14. Differential Equations

Ex. (1). Water at 100° C cools in 10 minutes to 85° C in a room temperature of 25° C. Find the temperature of water after 20 minutes.

Solution : Let θ be the temperature of the body at time t . Room temperature is 25° C.

According to Newton's law of cooling

$\frac{d\theta}{dt} \propto (\theta-25) \therefore \frac{d\theta}{dt} = -k(\theta-25)$ where k is a constant and $k > 0$.

i.e. $\frac{d\theta}{\theta-25} = -\,kdt$

On integrating we get,

$$\int \frac{d\theta}{\theta-25} = -k\int dt + C_1$$

$$\therefore \log|\theta-25| = -kt + C \quad \ldots \text{(I)}$$

Given that when t = 0, θ =100.

$$\log|100-25| = -\,k\,(0) + C \Rightarrow C = \log 75$$

$\therefore$ Equation (I) becomes

$$\log|\theta-25| = \text{kt} + \log 75 \Rightarrow \log|\theta-25| - \log 75 = -kt$$

$$\log\left|\frac{\theta-25}{75}\right| = -\,kt \quad \ldots \text{(II)}$$

Also, when t = 10, θ = 85. From (II) we get.

$$\log\left|\frac{85-25}{75}\right| = -\text{k}\,(10) \Rightarrow -\text{k} = \frac{1}{10}\log\left(\frac{60}{75}\right)$$

$\therefore$ Equation (II) becomes

$$\log\left|\frac{\theta-25}{75}\right| = \frac{t}{10}\log\left(\frac{4}{5}\right) \quad \ldots \text{(III)}$$

We have to find θ , when t = 20. so, put in (III).

$$\log\left|\frac{\theta-25}{75}\right| = \frac{20}{10}\log\left(\frac{4}{5}\right) \Rightarrow \log\left|\frac{\theta-25}{75}\right| = 2\log\left(\frac{4}{5}\right)$$

$$\log\left|\frac{\theta-25}{75}\right| = \log\left(\frac{4}{5}\right)^2 \Rightarrow \left|\frac{\theta-25}{75}\right| = \left(\frac{4}{5}\right)^2$$

$$\therefore \theta - 25 = \frac{4}{5}\times\frac{4}{5}\times 75 \Rightarrow \theta - 25 = 4\times 4\times 3 = 48$$

$$\therefore \theta = 73$$

Therefore the temperature of water after 20 minutes will be 73°C

Ex. (2). The population of a town increases at a rate proportional to the population at that time. If the population increases from 40,000 to 60,000 in 40 years, what will be the population of the town in another 20 years ? (Given : $\sqrt{1.5} = 1.2247$)

Solution : Let P be the population of a town at time t. Given that the population of the town increases at a rate proportional to the population at that time.

$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$, where k is a constant.

i.e. $\frac{dP}{P} = kdt$

On integrating both sides we get,

$\int \frac{dP}{P} = k\int dt + C$

i.e. $\log P = kt + C$ .... (I)

Given that when $t = 0$, $P = 40{,}000$.

i.e. $\log 40{,}000 = k(0) + C \Rightarrow C = \log 40{,}000$

$\therefore$ Equation (I) becomes

i.e. $\log P = kt + \log 40{,}000 \Rightarrow \log P - \log 40{,}000 = kt$

$\log\left(\frac{P}{40{,}000}\right) = kt$ .... (II)

Also, when $t = 40$, $P = 60{,}000$. So, from (II) we get.

$\log\left(\frac{60{,}000}{40{,}000}\right) = k(40)$ $\therefore k = \frac{1}{40}\log\left(\frac{3}{2}\right)$ ....(III)

$\therefore$ Equation (II) becomes

$$\log\left(\frac{P}{40{,}000}\right) = \frac{1}{40}\log\left(\frac{3}{2}\right) \times t$$

So, population in another 20 years means, when $t = 40 + 20 = 60$ years. From (III) we get,

$\log\left(\frac{P}{40{,}000}\right) = \frac{60}{40}\log\left(\frac{3}{2}\right) \Rightarrow \log\left(\frac{P}{40{,}000}\right) = \frac{3}{2}\log\left(\frac{3}{2}\right)$

$\log\left(\frac{P}{40{,}000}\right) = \log\left(\frac{3}{2}\right)^{\frac{3}{2}} \Rightarrow \frac{P}{40{,}000} = \left(\frac{3}{2}\right)^{\frac{3}{2}}$

$P = 40{,}000\left(\sqrt{1.5}\right)^3 = 40{,}000 \times \left(\sqrt{1.5}\right)^3$

$= 4 \times 10000 \times (1.5) \times \left(\sqrt{1.5}\right) = 6 \times 10000 \times (1.2247)$

$P = 6 \times 12247 = 73{,}482$

$\therefore$ Population in another 20 years is 73,482

Ex. (3). The radioactive isotope Indium-111 is often used for diagnosis and imaging in nuclear medicine and treatment. The rate of decay of an Indium isotope is proportional to the mass of isotope present at that time. Its half life is $2\frac{4}{5}$ days. What was the initial mass of the isotope before decay, if the mass in 2 weeks was 5 g?

Solution : Let M be the mass of isotope present at time t. Given that the rate of decay of an isotope is proportional to the mass of isotope present at that time.

$\frac{dM}{dt} \propto M \Rightarrow \frac{dM}{dt} =$ $-KM$, where k is a constant and $k > 0$.

i.e. $\frac{dM}{M} = -K\, dt$

On integrating both sides we get,

$\int \frac{dM}{M} = -K \int dt + C$

i.e. $\log |M| = -K\, t + C$ ... (I)

Let M_0 be the initial mass i.e. when $t = 0$.

i.e. $\log |M_0| = -k(0) + C \Rightarrow C = \log M_0$

$\therefore$ Equation (I) becomes

$\log M = -kt + \log M_0$

$\log \left(\frac{M}{M_0}\right) = -kt$ ... (II)

Given that, half life is $2\frac{4}{5}$ days i.e. in $2\frac{4}{5}$ days isotope decays to half of its original mass. i.e. when $t = 2\frac{4}{5} = \frac{14}{5}, M = \frac{M_0}{2}$. So, from (II) we get.

$\log \left(\frac{M_0/2}{M_0}\right) = -k\left(\frac{14}{5}\right)$

$-k = \frac{5}{14} \log \left(\frac{1}{2}\right)$

$\therefore$ Equation (II) becomes

$\log \left(\frac{M}{M_0}\right) = \frac{(5)t}{14} \log \left(\frac{1}{2}\right)$ ... (III)

Given that, the mass in 2 weeks was 5 g,

i.e. $t = 2$ weeks $= 14$ days, $M = 5\, g$ from (III) we get,

$\log \left(\frac{5}{M_0}\right) = \frac{5 \times 2}{14} \log\left(\frac{1}{2}\right)$

$\log \left(\frac{5}{M_0}\right) = 5 \log \left(\frac{1}{2}\right) = \log \left(\frac{1}{2}\right)^5$

$\therefore \frac{5}{M_0} = \left(\frac{1}{2}\right)^{(5)} = \frac{1}{32} \Rightarrow M_0 = (5) \times (32) =$ 160 g

$\therefore$ Initial mass of the isotope before decay is 160 g.

Ex. (4). Find the equation of the curve that passes through the point (1,2) and has at every point $\frac{dy}{dx} = -\frac{2xy}{1+x^2}$.

Solution : Given that, $\frac{dy}{dx} = -\frac{2xy}{1+x^2}$

i.e. $\frac{1}{y}\,dy = -\frac{2x}{1+x^2}\,dx$

Integrating both sides w. r. t. x., we get,

$\int\frac{1}{y}\,dy = -\int\frac{2x}{1+x^2}\,dx$

$\log|y| = -\log|1+x^2| + \log C$ ...$[\because \frac{d}{dx}(1+x^2) = 2x]$

$\log|y| + \log|1+x^2| = \log C$

$\log|y(1+x^2)| = \log C$

$y(1+x^2) = C$ ... (I)

Given that the curve passes through (1,2) . So, put $x = 1, y = 2$ in (I)

$2(1 + 1) = C \Rightarrow C = 4$

Put $C =$ in (I), we get,

$y(1+x^2) = 4$ is the required equation of the curve.

Ex. (5). Water tank is being emptied in such a way that the rate at which water is flowing out is proportional to the amount left in at that instant. If half the water flows out in 7 minutes. Initially the tank is filled with 8000 liters of water. Find the amount of water left after 21 minutes.

Solution: Let the amount of water be x lit. at time t

$\therefore \frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx$ k is a constant

$\therefore \frac{1}{x}\,dx = -k\,dt$ on integrating $\int\frac{1}{x}\,dx = -k\int 1\,dt + c$

$\therefore \log|x| = -kt + c$ — (I)

Since initially the tank is filled with 8000 lit.

$\therefore \log|8000| = -k(0) + c$

$c = \log 8000$ — Put in (I)

$\therefore \log x = -kt + \log|8000|$

$\log x - \log|8000| = -kt$ — II

Since half water flowes out in 7 min t=7

$\therefore \log\left|\frac{4000}{8000}\right| = -k \times 7$

$$-k = \frac{1}{7}\log\left|\frac{1}{2}\right| \quad \text{put in eq}^n \text{ (II)}$$

$$\log\left|\frac{x}{8000}\right| = \frac{t}{7}\log\left|\frac{1}{2}\right|$$

when $t = 21$, $x = ?$

$$\therefore \log\left|\frac{x}{8000}\right| = \frac{21}{7}\log\left|\frac{1}{2}\right| = 3\log\frac{1}{2}$$

$$\therefore \log\left|\frac{x}{8000}\right| = \log\left|\left(\frac{1}{2}\right)^3\right|$$

$$\therefore \frac{x}{8000} = \frac{1}{8}$$

$$\therefore x = \frac{8000}{8} = 1000$$

The amount of water is 1000 liter left after 21 minutes.

Ex. (6). The rate of decay of an Iodine-123 isotope is proportional to the mass of isotope present at that time. The initial mass of the isotope was 200 g. Determine the mass of Iodine present after 39 days if the half life period of the Iodine isotope is approximately 13 hours.

Solution : Let the amount of isotope be x gm at t

$$\therefore \frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx \quad k \text{ is constant}$$

$$\therefore \frac{dx}{x} = -k\,dt \quad \text{on integrating}$$

$$\int \frac{1}{x}dx = -k\int 1\,dt + c$$

$$\therefore \log|x| = -kt + c \quad \text{— I}$$

since the initial mass of the isotope was 200g

when $t = 0$, $x = 200$

Put in I

$\log|200| = -K(0) + c$

$\therefore c = \log|200|$ Put in I

$\log x = -Kt + \log|200|$

$\log x - \log|200| = -Kt$

$\log\left|\frac{x}{200}\right| = -Kt$ — II

Now when $t = 13$, $x = 100$ g

eqn II becomes

$\log\left|\frac{100}{200}\right| = -K \times 13$

$-K = \frac{1}{13}\log\left|\frac{1}{2}\right|$ Put in II

$\log\left|\frac{x}{200}\right| = \frac{t}{13}\log\left|\frac{1}{2}\right|$

when $t = 39$ days $= 39 \times 24$ hours, $x = ?$

$\log\left|\frac{x}{200}\right| = \frac{39 \times 24}{13}\log\left|\frac{1}{2}\right| \Rightarrow \frac{x}{200} = \frac{1}{2^{72}}$

$\therefore x = \frac{200}{2^{72}} = \frac{200}{8 \times 2^{69}} = \frac{25}{2^{69}}$

The mass of isotope after 39 days is $\frac{25}{2^{69}}$ gm.

Ex. (7). The rate of reduction of a person's assets is proportional to the square root of the existing assets. If the assets dwindle from 25 lakhs to 6.25 lakhs in 2 years, in how many years, will the person be bankrupt ?

Solution : Let a person's assets be x

$\therefore \frac{dx}{dt} \propto \sqrt{x}$ as per given information

$\therefore \frac{dx}{dt} = -K\sqrt{x}$ K is constant

$\therefore \frac{dx}{\sqrt{x}} = -K\,dt$ integrating

$\int x^{-\frac{1}{2}}\,dx = -K\int 1\,dt + c$

$$\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -kt+c \Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = -kt+c$$

$2\sqrt{x} = -kt+c$ ⟵ I

when $t=0$, $x=25$

$2\sqrt{25} = -k(0)+c$ $\therefore C=10$ put in I

$\therefore 2\sqrt{x} = -kt+10$ — II

when $t=2$ years, $x = 6.25$ lakhs

eq^n II becomes

$2\sqrt{6.25} = -k\times 2+10$

$2(2.5) = -2k+10$

$5-10 = -2k$

$k = \frac{5}{2}$ put in II

$2\sqrt{x} = \frac{-5t}{2}+10$

when $x=0$, $t=?$

$2\sqrt{0} = \frac{-5t}{2}+10$

$5t = 20$

$t = 4$

Assets	25	6.25	0
years	0	2	4

The person will be bankrupt in next $\boxed{\text{2 years}}$ that means total 4 years from begning.

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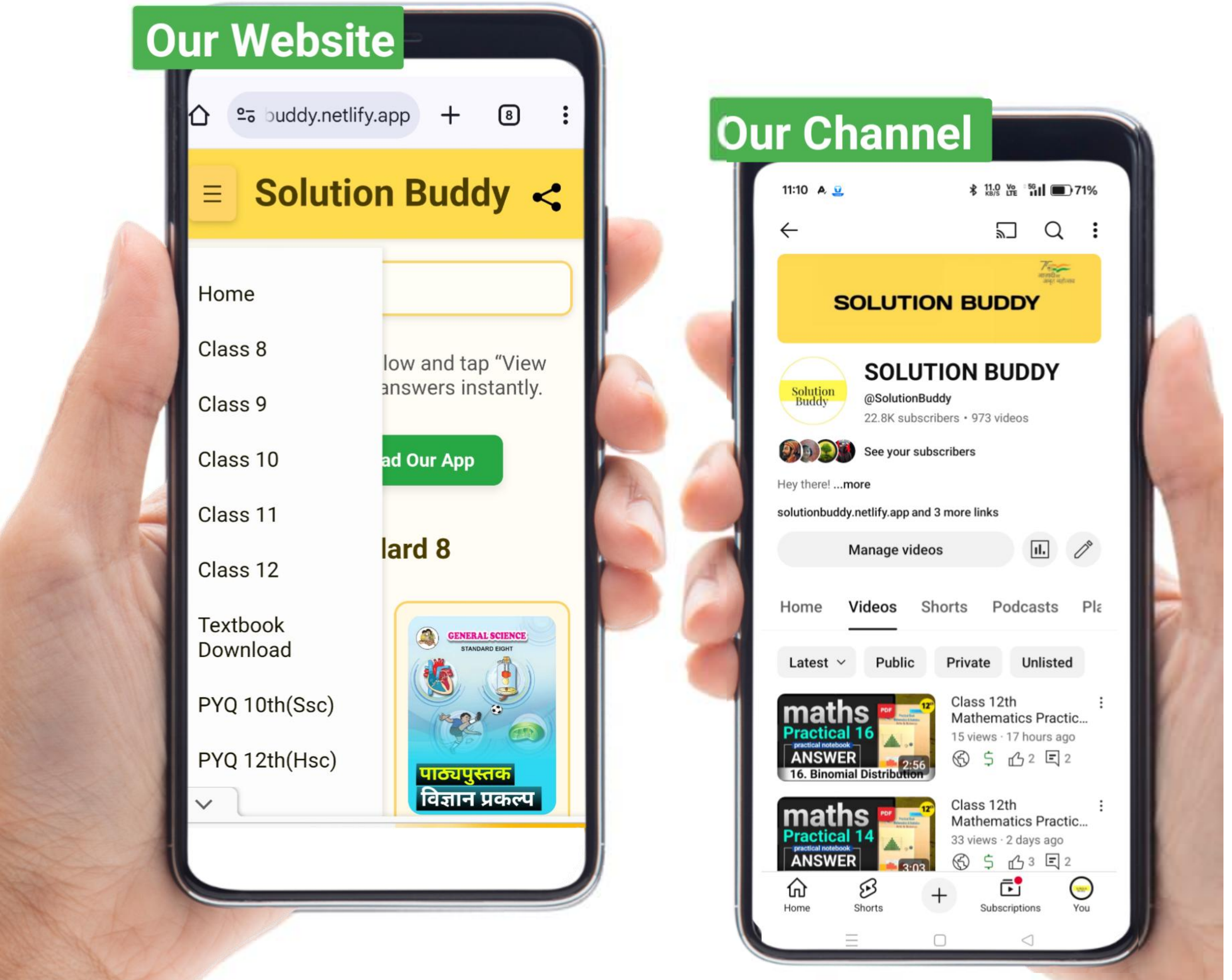

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