2. Matrices

Let us Recall

- If A and B are square matrices of the same order such that $AB = BA = I$ then A and B are called inverses of each other. We denote inverse of A by A^{-1}.
- If A is a non singular matrix then $A^{-1} = \frac{1}{|A|} adj A$.

Ex. (1) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ then find matrices X and Y such that

$AX = B$ and $YB = A$.

Solution : Consider the matrix equation $AX = B$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 2 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 1 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -3 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_3 \to -\frac{1}{3} R_3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$R_1 \to R_1 - R_3 \quad , \quad R_2 \to R_2 - \frac{3}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & \frac{7}{3} & \frac{8}{3} \\ \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 1 & \frac{7}{3} & \frac{8}{3} \\ \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Now consider the equation $YB = A$

$$Y \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1$$

$$Y \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$C_2 \to -1C_2$$

$$Y\begin{bmatrix}1&0&0\\1&1&2\\2&0&1\end{bmatrix}=\begin{bmatrix}1&2&-2\\0&-2&3\\1&0&-2\end{bmatrix}$$

$C_1 \to C_1 - C_2$, $C_3 \to C_3 - 2C_2$

$$Y\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=\begin{bmatrix}-1&2&-6\\2&-2&7\\1&0&-2\end{bmatrix}$$

$C_1 \to C_1 - 2C_3$

$$Y\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}11&2&-6\\-12&-2&7\\5&0&-2\end{bmatrix}$$

$$\therefore Y=\begin{bmatrix}11&2&-6\\-12&-2&7\\5&0&-2\end{bmatrix}$$

Ex. (2) Show that following system of equations has unique solution. Find its solution by the reduction method.

$x+y+z=2$,

$x-2y+z=8$,

$3x+y+z=4$

Solution : We write the given system of equations in matrix equation as :

AX = B , where $A=\begin{bmatrix}1&1&1\\1&-2&1\\3&1&1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}2\\8\\4\end{bmatrix}$

$$\begin{bmatrix}1&1&1\\1&-2&1\\3&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\8\\4\end{bmatrix}$$

$$|A|=\begin{vmatrix}1&1&1\\1&-2&1\\3&1&1\end{vmatrix}=1(-3)-1(-2)+1(7)=6$$

As $|A| \neq 0$, A is non-singular.

$\therefore$ Given system has unique solution.

$$\begin{bmatrix}1 & 1 & 1\\1 & -2 & 1\\3 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\8\\4\end{bmatrix}$$

$R_2 \to R_2 - R_1,\quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix}1 & 1 & 1\\0 & -3 & 0\\0 & -2 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\\-2\end{bmatrix}$$

$R_3 \to -\frac{1}{2}R_3$

$$\begin{bmatrix}1 & 1 & 1\\0 & -3 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\\1\end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{3}R_2$

$$\begin{bmatrix}1 & 1 & 1\\0 & -3 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\\3\end{bmatrix}$$

$\therefore\ x + y + z = 2$

$-3y = 6$

$z = 3$

$\therefore\ x = 1,\ y = -2,\ z = 3$

Ex. (3) If $A = \begin{bmatrix}1 & -1 & 2\\3 & 0 & 2\\1 & 0 & 3\end{bmatrix}$ then verify that $A \times (adjA) = (adjA) \times A = |A| \times I$

Solution : $A = \begin{bmatrix}1 & -1 & 2\\3 & 0 & 2\\1 & 0 & 3\end{bmatrix}$

$\therefore\ |A| = (1)\begin{vmatrix}0 & 2\\0 & 3\end{vmatrix} - (-1)\begin{vmatrix}3 & 2\\1 & 3\end{vmatrix} + (2)\begin{vmatrix}3 & 2\\1 & 3\end{vmatrix} = 1(0)+1(7)+2(0) = 7$

$\therefore\ |A| = 7$

Let us find minors and cofactors. $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0 \ , \ A_{12} = (-1)^3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -7 \ , A_{13} = (-1)^4 \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \quad A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \quad A_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} = -2 \quad A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4 \quad A_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$$

$\therefore$ The matrix of cofactors is $\begin{bmatrix} 0 & -7 & 0 \\ 3 & 1 & -1 \\ -2 & 4 & 3 \end{bmatrix}$

The transpose of the cofactor matrix is the adjoint of A.

$$\therefore \ adj\ A = \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A \times adj\ A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\therefore \ A \times adj\ A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I = |A| \times I \ \ldots. \ (1)$$

$$adj\ A \times A = \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\therefore \ adj\ A \times A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I = |A| \times I \ \ldots. \ (2)$$

From (1) and (2) we get $A \times adj\ A = (adj\ A) \times A = |A| \times I$

Ex. (4) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ then find A^{-1} by elementary column transformations.

Solution : $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

$\therefore \ |A| = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} = 1\begin{vmatrix} 1 & 5 \\ 4 & 7 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$

$= 1\ (-13) - 2(-3) + 3(2) = -1$

As $|A| \neq 0$, A is non singular.

$\therefore \ A^{-1}$ exist.

$A^{-1} \ A = I$

$$A^{-1}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1$

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -1C_2$

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_1 \to C_1 - C_2 \ , \ C_3 \to C_3 - 2C_2$

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -7 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_1 \to C_1 - 2C_3$

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore \; A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

Ex.(5) Show that matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible. Find its inverse by adjoint method.

Solution : $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore \; |A| = (0)\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - (0)\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 0 - 1 = -1$$

$\therefore \; |A| = -1$

As $|A| \neq 0$, A is invertible.

$\therefore \; A^{-1}$ exist.

Let us find minors and cofactors. $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \quad A_{12} = (-1)^3 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad A_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \cancel{0} \; -1$$

$$A_{21} = (-1)^3 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad A_{22} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{23} = (-1)^5 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{31} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{32} = (-1)^5 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad A_{33} = (-1)^6 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$\therefore$ The matrix of cofactors is $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

The transpose of the cofactor matrix is the adjoint of A.

$$\therefore \; adj\, A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{-1}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Ex. (6) If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}$ and AX = B then find X.

Solution :

$AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}$$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -15 \\ -9 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -15 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x + 2y + 3z \\ 3y + 6z \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -15 \\ -6 \end{bmatrix}$$

by equality of matrices

$x + 2y + 3z = -3 \quad -(\text{I})$

$3y + 6z = -15 \quad -(\text{II})$

$\boxed{z = -6}$

Put $z = -6$ in eq (II)

$3y + 6(-6) = -15$

$3y - 36 = -15$

$3y = -15 + 36$

$3y = 21$

$\boxed{y = 7}$

Now Put $y = 7$ and $z = -6$ in eq^n (I)

$x + 14 - 18 = -3$

$x - 4 = -3$

$x = -3 + 4$

$\boxed{x = 1}$

Ex. (7) If $A=\begin{bmatrix}1&0&0\\5&1&0\\1&3&1\end{bmatrix}$ then find matrix B such that AB = I. Verify that BA = I.

Solution :

$A=\begin{bmatrix}1&0&0\\5&1&0\\1&3&1\end{bmatrix}$

$|A|=\begin{vmatrix}1&0&0\\5&1&0\\1&3&1\end{vmatrix}$

$=1(1-0)-0(5-0)+0(15-1)$

$=1-0+0$

$|A|=1$

$\therefore |A| \neq 0$

A is non singular

$\therefore A^{-1}$ exist

Now we will find matrix B

Such that $AB=I$

multiplying by A^{-1} on b.s.

$A^{-1}AB=A^{-1}I$

$IB=A^{-1} \quad \because A^{-1}A=I$

$\boxed{\therefore B=A^{-1}}$ — (I)

Now $AA^{-1}=I$

$\begin{bmatrix}1&0&0\\5&1&0\\1&3&1\end{bmatrix}A^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$

$R_2 \to R_2-5R_1,\ R_3 \to R_3-R_1$

$\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}A^{-1}=\begin{bmatrix}1&0&0\\-5&1&0\\-1&0&1\end{bmatrix}$

$R_3 \to R_3-3R_2$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A^{-1}=\begin{bmatrix}1&0&0\\-5&1&0\\14&-3&1\end{bmatrix}$

$IA^{-1}=\begin{bmatrix}1&0&0\\-5&1&0\\14&-3&1\end{bmatrix}$

$\therefore B=\begin{bmatrix}1&0&0\\-5&1&0\\14&-3&1\end{bmatrix}$ from (I)

we know that

$A^{-1}A=I$

but $A^{-1}=B$

$\therefore BA=I$

Hence Verified.

Sign of Teacher :