# 1 COMPLEX NUMBERS

**Let's Study**

- A complex number (C.N.)
- Algebra of C.N.
- Geometrical Representation of C.N.
- Polar & Exponential form of C.N.
- De Moivre's Theorem.

**Let's Recall**

- Algebra of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane
- Trigonometric ratios

## Introduction:

Consider, the equation $x^2 + 1 = 0$. This equation has no solution in the set of real numbers because there is no real number whose square is $-1$. We need to extend the set of real numbers to a larger set, which would include solutions of such equations.

We introduce a symbol $i$ (greek letter iota) such that $i = \sqrt{-1}$ and $i^2 = -1$. $i$ is called as an **imaginary unit** or an **imaginary number.**

Swiss mathematician Leonard Euler (1707-1783) was the first mathematician to introduce the symbol $i$ with $i = \sqrt{-1}$ and $i^2 = -1$.

## 1.1 A Complex number :

### 1.1(a) Imaginary Number :

A number of the form b$i$, where b$\in$ R, b $\neq$ 0, $i = \sqrt{-1}$ is called an imaginary number.

**Ex :** $\sqrt{-25} = 5i, 2i, \frac{2}{7}i, -11i$ etc.

**Note:**

The number $i$ satisfies following properties,

i) $i \times 0 = 0$

ii) If a$\in$R, then $\sqrt{-a^2} = \sqrt{i^2a^2} = \pm ia$

iii) If a, b$\in$R, and a$i$ = b$i$ then a = b

### 1.1 (b) Complex Number :

**Definition :** A number of the form a+$i$b, where a, b$\in$ R and $i = \sqrt{-1}$ with $i^2 = -1$ is called a complex number and is usually denoted by z.

That is z = a+$i$b, a, b$\in$R and $i = \sqrt{-1}$

Here 'a' is called the real part of z and is denoted by **Re(z) or R(z).** 'b' is called imaginary part of z and is denoted by **Im(z) or I(z)**

The set of complex numbers is denoted by C

$\therefore$ C = {a+$i$b / a, b$\in$ R, and $i = \sqrt{-1}$ }

**Ex :**

| z | a+$i$b | Re($z$) | Im($z$) |
|---|---|---|---|
| $2+4i$ | $2+4i$ | 2 | 4 |
| $5i$ | $0+5i$ | 0 | 5 |
| $3-4i$ | $3-4i$ | 3 | $-4$ |
| $5+\sqrt{-16}$ | $5+4i$ | 5 | 4 |
| $2+\sqrt{5}i$ | $2+\sqrt{5}\,i$ | 2 | $\sqrt{5}\,i$ |
| $7+\sqrt{3}$ | $(7+\sqrt{3})+0i$ | $(7+\sqrt{3})$ | 0 |

**Note :**

1) A complex number whose real part is zero is called a purely imaginary number. Such a number is of the form $z = 0 + ib = ib$

2) A complex number whose imaginary part is zero is a real number.
$z = a + 0i = a$, is a real number.

3) A complex number whose both real and imaginary parts are zero is the zero complex number. $0 = 0 + 0i$

4) The set R of real numbers is a subset of the set C of complex numbers.

5) The real part and imaginary part cannot be combined to form single term. e.g. $2 + 3i \neq 5i$

## 1.2 Algebra of Complex Numbers :

### 1.2.1 Equality of two Complex Numbers :

**Definition :** Two complex numbers $z_1 = a+ib$ and $z_2 = c + id$ are said to be equal if their corresponding real and imaginary parts are equal.

i.e. $a + ib = c + id$ if $a = c$ and $b = d$

**Ex. :** i) If $x + iy = 4 + 3i$ then $x = 4$ and $y = 3$

**Ex. :** ii) If $7a + i(3a - b) = 21 - 3i$ then find a and b.

**Solution :** $7a + (3a - b)\,i = 21 - 3i$

By equality of complex numbers

$7a = 21 \qquad \therefore\ a = 3$

and $3a - b = -3 \quad \therefore\ 3(3) + 3 = b$

$\therefore\ 12 = b$

**Note :** The order relation (inequality) of complex number can not be defined. Hence, there does not exist a smaller or greater complex number than given complex number. We cannot say $i < 4$.

### 1.2.2 Conjugate of a Complex Number:

**Definition :** The conjugate of a complex number $z = a + ib$ is defined as $a - ib$ and is denoted by $\bar{z}$

**Ex :** 1)

| $z$ | $\bar{z}$ |
|---|---|
| $3 + 4i$ | $3 - 4i$ |
| $7i - 2$ | $-7i - 2$ |
| $3$ | $3$ |
| $5i$ | $-5i$ |
| $2 + \sqrt{3}$ | $2 + \sqrt{3}$ |
| $7 + \sqrt{5}\,i$ | $7 - \sqrt{5}\,i$ |

**2) Properties of $\bar{z}$**

1) $\overline{(\bar{z})} = z$

2) If $z = \bar{z}$, then z is purely real.

3) If $z = -\bar{z}$, then z is purely imaginary.

Now we define the four fundamental operations of addition, subtraction, multiplication and division of complex numbers.

### 1.2.3 Addition of complex numbers :

Let $z_1 = a+ib$ and $z_2 = c+id$

then $z_1 + z_2 = (a+ib) + (c+id)$

$= (a+c) + (b+d)\ i$

In other words, $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$

and $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$

**Ex.** 1) $(2 + 3i) + (4 + 3i) = (2+4) + (3+3)i$

$= 6 + 6i$

2) $(-2 + 5i) + (7 + 3i) + (6 - 4i)$

$= [(-2) + 7 + 6] + [5 + 3 + (-4)]i$

$= 11 + 4i$

**Properties of addition :** If $z_1, z_2, z_3$ are complex numbers then

i) $z_1 + z_2 = z_2 + z_1$ (commutative)

ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative)

iii) $z_1 + 0 = 0 + z_1 = z_1$ (identity)

iv) $z + \overline{z} = 2\text{Re}(z)$ (Verify)

v) $(\overline{z_1+z_2}) = \overline{z_1} + \overline{z_2}$

## 1.2.4 Scalar Multiplication :

If $z = a+ib$ is any complex number, then for every real number k, define $kz = ka + i\,(kb)$

**Ex.** 1) If $z = 7 + 3i$ then

$5z = 5(7 + 3i) = 35 + 15i$

2) $z_1 = 3 - 4i$ and $z_2 = 10 - 9i$ then

$2z_1 + 5z_2 = 2(3 - 4i) + 5(10 - 9i)$

$= 6 - 8i + 50 - 45i$

$= 56 - 53i$

**Note:** 1) $0.z = 0(a + ib) = 0 + 0i = 0$

## 1.2.5 Subtraction of complex numbers :

Let $z_1 = a+ib$, $z_2 = c+id$ then define

$z_1 - z_2 = z_1 + (-1)z_2 = (a+ib) + (-1)(c+id)$

$= (a+ib) + (-c - id)$

$= (a-c) + i\ (b-d)$

Hence, $\text{Re}(z_1 - z_2) = \text{Re}(z_1) - \text{Re}(z_2)$

$\text{Im}(z_1 - z_2) = \text{Im}(z_1) - \text{Im}(z_2)$

**Ex.** 1) $z_1 = 4+3i$, $z_2 = 2+i$

$\therefore z_1 - z_2 = (4+3i) - (2+i)$

$= (4-2) + (3-1)\,i$

$= 2 + 2i$

2) $z_1 = 7+i$, $z_2 = 4i$, $z_3 = -3+2i$

then $2z_1 - (5z_2 + 2z_3)$

$= 2(7+i) - [5(4i) + 2(-3+2i)]$

$= 14 + 2i - [20i - 6 + 4i]$

$= 14 + 2i - [-6 + 24i]$

$= 14 + 2i + 6 - 24i$

$= 20 - 22i$

### Properties of Subtraction :

1) $z - \overline{z} = 2\,\text{Im}(z)$ (Verify)

1) $(\overline{z_1 - z_2}) = \overline{z_1} - \overline{z_2}$

## 1.2.6 Multiplication of complex numbers:

Let $z_1 = a+ib$ and $z_2 = c+id$. We denote multiplication of $z_1$ and $z_2$ as $z_1.z_2$ and is given by

$z_1.z_2 = (a+ib)(c+id) = a(c+id)+ib(c+id)$

$= ac + adi + bci + i^2bd$

$= ac + (ad+bc)i - bd \qquad (\because i^2 = -1)$

$z_1.z_2 = (ac-bd) + (ad+bc)i$

**Ex. 1)** $z_1 = 2+3i$, $z_2 = 3-2i$

$\therefore z_1.z_2 = (2+3i)(3-2i) = 2(3-2i) + 3i(3-2i)$

$= 6 - 4i + 9i - 6i^2$

$= 6 - 4i + 9i + 6 \qquad (\because i^2 = -1)$

$= 12 + 5i$

**Ex. 2)** $z_1 = 2-7i$, $z_2 = 4-3i$, $z_3 = 1+i$ then

$(2\,z_1) . (z_2) . (z_3) = 2(2-7i) . (4-3i) . (1+i)$

$= (4-14i) . [4+4i-3i-3i^2]$

$= (4-14i) . [7+i]$

$= 28 + 4i - 98i - 14i^2$

$= 42 - 94i$

### Properties of Multiplication :

i) $z_1.z_2 = z_2.z_1$ (commutative)

ii) $(z_1.z_2).z_3 = z_1.(z_2.z_3)$ (associative)

iii) $(z_1.1) = 1.z_1 = z_1$ (identity)

iv) $(\overline{z_1.z_2}) = \overline{z_1} . \overline{z_2}$ (Verify)

v) If $z = a+ib$ $z.\overline{z} = a^2 + b^2$

**1.2.7. Powers of $i$ :** We have $\sqrt{-1} = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. Let us consider $i^n$, where $n$ is a positive integer and $n > 4$.

We divide $n$ by 4 and obtain the remainder '$r$'. Let $m$ be the quotient.

Then, $n = 4m + r$, where $0 \le r < 4$

$\therefore \quad i^n = i^{4m+r} = i^{4m}.\, i^r = (i^4)^m.\, i^r = 1.i^r = i^r$

Similarly,

$$i^{4m} = 1 = \frac{1}{i^{4m}} = i^{-4m}$$

$$i^2 = -1,\; i^3 = i \times i^2 = -i$$

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

The above equations help us to find $i^k$ for any integer $k$.

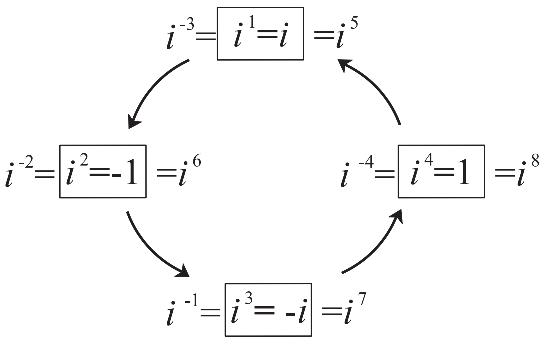



**Fig. 1.1**

e.g. (i) $i^{50} = (i^4)^{12}.\, i^2 = i^2 = -1$

(ii) $i^{318} = (i^4)^{79}.\, i^2 = -1$

(iii) $i^{999} = (i^4)^{249+3} = (i^4)^{249}.\, i^3 = -i$

**Remark :** In general,

$i^{4n} = 1, \qquad i^{4n+1} = i,$

$i^{4n+2} = -1, \qquad i^{4n+3} = -i$ where $n \in Z$

**1.2.8 Division of complex number :**

Let $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers such that $z_2 \neq 0$

Now,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \quad \text{where } z_2 \neq 0 \text{ i.e. } c+id \neq 0$$

Multiply and divide by conjugate of $z_2$.

$$\therefore \frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{ac-bd}{c^2+d^2}\right)i$$

Where $\left(\frac{ac+bd}{c^2+d^2}\right) \in R$ and $\left(\frac{ac-bd}{c^2+d^2}\right) \in R$

**Illustration :** If $z_1 = 3+2i$, & $z_2 = 1+i$,

then $\frac{z_1}{z_2} = \frac{3+2i}{1+i}$

By multiplying numerator and denominator with $\overline{z}_2 = 1 - i$

$$\frac{z_1}{z_2} = \frac{3+2i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-3i+2i-2i^2}{1+1}$$

$$= \frac{5-i}{2}$$

$$\therefore \frac{z_1}{z_2} = \frac{5}{2} - \frac{1}{2}i$$

**Properties of Division :**

1) $\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$

2) $\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$

## SOLVED EXAMPLES

**Ex. 1 :** Write $(1+2i)\ (1+3i)\ (2+i)^{-1}$ in the form $a+ib$

**Solution :**

$$(1+2i)\ (1+3i)\ (2+i)^{-1} = \frac{(1+2i)(1+3i)}{2+i}$$

$$= \frac{1+3i+2i+6i^2}{2+i} = \frac{-5+5i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-10+5i+10i-5i^2}{4-i^2} = \frac{-5+15i}{4+1} \quad (\because\ i^2=-1)$$

$$= \frac{-5+15i}{5} = -1 + 3i$$

**Ex. 2 :** (Activity) Express $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$ in the form of $(a + ib)$.

**Solution :** $i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$$\frac{1}{i} = \boxed{\phantom{00}},\ \frac{1}{i^2} = \boxed{\phantom{00}},\ \frac{1}{i^3} = \boxed{\phantom{00}},\ \frac{1}{i^4} = \boxed{\phantom{00}}$$

$$\therefore\ \frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$$

$$= 1\left(\boxed{\phantom{00}}\right) + 2\left(\boxed{\phantom{00}}\right) + 3\left(\boxed{\phantom{00}}\right) + 5\left(\boxed{\phantom{00}}\right)$$

$$= \boxed{\phantom{00}} + i\,\boxed{\phantom{00}}$$

$$\therefore\ a = \boxed{\phantom{00}},\ b = \boxed{\phantom{00}}$$

**Ex. 3 :** If a and b are real and
$(i^4+3i)a + (i-1)b + 5i^3 = 0$, find a and b.

**Solution :** $(i^4+3i)a + (i-1)b + 5i^3 = 0+0i$

i.e. $(1+3i)a + (i-1)b - 5i = 0+0i$

$\therefore$ $a + 3ai + bi - b - 5i = 0+0i$

i.e. $(a-b) + (3a+b-5)i = 0+0i$

By equality of complex numbers, we get

$a-b = 0$ and $3a+b-5 = 0$

$\therefore$ $a=b$ and $3a+b = 5$

$\therefore$ $3a+a = 5$

$\therefore$ $4a = 5$

$\therefore$ $a = \frac{5}{4}$

$\therefore$ $a = b = \frac{5}{4}$

**Ex. 4 :** If $x + 2i + 15i^6y = 7x + i^3\ (y+4)$ find $x + y$, given that $x, y \in$ R.

**Solution :**

$x + 2i + 15i^6y = 7x + i^3\ (y+4)$

$\therefore$ $x + 2i - 15y = 7x - (y+4)\ i$

$(\because i^6 = -1,\ i^3 = -i)$

$\therefore$ $x - 15y + 2i = 7x - (y+4)\ i$

Equating real and imaginary parts, we get

$x - 15y = 7x$ and $2 = -(y+4)$

$\therefore$ $-6x - 15y = 0$ .... (i) $\quad y+6 = 0$ ....(ii)

$\therefore$ $y = -6,\ x = 15$ [Solving (i) and (ii)]

$\therefore$ $x + y = 15 - 6 = 9$

**Ex. 5 :** Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = i.$

**Solution :**

$$\text{L.H.S.} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = \left(\frac{\sqrt{3}+i}{2}\right)^3$$

$$= \frac{\left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2 i + 3\sqrt{3}i^2 + (i)^3}{(2)^3}$$

$$= \frac{3\sqrt{3} + 9i - 3\sqrt{3} - i}{8}$$

$$= \frac{8i}{8}$$

$$= i$$

$$= \text{R.H.S.}$$

## EXERCISE 1.1

1) Simplify :

i) $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$

ii) $4\sqrt{-4} + 5\sqrt{-9} - 3\sqrt{-16}$

2) Write the conjugates of the following complex numbers

i) $3+i$ ii) $3-$ iii) $-\sqrt{5}-\sqrt{7}i$

iv) $-\sqrt{-5}$ v) $5i$ vi) $\sqrt{5}-i$

vii) $\sqrt{2}+\sqrt{3}\,i$ viii) $\cos\theta + i\sin\theta$

3) Find a and b if

i) $a + 2b + 2ai = 4 + 6i$

ii) $(a-b) + (a+b)i = a + 5i$

iii) $(a+b)(2 + i) = b + 1 + (10 + 2a)i$

iv) $abi = 3a - b + 12i$

v) $\dfrac{1}{a+ib} = 3 - 2i$

vi) $(a + ib)(1 + i) = 2 + i$

4) Express the following in the form of $a+ib$, $a, b \in R$ $i = \sqrt{-1}$. State the values of a and b.

i) $(1+2i)(-2+i)$ ii) $(1+i)(1-i)^{-1}$

iii) $\dfrac{i(4+3i)}{(1-i)}$ iv) $\dfrac{(2+i)}{(3-i)(1+2i)}$

v) $\left(\dfrac{1+i}{1-i}\right)^2$ vi) $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}$

vii) $(1+i)^{-3}$ viii) $\dfrac{2+\sqrt{-3}}{4+\sqrt{-3}}$

ix) $(-\sqrt{5}+2\sqrt{-4})+(1-\sqrt{-9})+(2+3i)(2-3i)$

x) $(2+3i)(2-3i)$ xi) $\dfrac{4i^8-3i^9+3}{3i^{11}-4i^{10}-2}$

5) Show that $(-1+\sqrt{3}i)^3$ is a real number.

6) Find the value of $\left(3+\dfrac{2}{i}\right)(i^6-i^7)(1+i^{11})$

7) Evaluate the following :

i) $i^{35}$ ii) $i^{888}$ iii) $i^{93}$ iv) $i^{116}$

v) $i^{403}$ vi) $\dfrac{1}{i^{58}}$ vii) $i^{-888}$

viii) $i^{30}+i^{40}+i^{50}+i^{60}$

8) Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

9) Find the value of

i) $i^{49}+i^{68}+i^{89}+i^{110}$

ii) $i+i^2+i^3+i^4$

10) Simplify : $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$

11) Find the value of $1 + i^2 + i^4 + i^6 + i^8 + ... + i^{20}$

12) Show that $1 + i^{10} + i^{100} - i^{1000} = 0$.

13) Is $(1 + i^{14} + i^{18} + i^{22})$ a real number? Justify your answer.

14) Evaluate : $\left(i^{37}+\dfrac{1}{i^{67}}\right)$

15) Prove that $(1+i)^4 \times \left(1+\dfrac{1}{i}\right)^4 = 16$.

16) Find the value of $\dfrac{i^6+i^7+i^8+i^9}{i^2+i^3}$

17) If $a = \dfrac{-1+\sqrt{3}i}{2}$, $b = \dfrac{-1-\sqrt{3}i}{2}$ then show that $a^2 = b$ and $b^2 = a$.

18) If $x+iy = (a+ib)^3$, show that $\dfrac{x}{a}+\dfrac{y}{b} = 4(a^2-b^2)$

19) If $\dfrac{a+3i}{2+ib} = 1-i$, show that $(5a-7b) = 0$.

20) If $x+iy = \sqrt{\dfrac{a+ib}{c+id}}$, prove that $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$

21) If $(a+ib) = \dfrac{1+i}{1-i}$, then prove that $(a^2+b^2) = 1$.

22) Show that $\left(\dfrac{\sqrt{7}+i\sqrt{3}}{\sqrt{7}-i\sqrt{3}}+\dfrac{\sqrt{7}-i\sqrt{3}}{\sqrt{7}+i\sqrt{3}}\right)$ is real.

23) If $(x+iy)^3 = y+vi$ the show that $\left(\dfrac{y}{x}+\dfrac{v}{y}\right) = 4(x^2-y^2)$

24) Find the value of $x$ and $y$ which satisfy the following equations ($x, y \in R$)

i) $(x+2y) + (2x-3y)\,i + 4i = 5$

ii) $\frac{x+1}{1+i} + \frac{y-1}{1-i} = i$

iii) $\frac{(x+iy)}{2+3i} + \frac{2+i}{2-3i} = \frac{9}{13}(1+i)$

iv) If $x(1+3i) + y(2-i) - 5 + i^3 = 0$, find $x+y$

v) If $x+2i+15i^6 y = 7x+i^3(y+4)$, find $x+y$

## 1.3 Square root of a complex number :

Consider $z = x+iy$ be any complex number

Let $\sqrt{x+iy} = a+ib$, a, b$\in$R

On squaring both the sides, we get

$x+iy = (a+ib)^2$

$x+iy = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we get

$x = (a^2-b^2)$ and $y = 2ab$

Solving these equations simultaneously, we can get the values of a and b.

### Solved Examples:

**Ex.1 :** Find the square root of $6+8i$.

**Solution :**

Let, $\sqrt{6+8i} = a+ib$ (a, b$\in$R)

On squaring both the sides, we get

$6+8i = (a+ib)^2$

$\therefore 6+8i = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we have

$6 = a^2-b^2$ .... (1)

$8 = 2ab$ .... (2)

$\therefore\ a = \frac{4}{b}$

$\therefore$ (1) becomes $6 = \left(\frac{4}{b}\right)^2 - b^2$

i.e. $6 = \frac{16}{b^2} - b^2$

$\therefore\ b^4+6b^2-16 = 0$ i.e. $(b^2)^2 + 6b^2 - 16 = 0$

put $b^2 = m$

$\therefore\ m^2+6m-16 = 0$

$\therefore\ (m+8)(m-2) = 0$

$\therefore\ m = -8$ or $m = 2$

i.e. $b^2 = -8$ or $b^2 = 2$

but b is a real number $\therefore\ b^2 \neq -8$

So, $b^2 = 2$ $\therefore\ b = \pm\sqrt{2}$

For, $b = \sqrt{2}$, $a = 2\sqrt{2}$

$\therefore\ \sqrt{6+8i} = 2\sqrt{2} + \sqrt{2}\,i = \sqrt{2}\,(2+i)$

For, $b = -\sqrt{2}$, $a = -2\sqrt{2}$

$\therefore\ \sqrt{6+8i} = -2\sqrt{2} - \sqrt{2}\ i = -\sqrt{2}\,(2+i)$

$\therefore\ \sqrt{6+8i} = \pm\sqrt{2}\,(2+i)$

**Ex. 2 :** Find the square root of $3 - 4i$

**Solution :**

Let $\sqrt{3-4i} = a+ib$ a, b$\in$R

On squaring both the sides, we have

$3 - 4i = (a+ib)^2$

$\therefore 3 - 4i = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we have

$a^2-b^2 = 3,\quad 2ab = -4$

As $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$(a^2+b^2)^2 = 3^2 + (-4)^2 = 9 + 16 = 25$

$(a^2+b^2)^2 = 5^2$

$\therefore\ a^2+b^2 = 5$

Solving $a^2+b^2 = 5$ and $a^2-b^2 = 3$ we get

$2a^2 = 8$

$a^2 = 4$

$\therefore\ a = \pm 2$

For $a = 2$;

$b = \frac{-4}{2a} = \frac{-4}{2(2)} = -1$

For, $a = -2$, $b = \frac{-4}{2(-2)} = 1$

$\therefore\ \sqrt{3-4i} = 2 - i$ or $-2 + i$

## 1.4 Fundamental Theorem of Algebra :

'A polynomial equation with real coefficients has at least one root' in C.

or 'A polynomial equation with complex coefficients and of degree $n$ has $n$ complex roots'.

### 1.4.1 Solution of a Quadratic Equation in complex number system :

Let the given equation be $ax^2 + bx + c = 0$ where $a, b, c \in \mathrm{R}$ and $a \neq 0$

$\therefore$ The solution of this quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence, the roots of the equation $ax^2 + bx + c = 0$

are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

The expression $(b^2 - 4ac) = \mathrm{D}$ is called the discriminant.

If $\mathrm{D} < 0$ then the roots of the given quadratic equation are complex.

**Note :** If $p + iq$ is the root of equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathrm{R}$ and $a \neq 0$ then $p - iq$ is also a solution of the given equation. Thus, complex roots occur in conjugate pairs.

### Solved Examples :

**Ex. 1 :** Solve $x^2 + x + 1 = 0$

**Solution :** Given equation is $x^2 + x + 1 = 0$

Comparing with $ax^2 + bx + c = 0$ we get

$a = 1, \quad b = 1, c = 1$

These roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2}$$

$\therefore$ Roots are $\frac{-1 + \sqrt{3}\, i}{2}$ and $\frac{-1 - \sqrt{3}\, i}{2}$

**Ex. 2 :** Solve $x^2 - (2\sqrt{3} + 3i)\, x + 6\sqrt{3}\, i = 0$

**Solution :** Given equation is

$x^2 - (2\sqrt{3} + 3i)\, x + 6\sqrt{3}\, i = 0$

The method of finding the roots of

$ax^2 + bx + c = 0$, is applicable even if a, b, c are complex numbers. where $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1$, $b = -(2\sqrt{3} + 3i)$, $c = 6\sqrt{3}\, i$

$b^2 - 4ac = [-(2\sqrt{3} + 3i)]^2 - 4 \times 1 \times 6\sqrt{3}\, i$

$= 12 - 9 + 12\sqrt{3}\, i - 24\sqrt{3}\, i$

$= 3 - 12\sqrt{3}\, i$

$= 3\,(1 - 4\sqrt{3}\, i)$

So, the given equation has complex roots. These roots are given by

$$x = \frac{\left(2\sqrt{3} + 3i\right) \pm \sqrt{3(1 - 4\sqrt{3}\, i)}}{2}$$

Now, we shall find $\sqrt{1 - 4\sqrt{3}\, i}$

Let $a + ib = \sqrt{1 - 4\sqrt{3}\, i}$

$\therefore \; a^2 - b^2 + 2iab = 1 - 4\sqrt{3}\, i$

$\therefore \; a^2 - b^2 = 1$ and $2ab = -4\sqrt{3}$

$a^2 - b^2 = 1$ and $ab = -2\sqrt{3}$

Consider $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4\, a^2 b^2$

$= 1 + 4\,(12) = 49$

$\therefore \; a^2 + b^2 = 7 \quad \ldots(1)$

and also $a^2 - b^2 = 1 \quad \ldots(2)$

Solving (1) and (2)

$2a^2 = 8 \qquad \therefore \; a^2 = 4$

$\therefore \; a = \pm 2$ and $b = \pm\sqrt{3}$

$\therefore$ We have four choices

$a = 2, \qquad b = \sqrt{3} \qquad \ldots (3)$

$a = 2, \qquad b = -\sqrt{3} \qquad \ldots (4)$

$a = -2, \qquad b = \sqrt{3} \qquad \ldots (5)$

$a = -2, \qquad b = -\sqrt{3} \qquad \ldots (6)$

With this combination, the condition $ab = -2\sqrt{3}$ should also be satisfied.

We can verify the values of $a$ and $b$ given by (4) and (5) satisfy the condition and that from (3) and (6) do not satisfy the condition.

Hence, $\quad a = 2, \qquad b = -\sqrt{3}$

and $\qquad a = -2, \qquad b = \sqrt{3}$

$\therefore \sqrt{1-4\sqrt{3}\,i} = \pm(2-\sqrt{3}\,i)$

$\therefore$ The roots are given by

$$x = \frac{\left(2\sqrt{3}+3i\right)\pm\sqrt{3}\left(2-\sqrt{3}\,i\right)}{2}$$

So, the roots are $2\sqrt{3}$ and $3i$.

**Ex. 3 :** Find the value of $x^3 - x^2 + 2x + 10$ when $x = 1 + \sqrt{3}\,i$.

**Solution :** Since $x = 1 + \sqrt{3}\,i$

$\therefore x - 1 = \sqrt{3}\,i$

squaring both sides, we get

$(x-1)^2 = (\sqrt{3}\,i)^2$

$\therefore x^2 - 2x + 1 = 3i^2$

i.e. $x^2 - 2x + 1 = -3$

$\therefore x^2 - 2x = -4 \qquad \ldots\ldots\ldots (I)$

$x^3 - x^2 + 2x + 10 = x^3 - (x^2 - 2x) + 10 \quad$ (By I)

$= x^3 - (-4) + 10 = x^3 + 14$

$= (1 + \sqrt{3}\,i)^3 + 14$

$= 1 + 3\sqrt{3}\,i - 9 - 3\sqrt{3}\,i + 14$

$= 6$

**Ex. 4 :** If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 64$.

**Solution :** $x = -5 + 2\sqrt{-4} = -5 + 2(2i) = -5 + 4i$

Let $P(x) = x^4 + 9x^3 + 35x^2 - x + 64$

Form the quadratic expression $q(x)$ with real coefficients and roots $\alpha = -5+4i$ and $\bar{\alpha} = -5-4i$.

$\alpha\bar{\alpha} = (-5+4i)(-5-4i) = 25 + 16 = 41$

$\alpha + \bar{\alpha} = (-5+4i) + (-5-4i) = -10$

Therefore $q(x) = x^2 + 10x + 41$

Now $q(\alpha) = \alpha^2 + 10\alpha + 41 = 0 \quad \ldots(1)$

We divide the given polynomial $p(x)$ by $q(x)$.

$$\begin{array}{r|l}
 & x^2 - x + 4 \\
\hline
x^2+10x+41 & x^4 + 9x^3 + 35x^2 - x + 64 \\
 & x^4 + 10x^3 + 41x^2 \\
\hline
 & -x^3 + 6x^2 - x \\
 & -x^3 - 10x^2 - 41x \\
\hline
 & 4x^2 + 40x + 64 \\
 & 4x^2 + 40x + 164 \\
\hline
 & -100
\end{array}$$

$\therefore \quad p(x) = (x^2 + 10x + 41)(x^2 - x + 4) - 100$

$\therefore \quad p(\alpha) = q(\alpha)(\alpha^2 - \alpha + 4) - 100$

$\therefore \quad p(\alpha) = 0 - 100 = -100 \qquad$ (By I)

## EXERCISE 1.2

1) Find the square root of the following complex numbers

i) $-8-6i$    ii) $7+24i$    iii) $1+4\sqrt{3}\,i$

iv) $3+2\sqrt{10}\,i$    v) $2(1-\sqrt{3}\,i)$

2) Solve the following quadratic equations.

i) $8x^2 + 2x + 1 = 0$

ii) $2x^2 - \sqrt{3}\,x + 1 = 0$

iii) $3x^2 - 7x + 5 = 0$

iv) $x^2 - 4x + 13 = 0$

3) Solve the following quadratic equations.

i) $x^2 + 3ix + 10 = 0$

ii) $2x^2 + 3ix + 2 = 0$

iii) $x^2 + 4ix - 4 = 0$

iv) $ix^2 - 4x - 4i = 0$

4) Solve the following quadratic equations.

i) $x^2 - (2+i)\,x - (1-7i) = 0$

ii) $x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}\,i = 0$

iii) $x^2 - (5-i)\,x + (18+i) = 0$

iv) $(2+i)x^2 - (5-i)\,x + 2\,(1-i) = 0$

5) Find the value of

*iii)* $x^3 - x^2 + x + 46$, if $x = 2+3i$.

iv) $2x^3 - 11x^2 + 44x + 27$, if $x = \dfrac{25}{3-4i}$ .

*v)* $x^3 + x^2 - x + 22$, if $x = \dfrac{5}{1-2i}$ .

*vi)* $x^4 + 9x^3 + 35x^2 - x + 4$, if $x = -5 + \sqrt{-4}$ .

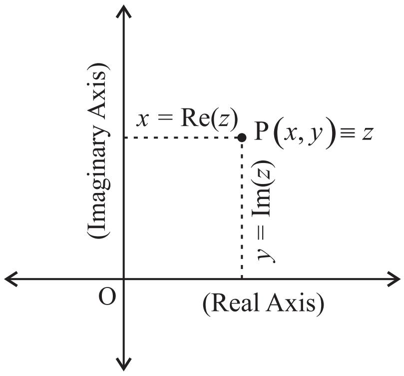
vii) $2x^4 + 5x^3 + 7x^2 - x + 41$, if $x = -2 - \sqrt{3}\,i$.

## 1.5 Argand Diagram or Complex Plane :

A complex number $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$ is expressed as a point in the plane whose co-ordinates are ordered pair $(x, y)$. Jean Robert Argand used the one to one correspondence between a complex number and the points in a the plane.

Let $z = x + iy$ be a complex number.

Then the point $P(x, y)$ represents the complex number $z = x + iy$ (fig.1.2) i.e. $x + iy \equiv (x, y)$, $x = Re(z)$ is represented on the X-axis. So, X-axis is called the real axis. Similarly, $y = Im(z)$ is represented on the Y-axis, so the Y-axis is called the **imaginary axis.**




Argand Diagram

**Fig. 1.2**

e.g. (1) $(1, 2) \equiv 1 + 2i$ (2) $-4+3i \equiv (-4, 3)$

(3) $(0, 0) \equiv 0 + 0i$ (4) $5+0i \equiv (5, 0)$

(5) $(0, -1) \equiv 0 - i$ (6) $-2-2i \equiv (-2, -2)$

A diagram which represents complex numbers by points in a plane with reference to the real and imaginary axes is called **Argand's diagram on complex plane.**

### 1.5.1 Modulus of z :

If $z = a+ib$ is a complex number then the modulus of $z$, denoted by $|\,z\,|$ or r, is defined as $|\,z\,| = \sqrt{a^2 + b^2}$ . (From fig. 1.3), point $P(a, b)$ represents the complex number $z = a+ib$.

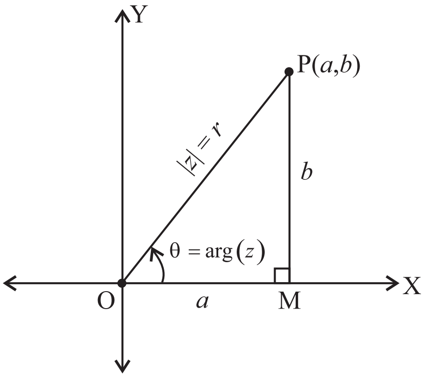
$$\therefore\ r = |\,z\,| = \sqrt{a^2 + b^2} = OP$$

Hence, modulus of $z$ is the distance of point P from the origin where the point P represents the complex number $z$ in the plane. e.g. For, $z = 4 + 3i$,

Modulus of $z = |\,z\,| = \sqrt{16+9} = \sqrt{25} = 5$

### 1.5.2 Argument of z :

OP makes an angle $\theta$ with positive direction of X-axis. $\theta$ is called the **argument** or **amplitude** of the complex number $z = a+ib$, denoted by arg(z).




Modulus and Argument of $z$

**Fig. 1.3**

$$\therefore \sin\theta = \frac{b}{r}, \; \cos\theta = \frac{a}{r}, \; r \neq 0$$

$$\therefore b = r\sin\theta, \; a = r\cos\theta$$

and $\tan\theta = \dfrac{b}{a}$, if $a \neq 0$

$$\therefore \theta = \tan^{-1} = \left(\frac{b}{a}\right) = \arg(z),$$

e.g. If $z = 2+2i$ then

$$\arg(z) = \theta = \tan^{-1}\left(\frac{2}{2}\right)$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

**Note :** If $\tan x = y$ then its inverse funtion is given by $x = \tan^{-1} y$ or $x = \text{arc} \tan y$

eg:

1) As $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$ then $\tan^{-1} = \left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

2) As $\tan\left(-\dfrac{\pi}{4}\right) = -\tan\dfrac{\pi}{4} = -1$ then

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

### 1.5.3 Argument of z in different quadrants/axes :

| $z = a + ib$ | Example | Quadrant/Axis | $\theta = \arg z$ $= \tan^{-1}\left(\frac{b}{a}\right)$, $(0 \le \theta < 2\pi)$ | from Example |
|---|---|---|---|---|
| $a > 0, b = 0$ | $z = 3$ | On positive real (X) axis | $\theta = 0$ | $\theta = 0$ |
| $a > 0, b > 0$ | $z = 1 + i$ | In quadrant I | $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $(0 < \theta < \frac{\pi}{2})$ | $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ |
| $a = 0, b < 0$ | $z = 5i$ | On Positive imaginary (Y) axis | $\theta = \frac{\pi}{2}$ | $\theta = \frac{\pi}{2}$ |
| $a < 0, b > 0$ | $z = -\sqrt{3} + i$ | In quadrant II | $\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ $(\frac{\pi}{2} < \theta < \pi)$ | $\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi$ $= \frac{-\pi}{6} + \pi = \frac{5\pi}{6})$ |
| $a < 0, b = 0$ | $z = -6$ | On negative real (X) axis | $\theta = \pi$ | $\theta = \pi$ |

| | | | | |
|---|---|---|---|---|
| $a < 0,\ b < 0$ | $z = -1 - \sqrt{3i}$ | In quadrant III | $\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ $(\pi < \theta < \frac{3\pi}{2})$ | $\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) + \pi$ $= \frac{\pi}{3} + \pi = \frac{4\pi}{3})$ |
| $a = 0,\ b < 0$ | $z = -2i$ | On negative imaginary (Y) axis | $\theta = \frac{3\pi}{2}$ | $\theta = \frac{3\pi}{2}$ |
| $a > 0,\ b < 0$ | $z = 1 - i$ | In quadrant IV | $\theta = \tan^{-1}\left(\frac{b}{a}\right) + 2\pi$ $(\frac{3\pi}{2} < \theta < 2\pi)$ | $\theta = \tan^{-1}\left(\frac{-1}{1}\right) + 2\pi$ $= \frac{-\pi}{4} + 2\pi = \frac{3\pi}{4})$ |

## SOLVED EXAMPLES

**Ex. 1 :** If $z = 1+3i$, find the modulus and amplitude of $z$.

**Solution :** $z = 1+3i$ here a=1, b=3 and a, b > 0

$\therefore\ |z| = \sqrt{1^2+3^2} = \sqrt{10}$

amp $z = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{3}{1}\right) = \tan^{-1}(3)$

**Ex. 2 :** Find the modulus, argument of the complex number $-7 + 24i$.

**Solution :** let $z = -7+24i$ a = −7, b = 24

$\therefore\ |z| = \sqrt{(7)^2+(24)^2} = \sqrt{625} = 25$

Here,

arg z $= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{24}{-7}\right)$

As a < 0, b > 0, $\theta$ lies in 2$^{nd}$ quadrant.

### Properties of modulus of complex numbers :

If $z_1, z_2, z_3$ are complex numbers, then

i) $|z| = 0 \Leftrightarrow z = 0$ i.e. Re $(z)$ = I$m$ $(z)$ = 0

ii) $|z| = |-z| = |\overline{z}| = |-\overline{z}|$

iii) $-|z| \le$ Re $(z) \le |z|$; $-|z| \le$ I$m$ $(z) \le |z|$

iv) $z\overline{z} = |z|^2$

v) $|z_1 z_2| = |z_1|\ |z_2|$

vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$

vii) $|z_1 + z_2| \le |z_1| + |z_2|$

viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\overline{z_2})$

ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}\,(z_1\overline{z_2})$

x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (a^2+b^2)\ (|z_1|^2 + |z_2|^2)$ where $a, b \in$ R

### Properties of arguments :

1) arg $(z_1 . z_2)$ = arg $z_1$ + arg $z_2$

2) arg$\left(\frac{z_1}{z_2}\right)$ = arg $z_1$ − arg $z_2$

3) arg $z$ + arg $\overline{z}$ = 0, arg $(z\overline{z}) = 0$

4) arg $\overline{z}$ = − arg $z$ = arg $\frac{1}{z}$

5) If arg $z = 0$ then $z$ is real

### 1.5.4 Polar form of a complex number :

Let the complex number $z$ = a+$i$b be represented by the point P(a, b) (see fig 1.4)

Let m∠XOP = $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and $l$(OP) = $r$ $= \sqrt{a^2+b^2} > 0$, then P($r$, $\theta$) are called the Polar Co-ordinates of P.

We call the origin as pole. (figure 1.4)

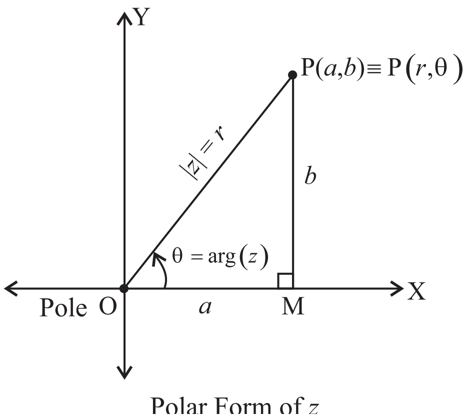



**Fig.1.4**

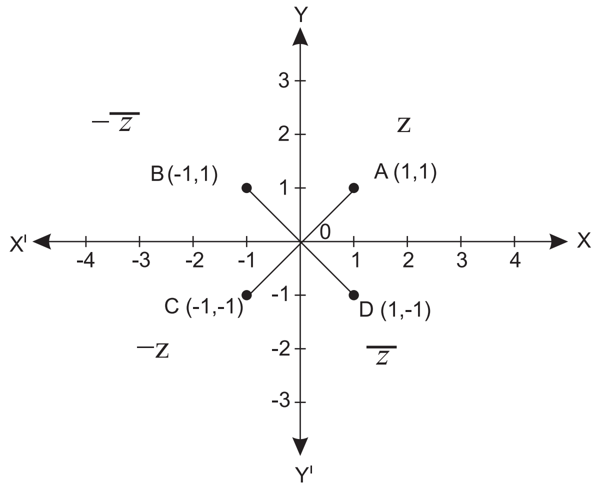



**Fig. 1.5**

As $a = r\cos\theta$, $b = r\sin\theta$

$\therefore$ $z = a + ib$ becomes

$z = r\cos\theta + ir\sin\theta$

$\therefore$ $z = r(\cos\theta + i\sin\theta)$

This is called polar form of complex number $z = a + ib$

### 1.5.5 Exponential form :

It is known and can be proved using special series that $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore$ $z = a + ib = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}$

where $r = |z|$ and $\theta = \arg z$ is called an exponential form of complex number.

### Solved Example:

**Ex. 1 :** Represent the complex numbers

$z = 1+i$, $\overline{z} = 1-i$, $-\overline{z} = -1+i$, $-z = -1-i$ in Argand's diagram and hence find their arguments from the figure.

**Solution :**

arg z is the angle made by the segment OA with the positive direction of the X-axis. (Fig.1.5)

Thus, θ together with r gives the position of the point A in the Argand's diagram.

Hence, from the figure 1.5,

$$\arg z = 45° = \frac{\pi}{4}, \quad \arg(-\overline{z}) = 135° = \frac{3\pi}{4}$$

$$\arg(-z) = 225° = \frac{5\pi}{4}, \quad \arg \overline{z} = 315° = \frac{7\pi}{4}$$

**Ex. 2 :** Represent the following complex numbers in the polar form and in the exponential form

i) $4+4\sqrt{3}\,i$ ii) $-2$ iii) $3i$ iv) $-\sqrt{3}+i$

**Solution :**

i) Let, $z = 4 + 4\sqrt{3}\,i$

$a = 4$, $b = 4\sqrt{3}$

$$r = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16+48} = \sqrt{64} = 8$$

As θ lies in quadrant I

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \text{ or } 60°$$

$\therefore$ The polar form of $z = 4 + 4\sqrt{3}\,i$ is

$z = r(\cos\theta + i\sin\theta)$

$z = 8(\cos 60° + i\sin 60°)$

$= 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore$ The exponential form of $z = 8\,e^{i\left(\frac{\pi}{3}\right)}$

ii) Let $z = -2$

$\therefore$ $a = -2,\ b = 0$

Hence, $r = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$

As point $(-2, 0)$ lies on negative real axis $\theta = \pi^c$ or $180°$

$\therefore$ The polar form of $z = 2(\cos 180° + i\sin 180°)$
$= 2(\cos\pi + i\sin\pi)$

$\therefore$ The exponential form of $z = 2e^{i\pi}$

iii) Let $z = 3i$

$a=0,\ b=3$

Hence, $r = \sqrt{0^2 + 3^2} = 3$

As point $(0, 3)$ lies on positive imaginary axis

$\theta = \frac{\pi^c}{2}$ or $90°$

$\therefore$ The polar form of $z = 3(\cos 90° + i\sin 90°)$

$= 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\therefore$ The exponential form of $z = 3e^{i\frac{\pi}{2}}$

iv) Let, $z = -\sqrt{3} + i$

$\therefore$ $a = -\sqrt{3},\quad b = 1$

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

As $(-\sqrt{3}, 1)$ lies in quadrant II

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \pi$

$= \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi$

$= \frac{5\pi}{6}$

$\therefore$ The polar form of

$z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$\therefore$ The exponential form of

$z = re^{i\theta} = 2\,e^{i\left(\frac{5\pi}{6}\right)}$

**Ex. 3 :** Express $z = \sqrt{2}.e^{\frac{3\pi}{4}i}$ in the $a + ib$ form.

**Solution:** $z = \sqrt{2}.e^{\frac{3\pi}{4}i} = r\,e^{i\theta}$

$\therefore$ $r = \sqrt{2},\ \theta = \frac{3\pi}{4}$

As the polar form of z is

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

By using allied angles results in trigonometry, we get

$\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ $z = \sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)\right]$

$= -1 + i$

**Ex. 4 :** Express (i) $3.e^{\frac{5\pi}{12}i} \times 4.e^{\frac{\pi}{12}i}$

ii) $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in $a + ib$ form

**Solution:** (i) $3.e^{\frac{5\pi}{12}i} \times 4.e^{\frac{\pi}{12}i}$

$$= (3\times 4)\ e^{\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)i}$$

$$= 12\ e^{\frac{6\pi}{12}i} = 12\ e^{\frac{\pi}{2}i}$$

$$= 12\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$= 12\ (0+i) = 12i$$

(ii) $$\frac{\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)} = \frac{\sqrt{2}.e^{\frac{\pi}{12}i}}{2e^{\frac{5\pi}{6}i}}$$

$$= \left(\frac{\sqrt{2}}{2}\right)e^{\left(\frac{\pi}{12}-\frac{5\pi}{6}\right)i} = \left(\frac{\sqrt{2}}{2}\right)e^{\left(-\frac{3\pi}{4}\right)i}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left[-\frac{1}{\sqrt{2}}+i\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$= -\frac{1}{2}-\frac{i}{2}$$

## EXERCISE 1.3

1) Find the modulus and amplitude for each of the following complex numbers.

i) $7-5i$ ii) $\sqrt{3}+\sqrt{2}i$ iii) $-8+15i$

iv) $-3(1-i)$ v) $-4-4i$ vi) $\sqrt{3}-i$

vii) 3 viii) $1+i$ iv) $1+i\sqrt{3}$

x) $(1+2i)^2\ (1-i)$

2) Find real values of θ for which $\left(\frac{4+3i\sin\theta}{1-2i\sin\theta}\right)$ is purely real.

3) If $z = 3+5i$ then represent the $z,\ \bar{z}, -z, -\bar{z}$ in Argand's diagram.

4) Express the following complex numbers in polar form and exponential form.

i) $-1+\sqrt{3}\,i$ ii) $-i$ iii) $-1$ iv) $\frac{1}{1+i}$

v) $\frac{1+2i}{1-3i}$ vi) $\frac{1+7i}{(2-i)^2}$

5) Express the following numbers in the form $x+iy$

i) $\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

ii) $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

iii) $7\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

iv) $e^{\frac{\pi}{3}i}$ v) $e^{\frac{-4\pi}{3}i}$ vi) $e^{\frac{5\pi}{6}i}$

6) Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.

7) Convert the complex number

$$z = \frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$$ in the polar form.

8) For $z = 2+3i$ verify the following :

i) $\overline{(\bar{z})} = z$ ii) $z\bar{z} = |z|^2$

iii) $(z+\bar{z})$ is real iv) $z-\bar{z} = 6i$

9) $z_1 = 1+i$, $z_2 = 2-3i$. Verify the following :

i) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

ii) $\overline{z_1-z_2} = \overline{z_1}-\overline{z_2}$

iii) $\overline{z_1 \cdot z_2} = \overline{z_1}\,.\,\overline{z_2}$

iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

## 1.6 De Moivres Theorem:

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then $z_1 . z_2 = (r_1 e^{i\theta_1})\,(r_2 e^{i\theta_2})$

$= r_1 . r_2\, e^{i(\theta_1 + \theta_2)}$

That is if two complex numbers are multiplied then their modulii get multiplied and arguments get added.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right).e^{i(\theta_1 - \theta_2)}$$

That is, if one complex numbers is divided by the other, then their modulii get divided and arguments get substracted.

In 1730, De Moivre propsoed the following theorem for finding the power of a complex number $z = r(\cos\theta + i\sin\theta)$, as $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)]$

for any $n \in z$.

The proof of this theorem can be given using the Method of Induction (Chapter 4).

### For example:

i) $(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$

ii) $(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$

iii) $(\cos\theta + i\sin\theta)^{\frac{2}{3}} = \cos\left(\frac{2}{3}\theta\right) + i\sin\left(\frac{2}{3}\theta\right)$

### Solved Examples:

**Ex. 1** Use De Moivres Theorom and simplify.

i) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

ii) $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

iii) $\dfrac{(\cos 5\theta + i\sin 5\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3}$

**Solution:** (i) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

$$= \cos\left(8\times\frac{\pi}{3}\right) + i\sin\left(8\times\frac{\pi}{3}\right)$$

$$= \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)$$

$$\left(\because \frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}\right)$$

$$= \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$$

$$= -\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

(ii) $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

$$= \cos\left(\frac{15\pi}{10}\right) - i\sin\left(\frac{15\pi}{10}\right)$$

$$= \cos\left(\frac{3\pi}{2}\right) - i\sin\left(\frac{3\pi}{2}\right)$$

$$= \cos\left(\pi + \frac{\pi}{2}\right) - i\sin\left(\pi + \frac{\pi}{2}\right)$$

$$= -\cos\frac{\pi}{2} - i\left(-\sin\frac{\pi}{2}\right)$$

$$= -0 - i(-1) = i$$

(iii) $\dfrac{(\cos 5\theta + i\sin 5\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3}$

$$= \frac{\cos 10\theta + i\sin 10\theta}{\cos 12\theta - i\sin 12\theta}$$

$$= \frac{\cos 10\theta + i\sin 10\theta}{\cos(-12\theta) + i\sin(-12\theta)}$$

$$= \cos[10\theta - (-12\theta)] + i\sin[10\theta - (-12\theta)]$$

$$= \cos 22\theta + i\sin 22\theta$$

**Ex. 2** Express $(1 + i)^4$ in $a + ib$ form.

**Solution :** Let $z = 1 + i$

$\therefore\ x = 1,\ y = 1$

$$r = \sqrt{1+1} = \sqrt{2},$$

$$\theta = \tan^{-1}\frac{x}{y}$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore\ z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$\therefore\ z^4 = (1+i)^4 = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^4$$

$$= \sqrt{2}\left[\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right]$$

$$= \sqrt{2}\left[\cos\pi + i\sin\pi\right]$$

$$= \sqrt{2}\left[-1 + i(0)\right]$$

$$= \sqrt{2}\left[-1\right] = -\sqrt{2} = -\sqrt{2} + 0i$$

## 1.7 Cube roots of unity :

Number 1 is often called unity. Let $x$ be the cube root of unity i.e. 1

$$\therefore\ x^3 = 1$$

$$\therefore\ x^3 - 1 = 0$$

$$\therefore\ (x-1)(x^2+x+1) = 0$$

$$\therefore\ x-1 = 0 \text{ or } x^2+x+1 = 0$$

$$\therefore\ x = 1 \text{ or } x = \frac{-1 \pm \sqrt{(1)^2 - 4\times1\times1}}{2\times1}$$

$$\therefore\ x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore\ x = 1 \text{ or } x = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ Cube roots of unity are

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

Among the three cube roots of unity, one is real and other two roots are complex conjugates of each other.

$$= \left(\frac{-1+i\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4}\left[(-1)^2 + 2\times(-1)\times i\sqrt{3} + (i\sqrt{3})^2\right]$$

$$= \frac{1}{4}(1 - 2i\sqrt{3} - 3)$$

$$= \frac{1}{4}(-2 - 2i\sqrt{3})$$

$$= \frac{-1-i\sqrt{3}}{2}$$

Similarly it can be verified that $\left(\frac{-1-i\sqrt{3}}{2}\right)^2 = \frac{-1+i\sqrt{3}}{2}$

Thus cube roots of unity are 1,

$$\frac{-1+i\sqrt{3}}{2}, \left(\frac{-1+i\sqrt{3}}{2}\right)^2$$

Let $\frac{-1+i\sqrt{3}}{2} = w$, then $\left(\frac{-1+i\sqrt{3}}{2}\right)^2 = w^2$

Hence, cube roots of unity are 1, $w$, $w^2$

where $w = \left(\frac{-1+i\sqrt{3}}{2}\right)$ and $w^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2$

Also note that $1 = e^{2\pi i}$, $w = e^{\frac{2\pi}{3}i}$, $w^2 = e^{\frac{4\pi}{3}i}$

### Properties of 1, $w$, $w^2$

i) $w$ is complex cube root of 1.

$\therefore\ w^3 = 1$

ii) $w^3 - 1 = 0$

i.e. $(w-1)(w^2+w+1) = 0$

$\therefore\ w = 1$ or $w^2+w+1 = 0$

but $w \neq 1$

$\therefore\ w^2+w+1 = 0$

iii) $w^2 = \frac{1}{w}$ and $\frac{1}{w^2} = w$

iv) $w^3 = 1$ so $w^{3n} = 1$

v) $w^4 = w^3 . w = w$ so $w^{3n+1} = w$

vi) $w^5 = w^2 . w^3 = w^2 . 1 = w^2$ So $w^{3n+2} = w^2$

vii) $\overline{w} = w^2$

viii) $\overline{w}^2 = w$

## 1.8 Set of points in complex plane

If z = $x + iy$ represents the variable point $P(x,y)$ and $z_1 = x_1 + iy_1$, represents the fixed point A $(x_1,y_1)$ then (i) $|z-z_1|$ represents the length of AP

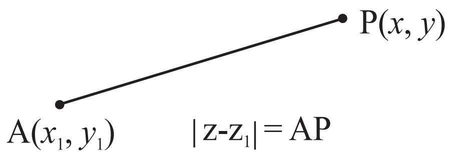



**Fig. 1.6**

(2) $|z-z_1| = a$ represents the circle with centre $A(x_1,y_1)$ and radius $a$.

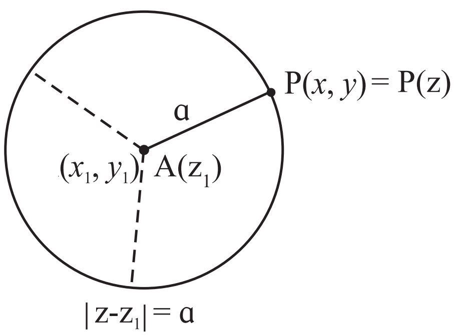



**Fig. 1.7**

(3) $|z-z_1| = |z-z_2|$ represents the perpendicular bisector of the line joining the points A and B.

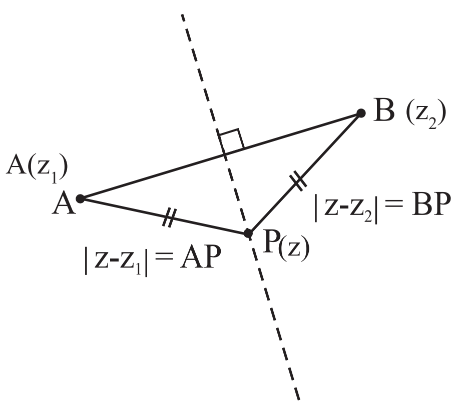



**Fig. 1.8**

**Illustration:**

For $z_1 = 2+3i$ , $z_2 = 1+i$ and $z = x+iy$

(i) $|z-z_1| = |(x+iy)-(2+3i)| = |x+iy-2-3i|$

$$= |(x-2)+i(y-3)| = \sqrt{(x-2)^2+(y-3)^2}$$

represents the distance between $(x,y)$ and (2,3).

(ii) If $|z-z_1| = 5$, then

$$|(x+iy)-(-1+i)| = |(x+1)+i(y-1)| = 5$$

$$\therefore \sqrt{(x+1)^2+(y-1)^2} = 5$$

$$\therefore (x+1)^2+(y-1)^2 = 5^2$$

represents the circle with centre (−1,1) and radius 5.

(iii) If $|z-z_1| = |z-z_2|$ then

$$|(x+iy)-(2+3i)| = |(x+iy)-(-1+i)|$$

$$\therefore |(x-2)+i(y-3)| = |(x+1)+i(y-1)|$$

$$\therefore (x-2)^2+(y-3)^2 = (x+1)^2+(y-1)^2$$

$$\therefore x^2-4x+4+y^2-6y+9$$

$$= x^2+2x+1+y^2-2y+1$$

$$\therefore -6x+4y+11 = 0 \quad i.e. \quad 6x+4y-11=0$$

represents the perpendicular bisector of line joining points (2,3) and (−1,1).

**SOLVED EXAMPLES**

**Ex. 1 :** If $w$ is a complex cube root of unity, then prove that

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = -1$

ii) $(1+w^2)^3 = -1$

iii) $(1-w+w^2)^3 = -8$

**Solution :** Given, $w$ is a complex cube root of unity.

$\therefore w^3 = 1$ Also $w^2+w+1 = 0$

$\therefore\ w^2+1 = -w$ and $w+1 = -w^2$

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = \dfrac{w+1}{w^2} = \dfrac{-w^2}{w^2} = -1$

ii) $(1+w^2)^3 = (-w)^3 = -w^3 = -1$

iii) $(1-w+w^2)^3 = (1+w^2-w)^3$

$= (-w-w)^3 \quad (\because 1+w^2 = -w)$

$= (-2w)^3 = -8\,w^3 = -8 \times 1 = -8$

**Ex. 2 :** If $w$ is a complex cube root of unity, then show that

i) $(1-w+w^2)^5 + (1+w-w^2)^5 = 32$

ii) $(1-w)(1-w^2)\,(1-w^4)(1-w^5) = 9$

**Solution :**

i) Since $w^3 = 1$

and $w \neq 1\ \therefore\ w^2+w+1 = 0$

Also $w^2+1 = -w,\ w^2+w = -1$

and $w+1 = -w^2$

Now, $(1-w+w^2)^5$

$= (-w-w)^5 \quad (\because 1+w^2 = -w)$

$= (-2w)^5$

$= -32w^5$

$(1+w-w^2)^5 = (-w^2-w^2)^5$

$= (-2w^2)^5$

$= -32w^{10}$

$\therefore\ (1-w+w^2)^5 + (1+w-w^2)^5 = -32w^5 - 32w^{10}$

$= -32w^5(1+w^2)$

$= -32w^5 \times (-w) = 32w^6 = 32(w^3)^2$

$= 32 \times (1)^2 = 32$

ii) $(1-w)(1-w^2)\,(1-w^4)(1-w^5)$

$= (1-w)(1-w^2)\,(1-w^3.w)(1-w^3.w^2)$

$= (1-w)(1-w^2)\,(1-w)(1-w^2)$

$= (1-w)^2(1-w^2)^2$

$= [(1-w)(1-w^2)]^2$

$= (1-w^2-w+w^3)^2$

$= [1-(w^2+w)+1]^2$

$= [1-(-1)+1]^2 = (1+1+1)^2 = (3)^2 = 9$

**Ex. 3 :** If $w$ is a complex cube root of unity such that $x$=a+b, $y$=a$w$+b$w^2$ and z=a$w^2$+b$w$, a, b$\in$R prove that

i) $x+y+z=0$ ii) $x^3+y^3+z^3=3(a^3+b^3)$

**Solution :** Since $w$ is a complex cube root of unity

$\therefore\ w^3 = 1$ and $w^2+w+1 = 0$ but $w \neq 1$ given

$\therefore\ w^2+1 = -w,\ w+1 = -w^2,\ w^3 = w^6 = 1$

i) $x+y+z = a+b+aw+bw^2+aw^2+bw$

$= a(1+w+w^2)+b(1+w+w^2)$

$= a.0 + b.0$

$= 0$

ii) $x^3 = a^3+3a^2b+3ab^2+b^3$

$y^3 = a^3w^3+3a^2bw^4+3ab^2w^5+b^3w^6$

$z^3 = a^3w^6+3a^2bw^5+3ab^2w^4+b^3w^3$

adding the above three equations

$x^3+y^3+z^3 = a^3(1+w^3+w^6)+3a^2b(1+w+w^2) + 3ab^2\,(1+w^2+w)+b^3(1+w^6+w^3)$

$= 3a^3 + 3a^2b(0) + 3ab^2(0) + 3b^3$

$= 3(a^3 + b^3)$

**Ex. 4 :** Prove that

$1+w^n+w^{2n} = 3$, if n is multiple of 3

$1+w^n+w^{2n} = 0$, if n is not multiple of 3, n$\in$N

**Solution :** for n, k$\in$N, if n is multiple of 3 then n=3k and if n is not multiple of 3 then n=3k+1 or 3k+2

$\therefore$ if n is multiple of 3

i.e. n=3k

then $1+w^n+w^{2n} = 1+w^{3k}+w^{2\times 3k}$

$= 1+(w^3)^k+(w^3)^{2k}$

$= 1+(1)^k+(1)^{2k}$

$=1+1+1$

$= 3$

If $n = 3k + r$ , $r = 1,2$.

As $w$ is complex root of Unity.

$w^r$, $r = 1,2$ is also complex root of Unity.

$\therefore\ 1 + w^r + w^{2r} = 0$

we have, $1+w^n+w^{2n}=0$, if n is not a multiple of 3.

## EXERCISE 1.4

1) Find the value of

   i) $w^{18}$ ii) $w^{21}$ iii) $w^{-30}$ iv) $w^{-105}$

2) If $w$ is a complex cube root of unity, show that

   i) $(2-w)(2-w^2) =7$

   ii) $(1+w-w^2)^6 = 64$

   iii) $(1+w)^3-(1+w^2)^3 = 0$

   iv) $(2+w+w^2)^3 -(1-3w+w^2)^3 = 65$

   v) $(3+3w+5w^2)^6 -(2+6w+2w^2)^3 = 0$

   vi) $\dfrac{a+bw+cw^2}{c+aw+bw^2} = w^2$

   vii) $(a + b) + (aw + bw^2) + (aw^2 + bw) = 0$

   viii) $(a - b)(a - bw)(a - bw^2) = a^3 - b^3$

   ix) $(a + b)^2 + (aw + bw^2)^2 + (aw^2 + bw)^2 = 6ab$

3) If $w$ is a complex cube root of unity, find the value of

   i) $w+\dfrac{1}{w}$ ii) $w^2+w^3+w^4$ iii) $(1+w^2)^3$

   iv) $(1-w-w^2)^3 + (1-w+w^2)^3$

   v) $(1+w)(1+w^2)(1+w^4)(1+w^8)$

4) If $\alpha$ and $\beta$ are the complex cube root of unity, show that

   (a) $\alpha^2+\beta^2+\alpha\beta = 0$

   (b) $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1} = 0$

5) If $x = a+b$, $y = \alpha a+\beta b$ and $z = a\beta+b\alpha$ where $\alpha$ and $\beta$ are the complex cube-roots of unity, show that $xyz = a^3+b^3$

6) Find the equation in cartesian coordinates of the locus of z if

   (i) $|z| = 10$ (ii) $|z - 3| = 2$

   (iii) $|z - 5 + 6i| = 5$ (iv) $|z + 8| = |z - 4|$

   (v) $|z - 2 - 2i| = |z + 2 + 2i|$

   (vi) $\dfrac{|z+3i|}{|z-6i|}=1$

7) Use De Moivres theorem and simplify the following

   i) $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

   ii) $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$

   iii) $\dfrac{\left(\cos\frac{7\pi}{13} + i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - i\sin\frac{4\pi}{13}\right)^6}$

8) Express the following in the form $a + ib$, $a, b \in R$, using De Moivre's theorem.

   i) $(1 - i)^5$ ii) $(1 + i)^6$ iii) $(1 - \sqrt{3}\, i)^4$

   iv) $(-2\sqrt{3} - 2i)^5$

## Let's Remember

- A number $a+ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is a complex number.
- Let $z_1 = a+ib$ and $z_2 = c+id$. Then

  $z_1 + z_2 = (a+c) + (b+d)i$

$z_1 - z_2 = (a-c) + (b-d)i$

$z_1 . z_2 = (ac-bd) + (ad+bc)i$

$$\frac{z_1}{z_2} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

- For any non-zero complex number $z = a+ib$
$$\frac{1}{z} = \frac{a}{a^2+b^2} + i\,\frac{-b}{a^2+b^2}$$

- For any positive integer k, $i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i$

- The conjugate of $z = a+ib$ is $\overline{z}$, is given by $\overline{z} = a-ib$

- The polar form of the complex number $z = x+iy$ is $r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$ where $r = \sqrt{x^2+y^2}$ is called modulus and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ ($\theta$ is called argument of z) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$.

- $w$ is complex cute root of unity then $1 + w + w^2 = 0$, $w^3 = 1$.

## MISCELLANEOUS EXERCISE - 1

**I) Select the correct answer from the given alternatives.**

1) If $n$ is an odd positive integer then the value of $1 + (i)^{2n} + (i)^{4n} + (i)^{6n}$ is :

A) $-4i$ B) 0 C) $4i$ D) 4

2) The value of is $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$ is equal to :

A) $-2$ B) 1 C) 0 D) $-1$

3) $\sqrt{-3}\,\sqrt{-6}$ is equal to

A) $-3\sqrt{2}$ B) $3\sqrt{2}$ C) $3\sqrt{2}\,i$ D) $-3\sqrt{2}\,i$

4) If $w$ is a complex cube root of unity, then the value of $w^{99}+w^{100}+w^{101}$ is :

A) $-1$ B) 1 C) 0 D) 3

5) If $z=r(\cos\theta+i\sin\theta)$, then the value of $\frac{z}{\overline{z}}+\frac{\overline{z}}{z}$

A) $\cos 2\theta$ B) $2\cos 2\theta$ C) $2\cos\theta$ D) $2\sin\theta$

6) If $w(\neq 1)$ is a cube root of unity and $(1+w)^7 = A + Bw$, then A and B are respectively the numbers

A) 0, 1 B) 1,1 C) 1, 0 D) $-1$,1

7) The modulus and argument of $(1+i\sqrt{3})^8$ are respectively

A) 2 and $\frac{2\pi}{3}$ B) 256 and $\frac{8\pi}{3}$

C) 256 and $\frac{2\pi}{3}$ D) 64 and $\frac{4\pi}{3}$

8) If $\arg(z) = \theta$, then $\arg\overline{(z)} =$

A) $-\theta$ B) $\theta$ C) $\pi-\theta$ D) $\pi+\theta$

9) If $-1+\sqrt{3}\,i = re^{i\theta}$, then $\theta =$ ................. .

A) $-\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $-\frac{\pi}{3}$ D) $\frac{2\pi}{3}$

10) If $z = x+iy$ and $|z-zi| = 1$ then

A) $z$ lies on $x$-asis B) $z$ lies on $y$-asis

D) $z$ lies on a rectangle C) $z$ lies on a circle

**II) Answer the following.**

1) Simplify the following and express in the form a+$i$b.

i) $3+\sqrt{-64}$ ii) $(2i^3)^2$ iii) $(2+3i)(1-4i)$

iv) $\frac{5}{2}i(-4-3i)$ v) $(1+3i)^2(3+i)$ vi) $\frac{4+3i}{1-i}$

vii) $(1+\frac{2}{i})(3+\frac{4}{i})(5+i)^{-1}$ viii) $\frac{\sqrt{5}+\sqrt{3}i}{\sqrt{5}-\sqrt{3}i}$

ix) $\frac{3i^5+2i^7+i^9}{i^6+2i^8+3i^{18}}$ x) $\frac{5+7i}{4+3i} + \frac{5+7i}{4-3i}$

2) Solve the following equations for $x, y \in R$

i) $(4-5i)x + (2+3i)y = 10-7i$

ii) $\dfrac{x+iy}{2+3i} = 7-i$

iii) $(x+iy)\ (5+6i) = 2+3i$

iv) $2x+i^9y\ (2+i) = xi^7+10i^{16}$

3) Evaluate i) $(1-i+i^2)^{-15}$ ii) $(i^{131}+i^{49})$

4) Find the value of

i) $x^3+2x^2-3x+21$, if $x = 1+2i$.

ii) $x^4+9x^3+35x^2-x+164$, if $x = -5+4i$.

5) Find the square roots of

i) $-16+30i$ ii) $15-8i$ iii) $2+2\sqrt{3}\,i$

iv) $18i$ v) $3-4i$ vi) $6+8i$

6) Find the modulus and amplitude of each complex number and express it in the polar form.

i) $8+15i$ ii) $6-i$ iii) $\dfrac{1+\sqrt{3}i}{2}$ iv) $\dfrac{-1-i}{\sqrt{2}}$

v) $2i$ vi) $-3i$ vii) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

7) Represent $1+2i$, $2-i$, $-3-2i$, $-2+3i$ by points in Argand's diagram.

8) Show that $z = \dfrac{5}{(1-i)(2-i)(3-i)}$ is purely imaginary number.

9) Find the real numbers $x$ and $y$ such that

$$\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{-1+8i}$$

10) Show that $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{10} + \left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)^{10} = 0$

11) Show that $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8 = 2.$

12) Convert the complex numbers in polar form and also in exponential form.

i) $z = \dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

ii) $z = -6+\sqrt{2}\,i$

iii) $\dfrac{-3}{2} + \dfrac{3\sqrt{3}\,i}{2}$

13) If $x+iy = \dfrac{a+ib}{a-ib}$, prove that $x^2+y^2 = 1$.

14) Show that $z = \left(\dfrac{-1+\sqrt{-3}}{2}\right)^3$ is a rational number.

15) Show that $\dfrac{1-2i}{3-4i} + \dfrac{1+2i}{3+4i}$ is real.

16) Simplify i) $\dfrac{i^{29}+i^{39}+i^{49}}{i^{30}+i^{40}+i^{50}}$ ii) $\left(i^{65} + \dfrac{1}{i^{145}}\right)$

iii) $\dfrac{i^{238}+i^{236}+i^{234}+i^{232}+i^{230}}{i^{228}+i^{226}+i^{224}+i^{222}+i^{220}}$

17) Simplify $\left[\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right]\left[\dfrac{3+4i}{2-4i}\right]$

18) If $\alpha$ and $\beta$ are complex cube roots of unity, prove that $(1-\alpha)(1-\beta)\ (1-\alpha^2)(1-\beta^2) = 9$

19) If $w$ is a complex cube root of unity, prove that $(1-w+w^2)^6+(1+w-w^2)^6 = 128$

20) If $w$ is the cube root of unity then find the value of

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^{18} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{18}$$

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