

Trigonometric Functions

EXERCISE 3.1 [PAGE 75]

Exercise 3.1 | Q 1.1 | Page 75

Find the principal solution of the following equation:

$$\cos\theta = 1/2$$

Solution:

We know that, $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\cos(2\pi - \theta) = \cos\theta$

$$\therefore \cos \frac{\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos \frac{5\pi}{3}$$

$$\therefore \cos \frac{\pi}{3} = \cos \frac{5\pi}{3} = \frac{1}{2}, \text{ where}$$

$$0 < \frac{\pi}{3} < 2\pi \text{ and } 0 < \frac{5\pi}{3} < 2\pi$$

$$\therefore \cos \theta = \frac{1}{2} \text{ gives } \cos \theta = \cos \frac{\pi}{3} = \cos \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$

Hence, the required principal solutions are

$$\theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}.$$

Exercise 3.1 | Q 1.2 | Page 75

Find the principal solution of the following equation:

$$\sec\theta = 2/\sqrt{3}$$

Solution:

$$\theta = \frac{\pi}{6} \text{ and } \theta = \frac{11\pi}{6}$$

Solution is not available.

Exercise 3.1 | Q 1.3 | Page 75

Find the principal solution of the following equation :

$$\cot \theta = \sqrt{3}$$

Solution:

The given equation is $\cot \theta = \sqrt{3}$ which is same as $\tan \theta = \frac{1}{\sqrt{3}}$.

We know that,

$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan(\pi + \theta) = \tan \theta$$

$$\therefore \tan \frac{\pi}{6} = \tan \left(\pi + \frac{\pi}{6} \right) = \tan \frac{7\pi}{6}$$

$$\therefore \tan \frac{\pi}{6} = \tan \frac{7\pi}{6} = \frac{1}{\sqrt{3}}, \text{ where}$$

$$0 < \frac{\pi}{6} < 2\pi \text{ and } 0 < \frac{7\pi}{6} < 2\pi$$

$$\therefore \cot \theta = \sqrt{3}, \text{ i.e. } \tan \theta = \frac{1}{\sqrt{3}} \text{ gives}$$

$$\tan \theta = \tan \frac{\pi}{6} = \tan \frac{7\pi}{6}$$

$$\therefore \theta = \frac{\pi}{6} \text{ and } \theta = \frac{7\pi}{6}$$

Hence, the required principal solution are

$$\theta = \frac{\pi}{6} \text{ and } \theta = \frac{7\pi}{6}.$$

Exercise 3.1 | Q 1.4 | Page 75

Find the principal solution of the following equation:

$$\cot\theta = 0$$

Solution:

$$\theta = \frac{\pi}{2} \text{ and } \theta = \frac{3\pi}{2}$$

Solution is not available

Exercise 3.1 | Q 2.1 | Page 75

Find the principal solution of the following equation:

$$\sin \theta = -1/2$$

Solution:

We now that,

$$\sin \frac{\pi}{6} = \frac{1}{2} \text{ and } \sin(\pi + \theta) = -\sin \theta,$$

$$\sin(2\pi - \theta) = -\sin\theta.$$

$$\therefore \sin\left(\pi + \frac{\pi}{6}\right) = -\frac{\sin \pi}{6} = -\frac{1}{2}$$

$$\text{and } \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\therefore \sin \frac{7\pi}{6} = \sin \frac{11\pi}{6} = -\frac{1}{2}, \text{ where}$$

$$0 < \frac{7\pi}{6} < 2\pi \text{ and } 0 < \frac{11\pi}{6} < 2\pi$$

$$\therefore \sin \theta = -\frac{1}{2} \text{ gives,}$$

$$\sin \theta = \sin \frac{7\pi}{6} = \sin \frac{11\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{6} \text{ and } \theta = \frac{11\pi}{6}$$

Hence, the required principal solutions are

$$\theta = \frac{7\pi}{6} \text{ and } \theta = \frac{11\pi}{6}.$$

Exercise 3.1 | Q 2.2 | Page 75

Find the principal solution of the following equation:

$$\tan \theta = -1$$

Solution:

We know that,

$$\tan \frac{\pi}{4} = 1 \text{ and } \tan(\pi - \theta) = -\tan \theta,$$

$$\tan(2\pi - \theta) = -\tan \theta$$

$$\therefore \tan\left(\pi - \frac{\pi}{4}\right) = -\tan \frac{\pi}{4} = -1$$

$$\text{and } \tan\left(2\pi - \frac{\pi}{4}\right) = -\tan \frac{\pi}{4} = -1$$

$$\therefore \tan \frac{3\pi}{4} = \tan \frac{7\pi}{4} = -1, \text{ where}$$

$$0 < \frac{3\pi}{4} < 2\pi \text{ and } 0 < \frac{7\pi}{4} < 2\pi$$

$\therefore \tan \theta = -1$ gives,

$$\tan \theta = \tan \frac{3\pi}{4} = \tan \frac{7\pi}{4}$$

$$\therefore \theta = \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4}$$

Hence, the required principal solutions are

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4}.$$

Exercise 3.1 | Q 2.3 | Page 75

Find the principal solution of the following equation:

$$\sqrt{3}\operatorname{cosec}\theta + 2 = 0$$

Solution:

$$\theta = \frac{4\pi}{3} \text{ and } \theta = \frac{5\pi}{3}.$$

The solution is not available.

Exercise 3.1 | Q 3.1 | Page 75

Find the general solution of the following equation:

$$\sin \theta = 1/2.$$

Solution:

The general solution of $\sin \theta = \sin \alpha$ is

$$\theta = n\pi + (-1)^n \alpha, n \in \mathbb{Z}$$

Now,

$$\sin \theta = \frac{1}{2} = \sin \frac{\pi}{6} \quad \dots \left[\because \sin \frac{\pi}{6} = \frac{1}{2} \right]$$

\therefore the required general solution is $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}.$

Exercise 3.1 | Q 3.2 | Page 75

Find the general solution of the following equation :

$$\cos \theta = \sqrt{3}/2$$

Solution:

The general solution of $\cos \theta = \cos \alpha$ is

$$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$$

Now,

$$\cos \theta = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6} \quad \dots \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \right]$$

\therefore the required general solution is

$$\theta = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 3.3 | Page 75

Find the general solution of the following equation:

$$\tan \theta = 1/\sqrt{3}$$

Solution:

The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}$$

Now,

$$\tan \theta = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6} \quad \dots \left[\because \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \right]$$

\therefore the required general solution is

$$\theta = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 3.4 | Page 75

Find the general solution of the following equation:

$$\cot \theta = 0.$$

Solution:

The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}$$

$$\text{Now, } \cot \theta = 0$$

$\therefore \tan \theta$ does not exist

$$\therefore \tan \theta = \tan \frac{\pi}{2} \quad \dots \left[\because \tan \frac{\pi}{2} \text{ does not exist} \right]$$

\therefore the required general solution is

$$\theta = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 4.1 | Page 75

Find the general solution of the following equation:

$$\sec \theta = \sqrt{2}.$$

Solution:

The general solution of $\cos \theta = \cos \alpha$ is

$$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$$

Now,

$$\sec \theta = \sqrt{2}$$

$$\therefore \cos \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \cos \theta = \cos \frac{\pi}{4} \quad \dots \left[\because \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right]$$

\therefore the required general solution is

$$\theta = 2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 4.2 | Page 75

Find the general solution of the following equation:

$$\operatorname{cosec} \theta = -\sqrt{2}.$$

Solution: The general solution of $\sin \theta = \sin \alpha$ is

$$\theta = n\pi + (-1)^n \alpha, n \in \mathbb{Z}.$$

Now,

$$\operatorname{Cosec} \theta = -\sqrt{2}$$

$$\therefore \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \sin \theta = -\sin \frac{\pi}{4} \quad \dots \left[\because \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right]$$

$$\therefore \sin \theta = \sin \left(\pi + \frac{\pi}{4} \right) \quad \dots [\because \sin(\pi + \theta) = -\sin \theta]$$

$$\therefore \sin \theta = \sin \frac{5\pi}{4}$$

\therefore the required general solution is

$$\theta = n\pi + (-1)^n \left(\frac{5\pi}{4} \right), n \in \mathbb{Z}.$$

Exercise 3.1 | Q 4.3 | Page 75

Find the general solution of the following equation:

$$\tan \theta = -1$$

Solution:

The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}.$$

Now, $\tan \theta = -1$

$$\therefore \tan \theta = -\tan \frac{\pi}{4} \quad \dots \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\therefore \tan \theta = \tan \left(\pi - \frac{\pi}{4} \right) \quad \dots [\because \tan(\pi - \theta) = -\tan \theta]$$

$$\therefore \tan \theta = \tan \frac{3\pi}{4}$$

\therefore the required general solution is

$$\theta = n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 5.1 | Page 75

Find the general solution of the following equation:

$$\sin 2\theta = 1/2$$

Solution:

The general solution of $\sin \theta = \sin \alpha$ is

$$\theta = n\pi + (-1)^n \alpha, n \in \mathbb{Z}.$$

Now,

$$\sin 2\theta = \frac{1}{2}$$

$$\therefore \sin 2\theta = \sin \frac{\pi}{6} \quad \dots \left[\because \sin \frac{\pi}{6} = \frac{1}{2} \right]$$

\therefore the required general solution is given by

$$2\theta = n\pi + (-1)^n \left(\frac{\pi}{6} \right), n \in \mathbb{Z}.$$

$$\text{i.e. } \theta = \frac{n\pi}{2} + (-1)^n \left(\frac{\pi}{12} \right), n \in \mathbb{Z}.$$

Exercise 3.1 | Q 5.2 | Page 75

Find the general solution of the following equation:

$$\tan 2\theta/3 = \sqrt{3}.$$

Solution:

The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}$$

Now,

$$\tan \frac{2\theta}{3} = \sqrt{3}.$$

$$\therefore \tan \frac{2\theta}{3} = \tan \frac{\pi}{3} \quad \dots \left[\because \tan \frac{\pi}{3} = \sqrt{3} \right]$$

\therefore the required general solution is given by

$$\frac{2\theta}{3} = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}.$$

$$\text{i.e. } \theta = \frac{3n\pi}{2} + \frac{\pi}{2}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 5.3 | Page 75

Find the general solution of the following equation:

$$\cot 4\theta = -1$$

Solution: The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}$$

Now,

$$\cot 4\theta = -1$$

$$\therefore \tan 4\theta = -1$$

$$\therefore \tan 4\theta = -\tan \frac{\pi}{4} \quad \dots \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\therefore \tan 4\theta = \tan \left(\pi - \frac{\pi}{4} \right) \quad \dots [\because \tan(\pi - \theta) = -\tan \theta]$$

$$\therefore \tan 4\theta = \tan \frac{3\pi}{4}$$

\therefore the required general solution is given by

$$4\theta = n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}$$

$$\text{i.e. } \theta = \frac{n\pi}{4} + \frac{3\pi}{16}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 6.1 | Page 75

Find the general solution of the following equation:

$$4\cos^2\theta = 3.$$

Solution:

The general solution of $\cos^2\theta = \cos^2\alpha$ is

$$\theta = n\pi \pm \alpha, n \in \mathbb{Z}.$$

$$\text{Now, } 4\cos^2\theta = 3$$

$$\therefore \cos^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\therefore \cos^2\theta = \left(\cos \frac{\pi}{6}\right)^2 \quad \dots \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \right]$$

$$\therefore \cos^2\theta = \cos^2 \frac{\pi}{6}$$

\therefore the required general solution is given by

$$\theta = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 6.2 | Page 75

Find the general solution of the following equation:

$$4\sin^2\theta = 1.$$

Solution:

The general solution of $\sin^2\theta = \sin^2\alpha$ is

$$\theta = n\pi \pm \alpha, n \in \mathbb{Z}.$$

$$\text{Now, } 4 \sin^2\theta = 1$$

$$\therefore \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\therefore \sin^2\theta = \left(\sin \frac{\pi}{6}\right)^2 \dots \left[\because \sin \frac{\pi}{6} = \frac{1}{2}\right]$$

$$\therefore \sin^2\theta = \sin^2 \frac{\pi}{6}$$

$$\therefore \text{the required general solution is } \theta = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 6.3 | Page 75

Find the general solution of the following equation:

$$\cos 4\theta = \cos 2\theta$$

Solution: The general solution of $\cos \theta = \cos \alpha$ is
 $\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$

\therefore the general solution of $\cos 4\theta = \cos 2\theta$ is given by

$$4\theta = 2n\pi \pm 2\theta, n \in \mathbb{Z}$$

Taking positive sign, we get

$$4\theta = 2n\pi + 2\theta, n \in \mathbb{Z}$$

$$\therefore 2\theta = 2n\pi, n \in \mathbb{Z}$$

$$\therefore \theta = n\pi, n \in \mathbb{Z}$$

Taking negative sign, we get

$$4\theta = 2n\pi - 2\theta, n \in \mathbb{Z}$$

$$\therefore 6\theta = 2n\pi, n \in \mathbb{Z}$$

$$\therefore \theta = \frac{n\pi}{3}, n \in \mathbb{Z}$$

Hence, the required general solution is

$$\theta = \frac{n\pi}{3}, n \in \mathbb{Z} \text{ or } \theta = n\pi, n \in \mathbb{Z}.$$

Alternative Method:

$$\cos 4\theta = \cos 2\theta$$

$$\therefore \cos 4\theta - \cos 2\theta = 0$$

$$\therefore -2 \sin\left(\frac{4\theta + 2\theta}{2}\right) \cdot \sin\left(\frac{4\theta - 2\theta}{2}\right) = 0$$

$$\therefore \sin 3\theta \cdot \sin \theta = 0$$

$$\therefore \text{either } \sin 3\theta = 0 \text{ or } \sin \theta = 0$$

The general solution of $\sin \theta = 0$ is $\theta = n\pi, n \in \mathbb{Z}$.

\therefore the required general solution is given by

$$3\theta = n\pi, n \in \mathbb{Z} \text{ or } \theta = n\pi, n \in \mathbb{Z}$$

$$\text{i.e. } \theta = n\pi/3, n \in \mathbb{Z} \text{ or } \theta = n\pi, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 7.1 | Page 75

Find the general solution of the following equation:

$$\sin \theta = \tan \theta.$$

Solution:

$$\sin \theta = \tan \theta$$

$$\therefore \sin \theta = \frac{\sin \theta}{\cos \theta}$$

$$\begin{aligned}
&\therefore \sin\theta \cos\theta = \sin\theta \\
&\therefore \sin\theta \cos\theta - \sin\theta = 0 \\
&\therefore \sin\theta (\cos\theta - 1) = 0 \\
&\therefore \text{either } \sin\theta = 0 \text{ or } \cos\theta - 1 = 0 \\
&\therefore \text{either } \sin\theta = 0 \text{ or } \cos\theta = 1 \\
&\therefore \text{either } \sin\theta = 0 \text{ or } \cos\theta = \cos 0 \quad \dots [\because \cos 0 = 1]
\end{aligned}$$

The general solution of $\sin \theta = 0$ is $\theta = n\pi$, $n \in \mathbb{Z}$ and $\cos\theta = \cos \alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in \mathbb{Z}$.

$$\begin{aligned}
&\therefore \text{the required general solution is given by} \\
&\theta = n\pi, n \in \mathbb{Z} \text{ or } \theta = 2n\pi \pm 0, n \in \mathbb{Z} \\
&\therefore \theta = n\pi, n \in \mathbb{Z} \text{ or } \theta = 2n\pi, n \in \mathbb{Z}.
\end{aligned}$$

Exercise 3.1 | Q 7.2 | Page 75

Find the general solution of the following equation:

$$\tan^3\theta = 3 \tan\theta.$$

Solution: $\tan^3\theta = 3 \tan\theta$

$$\begin{aligned}
&\therefore \tan^3\theta - 3 \tan\theta = 0 \\
&\therefore \tan\theta (\tan^2\theta - 3) = 0 \\
&\therefore \text{either } \tan\theta = 0 \text{ or } \tan^2\theta - 3 = 0 \\
&\therefore \text{either } \tan\theta = 0 \text{ or } \tan^2\theta = 3 \\
&\therefore \text{either } \tan\theta = 0 \text{ or } \tan^2\theta = (\sqrt{3})^2
\end{aligned}$$

$$\begin{aligned}
&\therefore \text{either } \tan \theta = 0 \text{ or } \tan^2\theta = \left(\tan \frac{\pi}{3}\right)^2 \quad \dots \left[\because \tan \frac{\pi}{3} = \sqrt{3}\right] \\
&\therefore \text{either } \tan\theta = 0 \text{ or } \tan^2\theta = \tan^2 \frac{\pi}{3}
\end{aligned}$$

The general solution of

$$\tan\theta = 0 \text{ is } \theta = n\pi, n \in \mathbb{Z} \text{ and}$$

$$\tan^2\theta = \tan^2\alpha \text{ is } \theta = n\pi \pm \alpha, n \in \mathbb{Z}.$$

$$\begin{aligned}
&\therefore \text{the required general solution is given by} \\
&\theta = n\pi, n \in \mathbb{Z} \text{ or } \theta = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}.
\end{aligned}$$

Exercise 3.1 | Q 7.3 | Page 75

Find the general solution of the following equation:

$$\cos \theta + \sin \theta = 1.$$

Solution:

$$\cos \theta + \sin \theta = 1$$

Dividing both sides by $\sqrt{(1)^2 + (1)^2} = \sqrt{2}$, we get

$$\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta = \cos \frac{\pi}{4}$$

$$\therefore \cos \left(\theta - \frac{\pi}{4} \right) = \cos \frac{\pi}{4} \dots (1)$$

The general solution of

$$\cos \theta = \cos \alpha \text{ is } \theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$$

\therefore the general solution of (1) is given by

$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

Taking positive sign, we get

$$\theta - \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$$

Taking negative sign, we get,

$$\theta - \frac{\pi}{4} = 2n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi, n \in \mathbb{Z}$$

\therefore the required general solution is

$$\theta = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z} \text{ or } \theta = 2n\pi, n \in \mathbb{Z}.$$

Alternative Method:

$$\cos\theta + \sin\theta = 1$$

$$\therefore \sin\theta = 1 - \cos\theta$$

$$\therefore 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin^2 \frac{\theta}{2}$$

$$\therefore 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2} = 0$$

$$\therefore 2 \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) = 0$$

$$\therefore 2 \sin \frac{\theta}{2} = 0 \text{ or } \cos \frac{\theta}{2} - \sin \frac{\theta}{2} = 0$$

$$\therefore \sin \frac{\theta}{2} = 0 \text{ or } \sin \frac{\theta}{2} = \cos \frac{\theta}{2}$$

$$\therefore \sin \frac{\theta}{2} = 0 \text{ or } \tan \frac{\theta}{2} = 1 \quad \dots \left[\because \cos \frac{\theta}{2} \neq 0 \right]$$

$$\therefore \sin \frac{\theta}{2} = 0 \text{ or } \tan \frac{\theta}{2} = \tan \frac{\pi}{4} \quad \dots \left[\because \tan \frac{\pi}{4} = 1 \right]$$

The general solution of $\sin \theta = 0$ is $\theta = n\pi$, $n \in \mathbb{Z}$ and $\tan \theta = \tan \alpha$ is $\theta = n\pi + \alpha$, $n \in \mathbb{Z}$.

\therefore the required general solution is

$$\frac{\theta}{2} = n\pi, n \in \mathbb{Z} \text{ or } \frac{\theta}{2} = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\text{i.e. } \theta = 2n\pi, n \in \mathbb{Z} \text{ or } \theta = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}.$$

Exercise 3.1 | Q 8.1 | Page 75

State whether the following equation have solution or not?

$$\cos 2\theta = -1$$

Solution: $\cos 2\theta = -1$

Since $-1 \leq \cos\theta \leq 1$ for any θ ,
 $\cos 2\theta = -1$ has solution.

Exercise 3.1 | Q 8.2 | Page 75

State whether the following equation has a solution or not?

$$\cos^2\theta = -1.$$

Solution: $\cos^2\theta = -1$

This is not possible because $\cos^2\theta \geq 0$ for any θ .

$\therefore \cos^2\theta = -1$ does not have any solution.

Exercise 3.1 | Q 8.3 | Page 75

State whether the following equation has a solution or not?

$$2\sin\theta = 3$$

Solution: $2\sin\theta = 3$

$$\therefore \sin\theta = 3/2$$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ .

$\therefore 2 \sin\theta = 3$ does not have any solution.

Exercise 3.1 | Q 8.4 | Page 75

State whether the following equation have solution or not?

$$3 \tan\theta = 5$$

Solution: $3 \tan\theta = 5$

$$\therefore \tan\theta = 5/3$$

This is possible because $\tan\theta$ is any real number.

$\therefore 3 \tan\theta = 5$ has solution.

EXERCISE 3.2 [PAGE 88]

Exercise 3.2 | Q 1.1 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are :

$$\left(\sqrt{2}, \frac{\pi}{4}\right)$$

Solution:

Here, $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

Let the cartesian coordinates be (x,y)

$$\text{Then, } x = r \cos \theta = \sqrt{2} \cos \frac{\pi}{4} = \sqrt{2} \left(\frac{1}{\sqrt{2}} \right) = 1$$

$$y = r \sin \theta = \sqrt{2} \sin \frac{\pi}{4} = \sqrt{2} \left(\frac{1}{\sqrt{2}} \right) = 1$$

\therefore the cartesian coordinates of the given point are (1, 1).

Exercise 3.2 | Q 1.2 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are :
(4, $\pi/2$)

Solution:

The cartesian coordinates of the given point are (0, 4).

Solution is not available.

Exercise 3.2 | Q 1.3 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are:

$$\left(\frac{3}{4}, \frac{3\pi}{4} \right)$$

Solution:

Here, $r = \frac{3}{4}$ and $\theta = \frac{3\pi}{4}$

Let the cartesian coordinates be (x, y)

Then,

$$x = r \cos \theta = \frac{3}{4} \cos \frac{3\pi}{4} = \frac{3}{4} \cos \left(\pi - \frac{\pi}{4} \right)$$

$$= -\frac{3}{4} \cos \frac{\pi}{4} = -\frac{3}{4} \times \frac{1}{\sqrt{2}} = -\frac{3}{4\sqrt{2}}$$

$$y = r \sin \theta = \frac{3}{4} \sin \frac{3\pi}{4} = \frac{3}{4} \sin \left(\pi - \frac{\pi}{4} \right) \\ = \frac{3}{4} \sin \frac{\pi}{4} = \frac{3}{4} \times \frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$$

∴ The cartesian coordinates of the given point are $\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}} \right)$.

Exercise 3.2 | Q 1.4 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are:

$$\left(\frac{1}{2}, \frac{7\pi}{3} \right)$$

Solution:

$$\text{Here, } r = \frac{1}{2} \text{ and } \theta = \frac{7\pi}{3}$$

Let the cartesian coordinates be (x, y)

Then,

$$x = r \cos \theta = \frac{1}{2} \cos \frac{7\pi}{3} = \frac{1}{2} \cos \left(2\pi + \frac{\pi}{3} \right) \\ = \frac{1}{2} \cos \frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$y = r \sin \theta = \frac{1}{2} \sin \frac{7\pi}{3} = \frac{1}{2} \sin \left(2\pi + \frac{\pi}{3} \right) \\ = \frac{1}{2} \sin \frac{\pi}{3} = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

∴ The cartesian coordinates of the given point are $\left(\frac{1}{4}, \frac{\sqrt{3}}{4} \right)$

Exercise 3.2 | Q 2.1 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$(\sqrt{2}, \sqrt{2})$

Solution:

Here $x = \sqrt{2}$ and $y = \sqrt{2}$

\therefore the point lies in the first quadrant.

Let the polar coordinates be (r, θ)

$$\text{Then, } r^2 = x^2 + y^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2 + 2 = 4$$

$$\therefore r = 2 \quad \dots [\because r > 0]$$

$$\cos \theta = \frac{x}{r} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

and

$$\sin \theta = \frac{y}{r} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Since the point lies in the first quadrant and

$$0 \leq \theta < 2\pi, \tan \theta = 1 = \tan \frac{\pi}{4}$$

$$\therefore \theta = \frac{\pi}{4}$$

\therefore the polar coordinates of the given point are $\left(2, \frac{\pi}{4}\right)$.

Exercise 3.2 | Q 2.2 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$\left(0, \frac{1}{2}\right)$

Solution: Here $x = 0$ and $y = 2$

\therefore the point lies on the positive side of Y-axis.

Let the polar coordinates be (r, θ)

Then, $r^2 = x^2 + y^2$

$$= (0)^2 + \left(\frac{1}{2}\right)^2$$

$$= 0 + \frac{1}{4}$$

$$= \frac{1}{4}$$

$$\therefore r = \frac{1}{2} \quad \dots[\because r > 0]$$

$$\cos \theta = \frac{x}{r} = \frac{0}{\frac{1}{2}} = 0$$

and

$$\sin \theta = \frac{y}{r} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Since, the point lies on the positive side of Y-axis and

$$0 \leq \theta < 2\pi$$

$$\cos \theta = 0 = \cos \frac{\pi}{2} \quad \text{and} \quad \sin \theta = 1 = \sin \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{2}$$

\therefore the polar coordinates of the given point are $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

Exercise 3.2 | Q 2.3 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$(1, -\sqrt{3})$

Solution: Here $x = 1$ and $y = -\sqrt{3}$

\therefore the point lies in the fourth quadrant.

Let the polar coordinates be (r, θ) .

$$\text{Then } r^2 = x^2 + y^2 = (1)^2 + (-\sqrt{3})^2 = 1 + 3 = 4$$

$$\therefore r = 2 \quad \dots [\because r > 0]$$

$$\cos \theta = \frac{x}{r} = \frac{1}{2}$$

$$\text{and } \sin \theta = \frac{y}{r} = -\frac{\sqrt{3}}{2}$$

$$\therefore \tan \theta = -\sqrt{3}$$

Since, the point lies in the fourth quadrant and $0 \leq \theta < 2\pi$.

$$\tan \theta = -\sqrt{3} = -\tan \frac{\pi}{3}$$

$$= \tan \frac{5\pi}{3}$$

$$\therefore \theta = \frac{5\pi}{3}$$

\therefore The polar coordinates of the given point are $\left(2, \frac{5\pi}{3}\right)$.

Exercise 3.2 | Q 2.4 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right).$$

Solution: The polar coordinates of the given point are $(3, \pi/3)$.

Solution is not available.

Exercise 3.2 | Q 3 | Page 88

In $\triangle ABC$, if $\angle A = 45^\circ$, $\angle B = 60^\circ$ then find the ratio of its sides.

Solution: By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\therefore \frac{a}{b} = \frac{\sin A}{\sin B} \text{ and } \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\therefore a : b : c = \sin A : \sin B : \sin C$$

Given $\angle A = 45^\circ$ and $\angle B = 60^\circ$

$$\because \angle A + \angle B + \angle C = 180^\circ$$

$$\therefore 45^\circ + 60^\circ + \angle C = 180^\circ$$

$$\therefore \angle C = 180^\circ - 105^\circ = 75^\circ$$

$$\text{Now, } \sin A = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin B = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\text{and } \sin C = \sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2(\sqrt{2})} + \frac{1}{2(\sqrt{2})}$$

$$= \frac{\sqrt{3} + 1}{2(\sqrt{2})}$$

\therefore the ratio of the sides of $\triangle ABC$

$$= a : b : c$$

$$= \sin A : \sin B : \sin C$$

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore a : b : c = 2 : \sqrt{6} : (\sqrt{3} + 1).$$

Exercise 3.2 | Q 4 | Page 88

In ΔABC , prove that $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right) \cos \frac{A}{2}$.

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\begin{aligned} \text{R.H.S.} &= \left(\frac{b-c}{a}\right) \cos \frac{A}{2} \\ &= \left(\frac{k \sin B - k \sin C}{k \sin A}\right) \cos \frac{A}{2} \\ &= \left(\frac{\sin B - \sin C}{\sin A}\right) \cos \frac{A}{2} \\ &= \frac{2 \cos\left(\frac{B+C}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)}{2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}} \cdot \cos \frac{A}{2} \\ &= \frac{\cos\left(\frac{B+C}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)}{\sin \frac{A}{2}} \\ &= \frac{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)}{\sin \frac{A}{2}} \quad \dots [\because A + B + C = \pi] \\ &= \frac{\sin \frac{A}{2} \cdot \sin\left(\frac{B-C}{2}\right)}{\frac{\sin A}{2}} \end{aligned}$$

$$= \sin\left(\frac{B - C}{2}\right)$$

$$= \text{L.H.S.}$$

Exercise 3.2 | Q 5 | Page 88

With the usual notations prove that $2\left\{a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right\} = a - b + c$.

Solution:

$$\text{L.H.S.} = 2\left\{a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right\}$$

$$= a\left(2 \sin^2 \frac{C}{2}\right) + c\left(2 \sin^2 \frac{A}{2}\right)$$

$$= a(1 - \cos C) + c(1 - \cos A)$$

$$= a\left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right] + c\left[1 - \frac{b^2 + c^2 - a^2}{2bc}\right] \quad \dots[\text{By cosine rule}]$$

$$= a\left[\frac{2ab - a^2 - b^2 + c^2}{2ab}\right] + c\left[\frac{2bc - b^2 - c^2 + a^2}{2bc}\right]$$

$$= \frac{2ab - a^2 - b^2 + c^2}{2b} + \frac{2bc - b^2 - c^2 + a^2}{2b}.$$

$$= \frac{2ab - a^2 - b^2 + c^2 + 2bc - b^2 - c^2 + a^2}{2b}.$$

$$= \frac{2ab - 2b^2 + 2bc}{2b}$$

$$= a - b + c$$

$$= \text{R.H.S.}$$

Exercise 3.2 | Q 6 | Page 88

In ΔABC , prove that $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$

Solution: By the sine rule,

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\text{L.H.S.} = a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B)$$

$$= a^3 (\sin B \cos C - \cos B \sin C) + b^3 (\sin C \cos A - \cos C \sin A) + c^3 (\sin A \cos B - \cos A \sin B)$$

$$= a^3 \left(\frac{b}{k} \cos C - \frac{c}{k} \cos B \right) + b^3 \left(\frac{c}{k} \cos A - \frac{a}{k} \cos C \right) + c^3 \left(\frac{a}{k} \cos B - \frac{b}{k} \cos A \right)$$

$$= \frac{1}{k} [a^3 b \cos C - a^3 c \cos B + b^3 c \cos A - b^3 a \cos C + c^3 a \cos B - c^3 b \cos A]$$

$$= \frac{1}{k} \left[a^3 b \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - a^3 c \left(\frac{c^2 + a^2 - b^2}{2ca} \right) + b^3 c \left(\frac{b^2 + c^2 - a^2}{2bc} \right) - ab^3 \left(\frac{a^2 + b^2 - c^2}{2ab} \right) + ac^3 \left(\frac{c^2 + a^2 - b^2}{2ca} \right) - bc^3 \left(\frac{b^2 + c^2 - a^2}{2bc} \right) \right] \dots [\text{By cosine rule}]$$

$$= \frac{1}{2k} [a^2(a^2 + b^2 - c^2) - a^2(a^2 + c^2 - b^2) + b^2(b^2 + c^2 - a^2) - b^2(a^2 + b^2 - c^2) + c^2(c^2 + a^2 - b^2) - c^2(b^2 + c^2 - a^2)]$$

$$= \frac{1}{2k} [a^4 + a^2 b^2 - a^2 c^2 - a^4 - a^2 c^2 + a^2 b^2 + b^4 + b^2 c^2 - a^2 b^2 - a^2 b^2 - b^4 + b^2 c^2 + c^4 + a^2 c^2 - b^2 c^2 - b^2 c^2 - c^4 + a^2 c^2]$$

$$= \frac{1}{2k} (0)$$

$$= 0$$

$$= \text{R.H.S.}$$

Exercise 3.2 | Q 7 | Page 88

In ΔABC , if $\cot A, \cot B, \cot C$ are in A.P. then show that a^2, b^2, c^2 are also in A.P.

Solution:

By the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$\therefore \sin A = ka, \sin B = kb, \sin C = kc \dots (1)$$

Now, $\cot A, \cot B, \cot C$ are in A.P.

$$\therefore \cot C - \cot B = \cot B - \cot A$$

$$\therefore \cot A + \cot C = 2\cot B$$

$$\therefore \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = 2\cot B$$

$$\therefore \frac{\sin C \cos A + \sin A \cos C}{\sin A \cdot \sin C} = 2\cot B$$

$$\therefore \frac{\sin(A + C)}{\sin A \cdot \sin C} = 2\cot B$$

$$\therefore \frac{\sin(\pi - B)}{\sin A \cdot \sin C} = 2\cot B \quad \dots[\because A + B + C = \pi]$$

$$\therefore \frac{\sin B}{\sin A \cdot \sin C} = \frac{2 \cos B}{\sin B}$$

$$\therefore \frac{k^2 b^2}{(ka)(kc)} = 2 \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$\therefore \frac{b^2}{ac} = \frac{a^2 + c^2 - b^2}{ac}$$

$$\therefore b^2 = a^2 + c^2 - b^2$$

$$\therefore 2b^2 = a^2 + c^2$$

Hence, a^2, b^2, c^2 are in A.P.

Exercise 3.2 | Q 8 | Page 88

In $\triangle ABC$, if $a \cos A = b \cos B$ then prove that the triangle is either a right angled or an isosceles triangle.

Solution: Using the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k$$

$$a = k \sin A \text{ and } b = k \sin B$$

$$\therefore a \cos A = b \cos B \text{ gives}$$

$$k \sin A \cos A = k \sin B \cos B$$

$$\therefore 2 \sin A \cos A = 2 \sin B \cos B$$

$$\therefore \sin 2A = \sin 2B$$

$$\therefore \sin 2A - \sin 2B = 0$$

$$\therefore 2 \cos(A + B) \cdot \sin(A - B) = 0$$

$$\therefore 2 \cos(\pi - C) \cdot \sin(A - B) = 0 \quad \dots [\because A + B + C = \pi]$$

$$\therefore -2 \cos C \cdot \sin(A - B) = 0$$

$$\therefore \cos C = 0 \text{ OR } \sin(A - B) = 0$$

$$\therefore C = 90^\circ \text{ OR } A - B = 0$$

$$\therefore C = 90^\circ \text{ OR } A = B$$

\therefore the triangle is either rightangled or an isosceles triangle.

Exercise 3.2 | Q 9 | Page 88

With usual notations prove that $2(bc \cos A + ac \cos B + ab \cos C) = a^2 + b^2 + c^2$.

Solution:

$$\text{L.H.S.} = 2(bc \cos A + ac \cos B + ab \cos C)$$

$$= 2bc \cos A + 2ac \cos B + 2ab \cos C$$

$$= 2bc \left(\frac{b^2 + c^2 - a^2}{2bc} \right) + 2ac \left(\frac{c^2 + a^2 - b^2}{2ca} \right) + 2ab \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \quad \dots [\text{By cosine rule}]$$

$$= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$$

$$= a^2 + b^2 + c^2$$

$$= \text{R.H.S.}$$

Exercise 3.2 | Q 10.1 | Page 88

In $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$ then find the values of $\cos A$

Solution: Given: $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(24)^2 + (30)^2 - (18)^2}{2(24)(30)} \\ &= \frac{576 + 900 - 324}{1440} \\ &= \frac{1152}{1440} \\ &= \frac{4}{5}.\end{aligned}$$

Exercise 3.2 | Q 10.2 | Page 88

In $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$ then find the values of $\sin A/2$.

Solution: Given: $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$\begin{aligned}\sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ &= \sqrt{\frac{(36-24)(36-30)}{(24)(30)}} \\ &= \sqrt{\frac{12 \times 6}{24 \times 30}}\end{aligned}$$

$$= \sqrt{\frac{1}{10}}$$

$$= \frac{1}{\sqrt{10}}.$$

Exercise 3.2 | Q 10.3 | Page 88

In $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$ then find the values of $\cos A/2$

Solution: Given: $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{36(36-18)}{(24)(30)}}$$

$$= \sqrt{\frac{36 \times 18}{24 \times 30}}$$

$$= \sqrt{\frac{9}{10}}$$

$$= \frac{3}{\sqrt{10}}.$$

Exercise 3.2 | Q 10.4 | Page 88

In $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$ then find the values of $\tan A/2$

Solution: Given : $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$$

$$= \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}}$$

$$= \frac{1}{3}.$$

Exercise 3.2 | Q 10.5 | Page 88

In ΔABC , if $a = 18$, $b = 24$, $c = 30$ then find the values of $A(\Delta ABC)$

Solution:

Given: $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{36(36-18)(36-24)(36-30)}$$

$$= \sqrt{36 \times 18 \times 12 \times 6}$$

$$= \sqrt{36 \times 18 \times 4 \times 18}$$

$$= 6 \times 18 \times 2$$

$$= 216 \text{ sq units.}$$

Exercise 3.2 | Q 10.6 | Page 88

In ΔABC , if $a = 18$, $b = 24$, $c = 30$ then find the values of $\sin A$

Solution: Given : $a = 18$, $b = 24$ and $c = 30$

$$\therefore 2s = a + b + c$$

$$= 18 + 24 + 30$$

$$= 72$$

$$\therefore s = 36$$

$$216 = \frac{1}{2}(24)(30) \sin A$$

$$\therefore \sin A = \frac{216}{12 \times 30}$$

$$= \frac{216}{360}$$

$$= \frac{3}{5}.$$

Exercise 3.2 | Q 11 | Page 88

In ΔABC prove that $(b+c-a)\tan \frac{A}{2} = (c+a-b)\tan \frac{B}{2} = (a+b-c)\tan \frac{C}{2}$.

Solution:

$$\begin{aligned} & (b + c - a) \tan \frac{A}{2} \\ &= (a + b + c - 2a) \cdot \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \\ &= (2s - 2a) \cdot \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \\ &= 2\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \dots (1) \end{aligned}$$

$$\begin{aligned}
& (c + a - b) \tan \frac{B}{2} \\
&= (a + b + c - 2b) \cdot \sqrt{\frac{(s - a)(s - c)}{s(s - b)}} \\
&= (2s - 2b) \cdot \sqrt{\frac{(s - a)(s - c)}{s(s - b)}} \\
&= 2\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \quad \dots(2)
\end{aligned}$$

$$\begin{aligned}
& (a + b - c) \tan \frac{C}{2} \\
&= (a + b + c - 2c) \cdot \sqrt{\frac{(s - a)(s - b)}{s(s - c)}} \\
&= (2s - 2c) \cdot \sqrt{\frac{(s - a)(s - b)}{s(s - c)}} \\
&= 2\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \quad \dots(3)
\end{aligned}$$

From (1), (2) and (3), we get

$$(b + c - a) \tan \frac{A}{2} = (c + a - b) \tan \frac{B}{2} = (a + b - c) \tan \frac{C}{2}.$$

Exercise 3.2 | Q 12 | Page 88

$$\text{In } \triangle ABC \text{ prove that } \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = \frac{[A(\triangle ABC)]^2}{abcs}$$

Solution:

L.H.S.

$$\begin{aligned} &= \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \\ &= \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-a)(s-c)}{ac}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= \sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}} \\ &= \frac{(s-a)(s-b)(s-c)}{abc} \\ &= \frac{s(s-a)(s-b)(s-c)}{abcs} \\ &= \frac{[A(\Delta ABC)]^2}{abcs} \quad \dots \left[\because A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)} \right] \\ &= \text{R.H.S.} \end{aligned}$$

EXERCISE 3.3 [PAGES 102 - 103]

Exercise 3.3 | Q 1.1 | Page 102

Find the principal value of the following: $\sin^{-1}(1/2)$

Solution:

The principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{2}\right) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$$\therefore \sin \alpha = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\therefore \alpha = \frac{\pi}{6} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \right]$$

\therefore the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

Exercise 3.3 | Q 1.2 | Page 102

Find the principal value of the following: $\operatorname{cosec}^{-1}(2)$

Solution:

The principal value branch of $\operatorname{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(2) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}, \alpha \neq 0$.

$$\therefore \operatorname{cosec} \alpha = 2 = \operatorname{cosec} \frac{\pi}{6}$$

$$\therefore \alpha = \frac{\pi}{6} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \right]$$

\therefore the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Exercise 3.3 | Q 1.3 | Page 102

Find the principal value of the following: $\tan^{-1}(-1)$

Solution:

The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Let $\tan^{-1}(-1) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$$\therefore \tan \alpha = -1 = -\tan \frac{\pi}{4}$$

$$\therefore \tan \alpha = \tan\left(-\frac{\pi}{4}\right) \quad \dots [\because \tan(-\theta) = -\tan \theta]$$

$$\therefore \alpha = -\frac{\pi}{4} \quad \dots \left[\because -\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2} \right]$$

\therefore the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Exercise 3.3 | Q 1.4 | Page 102

Find the principal value of the following: $\tan^{-1}(-\sqrt{3})$

Solution:

The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-\sqrt{3}) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$$\therefore \tan \alpha = -\sqrt{3} = -\tan \frac{\pi}{3}$$

$$\therefore \tan \alpha = \tan\left(-\frac{\pi}{3}\right) \quad \dots [\because \tan(-\theta) = -\tan \theta]$$

$$\therefore \alpha = -\frac{\pi}{3} \quad \dots \left[\because -\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2} \right]$$

\therefore the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

Exercise 3.3 | Q 1.5 | Page 102

Find the principal value of the following: $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Solution:

The principal value branch of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$$\therefore \sin \alpha = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$$\therefore \alpha = \frac{\pi}{4} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2} \right]$$

\therefore the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

Exercise 3.3 | Q 1.6 | Page 102

Find the principal value of the following: $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

The principal value branch of $\cos^{-1}x$ $[0, \pi]$.

Let $\cos^{-1}\left(-\frac{1}{2}\right) = \alpha$, where $0 \leq \alpha \leq \pi$

$$\therefore \cos \alpha = -\frac{1}{2} = -\cos \frac{\pi}{3}$$

$$\therefore \cos \alpha = \cos\left(\pi - \frac{\pi}{3}\right) \quad \dots[\because \cos(\pi - \theta) = -\cos\theta]$$

$$\therefore \cos \alpha = \cos \frac{2\pi}{3}$$

$$\therefore \alpha = \frac{2\pi}{3} \quad \dots\left[\because 0 \leq \frac{2\pi}{3} \leq \pi\right]$$

\therefore the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Exercise 3.3 | Q 2.1 | Page 102

Evaluate the following:

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

Solution:

$$\text{Let } \tan^{-1}(1) = \alpha, \text{ where } \frac{-\pi}{2} < \alpha < \frac{\pi}{2}$$

$$\therefore \tan \alpha = 1 = \tan \frac{\pi}{4}$$

$$\therefore \alpha = \frac{\pi}{4} \quad \dots \left[\because \frac{-\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2} \right]$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4} \quad \dots(1)$$

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right) = \beta, \text{ where } 0 \leq \beta \leq \pi$$

$$\therefore \cos \beta = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\therefore \beta = \frac{\pi}{3} \quad \dots \left[\because 0 < \frac{\pi}{3} < \pi \right]$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \dots(2)$$

$$\therefore \sin \gamma = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\therefore \gamma = \frac{\pi}{6} \quad \dots \left[\because \frac{-\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \right]$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \dots(3)$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6} \quad \dots[\text{By (1), (2) and (3)}]$$

$$= \frac{3\pi + 4\pi + 2\pi}{12}$$

$$\begin{aligned}
 &= \frac{9\pi}{12} \\
 &= \frac{3\pi}{4}.
 \end{aligned}$$

Exercise 3.3 | Q 2.2 | Page 102

Evaluate the following:

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

Solution:

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right) = \alpha, \text{ where } 0 \leq \alpha \leq \pi$$

$$\therefore \cos \alpha = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\therefore \alpha = \frac{\pi}{3} \quad \dots \left[\because 0 < \frac{\pi}{3} < \pi \right]$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \dots(1)$$

$$\text{Let } \sin^{-1}\left(\frac{1}{2}\right) = \beta, \text{ where } -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$$

$$\therefore \sin \beta = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\therefore \beta = \frac{\pi}{6} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \right]$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \dots(2)$$

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \text{and} \quad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}.$$

Exercise 3.3 | Q 2.3 | Page 102

Evaluate the following:

$$\tan^{-1} \sqrt{3} - \sec^{-1}(-2)$$

Solution:

$$\text{Let } \tan^{-1}(\sqrt{3}) = \alpha, \text{ where } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

$$\therefore \tan \alpha = \sqrt{3} = \tan \frac{\pi}{3}$$

$$\therefore \alpha = \frac{\pi}{3} \quad \dots \left[\because -\frac{\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2} \right]$$

$$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \quad \dots(1)$$

$$\text{Let } \sec^{-1}(-2) = \beta, \text{ where } 0 \leq \beta \leq \pi, \beta \neq \frac{\pi}{2}$$

$$\therefore \sec \beta = -2 = -\sec \frac{\pi}{3}$$

$$\therefore \sec \beta = \sec \left(\pi - \frac{\pi}{3} \right) \quad \dots [\because \sec(\pi - \theta) = -\sec \theta]$$

$$\therefore \sec \beta = \sec \frac{2\pi}{3}$$

$$\therefore \beta = \frac{2\pi}{3} \quad \dots \left[\because 0 \leq \frac{2\pi}{3} \leq \pi \right]$$

$$\therefore \sec^{-1}(-2) = \frac{2\pi}{3} \quad \dots (2)$$

$$\begin{aligned} &\therefore \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) \\ &= \frac{\pi}{3} - \frac{2\pi}{3} \quad \dots [\text{By (1) and (2)}] \\ &= -\frac{\pi}{3}. \end{aligned}$$

Exercise 3.3 | Q 2.4 | Page 103

Evaluate the following:

$$\operatorname{cosec}^{-1}(-\sqrt{2}) + \cot^{-1}(\sqrt{3})$$

Solution:

$$\text{Let } \operatorname{cosec}^{-1}(-\sqrt{2}) = \alpha, \text{ where } \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$$

$$\therefore \operatorname{cosec} \alpha = -\sqrt{2} = -\operatorname{cosec} \frac{\pi}{4}$$

$$\therefore \operatorname{cosec} \alpha = \operatorname{cosec} \left(-\frac{\pi}{4} \right) \quad \dots [\because \operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta]$$

$$\therefore \alpha = -\frac{\pi}{4} \quad \dots \left[\because \frac{-\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2} \right]$$

$$\therefore \operatorname{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4} \quad \dots (1)$$

Let $\cot^{-1}(\sqrt{3}) = \beta$, where $0 < \beta < \pi$

$$\therefore \cot \beta = \sqrt{3} = \cot \frac{\pi}{6}$$

$$\therefore \beta = \frac{\pi}{6} \quad \dots \left[\because 0 < \frac{\pi}{6} < \pi \right]$$

$$\therefore \cot^{-1}(\sqrt{3}) = \frac{\pi}{6} \quad \dots (2)$$

$$\begin{aligned} \therefore \operatorname{cosec}^{-1}(-\sqrt{2}) + \cot^{-1}(\sqrt{3}) \\ &= -\frac{\pi}{4} + \frac{\pi}{6} \quad \dots [\text{By (1) and (2)}] \\ &= \frac{-3\pi + 2\pi}{12} \\ &= -\frac{\pi}{12}. \end{aligned}$$

Exercise 3.3 | Q 3.1 | Page 103

Prove the following:

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3 \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\pi}{4}$$

Solution:

$$\text{Let } \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha, \text{ where } -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$$

$$\therefore \sin \alpha = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$$\therefore \alpha = \frac{\pi}{4} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2} \right]$$

$$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \quad \dots(1)$$

$$\text{Let } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \beta, \text{ where } -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$$

$$\therefore \sin \beta = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$$

$$\therefore \beta = \frac{\pi}{3} \quad \dots \left[\because -\frac{\pi}{2} \leq \frac{\pi}{3} \leq \frac{\pi}{2} \right]$$

$$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \quad \dots(2)$$

$$\text{L.H.S.} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3 \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} - 3\left(\frac{\pi}{3}\right) \quad \dots[\text{By (1) and (2)}]$$

$$= \frac{\pi}{4} - \pi$$

$$= -\frac{3\pi}{4}$$

$$= \text{R.H.S.}$$

Exercise 3.3 | Q 3.2 | Page 103

Prove the following:

$$\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$$

Solution:

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) = \alpha, \text{ where } -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$$

$$\therefore \sin \alpha = -\frac{1}{2} = -\sin \frac{\pi}{6}$$

$$\therefore \sin \alpha = \sin\left(-\frac{\pi}{6}\right) \quad \dots[\because \sin(-\theta) = -\sin \theta]$$

$$\therefore \alpha = -\frac{\pi}{6} \quad \dots\left[\because -\frac{\pi}{2} \leq -\frac{\pi}{6} \leq \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \quad \dots(1)$$

$$\text{Let } \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \beta, \text{ where } 0 \leq \beta \leq \pi$$

$$\therefore \cos \beta = -\frac{\sqrt{3}}{2} = -\cos \frac{\pi}{6}$$

$$\therefore \cos \beta = \cos\left(\pi - \frac{\pi}{6}\right) \quad \dots[\because \cos(\pi - \theta) = -\cos \theta]$$

$$\therefore \cos \beta = \cos \frac{5\pi}{6}$$

$$\therefore \beta = \frac{5\pi}{6} \quad \dots\left[\because 0 \leq \frac{5\pi}{6} \leq \pi\right]$$

$$\therefore \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6} \quad \dots(2)$$

$$\text{Let } \cos^{-1}\left(-\frac{1}{2}\right) = Y, \text{ where } 0 \leq Y \leq \pi$$

$$\therefore \cos Y = -\frac{1}{2} = -\cos \frac{\pi}{3}$$

$$\therefore \cos Y = \cos\left(\pi - \frac{\pi}{3}\right) \quad \dots[\because \cos(\pi - \theta) = -\cos \theta]$$

$$\therefore \cos Y = \cos \frac{2\pi}{3}$$

$$\therefore Y = \frac{2\pi}{3} \quad \dots\left[\because 0 \leq \frac{2\pi}{3} \leq \pi\right]$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \quad \dots(3)$$

$$\text{L.H.S.} = \sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\pi}{6} + \frac{5\pi}{6} \quad \dots[\text{By (1) and (2)}]$$

$$= \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$= \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$= \cos^{-1}\left(-\frac{1}{2}\right) \quad \dots[\text{By (3)}]$$

$$= \text{R.H.S.}$$

Exercise 3.3 | Q 3.3 | Page 103

Prove the following:

$$\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

Solution:

$$\text{Let } \sin^{-1}\left(\frac{3}{5}\right) = x, \cos^{-1}\left(\frac{12}{13}\right) = y \text{ and } \sin^{-1}\left(\frac{56}{65}\right) = z.$$

$$\text{Then } \sin x = \frac{3}{5}, \text{ where } 0 < x < \frac{\pi}{2}$$

$$\cos y = \frac{12}{13}, \text{ where } 0 < y < \frac{\pi}{2}$$

$$\text{and } \sin z = \frac{56}{65}, \text{ where } 0 < z < \frac{\pi}{2}$$

$$\therefore \cos x > 0, \sin y > 0$$

$$\text{Now, } \cos x = \sqrt{1 - \sin^2 x}$$

$$= \sqrt{1 - \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \sin y = \sqrt{1 - \cos^2 y}$$

$$= \sqrt{1 - \frac{144}{169}}$$

$$= \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\text{We have to prove, that, } x + y = z$$

$$\text{Now, } \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$$

$$\therefore \sin(x + y) = \sin z$$

$$\therefore x + y = z$$

$$\text{Hence, } \sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right).$$

Exercise 3.3 | Q 3.4 | Page 103

Prove the following:

$$\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

Solution:

$$\text{Let } \cos^{-1}\left(\frac{3}{5}\right) = x$$

$$\therefore \cos x = \frac{3}{5}, \text{ where } 0 < x < \frac{\pi}{2}$$

$$\therefore \sin x > 0$$

Now,

$$\sin x = \sqrt{1 - \cos^2 x}$$

$$= \sqrt{1 - \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}}$$

$$= \frac{4}{5}$$

$$\therefore x = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\therefore \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{4}{5}\right) \quad \dots(1)$$

$$\begin{aligned} \text{L.H.S.} &= \cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) \\ &= \sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) \quad \dots[\text{By (1)}] \\ &= \frac{\pi}{2} \quad \dots\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right] \\ &= \text{R.H.S.} \end{aligned}$$

Exercise 3.3 | Q 3.5 | Page 103

Prove the following:

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

Solution:

$$\begin{aligned} \text{L.H.S.} &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \\ &= \tan^{-1}\left[\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}\right] \\ &= \tan^{-1}\left(\frac{3+2}{6-1}\right) \\ &= \tan^{-1}(1) \\ &= \tan^{-1}\left(\tan \frac{\pi}{4}\right) \\ &= \frac{\pi}{4} \\ &= \text{R.H.S.} \end{aligned}$$

Exercise 3.3 | Q 3.6 | Page 103

Prove the following:

$$2 \tan^{-1} \left(\frac{1}{3} \right) = \tan^{-1} \left(\frac{3}{4} \right)$$

Solution:

$$\begin{aligned} \text{L.H.S.} &= 2 \tan^{-1} \left(\frac{1}{3} \right) \\ &= \tan^{-1} \left[\frac{2 \left(\frac{1}{3} \right)}{1 - \left(\frac{1}{3} \right)^2} \right] \dots \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right] \\ &= \tan^{-1} \left[\frac{\left(\frac{2}{3} \right)}{1 - \frac{1}{9}} \right] \\ &= \tan^{-1} \left(\frac{2}{3} \times \frac{9}{8} \right) \\ &= \tan^{-1} \left(\frac{3}{4} \right) \\ &= \text{R.H.S.} \end{aligned}$$

Alternative Method:

$$\text{L.H.S.} = 2 \tan^{-1} \left(\frac{1}{3} \right) = \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{3} \right)$$

$$= \tan^{-1} \left[\frac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3} \times \frac{1}{3}} \right]$$

$$= \tan^{-1} \left(\frac{3+3}{9-1} \right)$$

$$= \tan^{-1} \left(\frac{6}{8} \right)$$

$$= \tan^{-1} \left(\frac{3}{4} \right)$$

$$= \text{R.H.S.}$$

Exercise 3.3 | Q 3.7 | Page 103

Prove the following:

$$\tan^{-1} \left[\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right] = \frac{\pi}{4} + \theta, \quad \text{if } \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right)$$

Solution:

$$\text{L.H.S.} = \tan^{-1} \left[\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right]$$

$$= \tan^{-1} \left[\frac{\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}} \right]$$

$$= \tan^{-1} \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right)$$

$$= \tan^{-1} \left[\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right]$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \theta \right) \right]$$

$$= \frac{\pi}{4} + \theta \quad \dots [\because \tan^{-1}(\tan \theta) = \theta]$$

$$= \text{R.H.S.}$$

Exercise 3.3 | Q 3.8 | Page 103

Prove the following:

$$\tan^{-1} \left[\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \right] = \frac{\theta}{2}, \text{ if } \theta \in (-\pi, \pi).$$

Solution:

$$\begin{aligned} \frac{1 - \cos \theta}{1 + \cos \theta} &= \frac{2 \sin^2 \left(\frac{\theta}{2} \right)}{2 \cos^2 \left(\frac{\theta}{2} \right)} \\ &= \tan^2 \left(\frac{\theta}{2} \right) \\ \therefore \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} &= \sqrt{\tan^2 \left(\frac{\theta}{2} \right)} \\ &= \tan \left(\frac{\theta}{2} \right) \\ \therefore \text{L.H.S.} &= \tan^{-1} \left[\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \right] \\ &= \tan^{-1} \left[\tan \left(\frac{\theta}{2} \right) \right] \\ &= \frac{\theta}{2} \quad \dots [\because \tan^{-1}(\tan \theta) = \theta] \\ &= \text{R.H.S.} \end{aligned}$$

MISCELLANEOUS EXERCISE 3 [PAGES 106 - 108]**Miscellaneous exercise 3 | Q 1.01 | Page 106**

Select the correct option from the given alternatives:

The principal solutions of equation $\sin \theta = -\frac{1}{2}$ are

Options

$$\frac{5\pi}{6}, \frac{\pi}{6}$$

$$\frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{\pi}{6}, \frac{7\pi}{6}$$

$$\frac{7\pi}{6}, \frac{\pi}{3}$$

Solution:

The principal solutions of equation $\sin \theta = -\frac{1}{2}$ are $\frac{7\pi}{6}, \frac{11\pi}{6}$.

Miscellaneous exercise 3 | Q 1.02 | Page 106

Select the correct option from the given alternatives:

The principal solutions of equation $\cot \theta = \sqrt{3}$ are

Options

$$\frac{\pi}{6}, \frac{7\pi}{6}$$

$$\frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{\pi}{6}, \frac{8\pi}{6}$$

$$\frac{7\pi}{6}, \frac{\pi}{3}$$

Solution:

The principal solutions of equation $\cot \theta = \sqrt{3}$ are $\frac{\pi}{6}, \frac{7\pi}{6}$

Miscellaneous exercise 3 | Q 1.03 | Page 106

Select the correct option from the given alternatives:

The general solution of $\sec x = \sqrt{2}$ is

Options

$$2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$2n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$$

$$n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$$

$$2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

Solution:

The general solution of $\sec x = \sqrt{2}$ is $2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$.

Miscellaneous exercise 3 | Q 1.04 | Page 106

Select the correct option from the given alternatives:

If $\cos p\theta = \cos q\theta$, $p \neq q$, then,

Options

$$\theta = \frac{2n\pi}{p \pm q}$$

$$\theta = 2n\pi$$

$$\theta = 2n\pi \pm p$$

$$\theta = n\pi \pm q$$

Solution:

$$\text{If } \cos p\theta = \cos q\theta, p \neq q, \text{ then, } \theta = \frac{2n\pi}{p \pm q}$$

Miscellaneous exercise 3 | Q 1.05 | Page 106

Select the correct option from the given alternatives:

If polar coordinates of a point are $\left(2, \frac{\pi}{4}\right)$, then its cartesian coordinates are

Options

$$\left(2, \sqrt{2}\right)$$

$$\left(\sqrt{2}, 2\right)$$

$$(2, 2)$$

$$\left(\sqrt{2}, \sqrt{2}\right)$$

Solution:

If polar coordinates of a point are $\left(2, \frac{\pi}{4}\right)$, then its cartesian coordinates are $(\sqrt{2}, \sqrt{2})$.

Miscellaneous exercise 3 | Q 1.06 | Page 106

Select the correct option from the given alternatives:

If $\sqrt{3}\cos x - \sin x = 1$, then general value of x is

Options

$$2n\pi \pm \frac{\pi}{3}$$

$$2n\pi \pm \frac{\pi}{6}$$

$$2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$$

$$n\pi + (-1)^n \frac{\pi}{3}$$

Solution:

If $\sqrt{3}\cos x - \sin x = 1$, then general value of x is $2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

Miscellaneous exercise 3 | Q 1.07 | Page 107

Select the correct option from the given alternatives:

In ΔABC if $\angle A = 45^\circ$, $\angle B = 60^\circ$, then the ratio of its sides are

Options

$$2 : \sqrt{6} : \sqrt{3} + 1$$

$$\sqrt{2} : 2 : \sqrt{3} + 1$$

$$2\sqrt{2} : \sqrt{2} : \sqrt{3}$$

$$2 : 2\sqrt{2} : \sqrt{3} + 1$$

Solution: In ΔABC if $\angle A = 45^\circ$, $\angle B = 60^\circ$, then the ratio of its sides are **2: $\sqrt{6}$: $\sqrt{3} + 1$** .

Miscellaneous exercise 3 | Q 1.08 | Page 107

Select the correct option from the given alternatives:

In ΔABC if $c^2 + a^2 - b^2 = ac$, then $\angle B = \underline{\hspace{2cm}}$

Options

$\frac{\pi}{4}$

$\frac{\pi}{3}$

$\frac{\pi}{2}$

$\frac{\pi}{6}$

Solution:

In ΔABC if $c^2 + a^2 - b^2 = ac$, then $\angle B = \frac{\pi}{3}$

Miscellaneous exercise 3 | Q 1.09 | Page 107

Select the correct option from the given alternatives:

In ΔABC , $ac \cos B - bc \cos A = \underline{\hspace{2cm}}$

1. $a^2 - b^2$

2. $b^2 - c^2$

3. $c^2 - a^2$

4. $a^2 - b^2 - c^2$

Solution: In ΔABC , $ac \cos B - bc \cos A = a^2 - b^2$.

Miscellaneous exercise 3 | Q 1.1 | Page 107

Select the correct option from the given alternatives:

If in a triangle, the angles are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then A is equal to

1. 30°
2. 60°
3. 75°
4. 45°

Solution: If in a triangle, the angles are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then A is equal to **75°** .

Miscellaneous exercise 3 | Q 1.11 | Page 107

Select the correct option from the given alternatives:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \underline{\hspace{2cm}}.$$

Options

$$\frac{7\pi}{6}$$

$$\frac{5\pi}{6}$$

$$\frac{\pi}{6}$$

$$\frac{3\pi}{2}$$

Solution:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}.$$

Miscellaneous exercise 3 | Q 1.12 | Page 107

Select the correct option from the given alternatives:

The value of $\cot(\tan^{-1}2x + \cot^{-1}2x)$ is

1. 0

2. $2x$
3. $\pi + 2x$
4. $\pi - 2x$

Solution: The value of $\cot (\tan^{-1}2x + \cot^{-1}2x)$ is 0.

Miscellaneous exercise 3 | Q 1.13 | Page 107

Select the correct option from the given alternatives:

The principal value of $\sin^{-1} \left(-\frac{\sqrt{3}}{2} \right)$ is

Options

$$\left(-\frac{2\pi}{3} \right)$$

$$\frac{4\pi}{3}$$

$$\frac{5\pi}{3}$$

$$-\frac{\pi}{3}$$

Solution:

The principal value of $\sin^{-1} \left(-\frac{\sqrt{3}}{2} \right)$ is $-\frac{\pi}{3}$.

Miscellaneous exercise 3 | Q 1.14 | Page 107

Select the correct option from the given alternatives:

If $\sin^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \sin^{-1} \alpha$, then $\alpha = \underline{\hspace{2cm}}$

1. 63/65

2. 62/65

3. 61/65

4. 60/65

Solution:

$$\text{If } \sin^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \sin^{-1} \alpha, \text{ then } \alpha = \frac{63}{65}.$$

Miscellaneous exercise 3 | Q 1.15 | Page 107

Select the correct option from the given alternatives:

If $\tan^{-1}(2x) + \tan^{-1}(3x) = \pi/4$, then $x =$ _____

1. - 1

2. 16

3. 26

4. 32

Solution:

$$\text{If } \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}, \text{ then } x = \frac{1}{6}$$

Miscellaneous exercise 3 | Q 1.16 | Page 108

Select the correct option from the given alternatives:

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \text{_____}$$

Options

$$\tan^{-1}\left(\frac{4}{5}\right)$$

$$\frac{\pi}{2}$$

$$1$$

$$\frac{\pi}{4}$$

Solution:

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}.$$

Miscellaneous exercise 3 | Q 1.17 | Page 108

Select the correct option from the given alternatives:

$$\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = \text{_____}$$

Options

$$\frac{17}{7}$$

$$-\frac{17}{7}$$

$$\frac{7}{17}$$

$$-\frac{7}{17}$$

Solution:

$$\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = -\frac{7}{17}.$$

Miscellaneous exercise 3 | Q 1.18 | Page 108

Select the correct option from the given alternatives:

The principal value branch of $\sec^{-1}x$ is

Options

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

$$[0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$(0, \pi)$$

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Solution:

The principal value branch of $\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

Miscellaneous exercise 3 | Q 1.19 | Page 108

Select the correct option from the given alternatives:

$$\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right] = \text{_____}$$

Options

$$\frac{1}{\sqrt{2}}$$

$$\frac{\sqrt{3}}{2}$$

$$\frac{1}{2}$$

$$\frac{\pi}{4}$$

Solution:

$$\cos \left[\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} \right] = \frac{1}{\sqrt{2}}$$

Miscellaneous exercise 3 | Q 1.2 | Page 108

Select the correct option from the given alternatives:

If $\tan \theta + \tan 2\theta + \tan 3\theta = \tan \theta \cdot \tan 2\theta \cdot \tan 3\theta$, then the general value of the θ is

1. $n\pi$
2. $n\pi/6$
3. $n\pi \pm \pi/4$
4. $n\pi/2$

Solution: If $\tan \theta + \tan 2\theta + \tan 3\theta = \tan \theta \cdot \tan 2\theta \cdot \tan 3\theta$, then the general value of the θ is $n\pi/6$

Miscellaneous exercise 3 | Q 1.21 | Page 108

Select the correct option from the given alternatives:

In any $\triangle ABC$, if $a \cos B = b \cos A$, then the triangle is

1. equilateral triangle
2. isosceles triangle
3. scalene
4. right-angled

Solution: In any $\triangle ABC$, if $a \cos B = b \cos A$, then the triangle is **isosceles triangle**.

MISCELLANEOUS EXERCISE 3 [PAGES 108 - 111]

Miscellaneous exercise 3 | Q 1.1 | Page 108

Find the principal solutions of the following equation:

$$\sin 2\theta = -1/2$$

Solution:

$$\sin 2\theta = -\frac{1}{2}$$

Since, $\theta \in (0, 2\pi)$, $2\theta \in (0, 4\pi)$

$$\begin{aligned}\sin 2\theta &= -\frac{1}{2} = -\sin \frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right) \\ &= \sin\left(3\pi + \frac{\pi}{6}\right) = \sin\left(4\pi - \frac{\pi}{6}\right) \dots\dots[\because \sin(\pi + \theta) = \sin(2\pi - \theta) = \sin(3\pi + \theta) = \sin(4\pi - \theta) = -\sin \theta]\end{aligned}$$

$$\therefore \sin 2\theta = \sin \frac{7\pi}{6} = \sin \frac{11\pi}{6} = \sin \frac{19\pi}{6} = \sin \frac{23\pi}{6}$$

$$\therefore 2\theta = \frac{7\pi}{6} \text{ or } 2\theta = \frac{11\pi}{6} \text{ or } 2\theta = \frac{19\pi}{6} \text{ or } 2\theta = \frac{23\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{12} \text{ or } \theta = \frac{11\pi}{12} \text{ or } \theta = \frac{19\pi}{12} \text{ or } \theta = \frac{23\pi}{12}$$

Hence, the required principal solutions are

$$\left\{ \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12} \right\}.$$

Miscellaneous exercise 3 | Q 1.2 | Page 108

Find the principal solutions of the following equation:

$$\tan 3\theta = -1$$

Solution:

$$\tan 3\theta = -1$$

Since, $\theta \in (0, 2\pi)$, $3\theta \in (0, 6\pi)$

$$\tan 3\theta = -1 = -\tan \frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\left(3\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(4\pi - \frac{\pi}{4}\right) = \tan\left(5\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(6\pi - \frac{\pi}{4}\right) \dots [\because \tan(\pi - \theta) = \tan(2\pi - \theta) = \tan(3\pi - \theta) = \tan(4\pi - \theta) = \tan(5\pi - \theta) = \tan(6\pi - \theta) = -\tan \theta]$$

$$\tan 3\theta = -1$$

Since, $\theta \in (0, 2\pi)$, $3\theta \in (0, 6\pi)$

$$\tan 3\theta = -1 = -\tan \frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\left(3\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(4\pi - \frac{\pi}{4}\right) = \tan\left(5\pi - \frac{\pi}{4}\right)$$

Miscellaneous exercise 3 | Q 1.3 | Page 108

Find the principal solutions of the following equation:

$$\cot \theta = 0$$

Solution:

$$\cot \theta = 0$$

Since $\theta \in (0, 2\pi)$

$$\therefore \cot \theta = 0 = \cot \frac{\pi}{2} = \cot\left(\pi + \frac{\pi}{2}\right) \dots [\because \cot(\pi + \theta) = \cot \theta]$$

$$\therefore \cot \theta = \cot \frac{\pi}{2} = \cot \frac{3\pi}{2}$$

$$\therefore \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}$$

Hence, the required principal solutions are $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

Miscellaneous exercise 3 | Q 2.1 | Page 108

Find the principal solutions of the following equation:

$$\sin 2\theta = -1/\sqrt{2}.$$

Solution:

$$\left\{ \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8} \right\}$$

Miscellaneous exercise 3 | Q 2.2 | Page 108

Find the principal solutions of the following equation:

$$\tan 5\theta = -1$$

Solution:

$$\left\{ \frac{3\pi}{20}, \frac{7\pi}{20}, \frac{11\pi}{20}, \frac{15\pi}{20}, \frac{19\pi}{20}, \frac{23\pi}{20}, \frac{27\pi}{20}, \frac{31\pi}{20}, \frac{35\pi}{20}, \frac{39\pi}{20} \right\}$$

Miscellaneous exercise 3 | Q 2.3 | Page 108

Find the principal solutions of the following equation:

$$\cot 2\theta = 0.$$

Solution:

$$\left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}.$$

Miscellaneous exercise 3 | Q 3.1 | Page 109

State whether the following equation has a solution or not?

$$\cos 2\theta = 1/3$$

Solution:

$$\cos 2\theta = \frac{1}{3}$$

since $\frac{1}{3} \leq \cos \theta \leq 1$ for any θ

$\cos 2\theta = \frac{1}{3}$ has solution.

Miscellaneous exercise 3 | Q 3.2 | Page 109

State whether the following equation has a solution or not?

$$\cos^2\theta = -1.$$

Solution: $\cos^2\theta = -1$

This is not possible because $\cos^2\theta \geq 0$ for any θ .

$\therefore \cos^2\theta = -1$ does not have any solution.

Miscellaneous exercise 3 | Q 3.3 | Page 109

State whether the following equation has a solution or not?

$$2\sin\theta = 3$$

Solution: $2\sin\theta = 3$

$$\therefore \sin\theta = 3/2$$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ .

$\therefore 2\sin\theta = 3$ does not have any solution.

Miscellaneous exercise 3 | Q 3.4 | Page 109

State whether the following equation has a solution or not?

$$3\sin\theta = 5.$$

Solution: $\therefore \sin\theta = 5/3$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ .

$\therefore 3\sin\theta = 5$ does not have any solution.

Miscellaneous exercise 3 | Q 4.1 | Page 109

Find the general solutions of the following equation:

$$\tan\theta = -\sqrt{3}$$

Solution:

The general solution of $\tan \theta = \tan \alpha$ is

$$\theta = n\pi + \alpha, n \in \mathbb{Z}.$$

$$\text{Now, } \tan \theta = -\sqrt{3}$$

$$\therefore \tan \theta = -\tan \frac{\pi}{3} \dots \left[\because \tan \frac{\pi}{3} = \sqrt{3} \right]$$

$$\therefore \tan \theta = \tan \left(\pi - \frac{\pi}{3} \right) \dots \left[\because \tan(\pi - \theta) = -\tan \theta \right]$$

$$\therefore \tan \theta = \tan \frac{2\pi}{3}$$

\therefore the required general solution is

$$\therefore \theta = n\pi + \frac{2\pi}{3}, n \in \mathbb{Z}$$

Miscellaneous exercise 3 | Q 4.2 | Page 109

Find the general solutions of the following equation:

$$\tan^2 \theta = 3$$

Solution: The general solution of $\tan^2 \theta = \tan^2 \alpha$ is $\theta = n\pi \pm \alpha, n \in \mathbb{Z}$.

$$\text{Now, } \tan^2 \theta = 3 = \left(\sqrt{3} \right)^2$$

$$\therefore \tan^2 \theta = \left(\tan \frac{\pi}{3} \right)^2 \dots \left[\because \tan \frac{\pi}{3} = \sqrt{3} \right]$$

$$\therefore \tan^2 \theta = \tan^2 \frac{\pi}{3}$$

\therefore the required general solution is

$$\therefore \theta = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}.$$

Miscellaneous exercise 3 | Q 4.3 | Page 109

Find the general solutions of the following equation:

$$\sin \theta - \cos \theta = 1$$

Solution: $\sin \theta - \cos \theta = 1$

$$\cos \theta - \sin \theta = -1$$

Dividing both sides by $\sqrt{(1)^2 + (-1)^2} = \sqrt{2}$, we get

$$\frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \cos \frac{\pi}{4} \cos \theta - \sin \frac{\pi}{4} \sin \theta = -\cos \frac{\pi}{4}$$

$$\therefore \cos \left(\theta - \frac{\pi}{4} \right) = \cos \left(\pi - \frac{\pi}{4} \right) \dots [\because \cos(\pi - \theta) = -\cos \theta]$$

$$\therefore \cos \left(\theta - \frac{\pi}{4} \right) = \cos \frac{3\pi}{4} \dots (1)$$

The general solution of $\cos \theta = \cos \alpha$ is

$$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$$

\therefore the general solution of (1) is given by

$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}, n \in \mathbb{Z}$$

Taking positive sign, we get

$$\theta - \frac{\pi}{4} = 2n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi + \pi = (2n + 1)\pi, n \in \mathbb{Z}$$

Taking negative sign, we get

$$\theta - \frac{\pi}{4} = 2n\pi - \frac{3\pi}{4}, n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$$

\therefore the required general solution is

$$\theta = (2n + 1)\pi, n \in \mathbb{Z} \text{ or } \theta = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$$

Miscellaneous exercise 3 | Q 4.4 | Page 109

Find the general solutions of the following equation:

$$\sin^2 \theta - \cos^2 \theta = 1$$

Solution: $\sin^2 \theta - \cos^2 \theta = 1$

$$\therefore \cos^2 \theta - \sin^2 \theta = -1$$

$$\therefore \cos 2\theta = \cos \pi \quad \dots(1)$$

The general solution of $\cos \theta = \cos \alpha$ is

$$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$$

\therefore the general solution of (1) is given by

$$2\theta = 2n\pi \pm \pi, n \in \mathbb{Z}.$$

$$\therefore \theta = n\pi \pm \pi/2, n \in \mathbb{Z}$$

Miscellaneous exercise 3 | Q 5 | Page 109

In ΔABC , prove that $\cos\left(\frac{A - B}{2}\right) = \left(\frac{a + b}{c}\right) \sin \frac{C}{2}$.

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\begin{aligned} \text{RHS} &= \left(\frac{a+b}{c} \right) \sin \frac{C}{2} \\ &= \left(\frac{k \sin A + k \sin B}{k \sin C} \right) \sin \frac{C}{2} \\ &= \left(\frac{\sin A + \sin B}{\sin C} \right) \sin \frac{C}{2} \\ &= \frac{2 \sin \left(\frac{A+B}{2} \right) \cdot \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cdot \cos \frac{C}{2}} \cdot \sin \frac{C}{2} \\ &= \frac{\sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}}{\cos \frac{C}{2}} \\ &= \frac{\sin \left(\frac{\pi}{2} - \frac{C}{2} \right) \cdot \cos \frac{A-B}{2}}{\cos \frac{C}{2}} \quad \dots [\because A + B + C = \pi] \\ &= \frac{\cos \frac{C}{2} \cdot \cos \frac{A-B}{2}}{\cos \frac{C}{2}} \\ &= \cos \left(\frac{A-B}{2} \right) \\ &= \text{LHS} \end{aligned}$$

With the usual notations, prove that $\frac{\sin(A - B)}{\sin(A + B)} = \frac{a^2 - b^2}{c^2}$

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\begin{aligned} \text{RHS} &= \frac{a^2 - b^2}{c^2} = \frac{k^2 \sin^2 A - k^2 \sin^2 B}{k^2 \sin^2 C} \\ &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C} \\ &= \frac{(\sin A + \sin B)(\sin A - \sin B)}{[\sin\{\pi - (A + B)\}]^2} \quad \dots [\because A + B + C = \pi] \\ &= \frac{2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) \times 2 \cos\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}{\sin^2(A + B)} \\ &= \frac{2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A+B}{2}\right) \times 2 \sin\left(\frac{A-B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)}{\sin^2(A + B)} \\ &= \frac{\sin(A + B) \cdot \sin(A - B)}{\sin^2(A + B)} \\ &= \frac{\sin(A - B)}{\sin(A + B)} = \text{LHS} \end{aligned}$$

Miscellaneous exercise 3 | Q 7 | Page 109

In $\triangle ABC$, prove that $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

Solution:

$$\begin{aligned}
\text{LHS} &= (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2 \\
&= (a^2 + b^2 - 2ab) \cos^2 \frac{C}{2} + (a^2 + b^2 + 2ab) \sin^2 \frac{C}{2} \\
&= (a^2 + b^2) \cos^2 \frac{C}{2} - 2ab \cos^2 \frac{C}{2} + (a^2 + b^2) \sin^2 \frac{C}{2} + 2ab \sin^2 \frac{C}{2} \\
&= (a^2 + b^2) \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) \\
&= a^2 + b^2 - 2ab \cos C \\
&= c^2 = \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 8 | Page 109

In ΔABC , if $\cos A = \sin B - \cos C$ then show that it is a right-angled triangle.

Solution:

$$\cos A = \sin B - \cos C$$

$$\therefore \cos A + \cos C = \sin B$$

$$\therefore 2 \cos \left(\frac{A + C}{2} \right) \cdot \cos \left(\frac{A - C}{2} \right) = \sin B$$

$$\therefore 2 \cos \left(\frac{\pi}{2} - \frac{B}{2} \right) \cdot \cos \left(\frac{A - C}{2} \right) = \sin B \dots [\because A + B + C = \pi]$$

$$\therefore 2 \sin \frac{B}{2} \cdot \cos \left(\frac{A - C}{2} \right) = 2 \sin \frac{B}{2} \cdot \cos \frac{B}{2}$$

$$\therefore \cos \left(\frac{A - C}{2} \right) = \cos \frac{B}{2}$$

$$\therefore \frac{A - C}{2} = \frac{B}{2}$$

$$\therefore A - C = B$$

$$\therefore A = B + C$$

$$\therefore A + B + C = 180^\circ \text{ gives}$$

$$\therefore A + A = 180^\circ$$

$$\therefore 2A = 180^\circ$$

$$\therefore A = 90^\circ$$

$\therefore \Delta ABC$ is a right angled triangle.

Miscellaneous exercise 3 | Q 9 | Page 109

If $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution: By sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$\therefore \sin A = ka, \sin B = kb, \sin C = kc$$

$$\text{Now, } \frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$$

$$\therefore \sin A \cdot \sin(B - C) = \sin C \cdot \sin(A - B)$$

$$\therefore \sin[\pi - (B + C)] \cdot \sin(B - C)$$

$$= \sin[\pi - (A + B)] \cdot \sin(A - B) \quad \dots[\because A + B + C = \pi]$$

$$\therefore \sin(B + C) \cdot \sin(B - C) = \sin(A + B) \cdot \sin(A - B)$$

$$\therefore \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$$\therefore 2 \sin^2 B = \sin^2 A + \sin^2 C$$

$$\therefore 2k^2 b^2 = k^2 a^2 + k^2 c^2$$

$$\therefore 2b^2 = a^2 + c^2$$

Hence, a^2, b^2, c^2 are in A.P.

Miscellaneous exercise 3 | Q 10 | Page 109

Solve the triangle in which $a = (\sqrt{3}+1)$, $b = (\sqrt{3}-1)$ and $\angle C = 60^\circ$.

Solution:

Given: $a = (\sqrt{3} + 1)$, $b = (\sqrt{3} - 1)$ and $\angle C = 60^\circ$

By cosine rule,

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C \\&= (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 - 2(\sqrt{3} + 1)(\sqrt{3} - 1)\cos 60^\circ \\&= 3 + 1 + 2\sqrt{3} + 3 + 1 - 2\sqrt{3} - 2(3 - 1)\left(\frac{1}{2}\right) \\&= 8 - 2 = 6\end{aligned}$$

$$\therefore c = \sqrt{6} \quad \dots[\because c > 0]$$

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{\sqrt{3} + 1}{\sin A} = \frac{\sqrt{3} - 1}{\sin B} = \frac{\sqrt{6}}{\sin 60^\circ}$$

$$\therefore \frac{\sqrt{3} + 1}{\sin A} = \frac{\sqrt{3} - 1}{\sin B} = \frac{\sqrt{6}}{\sqrt{3}/2} = 2\sqrt{2}$$

$$\therefore \sin A = \frac{\sqrt{3} + 1}{2\sqrt{2}} \text{ and } \sin B = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\therefore \sin A = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \text{ and } \sin B = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$\therefore \sin A = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$\therefore \text{ and } \sin B = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$\therefore \sin A = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \text{ and } \sin B = \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$$

$$\therefore \sin A = \sin (60^\circ + 45^\circ) = \sin 105^\circ$$

$$\text{and } \sin B = \sin (60^\circ - 45^\circ) = \sin 15^\circ$$

$$\therefore A = 105^\circ \text{ and } B = 15^\circ$$

$$\text{Hence, } A = 105^\circ, B = 15^\circ \text{ and } C = \sqrt{6} \text{ units}$$

Miscellaneous exercise 3 | Q 11.1 | Page 109

In any ΔABC , prove the following:

$$a \sin A - b \sin B = c \sin (A - B)$$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\text{LHS} = a \sin A - b \sin B$$

$$= k \sin A \cdot \sin A - k \sin B \cdot \sin B$$

$$= k (\sin^2 A - \sin^2 B)$$

$$= k (\sin A + \sin B)(\sin A - \sin B)$$

$$= k \times 2 \sin \left(\frac{A+B}{2} \right) \cdot \cos \left(\frac{A-B}{2} \right) \times 2 \cos \left(\frac{A+B}{2} \right) \cdot \sin \left(\frac{A-B}{2} \right)$$

$$= k \times 2 \sin \left(\frac{A+B}{2} \right) \cdot \cos \left(\frac{A+B}{2} \right) \times 2 \sin \left(\frac{A-B}{2} \right) \cdot \cos \left(\frac{A-B}{2} \right)$$

$$= k \times \sin (A+B) \times \sin (A-B)$$

$$= k \sin (\pi - C) \cdot \sin (A - B) \quad \dots [\because A + B + C = \pi]$$

$$= k \sin C \cdot \sin (A - B)$$

$$= c \sin (A - B)$$

$$= \text{RHS.}$$

Miscellaneous exercise 3 | Q 11.2 | Page 109

In any ΔABC , prove the following:

$$\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{c - b \cos A}{b - c \cos A} \\ &= \frac{c - b \left(\frac{b^2 + c^2 - a^2}{2bc} \right)}{b - c \left(\frac{b^2 + c^2 - a^2}{2bc} \right)} \\ &= \frac{c - \left(\frac{b^2 + c^2 - a^2}{2c} \right)}{b - \left(\frac{b^2 + c^2 - a^2}{2b} \right)} \\ &= \frac{\frac{2c^2 - b^2 - c^2 + a^2}{2c}}{\frac{2b^2 - b^2 - c^2 + a^2}{2b}} \\ &= \frac{\left(\frac{c^2 + a^2 - b^2}{2c} \right)}{\left(\frac{a^2 + b^2 - c^2}{2b} \right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\frac{c^2 + a^2 - b^2}{2ca} \right)}{\left(\frac{a^2 + b^2 - c^2}{2ab} \right)} \\
&= \frac{\cos B}{\cos C} \\
&= \text{RHS.}
\end{aligned}$$

Miscellaneous exercise 3 | Q 11.3 | Page 109

In any ΔABC , prove the following:

$$a^2 \sin(B - C) = (b^2 - c^2) \sin A.$$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\text{RHS} = (b^2 - c^2) \sin A$$

$$= (k^2 \sin^2 B - k^2 \sin^2 C) \sin A$$

$$= k^2 (\sin^2 B - \sin^2 C) \sin A$$

$$= k^2 (\sin B + \sin C)(\sin B - \sin C) \sin A$$

$$= k^2 \times 2 \sin\left(\frac{B+C}{2}\right) \cdot \cos\left(\frac{B-C}{2}\right) \times 2 \cos\left(\frac{B+C}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right) \times \sin A$$

$$= k^2 \times 2 \sin\left(\frac{B+C}{2}\right) \cdot \cos\left(\frac{B+C}{2}\right) \times 2 \sin\left(\frac{B-C}{2}\right) \cdot \cos\left(\frac{B-C}{2}\right) \times \sin A$$

$$= k^2 \times \sin(B+C) \times \sin(B-C) \times \sin A$$

$$= k^2 \cdot \sin(\pi - A) \cdot \sin(B-C) \cdot \sin A \quad \dots[\because A+B+C=\pi]$$

$$= k^2 \cdot \sin A \cdot \sin(B-C) \cdot \sin A$$

$$= (k \sin A)^2 \cdot \sin(B-C)$$

$$= a^2 \sin (B - C)$$

$$= \text{LHS}$$

Miscellaneous exercise 3 | Q 11.4 | Page 109

In any ΔABC , prove the following:

$$ac \cos B - bc \cos A = a^2 - b^2$$

Solution: LHS = $ac \cos B - bc \cos A = a^2 - b^2$

$$\text{LHS} = ac \cos B - bc \cos A = a^2 - b^2$$

$$= ac \left(\frac{c^2 + a^2 - b^2}{2ca} \right) - bc \left(\frac{b^2 + c^2 - a^2}{2bc} \right)$$

$$= \frac{1}{2} (c^2 + a^2 - b^2) - \frac{1}{2} (b^2 + c^2 - a^2)$$

$$= \frac{1}{2} (c^2 + a^2 - b^2 - b^2 - c^2 + a^2)$$

$$= \frac{1}{2} (2a^2 - 2b^2)$$

$$= a^2 - b^2$$

$$= \text{RHS}$$

Miscellaneous exercise 3 | Q 11.5 | Page 109

In any ΔABC , prove the following:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \\
&= \frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{b} + \frac{\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c} \\
&= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} \\
&= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{2abc} \\
&= \frac{a^2+b^2+c^2}{2abc} \\
&= \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 11.6 | Page 109

In any ΔABC , prove the following:

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution:

By sine rule,

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b} \\
\therefore \frac{\sin^2 A}{a^2} &= \frac{\sin^2 B}{b^2} \quad \dots(1)
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} \\
&= \frac{1 - 2\sin^2 A}{a^2} - \frac{1 - 2\sin^2 B}{b^2} \\
&= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2} \\
&= \frac{1}{a^2} - \frac{1}{b^2} - 2\left(\frac{\sin^2 A}{a^2} - \frac{\sin^2 B}{b^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a^2} - \frac{1}{b^2} - 2 \left(\frac{\sin^2 B}{b^2} - \frac{\sin^2 B}{b^2} \right) \dots [\text{By (1)}] \\
&= \frac{1}{a^2} - \frac{1}{b^2} - 2 \times 0 \\
&= \frac{1}{a^2} - \frac{1}{b^2} \\
&= \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 11.7 | Page 109

In any ΔABC , prove the following:

$$\frac{b - c}{a} = \frac{\tan \frac{B}{2} - \tan \frac{C}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}}$$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\begin{aligned}
\text{LHS} &= \frac{b - c}{a} \\
&= \frac{k \sin B - k \sin C}{k \sin A} \\
&= \frac{\sin B - \sin C}{\sin A} \\
&= \frac{\sin B - \sin C}{\sin \{ \pi - (B + C) \}} \dots [\because A + B + C = \pi] \\
&= \frac{\sin B - \sin C}{\sin(B + C)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2 \cos \left(\frac{B+C}{2} \right) \cdot \sin \left(\frac{B-C}{2} \right)}{2 \sin \left(\frac{B+C}{2} \right) \cdot \cos \left(\frac{B+C}{2} \right)} \\
&= \frac{\sin \frac{B-C}{2}}{\sin \frac{B+C}{2}} \\
&= \frac{\sin \left(\frac{B}{2} - \frac{C}{2} \right)}{\sin \left(\frac{B}{2} + \frac{C}{2} \right)} \\
&= \frac{\sin \frac{B}{2} \cos \frac{C}{2} - \cos \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}} \\
&= \frac{\frac{\sin \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} - \frac{\cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}}{\frac{\sin \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}} \\
&= \frac{\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} - \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}}{\frac{\sin \frac{B}{2}}{\cos \frac{C}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}} \\
&= \frac{\tan \frac{B}{2} - \tan \frac{C}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}}
\end{aligned}$$

= RHS.

Miscellaneous exercise 3 | Q 12 | Page 109

In ΔABC , if a, b, c are in A.P., then show that $\cot A/2, \cot B/2, \cot C/2$ are also in A.P.

Solution: a, b, c are in A.P.

$$\therefore 2b = a + c \quad \dots(1)$$

Now,

$$\begin{aligned}
& \cot \frac{A}{2} + \cot \frac{C}{2} \\
&= \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \\
&= \frac{\cos \frac{A}{2} \cdot \sin \frac{C}{2} + \sin \frac{A}{2} \cdot \cos \frac{C}{2}}{\sin \frac{A}{2} \cdot \sin \frac{C}{2}} \\
&= \frac{\sin \left(\frac{A}{2} + \frac{C}{2} \right)}{\sin \frac{A}{2} \cdot \sin \frac{C}{2}} \\
&= \frac{\sin \left(\frac{\pi}{2} - \frac{B}{2} \right)}{\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}} \quad \dots [\because A + B + C = \pi] \\
&= \frac{\cos \frac{B}{2}}{\left(\frac{s-b}{b} \right) \cdot \sqrt{\frac{(s-c)(s-a)}{ca}}} \\
&= \frac{b \cos \frac{B}{2}}{(s-b) \cdot \sin \frac{B}{2}} \\
&= \frac{b}{s-b} \cdot \cot \frac{B}{2} \\
&= \frac{b}{\left(\frac{a+b+c}{2} - b \right)} \cdot \cot \frac{B}{2} \quad \dots [\because 2s = a + b + c] \\
&= \left(\frac{2b}{a+c-b} \right) \cdot \cot \frac{B}{2} \\
&= \frac{2b}{(2b-b)} \cdot \cot \frac{B}{2} \quad \dots [\text{By (1)}]
\end{aligned}$$

$$= \frac{2b}{b} \cdot \cot \frac{B}{2}$$

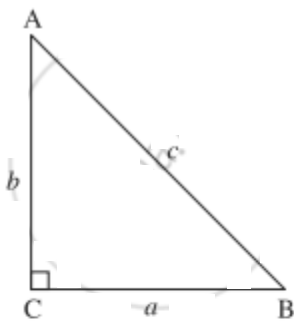
$$\therefore \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$$

Hence, $\cot \frac{A}{2}$, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ are in A.P.

Miscellaneous exercise 3 | Q 13 | Page 109

In ΔABC , if $\angle C = 90^\circ$, then prove that $\sin (A - B) = \frac{a^2 - b^2}{a^2 + b^2}$

Solution:



In ΔABC , if $\angle C = 90^\circ$

$$\therefore c^2 = a^2 + b^2 \quad \text{.....(1)}$$

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin 90^\circ}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = c \quad \text{.....}[\because \sin 90^\circ = 1]$$

$$\therefore \sin A = \frac{a}{c} \quad \text{and} \quad \sin B = \frac{b}{c} \quad \dots(2)$$

$$\text{LHS} = \sin (A - B)$$

$$= \sin A \cos B - \cos A \sin B$$

$$= \frac{a}{c} \cos B - \frac{b}{c} \cos A \quad \dots[\text{By (2)}]$$

$$= \frac{a}{c} \left(\frac{c^2 + a^2 - b^2}{2ca} \right) - \frac{b}{c} \left(\frac{b^2 + c^2 - a^2}{2bc} \right)$$

$$= \frac{c^2 + a^2 - b^2}{2c^2} - \frac{b^2 + c^2 - a^2}{2c^2}$$

$$= \frac{c^2 + a^2 - b^2 - b^2 - c^2 + a^2}{2c^2}$$

$$= \frac{2a^2 - 2b^2}{2c^2}$$

$$= \frac{a^2 - b^2}{c^2}$$

$$= \frac{a^2 - b^2}{a^2 + b^2} \quad \dots[\text{By (1)}]$$

$$= \text{RHS.}$$

Miscellaneous exercise 3 | Q 14 | Page 110

In ΔABC , if $\frac{\cos A}{a} = \frac{\cos B}{b}$, then show that it is an isosceles triangle.

Solution:

$$\text{Given: } \frac{\cos A}{a} = \frac{\cos B}{b} \quad \dots(1)$$

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k$$

$$\therefore a = k \sin A, b = k \sin B$$

\therefore (1) gives,

$$\frac{\cos A}{k \sin A} = \frac{\cos B}{k \sin B}$$

$$\therefore \frac{\cos A}{\sin A} = \frac{\cos B}{\sin B}$$

$$\therefore \sin A \cos B = \cos A \sin B$$

$$\therefore \sin A \cos B - \cos A \sin B = 0$$

$$\therefore \sin (A - B) = 0 = \sin 0$$

$$\therefore A - B = 0$$

$$\therefore A = B$$

\therefore the triangle is an isosceles triangle.

Miscellaneous exercise 3 | Q 15 | Page 110

In ΔABC , if $\sin^2 A + \sin^2 B = \sin^2 C$, then show that the triangle is a right-angled triangle.

Solution:

By sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$\therefore \sin A = ka, \sin B = kb, \sin C = kc$$

$$\therefore \sin^2 A + \sin^2 B = \sin^2 C$$

$$\therefore k^2 a^2 + k^2 b^2 = k^2 c^2$$

$$\therefore a^2 + b^2 = c^2$$

$\therefore \Delta ABC$ is a rightangled triangle, rightangled at C.

Miscellaneous exercise 3 | Q 16 | Page 110

In ΔABC , prove that $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0$.

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

$$\text{LHS} = a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B)$$

$$= k^2 \sin^2 A [(1 - \sin^2 B) - (1 - \sin^2 C)] + k^2 \sin^2 B [(1 - \sin^2 C) - (1 - \sin^2 A)] + k^2 \sin^2 C [(1 - \sin^2 A) - (1 - \sin^2 B)]$$

$$= k^2 \sin^2 A (\sin^2 C - \sin^2 B) + k^2 \sin^2 B (\sin^2 A - \sin^2 C) + k^2 \sin^2 C (\sin^2 B - \sin^2 A)$$

$$= k^2 (\sin^2 A \sin^2 C - \sin^2 A \sin^2 B + \sin^2 A \sin^2 B - \sin^2 B \sin^2 C + \sin^2 B \sin^2 C - \sin^2 A \sin^2 C)$$

$$= k^2 (0)$$

$$= 0$$

$$= \text{RHS.}$$

Miscellaneous exercise 3 | Q 17 | Page 110

With the usual notations, show that

$$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k \sin A, b = k \sin B, c = k \sin C$$

Now,

$$(c^2 - a^2 + b^2) \tan A = (c^2 - a^2 + b^2) \cdot \frac{\sin A}{\cos A}$$

$$= (c^2 + b^2 - a^2) \times \frac{ka}{\left(\frac{c^2 + b^2 - a^2}{2bc}\right)}$$

$$= (c^2 + b^2 - a^2) \times \frac{2kabc}{c^2 + b^2 - a^2}$$

$$= 2kabc \quad \dots\dots(1)$$

$$(a^2 - b^2 + c^2) \tan B = (a^2 - b^2 + c^2) \cdot \frac{\sin B}{\cos B}$$

$$= (a^2 + c^2 - b^2) \times \frac{kb}{\left(\frac{a^2 + c^2 - b^2}{2ac}\right)}$$

$$= (a^2 + c^2 - b^2) \times \frac{2kabc}{a^2 + c^2 - b^2}$$

$$= 2kabc \quad \dots\dots(2)$$

$$\begin{aligned}
&= (a^2 + c^2 - b^2) \times \frac{kb}{\left(\frac{a^2+c^2-b^2}{2ac}\right)} \\
&= (a^2 + c^2 - b^2) \times \frac{2kabc}{a^2 + c^2 - b^2} \\
&= 2kabc \quad \dots(2)
\end{aligned}$$

$$\begin{aligned}
(b^2 - c^2 + a^2) \tan C &= (b^2 - c^2 + a^2) \cdot \frac{\sin C}{\cos C} \\
&= (a^2 + b^2 - c^2) \times \frac{kc}{\left(\frac{a^2+b^2-c^2}{2ab}\right)} \\
&= (a^2 + b^2 - c^2) \times \frac{2kabc}{a^2 + b^2 - c^2} \\
&= 2kabc \quad \dots(3)
\end{aligned}$$

From (1), (2) and (3), we get

$$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$$

Miscellaneous exercise 3 | Q 18 | Page 110

In ΔABC , if $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$, then prove that a, b, c are in A.P.

Solution:

$$\begin{aligned}
a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} &= \frac{3b}{2} \\
\therefore a \left(\frac{1 + \cos C}{2} \right) + c \left(\frac{1 + \cos A}{2} \right) &= \frac{3b}{2} \\
\therefore \frac{1}{2} (a + a \cos C + c + c \cos A) &= \frac{3b}{2}
\end{aligned}$$

$$\therefore a + c + (a \cos C + c \cos A) = 3b$$

$$\therefore a + c + b = 3b \quad \dots[\because a \cos C + c \cos A = b]$$

$$\therefore a + c = 2b$$

Hence, a, b, c are in A.P.

Miscellaneous exercise 3 | Q 19 | Page 110

$$\text{Show that } 2 \sin^{-1} \left(\frac{3}{5} \right) = \tan^{-1} \left(\frac{24}{7} \right)$$

Solution:

$$\text{Let } 2 \sin^{-1} \left(\frac{3}{5} \right) = x$$

$$\text{Then } \sin x = \frac{3}{5}, \text{ where } 0 < x < \frac{\pi}{2}$$

$$\therefore \cos x > 0$$

$$\text{Now, } \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\therefore x = \tan^{-1} \left(\frac{3}{4} \right)$$

$$\therefore \sin^{-1} \left(\frac{3}{5} \right) = \tan^{-1} \left(\frac{3}{4} \right)$$

$$\text{Now, LHS} = 2 \sin^{-1} \left(\frac{3}{5} \right) = 2 \tan^{-1} \left(\frac{3}{4} \right)$$

$$\begin{aligned}
&= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right) \\
&= \tan^{-1}\left[\frac{\frac{3}{4} + \frac{3}{4}}{1 - \frac{3}{4} \times \frac{3}{4}}\right] = \tan^{-1}\left[\frac{12 + 12}{16 - 9}\right] \\
&= \tan^{-1}\left(\frac{24}{7}\right) = \text{RHS}
\end{aligned}$$

Alternative Method:

$$\begin{aligned}
\text{LHS} &= 2 \sin^{-1}\left(\frac{3}{5}\right) = 2 \tan^{-1}\left(\frac{3}{4}\right) \\
&= \tan^{-1}\left[\frac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2}\right] \dots \left[\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right] \\
&= \tan^{-1}\left[\frac{\frac{3}{2}}{1 - \left(\frac{9}{16}\right)}\right] \\
&= \tan^{-1}\left(\frac{3}{2} \times \frac{16}{7}\right) \\
&= \tan^{-1}\left(\frac{24}{7}\right) \\
&= \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 20 | Page 110

Show that

$$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Solution:

$$\begin{aligned}
\text{LHS} &= \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) \\
&= \tan^{-1}\left[\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}}\right] + \tan^{-1}\left[\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}}\right] \\
&= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right) \\
&= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right) \\
&= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right) \\
&= \tan^{-1}\left[\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}}\right] \\
&= \tan^{-1}\left(\frac{138+187}{391-66}\right) = \tan^{-1}\left(\frac{325}{325}\right) \\
&= \tan^{-1}(1) = \tan^{-1}\left(\tan \frac{\pi}{4}\right) \\
&= \frac{\pi}{4} \\
&= \text{RHS.}
\end{aligned}$$

Miscellaneous exercise 3 | Q 21 | Page 110

Prove that $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$, if $x \in [0, 1]$

Solution:

$$\text{Let } \tan^{-1} \sqrt{x} = y$$

$$\therefore \tan y = \sqrt{x}$$

$$\therefore x = \tan^2 y$$

Now,

$$\text{RHS} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{1-\tan^2 y}{1+\tan^2 y} \right)$$

$$= \frac{1}{2} \cos^{-1}(\cos 2y)$$

$$= \frac{1}{2}(2y) = y$$

$$= \tan^{-1} \sqrt{x}$$

$$= \text{LHS.}$$

Miscellaneous exercise 3 | Q 22 | Page 110

$$\text{Show that } \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) = \frac{9}{4} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

Solution:

We have to show that

$$\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) = \frac{9}{4} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

i.e. to show that,

$$\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$$

$$\text{Let } \sin^{-1}\left(\frac{1}{3}\right) = x$$

$$\therefore \sin x = \frac{1}{3}, \text{ where } 0 < x < \frac{\pi}{3}$$

$$\therefore \cos x > 0$$

$$\text{Now, } \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \left(\frac{2\sqrt{2}}{3}\right)$$

$$\therefore x = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\therefore \sin^{-1}\left(\frac{1}{3}\right) = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) \quad \dots(1)$$

$$\therefore \text{LHS} = \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{9}{4} \left[\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \right]$$

$$= \frac{9}{4} \left[\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \right] \dots[\text{By (1)}]$$

$$= \frac{9}{4} \left(\frac{\pi}{2} \right) \dots \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$$

$$= \frac{9\pi}{8}$$

$$= \text{RHS.}$$

Miscellaneous exercise 3 | Q 23 | Page 110

Show that $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$, for $-\frac{1}{\sqrt{2}} \leq x \leq 1$

Solution:

$$\text{LHS} = \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

Put $x = \cos \theta$

$$\therefore \theta = \cos^{-1} x$$

$$\therefore \text{LHS} = \tan^{-1} \left(\frac{\sqrt{1+\cos \theta} - \sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta} + \sqrt{1-\cos \theta}} \right)$$

$$= \tan^{-1} \left[\frac{\sqrt{2 \cos^2 \left(\frac{\theta}{2} \right)} - \sqrt{2 \sin^2 \left(\frac{\theta}{2} \right)}}{\sqrt{2 \cos^2 \left(\frac{\theta}{2} \right)} + \sqrt{2 \sin^2 \left(\frac{\theta}{2} \right)}} \right]$$

$$= \tan^{-1} \left[\frac{\sqrt{2} \cos \left(\frac{\theta}{2} \right) - \sqrt{2} \sin \left(\frac{\theta}{2} \right)}{\sqrt{2} \cos \left(\frac{\theta}{2} \right) + \sqrt{2} \sin \left(\frac{\theta}{2} \right)} \right]$$

$$= \tan^{-1} \left[\frac{\frac{\sqrt{2} \cos \left(\frac{\theta}{2} \right)}{\sqrt{2} \cos \left(\frac{\theta}{2} \right)} - \frac{\sqrt{2} \sin \left(\frac{\theta}{2} \right)}{\sqrt{2} \cos \left(\frac{\theta}{2} \right)}}{\frac{\sqrt{2} \cos \left(\frac{\theta}{2} \right)}{\sqrt{2} \cos \left(\frac{\theta}{2} \right)} + \frac{\sqrt{2} \sin \left(\frac{\theta}{2} \right)}{\sqrt{2} \cos \left(\frac{\theta}{2} \right)}} \right]$$

$$= \tan^{-1} \left[\frac{1 - \tan \left(\frac{\theta}{2} \right)}{1 + \tan \left(\frac{\theta}{2} \right)} \right]$$

$$= \tan^{-1} \left[\frac{\tan \frac{\pi}{4} - \tan \left(\frac{\theta}{2} \right)}{1 + \tan \frac{\pi}{4} \cdot \tan \left(\frac{\theta}{2} \right)} \right] \dots \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\begin{aligned}
&= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right] \\
&= \frac{\pi}{4} - \frac{\theta}{2} \\
&= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \quad \dots [\because \theta = \cos^{-1} x] \\
&= \text{RHS.}
\end{aligned}$$

Miscellaneous exercise 3 | Q 24 | Page 110

If $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1$, then find the value of x .

Solution:

$$\begin{aligned}
&\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1 \\
&\therefore \sin^{-1} \frac{1}{5} + \cos^{-1} x = \sin^{-1} (1) \\
&\therefore \sin^{-1} \frac{1}{5} + \cos^{-1} x = \sin^{-1} \left(\sin \frac{\pi}{2} \right) \\
&\therefore \sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2} \\
&\therefore x = \frac{1}{5} \quad \dots \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]
\end{aligned}$$

Miscellaneous exercise 3 | Q 25 | Page 110

If $\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) = \frac{\pi}{4}$, find the value of x .

Solution:

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\therefore \tan^{-1} \left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} \right] = \frac{\pi}{4}$$

$$\therefore \frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2) - (x-1)(x+1)} = \tan \frac{\pi}{4}$$

$$\therefore \frac{(x^2 + x - 2) + (x^2 - x - 2)}{(x^2 - 4) - (x^2 - 1)} = 1$$

$$\therefore \frac{x^2 + x - 2 + x^2 - x - 2}{x^2 - 4 - x^2 + 1} = 1$$

$$\therefore \frac{2x^2 - 4}{-3} = 1$$

$$\therefore 2x^2 - 4 = -3$$

$$\therefore 2x^2 = 1$$

$$\therefore x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}.$$

Miscellaneous exercise 3 | Q 26 | Page 110

If $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$, then find the value of x .

Solution:

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\therefore \tan^{-1} \left[\frac{2 \cos x}{1 - \cos^2 x} \right] = \tan^{-1}(2 \operatorname{cosec} x) \quad \dots \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right]$$

$$\therefore \frac{2 \cos x}{1 - \cos^2 x} = 2 \operatorname{cosec} x$$

$$\therefore \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\therefore \cos x = \sin x$$

$$\therefore x = \frac{\pi}{4} \quad \dots \left[\because \sin \frac{\pi}{4} = \cos \frac{\pi}{4} \right]$$

Miscellaneous exercise 3 | Q 27 | Page 110

$$\text{Solve: } \tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} (\tan^{-1} x), \text{ for } x > 0.$$

Solution:

$$\tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} (\tan^{-1} x)$$

$$\therefore 2 \tan^{-1} \left(\frac{1-x}{1+x} \right) = (\tan^{-1} x)$$

$$\therefore \tan^{-1} \left[\frac{2 \left(\frac{1-x}{1+x} \right)}{1 - \left(\frac{1-x}{1+x} \right)^2} \right] = \tan^{-1} x \quad \dots \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$\therefore \frac{2 \left(\frac{1-x}{1+x} \right) (1+x)^2}{(1+x)^2 - (1-x)^2} = x$$

$$\therefore \frac{2(1-x)(1+x)}{(1+2x+x^2) - (1-2x+x^2)} = x$$

$$\therefore \frac{2(1-x^2)}{1+2x+x^2-1+2x-x^2} = x$$

$$\therefore \frac{2-2x^2}{4x} = x$$

$$\therefore 2 - 2x^2 = 4x^2$$

$$\therefore 6x^2 = 2$$

$$\therefore x^2 = \frac{1}{3}$$

$$\therefore x = \frac{1}{\sqrt{3}} \quad \therefore \dots [\because x > 0]$$

Miscellaneous exercise 3 | Q 28 | Page 110

If $\sin^{-1}(1 - x) - 2 \sin^{-1}x = \pi/2$, then find the value of x .

Solution:

$$\sin^{-1}(1 - x) - 2 \sin^{-1}x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(1 - x) = \frac{\pi}{2} + 2 \sin^{-1}x$$

$$\therefore 1 - x = \sin \left(\frac{\pi}{2} + 2 \sin^{-1}x \right)$$

$$\therefore 1 - x = \cos (2 \sin^{-1}x) \quad \therefore \dots \left[\because \sin \left(\frac{\pi}{2} + \theta \right) = \cos \theta \right]$$

$$\therefore 1 - x = 1 - 2[\sin(\sin^{-1}x)]^2 \quad \therefore \dots [\because \cos 2\theta = 1 - 2 \sin^2\theta]$$

$$\therefore 1 - x = 1 - 2x^2$$

$$\therefore 2x^2 - x = 0$$

$$\therefore x(2x - 1) = 0$$

$$\therefore x = 0 \text{ or } x = \frac{1}{2}$$

$$\text{When } x = \frac{1}{2}$$

$$\begin{aligned}
\text{LHS} &= \sin^{-1}\left(1 - \frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right) \\
&= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right) \\
&= -\sin^{-1}\left(\frac{1}{2}\right) \\
&= -\sin^{-1}\left(\sin \frac{\pi}{6}\right) \\
&= -\frac{\pi}{6} \neq \frac{\pi}{2} \\
\therefore x &\neq \frac{1}{2}
\end{aligned}$$

Hence, $x = 0$.

Miscellaneous exercise 3 | Q 29 | Page 110

If $\tan^{-1}2x + \tan^{-1}3x = \pi/4$, then find the value of x .

Solution:

$$\begin{aligned}
\tan^{-1}2x + \tan^{-1}3x &= \frac{\pi}{4} \\
\therefore \tan^{-1}\left(\frac{2x + 3x}{1 - 2x \times 3x}\right) &= \frac{\pi}{4}, \text{ where } 2x > 0, 3x > 0 \\
\therefore \frac{5x}{1 - 6x^2} &= \tan \frac{\pi}{4} = 1 \\
\therefore 5x &= 1 - 6x^2 \\
\therefore 6x^2 + 5x - 1 &= 0 \\
\therefore 6x^2 + 6x - x - 1 &= 0 \\
\therefore 6x(x + 1) - 1(x + 1) &= 0
\end{aligned}$$

$$\therefore (x + 1)(6x - 1) = 0$$

$$\therefore x = -1 \text{ or } x = 1/6$$

$$\text{But } x > 0 \therefore x \neq -1$$

$$\text{Hence, } x = 1/6$$

Miscellaneous exercise 3 | Q 30 | Page 110

$$\text{Show that } \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{2}{9}.$$

Solution:

$$\begin{aligned} \text{LHS} &= \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{4} \\ &= \tan^{-1} \left[\frac{\frac{1}{2} - \frac{1}{4}}{1 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)} \right] \\ &= \tan^{-1} \left(\frac{4 - 2}{8 + 1} \right) \\ &= \tan^{-1} \left(\frac{2}{9} \right) = \text{RHS.} \end{aligned}$$

Miscellaneous exercise 3 | Q 31 | Page 110

$$\text{Show that } \cot^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{3} = \cot^{-1} \frac{3}{4}.$$

Solution:

$$\begin{aligned} \text{LHS} &= \cot^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{3} \\ &= \tan^{-1} 3 - \tan^{-1} \frac{1}{3} \dots \left[\because \cot^{-1} x = \tan^{-1} \left(\frac{1}{x} \right) \right] \\ &= \tan^{-1} \left[\frac{3 - \frac{1}{3}}{1 + 3\left(\frac{1}{3}\right)} \right] \end{aligned}$$

$$\begin{aligned}
&= \tan^{-1} \left[\frac{\frac{8}{3}}{1+1} \right] \\
&= \tan^{-1} \left(\frac{4}{3} \right) \\
&= \cot^{-1} \left(\frac{3}{4} \right) \dots \left[\tan^{-1} x = \cot^{-1} \left(\frac{1}{x} \right) \right] \\
&= \text{RHS.}
\end{aligned}$$

Miscellaneous exercise 3 | Q 32 | Page 110

Show that $\tan^{-1} \frac{1}{2} = \frac{1}{3} \tan^{-1} \frac{11}{2}$

Solution:

We have to show that

$$\tan^{-1} \frac{1}{2} = \frac{1}{3} \tan^{-1} \frac{11}{2}$$

i.e. to show that $3 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{11}{2}$

$$\text{LHS} = 3 \tan^{-1} \frac{1}{2}$$

$$= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} \left[\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2} \right)^2} \right] + \tan^{-1} \frac{1}{2} \dots \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$= \tan^{-1} \left[\frac{1}{\frac{3}{4}} \right] + \tan^{-1} \frac{1}{2}$$

$$\begin{aligned}
&= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{2} \\
&= \tan^{-1} \left[\frac{\frac{4}{3} + \frac{1}{2}}{1 - \frac{4}{3} \times \frac{1}{2}} \right] \\
&= \tan^{-1} \left(\frac{8+3}{6-4} \right) \\
&= \tan^{-1} \left(\frac{11}{2} \right) = \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 33 | Page 111

Show that $\cos^{-1} \frac{\sqrt{3}}{2} + 2 \sin^{-1} \frac{\sqrt{3}}{2} = \frac{5\pi}{6}$.

Solution:

$$\begin{aligned}
\text{LHS} &= \cos^{-1} \frac{\sqrt{3}}{2} + 2 \sin^{-1} \frac{\sqrt{3}}{2} \\
&= \cos^{-1} \left(\cos \frac{\pi}{6} \right) + 2 \sin^{-1} \left(\sin \frac{\pi}{3} \right) \dots \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3} \right] \\
&= \frac{\pi}{6} + 2 \left(\frac{\pi}{3} \right) \dots \left[\because \sin^{-1}(\sin x) = x, \cos^{-1}(\cos x) = x \right] \\
&= \frac{\pi}{6} + \frac{2\pi}{3} \\
&= \frac{5\pi}{6} = \text{RHS.}
\end{aligned}$$

Miscellaneous exercise 3 | Q 34 | Page 111

Show that $2 \cot^{-1} \frac{3}{2} + \sec^{-1} \frac{13}{12} = \frac{\pi}{2}$

Solution:

$$\begin{aligned}
2 \cot^{-1} \frac{3}{2} &= 2 \tan^{-1} \frac{2}{3} \dots \left[\because \cot^{-1} x = \tan^{-1} \left(\frac{1}{x} \right) \right] \\
&= \tan^{-1} \left[\frac{2 \times \frac{2}{3}}{1 - \left(\frac{2}{3} \right)^2} \right] \dots \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right] \\
&= \tan^{-1} \left[\frac{\frac{4}{3}}{1 - \frac{4}{9}} \right] \\
&= \tan^{-1} \left(\frac{4}{3} \times \frac{9}{5} \right) = \tan^{-1} \frac{12}{5} \dots (1)
\end{aligned}$$

$$\text{Let } \sec^{-1} \frac{13}{12} = \alpha$$

$$\text{Then, } \sec \alpha = \frac{13}{12}, \text{ where } 0 < \alpha < \frac{\pi}{2}$$

$$\therefore \tan \alpha > 0$$

$$\text{Now, } \tan \alpha = \sqrt{\sec^2 \alpha - 1}$$

$$= \sqrt{\frac{169}{144} - 1} = \sqrt{\frac{25}{144}} = \frac{5}{12}$$

$$\therefore \alpha = \tan^{-1} \frac{5}{12} = \cot^{-1} \frac{12}{5} \dots \left[\because \tan^{-1} x = \cot^{-1} \left(\frac{1}{x} \right) \right]$$

$$\therefore \sec^{-1} \frac{13}{12} = \cot^{-1} \frac{12}{5} \dots (2)$$

Now,

$$\text{LHS} = 2 \cot^{-1} \frac{3}{2} + \sec^{-1} \frac{13}{12}$$

$$= \tan^{-1} \frac{12}{5} + \cot^{-1} \frac{12}{5} \dots [\text{By (1) and (2)}]$$

$$= \frac{\pi}{2} \dots \left[\because \tan^{-1} x + \cot^{-1} x - \frac{\pi}{2} \right]$$

= RHS.

Miscellaneous exercise 3 | Q 35.1 | Page 111

Prove the following:

$$\cos^{-1} x = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right), \text{ if } x > 0$$

Solution:

$$\text{Let } \cos^{-1} x = \alpha$$

Then, $\cos \alpha = x$, where $0 < \alpha < \pi$

$$\text{Since, } x > 0, 0 < \alpha < \frac{\pi}{2}$$

$$\therefore \sin \alpha > 0, \cos \alpha > 0$$

$$\text{Now, } \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \tan^{-1} \left(\frac{\sqrt{1-\cos^2 \alpha}}{\cos \alpha} \right)$$

$$= \tan^{-1} \left(\frac{\sqrt{\sin^2 \alpha}}{\cos \alpha} \right)$$

$$= \tan^{-1} (\tan \alpha)$$

$$= \alpha = \cos^{-1} x$$

$$\text{Hence, } \cos^{-1} x = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right), \text{ if } x > 0$$

Miscellaneous exercise 3 | Q 35.2 | Page 111

Prove the following:

$$\cos^{-1} x = \pi + \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right), \text{ if } x < 0$$

Solution:

$$\text{Let } \cos^{-1} x = \alpha$$

$$\text{Then, } \cos \alpha = x, \text{ where } 0 < \alpha < \pi$$

$$\text{Since, } x < 0, \frac{\pi}{2} < \alpha < \pi$$

$$\text{Now, } \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \tan^{-1} \left(\frac{\sqrt{1-\cos^2 \alpha}}{\cos \alpha} \right)$$

$$= \tan^{-1} (\tan \alpha) \quad \dots(1)$$

But $\frac{\pi}{2} < \alpha < \pi$, therefore inverse of tangent does not exist.

$$\text{Consider, } \frac{\pi}{2} - \pi < \alpha - \pi < \pi - \pi,$$

$$\therefore -\frac{\pi}{2} < \alpha - \pi < 0$$

$$\text{and } \tan(\alpha - \pi) = \tan[-(\pi - \alpha)]$$

$$= -\tan(\pi - \alpha) \quad \dots[\because \tan(-\theta) = -\tan \theta]$$

$$= -(-\tan \alpha) = \tan \alpha$$

\therefore from (1), we get

$$\tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \tan^{-1} [\tan(\alpha - \pi)]$$

$$= \alpha - \pi \quad \dots[\because \tan^{-1}(\tan x) = x]$$

$$= \cos^{-1} x - \pi$$

$$\therefore \cos^{-1} x = \pi + \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right), \text{ if } x < 0$$

Miscellaneous exercise 3 | Q 36 | Page 111

If $|x| < 1$, then prove that

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) = \sin^{-1} \left(\frac{2x}{1+x^2} \right) = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

Solution:

$$\text{Let } \tan^{-1} x = y$$

$$\text{Then, } x = \tan y$$

$$\text{Now, } \tan^{-1} \left(\frac{2x}{1-x^2} \right) = \tan^{-1} \left(\frac{2 \tan y}{1 - \tan^2 y} \right)$$

$$= \tan^{-1}(\tan 2y)$$

$$= 2y$$

$$= 2 \tan^{-1} x \quad \dots\dots(1)$$

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = \sin^{-1} \left(\frac{2 \tan y}{1 + \tan^2 y} \right)$$

$$= \sin^{-1}(\sin 2y)$$

$$= 2y$$

$$= 2 \tan^{-1} x \quad \dots\dots(2)$$

$$\begin{aligned}
\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) &= \cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right) \\
&= \cos^{-1}(\cos 2y) \\
&= 2y \\
&= 2 \tan^{-1} x \quad \dots\dots(3)
\end{aligned}$$

From (1), (2) and (3), we get

$$2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Miscellaneous exercise 3 | Q 37 | Page 111

If x, y, z are positive, then prove that

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right) = 0$$

Solution:

$$\begin{aligned}
\text{LHS} &= \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right) \\
&= \tan^{-1} x - \tan^{-1} y + \tan^{-1} y - \tan^{-1} z + \tan^{-1} z - \tan^{-1} x \quad \dots\dots[\because x > 0, y > 0, z > 0] \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

Miscellaneous exercise 3 | Q 38 | Page 111

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then show that $xy + yz + zx = 1$

Solution:

$$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$$

$$\therefore \tan^{-1} \left(\frac{x+y}{1-xy} \right) + \tan^{-1} z = \frac{\pi}{2}$$

$$\therefore \tan^{-1} \left[\frac{\frac{x+y}{1-xy} + z}{1 - \left(\frac{x+y}{1-xy} \right) z} \right] = \frac{\pi}{2}$$

$$\therefore \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-xz-yz} \right] = \frac{\pi}{2}$$

$$\therefore \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan \frac{\pi}{2}, \text{ which does not exist}$$

$$\therefore 1 - xy - yz - zx = 0$$

$$\therefore xy + yz + zx = 1$$

Miscellaneous exercise 3 | Q 39 | Page 111

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, then show that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution: $0 \leq \cos^{-1} x \leq \pi$ and

$$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$$

$$\therefore \cos^{-1} x = \pi, \cos^{-1} y = \pi \text{ and } \cos^{-1} z = \pi$$

$$\therefore x = y = z = \cos \pi = -1$$

$$\therefore x^2 + y^2 + z^2 + 2xyz$$

$$= (-1)^2 + (-1)^2 + (-1)^2 + 2(-1)(-1)(-1)$$

$$= 1 + 1 + 1 - 2$$

$$= 3 - 2$$

$$= 1.$$