

Chapter 1: Differentiation

EXERCISE 1.1 [PAGES 11 - 13]

Exercise 1.1 | Q 1.1 | Page 11

Differentiate the following w.r.t.x: $(x^3 - 2x - 1)^5$

SOLUTION

Method 1 :

$$\text{Let } y = (x^3 - 2x - 1)^5$$

Put $u = x^3 - 2x - 1$. Then $y = u^5$

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^5) = 5u^4$$

$$= 5(x^3 - 2x - 1)^4$$

and

$$\frac{du}{dx} = \frac{d}{dx}(x^3 - 2x - 1)$$

$$= 3x^2 - 2x - 1 - 0$$

$$= 3x^2 - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5(x^3 - 2x - 1)^4(3x^2 - 2)$$

$$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$$

Method 2 :

$$\text{Let } y = (x^3 - 2x - 1)^5$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x - 1)^5$$

$$\begin{aligned}
&= 5(x^3 - 2x - 1)4 \times \frac{d}{dx}(x^3 - 2x - 1) \\
&= 5(x^3 - 2x - 1)^4 \times (3x^2 - 2 \times 1 - 0) \\
&= 5(3x^2 - 2)(x^3 - 2x - 1)^4.
\end{aligned}$$

Note : Out of the two methods given above, we will use Method 2 for solving the remaining problems.

Exercise 1.1 | Q 1.2 | Page 11

Differentiate the following w.r.t.x: $(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{5}{2}}$

SOLUTION

$$\text{Let } y = (2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{5}{2}}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{5}{2}} \\
&= \frac{5}{2}(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{5}{2}-1} \times \frac{d}{dx}(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5) \\
&= \frac{5}{2}(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{3}{2}} \times \left(2 \times \frac{3}{2}x^{\frac{3}{2}-1} - 3 \times \frac{4}{3}x^{\frac{4}{3}-1} - 0\right) \\
&= \frac{5}{2}(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{3}{2}}(3x^{\frac{1}{2}} - 4x^{\frac{1}{3}}) \\
&= \frac{5}{2}(3\sqrt{x} - 4\sqrt[3]{x})(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5)^{\frac{3}{2}}.
\end{aligned}$$

Exercise 1.1 | Q 1.3 | Page 11

Differentiate the following w.r.t.x: $\sqrt{x^2 + 4x - 7}$

$$y = \sqrt{x^2 + 4x - 7} \left[\sqrt{x} = \frac{1}{2\sqrt{x}} \right]$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{x^2 + 4x - 7}} \cdot \frac{d}{dx}(x^2 + 4x - 7) \\ &= \frac{1}{2\sqrt{x^2 + 4x - 7}} \left(\frac{d}{dx}x^2 + \frac{d}{dx}4x - \frac{d}{dx}7 \right) \\ &= \frac{1}{2\sqrt{x^2 + 4x - 7}} \cdot (2x + 4 - 0) \\ &= \frac{2(x + 2)}{2\sqrt{x^2 + 4x - 7}} \\ &= \frac{(x + 2)}{\sqrt{x^2 + 4x - 7}}.\end{aligned}$$

Exercise 1.1 | Q 1.4 | Page 11

Differentiate the following w.r.t.x: $\sqrt{x^2 + \sqrt{x^2 + 1}}$

SOLUTION

$$\text{Let } y = \sqrt{x^2 + \sqrt{x^2 + 1}}$$

Differentiating w.r.t. x,we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(x^2 + \sqrt{x^2 + 1} \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left(x^2 + \sqrt{x^2 + 1} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(x^2 + \sqrt{x^2 + 1} \right) \\ &= \frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} \cdot \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(\sqrt{x^2 + 1}) \right] \\ &= \frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} \cdot \left[2x + \frac{1}{2\sqrt{x^2 + 1}} \cdot \frac{d}{dx}(x^2 + 1) \right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} \cdot \left[2x + \frac{1}{2\sqrt{x^2 + 1}}(2x + 0) \right] \\
&= \frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} \cdot \left[2x + \frac{x}{\sqrt{x^2 + 1}} \right] \\
&= \frac{1}{2\sqrt{x^2 + \sqrt{x^2 + 1}}} \cdot \left[\frac{2x\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right] \\
&= \frac{x(2\sqrt{x^2 + 1} + 1)}{2\sqrt{x^2 + 1} \cdot \sqrt{x^2 + \sqrt{x^2 + 1}}}.
\end{aligned}$$

Exercise 1.1 | Q 1.5 | Page 11

Differentiate the following w.r.t.x: $\frac{3}{5\sqrt[3]{(2x^2 - 7x - 5)^5}}$

SOLUTION

$$\text{Let } y = \frac{3}{5\sqrt[3]{(2x^2 - 7x - 5)^5}}$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3d}{5dx} (2x^2 - 7x - 5)^{-\frac{5}{3}} \\
&= \frac{3}{5} \times \left(-\frac{5}{3}\right) (2x^2 - 7x - 5)^{-\frac{5}{3}-1} \cdot \frac{d}{dx} (2x^2 - 7x - 5) \\
&= -(2x^2 - 7x - 5)^{-\frac{8}{3}} \cdot (2 \times 2x - 7 \times 1 - 0) \\
&= -\frac{4x - 7}{(2x^2 - 7x - 5)^{\frac{8}{3}}}.
\end{aligned}$$

Exercise 1.1 | Q 1.6 | Page 11

Differentiate the following w.r.t.x: $\left(\sqrt{3x - 5} - \frac{1}{\sqrt{3x - 5}}\right)^5$

SOLUTION

$$\text{Let } y = \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^5$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^5 \\&= 5 \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \cdot \frac{d}{dx} \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right) \\&= 5 \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \cdot \left[\frac{d}{dx} (3x-5)^{\frac{1}{2}} - \frac{d}{dx} (3x-5)^{-\frac{1}{2}} \right] \\&= 5 \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \times \left[\frac{1}{2} (3x-5)^{-\frac{1}{2}} \cdot \frac{d}{dx} (3x-5) - \left(-\frac{1}{2} \right) (3x-5)^{-\frac{3}{2}} \cdot \frac{d}{dx} (3x-5) \right] \\&= 5 \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \times \left[\frac{1}{2\sqrt{3x-5}} \cdot (3 \times 1 - 0) + \frac{1}{2(3x-5)^{\frac{3}{2}}} \cdot (3 \times 1 - 0) \right] \\&= \frac{15}{2} \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \cdot \left[\frac{3}{2\sqrt{3x-5}} + \frac{3}{2(3x-5)^{\frac{3}{2}}} \right] \\&= \frac{15}{2} \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4 \cdot \left[\frac{3x-5+1}{(3x-5)^{\frac{3}{2}}} \right] \\&= \frac{15(3x-4)}{2(3x-5)^{\frac{3}{2}}} \left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}} \right)^4.\end{aligned}$$

Exercise 1.1 | Q 2.01 | Page 12

Differentiate the following w.r.t. x: $\cos(x^2 + a^2)$

SOLUTION

Let $y = \cos(x^2 + a^2)$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\cos(x^2 + a^2)] \\ &= -\sin(x^2 + a^2) \cdot \frac{d}{dx}(x^2 + a^2) \\ &= -\sin(x^2 + a^2) \cdot (2x + 0) \\ &= -2x \sin(x^2 + a^2).\end{aligned}$$

Exercise 1.1 | Q 2.02 | Page 12

Differentiate the following w.r.t. x: $\sqrt{e^{(3x+2)+5}}$

SOLUTION

Let $y = \sqrt{e^{(3x+2)+5}}$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [e^{(3x+2)+5}]^{\frac{1}{2}} \\ &= \frac{1}{2} [e^{(3x+2)+5}]^{-\frac{1}{2}} \cdot \frac{d}{dx} [e^{(3x+2)+5}] \\ &= \frac{1}{2\sqrt{e^{(3x+2)+5}}} \cdot \left[e^{(3x+2)} \cdot \frac{d}{dx}(3x+2) + 0 \right] \\ &= \frac{1}{2\sqrt{e^{(3x+2)+5}}} \cdot \left[e^{(3x+2)} \cdot (3 \times 1 + 0) \right] \\ &= \frac{3e^{(3x+2)}}{2\sqrt{e^{(3x+2)+5}}}.\end{aligned}$$

Exercise 1.1 | Q 2.03 | Page 12

Differentiate the following w.r.t. x: $\log \left[\tan \left(\frac{x}{2} \right) \right]$

SOLUTION

$$\text{Let } y = \log \left[\tan \left(\frac{x}{2} \right) \right]$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \log \left[\tan \left(\frac{x}{2} \right) \right] \\ &= \frac{1}{\tan \left(\frac{x}{2} \right)} \cdot \frac{d}{dx} \left[\tan \left(\frac{x}{2} \right) \right] \\ &= \frac{1}{\tan \left(\frac{x}{2} \right)} \cdot \sec^2 \left(\frac{x}{2} \right) \cdot \frac{d}{dx} \left(\frac{x}{2} \right) \\ &= \frac{\cos \left(\frac{x}{2} \right)}{\sin \left(\frac{x}{2} \right)} \cdot \frac{1}{\cos^2 \left(\frac{x}{2} \right)} \cdot \frac{1}{2} \times 1 \\ &= \frac{1}{2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right)} \\ &= \frac{1}{\sin x} \\ &= \operatorname{cosec} x.\end{aligned}$$

Exercise 1.1 | Q 2.04 | Page 12

Differentiate the following w.r.t.x: $\sqrt{\tan \sqrt{x}}$

SOLUTION

$$\text{Let } y = \sqrt{\tan \sqrt{x}}$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{\tan \sqrt{x}} \right) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \cdot \frac{d}{dx} (\tan \sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \times \sec^2 \sqrt{x} \cdot \frac{d}{dx} (\sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \times \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} \\ &= \frac{\sec^2 \sqrt{x}}{4\sqrt{x}\sqrt{\tan \sqrt{x}}}.\end{aligned}$$

Exercise 1.1 | Q 2.05 | Page 12

Differentiate the following w.r.t.x: $\cot^3[\log(x^3)]$

SOLUTION

Let $y = \cot^3[\log(x^3)]$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\cot(\log x^3)]^3 \\&= 3 [\cot(\log x^3)]^2 \cdot \frac{d}{dx} [\cot(\log x^3)] \\&= 3 \cot^2[\log(x^3)] \cdot [-\operatorname{cosec}^2(\log x^3)] \cdot \frac{d}{dx} (\log x^3) \\&= -3 \cot^2[\log(x^3)] \cdot \operatorname{cosec}^2[\log(x^3)] \cdot 3 \frac{d}{dx} (\log x) \\&= -3 \cot^2[\log(x^3)] \cdot \operatorname{cosec}^2[\log(x^3)] \cdot 3 \times \frac{1}{x} \\&= \frac{-9 \operatorname{cosec}^2[\log(x^3)] \cdot \cot^2[\log(x^3)]}{x}\end{aligned}$$

Exercise 1.1 | Q 2.06 | Page 12

Differentiate the following w.r.t. x: $5^{\sin^3 x + 3}$

SOLUTION

Let $y = 5^{\sin^3 x + 3}$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (5^{\sin^3 x + 3}) \\&= 5^{\sin^3 x + 3} \cdot \log 5 \cdot \frac{d}{dx} (\sin^3 x + 3) \\&= 5^{\sin^3 x + 3} \cdot \log 5 \cdot \left[3 \sin^2 x \cdot \frac{d}{dx} (\sin x) + 0 \right] \\&= 5^{\sin^3 x + 3} \cdot \log 5 \cdot [3 \sin^2 x \cos x] \\&= 3 \sin^2 x \cos x \cdot 5^{\sin^3 x + 3} \cdot \log 5.\end{aligned}$$

Exercise 1.1 | Q 2.07 | Page 12

Differentiate the following w.r.t.x: $\text{cosec}(\sqrt{\cos x})$

SOLUTION

$$\text{Let } y = \text{cosec}(\sqrt{\cos x})$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\text{cosec}(\sqrt{\cos x})] \\&= -\text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{d}{dx} \sqrt{\cos x} \\&= -\text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{1}{2\sqrt{\cos x}} \cdot \frac{d}{dx} (\cos x) \\&= -\text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x) \\&= \frac{\sin x \cdot \text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x})}{2\sqrt{\cos x}}.\end{aligned}$$

Exercise 1.1 | Q 2.08 | Page 12

Differentiate the following w.r.t.x: $\log[\cos(x^3 - 5)]$

SOLUTION

$$\text{Let } y = \log[\cos(x^3 - 5)]$$

Differentiating w.r.t. x, we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \{\log[\cos(x^3 - 5)]\} \\&= \frac{1}{\cos(x^3 - 5)} \cdot \frac{d}{dx} [\cos(x^3 - 5)] \\&= \frac{1}{\cos(x^3 - 5)} \cdot [-\sin(x^3 - 5)] \cdot \frac{d}{dx} (x^3 - 5)\end{aligned}$$

$$\begin{aligned}
 &= -\tan(x^3 - 5) \times (3x^2 - 0) \\
 &= -3x^2 \tan(x^3 - 5).
 \end{aligned}$$

Exercise 1.1 | Q 2.09 | Page 12

Differentiate the following w.r.t.x: $e^{3 \sin^2 x - 2 \cos^2 x}$

SOLUTION

$$\text{Let } y = e^{3 \sin^2 x - 2 \cos^2 x}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left[e^{3 \sin^2 x - 2 \cos^2 x} \right] \\
 &= e^{3 \sin^2 x - 2 \cos^2 x} \cdot \frac{d}{dx} (3 \sin^2 x - 2 \cos^2 x) \\
 &= e^{3 \sin^2 x - 2 \cos^2 x} \cdot \left[3 \frac{d}{dx} (\sin x)^2 - 2 \frac{d}{dx} (\cos^2 x) \right] \\
 &= e^{3 \sin^2 x - 2 \cos^2 x} \cdot \left[3 \times 2 \sin x \cdot \frac{d}{dx} (\sin x) - 2 \times 2 \cos x \cdot \frac{d}{dx} (\cos x) \right] \\
 &= e^{3 \sin^2 x - 2 \cos^2 x} \cdot [6 \sin x \cos x - 4 \cos x (-\sin x)] \\
 &= e^{3 \sin^2 x - 2 \cos^2 x} \cdot (10 \sin x \cos x) \\
 &= 5(2 \sin x \cos x) \cdot e^{3 \sin^2 x - 2 \cos^2 x} \\
 &= 5 \sin 2x \cdot e^{3 \sin^2 x - 2 \cos^2 x}.
 \end{aligned}$$

Exercise 1.1 | Q 2.1 | Page 12

Differentiate the following w.r.t.x: $\cos^2[\log(x^2 + 7)]$

SOLUTION

Let $y = \cos^2[\log(x^2 + 7)]$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \{ \cos[\log(x^2 + 7)] \}^2 \\&= 2 \cos[\log(x^2 + 7)] \cdot \frac{d}{dx} \{ \cos[\log(x^2 + 7)] \} \\&= 2 \cos[\log(x^2 + 7)] \cdot \{-\sin[\log(x^2 + 7)]\} \cdot \frac{d}{dx} [\log(x^2 + 7)] \\&= -2 \sin[\log(x^2 + 7)] \cdot \cos[\log(x^2 + 7)] \times \frac{1}{x^2 + 7} \cdot \frac{d}{dx}(x^2 + 7) \\&= -\sin[2\log(x^2 + 7)] \times \frac{1}{x^2 + 7} \cdot (2x + 0) \quad \dots [\because 2\sin x \cdot \cos x = \sin 2x] \\&= \frac{-2x \cdot \sin[2\log(x^2 + 7)]}{x^2 + 7}.\end{aligned}$$

Exercise 1.1 | Q 2.11 | Page 12

Differentiate the following w.r.t.x: $\tan[\cos(\sin x)]$

SOLUTION

Let $y = \tan[\cos(\sin x)]$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \{ \tan[\cos(\sin x)] \} \\&= \sec^2[\cos(\sin x)] \cdot \frac{d}{dx} [\cos(\sin x)] \\&= \sec^2[\cos(\sin x)] \cdot [-\sin(\sin x)] \cdot \frac{d}{dx} (\sin x) \\&= -\sec^2[\cos(\sin x)] \cdot \sin(\sin x) \cdot \cos x.\end{aligned}$$

Exercise 1.1 | Q 2.12 | Page 12

Differentiate the following w.r.t.x: $\sec[\tan(x^4 + 4)]$

SOLUTION

$$\text{Let } y = \sec[\tan(x^4 + 4)]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \{ \sec[\tan(x^4 + 4)] \}$$

$$= \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)] \cdot \frac{d}{dx} [\tan(x^4 + 4)]$$

$$= \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)] \cdot \sec^2(x^4 + 4) \cdot \frac{d}{dx}(x^4 + 4)$$

$$= \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)] \cdot \sec^2(x^4 + 4) \cdot (4x^3 + 0)$$

$$= 4x^3 \cdot \sec^2(x^4 + 4) \cdot \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)].$$

Exercise 1.1 | Q 2.13 | Page 12

Differentiate the following w.r.t.x: $e^{\log[(\log x)^2 - \log x^2]}$

SOLUTION

$$\text{Let } y = e^{\log[(\log x)^2 - \log x^2]}$$

$$= (\log x)^2 - \log x^2 \quad \dots [\because e^{\log x} = x]$$

$$= (\log x)^2 - 2\log x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [(\log x)^2 - 2\log x]$$

$$= \frac{d}{dx} (\log x)^2 - 2 \frac{d}{dx} (\log x)$$

$$= 2\log x \cdot \frac{d}{dx} (\log x) - 2 \times \frac{1}{x}$$

$$\begin{aligned}
 &= 2 \log x \times \frac{1}{x} - \frac{2}{x} \\
 &= \frac{2 \log x}{x} - \frac{2}{x}.
 \end{aligned}$$

Exercise 1.1 | Q 2.14 | Page 12

Differentiate the following w.r.t.x: $\sin \sqrt{\sin \sqrt{x}}$

SOLUTION

$$\text{Let } y = \sin \sqrt{\sin \sqrt{x}}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\sin \sqrt{\sin \sqrt{x}} \right) \\
 &= \cos \sqrt{\sin \sqrt{x}} \cdot \frac{d}{dx} \left(\sqrt{\sin \sqrt{x}} \right) \\
 &= \cos \sqrt{\sin \sqrt{x}} \times \frac{1}{2\sqrt{\sin \sqrt{x}}} \cdot \frac{d}{dx} (\sin \sqrt{x}) \\
 &= \frac{\cos \sqrt{\sin \sqrt{x}}}{2\sqrt{\sin \sqrt{x}}} \times \cos \sqrt{x} \cdot \frac{d}{dx} (\sqrt{x}) \\
 &= \frac{\cos \sqrt{\sin \sqrt{x}} \cdot \cos \sqrt{x}}{2\sqrt{\sin \sqrt{x}}} \times \frac{1}{2\sqrt{x}} \\
 &= \frac{\cos \sqrt{\sin \sqrt{x}} \cdot \cos \sqrt{x}}{4\sqrt{x} \cdot \sqrt{\sin \sqrt{x}}}.
 \end{aligned}$$

Exercise 1.1 | Q 2.15 | Page 12

Differentiate the following w.r.t.x: $\log [\sec(e^{x^2})]$

SOLUTION

$$\text{Let } y = \log \left[\sec(e^{x^2}) \right]$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \log \left[\sec(e^{x^2}) \right] \\ &= \frac{1}{\sec(e^{x^2})} \cdot \frac{d}{dx} \left[\sec(e^{x^2}) \right] \\ &= \frac{1}{\sec(e^{x^2})} \cdot \sec(e^{x^2}) \tan(e^{x^2}) \cdot \frac{d}{dx}(e^{x^2}) \\ &= \tan(e^{x^2}) \cdot e^{x^2} \cdot \frac{d}{dx}(x^2) \\ &= \tan(e^{x^2}) \cdot e^{x^2} \cdot 2x \\ &= 2x \cdot e^{x^2} \tan(e^{x^2}).\end{aligned}$$

Exercise 1.1 | Q 2.16 | Page 12

Differentiate the following w.r.t.x: $\log_e^2(\log x)$

SOLUTION

$$\text{Let } y = \log_e^2(\log x)$$

$$= \frac{\log(\log x)}{\log e^2}$$

$$= \frac{\log(\log x)}{2 \log e}$$

$$= \frac{\log(\log x)}{2} \quad \dots [\because \log e = 1]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2} \frac{d}{2dx} [\log(\log x)]$$

$$= \frac{1}{2} \times \frac{1}{\log x} \cdot \frac{d}{dx} (\log x)$$

$$= \frac{1}{2 \log x} \times \frac{1}{x}$$

$$= \frac{1}{2x \log x}.$$

Exercise 1.1 | Q 2.17 | Page 12

Differentiate the following w.r.t.x: $[\log \{\log(\log x)\}]^2$

SOLUTION

$$\text{Let } y = [\log \{\log(\log x)\}]^2$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\log\{\log(\log x)\}]^2 \\&= 2 \cdot \log\{\log(\log x)\} \times \frac{d}{dx} [\log\{\log(\log x)\}] \\&= 2 \cdot \log\{\log(\log x)\} \times \frac{1}{\log(\log x)} \cdot \frac{d}{dx} [\log(\log x)] \\&= 2 \cdot \log\{\log(\log x)\} \times \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \times \frac{d}{dx} (\log x) \\&= 2 \cdot \log\{\log(\log x)\} \times \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \times \frac{1}{x} \\&= 2 \cdot \left[\frac{\log\{\log(\log x)\}}{x \cdot \log x \cdot \log(\log x)} \right].\end{aligned}$$

Exercise 1.1 | Q 2.18 | Page 12

Differentiate the following w.r.t.x: $\sin^2 x^2 - \cos^2 x^2$

SOLUTION

$$\text{Let } y = \sin^2 x^2 - \cos^2 x^2$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\sin^2 x^2 - \cos^2 x^2] \\&= \frac{d}{dx} (\sin x^2)^2 - \frac{d}{dx} (\cos x^2)^2 \\&= 2\sin x^2 \cdot \frac{d}{dx} (\sin x^2) - 2\cos x^2 \cdot \frac{d}{dx} (\cos x^2) \\&= 2\sin x^2 \cdot \cos x^2 \cdot \frac{d}{dx} (x^2) - 2\cos x^2 \cdot (-\sin x^2) \cdot \frac{d}{dx} (x^2) \\&= 2\sin x^2 \cdot \cos x^2 \cdot 2x + 2\cos x^2 \cdot \sin x^2 \cdot 2x \\&= 4x (2\sin x^2 \cdot \cos x^2) \\&= 4x \sin(2x^2).\end{aligned}$$

Exercise 1.1 | Q 3.01 | Page 12

Differentiate the following w.r.t.x: $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

SOLUTION

$$\text{Let } y = (x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4] \\&= \frac{d}{dx} (x^2 + 4x + 1)^3 + \frac{d}{dx} (x^3 - 5x - 2)^4 \\&= 3(x^2 + 4x + 1)^2 \cdot \frac{d}{dx} (x^2 + 4x + 1) + 4(x^3 - 5x - 2)^3 \cdot \frac{d}{dx} (x^3 - 5x - 2) \\&= 3(x^2 + 4x + 1)^2 \cdot (2x + 4 \times 1 + 0) + 4(x^3 - 5x - 2)^3 \cdot (3x^2 - 5 \times 1 - 0) \\&= 6(x + 2)(x^2 + 4x + 1)^2 + 4(3x^2 - 5)(x^3 - 5x - 2)^3.\end{aligned}$$

Exercise 1.1 | Q 3.02 | Page 12

Differentiate the following w.r.t.x: $(1 + 4x)^5 (3 + x - x^2)^8$

SOLUTION

$$\text{Let } y = (1 + 4x)^5 (3 + x - x^2)^8$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (1 + 4x)^5 (3 + x - x^2)^8 \\&= (1 + 4x)^5 \cdot \frac{d}{dx} (3 + x - x^2)^8 + (3 + x - x^2)^8 \cdot \frac{d}{dx} (1 + 4x)^5 \\&= (1 + 4x)^5 \times 8(3 + x - x^2)^7 \cdot \frac{d}{dx} (3 + x - x^2) + (3 + x - x^2)^8 \times 5(1 + 4x)^4 \cdot \frac{d}{dx} (1 + 4x) \\&= 8(1 + 4x)^5 (3 + x - x^2)^7 \cdot (0 + 1 - 2x) + 5(1 + 4x)^4 (3 + x - x^2)^8 \cdot (0 + 4 \times 1) \\&= 8(1 - 2x)(1 + 4x)^5 (3 + x - x^2)^7 + 20(1 + 4x)^4 (3 + x - x^2)^8.\end{aligned}$$

Exercise 1.1 | Q 3.03 | Page 12

Differentiate the following w.r.t.x: $\frac{x}{\sqrt{7 - 3x}}$

SOLUTION

$$\text{Let } y = \frac{x}{\sqrt{7 - 3x}}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x}{\sqrt{7 - 3x}} \right) \\&= \frac{\sqrt{7 - 3x} \cdot \frac{d}{dx}(x) - x \frac{d}{dx}(\sqrt{7 - 3x})}{(\sqrt{7 - 3x})^2} \\&= \frac{\sqrt{7 - 3x} \times 1 - x \times \frac{1}{2\sqrt{7-3x}} \cdot \frac{d}{dx}(7 - 3x)}{7 - 3x} \\&= \frac{\sqrt{7 - 3x} - \frac{x}{2\sqrt{7-3x}}(0 - 3 \times 1)}{7 - 3x} \\&= \frac{2(7 - 3x) + 3x}{2(7 - 3x)^{\frac{3}{2}}} \\&= \frac{14 - 6x + 3x}{2(7 - 3x)^{\frac{3}{2}}} \\&= \frac{14 - 3x}{2(7 - 3x)^{\frac{3}{2}}}.\end{aligned}$$

[Exercise 1.1 | Q 3.04 | Page 12](#)

Differentiate the following w.r.t.x: $\frac{(x^3 - 5)^5}{(x^3 + 3)^3}$

SOLUTION

$$\text{Let } y = \frac{(x^3 - 5)^5}{(x^3 + 3)^3}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{(x^3 - 5)^5}{(x^3 + 3)^3} \right] \\ &= \frac{(x^3 + 3)^3 \cdot \frac{d}{dx}(x^3 - 5)^5 - (x^3 - 5)^5 \cdot \frac{d}{dx}(x^3 + 3)^3}{[(x^3 + 3)^3]^2} \\ &= \frac{(x^3 + 3)^3 \times 5(x^3 - 5)^4 \cdot \frac{d}{dx}(x^3 - 5) - (x^3 - 5)^5 \times 3(x^3 + 3)^2 \cdot \frac{d}{dx}(x^3 + 3)}{(x^3 + 3)^6} \\ &= \frac{5(x^3 + 3)^3(x^3 - 5)^4 \cdot (3x^3 - 0) - 3(x^3 - 5)^5(x^3 + 3)^2 \cdot (3x^2 + 0)}{(x^3 + 3)^6} \\ &= \frac{3x^2(x^3 + 3)^2(x^3 - 5)^4[5(x^3 + 3) - 3(x^3 - 5)]}{(x^3 + 3)^6} \\ &= \frac{3x^2(x^3 - 5)^4(5x^3 + 15 - 3x^3 + 15)}{(x^3 + 3)^4} \\ &= \frac{3x^2(x^3 - 5)^4(2x^3 + 30)}{(x^3 + 3)^4} \\ &= \frac{6x^2(x^2 + 15)(x^3 - 5)^4}{(x^3 + 3)^4}. \end{aligned}$$

Exercise 1.1 | Q 3.05 | Page 12

Differentiate the following w.r.t.x: $(1 + \sin^2 x)^2 (1 + \cos^2 x)^3$

SOLUTION

Let $y = (1 + \sin^2 x)^2 (1 + \cos^2 x)^3$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[(1 + \sin^2 x)^2 (1 + \cos^2 x)^3 \right] \\&= (1 + \sin^2 x)^2 \cdot \frac{d}{dx} (1 + \cos^2 x)^3 + (1 + \cos^2 x)^3 \cdot \frac{d}{dx} (1 + \sin^2 x)^2 \\&= (1 + \sin^2 x)^2 \times 3(1 + \cos^2 x)^2 \cdot \frac{d}{dx} (1 + \cos^2 x) + (1 + \cos^2 x)^3 \times 2(1 + \sin^2 x) \cdot \frac{d}{dx} (1 + \sin^2 x) \\&= 3(1 + \sin^2 x)^2 (1 + \cos^2 x)^2 \cdot \left[0 + 2 \cos x \cdot \frac{d}{dx} (\cos x) \right] + 2(1 + \sin^2 x) (1 + \cos^2 x)^3 \cdot \left[0 + 2 \sin x \cdot \frac{d}{dx} (\sin x) \right] \\&= 3(1 + \sin^2 x)^2 (1 + \cos^2 x)^2 [2\cos x(-\sin x)] + 2(1 + \sin^2 x) (1 + \cos^2 x)^3 [2\sin x \cdot \cos x] \\&= 3(1 + \sin^2 x)^2 (1 + \cos^2 x)^2 (-\sin 2x) + 2(1 + \sin^2 x) (1 + \cos^2 x)^3 (\sin 2x) \\&= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 [-3(1 + \sin^2 x) + 2(1 + \cos^2 x)] \\&= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 (-3 - 3\sin^2 x + 2 + 2\cos^2 x) \\&= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 [-1 - 3\sin^2 x + 2(1 - \sin^2 x)] \\&= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 (-1 - 3\sin^2 x + 2 - 2\sin^2 x) \\&= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 (1 - 5\sin^2 x).\end{aligned}$$

Exercise 1.1 | Q 3.06 | Page 12

Differentiate the following w.r.t.x: $\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}$

SOLUTION

$$\text{Let } y = \sqrt{\cos x} + \sqrt{\cos \sqrt{x}}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\sqrt{\cos x} + \sqrt{\cos \sqrt{x}} \right] \\&= \frac{d}{dx} (\cos x)^{\frac{1}{2}} + \frac{d}{dx} (\cos \sqrt{x})^{\frac{1}{2}} \\&= \frac{1}{2} (\cos x)^{-\frac{1}{2}} \cdot \frac{d}{dx} (\cos x) + \frac{1}{2} (\cos \sqrt{x})^{-\frac{1}{2}} \cdot \frac{d}{dx} (\cos \sqrt{x}) \\&= \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x) + \frac{1}{2\sqrt{\cos \sqrt{x}}} \times (-\sin \sqrt{x}) \cdot \frac{d}{dx} (\sqrt{x}) \\&= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin \sqrt{x}}{2\sqrt{\cos \sqrt{x}}} \times \frac{1}{2\sqrt{x}} \\&= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin \sqrt{x}}{4\sqrt{x}\sqrt{\cos \sqrt{x}}}.\end{aligned}$$

Exercise 1.1 | Q 3.07 | Page 12

Differentiate the following w.r.t.x: $\log(\sec 3x + \tan 3x)$

SOLUTION

Let $y = \log(\sec 3x + \tan 3x)$

Differentiating w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\log(\sec 3x + \tan 3x)] \\&= \frac{1}{\sec 3x + \tan 3x} \cdot \frac{d}{dx} (\sec 3x + \tan 3x) \\&= \frac{1}{\sec 3x + \tan 3x} \times \left[\frac{d}{dx}(\sec 3x) + \frac{d}{dx}(\tan 3x) \right] \\&= \frac{1}{\sec 3x + \tan 3x} \times \left[\sec 3x \tan 3x \cdot \frac{d}{dx}(3x) + \sec^2 3x \cdot \frac{d}{dx}(3x) \right] \\&= \frac{1}{\sec 3x + \tan 3x} \times [\sec 3x \tan 3x \times 3 + \sec^2 3x \times 3] \\&= \frac{3 \sec 3x (\tan 3x + \sec 3x)}{\sec 3x + \tan 3x} \\&= 3 \sec 3x.\end{aligned}$$

Exercise 1.1 | Q 3.08 | Page 12

Differentiate the following w.r.t. x : $\frac{1 + \sin x^\circ}{1 - \sin x^\circ}$

SOLUTION

$$\begin{aligned} \text{Let } y &= \frac{1 + \sin x^\circ}{1 - \sin x^\circ} \\ &= \frac{1 + \sin\left(\frac{\pi x}{180}\right)}{1 - \sin\left(\frac{\pi x}{180}\right)} \quad \dots \left[\because x^\circ = \left(\frac{\pi x}{180}\right)^\circ \right] \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{1 + \sin\left(\frac{\pi x}{180}\right)}{1 - \sin\left(\frac{\pi x}{180}\right)} \right] \\ &= \frac{\left[1 - \sin\left(\frac{\pi x}{180}\right)\right] \cdot \frac{d}{dx} \left[1 + \sin\left(\frac{\pi x}{180}\right)\right] - \left[1 + \sin\left(\frac{\pi x}{180}\right)\right] \cdot \frac{d}{dx} \left[1 - \sin\left(\frac{\pi x}{180}\right)\right]}{\left[1 - \sin\left(\frac{\pi x}{180}\right)\right]^2} \\ &= \frac{\left[1 - \sin\left(\frac{\pi x}{180}\right)\right] \cdot \left[0 + \cos\left(\frac{\pi x}{180}\right) \cdot \frac{d}{dx} \left(\frac{\pi x}{180}\right)\right] - \left[1 + \sin\left(\frac{\pi x}{180}\right)\right] \cdot \left[0 - \cos\left(\frac{\pi x}{180}\right) \cdot \frac{d}{dx} \left(\frac{\pi x}{180}\right)\right]}{\left[1 - \sin\left(\frac{\pi x}{180}\right)\right]^2} \\ &= \frac{(1 - \sin x^\circ) \left[(\cos x^\circ) \times \frac{\pi}{180} \times 1\right] - (1 + \sin x^\circ) \left[-(\cos x^\circ) \times \frac{\pi}{180} \times 1\right]}{(1 - \sin x^\circ)^2} \\ &= \frac{\frac{\pi}{180} \cos x^\circ (1 - \sin x^\circ + 1 + \sin x^\circ)}{(1 - \sin x^\circ)^2} \\ &= \frac{\pi \cos x^\circ}{90(1 - \sin x^\circ)^2}. \end{aligned}$$

Exercise 1.1 | Q 3.09 | Page 12

Differentiate the following w.r.t.x: $\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$

SOLUTION

$$\text{Let } y = \cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right) \right] \\ &= \frac{d}{dx} \left[\cot\left(\frac{\log x}{2}\right) \right] - \frac{d}{dx} \left[\log\left(\frac{\cot x}{2}\right) \right] \\ &= -\operatorname{cosec}^2\left(\frac{\log x}{2}\right) \cdot \frac{d}{dx}\left(\frac{\log x}{2}\right) - \frac{1}{\left(\frac{\cot x}{2}\right)} \cdot \frac{d}{dx}\left(\frac{\cot x}{2}\right) \\ &= -\operatorname{cosec}^2\left(\frac{\log x}{2}\right) \times \frac{1}{2} \times \frac{1}{x} - \frac{2}{\cot x} \times \frac{1}{2} \times (-\operatorname{cosec}^2 x) \\ &= -\frac{\operatorname{cosec}^2\left(\frac{\log x}{2}\right)}{2x} + \tan x \cdot \operatorname{cosec}^2 x.\end{aligned}$$

Exercise 1.1 | Q 3.1 | Page 12

Differentiate the following w.r.t.x: $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

SOLUTION

$$\text{Let } y = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

$$= \frac{e^{2x} - \frac{1}{e^{2x}}}{e^{2x} + \frac{1}{e^{2x}}}$$

$$= \frac{e^{4x} - 1}{e^{4x} + 1}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{e^{4x} - 1}{e^{4x} + 1} \right)$$

$$= \frac{(e^{4x} + 1) \cdot \frac{d}{dx}(e^{4x} - 1) - (e^{4x} - 1) \cdot \frac{d}{dx}(e^{4x} + 1)}{(e^{4x} + 1)^2}$$

$$= \frac{(e^{4x} + 1)[e^{4x} \cdot \frac{d}{dx}(4x) - 0] - (e^{4x} - 1)[e^{4x} \cdot \frac{d}{dx}(4x) + 0]}{(e^{4x} + 1)^2}$$

$$= \frac{(e^{4x} + 1) \cdot e^{4x} \times 4 - (e^{4x} - 1) \cdot e^{4x} \times 4}{(e^{4x} + 1)^2}$$

$$= \frac{4e^{4x}(e^{4x} + 1 - e^{4x} + 1)}{(e^{4x} + 1)^2}$$

$$= \frac{8e^{4x}}{(e^{4x} + 1)^2}.$$

Exercise 1.1 | Q 3.11 | Page 12

Differentiate the following w.r.t.x: $\frac{e^{\sqrt{x}} + 1}{e^{\sqrt{x}} - 1}$

SOLUTION

$$\text{Let } y = \frac{e^{\sqrt{x}} + 1}{e^{\sqrt{x}} - 1}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{e^{\sqrt{x}} + 1}{e^{\sqrt{x}} - 1} \right) \\&= \frac{\left(e^{\sqrt{x}} - 1 \right) \frac{d}{dx} \left(e^{\sqrt{x}} + 1 \right) - \left(e^{\sqrt{x}} + 1 \right) \frac{d}{dx} \left(e^{\sqrt{x}} - 1 \right)}{\left(e^{\sqrt{x}} - 1 \right)^2} \\&= \frac{\left(e^{\sqrt{x}} - 1 \right) \left[e^{\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) + 0 \right] - \left(e^{\sqrt{x}} + 1 \right) \left[e^{\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) - 0 \right]}{\left(e^{\sqrt{x}} - 1 \right)^2} \\&= \frac{\left(e^{\sqrt{x}} - 1 \right) \left[e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} \right] - \left(e^{\sqrt{x}} + 1 \right) \left[e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} \right]}{\left(e^{\sqrt{x}} - 1 \right)^2} \\&= \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}} \left(e^{\sqrt{x}} - 1 - e^{\sqrt{x}} - 1 \right)}{\left(e^{\sqrt{x}} - 1 \right)^2} \\&= \frac{-e^{\sqrt{x}}}{\sqrt{x} \left(e^{\sqrt{x}} - 1 \right)^2}.\end{aligned}$$

Exercise 1.1 | Q 3.12 | Page 12

Differentiate the following w.r.t.x: $\log[\tan^3x \cdot \sin^4x \cdot (x^2 + 7)^7]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \log[\tan^3 x \cdot \sin^4 x \cdot (x^2 + 7)^7] \\ &= \log \tan^3 x + \log \sin^4 x + \log(x^2 + 7)^7 \\ &= 3 \log \tan x + 4 \log \sin x + 7 \log(x^2 + 7) \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [3 \log \tan x + 4 \log \sin x + 7 \log(x^2 + 7)] \\ &= 3 \frac{d}{dx} (\log \tan x) + 4 \frac{d}{dx} (\log \sin x) + 7 \frac{d}{dx} [\log(x^2 + 7)] \\ &= 3 \times \frac{1}{\tan x} \cdot \frac{d}{dx} (\tan x) + 4 \times \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) + 7 \times \frac{1}{x^2 + 7} \cdot \frac{d}{dx} (x^2 + 7) \\ &= 3 \times \frac{1}{\tan x} \cdot \sec^2 x + 4 \times \frac{1}{\sin x} \cdot \cos x + 7 \times \frac{1}{x^2 + 7} \cdot (2x + 0) \\ &= 3 \times \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} + 4 \cot x + \frac{14x}{x^2 + 7} \\ &= \frac{6}{2 \sin x \cos x} + 4 \cot x + \frac{14x}{x^2 + 7} \\ &= \frac{6}{\sin 2x} + 4 \cot x + \frac{14x}{x^2 + 7} \\ &= 6 \operatorname{cosec} 2x + 4 \cot x + \frac{14x}{x^2 + 7}. \end{aligned}$$

Exercise 1.1 | Q 3.13 | Page 12

Differentiate the following w.r.t.x: $\log \left(\sqrt{\frac{1 - \cos 3x}{1 + \cos 3x}} \right)$

SOLUTION

$$\text{Let } y = \log \left(\sqrt{\frac{1 - \cos 3x}{1 + \cos 3x}} \right)$$

$$= \log \left(\sqrt{\frac{2 \sin^3 \left(\frac{3x}{2} \right)}{2 \cos^2 \left(\frac{3x}{2} \right)}} \right)$$

$$= \log \tan \left(\frac{3x}{2} \right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[\log \tan \left(\frac{3x}{2} \right) \right]$$

$$= \frac{1}{\tan \left(\frac{3x}{2} \right)} \times \frac{d}{dx} \left[\tan \left(\frac{3x}{2} \right) \right]$$

$$= \frac{1}{\tan \left(\frac{3x}{2} \right)} \times \sec^2 \left(\frac{3x}{2} \right) \cdot \frac{d}{dx} \left(\frac{3x}{2} \right)$$

$$= \frac{\cos \left(\frac{3x}{2} \right)}{\sin \left(\frac{3x}{2} \right)} \times \frac{1}{\cos^2 \left(\frac{3x}{2} \right)} \times \frac{3}{2} \times 1$$

$$= 3 \times \frac{1}{2 \sin \left(\frac{3x}{2} \right) \cos \left(\frac{3x}{2} \right)}$$

$$= 3 \times \frac{1}{\sin 3x}$$

$$= 3 \operatorname{cosec} 3x.$$

Exercise 1.1 | Q 3.14 | Page 12

Differentiate the following w.r.t.x: $\left(\sqrt{\frac{1 + \cos \left(\frac{5x}{2} \right)}{1 - \cos \left(\frac{5x}{2} \right)}} \right)$

SOLUTION

Using

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\log a^b = b \log a$$

$$y = \log\left(\sqrt{1 + \cos\left(\frac{5x}{2}\right)}\right) - \log\left(\sqrt{1 - \cos\left(\frac{5x}{2}\right)}\right)$$

$$y = \log\left(1 + \cos\left(\frac{x}{2}\right)^{\frac{1}{2}}\right) - \log\left(1 - \cos\left(\frac{5x}{2}\right)\right)^{\frac{1}{2}}$$

$$y = \frac{1}{2} \log\left[1 + \cos\left(\frac{5x}{2}\right)\right] - \frac{1}{2} \log\left[\left(1 - \cos\left(\frac{5x}{2}\right)\right)\right]$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2} \frac{1}{1 + \cos\left(\frac{5x}{2}\right)} \frac{d}{dx}\left(1 + \cos\left(\frac{5x}{2}\right)\right) - \frac{1}{2} \times \frac{1}{1 - \cos\left(\frac{5x}{2}\right)} \frac{d}{dx}\left(1 - \cos\left(\frac{5x}{2}\right)\right)$$

$$= \frac{1}{2(1 + \cos\left(\frac{5x}{2}\right))} \left(-\sin\left(\frac{5x}{2}\right) \cdot \frac{5}{2} - \frac{1}{2(1 - \cos\left(\frac{5x}{2}\right))} \left(\sin\left(\frac{5x}{2}\right) \cdot \frac{5}{2}\right)\right)$$

$$= \frac{-5 \sin\left(\frac{5x}{2}\right)}{4(1 + \cos\left(\frac{5x}{2}\right))} - \frac{5 \sin\left(\frac{5x}{2}\right)}{4(1 - \cos\left(\frac{5x}{2}\right))}$$

$$= \frac{-5}{4} \sin\left(\frac{5x}{2}\right) \left[\frac{1}{1 + \cos\left(\frac{5x}{2}\right)} + \frac{1}{1 - \cos\left(\frac{5x}{2}\right)} \right]$$

$$= \frac{\frac{-5}{2} \sin\left(\frac{5x}{2}\right) [1 - \cos\left(\frac{5x}{2}\right) + 1 + \cos\left(\frac{5x}{2}\right)]}{[1 - \cos^2\left(\frac{5x}{2}\right)]}$$

$$= \frac{-5}{4} \sin\left(\frac{5x}{2}\right) \times \frac{2}{\sin^2\left(\frac{5x}{2}\right)} \quad \dots [\because 1 - \cos^2 x = \sin^2 x]$$

$$= \frac{-5}{4} \frac{1}{\sin\left(\frac{5x}{2}\right)}$$

$$- \frac{5}{2} \times \text{cosec} x$$

$$- \frac{5}{2} \text{cosec}\left(\frac{5x}{2}\right).$$

Exercise 1.1 | Q 3.15 | Page 12

Differentiate the following w.r.t.x: $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right) \\ &= \log\left(\sqrt{\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}}\right) \\ &= \log\left(\sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}}\right) \\ &= \log\left(\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}\right) \\ &= \log\left(\frac{1-\sin x}{\cos x}\right) \\ &= \log\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\ &= \log(\sec x - \tan x) \end{aligned}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\log(\sec x - \tan x)]$$

$$\begin{aligned}
&= \frac{1}{\sec x - \tan x} \cdot \frac{d}{dx} (\sec x - \tan x) \\
&= \frac{1}{\sec x - \tan x} \times (\sec x \tan x - \sec^2 x) \\
&= \frac{-\sec x (\sec x - \tan x)}{\sec x - \tan x} \\
&= -\sec x.
\end{aligned}$$

Exercise 1.1 | Q 3.16 | Page 12

Differentiate the following w.r.t.x: $\log \left[4^{2x} \left(\frac{x^2 + 5}{\sqrt{2x^3 - 4}} \right)^{\frac{3}{2}} \right]$

SOLUTION

$$\begin{aligned}
\text{Let } y &= \log \left[4^{2x} \left(\frac{x^2 + 5}{\sqrt{2x^3 - 4}} \right)^{\frac{3}{2}} \right] \\
&= \log 4^{2x} + \log \left(\frac{x^2 + 5}{\sqrt{2x^3 - 4}} \right)^{\frac{3}{2}} \\
&= 2x \log 4 + \frac{3}{2} \log \left(\frac{x^2 + 5}{\sqrt{2x^3 - 4}} \right) \\
&= 2x \log 4 + \frac{3}{2} [\log(x^2 + 5) - \log(2x^3 - 4)^{\frac{1}{2}}] \\
&= 2x \log 4 + \frac{3}{2} [\log(x^2 + 5) - \log(2x^3 - 4)]
\end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[2x \log 4 + \frac{3}{2} \log(x^2 + 5) - \frac{3}{4} \log(2x^3 - 4) \right] \\
&= (2 \log 4) \frac{d}{dx}(x) + \frac{3}{2} \frac{d}{dx} [\log(x^2 + 5)] - \frac{3}{4} \frac{d}{dx} [\log(2x^3 - 4)]
\end{aligned}$$

$$\begin{aligned}
&= (2 \log 4) \times 1 + \frac{3}{2} \times \frac{1}{x^2 + 5} \cdot \frac{d}{dx}(x^2 + 5) - \frac{3}{4} \times \frac{1}{2x^3 - 4} \cdot \frac{d}{dx}(2x^3 - 4) \\
&= 2 \log 4 + \frac{3}{2(x^2 + 5)} \times (2x + 0) - \frac{3}{4(2x^3 - 4)} \times (2 \times 3x^2 - 0) \\
&= 2 \log 4 + \frac{3x}{x^2 + 5} - \frac{9x^2}{2(2x^3 - 4)}.
\end{aligned}$$

Exercise 1.1 | Q 3.17 | Page 12

Differentiate the following w.r.t.x: $\log \left[\frac{e^{x^2}(5 - 4x)^{\frac{3}{2}}}{\sqrt[3]{7 - 6x}} \right]$

SOLUTION

$$\text{Let } y = \log \left[\frac{e^{x^2}(5 - 4x)^{\frac{3}{2}}}{\sqrt[3]{7 - 6x}} \right]$$

Using

$$\log(A \cdot B) = \log A + \log B$$

$$\begin{aligned}
y &= \log e^{x^2} + \log \left(\frac{(5 - 4x)^{\frac{3}{2}}}{\sqrt[3]{7 - 6x}} \right) \\
&= \log e^{x^2} + \log(5 - 4x)^{\frac{3}{2}} - \log(\sqrt[3]{7 - 6x}) \\
&= x^2 \log e + \frac{3}{2} \log(5 - 4x) - \log(7 - 6x)^{\frac{1}{3}} \\
&= x^2 + \frac{3}{2} \log(5 - 4x) - \frac{1}{3} \log(7 - 6x)
\end{aligned}$$

Now,

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} x^2 + \frac{3}{2} \frac{d}{dx} \log(5 - 4x) - \frac{1}{3} \frac{d}{dx} \log(7 - 6x) \\
&= 2x + \frac{3}{2} \frac{1}{5 - 4x} (-4) - \frac{1}{3} \frac{1}{(7 - 6x)} x(-6)
\end{aligned}$$

$$= 2x - \frac{6}{(5-4x)} + \frac{2}{(7-6x)}$$

$$2x - \frac{6}{5-4x} + \frac{2}{7-6x}.$$

Exercise 1.1 | Q 3.18 | Page 12

Differentiate the following w.r.t.x:

$$\log \left[\frac{a^{\cos x}}{(x^2 - 3)^3 \log x} \right]$$

SOLUTION

$$\text{Let } y = \log \left[\frac{a^{\cos x}}{(x^2 - 3)^3 \log x} \right]$$

$$= \log a^{\cos x} - \log(x^2 - 3)^3 - \log(\log x)$$

$$= (\cos x)(\log a) - 3\log(x^2 - 3) - \log(\log x)$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [(\cos x)(\log a) - 3\log(x^2 - 3) - \log(\log x)] \\ &= (\log a) \cdot \frac{d}{dx}(\cos x) - 3 \frac{d}{dx}[\log(x^2 - 3)] - \frac{d}{dx}[\log(\log x)] \\ &= (\log a)(-\sin x) - 3 \times \frac{1}{x^2 - 3} \cdot \frac{d}{dx}(x^2 - 3) - \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \\ &= -(\sin x)(\log a) - \frac{3}{x^2 - 3} \times (2x - 0) - \frac{1}{\log x} \times \frac{1}{x} \\ &= -(\sin x)(\log a) - \frac{6x}{x^2 - 3} - \frac{1}{x \log x}. \end{aligned}$$

Exercise 1.1 | Q 3.19 | Page 12

Differentiate the following w.r.t.x: $y = (25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

SOLUTION

$$\begin{aligned}y &= (25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)} \\&= 5^{2\log_5(\sec x)} - 4^{2\log_4(\tan x)} \\&= 5^{2\log_5(\sec^2 x)} - 4^{2\log_4(\tan^2 x)} \\&= \sec^2 x - \tan^2 x \quad \dots [\because a^{\log_a x} = x] \\&\therefore y = 1\end{aligned}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(1) = 0.$$

Exercise 1.1 | Q 3.2 | Page 12

Differentiate the following w.r.t.x: $\frac{(x^2 + 2)^4}{\sqrt{x^2 + 5}}$

SOLUTION

$$\text{Let } y = \frac{(x^2 + 2)^4}{\sqrt{x^2 + 5}}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{(x^2 + 2)^4}{\sqrt{x^2 + 5}} \right] \\ &= \frac{\sqrt{x^2 + 5} \cdot \frac{d}{dx} (x^2 + 2)^4 - (x^2 + 2)^4 \cdot \frac{d}{dx} (\sqrt{x^2 + 5})}{(\sqrt{x^2 + 5})^2} \\ &= \frac{\sqrt{x^2 + 5} \times 4(x^2 + 2)^3 \cdot \frac{d}{dx} (x^2 + 2) - (x^2 + 2)^4 \times \frac{1}{2\sqrt{x^2 + 5}} \cdot \frac{d}{dx} (x^2 + 5)}{x^2 + 5} \\ &= \frac{\sqrt{x^2 + 5} \times 4(x^2 + 2)^3 \cdot (2x + 0) - \frac{(x^2 + 2)^4}{2\sqrt{x^2 + 5}} \times (2x + 0)}{x^2 + 5} \\ &= \frac{8x(x^2 + 5)(x^2 + 2)^3 - x(x^2 + 2)^4}{(x^2 + 5)^{\frac{3}{2}}} \\ &= \frac{x(x^2 + 2)^3 [8(x^2 + 5) - (x^2 + 2)]}{(x^2 + 5)^{\frac{3}{2}}} \\ &= \frac{x(x^2 + 2)^3 (8x^2 + 40 - x^2 - 2)}{(x^2 + 5)^{\frac{3}{2}}} \\ &= \frac{x(x^2 + 2)^3 (7x^2 + 38)}{(x^2 + 5)^{\frac{3}{2}}}. \end{aligned}$$

Exercise 1.1 | Q 4.1 | Page 12

A table of values of f, g, f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
---	------	------	-------	-------

2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

If $r(x) = f[g(x)]$ find $r'(2)$.

SOLUTION

$$\begin{aligned}
 r(x) &= f[g(x)] \\
 \therefore r'(x) &= \frac{d}{dx} f[g(x)] \\
 &= f'[g(x)] \cdot \frac{d}{dx} [g(x)] \\
 &= f'[g(x)] \cdot [g'(x)] \\
 \therefore r'(2) &= f'[g(2)] \cdot g'(2) \\
 &= f'(6) \cdot g'(2) \quad \dots [\because g(x) = 6, \text{ when } x = 2] \\
 &= -4 \times 4 \quad \dots [\text{From the table}] \\
 &= -16.
 \end{aligned}$$

Exercise 1.1 | Q 4.2 | Page 12

A table of values of f , g , f' and g' is given :

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

If $R(x) = g[3 + f(x)]$ find $R'(4)$.

SOLUTION

$$\begin{aligned} R(x) &= g[3 + f(x)] \\ \therefore R'(x) &= \frac{d}{dx}\{g[3 + f(x)]\} \\ &= g'[3 + f(x)] \cdot \frac{d}{dx}[3 + f(x)] \\ &= g'[3 + f(x)] \cdot [0 + f'(x)] \\ &= g'[3 + f(x)] \cdot f'(x) \\ \therefore R'(4) &= g'[3 + f(4)] \cdot f'(4) \\ &= g'[3 + 3] \cdot f'(4) \quad \dots [\because f(x) = 3, \text{ when } x = 4] \\ &= g'(6) \cdot f'(4) \\ &= 7 \times 5 \quad \dots [\text{From the table}] \\ &= 35. \end{aligned}$$

Exercise 1.1 | Q 4.3 | Page 12

A table of values of f , g , f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

If $s(x) = f[9 - f(x)]$ find $s'(4)$.

SOLUTION

$$s(x) = f[9 - f(x)]$$

$$\therefore s'(x) = \frac{d}{dx} \{f[9 - f(x)]\}$$

$$= f'[9 - f(x)]. \frac{d}{dx} [9 - f(x)]$$

$$= f'[9 - f(x)].[0 - f'(x)]$$

$$= -f'[9 - f(x)].f'(x)$$

$$\therefore s'(4) = -f'[9 - f(4)].f'(4)$$

$$= -f'[9 - 3].f'(4) \quad \dots [\because f(x) = 3, \text{ when } x = 4]$$

$$= -f'(6).f'(4)$$

$$= -(-4)(5) \quad \dots [\text{From the table}]$$

$$= 20.$$

Exercise 1.1 | Q 4.4 | Page 12

A table of values of f , g , f' and g' is given :

x	f(x)	g(x)	f'(x)	g'(x)
2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

If $S(x) = g[g(x)]$ find $S'(6)$.

SOLUTION

$$S(x) = g[g(x)]$$

$$\begin{aligned}\therefore S'(x) &= \frac{d}{dx} g[g(x)] \\&= g'[g(x)] \cdot \frac{d}{dx}[g(x)] \\&= g'[g(x)] \cdot g'(x) \\ \therefore S'(6) &= g'[g(6)] \cdot g'(6) \\&= g'(2) \cdot g'(6) \quad \dots [\because g(x) = 2, \text{ when } x = 6] \\&= 4 \times 7 \quad \dots [\text{From the table}] \\&= 28.\end{aligned}$$

Exercise 1.1 | Q 5 | Page 12

Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$ and $y = f[g(x)]$, then $\left[\frac{dy}{dx} \right]_{x=2} = ?$

$$\begin{aligned}y &= f[g(x)] \\ \therefore \frac{dy}{dx} &= \frac{d}{dx} \{f[g(x)]\} \\&= f'[g(x)] \cdot \frac{d}{dx}[g(x)] \\&= f'[g(x)] \cdot g'(x) \\ \therefore \left[\frac{dy}{dx} \right]_{x=2} &= f'[g(2)] \cdot g'(2) \\&= f'(3) \cdot g'(2) \quad \dots [\because g(2) = 3] \\&= -1 \times 5 \quad \dots (\text{Given}) \\&= -5.\end{aligned}$$

Exercise 1.1 | Q 6 | Page 12

If $h(x) = \sqrt{4f(x) + 3g(x)}$, $f(1) = 4$, $g(1) = 3$, $f'(1) = 3$, $g'(1) = 4$, find $h'(1)$.

SOLUTION

Given : $f(1) = 4$, $g(1) = 3$, $f'(1) = 3$, $g'(1) = 4$... (1)

$$\text{Now, } h(x) = \sqrt{4f(x) + 3g(x)}$$

$$\therefore h'(x) = \frac{d}{dx} \left[\sqrt{4f(x) + 3g(x)} \right]$$

$$= \frac{1}{2\sqrt{4f(x) + 3g(x)}} \cdot \frac{d}{dx} [4f(x) + 3g(x)]$$

$$= \frac{1}{2\sqrt{4f(x) + 3g(x)}} \cdot [4f'(x) + 3g'(x)]$$

$$\therefore h'(1) = \frac{1}{2\sqrt{4f(1) + 3g(1)}} \cdot [4f'(1) + 3g'(1)]$$

$$= \frac{1}{2\sqrt{4 \times 4 + 3 \times 3}} \times [4 \times 3 + 3 \times 4] \quad \text{...[By (1)]}$$

$$= \frac{1}{2\sqrt{25}} \times 24$$

$$= \frac{1}{2 \times 5} \times 24$$

$$= \frac{12}{5}.$$

Exercise 1.1 | Q 7 | Page 12

Find the x co-ordinates of all the points on the curve $y = \sin 2x - 2 \sin x$, $0 \leq x < 2\pi$, where $dy/dx = 0$.

SOLUTION

$$y = \sin 2x - 2 \sin x, 0 \leq x < 2\pi$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin 2x - 2 \sin x)$$

$$= \frac{d}{dx}(\sin 2x) - 2 \frac{d}{dx}(\sin x)$$

$$= \cos 2x \cdot \frac{d}{dx}(2x) - 2 \cos x$$

$$= \cos^2 x \times 2 - 2 \cos x$$

$$= 2(2 \cos^2 x - 1) - 2 \cos x$$

$$= 4 \cos^2 x - 2 - 2 \cos x$$

$$= 4 \cos^2 x - 2 \cos x - 2$$

$$\text{If } \frac{dy}{dx} = 0, \text{ then } 4 \cos^2 x - 2 \cos x - 2 = 0$$

$$\therefore 4 \cos^2 x - 4 \cos x - 2 \cos x - 2 = 0$$

$$\therefore 4 \cos x (\cos x - 1) + 2 (\cos x - 1) = 0$$

$$\therefore (\cos x - 1)(4 \cos x + 2) =$$

$$\therefore \cos x - 1 = 0 \text{ or } 4 \cos x + 2 = 0$$

$$\therefore \cos x = 1 \text{ or } \cos x = -\frac{1}{2}$$

$$\therefore \cos x = \cos 0$$

or

$$\cos x = -\cos \frac{\pi}{3}$$

$$= \cos\left(\pi - \frac{\pi}{3}\right)$$

$$= \frac{\cos 2\pi}{3}$$

or

$$\cos x = -\cos \frac{\pi}{3}$$

$$\begin{aligned}
 &= \cos\left(\pi - \frac{\pi}{3}\right) \\
 &= \frac{\cos 4\pi}{3} \quad \dots [\because 0 \leq x < 2\pi] \\
 \therefore x &= 0 \text{ or } x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}. \\
 \therefore x &= 0 \text{ or } \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}.
 \end{aligned}$$

Exercise 1.1 | Q 8 | Page 13

Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]:

"Let $f(x) = x^2 + 5$ and $g(x) = e^x + 3$ then
 $f[g(x)] = \dots$ and $g[f(x)] = \dots$

Now $f'(x) = \dots$ and $g'(x) = \dots$

The derivative of $f[g(x)]$ w. r. t. x in terms of f and g is \dots

Therefore $\frac{d}{dx}[f[g(x)]] = \dots$ and

$$\left[\frac{d}{dx}[f[g(x)]] \right]_{x=0} = \dots$$

The derivative of $g[f(x)]$ w. r. t. x in terms of f and g is

Therefore $\frac{d}{dx}[g[f(x)]] = \dots$ and

$$\left[\frac{d}{dx}[g[f(x)]] \right]_{x=-1} = \dots$$

Hint basket : $\{f'[g(x)] \cdot g'(x), 2e^{2x} + 6e^x, 8, g'[f(x)] \cdot f'(x), 2xe^{x^2+5}, -2e^6, e^{2x} + 6e^x + 14, e^{x^2+5} + 3, 2x, e^x\}$

SOLUTION

$$f[g(x)] = e^{2x} + 6e^x + 14$$

$$g[f(x)] = e^{x^2+5} + 3$$

$$f'(x) = 2x, \quad g'f(x) = e^x$$

The derivative of $f[g(x)]$ w. r. t. x in terms of f and g is $f'[g(x)].g'(x)$.

$$\therefore \frac{d}{dx} \{f[g(x)]\} = 2e^{2x} + 6e^x \text{ and } \frac{d}{dx} \{f[g(x)]\}_{x=0} = 8$$

The derivative of $g[f(x)]$ w. r. t. x in terms of f and g is $g'[f(x)].f'(x)$.

$$\therefore \frac{d}{dx} \{g[f(x)]\} = 2e^{x^2+5} \text{ and}$$

$$\frac{d}{dx} \{g[f(x)]\}_{x=-1} = -2e^6.$$

EXERCISE 1.2 [PAGES 29 - 30]

Exercise 1.2 | Q 1.1 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = \sqrt{x}$

SOLUTION

$$y = \sqrt{y}$$

We have to find the inverse function of $y = f(x)$, i.e. x in terms of y .

From (1),

$$y^2 = x$$

$$\therefore x = y^2$$

$$\therefore x = f^{-1}(y) = y^2$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(y^2) = 2y$$

$$= 2\sqrt{x} \quad \dots[\text{By (1)}]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{2\sqrt{x}}.$$

Exercise 1.2 | Q 1.2 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = \sqrt{2 - \sqrt{x}}$

SOLUTION

$$y = \sqrt{2 - \sqrt{x}} \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$y^2 = 2 - \sqrt{x} \quad \therefore \sqrt{x} = 2 - y^2$$

$$\therefore x = (2 - y^2)^2$$

$$\therefore x = f^{-1}(y) = (2 - y^2)^2$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy} (2 - y^2)^2$$

$$= 2(2 - y^2) \cdot \frac{d}{dy} (2 - y^2)$$

$$= 2(2 - y^2)(0 - 2y)$$

$$= -4y(2 - y^2)$$

$$= -4\sqrt{2 - \sqrt{x}}(2 - 2 + \sqrt{x}) \quad \dots[\text{By (1)}]$$

$$= -4\sqrt{x}\sqrt{2 - \sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{4\sqrt{x}\sqrt{2 - \sqrt{x}}}.$$

Exercise 1.2 | Q 1.3 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = \sqrt[3]{x-2}$

SOLUTION

$$y = \sqrt[3]{x-2} \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$y^3 = x - 2$$

$$\therefore x = y^3 + 2$$

$$\therefore x = f^{-1}(y) = y^3 + 2$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy} (y^3 + 2)$$

$$= 3y^2 + 0 = 3y^2$$

$$= 3\left(\sqrt[3]{(x-2)}\right)^2 \quad \dots[\text{By (1)}]$$

$$= 3(x-2)^{\frac{2}{3}}$$

$$= 3 \cdot \left(\sqrt[3]{(x-2)^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{3\sqrt[3]{(x-2)^2}}, x > 2.$$

Exercise 1.2 | Q 1.4 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = \log(2x - 1)$

SOLUTION

$$y = \log(2x - 1) \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$2x - 1 = e^y$$

$$\therefore 2x = e^y + 1$$

$$\therefore x = f^{-1}(y)$$

$$= \frac{1}{2}(e^y + 1)$$

$$\therefore \frac{dx}{dy} = \frac{1}{2} \frac{d}{dy}(e^y + 1)$$

$$= \frac{1}{2}(e^y + 0)$$

$$= \frac{1}{2}e^y$$

$$= \frac{1}{2}e^{\log(2x-1)} \quad \dots[\text{By (1)}]$$

$$= \frac{1}{2}(2x - 1) \quad \dots[\because e^{\log x} = x]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{2}{2x - 1}.$$

Exercise 1.2 | Q 1.5 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = 2x + 3$

SOLUTION

$$y = 2x + 3 \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$2x = y - 3$$

$$\therefore x = \frac{y - 3}{2}$$

$$\therefore x = f^{-1}(y)$$

$$= \frac{y - 3}{2}$$

$$\therefore \frac{dx}{dy} = \frac{1}{2} \frac{d}{dy} (y - 3)$$

$$= \frac{1}{2} (1 - 0)$$

$$= \frac{1}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{\left(\frac{1}{2}\right)}$$

$$= 2.$$

Exercise 1.2 | Q 1.6 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = e^x - 3$

SOLUTION

$$y = e^x - 3 \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$e^x = y + 3$$

$$\therefore x = \log(y + 3)$$

$$\therefore x = f^{-1}(y) = \log(y + 3)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dy}[\log(y + 3)]$$

$$= \frac{1}{y+3} \cdot \frac{d}{dy}(y+3)$$

$$= \frac{1}{y+3} \cdot (1+0)$$

$$= \frac{1}{y+3}$$

$$= \frac{1}{e^x - 3 + 3} \quad \dots[\text{By (1)}]$$

$$= \frac{1}{e^x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{\left(\frac{1}{e^x}\right)}$$

$$= e^x.$$

Exercise 1.2 | Q 1.7 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y = e^{2x-3}$

SOLUTION

$$y = e^{2x-3} \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$2x - 3 = \log y$$

$$\therefore 2x = \log y + 3$$

$$\therefore x = f^{-1}(y)$$

$$= \frac{1}{2}(\log y + 3)$$

$$\therefore \frac{dx}{dy} = \frac{1}{2} \frac{d}{dy} (\log y + 3)$$

$$= \frac{1}{2} \left(\frac{1}{y} + 0 \right)$$

$$= \frac{1}{2y}$$

$$= \frac{1}{2e^{2x-3}} \quad \dots[\text{By (1)}]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy} \right)}$$

$$= \frac{1}{\left(\frac{1}{2e^{2x-3}} \right)}$$

$$= 2e^{2x-3}.$$

Exercise 1.2 | Q 1.8 | Page 29

Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following: $y =$

$$\log_2 \left(\frac{x}{2} \right)$$

SOLUTION

$$y = \log_2\left(\frac{x}{2}\right) \quad \dots(1)$$

We have to find the inverse function of $y = f(x)$, i.e x in terms of y .

From (1),

$$\frac{x}{2} = 2^y$$

$$\therefore x = 2 \cdot 2^y = 2^{y+1}$$

$$\therefore x = f^{-1}(y) = 2^{y+1}$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(2^{y+1})$$

$$= 2^{y+1} \cdot \log 2 \cdot \frac{d}{dy}(y+1)$$

$$= 2^{y+1} \cdot \log 2 \cdot (1 + 0)$$

$$= 2^{y+1} \cdot \log 2$$

$$= 2^{\log_2\left(\frac{x}{2}\right)+1} \cdot \log 2 \quad \dots[\text{By (1)}]$$

$$= 2^{\log_2\left(\frac{x}{2}\right)+\log_2 2} \cdot \log 2$$

$$= 2^{\log_2\left(\frac{x}{2} \times 2\right)} \cdot \log 2$$

$$= 2^{\log_2 x} \cdot \log 2$$

$$= x \log 2 \quad \dots[\because a^{\log_a x} = x]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{x \log 2}.$$

Find the derivative of the inverse function of the following : $y = x^2 \cdot e^x$

SOLUTION

$$y = x^2 \cdot e^x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2 \cdot e^x) \\ &= x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2) \\ &= x^2 \cdot e^x + e^x \cdot 2x \\ &= xe^x(x + 2)\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dx}{dy} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\ &= \frac{1}{xe^x(x + 2)}.\end{aligned}$$

Exercise 1.2 | Q 2.2 | Page 29

Find the derivative of the inverse function of the following : $y = x \cos x$

SOLUTION

$$y = x \cos x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x \cos x) \\ &= x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x) \\ &= x(-\sin x) + \cos x \cdot 1 \\ &= \cos x - x \sin x\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dx}{dy} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\ &= \frac{1}{\cos x - x \sin x}.\end{aligned}$$

Exercise 1.2 | Q 2.3 | Page 29

Find the derivative of the inverse function of the following : $y = x \cdot 7^x$

SOLUTION

$$y = x \cdot 7^x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x \cdot 7^x) \\ &= x \frac{d}{dx}(7^x) + 7^x \frac{d}{dx}(x) \\ &= x \cdot 7^x \log 7 + 7^x \times 1 \\ &= 7^x(x \log 7 + 1)\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\ &= \frac{1}{7^x(x \log 7 + 1)}.\end{aligned}$$

Exercise 1.2 | Q 2.4 | Page 29

Find the derivative of the inverse function of the following : $y = x^2 + \log x$

SOLUTION

$$y = x^2 + \log x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2 + \log x) \\ &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\log x) \\ &= 2x + \frac{1}{x} \\ &= \frac{2x^2 + 1}{x}\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dx}{dy} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\ &= \frac{1}{\left(\frac{2x^2+1}{x}\right)} \\ &= \frac{x}{2x^2 + 1}.\end{aligned}$$

Exercise 1.2 | Q 2.5 | Page 29

Find the derivative of the inverse function of the following : $y = x \log x$

SOLUTION

$$y = x \log x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x \log x) \\&= x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \\&= x \times \frac{1}{x} + (\log x) \times 1 \\&= 1 + \log x\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dx}{dy} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\&= \frac{1}{1 + \log x}.\end{aligned}$$

Exercise 1.2 | Q 3.1 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. $y = x^5 + 2x^3 + 3x$, at $x = 1$

SOLUTION

$$y = x^5 + 2x^3 + 3x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^5 + 2x^3 + 3x)$$

$$= 5x^4 + 2 \times 3x^2 + 3 \times 1$$

$$= 5x^4 + 6x^2 + 3$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{5x^4 + 6x^2 + 3}$$

$$\text{At } x = 1, \frac{dx}{dy}$$

$$= \frac{1}{(5x^4 + 6x^2 + 3)_{at \ x=1}}$$

$$= \frac{1}{5(1)^4 + 6(1)^2 + 3}$$

$$= \frac{1}{5 + 6 + 3}$$

$$= \frac{1}{14}.$$

Exercise 1.2 | Q 3.2 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. $y = e^x + 3x + 2$

SOLUTION

$$y = e^x + 3x + 2$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}(e^x + 3x + 2)$$

$$= e^x + 3 \times 1 + 0$$

$$= e^x + 3$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{e^x + 3}$$

$$\text{At } x = 0, \frac{dx}{dy}$$

$$= \frac{1}{(e^x + 3)_{atx=0}}$$

$$= \frac{1}{e^0 + 3}$$

$$= \frac{1}{1 + 3}$$

$$= \frac{1}{4}.$$

Exercise 1.2 | Q 3.3 | Page 29

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. $y = 3x^2 + 2\log x^3$

SOLUTION

$$y = 3x^2 + 2\log x^3$$

$$= 3x^2 + 6\log x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} (3x^2 + 6 \log x)$$

$$= 3 \times 2x + 6 \times \frac{1}{x}$$

$$= 6x + \frac{6}{x}$$

$$= \frac{6x^2 + 6}{x}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{\left(\frac{6x^2+6}{x}\right)}$$

$$= \frac{x}{6x^2 + 6}$$

At $x = 1$, $\frac{dx}{dy}$

$$= \left(\frac{x}{6x^2 + 6} \right)_{at \ x=1}$$

$$= \frac{1}{6(1)^2 + 6}$$

$$= \frac{1}{12}.$$

Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them. $y = \sin(x - 2) + x^2$

SOLUTION

$$y = \sin(x - 2) + x^2$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\sin(x - 2) + x^2] \\ &= \frac{d}{dx} [\sin(x - 2)] + \frac{d}{dx} (x^2) \\ &= \cos(x - 2) \cdot \frac{d}{dx}(x - 2) + 2x \\ &= \cos(x - 2) \cdot (1 - 0) + 2x \\ &= \cos(x - 2) + 2x\end{aligned}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\begin{aligned}\frac{dx}{dy} &= \frac{1}{\left(\frac{dy}{dx}\right)} \\ &= \frac{1}{\cos(x - 2) + 2x}\end{aligned}$$

$$\text{At } x = 2, \frac{dx}{dy}$$

$$\begin{aligned}&= \left(\frac{1}{[\cos(x - 2) + 2x]} \right)_{\text{at } x=2} \\ &= \frac{1}{\cos(2 - 2) + 2 \cdot 2} \\ &= \frac{1}{1 + 4} \\ &= \frac{1}{5}.\end{aligned}$$

Exercise 1.2 | Q 4 | Page 29

If $f(x) = x^3 + x - 2$, find $(f^{-1})'(-2)$.

SOLUTION

$$f(x) = x^3 + x - 2 \quad \dots(1)$$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(x^3 + x - 2)$$

$$= 3x^2 + 1 = 0$$

$$= 3x^2 + 1$$

We know that

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \dots(2)$$

From (1), $y = f(x) = -2$, when $x = 0$

\therefore from (2), $(f^{-1})'(-2)$

$$= \frac{1}{f'(0)}$$

$$= \frac{1}{(3x^2 + 1)_{at x=0}}$$

$$= \frac{1}{3(0) + 1}$$

$$= 1.$$

Exercise 1.2 | Q 5.1 | Page 29

Using derivative, prove that: $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

SOLUTION

$$\text{Let } f(x) = \tan^{-1}x + \cot^{-1}x \quad \dots(1)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}f'(x) &= \frac{d}{dx} (\tan^{-1}x + \cot^{-1}x) \\&= \frac{d}{dx} (\tan^{-1}x) + \frac{d}{dx} (\cot^{-1}x) \\&= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\&= 0\end{aligned}$$

Since $f'(x) = 0$, $f(x)$ is a constant function.

Let $f(x) = k$.

For any value of x, $f(x) = k$

Let $x = 0$.

$$\text{Then } f(0) = k \quad \dots(2)$$

$$\text{From (1), } f(0) = \tan^{-1}(0) + \cot^{-1}(0)$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore k = \frac{\pi}{2} \quad \dots[\text{By (2)}]$$

$$\therefore f(x) = k = \frac{\pi}{2}$$

$$\text{Hence, } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}. \quad \dots[\text{By (1)}]$$

Exercise 1.2 | Q 5.2 | Page 29

$$\text{Using derivative, prove that: } \sec^{-1}x + \cosec^{-1}x = \frac{\pi}{2} \quad \dots[\text{for } |x| \geq 1]$$

SOLUTION

Let $f(x) = \sec^{-1}x + \operatorname{cosec}^{-1}x$ for $|x| \geq 1$...(1)

Differentiating w.r.t. x, we get

$$\begin{aligned}f'(x) &= \frac{d}{dx} (\sec^{-1} x + \operatorname{cosec}^{-1} x) \\&= \frac{d}{dx} (\sec^{-1} x) + \frac{d}{dx} (\operatorname{cosec}^{-1} x) \\&= \frac{1}{x\sqrt{x^2 - 1}} - \frac{1}{x\sqrt{x^2 - 1}} \\&= 0\end{aligned}$$

Since, $f'(x) = 0$, $f(x)$ is a constant function.

Let $f(x) = k$.

For any value of x , $f(x) = k$, where $|x| > 1$

Let $x = 2$.

Then, $f(2) = k$... (2)

From (1), $f(2) = \sec^{-1}(2) + \operatorname{cosec}^{-1}(2)$

$$= \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\therefore k = \frac{\pi}{2} \quad \dots[\text{By (2)}]$$

$$\therefore f(x) = k = \frac{\pi}{2}$$

$$\text{Hence, } \sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}. \quad \dots[\text{By (1)}]$$

Exercise 1.2 | Q 6.01 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}(\log x)$

SOLUTION

Let $y = \tan^{-1}(\log x)$

Differentiating w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(\log x)] \\ &= \frac{1}{1 + (\log x)^2} \cdot \frac{d}{dx} (\log x) \\ &= \frac{1}{1 + (\log x)^2} \times \frac{1}{x} \\ &= \frac{1}{x[1 + (\log x)^2]}.\end{aligned}$$

Exercise 1.2 | Q 6.02 | Page 29

Differentiate the following w.r.t. x : $\text{cosec}^{-1}(e^{-x})$

SOLUTION

Let $y = \text{cosec}^{-1}(e^{-x})$

Differentiating w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\text{cosec}^{-1}(e^{-x})] \\ &= \frac{-1}{e^{-x}\sqrt{(e^{-x})^2 - 1}} \cdot \frac{d}{dx}(e^{-x}) \\ &= \frac{1}{e^{-x}\sqrt{e^{-2}x - 1}} \times e^{-x} \cdot \frac{d}{dx}(-x) \\ &= \frac{-1}{\sqrt{e^{-2}x - 1}} \times -1 \\ &= \frac{1}{\sqrt{\frac{1}{e^{2x}} - 1}}\end{aligned}$$

$$= \frac{e^x}{\sqrt{1 - e^{2x}}}.$$

Exercise 1.2 | Q 6.03 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(x^3)$

SOLUTION

Let $y = \cot^{-1}(x^3)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\cot^{-1}(x^3)]$$

$$= \frac{-1}{1 + (x^3)^2} \cdot \frac{d}{dx} (x^3)$$

$$= \frac{-1}{1 + x^6} \times 3x^2$$

$$= \frac{-3x^2}{1 + x^6}.$$

Exercise 1.2 | Q 6.04 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(4^x)$

SOLUTION

Let $y = \cot^{-1}(4^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\cot^{-1}(4^x)]$$

$$= \frac{-1}{1 + (4^x)^2} \cdot \frac{d}{dx} (4^x)$$

$$= \frac{-1}{1 + 4^{2x}} \times 4^x \log 4$$

$$= \frac{4^x \log 4}{1 + 4^{2x}}.$$

Exercise 1.2 | Q 6.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}(\sqrt{x})$

SOLUTION

Let $y = \tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\tan^{-1}(\sqrt{x})]$$

$$= \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{1 + x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1 + x)}.$$

Exercise 1.2 | Q 6.06 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

SOLUTION

$$\text{Let } y = \sin^{-1} \left(\sqrt{\frac{1+x^2}{2}} \right)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\sin^{-1} \left(\sqrt{\frac{1+x^2}{2}} \right) \right] \\&= \frac{1}{\sqrt{1 - \left(\sqrt{\frac{1+x^2}{2}} \right)^2}} \cdot \frac{d}{dx} \left(\sqrt{\frac{1+x^2}{2}} \right) \\&= \frac{1}{\sqrt{\left(1 - \frac{1+x^2}{2} \right)}} \times \frac{1}{\sqrt{2}} \frac{d}{dx} \left(\sqrt{1+x^2} \right) \\&= \frac{\sqrt{2}}{\sqrt{2-1-x^2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx} (1+x^2) \\&= \frac{1}{\sqrt{1-x^2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot (0+2x) \\&= \frac{x}{\sqrt{(1-x^2)(1+x^2)}} \\&= \frac{x}{\sqrt{1-x^4}}.\end{aligned}$$

Exercise 1.2 | Q 6.07 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}(1-x^2)$

SOLUTION

Let $y = \cos^{-1}(1 - x^2)$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} [\cos^{-1}(1 - x^2)] \\ &= \frac{-1}{\sqrt{1 - (1 - x^2)^2}} \cdot \frac{d}{dx} (1 - x^2) \\ &= \frac{-1}{\sqrt{1 - (1 - 2x^2 + x^4)}} \cdot (0 - 2x) \\ &= \frac{2x}{\sqrt{2x^2 - x^4}} \\ &= \frac{2x}{x\sqrt{2 - x^2}} \\ &= \frac{2}{\sqrt{2 - x^2}}.\end{aligned}$$

Exercise 1.2 | Q 6.08 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

SOLUTION

$$\text{Let } y = \sin^{-1}\left(x^{\frac{3}{2}}\right)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\sin^{-1}\left(x^{\frac{3}{2}}\right) \right] \\ &= \frac{1}{\sqrt{1 - \left(x^{\frac{3}{2}}\right)^2}} \cdot \frac{d}{dx} \left(x^{\frac{3}{2}}\right) \\ &= \frac{1}{\sqrt{1 - x^3}} \times \frac{3}{2} x^{\frac{1}{2}} \\ &= \frac{3\sqrt{x}}{2\sqrt{1 - x^3}}.\end{aligned}$$

Exercise 1.2 | Q 6.09 | Page 29

Differentiate the following w.r.t. x : $\cos^3[\cos^{-1}(x^3)]$

SOLUTION

$$\text{Let } y = \cos^3[\cos^{-1}(x^3)]$$

$$= [\cos(\cos^{-1}x^3)]^3$$

$$= (x^3)^3$$

$$= x^9$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (x^9) \\ &= 9x^8.\end{aligned}$$

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4[\sin^{-1}(\sqrt{x})]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \sin^4 [\sin^{-1}(\sqrt{x})] \\ &= \{\sin[\sin^{-1}(\sqrt{x})]\}^4 \\ &= (\sqrt{x})^4 \\ &= x^2 \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2) \\ &= 2x. \end{aligned}$$

Exercise 1.2 | Q 6.02 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}(e^{-x})$

SOLUTION

$$\text{Let } y = \operatorname{cosec}^{-1}(e^{-x})$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\operatorname{cosec}^{-1}(e^{-x})] \\ &= \frac{-1}{e^{-x}\sqrt{(e^{-x})^2 - 1}} \cdot \frac{d}{dx}(e^{-x}) \\ &= \frac{1}{e^{-x}\sqrt{e^{-2x}-1}} \times e^{-x} \cdot \frac{d}{dx}(-x) \\ &= \frac{-1}{\sqrt{e^{-2x}-1}} \times -1 \\ &= \frac{1}{\sqrt{\frac{1}{e^{2x}} - 1}} \\ &= \frac{e^x}{\sqrt{1-e^{2x}}}. \end{aligned}$$

Exercise 1.2 | Q 6.03 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(x^3)$

SOLUTION

$$\text{Let } y = \cot^{-1}(x^3)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\cot^{-1}(x^3)]$$

$$= \frac{-1}{1 + (x^3)^2} \cdot \frac{d}{dx} (x^3)$$

$$= \frac{-1}{1 + x^6} \times 3x^2$$

$$= \frac{-3x^2}{1 + x^6}.$$

Exercise 1.2 | Q 6.04 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}(4^x)$

SOLUTION

$$\text{Let } y = \cot^{-1}(4^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\cot^{-1}(4^x)]$$

$$= \frac{-1}{1 + (4^x)^2} \cdot \frac{d}{dx} (4^x)$$

$$= \frac{-1}{1 + 4^{2x}} \times 4^x \log 4$$

$$= \frac{4^x \log 4}{1 + 4^{2x}}.$$

Exercise 1.2 | Q 6.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1}(\sqrt{x})$

SOLUTION

$$\text{Let } y = \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\tan^{-1}(\sqrt{x})]$$

$$= \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{1 + x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1 + x)}.$$

Exercise 1.2 | Q 6.06 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

SOLUTION

$$\text{Let } y = \sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right) \right]$$

$$= \frac{1}{\sqrt{1 - \left(\sqrt{\frac{1+x^2}{2}}\right)^2}} \cdot \frac{d}{dx} \left(\sqrt{\frac{1+x^2}{2}}\right)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\left(1 - \frac{1+x^2}{2}\right)}} \times \frac{1}{\sqrt{2}} \frac{d}{dx} \left(\sqrt{1+x^2}\right) \\
&= \frac{\sqrt{2}}{\sqrt{2}-1-x^2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx} (1+x^2) \\
&= \frac{1}{\sqrt{1-x^2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot (0+2x) \\
&= \frac{x}{\sqrt{(1-x^2)(1+x^2)}}
\end{aligned}$$

Exercise 1.2 | Q 6.07 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1}(1-x^2)$

SOLUTION

$$\text{Let } y = \cos^{-1}(1-x^2)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
&\frac{dy}{dx} [\cos^{-1}(1-x^2)] \\
&= \frac{-1}{\sqrt{1-(1-x^2)^2}} \cdot \frac{d}{dx} (1-x^2) \\
&= \frac{-1}{\sqrt{1-(1-2x^2+x^4)}} \cdot (0-2x) \\
&= \frac{2x}{\sqrt{2x^2-x^4}} \\
&= \frac{2x}{x\sqrt{2-x^2}} \\
&= \frac{2}{\sqrt{2-x^2}}
\end{aligned}$$

Exercise 1.2 | Q 6.08 | Page 29

Differentiate the following w.r.t. x : $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

SOLUTION

$$\text{Let } y = \sin^{-1}\left(x^{\frac{3}{2}}\right)$$

Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\sin^{-1}\left(x^{\frac{3}{2}}\right) \right] \\ &= \frac{1}{\sqrt{1 - \left(x^{\frac{3}{2}}\right)^2}} \cdot \frac{d}{dx} \left(x^{\frac{3}{2}}\right) \\ &= \frac{1}{\sqrt{1 - x^3}} \times \frac{3}{2} x^{\frac{1}{2}} \\ &= \frac{3\sqrt{x}}{2\sqrt{1 - x^3}}. \end{aligned}$$

Exercise 1.2 | Q 6.09 | Page 29

Differentiate the following w.r.t. x : $\cos^3[\cos^{-1}(x^3)]$

SOLUTION

$$\text{Let } y = \cos^3[\cos^{-1}(x^3)]$$

$$= [\cos(\cos^{-1}x^3)]^3$$

$$= (x^3)^3$$

$$= x^9$$

Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (x^9) \\ &= 9x^8. \end{aligned}$$

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4[\sin^{-1}(\sqrt{x})]$

Exercise 1.2 | Q 6.1 | Page 29

Differentiate the following w.r.t. x : $\sin^4[\sin^{-1}(\sqrt{x})]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \sin^4[\sin^{-1}(\sqrt{x})] \\ &= \{\sin[\sin^{-1}(\sqrt{x})]\}^4 \\ &= (\sqrt{x})^4 \\ &= x^2 \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2) \\ &= 2x. \end{aligned}$$

Exercise 1.2 | Q 7.01 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1}[\cot(e^{x^2})]$

SOLUTION

$$\text{Let } y = \cot^{-1}[\cot(e^{x^2})] = e^{x^2}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(e^{x^2}) \\ &= e^{x^2} \cdot \frac{d}{dx}(x^2) \\ &= e^{x^2} \times 2x \\ &= 2xe^2. \end{aligned}$$

Exercise 1.2 | Q 7.02 | Page 29

Differentiate the following w.r.t. x : $\operatorname{cosec}^{-1}\left[\frac{1}{\cos(5x)}\right]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \operatorname{cosec}^{-1} \left[\frac{1}{\cos(5^x)} \right] \\ &= \operatorname{cosec}^{-1} [\sec(5^x)] \\ &= \operatorname{cosec}^{-1} \left[\operatorname{cosec} \left(\frac{\pi}{2} - 5^x \right) \right] \\ &= \frac{\pi}{2} - 5^x \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - 5^x \right) \\ &= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{d}{dx} (5^x) \\ &= 0 - 5^x \cdot \log 5 \\ &= -5^x \cdot \log 5. \end{aligned}$$

Exercise 1.2 | Q 7.03 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1} \left(\sqrt{\frac{1 + \cos x}{2}} \right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cos^{-1} \left(\sqrt{\frac{1 + \cos x}{2}} \right) \\ &= \cos^{-1} \left(\sqrt{\frac{2 \cos^2 \left(\frac{x}{2} \right)}{2}} \right) \\ &= \cos^{-1} \left[\cos \left(\frac{x}{2} \right) \right] \\ &= \frac{x}{2} \end{aligned}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{2} \right)$$

$$= \frac{1}{2} \frac{d}{dx}(x)$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}.$$

Exercise 1.2 | Q 7.04 | Page 29

Differentiate the following w.r.t. x : $\cos^{-1} \left(\frac{\sqrt{1 - \cos(x^2)}}{2} \right)$

SOLUTION

$$\text{Let } y = \cos^{-1} \left(\frac{\sqrt{1 - \cos(x^2)}}{2} \right)$$

$$= \cos^{-1} \left(\sqrt{\frac{2 \sin^2 \left(\frac{x^2}{2} \right)}{2}} \right)$$

$$= \cos^{-1} \left[\sin \left(\frac{x^2}{2} \right) \right]$$

$$= \cos^{-1} \left[\cos \left(\frac{\pi}{2} - \frac{x^2}{2} \right) \right]$$

$$= \frac{\pi}{2} - \frac{x^2}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - \frac{x^2}{2} \right)$$

$$= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{1}{2} \frac{d}{dx} (x^2)$$

$$= 0 - \frac{1}{2} \times 2x$$

$$= -x.$$

Exercise 1.2 | Q 7.05 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1} \left[\frac{1 - \tan(\frac{x}{2})}{1 + \tan(\frac{x}{2})} \right]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \tan^{-1} \left[\frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)} \right] \\ &= \tan^{-1} \left[\frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right) \cdot \tan\left(\frac{x}{2}\right)} \right] \dots \left[\because \tan \frac{\pi}{4} = 1 \right] \\ &= \tan^{-1} \left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \right] \\ &= \frac{\pi}{4} - \frac{x}{2} \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{4} - \frac{x}{2} \right) \\ &= \frac{d}{dx} \left(\frac{\pi}{4} \right) - \frac{1}{2} \frac{d}{dx}(x) \\ &= 0 - \frac{1}{2} \times 1 \\ &= -\frac{1}{2}. \end{aligned}$$

Exercise 1.2 | Q 7.06 | Page 29

Differentiate the following w.r.t. x : $\cosec^{-1} \left(\frac{1}{4 \cos^3 2x - 3 \cos 2x} \right)$

$$\begin{aligned}
\text{Let } y &= \operatorname{cosec}^{-1} \left(\frac{1}{4 \cos^3 2x - 3 \cos 2x} \right) \\
&= \operatorname{cosec}^{-1} \left(\frac{1}{\cos 6x} \right) \quad \dots [\because \cos 3x = 4\cos^3 x - 3\cos x] \\
&= \operatorname{cosec}^{-1}(\sec 6x) \\
&= \operatorname{cosec}^{-1} \left[\operatorname{cosec} \left(\frac{\pi}{2} - 6x \right) \right] \\
&= \frac{\pi}{2} - 6x
\end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - 6x \right) \\
&= \frac{d}{dx} \left(\frac{\pi}{2} \right) - 6 \frac{d}{dx} (x) \\
&= 0 - 6 \times 1 \\
&= -6.
\end{aligned}$$

Exercise 1.2 | Q 7.07 | Page 29

Differentiate the following w.r.t. x : $\tan^{-1} \left[\frac{1 + \cos \left(\frac{x}{3} \right)}{\sin \left(\frac{x}{3} \right)} \right]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \tan^{-1} \left[\frac{1 + \cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)} \right] \\ &= \tan^{-1} \left[\frac{2 \cos^2\left(\frac{x}{6}\right)}{2 \sin\left(\frac{x}{6}\right) \cos\left(\frac{x}{6}\right)} \right] \\ &= \tan^{-1} \left[\cot\left(\frac{x}{6}\right) \right] \\ &= \tan^{-1} \left[\tan\left(\frac{\pi}{2} - \frac{x}{6}\right) \right] \\ &= \frac{\pi}{2} - \frac{x}{6} \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - \frac{x}{6} \right) \\ &= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{1}{6} \frac{d}{dx}(x) \\ &= 0 - \frac{1}{6} \times 1 \\ &= -\frac{1}{6}. \end{aligned}$$

Exercise 1.2 | Q 7.08 | Page 29

Differentiate the following w.r.t. x : $\cot^{-1} \left(\frac{\sin 3x}{1 + \cos 3x} \right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cot^{-1} \left(\frac{\sin 3x}{1 + \cos 3x} \right) \\ &= \cot^{-1} \left[\frac{2 \sin \left(\frac{3x}{2} \right) \cos \left(\frac{3x}{2} \right)}{2 \cos^2 \left(\frac{3x}{2} \right)} \right] \\ &= \cot^{-1} \left[\tan \left(\frac{3x}{2} \right) \right] \\ &= \cot^{-1} \left[\cot \left(\frac{\pi}{2} - \frac{3x}{2} \right) \right] \\ &= \frac{\pi}{2} - \frac{3x}{2} \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - \frac{3x}{2} \right) \\ &= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{3}{2} \frac{dx}{dx} \\ &= 0 - \frac{3}{2} \times 1 \\ &= -\frac{3}{2}. \end{aligned}$$

Exercise 1.2 | Q 7.09 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1} \left(\frac{\cos 7x}{1 + \sin 7x} \right)$

SOLUTION

$$\begin{aligned}
\text{Let } y &= \tan^{-1} \left(\frac{\cos 7x}{1 + \sin 7x} \right) \\
&= \tan^{-1} \left[\frac{\sin \left(\frac{\pi}{2} - 7x \right)}{1 + \cos \left(\frac{\pi}{2} - 7x \right)} \right] \\
&= \tan^{-1} \left[\frac{2 \sin \left(\frac{\pi}{4} - \frac{7x}{2} \right) \cdot \cos \left(\frac{\pi}{4} - \frac{7x}{2} \right)}{2 \cos^2 \left(\frac{\pi}{4} - \frac{7x}{2} \right)} \right] \\
&= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{7x}{2} \right) \right] \\
&= \frac{\pi}{4} - \frac{7x}{2}
\end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{4} - \frac{7x}{2} \right) \\
&= \frac{d}{dx} \left(\frac{\pi}{4} \right) - \frac{7}{2} \frac{d}{dx}(x) \\
&= 0 - \frac{7}{2} \times 1 \\
&= -\frac{7}{2}.
\end{aligned}$$

Exercise 1.2 | Q 7.1 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1} \left(\sqrt{\frac{1 + \cos x}{1 - \cos x}} \right)$

SOLUTION

$$\text{Let } y = \tan^{-1} \left(\sqrt{\frac{1 + \cos x}{1 - \cos x}} \right)$$

$$= \tan^{-1} \left[\sqrt{\frac{2 \cos^2 \left(\frac{x}{2} \right)}{2 \sin^2 \left(\frac{x}{2} \right)}} \right]$$

$$= \tan^{-1} \left[\cot \left(\frac{x}{2} \right) \right]$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{2} - \frac{x}{2} \right) \right]$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

Differentiate the following w.r.t. x :

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - \frac{x}{2} \right)$$

$$= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{1}{2} \frac{d}{dx} (x)$$

$$= 0 - \frac{1}{2} \times 1$$

$$= -\frac{1}{2}.$$

Exercise 1.2 | Q 7.11 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1} (\cosec x + \cot x)$

SOLUTION

$$\text{Let } y = \tan^{-1} (\csc x + \cot x)$$

$$\begin{aligned}&= \tan^{-1} \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x} \right) \\&= \tan^{-1} \left(\frac{1 + \cos x}{\sin x} \right) \\&= \tan^{-1} \left[\frac{2 \cos^2 \left(\frac{x}{2} \right)}{2 \sin \left(\frac{x}{2} \right) \cdot 2 \cos \left(\frac{x}{2} \right)} \right] \\&= \tan^{-1} \left[\cot \left(\frac{x}{2} \right) \right] \\&= \tan^{-1} \left[\tan \left(\frac{\pi}{2} - \frac{x}{2} \right) \right] \\&= \frac{\pi}{2} - \frac{x}{2}\end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - \frac{x}{2} \right) \\&= \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{1}{2} \frac{d}{dx}(x) \\&= 0 - \frac{1}{2} \times 1 \\&= -\frac{1}{2}.\end{aligned}$$

Exercise 1.2 | Q 7.12 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1} \left[\frac{\sqrt{1 + \sin \left(\frac{4x}{3} \right)} + \sqrt{1 - \sin \left(\frac{4x}{3} \right)}}{\sqrt{1 + \sin \left(\frac{4x}{3} \right)} - \sqrt{1 - \sin \left(\frac{4x}{3} \right)}} \right]$

SOLUTION

$$\text{Let } y = \cot^{-1} \left[\frac{\sqrt{1 + \sin \left(\frac{4x}{3} \right)} + \sqrt{1 - \sin \left(\frac{4x}{3} \right)}}{\sqrt{1 + \sin \left(\frac{4x}{3} \right)} - \sqrt{1 - \sin \left(\frac{4x}{3} \right)}} \right]$$

$$= 1 + \sin \left(\frac{4x}{3} \right)$$

$$= 1 + \cos \left(\frac{\pi}{2} - \frac{4x}{3} \right)$$

$$= 2 \cos^2 \left(\frac{\pi}{4} - \frac{2x}{3} \right)$$

$$\therefore \sqrt{1 + \sin \left(\frac{4x}{3} \right)} = \sqrt{2} \cos \left(\frac{\pi}{4} - \frac{2x}{3} \right)$$

$$\text{Also, } 1 - \sin \left(\frac{4x}{3} \right)$$

$$= 1 - \cos \left(\frac{\pi}{2} - \frac{4x}{3} \right)$$

$$= 2 \sin^2 \left(\frac{\pi}{4} - \frac{2x}{3} \right)$$

$$= 2 \sin^2 \left(\frac{\pi}{4} - \frac{2x}{3} \right)$$

$$\therefore \sqrt{1 - \sin \left(\frac{4x}{3} \right)} = \sqrt{2} \sin \left(\frac{\pi}{4} - \frac{2x}{3} \right)$$

$$\therefore \underline{\sqrt{1 + \sin \left(\frac{4x}{3} \right)} + \sqrt{1 - \sin \left(\frac{4x}{3} \right)}}$$

$$\begin{aligned}
& \dots \frac{\sqrt{1 + \sin\left(\frac{4x}{3}\right)} - \sqrt{1 - \sin\left(\frac{4x}{3}\right)}}{\sqrt{2} \cos\left(\frac{\pi}{4} - \frac{2x}{3}\right) + \sqrt{2} \sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)} \\
&= \frac{\sqrt{2} \cos\left(\frac{\pi}{4} - \frac{2x}{3}\right) + \sqrt{2} \sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)}{\sqrt{2} \cos\left(\frac{\pi}{4} - \frac{2x}{3}\right) - \sqrt{2} \sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)} \\
&= \frac{\cos\left(\frac{\pi}{4} - \frac{2x}{3}\right) + \sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)}{\cos\left(\frac{\pi}{4} - \frac{2x}{3}\right) - \sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)} \\
&= \frac{1 + \tan\left(\frac{\pi}{4} - \frac{2x}{3}\right)}{1 - \tan\left(\frac{\pi}{4} - \frac{2x}{3}\right)} \dots [\text{Dividing by } \cos\left(\frac{\pi}{4} - \frac{2x}{3}\right)] \\
&= \frac{\tan \frac{\pi}{4} + \tan\left(\frac{\pi}{4} - \frac{2x}{3}\right)}{1 - \tan \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4} - \frac{2x}{3}\right)} \dots [\because \tan \frac{\pi}{4} = 1] \\
&= \tan\left[\frac{\pi}{4} + \frac{\pi}{4} - \frac{2x}{3}\right] \\
&= \tan\left(\frac{\pi}{2} - \frac{2x}{3}\right) \\
&= \cot\left(\frac{2x}{3}\right) \\
&\therefore y = \cot^{-1}\left[\cot\left(\frac{2x}{3}\right)\right] = \frac{2x}{3}
\end{aligned}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{3}\right)$$

$$= \frac{2}{3} \frac{d}{dx}(x)$$

$$= \frac{2}{3} \times 1$$

$$= \frac{2}{3}.$$

Exercise 1.2 | Q 8.1 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{4 \sin x + 5 \cos x}{\sqrt{41}}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \sin^{-1}\left(\frac{4 \sin x + 5 \cos x}{\sqrt{41}}\right) \\ &= \sin^{-1}\left[\left(\sin x\right)\left(\frac{4}{\sqrt{41}}\right) + \left(\cos x\right)\left(\frac{5}{\sqrt{41}}\right)\right] \end{aligned}$$

$$\text{Since, } \left(\frac{4}{\sqrt{41}}\right)^2 + \left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41} + \frac{25}{41} = 1,$$

we can write, $\frac{4}{\sqrt{41}} = \cos \infty$ and $\frac{5}{\sqrt{41}} = \sin \infty$.

$$\therefore y = \sin^{-1}(\sin x \cos \infty + \cos x \sin \infty)$$

$$= \sin^{-1}[\sin(x + \infty)]$$

$= x + \infty$, where ∞ is a constant

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(x + \infty)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(\infty)$$

$$= 1 + 0$$

$$= 1.$$

Exercise 1.2 | Q 8.2 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{\sqrt{3} \cos x - \sin x}{2}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cos^{-1} \left(\frac{\sqrt{3} \cos x - \sin x}{2} \right) \\ &= \cos^{-1} \left[(\cos x) \left(\frac{\sqrt{3}}{2} \right) - (\sin x) \left(\frac{1}{2} \right) \right] \\ &= \cos^{-1} \left(\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} \right) \dots \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \sin \frac{\pi}{6} = \frac{1}{2} \right] \\ &= \cos^{-1} \left[\cos \left(x + \frac{\pi}{6} \right) \right] \\ &= x + \frac{\pi}{6} \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(x + \frac{\pi}{6} \right) \\ &= \frac{d}{dx} (x) + \frac{d}{dx} \left(\frac{\pi}{6} \right) \\ &= 1 + 0 \\ &= 1. \end{aligned}$$

Exercise 1.2 | Q 8.3 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1} \left(\frac{\cos \sqrt{x} + \sin \sqrt{x}}{\sqrt{2}} \right)$

SOLUTION

$$\begin{aligned}y &= \sin^{-1} \left(\frac{\cos \sqrt{x} + \sin \sqrt{x}}{\sqrt{2}} \right) \\&= \sin^{-1} \left(\frac{1}{\sqrt{2}} \cos \sqrt{x} + \frac{1}{\sqrt{2}} \sin \sqrt{x} \right)\end{aligned}$$

Put,

$$\frac{1}{\sqrt{2}} = \sin x$$

$$\frac{1}{\sqrt{2}} = \cos \alpha$$

Also,

$$\sin^2 \alpha + \cos^2 \alpha = \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{\sqrt{2}} \right)^2 = 1$$

And,

$$\tan \alpha = 1$$

$$\therefore \alpha = \tan^{-1} 1$$

$$\begin{aligned}y &= \sin^{-1} (\sin \alpha \cdot \cos \sqrt{x} + \cos \alpha \cdot \sin(\sqrt{x})) \\&= \sin^{-1} (\sin(\alpha + \sqrt{x}))\end{aligned}$$

$$y = \alpha + \sqrt{x}$$

$$y = \tan^{-1}(1) + \sqrt{x}$$

$$y = \alpha + \sqrt{x}$$

$$y = \tan^{-1}(1) + \sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} (\tan^{-1} + \sqrt{x})$$

$$= 0 + \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

Exercise 1.2 | Q 8.4 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{3 \cos 3x - 4 \sin 3x}{5}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cos^{-1}\left(\frac{3 \cos 3x - 4 \sin 3x}{5}\right) \\ &= \cos^{-1}\left[\left(\cos 3x\right)\left(\frac{3}{5}\right) - \left(\sin 3x\right)\left(\frac{4}{5}\right)\right] \\ \text{Since, } \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 &= \frac{9}{25} + \frac{16}{25} = 1 \end{aligned}$$

we can write, $\frac{3}{5} = \cos \infty$ and $\frac{4}{5} = \sin \infty$.

$$\therefore y = \cos^{-1}(\cos 3x \cos \infty - \sin 3x \sin \infty)$$

$$\begin{aligned} &= \cos^{-1}[\cos(3x + \infty)] \\ &= 3x + \infty, \text{ where } \infty \text{ is a constant} \end{aligned}$$

Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3x + \infty) \\ &= 3 \frac{d}{dx}(x) + \frac{d}{dx}(\infty) \\ &= 3 \times 1 + 0 \\ &= 3. \end{aligned}$$

Exercise 1.2 | Q 8.5 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left[\frac{3 \cos(e^x) + 2 \sin(e^x)}{\sqrt{13}}\right]$

SOLUTION

$$\begin{aligned}y &= \cos^{-1} \left(\frac{3 \cos(e^x) + 2 \sin(e^x)}{\sqrt{13}} \right) \\&= \cos^{-1} \left(\cos(e^x) \cdot \frac{3}{\sqrt{13}} + \sin(e^x) \cdot \frac{2}{\sqrt{13}} \right)\end{aligned}$$

Put,

$$\frac{3}{\sqrt{13}} = \cos x$$

$$\frac{2}{\sqrt{3}} = \sin x$$

Also,

$$\sin^2 \alpha + \cos^2 \alpha = \frac{9}{13} + \frac{4}{13} = 1$$

And,

$$\tan \alpha = \frac{\sin x}{\cos x} = \frac{2}{3}$$

$$\therefore \alpha = \tan^{-1} \left(\frac{2}{3} \right)$$

$$y = \cos^{-1} (\cos e^x \cdot \cos \alpha + \sin e^x \cdot \sin \alpha)$$

$$y = \cos^{-1} (\cos e^x - \alpha) \quad \because \cos^{-1} x \cdot (\cos x) = x$$

$$y = e^x - \alpha$$

$$= e^x = \tan^{-1} \left(\frac{2}{3} \right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(e^x - \tan^{-1} \left(\frac{2}{3} \right) \right)$$

$$= e^x - 0$$

$$= e^x.$$

Exercise 1.2 | Q 8.5 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1} \left[\frac{3 \cos(e^x) + 2 \sin(e^x)}{\sqrt{13}} \right]$

SOLUTION

$$y = \cos^{-1} \left(\frac{3 \cos(e^x) + 2 \sin(e^x)}{\sqrt{13}} \right)$$

$$= \cos^{-1} \left(\cos(e^x) \cdot \frac{3}{\sqrt{13}} + \sin(e^x) \cdot \frac{2}{\sqrt{13}} \right)$$

Put,

$$\frac{3}{\sqrt{13}} = \cos x$$

$$\frac{2}{\sqrt{13}} = \sin x$$

Also,

$$\sin^2 \alpha + \cos^2 \alpha = \frac{9}{13} + \frac{4}{13} = 1$$

And,

$$\tan \alpha = \frac{\sin x}{\cos x} = \frac{2}{3}$$

$$\therefore \alpha = \tan^{-1} \left(\frac{2}{3} \right)$$

$$y = \cos^{-1}(\cos e^x \cdot \cos \alpha + \sin e^x \cdot \sin \alpha)$$

$$y = \cos^{-1}(\cos e^x - \alpha) \quad \because \cos^{-1} x \cdot (\cos x) = x$$

$$y = e^x - \alpha$$

$$= e^x = \tan^{-1} \left(\frac{2}{3} \right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(e^x - \tan^{-1} \left(\frac{2}{3} \right) \right)$$

$$= e^x - 0$$

$$= e^x.$$

Exercise 1.2 | Q 8.6 | Page 30

Differentiate the following w.r.t. x : $\cosec^{-1} \left[\frac{10}{6 \sin(2^x) - 8 \cos(2^x)} \right]$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cosec^{-1} \left[\frac{10}{6 \sin(2^x) - 8 \cos(2^x)} \right] \\ &= \sin^{-1} \left[\frac{6 \sin(2^x) - 8 \cos(2^x)}{10} \right] \dots \left[\because \cosec^{-1} x = \sin^{-1} \left(\frac{1}{x} \right) \right] \\ &= \sin^{-1} \left[\{\sin(2^x)\} \left(\frac{6}{10} \right) - \{\cos(2^x)\} \left(\frac{8}{10} \right) \right] \end{aligned}$$

$$\text{Since, } \left(\frac{6}{10} \right)^2 + \left(\frac{8}{10} \right)^2 = \frac{36}{100} + \frac{64}{100} = 1,$$

$$\text{we can write, } \frac{6}{10} = \cos \infty \text{ and } \frac{8}{10} = \sin \infty.$$

$$\therefore y = \sin^{-1} [\sin(2^x) \cdot \cos \infty - \cos(2^x) \cdot \sin \infty]$$

$$= \sin^{-1} [\sin(2^x - \infty)]$$

$$= 2^x - \infty, \text{ where } \infty \text{ is a constant}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} (2^x - \infty)$$

$$= \frac{d}{dx} (2^x) - \frac{d}{dx} (\infty)$$

$$= 2^x \cdot \log 2 - 0$$

$$= 2^x \cdot \log 2.$$

Exercise 1.2 | Q 9.01 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

SOLUTION

$$\text{Let } y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$

$$\therefore y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$= 2 \frac{d}{dx}(\tan^{-1}x)$$

$$= 2 \times \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}.$$

Exercise 1.2 | Q 9.02 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

SOLUTION

$$\text{Let } y = \tan^{-1} \left(\frac{2x}{1 - x^2} \right)$$

Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$

$$\therefore y = \tan^{-1} \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx} (2 \tan^{-1} x)$$

$$= 2 \frac{d}{dx} (\tan^{-1} x)$$

$$= 2 \times \frac{1}{1 + x^2}$$

$$= \frac{2}{1 + x^2}.$$

Exercise 1.2 | Q 9.03 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$

SOLUTION

$$\text{Let } y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$

$$\therefore y = \sin^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$= \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1} \left[\sin \left(\frac{\pi}{2} - 2\theta \right) \right]$$

$$= \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2 \tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - 2 \tan^{-1} x \right)$$

$$= \frac{d}{dx} \left(\frac{\pi}{2} \right) - 2 \frac{d}{dx} (\tan^{-1} x)$$

$$= 0 - 2 \times \frac{1}{1+x^2}$$

$$= \frac{-2}{1+x^2}.$$

Exercise 1.2 | Q 9.04 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1} \left(2x\sqrt{1-x^2} \right)$

SOLUTION

$$\text{Let } y = \sin^{-1} \left(2x \sqrt{1 - x^2} \right)$$

Put $x = \sin\theta$.

Then $\theta = \sin^{-1} x$

$$\therefore y = \sin^{-1} \left(2 \sin \theta \sqrt{1 - \sin^2 \theta} \right)$$

$$= \sin^{-1}(2\sin\theta \cos\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\sin^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1} x)$$

$$= 2 \frac{d}{dx} (\sin^{-1} x)$$

$$= 2 \times \frac{1}{\sqrt{1 - x^2}}$$

$$= \frac{2}{\sqrt{1 - x^2}}$$

We can also put $x = \cos\theta$.

Then $\theta = \cos^{-1} x$

$$\therefore y = \sin^{-1} \left(2 \cos \theta \sqrt{1 - \cos^2 \theta} \right)$$

$$= \sin^{-1}(2\cos\theta \sin\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\cos^{-1} x$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (2 \cos^{-1} x) \\
 &= 2 \frac{d}{dx} (\cos^{-1} x) \\
 &= 2 \times \frac{-1}{\sqrt{1-x^2}} \\
 &= \frac{-2}{\sqrt{1-x^2}}
 \end{aligned}$$

Hence, $\frac{dy}{dx} = \pm \frac{2}{\sqrt{1-x^2}}$.

Exercise 1.2 | Q 9.05 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}(3x - 4x^3)$

SOLUTION

Let $y = \cos^{-1}(3x - 4x^3)$

Put $x = \sin\theta$.

Then $\theta = \sin^{-1}x$

$$\begin{aligned}
 \therefore y &= \cos^{-1}(3\sin\theta - 4\sin^3\theta) \\
 &= \cos^{-1}(\sin 3\theta)
 \end{aligned}$$

$$\begin{aligned}
 &= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 3\theta\right)\right] \\
 &= \frac{\pi}{2} - 3\theta \\
 &= \frac{\pi}{2} - 3\sin^{-1}x
 \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\pi}{2} - 3\sin^{-1}x \right) \\
 &= \frac{dy}{dx} \left(\frac{\pi}{2} \right) - 3 \frac{d}{dx} (\sin^{-1}x)
 \end{aligned}$$

$$= 0 - 3 \times \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-3}{\sqrt{1-x^2}}.$$

Exercise 1.2 | Q 9.06 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

SOLUTION

$$\text{Let } y = \cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \cos^{-1}\left[\frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}\right]$$

$$= \cos^{-1}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right)$$

Put $e^x = \tan\theta$.

Then $\theta = \tan^{-1}(e^x)$

$$\therefore y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$

$$= \cos^{-1}\left[-\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)\right]$$

$$= \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta$$

$$= \pi - 2\tan^{-1}(e^x)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} [\pi - 2 \tan^{-1}(e^x)] \\
&= \frac{d}{dx}(\pi) - 2 \frac{d}{dx} [\tan^{-1}(e^x)] \\
&= 0 - 2 \times \frac{1}{1 + (e^x)^2} \cdot \frac{d}{dx}(e^x) \\
&= \frac{-2}{1 + e^{2x}} \times e^x \\
&= \frac{2e^x}{1 + e^{2x}}.
\end{aligned}$$

Exercise 1.2 | Q 9.07 | Page 30

Differentiate the following w.r.t. x : $\cos^{-1} \frac{(1 - 9^x)}{(1 + 9^x)}$

SOLUTION

$$\text{Let } y = \cos^{-1} \frac{(1 - 9^x)}{(1 + 9^x)}$$

$$= \cos^{-1} \left[\frac{1 - (3^x)^2}{1 + (3^x)^2} \right]$$

Put $3x = \tan\theta$.

Then $\theta = \tan^{-1}(3^x)$

$$\therefore y = \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}(3^x)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} [2 \tan^{-1}(3^x)] \\
&= 2 \frac{d}{dx} [\tan^{-1}(3^x)] \\
&= 2 \times \frac{1}{1 + (3^x)^2} \cdot \frac{d}{dx} (3^x) \\
&= \frac{2}{1 + 3^{2x}} \times 3^x \log 3 \\
&= \frac{2 \cdot 3^x \log 3}{1 + 3^{2x}}.
\end{aligned}$$

Exercise 1.2 | Q 9.08 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1 - 2^{4x}}\right)$

SOLUTION

$$\begin{aligned}
\text{Let } y &= \sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1 - 2^{4x}}\right) \\
&= \sin^{-1}\left[\frac{4^x \cdot 4^{\frac{1}{2}}}{1 + (2^2)^{2x}}\right] \\
&= \sin^{-1}\left(\frac{2 \cdot 4^x}{1 + 4^{2x}}\right)
\end{aligned}$$

Put $4^x = \tan\theta$,

Then $\theta = \tan^{-1}(4^x)$

$$\begin{aligned}
\therefore y &= \sin^{-1}\left(\frac{2 \tan\theta}{1 + \tan\theta}\right) \\
&= \sin^{-1}(\sin 2\theta) \\
&= 2\theta
\end{aligned}$$

$$= 2\tan^{-1}(4^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [2\tan^{-1}(4^x)]$$

$$= 2 \times \frac{1}{1 + (4^x)^2} \cdot \frac{d}{dx}(4^x)$$

$$= \frac{2}{1 + 4^{2x}} \times 4^x \log 4$$

$$= \frac{2 \cdot 4^x \log 4}{1 + 4^{2x}}$$

Note: The answer can also be written as :

$$\frac{dy}{dx} = \frac{4^{\frac{1}{2}} \cdot 4^x \log 4}{1 + 4^{2x}}$$

$$= \frac{4^{x+\frac{1}{2}} \cdot \log 4}{1 + 4^{2x}}.$$

Exercise 1.2 | Q 9.09 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1} \left(\frac{1 - 25x^2}{1 + 25x^2} \right)$

SOLUTION

$$\text{Let } y = \sin^{-1} \left(\frac{1 - 25x^2}{1 + 25x^2} \right)$$

$$= \sin^{-1} \left[\frac{1 - (5x)^2}{1 + (5x)^2} \right]$$

Put $5x = \tan\theta$.

Then $\theta = \tan^{-1}(5x)$

$$\therefore y = \sin^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$= \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1} \left[\sin \left(\frac{\pi}{2} - 2\theta \right) \right]$$

$$= \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2 \tan^{-1}(5x)$$

Differentiating w.r.t. x, we get

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left[\frac{\pi}{2} - 2 \tan^{-1}(5x) \right]$$

$$= \frac{d}{dx} \left(\frac{\pi}{2} \right) - 2 \frac{d}{dx} [\tan^{-1}(5x)]$$

$$= 0 - 2 \times \frac{1}{1 + (5)^2} \cdot \frac{d}{dx}(5x)$$

$$= \frac{-2}{1 + 25x^2} \times 5$$

$$= \frac{-10}{1 + 25x^2}.$$

Exercise 1.2 | Q 9.1 | Page 30

Differentiate the following w.r.t. x : $\sin^{-1} \left(\frac{1-x^3}{1+x^3} \right)$

SOLUTION

$$\text{Let } y = \sin^{-1} \left(\frac{1-x^3}{1+x^3} \right)$$

$$= \sin^{-1} \left[\frac{1 - \left(\frac{x^3}{2} \right)^2}{1 + \left(\frac{x^3}{2} \right)^2} \right]$$

Put $x^{\frac{3}{2}} = \tan \theta$.

$$\text{Then } \theta = \tan^{-1} \left(x^{\frac{3}{2}} \right)$$

$$\therefore y = \sin^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$= \sin^{-1}(\cos 2\theta)$$

$$= \left[\sin \left(\frac{\pi}{2} - 2\theta \right) \right]$$

$$= \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2 \tan^{-1} \left(x^{\frac{3}{2}} \right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{\pi}{2} - 2 \tan^{-1} \left(x^{\frac{3}{2}} \right) \right]$$

$$= \frac{d}{dx} \left(\frac{\pi}{2} \right) - 2 \frac{d}{dx} \left[\tan^{-1} \left(x^{\frac{3}{2}} \right) \right]$$

$$= 0 - 2 \times \frac{1}{1 + \left(x^{\frac{3}{2}} \right)^2} \cdot \frac{d}{dx} \left(x^{\frac{3}{2}} \right)$$

$$\begin{aligned}
&= \frac{2}{1+x^3} \times \frac{3}{2} x^{\frac{1}{2}} \\
&= -\frac{3\sqrt{x}}{1+x^3}.
\end{aligned}$$

Exercise 1.2 | Q 9.11 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$

SOLUTION

$$\text{Let } y = \tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$$

$$\text{Put } x^{\frac{5}{2}} = \tan\theta.$$

$$\text{Then } \theta = \tan^{-1}\left(x^{\frac{5}{2}}\right)$$

$$\therefore y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta))$$

$$= 2\theta$$

$$= 2\tan^{-1}\left(x^{\frac{5}{2}}\right)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[2\tan^{-1}\left(x^{\frac{5}{2}}\right) \right] \\
&= 2 \frac{d}{dx} \left[\tan^{-1}\left(x^{\frac{5}{2}}\right) \right] \\
&= 2 \times \frac{1}{1+\left(x^{\frac{5}{2}}\right)^2} \cdot \frac{d}{dx} \left(x^{\frac{5}{2}} \right) \\
&= \frac{2}{1+x^5} \times \frac{5}{2} x^{\frac{3}{2}}
\end{aligned}$$

$$= \frac{5x\sqrt{x}}{1+x^5}.$$

Exercise 1.2 | Q 9.12 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) \\ &= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \dots [\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)] \\ &= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-1 \times \sqrt{x}}\right) \\ &= \tan^{-1}(1) + \tan^{-1}(\sqrt{x}) \dots \left[\because \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} x + \tan^{-1} y\right] \\ &= \frac{\pi}{4} + \tan^{-1}(\sqrt{x}) \end{aligned}$$

Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[\frac{\pi}{4} + \tan^{-1}(\sqrt{x})\right] \\ &= \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}[\tan^{-1}(\sqrt{x})] \\ &= 0 + \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x}) \\ &= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}(1+x)}. \end{aligned}$$

Exercise 1.2 | Q 10.1 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{8x}{1 - 15x^2}\right)$

SOLUTION

$$\text{Let } y = \tan^{-1}\left(\frac{8x}{1 - 15x^2}\right)$$

$$= \tan^{-1}\left[\frac{5x + 3x}{1 - (5x)(3x)}\right]$$

$$= \tan^{-1}(5x) + \tan^{-1}(3x)$$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx} [\tan^{-1}(5x) + \tan^{-1}(3x)]$$

$$= \frac{d}{dx} [\tan^{-1}(5x)] + \frac{d}{dx} [\tan^{-1}(3x)]$$

$$= \frac{1}{1 + (5x)^2} \cdot \frac{d}{dx}(5x) + \frac{1}{1 + (3x)^2} \cdot \frac{d}{dx}(3x)$$

$$= \frac{1}{1 + 25x^2} \times 5 + \frac{1}{1 + 9x^2} \times 3$$

$$= \frac{5}{1 + 25x^2} + \frac{3}{1 + 9x^2}.$$

Exercise 1.2 | Q 10.2 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{1 + 35x^2}{2x}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cot^{-1} \left(\frac{1 + 35x^2}{2x} \right) \\ &= \tan^{-1} \left(\frac{2x}{1 + 35x^2} \right) \dots \left[\because \cot^{-1} x = \tan^{-1} \left(\frac{1}{x} \right) \right] \\ &= \tan^{-1} \left[\frac{7x - 5x}{1 + (7x)(5x)} \right] \\ &= \tan^{-1}(7x) - \tan^{-1}(5x) \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(7x) - \tan^{-1}(5x)] \\ &= \frac{d}{dx} [\tan^{-1}(x)] - \frac{d}{dx} [\tan^{-1}(5x)] \\ &= \frac{1}{1 + (7)^2} \cdot \frac{d}{dx}(7x) - \frac{1}{1 + (5x)^2} \cdot \frac{d}{dx}(5x) \\ &= \frac{1}{1 + 49x^2} \times 7 - \frac{1}{1 + 25x^2} \times 5 \\ &= \frac{7}{1 + 49x^2} - \frac{5}{1 + 25x^2}. \end{aligned}$$

Exercise 1.2 | Q 10.3 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1} \left(\frac{2\sqrt{x}}{1 + 3x} \right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \tan^{-1} \left(\frac{2\sqrt{x}}{1 + 3x} \right) \\ &= \tan^{-1} \left[\frac{3\sqrt{x} - \sqrt{x}}{1 + (3\sqrt{x})(\sqrt{x})} \right] \end{aligned}$$

$$= \tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(\sqrt{x}) - \tan^{-1}(\sqrt{x})] \\ &= \frac{d}{dx} \left[\tan^{-1}(3\sqrt{x}) - \frac{d}{dx} [\tan^{-1}(\sqrt{x})] \right] \\ &= \frac{1}{1 + (3\sqrt{x})^2} \cdot \frac{d}{dx} (3\sqrt{x}) - \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} (\sqrt{x}) \\ &= \frac{1}{1 + 9x} \times 3 \times \frac{1}{2\sqrt{x}} - \frac{1}{1 + x} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \left[\frac{3}{1 + 9x} - \frac{1}{1 + x} \right].\end{aligned}$$

Exercise 1.2 | Q 10.4 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1} \left[\frac{2^x + 2}{1 - 3(4^x)} \right]$

SOLUTION

$$\text{Let } y = \tan^{-1} \left[\frac{2^x + 2}{1 - 3(4^x)} \right]$$

$$= \tan^{-1} \left[\frac{2^2 \cdot 2^x}{1 - 3(4^x)} \right]$$

$$= \tan^{-1} \left[\frac{4 \cdot 2^x}{1 - 3(4^x)} \right]$$

$$= \tan^{-1} \left[\frac{3 \cdot 2^x + 2^x}{1 - (3 \cdot 2^x \times 2^x)} \right]$$

$$= \tan^{-1}(3 \cdot 2^x) + \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(3.2^x) + \tan^{-1}(2^x)] \\&= \frac{d}{dx} [\tan^{-1}(3.3x^x)] + \frac{d}{dx} [\tan^{-1}(2^x)] \\&= \frac{1}{1 + (3.2^x)^2} \cdot \frac{d}{dx} (3.2^x) + \frac{1}{1 + (2^x)^2} \cdot \frac{d}{dx} (2^x) \\&= \frac{1}{1 + 9(2^{2x})} \times 3 \times 2^x \log 2 + \frac{1}{1 + 2^{2x}} \times 2^x \log 2 \\&= 2^x \log 2 \left[\frac{3}{1 + 9(2^{2x})} + \frac{1}{1 + 2^{2x}} \right].\end{aligned}$$

Exercise 1.2 | Q 10.5 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{2^x}{1 + 2^{2x+1}}\right)$

SOLUTION

$$\text{Let } y = \tan^{-1}\left(\frac{2^x}{1 + 2^{2x+1}}\right)$$

$$= \tan^{-1}\left[\frac{2 \cdot 2^x - 2^x}{1 + (2 \cdot 2^x)(2^x)}\right]$$

$$= \tan^{-1}(2 \cdot 2^x) - \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(2 \cdot 2^x) - \tan^{-1}(2^x)] \\&= \frac{d}{dx} [\tan^{-1}(2 \cdot 2^x)] - \frac{d}{dx} [\tan^{-1}(2^x)] \\&= \frac{1}{1 + (2 \cdot 2^x)^2} \cdot \frac{d}{dx} (2 \cdot 2^x) - \frac{1}{1 + (2^x)^2} \cdot \frac{d}{dx} (2^x)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 + 4(2^{2x})} \times 2 \times 2^x \log 2 - \frac{1}{1 + 2^{2x}} \times 2^x \log 2 \\
&= 2^x \log 2 \left[\frac{2}{1 + 4(2^{2x})} - \frac{1}{1 + 2^{2x}} \right].
\end{aligned}$$

Exercise 1.2 | Q 10.6 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1}\left(\frac{a^2 - 6x^2}{5ax}\right)$

SOLUTION

$$\begin{aligned}
&\text{Let } y = \cot^{-1}\left(\frac{a^2 - 6x^2}{5ax}\right) \\
&= ta^{-1}\left(\frac{5ax}{a^2 - 6x^2}\right) \dots \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right] \\
&= \tan^{-1}\left[\frac{5\left(\frac{x}{a}\right)}{1 - 6\left(\frac{x}{a}\right)^2}\right] \dots [\text{Dividing by } a^2] \\
&= \tan\left[\frac{3\left(\frac{x}{a}\right) + 2\left(\frac{x}{a}\right)}{1 - 3\left(\frac{x}{a}\right) \times 2\left(\frac{x}{a}\right)}\right] \\
&= \tan^{-1}\left(\frac{3x}{a}\right) + \tan^{-1}\left(\frac{2x}{a}\right)
\end{aligned}$$

Differentiating w.r.t. x , we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[\tan^{-1}\left(\frac{3x}{a}\right) + \tan^{-1}\left(\frac{2x}{a}\right) \right] \\
&= \frac{d}{dx} \left[\tan^{-1}\left(\frac{3x}{a}\right) \right] + \frac{d}{dx} \left[\tan^{-1}\left(\frac{2x}{a}\right) \right] \\
&= \frac{1}{1 + \left(\frac{9x^2}{a^2}\right)} \times \frac{3}{a} \times 1 + \frac{1}{1 + \left(\frac{4x^2}{a^2}\right)} \times \frac{2}{a} \times 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^2}{a^2 + 9x^2} \times \frac{3}{a} + \frac{a^2}{a^2 + 4x^2} \times \frac{2}{a} \\
&= \frac{3a}{a^2 + 9x^2} + \frac{2a}{a^2 + 4x^2}.
\end{aligned}$$

Exercise 1.2 | Q 10.7 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

SOLUTION

$$\begin{aligned}
\text{Let } y &= \tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right) \\
&= \tan^{-1}\left[\frac{\frac{a}{b} + \tan x}{1 - \frac{a}{b} \cdot \tan x}\right] \\
&= \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(\tan x) \\
&= \tan^{-1}\left(\frac{a}{b}\right) + x
\end{aligned}$$

Differentiating w.r.t. x , we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[\tan^{-1}\left(\frac{a}{b}\right) + x \right] \\
&= \frac{d}{dx} \left[\tan^{-1}\left(\frac{a}{b}\right) \right] + \frac{d}{dx}(x) \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

Exercise 1.2 | Q 10.8 | Page 30

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{5-x}{6x^2-5x-3}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \tan^{-1} \left(\frac{5-x}{6x^2 - 5x - 3} \right) \\ &= \tan^{-1} \left[\frac{5-x}{1 + (6x^2 - 5x - 4)} \right] \\ &= \tan^{-1} \left[\frac{(2x-1) - (3x-4)}{1 + (2x+1)(3x-4)} \right] \\ &= \tan^{-1}(2x+1) - \tan^{-1}(3x-4) \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(2x+1)] - \frac{d}{dx} [\tan^{-1}(3x-4)] \\ &= \frac{1}{1 + (2x+1)^2} \cdot \frac{d}{dx}(2x+1) - \frac{1}{1 + (3x-4)^2} \cdot \frac{d}{dx}(3x-4) \\ &= \frac{1}{1 + (2x+1)^2} \cdot (2 \times 1 + 0) - \frac{1}{1 + (3x-4)^2} \cdot (3 \times 1 - 0) \\ &= \frac{2}{1 + (2x+1)^2} - \frac{3}{1 + (3x-4)^2}. \end{aligned}$$

Exercise 1.2 | Q 10.9 | Page 30

Differentiate the following w.r.t. x : $\cot^{-1} \left(\frac{4-x-2x^2}{3x+2} \right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \cot^{-1} \left(\frac{4-x-2x^2}{3x+2} \right) \\ &= \tan^{-1} \left(\frac{3x+2}{4-x-2x^2} \right) \dots \left[\because \cot^{-1} x = \tan^{-1} \left(\frac{1}{x} \right) \right] \\ &= \tan^{-1} \left[\frac{3x+2}{1-(2x^2+x-3)} \right] \\ &= \tan^{-1} \left[\frac{(2x+3)+(x-1)}{1-(2x+3)(x-1)} \right] \\ &= \tan^{-1}(2x+3) + \tan^{-1}(x-1) \end{aligned}$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(2x+3) + \tan^{-1}(x-1)] \\ &= \frac{d}{dx} [\tan^{-1}(2x+3)] + \frac{d}{dx} [\tan^{-1}(x-1)] \\ &= \frac{1}{1+(2x+3)^2} \cdot \frac{d}{dx}(2x+3) + \frac{1}{1+(x-1)^2} \cdot \frac{d}{dx}(x-1) \\ &= \frac{1}{1+(2x+3)^2} \cdot (2 \times 1 + 0) + \frac{1}{1+(x-1)^2} \cdot (1 - 0) \\ &= \frac{2}{1+(2x+3)^2} + \frac{1}{1+(x-1)^2}. \end{aligned}$$

EXERCISE 1.3 [PAGES 39 - 40]

Exercise 1.3 | Q 1.1 | Page 39

Differentiate the following w.r.t. x : $\frac{(x+1)^2}{(x+2)^3(x+3)^4}$

SOLUTION

$$\text{Let } y = \frac{(x+1)^2}{(x+2)^3(x+3)^4}$$

$$\text{Then, } \log y = \log \left[\frac{(x+1)^2}{(x+2)^3(x+3)^4} \right]$$

$$= \log(x+1)^2 - \log(x+2)^3 - \log(x+3)^4$$

$$= 2\log(x+1) - 3\log(x+2) - 4\log(x+3)$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= 2 \frac{d}{dx} [\log(x+1)] - 3 \frac{d}{dx} [\log(x+2)] - 4 \frac{d}{dx} [\log(x+3)] \\ &= 2 \times \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) - 3 \times \frac{1}{x+2} \cdot \frac{d}{dx}(x+2) - 4 \times \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) \\ &= \frac{2}{x+1} \cdot (1+0) - \frac{3}{x+2} \cdot (1+0) - \frac{4}{x+3} \cdot (1+0) \\ \therefore \frac{dy}{dx} &= y \left[\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3} \right] \\ &= \frac{(x+1)^2}{(x+2)^2(x+3)^4} \cdot \left[\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3} \right]. \end{aligned}$$

Exercise 1.3 | Q 1.2 | Page 39

Differentiate the following w.r.t. x : $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

SOLUTION

$$\text{Let } y = \sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$$

$$\text{Then } \log y = \log \left[\frac{4x-1}{(2x+3)(5-2x)^2} \right]^{\frac{1}{3}}$$

$$= \frac{1}{3} \log \left[\frac{4x-1}{(2x+3)(5-2x)^2} \right]$$

$$= \frac{1}{3} [\log(4x-1) - \log(2x+3)(5-2x)^2]$$

$$= \frac{1}{3} \log(4x-1) - \frac{1}{3} \log(2x+3) - \frac{2}{3} \log(5-2x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{3} \frac{d}{dx} [\log(4x-1)] - \frac{1}{3} \frac{d}{dx} [\log(2x+3)] - \frac{2}{3} \frac{d}{dx} [\log(5-2x)]$$

$$= \frac{1}{3} \times \frac{1}{4x-1} \cdot \frac{d}{dx}(4x-1) - \frac{1}{3} \times \frac{1}{2x+3} \cdot \frac{d}{dx}(2x+3) - \frac{2}{3} \times \frac{1}{5-2x} \cdot \frac{d}{dx}(5-2x)$$

$$= \frac{1}{3(4x-1)} \cdot (4 \times 1 - 0) - \frac{1}{3(2x+3)} \cdot (2 \times 1 + 0) - \frac{2}{3(5-2x)} \cdot (0 - 2 \times 1)$$

$$\therefore \frac{dy}{dx} = y \left[\frac{4}{3(4x-1)} - \frac{2}{3(2x+3)} + \frac{4}{3(5-2x)} \right]$$

$$= \sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}} \left[\frac{4}{3(4x-1)} - \frac{2}{3(2x+3)} + \frac{4}{3(5-2x)} \right].$$

Exercise 1.3 | Q 1.3 | Page 39

Differentiate the following w.r.t. x : $(x^2 + 3)^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2}$

SOLUTION

$$\text{Let } y = (x^2 + 3)^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2}$$

$$\text{Then } \log y = \log [x^2 + 3]^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2}$$

$$= \log(x^2 + 3)^{\frac{3}{2}} + \log \sin^3 2x + \log 2^{x^2}$$

$$= \frac{3}{2} \log(x^2 + 3) + 3 \log(\sin 2x) + x^2 \cdot \log 2$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{3}{2} \frac{d}{dx} [\log(x^2 + 3)] + 3 \frac{d}{dx} [\log(\sin 2x)] + \log 2 \cdot \frac{d}{dx}(x^2) \\&= \frac{3}{2} \times \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) + 3 \times \frac{1}{\sin 2x} \cdot \frac{d}{dx}(\sin 2x) + \log 2 \times 2x \\&= \frac{3}{2(x^2 + 3)} \cdot (2x + 0) + \frac{3}{\sin 2x} \times \cos 2x \cdot \frac{d}{dx}(2x) + 2x \log 2 \\&= \frac{6x}{2(x^2 + 3)} + 3 \cot 2x \times 2 + 2x \log 2 \\ \therefore \frac{dy}{dx} &= y \left[\frac{3x}{x^2 + 3} + 6 \cot 2x + 2x \log 2 \right] \\&= (x^2 + 3)^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2} \left[\frac{3x}{x^2 + 3} + 6 \cot 2x + 2x \log 2 \right].\end{aligned}$$

Exercise 1.3 | Q 1.4 | Page 39

Differentiate the following w.r.t. x : $\frac{(x^2 + 2x + 2)^{\frac{3}{2}}}{(\sqrt{x} + 3)^3 (\cos x)^x}$

SOLUTION

$$\text{Let } y = \frac{(x^2 + 2x + 2)^{\frac{3}{2}}}{(\sqrt{x} + 3)^3 (\cos x)^x}$$

$$\text{Then } \log y = \log \left[\frac{(x^2 + 2x + 2)^{\frac{3}{2}}}{(\sqrt{x} + 3)^3 (\cos x)^x} \right]$$

$$= \log(x^2 + 2x + 2)^{\frac{3}{2}} - \log(\sqrt{x} + 3)^3 (\cos x)^x$$

$$= \frac{3}{2} \log(x^2 + 2x + 2) - 3 \log(\sqrt{x} + 3) - x \log(\cos x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{2} \frac{d}{dx} [\log(x^2 + 2x + 2)] - 3 \frac{d}{dx} [\log(\sqrt{x} + 3)] - \frac{d}{dx} [x \log(\cos x)]$$

=

$$\frac{3}{2} \times \frac{1}{x^2 + 2x + 2} \cdot \frac{d}{dx}(x^2 + 2x + 2) - 3 \times \frac{1}{\sqrt{x} + 3} \cdot \frac{d}{dx}(\sqrt{x} + 3) - \left\{ x \frac{d}{dx}[\log(\cos x)] + \log(\cos x) \cdot \frac{d}{dx}(x) \right\}$$

=

$$\frac{3}{2(x^2 + 2x + 2)} \times (2x + 2 \times 1 + 0) - \frac{3}{\sqrt{x} + 3} \times \left(\frac{1}{2\sqrt{x} + 0} \right) - x \times \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \log(\cos x) \times 1 \}$$

$$= \frac{3(2x + 2)}{2(x^2 + 2x + 2)} - \frac{3}{2\sqrt{x}(\sqrt{x} + 3)} - \left\{ x \times \frac{1}{\cos x} \cdot (-\sin x) + \log(\cos x) \right\}$$

$$\therefore \frac{dy}{dx} = y \left[\frac{3(x + 1)}{x^3 + 2x + 2} - \frac{3}{2\sqrt{x}(\sqrt{x} + 3)} + x \tan x - \log(\cos x) \right]$$

$$= \frac{(x^2 + 2x + 2)^{\frac{3}{2}}}{(\sqrt{x} + 3)^3 (\cos x)^x} \left[\frac{3(x + 1)}{x^2 + 2x + 2} - \frac{3}{2\sqrt{x}(\sqrt{x} + 3)} + x \tan x - \log(\cos x) \right].$$

Exercise 1.3 | Q 1.5 | Page 39

Differentiate the following w.r.t. x : $\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$

SOLUTION

$$\text{Let } y = \frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$$

$$\text{Then } \log y = \log \left[\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x} \right]$$

$$= \log x^5 + \log \tan^3 4x - \log \sin^2 3x$$

$$= 5 \log x + 3 \log (\tan 4x) - 2 \log (\sin 3x)$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= 5 \frac{d}{dx}(\log x) + 3 \frac{d}{dx}[\log(\tan 4x)] - 2 \frac{d}{dx}[\log(\sin 3x)] \\ &= 5 \times \frac{1}{x} + 3 \times \frac{1}{\tan 4x} \cdot \frac{d}{dx}(\tan 4x) - 2 \times \frac{1}{\sin 3x} \cdot \frac{d}{dx}(\sin 3x) \\ &= \frac{5}{x} + 3 \times \frac{1}{\tan 4x} \times \sec^2 4x \cdot \frac{d}{dx}(4x) - 2 \times \frac{1}{\sin 3x} \times \cos 3x \cdot \frac{d}{dx}(3x) \\ &= \frac{5}{x} + 3 \cdot \frac{\cos 4x}{\sin 4x} \times \frac{1}{\cos^2 4x} \times 4 - 2 \cot 3x \times 3 \\ &= \frac{5}{x} + \frac{24}{2 \sin 4x \cdot \cos 4x} - 6 \cot 3x \\ \therefore \frac{dy}{dx} &= y \left[\frac{5}{x} + \frac{24}{\sin 8x} - 6 \cot 3x \right] \\ &= \frac{x^5 \cdot \tan^3 4x}{\sin^2 3x} \left[\frac{5}{x} + 24 \operatorname{cosec} 8x - 6 \cot 3x \right]. \end{aligned}$$

Exercise 1.3 | Q 1.6 | Page 39

Differentiate the following w.r.t. x : $x^{\tan^{-1} x}$

SOLUTION

Let $y = x^{\tan^{-1} x}$

Then $\log y = \log(x^{\tan^{-1} x}) = (\tan^{-1} x)(\log x)$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx} [(\tan^{-1} x)(\log x)] \\&= (\tan^{-1} x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\tan^{-1} x) \\&= (\tan^{-1} x) \times \frac{1}{x} + (\log x) \times \frac{1}{1+x^2} \\ \therefore \frac{dy}{dx} &= y \left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2} \right] \\&= x^{\tan^{-1} x} \left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2} \right].\end{aligned}$$

Exercise 1.3 | Q 1.7 | Page 39

Differentiate the following w.r.t. x : $(\sin x)^x$

SOLUTION

Let $y = (\sin x)^x$

Then $\log y = \log(\sin x)^x = x \cdot \log(\sin x)$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx}[x \cdot \log(\sin x)] \\&= x \cdot \frac{d}{dx}[\log(\sin x)] + \log(\sin x) \cdot \frac{d}{dx}(x) \\&= x \times \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log(\sin x) \times 1\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= y \left[x \times \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \right] \\ &= (\sin x)^x [x \cot x + \log(\sin x)].\end{aligned}$$

Exercise 1.3 | Q 1.8 | Page 39

Differentiate the following w.r.t. x : $(\sin x^x)$

SOLUTION

Let $y = (\sin x^x)$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{d}{dx} [(\sin x^x)] \\ \therefore \frac{dy}{dx} &= \cos(x^x) \cdot \frac{d}{dx}(x^x) \quad \dots(1)\end{aligned}$$

Let $u = x^x$

Then $\log u = \log x^x = x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x \cdot \log x) \\ &= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \\ &= x \times \frac{1}{x} + (\log x) \times 1 \\ \therefore \frac{du}{dx} &= u(1 + \log x) \\ \therefore \frac{d}{dx}(x^x) &= x^x(1 + \log x) \quad \dots(2)\end{aligned}$$

From (1) and (2), we get

$$\frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \log x).$$

Exercise 1.3 | Q 2.1 | Page 40

Differentiate the following w.r.t. x : $x^e + x^x + e^x + e^e$

SOLUTION

$$\text{Let } y = x^e + x^x + e^x + e^e$$

$$\text{Let } u = x^x$$

$$\text{Then } \log u = \log x^x = x \log x$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x \log x) \\ &= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \\ &= x \times \frac{1}{x} + (\log x)(1) \\ \therefore \frac{du}{dx} &= u(1 + \log x) = x^x (1 + \log x) \quad \dots(1) \end{aligned}$$

$$\text{Now, } y = x^e + u + e^x + e^e$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(x^e) + \frac{du}{dx} + \frac{d}{dx}(e^x) + \frac{d}{dx}(e^e) \\ &= ex^{e-1} + x^x (1 + \log x) + e^x + 0 \quad \dots[\text{By (1)}] \\ &= ex^{e-1} + x^x (1 + \log x) + e^x \\ &= ex^{e-1} + e^x + x^x (1 + \log x). \end{aligned}$$

Exercise 1.3 | Q 2.2 | Page 40

Differentiate the following w.r.t. x : $x^{x^x} + e^{x^x}$

SOLUTION

Let $y = x^{x^x} + e^{x^x}$

Put $u = x^{x^x}$ and $v = e^{x^x}$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Take $u = x^{x^x}$

$$\therefore \log u = \log x^{x^x} = x^x \cdot \log x$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x^x \cdot \log x) \\ &= x^x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^x) \\ &= x^x \times \frac{1}{x} + (\log x) \cdot \frac{d}{dx}(x^3) \quad \dots(2) \end{aligned}$$

To find $\frac{d}{dx}(x^3)$

Let $\omega = x^x$.

Then $\log \omega = x \log x$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{1}{\omega} \cdot \frac{d\omega}{dx} &= \frac{d}{dx}(x \log x) \\ &= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \\ &= x \times \frac{1}{x} + (\log x) \times 1 \\ \therefore \frac{d\omega}{dx} &= \omega(1 + \log x) \\ \therefore \frac{d}{dx}(x^x) &= x^x(1 + \log x) \quad \dots(3) \\ \therefore \text{from (2),} \end{aligned}$$

$$\begin{aligned}
 \frac{1}{u} \cdot \frac{du}{dx} &= x^x \times \frac{1}{x} + (\log x) \cdot x^x(1 + \log x) \\
 \therefore \frac{du}{dx} &= y \left[x^x \times \frac{1}{x} + (\log x) \cdot x^x(1 + \log x) \right] \\
 &= x^{x^x} \cdot x^x \left[\frac{1}{x} + (\log x) \cdot (1 + \log x) \right] \\
 &= x^{x^x} \cdot x^x \cdot \log x \left[1 + \log x + \frac{1}{x \log x} \right] \quad \dots(4)
 \end{aligned}$$

Also, $v = e^{x^x}$

$$\begin{aligned}
 \therefore \frac{dv}{dx} &= \frac{d}{dx}(e^{x^x}) \\
 &= e^{x^x} \cdot \frac{d}{dx}(e^{x^x}) \\
 &= e^{x^x} \cdot x^x(1 + \log x) \quad \dots(5)
 \end{aligned}$$

From (1), (4) and (5), we get \dots [By (3)]

$$\frac{dy}{dx} = x^{x^x} \cdot x^x \cdot \log x \left[1 + \log x + \frac{1}{x \log x} \right] + e^{x^x} \cdot x^x(1 + \log x).$$

Exercise 1.3 | Q 2.3 | Page 40

Differentiate the following w.r.t. x : $(\log x)^x - (\cos x)^{\cot x}$

SOLUTION

Let $y = (\log x)^x - (\cos x)^{\cot x}$

Put $u = (\log x)^x$ and $v = (\cos x)^{\cot x}$

Then $y = u - v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad \dots(1)$$

Take $u = (\log x)^x$

$$\therefore \log u = \log(\log x)^x = x \cdot \log(\log x)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
& \frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx} [x \cdot \log(\log x)] \\
&= x \frac{d}{dx} [\log(\log x)] + \log(\log x) \cdot \frac{d}{dx}(x) \\
&= x \times \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) + \log(\log x) \times 1 \\
&= x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x) \\
&\therefore \frac{du}{dx} = u \left[\frac{1}{\log x} + \log(\log x) \right] \\
&= (\log x)x \left[\frac{1}{\log x} + \log(\log x) \right] \quad \dots(2)
\end{aligned}$$

Also $v = (\cos x)^{\cot x}$

$$\therefore \log v = \log(\cos x)^{\cot x} = (\cot x) \cdot (\log \cos x)$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
\frac{1}{v} = \frac{dv}{dx} &= \frac{d}{dx} [(\cot x) \cdot \log(\cos x)] \\
&= (\cot x) \cdot \frac{d}{dx} (\log \cos x) + (\log \cos x) \cdot \frac{d}{dx} (\cot x) \\
&= \cot x \times \frac{1}{\cos x} \cdot \frac{d}{dx} (\cos x) + (\log \cos x) (-\operatorname{cosec}^2 x) \\
&= \cot x \times \frac{1}{\cos x} \times (-\sin x) - (\operatorname{cosec}^2 x) (\log \cos x) \\
&\therefore \frac{dv}{dx} = v \left[\frac{1}{\tan x} \times (-\tan x) - (\operatorname{cosec}^2 x) (\log \cos x) \right] \\
&= -(\cos x)^{\cot x} [1 + (\operatorname{cosec}^2 x) (\log \cos x)]
\end{aligned}$$

From (1), (2) and (3), we get

$$\therefore \frac{dv}{dx} = (\log x)^x \left[\frac{1}{\log x} + \log(\log x) \right] + (\cos x)^{\cot x} [1 + (\operatorname{cosec}^2 x) (\log \cos x)].$$

Exercise 1.3 | Q 2.4 | Page 40

Differentiate the following w.r.t. x : $x^{x^x} + (\log x)^{\sin x}$

SOLUTION

Let $y = x^{x^x} + (\log x)^{\sin x}$

Put $u = x^{e^x}$ and $v = (\log x)^{\sin x}$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Take $u = x^{e^x}$

$$\therefore \log u = \log x^{e^x} = e^x \cdot \log x$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(e^x \log x) \\ &= e^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(e^x) \\ &= e^x \cdot \frac{1}{x} + (\log x)(e^x) \\ \therefore \frac{du}{dx} &= y \left[\frac{e^x}{x} + e^x \cdot \log x \right] \\ &= e^x \cdot x^{e^x} \left[\frac{1}{x} + \log x \right] \quad \dots(2) \end{aligned}$$

Also, $v = (\log x)^{\sin x}$

$$\therefore \log v = \log(\log x)^{\sin x} = (\sin x) \cdot (\log \log x)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{1}{v} \cdot \frac{dv}{dx} &= \frac{d}{dx}[(\sin x) \cdot (\log \log x)] \\ &= (\sin x) \cdot \frac{d}{dx}[(\log \log x) + (\log \log x) \cdot \frac{d}{dx}(\sin x)] \end{aligned}$$

$$\begin{aligned}
&= \sin x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) + (\log \log x) \cdot (\cos x) \\
\therefore \frac{dv}{dx} &= v \left[\frac{\sin x}{\log x} \times \frac{1}{x} + (\cos x)(\log \log x) \right] \\
&= (\log x)^{\sin x} \left[\frac{\sin x}{x} \log x + (\cos x)(\log \log x) \right] \quad \dots(2)
\end{aligned}$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = e^x \cdot x^{e^x} \left[\frac{1}{x} + \log x \right] + (\log x)^{\sin x} \left[\frac{\sin x}{x \log x} + (\cos x)(\log \log x) \right].$$

Exercise 1.3 | Q 2.5 | Page 40

Differentiate the following w.r.t. x : $e^{\tan x} + (\log x)^{\tan x}$

SOLUTION

$$\text{Let } y = e^{\tan x} + (\log x)^{\tan x}$$

$$\text{Put } u = (\log x)^{\tan x}$$

$$\therefore \log u = \log (\log x)^{\tan x} = (\tan x)(\log \log x)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx} [(\tan x)(\log \log x)] \\
&= (\tan x) \cdot \frac{d}{dx}(\log \log x) + (\log \log x) \cdot \frac{d}{dx}(\tan x) \\
&= \tan x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) + (\log \log x)(\sec^2 x) \\
&= \tan x \times \frac{1}{\log x} \times \frac{1}{x} + (\log \log x)(\sec^2 x) \\
\therefore du'/dx &= u \left[\frac{\tan x}{x \log x} + (\log \log x)(\sec^2 x) \right]
\end{aligned}$$

$$= (\log x)^{\tan x} [\tan x(x \log x) + (\log \log x)(\sec^2 x)]$$

Now, $y = e^{\tan x} + u$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^{\tan x}) + \frac{du}{dx}$$

$$= e^{\tan x} \cdot \frac{d}{dx}(\tan x) + \frac{du}{dx}$$

$$= e^{\tan x} \cdot \sec^2 x + (\log x)^{\tan x} \left[\frac{\tan x}{x \log x} + (\log \log x)(\sec^2 x) \right].$$

Exercise 1.3 | Q 2.6 | Page 40

Differentiate the following w.r.t. x : $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

SOLUTION

Let $y = (\sin x)^{\tan x} + (\cos x)^{\cot x}$

Put $u = (\sin x)^{\tan x}$ and $v = (\cos x)^{\cot x}$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Take $u = (\sin x)^{\tan x}$

$$\therefore \log u = \log(\sin x)^{\tan x} = (\tan x)(\log \sin x)$$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}[(\tan x)(\log \sin x)]$$

$$= (\tan x) \cdot \frac{d}{dx}(\log \sin x) + (\log \sin x) \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{\tan x}{\sin x} \cdot \frac{d}{dx}(\sin x) + (\log \sin x)(\sec^2 x)$$

$$= \frac{\sin x}{\cos x} \cdot \cos x + (\sec^2 x)(\log \sin x)$$

$$= 1 + (\sec^2 x)(\log \sin x)$$

$$\therefore \frac{du}{dx} = y[1 + (\sec^2 x)(\log \sin x)]$$

$$= (\sin x)^{\tan x}[1 + (\sec^2 x)(\log \sin x)] \quad \dots(2)$$

$$\text{Also, } v = (\cos x)^{\cot x}$$

$$\therefore \log v = \log(\cos x)^{\cot x} = (\cot x).(\log \cos x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} [(\cot x).(\log \cos x)]$$

$$= (\cot x) \cdot \frac{d}{dx}(\log \cos x) + (\log \cos x) \cdot \frac{d}{dx}(\cot x)$$

$$= \cot x \times \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + (\log \cos x) \cdot (-\operatorname{cosec}^2 x)$$

$$= \cot x \times \frac{1}{\cos x} \times (-\sin x) - (\operatorname{cosec}^2 x)(\log \cos x)$$

$$\therefore \frac{dv}{dx} = v \left[\frac{1}{\tan x} \times (-\tan x) - (\operatorname{cosec}^2 x)(\log \cos x) \right]$$

$$= -(\cos x)^{\cot x} [1 + (\operatorname{cosec}^2 x)(\log \cos x)] \quad \dots(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = (\sin x)^{\tan x} [1 + (\sec^2 x)(\log \sin x)] - (\cos x)^{\cot x} [1 + (\operatorname{cosec}^2 x)(\log \cos x)].$$

Exercise 1.3 | Q 2.7 | Page 40

Differentiate the following w.r.t. x : $10^{x^x} + x^{x(10)} + x^{10x}$

SOLUTION

$$\text{Let } y = 10^{x^x} + x^{x^{10}} + x^{10x}$$

$$\text{Put } u = 10^{x^x}, v = x^{x^{10}} \text{ and } \omega = x^{10x}$$

$$\text{Then } y = u + v + \omega$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{d\omega}{dx} \quad \dots(1)$$

$$\text{Take, } u = 10^{x^x}$$

$$\therefore \frac{du}{dx} = \frac{d}{dx}(10^{x^x})$$

$$= 10^{x^x} \cdot \log 10 \cdot \frac{d}{dx}(x^x)$$

$$\text{To find } \frac{d}{dx}(x^x)$$

$$\text{Let } z = x^x$$

$$\therefore \log z = \log x^x = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{z} \cdot \frac{dz}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x)(1)$$

$$\therefore \frac{dz}{dx} = z(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x)$$

$$\therefore \frac{du}{dx} = 10^{x^x} \cdot \log 10 \cdot x^x(1 + \log x) \quad \dots(2)$$

$$\text{Take, } v = x^{x^{10}}$$

$$\therefore \log v = \log x^{x^{10}} = x^{10} \cdot \log x$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
& \frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} (x^{10} \log x) \\
&= x^{10} \cdot \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (x^{10}) \\
&= x^{10} \times \frac{1}{x} + (\log x)(10x^9) \\
\therefore \frac{dv}{dx} &= v[x^9 + 10x^9 \log x] \\
\therefore \frac{dv}{dx} &= x^{x^{10}} \cdot x^9(1 + 10 \log x) \quad \dots(3) \\
\text{Also, } \omega &= x^{10x} \\
\therefore \log \omega &= \log x^{10x} = 10x \cdot \log x
\end{aligned}$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
& \frac{1}{\omega} \cdot \frac{d\omega}{dx} = \frac{d}{dx} (10^x \cdot \log x) \\
&= 10^x \cdot \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (10^x) \\
&= 10^x \times \frac{1}{x} + (\log x)(10^x \cdot \log 10) \\
\therefore \frac{d\omega}{dx} &= \omega \left[\frac{10^x}{x} + 10^x \cdot (\log x)(\log 10) \right] \\
\therefore \frac{d\omega}{dx} &= x^{10x} \cdot 10^x \left[\frac{1}{x} + (\log x)(\log 10) \right] \dots(4)
\end{aligned}$$

From (1),(2),(3) and (4), we get

$$\frac{dy}{dx} = 10^{x+x} \cdot \log 10 \cdot x^x (1 + \log x) + x^{x^{10}} \cdot x^9 (1 + 10 \log x) + x^{10x} \cdot 10^x \left[\frac{1}{x} + (\log x)(\log 10) \right].$$

Exercise 1.3 | Q 2.8 | Page 40

Differentiate the following w.r.t. x : $[(\tan x)^{\tan x}]^{\tan x}$ at $x = \frac{\pi}{4}$

SOLUTION

$$\begin{aligned} \text{Let } y &= [(\tan x)^{\tan x}]^{\tan x} \\ \therefore \log y &= \log [(\tan x)^{\tan x}] \tan x \\ &= \tan x \cdot \log(\tan x) \\ &= \tan x \cdot \tan x \log(\tan x) \\ &= (\tan x)^2 \cdot \log(\tan x) \end{aligned}$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx} [(\tan x)^2 \cdot \log(\tan x)] \\ &= (\tan x)^2 \cdot \frac{d}{dx} (\log \tan x) + (\log \tan x) \cdot \frac{d}{dx} (\tan x)^2 \\ &= (\tan x)^2 \cdot \frac{1}{\tan x} \cdot \frac{d}{dx} (\tan x) + (\log \tan x) \times 2 \tan x \cdot \frac{d}{dx} (\tan x) \\ &= (\tan x)^2 \times \frac{1}{\tan x} \cdot \sec^2 x + (\log \tan x) \times 2 \tan x \sec^2 x \\ \therefore \frac{dy}{dx} &= y [(\tan x)(\sec^2 x) + (\log \tan x)(2 \tan x \sec^2 x)] \\ &= [(\tan x)^{\tan x}]^{\tan x} \cdot (\tan x \sec^2 x) [1 + 2 \log \tan x] \end{aligned}$$

If $x = \frac{\pi}{4}$, then

$$\begin{aligned} \frac{dy}{dx} &= \left[\left(\frac{\tan \pi}{4} \right)^{\tan \frac{\pi}{4}} \right]^{\tan \frac{\pi}{4}} \left(\tan \frac{\pi}{4} \sec^2 \frac{\pi}{4} \right) \left[1 + 2 \log \tan \frac{\pi}{4} \right] \\ &= [(1)^1]^1 \cdot \left[1 (\sqrt{2})^2 \right] [1 + 2 \log 1] \\ &= 1 \times 2 \times 1 \quad \dots [\because \log 1 = 0] \\ &= 2. \end{aligned}$$

Exercise 1.3 | Q 3.01 | Page 40

Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = \sqrt{a}$

SOLUTION

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}.$$

Exercise 1.3 | Q 3.02 | Page 40

Find $\frac{dy}{dx}$ if $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$

SOLUTION

$$x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$$

$$\therefore x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$$

Differentiating both sides w.r.t. x, we get

$$\frac{3}{2} \cdot x^{\frac{1}{2}} + \frac{3}{2} \cdot y^{\frac{1}{2}} \frac{dy}{dx} = 0$$

$$\therefore \frac{3}{2} \cdot y^{\frac{1}{2}} \frac{dy}{dx} = -\frac{3}{2} x^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{-x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$$

$$= -\sqrt{\frac{x}{y}}.$$

Exercise 1.3 | Q 3.03 | Page 40

Find $\frac{dy}{dx}$ if $x + \sqrt{xy} + y = 1$

SOLUTION

$$x + \sqrt{xy} + y = 1$$

Differentiating both sides w.r.t. x, we get

$$1 + \frac{1}{2\sqrt{xy}} \cdot \frac{d}{dx}(xy) + \frac{dy}{dx} = 0$$

$$\therefore 1 + \frac{1}{2\sqrt{x}} \cdot \left[x \frac{dy}{dx} + y \times 1 \right] + \frac{dy}{dx} = 0$$

$$\therefore 1 + \frac{1}{2} \sqrt{\frac{x}{y}} \frac{dy}{dx} + \frac{1}{2} \sqrt{\frac{y}{x}} + \frac{dy}{dx} = 0$$

$$\therefore \left(\frac{1}{2} \sqrt{\frac{x}{y}} + 1 \right) \frac{dy}{dx} = -\frac{1}{2} \sqrt{\frac{y}{x}} - 1$$

$$\therefore \left(\frac{\sqrt{x} + 2\sqrt{y}}{2\sqrt{y}} \right) \frac{dy}{dx} = \frac{-\sqrt{y} - 2\sqrt{y}}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{-\sqrt{y}(2\sqrt{x} + \sqrt{y})}{\sqrt{x}(\sqrt{x} + 2\sqrt{y})}.$$

Exercise 1.3 | Q 3.04 | Page 40

Find $\frac{dy}{dx}$ if $x^3 + x^2y + xy^2 + y^3 = 81$

SOLUTION

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Differentiating both sides w.r.t. x, we get

$$3x^2 + x^2 \frac{dy}{dx} + y \frac{d}{dx}(x^2) + x \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x) + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore 3x^2 + x^2 \frac{dy}{dx} + y \times 2x + x \times 2y \frac{dy}{dx} + y^2 \times 1 + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore 3x^2 + x^2 \frac{dy}{dx} + 2xy + 2xy \frac{dy}{dx} + y^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore (x^2 + 2xy + 3y^2) \frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\therefore \frac{dy}{dx} = \frac{(-3x^2 - 2xy - y^2)}{x^2 + 2xy + 3y^2}.$$

Exercise 1.3 | Q 3.05 | Page 40

$$\text{Find } \frac{dy}{dx} \text{ if } x^2y^2 - \tan^{-1}(\sqrt{x^2 + y^2}) = \cot^{-1}(\sqrt{x^2 + y^2})$$

SOLUTION

$$x^2y^2 - \tan^{-1}(\sqrt{x^2 + y^2}) = \cot^{-1}(\sqrt{x^2 + y^2})$$

$$\therefore x^2y^2 - \tan^{-1}(\sqrt{x^2 + y^2}) + \cot^{-1}(\sqrt{x^2 + y^2})$$

$$\therefore x^2y^2 = \frac{\pi}{2} \quad \dots [\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}]$$

Differentiating both sides w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore x^2 \times 2y \frac{dy}{dx} + y^2 \times 2x = 0$$

$$\therefore 2x^2y \frac{dy}{dx} = -2xy^2$$

$$\therefore x \frac{dy}{dx} = -y$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}.$$

Exercise 1.3 | Q 3.06 | Page 40

Find $\frac{dy}{dx}$ if $xe^y + ye^x = 1$

SOLUTION

$$xe^y + ye^x = 1$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(xe^y) + \frac{d}{dx}(ye^x) = 0$$

$$\therefore x \cdot \frac{d}{dx}(e^y) + e^y \cdot \frac{d}{dx}(x) + y \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{dy}{dx} = 0$$

$$\therefore x \cdot e^y \frac{dy}{dx} + e^y \times 1 + y \times e^x + e^x \frac{dy}{dx} = 0$$

$$\therefore (e^x + xe^y) \frac{dy}{dx} = -e^y - ye^x$$

$$\therefore \frac{dy}{dx} = -\left(\frac{e^y + ye^x}{e^x + xe^y}\right).$$

Exercise 1.3 | Q 3.07 | Page 40

Find $\frac{dy}{dx}$ if $e^{x+y} = \cos(x-y)$

SOLUTION

$$e^{x+y} = \cos(x-y)$$

Differentiating both sides w.r.t. x , we get

$$e^{x+y} \cdot \frac{d}{dx}(x+y) = -\sin(x-y) \cdot \frac{d}{dx}(x-y)$$

$$\therefore e^{x+y} \left(1 + \frac{dy}{dx} \right) = -\sin(x-y) \cdot \frac{dy}{dx} (x-y)$$

$$\therefore e^{x+y} + e^{x+y} \cdot \frac{dy}{dx} = -\sin(x-y) \left(1 - \frac{dy}{dx} \right)$$

$$\therefore [e^{x+y} - \sin(x-y)] \frac{dy}{dx} = -\sin(x-y) - e^{x+y}$$

$$\therefore \frac{dy}{dx} = - \left[\frac{\sin(x-y) + e^{x+y}}{e^{x+y} - \sin(x-y)} \right] = \frac{\sin(x-y) + e^{x+y}}{\sin(x-y) - e^{x+y}}.$$

Exercise 1.3 | Q 3.08 | Page 40

Find $\frac{dy}{dx}$ if $\cos(xy) = x + y$

SOLUTION

$$\cos(xy) = x + y$$

Differentiating both sides w.r.t. x , we get

$$-\sin(xy) \cdot \frac{d}{dx}(xy) = 1 + \frac{dy}{dx}$$

$$\therefore -\sin(xy) \left[x \frac{dy}{dx} + y \frac{d}{dx}(x) \right] = 1 + \frac{dy}{dx}$$

$$\therefore -\sin(xy) \left[x \frac{dy}{dx} + y \times 1 \right] = 1 + \frac{dy}{dx}$$

$$\therefore -x \sin(xy) \frac{dy}{dx} - y \sin(xy) = 1 + \frac{dy}{dx}$$

$$\therefore -\frac{dy}{dx} - \sin(xy) \frac{dy}{dx} = 1 + y \sin(xy)$$

$$\therefore -[1 + x \sin(xy)] \frac{dy}{dx} = 1 + y \sin(xy)$$

$$\therefore \frac{dy}{dx} = \frac{-[1 + y \sin(xy)]}{1 + x \sin(xy)}.$$

Exercise 1.3 | Q 3.09 | Page 40

Find $\frac{dy}{dx}$ if $e^{x-y} = \frac{x}{y}$

SOLUTION

$$e^{x-y} = \frac{x}{y}$$

$$\therefore e^{x-y} = \log\left(\frac{x}{y}\right) \quad \dots [\because e^x = y \Rightarrow x = \log y]$$

$$\therefore e^{x-y} = \log x - \log y$$

Differentiating both sides w.r.t. x, we get

$$e^{x-y} \cdot \frac{d}{dx}(x-y) = \frac{1}{x} - \frac{1}{y} \frac{dy}{dx}$$

$$\therefore e^{x-y} \left(1 - \frac{dy}{dx}\right) = \frac{1}{x} - \frac{1}{y} \frac{dy}{dx}$$

$$\therefore e^{x-y} - e^{x-y} \frac{dy}{dx} = \frac{1}{x} - \frac{1}{y} \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{y} - e^{x-y}\right) \frac{dy}{dx} = \frac{1}{x} - e^{x-y}$$

$$\left(\frac{1 - ye^{x-y}}{y}\right) \frac{dy}{dx} = \frac{1 - xe^{x-y}}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(1 - xe^{x-y})}{(x(1 - ye^{x-y}))}.$$

Exercise 1.3 | Q 4.1 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $x^7 \cdot y^5 = (x + y)^{12}$

SOLUTION

$$x^7 \cdot y^5 = (x + y)^{12}$$

$$\therefore (\log x^7 \cdot y^5) = \log(x + y)^{12}$$

$$\therefore \log x^7 + \log y^5 = \log(x + y)^{12}$$

$$\therefore 7\log x + 5\log y = 12\log(x + y)$$

Differentiating both sides w.r.t. x, we get

$$7 \times \frac{1}{x} + 5 \times \frac{1}{y} \cdot \frac{dy}{dx} = 12 \times \frac{1}{x + y} \cdot \frac{d}{dx}(x + y)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{dy}{dx} = \frac{12}{x + y} \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{y}{dx} = \frac{12}{x + y} + \frac{12}{x + y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{5}{y} - \frac{12}{x + y}\right) \frac{dy}{dx} = \frac{12}{x + y} - \frac{7}{x}$$

$$\therefore \left[\frac{5x + 5y - 12y}{y(x + y)}\right] \frac{dy}{dx} = \frac{12x - 7x - 7y}{x(x + y)}$$

$$\therefore \left[\frac{5x - 7y}{y(x + y)}\right] \frac{dy}{dx} = \frac{5x - 7y}{x(x + y)}$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.2 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $x^p y^4 = (x + y)^{p+4}$, $p \in \mathbb{N}$

SOLUTION

$$x^p y^4 = (x + y)^{p+4}$$

Taking log

$$\log(x^p y^4) = \log(x + y)^{p+4}$$

$$\log x^p + \log y^4 = (p + 4) \log(x + y)$$

$$p \log x + 4 \log y = (p + 4) \log(x + y)$$

Differentiating both sides w.r.t. x, we get

$$p \cdot \frac{d}{dx} \log x + 4 \cdot \frac{d}{dx} \log y = (p + 4) \frac{d}{dx} \log(x + y)$$

$$\frac{p}{x} + 4 \frac{1}{y} \frac{dy}{dx} = (p + 4) \frac{1}{x + y} \left(1 + \frac{dy}{dx} \right)$$

$$\frac{p}{4} + \frac{4}{y} \frac{dy}{dx} = \frac{(p + 4)}{(x + y)} + \frac{p + 4}{(x + y)} \frac{dy}{dx}$$

$$\frac{dy}{dx} \left[\frac{4}{y} - \frac{(p + 4)}{(x + y)} \right] = \frac{p + 4}{x + y} - \frac{p}{x}$$

$$\frac{dy}{dx} \left[\frac{4(x + y) - y(p + 4)}{y(x + y)} \right] = \frac{x(p + 4) - p(x + y)}{x(x + y)}$$

$$\frac{dy}{dx} \left[\frac{4x + 4y - py - 4y}{y(x + y)} \right] = \frac{px + 4x - px - py}{x(x + y)}$$

$$\frac{dy}{dx} \left[\frac{4x - py}{y} \right] = \frac{4x - py}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.3 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $\sec\left(\frac{x^5 + y^5}{x^5 - y^5}\right) = a^2$

SOLUTION

$$\sec\left(\frac{x^5 + y^5}{x^5 - y^5}\right) = a^2$$

$$\therefore \frac{x^5 + y^5}{x^5 - y^5} = \sec^{-1}(a^2) = k$$

$$\therefore x^5 + y^5 = kx^5 - ky^5$$

$$\therefore (1 + k)y^5 = (k - 1)x^5$$

$$\therefore \frac{y^5}{x^5} = \frac{k - 1}{k + 1}$$

$$\therefore \frac{y}{x} = \left(\frac{k - 1}{k + 1}\right)^{\frac{1}{5}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

$$\therefore \frac{x \cdot \frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x \frac{dy}{dx} - y \times 1 = 0$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

Alternative Method :

$$\sec\left(\frac{x^5 + y^5}{x^5 - y^5}\right) = a^2$$

$$\therefore \frac{x^5 + y^5}{x^5 - y^5} = \sec^{-1}a^2 = k \quad \dots(\text{Say})$$

$$\therefore x^5 + y^5 = kx^5 - ky^5$$

$$\therefore (1 + k)y^5 = (k - 1)x^5$$

$$\therefore \frac{y^5}{x^5} = \frac{k-1}{k+1} \quad \dots(1)$$

$$\therefore y^5 = k'x^5, \text{ where } k' = \frac{k-1}{k+1}$$

Differentiating both sides w.r.t. x, we get

$$5y^4 \frac{dy}{dx} = k' \times 5x^4$$

$$\therefore \frac{dy}{dx} = k' \cdot \frac{x^4}{y^4}$$

$$\therefore \frac{dy}{dx} = \left(\frac{k-1}{k+1} \right) \cdot \frac{x^4}{y^4}$$

$$= \frac{y^5}{x^5} \times \frac{x^4}{y^4} \quad \dots[\text{By (1)}]$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.4 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $\tan^{-1} \left(\frac{3x^2 - 4y^2}{3x^2 + 4y^2} \right) = a^2$

SOLUTION

$$\tan^{-1} \left(\frac{3x^2 - 4y^2}{3x^2 + 4y^2} \right) = a^2$$

$$\therefore \frac{3x^2 - 4y^2}{3x^2 + 4y^2} = \tan a^2 = k \quad \dots(\text{Say})$$

$$\therefore 3x^2 - 4y^2 = 3kx^2 + 4ky^2$$

$$\therefore (4k + 4)y^2 = (3 - 3k)x^2$$

$$\therefore \frac{y^2}{x^2} = \frac{3 - 3k}{4k + 4}$$

$$\therefore \frac{y}{x} = \sqrt{\frac{3 - 3k}{4k + 4}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

$$\therefore \frac{x \frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x \frac{dy}{dx} - y \times 1 = 0$$

$$\therefore x \cdot \frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.5 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $\cos^{-1}\left(\frac{7x^4 + 5y^4}{7x^4 - 5y^4}\right) = \tan^{-1} a$

SOLUTION

$$\cos^{-1}\left(\frac{7x^4 + 5y^4}{7x^4 - 5y^4}\right) = \tan^{-1} a$$

$$\frac{7x^4 + 5y^4}{7x^4 - 5y^4} = \cos(\tan^{-1} a) = b$$

$$\frac{7x^4 + 5y^4}{7x^4 - 5y^4} = b$$

$$7x^4 + 5y^4 = b(7x^4 - 5y^4)$$

$$7x^4 + 5y^4 = 7bx^4 - 5by^4$$

$$5y^4 + 5by^4 = 7bx^4 - 7x^4$$

$$5y^4(1 + b) = 7x^4(b - 1)$$

$$\frac{5y^4}{7x^4} = \frac{b-1}{1+b}$$

$$\frac{y^4}{x^4} = \frac{7(b-1)}{5(1+b)} = x$$

$$\frac{y^4}{x^4} = c \dots (1)$$

$$y^4 = cx^4$$

Differentiating both sides w.r.t. x, we get

$$4 \cdot y^3 \frac{dy}{dx} = c \cdot 4x^3$$

$$\frac{dy}{dx} = \frac{c \cdot 4x^3}{4y^3}$$

$$\frac{dy}{dx} = \frac{c \cdot x^3}{y^3}$$

$$\frac{dy}{dx} = \frac{y^4}{x^4} \cdot \frac{x^3}{y^3} \quad \dots \text{from..(1)}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.6 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$

SOLUTION

$$\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$$

$$\therefore \frac{x^{20} - y^{20}}{x^{20} + y^{20}} = e^{20} = k \quad \dots (\text{Say})$$

$$\therefore x^{20} - y^{20} = kx^{20} + ky^{20}$$

$$\therefore (1 + k)y^{20} = kx^{20} + ky^{20}$$

$$\therefore \frac{y^{20}}{x^{20}} = \frac{1 - k}{1 + k}$$

$$\therefore \frac{y}{x} = \left(\frac{1 - k}{1 + k}\right)^{\frac{1}{20}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

$$\therefore \frac{x \frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x \frac{dy}{dx} - y \times 1 = 0$$

$$\therefore x \frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 4.7 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $e^{\frac{x^7 - y^7}{x^7 + y^7}} = a$

SOLUTION

$$e^{\frac{x^7 - y^7}{x^7 + y^7}} = a$$

$$\therefore \frac{x^7 - y^7}{x^7 + y^7} = \log a = k \quad \dots(\text{Say})$$

$$\therefore x^7 - y^7 = kx^7 + ky^7$$

$$\therefore (1 + k)y^7 = (1 - k)x^7$$

$$\therefore \frac{y^7}{x^7} = \frac{1 - k}{1 + k}$$

$$\therefore \frac{y}{x} = \left(\frac{1 - k}{1 + k} \right)^{\frac{1}{7}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx} \left(\frac{y}{x} \right) = 0$$

$$\therefore \frac{x \frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x \frac{dy}{dx} - y \times 1 = 0$$

$$\therefore x \frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

Exercise 1.3 | Q 4.8 | Page 40

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants : $\sin\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = a^3$

SOLUTION

$$\sin\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = a^3$$

$$\frac{x^3 - y^3}{x^3 + y^3} = \sin a^3 = b$$

$$\frac{x^3 - y^3}{x^3 + y^3} = b$$

$$x^3 - y^3 = b(x^3 + y^3)$$

$$x^3 - y^3 = bx^3 + by^3$$

$$x^3 - bx^3 = by^3 + y^3$$

$$x^3(1 - b) = y^3(b + 1)$$

$$\frac{y^3}{x^3} = \frac{1 - b}{1 + b} = e$$

$$\frac{y^3}{x^3} = c \quad \dots\dots(1)$$

$$y^3 = cx^3$$

Differentiating both sides w.r.t. x, we get

$$3y^2 \frac{dy}{dx} = c \cdot 3x^2$$

$$\frac{y^2 dy}{dx} = cx^2$$

$$\frac{dy}{dx} c \frac{x^2}{y^2}$$

$$\frac{dy}{dx} = \frac{y^3}{x^3} \cdot \frac{x^2}{y^2} \quad \dots\text{from}(1)$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

Exercise 1.3 | Q 5.01 | Page 40

If $\log(x + y) = \log(xy) + p$, where p is a constant, then prove that $\frac{dy}{dx} = \frac{-y^2}{x^2}$.

SOLUTION

$$\log(x + y) = \log(xy) + p$$

$$\therefore \log(x + y) = \log x + \log y + p$$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{x+y} \cdot \frac{d}{dx}(x+y) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore \frac{1}{x+y} \left(1 + \frac{dy}{dx} \right) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \frac{1}{x+y} + \frac{1}{x+y} \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{x+y} - \frac{1}{y} \right) \frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+y}$$

$$\therefore \left[\frac{y-x-y}{y(x+y)} \right] \frac{dy}{dx} = \frac{x+y-x}{x(x+y)}$$

$$\therefore \left[\frac{-x}{y(x+y)} \right] \frac{dy}{dx} = \frac{y}{x(x+y)}$$

$$\therefore \left(-\frac{x}{y} \right) \frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{y^2}{x^2}.$$

Exercise 1.3 | Q 5.02 | Page 40

If $\log_{10} \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 2$, show that $\frac{dy}{dx} = -\frac{99x^2}{101y^2}$

SOLUTION

$$\log_{10} \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 2$$

$$\therefore \frac{x^3 - y^3}{x^3 + y^3} = 10^2 = 100$$

$$\therefore x^3 - y^3 = 100x^3 + 100y^3$$

$$\therefore 101y^3 = -99x^3$$

$$\therefore y^3 = \frac{-99}{101}x^3$$

Differentiating both sides w.r.t. x, we get

$$3y^2 \frac{dy}{dx} = \frac{-99}{101} \times 3x^2$$

$$\therefore \frac{dy}{dx} = -\frac{99x^2}{101y^2}.$$

Exercise 1.3 | Q 5.03 | Page 40

If $\log_5 \left(\frac{x^4 + y^4}{x^4 - y^4} \right) = 2$, show that $\frac{dy}{dx} = \frac{12x^3}{13y^3}$.

SOLUTION

$$\log_5 \left(\frac{x^4 + y^4}{x^4 - y^4} \right) = 2$$

$$\frac{x^4 + y^4}{x^4 - y^4} = 5^2$$

$$\frac{x^4 + y^4}{x^4 - y^4} = 25$$

$$x^4 + y^4 = 25x^4 - 25y^4$$

$$26y^4 = 24x^4$$

Differentiating both sides w.r.t. x, we get

$$26 \frac{dy}{dx} y^4 = 24 \frac{d}{dx} x^4$$

$$26.4y^3 \cdot \frac{dy}{dx} = 24.4 \cdot x^3$$

$$26y^3 \cdot \frac{dy}{dx} = 24 \cdot x^3$$

$$\frac{dy}{dx} = \frac{24x^3}{26y^3}$$

$$\frac{dx}{dy} = \frac{12x^3}{13y^3}$$

Exercise 1.3 | Q 5.04 | Page 40

If $e^x + e^y = e^{x+y}$, then show that $\frac{dy}{dx} = -e^{y-x}$.

SOLUTION

$$e^x + e^y = e^{x+y} \quad \dots(1)$$

Differentiating both sides w.r.t. x, we get

$$e^x + e^y \cdot \frac{dy}{dx} = e^{x+y} \cdot \frac{d}{dx}(x+y)$$

$$\therefore e^x + e^y \cdot \frac{dy}{dx} = e^{x+y} \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\therefore e^x + e^y \frac{dy}{dx} = e^{x+y} + e^{x+y} \frac{dy}{dx}$$

$$\therefore (e^y - e^{x+y}) \frac{dy}{dx} = e^{x+y} - e^x$$

$$\therefore \frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

$$\begin{aligned}
 &= \frac{e^x + e^y - e^x}{e^y - e^x - e^y} \quad \dots[\text{By (1)}] \\
 &= \frac{e^y}{-e^x} \\
 &= -e^{y-x}.
 \end{aligned}$$

Exercise 1.3 | Q 5.05 | Page 40

If $\sin^{-1}\left(\frac{x^5 - y^5}{x^5 + y^5}\right) = \frac{\pi}{6}$, show that $\frac{dy}{dx} = \frac{x^4}{3y^4}$

SOLUTION

$$\begin{aligned}
 \sin^{-1}\left(\frac{x^5 - y^5}{x^5 + y^5}\right) &= \frac{\pi}{6} \\
 \therefore \frac{x^5 - y^5}{x^5 + y^5} &= \sin \frac{\pi}{6} = \frac{1}{2} \\
 \therefore 2x^5 - 2y^5 &= x^5 + y^5 \\
 \therefore 3y^5 &= x^5
 \end{aligned}$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
 3 \times 5y^4 \frac{dy}{dx} &= 5x^4 \\
 \therefore \frac{dy}{dx} &= \frac{x^4}{3y^4}.
 \end{aligned}$$

Exercise 1.3 | Q 5.06 | Page 40

If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

SOLUTION

$$x^y = e^{x-y}$$

$$\therefore \log xy = \log e^{x-y}$$

$$\therefore y \log x = (x - y) \log e$$

$$\therefore y \log x = x - y \quad \dots [\because \log e = 1]$$

$$\therefore y + y \log x = x \quad \therefore y(1 + \log x) = x$$

$$\therefore y = \frac{x}{1 + \log x}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{1 + \log x} \right)$$

$$= \frac{(1 + \log x) \cdot \frac{d}{dx}(x) - x \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x) \cdot 1 - x \left(0 + \frac{1}{x} \right)}{(1 + \log x)^2}$$

$$= \frac{1 + \log x - 1}{(1 + \log x)^2}$$

$$= \frac{\log x}{(1 + \log x)^2}.$$

Exercise 1.3 | Q 5.07 | Page 40

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, then show that $\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$.

SOLUTION

$$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$$

$$\therefore y^2 = \cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}$$

$$\therefore y^2 = \cos x + y$$

Differentiating both sides w.r.t. x, we get

$$2y \frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\therefore (1 - 2y) \frac{dy}{dx} = \sin x$$

$$\therefore \frac{dy}{dx} = \frac{\sin x}{1 - 2y}.$$

Exercise 1.3 | Q 5.08 | Page 40

If $y = \sqrt{\log x + \log x + \sqrt{\log x + \dots \infty}}$, show that $\frac{dy}{dx} = \frac{1}{x(2y - 1)}$.

SOLUTION

$$y = \sqrt{\log x + \log x + \sqrt{\log x + \dots \infty}}$$

$$\therefore y^2 = \log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}$$

$$\therefore y^2 = \log x + y$$

Differentiating both sides w.r.t. x , we get

$$2y \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\therefore (2y - 1) \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x(2y - 1)}.$$

Exercise 1.3 | Q 5.09 | Page 40

If $y = x^{x^{\dots \infty}}$, show that $\frac{dy}{dx} = \frac{y^2}{x(1 - \log y)}$.

SOLUTION

$$\begin{aligned}y &= x^{x^{x^{\dots^{\infty}}}}} \\ \therefore \log y &= \log\left(x^{x^{x^{\dots^{\infty}}}}\right) \\ &= \log x^{x^{x^{\dots^{\infty}}}} \cdot \log x \\ \therefore \log y &= y \log x \quad \dots(1)\end{aligned}$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= y \cdot \frac{d}{dx}(\log x) + (\log x) \frac{dy}{dx} \\ \therefore \frac{1}{y} \frac{dy}{dx} &= y \times \frac{1}{x} + (\log x) \frac{dy}{dx} \\ \therefore \left(\frac{1}{y} - \log x\right) \frac{dy}{dx} &= \frac{y}{x} \\ \therefore \left(\frac{1 - y \log x}{y}\right) \frac{dy}{dx} &= \frac{y}{x} \\ \therefore \frac{dy}{dx} &= \frac{y^2}{x(1 - y \log x)} \\ \therefore \frac{dy}{dx} &= \frac{y^2}{x(1 - y \log)} \quad \dots[\text{By (1)}]\end{aligned}$$

Exercise 1.3 | Q 5.1 | Page 40

If $e^y = y^x$, then show that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$.

SOLUTION

$$e^y = y^x$$

$$\therefore \log e^y = \log y^x$$

$$\therefore y \log e = x \log y$$

$$\therefore y = x \log y \quad \dots [\because \log e = 1] \quad \dots (1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx} = x \times \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \times 1$$

$$\therefore \frac{dy}{dx} = \frac{x}{y} \frac{dy}{dx} + \log y$$

$$\therefore \left(1 - \frac{x}{y}\right) \frac{dy}{dx} = \log y$$

$$\therefore \left(\frac{y-x}{y}\right) \frac{dy}{dx} = \log y$$

$$\therefore \frac{dy}{dx} = \frac{y \log y}{y-x}$$

$$= \frac{y \log y}{y - \left(\frac{y}{\log y}\right)} \quad \dots [\text{By (1)}]$$

$$\therefore \frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}.$$

Alternative Method :

$$ey = yx$$

$$\therefore \log ey = \log yx$$

$$\therefore y \log e = x \log y$$

$$\therefore y = x \log y \quad \dots [\because \log e = 1]$$

$$\therefore x = \frac{y}{\log y}$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy} \left(\frac{y}{\log y} \right) \\ &= \frac{(\log y) \cdot \frac{d}{dy}(y) - y \cdot \frac{d}{dy}(\log y)}{(\log y)^2} \\ &= \frac{(\log y) \times 1 - y \times \frac{1}{y}}{(\log y)^2} \\ &= \frac{\log y - 1}{(\log y)^2} \\ \therefore \frac{dy}{dx} &= \frac{1}{\left(\frac{dx}{dy} \right)} = \frac{(\log y)^2}{\log y - 1}.\end{aligned}$$

EXERCISE 1.4 [PAGES 48 - 49]

Exercise 1.4 | Q 1.1 | Page 48

Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$

SOLUTION

$$x = at^2, y = 2at$$

Differentiating x and y w.r.t. x, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(at^2) = a \frac{d}{dt}(t^2) \\ &= a \times 2t = 2at\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(2at) = 2a \frac{d}{dt}(t) \\ &= 2a \times 1 = 2a\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

Exercise 1.4 | Q 1.2 | Page 48

Find $\frac{dy}{dx}$ if $x = a \cot \theta$, $y = b \operatorname{cosec} \theta$

SOLUTION

$$a \cot \theta, y = b \operatorname{cosec} \theta$$

Differentiating x and y w.r.t. x, we get

$$\begin{aligned}\frac{dx}{d\theta} &= a \frac{d}{d\theta} (\cot \theta) = a(-\operatorname{cosec}^2 \theta) \\ &= -b \operatorname{cosec}^2 \theta\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= b \frac{d}{d\theta} (\operatorname{cosec} \theta) = b(-\operatorname{cosec} \theta \cot \theta) \\ &= -b \operatorname{cosec} \theta \cot \theta\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b \operatorname{cosec} \theta}{-a \operatorname{cosec}^2 \theta}$$

$$= \frac{b}{a} \cdot \frac{\cot \theta}{\operatorname{cosec} \theta}$$

$$= \frac{b}{a} \times \frac{\cos \theta}{\sin \theta} \times \sin \theta$$

$$= \left(\frac{b}{a}\right) \cos \theta.$$

Exercise 1.4 | Q 1.3 | Page 48

Find $\frac{dy}{dx}$, if : $x = \sqrt{a^2 + m^2}$, $y = \log(a^2 + m^2)$

SOLUTION

$$x = \sqrt{a^2 + m^2}, y = \log(a^2 + m^2)$$

Differentiating x and y w.r.t. x, we get

$$\begin{aligned}\frac{dx}{dm} &= \frac{d}{dm} \left(\sqrt{a^2 + m^2} \right) \\ &= \frac{1}{2\sqrt{a^2 + m^2}} \cdot \frac{d}{dm} (a^2 + m^2) \\ &= \frac{1}{2\sqrt{a^2 + m^2}} \times (0 + 2m) = \frac{m}{\sqrt{a^2 + m^2}}\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dm} &= \frac{d}{dm} [\log(a^2 + m^2)] \\ &= \frac{1}{a^2 + m^2} \cdot \frac{d}{dm} (a^2 + m^2) \\ &= \frac{1}{a^2 + m^2} \times (0 + 2m) = \frac{2m}{a^2 + m^2} \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dm} \right)}{\left(\frac{dx}{dm} \right)}\end{aligned}$$

$$\begin{aligned}&= \frac{\left(\frac{2m}{a^2 + m^2} \right)}{\left(\frac{m}{\sqrt{a^2 + m^2}} \right)} \\ &= \frac{2}{\sqrt{a^2 + m^2}}.\end{aligned}$$

Exercise 1.4 | Q 1.4 | Page 48

Find $\frac{dy}{dx}$, if : $x = \sin\theta$, $y = \tan\theta$

SOLUTION

$$x = \sin\theta, y = \tan\theta$$

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \sec^2\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{\sec^2\theta}{\cos\theta}$$

$$= \sec^3\theta.$$

Exercise 1.4 | Q 1.5 | Page 48

Find $\frac{dy}{dx}$, if : $x = a(1 - \cos\theta)$, $y = b(\theta - \sin\theta)$

SOLUTION

$$x = a(1 - \cos\theta), y = b(\theta - \sin\theta)$$

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{d\theta} = a \frac{d}{d\theta}(1 - \cos\theta)$$

$$= a[0 - (-\sin\theta)] = a \sin\theta$$

and

$$\frac{dy}{d\theta} = b \frac{d}{d\theta}(\theta - \sin\theta)$$

$$= b(1 - \cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b(1 - \cos\theta)}{a \sin\theta}$$

$$= \frac{b \times \sin^2\left(\frac{\theta}{2}\right)}{a \times 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)}$$

$$= \left(\frac{b}{a}\right) \tan\left(\frac{\theta}{2}\right).$$

Exercise 1.4 | Q 1.6 | Page 48

Find $\frac{dy}{dx}$, if : $x = \left(t + \frac{1}{t}\right)$, $y = a\left(t + \frac{1}{t}\right)$, where $a > 0, a \neq 1, t \neq 0$.

SOLUTION

$$x = \left(t + \frac{1}{t} \right), y = a \left(t + \frac{1}{t} \right)^a \quad \dots(1)$$

Differentiating x and y w.r.t. x, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt} \left(t + \frac{1}{t} \right)^a \\ &= a \left(t + \frac{1}{t} \right)^{a-1} \cdot \frac{d}{dt} \left(t + \frac{1}{t} \right)\end{aligned}$$

$$= a \left(t + \frac{1}{t} \right)^{a-1} \cdot \left(1 - \frac{1}{t^2} \right)$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt} \left[a^{(t+\frac{1}{t})} \right] \\ &= a^{(t+\frac{1}{t})} \cdot \log a \cdot \frac{d}{dt} \left(t + \frac{1}{t} \right) \\ &= a^{(t+\frac{1}{t})} \cdot \log a \cdot \left(1 - \frac{1}{t^2} \right)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)}$$

$$= \frac{a^{(t+\frac{1}{t})} \cdot \log a \cdot \left(1 - \frac{1}{t^2} \right)}{a \left(t + \frac{1}{t} \right)^{a-1} \cdot \left(1 - \frac{1}{t^2} \right)}$$

$$= \frac{a^{t+\frac{1}{t}} \cdot \log a \cdot \left(t + \frac{1}{t} \right)}{a \cdot \left(t + \frac{1}{t} \right)^a}$$

$$\begin{aligned}
 &= \frac{y \log a \cdot \left(\frac{t^2+1}{t} \right)}{ax} \quad \dots [\text{By (1)}] \\
 &= \frac{y(t^2 + 1) \log a}{axt}.
 \end{aligned}$$

Exercise 1.4 | Q 1.7 | Page 48

Find $\frac{dy}{dx}$, if : $x = \cos^{-1}\left(\frac{2t}{1+t^2}\right)$, $y = \sec^{-1}\left(\sqrt{1+t^2}\right)$

SOLUTION

$$x = \cos^{-1}\left(\frac{2t}{1+t^2}\right), y = \sec^{-1}\left(\sqrt{1+t^2}\right)$$

Put $t = \tan\theta$.

Then $\theta = \tan^{-1}t$.

$$\therefore x = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right), y = \sec^{-1}\left(\sqrt{1+\tan^2\theta}\right)$$

$$\therefore x = \cos^{-1}(\sin 2\theta), y = \sec^{-1}\left(\sqrt{\sec^2\theta}\right)$$

$$\therefore x = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right], y = \sec^{-1}(\sec\theta)$$

$$\therefore x = \frac{\pi}{2} - 2\theta, y = \theta$$

$$\therefore x = \frac{\pi}{2} - 2\tan^{-1}t, y = \tan^{-1}t$$

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{\pi}{2}\right) - 2\frac{d}{dt}(\tan^{-1}t)$$

$$= 0 - 2 \times \frac{1}{1+t^2}$$

$$= \frac{-2}{1+t^2}$$

and

$$\frac{dy}{dt} = \frac{d}{dt}(\tan^{-1} t)$$

$$= \frac{1}{1+t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)}$$

$$= \frac{\left(\frac{1}{1+t^2} \right)}{\left(\frac{-2}{1+t^2} \right)}$$

$$= -\frac{1}{2}.$$

Exercise 1.4 | Q 1.8 | Page 48

Find $\frac{dy}{dx}$, if : $x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$.

SOLUTION

$$x = \cos^{-1}(4t^3 - 3t), y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$$

Put $t = \cos\theta$.

Then $\theta = \cos^{-1}t$.

$$\therefore x = \cos^{-1}(4\cos^3\theta - 3\cos\theta),$$

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$\therefore x = \cos^{-1}(\cos 3\theta), y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$\therefore x = 3\theta \text{ and } y = \theta$$

$$\therefore x = 3\cos^{-1}t \text{ and } y = \cos^{-1}t$$

Differentiating x and y w.r.t. x, we get

$$\frac{dx}{dt} = 3 \frac{d}{dt} (\cos^{-1})$$

$$= 3 \times \frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{-3}{\sqrt{1-t^2}}$$

and

$$\frac{dy}{dt} = \frac{d}{dt} (\cos^{-1} t)$$

$$= \frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)}$$

$$= \frac{\left(\frac{-1}{\sqrt{1-t^2}} \right)}{\left(\frac{-3}{\sqrt{1-t^2}} \right)}$$

$$= \frac{1}{3}.$$

Alternative Method :

$$x = \cos^{-1} (4t^3 - 3t), t = \tan^{-1} \left(\frac{\sqrt{1-t^2}}{t} \right)$$

Put $t = \cos\theta$.

$$\text{Then } x = \cos^{-1}(4\cos^3\theta - 3\cos\theta),$$

$$y = \tan^{-1} \left(\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta} \right)$$

$$\therefore x = \cos^{-1}(\cos 3\theta), y = \tan^{-1} \left(\frac{\sin \theta}{\cos \theta} \right) = \tan^{-1}(\tan \theta)$$

$$\therefore x = 3\theta, y = \theta$$

$$\therefore x = 3y$$

$$\therefore y = \frac{1}{3}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{3} \frac{d}{dx}(x)$$

$$= \frac{1}{3} \times 1$$

$$= \frac{1}{3}.$$

Exercise 1.4 | Q 2.1 | Page 48

Find $\frac{dy}{dx}$ if : $x = \operatorname{cosec}^2 \theta, y = \cot^3 \theta$ at $\theta = \frac{\pi}{6}$

SOLUTION

$$x = \operatorname{cosec}^2 \theta, y = \cot^3 \theta$$

Differentiating x and y w.r.t. θ , we get

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{d}{d\theta} (\operatorname{cosec} \theta)^2 = 2 \operatorname{cosec} \theta \cdot \frac{d}{d\theta} (\operatorname{cosec} \theta) \\ &= 2 \operatorname{cosec} \theta (-\operatorname{cosec} \theta \cot \theta) \\ &= -2 \operatorname{cosec}^2 \theta \cot \theta\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta} (\cot \theta)^3 = 3 \cot^2 \theta \cdot \frac{d}{d\theta} (\cot \theta) \\ &= 3 \cot^2 \theta \cdot (-\operatorname{cosec}^2 \theta) \\ &= -3 \cot^2 \theta \operatorname{cosec}^2 \theta\end{aligned}$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-3 \cot^2 \theta \cdot \csc^2 \theta}{-2 \cosec^2 \theta \cdot \cot \theta} \\
&= \frac{3}{2} \cot \theta \\
\therefore \left(\frac{dy}{dx}\right)_{\text{at } \theta=\frac{\pi}{6}} &= \frac{3}{2} \cot \frac{\pi}{6} \\
&= \frac{3}{2} \cot \frac{\pi}{6} \\
&= \frac{3\sqrt{3}}{2}.
\end{aligned}$$

Exercise 1.4 | Q 2.2 | Page 48

Find $\frac{dy}{dx}$ if : $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{3}$

SOLUTION

$$a \cos^3 \theta, y = a \sin^3 \theta$$

Differentiating x and y w.r.t. θ , we get

$$\begin{aligned}
\frac{dx}{d\theta} &= a \frac{d}{d\theta} (\cos \theta)^3 \\
&= a \times 3 \cos^2 \theta \cdot \frac{d}{d\theta} (\cos \theta) \\
&= 3a \cos^2 \theta (-\sin \theta) \\
&= -3a \cos^2 \theta \sin \theta
\end{aligned}$$

and

$$\frac{dy}{d\theta} = a \frac{d}{d\theta} (\sin \theta)^3$$

$$= a \times 3 \sin^2 \theta \cdot \frac{d}{d\theta}(\sin \theta)$$

$$= 3a \sin^2 \theta \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)}$$

$$= \frac{3a \sin^2 \cos \theta}{-3a \cos^2 \theta \sin \theta}$$

$$= -\tan \theta$$

$$\therefore \left(\frac{dx}{dy} \right)_{\text{at } \theta = -\frac{\pi}{3}}$$

$$= -\tan \frac{\pi}{3}$$

$$= -\sqrt{3}.$$

Exercise 1.4 | Q 2.3 | Page 48

Find $\frac{dy}{dx}$ if : $x = t^2 + t + 1$, $y = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$ at $t = 1$

SOLUTION

$$x = t^2 + t + 1, y = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(t^2 + t + 1)$$

$$= 2t + 1 + 0 = 2t + 1$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left[\sin\left(\frac{\pi t}{2}\right)\right] + \frac{d}{dt}\left[\cos\left(\frac{\pi t}{2}\right)\right] \\ &= \cos\left(\frac{\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{\pi t}{2}\right) + \left[-\sin\left(\frac{\pi t}{2}\right)\right] \cdot \frac{d}{dt}\left(\frac{\pi t}{2}\right) \\ &= \cos\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1 - \sin\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1 \\ &= \frac{\pi}{2} \left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right) \right] \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\ &= \frac{\frac{\pi}{2} \left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right) \right]}{2t + 1}\end{aligned}$$

$$\begin{aligned}\therefore \left(\frac{dx}{dy}\right)_{\text{at } t=1} &= \frac{\frac{\pi}{2} \left[\cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right]}{2(1) + 1} \\ &= \frac{\frac{\pi}{2}(0 - 1)}{3} \\ &= -\frac{\pi}{6}.\end{aligned}$$

Exercise 1.4 | Q 2.4 | Page 48

Find $\frac{dy}{dx}$ if : $x = 2 \cot t + \cos 2t$, $y = 2 \sin t - \sin 2t$ at $t = \frac{\pi}{4}$

SOLUTION

$$x = 2 \cot t + \cos 2t, y = 2 \sin t - \sin 2t$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(2 \cos t + \cos 2t) \\&= 2 \frac{d}{dt}(\cos t) + \frac{d}{dt}(\cos 2t) \\&= 2(-\sin t) + (-\sin 2t) \cdot \frac{d}{dt}(2t) \\&= 2 \sin t - \sin 2t \times 2 \times 1\end{aligned}$$

$$= -2 \sin t - 2 \sin 2t$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dt}(2 \sin t - \sin 2t) \\&= 2 \frac{d}{dt}(\sin t) - \frac{d}{dt}(\sin 2t) \\&= 2 \cos t - \cos 2t \cdot \frac{d}{dt}(2t) \\&= 2 \cos t - \cos 2t \times 2 \times 1 \\&= 2 \cos t - 2 \cos 2t\end{aligned}$$

$$\begin{aligned}
& \therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\
&= \frac{2 \cos t - 2 \cos 2t}{-2 \sin t - 2 \sin 2t} \\
&= \frac{\cos t - \cos 2t}{-\sin t - \sin 2t} \\
&\therefore \left(\frac{dy}{dx}\right) \text{ at } t = \frac{\pi}{4} \\
&= \frac{\cos \frac{\pi}{4} - \cos \frac{\pi}{2}}{-\sin \frac{\pi}{4} - \sin \frac{\pi}{2}} \\
&= \frac{\frac{1}{\sqrt{2}} - 0}{-\frac{1}{\sqrt{2}} - 1} \\
&= \frac{-1}{1 + \sqrt{2}} \\
&= \frac{-1}{1 + \sqrt{2}} + \frac{1 - \sqrt{2}}{1 - \sqrt{2}} \\
&\quad - (1 - \sqrt{2}) \\
&= 1 - \sqrt{2}.
\end{aligned}$$

Exercise 1.4 | Q 2.5 | Page 48

Find $\frac{dy}{dx}$ if : $x = t + 2\sin(\pi t)$, $y = 3t - \cos(\pi t)$ at $t = \frac{1}{2}$

SOLUTION

$$x = t + 2\sin(\pi t), y = 3t - \cos(\pi t)$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}[t + 2\sin(\pi t)]$$

$$= \frac{d}{dt}(t) + 3 \cdot \frac{d}{dt}[\sin(\pi t)]$$

$$= 1 + 2 \times \cos(\pi t) \cdot \frac{d}{dx}(\pi t)$$

$$= 1 + 2\cos(\pi t) \times \pi \times 1$$

$$= 1 + 2\pi \cos(\pi t)$$

and

$$\frac{dy}{dt} = \frac{d}{dt}[3t - \cos(\pi t)]$$

$$= 3 \times 1 - [-\sin(\pi t)] \cdot \frac{d}{dt}(\pi t)$$

$$= 3 + \sin(\pi t) \times \pi \times 1$$

$$= 3 + \pi \sin(\pi t)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{3 + \pi \sin(\pi t)}{1 + 2\pi \cos(\pi t)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{1}{2}}$$

$$= \frac{3 + \sin\left(\frac{\pi}{2}\right)}{1 + 2\pi \cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{3 + \pi \times 1}{1 + 2\pi(0)}$$

$$= 3 + \pi.$$

Exercise 1.4 | Q 3.1 | Page 48

If $x = a\sqrt{\sec \theta - \tan \theta}$, $y = a\sqrt{\sec \theta + \tan \theta}$, show that $\frac{dy}{dx} = \frac{y}{x}$.

SOLUTION

$$x = a\sqrt{\sec \theta - \tan \theta}, y = a\sqrt{\sec \theta + \tan \theta}$$

$$\therefore \frac{x}{a} = \sqrt{\sec \theta - \tan \theta}, y = a\sqrt{\sec \theta + \tan \theta}$$

$$\therefore \sec \theta - \tan \theta = \frac{x^2}{a^2} \quad \dots(1)$$

$$\sec \theta + \tan \theta = \frac{y^2}{a^2} \quad \dots(2)$$

Adding (1) and (2), we get

$$2\sec \theta = \frac{x^2}{a^2} + \frac{y^2}{a^2}$$

$$= \frac{x^2 + y^2}{a^2}$$

$$\therefore \sec \theta = \frac{x^2 + y^2}{2a^2}$$

Subtracting (1) from (2), we get

$$2\tan \theta = \frac{y^2}{a^2} - \frac{x^2}{a^2}$$

$$= \frac{y^2 - x^2}{a^2}$$

$$\therefore \tan\theta = \frac{y^2 - x^2}{2a^2}$$

$\therefore \sec^2\theta - \tan^2\theta = 1$ gives,

$$\left(\frac{x^2 + y^2}{a^2}\right)^2 - \left(\frac{y^2 - x^2}{2a^2}\right)^2 = 1$$

$$\therefore (x^2 + y^2)^2 - (y^2 - x^2)^2 = 4a^2$$

$$\therefore (4 + 2x^2y^2 + y^4) - (y^4 - 2x^2y^2 + x^4) = 4a^4$$

$$\therefore 4x^2y^2 = 4a^4$$

$$\therefore x^2y^2 = a^4$$

Differentiating both sides w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^3) = 0$$

$$\therefore x^2 \times 2y \frac{dy}{dx} + y^2 \times 3x^2 = 0$$

$$\therefore 2x^2y \frac{dy}{dx} = -3x^2y^2$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2y^2}{2x^2y} = \frac{-3y^2}{2}$$

Exercise 1.4 | Q 3.2 | Page 48

If $x = e^{\sin 3t}$, $y = e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

SOLUTION

$$x = e^{\sin 3t}, y = e^{\cos 3t}$$

$$\therefore \log x = \log e^{\sin 3t}, \log y = \log e^{\cos 3t}$$

$$\therefore \log x = (\sin 3t)(\log e), \log y = (\cos 3t)(\log e)$$

$$\therefore \log x = \sin 3t, \log y = \cos 3t \dots (1) \quad [\because \log e = 1]$$

Differentiating both sides w.r.t. t, we get

$$\frac{1}{x} \cdot \frac{dx}{dt} = \frac{d}{dt}(\sin 3t) = \cos 3t \cdot \frac{d}{dt}(3t)$$

$$= \cos 3t \times 3 = 3 \cos 3t$$

and

$$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{d}{dt}(\cos 3t) = -\sin 3t \cdot \frac{d}{dx}(3t)$$

$$= -\sin 3t \times 3 = 3 \cos 3t$$

$$\therefore \frac{dx}{dt} = 3x \cos 3t \text{ and } \frac{dy}{dt} = -3y \sin 3t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{-3y \sin 3t}{3x \cos 3t}$$

$$= \frac{-ys \sin 3t}{x \cos 3t}$$

$$= \frac{-y \log x}{x \log y}. \quad \dots[\text{By (1)}]$$

Exercise 1.4 | Q 3.3 | Page 48

If $x = \frac{t+1}{t-1}, y = \frac{1-t}{t+1}$, then show that $y^2 - \frac{dy}{dx} = 0$.

SOLUTION

$$x = \frac{t+1}{t-1}, y = \frac{1-t}{t+1}$$

$$\therefore y = \frac{1}{\left(\frac{t+1}{1-t}\right)} = \frac{-1}{\left(\frac{t+1}{t-1}\right)}$$

$$\therefore y = -\frac{1}{x}$$

$$\therefore xy = -1 \quad \dots(1)$$

Differentiating both sides w.r.t. t, we get

$$x \frac{dy}{dx} + \frac{d}{dx}(x) = 0$$

$$\therefore x \frac{dy}{dx} + y \times 1 = 0$$

$$\therefore -\frac{1}{y} \frac{dy}{dx} + y = 0 \quad \dots[\text{By (1)}]$$

$$\therefore -\frac{dy}{dx} + y^2 = 0$$

$$\therefore y^2 - \frac{dy}{dx} = 0.$$

Exercise 1.4 | Q 3.4 | Page 48

If $x = a \cos^3 t$, $y = a \sin^3 t$, show that $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^3$.

SOLUTION

$$x = a \cos^3 t, y = a \sin^3 t$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= a \frac{d}{dt} (\cos t)^3 = a \cdot 3(\cos t)^2 \frac{d}{dt} (\cos t) \\ &= 3a \cos^2 t (-\sin t) = -3a \cos^2 t \sin t\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= a \frac{d}{dt} (\sin t)^3 \\ &= a \cdot 3(\sin t)^2 \frac{d}{dt} (\sin t) \\ &= 3a \sin^2 t \cos t\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t}$$

$$= -\frac{\sin t}{\cos t} \quad \dots(1)$$

Now, $x = a \cos^3 t$

$$\therefore \cos^3 t = \frac{x}{a}$$

$$\therefore \cos t = \left(\frac{x}{a}\right)^{\frac{1}{3}}$$

$$y = a \sin^3 t$$

$$\therefore \sin^3 t = \frac{y}{a}$$

$$\therefore \cos^3 t = \left(\frac{y}{a}\right)^{\frac{1}{3}}$$

$$\therefore \text{from (1), } \frac{dy}{dx} = -\frac{\frac{y^{\frac{1}{3}}}{a^{\frac{1}{3}}}}{\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}}$$

$$= -\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

Alternative Method :

$$x = a \cos^3 t, y = a \sin^3 t$$

$$\therefore \cos^3 t = \frac{x}{a}, \sin^3 t = \frac{y}{a}$$

$$\therefore \cos t = \left(\frac{x}{a}\right)^{\frac{1}{3}}, \sin t = \left(\frac{y}{a}\right)^{\frac{1}{3}}$$

$$\therefore \cos^2 t + \sin^2 t = 1 \text{ gives}$$

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

Differentiating both sides w.r.t. t, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}, \frac{dy}{dx} = 0$$

$$\therefore \frac{2}{3}y^{\frac{-1}{3}} \frac{dy}{dx} = -\frac{2}{3}x^{\frac{-1}{3}}$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-\frac{1}{2}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

Exercise 1.4 | Q 3.5 | Page 48

If $x = 2\cos^4(t + 3)$, $y = 3\sin^4(t + 3)$, show that $\frac{dy}{dx} = -\sqrt{\frac{3y}{2x}}$.

SOLUTION

$$x = 2\cos^4(t + 3), y = 3\sin^4(t + 3)$$

$$\therefore \cos^4(t + 3) = \frac{x}{2}, \sin^4(t + 3) = \frac{y}{3}$$

$$\therefore \cos^2(t + 3) = \sqrt{\frac{x}{2}}, \sin^2(t + 3) = \sqrt{\frac{y}{3}}$$

$$\therefore \cos^2(t + 3) + \sin^2(t + 3) = 1$$

$$\therefore \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} = 1$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}
 & \frac{1}{\sqrt{2}} \frac{d}{dx}(\sqrt{x}) + \frac{1}{\sqrt{3}} \frac{d}{dx}(\sqrt{y}) = 0 \\
 \therefore & \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{3}} \times \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0 \\
 \therefore & \frac{1}{2\sqrt{3}\sqrt{y}} \cdot \frac{dy}{dx} = -\frac{1}{2\sqrt{2}\sqrt{x}} \\
 \therefore & \frac{dy}{dx} = -\frac{\sqrt{3}\sqrt{y}}{\sqrt{2}\sqrt{x}} \\
 & = -\sqrt{\frac{3y}{2x}}
 \end{aligned}$$

Exercise 1.4 | Q 3.6 | Page 48

If $x = \log(1 + t^2)$, $y = t - \tan^{-1}t$, show that $\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$.

SOLUTION

$$x = \log(1 + t^2), y = t - \tan^{-1}t$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt} [\log(1 + t^2)]$$

$$= \frac{1}{1 + t^2} \cdot \frac{d}{dt}(1 - t^2)$$

$$= \frac{1}{1 + t^2} \times (0 + 2t)$$

$$= \frac{2t}{1 + t^2}$$

and

$$\frac{dy}{dt} = \frac{d}{dt}(t) - \frac{d}{dt}(\tan^{-1}t)$$

$$\begin{aligned}
 &= 1 - \frac{1}{1+t^2} \\
 &= \frac{1+t^2-1}{1+t^2} \\
 &= \frac{t^2}{1+t^2}
 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\left(\frac{t^2}{1+t^2}\right)}{\left(\frac{2t}{1+t^2}\right)}$$

$$= \frac{t}{2}$$

$$\text{Now, } x = \log(1+t^2)$$

$$\therefore 1+t^2 = e^x$$

$$\therefore t^2 = e^x - 1$$

$$\therefore t = \sqrt{e^x - 1}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}.$$

Exercise 1.4 | Q 3.7 | Page 48

If $x = \sin^{-1}(e^t)$, $y = \sqrt{1 - e^{2t}}$, show that $\sin x + \frac{dy}{dx} = 0$

SOLUTION

$$x = \sin^{-1}(e^t), y = \sqrt{1 - e^{2t}}$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt} [\sin^{-1}(e^t)] \\ &= \frac{1}{\sqrt{1 - (e^t)^2}} \cdot \frac{d}{dt}(e^t) \\ &= \frac{1}{\sqrt{1 - e^{2t}}} \times e^2 = \frac{e^2}{\sqrt{1 - e^{2t}}}\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt} (\sqrt{1 - e^2}) \\ &= \frac{1}{2\sqrt{1 - e^{2t}}} \cdot \frac{d}{dt}(1 - e^{2t}) \\ &= \frac{1}{2\sqrt{1 - e^{2t}}} \times (0 - e^{2t} \times 2) \\ &= \frac{-e^{2t}}{\sqrt{1 - e^{2t}}}\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\ &= \frac{\left(\frac{-e^{2t}}{\sqrt{1-e^{2t}}}\right)}{\left(\frac{e^t}{\sqrt{1-e^{2t}}}\right)} \\ &= -e^t \\ &= -\sin x \quad \dots [\because x = \sin^{-1}(e^t)] \\ \therefore \sin x + \frac{dy}{dx} &= 0.\end{aligned}$$

Exercise 1.4 | Q 3.8 | Page 48

If $x = \frac{2bt}{1+t^2}$, $y = a\left(\frac{1-t^2}{1+t^2}\right)$, show that $\frac{dy}{dx} = -\frac{b^2y}{a^2x}$.

SOLUTION

$$x = \frac{2bt}{1+t^2}, y = a\left(\frac{1-t^2}{1+t^2}\right)$$

Put $t = \tan\theta$.

$$\text{Then } x = b\left(\frac{2\tan\theta}{1+\tan\theta}\right), y = a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$\therefore x = b \sin 2\theta, y = a \cos 2\theta$$

$$\therefore \frac{x}{b} = \sin 2\theta, \frac{y}{a} = \cos 2\theta$$

$$\therefore \left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = \sin^2 2\theta + \cos^2 2\theta$$

$$\therefore \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Differentiating x and y w.r.t. y, we get

$$\frac{1}{b^2} \times 2x \frac{dx}{dy} + \frac{1}{a^2} \times 2y = 0$$

$$\therefore \frac{2xdx}{b^2dy} = \frac{-2y}{a^2}$$

$$\therefore \frac{dx}{dy} = -\frac{b^2y}{a^2x}.$$

Exercise 1.4 | Q 4.1 | Page 49

Differentiate $x \sin x$ w.r.t. $\tan x$.

SOLUTION

Let $u = x \sin x$ and $v = \tan x$.

Then we want to find $\frac{du}{dv}$

Differentiating u and v w.r.t. x , we get

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}(x \sin x) \\&= x \frac{d}{dx}(\sin x) + (\sin x) \cdot \frac{d}{dx}(x) \\&= x \cos x + (\sin x) \times 1 \\&= x \cos x + \sin x \\&\text{and} \\ \frac{dv}{dx} &= \frac{d}{dx}(\tan x) = \sec^2 x \\ \therefore \frac{du}{dv} &= \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\&= \frac{x \cos x + \sin x}{\sec^2 x}.\end{aligned}$$

Exercise 1.4 | Q 4.2 | Page 49

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

SOLUTION

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and

$v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then we want to find $\frac{du}{dv}$.

Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$.

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

$$\therefore \frac{du}{dx} = 2 \frac{d}{dx} (\tan^{-1} x)$$

$$= 2 \times \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

$$\text{Also, } v = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

$$\therefore \frac{dv}{dx} = 2 \frac{d}{dx} (\tan^{-1} x)$$

$$= 2 \times \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{2}{1+x^2}\right)}$$

$$= 1.$$

Alternative Method :

$$\text{Let } u = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{ and } v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\text{Then we want to find } \frac{du}{dv}$$

$$\text{Put } x = \tan\theta.$$

$$\text{Then } u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

and

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$\therefore u = v$$

Differentiating both sides w.r.t. v, we get

$$\frac{du}{dv} = 1.$$

Exercise 1.4 | Q 4.3 | Page 49

Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t. $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

SOLUTION

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and

$$v = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right).$$

Then we want to find $\frac{du}{dv}$.

Put $x = \cos\theta$.

Then $\theta = \cos^{-1}x$.

$$\therefore u = \tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \frac{\pi}{2} - \theta$$

$$= \frac{\pi}{2} - \cos^{-1}x$$

$$\therefore \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(\cos^{-1}x)$$

$$= 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{-x^2}}$$

$$v = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

$$= \cos^{-1}(2x^2 - 1)$$

$$= \cos^{-1}(2\cos^2\theta - 1)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$\begin{aligned}
&= 2\theta \\
&= 2 \cos^{-1} x \\
\therefore \frac{dv}{dx} &= 2 \cdot \frac{d}{dx} (\cos^{-1} x) \\
&= \frac{-2}{\sqrt{1-x^2}} \\
\therefore \frac{du}{dv} &= \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\
&= \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-2} \\
&= -\frac{1}{2}.
\end{aligned}$$

Exercise 1.4 | Q 4.4 | Page 49

Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. $\tan^{-1} x$.

SOLUTION

Let $u = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $v = \tan^{-1} x$.

Then we want to find $\frac{du}{dv}$.

Put $x = \tan\theta$.

Then $= \tan^{-1} x$.

$$\therefore u = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$\begin{aligned}
 \therefore u &= 2\tan^{-1}x \\
 \therefore \frac{du}{dx} &= 2 \cdot \frac{d}{dx}(\tan^{-1} x) \\
 &= 2 \times \frac{1}{1+x^2} \\
 &= \frac{2}{1+x^2}
 \end{aligned}$$

Also, $v = \tan^{-1}x$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\begin{aligned}
 \therefore \frac{du}{dv} &= \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\
 &= \frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)} \\
 &= 2.
 \end{aligned}$$

Exercise 1.4 | Q 4.5 | Page 49

Differentiate 3^x w.r.t. $\log_x 3$.

SOLUTION

Let $u = 3^x$ and $v = \log_x 3$.

Then we want to find $\frac{du}{dv}$.

Differentiating u and v w.r.t. x , we get

$$\frac{du}{dv} = \frac{d}{dx}(3^x)$$

$$= 3^x \cdot \log 3$$

and

$$\frac{dv}{dx} = \frac{d}{dx}(\log_x 3)$$

$$= \frac{d}{dx} \left(\frac{\log 3}{\log x} \right)$$

$$= \log 3 \cdot \frac{d}{dx} (\log x)^{-1}$$

$$= (\log 3)(-1)(\log x)^{-2} \cdot \frac{d}{dx} (\log x)$$

$$= \frac{-\log 3}{(\log x)^2} \times \frac{1}{x}$$

$$= \frac{-\log 3}{x(\log x)^2}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx} \right)}{\left(\frac{dv}{dx} \right)}$$

$$= \frac{3^x \cdot \log 3}{\left[\frac{-\log 3}{x(\log x)^2} \right]}$$

$$= -x(\log x)^2 \cdot 3^x.$$

Exercise 1.4 | Q 4.6 | Page 49

Differentiate $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$ w.r.t. $\sec^{-1} x$.

SOLUTION

Let $u = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$ and $v = \sec^{-1} x$.

Then we want to find $\frac{du}{dv}$.

Differentiate u and v w.r.t. x , we get

$$\frac{du}{dx} = \frac{d}{dx} \left[\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) \right]$$

$$\begin{aligned} \frac{\cos x}{1 + \sin x} &= \frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)} \\ &= \frac{2 \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \cdot \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} \end{aligned}$$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$\therefore \frac{du}{dx} = \frac{d}{dx} \left[\tan^{-1} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \right]$$

$$= \frac{d}{dx} \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

$$= \frac{d}{dx} \left(\frac{\pi}{4} \right) - \frac{1}{2} \frac{d}{dx}(x)$$

$$= 0 - \frac{1}{2} \times 1$$

$$= -\frac{1}{2}$$

and

$$\frac{dv}{dx} = \frac{d}{dx} (\sec^{-1} x)$$

$$\begin{aligned}
&= \frac{1}{x\sqrt{x^2 - 1}} \\
\therefore \frac{du}{dx} &= \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\
&= \frac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{x\sqrt{x^2 - 1}}\right)} \\
&= -\frac{x\sqrt{x^2 - 1}}{2}.
\end{aligned}$$

Exercise 1.4 | Q 4.7 | Page 49

Differentiate x^x w.r.t. $x^{\sin x}$.

SOLUTION

Let $u = x^x$ and $v = x^{\sin x}$

Then we want to find $\frac{du}{dx}$.

Take, $u = x^x$

$\therefore \log u \log x x = x \log x$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x \log x) \\
&= x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \\
&= x \times \frac{1}{x} + (\log x) \times 1 \\
\therefore \frac{du}{dx} &= u(1 + \log x) \\
&= x^x(1 + \log x)
\end{aligned}$$

Also, $v = x^{\sin x}$

$$\therefore \log v = \log x^{\sin x} = (\sin x)(\log x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} [(\sin x)(\log x)]$$

$$= (\sin x) \cdot \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (\sin x)$$

$$= (\sin x) \times \frac{1}{x} + (\log x)(\cos x)$$

$$\therefore \frac{dv}{dx} = v \left[\frac{\sin x}{x} + (\log x)(\cos x) \right]$$

$$= x^{\sin x} \left[\frac{\sin x}{x} + (\log x)(\cos x) \right]$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx} \right)}{\left(\frac{dv}{dx} \right)}$$

$$= \frac{x^x(1 + \log x)}{x^{\sin x} \left[\frac{\sin x}{x} + (\log x)(\cos x) \right]}$$

$$= \frac{x(1 + \log x) \times x}{x^{\sin x} [\sin x + x \cos x, \log x]}$$

$$= \frac{(1 + \log x) \cdot x^{x+1-\sin x}}{\sin x + x \cos x, \log x}.$$

Exercise 1.4 | Q 4.8 | Page 49

Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$ w.r.t. $\tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right)$.

SOLUTION

$$\text{Let } u = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$$

and

$$v = \tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right).$$

Then we want to find $\frac{du}{dv}$

$$u = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$$

Put $x = \tan \theta$.

Then $\theta = \tan^{-1} x$

and

$$\frac{\sqrt{1+x^2} - 1}{x} = \frac{\sqrt{1+\tan^2 \theta} - 1}{\tan \theta}$$

$$= \frac{\sec \theta - 1}{\tan \theta}$$

$$\begin{aligned}
&= \frac{\frac{1}{\cos \theta} - 1}{\left(\frac{\sin \theta}{\cos \theta}\right)} \\
&= \frac{1 - \cos \theta}{\sin \theta} \\
&= \frac{2 \sin^2\left(\frac{\theta}{2}\right)}{2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)} \\
&= \tan\left(\frac{\theta}{2}\right) \\
\therefore u &= \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x \\
\therefore \frac{du}{dx} &= \frac{1}{2} \frac{d}{dx}(\tan^{-1} x) \\
&= \frac{1}{2} \times \frac{1}{1+x^2} \\
&= \frac{1}{2(1+x^2)}
\end{aligned}$$

$$v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$$

Put $x = \sin \theta$.

Then $\theta = \sin^{-1} x$

and

$$\frac{2x\sqrt{1-x^2}}{1-2x^2}$$

$$= \frac{2 \sin \theta \sqrt{1 - \sin^2 \theta}}{1 - 2 \sin^2 \theta}$$

$$= \frac{2 \sin \theta \cos \theta}{1 - 2 \sin^2 \theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan 2\theta$$

$$\therefore v = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$= 2\sin^{-1}x$$

$$\therefore \frac{dv}{dx} = 2 \frac{d}{dx} (\sin^{-1} x)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{\left(\frac{du}{dx} \right)}{\left(\frac{dv}{dx} \right)}$$

$$= \frac{\left[\frac{1}{2(1+x^2)} \right]}{\left(\frac{2}{\sqrt{1-x^2}} \right)}$$

$$= \frac{1}{2(1+x^2)} \times \frac{\sqrt{1-x^2}}{2}$$

$$= \frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

EXERCISE 1.5 [PAGE 60]

Exercise 1.5 | Q 1.1 | Page 60

Find the second order order derivatives of the following : $2x^5 - 4x^3 - \frac{2}{x^2} - 9$

SOLUTION

$$\text{Let } y = 2x^5 - 4x^3 - \frac{2}{x^2} - 9$$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{d}{dx} \left(2x^5 - 4x^3 - \frac{2}{x^2} - 9 \right) \\ &= 2 \frac{d}{dx}(x^5) - 4 \frac{d}{dx}(x^3) - 2 \frac{d}{dx}(x^{-2}) - \frac{d}{dx}(9)\end{aligned}$$

$$= 2 \times 5x^4 - 4 \times 3x^2 - 2(-2)x^{-3} - 0$$

$$= 10x^4 - 12x^2 + 4x^{-3}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} (10x^4 - 12x^2 + 4x^{-3}) \\ &= 10 \frac{d}{dx}(x^4) - 12 \frac{d}{dx}(x^2) + 4 \frac{d}{dx}(x^{-3}) \\ &= 10 \times 4x^3 - 12 \times 2x + 4(-3)x^{-4} \\ &= 40x^3 - 24x - \frac{12}{x^4}.\end{aligned}$$

Exercise 1.5 | Q 1.2 | Page 60

Find the second order derivatives of the following : $e^{2x} \cdot \tan x$

SOLUTION

Let $y = e^{2x} \cdot \tan x$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{2x} \cdot \tan x)$$

$$= e^{2x} \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(e^{2x})$$

$$= e^{2x} \times \sec^2 x + \tan x \times e^{2x} \cdot \frac{d}{dx}(2x)$$

$$= e^{2x} \cdot \sec^2 x + e^{2x} \cdot \tan x \times 2$$

$$= e^{2x} (\sec^2 x + 2\tan x)$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^{2x}(\sec^2 x + 2\tan x)]$$

$$= e^{2x} \cdot \frac{d}{dx}(\sec^2 x + 2\tan x) + (\sec^2 x + 2\tan x) \frac{d}{dx}(e^{2x})$$

$$= e^{2x} \left[\frac{d}{dx}(\sec x)^2 + 2 \frac{d}{dx}(\tan x) \right] + (\sec^2 x + 2\tan x) \times e^{2x} \cdot \frac{d}{dx}(2x)$$

$$= e^{2x} \left[2\sec x \cdot \frac{d}{dx}(\sec x) + 2\sec^2 x \right] + (\sec^2 x + 2\tan x) e^{2x} \times 2$$

$$= e^{2x}(2\sec x \cdot \sec x \tan x + 2\sec^2 x) + 2e^{2x}(\sec^2 x + 2\tan x)$$

$$= 2e^{2x}(\sec^2 x \tan x + \sec^2 x + \sec^2 x + 2\tan x)$$

$$= 2e^{2x}[\sec^2 x(\tan x + 1) + 1 + \tan^2 x + 2\tan x]$$

$$= 2e^{2x}[\sec^2 x(1 + \tan x) + (1 + \tan x)^2]$$

$$= 2e^{2x}[(1 + \tan x)(\sec^2 x + 1 + \tan x)]$$

$$= 2e^{2x}[(1 + \tan x)(1 + \tan^2 x + 1 + \tan x)]$$

$$= 2e^{2x}(1 + \tan x)(2 + \tan x + \tan^2 x).$$

Exercise 1.5 | Q 1.3 | Page 60

Find the second order derivatives of the following : $e^{4x} \cdot \cos 5x$

SOLUTION

Let $y = e^{4x} \cdot \cos 5x$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{d}{dx}(e^{4x} \cdot \cos 5x) \\ &= e^{4x} \cdot \frac{d}{dx}(\cos 5x) + \cos 5x \cdot \frac{d}{dx}(e^{4x}) \\ &= e^4 x \cdot (-\sin 5x) \cdot \frac{d}{dx}(5x) + \cos 5x \times e^{4x} \cdot \frac{d}{dx}(4x) \\ &= -e^{4x} \cdot \sin 5x \times 5 + e^{4x} \cos 5x \times 4\end{aligned}$$

$$= e^{4x} (4 \cos 5x - 5 \sin 5x)$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}[e^{4x}(4 \cos 5x - 5 \sin 5x)] \\ &= e^{4x} \frac{d}{dx}(4 \cos 5x - 5 \sin 5x) + (4 \cos 5x - 5 \sin 5x) \cdot \frac{d}{dx}(4x) \\ &= e^{4x} [4(-\sin 5x) \cdot \frac{d}{dx}(5x) - 5 \cos 5x \cdot \frac{d}{dx}(5x)] + (4 \cos 5x - 5 \sin 5x) \times e^{4x} \cdot \frac{d}{dx}(4x) \\ &= e^{4x} [-4 \sin 5x \times 5 - 5 \cos 5x \times 5] + (4 \cos 5x - 5 \sin 5x) e^{4x} \times 4 \\ &= e^{4x} (-20 \sin 5x - 25 \cos 5x + 16 \cos 5x - 5 \sin 5x) \\ &= e^{4x} (-9 \cos 5x - 40 \sin 5x) \\ &= -e^{4x} (9 \cos 5x + 40 \sin 5x).\end{aligned}$$

Exercise 1.5 | Q 1.4 | Page 60

Find the second order derivatives of the following : $x^3 \cdot \log x$

SOLUTION

Let $y = x^3 \cdot \log x$

$$\begin{aligned}\text{Then, } \frac{dy}{dx} &= \frac{d}{dx}(x^3 \cdot \log x) \\&= x^3 \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^3) \\&= x^3 \times \frac{1}{x} + (\log x) \times 3x^2 \\&= x^2 + 3x^2 \log x \\&= x^2(1 + 3 \log x) \\&\text{and} \\&\frac{d^2y}{dx^2} = \frac{d}{dx}[x^2(1 + 3 \log x)] \\&= x^2 \cdot \frac{d}{dx}(1 + 3 \log x) + (1 + 3 \log x) \times 2x \\&= x^2 \left(0 + 3 \times \frac{1}{x}\right) + (1 + 3 \log x) \times 2x \\&= 3x + 2x + 6x \log x \\&= 5x + 6x \log x \\&= x(5 + 6 \log x).\end{aligned}$$

Exercise 1.5 | Q 1.5 | Page 60

Find the second order derivatives of the following : $\log(\log x)$

SOLUTION

Let $y = \log(\log x)$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} [\log(\log x)]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx} (\log x)$$

$$= \frac{1}{\log x} \times \frac{1}{x} = \frac{1}{x \log x}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (x \log x)^{-1}$$

$$= (-1)(x \log x)^{-2} \cdot \frac{d}{dx} (x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[x \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (x) \right]$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[x \times \frac{1}{x} + (\log x) \times 1 \right]$$

$$= -\frac{1 + \log x}{(x \log x)^2}.$$

Exercise 1.5 | Q 2.1 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

SOLUTION

$$x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$$

Differentiating x and y w.r.t. θ , we get

$$\begin{aligned}\frac{dx}{d\theta} &= a \frac{d}{d\theta}(\theta - \sin \theta) \\ &= a(1 - \cos \theta)\end{aligned}\quad \dots(1)$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= a \frac{d}{d\theta}(1 - \cos \theta) \\ &= a[0 - (-\sin \theta)] \\ &= a \sin \theta\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \\ &= \frac{a \sin \theta}{a(1 - \cos \theta)} \\ &= \frac{2 \sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)}{2 \sin^2\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\cot\left(\frac{\theta}{2}\right) \right] \\ &= \frac{d}{dx} \left[\cot\left(\frac{\theta}{2}\right) \right] \cdot \frac{d\theta}{dx} \\ &= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \cdot \frac{d}{d\theta}\left(\frac{\theta}{2}\right) \times \frac{1}{\left(\frac{dx}{d\theta}\right)} \\ &= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \times \frac{1}{2} \times \frac{1}{a(1 - \cos \theta)} \quad \dots[\text{by (1)}]\end{aligned}$$

$$= -\frac{1}{2a} \operatorname{cosec}^2\left(\frac{\theta}{2}\right) \times \frac{1}{2 \sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{1}{4a} \cdot \operatorname{cosec}^4\left(\frac{\theta}{2}\right).$$

Exercise 1.5 | Q 2.2 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : $x = 2at^2$, $y = 4at$

SOLUTION

$$x = 2at^2, y = 4at$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(2at^2) \\ &= 2a \cdot \frac{d}{dt}(t^2) \\ &= 2a \times 2t = 4at \quad \dots(1)\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(4at) \\ &= 4a \frac{d}{dt}(t) \\ &= 4a \times 1 = 4a\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{4a}{4at} = \frac{1}{t}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right)$$

$$\begin{aligned}
&= \frac{d}{dt} (t^{-1}) \times \frac{dt}{dx} \\
&= 1(t)^{-2} \times \frac{1}{\left(\frac{dx}{dt}\right)} \\
&= \frac{1}{t^2} \times \frac{1}{4at} \quad \dots[\text{By (1)}] \\
&= -\frac{1}{4at^3}.
\end{aligned}$$

Exercise 1.5 | Q 2.3 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : $x = \sin\theta$, $y = \sin^3\theta$ at $\theta = \frac{\pi}{2}$

SOLUTION

$$x = \sin\theta, y = \sin^3\theta$$

Differentiating x and y w.r.t. θ , we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta} (\sin\theta) = \cos\theta \quad \dots(1)$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (\sin\theta)^3$$

$$= 3(\sin\theta)^2 \cdot \frac{d}{d\theta} (\sin\theta)$$

$$= 3\sin^2\theta \cdot \cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{3\sin^2\theta \cos\theta}{\cos\theta}$$

$$= 3\sin^2\theta$$

and

$$\frac{d^2y}{dx^2} = 3 \frac{d}{dx} (\sin \theta)^2$$

$$= 3 \frac{d}{d\theta} (\sin \theta)^2 \times \frac{d\theta}{dx}$$

$$= 3 \times 2 \sin \theta \frac{d}{d\theta} (\sin \theta) \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= 6 \sin \theta \cdot \cos \theta \times \frac{1}{\cos \theta} \quad \dots [\text{By (1)}]$$

$$= 6 \sin \theta$$

$$\therefore \left(\frac{d^2y}{dx^2} \right)_{\text{at } \theta = \frac{\pi}{2}}$$

$$= 6 \sin \frac{\pi}{2}$$

$$= 6 \times 1$$

$$= 6.$$

Alternative Method :

$$x = \sin \theta, y = \sin^3 \theta$$

$$\therefore y = x^3$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3) = 3x^2$$

$$\therefore \frac{d^2y}{dx^2} = 3 \frac{d}{dx} (x^2)$$

$$= 3 \times 2x$$

$$= 6x$$

$$\text{If } \theta = \frac{\pi}{2}, \text{ then } x = \sin \frac{\pi}{2} = 1$$

$$\therefore \left(\frac{d^2y}{dx^2} \right)_{\text{at } \theta = \frac{\pi}{2}}$$

$$= \left(\frac{d^2y}{dx^2} \right)_{at x=1}$$

$$= 6(1)$$

$$= 6.$$

Exercise 1.5 | Q 2.4 | Page 60

Find $\frac{d^2y}{dx^2}$ of the following : $x = a \cos \theta$, $y = b \sin \theta$ at $\theta = \frac{\pi}{4}$.

SOLUTION

$$x = a \cos \theta, y = b \sin \theta$$

Differentiating x and y w.r.t. θ , we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta)$$

$$= a \frac{d}{d\theta}(\cos \theta)$$

$$= a(-\sin \theta)$$

$$= -a \sin \theta \quad \dots(1)$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b \sin \theta)$$

$$= b \frac{d}{dx}(\sin \theta)$$

$$= b \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)}$$

$$= \frac{b \cos \theta}{-a \sin \theta}$$

$$= \left(-\frac{b}{a} \right) \cot \theta$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\left(-\frac{b}{a} \right) \cot \theta \right] \\ &= -\frac{b}{a} \cdot \frac{d}{d\theta} (\cot \theta) \times \frac{d\theta}{dx} \\ &= \left(-\frac{b}{a} \right) (-\operatorname{cosec}^2 \theta) \times \frac{1}{\left(\frac{dx}{d\theta} \right)}\end{aligned}$$

$$= \left(\frac{b}{a} \right) \operatorname{cosec}^2 \theta \times \frac{1}{-a \sin \theta} \quad ..[\text{By (1)}]$$

$$= \left(-\frac{b}{a^2} \right) \operatorname{cosec}^3 \theta$$

$$\therefore \left(\frac{d^2y}{dx^2} \right)_{\text{at } \theta = \frac{\pi}{4}}$$

$$= \left(-\frac{b}{a^2} \right) \operatorname{cosec}^3 \frac{\pi}{4}$$

$$= \frac{-b}{a^2} \times (\sqrt{2})^3$$

$$= -\frac{2\sqrt{2}b}{a^2}.$$

Exercise 1.5 | Q 3.01 | Page 60

If $x = at^2$ and $y = 2at$, then show that $xy \frac{d^2y}{dx^2} + a = 0$.

SOLUTION

$$x = at^2 \text{ and } y = 2at \quad \dots(1)$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(at^2) \\ &= a \frac{d}{dt}(t^2) \\ &= a \times 2t \\ &= 2at \quad \dots(2)\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(2at) \\ &= 2a \frac{d}{dt}(t) \\ &= 2a \times 1 \\ &= 2a\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$
$$= \frac{2a}{2at} = \frac{1}{t}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{t}\right) \\ &= \frac{d}{dt}(t^{-1}) \cdot \frac{dt}{dx} \\ &= (-1)t^{-2} \cdot \frac{1}{\left(\frac{dx}{dt}\right)} \\ &= \frac{-1}{t^2} \times \frac{1}{2at} \quad \dots[\text{By (2)}]\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2at^3} \\
 \therefore \frac{d^2y}{dx^2} &= -\frac{1}{(at^2)(2at) \times a} \\
 &= -\frac{a}{xy} \quad \dots[\text{By (1)}] \\
 \therefore xy \frac{d^2y}{dx^2} &= -a \\
 \therefore xy \frac{d^2y}{dx^2} + a &= 0.
 \end{aligned}$$

Exercise 1.5 | Q 3.02 | Page 60

If $y = e^{m \tan^{-1} x}$, show that $(1 + x^2) \frac{d^2y}{dx^2} + (2x - m) \frac{dy}{dx} = 0$.

SOLUTION

$$\begin{aligned}
 y &= e^{m \tan^{-1} x} \quad \dots(1) \\
 \therefore \frac{dy}{dx} &= \frac{d}{dx} \left(e^{m \tan^{-1} x} \right) \\
 &= e^{m \tan^{-1} x} \cdot \frac{d}{dx} (m \tan^{-1} x) \\
 &= e^{m \tan^{-1} x} \times m \times \frac{1}{1 + x^2} \\
 \therefore (1 + x^2) \frac{dy}{dx} &= my \quad \dots[\text{By (1)}]
 \end{aligned}$$

Differentiating again w.r.t. x, we get

$$\begin{aligned}
 (1 + x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (1 + x^2) &= m \frac{dy}{dx} \\
 \therefore (1 + x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} (0 + 2x) &= m \frac{dy}{dx}
 \end{aligned}$$

$$\therefore (1 + x^2) \frac{d^2y}{dx^2} + 2x \cdot \frac{dy}{dx} = m \frac{dy}{dx}.$$

$$\therefore (1 + x^2) \frac{d^2y}{dx^2} + (2x - m) \frac{dy}{dx} = 0.$$

Exercise 1.5 | Q 3.03 | Page 60

If $x = \cos t$, $y = e^{mt}$, show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0$.

SOLUTION

$$x = \cos t, y = e^{mt}$$

$$\therefore t = \cos^{-1} x \text{ and } y = e^{m \cos^{-1} x} \quad \dots(1)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(e^{m \cos^{-1} x} \right)$$

$$= e^{m \cos^{-1} x} \cdot \frac{d}{dx} (m \cos^{-1} x)$$

$$= e^{m \cos^{-1} x} \times m \times \frac{-1}{\sqrt{1 - x^2}}$$

$$\therefore \sqrt{1 - x^2} \cdot \frac{dy}{dx} = -my \quad \dots[\text{By (1)}]$$

$$\therefore (1 - x^2) \left(\frac{dy}{dx} \right)^2 = m^2 y^2$$

Differentiating again w.r.t. x , we get

$$(1 - x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 + \left(\frac{dy}{dx} \right)^2 \cdot \frac{d}{dx} (1 - x^2) = m^2 \cdot \frac{d}{dx} (y^2)$$

$$\therefore (1 - x^2) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 (0 - 2x) = m^2 \times 2y \frac{dy}{dx}$$

Cancelling $2 \frac{dy}{dx}$ throughout, we get

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = m^2y$$

$$\therefore (1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2y = 0.$$

Exercise 1.5 | Q 3.04 | Page 60

If $y = x + \tan x$, show that $\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0$.

SOLUTION

$$y = x + \tan x$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}(x + \tan x) \\ &= 1 + \sec^2 x\end{aligned}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(1 + \sec x)^2 \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(\sec x)^2 \\ &= 2\sec x \cdot \sec x \tan x \\ &= 2 \sec^2 x \tan x \\ \therefore \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x &= \cos^2 x (2 \sec^2 x \tan x) - 2(x + \tan x) + 2x \\ &= \cos^2 x \times \frac{2}{\cos^2 x} \times \tan x - 2x - 2 \tan x + 2x \\ &= 2 \tan x - 2 \tan x \\ \therefore \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x &= 0.\end{aligned}$$

Exercise 1.5 | Q 3.05 | Page 60

If $y = e^{ax} \cdot \sin(bx)$, show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

SOLUTION

$$y = e^{ax} \cdot \sin(bx) \quad \dots(1)$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}[e^{ax} \cdot \sin(bx)] \\ &= e^{ax} \cdot \frac{d}{dx}[\sin(bx)] + \sin(bx) \cdot \frac{d}{dx}(e^{ax}) \\ &= e^{ax} \cdot \cos(bx) \cdot \frac{d}{dx}(bx) + \sin(bx) \times e^{ax} \cdot \frac{d}{dx}(ax) \\ &= e^{ax} \cdot \cos(bx) \times b + e^{ax} \cdot \sin(bx) \times a\end{aligned}$$

$$\therefore y_1 = e^{ax} [b \cos(bx) + a \sin(bx)] \quad \dots(2)$$

Differentiating again w.r.t. x, we get

$$\begin{aligned}\frac{dy_1}{dx} &= \frac{d}{dx}[e^{ax} \{b \cos(bx) + a \sin(bx)\}] \\ &= e^{ax} \cdot \frac{d}{dx}[b \cos(bx) + a \sin(bx)] + [b \cos(bx) + a \sin(bx)] \cdot \frac{d}{dx}(e^{ax}) \\ &= e^{ax} \cdot \left[b \{-\sin(bx)\} \cdot \frac{d}{dx}(bx) + a \cos(bx) \cdot \frac{d}{dx}(bx) \right] + [b \cos(bx) + a \sin(bx)] \times e^{ax} \cdot \frac{d}{dx}(ax) \\ &= e^{ax} [-b \sin(bx) \times b + a \cos(bx) \times b] + [b \cos(bx) + a \sin(bx)] e^{ax} \times a \\ &= e^{ax} [-b^2 \sin(bx) + ab \cos(bx) + a^2 \sin(bx)] \\ \therefore y_2 &= e^{ax} [-b^2 \sin(bx) + 2ab \cos(bx) + a^2 \sin(bx)] \quad \dots(3) \\ \therefore y^2 - 2ay_1 + (a^2 + b^2)y &= \\ &= e^{ax} [-b^2 \sin(bx) + 2ab \cos(bx) + a^2 \sin(bx)] - 2a \cdot e^{ax} [b \cos(bx) + a \sin(bx)] + (a^2 + b^2) e^{ax} \sin(bx) \quad \dots[\text{By (1), (2) and (3)}] \\ &= e^{ax} [-b^2 \sin(bx) + 2ab \cos(bx) + a^2 \sin(bx) - 2ab \cos(bx) - 2a^2 \sin(bx) + a^2 \sin(bx) + b^2 \sin(bx)] \\ &= e^{ax} \times 0 \\ \therefore y_2 - 2ay_1 + (a^2 + b^2)y &= 0.\end{aligned}$$

Exercise 1.5 | Q 3.06 | Page 60

If $\sec^{-1} \left(\frac{7x^3 - 5y^3}{7^3 + 5y^3} \right) = m$, show $\frac{d^2y}{dx^2} = 0$.

SOLUTION

$$\sec^{-1} \left(\frac{7x^3 - 5y^3}{7^3 + 5y^3} \right) = m$$

$$\therefore \frac{7x^3 - 5y^3}{7^3 + 5y^3} = \sec m = k \quad \dots (\text{Say})$$

$$\therefore 7x^3 - 5y^3 = 7kx^3 + 5ky^3$$

$$\therefore (5k + 5)y^3 = (7 - 7k)x^2$$

$$\therefore \frac{y^3}{x^3} = \frac{7 - 7k}{5k + 5}$$

$$\therefore \frac{y}{x} = \left(\frac{7 - 7k}{5k + 5} \right)^{\frac{1}{3}} = p, \text{ where } p \text{ is a constant}$$

$$\therefore \frac{d}{dx} \left(\frac{y}{x} \right) = \frac{d}{dx}(p)$$

$$\therefore \frac{x \frac{dy}{dx} - y \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x \frac{dy}{dx} - y \times 1 = 0$$

$$\therefore x \frac{dy}{dx} = y$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{y}{x} && \dots(1) \\
 \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{y}{x} \right) \\
 &= \frac{x \frac{dy}{dx} - y \frac{d}{dx}(x)}{x^2} \\
 &= \frac{x \left(\frac{y}{x} \right) - y \times 1}{x^2} && \dots[\text{By (1)}] \\
 &= \frac{y - y}{x^2} \\
 &= \frac{0}{x^2} \\
 &= 0
 \end{aligned}$$

Note : $\frac{dy}{dx} = \frac{y}{x}$, where $\frac{y}{x} = p$.

$\therefore \frac{dy}{dx} = p$, where p is a constant.

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(p) = 0.$$

Exercise 1.5 | Q 3.07 | Page 60

If $2y = \sqrt{x+1} + \sqrt{x-1}$, show that $4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

SOLUTION

$$2y = \sqrt{x+1} + \sqrt{x-1} \quad \dots(1)$$

Differentiating both sides w.r.t. x , x we get

$$\begin{aligned}
 \therefore 2 \frac{dy}{dx} &= \frac{d}{dx}(\sqrt{x+1}) + \frac{d}{dx}(\sqrt{x-1}) \\
 &= \frac{1}{2\sqrt{x+1}}(1+0) + \frac{1}{2\sqrt{x-1}}(1-0)
 \end{aligned}$$

$$\begin{aligned}\therefore 2 \frac{dy}{dx} &= \frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x+1}} \\ &= \frac{\sqrt{x+1} + \sqrt{x+1}}{2\sqrt{x+1} \cdot \sqrt{x-1}} \\ &= \frac{2y}{2\sqrt{x^2-1}}\end{aligned}$$

...[By (1)]

$$\therefore 2\sqrt{x^2-1} \frac{dy}{dx} = y$$

$$\therefore 4(x^2-1) \cdot \left(\frac{dy}{dx}\right)^2 = y^2$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}4(x^2-1) \frac{d}{dx} \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \cdot \frac{d}{dx}[4(x^2-1)] &= 2y \frac{dy}{dx} \\ \therefore 4(x^2-1) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \cdot 4(2x) &= 2y \left(\frac{dy}{dx}\right)\end{aligned}$$

Cancelling $2 \frac{dy}{dx}$ on both sides, we get

$$\begin{aligned}4(x^2-1) \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} &= y \\ \therefore 4(x^2-1) \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - y &= 0 \\ \therefore 4(x^2-1)y_2 + 4xy_1 - y &= 0.\end{aligned}$$

Exercise 1.5 | Q 3.08 | Page 60

If $y = \log(x + \sqrt{x^2 + a^2})^m$, show that $(x^2 + a^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$.

SOLUTION

$$\begin{aligned}
y &= \log \left(x + \sqrt{x^2 + a^2} \right)^m \\
&= m \log \left(x + \sqrt{x^2 + a^2} \right) \\
\therefore \frac{dy}{dx} &= m \frac{d}{dx} \left[\log \left(x + \sqrt{x^2 + a^2} \right) \right] \\
&= m \times \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{d}{dx} \left(x + \sqrt{x^2 + a^2} \right) \\
&= \frac{m}{x + \sqrt{x^2 + a^2}} \times \left[1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot \frac{d}{dx} (x^2 + a^2) \right] \\
&= \frac{m}{x + \sqrt{x^2 + a^2}} \times \left[1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot (2x + 0) \right] \\
&= \frac{m}{x + \sqrt{x^2 + a^2}} \times \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \\
\therefore \frac{dy}{dx} &= \frac{m}{\sqrt{x^2 + a^2}} \\
\therefore \sqrt{x^2 + a^2} \frac{dy}{dx} &= m
\end{aligned}$$

$$\therefore (x^2 + a^2) \left(\frac{dy}{dx} \right)^2 = m^2$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
(x^2 + a^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 + \left(\frac{dy}{dx} \right)^2 \cdot \frac{d}{dx} (x^2 + a^2) &= \frac{d}{dx} (m^2) \\
\therefore (x^2 + a^2) \times 2 \frac{dy}{dx} \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \left(\frac{dy}{dx} \right)^2 \times (2x + 0) &= 0 \\
\therefore (x^2 + a^2) \cdot 2 \frac{dy}{dx} \frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx} \right)^2 &= 0
\end{aligned}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(x^2 + a^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0.$$

Exercise 1.5 | Q 3.09 | Page 60

If $y = \sin(m \cos^{-1} x)$, then show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$.

SOLUTION

$$y = \sin(m \cos^{-1} x)$$

$$\therefore \sin^{-1} y = m \cos^{-1} x$$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{\sqrt{1 - y^2}} \cdot \frac{dy}{dx} = m \times \frac{-1}{\sqrt{1 - x^2}}$$

$$\therefore \sqrt{1 - x^2} \cdot \frac{dy}{dx} = -2\sqrt{1 - y^2}$$

$$\therefore (1 - x^2) \left(\frac{dy}{dx} \right)^2 = m^2 (1 - y^2)$$

$$\therefore (1 - x^2) \left(\frac{dy}{dx} \right)^2 = m^2 - m^2 y^2$$

Differentiating both sides w.r.t. x , we get

$$(1 - x^2) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 + \left(\frac{dy}{dx} \right)^2 \cdot \frac{d}{dx} (1 - x^2) = 0 - m^2 \cdot \frac{d}{dx} (y^2)$$

$$\therefore (1 - x^2) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx} \right)^2 = -2m^2 y \frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = -m^2y$$

$$\therefore (1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2y = 0.$$

Exercise 1.5 | Q 3.1 | Page 60

If $y = \log(\log 2x)$, show that $xy_2 + y_1(1 + xy_1) = 0$.

SOLUTION

$$y = \log(\log 2x)$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}[\log(\log 2x)] \\ &= \frac{1}{\log 2x} \cdot \frac{d}{dx}(\log 2x) \\ &= \frac{1}{\log 2x} \times \frac{1}{2x} \cdot \frac{d}{dx}(2x) \\ &= \frac{1}{\log 2x} \times \frac{1}{2x} \times 2 \\ \therefore \frac{dy}{dx} &= \frac{1}{x \log 2x} \\ \therefore (\log 2x) \cdot \frac{dy}{dx} &= \frac{1}{x} \quad \dots(1)\end{aligned}$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}(\log 2x) \cdot \frac{d}{dx} \left(\frac{dx}{dy} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\log 2x) &= \frac{d}{dx} \left(\frac{1}{x} \right) \\ \therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \cdot \frac{d}{dx}(2x) &= -\frac{1}{x^2}\end{aligned}$$

$$\begin{aligned}\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \times 2 &= -\frac{1}{x^2} \\ \therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} &= \frac{1}{x} \cdot \frac{1}{x} \\ \therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \left[(\log 2x) \cdot \frac{dy}{dx} \right] \frac{dy}{dx} &= -\frac{1}{x} \left[(\log 2x) \cdot \frac{dy}{dx} \right] \quad \dots[\text{By (1)}]\end{aligned}$$

Dividing throughout by $\log 2x$, we get

$$\begin{aligned}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 &= -\frac{1}{x} \frac{dy}{dx} \\ \therefore x \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 &= -\frac{dy}{dx} \\ \therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} + x \left(\frac{dy}{dx} \right)^2 &= 0 \\ \therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(1 + x \frac{dy}{dx} \right) &= 0 \\ \therefore xy_2 + y_1 (1 + xy_1) &= 0.\end{aligned}$$

Exercise 1.5 | Q 3.11 | Page 60

If $x^2 + 6xy + y^2 = 10$, show that $\frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}$.

SOLUTION

$$x^2 + 6xy + y^2 = 10 \quad \dots(1)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}2x + 6 \left[x \frac{dy}{dx} + y \cdot \frac{d}{dx}(x) \right] + 2y \frac{dy}{dx} &= 0 \\ \therefore 2x + 6x \frac{dy}{dx} + 6y \times 1 + 2y \frac{dy}{dx} &= 0\end{aligned}$$

$$\therefore (6x + 2y) \frac{dy}{dx} = -2x - 6y$$

$$\therefore \frac{dy}{dx} = \frac{-2(x + 3y)}{2(3x + y)} = -\left(\frac{x + 3y}{3x + y}\right) \quad \dots(2)$$

$$\begin{aligned} & \therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x + 3y}{3x + y}\right) \\ &= -\left[\frac{(3x + y) \cdot \frac{d}{dx}(x + 3y) - (x + 3y) \cdot \frac{d}{dx}(3x + y)}{(3x + y)^2} \right] \\ &= -\left[\frac{(3x + y)\left(1 + 3\frac{dy}{dx}\right) - (x + 3y)\left(3 + \frac{dy}{dx}\right)}{(3x + y)^2} \right] \\ &= \frac{1}{(3x + y)^2} \left[-(3x + y) \left\{ 1 - \frac{3(x + 3y)}{3x + y} \right\} + (x + 3y) \left(3 - \frac{x + 3y}{3x + y} \right) \right] \quad \dots[\text{By (2)}] \\ &= \frac{1}{(3x + y)^2} \left[-(3x + y) \left[-(3x + y) \left(\frac{3x + y - 3x - 9y}{3x + y} \right) + (x + 3y) \left(\frac{9x + 3y - x - 3y}{3x + y} \right) \right] \right] \\ &= \frac{1}{(3x + y)^2} \left[8y + \frac{(x + 3y)(8x)}{3x + y} \right] \\ &= \frac{1}{(3x + y)^2} \left[\frac{(8y(3x + y) + (x + 3y)8x)}{3x + y} \right] \\ &= \frac{24xy + 8y^2 + 8x^2 + 24xy}{(3x + y)^2} \\ &= \frac{8x^2 + 48xy + 8y^2}{(3x + y)^3} \\ &= \frac{8(x^2 + 6xy + y^2)}{(3x + y)^3} \\ &= \frac{8(10)}{(3x + y)^3} \quad \dots[\text{By (1)}] \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}.$$

Exercise 1.5 | Q 3.12 | Page 60

If $x = a \sin t - b \cos t$, $y = a \cos t + b \sin t$, show that $\frac{d^2y}{dx^2} = -\frac{x^3 + y^2}{y^3}$.

SOLUTION

$$x = a \sin t - b \cos t, y = a \cos t + b \sin t$$

Differentiating x and y w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= a \frac{d}{dx}(\sin t) - b \frac{d}{dt}(\cos t) \\ &= a \cos t - b(-\sin t) \\ &= a \cos t + b \sin t\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= a \frac{d}{dx}(\cos t) - b \frac{d}{dt}(\sin t) \\ &= a(-\sin t) + b \cos t \\ &= -a \sin t + b \cos t\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\ &= \frac{-a \sin t + b \cos t}{a \cos t + b \sin t} \\ &= -\left(\frac{a \sin t - b \cos t}{a \cos t + b \sin t}\right) \\ \therefore \frac{dy}{dx} &= -\frac{x}{y} \quad \dots(1)\end{aligned}$$

$$\begin{aligned}
\therefore \frac{d^2y}{dx^2} &= -\frac{d}{dx} \left(\frac{x}{y} \right) \\
&= -\left[\frac{y \frac{d}{dx}(x) - x \frac{dy}{dx}}{y^2} \right] \\
&= -\left[\frac{y \times 1 - x \left(-\frac{x}{y} \right)}{y^2} \right] \quad \dots[\text{By (1)}] \\
&= -\left[\frac{y^2 + x^2}{y^3} \right] \\
\therefore \frac{d^2y}{dx^2} &= -\frac{x^2 + y^2}{y^2}.
\end{aligned}$$

Exercise 1.5 | Q 4.01 | Page 60

Find the n^{th} derivative of the following : $(ax + b)^m$

SOLUTION

Let $y = (ax + b)^m$

$$\begin{aligned}
\text{Then } \frac{dy}{dx} &= \frac{d}{dx} (ax + b)^m \\
&= m(ax + b)^{m-1} \cdot \frac{d}{dx} (ax + b) \\
&= m(ax + b)^{m-1} \times (a \times 1 + 0) \\
&= am(ax + b)^{m-1} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx} [am(ax + b)^{m-1}] \\
&= am \frac{d}{dx} (ax + b)^{m-1}
\end{aligned}$$

$$= am(m-1)(ax+b)^{m-2} \cdot \frac{d}{dx}(ax+b)$$

$$= am(m-1)(ax+b)^{m-2} \times (a \times 1 + 0)$$

$$= a^2 m(m-1) (ax+b)^{m-2}$$

$$\frac{d^2y}{dx^3} = \frac{d}{dx} \left[a^2 m(m-1) \frac{d}{dx} (ax+b)^{m-2} \right]$$

$$= a^2 m(m-1) \frac{d}{dx} (ax+b)^{m-2}$$

$$= a^m(m-1)(m-2)(ax+b)^{m-3} \frac{d}{dx}(ax+b)$$

$$= a^2 m(m-1)(m-2)(ax+b)^{m-3} \times (a \times 1 + 0)$$

$$= a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

In general, the nth order derivative is given by

$$\frac{d^n y}{dx^n} = a^n m(m-1)(m-2) \dots (m-n+1)(ax+b)^{m-n}$$

Case (i) : if $m > 0, m > n$, then

$$\frac{d^n y}{dx^n} = \frac{(a^n \cdot m(m-1)(m-2) \dots (m-n+1)(m-n) \dots 3.2.1)}{(m-n)(m-n-1) \dots 3.2.1} \times (ax+b)^{m-n}$$

$$\therefore \frac{d^2 y}{dx^n} = \frac{(a^n \cdot m!(ax+b)^{m-n})}{(m-n)!}, \text{if } m > 0, m > n.$$

Case (ii) : if $m > 0$ and $m < n$, then its mth order derivative is a constant and every derivatives after mth order are zero.

$$\therefore \frac{d^n y}{dx^n} = 0, \text{if } m > 0, m = n.$$

Case (iii) : If $m > 0, m = n$, then

$$\frac{d^n y}{dx^n} = a^n \cdot n(n-1)(n-2) \dots (n-n+1)(ax+b)^{n-n}$$

$$= a^n \cdot n(n-1)(n-2) \dots 1 \cdot (ax+b)^0$$

$$\therefore \frac{d^n y}{dx^n} = a^n \cdot n!, \text{if } m > 0, m = n.$$

Exercise 1.5 | Q 4.02 | Page 60

Find the n^{th} derivative of the following : $\frac{1}{x}$

SOLUTION

$$\text{Let } y = \frac{1}{x}$$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$= \frac{1}{x^2}$$

$$= \frac{(-1)^1 1!}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(-\frac{1}{x^2} \right)$$

$$= 1 \frac{d}{dx} (x^{-2})$$

$$= (-1)(-2)x^{-3}$$

$$= \frac{(-1)^{2,1} \cdot 2}{x^3}$$

$$= \frac{(-1)^2 2!}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left[\frac{(-1)^2 \cdot 2!}{x^3} \right]$$

$$= (-1)^2 \cdot 2! \frac{d}{dx} (x^{-3})$$

$$= (-1)^2 \cdot 2! \cdot (-3)x^{-4}$$

$$= \frac{(-1)^3 \times 3 \cdot 2!}{x^4}$$

$$= \frac{(-1)^3 \cdot 3!}{x^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n!}{x^{n+1}}.$$

Exercise 1.5 | Q 4.03 | Page 60

Find the n^{th} derivative of the following : e^{ax+b}

SOLUTION

Let $y = e^{ax+b}$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} (e^{ax+b})$$

$$= e^{ax+b} \cdot \frac{d}{dx} (ax + b)$$

$$= e^{ax+b} \times (a \times 1 + 0)$$

$$= ae^{ax+b}$$

$$\frac{d^2 y}{dx^3} = \frac{d}{dx} (ae^{ax+b})$$

$$= a \cdot \frac{d}{dx} (ax + b)$$

$$= ae^{ax+b} \times (a \times 1 + 0)$$

$$= a^2 \cdot e^{ax+b}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} [a^2 e^{ax+b}]$$

$$= a^2 \frac{d}{dx} (e^{ax+b})$$

$$= a^2 e^{ax+b} \cdot \frac{d}{dx} (ax + b)$$

$$= a^2 e^{ax+b} \times (a \times 1 + 0)$$

$$= a^3 \cdot e^{ax+b}$$

In general, the nth order derivative is given by

$$\frac{d^n y}{dx^n} = a^n \cdot e^{ax+b}.$$

Exercise 1.5 | Q 4.04 | Page 60

Find the nth derivative of the following : a^{px+q}

SOLUTION

Let $y = a^{px+q}$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} (a^{px+q})$$

$$= a^{px+q} \log a \cdot \frac{d}{dx} (px + q)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} [p \log a \cdot a^{px+q}]$$

$$= p \log a \cdot \frac{d}{dx} (a^{px+q})$$

$$\begin{aligned}
&= p \log a \cdot a^{px+q} \cdot \log a \cdot \frac{d}{dx}(px + q) \\
&= p \log a \cdot a^{px+q} \cdot \log a \times (p \times 1 + 0) \\
&= p^2 \cdot (\log a)^2 \cdot a^{px+q} \\
\frac{d^3y}{dx^3} &= \frac{d}{dx} \left[p^2 \cdot (\log a)^2 \cdot a^{px+q} \right] \\
&= p^2 \cdot (\log a)^2 \cdot \frac{d}{dx} (a^{px+q}) \\
&= p^2 \cdot (\log a)^2 \cdot a^{px+q} \cdot \log a \cdot \frac{d}{dx}(px + q) \\
&= p^2 \cdot (\log a)^3 \cdot a^{px+q} \times (p \times 1 + 0) \\
&= p^3 \cdot (\log a)^3 \cdot a^{px+q}
\end{aligned}$$

In general, the nth order derivative is given by

$$\frac{d^n y}{dx^n} = p^n \cdot (\log a)^n \cdot a^{px+q}.$$

Exercise 1.5 | Q 4.05 | Page 60

Find the nth derivative of the following : log (ax + b)

Let $y = \log (ax + b)$

Then $\frac{dy}{dx} = \frac{d}{dx} [\log(ax + b)]$

$$= \frac{1}{ax + b} \cdot \frac{d}{dx}(ax + b)$$

$$= \frac{1}{ax + b} \times (a \times 1 + 0)$$

$$= \frac{a}{ax + b}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{a}{ax + b} \right)$$

$$\begin{aligned}
&= a \frac{d}{dx} (ax + b)^{-1} \\
&= a(-1)(ax + b)^{-2} \cdot \frac{d}{dx} (ax + b) \\
&= \frac{(-1)a}{(ax + b)^2} \times (a \times 1 + 0) \\
&= \frac{(-1)a}{(ax + b)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d^3y}{dx^3} &= \frac{d}{dx} \left[\frac{(-1)^1 a^2}{(ax + b)^2} \right] \\
&= (-1)^1 a^2 \cdot \frac{d}{dx} (ax + b)^{-2} \\
&= (-1)^1 a^2 \cdot (-2)(ax + b)^{-3} \cdot \frac{d}{dx} (ax + b) \\
&= \frac{(-1)^2 \cdot 1.2 \cdot a^2}{(ax + b)^3} \times (a \times 1 + 0) \\
&= \frac{(-1) \cdot 2.2! a^3}{(ax + b)^3}
\end{aligned}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)! a^n}{(ax + b)^n}$$

Exercise 1.5 | Q 4.06 | Page 60

Find the n^{th} derivative of the following : $\cos x$

SOLUTION

Let $y = \cos x$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(\cos x)$$

$$= -\sin x$$

$$= \cos\left(\frac{\pi}{2} + x\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\sin x)$$

$$= -\cos x$$

$$= \cos(\pi + x)$$

$$= \cos\left(\frac{2\pi}{2} + x\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(-\cos x)$$

$$= -\frac{d}{dx}(\cos x)$$

$$= -(-\sin x)$$

$$= \sin x$$

$$= \cos\left(\frac{3\pi}{2} + x\right)$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \cos\left(\frac{n\pi}{2} + x\right).$$

[Exercise 1.5 | Q 4.07 | Page 60](#)

Find the n^{th} derivative of the following : $\sin(ax + b)$

SOLUTION

Let $y = \sin(ax + b)$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} [\sin(ax + b)]$$

$$= \cos(ax + b) \cdot \frac{d}{dx}(ax + b)$$

$$= \cos(ax + b) \times (a \times 1 + 0)$$

$$= a \sin\left[\frac{\pi}{2} + (ax + b)\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} [a \cos(ax + b)]$$

$$= a \frac{d}{dx} [\cos(ax + b)]$$

$$= a[-\sin(ax + b)] \cdot \frac{d}{dx}(ax + b)$$

$$= a[-\sin(ax + b)] \times (a \times 1 + 0)$$

$$= a^2 \cdot \sin[\pi + (ax + b)]$$

$$= a^2 \cdot \sin\left[\frac{2\pi}{2} + (ax + b)\right]$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} [-a^2 \sin(ax + b)]$$

$$= -a^2 \frac{d}{dx} [\sin(ax + b)]$$

$$= -a^2 \cdot \cos(ax + b) \cdot \frac{d}{dx}(ax + b)$$

$$= -a^2 \cdot \cos(ax + b) \times (a \times 1 + 0)$$

$$= a^3 \cdot \sin\left[\frac{3\pi}{2} + (ax + b)\right]$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = a^n \cdot \sin \left[\frac{n\pi}{2} + (ax + b) \right].$$

Exercise 1.5 | Q 4.08 | Page 60

Find the n^{th} derivative of the following : $\cos(3 - 2x)$

SOLUTION

Let $y = \cos(3 - 2x)$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} [\cos(3 - 2x)]$$

$$= \cos(3 - 2x) \cdot \frac{d}{dx}(3 - 2x)$$

$$= \cos(3 - 2x) \times (a \times 1 + 0)$$

$$= a \cos \left[\frac{\pi}{2} + (3 - 2x) \right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} [a \cos(3 - 2x)]$$

$$= a \frac{d}{dx} [\cos(3 - 2x)]$$

$$= a[-\cos(3 - 2x)] \cdot \frac{d}{dx}(3 - 2x)$$

$$= a[-\cos(3 - 2x)] \times (a \times 1 + 0)$$

$$= a^2 \cdot \cos[\pi + (3 - 2x)]$$

$$= a^2 \cdot \cos \left[\frac{2\pi}{2} + (3 - 2x) \right]$$

$$\begin{aligned}
\frac{d^3y}{dx^3} &= \frac{d}{dx} [-a^2 \cos(ax + b)] \\
&= -a^2 \frac{d}{dx} [\cos(3 - 2x)] \\
&= -a^2 \cdot \cos(3 - 2x) \cdot \frac{d}{dx}(3 - 2x) \\
&= -a^2 \cdot \cos(3 - 2x) \times (a \times 1 + 0) \\
&= a^3 \cdot \cos\left[\frac{3\pi}{2} + (3 - 2x)\right]
\end{aligned}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = (-2)^n \cos\left[\frac{n\pi}{2} + (3 - 2x)\right].$$

Exercise 1.5 | Q 4.09 | Page 60

Find the n^{th} derivative of the following : $\log(2x + 3)$

SOLUTION

Let $y = \log(2x + 3)$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} [\log(2x + 3)]$$

$$= \frac{1}{2x + 3} \cdot \frac{d}{dx}(2x + 3)$$

$$= \frac{1}{2x + 3} \times (a \times 1 + 0)$$

$$= \frac{a}{2x + 3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{2x + 3}\right)$$

$$= a \frac{d}{dx}(2x + 3)^{-1}$$

$$= a(-1)(2x+3)^{-2} \cdot \frac{d}{dx}(2x+3)$$

$$= \frac{(-1)a}{(2x+3)^2} \times (a \times 1 + 0)$$

$$= \frac{(-1)a}{(2x+3)^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left[\frac{(-1)^1 a^2}{(2x+3)^2} \right]$$

$$= (-1)^1 a^2 \cdot \frac{d}{dx} (2x+3)^{-2}$$

$$= (-1)^1 a^2 \cdot (-2)(2x+3)^{-3} \cdot \frac{d}{dx} (2x+3)$$

$$= \frac{(-1)^2 \cdot 1 \cdot 2 \cdot a^2}{(2x+3)^3} \times (a \times 1 + 0)$$

$$= \frac{(-1) \cdot 2 \cdot 2! a^3}{(2x+3)^3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)! 2^n}{(2x+3)^n}.$$

Exercise 1.5 | Q 4.1 | Page 60

Find the n^{th} derivative of the following : $\frac{1}{3x-5}$

SOLUTION

$$\text{Let } y = \frac{1}{3x - 5}$$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(3x - 5)$$

$$= -1(3x - 5)^{-2} \cdot \frac{d}{dx}(3x - 5)$$

$$= \frac{-1}{(3x - 5)^2} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^1 \cdot 3}{(3x - 5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{(-1)^1 \cdot 3}{(3x - 5)^2} \right]$$

$$= (-1)^1 \cdot 3 \frac{d}{dx}(3x - 5)^{-2}$$

$$= (-1)^{-1} \cdot 3 \cdot (-2)(3x - 5)^{-3} \cdot \frac{d}{dx}(3x - 5)$$

$$= \frac{(-1)^2 \cdot 3 \cdot 2}{(3x - 5)^3} \times (3 \times 1 - 0)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left[\frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x - 5)^3} \right]$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot \frac{d}{dx}(3x - 5)^{-3}$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot (-3)(3x - 5)^{-4} \cdot \frac{d}{dx}(3x - 5)$$

$$= \frac{(-1)^3 \times 3 \cdot 2! \times 3^2}{(3x - 5)^4} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^3 \times 3! \times 3^3}{(3x - 5)^4}$$

In general, the nth order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n! \cdot 3^n}{(3x - 5)^{n+1}}.$$

Exercise 1.5 | Q 4.11 | Page 60

Find the nth derivative of the following : $y = e^{ax} \cdot \cos(bx + c)$

SOLUTION

$$y = e^{ax} \cdot \cos(bx + c)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax} \cdot \cos(bx + c)]$$

$$= e^{ax} \cdot \frac{d}{dx}[\cos(bx + c)] + \cos(bx + c) \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot [-\sin(bx + c)] \cdot \frac{d}{dx}(bx + c) + \cos(bx + c) \cdot e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= -e^{ax} \sin(bx + c) \times (b \times 1 + 0) + e^{ax} \cos(bx + c) \times a \times 1$$

$$= e^{ax} [a \cos(bx + c) - b \sin(bx + c)]$$

$$= e^{ax} \cdot \sqrt{a^2 + b^2} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos(bx + c) - \frac{b}{\sqrt{a^2 + b^2}} \sin(bx + c) \right]$$

$$\text{Let } \frac{a}{\sqrt{a^2 + b^2}} = \cos x \text{ and } \frac{b}{\sqrt{a^2 + b^2}} = \sin x$$

$$\text{Then } \tan \infty = \frac{b}{a}$$

$$\therefore \infty = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\begin{aligned}
& \therefore \frac{dy}{dx} = e^{ax} \cdot \sqrt{a^2 + b^2} [\cos \infty \cdot \cos(bx + c) - \sin \infty \cdot \sin(bx + c)] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(bx + c + x) \\
&\frac{d^2y}{dx^2} = \frac{d}{dx} \left[e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(bx + c + \infty) \right] \\
&= (a^2 + b^2)^{\frac{1}{2}} \cdot \frac{d}{dx} [e^{ax} \cdot \cos(bx + c + \infty)] \\
&= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \cdot \frac{d}{dx} \{ \cos(bx + c + \infty) \} + \cos(bx + c + \infty) \cdot \frac{d}{dx} (e^{ax}) \right] \\
&= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \cdot \{- \sin(bx + c + \infty)\} \cdot \frac{d}{dx} (bx + c + \infty) + \cos(bx + c + \infty) \cdot e^{ax} \cdot \frac{d}{dx} (ax) \right] \\
&= (a^2 + b^2)^{\frac{1}{2}} [-e^{ax} \sin(bx + c + \infty) \times (b \times 1 + 0 + 0) + \cos(bx + c + \infty) \cdot e^{ax} \times a \times 1] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} [a \cos(bx + c + \infty) - b \sin(bx + c + \infty)] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos(bx + c + \infty) - \frac{b}{\sqrt{a^2 + b^2}} \sin(bx + c + \infty) \right] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} [\cos \infty \cdot \cos(bx + c + \infty) - \sin \infty \cdot \sin(bx + c + \infty)] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(bx + c + \infty + \infty) \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(bx + c + 2\infty)
\end{aligned}$$

Similarly.

$$\frac{d^3y}{dx^3} = e^{ax} \cdot (a^2 + b^2)^{\frac{3}{2}} \cdot \cos(bx + c + 3\infty)$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = e^{ax} \cdot (a^2 + b^2)^{\frac{n}{2}} \cdot \cos(bx + c + n\infty),$$

$$\text{Where } \infty = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\therefore \frac{d^n y}{dx^n} = e^{ax} \cdot (a^2 + b^2)^{\frac{n}{2}} \cdot \cos \left[bx + c + n \tan^{-1} \left(\frac{b}{a} \right) \right].$$

Exercise 1.5 | Q 4.12 | Page 60

Find the n^{th} derivative of the following : $y = e^{8x} \cdot \cos(6x + 7)$

SOLUTION

$$y = e^{8x} \cdot \cos(6x + 7)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax} \cdot \cos(6x + 7)]$$

$$= e^{ax} \cdot \frac{d}{dx}[\cos(6x + 7)] + \cos(6x + 7) \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot [-\sin(6x + 7)] \cdot \frac{d}{dx}(6x + 7) + \cos(6x + 7) \cdot e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= -e^{ax} \sin(6x + 7) \times (b \times 1 + 0) + e^{ax} \cos(6x + 7) \times a \times 1$$

$$= e^{ax} [a \cos(6x + 7) - b \sin(6x + 7)]$$

$$= e^{ax} \cdot \sqrt{a^2 + b^2} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos(6x + 7) - \frac{b}{\sqrt{a^2 + b^2}} \sin(6x + 7) \right]$$

$$\text{Let } \frac{a}{\sqrt{a^2 + b^2}} = \cos x \text{ and } \frac{b}{\sqrt{a^2 + b^2}} = \sin x$$

$$\text{Then } \tan \infty = \frac{b}{a}$$

$$\therefore \infty = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\therefore \frac{dy}{dx} = e^{ax} \cdot \sqrt{a^2 + b^2} [\cos \infty \cdot \cos(bx + c) - \sin \infty \cdot \sin(bx + c)]$$

$$= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(6x + 7 + x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(6x + 7 + \infty) \right]$$

$$\begin{aligned}
&= (a^2 + b^2)^{\frac{1}{2}} \cdot \frac{d}{dx} [e^{ax} \cdot \cos(6x + 7 + \infty)] \\
&= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \cdot \frac{d}{dx} \{ \cos(6x + 7 + \infty) \} + \cos(6x + 7 + \infty) \cdot \frac{d}{dx} (e^{ax}) \right] \\
&= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \cdot \{-\sin(6x + 7 + \infty)\} \cdot \frac{d}{dx} (6x + 7 + \infty) + \cos(6x + 7 + \infty) \cdot e^{ax} \cdot \frac{d}{dx} (ax) \right] \\
&= (a^2 + b^2)^{\frac{1}{2}} [-e^{ax} \sin(6x + 7 + \infty) \times (b \times 1 + 0 + 0) + \cos(6x + 7 + \infty) \cdot e^{ax} \times a \times 1] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} [a \cos(6x + 7 + \infty) - b \sin(6x + 7 + \infty)] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos(6x + 7 + \infty) = \frac{b}{\sqrt{a^2 + b^2}} \sin(6x + 7 + \infty) \right] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} [\cos \infty \cdot \cos(6x + 7 + \infty) - \sin \infty \cdot \sin(6x + 7 + \infty)] \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(6x + 7 + \infty + \infty) \\
&= e^{ax} \cdot (a^2 + b^2)^{\frac{1}{2}} \cdot \cos(6x + 7 + 2\infty)
\end{aligned}$$

Similarly,

$$\frac{d^3y}{dx^3} = e^{ax} \cdot (a^2 + b^2)^{\frac{3}{2}} \cdot \cos(6x + 7 + 3\infty)$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = e^{ax} \cdot (a^2 + b^2)^{\frac{n}{2}} \cdot \cos(6x + 7 + n\infty),$$

$$\text{Where } \infty = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\therefore \frac{d^n y}{dx^n} = e^{8x} \cdot (10)^n \cdot \cos \left[6x + 7 + n \tan^{-1} \left(\frac{3}{4} \right) \right].$$

MISCELLANEOUS EXERCISE 1 [PAGES 61 - 63]

[Miscellaneous Exercise 1 | Q 1 | Page 61](#)

Choose the correct option from the given alternatives :

Let $f(1) = 3$, $f'(1) = -\frac{1}{3}$, $g(1) = -4$ and $g'(1) = -\frac{8}{3}$. The derivative of $\sqrt{[f(x)]^2 + [g(x)]^2}$ w.r.t. x at $x = 1$ is

$$\begin{array}{r}
 -\frac{29}{15} \\
 -\frac{7}{3} \\
 \hline
 \frac{31}{15} \\
 \hline
 \frac{29}{15}
 \end{array}$$

SOLUTION

$$\frac{29}{15}$$

[Hint : Let $y = \sqrt{[f(x)]^2 + [g(x)]^2}$

$$\text{Then } \frac{dy}{dx} = \frac{1}{\sqrt{[f(x)]^2 + [g(x)]^2}} \cdot [2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)]$$

$$\begin{aligned}
 \therefore \left(\frac{dy}{dx} \right)_{at x=1} &= \frac{1}{\sqrt{[f(x)]^2 + [g(x)]^2}} \cdot [2f(1) \cdot f'(1) + 2g(1) \cdot g'(1)] \\
 &= \frac{1}{2\sqrt{9+16}} \times \left[2(3)\left(-\frac{1}{3}\right) + 2(-4)\left(-\frac{8}{3}\right) = \frac{29}{15} \right].
 \end{aligned}$$

Miscellaneous Exercise 1 | Q 2 | Page 62

Choose the correct option from the given alternatives :

If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$, is equal to

$$\frac{1}{2}$$

$$1$$

$$\frac{1}{\sqrt{2}}$$

SOLUTION

$$\frac{1}{\sqrt{2}}$$

[Hint : $\frac{dy}{dx} = \sec(\tan^{-1} x) \cdot \tan(\tan^{-1} x) \times \frac{1}{1+x^2}$

$$\begin{aligned}\therefore \left(\frac{dy}{dx} \right)_{at x=1} &= \sec(\tan^{-1} 1) \times 1 \times \frac{1}{1+1^2} \\ &= \sec \frac{\pi}{4} \times \frac{1}{2} = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}.\end{aligned}$$

Miscellaneous Exercise 1 | Q 3 | Page 62

Choose the correct option from the given alternatives :

If $f(x) = \sin^{-1} \left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}} \right)$, which of the following is not the derivative of $f(x)$?

$$\frac{2 \cdot 4^x \cdot \log 4}{1+4^{2x}}$$

$$\frac{4^{x+1} \cdot \log 2}{1+4^{2x}}$$

$$\frac{4^{x+1} \cdot \log 4}{1+4^{2x}}$$

$$\frac{2^{2(x+1)} \cdot \log 2}{1+2^{4x}}$$

SOLUTION

$$\frac{4^{x+1} \log 4}{1+4^{4x}}$$

[Hint : Put $4^x = \tan \theta$. Then $\theta = \tan^{-1}(4^x)$

$$\therefore f(x) = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan \theta} \right)$$

$$\begin{aligned}
&= \sin^{-1}(\sin 2\theta) \\
&= 2\theta \\
&= 2\tan^{-1}(4^x) \\
\therefore f'(x) &= 2 \times \frac{1}{1 + (4^x)^2} \times 4^x \log 4 \\
&= \frac{2 \cdot 4^x \cdot \log 4}{1 + 4^{2x}} \quad \dots (a) \\
&= \frac{2 \cdot 4^x \cdot 2 \log 2}{1 + 4^{2x}} \\
&= \frac{4^{x+1} \cdot \log 2}{1 + 4^{2x}} \quad \dots (b) \\
&= \frac{(2^2)^{x+1} \cdot \log 2}{1 + 2^{4x}} \\
&= \frac{2^{2(x+1)} \cdot \log 2}{1 + 2^{4x}} \quad \dots (d)
\end{aligned}$$

Miscellaneous Exercise 1 | Q 4 | Page 62

Choose the correct option from the given alternatives :

If $x^y = y^x$, then $\frac{dy}{dx} = \dots \dots \dots$

$$\frac{x(x \log y - y)}{y(y \log x - x)}$$

$$\frac{y(x \log y - y)}{x(y \log x - x)}$$

$$\frac{x(y \log x - x)}{y^2(1 - \log x)}$$

$$\frac{y^2(1 - \log y)}{x^2(1 - \log y)}$$

$$\frac{y(1 - \log x)}{x(1 - \log y)}$$

SOLUTION

$$\frac{y(x \log y - y)}{x(y \log x - x)}$$

$$x^y = y^x \quad \therefore y \log x = x \log y$$

$$\therefore y \times \frac{1}{x} + (\log x) \frac{dy}{dx} = x \times \frac{1}{y} \frac{dy}{dx} + \log y$$

$$\therefore \left(\log x - \frac{x}{y} \right) \frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\therefore \left(\frac{y \log x - x}{y} \right) \frac{dy}{dx} = \frac{x \log y - y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

Miscellaneous Exercise 1 | Q 5 | Page 62

Choose the correct option from the given alternatives :

If $y = \sin(2\sin^{-1} x)$, then $\frac{dy}{dx} = \dots\dots$

$$\frac{2 - 4x^2}{\sqrt{1 - x^2}}$$

$$\frac{2 + 4x^2}{\sqrt{1 - x^2}}$$

$$\frac{4x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

$$\frac{\sqrt{1 - x^2}}{1 - 2x^2}$$

SOLUTION

$$\frac{2 - 4x^2}{\sqrt{1 - x^2}}$$

[Hint : $y = \sin(2\sin^{-1} x)$

Put $x = \sin\theta$. Then $\theta = \sin^{-1} x$

$\therefore y = \sin 2\theta$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2 \cos 2\theta \times \frac{1}{\sqrt{1 - x^2}} \\ &= \frac{2(1 - 2\sin^2 \theta)}{\sqrt{1 - x^2}} = \frac{2 - 4x^2}{\sqrt{1 - x^2}}.\end{aligned}$$

Miscellaneous Exercise 1 | Q 6 | Page 62

Choose the correct option from the given alternatives :

If $y = \tan^{-1} \left(\frac{x}{1 + \sqrt{1 - x^2}} \right) + \sin \left[2 \tan^{-1} \left(\sqrt{\frac{1 - x}{1 + x}} \right) \right]$ then $\frac{dy}{dx} = \dots$

$\frac{x}{\sqrt{1 - x^2}}$

$\frac{1 - 2x}{\sqrt{1 - x^2}}$

$\frac{1 - 2x}{\sqrt{1 - x^2}}$

$\frac{2\sqrt{1 - x^2}}{1 - 2x^2}$

$\frac{1 - 2x^2}{\sqrt{1 - x^2}}$

SOLUTION

$$\frac{1-2x}{2\sqrt{1-x^2}}$$

$$y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$$

Put $x = \cos \theta$. Then $\theta = \cos^{-1} x$

$$\begin{aligned}\therefore y &= \tan^{-1}\left(\frac{\cos \theta}{1+\sqrt{1-\cos^2 \theta}}\right) + \sin\left[2\tan^{-1}\sqrt{\frac{1-\cos \theta}{1+\cos \theta}}\right] \\ &= \tan^{-1}\left(\frac{\cos \theta}{1+\sin \theta}\right) + \sin\left[2\tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}\right] \\ &= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}\right] + \sin\left[2\tan^{-1}\left(\tan \frac{\theta}{2}\right)\right] \\ &= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right] + \sin\left(2\times\frac{\theta}{2}\right) \\ &= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\sin\theta\right] \\ &= \frac{\pi}{4}-\frac{\theta}{2}+\sqrt{1-\cos^2\theta} \\ &= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x+\sqrt{1-x^2} \\ \therefore \frac{dy}{dx} &= 0-\frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{1-x^2}}\times(-2x)\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\
 &= \frac{1-2x}{2\sqrt{1-x^2}}.
 \end{aligned}$$

Miscellaneous Exercise 1 | Q 7 | Page 62

Choose the correct option from the given alternatives :

If y is a function of x and $\log(x+y) = 2xy$, then the value of $y'(0) = \dots$

- 2
- 0
- 1
- 1**

SOLUTION

1

[Hint : $\log(x+y) = 2xy \dots(1)$

$$\begin{aligned}
 \therefore \frac{1}{x+y} \cdot \left(1 + \frac{dy}{dx}\right) &= 2x \frac{dy}{dx} + 2y \\
 \therefore \left(\frac{1}{x+y} - 2x\right) \frac{dy}{dx} &= 2y - \frac{1}{x+y} \\
 \therefore \frac{dy}{dx} &= \frac{2y(x+y) - 1}{1 - 2x(x+y)}
 \end{aligned}$$

If $x = 0$, then from (1),

$$\log y = 0 = \log 1$$

$$\therefore y = 1$$

$$\therefore y'(0) = \frac{(2(1)(0+1) - 1)}{1 - 2(0)(0+1)} = 1.$$

Choose the correct option from the given alternatives :

If g is the inverse of function f and $f'(x) = \frac{1}{1+x^7}$, then the value of $g'(x)$ is equal to :

$$\frac{1 + x^7}{1 + [g(x)]^7}$$

$$1 + [g(x)]^7$$

$$7x^6$$

SOLUTION

$$1 + [g(x)]^7$$

[Hint : Since g is the inverse of f , $f^{-1}(x) = g(x)$

$$\therefore f[f^{-1}(x)] = f[g(x)] = x$$

$$\therefore f'[g(x)] \cdot \frac{d}{dx}[g(x)] = 1$$

$$\therefore f'[g(x)] \times g'(x) = 1$$

$$\therefore g'(x) = \frac{1}{f'[g(x)]}, \text{ where } f'(x) = \frac{1}{1+x^7}$$

$$\therefore g'(x) = 1 + [g(x)]^7.$$

Choose the correct option from the given alternatives :

If $x\sqrt{y+1} + y\sqrt{x+1} = 0$ and $x \neq y$, then $\frac{dy}{dx} = \dots$

$$-\frac{1}{(1+x)^2}$$

$$\frac{(1+x)^2}{x} - \frac{x}{x+1}$$

SOLUTION

$$-\frac{1}{(1+x)^2}$$

$$[\text{Hint : } x\sqrt{y-1} = -y\sqrt{x+1}]$$

$$\therefore x^2(y+1) = y^2(x+1)$$

$$\therefore x^2y + x^2 = xy^2 + y^2$$

$$\therefore x^2 - y^2 = xy^2 - x^2y$$

$$\therefore (x-y)(x+y) = -xy(x-y)$$

$$\therefore x+y = -xy \quad \dots [\because x \neq y] \dots (1)$$

Differentiating both sides w.r.t. x, we get

$$1 + \frac{dy}{dx} = -x \frac{dy}{dx} - y$$

$$\therefore (1+x) \frac{dy}{dx} = -1 - y$$

$$\therefore (1+x) \frac{dy}{dx}$$

$$= -(1+x)(1+y)$$

$$= -(1+x+y+xy)$$

$$= -(1-xy+xy) \quad \dots [\text{By (1)}]$$

$$= -1$$

$$\therefore \frac{dy}{dx} = -\frac{1}{(1+x)^2}.$$

Choose the correct option from the given alternatives :

If $y \tan^{-1} \left(\sqrt{\frac{a-x}{a+x}} \right)$, where $-a < x < a$, then $\frac{dy}{dx} = \dots$

$\frac{x}{\sqrt{a^2 - x^2}}$
 $\frac{1}{\sqrt{a^2 - x^2}}$
 $-\frac{1}{2\sqrt{a^2 - x^2}}$
 $\frac{1}{2\sqrt{a^2 - x^2}}$

SOLUTION

$$-\frac{1}{2\sqrt{a^2 - x^2}}$$

[Hint : Put $x = a \cos \theta$].

Choose the correct option from the given alternatives :

If $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$, then $\left(\frac{d^2y}{dx^2} \right)_{\theta=\frac{\pi}{4}} = \dots$

$\frac{8\sqrt{2}}{a\pi}$
 $-\frac{8\sqrt{2}}{a\pi}$
 $\frac{8\sqrt{2}}{4\sqrt{2}}$
 $\frac{a\pi}{8\sqrt{2}}$

SOLUTION

$$\frac{8\sqrt{2}}{a\pi}$$

$$[\text{Hint: } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{a(\cos\theta + \theta\sin\theta - \cos\theta)}{a(-\sin\theta + \theta\cos\theta + \sin\theta)}$$

$$= \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$= \tan\theta$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta)$$

$$= \frac{d}{d\theta}(\tan\theta) \times \frac{d\theta}{dx}$$

$$= \sec^2\theta \times \frac{1}{a\theta\cos\theta}$$

$$= \frac{1}{a\theta} \cdot \sec^3\theta$$

$$\therefore \left(\frac{d^2y}{dx^2} \right)_{\text{at } \theta=\frac{\pi}{4}} = \frac{1}{a\left(\frac{\pi}{4}\right)} \left(\sec \frac{\pi}{4} \right)^3$$

$$= \frac{4}{a\pi} \times (\sqrt{2})^3$$

$$= \frac{8\sqrt{2}}{a\pi} \Bigg].$$

Choose the correct option from the given alternatives :

If $y = a \cos(\log x)$ and $A \frac{d^2y}{dx^2} + B \frac{dy}{dx} + C = 0$, then the values of A, B, C are

$x^2, -x, -y$

x^2, x, y

$x^2, x, -y$

$x^2, -x, y$

SOLUTION

x^2, x, y

[Hint : $y = a \cos (\log x)$... (1)]

$$\therefore \frac{dy}{dx} = a[-\sin(\log x)] \times \frac{1}{x}$$

$$\therefore x \frac{dy}{dx} = -a \sin(\log x)$$

$$\therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} = -a \cos(\log x) \times \frac{1}{x}$$

$$\therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} = -y \quad \dots [\text{By (1)}]$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Comparing this with $A \frac{d^2y}{dx^2} + B \frac{dy}{dx} + C = 0$, we get $A = x^2$, $B = x$, $C = y$.

MISCELLANEOUS EXERCISE 1 [PAGES 63 - 64]

Solve the following :

$$f(x) = -x, \text{ for } -2 \leq x < 0$$

$$= 2x, \text{ for } 0 \leq x < 2$$

$$= \frac{18-x}{4}, \text{ for } 2 < x \leq 7$$

$$g(x) = 6 - 3x, \text{ for } 0 \leq x < 2$$

$$= \frac{2x-4}{3}, \text{ for } 2 < x \leq 7$$

Let $u(x) = f[g(x)]$, $v(x) = g[f(x)]$ and $w(x) = g[g(x)]$. Find each derivative at $x = 1$, if it exists i.e. find $u'(1)$, $v'(1)$ and $w'(1)$. If it doesn't exist, then explain why?

SOLUTION

$$u(x) = f[g(x)]$$

$$\therefore u'(x) = \frac{d}{dx} \{f[g(x)]\}$$

$$= f'[g(x)] \cdot \frac{d}{dx}[g(x)]$$

$$= f'[g(x)] \times g'(x)$$

$$\therefore u'(1) = f'[g(1)] \times g'(1)$$

$$= f'(3) \times g'(1) \quad \dots(1)$$

$$\dots [\because g(x) = 6 - 3x, 0 \leq x \leq 2]$$

$$\text{Now, } f(x) = \frac{18-x}{4}, \text{ for } 2 < x \leq 7$$

$$\text{and } g(x) = 6 - 3x, \text{ for } 0 < x \leq 2$$

$$\therefore f'(x) = \frac{1}{4}(0-1) = -\frac{1}{4}, \text{ for } 2 < x \leq 7$$

$$\text{and } g'(x) = 0 - 3(1) = -3, \text{ for } 0 < x \leq 2$$

$$\therefore f'(3) = -\frac{1}{4} \text{ and } g'(1) = -3$$

\therefore from (1),

$$u'(1) = -\frac{1}{4}(-3) = \frac{3}{4}$$

$$\text{Now, } v(x) = g[f(x)]$$

$$\therefore v'(x) = \frac{d}{dx} \{g[f(x)]\}$$

$$= g'[f(x)] \cdot \frac{d}{dx}[f(x)]$$

$$= g'[f(x)] \times f'(x)$$

$$\therefore v'(1) = g'[f(1)] \times f'(x)$$

$$= g'(2) \times f'(1) \quad \dots(2)$$

$$\dots [\because f(x) = 2x, 0 \leq x \leq 2]$$

$$\text{Now, } g(x) = 6 - 3x, \text{ for } 0 \leq x \leq 2$$

$$= \frac{2x - 4}{3}, \text{ for } 2 < x \leq 7$$

$$\therefore g''(x) = 0 - 3 \times 1 = -3, \text{ for } 0 \leq x \leq 2$$

$$\text{and } g'(x) = \frac{1}{3}(2 \times 1 - 0) = \frac{2}{3}, \text{ for } 2 < x \leq 7$$

$$\therefore Lg'(2) \neq Rg'(2)$$

$$\therefore g'(2) \text{ does not exist}$$

\therefore from (2),

$$v'(1) \text{ does not exist}$$

$$\text{Also, } w(x) = g[g(x)]$$

$$\therefore w'(x) = \frac{d}{dx} \{g[g(x)]\}$$

$$= g'[g(x)] \cdot \frac{d}{dx}[g(x)]$$

$$= g'[g(x)] \times g'(x)$$

$$\therefore w'(1) = g'[g(1)] \times g'(x)$$

$$= g'(3) \times g'(1) \quad \dots(3)$$

$$\dots [\because g(x) = 6 - 3x, 0 \leq x \leq 2]$$

$$\text{Now, } g(x) = 6 - 3x, \text{ for } 0 \leq x \leq 2$$

$$= \frac{2x - 4}{3}, \text{ for } 2 < x \leq 7$$

$$\therefore g'(x) = 0 - 3 \times 1 = -3, \text{ for } 0 \leq x \leq 2$$

$$\text{and } g'(x) = \frac{1}{3}(2 \times 1 - 0) = \frac{2}{3}, \text{ for } 2 \leq x \leq 7$$

$$\therefore g(3) = \frac{2}{3} \text{ and } g'(1) = -3$$

\therefore from (3),

$$w'(1) = \frac{2}{3}(-3) = -2.$$

$$\text{Hence, } u'(1) = \frac{3}{4}, v'(1) \text{ does not exist and } w'(1) = -2.$$

Miscellaneous Exercise 1 | Q 2 | Page 63

Solve the following :

The values of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ are given in the following table :

x	f(x)	g(x)	f'(x)	fg'(x)
-1	3	2	-3	4
2	2	-1	-5	-4

Match the following :

A Group – Function	B Group – Derivative
(A) $\frac{d}{dx}[f(g(x))] \text{ at } x = -1$	1. -16
(B) $\frac{d}{dx}[g(f(x) - 1)] \text{ at } x = -1$	2. 20
(C) $\frac{d}{dx}[f(f(x) - 3)] \text{ at } x = 2$	3. -20
(D) $\frac{d}{dx}[g(g(x))] \text{ at } x = 2$	5. 12

SOLUTION

$$\begin{aligned} (\text{A}) \quad & \frac{d}{dx} [f(g(x))] \\ &= f'(g(x)) \cdot \frac{d}{dx} (g(x)) \\ &= f'(g(x)) \times g'(x) \end{aligned}$$

$$\begin{aligned} &\therefore \frac{d}{dx} [f(g(x))] \text{ at } x = -1 \\ &= f'(g(-1)) \times g'(-1) \\ &= f'(2) \times g'(-1) \quad \dots [\because g(x) = 2, \text{ when } x = -1] \\ &= -5 \times 4 \\ &= -20 \end{aligned}$$

$$\begin{aligned} (\text{B}) \quad & \frac{d}{dx} [g(f(x) - 1)] \\ &= g'(f(x) - 1) \cdot \frac{d}{dx} [f(x) - 1] \\ &= g'(f(x) - 1) \times [f'(x) - 0] \end{aligned}$$

$$\begin{aligned} &\therefore \frac{d}{dx} [gf(x) - 1] \text{ at } x = -1 \\ &= g'(f(-1) - 1) \times f'(-1) \\ &= g'(3 - 1) \times f'(-1) \quad \dots [\because f(x) = 3, \text{ when } x = -1] \\ &= g'(2) \times f'(-1) \\ &= (-4)(-3) \\ &= 12 \end{aligned}$$

$$\begin{aligned} (\text{C}) \quad & \frac{d}{dx} [f(f(x) - 3)] \\ &= f'(f(x) - 3) \cdot \frac{d}{dx} [f(x) - 3] \\ &= f'(f(x) - 3) \times [f'(x) - 0] \end{aligned}$$

$$\begin{aligned}
& \therefore \frac{d}{dx} [f(f(x) - 3)] \text{ at } x = 2 \\
&= f''(f(2) - 3) \times f'(2) \\
&= f'(2 - 3) \times f'(2) \quad \dots [\because f(x) = 2, \text{ when } x = 2] \\
&= f'(-1) \times f'(2) \\
&= (-3)(-5) \\
&= 15
\end{aligned}$$

$$\begin{aligned}
(D) \quad & \frac{d}{dx} [g(g(x))] \\
&= g'(g(x)) \cdot \frac{d}{dx} [g(x)] \\
&= g'(g(x)) \times g'(x)
\end{aligned}$$

$$\begin{aligned}
& \therefore \frac{d}{dx} [g(g(x))] \text{ at } x = 2 \\
&= g'(g(2)) \times g'(2) \\
&= g'(-1) \times g'(2) \quad \dots [\because g(x) = -1 \text{ at } x = 2] \\
&= 4(-4) \\
&= -16
\end{aligned}$$

Hence, (A) $\rightarrow 3$, (B) $\rightarrow 5$, (C) $\rightarrow 4$, (D) $\rightarrow 1$.

Miscellaneous Exercise 1 | Q 3 | Page 63

Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1		5	$\frac{1}{3}$
1	3	-4	$-\frac{1}{3}$	$-\frac{8}{3}$

- (i) The derivative of $f[g(x)]$ w.r.t. x at $x = 0$ is
- (ii) The derivative of $g[f(x)]$ w.r.t. x at $x = 0$ is
- (iii) The value of $\left[\frac{d}{dx} [x^{10} + f(x)]^{-2} \right]_{x=1}$ is
- (iv) The derivative of $f[(x + g(x))]$ w.r.t. x at $x = 0$ is ...

SOLUTION

$$\begin{aligned}
 & \text{(i)} \quad \frac{d}{dx} \{f[g(x)]\} \\
 &= f'[g(x)] \cdot \frac{d}{dx} [g(x)] \\
 &= f'[g(x)] x g'(x) \\
 &\therefore \frac{d}{dx} \{f[g(x)]\} \text{ at } x = 0 \\
 &= f'[g(0)] x g'(0) \\
 &= f'(1) x g'(0) \quad \dots [\because g(x) = 1 \text{ at } x = 0]
 \end{aligned}$$

$$= -\frac{1}{3} \times \frac{1}{3}$$

$$= -\frac{1}{9}.$$

$$\begin{aligned}
 & \text{(ii)} \quad \frac{d}{dx} \{g[f(x)]\} \\
 &= g'[f(x)] \cdot \frac{d}{dx} [f(x)] \\
 &= g'[f(x)] x f'(x) \\
 &\therefore \frac{d}{dx} \{f[g(x)]\} \text{ at } x = 0
 \end{aligned}$$

$$= g'(1)x f'(0) \quad \dots [\because f(x) = 1 \text{ at } x = 0]$$

$$= -\frac{8}{3} \times 5$$

$$= -\frac{40}{3}.$$

$$(iii) \frac{d}{dx} [x^{10} + f(x)]^{-2}$$

$$= 2[x^{10} + f(x)]^{-3} \cdot \frac{d}{dx} [x^{10} + f(x)]$$

$$= -2[x^{10} + f(x)]^{-3} \times [10x^9 + f'(x)]$$

$$\therefore \left\{ \frac{d}{dx} [x^{10} f(x)]^{-2} \right\}_{at x=1}$$

$$= -2[1^{10} + f(1)]^{-3} \times [10(1)^9 + f'(1)]$$

$$= \frac{-2}{(1+3)^3} \times \left[10 + \left(-\frac{1}{3} \right) \right] \quad \dots [\because f(x) = 3 \text{ at } x = 1]$$

$$= \frac{-2}{64} \times \frac{29}{3}$$

$$= -\frac{29}{96}.$$

$$(iv) \frac{d}{dx} [f(x + g(x))]$$

$$= f'(x + g(x)) \cdot \frac{d}{dx} [x + g(x)]$$

$$= f'(x + g(x)) \times [1 + g'(x)]$$

$$\therefore \left\{ \frac{d}{dx} [f(x + g(x))] \right\}_{at x=0}$$

$$= f'(0 + g(0))x[1 + g'(0)]$$

$$= f'(1) \cdot [1 + g'(0)] \quad \dots [\because g(x) = 1 \text{ at } x = 0]$$

$$\begin{aligned}
 &= -\frac{1}{3} \left[1 + \frac{1}{3} \right] \\
 &= -\frac{1}{3} \times \frac{4}{3} \\
 &= -\frac{4}{9}.
 \end{aligned}$$

Miscellaneous Exercise 1 | Q 4.1 | Page 64

Differentiate the following w.r.t. x : $\sin \left[2 \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right]$

SOLUTION

$$\text{Let } y = \sin \left[2 \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right]$$

Put $x = \cos \theta$. Then $\theta = \cos^{-1} x$ and

$$\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-\cos \theta}{1+\cos \theta}}$$

$$= \sqrt{\frac{2 \sin^2 \left(\frac{\theta}{2} \right)}{2 \cos^2 \left(\frac{\theta}{2} \right)}}$$

$$= \sqrt{\tan^2 \left(\frac{\theta}{2} \right)}$$

$$= \tan \left(\frac{\theta}{2} \right)$$

$$\therefore \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right)$$

$$= \tan^{-1} \left[\tan \left(\frac{\theta}{2} \right) \right]$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2} \cos^{-1} x$$

$$\therefore y = \sin \left[2 \times \frac{1}{2} \cos^{-1} x \right]$$

$$= \sin (\cos^{-1} x)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} [\sin(\cos^{-1} x)]$$

$$= \cos(\cos^{-1} x) \cdot \frac{d}{dx} (\cos^{-1} x)$$

$$= x \times \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-x}{\sqrt{1-x^2}}.$$

Miscellaneous Exercise 1 | Q 4.2 | Page 64

Differentiate the following w.r.t. x : $\sin^2 \left[\cot^{-1} \left(\sqrt{\frac{1+x}{1-x}} \right) \right]$

SOLUTION

$$\text{Let } y = \sin^2 \left[\cot^{-1} \left(\sqrt{\frac{1+x}{1-x}} \right) \right]$$

Put $x = \cos\theta$. Then $\theta = \cos^{-1}x$ and

$$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$= \sqrt{\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}}$$

$$= \sqrt{\cot^2\left(\frac{\theta}{2}\right)}$$

$$= \cot\left(\frac{\theta}{2}\right)$$

$$\therefore \cot^{-1} \sqrt{\frac{1+x}{1-x}}$$

$$= \cot^{-1} \left[\cot\left(\frac{\theta}{2}\right) \right]$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2} \cos^{-1} x$$

$$\therefore y = \sin^2 \left(\frac{1}{2} \cos^{-1} x \right)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left[\sin \left(\frac{1}{2} \cos^{-1} x \right) \right]^2$$

$$= 2 \sin \left(\frac{1}{2} \cos^{-1} x \right) \cdot \frac{d}{dx} \sin \left(\frac{1}{2} \cos^{-1} x \right)$$

$$\begin{aligned}
&= 2 \sin\left(\frac{1}{2}\cos^{-1}x\right) \cdot \cos\left(\frac{1}{2}\cos^{-1}x\right) \cdot \frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x\right) \\
&= \sin\left[2\left(\frac{1}{2}\cos^{-1}x\right)\right] \times \frac{1}{2} \cdot \frac{d}{dx}(\cos^{-1}x) \\
&= \sin(\cos^{-1}x) \times \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}} \\
&= \sin(\sin^{-1}\sqrt{1-x^2}) \times \frac{-1}{2\sqrt{1-x^2}} \quad \dots [\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}] \\
&= \sqrt{1-x^2} \times \frac{-1}{2\sqrt{1-x^2}} \\
&= -\frac{1}{2}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 4.3 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$

SOLUTION

$$\text{Let } y = \tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$$

$$= \tan^{-1}\left[\frac{3\sqrt{x}-x\sqrt{x}}{1-3x}\right]$$

$$\text{Put } \sqrt{x} = \tan \theta. \text{ Then } \theta = \tan^{-1}(\sqrt{x})$$

$$\therefore y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$\begin{aligned}
&= 3 \tan^{-1}(\sqrt{x}) \\
\therefore \frac{dy}{dx} &= 3 \frac{d}{dx} [\tan^{-1}(\sqrt{x})] \\
&= 3 \times \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} (\sqrt{x}) \\
&= \frac{3}{1+x} \times \frac{1}{2\sqrt{x}} \\
&= \frac{3}{2\sqrt{x}(1+x)}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 4.4 | Page 64

Differentiate the following w.r.t. x : $\cos^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{2}\right)$

SOLUTION

$$\text{Let } y = \cos^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{2}\right)$$

Put $x = \cos\theta$. Then $\theta = \cos^{-1}x$ and

$$\begin{aligned}
&\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{2}\right) \\
&= \left(\frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{2}\right) \\
&= \frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)} - \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \\
&= \left[\cos \frac{\theta}{2}\right]\left(\frac{1}{\sqrt{2}}\right) - \left[\sin \frac{\theta}{2}\right]\left(\frac{1}{\sqrt{2}}\right) \\
&= \cos \frac{\theta}{2} \cdot \cos \frac{\pi}{4} - \sin \frac{\theta}{2} \cdot \sin \frac{\pi}{4} \\
&= \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \\
\therefore y &= \cos^{-1}\left[\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right] \\
&= \frac{\theta}{2} + \frac{\pi}{4} \\
&= \frac{1}{2}\cos^{-1}x + \frac{\pi}{4} \\
\therefore \frac{dy}{dx} &= \frac{1}{2}\frac{d}{dx}(\cos^{-1}x) + \frac{d}{dx}\left(\frac{\pi}{4}\right) \\
&= \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}} + 0 \\
&= \frac{-1}{2\sqrt{1-x^2}}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 4.5 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left(\frac{x}{1+6x^2}\right) + \cot^{-1}\left(\frac{1-10x^2}{7x}\right)$

SOLUTION

$$\begin{aligned} \text{Let } y &= \tan^{-1}\left(\frac{x}{1+6x^2}\right) + \cot^{-1}\left(\frac{1-10x^2}{7x}\right) \\ &= \tan^{-1}\left(\frac{x}{1+6x^2}\right) + \cot^{-1}\left(\frac{7x}{1-10x^2}\right) \dots \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right] \\ &= \tan^{-1}\left[\frac{3x-2x}{1+(3)(2x)}\right] + \tan^{-1}\left[\frac{5x+2x}{1-(5x)(2x)}\right] \\ &= \tan^{-1}3x - \tan^{-1}2x + \tan^{-1}5x + \tan^{-1}2x \\ &= \tan^{-1}3x + \tan^{-1}5x \\ \therefore \frac{dy}{dx} &[\tan^{-1}3x + \tan^{-1}5x] \\ &= \frac{d}{dx}(\tan^{-1}3x) + \frac{d}{dx}(\tan^{-1}5x) \\ &= \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x) + \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x) \\ &= \frac{1}{1+9x^2} \times 3 \times 1 + \frac{1}{1+25x^2} \times 5 \times 1 \\ &= \frac{3}{1+9x^2} + \frac{5}{1+25x^2}. \end{aligned}$$

Miscellaneous Exercise 1 | Q 4.6 | Page 64

Differentiate the following w.r.t. x : $\tan^{-1}\left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$

SOLUTION

$$\text{Let } y = \tan^{-1} \left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}} \right]$$

Put $x = \tan \theta$. Then $\theta = \tan^{-1} x$

$$\therefore \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} = \frac{\sqrt{1+\tan^2 \theta} + \tan \theta}{\sqrt{1+\tan^2 \theta} - \tan \theta}$$

$$= \frac{\sec \theta + \tan \theta}{\sec \theta - \tan \theta}$$

$$= \frac{\left(\frac{1}{\cos \theta} \right) + \left(\frac{\sin \theta}{\cos \theta} \right)}{\left(\frac{1}{\cos \theta} \right) - \left(\frac{\sin \theta}{\cos \theta} \right)}$$

$$= \frac{1 + \sin \theta}{1 - \sin \theta}$$

$$= \frac{1 - \cos\left(\frac{\pi}{2} + \theta\right)}{1 + \cos\left(\frac{\pi}{2} + \theta\right)}$$

$$= \frac{2 \sin^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$$

$$= \tan^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\therefore \sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\begin{aligned}
\therefore y &= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right] \\
&= \frac{\pi}{4} + \frac{\theta}{2} \\
&= \frac{\pi}{4} + \frac{1}{2} \tan^{-1} x \\
\therefore \frac{d}{dx} \left(\frac{\pi}{4} \right) + \frac{1}{2} \frac{d}{dx} (\tan^{-1} x) &= 0 + \frac{1}{2} \times \frac{1}{1+x^2} \\
&= \frac{1}{2(1+x^2)}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 5.1 | Page 64

If $\sqrt{y+x} + \sqrt{y-x} = c$, show that $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$.

SOLUTION

$$\sqrt{y+x} + \sqrt{y-x} = c$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\frac{1}{2\sqrt{y+x}} \cdot \frac{d}{dx}(y+x) + \frac{1}{2\sqrt{y-x}} \cdot \frac{d}{dx}(y-x) &= 0 \\
\therefore \frac{1}{\sqrt{y+x}} \cdot \left(\frac{dy}{dx} + 1 \right) + \frac{1}{\sqrt{y-x}} \cdot \left(\frac{dy}{dx} - 1 \right) &= 0 \\
\therefore \frac{1}{\sqrt{y+x}} \cdot \frac{dy}{dx} + \frac{1}{\sqrt{y+x}} + \frac{1}{\sqrt{y-x}} \cdot \frac{dy}{dx} - \frac{1}{\sqrt{y-x}} &= 0 \\
\therefore \left(\frac{1}{\sqrt{y+x}} + \frac{1}{\sqrt{y-x}} \right) \frac{dy}{dx} &= \frac{1}{\sqrt{y-x}} - \frac{1}{\sqrt{y+x}}
\end{aligned}$$

$$\begin{aligned}
& \therefore \left[\frac{\sqrt{y-x} + \sqrt{y+x}}{\sqrt{y+x} \cdot \sqrt{y-x}} \right] \frac{dy}{dx} = \frac{\sqrt{y+x} + \sqrt{y-x}}{\sqrt{y-x} \cdot \sqrt{y+x}} \\
& \therefore \frac{dy}{dx} = \frac{\sqrt{y+x} + \sqrt{y-x}}{\sqrt{y+x} \cdot \sqrt{y-x}} \\
& = \frac{\sqrt{y+x} + \sqrt{y-x}}{\sqrt{y+x} + \sqrt{y-x}} \times \frac{\sqrt{y+x} + \sqrt{y-x}}{\sqrt{y+x} - \sqrt{y-x}} \\
& = \frac{(\sqrt{y+x} - \sqrt{y-x})^2}{(y+x) - (y-x)} \\
& = \frac{y+x + y-x - 2\sqrt{y+x} \cdot \sqrt{y-x}}{y+x - y+x} \\
& = \frac{2y - 2\sqrt{y^2 - x^2}}{2x} \\
& = \frac{2y}{2x} - \frac{2\sqrt{y^2 - x^2}}{2x} \\
& = \frac{y}{x} - \sqrt{\frac{y^2 - x^2}{x^2}} \\
& \therefore \frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 5.2 | Page 64

If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, then show that $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

SOLUTION

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$$

$$\therefore y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$$

Differentiating both sides w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\sqrt{1-x^2}) + \sqrt{1-x^2} \cdot \frac{dy}{dx} + x \cdot \frac{d}{dx}(\sqrt{1-y^2}) + \sqrt{1-y^2} \cdot \frac{d}{dx}(x) = 0$$

$$\therefore y \times \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2) + \sqrt{1-x^2} \cdot \frac{dy}{dx} + x \times \frac{1}{2\sqrt{1-y^2}} \cdot \frac{d}{dx}(1-y^2) + \sqrt{1-y^2} \times 1 = 0$$

$$\therefore \frac{y}{2\sqrt{1-x^2}} \times (0-2x) + \sqrt{1-x^2} \cdot \frac{dy}{dx} + \frac{x}{2\sqrt{1-y^2}} \times \left(0 - 2y \frac{dy}{dx}\right) + \sqrt{1-y^2} = 0$$

$$\therefore \frac{-xy}{\sqrt{1-x^2}} + \sqrt{1-x^2} \cdot \frac{dy}{dx} - \frac{xy}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} + \sqrt{1-y^2} = 0$$

$$\therefore \left(\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right) \frac{dy}{dx} = \frac{xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}$$

$$\therefore \left[\frac{\sqrt{1-x^2} \cdot \sqrt{1-y^2} - xy}{\sqrt{1-y^2}} \right] \frac{dy}{dx} = \frac{xy - \sqrt{1-x^2} \cdot \sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\therefore \frac{1}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}.$$

Miscellaneous Exercise 1 | Q 5.3 | Page 64

If $x \sin(a+y) + \sin a \cdot \cos(a+y) = 0$, then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

SOLUTION

$$x \sin(a+y) + \sin a \cdot \cos(a+y) = 0 \quad \dots(1)$$

Differentiating w.r.t. x, we get

$$\begin{aligned}
& x \frac{d}{dx} [\sin(a+y)] + \sin(a+y) \cdot \frac{d}{dx}(x) + (\sin a) \cdot \frac{d}{dx} [\cos(a+y)] = 0 \\
\therefore & x \cos(a+y) \cdot \frac{d}{dx}(a+y) + \sin(a+y) \times 1 + (\sin a) [-\sin(a+y)] \cdot \frac{d}{dx}(a+y) = 0 \\
\therefore & x \cos(a+y) \cdot \left(0 + \frac{dy}{dx}\right) + \sin(a+y) - \sin a \cdot \sin(a+y) \left(0 + \frac{dy}{dx}\right) = 0 \\
\therefore & x \cos(a+y) \frac{dy}{dx} + \sin(a+y) - \sin a \cdot \sin(a+y) \frac{dy}{dx} = 0 \\
\therefore & \sin a \cdot \sin(a+y) \frac{dy}{dx} - x \cos(a+y) \frac{dy}{dx} = \sin(a+y) \\
\therefore & [\sin a \cdot \sin(a+y) - x \cos(a+y)] \frac{dy}{dx} = \sin(a+y) \\
\therefore & \frac{dy}{dx} = \frac{\sin(a+y)}{\sin a \cdot \sin(a+y) - x \cos(a+y)}
\end{aligned}$$

From (1),

$$x = \frac{-\sin a \cdot \cos(a+y)}{\sin(a+y)}$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{\sin(a+y)}{\sin a \cdot \sin(a+y) + \frac{\sin a \cdot \cos(a+y)}{\sin(a+y)} \cdot \cos(a+y)} \\
&= \frac{\sin^2(a+y)}{\sin a \cdot \sin^2(a+y) + \sin a \cdot \cos^2(a+y)} \\
&= \frac{\sin^2(a+y)}{\sin a [\sin^2(a+y) + \cos^2(a+y)]} \\
\therefore \frac{dy}{dx} &= \frac{\sin^2(a+y)}{\sin a}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 5.4 | Page 64

If $\sin y = x \sin(a+y)$, then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

SOLUTION

$$\sin y = x \sin(a + y)$$

$$\Rightarrow x = \frac{\sin y}{\sin(a + y)} \quad \dots(i)$$

Differentiating (i) w.r.t.x,

$$\begin{aligned} \Rightarrow 1 &= \frac{\sin(a + y) \cdot \left(\frac{d}{dx} \sin y \right) - \sin y \cdot \left(\frac{d}{dx} \sin(a + y) \right)}{\sin^2(a + y)} \\ \Rightarrow \sin(a + y) \cdot \cos y \frac{d}{dx} y &- \sin y \cdot \cos(a + y) \cdot \frac{d}{dx} &= \sin^2(a + y) \\ \Rightarrow \frac{d}{dx} [\sin(a + y) \cdot \cos y &- \sin y \cdot \cos(a + y)] = \sin^2(a + y) \\ \Rightarrow \frac{dy}{dx} [\sin(a + y) \cdot \cos y &- \sin y \cdot \cos(a + y)] = \sin^2(a + y) \\ \Rightarrow \frac{dy}{dx} &= \frac{\sin^2(a + y)}{\sin a} \end{aligned}$$

Hence proved.

Miscellaneous Exercise 1 | Q 5.5 | Page 64

If $x = e^{\frac{x}{y}}$, then show that $\frac{dy}{dx} = \frac{x - y}{x \log x}$

SOLUTION

$$x = e^{\frac{x}{y}}$$

$$\therefore \frac{x}{y} = \log x \quad \dots(1)$$

$$\therefore y = \frac{x}{\log x}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{\log x} \right)$$

$$\begin{aligned}
&= \frac{(\log x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\log x)}{(\log x)} \\
&= \frac{(\log x) \times 1 - x \times \frac{1}{x}}{(\log x)^2} \\
&= \frac{\log x - 1}{(\log x)(\log x)} \\
&= \frac{\frac{x}{y} - 1}{\left(\frac{x}{y}\right)(\log x)} \quad \dots[\text{By (1)}] \\
&= \frac{x - y}{x \log x}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 5.6 | Page 64

If $y = f(x)$ is a differentiable function of x , then show that $\frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3} \cdot \frac{d^2y}{dx^2}$.

SOLUTION

If $y = f(x)$ is a differentiable function of x such that inverse function $x = f^{-1}(y)$ exists, then

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}, \text{ where } \frac{dy}{dx} \neq 0$$

$$\therefore \frac{d^2x}{dy^2} = \frac{d}{dy} \left(\frac{dx}{dy} \right)$$

$$= \frac{d}{dy} \left[\frac{1}{\left(\frac{dy}{dx}\right)} \right]$$

$$= \frac{d}{dx} \left(\frac{dy}{dx} \right)^{-1} \times \frac{dx}{dy}$$

$$\begin{aligned}
&= -1 \left(\frac{dy}{dx} \right)^{-2} \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) \times \frac{1}{\left(\frac{dy}{dx} \right)} \\
&= - \left(\frac{dy}{dx} \right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx} \right)^{-1} \\
\therefore \frac{d^2x}{dy^2} &= - \left(\frac{dy}{dx} \right)^{-3} \cdot \frac{d^2y}{dx^2}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 6.1 | Page 64

Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ w.r.t. $\tan^{-1} \left(\sqrt{\frac{2x\sqrt{1+x^2}}{1-2x^2}} \right)$.

SOLUTION

Let $u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$

and

$$v = \tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right).$$

Then we want to find $\frac{du}{dv}$

$$u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$$

Put $x = \tan\theta$.

Then $\theta = \tan^{-1} x$

and

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \frac{\sec \theta - 1}{\tan \theta}$$

$$= \frac{\frac{1}{\cos \theta} - 1}{\left(\frac{\sin \theta}{\cos \theta}\right)}$$

$$= \frac{1 - \cos \theta}{\sin \theta}$$

$$= \frac{2 \sin^2\left(\frac{\theta}{2}\right)}{2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

$$\therefore u = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \frac{du}{dx} = \frac{1}{2} \frac{d}{dx} (\tan^{-1} x)$$

$$= \frac{1}{2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

$$v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$$

Put $x = \sin \theta$.

Then $\theta = \sin^{-1} x$

and

$$\frac{2x\sqrt{1-x^2}}{1-2x^2}$$

$$= \frac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan 2\theta$$

$$\therefore v = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$= 2\sin^{-1} x$$

$$\therefore \frac{dv}{dx} = 2 \frac{d}{dx} (\sin^{-1} x)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left[\frac{1}{2(1+x^2)}\right]}{\left(\frac{2}{\sqrt{1-x^2}}\right)}$$

$$= \frac{1}{2(1+x^2)} \times \frac{\sqrt{1-x^2}}{2}$$

$$= \frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

Miscellaneous Exercise 1 | Q 6.2 | Page 64

Differentiate $\log \left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} \right]$ w.r.t. $\cos(\log x)$.

SOLUTION

Let $y = \log \left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} \right]$ and $v = \cos(\log x)$

Then we want to find $\frac{du}{dv}$.

$$u = \log \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} \times \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} \right)$$

$$= \log \left[\frac{(\sqrt{1+x^2}+x)^2}{1+x^2-x^2} \right]$$

$$= 2 \log(\sqrt{1+x^2}+x)$$

$$\therefore \frac{du}{dx} = 2 \frac{d}{dx} [\log(\sqrt{1+x^2}+x)]$$

$$= \frac{2}{\sqrt{1+x^2}+x} \cdot \frac{d}{dx} (\sqrt{1+x^2}+x)$$

$$= \frac{2}{\sqrt{1+x^2}+x} \cdot \left[\frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx} (1+x^2) + 1 \right]$$

$$\begin{aligned}
&= \frac{2}{\sqrt{1+x^2}+x} \cdot \left[\frac{2x}{2\sqrt{1+x^2}} + 1 \right] \\
&= \frac{2}{\sqrt{1+x^2}+x} \left(\frac{x}{\sqrt{1+x^2}} + 1 \right) \\
&= \frac{2(x + \sqrt{1+x^2})}{(\sqrt{1+x^2}+x)\sqrt{1+x^2}}
\end{aligned}$$

$$= \frac{2}{\sqrt{1+x^2}}$$

$$\frac{dv}{dx} = \frac{d}{dx} [\cos(\log x)]$$

$$= -\sin(\log x) \frac{d}{dx} (\log x)$$

$$= [-\sin(\log x)] \times \frac{1}{x}$$

$$= \frac{-\sin(\log x)}{x}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$\begin{aligned}
&\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\
&= \frac{\left(\frac{2}{\sqrt{1+x^2}}\right)}{\left[\frac{(-\sin(\log x))}{x}\right]} \\
&= \frac{-2x}{\sqrt{1+x^2 \cdot \sin(\log x)}}.
\end{aligned}$$

Miscellaneous Exercise 1 | Q 6.3 | Page 64

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$.

SOLUTION

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$

Then we want to find $\frac{du}{dv}$.

Put $s = \tan\theta$. Then $\theta = \tan^{-1}x$.

$$\text{Also, } \frac{\sqrt{1+x^2}-1}{x}$$

$$= \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \frac{\sec\theta-1}{\tan\theta}$$

$$= \frac{\frac{1}{\cos\theta}-1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{2 \sin^2\left(\frac{\theta}{2}\right)}{2 \sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

and

$$\frac{1 + \sqrt{1 + x^2}}{2\sqrt{1 + x^2}}$$

$$= \frac{1 + \sqrt{1 + \tan^2 \theta}}{2\sqrt{1 + \tan^2 \theta}}$$

$$= \frac{1 + \sec \theta}{2 \sec \theta}$$

$$= \frac{1 + \frac{1}{\cos \theta}}{\left(\frac{2}{\cos \theta}\right)}$$

$$= \frac{1 + \cos \theta}{2}$$

$$= \frac{2 \cos^2\left(\frac{\theta}{2}\right)}{2}$$

$$= \cos^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sqrt{\frac{1 + \sqrt{1 + x^2}}{2\sqrt{1 + x^2}}} = \cos\left(\frac{\theta}{2}\right)$$

$$\therefore u = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right] \text{ and } v = \cos^{-1}\left[\cos\left(\frac{\theta}{2}\right)\right]$$

$$\therefore u = \left(\frac{\theta}{2}\right) \text{ and } v = \left(\frac{\theta}{2}\right)$$

$$\therefore u = \frac{1}{2} \tan^{-1} x \text{ and } v = \frac{1}{2} \tan^{-1} x$$

Differentiating u and v w.r.t. x , we get

$$\frac{du}{dx} = \frac{1}{2} \frac{d}{dx} (\tan^{-1} x)$$

$$= \frac{1}{2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

and

$$\frac{dv}{dx} = \frac{1}{2} \frac{d}{dx} (\tan^{-1} x)$$

$$= \frac{1}{2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

$$\therefore \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\frac{1}{2(1+x^2)}}{\frac{1}{2(1+x^2)}} = 1.$$

$$\text{Remark : } u = \frac{1}{2} \tan^{-1} x \text{ and } v = \frac{1}{2} \tan^{-1} x$$

$$\therefore u = v$$

$$\therefore \frac{du}{dv} = \frac{d}{dv}(v) = 1.$$

Miscellaneous Exercise 1 | Q 7.1 | Page 64

$$\text{If } y^2 = a^2 \cos^2 x + b^2 \sin^2 x, \text{ show that } y + \frac{d^2y}{dx^2} = \frac{a^2 b^2}{y^3}$$

SOLUTION

$$y^2 = a^2 \cos^2 x + b^2 \sin^2 x \quad \dots(1)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} 2y \frac{dy}{dx} &= a^2 \frac{d}{dx} (\cos x)^2 + b^2 \frac{d}{dx} (\sin x)^2 \\ &= a^2 \times 2 \cos x \cdot \frac{d}{dx} (\cos x) + b^2 \times 2 \sin x \cdot \frac{d}{dx} (\sin x) \\ &= a^2 \times 2 \cos x (-\sin x) + b^2 \times 2 \sin x \cos x \\ &= (b^2 - a^2) \sin 2x \\ \therefore y \frac{dy}{dx} &= \left(\frac{b^2 - a^2}{2} \right) \sin 2x \quad \dots(2) \end{aligned}$$

Differentiating again w.r.t. x , we get

$$\begin{aligned} y \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{dy}{dx} &= \left(\frac{b^2 - a^2}{2} \right) \cdot \frac{d}{dx} (\sin 2x) \\ \therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 &= \left(\frac{b^2 - a^2}{2} \right) \times \cos 2x \times 2 \\ \therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 &= (b^2 - a^2) \cos 2x \\ \therefore y^3 \frac{d^2y}{dx^2} + y^2 \left(\frac{dy}{dx} \right)^2 &= y^2 (b^2 - a^2) \cos 2x \\ \therefore y^3 \frac{d^2y}{dx^2} &= y^2 (b^2 - a^2) \cos 2x - y^2 \left(\frac{dy}{dx} \right)^2 \\ \therefore y^4 + y^3 \frac{d^2y}{dx^2} &= y^2 (b^2 - a^2) \cos 2x - y^2 \left(\frac{dy}{dx} \right)^2 + y^4 \\ &= (a^2 \cos^2 x + b^2 \sin^2 x)(b^2 - a^2)(\cos^2 x - \sin^2 x) - [(b^2 - a^2) \sin x \cos x]^2 + (a^2 \cos^2 x + b^2 \sin^2 x)^2 \quad \dots[\text{By (1) and (2)}] \end{aligned}$$

$$\begin{aligned}
&= (a^2b^2\cos^2x - a^4\cos^2x + b^4\sin^2x - a^2b^2\sin^2x) \times (\cos^2x - \sin^2x) - (b^4\sin^2x\cos^2x + \\
&\quad a^4\sin^2x\cos^2x - 2a^2b^2\sin^2x\cos^2x) + (a^4\cos^4x + b^4\sin^4x + b^4\sin^4x + 2a^2b^2\sin^2x\cos^2x) \\
&= a^2b^2\cos 4x - a^2b^2\sin^2x\cos^2x - a^4\cos^4x + a^4\sin^2x\cos^2x + b^4\sin^2x\cos^2x - b^4\sin^2x\cos^2x - \\
&\quad a^4\sin^2x\cos^2x + 2a^2b^2\sin^2x\cos^2x + a^4\cos^4x + b^4\sin^4x + 2a^2b^2\sin^2x\cos^2x \\
&= a^2b^2\cos^4x + 2a^2b^2\sin^2x\cos^2x + a^2b^2\sin^4x \\
&= a^2b^2(\sin^4x + 2\sin^2x\cos^2x + \cos^4x)
\end{aligned}$$

$$\therefore y^4 + y^3 \frac{d^2y}{dx^2} = a^2b^2 \quad \dots [\because \sin^2x + \cos^2x = 1]$$

$$\therefore y^3 \left(y + \frac{d^2y}{dx^2} \right) = a^2b^2$$

$$\therefore y + \frac{d^2y}{dx^2} = \frac{a^2b^2}{y^3}.$$

Miscellaneous Exercise 1 | Q 7.2 | Page 64

If $\log y = \log(\sin x) - x^2$, show that $\frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (4x^2 + 3)y = 0$.

SOLUTION

$$\log y = \log(\sin x) - x^2$$

$$\therefore \log y = \log(\sin x) - \log e^{x^2}$$

$$\therefore \log y = \log \left(\frac{\sin x}{e^{x^2}} \right)$$

$$\therefore y = \frac{\sin x}{e^{x^2}}$$

$$\therefore e^{x^2} \cdot y = \sin x \quad \dots (1)$$

Differentiating both sides w.r.t. x, we get

$$e^{x^2} \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx} e^{x^2} = \frac{d}{dx}(\sin x)$$

$$\therefore e^{x^2} \cdot \frac{dy}{dx} + y \cdot e^{x^2} \cdot \frac{d}{dx}(x^2) = \cos x$$

$$\therefore e^{x^2} \cdot \frac{dy}{dx} + y \cdot e^{x^2} \times 2x = \cos x$$

$$\therefore e^{x^2} \left(\frac{dy}{dx} + 2xy \right) = \cos x$$

Differentiating again w.r.t. x, we get

$$e^{x^2} \cdot \frac{d}{dx} \left(\frac{dy}{dx} + 2xy \right) + \left(\frac{dy}{dx} + 2xy \right) \cdot \frac{d}{dx} (e^{x^2}) = \frac{d}{dx} (\cos x)$$

$$\therefore e^{x^2} \left[\frac{d^2y}{dx^2} + 2 \left(x \frac{dy}{dx} + y \times 1 \right) \right] + \left(\frac{dy}{dx} + 2xy \right) \cdot e^{x^2} \cdot \frac{d}{dx} (x^2) = -\sin x$$

$$\therefore e^{x^2} \left[\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y \right] + \left(\frac{dy}{dx} + 2xy \right) \cdot e^{x^2} \times 2x = -\sin x$$

$$\therefore e^{x^2} \left[\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2y + 2x \frac{dy}{dx} + 4x^2y \right]$$

$$= -e^{x^2} \cdot y \quad \dots [\text{By (1)}]$$

$$\therefore \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 4x^2y + y = -y$$

$$\therefore \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (4x^2 + 3)y = 0.$$

Miscellaneous Exercise 1 | Q 7.3 | Page 64

If $x = a \cos \theta$, $y = b \sin \theta$, show that $a^2 \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] + b^2 = 0$.

SOLUTION

$$x = a \cos \theta, y = b \sin \theta$$

Differentiating x and y w.r.t. θ , we get

$$\frac{dx}{d\theta} = a \frac{d}{d\theta} (\cos \theta) = a(-\sin \theta) = -\sin \theta \quad \dots (1)$$

and

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \\
&= \frac{b \cos \theta}{-a \sin \theta} \\
&= \left(-\frac{b}{a}\right) \cot \theta \\
\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\left(-\frac{b}{a}\right) \cot \theta \right] \\
&= \left(-\frac{b}{a}\right) \cdot \frac{d}{d\theta} (\cot \theta) \cdot \frac{d\theta}{dx} \\
&= \left(-\frac{b}{a}\right) \cdot (-\operatorname{cosec}^2 \theta) \times \frac{1}{\left(\frac{dx}{d\theta}\right)} \\
&= \left(-\frac{b}{a}\right) \operatorname{cosec}^2 \theta \times \frac{1}{-a \sin \theta} \quad \dots[\text{By (1)}] \\
&= \left(-\frac{b}{a^2}\right) \operatorname{cosec}^3 \theta \\
\therefore a^2 \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] + b^2 &= a^2 \left[b \sin \theta \cdot \left(-\frac{b}{a^2}\right) \operatorname{cosec}^3 \theta + \left\{ \left(-\frac{b}{a}\right) \cot \theta \right\}^2 \right] + b^2 \\
&= a^2 \left[-\frac{b^2}{a^2} \operatorname{cosec}^2 \theta + \frac{b^2}{a^2} \cot^2 \theta \right] + b^2 \\
&= a^2 \left(-\frac{b^2}{a^2} \right) (\operatorname{cosec}^2 \theta - \cot^2 \theta) + b^2
\end{aligned}$$

$$= -b^2 + b^2 \quad \dots [\because \cosec^2\theta - \cot^2\theta = 1]$$

$$\therefore a^2 \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] + b^2 = 0.$$

Miscellaneous Exercise 1 | Q 7.4 | Page 64

If $y = A \cos(\log x) + B \sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.

SOLUTION

$$y = A \cos(\log x) + B \sin(\log x) \quad \dots (1)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= A \frac{d}{dx}[\cos(\log x)] + B \frac{d}{dx}[\sin(\log x)] \\&= A[-\sin(\log x)] \cdot \frac{d}{dx}(\log x) + B \cos(\log x) \cdot \frac{d}{dx}(\log x) \\&= A \sin(\log x) \times \frac{1}{x} B \cos(\log x) \times \frac{1}{x} \\ \therefore x \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) &= -A \frac{d}{dx}[\sin(\log x)] + B \frac{d}{dx}[\cos(\log x)] \\ \therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 &= -A \cos(\log x) \cdot \frac{d}{dx}(\log x) + B[-\sin(\log x)] \cdot \frac{d}{dx}(\log x) \\ \therefore xy_2 + y_1 &= -A \cos(\log x) \times \frac{1}{x} - B \sin(\log x) \times \frac{1}{x} \\ \therefore x^2y_2 + xy_1 &= -[A \cos(\log x) + B \sin(\log x)] \quad \dots [\text{By (1)}] \\ \therefore x^2y_2 + xy_1 + y &= 0.\end{aligned}$$

Miscellaneous Exercise 1 | Q 7.5 | Page 64

If $y = Ae^{mx} + Be^{nx}$, show that $y_2 - (m+n)y_1 + mny = 0$.

SOLUTION

$$y = Ae^{mx} + Be^{nx}$$

Differentiating w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= A \frac{d}{dx}(e^{mx}) + B \frac{d}{dx}(e^{nx}) \\ &= Ae^{mx} \cdot \frac{d}{dx}(mx) + Be^{nx} \cdot \frac{d}{dx}(nx) \\ &= Ae^{mx} \cdot m + Be^{nx} \cdot n \\ &= y_1 = mAe^{mx} + nBe^{nx} \quad \dots(2)\end{aligned}$$

Differentiating again w.r.t. x, we get

$$\begin{aligned}y_2 &= mA \frac{d}{dx}(e^{mx}) + nB \frac{d}{dx}(e^{nx}) \\ &= mAe^{mx} \cdot \frac{d}{dx}(mx) + nBe^{nx} \cdot \frac{d}{dx}(nx) \\ &= mAe^{mx} \cdot m + nBe^{nx} \cdot n \\ &\therefore y_2 = m^2Ae^{mx} + n^2Be^{nx} \quad \dots(3)\end{aligned}$$

$$\begin{aligned}\therefore y_2 - (m+n)y_1 + mny &= (m^2Ae^{mx} + n^2Be^{nx}) - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx}) \quad \dots[\text{By (1), (2), (3)}] \\ &= m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx} - mnBe^{mx} - n^2Be^{nx} + mnAe^{mx} + mnBe^{nx} \\ &= 0 \\ \therefore y_2 - (m+n)y_1 + mny &= 0.\end{aligned}$$