14. Assignment Problems

A. Activities

1) Consider the following cost matrix.

Jobs	Machines			
	M_1	M_2	M_3	M_4
	Manufacturing cost (₹)			
J_1	40	50	60	65
J_2	30	38	46	48
J_3	25	33	41	43
J_4	39	45	51	59

Step-1 : Subtract the smallest element in each row from every element in its row.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & \boxed{10} & 20 & \boxed{25} \\ \boxed{0} & 8 & \boxed{16} & 18 \\ 0 & \boxed{8} & \boxed{16} & 18 \\ \boxed{0} & 6 & 12 & \boxed{20} \end{bmatrix} \end{array}$$

Step-2 : Subtract the smallest element in each ...... column of the assignment matrix obtained in step-1 from every element in that ...column.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & \boxed{4} & \boxed{8} & 7 \\ \boxed{0} & 2 & 4 & \boxed{0} \\ \boxed{0} & 2 & \boxed{4} & 0 \\ 0 & \boxed{0} & 0 & \boxed{2} \end{bmatrix} \end{array}$$

Step-3 : Cover all zeros with minimum number of straight lines.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 4 & 8 & 7 \\ 0 & 2 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{array}$$

As the number of lines required to cover all zeros is less than the number of rows/columns, optimal solution has not reached.

Step-4 : Select the ...... smallest element not covered by the lines, subtract it from each uncovered element and ...... odd it to elements which are at the intersection of the lines.

$$
\begin{array}{c|cccc}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & 0 & \boxed{2} & 6 & \boxed{7} \\
J_2 & \boxed{0} & \boxed{0} & \boxed{2} & 0 \\
J_3 & 0 & 0 & \boxed{2} & \boxed{0} \\
J_4 & \boxed{2} & 0 & 0 & 4
\end{array}
$$

Step-5 : Cover all zeros by minimum number of straight lines.

$$
\begin{array}{c|cccc}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & 0 & \boxed{2} & \boxed{6} & \boxed{7} \\
J_2 & \boxed{0} & 0 & 2 & 0 \\
J_3 & 0 & 0 & 2 & \boxed{0} \\
J_4 & \boxed{2} & 0 & \boxed{0} & 4
\end{array}
$$

Since the number of lines covering all zeros is equal to the number of rows/columns, hence the optimal solution has reached.

$$
\begin{array}{c|cccc}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & \boxed{0} & 2 & \boxed{6} & \boxed{7} \\
J_2 & \cancel{0} & \boxed{0} & 2 & 0 \\
J_3 & \cancel{0} & \boxed{\cancel{0}} & \boxed{2} & \boxed{0} \\
J_4 & \boxed{2} & \cancel{0} & \boxed{0} & \boxed{4}
\end{array}
$$

OR

$$
\begin{array}{c|cccc}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & \boxed{0} & 2 & \boxed{6} & 7 \\
J_2 & \cancel{0} & \cancel{0} & 2 & \boxed{0} \\
J_3 & \cancel{0} & \boxed{0} & 2 & \cancel{0} \\
J_4 & \boxed{2} & \cancel{0} & \boxed{0} & \boxed{4}
\end{array}
$$

Hence the optimal assignment and optimal cost can be computed as

Jobs	Machine	Cost
J_1	M_1	40
J_2	M_2	38
J_3	M_4	43
J_4	M_3	51
Total (minimum) cost = ₹ 172		

2) We are given the performance matrix.

subordinates	Jobs		
	I	II	III
A	7	3	5
B	2	7	4
C	6	5	3
D	3	4	7

Step-1 : Add column IV with all elements zero so that the problem is balanced.

$$\begin{array}{cc} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 7 & 3 & 5 & \boxed{0} \\ 2 & 7 & 4 & \boxed{0} \\ 6 & 5 & 3 & \boxed{0} \\ 3 & 4 & 7 & \boxed{0} \end{bmatrix} \end{array}$$

Step-2 : Subtract all the elements from the highest element (7). Thus the matrix formed is to be minimized.

$$\begin{array}{cc} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \boxed{0} & 4 & \boxed{2} & 7 \\ 5 & \boxed{0} & 3 & \boxed{7} \\ 1 & \boxed{2} & \boxed{4} & 7 \\ \boxed{4} & 3 & 0 & \boxed{7} \end{bmatrix} \end{array}$$

Step-3 : Minimum element of each row is subtracted from every element in that row.

$$\begin{array}{cc} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & \boxed{4} & 2 & \boxed{7} \\ \boxed{5} & 0 & \boxed{3} & 7 \\ \boxed{0} & 1 & 3 & \boxed{6} \\ 4 & \boxed{3} & \boxed{0} & 7 \end{bmatrix} \end{array}$$

Step-4 : Minimum element in each column is subtracted from every element in that column.

$$\begin{array}{cc} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \boxed{0} & 4 & 2 & \boxed{1} \\ 5 & \boxed{0} & \boxed{3} & 1 \\ 0 & \boxed{1} & \boxed{3} & 0 \\ \boxed{4} & 3 & 0 & \boxed{1} \end{bmatrix} \end{array}$$

The optimal assignment is

subordinates	Jobs	Units
A	$\boxed{I}$	7
$\boxed{B}$	II	$\boxed{7}$
C	$\boxed{IV}$	$\boxed{0}$
$\boxed{D}$	III	7
	Total	21

∴ subordinate $\boxed{C}$ will not be given any job.

3) Given the following matrix.

$$\begin{array}{c c} & \begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{array} & \begin{bmatrix} 5 & 3 & 4 & 7 & 1 \\ 2 & 3 & 7 & 6 & 5 \\ 4 & 1 & 5 & 2 & 4 \\ 6 & 8 & 1 & 2 & 3 \\ 4 & 2 & 5 & 7 & 1 \end{bmatrix} \end{array}$$

Step-1 : Select the smallest element in each row and subtract it from every element in its row.

$$\begin{array}{c c} & \begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{array} & \begin{bmatrix} 4 & \boxed{2} & 3 & \boxed{6} & 0 \\ \boxed{0} & 1 & \boxed{5} & 4 & \boxed{3} \\ \boxed{3} & \boxed{0} & 4 & 1 & \boxed{3} \\ 5 & 7 & \boxed{0} & \boxed{1} & 2 \\ \boxed{3} & 1 & \boxed{4} & 6 & \boxed{0} \end{bmatrix} \end{array}$$

Step-2 : Select the smallest element in each column and subtract it from every element in its column.

$$\begin{array}{c c} & \begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{array} & \begin{bmatrix} \boxed{4} & 2 & \boxed{3} & 5 & \boxed{0} \\ 0 & \boxed{1} & 5 & \boxed{3} & 3 \\ 3 & 0 & \boxed{4} & \boxed{0} & 3 \\ \boxed{5} & \boxed{7} & 0 & 0 & \boxed{2} \\ 3 & \boxed{1} & 4 & \boxed{5} & 0 \end{bmatrix} \end{array}$$

Step-3 : Here the number of lines required to cover the zero is less than the number of columns/rows of matrix. Select the smallest element among all the uncovered elements. Subtract this smallest element from all the uncovered elements and ...... it to the element which lines at the ...... of two lines. The reduced matrix is

$$
\begin{array}{c}
 \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\
\begin{bmatrix}
\boxed{4} & 1 & \boxed{2} & 4 & \boxed{0} \\
0 & \boxed{0} & 4 & \boxed{2} & 3 \\
\boxed{4} & 0 & \boxed{4} & 0 & \boxed{4} \\
6 & \boxed{7} & 0 & \boxed{0} & 3 \\
\boxed{3} & 0 & \boxed{3} & 4 & \boxed{0}
\end{bmatrix}
\end{array}
$$

Step-4 : Minimum number of lines required to cover all zeros is equal to order of the matrix, optimum solution has reached.

$$
\begin{array}{c}
 \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\
\begin{bmatrix}
4 & \boxed{1} & 2 & \boxed{4} & \boxed{0} \\
\boxed{0} & \cancel{0} & \boxed{4} & 2 & \boxed{3} \\
\boxed{4} & \cancel{0} & 4 & \boxed{0} & 4 \\
6 & 7 & \boxed{0} & \cancel{0} & \boxed{3} \\
\boxed{3} & \boxed{0} & 3 & \boxed{4} & \cancel{0}
\end{bmatrix}
\end{array}
$$

The optimal assignement schedule is

$H_1 \rightarrow \boxed{R_5}$

$\boxed{H_2} \rightarrow R_1$

$H_3 \rightarrow \boxed{R_4}$

$\boxed{H_4} \rightarrow R_3$

$H_5 \rightarrow \boxed{R_2}$

B. Solve the Following

Q.1. A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. The time required by each subordinate to perform each task is given in the following effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

	Subordinates			
Tasks	1	2	3	4
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Step 1: Subtract the smallest element in each row from every element in its row.

Task.	Subordinates			
	1	2	3	4.
A.	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0.

Step 2: Subtract the smallest element in each column of the assignment matrix obtained in step-1 from every element in that column

Task.	Subordinates.			
	1	2	3	4.
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

Step 3: Since the number of straight lines covering all zero is equal to number of row/column, the optimum solution has reached. The optimal assignment can be made as.

Task.	Subordinates			
	1	2	3	4.
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D.	9.	12	14.	0

Hence the optimum assignment schedule is obtained as follows.

Task	Subordinates	Time
A	1	3
B	3	4
C	2	19
D	4	10
Total.		41

$\therefore$ Minimum Time = 41 hours.

Q.2. There are four capsulation machines available in a pharmaceutical company, namely C_1, C_2, C_3, C_4 and company has five types of antibiotic products A,B,C,D and E to be filled in capsules. The cost of performance of various products on different capsulation machines is given below in the matrix:

Capsulation Machines	Antibiotics Products				
	A	B	C	D	E
C_1 (M_1)	27	18	-	20	21
C_2 (M_2)	31	24	21	12	17
C_3 (M_3)	20	17	20	-	16
C_4 (M_4)	21	28	20	16	27

Find the optimal assignments of antibiotic products to different capsulation machines if capsule of product C cannot be filled on machine C_1 and capsule of product D cannot be filled on machine C_3.

Sol^n: Step 1: We replace - by very large number ∞. No. of antibiotic product and no of capsulation machines are not same. It is balanced by introduction of dummy machines C_5 with zero cost.

Capsulation machines	Antibiotics Products				
	A	B	C	D	E
C_1	27	18	∞	20	21
C_2	31	24	21	12	17
C_3	20	17	20	∞	16
C_4	21	28	20	16	27
C_5	0	0	0	0	0

Step 2: Subtract the smallest element in each row from every element in that row.

Capsulation machines	Antibiotic Products				
	A	B	C	D	E
C_1	9	0	∞	2	3
C_2	19	12	9	0	5
C_3	04	1	4	∞	0
C_4	05	12	4	0	11
C_5	0	0	0	0	0

Step 3: Since there is one zero in each column, the reduced matrix obtained in step 2 remains unchanged.

Step 4: Since the no of straight lines covering all zero is not equal to the no of rows/columns, the optimal solution has not reached.

Step 5: Subtract the smallest element 4 among the uncovered element from each of the uncovered element and add it to the element at the intersection of two lines.

Capsulation machines	Antibiotic Products				
	A	B	C	D	E
C_1	9	0	∞	6	3
C_2	15	8	5	0	1
C_3	4	1	4	∞	0
C_4	1	8	0	0	7
C_5	0	0	0	4	0

Step 6: The no of straight lines covering all zero is equal to the no of rows/columns, the optimal solution has reached. The optimal assignment can be made as above. Hence optimal schedule is.

Anti products	Capsulat-	cost (₹)
A	C_5	0
B	C_1	18
C	C_4	20
D	C_2	12
E	C_3	16

Total minimum cost = ₹66.

Q.3. Solve the following unbalance assignment problem for maximising total profit.

Operators	Jobs		
	1	2	3
1	6	2	5
2	2	5	8
3	7	8	6

Step 1: Since it is a maximization Problem, subtract each of the element in the table from the largest element i.e 8.

operators	Jobs.		
	1	2	3
1	2	6	3
2	6	3	0
3	1	0	2

Step 2:=> Row minimum

operators.	Jobs.		
	1	2	3
1	0	4	1
2	6	3	0
3	1	0	2

Step 3:=> Column minimum

Here each column contain '0'. Therefore matrix obtained by column min is same as above matrix.

Sign of Teacher :

Step 4:=> Draw min no. of vertical and horizontal lines to cover all '0'

operators.	Jobs.		
	1	2	3.
1	0	4	1
2	6	3	0
3.	1	0	2

Step 5:=> No of straight lines covering all zeros. is equal to no of rows/columns. the optimal Soln has reached assignment as follows.

operators	Jobs.		
	1	2	3.
1	0	4	1
2	6	3	0
3	1	0	2

optimal schedule is as follows.

operators.	Jobs	Profit (₹)
1	1	6
2	3	8
3	2	8.

Maximum profit = ₹22.