4. Trigonometric Functions – II

Let us Recall

- Inverse Trigonometric functions :

(i) $\sin\left(\sin^{-1} x\right) = x$, for $x \in [-1,1]$

(ii) $\sin^{-1}(\sin y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\cos\left(\cos^{-1} x\right) = x$, for $x \in [-1,1]$

(iv) $\cos^{-1}(\cos y) = y$, for $y \in [0,\pi]$

(v) $\tan\left(\tan^{-1} x\right) = x$, for $x \in R$ (vi) $\tan^{-1}(\tan y) = y$, for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\sec\left(\sec^{-1} x\right) = x$, for $x \in R - (-1,1)$

(viii) $\sec^{-1}(\sec y) = y$, for $y \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

(ix) $\cot\left(\cot^{-1} x\right) = x$, for $x \in R$ (x) $\cot^{-1}(\cot y) = y$, for $y \in (0,\pi)$

(xi) $\operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right) = x$, for $x \in R - (-1,1)$

(xii) $\operatorname{cosec}^{-1}(\operatorname{cosec} y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Ex. (1) Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$ *if* $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

Solution : Let $\sin^{-1} x = \theta$

$\therefore \quad \sin\theta = x, \quad x \in [-1,1]$,

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Given $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$,

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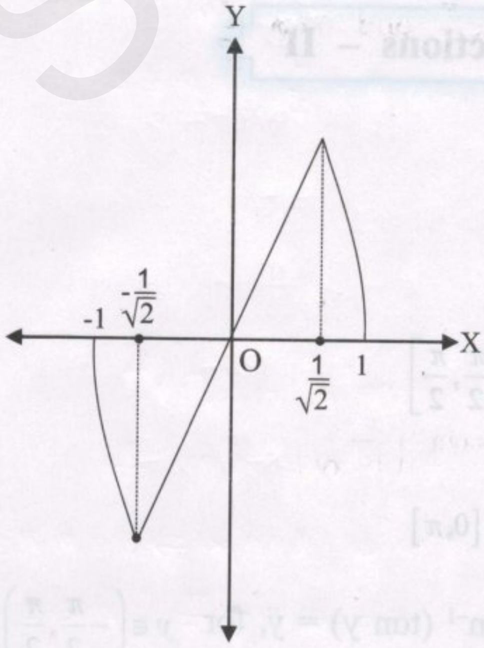

$$\therefore \quad \sin\left(-\frac{\pi}{4}\right) \le \sin\theta \le \sin\left(\frac{\pi}{4}\right)$$

$$\therefore \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

$$\therefore \quad -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

$$\therefore \quad 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{L.H.S} = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\left(2\sin\theta\cos\theta\right)$$

$$= \sin^{-1}\left(\sin 2\theta\right)$$

$$= 2\theta \quad \text{As } 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= 2\sin^{-1} x = \text{R.H.S.}$$

Ex. (2) If $x > 0, y > 0$ then prove that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution : Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$

$\therefore \quad \tan\theta = x, \ \tan\phi = y$

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As $x > 0$ and $y > 0$, we have $0 << \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$.

$\therefore \quad -\frac{\pi}{2} < -\phi < 0$

$\therefore \quad -\frac{\pi}{2} < \theta - \phi < \frac{\pi}{2} \qquad \ldots(1)$

Also $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \dfrac{x-y}{1+xy} \qquad \ldots(2)$

From (1) and (2) we get $\theta - \phi = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

$\therefore \quad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Ex. (3) Prove that for all $x \in R$

(a) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$ (b) $\tan^{-1}(-x) = -\tan^{-1}(x)$

Solution : (a) To prove that $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

Let $\cot^{-1}(-x) = \theta$

$\therefore \cot\theta = -x$, Where $-x \in R, 0 < \theta < \pi$

$\therefore -\cot\theta = x$

$\therefore \cot(\pi - \theta) = x, x \in R$

Since, $0 < \theta < \pi$

$\therefore \quad -\pi < -\theta < 0$

$\therefore 0 < \pi - \theta < \pi$.

Which implies $\cot(\pi - \theta) = x$ and $x \in R, 0 < \pi - \theta < \pi$

$\therefore \pi - \theta = \cot^{-1}x$

$\therefore \theta = \pi - \cot^{-1}x$

$\therefore \cot^{-1}(-x) = \pi - \cot^{-1}x$

(b) To prove that $\tan^{-1}(-x) = -\tan^{-1}(x)$

Let $\tan^{-1}(-x) = \theta$

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$\therefore \tan\theta = -x$ where $-x \in R, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\therefore -\tan\theta = x$

$\therefore \tan(-\theta) = x$

$\therefore \quad \tan\theta = x, \quad x \in R$ *and* $-\frac{\pi}{2} < -\theta < \frac{\pi}{2}$

$\therefore -\theta = \tan^{-1} x$

$-\tan^{-1}x = \tan^{-1} x$

$\therefore \tan^{-1}(-x) = -\tan^{-1} x$

Ex. (4) Prove that : $\tan^{-1}\left[\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right] = \frac{\pi}{4} + \theta$ if $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution : $L.H.S. = \tan^{-1}\left[\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right]$

$$= \tan^{-1}\left[\frac{1 + \tan\theta}{1 - \tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\tan\left(\frac{\pi}{4}\right) + \tan\theta}{1 - \tan\left(\frac{\pi}{4}\right)\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right] \quad \text{- - - - - - - - - - - - - - - - - - (1)}$$

We have, $\tan^{-1}(\tan\theta) = \theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. - - - - - - - - - - - - - - - - (2)

Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$\therefore -\frac{\pi}{4} + \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{\pi}{4} + \frac{\pi}{4}$

$\therefore 0 < \theta + \frac{\pi}{4} < \frac{\pi}{2}$

From equation (1) we get,

$L.H.S. = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$

$= \frac{\pi}{4} + \theta$ From equation (2).

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$= \text{R. H. S}$

Thus, $\tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right]=\frac{\pi}{4}+\theta$ for $\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

Ex. (5) If $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ then find the value of x.

Solution :

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

Put $x=\sin y$

$\sin^{-1}(1-\sin y)-2\sin^{-1}(\sin y)=\frac{\pi}{2}$

$\therefore \sin^{-1}(1-\sin y)=\frac{\pi}{2}+2y$

$\therefore 1-\sin y=\sin\left(\frac{\pi}{2}+2y\right)$

$1-\sin y=\cos 2y$

$1-\sin y=1-2\sin^2 y$

$1-\sin y-1+2\sin^2 y=0$

$2\sin^2 y-\sin y=0$

$\sin y\,(2\sin y-1)=0$

$\sin y=0$ or $2\sin y-1=0$

$\sin y=0$ or $\sin y=\frac{1}{2}$

$\therefore x=0$ or $x=\frac{1}{2}$ $\because x=\sin y$

Let $x=\frac{1}{2}$

$\text{LHS}=\sin^{-1}(1-x)-2\sin^{-1}x$

$=\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{6}-2\times\frac{\pi}{6}$

$=-\frac{\pi}{6}$

$\therefore \text{LHS}\neq\text{RHS}$

$\therefore x=\frac{1}{2}$ is not sol^n

$\boxed{\therefore x=0}$

Ex. (6) Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=-2\pi+2\cos^{-1}x$ *if* $-1\le x\le-\frac{1}{\sqrt{2}}$

Solution :

we know that

$\sin^{-1}(\sin x)=x,\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$

$\text{LHS}=\sin^{-1}(2x\sqrt{1-x^2}$ —(I)

put $x=\sin\theta\Rightarrow\theta=\sin^{-1}x$

$-1\le x\le 1,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\text{LHS}=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$=\sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$=\sin^{-1}(2\sin\theta\cdot\cos\theta)$

$=\sin^{-1}(\sin 2\theta)$

where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

multiplying by 2 b.s

$-\pi\le 2\theta\le\pi$

Here $x=2\theta$ does not Satisfy $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

we have to find the value of x for which

$-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

$\therefore \text{LHS}=\sin^{-1}(\sin 2\theta)$

$=\sin^{-1}\left[-\sin(\pi+2\theta)\right]$

$=\sin^{-1}\left[\sin\left[-(\pi+2\theta)\right]\right]$

$=\sin^{-1}\left[\sin(-\pi-2\theta)\right]$

$-1\le x\le-\frac{1}{2}$

$-\sin\frac{\pi}{2}\le\sin\theta\le-\sin\frac{\pi}{4}$

$\therefore -\frac{\pi}{2}\le\theta\le-\frac{\pi}{4}$

$-\pi\le 2\theta\le-\frac{\pi}{2}$

$\frac{\pi}{2}\le-2\theta\le\pi$

$-\pi+\frac{\pi}{2} \leq -\pi-2\theta \leq -\pi+\pi$

$-\frac{\pi}{2} \leq -\pi-2\theta \leq 0$

$\therefore LHS = \sin^{-1}(\sin(-\pi-2\theta))$

where $-\frac{\pi}{2} \leq -\pi-2\theta \leq 0$

$LHS = -\pi-2\theta$ from -(I)

$LHS = -\pi-2\sin^{-1}x$

$= -\pi-2\left(\frac{\pi}{2}-\cos^{-1}x\right)$

$\left(\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$

$LHS = -\pi-\pi+2\cos^{-1}x$

$= -2\pi+2\cos^{-1}x$

$\therefore LHS = RHS$

Ex. (7) Prove that : $\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\frac{\theta}{2}$, if $\theta \in (-\pi, 0)$

Solution :

$LHS = \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)$

$= \tan^{-1}\left(\sqrt{\tan^2\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(\pm\tan\frac{\theta}{2}\right)$

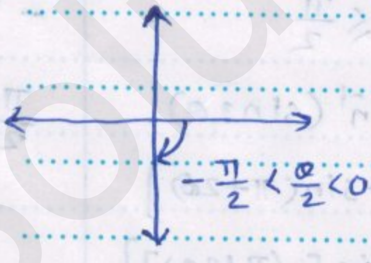
since $\theta \in (-\pi, 0)$

$\therefore -\pi < \theta < 0$

dividing by 2 on b.s

$-\frac{\pi}{2} < \frac{\theta}{2} < 0$

$\frac{\theta}{2}$ is in IV quadrant



$\therefore LHS = \tan^{-1}\left(-\tan\frac{\theta}{2}\right)$

$= -\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\therefore \tan^{-1}-x = -\tan^{-1}x$

$LHS = -\frac{\theta}{2}$

$\therefore LHS = RHS$

Sign of Teacher :

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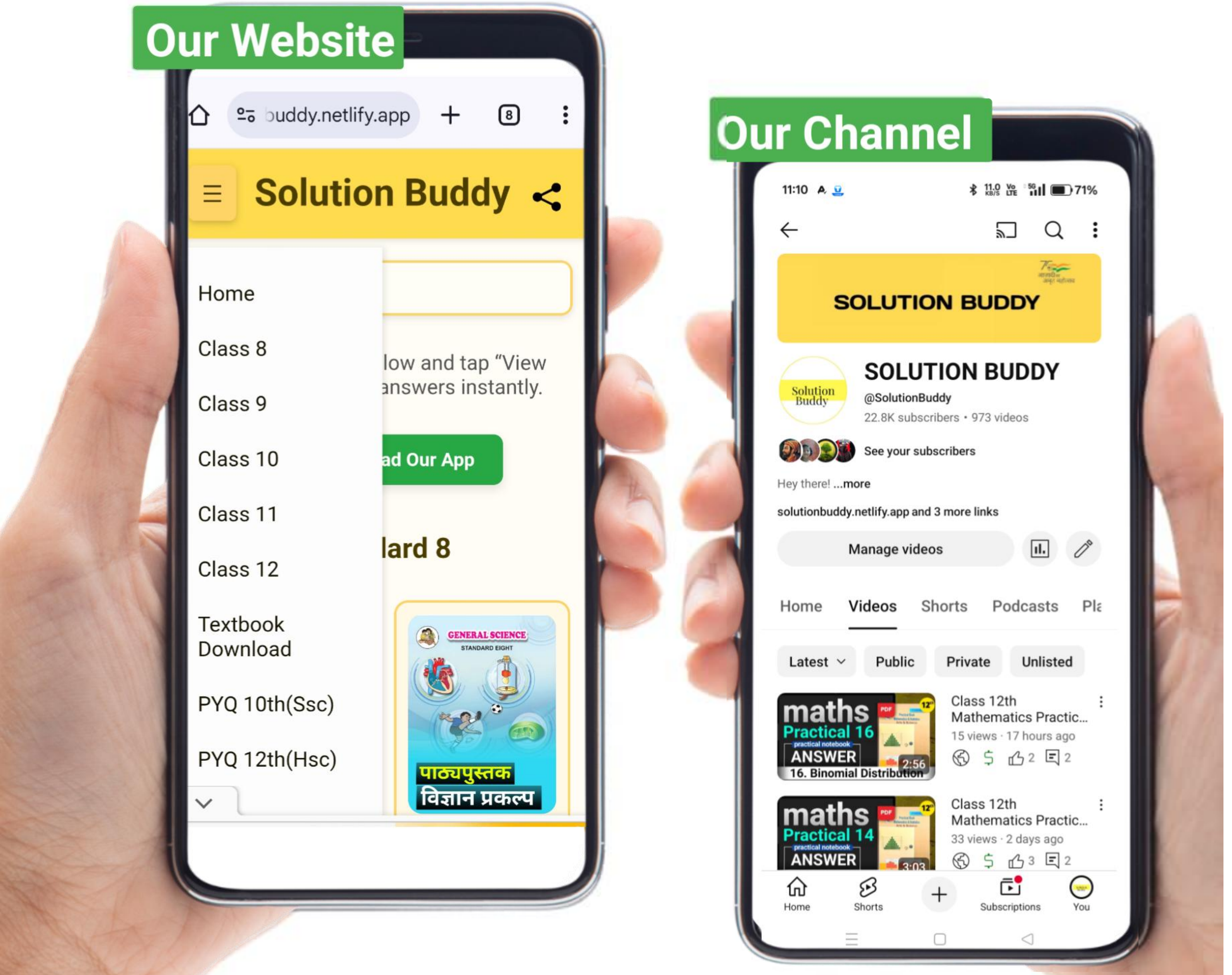

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