12. Definite Integration - II

Ex. (1) Evaluate $\int_0^2 |4x-5|\,dx$

Solution : $|4x-5| = \begin{cases} 4x-5, & \text{if } x \geq \frac{5}{4} \\ -(4x-5), & \text{if } x < \frac{5}{4} \end{cases}$

$$\int_0^2 |4x-5|\,dx = \int_0^{\frac{5}{4}} |4x-5|\,dx + \int_{\frac{5}{4}}^2 |4x-5|\,dx$$

$$= \int_0^{\frac{5}{4}} -(4x-5)\,dx + \int_{\frac{5}{4}}^2 (4x-5)\,dx$$

$$= \int_0^{\frac{5}{4}} (-4x+5)\,dx + \int_{\frac{5}{4}}^2 (4x-5)\,dx$$

$$= \left[\frac{-4x^2}{2} + 5x\right]_0^{\frac{5}{4}} + \left[\frac{4x^2}{2} - 5x\right]_{\frac{5}{4}}^2$$

$$= \left\{\left[\left[-2\left(\tfrac{5}{4}\right)^2 + 5\left(\tfrac{5}{4}\right)\right] - [0]\right]\right\} + \left\{\left[2(2)^2 - 5(2)\right] - \left[2\left(\tfrac{5}{4}\right)^2 - 5\left(\tfrac{5}{4}\right)\right]\right\}$$

$$= \frac{-50}{16} + \frac{25}{4} - 2 - \frac{50}{16} + \frac{25}{4}$$

$$= \frac{-50+100-32-50+100}{16} = \frac{68}{16} = \frac{17}{4}$$

$$\therefore \int_0^2 |4x-5|\,dx = \frac{17}{4}$$

Ex. (2) Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx$

Solution : $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx = \int_0^{\pi} \frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\,dx$

$$= \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}\,dx$$

$$I = \int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$$

$$I = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x}\, dx - I$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x}\, dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx$$

$$2I = \pi \int_0^{\pi} \left(1 - \frac{1}{1+\sin x}\right) dx$$

$$2I = \pi \int_0^{\pi} \left(1 - \frac{1-\sin x}{1-\sin^2 x}\right) dx$$

$$2I = \pi \int_0^{\pi} \left(1 - \frac{1-\sin x}{\cos^2 x}\right) dx$$

$$2I = \pi \int_0^{\pi} \left(1 - \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$2I = \pi \int_0^{\pi} \left(1 - \sec^2 x + \sec x \tan x\right) dx$$

$$2I = \pi \left[x - \tan x + \sec x\right]_0^{\pi}$$

$$2I = \pi \left[\left(\pi - \tan \pi + \sec \pi\right) - \left(0 - \tan 0 + \sec 0\right)\right]$$

$$2I = \pi \left[\pi - 0 - 1 - 0 - 1\right] = \pi\ (\pi - 2)$$

$$I = \frac{\pi}{2}(\pi - 2) \quad = \left(\frac{\pi^2}{2} - \pi\right)$$

Ex. (3) $\int_{-3}^{3} \frac{x+x^2}{25-x^2}\, dx$

Solution : $I = \int_{-3}^{3} \frac{x+x^2}{25-x^2}\, dx$

$$I = \int_{-3}^{3} \frac{x}{25-x^2}\, dx + \int_{-3}^{3} \frac{x^2}{25-x^2}\, dx \quad ...(1)$$

Let $f(x)=\dfrac{x}{25-x^2}$

$$f(-x)=\frac{-x}{25-(-x)^2}=-\frac{x}{25-x^2}=-f(x)$$

$\therefore f(x)$ is an odd function.

$$\therefore \int_{-3}^{3}\frac{x}{25-x^2}dx=0 \quad .....(2)$$

Let $g(x)=\dfrac{x^2}{25-x^2}$

$$g(-x)=\frac{(-x)^2}{25-(-x)^2}=\frac{x^2}{25-x^2}=g(x)$$

$\therefore g(x)$ is an even function.

$$\therefore \int_{-3}^{3}\frac{x^2}{25-x^2}dx=2\int_{0}^{3}\frac{x^2}{25-x^2}dx \quad .....(3)$$

From (1) , (2) and (3) we get,

$$I=2\int_0^3\frac{x^2}{25-x^2}\,dx$$

$$=2\int_0^3\frac{25-25+x^2}{25-x^2}\,dx$$

$$=2\int_0^3\frac{25-\left(25-x^2\right)}{25-x^2}\,dx$$

$$=2\int_0^3\frac{25}{25-x^2}-1\,dx$$

$$=2\left[\frac{25}{2(5)}\log\left(\frac{5+x}{5-x}\right)-x\right]_0^3$$

$$=2\left[\left(\frac{5}{2}\log\left(\frac{5+3}{5-3}\right)-3\right)-\left(\frac{5}{2}\log\left(\frac{5+0}{5-0}\right)-0\right)\right]$$

$$=2\left[\frac{5}{2}\log\left(\frac{8}{2}\right)-3-\frac{5}{2}\log(1)\right]$$

$$=2\left[\frac{5}{2}\log(4)-3\right]$$

$$=5\log 4-6$$

$$\therefore \int_{-3}^{3}\frac{x+x^2}{25-x^2}\,dx=5\log 4-6$$

Ex. (4) Evaluate $\int_0^2 \left([x]+|x-1|\right) dx$

Solution :

$$I = \int_0^2 ([x] + |x-1|)\,dx$$

using $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \quad (a<b<c)$

$$\therefore I = \int_0^1 ([x] + |x-1|)\,dx + \int_1^2 [x] + |x-1|)\,dx$$

$[x]$ $0 - 1$ $\qquad |x| = x,\ x>0$

$[x] = 0$ $\qquad = -x,\ x<0$

$1 - 2$

$[x] = 1$ $\qquad |x-1| = x-1,\ x-1>0$

$\qquad = -(x-1),\ x-1<0$

$\qquad |x-1| = x-1,\ x>1$

$\qquad = 1-x,\ x<1$

$$I = \int_0^1 (0+1-x)\,dx + \int_1^2 (1+x-1)\,dx$$

$$= \int_0^1 (1-x)\,dx + \int_1^2 x\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2$$

$$= 1 - \frac{1}{2} - 2\left(0 - \frac{0}{2}\right) + \frac{4}{2} - \frac{1}{2} \Rightarrow \frac{1}{2} + 2 - \frac{1}{2}$$

$$I = 2$$

Ex. (5) Evaluate $\int_e^{e^2} \frac{1}{\log x} - \frac{1}{(\log x)^2}\,dx$

Solution :

$$I = \int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$$

put $\log x = t \Rightarrow x = e^t$

$\therefore dx = e^t\,dt$

$t = \log x$

when $x = e$, $t = \log e = 1$

when $x = e^2$, $t = \log e^2 = 2$

$$\therefore\ I = \int_1^2 \left[\frac{1}{t} - \frac{1}{t^2}\right]$$

$$= \int_1^2 e^t\left[\frac{1}{t} - \left(-\frac{1}{t^2}\right)\right] dt$$

$$= \left[e^t\left(\frac{1}{t}\right)\right]_1^2 \quad \because \int e^x[f(x)+f'(x)]dx = e^x f(x)+c$$

$$= e^2\left(\frac{1}{2}\right) - e^1\left(\frac{1}{1}\right)$$

$$= \frac{e^2}{2} - e$$

$$I = \frac{e^2-2e}{2}$$

$$= \frac{e(e-2)}{2}$$

Ex. (7) Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x \cos ecx}\, dx$

Solution :

using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

changing $x \to \pi - x$

$$I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\operatorname{cosec}(\pi-x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi-x)(-\tan x)}{(-\sec x)\operatorname{cosec} x}\,dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx$$

$$= \int_0^{\pi} \frac{\pi\tan x - x\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx$$

$$= \int_0^{\pi} \frac{\pi\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx - \int_0^{\pi} \frac{x\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx$$

$$I = \int_0^{\pi} \frac{\pi\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx - I \quad \text{by } I$$

$$2I = \pi\int_0^{\pi} \frac{\sin x}{\cos x}\times\cos x\times\sin x\,dx$$

$$2I = \pi\int_0^{\pi} \sin^2 x\,dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi} 2\sin^2 x\,dx$$

$$\because \cos 2\theta = 1 - 2\sin^2\theta$$

$$I = \frac{\pi}{4}\int_0^{\pi} (1-\cos 2x)\,dx$$

$$I = \frac{\pi}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$I = \frac{\pi}{4}\left[\pi - \frac{\sin 2\pi}{2} - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{\pi}{4} \times \pi$$

$$I = \frac{\pi^2}{4}$$

Ex. (8) Evaluate $\int_0^3 x\,[x]\,dx$, where $[x]$ denote greatest integer function not greater than x.

Solution :

$0-1 \ [x] = 0, \quad 1-2 \ [x] = 1, \quad 2-3 \ [x] = 2$

$$I = \int_0^3 x[x]\,dx$$

if $a<b<c$ then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

$$I = \int_0^1 x[x]\,dx + \int_1^2 x[x]\,dx + \int_2^3 x[x]\,dx \quad (0<1<2<3)$$

$$= \int_0^1 x(0)\,dx + \int_1^2 x(1)\,dx + \int_2^3 x(2)\,dx$$

$$= 0 + \int_1^2 x\,dx + \int_2^3 2x\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^2 + \left[\frac{2x^2}{2}\right]_2^3$$

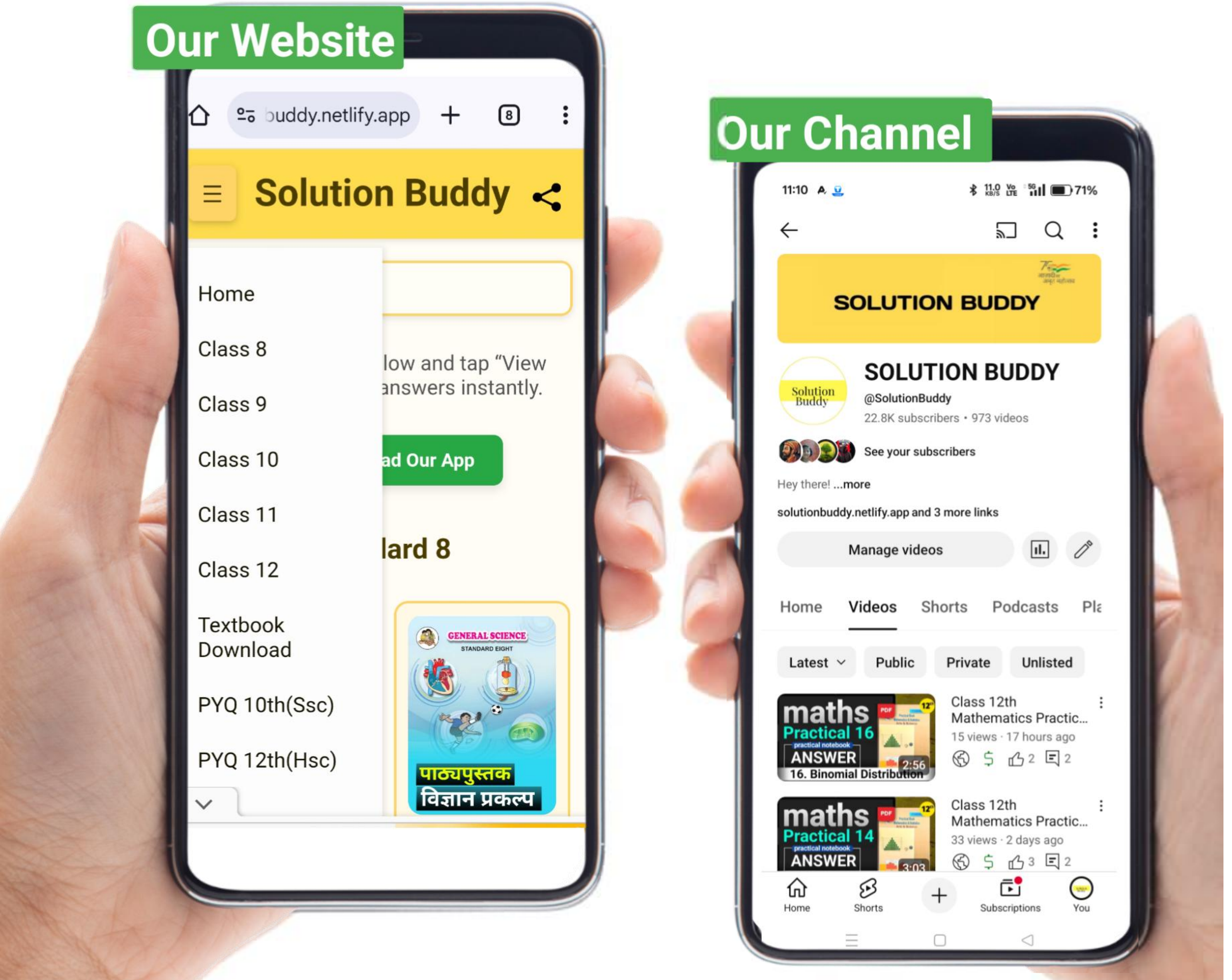
$$= \left[\frac{x^2}{2}\right]_1^2 + \left[x^2\right]_2^3$$

$$= \frac{4}{2} - \frac{1}{2} + 9 - 4$$

$$= 7 - \frac{1}{2}$$

$$\therefore I = \frac{13}{2}$$

Sign of Teacher :



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