3. Solution of Linear Equations using Matrices

A. Activities

Carry out the following activities to solve the equations.

1. $x - y + z = 1$, $2x - y = 1$, $3x + 3y - 4z = 2$

Solution : The given equation can be written as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 3 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ \boxed{0} & \boxed{1} & \boxed{-2} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \text{where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

By $R_3 - 6R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & \boxed{1} & -2 \\ \boxed{0} & \boxed{0} & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \boxed{5} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{x-y+z} \\ y - 2z \\ \boxed{5z} \end{bmatrix} = \begin{bmatrix} \boxed{1} \\ \boxed{-1} \\ \boxed{5} \end{bmatrix}$$

By equality of matrices

$\boxed{x} - \boxed{y} + \boxed{z} = 1$ .... (1)

$\boxed{y} - 2\boxed{z} = \boxed{-1}$ .... (2)

$\boxed{5} z = \boxed{5}$ .... (3)

From (3) $z = \boxed{1}$, from (2), we get $y = \boxed{1}$

Substituting $y = \boxed{}$, $z = \boxed{}$ in (1), we get $x = \boxed{1}$

2. The cost of 4 pencils, 3 pens and 2 erasors is ₹ 60. The cost of 2 pencils, 4 pens and 6 erasors is ₹ 90. The cost of 6 pencils, 2 pens and 3 erasors is ₹ 70. Find the cost of each item, using the following activity.

Solution : Let the cost of 1 pencil, 1 pen and 1 erasor be ₹ x, ₹ y, ₹ z respectively.

Then $4x + \boxed{3} y + \boxed{2} z = \boxed{60}$

$x + \boxed{2}\, 4y + \boxed{3} z = \boxed{45}$

$6x + \boxed{2} y + \boxed{3} z = 70$

These equation can be written as

$$\begin{bmatrix} 4 & \boxed{3} & 2 \\ \boxed{1} & 2 & \boxed{3} \\ 6 & \boxed{2} & 3 \end{bmatrix} X = \begin{bmatrix} 60 \\ \boxed{45} \\ 70 \end{bmatrix} \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$ $\begin{bmatrix} \boxed{1} & 2 & \boxed{3} \\ 4 & \boxed{3} & 2 \\ 6 & \boxed{2} & 3 \end{bmatrix} X = \begin{bmatrix} \boxed{45} \\ 60 \\ 70 \end{bmatrix}$

By $R_2 - 4R_1$ and $R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ \boxed{0} & \boxed{-5} & \boxed{-10} \\ \boxed{0} & \boxed{-10} & \boxed{-15} \end{bmatrix} X = \begin{bmatrix} 45 \\ \boxed{-120} \\ \boxed{-200} \end{bmatrix}, \text{ By } R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ \boxed{0} & \boxed{0} & \boxed{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ -120 \\ \boxed{40} \end{bmatrix}$$

$\therefore x + 2y + 3z = 45$ .... (1) $\quad -5y - 10z = -120$ .... (2)

$\boxed{0}\,x + \boxed{0}\,y + \boxed{5}\,z = \boxed{40}$ .... (3) we get $z = \boxed{8}$

From (2) $y = \boxed{8}$, from (1) $x = \boxed{5}$

3. Express the following equation in matrix from and sovle them using method of inversion by completing the following activity.
$x + 2y + 3z = 6$, $x + y + 5z = 7$, $2x + 4y + 7z = 13$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$, $|A| = -1 \neq 0$ A^{-1} exists.

$A_{11} = (-1)^{1+1} M_{11} = (1) \begin{vmatrix} 1 & 5 \\ 4 & 7 \end{vmatrix} = 7 - 20 = -13$

$A_{12} = (-1)^{1+2} M_{12} = (-1) \begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} = \boxed{3}$

$A_{13} = (-1)^{1+3} M_{13} = (1) \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = \boxed{2}$

$A_{21} = (-1)^{2+1} M_{21} = (-1) \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = \boxed{-2}$

$A_{22} = (-1)^{2+2} \boxed{M_{22}} = (-1)^1 \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = \boxed{1}$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^1 \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \boxed{0}$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$

$A_{32} = (-1)^{3+2} M_{32} = (-1) \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -2$

$A_{33} = (-1)^{3+3} M_{33} = (1) \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = \boxed{1} - \boxed{2} = \boxed{-1}$

$$\text{adj } A = \begin{bmatrix} \boxed{-13} & -2 & \boxed{7} \\ 3 & \boxed{1} & -2 \\ \boxed{2} & 0 & \boxed{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \boxed{13} & 2 & \boxed{-7} \\ -3 & -1 & 2 \\ \boxed{-2} & 0 & \boxed{1} \end{bmatrix}$$

Now $AX = B$

Premultiplying by A^{-1} we get,

$A^{-1}A\,X = A^{-1}B$

$I\,X = A^{-1}B$

$$X = \begin{bmatrix} \boxed{13} & 2 & \boxed{-7} \\ -3 & \boxed{-1} & 2 \\ \boxed{-2} & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 13 \end{bmatrix}$$

$$X = \begin{bmatrix} \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{bmatrix}$$

$\therefore x = \boxed{1}, y = \boxed{1}, z = \boxed{1}$

B. Solve the Following

Q.1 The cost of 4 kg potato, 3 kg wheat, and 2 kg rice is Rs. 60. The cost of 1 kg potato, 2 kg wheat, and 3 kg rice is Rs. 45. The cost of 6 kg potato, 2 kg wheat, and 3 kg rice is Rs. 70. Find the per kg cost of each item by matrix method.

Let ₹ x be the cost of 1 kg potato.

₹ y be the cost of 1 kg wheat.

₹ z be the cost of 1 kg rice.

$\therefore$ From the 1st condition, $4x + 3y + 2z = 60$ —(1)

From the 2nd condition, $x + 2y + 3z = 45$ —(2)

& from the 3rd condition, $6x + 2y + 3z = 70$ —(3)

$\therefore$ The matrix form of these eqns is

$$\begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 60 \\ 70 \end{bmatrix}$

By $R_2 \to R_2 - 4R_1$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ -120 \\ 70 \end{bmatrix}$

By $R_3 \to R_3 - 6R_1$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & -10 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ -120 \\ -200 \end{bmatrix}$

By $R_2 \to (-1/5)R_2$ and $R_3 \to (-1/5)R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 24 \\ 40 \end{bmatrix}$$

By $R_3 \to R_3 - 2R_2$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 24 \\ -8 \end{bmatrix}$

$\therefore$ $x + 2y + 3z = 45$ — (4) , $y + 2z = 24$ —(5)

and $-z = -8 \Rightarrow \boxed{z = 8}$

$\therefore$ Substituting $z = 8$ in eq^n (5), $y + 2(8) = 24 \Rightarrow \boxed{y = 8}$

$\therefore$ from (4), $x + 2(8) + 3(8) = 45$

$\Rightarrow x = 45 - 16 - 24 \Rightarrow \boxed{x = 5}$

Hence, the cost of 1 kg potato = ₹5, cost of 1 kg wheat = ₹8 and cost of 1 kg rice = ₹8.

Q.2 Solve the following equations by the method of inversion.

(i) $x + y = 1, y + z = \frac{5}{3}, z + x = \frac{4}{3}$.

[Find A^{-1} using row transformations.]

The give equations can be written in the matrix form as,

$AX = B$

Premultiplying by A^{-1},

$A^{-1}AX = A^{-1}B$

$\Rightarrow X = A^{-1}B$ —— (1)

Consider $AA^{-1} = I$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 3 & 0 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 - 3R_1$ and $R_2 \to (1/3)R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

By $R_3 \to (1/6)R_3$; ~~R_2~~

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ -1/2 & 1/6 & 1/6 \end{bmatrix}$$

By $R_2 \to R_2 - R_3$, $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & 1/6 & -1/6 \\ -1/2 & 1/6 & 1/6 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & 1/6 & -1/6 \\ -1/2 & 1/6 & 1/6 \end{bmatrix}$$

$\therefore$ from eqn (1)

$$X = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & 1/6 & -1/6 \\ -1/2 & 1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 - 5/6 + 4/6 \\ 1/2 + 5/6 - 4/6 \\ -1/2 + 5/6 + 4/6 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

$\therefore x = 1/3 , y = 2/3 , z = 1$

(ii) $x + y + z = 6$, $3x - y + 3z = 10$, $5x + 5y - 4z = 3$.

[Find A^{-1} using adjoint method.]

The matrix form of above eqns is $AX = B$ —— (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$

Now to find A^{-1} using adjoint method,

$|A| = 1(9-15)-1(-12-15)+1(15+5) = 36 \neq 0$

$\therefore A^{-1}$ exists.

$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} -1 & 3 \\ 5 & -4 \end{vmatrix} = 4-15 = -11$

$A_{12} = (-1)^{1+2} M_{12} = (-1)\begin{vmatrix} 3 & 3 \\ 5 & -4 \end{vmatrix} = -(-12-15) = 27$

$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 3 & -1 \\ 5 & 5 \end{vmatrix} = 15-(-5) = 20$

$A_{21} = (-1)^{2+1} M_{21} = (-1)\begin{vmatrix} 1 & 1 \\ 5 & -4 \end{vmatrix} = -(-4-5) = 9$

$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 1 \\ 5 & -4 \end{vmatrix} = -9$

$A_{23} = (-1)^{2+3} M_{23} = (-1)\begin{vmatrix} 1 & 1 \\ 5 & 5 \end{vmatrix} = -(5-5) = 0$

$A_{31} = (-1)^{1+3} M_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3-(-1) = 4$

$A_{32} = (-1)^{3+2} M_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = -(3-3) = 0$

$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1-3 = -4$

$\therefore$ Cofactor matrix $= \begin{bmatrix} -11 & 27 & 20 \\ 9 & -9 & 0 \\ 4 & 0 & -4 \end{bmatrix}$.

Now, $\text{adj}(A) = \begin{bmatrix} -11 & 9 & 4 \\ 27 & -9 & 0 \\ 20 & 0 & -4 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{36}\begin{bmatrix} -11 & 9 & 4 \\ 27 & -9 & 0 \\ 20 & 0 & -4 \end{bmatrix}$

Premultiplying eqn ① by A^{-1}, we get, $X = A^{-1}B$

$\therefore X = \frac{1}{36}\begin{bmatrix} -11 & 9 & 4 \\ 27 & -9 & 0 \\ 20 & 0 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} -66+90+12 \\ 162-90+0 \\ 120-0-12 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 \\ 72 \\ 108 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore x=1,\ y=2\ \&\ z=3$

Q.3 Solve the following equations by the method of reduction.
$x + 3y + 2z = 6,\ 3x - 2y + 5z = 5,\ 2x - 3y + 6z = 7.$

The matrix form of above given equation is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

By $R_2 \to R_2 - 3R_1$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -11 & -1 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \\ 7 \end{bmatrix}$$

By $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -11 & -1 \\ 0 & -9 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \\ -5 \end{bmatrix}$$

By $R_3 \to R_3 + 2R_2$,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -11 & -1 \\ 0 & -31 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \\ -31 \end{bmatrix}$$

$\therefore\ x + 3y + 2z = 6 \quad \text{—— (1)}$

$-11y - z = -13 \quad \text{—— (2)}$

$-31y = -31$

$\Rightarrow \boxed{y = 1}$

from (2), $-11(1) - z = -13$

$\therefore\ z = -11 + 13 = 2$

$\therefore \boxed{z = 2}$

from (1), $x + 3(1) + 2(2) = 6$

$\therefore\ x = 6 - 7$

$\therefore \boxed{x = -1}$

$\therefore\ x = -1,\ y = 1,\ z = 2$

Sign of Teacher :