + 0.0005.$

Remark: The answer can also be given as :

$\cot^{-1}(0.999) \doteqdot \frac{3.1416}{4} + 0.0005$

$\doteqdot 0.7854 + 0.0005$

$= 0.7859.$

Exercise 2.2 | Q 3.3 | Page 75

Find the approximate values of : $\tan^{-1}(1.001)$

SOLUTION

Let $f(x) = \tan^{-1}x$

$\therefore f'(x) = \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

Take a = 1 and h = 0.001

Then $f(a) = f(1) = \tan^{-1}1 = \frac{\pi}{4}$

and $f'(a) = f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore \tan^{-1}1\ (1.001)$

$= f(1.001)$

$= f(1 + 0.001)$

$= f(1) + (0.001).f'(1)$

$\doteqdot \frac{\pi}{4} + (0.001) \times \frac{1}{2}$

$= \frac{\pi}{4} + 0.0005$

$\therefore \tan^{-1}(1.001) \doteqdot \frac{\pi}{4} + 0.0005.$

Remark: the answer can also be given as :

$\tan^{-1}(1.001) \doteqdot f(1) + (0.001).f'(1)$

$\doteqdot \frac{\pi}{4} + (0.001) \times \frac{1}{2}$

$\doteqdot \frac{3.1416}{4} + 0.0005$

$\doteqdot 0.7854 + 0.0005$

$= 0.7859.$

Exercise 2.2 | Q 4.1 | Page 75

Find the approximate values of : $e^{0.995}$, given that e = 2.7183.

SOLUTION

Let f(x) = ex.

Then $f'(x) = \frac{d}{dx}(e^x) = e^x$

Take a = 1 and h = – 0.005.

Then f(a) = f(1) = e = 2.7183

and f'(a) = f'(1) = e = 2.7183

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore e.^{0995} = f(0.995)$

$= f(1 - 0.005)$

$\doteqdot f(1) - (0.005).f'(1)$

$\doteqdot 2.7183 - 0.005 \times 2.7183$

$\doteqdot 2.7183 - 0.01359$

$= 2.70471$

$\therefore e^{0.995} \doteqdot 2.70471.$

Exercise 2.2 | Q 4.2 | Page 75

Find the approximate values of : $e^{2.1}$, given that $e^2 = 7.389$

SOLUTION

Let $f(x) = e^x$

Then $f'(x) = \dfrac{d}{dx}(e^x) = e^x$

Take a = 2 and h = 0.1

Then $f(a) = f(2) = e^2 = 7.389$

$f'(a) = f'(2) = e^2 = 7.389$

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore e^{2.1} = f(2.1)$

$= f(2 + 0.1)$

$\doteqdot f(2) + (0.1).f'(2)$

$\doteqdot 7.389 + 0.1 \times 7.389$

$\doteqdot 7.389 + 0.7389$

$= 8.1279$

$\therefore e^{2.1} \doteqdot 8.1279.$

Exercise 2.2 | Q 4.3 | Page 75

Find the approximate values of : $3^{2.01}$, given that $\log 3 = 1.0986$

SOLUTION

Let $f(x) = 3^x$

Then $f'(x) = \dfrac{d}{dx}(3^x) = 3^x . \log 3$

Take a = 2 and h = 0.01

Then $f(a) = f(2) = 3^2 = 9$

and f'(a) = f'(2) = 3^2.log3

= 9 x 1.0986

= 9.8874

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $3^{2.01}$ = f(2.01)

= f(2 + 0.01)

≑ f(2) + (0.01).f'(2)

≑ 9 + 0.01 x 9.8874

≑ 9 + 0.098874

= 9.098874

∴ $3^{2.01}$ ≑ 9.098874.

Exercise 2.2 | Q 5.1 | Page 75

Find the approximate values of : $\log_e(101)$, given that $\log_e 10 = 2.3026$.

SOLUTION

Let f(x) - $\log_e x$.

Then f'(x) = $\frac{1}{x}$

Take a = 100 and h = 1. then

f(a) = f(100)

= $\log_e 100$

= 2 $\log_e 10$

= 2 x 2.3026

= 4.6052

f'(a) = f'(100)

= $\frac{1}{100}$

= 0.01

The formula for a approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore \log_e 101 = f(101)$

$= f(100 + 1)$

$\doteqdot f(100) + 1.f'(100)$

$\doteqdot 4.6052 + 1 \times 0.01$

$= 4.6152$

$\log_e(101) \doteqdot 4.6152.$

Exercise 2.2 | Q 5.2 | Page 75

Find the approximate values of : $\log_e(9.01)$, given that log 3 = 1.0986.

SOLUTION

Let $f(x) - \log_e x$.

Then $f'(x) = \dfrac{1}{x}$

Take a = 100 and h = 1. then

$f(a) = f(100)$

$= \log_e 100$

$= 3 \log_e 10$

$= 3 \times 1.0986$

$= 3.2958$

$f'(a) = f'(100)$

$= \dfrac{1}{100}$

$= 0.01$

The formula for a approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore \log_e 9.01 = f(9.01)$

$= f(900 + 1)$

$\doteqdot f(100) + 1.f'(100)$

$\doteqdot 3.2958 + 1 \times 0.01$

$= 2.1983$

$\log_e(901) \doteqdot 2.1983.$

Exercise 2.2 | Q 5.3 | Page 75

Find the approximate values of : $\log_{10}(1016)$, given that $\log_{10}e = 0.4343$.

SOLUTION

Let $f(x) = \log_{10}x = \dfrac{\log_e x}{\log_e 10}$

$= (\log_{10}e)(\log x)$

$= (.4343) \log x$

Then $f'(x) = (0.4343). \dfrac{d}{dx}(\log x)$

$= \dfrac{0.4343}{x}$

Take $a = 1000$ and $h = 16$. then

$f(a) = f(1000)$

$= \log_{10}1000$

$= \log_{10}10^3$

$= 3$

$f'(a) = f'(1000)$

$= \dfrac{0.4343}{1000}$

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore \log_{10}1016 = f(1016)$

= f(1000 + 16)

≑ f(1000) + 16.f'(1000)

$$\doteqdot 3 + 16\ \frac{0.4343}{1000}$$

≑ 3 + 0.0069488

≑ 3.006949

∴ $\log_{10}1016 \doteqdot 3.006949$.

Exercise 2.2 | Q 6.1 | Page 75

Find the approximate values of : $f(x) = x^3 - 3x + 5$ at x = 1.99.

SOLUTION

$f(x) = x^3 - 3x + 5$

$$\therefore f'(x) = \frac{d}{dx}\left(x^3 - 3x + 5\right)$$

$= 3x^2 - 3 \times 1 + 0$

$= 3x^2 - 3$

Take a = 2, h = – 0.01

Then f(a)

= f(2)

$= (2)^3 - 3(2) + 5$

= 8 – 6 + 5

= 7

f'(a) = f'(2)

$= 3(2)^2 - 3$

= 12 – 3

= 9

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore f(1.99) = f(2 - 0.01)$

$\doteqdot f(2) - (0.01).f'(2)$

$\doteqdot 7 - 0.01 \times 9$

$= 7 - 0.09$

$= 6.91$

$\therefore f(1.99) \doteqdot 6.91.$

Exercise 2.2 | Q 6.2 | Page 75

Find the approximate values of : $f(x) = x^3 + 5x^2 - 7x + 10$ at $x = 1.12$.

SOLUTION

$f(x) = x^3 + 5x^2 - 7x + 10$

$\therefore f'(x) = \frac{d}{dx}\left(x^3 + 5x^2 - 7x + 10\right)$

$= 3x^2 + 5 \times 2x - 7 \times 1 + 0$

$= 3x^2 + 10x - 7$

Take a = 1, h = 0.12

Then f(a) = f(1)

$= (1)^3 + 5(1)^2 - 7(1) + 10$

$= 1 + 5 - 7 + 10$

$= 9$

and

$f'(a) = f'(1)$

$= 3(1)^2 + 10(1) - 7$

$= 3 + 10 - 7$

$= 6$

The formula for approximation is

$f(a + h) \doteqdot f(a) + h.f'(a)$

$\therefore f(1.12) = f(1 + 0.12)$

$\doteqdot f(1) + (0.12).f'(1)$

$\doteqdot 9 + 0.12 \times 6$

$\doteqdot 9 + 0.72$

$= 9.72$

$\therefore f(1.12) \doteqdot 9.72.$

EXERCISE 2.3 [PAGE 80]

Exercise 2.3 | Q 1.1 | Page 80

Check the validity of the Rolle's theorem for the following functions : $f(x) = x^2 - 4x + 3$, $x \in [1, 3]$

SOLUTION

The function f given as $f(x) = x^2 - 4x + 3$ is polynomial function. Hence, it is continuous om [1, 3] and differentiable on (1, 3).
Now,

f(1)
$= 1^2 - 4(1) + 3$
$= 1 - 4 + 3$
$= 0$

and
f(3)
$= 3^2 - 4(3) + 3$
$= 9 - 12 + 3$
$= 0$

$\therefore f(1) = f(3)$
Thus, the function f satisfies all the conditions of the Rolle's theorem.

Exercise 2.3 | Q 1.2 | Page 80

Check the validity of the Rolle's theorem for the following functions : $f(x) = e^{-x} \sin x$, $x \in [0, \pi]$.

SOLUTION

The function e^{-x} and sin x are continuous and differentiable on their domains.
$\therefore f(x) = e^{-x} \sin x$ is continuous on $[0, \pi]$

and differentiable on $(0, \pi)$.

Now,
$f(0) = e^{\circ} \sin 0 = 1 \times 0 = 0$

and
$f(\pi) = e^{-\pi} . \sin \pi = e^{-\pi} \times 0 = 0$

$\therefore f(0) = f(\pi)$
Thus, the function f satisfies all the conditions of the Rolle's theorem.

Exercise 2.3 | Q 1.3 | Page 80

Check the validity of the Rolle’s theorem for the following functions : $f(x) = 2x^2 - 5x + 3$, $x \in [1, 3]$.

SOLUTION

The function f given as $f(x) = 2x^2 - 5x + 3$ is a polynomial function. Hence, it is continuous on [1, 3] and differentiable on (1, 3).

Now,
f(1)
$= 2(1)^2 - 5(1) + 3$
$= 2 - 5 + 3$
$= 0$
and

f(3)
$= 2(3)^2 - 5(3) + 3$
$= 18 - 15 + 3$
$= 6$

$\therefore f(1) \neq f(3)$
Hence, the conditions of the Rolle's theorem are not satisfied.

Exercise 2.3 | Q 1.4 | Page 80

Check the validity of the Rolle’s theorem for the following functions : $f(x) = \sin x - \cos x + 3$, $x \in [0, 2\pi]$.

SOLUTION

The functions sin x, cos x and 3 are continuous and differentiable on their domains.
$\therefore f(x) = \sin x - \cos x + 3$ is continuous n $[0, 2\pi]$ and dfferentiable o $(0, 2\pi)$.
Now,
f(0)
$= \sin 0 - \cos 0 + 3$
$= 0 - 1 + 3$

= 2
and
f(2π)
= sin 2π – cos 2π + 3
= 0 – 1 + 3
= 2
∴ f(0) = f(2π)
Thus, the function f satisfies all the conditions of the Rolle's theorem.

Exercise 2.3 | Q 1.5 | Page 80

Check the validity of the Rolle’s theorem for the following functions : f(x = x2, if 0 ≤ x ≤ 2
= 6 – x, if 2 < x ≤ 6.

SOLUTION

f(x = x2, if 0 ≤ x ≤ 2
= 6 – x, if 2 < x ≤ 6

∴ f'(x) = $\frac{d}{dx}\left(x^2\right)$ = 2x, if 0 ≤ x ≤ 2

= $\frac{d}{dx}(6 - x)$ = – 1, if 2 < x ≤ 6

∴ Lf'(2) = 2(2) = 4
and Rf'(2) = – 1
∴ Lf'(2) ≠ Rf'(2)
∴ f is not differentiable at x = 2 and 2 ∈ (0, 6).
∴ f is not differentiable at all the points on 0 6.
Hence, the conditions o Rolle's theorem are not satisfied.

Exercise 2.3 | Q 1.6 | Page 80

Check the validity of the Rolle’s theorem for the following functions : f(x) =
$x^{\frac{2}{3}}, x \in [-1, 1]$.

SOLUTION

$f(x) = x^{\frac{2}{3}}$

$\therefore f'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$

$= \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{2}{3\sqrt[3]{x}}$

This does not exist at x = 0 and 0 ∈ (– 1, 1)

∴ f is not differentiable on the interval (– 1, 1).

Hence, the conditions of Rolle's theorem are not satisfied.

Exercise 2.3 | Q 2 | Page 80

Given an interval [a, b] that satisfies hypothesis of Rolle's theorem for the function $f(x) = x^4 + x^2 - 2$. It is known that a = – 1. Find the value of b.

SOLUTION

$f(x) = x^4 + x^2 - 2$

Since the hypothesis of Rolle's theorem are satisfied by f in the interval [a, b], we have f(a) = f(b), where a = – 1

Now, f(a)
= f(– 1)
= (– 1)4 + (– 1)2 – 2
= 1 + 1 – 2
= 0
and f(b)

$= b^4 + b^2 - 2$
∴ f(a) = f(b) gives
$0 = b^4 + b^2 - 2$
i.e. $b^4 + b^2 - 2 = 0$.
Since, b = 1 satisfies this equation, b = 1 is one of the root of this equation.
Hence, b = 1.

Exercise 2.3 | Q 3.1 | Page 80
Verify Rolle's theorem for the following functions : f(x) = sin x + cos x + 7, x ∈ [0, 2π]

SOLUTION

The functions sin x, cos x and 7 are continuous and differentiable on their domains.

∴ f(x) = sin x + cos x + 7 is continuous on [0, 2π] and differentiable on (0, 2π)
Now, f(0)

= sin 0 + cos 0 + 7
= 0 + 1 + 7 = 8

and f(2π)
= sin π + cos π + 7
= 0 + 1 + 7 = 8

∴ f(0) = f(2π)
Thus, the function f satisfies all the conditions of Rolle's theorem.
∴ there exists c∈ (0, 2π) such that f'(c) = 0.
Now , f(x) = sin x + cos x + 7

∴ f'(x) = $\frac{d}{dx}(\sin x + \cos x + 7)$

= cos x – sin x + 0

= cos x – sin x

∴ f'(c) = cos c – ic

∴ f'(c) = 0 gives, cos c – sin c = 0

∴ cos c = sin c

∴ c = $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, ...

But $\frac{\pi}{4}, \frac{5\pi}{4} \in (0, 2\pi)$

∴ c = $\frac{\pi}{4}$ or $\frac{5\pi}{4}$

Hence, the Rolle's theorem is verified.

Exercise 2.3 | Q 3.2 | Page 80

Verify Rolle’s theorem for the following functions : f(x) = $\sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$

SOLUTION

The function f(x) = $\sin\left(\frac{x}{2}\right)$ is continuous on [0, 2π] and differetiable on (0, 2π).

Now, f(0) = sin0 = 0

and f(2π) = sin π = 0

∴ f(0) = f(2π)

Thus, the functiontisfies all the conditions of Rolle's theorem.

∴ there exists c ∈ (0, 2π) such that f'(c) = 0.

Now, f(x) = $\sin\left(\frac{x}{2}\right)$

∴ f'(x) = $\frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\right]$

= $\cos\left(\frac{x}{2}\right) . \frac{d}{dx}\left(\frac{x}{2}\right)$

= $\cos\left(\frac{x}{2}\right) . \frac{1}{2} = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

∴ f'(c) = $\frac{1}{2}\cos\left(\frac{c}{2}\right)$

∴ f'(c) = 0 gives $\frac{1}{2}\cos\left(\frac{c}{2}\right)$ = 0

∴ $\cos\left(\frac{c}{2}\right)$ = 0

∴ $\frac{\cos c}{2} = \cos\ \frac{\pi}{2} = \cos\ \frac{3\pi}{2} = \cos\ \frac{5\pi}{2}$ = ...

∴ $\frac{c}{2}\frac{i}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, ...

∴ c = π, 3π 5π, ...

But π ∈ (0, 2π)

$\therefore$ c = π

Hence, the Rolle’s theorem is verified.

Exercise 2.3 | Q 3.3 | Page 80

Verify Rolle’s theorem for the following functions : f(x) = x^2 – 5x + 9, x ∈ 1, 4].

SOLUTION

The function f gives as f(x) = x2 – 5x + 9 s a polynomial function. Hence it is continuous on [1, 4] and differentiable on (1, 4).
Now, f(1)
= 1^2 – 5(1) + 9
= 1 – 5 + 9
= 5
and f(4)
= 4^2 – 5(4) + 9
= 16 – 20 + 9
= 5
$\therefore$ f(1) = f(4)
Thus, the function f satisfies all the conditions of the Rolle's theorem.
$\therefore$ there exists c ∈ (1, 4) such that f'(c) = 0.
Now, f(x) = x^2 – 5x + 9

$\therefore$ f'(x) = $\frac{d}{dx}\left(x^2 - 5x + 9\right)$

= 2x – 5 x 1 + 0

= 2x – 5

$\therefore$ f'(c) = 2c – 5

$\therefore$ f'(c) = 0 gives, 2c – 5 = 0

$\therefore$ c = $\frac{5}{2} \in (1, 4)$

Hence, the Rolle's theorem is verified.

Exercise 2.3 | Q 4 | Page 80

If Rolle's theorem holds for the function f(x) = $x^3 + px^2 + qx + 5$, x ∈ [1, 3] with c = $2 + \frac{1}{\sqrt{3}}$, find the values of p and q.

SOLUTION

The Rol's theorem hold for the function f(x) = $x^3 + px^2 + qx + 5$, x ∈ [1, 3]
∴ f(1) = f(3)

∴ $1^3 + p(1)^2 + q(1) + 5 = 3^3 + p(3)^2 + q(3) + 5$
∴ 1 + p + q + 5 = 27 + 9p + 3q + 5

∴ 8p + 2q = – 26
∴ 4p + q = – 13 ...(1)

Also, there exists at least one point c ∈ (1, 3) such that (c) = 0.

Now, f'(x) = $\frac{d}{dx}\left(x^3 + px^2 + qx + 5\right)$

= $3x^2$ + p x 2x + q x 1 + 0

= $3x^2$ + 2px + q

∴ f'(c) = $3c^2$ + 2pc + q, where c = $2 + \frac{1}{\sqrt{3}}$

∴ f'(c) = $3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2p\left(2 + \frac{1}{\sqrt{3}}\right) + q$

= $3\left(4 + \frac{4}{\sqrt{3}} + \frac{1}{3}\right) + 4p + \frac{2p}{\sqrt{3}} + q$

= $12 + \frac{12}{\sqrt{3}} + 1 + 4p + \frac{2p}{\sqrt{3}} + q$

= $4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}}$

But f'(c) = 0

∴ $4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}}$ = 0

$\therefore \left(4\sqrt{3}+2\right)p+\sqrt{3}q+\left(13\sqrt{3}+12\right) = 0$

$\therefore \left(4\sqrt{3}+2\right)p+\sqrt{3}q=-13\sqrt{3}-12$ ...(2)

Multiplying equation (1) by $\sqrt{3}$, we get

$4\sqrt{3}p+\sqrt{3}q=-13\sqrt{3}$

Subtracting this equation from (2), we get

2p = – 12

∴ p = – 6

∴ from (1), 4(– 6) + q = – 13

∴ q = 11

Hence, p = – 6 and q = 11.

Exercise 2.3 | Q 5 | Page 80

If Rolle's theorem holds for the function f(x) = (x –2) log x, x ∈ [1, 2], show that the equation x log x = 2 – x is satisfied by at least one value of x in (1, 2).

SOLUTION

The Rolle's theorem holds for the function f(x) = (x – 2) log x, x ∈ [1, 2].

∴ there exists t least onee real number c ∈ (1, 2) such that f'(c) 0.

Now, f(x) = (x – 2) log x

$\therefore$ f'(x) = $\frac{d}{dx}[(x-2)\log x]$

= $(x-2).\frac{\mathrm{d}}{\mathrm{dx}}(\log x)+\log x.\frac{d}{dx}(x-2)$

= $(x-2)\times\frac{1}{x}+(\log x)(1-0)$

= $1-\frac{2}{x}+\log x$

$\therefore$ f'(c) = $1-\frac{2}{c}+\log c$

$\therefore$ f'(c) = 0 gives $1 - \dfrac{2}{c} + \log c = 0$

$\therefore$ c – 2 + c log c = 0

$\therefore$ c log c = 2 – c, where c $\in$ (1, 2)

$\therefore$ c satisfies the equation x log x = 2 – x, c $\in$ (1, 2).

Hence, the equation log x = 2 – x is satisfied by at least one value of x i(1, 2).

Exercise 2.3 | Q 6 | Page 80

The function f(x) = $= x(x+3)e^{-\frac{x}{2}}$ satisfies all the conditions of Rolle's theorem on [– 3, 0]. Find the value of c such that f'(c) = 0.

SOLUTION

The function f(x) satisfies all the conditions of Rolle's theorem on [– 3, 0] such that f'(c) = 0.

Now, f(x) = $x(x+3)e^{-\frac{x}{2}}$

$= \left(x^2 + 3x\right)e^{-\frac{x}{2}}$

$\therefore$ f'(x) = $\dfrac{d}{dx}\left[\left(x^2+3x\right)e^{-\frac{x}{2}}\right]$

$= \left(x^2+3x\right).\dfrac{d}{dx}\left(e^{-\frac{x}{2}}\right) + e^{-\frac{x}{2}}.\dfrac{d}{dx}\left(x^2+3x\right)$

$= \left(x^2+3x\right).e^{-\frac{x}{2}}.\dfrac{d}{dx}\left(-\dfrac{x}{2}\right) + e^{-\frac{x}{2}} \times (2x + 3 \times 1)$

$= \left(x^2+3x\right).e^{-\frac{x}{2}} \times -\dfrac{1}{2} + e^{-\frac{x}{2}}(2x+3)$

$= e^{-\frac{x}{2}}\left[(2x+3) - \dfrac{x^2+3x}{2}\right]$

$= e^{-\frac{x}{2}}\left[\dfrac{4x+6-x^2-3x}{2}\right]$

$$= \frac{e^{-\frac{x}{2}}}{2}\left(6 + x - x^2\right)$$

$$= \frac{e^{-\frac{x}{2}}}{2}(3 - x)(2 + x)$$

$$= \frac{e^{-\frac{x}{2}}}{2}\left(6 + x - x^2\right)$$

$$= \frac{e^{-\frac{x}{2}}}{2}(3 - x)(2 + x)$$

$\therefore$ f'(c) = $\frac{e^{-\frac{c}{2}}}{2}(3 - c)(2 + c)$

$\therefore$ f'(c) = 0 gives $\frac{e^{-\frac{c}{2}}}{2}(3 - c)(2 + c)$ = 0

$\therefore$ (3 – c)(2 + c) = 0 ...$\left[\because \frac{e^{-\frac{c}{2}}}{2} \neq 0\right]$

$\therefore$ (3 – c) = 0 or (2 + c) = 0

$\therefore$ c = 3 or c = – 2

But $3 \notin (-3, 0)$

$\therefore$ c ≠ 3

Hence, c = – 2.

Exercise 2.3 | Q 7.1 | Page 80

Verify Lagrange's mean value theorem for the following functions : f(x) = log x on [1, e].

SOLUTION

The function f given as f(x) = log x is a logarithmic function which is continuous for all positive real numbers.

Hence, it is continuous on [1, e] and differentiable on (1, e).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

$\therefore$ there exists c ∈ (1, e) such that f'(c) = $\dfrac{f(e) - f(1)}{e - 1}$ ...(1)

Now, f(x) = log x

∴ f(1) = log 1 = 0

and

f(e) = log e = 1

Also, f'(x) = $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

∴ f'(c) = $\dfrac{1}{c}$

∴ from (1), $\dfrac{1}{c}$

= $\dfrac{1 - 0}{e - 1}$

= $\dfrac{1}{e - 1}$

∴ c = e – 1 ∈ (1, e)

Hence, Lagrange's mean value theorem is verified.

Exercise 2.3 | Q 7.2 | Page 80

Verify Lagrange's mean value theorem for the following functions : f(x) = (x – 1)(x – 2)(x – 3) on [0, 4].

SOLUTION

The function f given as f(x) = (x – 1)(x – 2)(x – 3)

= (x – 1)(x^2 – 5x + 6)
= x^3 – 5x^2 + 6x – x2 + 5x – 6

= x^3 – 6x^2 + 11x – 6 is a polynomial function.
Hence, it is continuous on [0, 4] and differentiable on (0, 4).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.
∴ there exists c ∈ (0, 4) such that

f'(c) = $\dfrac{f(4) - f(0)}{4 - 0}$ ...(1)

Now, f(x) = (x – 1)(x – 2)(x – 3)

∴ f(0)

= (0 – 1)(0 – 2)(0 – 3)

= (– 1)(– 2)(– 3)

= – 6

and

f(4)

= (4 – 1)(4 – 2)(4 – 3)

= (3)(2)(1)

= 6

Also, f'(x) = $\dfrac{d}{dx}\left(x^3 - 6x^2 + 11x - 6\right)$

= $3x^2$ – 6 x 2x + 11 x 1 – 0

= $3x^2$ – 12x + 11

∴ f'(c) = $3c^2$ – 12x + 11

∴ from (1), $3c^2$ – 12c + 11 = $\dfrac{6 - (-6)}{4}$

∴ $3c^2$ – 12c + 11 = 3

∴ $3c^2$ – 12c + 8 = 0.

∴ c = $\dfrac{12 \pm \sqrt{144 - 4(3)(8)}}{2(3)}$

∴ c = $\dfrac{12 \pm \sqrt{48}}{6}$

= $\dfrac{12 \pm 4\sqrt{3}}{6}$

$\therefore$ c = $2 \pm \frac{2}{\sqrt{3}} \in (0, 4)$

Hence, Lagrange's mean value theorem is verified.

Exercise 2.3 | Q 7.3 | Page 80

Verify Lagrange's mean value theorem for the following functions :

$x^2 - 3x - 1, x \in \left[\frac{-11}{7}, \frac{13}{7}\right]$.

SOLUTION

The function f given as f(x) = x^2 – 3x = 1 is a polynomial function. Hence, it is continuous on $\left[\frac{-11}{7}, \frac{13}{7}\right]$ and

differentiable on $\left(\frac{-11}{7}, \frac{13}{7}\right)$.

Thus, the function f satisfies the conditions of LMVT.

$\therefore$ there exists $c \in \left(\frac{-11}{7}, \frac{13}{7}\right)$ such that

f'(c) = $\frac{f\left(\frac{13}{7}\right) - f\left(\frac{-11}{7}\right)}{\frac{13}{7} - \left(\frac{-11}{7}\right)}$ ...(1)

Now, f(x) = x^2 – 3x = 1

$$\therefore f\left(\frac{-11}{7}\right) = \left(\frac{-11}{7}\right)^2 - 3\left(\frac{-11}{7}\right) - 1$$

$$= \frac{121}{49} + \frac{33}{7} - 1$$

$= \dfrac{121 + 231 - 49}{49}$

$= \dfrac{303}{49}$

and

$f\left(\dfrac{13}{7}\right) = \left(\dfrac{13}{7}\right)^2 - 3\left(\dfrac{13}{7}\right) - 1$

$= \dfrac{169}{49} - \dfrac{39}{7} - 1$

$= \dfrac{169 - 273 - 49}{49}$

$= \dfrac{-153}{49}$

Also, f'(x) = $\dfrac{d}{dx}\left(x^2 - 3x - 1\right)$

= 2x – 3 x – 0

= 2x – 3

∴ f'(c) = 2c – 3

∴ from (1), 2c – 3 = $\dfrac{\frac{-153}{49} - \frac{303}{49}}{\frac{13}{7}\ \frac{11}{7}}$

∴ 2c – 3 = $-\dfrac{456}{49} \times \dfrac{7}{24}$

$= \dfrac{-57}{21}$

$\therefore$ 2c = $\dfrac{-57}{21} + 3$

= $\dfrac{-57+63}{21}$

= $\dfrac{6}{21}$

= $\dfrac{2}{7}$

$\therefore$ c = $\dfrac{1}{7} \in \left(\dfrac{-11}{7}, \dfrac{13}{7}\right)$

Hence, Lagrange's mean value theorem is verified.

Exercise 2.3 | Q 7.4 | Page 80

Verify Lagrange's mean value theorem for the following functions : f(x) = 2x – x^2, x ∈ [0, 1].

SOLUTION

The function f given as f(x) = 2x – x^2 is a polynomial function. Hence, it is continuous on [0, 1] and differentiable on (0, 1).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

$\therefore$ there exists c ∈ (0, 1) such that

f'(c) = $\dfrac{f(1) - f(0)}{1 - 0}$ ...(1)

Now, f(x) = 2x – x^2

$\therefore$ f(0) = 0 – 0 = 0

and

f(1) = 2(1) – 1^2 = 1

Also, f'(x) = $\dfrac{d}{dx}(2x - x^2)$

= 2 x 1 – 2x

= 2 – 2x

$\therefore$ f'(c) = 2 – 2c

$\therefore$ from (1), 2 – 2c

$= \dfrac{1-0}{1}$

= 1

$\therefore$ 2c = 1

$\therefore$ c = $\dfrac{1}{2} \in (0, 1)$

Hence, Lagrange's mean value theorem is verified.

Exercise 2.3 | Q 7.5 | Page 80

Verify Lagrange's mean value theorem for the following functions : f(x) =

$= \dfrac{x-1}{x-3}$ on [4, 5].

SOLUTION

The function f given as f(x) = $\dfrac{x-1}{x-3}$ is a rational function which is continuous except at x = 3.

But $3 \notin [4, 5]$

Hence, it is continuous on [4, 5] and differentiable on (4, 5).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

$\therefore$ there exists c ∈ (4, 5) such that

f'(c) = $\dfrac{f(5)-f(4)}{5-4}$ ...(1)

Now, f(x) = $\dfrac{x-1}{x-3}$

$\therefore$ f(4) = $\dfrac{4-1}{4-3} = \dfrac{3}{1}$ = 3

and f(5) = $\dfrac{5-1}{5-3} = \left.\dfrac{4}{2}\right)$ = 2

Also, f'(x) = $\dfrac{d}{dx}\left(\dfrac{x-1}{x-3}\right)$

$= \dfrac{(x-3).\frac{d}{dx}(x-1)-(x-1).\frac{d}{dx}(x-3)}{(x-3)^2}$

$= \dfrac{(x-3)\times(1-0)-(x-1)\times(1-0)}{(x-3)^2}$

$= \dfrac{x-3-x+1}{(x-3)^2}$

$= \dfrac{-2}{(x-3)^2}$

$\therefore$ f'(c) = $\dfrac{-2}{(c-3)^2}$

$\therefore$ from (1), $\dfrac{-2}{(c-3)^2}$

$= \dfrac{2-3}{1}$

= – 1

$\therefore (c-3)^2 = 2$

$\therefore c - 3 = \pm\sqrt{2}$

$\therefore c = 3 \pm \sqrt{2}$

But $\left(3-\sqrt{2}\right) \notin (4,5)$

$\therefore c \neq 3 - \sqrt{2}$

$\therefore c \neq 3 - \sqrt{2}$

$\therefore c = 3 + \sqrt{2} \in (4,5)$

Hence, Lagrange's mean value theorem is verified.

Exercise 2.4 | Q 1.1 | Page 89

Test whether the following functions are increasing or decreasing : $f(x) = x^3 - 6x^2 + 12x - 16$, $x \in R$.

SOLUTION

f(x) = $x^3 - 6x^2 + 12x - 16$

$\therefore$ f'(x) = $\frac{d}{dx}\left(x^3 - 6x^2 + 12x - 16\right)$

$= 3x^2 - 6 \times 2x + 12 \times 1 - 0$

$= 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x - 2)^2 \geq 0$ for all $x \in R$

$\therefore$ f'(x) ≥ 0 for all $x \in R$

$\therefore$ f is increasing for all $x \in R$.

Exercise 2.4 | Q 1.2 | Page 89

Test whether the following functions are increasing or decreasing : f(x) = $2 - 3x + 3x^2 - x^3$, $x \in R$.

SOLUTION

f(x) = $2 - 3x + 3x^2 - x^3$

$\therefore$ f'(x) = $\frac{d}{dx}\left(2 - 3x + 3x^2 - x^3\right)$

$= 0 - 3 \times 1 + 3 \times 2x - 3x^2$

$= -3 + 6x - 3x^2$

$= -3(x^2 - 2x + 1)$

$= -3(x - 1)^2 \leq 0$ for all $x \in R$

$\therefore$ f'(x) ≤ 0 for all $x \in R$

$\therefore$ f is decreasing for all $x \in R$.

Exercise 2.4 | Q 1.3 | Page 89

Test whether the following functions are increasing or decreasing : f(x) = 1/x
1x, $x \in R$, $x \neq 0$.

SOLUTION

f(x) = $\frac{1}{x}$

∴ f'(x) = $\frac{d}{dx}\left(x - \frac{1}{x}\right)$

= $1 - \left(\frac{-1}{x^2}\right)$

= $1 + \frac{1}{x^2}$ > 0 for all x ∈ R , x ≠ 0

∴ f'(x) > 0 for all x ∈ R, where x ≠ 0

∴ f is increasing for all x ∈ R, where x ≠ 0.

Exercise 2.4 | Q 2.1 | Page 89

Find the values of x for which the following functions are strictly increasing : f(x) = $2x^3 - 3x^2 - 12x + 6$

SOLUTION

f(x) = $2x^3 - 3x^2 - 12x + 6$

∴ f'(x) = $\frac{d}{dx}\left(2x^3 - 3x^2 - 12x + 6\right)$

= $2 \times 3x^2 - 3 \times 2x - 12 \times 1 + 0$

= $6x^2 - 6x - 12$

= $6(x^2 - x - 2)$

f is strictly increasing if f'(x) > 0

i.e. if $6(x^2 - x - 2) > 0$

i.e. if $x^2 - x - 2 > 0$

i.e. if $x^2 - x > 2$

i.e. if $x^2 - x + \frac{1}{4} > 2 + \frac{1}{4}$

i.e. if $\left(x - \frac{1}{2}\right)^2 > \frac{9}{4}$

i.e. if $x - \frac{1}{2} > \frac{3}{2}$ or $x - \frac{1}{2} < -\frac{3}{2}$

i.e. if x > 2 or x < – 1

∴ f is strictly increasing if x < – 1 or x > 2.

Exercise 2.4 | Q 2.2 | Page 89

Find the values of x for which the following functions are strictly increasing : f(x) = 3 + 3x – $3x^2$ + x^3

SOLUTION

f(x) = 3 + 3x – $3x^2$ + x^3

∴ f'(x) = $\frac{d}{dx}\left(3 + 3x - 3x^2 + x^3\right)$

= 0 + 3 x 1 – 3 x 2x + $3x^2$

= 3 – 6x + $3x^2$

= 3(x^2 – 2x + 1)

f is strictly increasing if f'(x) > 0

i.e. if 3(x^2 – 2x + 1) > 0

i.e. if x^2 – 2x + 1 > 0

i.e. if $(x – 1)^2$ > 0

This is possible if x ∈ R and x ≠ 1

i.e. x ∈ R – { 1 }

∴ f is strictly increasing if x ∈ R – { 1 }.

Exercise 2.4 | Q 2.3 | Page 89

Find the values of x for which the following functions are strictly increasing : $f(x) = x^3 - 6x^2 - 36x + 7$

SOLUTION

$f(x) = x^3 - 6x^2 - 36x + 7$

$\therefore f'(x) = \frac{d}{dx}\left(x^3 - 6x^2 - 36x + 7\right)$

$= 3x^2 - 6 \times 2x - 36 \times 1 + 0$

$= 3x^2 - 12x - 36$

$= 3(x^2 - 4x - 12)$

f is strictly increasing if $f'(x) > 0$

i.e. if $3(x^2 - 4x - 12) > 0$

i.e. if $x^2 - 4x - 12 > 0$

i.e.if $x^2 - 4x > 12$

i.e. if $x^2 - 4x + 4 > 12 + 4$

i.e. if $(x - 2)^2 > 16$

i.e. if $x - 2 > 4$ or $x - 2 < -4$

i.e if $x > 6$ or $x < -2$

$\therefore$ f is strictly increasing if $x < -2$ or $x > 6$.

Exercise 2.4 | Q 3.1 | Page 89

Find the values of x for which the following functions are strictly decreasing : $f(x) = 2x^3 - 3x^2 - 12x + 6$

SOLUTION

f(x) = $2x^3 - 3x^2 - 12x + 6$

$\therefore$ f'(x) = $\frac{d}{dx}\left(2x^3 - 3x^2 - 12x + 6\right)$

= $2 \times 3x^2 - 3 \times 2x - 12 \times 1 + 0$

= $6x^2 - 6x - 12$

= $6(x^2 - x - 2)$

f is strictly decreasing if f'(x) < 0

i.e. if $6(x^2 - x - 2) < 0$

i.e. if $x^2 - x - 2 < 0$

i.e. if $x^2 - x < 2$

i.e. if $x^2 - x + \frac{1}{4} < 2 + \frac{1}{4}$

i.e. if $\left(x - \frac{1}{2}\right)^2 < \frac{9}{4}$

i.e. if $-\frac{3}{2} < x - \frac{1}{2} < \frac{3}{2}$

i.e. if $-\frac{3}{2} + \frac{1}{2} < x - \frac{1}{2} + \frac{1}{2} < \frac{3}{2} + \frac{1}{2}$

i.e. if $-1 < x < 2$

$\therefore$ f is strictly decreasing if $-1 < x < 2$.

Exercise 2.4 | Q 3.2 | Page 89

Find the values of x for which the following functions are strictly decreasing : f(x) =

= $x + \frac{25}{x}$

SOLUTION

f(x) = $x + \dfrac{25}{x}$

∴ f'(x) = $\dfrac{d}{dx}\left(x + \dfrac{25}{x}\right)$

= $1 + 25(-1)x^{-2}$

= $1 - \dfrac{25}{x^2}$

f is strictly decreasing if f'(x) < 0

i.e. if $1 - \dfrac{25}{x^2} < 0$

i.e. if $1 < \dfrac{25}{x^2}$

i.e. if $x^2 < 25$

i.e. if $-5 < x < 5,\ x \neq 0$

i.e. if $x \in (-5, 5) - \{0\}$

∴ f is strictly decreasing if $x \in (-5, 5) - \{0\}$.

Exercise 2.4 | Q 3.3 | Page 89

Find the values of x for which the following functions are strictly decreasing : $f(x) = x^3 - 9x^2 + 24x + 12$

SOLUTION

f(x) = $x + \dfrac{25}{x}$

∴ f'(x) = $\dfrac{d}{dx}\left(x + \dfrac{25}{x}\right)$

= $1 + 25(-1)x^{-2}$

$= \mathbf{1 - \frac{25}{x^2}}$

f is strictly decreasing if f'(x) < 0

i.e. if $\mathbf{1 - \frac{25}{x^2} < 0}$

i.e. if $\mathbf{1 < \frac{25}{x^2}}$

i.e. if $x^2 < 25$

i.e. if $-5 < x < 5, x \neq 0$

i.e. if $x \in (-5, 5) - \{0\}$

∴ f is strictly decreasing if $x \in (-5, 5) - \{0\}$.

$f(x) = x^3 - 9x^2 + 24x + 12$

∴ $f'(x) = \mathbf{\frac{d}{dx}\left(x^3 - 9x^2 + 24x + 12\right)}$

$= 3x^2 - 9 \times 2x + 24 \times 1 + 0$

$= 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8)$

f is strictly decreasing if f'(x) < 0

i.e. if $3(x^2 - 6x + 8) < 0$

i.e. if $x^2 - 6x + 8 < 0$

i.e. if $x^2 - 6x < -8$

i.e. if $x^2 - 6x + 9 < -8 + 9$

i.e. if $(x - 3)^2 < 1$

i.e. if $-1 < x - 3 < 1$

i.e. if $-1 + 3 < x - 3 + 3 < 1 + 3$

i.e. if $2 < x < 4$

i.e., if $x \in (2, 4)$

$\therefore$ f is strictly decreasing if $x \in (2, 4)$.

Exercise 2.4 | Q 4 | Page 90

Find the values of x for which the function $f(x) = x^3 - 12x^2 - 144x + 13$ (a) increasing (b) decreasing

SOLUTION

$f(x) = x^3 - 12x^2 - 144x + 13$

$\therefore f'(x) = \frac{d}{dx}\left(x^3 - 12x^2 - 144x + 13\right)$

$= 3x^2 - 12 \times 2x - 144 \times 1 + 0$

$= 3x^2 - 24x - 144$

$= 3(x^2 - 8x - 48)$

(a) if is increasing if $f'(x) \geq 0$

i.e. if $3(x^2 - 8x - 48) \geq 0$

i.e. if $x^2 - 8x - 48 \geq 0$

i.e. if $x^2 - 8x \geq 48$

i.e. if $x^2 - 8x + 16 \geq 48 + 16$

i.e. if $(x - 4)^2 \geq 64$

i.e. if $x - 4 \geq 8$ or $x - 4 \leq -8$

i.e. if $x \geq 12$ or $x \leq -4$

$\therefore$ f is increasing if $x \leq -4$ or $x \geq 12$,

i.e. $x \in (-\infty, -4] \cup [12, \infty)$.

(b) f is decreasing if $f'(x) \le 0$

i.e. if $3(x^2 - 8x - 48) \le 0$

i.e. if $x^2 - 8x - 48 \le 0$

i.e. if $x^2 - 8x \le 48$

i.e. if $x^2 - 8x + 16 \le 48 + 16$

i.e. if $(x - 4)^2 \le 64$

i.e. if $-8 \le x - 4 \le 8$

i.e. if $-4 \le x \le 12$

$\therefore$ f is decreasing if $-4 \le x \le 12$, i.e. $x \in [-4, 12]$.

Exercise 2.4 | Q 5 | Page 90

Find the values of x for which $f(x) = 2x^3 - 15x^2 - 144x - 7$ is (a) strictly increasing (b) strictly decreasing.

SOLUTION

$f(x) = 2x^3 - 15x^2 - 144x - 7$

$\therefore f'(x) = \dfrac{d}{dx}\left(2x^3 - 15x^2 - 144x - 7\right)$

$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$

$= 6x^2 - 30x - 144$

$= 6(x^2 - 5x - 24)$

(a) f is strictly increasing if $f'(x) > 0$

i.e. if $6(x^2 - 5x - 24) > 0$

i.e. if $x^2 - 5x - 24 > 0$

i.e. if $x^2 - 5x > 24$

i.e. if $x^2 - 5x + \dfrac{25}{4} > 24 + \dfrac{25}{4}$

i.e. if $\left(x-\frac{5}{2}\right)^{2}>\frac{121}{4}$

i.e. if $x-\frac{5}{2}>\frac{11}{2}$ or $x-\frac{5}{2}<-\frac{11}{2}$

i.e. if x > 8 or x < – 3

∴ f is strictly increasing, if x < – 3 or x > 8.

(b) f is strictly decreasing if f'(x) < 0

i.e. if $6(x^2 - 5x - 24) < 0$

i.e. if $x^2 - 5x - 24 < 0$

i.e. if $x^2 - 5x < 24$

i.e. if $x^{2}-5x+\frac{25}{4}<24+\frac{25}{4}$

i.e. if $\left(x-\frac{5}{2}\right)^{2}<\frac{121}{4}$

i.e. if $x-\frac{5}{2}<\frac{11}{2}$ or $x-\frac{5}{2}<-\frac{11}{2}$

i.e. if $-\frac{11}{2}+\frac{5}{2}<x-\frac{5}{2}+\frac{5}{2}<\frac{11}{2}+\frac{5}{2}$

i.e. if – 3 < x < 8

∴ f is strictly decreasing, if – 3 < x < 8.

Exercise 2.4 | Q 6 | Page 90

Find the values of x for which f(x) = $\frac{x}{x^{2}+1}$ is (a) strictly increasing (b) decreasing.

SOLUTION

f(x) = $\frac{x}{x^2+1}$

$\therefore$ f'(x) = $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

= $\frac{(x^2+1).\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

= $\frac{(x^2+1)(1) - x(2x+0)}{(x^2+1)^2}$

= $\frac{x^2+1-2x^2}{(x^2+1)^2}$

= $\frac{1-x^2}{(x^2+1)^2}$

(a) f is strictly increasing if f'(x) > 0

i.e. if $\frac{1-x^2}{(x^2+1)^2} > 0$

i.e. if $1 - x^2 > 0$ ...[$\because (x^2 + 1)^2 > 0$]

i.e. if $1 > x^2$

i.e. if $x^2 < 1$

i.e. if $-1 < x < 1$

$\therefore$ f is strictly increasing if $-1 < x < 1$

(b) f is strictly increasing if f'(x) < 0

i.e. if $\frac{1-x^2}{(x^2+1)^2} < 0$

i.e. if $1 - x^2 < 0$ ...[$\because (x^2 + 1)^2 > 0$]

i.e. if $1 < x^2$

i.e. if $x^2 > 1$

i.e. if $x > 1$ or $x < -1$

$\therefore$ f is strictly increasing if $x < -1$ or $x > 1$

i.e. $x \in (-\infty, -1) \cup (1, \infty)$.

Exercise 2.4 | Q 7 | Page 90

show that f(x) = $3x + \frac{1}{3x}$ is increasing in $\left(\frac{1}{3}, 1\right)$ and decreasing in $\left(\frac{1}{9}, \frac{1}{3}\right)$.

SOLUTION

f(x) = $3x + \frac{1}{3x}$

$\therefore$ f'(x) = $3\frac{d}{dx}(x) + \frac{1}{3}\frac{d}{dx}\left(x^{-1}\right)$

= $3 \times 1 + \frac{1}{3}(-1)x^{-2}$

= $3 - \frac{1}{3x^2}$

Now, f is increasing if f'(x) > 0 and is decreasing if f'(x) < 0.

Let $x \in \left(\frac{1}{3}, 3\right)$.

Then $\frac{1}{3} < x < 1$

$\therefore \frac{1}{9} < x^2 < 1$

$\therefore \frac{1}{3} < 3x^2 < 3$

$\therefore 3 > \frac{1}{3x^2} > \frac{1}{3}$

$\therefore -3 < -\frac{1}{3x^2} < -\frac{1}{3}$

$$\therefore 3-3<3-\frac{1}{3x^2}<3-\frac{1}{3}$$

$$\therefore 0<f\prime(x)<\frac{8}{3}$$

$\therefore$ f'(x) > 0 for all x ∈ $\left(\frac{1}{3},1\right)$

$\therefore$ f is increasing in rhe interval $\left(\frac{1}{3},1\right)$

Let x ∈ $\left(\frac{1}{9},\frac{1}{3}\right)$.

Then $\frac{1}{9}<x<\frac{1}{3}$

$$\therefore \frac{1}{81}<x^2<\frac{1}{9}$$

$$\therefore \frac{1}{27}<3x^2<\frac{1}{3}$$

$$\therefore 27>\frac{1}{3x^2}>3$$

$$\therefore -27<-\frac{1}{3x^2}<-3$$

$$\therefore 3-27<3-\frac{1}{3x^2}<3-3$$

$\therefore$ – 24 < f'(x) < 0

$\therefore$ f'(x) < 0 for all x ∈ $\left(\frac{1}{9},\frac{1}{3}\right)$

$\therefore$ f is decreasing in the interval $\left(\frac{1}{9},\frac{1}{3}\right)$.

Exercise 2.4 | Q 8 | Page 90

Show that f(x) = x – cos x is increasing for all x.

SOLUTION

f(x) = x − cos x

$\therefore$ f'(x) = $\frac{d}{dx}(x - \cos x)$

= 1 − (− sin x)

= 1 + sin x

Now, − 1 ≤ sin x ≤ 1 for all x ∈ R

$\therefore$ − 1 + 1 ≤ 1 + sin x ≤ 1 for all x ∈ R

$\therefore$ 0 ≤ f'(x) ≤ 1 for all x ∈ R

$\therefore$ f'(x) ≥ 0 for all x ∈ R

$\therefore$ f is increasing for all x.

Exercise 2.4 | Q 9.1 | Page 90

Find the maximum and minimum of the following functions : $y = 5x^3 + 2x^2 - 3x$

SOLUTION

$y = 5x^3 + 2x^2 - 3x$

$\therefore d\frac{dy}{dx} = \frac{d}{dx}\left(5x^3 + 2x^2 - 3x\right)$

= 5 x 3x2 + 2 x 2x − 3 x 1

= 15x2 + 4x − 3

and

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(15x^2 + 4x - 3\right)$

= 15 x 2x + 4 x 1 − 0

= 30x + 4

$\frac{dy}{dx}$ = 0 gives $15x^2 + 4x - 3 = 0$

= 15 x 2x + 4 x 1 – 0

= 30x + 4

$\frac{dy}{dx}$ = 0 gives 15x^2 + 4x – 3 = 0

∴ 15x^2 + 9x – 5x – 3 = 0

∴ 3x(5x + 3) – 1(5x + 3) = 0

∴ (5x + 3)(3x – 1) = 0

∴ x = $-\frac{3}{5}$ or $x = \frac{1}{3}$

∴ the roots of $\frac{dy}{dx} = 0$ are $x_1 = -\frac{3}{5}$ and $x_2 = \frac{1}{3}$.

Method 1 (second Derivative Test) :

(a) $\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=\frac{3}{5}}$

= $30\left(-\frac{3}{5}\right) + 4$

= – 14 < 0

∴ by the second derivative test, y is maximum at x = $-\frac{3}{5}$ and maximum value of y at x = $-\frac{3}{5}$

= $5\left(-\frac{3}{5}\right)^3 + 2\left(-\frac{3}{5}\right)^2 - 3\left(-\frac{3}{5}\right)$

= $\frac{-27}{25} + \frac{18}{25} + \frac{9}{5} = \frac{36}{25}$

(b) $\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=\frac{1}{3}}$

= $30\left(\frac{1}{3}\right) + 4$

= 14 > 0

∴ by the second derivative test, y is minimum at x = $\frac{1}{3}$ and minimum value of y at y = $\frac{1}{3}$

= $5\left(\frac{1}{3}\right)^3 + 2\left(-\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)$

= $\frac{5}{27} + \frac{2}{9} - 1$

= $-\frac{16}{27}$

Hence, the function has maximum value $\frac{36}{25}$ at x = $-\frac{3}{5}$ and minimum value $-\frac{16}{27}$ at $x = \frac{1}{3}$.

Method 2 (second Derivative Test) :

(a) $\frac{dy}{dx}$ = $15x^2 + 4x - 3$

= (5x + 3)(3x −1)

Consider x = $-\frac{3}{5}$

Let h be a small positive number. Then

$\left(\frac{dy}{dx}\right)_{\text{at } x=-\frac{3}{5}-h}$

= $\left[5\left(-\frac{3}{5} - h\right) + 3\right]\left[3\left(-\frac{3}{5} - h\right) - 1\right]$

= $(-3 - 5h + 3)\left(-\frac{9}{5} - 3h - 1\right)$

= $-5h\left(-\frac{14}{5} - 3h\right)$

= $5h\left(\frac{14}{5} + 3h\right) > 0$

and

$$\left(\frac{dy}{dx}\right)_{\text{at} x=-\frac{3}{5}+h}$$

$$= \left[5\left(-\frac{3}{5}+h\right)+3\right]\left[3\left(-\frac{3}{5}+h\right)-1\right]$$

$$= (3+5h+3)\left(-\frac{9}{5}+3h-1\right)$$

$$= 5h\left(3h-\frac{14}{5}\right) < 0,$$

as h is small positive number.

∴ by the first derivative test, y is maximum at x = $-\frac{3}{5}$ and maxximum value of y at x = $-\frac{3}{5}$

$$= 5\left(-\frac{3}{5}\right)^3+2\left(-\frac{3}{5}\right)^2-3\left(-\frac{3}{5}\right)$$

$$= -\frac{27}{25}+\frac{18}{25}+\frac{9}{5}=\frac{36}{25}$$

(b) $\frac{dy}{dx}$ = $15x^2 + 4x - 3$

= (5x + 3)(3x – 1)

Consider x = $\frac{1}{3}$

Let h be a small positive number. Then

$$\left(\frac{dy}{dx}\right)_{\text{at } x=\frac{1}{3}-h}$$

$$= \left[5\left(\frac{1}{3}-h\right)+3\right]\left[3\left(\frac{1}{3}-h\right)-1\right]$$

$$= \left(\frac{5}{3}-5h+3\right)(1-3h-1)$$

= $5h\left(3h-\frac{14}{5}\right)<0$,

as h is small positive number.

∴ by the first derivative test, y is maximum at x = $-\frac{3}{5}$ and maxximum value of y at x = $-\frac{3}{5}$

= $5\left(-\frac{3}{5}\right)^3+2\left(-\frac{3}{5}\right)^2-3\left(-\frac{3}{5}\right)$

= $-\frac{27}{25}+\frac{18}{25}+\frac{9}{5}=\frac{36}{25}$

(b) $\frac{dy}{dx}$ = $15x^2 + 4x - 3$

= (5x + 3)(3x – 1)

Consider x = $\frac{1}{3}$

Let h be a small positive number. Then

$\left(\frac{dy}{dx}\right)_{\text{at } x=\frac{1}{3}-h}$

= $\left[5\left(\frac{1}{3}-h\right)+3\right]\left[3\left(\frac{1}{3}-h\right)-1\right]$

= $\left(\frac{5}{3}-5h+3\right)(1-3h-1)$

= $\left(\frac{14}{5}-5h\right)(-3h)$ < 0, as h is small positive number

and

$\left(\frac{dy}{dx}\right)_{\text{at} x=\frac{1}{3}+h}$

= $\left[5\left(\frac{1}{3}+h\right)+3\right]\left[3\left(\frac{1}{3}+h\right)-1\right]$

$= \left(\frac{5}{3} + 5h + 3\right)(1 + 3h - 1)$

$= \left(\frac{14}{3} + 5h\right)(3h) > 0$

$\therefore$ by the first derivative test, y is minimum at x = $\frac{1}{3}$ and minimum value of y at x = $\frac{1}{3}$

$= 5\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)$

$= \frac{5}{27} + \frac{2}{9} - 1$

$= \frac{-16}{27}$

Hence, the function has maximum value $\frac{36}{25}$ at $x = -\frac{3}{5}$ and minimum value $-\frac{16}{27}$ at $x = \frac{1}{3}$.

Exercise 2.4 | Q 9.2 | Page 90

Find the maximum and minimum of the following functions : f(x) = $2x^3 - 21x^2 + 36x - 20$

SOLUTION

f(x) = $2x^3 - 21x^2 + 36x - 20$

$\therefore$ f'(x) = $\frac{d}{dx}\left(2x^3 - 21x^2 + 36x - 20\right)$

= $2 \times 3x^2 - 21 \times 2x + 36 \times 1 - 0$

= $6x^2 - 42x + 36$

and

f'(x) = $\frac{d}{dx}\left(6x^2 - 42x + 36\right)$

= $6 \times 2x - 42 \times 1 + 0$

= $12x - 42$

f'(x) = 0 gives $6x^2 - 42x + 36 = 0$

$\therefore$ x2 – 7x + 6 = 0

∴ $(x - 1)(x - 6) = 0$

∴ the roots of $f'(x) = 0$ are $x_1 = 1$ and $x_2 = 6$.

Method 1 (Second Derivative Test) :

(a) $f'(1) = 12(1) - 42 = - 30 < 0$

∴ by the second derivative test , f has maximum at x = 1 and maximum value of f at x = 1 = f(1)

$= 2(1)^3 - 21(1)^2 + 36(1) - 20$
$= 2 - 21 + 36 - 20$
$= - 3$

(b) $f'(6) = 12(6) - 42 = 30 > 0$
∴ by the second derivative test , f has minimum at x = 6 and minimum value of f at x = 6 = f(6)

$= 2(6)^3 - 21(6)^2 + 36(6) - 20$
$= 432 - 756 + 216 - 20$

$= - 128.$
Hence, the function f has maximum value – 3 at x = 1 and minimum value – 128 at x = 6.

Method 2 (Second Derivative Test) :

(a) $f'(x) = 6(x - 1)(x - 6)$
Consider x = 1

Let h be a small positive number. Then
$f'(1 - h)$
$= 6(1 - h - 1)(1 - h - 6)$

$= 6(- h)(- 5 - h)$
$= 6h(5 + h) > 0$

and
$f'(1 + h)$
$= 6(1 + h - 1)(1 + h - 6)$
$= 6h(h - 5) < 0,$

as h is small positive number.
∴ by the first derivative test, f has maximum at x = 1 and maximum value of f at x = 1 = f(1)

$= 2(1)^3 - 21(1)^2 + 36(1) - 20$
$= 2 - 21 + 36 - 20$
$= -3$

(b) $f'(x) = 6(x - 1)(x - 6)$
Consider $x = 6$
Let h be a small positive number. Then
$f'(6 - h)$
$= 6(6 - h - 1)(6 - h - 6)$
$= 6(5 - h)(- h)$
$= 6h(5 - h) < 0$,

as h is small positive number
and
$f'(6 + h)$

$= 6(6 + h - 1)(6 + h - 6)$
$= 6(5 + h)(h) < 0$,

∴ by the first derivative test, f has minimum at x = 6 and minimum value of f at x = 6
$= f(6)$
$= 2(6)^3 - 21(6)^2 + 36(16) - 20$

$= 432 - 756 + 216 - 20$
$= -128$
Hence, the function f has maximum value – 3 at=1 and minimum value – 128 at x = 6.

Note : Out of the two methods, given above, we will use the second derivative test for the remaining problems.

Exercise 2.4 | Q 9.3 | Page 90

Find the maximum and minimum of the following functions : $f(x) = x^3 - 9x^2 + 24x$
SOLUTION

$f(x) = x^3 - 9x^2 + 24x$

$\therefore f'(x) = \dfrac{d}{dx}\left(x^3 - 9x^2 + 24x\right)$

$= 3x^2 - 9 \times 2x + 24 \times 1$

$= 3x^2 - 18x + 24$

and

f''(x) = $\frac{d}{dx}\left(3x^2 - 18x + 24\right)$

= 3 x 2x – 18 x 1 + 0

= 6x – 18

f''(x) = 0 gives $3x^2 - 18x + 24 = 0$

∴ $x^2 - 6x + 8 = 0$

∴ (x – 2)(x – 4) = 0

∴ the roots of f'(x) = 0 are $x_1 = 2$ and $x_2 = 4$.

(a) f''(2)

= 6(2) – 18

= – 6 < 0

∴ by the second derivaative test, f has maximum at x = 2 and maximum value of at x = 2

= f(2)

= $(2)^3 - 9(2)^2 + 24(2)$

= 8 – 36 + 48

= 20

(b) f''(4) = 6(4) – 18 = 6 > 0

∴ by the second derivative test, f has maximum at x = 4 and maximum value of at x = 4

= f(4)

= $(4)^3 - 9(4)^2 + 24(4)$

= 64 – 144 + 96

= 16.

Hence, the function f has maximum value 20 at x = 2 and minimum value 16 at x = 4.

Exercise 2.4 | Q 9.4 | Page 90

Find the maximum and minimum of the following functions : f(x) = $x^2 + \frac{16}{x^2}$

SOLUTION

f(x) = $x^2 + \frac{16}{x^2}$

$\therefore$ f'(x) = $\frac{d}{dx}\left(x^2\right) + 16\frac{d}{dx}\left(x^{-2}\right)$

= $2x + 16(-2)x^{-3}$

= $2x - \frac{32}{x^3}$

and

f''(x) = $\frac{d}{dx}(2x) - 32\frac{d}{dx}\left(x^{-3}\right)$

= $2 \times 1 - 32(-3)x^{-4}$

= $2 + \frac{96}{x^4}$

f'(x) = 0 gives $2x - \frac{32}{x^3}$ = 0

$\therefore 2x^4 - 32 = 0$

$\therefore x^4 = 16$

$\therefore x = \pm 2$

$\therefore$ the roots of f'(x) = 0 are $x_1 = 2$ and $x_2 = -2$

(a) f''(2) = $2 + \frac{96}{(2)^4}$ = 8 > 0

$\therefore$ by the second derivative test, f has minimum at x = 2 and minimum value of f at x = 2

= f(2) = $(2)^2 + \frac{16}{(2)^2}$

= 4 + 4

= 8

(b) f''(– 2) = $2 + \frac{96}{(-2)^4}$ = 8 > 0

∴ by the second derivative test, f has minimum at x = – 2 and minimum value of f at x = – 2

= f(– 2)

= $(-2)^2 + \frac{16}{(-2)^2}$

= 4 + 4

= 8

Hence, the function f has minimum value 8 at x = ± 2.

Exercise 2.4 | Q 9.5 | Page 90

Find the maximum and minimum of the following functions : f(x) = x log x

SOLUTION

f(x) = x log x

∴ f'(x) = $\frac{d}{dx}(x \log x)$

= $x . \frac{d}{dx}(\log x) + \log x . \frac{d}{dx}(x)$

= $x \times \frac{1}{x} + (\log x) \times 1$

= 1 + log x

and

f''(x) = $\frac{d}{dx}(1 + \log x)$

= $0 + \frac{1}{x} = \frac{1}{x}$

Now, f'(x) = 0, if 1 + log x = 0

i.e. if log x = – 1 = – log e

i.e. if log x = $\log\left(e^{-1}\right) = \dfrac{\log(1)}{e}$

i.e. if x = $\dfrac{1}{e}$

When $x = (1)e, f(x) = \dfrac{1}{\left(\frac{1}{e}\right)}$ = e > 0

∴ by the second derivative test, f is minimum at x = $\dfrac{1}{e}$.

Minimum value of f at x = $\dfrac{1}{e}$

= $\dfrac{1}{e}\log\left(\dfrac{1}{e}\right)$

= $\dfrac{1}{e}.\log\left(e^{-1}\right)$

= $\dfrac{1}{e}.(-1)\log e$

= $-\dfrac{1}{e}$. ...[∵ log e = 1]

Exercise 2.4 | Q 9.6 | Page 90

Find the maximum and minimum of the following functions : f(x) = $\dfrac{\log x}{x}$

SOLUTION

f(x) = $\dfrac{\log x}{x}$

∴ f'(x) = $\dfrac{d}{dx}\left(\dfrac{\log x}{x}\right)$

= $\dfrac{x\frac{d}{x}(\log x) - \log x\frac{d}{dx}(x)}{x^2}$

= $\dfrac{x\left(\frac{1}{x}\right) - (\log x)(1)}{x^2}$

= $\dfrac{1 - \log x}{x^2}$

and

f''(x) = $\dfrac{d}{dx}\left(\dfrac{1 - \log x}{x^2}\right)$

= $\dfrac{x^2\frac{d}{dx}(1 - \log x) - (1 - \log x)\frac{d}{dx}\left(x^2\right)}{x^4}$

= $\dfrac{x^2\left(0 - \frac{1}{x}\right) - (1 - \log x) \times 2x}{x^4}$

= $\dfrac{-x - 2x + 2x\log x}{x^4}$

= $\dfrac{x(2\log x - 3)}{x^4}$

∴ f''(x) = $\dfrac{2\log x - 3}{x^3}$

Now, f'(x) = 0 if $\dfrac{1 - \log x}{x^2}$ = 0

i.e. if 1 – log x = 0, i.e. if log x = 1 = log e

i.e. if x = e

and

f"(e) = $\frac{2\log e - 3}{e^3}$

= $\frac{-1}{e^3} < 0$ ...[∵ log e = 1]

∴ by the second derivative test, f(x) is maximum at x = e.

Maximum value of f at x = e

= $\frac{\log e}{e}$

= $\frac{1}{e}$. ...[∵ log e = 1]

Exercise 2.4 | Q 10 | Page 90

Divide the number 30 into two parts such that their product is maximum.

SOLUTION

Let the first part of 30 – x.

∴ their product

= x(30 – x)

= 30x – x^2

= f'(x) ...(Say)

∴ f(x) = $\frac{d}{dx}(30 - 2x)$

= 0 – 2 x 1

= – 2

The root of the equation f'(x) = 0

i.e. 30 – 2x = 0 is x = 15

and f"(15) = – 2 < 0

∴ by the second derivative test, f is maximum at x = 15.

Hence, the required parts of 30 are 15 and 15.

Exercise 2.4 | Q 11 | Page 90

Divide the number 20 into two parts such that sum of their squares is minimum.

SOLUTION

Let the first part of 20 be x.

Then the second part is 20 – x.

∴ sum of their squares

$= x^2 + (20 - x)^2$

$= f(x)$ ...(Say)

$\therefore f'(x) \ \frac{d}{dx}\left[x^2 + (20 - x)^2\right]$

$= 2x + 2(20 - x). \frac{d}{dx}(20 - x)$

$= 2x + 2(20 - x) \times (0 - 1)$

$= 2x - 40 + 2x$

$= 4x - 40$

and

$f''(x) = \frac{d}{dx}(4x - 40)$

$= 4 \times 1 - 0$

$= 4$

The root of the equation $f'(x) = 0$,

i.e. $4x - 40 = 0$ is $x = 10$

and $f''(10) = 4 > 0$

∴ by the second derivative test, f is minimum at x = 10.

Hence, the required parts of 20 are 10 and 10.

Exercise 2.4 | Q 12 | Page 90

A wire of length 36 metres is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.

SOLUTION

Let x metres and y metre be the length and breadth of the rectangle.

Then its perieter is $2(x + y) = 36$

$\therefore x + y = 18$
$\therefore y = 18 - x$
Area of the rectangle
$= xy$
$= x(18 - x)$
Let f(x)
$= x(18 - x)$

$= 18x - x^2$

$\therefore f'(x) = \frac{d}{dx}\left(18x - x^2\right)$

$= 18 - 2x$

and

$f''(x) = \frac{d}{dx}(18 - 2x)$

$= 0 - 2 \times 1$

$= -2$

Now, $f'(x) = 0$, if $18 - 2x = 0$

i.e. if $x = 9$

and $f''(9) - 2 < 0$

$\therefore$ by the second derivative test f has maximum value at $x = 9$.

When $x = 9$, $y = 18 - 9 = 9$

$\therefore x = 9\text{cm}, y = 9\text{cm}$

$\therefore$ the recangleis a square of side 9metres.

Exercise 2.4 | Q 13 | Page 90

A ball is thrown in the air. Its height at any time t is given by $h = 3 + 14t - 5t^2$. Find the maximum height it can reach.

SOLUTION

The height h at any t is given by h = 3 + 14t – 5t^2

$$\therefore \frac{\mathrm{dh}}{dt} = \frac{d}{dt}\left(3 + 14t - 5t^2\right)$$

= 0 + 14 x 1 – 5 x 2t

= 14 – 10t

and

$$\frac{d^2h}{dt^2} = \frac{d}{dt}(14 - 10t)$$

= 0 – 10 x 1

= – 10

The root of $\frac{\mathrm{dh}}{dt}$ = 0,

i.e. 14 – 10t = 0 is t = $\frac{14}{10} = \frac{7}{5}$

and

$$\left(\frac{d^2h}{dt^2}\right)_{\mathrm{at}t=\frac{7}{5}}$$ = 10 < 0

∴ by the second derivative test, h is maximum at t = $\frac{7}{5}$.

∴ maximum height = $\left(3 + 14t - 5t^2\right)_{\mathrm{at}t=\frac{7}{5}}$

$$= 3 + 14\left(\frac{7}{5}\right) - 5\left(\frac{7}{5}\right)^2$$

$$= 3 + \frac{98}{5} - \frac{245}{25}$$

$$= \frac{5 + 490 - 245}{25}$$

$$= \frac{320}{25}$$

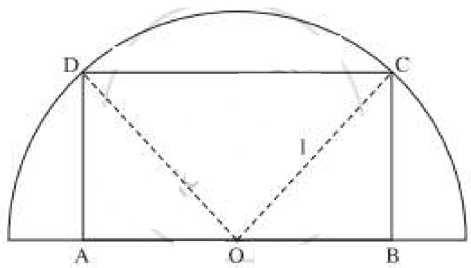
= 12.8

Hence, the maximum height the ball can reach = 12.8units.

Exercise 2.4 | Q 14 | Page 90

Find the largest size of a rectangle that can be inscribed in a semicircle of radius 1 unit, so that two vertices lie on the diameter.

SOLUTION



Let ABCD be the rectangle inscribed in a semicircle of radius 1 unit such that the vertices A and B lie on the diameter.
Let AB = DC = x and BC = AD = y.
Let O be the centre of the semicircle.
Join OC and OD. Then OC = OD = radius = 1.
Also, AD = BC and m∠A = m∠B = 90°.
∴ OA= OB

$$\therefore OB = \frac{1}{2}AB = \frac{x}{2}$$

In right angled triangle OBC,

$$OB^2 + BC^2 = OC^2$$

$$\therefore \left(\frac{x}{2}\right)^2 + y^2 = 1^2$$

$$\therefore y^2 = 1 - \frac{x^2}{4} = \frac{1}{4}\left(4 - x^2\right)$$

$$\therefore y = \frac{1}{2}\sqrt{4 - x^2} \qquad \ldots[\because y > 0]$$

Also of the triangle

$= xy$

$$= x.\frac{1}{2}\sqrt{4 - x^2}$$

Let f(x) = $\frac{1}{2} \times \sqrt{4-x^2}$

= $\frac{1}{2}\sqrt{4x2-x^4}$

∴ f'(x) = $\frac{1}{2}\frac{d}{dx}\left(\sqrt{4x^2-x^4}\right)$

= $\frac{1}{2} \times \frac{1}{2\sqrt{4x^2-x^4}} \times . \frac{d}{dx}\left(4x^2-x^4\right)$

= $\frac{1}{4\sqrt{4x^2-x^4}} \times \left(4 \times 2x - 4x^3\right)$

= $\frac{4x(2-x2)}{4x\sqrt{4-x^2}}$

= $\frac{2-x^2}{\sqrt{4-x^2}}$ ...[∵ x ≠ 0]

and

f''(x) = $\frac{d}{dx}\left(\frac{2-x^2}{\sqrt{4-x^2}}\right)$

= $\frac{d}{dx}\left[\frac{\left(4-x^2\right)-2}{\sqrt{4-x^2}}\right]$

= $\frac{d}{dx}\left[\sqrt{4-x^2} - \frac{2}{\sqrt{4-x^2}}\right]$

$= \frac{d}{dx}\left(\sqrt{4-x^2}\right) - 2\frac{d}{dx}\left(4-x^2\right)^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{4-x^2}} \cdot \frac{d}{dx}\left(4-x^2\right) - 2\left(-\frac{1}{2}\right)\left(4-x^2\right)^{-\frac{3}{2}} \cdot \frac{d}{dx}\left(4-x^2\right)$

$= \frac{1}{2\sqrt{4-+x^2}} \times (0-2x) + \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}} \times (0-2x)$

$= -\frac{x}{\sqrt{4-x^2}} - \frac{2x}{\left(4-x^2\right)^{\frac{3}{2}}}$

$= \frac{-x\left(4-x^2\right)-2x}{\left(4-x^2\right)^{\frac{3}{2}}}$

$= \frac{-4x+x^3-2x}{\left(4-x^2\right)^{\frac{3}{2}}}$

$= \frac{x^3-6x}{\left(4-x^2\right)^{\frac{3}{2}}}$

For maximum value of f(x),f'(x) = 0

$\therefore \frac{2-x^2}{\sqrt{4-x^2}} = 0$

$\therefore 2 - x^2 = 0$

$\therefore x^2 = 2$

$\therefore x = \sqrt{2}$ ...[$\because x > 0$]

Now, $f''\left(\sqrt{2}\right) = \frac{\left(\sqrt{2}\right)^3 - 6\sqrt{2}}{\left[4-\left(\sqrt{2}\right)^2\right]^{\frac{3}{2}}}$

= $\frac{-4\sqrt{2}}{2\sqrt{2}}$

= − 2 < 0

∴ by the second derivative test, f is maximum when x = $\sqrt{2}$

When $x = \sqrt{2}, y = \frac{1}{2}\sqrt{4 - x^2}$

= $\frac{1}{2}\sqrt{4 - 2}$

= $\frac{1}{2} \times \sqrt{2}$

Hence, the area of the rectangle is maximum (i.e. rectangle has the largest size) when its length is √2 units and breadth is 1/√2 unit.

Exercise 2.4 | Q 15 | Page 90

An open cylindrical tank whose base is a circle is to be constructed of metal sheet so as to contain a volume of πa^3 cu cm of water. Find the dimensions so that the quantity of the metal sheet required is minimum.

SOLUTION

Let x be the radius of the base, h be the height, V be the volume and S be the total surface area of the cylindrical tank.

Then V = πa^3 ...(Given)

∴ $\pi x^2 h = \pi a$

∴ h = $\frac{a^3}{x^2}$ ...(1)

Now, S = $2\pi x h + \pi x^2$

= $2\pi x\left(\frac{a^2}{x^2}\right) + \pi x^2$ ...[By (1)]

$= 2\pi \frac{a^2}{x} + \pi x^2$

$\therefore \frac{dS}{dx} = \frac{d}{dx}\left(\frac{2\pi a^3}{x} + \pi x^2\right)$

$= 2\pi a^3 (-1)x^{-2} + \pi \text{ x } 2\pi$

$= \frac{-2\pi a^3}{x^2} + 2\pi x$

and

$\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{-2\pi a^2}{x^2} + 2\pi x\right)$

$= -2\pi a^3(-2)x^{-3} + 2\pi \text{ x } 1$

$= \frac{4\pi a^3}{x^3} + 2\pi$

Now, $\frac{dS}{dx} = 0 \text{gives} \frac{-2\pi a^3}{x^2} + 2\pi x = 0$

$\therefore -2\pi a^3 + 2\pi x^3 = 0$

$\therefore 2\pi x^3 = 2\pi a^3$

$\therefore x = a$

and

$\left(\frac{d^2S}{dx^2}\right)_{\text{at } x=a}$

$= \frac{4\pi a^3}{a^3} + 2\pi$

$= 6\pi > 0$

$\therefore$ by the second derivative test,

S is minimum when x = a

When x = a, from (1)

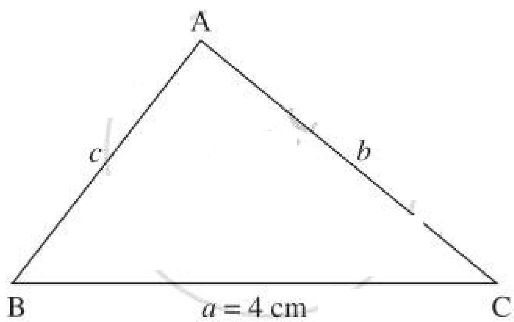
$$h = \frac{a^3}{a^2} = a$$

Hence, the quantity of metal sheet is n minimum when radius = height = a cm.

Exercise 2.4 | Q 16 | Page 90

The perimeter of a triangle is 10 cm. If one of the side is 4 cm. What are the other two sides of the triangle for its maximum area?

SOLUTION



Let ABC be the triangle such that the side BC = a = 4 cm. Also, the perimeter of the triangle is cm.
i.e. a + b + c = 10
∴ 2s = 10
∴ s = 5

Also, 4 + b + c = 10
∴ b + c = 6
∴ b = 6 – c

Let Δ be the area of the trangle.

$$\text{Then } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{5(5-4)(5-6+c)(5-c)}$$

$$= \sqrt{5(c-1)(5-c)}$$

$\therefore \Delta^2 = 5(c - 1)(5 - c)$

$= 5(5c - c^2 - 5 + c)$

$\therefore \Delta^2 = 5(- c^2 + 6c - 5)$

Differentiable both sidesww.r.t. c, we get

$$2\Delta \frac{d\Delta}{dc} = 5\frac{d}{\mathrm{dc}}(-c2 + +6c - 5)$$

$= 5 (- 2c + 6 \times 1 - 0)$

$= 5 (- 2c + 6)$

$$\therefore \frac{d\Delta}{\mathrm{dc}} = \frac{5(-c+3)}{\Delta}$$

and

$$\frac{d^2\Delta}{dc^2} = 5\frac{d}{\mathrm{dc}}\left(\frac{-c+3}{\Delta}\right)$$

$$= 5.\ \frac{\Delta\frac{d}{\mathrm{dc}}(-c+3) - (-c+3)\frac{d\Delta}{\mathrm{dc}}}{\Delta^2}$$

$$= 5.\ \frac{\Delta(-1+0) - (-c+3)\frac{d\Delta}{\mathrm{dc}}}{\Delta^2}$$

$$= \frac{5}{\Delta^2}\left(-\Delta - (c+3)\frac{d\Delta}{\mathrm{dc}}\right)$$

$$= \frac{-5}{\Delta^2}\left[\Delta + (c+3)\frac{d\Delta}{\mathrm{dc}}\right]$$

For maximum Δ, $\frac{d\Delta}{\mathrm{dc}} = 0$

$$\therefore \frac{5(-c+3)}{\Delta} = 0$$

$\therefore -\ c + 3 = 0$ ...[$\because \Delta \neq 0$]

$\therefore c = 3$

If c = 3,

$\Delta = \sqrt{5(3-1)(5-3)}$

$= 2\sqrt{3}$

$\therefore \left(\frac{d^2\Delta}{dc^2}\right)_{\text{at } c=3}$

$= \frac{-5}{4 \times 5}\left[2\sqrt{5} + (3+3)(0)\right]$

$= \frac{\sqrt{5}}{2} < 0$

∴ by the second derivative test, Δ is maximum when c= 3.

When c = 3, = 6 – c = 6 – 3 = 3

Hence, the area of the triangle is maximum when the other two sides are 3cm and 3cm.

Exercise 2.4 | Q 17 | Page 90

A box with a square base is to have an open top. The surface area of the box is 192 sq cm. What should be its dimensions in order that the volume is largest?

SOLUTION

Let x cm be the side of square base and h cm be its height.

Then $x^2 + 4xh = 192$

$\therefore h = \frac{192 - x^2}{4x}$ ...(1)

Let V = $x^2 h = x^2\left(\frac{192 - x^2}{4x}\right)$ ...[By (1)]

$\therefore V = \frac{1}{4}\left(192x - x^3\right)$

$= \frac{dV}{dx} = \frac{1}{4}\frac{d}{dx}\left(192x - x^3\right)$

$= \frac{1}{4}\left(192 \times 1 - 3x^2\right)$

$= \frac{3}{4}(64 - x^2)$

and

$\frac{d^2V}{dx^2} - \frac{3d}{4dx}(64 - x^2)$

$= \frac{3}{4}(0 - 2x)$

$= -\frac{3}{2}x$

For maximum V, $\frac{dV}{dx} = 0$

$\therefore \frac{3}{4}(64 - x^2) = 0$

$\therefore x^2 = 64$

$\therefore x = 8$ ...[$\because x > 0$]

and

$\left(\frac{d^2V}{dx^2}\right)_{\text{at}x=8}$

$= -\frac{3}{2} \times 8$

$= -12 < 0$

$\therefore$ by the second derivative test, V is maximum at x = 8.

If x = 8,

$h = \frac{192 - 64}{4(8)}$

$= \frac{128}{32}$

= 4

Hence, the volume of the box is largest, when the side of square base is 8cm and its height is 4cm.

Exercise 2.4 | Q 18 | Page 90

The profit function P(x) of a firm, selling x items per day is given by P(x) = (150 – x)x – 1625 . Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

SOLUTION

Profit function P(x) is given by

P(x) = (150 – x)x – 1625

= 150x – x^2 – 1625

$\therefore$ P'(x) = $\frac{d}{dx}\left(150x - x^2 - 1625\right)$

= 150 x 1 – 2x – 0

= 150 – 2x

and

P"(x) = $\frac{d}{dx}(150 - 2x)$

= 0 – 2 x 1

= 2

Now, P'(x) = 0 gives, 150 – 2x = 0

$\therefore$ x = 75

and

P"(75) = – 2 < 0

$\therefore$ by thecond derivative test, P(x) is maximum when x = 75

Maximum profit

= P(75)

= (150 – 75)75 – 1625

= 75 x 75 – 1625

= 4000

Hence, the profit will be maximum, if the manufacturer manufactures 75 items and maximum profit is 4000.

Exercise 2.4 | Q 20 | Page 90

Show that among rectangles of given area, the square has least perimeter.

SOLUTION

Let x be the length and y be the breadth of the rectangle whose area is A sq units (which is given as constant).

Then xy = A

$\therefore$ y = $\frac{A}{x}$ ...(1)

Let P be the perimeter of the retangle.

Then P = 2(x + y)

= $2\left(x+\frac{A}{x}\right)$ ...[By(1)]

$\therefore \frac{dP}{dx}=2.\frac{d}{dx}\left(x+\frac{A}{x}\right)$

= $2[1 + A(-1)x^{-2}]$

= $2\left(1-\frac{A}{x^2}\right)$

and

$\frac{d^2P}{dx^2}=2\frac{d}{dx}\left(1-\frac{A}{x^2}\right)$

= $2[0 - A(-1)x^{-3}]$

= $\frac{4A}{x^3}$

Now, $\frac{dp}{dx}$ $0, \text{gives} 2\left(1-\frac{A}{x^2}\right)$ = 0

$\therefore$ $x^2 - a = 0$

$\therefore$ $x^2 = A$

$\therefore$ x = $\sqrt{A}$ ...[$\because$ x > 0]

and

$$\left(\frac{d^2P}{dx^2}\right)_{\text{at } x=d\sqrt{\mathrm{A}}}$$

$$= \frac{4\mathrm{A}}{\left(\sqrt{\mathrm{A}}\right)^3} > 0$$

∴ P is minimum when x = $\sqrt{\mathrm{A}}$

If x = $\sqrt{\mathrm{A}}$, then $y = \frac{\mathrm{A}}{x} = \frac{\mathrm{A}}{\sqrt{\mathrm{A}}} = \sqrt{\mathrm{A}}$

∴ x = y

∴ rectangle is a square.

Hence, among rectangles of given area, the square has least perimeter.

Exercise 2.4 | Q 21 | Page 90

Show that the height of a closed right circular cylinder of given volume and least surface area is equal to its diameter.

SOLUTION

Let x be the radius of base, h be the height and S be the surface area of the closed right circular cylinder whose volume is V which is given to be constant.

Then $\pi r^2 h$ = V

∴ h = $\frac{\mathrm{V}}{\pi r^2} = \frac{\mathrm{A}}{x^2}$, ...(1)

where A = $\frac{\mathrm{V}}{\pi r^2}$, which is constant.

Now, S = $2\pi xh + 2\pi x^2$

$= 2\pi x\left(\frac{\mathrm{A}}{x^2}\right) + 2\pi x^2$ ...[By (1)]

$= \frac{2\pi\mathrm{A}}{x} + 2\pi x^2$

$\therefore \frac{dS}{dx} = \frac{d}{dx}\left(\frac{2\pi A}{x} + 2\pi x^2\right)$

$= 2\pi A(-1)x^{-2} + 2\pi \times 2x$

$= \frac{-2\pi A}{x^2} + 4\pi x$

and

$\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{-2\pi A}{x^2} + 4\pi x\right)$

$= -2\pi A(-2)x^{-3} + 4\pi \times 1$

$= \frac{4\pi A}{x^3} + 4\pi$

Now, $\frac{dS}{dx} = 0$ gives $\frac{-2\pi A}{x^2} + 4\pi x = 0$

$\therefore -2\pi + 4\pi x^3 = 0$

$\therefore 4\pi \times^3 = 2\pi A$

$\therefore x^3 = \frac{A}{2}$

$\therefore x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$

and

$\left(\frac{d^2S}{dx^2}\right)_{\text{at} x = \left(\frac{A}{2}\right)^{\frac{1}{3}}}$

$= \frac{4\pi A}{\left(\frac{A}{2}\right)} + 4\pi$

$= 12\pi > 0$

$\therefore$ by the second derivative test, S is minimum when $x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$

When $x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$, from (1),

$h = \frac{A}{\left(\frac{A}{2}\right)^{\frac{2}{3}}}$

$= 2^{\frac{2}{3}} . A^{\frac{1}{3}}$

$= 2. \left(\frac{A}{2}\right)^{\frac{1}{3}}$

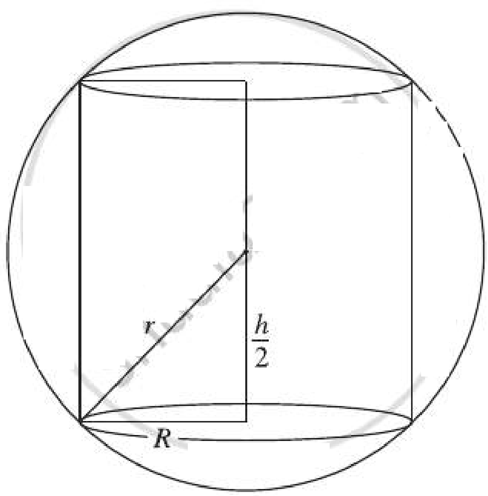
$\therefore h = 2x$

Hence, the surface area is least when height of the closed right circular cylinder is equal to its diameter.

Exercise 2.4 | Q 22 | Page 90

Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

SOLUTION



Let R be the radius and h be the height of the cylinder which is inscribed in a sphere of radius r cm.

Then from the figure,

$$R^2 + \left(\frac{h}{2}\right)^2 = r^2$$

$$\therefore R^2 = r^2 - \frac{h^2}{4} \quad ...(1)$$

Let V be the volume of the cylinder.

Then $V = \pi R^2 h$

$$= \pi\left(r^2 - \frac{h^2}{4}\right)h \quad ...[\text{By } (1)]$$

$$= \pi\left(r^2 - \frac{h^3}{4}\right)$$

$$\therefore \frac{dV}{dh} = \pi\frac{d}{dh}\left(r^2 h - \frac{h^3}{4}\right)$$

$$= \pi\left(r^2 \times 1 - \frac{1}{4} \times 3h^2\right)$$

$$= \pi\left(r^2 - \frac{3}{4}h^2\right)$$

and

$$\frac{d^2V}{dh^2} = \pi\frac{d}{dh}\left(r^2 - \frac{3}{4}h^2\right)$$

$$= \pi\left(0 - \frac{3}{4} \times 2h\right)$$

$$= -\frac{3}{2}\pi h$$

Now, $\frac{dV}{dh} = 0$ gives, $\pi\left(r^2 - \frac{3}{4}h^2\right) = 0$

$$\therefore r^2 - \frac{3}{4}h^2 = 0$$

$$\therefore \frac{3}{4}h^2 = r^2$$

$\therefore h^2 = \frac{4r^2}{3}$

$\therefore h = \frac{2r}{\sqrt{3}}$ ...[$\because h > 0$]

and

$\left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\frac{2r}{\sqrt{3}}}$

$= -\frac{3}{2}\pi \times \frac{2r}{\sqrt{3}} < 0$

$\therefore$ V is maximum at $h = \frac{2r}{\sqrt{3}}$

If $h = \frac{2r}{\sqrt{3}}$, then from (1)

$R^2 = r^2 - \frac{1}{4} \times \frac{4r^2}{3} = \frac{2r^2}{3}$

$\therefore$ volumeof the largest cylinder

$= \pi \times \frac{2r^2}{3} \times \frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$ cu cm.

Hence, the volume of the largest cylinder inscribed in a sphere of radius 'r' cm = $\frac{4r^3}{3\sqrt{3}}$ cu cm.

Exercise 2.4 | Q 23 | Page 90

Show that y = $\log(1 + x) - \frac{2x}{2 + x}, x > -1$ is an increasing function on its domain.

SOLUTION

y = $\log(1+x) - \frac{2x}{2+x}, x > -1$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\log(1+x) - \frac{2x}{2+x}\right]$$

$$= \frac{1}{1+x} \cdot \frac{d}{dx}(1+x) - \frac{(2+x).\frac{d}{dx}(2x) - 2x.\frac{d}{dx}(2+x)}{(2+x)^2}$$

$$= \frac{1}{1+x} \times (0+1)\frac{(2+x) \times 2 - 2x(0+1)}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+4x+x^2-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} > 0 \text{ for all } x > -1$$

Hence, the given function is increasing function on its domain.

Exercise 2.4 | Q 24 | Page 90

Prove that y = $\frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function if $\theta \in \left[0, \frac{\pi}{2}\right]$

SOLUTION

$y = \frac{4\sin\theta}{2+\cos\theta} - \theta$

$\therefore \frac{dy}{d\theta} = \frac{d}{d\theta}\left[\frac{4\sin\theta}{2+\cos\theta} - \theta\right]$

$= \frac{d}{d\theta}\left(\frac{4\sin\theta}{2+\cos\theta}\right) - \frac{d}{d\theta}(\theta)$

$= \frac{(2+\cos\theta).\frac{d}{d\theta}(4\sin\theta) - 4\sin\theta.\frac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - 1$

$= \frac{(2+\cos\theta)(4\cos\theta) - (4\sin\theta)(0-\sin\theta)}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$

$= \frac{(8\cos\theta + 4) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$

$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$

$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$

$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$

Since, $\theta \in \left[0, \frac{\pi}{2}\right]$, $\cos\theta \geq 0$ Also, cos θ < 4

∴ 4 - cos θ > 0

∴ cos θ (4 - cosθ) ≥ 0

$$\therefore \frac{\cos\theta(4-\cos\theta)}{2+\cos\theta^2} \geq 0$$

$$\therefore \frac{dy}{d\theta} \geq 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right]$$

Hence, y is an increasing function if $\theta \in \left[0, \frac{\pi}{2}\right]$.

MISCELLANEOUS EXERCISE 2 [PAGE 92]

Miscellaneous Exercise 2 | Q 1 | Page 92

Choose the correct option from the given alternatives :

If the function f(x) = $ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theoreem in [1, 3] and $f'\left(2+\frac{1}{\sqrt{3}}\right) = 0$, then values of a and b are respectively

- **1, – 6**
- – 2, 1
- – 1, – 6
- – 1, 6

SOLUTION

1, – 6
[Hint : f(x) = $ax^3 + bx^2 + 11x - 6$ satisfies the conditions of Rolle's theorem in [1, 3]

∴ f(1) = f(3)
∴ $a(1)^3 + b(1)^2 + 11(1) - 6 = a^3 + b(3)^2 + 11(3) - 6$

∴ a + b + 11 = 27a + 9b + 33
∴ 26a + 8b = – 22

∴ 13a + 4b = – 11

Only a = 1, b = – 6 satisfy this equation].

Miscellaneous Exercise 2 | Q 2 | Page 92

Choose the correct option from the given alternatives :

If f(x) = $\dfrac{x^2 - 1}{x^2 + 1}$, for every real x, then the minimum ue of f is

1

0

– 1

2

SOLUTION

– 1

Miscellaneous Exercise 2 | Q 3 | Page 92

Choose the correct option from the given alternatives :

A ladder 5 m in length is resting against vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5m/sec. The length of the higher point of ladder when the foot of the ladder is 4.0 m away from the wall decreases at the rate of

1

2

2.5

– 1

SOLUTION

2

Miscellaneous Exercise 2 | Q 4 | Page 92

Choose the correct option from the given alternatives :

Let f(x) and g(x) be differentiable for 0 x < 1 such that f(0) = 0, g(0), f(1) = 6. Let there exist a real number c in (0, 1) such that f'(c) = 2g'(c), then the value of g(1) must be

1. 1
2. **3**
3. 2.5
4. – 1

SOLUTION

3

Miscellaneous Exercise 2 | Q 5 | Page 92

Choose the correct option from the given alternatives :

Let f(x) = $x^3 - 6x^2 + 9x + 18$, then f(x) is strictly decreasing in

- $(-\infty, 1)$
- $[3, \infty)$
- $(-\infty,] \cup [3, \infty)$
- **(1, 3)**

SOLUTION

(1, 3)

Miscellaneous Exercise 2 | Q 6 | Page 92

Choose the correct option from the given alternatives :

If x = – 1 and x = 2 are the extreme points of y = $\infty \log x + \beta x^2 + x$, then

- $\infty = -6, \beta = \frac{1}{2}$
- $\infty = -6, \beta = -\frac{1}{2}$
- $\infty = 2, \beta = -\frac{1}{2}$
- $\infty = 2, \beta = \frac{1}{2}$

SOLUTION

$\infty = 2, \beta = -\frac{1}{2}$

[Hint : y = $\infty \log x + \beta x^2 + x$

$\therefore \frac{dy}{dx} = \frac{\infty}{x} + \beta \times 2x + 1$

$= \frac{\infty}{x} + 2\beta x + 1$

f(x) has extreme values at x = – 1 and x = 2

$\therefore$ f'(– 1) = 0 and f(2) = 0

$\therefore \infty + 2\beta$ = 1

and

$\frac{\infty}{2} + 4\beta$ = – 1

By solving these two equations, we get

∞ = 2, β = $-\frac{1}{2}$].

Miscellaneous Exercise 2 | Q 7 | Page 92

Choose the correct option from the given alternatives :

The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1)

1. meets the curve again in second quadrant
2. does not meet the curve again
3. meets the curve again in third quadrant
4. **meets the curve again in fourth quadrant**

SOLUTION

meets the curve again in fourth quadrant

[Hint : $x^2 + 2xy - 3y^2 = 0$

$\therefore 2x + 2\left(\frac{x}{dx} + y \times 1\right) - 3 \times 2y\frac{dy}{dx}$ = 0

$\therefore (2x - 6y)\frac{dy}{dx} = -2x - 2y$

$\therefore \frac{dy}{dx} = \frac{-(x+y)}{x-3y}$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(1,1)} = \frac{-1(1+1)}{1-3}$ = 1

= slope of the tangent at (1, 1)

= slope of the normal at (1, 1) is – 1

= equation of the normal is

y – 1

$= -1(x - 1)$

$= -x + 1$

$\therefore x + y = 2$

$\therefore y = 2 - x$

Substituting $y = 2 - x$ in $x^2 + 2xy - 3y^2 = 0$, we get
$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$
$\therefore x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0$
$\therefore x^2 - 4x + 3 = 0$
$\therefore (x - 1)(x - 3) = 0$
$\therefore x = 1, x = 3$
When $x = 1, y = 2 - 1$
When $x = 3, y = 2 - 3 = -1$
$\therefore$ the normal at (1, 1) meets the curve at (3, 91) which is in the fourth quadrant].

Miscellaneous Exercise 2 | Q 8 | Page 92

Choose the correct option from the given alternatives :

The equation of the tangent to the curve $y = 1 - e^{\frac{x}{2}}$ at the point of intersection with Y-axis is

$x + 2y = 0$

$2x + y = 0$

$x - y = 2$

$x + y = 2$

SOLUTION

$x + 2y = 0$
[Hint: The point of intersection of the curve with Y-axis is the origin (0 0)].

Miscellaneous Exercise 2 | Q 9 | Page 92

Choose the correct option from the given alternatives :

If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is

(4, 4)

(– 1, 2)

(3, 6)

$\left(\frac{9}{4}, \frac{3}{8}\right)$

SOLUTION

$$\left(\frac{9}{4}, \frac{3}{8}\right)$$

[Hint : $y^2 = x(2 - x)^2 = x(4 - 4x + x^2) = x^3 - 4x^2 - 4x$

$$\therefore 2y\frac{dy}{dx} = 3x^2 - 8x + 4$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - 8x + 4}{2y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(1,1)} = \frac{3(1)^2 - 8(1) + 4}{2(1)} = \frac{1}{2}$$

= slope of the tangent at (1, 1)

∴ equation of the tangent at (1, 1) is

$$y - 1 = -\frac{1}{2}(x - 1)$$

∴ $2y - 2 = -x + 1$

∴ $x + 2y = 3$

Only the coordinates $\left(\frac{9}{4}, \frac{3}{8}\right)$ satisfy both the

equations $y^2 = x(2 - x)^2$ and $x + 2y = 3$

∴ P is $\left(\frac{9}{4}, \frac{3}{8}\right)$].

Miscellaneous Exercise 2 | Q 10 | Page 92

Choose the correct option from the given alternatives :

The approximate value of tan (44° 30°), given that 1° = 0.0175, is

1. 0.8952
2. 0.9528
3. 0.9285
4. **0.9825**

SOLUTION

0.9825

MISCELLANEOUS EXERCISE 2 [PAGES 93 - 94]

Miscellaneous Exercise 2 | Q 1 | Page 93

Solve the following : If the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$, intersect orthogonally, then prove that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

SOLUTION

Let $P(x_1, y_1)$ be the point of intersection of the curves.

$\therefore ax_1^2 + by_1^2 = 1$ and $a'x_1^2 + b'y_1^2 = 1$

$\therefore ax_1^2 + by_1^2 = a'x_1^2 + b'y_1^2$

$\therefore (a - a')x_1^2 = (b' - b)y_1^2$ ...(1)

Differentiating $ax^2 + by^2 = 1$ w.r.t. x, we get

$$a \times 2x + b \times 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-ax}{by}$$

$\therefore$ slope of the tangent at $(x_1, y_1) = m_1$

$$= \left(\frac{dy}{dx}\right)_{\text{at}(x_1,y_1)} = \frac{-a'x_1}{b'y_1}$$

Since. curves intersect each other orthogonally,

$m_1m_2 = -1$

$$\therefore \left(\frac{-ax_1}{by_1}\right)\left(\frac{-a'x_1}{b'y_1}\right) = -1$$

$$\therefore \frac{aa\prime x_1^2}{b\,b\prime y_1^2} = -1$$

$$\therefore \frac{aa\prime}{bb\prime} = \frac{-y_1^2}{x_1^2}$$

$$\therefore \frac{aa\prime}{bb\prime} = \left(\frac{a-a\prime}{b-b\prime}\right) \qquad \text{...[By (1)]}$$

$$\therefore \frac{a-a\prime}{aa\prime} = \frac{a-a\prime}{bb\prime}$$

$$\therefore \frac{1}{a}\prime - \frac{1}{a} = \frac{1}{b}\prime - \frac{1}{b}$$

$$\therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{a}\prime - \frac{1}{b}\prime.$$

Miscellaneous Exercise 2 | Q 2 | Page 93

Solve the following : Determine the area of the triangle formed by the tangent to the graph of the function $y = 3 - x^2$ drawn at the point (1, 2) and the coordinate axes.

SOLUTION

$y = 3 - x^2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(3 - x^2\right) = 0 - 2x = -2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{at(1,2)} = -2(1) = -2$$

= slope of the tangent at (1 2)

∴ equation of the tangent at (1, 2) is

$y - 2 = 2(x - 1)$

∴ $y - 2 = -2x + 2$

∴ $2x + y = 4$

Let this tangent cuts the coordinate axes at A (a, 0) and B (0, b).

$\therefore$ 2a + 0 = 4 and 2(0) + b = 4

$\therefore$ a = 2 and b = 4

area of required triangle

= $\frac{1}{2} \times l(\text{OA}) \times l(\text{OB})$

= $\frac{1}{2}ab$

= $\frac{1}{2}(2)(4)$

= 4 sq units.

Miscellaneous Exercise 2 | Q 3 | Page 93

Solve the following : Find the equation of the tangent and normal drawn to the curve $y^4 - 4x^4 - 6xy = 0$ at the point M (1, 2).

SOLUTION

$y^4 - 4x^4 - 6xy = 0$

Differentiating both sides w.r.t x, we get

$$4y^2\frac{dy}{d} - 4 \times 4x^3 - 6\left[x\frac{dy}{dx} + y.\frac{d}{dx}(x)\right] = 0$$

$$\therefore 4y^3\frac{dy}{dx} - 16x^3 - 6x\frac{dy}{dx} - 6y \times 1 = 0$$

$$\therefore \left(4y^3 - 6x\right)\frac{dy}{dx} = 16x^3 + 6y$$

$$\therefore \frac{dy}{dx} = \frac{16x^3 + 6y}{4y^2 - 6x}$$

$$= \frac{8x^3 + 3y}{2y^3 - 3x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(1,2)} = \frac{8(1)^3 + 3(2)}{2(2)^3 - 3(1)}$$

$= \dfrac{8+6}{16-3}$

$= \dfrac{14}{13}$

= slope of the tangent at (1, 2)

∴ the equation of the tangent at M (1, 2) is

$y - 2 = \dfrac{14}{13}(x-1)$

∴ 13y – 26 = 14x – 14

∴ 14x – 13y + 12 = 0

The slope of normal at (1, 2)

$= \dfrac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(1,2)}}$

$= \dfrac{-1}{\left(\frac{14}{13}\right)}$

$= -\dfrac{13}{14}$

∴ the equation of normal at M (1, 2) is

$y - 2 = \dfrac{13}{14}(x-1)$

∴ 14y – 28 = 13x – 13

∴ 13x + 14y – 41 = 0

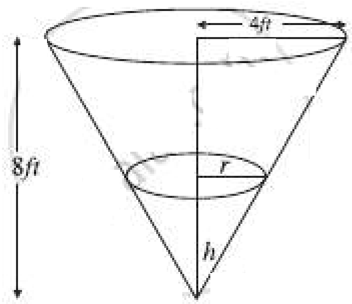
Hence, the equations of tangent and normal are

14x – 13y + 12 = 0 and 13x + 14y – 41 = 0 respecttively.

Miscellaneous Exercise 2 | Q 4 | Page 93

Solve the following : A water tank in the farm of an inverted cone is being emptied at the rate of 2 cubic feet per second. The height of the cone is 8 feet and the radius is 4 feet. Find the rate of change of the water level when the depth is 6 feet.

SOLUTION



Let r be the radius of base, h be the height and V be the volume of the water level at any time t. Since, the height of the cone is 8 feet and the radius s 4 feet,

$$\frac{r}{h} = \frac{4}{8} = \frac{1}{2}$$

$$\therefore r = \frac{h}{2} \qquad \text{...(1)}$$

Given : $\frac{dV}{dt} = \frac{2\text{cufeet}}{\text{sec}}$.

Now, $V = \frac{1}{3}r^2h$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \qquad \text{...[By (1)]}$$

$$\therefore V = \frac{\pi}{12}h^3$$

Differentiating w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \frac{dh}{dt}$$

$$= \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dv}{dt}$$

When h = 6, then

$$\frac{dh}{dt} = \frac{4}{\pi(6)^2} \times 2$$

$= \left(\frac{2}{9\pi}\right)\frac{\text{ft}}{\text{sec}}$

Hence, the rate of change of water level is $\left(\frac{2}{9\pi}\right)\frac{\text{ft}}{\text{sec}}$.

Miscellaneous Exercise 2 | Q 5 | Page 93

Solve the following : Find all points on the ellipse $9x^2 + 16y^2 = 400$, at which the y-coordinate is decreasing and the coordinate is increasing at the same rate.

SOLUTION

L P(x_1, y_1) be the point on the ellipse $9x^2 + 16y^2 = 400$, at which the y-coordinate decreasing and the x-coordinate is increasing at the same rate..

Then $-\left(\frac{dy}{dt}\right)_{\text{at}(x_1,y_1)} = \left(\frac{dx}{dy}\right)_{\text{at}(x_1,y_1)}$ ...(1)

Differentiating $9x^2 + 16y^2 = 400$ w.r.t. t, we get

$9 \times 2x\frac{dx}{dt} + 16 \times 2y\frac{dy}{dx} = 0$

$\therefore 9x\frac{dx}{dt} + 16y\frac{dy}{dt} = 0$

$\therefore 9x_1\left(\frac{dx}{dt}\right)_{\text{at}(x_1,y_1)} + 16y_1\left(\frac{dy}{dt}\right)_{\text{at}(x_1,y_1)} = 0$

$\therefore 9x_1\left(\frac{dx}{dt}\right)_{\text{at}(x_1,y_1)} - 16y_1\left(\frac{dy}{dt}\right)_{\text{at}(x_1,y_1)} = 0$ ..[By (1)]

$\therefore$ 9x1 −116y1 = 0 ...(2)

Now, (x1, y1) lies on the ellipse $9x^2 + 16y^2 = 400$

$\therefore 9x_1^2 + 16y_1^2 = 400$

From (2), $x_1 = \frac{16y_1}{9}$

Substitute $x_1 = \frac{16y_1}{9}$ in (3), we get

$\therefore 9\left(\dfrac{16y_1}{9}\right)^2 + 16_1^2$ = 400

$\therefore$ $16y_1^2 + 9y_1^2 = 225$

$\therefore$ $25y_1^2 = 225$

$\therefore$ $y_1^2 = 9$

$\therefore$ $y_1 = \pm 3$

When $y_1 = 3$, $x_1 = \dfrac{16(3)}{9} = \dfrac{16}{3}$

When $y_1 = -3$, $x_1 = \dfrac{16(-3)}{9} = -\dfrac{16}{3}$

Hence, the required points on the ellipse are

$\left(\dfrac{16}{3}, 3\right)$ and $\left(-\dfrac{16}{3}, -3\right)$.

Miscellaneous Exercise 2 | Q 6 | Page 93

Solve the following : Verify Rolle's theorem for the function f(x) $\dfrac{2}{e^x + e^{-x}}$ on [– 1, 1].

SOLUTION

The function e^X, e^{-X} and 2 are continuous and differentiable on their respective domains.

$\therefore$ f(x) = $\dfrac{2}{e^x + e^{-x}}$ is continuous on [– 1, 1] and differentiating on (–1, 1), because $e^X + e^{-X} \neq 0$ x ∈[– 1, 1].

Now f(– 1) = $\dfrac{2}{e^{-1} + e} = \dfrac{2}{e + e^{-1}}$

and

f(1) = $\dfrac{2}{e + e^{-1}}$

$\therefore$ f(c 1) = f(1)

Thus, the function f satisfies all the conditions of the Rolle's theorem.

$\therefore$ there exist c ∈ (–1, 1) such that f'(c) = 0

Now, f(x) = $\dfrac{2}{e + e^{-1}}$

∴ f(x) = $\dfrac{d}{dx}\left(\dfrac{2}{e^x + ^{-x}}\right)$

= $2\dfrac{\mathrm{d}}{dx}\left(e^x + e^{-1}\right)$

= $2(-1)\left(e^x + e^{-1}\right).\dfrac{d}{dx}\left(e^x + e^{-x}\right)$

= $\dfrac{-2}{\left(e^x + e^{-x}\right)^2} \times \left[e^x + e^{-x} - (-1)\right]$

= $\dfrac{-2\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$

∴ f'(c) = $\dfrac{-2\left(e^c - e^{-c}\right)}{\left(e^c + e^c\right)^2}$

∴ f'(c) = $0\text{gives}, \dfrac{-2\left(e^c - e^{-c}\right)}{\left(e^c + e^c\right)^2}$ = 0

∴ $e^c - e^{-c}$ 0

∴ $e^c = ^{-c} = \dfrac{1}{e^c}$

∴ $e^{2c} = 1 = e^{\circ}$

∴ 2c = 0

∴ c = 0 ∈(−1, 1)

Hence, Rolle's theorem is verified.

Miscellaneous Exercise 2 | Q 7 | Page 93

Solve the following : The position of a particle is given by the function s (t) = $2t^2 + 3t - 4$. Find the time t = c in the interval 0 ≤ t ≤ 4 when the instantaneous velocity of the particle equal to its average velocity in this interval.

SOLUTION

s (t) = $2t^2 + 3t - 4$

∴ $s(0) = 2(0)^2 + 3(0) - 4 = -4$

and

$s(4) = 2(4)^2 + 3(4) - 4 = 32 + 12 - 4 = 40$

∴ average velocity = $\frac{s(4) - s(0)}{4 - 0}$

= $\frac{40 - (-4)}{4}$

= 11

Also , instantaneus velocity = $\frac{\mathrm{ds}}{dt}$

= $\frac{d}{dt}\left(2t^2 + 3t - 4\right)$

= 2 x 2t + 3 x 1 – 0

= 4t + 3

∴ instantaneus velocity at t = c is $\left(\frac{\mathrm{ds}}{dt}\right)_{t=c} = 4c + 3$

When instantaneous velocity at t:c equal to its average velocity, we get

4x + 3 11

∴ 4c = 8

∴ c = 2 ∈[0, 4]

Hence, t = c = 2.

Miscellaneous Exercise 2 | Q 8 | Page 93

Find the approximate value of the function f(x) = $\sqrt{x^2 + 3x}$ at x = 1.02.

SOLUTION

f(x) = $\sqrt{x^2 + 3x}$

∴ f'(x) = $\frac{d}{dx}\left(\sqrt{x^2 + 3x}\right)$

= $\frac{1}{2\sqrt{x^2 + 3x}} \cdot \frac{d}{dx}\left(x^2 + 3x\right)$

= $\frac{1}{2\sqrt{x^2 + 3x}} \times (2x + 3 \times 1)$

= $\frac{2x + 3}{2\sqrt{x^2 + 1}}$

Take a = 1 and h = 0.02.

Then f(a) = f(1) = $\sqrt{1^2 + 3(1)}$ = 2

and

f'(a) = f'(1)

= $\frac{2(1) + 3}{2\sqrt{1^2 + 3(1)}}$

= $\frac{5}{2 \times 2}$

= $\frac{5}{4}$

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ f(1.02) = f(1 + 0.02)

≑ f(1) + (0.02)f'(1)

≑ $2 + 0.02 \times \frac{5}{4}$

≑ $\frac{8 + 0.1}{4}$

$= \dfrac{8.1}{4}$

= 2.025

∴ f1.02) ≑ 2.025.

Miscellaneous Exercise 2 | Q 9 | Page 93

Solve the following : Find the approximate value of $\cos^{-1}(0.51)$, given π = 3.1416, 2/√3 = 1.1547.

SOLUTION

Let $f(x) = \cos^{-1} x$.

Then $f'(x) = \dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$

Take a = 0.5 and h = 0.01

Then f(a) = f(0.5)

$= \cos^{-1}(0.5)$

$= \cos^{-1}\left(\cos \dfrac{\pi}{3}\right)$

$= \dfrac{\pi}{3}$

and

f'(a) = f'(0.5)

$= -\dfrac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$

$= -\dfrac{2}{\sqrt{3}}$

= − 1.1547

The formula for approximation is

f(a + h) ≑ f(a) + h.f'(a)

$\therefore \cos^{-1}(0.51) = f(0.51)$

= f(0.5 + 0.01)

≑ f(0.5) + (0.01)f'(0.5)

$$≑ \frac{\pi}{3} + 0.01 \times (-1.1547)$$

$$≑ \frac{3.1416}{3} - 0.011547$$

≑ 1.0472 - 0.01157 = 1.035653

$\therefore \cos^{-1}(0.51) ≑ 1.035653.$

Miscellaneous Exercise 2 | Q 10 | Page 93

Solve the following : Find the intervals on which the function $y = x^x$, (x > 0) is increasing and decreasing.

SOLUTION

$y = x^x$

$\therefore \log y = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(x \log x)$$

$$= x.\frac{d}{dx}(\log x) + (\log x).\frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{dy}{dx} = y(1 + \log x)$$

$= x^x(1 + \log x)$

y is increasing if $\frac{dy}{dx} \geq 0$

i.e. if $x^x (1 + \log x) \geq 0$

i.e. if $1 + \log x \geq 0$ ...[$\because x > 0$]

i.e. if $\log x \geq -1$

i.e. if $\log x \geq -\log e$ ...[$\because \log e = 1$]

i.e. if $\log x \geq \log \frac{1}{e}$

i.e. if $x \geq \frac{1}{e}$

$\therefore$ y is increasing in $\left[\frac{1}{e}, \infty\right)$

y is decreasing if $\frac{dy}{dx} \leq 0$

i.e. if $x^x (1 + \log x) \leq 0$

i.e. if $1 + \log x \leq 0$ ...[$\because x > 0$]

i.e. if $\log x \leq -1$

i.e. if $\log x \leq -\log e$ ...[$\because \log e = 1$]

i.e. if $\log x \leq \log \frac{1}{e}$

i.e. if $x \leq \frac{1}{e}$, where $x > 0$

$\therefore$ y is decreasing is $\left(0, \frac{1}{e}\right]$

Hence, the given function is increasing $\left[\frac{1}{e}, \infty\right)$

and decreasing in $\left(0, \frac{1}{e}\right]$.

Miscellaneous Exercise 2 | Q 11 | Page 93

Solve the following : Find the intervals on which the function f(x) = x/logx is increasing and decreasing.

SOLUTION

f(x) = $\frac{x}{\log x}$

$\therefore$ f'(x) = $\frac{d}{dx}\left(\frac{x}{\log x}\right)$

= $\frac{(\log x).\frac{d}{dx}(x) - x.\frac{d}{dx}(\log x)}{(\log x)^2}$

= $\frac{(\log x) \times 1 - x \times \frac{1}{x}}{(\log x)^2}$

= $\frac{\log x - 1}{(\log x)^2}$

f is increasing if f'(x) ≥ 0

i.e. if $\frac{\log x - 1}{(\log x)^2} \geq 0$

i.e. f log x – 1 ≥ 0 ...[$\because$ $(\log x)^2 > 0$]

i.e. if log x ≥ 1

i.e. if log x ≥ log e ...[$\because$ log e = 1]

i.e. if x ≥ e

$\therefore$ f is increasing n $[e, \infty]$

f is decreasing if f'(x) ≤ 0

ie. if $\frac{\log x - 1}{(\log x)^2} \leq 0$

i.e. f log x – 1 ≤ 0 ...[$\because$ $(\log x)^2 > 0$]

i.e. if log x ≤ 1

i.e. if log x ≤ log e

i.e. if x ≤ e

Also, x > 0 and x ≠ 1 beause (x) = $\frac{x}{\log x}$ is not defined at = 1.

∴ f is decreasing in (0, e) – { 1 }

Hence is increasing in $[e, \infty)$ and

decreasing in (1, e].

Miscellaneous Exercise 2 | Q 12 | Page 93

Solve the following : An open box with a square base is to be made out of given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

SOLUTION

Let x be the side of square base and h be the height of the box.

Then $x^2 + 4xh = a^2$

∴ h = $\frac{a^2 - x^2}{4x}$ ...(1)

Let V be the volume of the box.

Then V = x^2h

∴ V = $x^2\left(\frac{a^2 - x^2}{4x}\right)$ ...[By (1)]

∴ V= $\frac{1}{4}\left(a^2x - x^3\right)$ ...(2)

∴ $\frac{\mathrm{dV}}{dx} = \frac{1}{4}\left(a^2x - x^3\right)$

= $\frac{1}{4}\left(a^2 \times 1 - 3x^2\right)$

= $\frac{1}{4}\left(a^2 - 3x^2\right)$

and

$$\frac{d^2V}{dx^2} = \frac{1}{4}.\frac{d}{dx}\left(a^2 - 3x^2\right)$$

$= \frac{1}{4}(0 - 3 \times 2x)$

$= -\frac{3}{2}x$

Now, $\frac{\mathrm{dV}}{dx} = 0$ gives $\frac{1}{4}\left(a^2 - 3x^2\right) = 0$

$\therefore a^2 - 3x^2 = 0$

$\therefore 3x^2 = a^2$

$\therefore x^2 = \frac{a^2}{3}$

$\therefore x = \frac{a}{\sqrt{3}}$ ...[$\because x > 0$]

and

$\left(\frac{d^2V}{dx^2}\right)_{\text{at} x = \frac{a}{\sqrt{3}}}$

$= -\frac{3}{2} \times \frac{a}{\sqrt{3}}$

$= -\frac{\sqrt{3}}{2}a < 0$

$\therefore$ V is maximum when $x = \frac{a}{\sqrt{3}}$

From (2), maximum volume = $\left[\frac{1}{4}\left(a^2x - x^3\right)\right]_{\text{at} x = \frac{a}{\sqrt{3}}}$

$= \frac{1}{4}\left(a^2 \times \frac{a}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right)$

$= \frac{1}{4}\left(\frac{2a^3}{3\sqrt{3}}\right.$

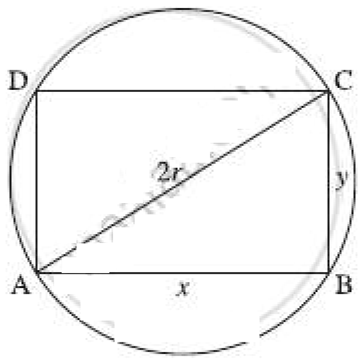
$= \dfrac{a^3}{6\sqrt{3}}$

Hence, the maximum volume of the box is $\dfrac{a^3}{6\sqrt{3}}$ cu unit.

Miscellaneous Exercise 2 | Q 13 | Page 93

Solve the following : Show that of all rectangles inscribed in a given circle, the square has the maximum area.

SOLUTION



Let ABCD be a rectangle inscribed in a circle of radius r. Let AB = x and BC = y.
Then $x^2 + y^2 = 4r^2$ … (1)

Area of rectangle = xy

$= x\sqrt{4r^2 - x^2}$ ...[By (1)]

Let $f(x) = x^2(4t^2 - x^2)$

$= 4r^2x^2 - x^4$

$\therefore f'(x) = \dfrac{d}{dx}\left(4r^2x^2 - x^4\right)$

$= 4r^2 \times 2x - 4x^3$

$= 8r^2x - 4x^3$

and

$f''(x) = \dfrac{d}{dx}\left(8r^2x - 4x^3\right)$

$= 8r^2 \times 1 - 4 \times 3x^2$

$= 8r^2 - 12x^2$

For maximum area, $f'(x) = 0$

$\therefore 8r^2x - 4x^3 = 0$

$\therefore 4x^3 = 8r^2x$

$\therefore x^2 = 2r^2$ ...[$\because x \neq 0$]

$\therefore x = \sqrt{2}r$ ...[$\because x > 0$]

and

$$f\left(\sqrt{2r}\right) = 8r^2 - 12\left(\sqrt{2r}\right)$$

$= -16r^2 < 0$

$\therefore$ f(x) is maximum when $x = \sqrt{2}r$

If $x = \sqrt{2}r$, then from (1),

$$\left(\sqrt{2r}\right)^2 + y^2 = 4r^2$$

$\therefore y^2 = 4r^2 - 2r^2 = 2r^2$

$\therefore y = \sqrt{2}r$ ...[$\because y > 0$]

$\therefore x = y$

$\therefore$ rectangle is a square.

Hence, amongst all rectangles inscribed in a circle, the square has maximum area.

Miscellaneous Exercise 2 | Q 14 | Page 93

Solve the following : Show that a closed right circular cylinder of given surface area has maximum volume if its height equals the diameter of its base.

SOLUTION

Let r be the radius of the base, h be the height and V be the volume of the closed right circular cylinder, whose surface area is a^2 sq units (which is given).

$\therefore 2\pi rh + 2\pi r^2 = a^2$

$\therefore 2\pi r (h + r) = a^2$

h = $\frac{a^2}{2\pi r} - r$ ...(1)

Now, V = $\pi r^2 h$

= $\pi r^2\left(\frac{a^2}{2\pi r} - r\right)$ ...[By (1)]

= $\frac{1}{2}a^2 r - \pi r^3$

$\therefore \frac{\mathrm{dV}}{\mathrm{dr}} = \frac{d}{dx}\left(\frac{1}{2}a^2 r - \pi r^2\right)$

= $\frac{1}{2}a^2 \times 1 - \pi \times 3r^2$

= $\frac{a^2}{2} - 3ar^2$

and

$\frac{d^2V}{dr^2} = \frac{d}{\mathrm{dr}}\left(\frac{a^2}{2} - 3\pi r^2\right)$

= 0 – 3π x 2r

= – 6πr

For maximum volume, $\frac{\mathrm{dV}}{\mathrm{dr}}$ = 0

$\therefore \frac{a^2}{2} - 3\pi r^2$ = 0

$\therefore$ $3\pi^2$ = $\frac{a^2}{2}$

$\therefore$ r^2 = $\frac{a^2}{6\pi}$

$\therefore$ r = $\frac{a}{\sqrt{6\pi}}$ ...[$\because$ r > 0]

and

$$\left(\frac{d^2V}{dr^2}\right)_{\text{at } x=\frac{a}{\sqrt{6x}}}$$

$$= -6\pi\left(\frac{a}{\sqrt{6\pi}}\right) < 0$$

$\therefore$ V is maximum when r = $\dfrac{a}{\sqrt{6\pi}}$

When r = $\dfrac{a}{\sqrt{6\pi}}$, then from (1),

$$h = \frac{a^2}{2\pi \times \frac{a}{\sqrt{6\pi}}} - \frac{a}{\sqrt{6\pi}}$$

$$= \frac{\sqrt{6\pi a}}{2\pi} - \frac{a}{\sqrt{6\pi}}$$

$$= \frac{6\pi a - 2\pi a}{2\pi\sqrt{6\pi}}$$

$$= \frac{4\pi a}{2\pi\sqrt{6\pi}}$$

$$= \frac{2a}{\sqrt{6\pi}}$$

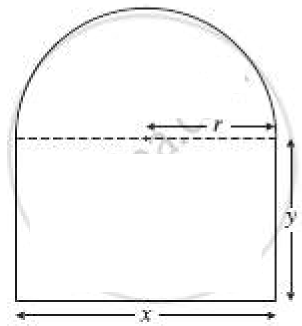
$\therefore$ h = 2r

Hence, the volume of the cylinder is maximum if its height is equal to the diameter of the base.

Miscellaneous Exercise 2 | Q 15 | Page 93

Solve the following : A window is in the form of a rectangle surmounted by a semicircle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

SOLUTION



Let x be the length, y be the breadth of the rectangle and r be the radius of the semicircle. Then perimeter of the window

Then perimeter of the window = x + 2y + πr, where x = 2r

This is given to be m
∴ 2r + 2y + πr = 30
2y = 30 – (π + 2)r

$$\therefore y = 15 - \frac{(\pi + 2)r}{2} \qquad ...(1)$$

The greatest possible amount of light may be admitted if the area of the window is maximum. Let A be the area of the window.

$$\text{Then } A = xy + \frac{\pi r^2}{2}$$

$$= 2yr + \frac{\pi r^2}{2} \qquad ...[\because x = 2r]$$

$$= 2r\left[15 - \frac{(\pi + 2)}{2}\right] + \frac{\pi r^2}{2} \qquad ...[\text{By } (1)]$$

$$= 30r - (\pi + 2)r^2 + \frac{\pi}{2}r^2$$

$$= 30r - \left(\pi + 2 - \frac{\pi}{2}\right)r^2$$

$$\therefore A = 30r - \left(\frac{\pi + 4}{2}\right)r^2$$

$$\therefore \frac{\mathrm{dA}}{\mathrm{dr}} = \frac{d}{\mathrm{dr}}\left[30r - \left(\frac{\pi + 4}{2}\right)r^2\right]$$

= $30 \times 1-\left(\frac{\pi+4}{2}\right) \times 2r$

= 30 – (π + 4)r

and

$\frac{d^2\mathrm{A}}{\mathrm{dr}}=\frac{d}{\mathrm{dr}}[30-(\pi+4)r]$

For maximum volume, $\frac{\mathrm{dV}}{\mathrm{dr}}$ = 0

∴ $\frac{a^2}{2}-3\pi r^2$ = 0

∴ 3π^2 = $\frac{a^2}{2}$

∴ r^2 = $\frac{a^2}{6\pi}$

∴ r = $\frac{a}{\sqrt{6\pi}}$ ...[∵ r > 0]

and

$\left(\frac{d^2V}{dr^2}\right)_{\text{at } x=\frac{a}{\sqrt{6\pi}}}$

= $-6\pi\left(\frac{a}{\sqrt{6\pi}}\right)<0$

∴ V is maximum when r = $\frac{a}{\sqrt{6\pi}}$

When r = $\frac{a}{\sqrt{6\pi}}$, then from (1),

$= \dfrac{30}{\pi + 4}$

Hence, the required dimensions of the window are as follows :

Length of rectangle = $\left(\dfrac{60}{\pi + 4}\right)$ metres,

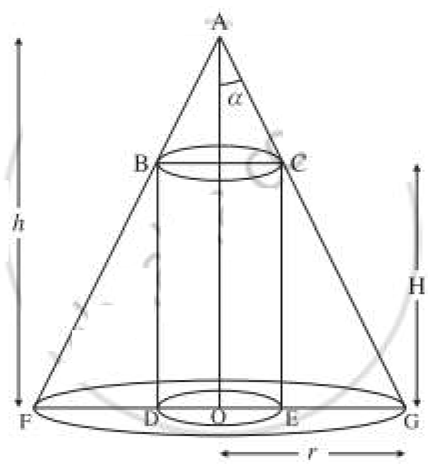
breadth of rectangle = $\left(\dfrac{30}{\pi + 4}\right)$ metres and

radius of the semicircle = $\left(\dfrac{30}{\pi + 4}\right)$ metres.

Miscellaneous Exercise 2 | Q 16 | Page 93

Solve the following : Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third of that of the cone.

SOLUTION



Given the right circular cone of fixed height h and semi-vertical angle ∞.

Let R be the radius of the base and H be the height of right circular cylinder that can be inscribed in the right circular cone.

In the figure, ∠GAO = ∞, OG = r, OA = h, OE = R, CE = H.

We have $\frac{r}{h} = \tan x$

$\therefore$ r = h tan x ...(1)

Since ΔAOG anf ΔCEG are similar.

$$\therefore \frac{\mathrm{AO}}{\mathrm{OG}} = \frac{\mathrm{CE}}{\mathrm{EG}} = \frac{\mathrm{CE}}{\mathrm{OG - OE}}$$

$$\therefore \frac{h}{r} = \frac{\mathrm{H}}{\mathrm{r - R}}$$

$\therefore$ H = $\frac{h}{r}(r - R)$

= $\frac{\mathrm{h}}{\mathrm{h\ tan\ oo}}(h \tan \infty - R)$ ...[By (1)]

$\therefore$ H = $\frac{1}{\tan \infty}(h \tan \infty - R)$ ...(2)

Let V be the volume of the cylinder

Then V = $\pi R^2 H = \frac{\pi R^2}{\tan \infty}(h \tan \infty - R)$

$\therefore$ V = $\pi R2h - \frac{\pi R^3}{\tan \infty}$

$$\therefore \frac{\mathrm{dV}}{\mathrm{dR}} = \frac{d}{\mathrm{dR}}\left(\pi R^2 h - \frac{\pi R^3}{\tan \infty}\right)$$

= $\pi R \times 2R - \frac{\pi}{\tan \infty} \times 3R^2$

= $2\pi Rh - \frac{3\pi R^2}{\tan \infty}$

and

$$\frac{d^2 V}{dR^2} = \frac{d}{\mathrm{dR}}\left(2\pi Rh - \frac{3\pi r^2}{\tan \infty}\right)$$

= $2\pi h \times 1 - \frac{3\pi}{\tan \infty} \times 2R$

$= 2\pi h - \dfrac{6\pi R}{\tan\infty}$

For maximum volume, $\dfrac{\mathrm{dV}}{\mathrm{dR}} = 0$

$\therefore 2\pi Rh - \dfrac{3\pi R^2}{\tan\infty} = 0$

$\therefore \dfrac{3\pi R2}{\tan\infty} = 2\pi Rh$

$\therefore$ R = $\dfrac{2h}{3}\tan\infty$ ...[$\because$ R ≠ 0]

and

$\left(\dfrac{d^2V}{dR^2}\right)_{\text{at } R=\frac{2h}{3}\tan x}$

$= 2\pi h - \dfrac{6\pi}{\tan\infty} \times \dfrac{2h}{3}\tan\infty$

= 2πh – 4πh = – 2πh < 0

$\therefore$ V is maximum when R = $\dfrac{2h}{3}\tan\infty$

When R = $\dfrac{2h}{3}\tan\infty$, then from (2), we get

H = $\dfrac{1}{\tan\infty}\left(h\tan\infty - \dfrac{2h}{3}\tan n\infty\right) = \dfrac{h}{3}$

Hence, the height of the right circular cylinder is one- third of that of the cone.

Miscellaneous Exercise 2 | Q 17 | Page 93

Solve the following:

A wire of length l is cut into two parts. One part is bent into a circle and the other into a square. Show that the sum of the areas of the circle and the square is the least, if the radius of the circle is half of the side of the square.

SOLUTION

Let r be the radius of the circle and x be the length of the side of the square. Then (circumference of the circle) + (perimeter of the square) = l

∴ $2\pi r + 4x = l$

∴ $r = \frac{l-4x}{2\pi}$

A = (area of the circle) + (area of the square)

$= \pi r^2 + x^2$

$= \pi\left(\frac{l-4x}{2\pi}\right)^2 + x^2$

$= x^2 + \frac{1}{4\pi}(l-4x)^2$

$=$ f(x) ...(Say)

Then $f'(x) = 2x + \frac{1}{4\pi} \times 2(l-4x)(-4)$

$= 2x - \frac{2}{\pi}(l-4x)$

and

$f''(x) = 2 - \frac{2}{\pi}(-4)$

$= 2 + \frac{8}{\pi}$

Now, $f'(x) = 0$ when $2x - \frac{2}{\pi}(l-4x) = 0$

i.e. when $2\pi x - 2l + 8x = 0$

i.e when $2(\pi + 4)x = 2l$

i.e. when $x = \frac{l}{\pi + 4}$

and

f" $\left(\dfrac{l}{\pi+4}\right) = 2 + \dfrac{8}{\pi} > 0$

∴ by the second derivative test, f has a minimum,

When x = $\dfrac{l}{\pi+4}$. For this value of x,

r = $\dfrac{l - 4\left(\frac{l}{\pi+4}\right)}{2\pi}$

= $\dfrac{\pi l + 4l - 4l}{2\pi(\pi+4)}$

= $\dfrac{l}{2(\pi+4)}$

Now, f'(x) = 0 when $2x - \dfrac{2}{\pi}(l - 4x)$ = 0

i.e. when 2πx – 2l + 8x = 0

i.e when 2(π + 4)x = 2l

i.e. when x = $\dfrac{l}{\pi+4}$

and

f" $\left(\dfrac{l}{\pi+4}\right) = 2 + \dfrac{8}{\pi} > 0$

∴ by the second derivative test, f has a minimum,

When x = $\dfrac{l}{\pi+4}$. For this value of x,

r = $\dfrac{l - 4\left(\frac{l}{\pi+4}\right)}{2\pi}$

= $\dfrac{\pi l + 4l - 4l}{2\pi(\pi+4)}$

= $\dfrac{l}{2(\pi+4)}$

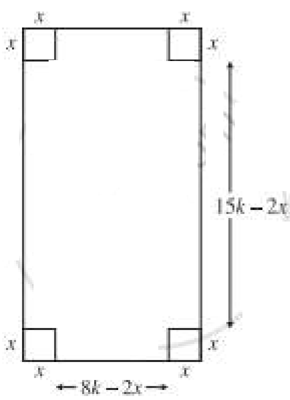
$= 1/x$

This shows that the sum of the areas of circle and square is least, when radius of the circle = (1/2) side of the square.

Miscellaneous Exercise 2 | Q 18 | Page 94

Solve the following : A rectangular sheet of paper of fixed perimeter with the sides having their lengths in the ratio 8 : 15 converted into an open rectangular box by folding after removing the squares of equal area from all corners. If the total area of the removed squares is 100, the resulting box has maximum volume. Find the lengths of the rectangular sheet of paper.

SOLUTION



The sides of the rectangular sheet of paper are in the ratio 8 : 15. Let the sides of the rectangular sheet of paper be 8k and 15k respectively.

Let x be the side of square which is removed from the corners of the sheet of paper. Then total area of removed squares is $4x^2$, which is given to be 100.

$\therefore 4x^2 = 100$
$\therefore x^2 = 25$
$\therefore x = 5$ ...[$\because x > 0$]

Now, length, breadth and the height of the rectangular box are $15k – 2x$, $8k – 2x$ and x respectively.

Let V be the volume of the box.

Then $V = (15k – 2x)(8k – 2x).x$

$\therefore V = (120k^2 – 16kx – 30kx + 4x^2).x$

$\therefore V = 4x^3 – 46kx^2 + 120k^2x$

$$\therefore \frac{dV}{dx} = \frac{d}{dx}\left(4x^2 - 46kx^2 + 120k^2x\right)$$

= 4 x $3x^2$ – 46k x x + $120k^2$ x 1

= $12x^2 - 92kx + 120k^2$

Since, volume is maximum when the square of side x = 5 is removed from the corners, $\left(\frac{dv}{dx}\right)_{\text{at } x=5} = 0$

$\therefore 12(5)^2 - 92k(5) + 120k^2 = 0$

$\therefore 60 - 92k + 24k^2 = 0$

$\therefore 6k^2 - 23k + 15 = 0$

$\therefore 6k^2 - 18k - 5k + 15 = 0$

$\therefore 6k(k - 3) - 5(k - 3) = 0$

$\therefore (k - 3)(6k - 5) = 0$

$\therefore$ k = or $k = \frac{5}{6}$

If k = $\frac{5}{6}$, then 8k – 10 < 0

$\therefore k \neq \frac{5}{6}$

$\therefore$ k = 3

$\therefore$ 8k = 8 x 3 = 24 and 15k = 15 x 3 = 45

Hence, the lengths of the rectangular sheet are 24 and 45.

Miscellaneous Exercise 2 | Q 19 | Page 94

Solve the following : Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is 4r/3.

SOLUTION

Let x be the radius of base and h be the height of the con which is inscribed in a sphere of radius r.

In the figure, AD = h and CD = x = BD

Since, ΔABD and ΔBDE are similar,

$$\frac{\mathrm{AD}}{\mathrm{BD}} = \frac{\mathrm{BD}}{\mathrm{DE}}$$

$\therefore$ BD^2 = AD-DE = AD.(AE – AD)

$\therefore$ x^2 = h(2r – h) ...(1)

Let V be the volume of the cone.

Then V = $\frac{1}{3}\pi x^2 h$

= $\frac{\pi}{3} h(2r - h)h$ ...[By (1)]

$\therefore$ V = $\frac{\pi}{3}\left(2rh^2 - h^3\right)$

$\therefore$ $\frac{\mathrm{dV}}{\mathrm{dh}} = \frac{\pi}{3}\frac{d}{\mathrm{dh}}\left(2rh^2 - h^3\right)$

= $\frac{\pi}{3}\left(2r \times 2h - 3h^2\right)$

= $\frac{\pi}{3}\left(4rh - 3h^2\right)$

and

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}.\frac{d}{\mathrm{dh}}\left(4rh - 3h^2\right)$$

= $\frac{\pi}{3}(4r \times 1 - 3 \times 2h)$

= $\frac{\pi}{3}(4r - 6h)$

For maximm volume, $\frac{\mathrm{dV}}{\mathrm{dh}}$ = 0

$\therefore$ $\frac{\pi}{3}\left(4rh - 3h^2\right)$ = 0

$\therefore 4rh = 3h^2$

$\therefore$ h = $\frac{4r}{3}$ ...[$\because$ h ≠ 0]

and

$$\left(\frac{d2V}{dh^2}\right)_{\text{at } h=\frac{4r}{3}}$$

$$= \frac{\pi}{3}\left(4r - 6 \times \frac{4r}{3}\right)$$

$$= \frac{\pi}{3}(4r - 8r)$$

$$= -\frac{4\pi r}{3} < 0$$

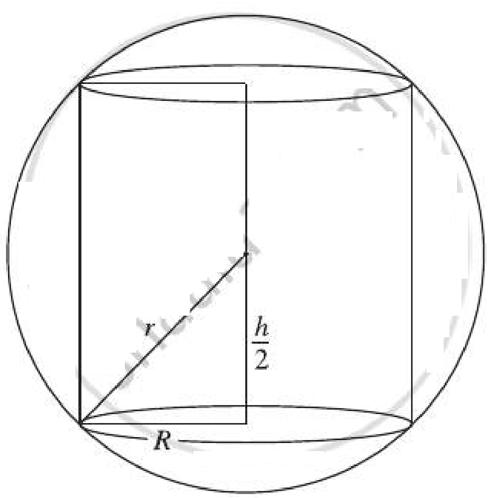
$\therefore$ V is maximum when h = $\frac{4r}{3}$

Hence, the attitude (i.e. height) of the right circular cone of maximum volume = $\frac{4r}{3}$.

Miscellaneous Exercise 2 | Q 20 | Page 94

Solve the following : Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is 2R/√3. Also, find the maximum volume.

SOLUTION



Let R be the radius and h be the height of the cylinder which is inscribed in a sphere of radius r cm.

Then from the figure,

$$R^2 + \left(\frac{h}{2}\right)^2 = r^2$$

$$\therefore R^2 = r^2 - \frac{h^2}{4} \quad ...(1)$$

Let V be the volume of the cylinder.

Then V = $\pi R^2 h$

$$= \pi\left(r^2 - \frac{h^2}{4}\right)h \quad ...[\text{By } (1)]$$

$$= \pi\left(r^2 - \frac{h^3}{4}\right)$$

$$\therefore \frac{dV}{dh} = \pi\frac{d}{dh}\left(r^2h - \frac{h^3}{4}\right)$$

$$= \pi\left(r^2 \times 1 - \frac{1}{4} \times 3h^2\right)$$

$$= \pi\left(r^2 - \frac{3}{4}h^2\right)$$

and

$$\frac{d^2V}{dh^2} = \pi\frac{d}{dh}\left(r^2 - \frac{3}{4}h^2\right)$$

$$= \pi\left(0 - \frac{3}{4} \times 2h\right)$$

$$= -\frac{3}{2}\pi h$$

Now, $\frac{dV}{dh} = 0$ gives, $\pi\left(r^2 - \frac{3}{4}h^2\right) = 0$

$$\therefore r^2 - \frac{3}{4}h^2 = 0$$

$$\therefore \frac{3}{4}h^2 = r^2$$

$\therefore$ h^2 = $\frac{4r^2}{3}$

$\therefore$ h = $\frac{2r}{\sqrt{3}}$ ...[$\because$ h > 0]

and

$\left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\frac{2r}{\sqrt{3}}}$

$= -\frac{3}{2}\pi \times \frac{2r}{\sqrt{3}} < 0$

$\therefore$ V is maximum at h = $\frac{2r}{\sqrt{3}}$

If h = $\frac{2r}{\sqrt{3}}$, then from (1)

R^2 = $r^2 - \frac{1}{4} \times \frac{4r^2}{3} = \frac{2r^2}{3}$

$\therefore$ volumeof the largest cylinder

$= \pi \times \frac{2r^2}{3} \times \frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$cu cm.

Hence, the volume of the largest cylinder inscribed in a sphere of radius 'r' cm = $\frac{4R^3}{3\sqrt{3}}$cu cm.

Miscellaneous Exercise 2 | Q 21 | Page 94

Solve the following : Find the maximum and minimum values of the function f(x) = cos^{2}x + sinx.

SOLUTION

f(x) = cos^{2}x + sin

$\therefore$ f'(x) = $\frac{d}{dx}\left(\cos^2 x + \sin x\right)$

$= 2\cos x . \frac{d}{dx}(\cos x) + \cos x$

= 2 cosx (– sin x) + cos x

= – sin 2x. + cos x

and

f"(x) = $\frac{d}{dx}(-\sin 2x + \cos x)$

= $-\cos 2x. \frac{d}{dx}(2x) - \sin x$

= $-\cos 2x \times 2 - sonx$

= – 2 cos 2x – son x

For extreme values of f(x), f'(x) = 0

∴ – sin 2x+ cos x = 0

∴ – 2 sin x cos x + cos x = 0

∴ cos x (– 2sin x + 1) = 0

∴ cos x = 0 or – 2 sin x + 1 = 0

∴ cos x = $\cos \frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin \frac{\pi}{6}$

∴ $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

(i) $f\left(\frac{\pi}{2}\right) = 2\cos \pi - \sin \frac{\pi}{2}$

= $-2(-1) - 1 = 1 > 0$

∴ by the second derivative test, f is minimum at x = $\frac{\pi}{2}$ and minimum value of f at x = $\frac{\pi}{2}$

= $f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$

(ii) $f\left(\frac{\pi}{6}\right) = -2\cos \frac{\pi}{3} - \sin \frac{\pi}{6}$

= $-2\left(\frac{1}{2}\right) - \frac{1}{2}$

= $-\frac{3}{2} < 0$

$\therefore$ by the second derivative test, f is maximum at x = $\frac{\pi}{6}$ and maxmuum value of f at x $\frac{\pi}{6}$

= $f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6}$

= $\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}$

= $\frac{5}{4}$

Hence, the maximum and minimum values of the 5 function f(x) are $\frac{5}{4}$ and 1 respectively.