3. Trigonometric Functions – I

Let us Recall

- A solution α of a trigonometric equation is called a principal solution if $0 \le \alpha < 2\pi$.
- The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.
- The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.
- The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.
- The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
- The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
- The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
- **The Sine Rule** : In Δ ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of Δ ABC.

  Following are the different forms of the Sine rule.

  (i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (ii) $a = 2R\sin A,\ \ b = 2R\sin B,\ \ c = 2R\sin C$

  (iii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$ (iv) $\frac{a}{b} = \frac{\sin A}{\sin B}, \frac{b}{c} = \frac{\sin B}{\sin C}$

  (v) $b\sin A = a\sin B, c\sin B = b\sin C, c\sin A = a\sin C$
- **The Cosine Rule** : In Δ ABC,

  $a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = c^2 + a^2 - 2ca\cos B$, $c^2 = a^2 + b^2 - 2ab\cos C$
- **The Projection Rule** : In Δ ABC,

  $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$
- **Half angle formulae** : In Δ ABC, if $a+b+c = 2s$ then

  (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ab}}$

  (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

  (iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$, $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Ex. (1) In Δ ABC, prove that $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0$.

Solution : Method I

We know that by Sine Rule, in Δ ABC, $\frac{a}{sinA}=\frac{b}{sinB}=\frac{c}{sinC}=2R$

$\therefore \quad \frac{sinA}{a}=\frac{sinB}{b}=\frac{sinC}{c}=k$

$\therefore \quad \sin A=ak, \quad \sin B=bk, \quad \sin C=ck$

By Cosine Rule, $b^2+c^2-a^2=2bc\,cosA$,

$c^2+a^2-b^2=2ca\,cosB$,

$a^2+b^2-c^2=2ab\,cosC$

Consider the expression, $a^3\sin(B-C)$,

$a^3\sin(B-C)=a^3(\sin B\cos C-\cos B\sin C)$

$=\ a^3(bk\cos C-ck\cos B)=ka^2(ab\cos C-ac\cos B)$

$=\frac{ka^2}{2}(2ab\,cosC-2ac\,cosB)\ =\frac{ka^2}{2}\left(\left(a^2+b^2-c^2\right)-\left(c^2+a^2-b^2\right)\right)$

$=\ \frac{ka^2}{2}(2b^2-2c^2)=ka^2b^2-ka^2c^2$

$\therefore \quad a^3\sin(B-C)=k^2b^2-k^2c^2 \quad \ldots(1)$

Similarly we can prove that

$b^3\sin(C-A)=kc^2b^2-ka^2b^2 \quad \ldots(2)$

$c^3\sin(A-B)=ka^2c^2-kb^2c^2 \quad \ldots(3)$

Adding (1), (2) and (3) , we get

$a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0$

Method II : By using identity $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

Consider the expression, $a^3\sin(B-C)$,

$a^3\sin(B-C)=a^2a\sin(B-C)$

$=a^2k\sin A\sin(B-C)$

$=a^2k\sin(B+C)\sin(B-C)$

$=a^2k\left(b^2-c^2\right)$

$\therefore \quad a^3\sin(B-C)\ =\ ka^2b^2-ka^2c^2 \quad \ldots(1)$

Similarly we can prove that

$b^3 \sin(C-A) = kc^2b^2 - ka^2b^2 \quad \ldots(2)$

$c^3 \sin(A-B) = ka^2c^2 - kb^2c^2 \ldots(3)$

Adding (1), (2) and (3) , we get

$a^3 \sin(B-C) + b^3 \sin(C-A) + c^3 \sin(A-B) = 0$

Ex. (2) In Δ ABC prove that :

$$\left(c^2+b^2-a^2\right)tanA = \left(a^2+c^2-b^2\right)tanB = \left(b^2+a^2-c^2\right)tanC$$

Solution : By Cosine Rule, $a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Consider the expression $\left(c^2+b^2-a^2\right)tanA$,

$$\left(c^2+b^2-a^2\right)\tan A = 2bc\cos A \times \frac{\sin A}{\cos A}$$

$= 2bc \times \sin A$

$= 2bc \times ak$ (by Sine Rule)

$= 2abck$

$\therefore \left(c^2+b^2-a^2\right)tanA = 2abck \quad \ldots. (1)$

Similarly we can prove that

$\left(a^2+c^2-b^2\right)tanB = 2abck \quad \ldots(2)$

$\left(b^2+a^2-c^2\right)tanC = 2abck \quad \ldots(3)$

From (1), (2) and (3) , we get

$$\left(c^2+b^2-a^2\right)tanA = \left(a^2+c^2-b^2\right)tanB = \left(b^2+a^2-c^2\right)tanC$$

Ex.(3) In Δ ABC, prove that $cot\left(\frac{A}{2}\right) + cot\left(\frac{B}{2}\right) + cot\left(\frac{C}{2}\right) = \left(\frac{a+b+c}{b+c-a}\right)cot\left(\frac{A}{2}\right)$

Solution : We know that $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$,

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

L.H.S. $= \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$$= \frac{1}{\tan\frac{A}{2}} + \frac{1}{\tan\frac{B}{2}} + \frac{1}{\tan\frac{C}{2}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-b)(s-a)}}$$

$$= \sqrt{\frac{s(s-a)^2}{(s-b)(s-c)(s-a)}} + \sqrt{\frac{s(s-b)^2}{(s-a)(s-c)(s-b)}} + \sqrt{\frac{s(s-c)^2}{(s-b)(s-a)(s-c)}}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}}\left\{\sqrt{(s-a)^2} + \sqrt{(s-b)^2} + \sqrt{(s-c)^2}\right\}$$

$$= \sqrt{\frac{s}{(s-a)(s-b)(s-c)}}\{(s-a)+(s-b)+(s-c)\}$$

$$= \sqrt{\frac{s}{(s-a)(s-b)(s-c)}}\{3s-(a+b+c)\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}}\{3s-2s\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \times s$$

$$= \sqrt{\frac{s}{(s-b)(s-c)}} \times \frac{s}{\sqrt{s-a}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \times \frac{s}{(s-a)}$$

$$= \frac{2s}{(2s-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{(a+b+c-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{b+c-a} \cot\frac{A}{2}$$

= R.H.S.

Ex.(4) If $0<2x<1$ and $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$ then find x.

Solution : Let $\sin^{-1}x=\theta$

$\therefore \sin\theta=x$ and $\cos\theta=\sqrt{1-\sin^2\theta}$

As $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

$\sin^{-1}2x=\frac{\pi}{3}-\sin^{-1}x$

$\therefore \sin^{-1}2x=\frac{\pi}{3}-\theta$

$\therefore 2x=\sin\left(\frac{\pi}{3}-\theta\right)$

$\therefore 2x=\sin\frac{\pi}{3}\cos\theta-\cos\frac{\pi}{3}\sin\theta$

$\therefore 2x=\frac{\sqrt{3}}{2}.\cos\theta-\frac{1}{2}.\sin\theta$

$\therefore 4x=\sqrt{3}\cos\theta-\sin\theta$

$\therefore 4x=\sqrt{3}\sqrt{1-\sin^2\theta}-x$

$\therefore 5x=\sqrt{3(1-x^2)}$

$\therefore 25x^2=3-3x^2$

$\therefore 28x^2=3$

$\therefore x=\pm\sqrt{\frac{3}{28}}$

But $0<2x<1$, $\therefore x=\sqrt{\frac{3}{28}}$

Ex.(5) Find the general solution of (a) $\sin\theta+\cos\theta+1=0$ (b) $\tan^3\theta-3\tan\theta=0$

Solution : (a) Given $\sin\theta+\cos\theta+1=0$ $\therefore \sin\theta+\cos\theta=-1$

Solution :

(a) Given $\sin\theta+\cos\theta+1=0$

$\therefore \sin\theta+\cos\theta=-1$

multiplying b.s. by $\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=-\frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta=-\cos\frac{\pi}{4}$

$\cos\left(\theta-\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)$

$\because -\cos\theta=\cos(\pi-\theta)$

$\therefore \cos\left(\theta-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right)$

$\cos\theta=\cos\alpha \Rightarrow \theta=2n\pi\pm\alpha, n\in Z$

$\therefore \theta-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4}, n\in Z$

$\therefore \theta-\frac{\pi}{4}=2n\pi+\frac{3\pi}{4}$

or $\theta-\frac{\pi}{4}=2n\pi-\frac{3\pi}{4}, n\in Z$

$\therefore \theta = 2n\pi + \frac{3\pi}{4} + \frac{\pi}{4}$ (or) $\theta = 2n\frac{3\pi}{4} + \frac{\pi}{4}$, $n \in Z$

$\therefore \theta = 2n\pi + \pi$ (or) $\theta = 2n\pi - \frac{\pi}{2}$, $n \in Z$

(b) $\tan^3\theta - 3\tan\theta = 0$

$\therefore \quad \tan\theta(\tan^2\theta - 3) = 0$

Solution :

Given $\tan^3\theta - 3\tan\theta = 0$

$\therefore \tan\theta(\tan^2\theta - 3) = 0$

$\tan\theta = 0$ or $\tan^2\theta - 3 = 0$

consider $\tan\theta = 0$

$\tan\theta = 0 \Rightarrow \theta = n\pi$, $n \in Z$

consider $\tan^2\theta - 3 = 0$

$\tan^2\theta = 3$

$\tan^2\theta = (\sqrt{3})^2$

$\tan^2\theta = \tan^2\frac{\pi}{3}$

$\tan^2\theta = \tan^2\alpha \Rightarrow \theta = m\pi \pm \alpha, m \in Z$

$\therefore \theta = m\pi \pm \frac{\pi}{3}$, $m \in Z$

$\therefore$ The required general sol^n are $\theta = n\pi$ or $\theta = m\pi \pm \frac{\pi}{3}$

$m, n \in Z$

Ex. (6) Using Cosine rule prove the Sine rule.

Solution : $\left(\frac{\sin A}{a}\right)^2 = \frac{\sin^2 A}{a^2}$

$= \frac{1-\cos^2 A}{a^2}$

$= \frac{1-\left[\frac{b^2+c^2-a^2}{2bc}\right]^2}{a^2}$

$= \frac{1-\frac{(b^2+c^2-a^2)^2}{(2bc)^2}}{a^2}$

$= \frac{\frac{(2bc)^2-(b^2+c^2-a^2)^2}{(2bc)^2}}{a^2}$

$= \frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4a^2b^2c^2}$

$= \frac{(b^2+2bc+c^2-a^2)(a^2-(b^2-2bc+c^2))}{4a^2b^2c^2}$

$= \frac{[(b+c)^2-a^2][a^2-(b-c)^2]}{4a^2b^2c^2}$

$= \frac{(a+b+c)(a+b-c)(b+c-a)(a-b+c)}{4a^2b^2c^2}$ —— (1)

See on Next Page

$$\left(\frac{\sin B}{b}\right)^2 = \frac{\sin^2 B}{b^2} = \frac{1-\cos^2 B}{b^2}$$

$$= \frac{1-\left[\frac{c^2+a^2-b^2}{2ca}\right]^2}{b^2}$$

$$= \frac{\frac{(2ca)^2-(c^2+a^2-b^2)^2}{(2ca)^2}}{b^2}$$

$$= \frac{(2ca+c^2+a^2-b^2)(2ca-c^2-a^2+b^2)}{4a^2b^2c^2}$$

$$= \frac{[(c+a-b)(c+a+b)(b+c-a)(b-c+a)]}{4a^2b^2c^2}$$

$$= \frac{(a+b+c)(a+b-c)(b+c-a)(a-b+c)}{4a^2b^2c^2} \quad \text{--- (II)}$$

IIIly

$$\left(\frac{\sin C}{c}\right)^2 = \frac{(a+b+c)(a+b-c)(b+c-a)(a-b+c)}{4a^2b^2c^2} \quad \text{--- (III)}$$

from (I) (II) and (III)

$$\left(\frac{\sin A}{a}\right)^2 = \left(\frac{\sin B}{b}\right)^2 = \left(\frac{\sin C}{c}\right)^2$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Ex. (7) Write principal solutions of $\tan 5\theta = -1$

Solution :

$\tan 5\theta = -1$

$\tan 5\theta = -\tan\frac{\pi}{4}$

$\tan 5\theta = \tan\left(\pi - \frac{\pi}{4}\right)$

$[\because -\tan\theta = \tan(\pi-\theta)]$

$\therefore \tan 5\theta = \tan\frac{3\pi}{4}$

$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$

$\therefore 5\theta = n\pi + \frac{3\pi}{4},\ n \in Z$

$\therefore \theta = \frac{n\pi}{5} + \frac{3\pi}{20},\ n \in Z$

put $n=0,\ \theta = \frac{3\pi}{20} \in [0, 2\pi]$

Put $n=1,\ \theta = \frac{\pi}{5} + \frac{3\pi}{20} = \frac{7\pi}{20} \in [0, 2\pi]$

Put $n=2,\ \theta = \frac{2\pi}{5} + \frac{3\pi}{20} = \frac{11\pi}{20} \in [0, 2\pi]$

Put $n=3,\ \theta = \frac{3\pi}{5} + \frac{3\pi}{20} = \frac{3\pi}{4} \in [0, 2\pi]$

Put $n=4,\ \theta = \frac{4\pi}{5} + \frac{3\pi}{20} = \frac{19\pi}{20} \in [0, 2\pi]$

Put $n=5,\ \theta = \frac{5\pi}{5} + \frac{3\pi}{20} = \frac{23\pi}{20} \in [0, 2\pi]$

Put $n=6,\ \theta = \frac{6\pi}{5} + \frac{3\pi}{20} = \frac{27\pi}{20} \in [0, 2\pi]$

Put $n=7,\ \theta = \frac{7\pi}{5} + \frac{3\pi}{20} = \frac{31\pi}{20} \in [0, 2\pi]$

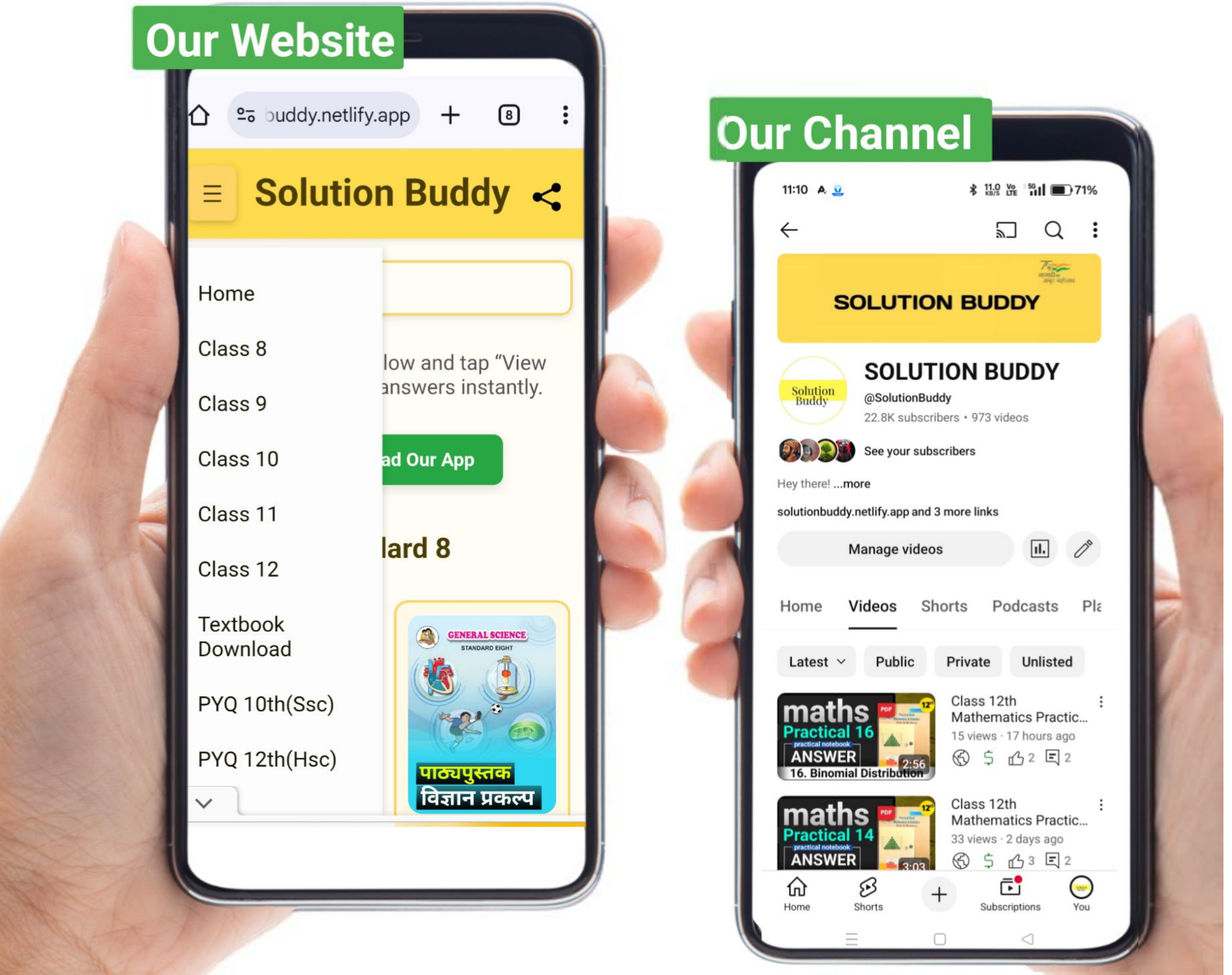
put $n=8,\ \theta = \frac{8\pi}{5} + \frac{3\pi}{20} = \frac{35\pi}{20} \in [0, 2\pi]$

put $n=9,\ \theta = \frac{9\pi}{5} + \frac{3\pi}{20} = \frac{39\pi}{20} \in [0, 2\pi]$

put $n=10,\ \theta = \frac{10\pi}{5} + \frac{3\pi}{20} = \frac{43\pi}{20} \notin [0, 2\pi]$

$\therefore$ The principal solutions of $\tan 5\theta = -1$ is $\left\{\frac{3\pi}{20}, \frac{7\pi}{20}, \frac{11\pi}{20}, \frac{3\pi}{4}, \frac{19\pi}{20}, \frac{23\pi}{20}, \frac{27\pi}{20}, \frac{31\pi}{20}, \frac{7\pi}{4}, \frac{39\pi}{20}\right\}$

Sign of Teacher :



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