4. Trigonometric Functions – II

Let us Recall

- Inverse Trigonometric functions :

(i) $\sin\left(\sin^{-1}x\right)=x$, for $x\in[-1,1]$

(ii) $\sin^{-1}(\sin y) = y$, for $y \in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(iii) $\cos\left(\cos^{-1}x\right)=x$, for $x\in[-1,1]$

(iv) $\cos^{-1}(\cos y) = y$, for $y \in [0,\pi]$

(v) $\tan\left(\tan^{-1}x\right)=x$, for $x\in R$ (vi) $\tan^{-1}(\tan y) = y$, for $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(vii) $\sec\left(\sec^{-1}x\right)=x$, for $x\in R-(-1,1)$

(viii) $\sec^{-1}(\sec y) = y$, for $y\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(ix) $\cot\left(\cot^{-1}x\right)=x$, for $x\in R$ (x) $\cot^{-1}(\cot y) = y$, for $y\in(0,\pi)$

(xi) $\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right)=x$, for $x\in R-(-1,1)$

(xii) $\operatorname{cosec}^{-1}(\operatorname{cosec} y) = y$, for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

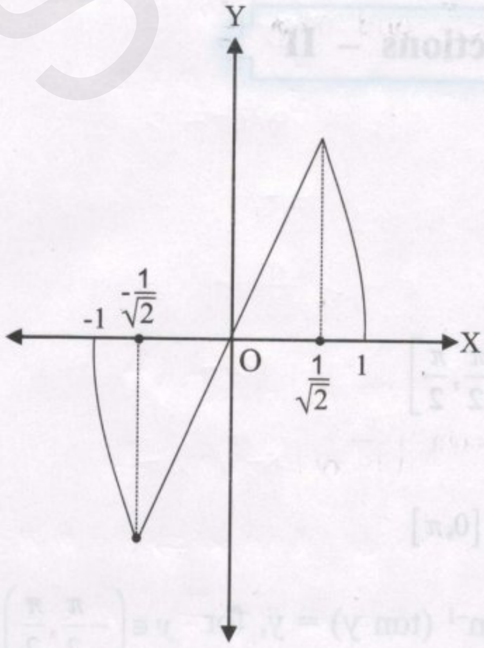
Ex. (1) Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$ *if* $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

Solution : Let $\sin^{-1}x=\theta$

$\therefore\ \sin\theta=x,\ x\in[-1,1],$

$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Given $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}},$



$\therefore \quad \sin\left(-\frac{\pi}{4}\right) \le \sin\theta \le \sin\left(\frac{\pi}{4}\right)$

$\therefore \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$\therefore \quad -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

$\therefore \quad 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

L.H.S $= \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta \quad$ As $\; 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= 2\sin^{-1} x \;=\;$ R.H.S.

Ex. (2) If $x > 0, y > 0$ then prove that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution : Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$

$\therefore \quad \tan\theta = x, \; \tan\phi = y$

As $x > 0$ and $y > 0$, we have $0 << \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$.

$\therefore \quad -\frac{\pi}{2} < -\phi < 0$

$\therefore \quad -\frac{\pi}{2} < \theta - \phi < \frac{\pi}{2} \qquad \ldots(1)$

Also $\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{x-y}{1+xy} \qquad \ldots(2)$

From (1) and (2) we get $\theta - \phi = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\therefore \quad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Ex. (3) Prove that for all $x \in R$

(a) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$ (b) $\tan^{-1}(-x) = -\tan^{-1}(x)$

Solution : (a) To prove that $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

Let $\cot^{-1}(-x) = \theta$

$\therefore \cot\theta = -x$, Where $-x \in R, 0 < \theta < \pi$

$\therefore -\cot\theta = x$

$\therefore \cot(\pi - \theta) = x, x \in R$

Since, $0 < \theta < \pi$

$\therefore \quad -\pi < -\theta < 0$

$\therefore 0 < \pi - \theta < \pi$.

Which implies $\cot(\pi - \theta) = x$ and $x \in R, 0 < \pi - \theta < \pi$

$\therefore \pi - \theta = \cot^{-1}x$

$\therefore \theta = \pi - \cot^{-1}x$

$\therefore \cot^{-1}(-x) = \pi - \cot^{-1}x$

(b) To prove that $\tan^{-1}(-x) = -\tan^{-1}(x)$

Let $\tan^{-1}(-x) = \theta$

$\therefore \tan \theta = -x$ where $-x \in R, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\therefore -\tan\theta = x$

$\therefore \tan(-\theta) = x$

$\therefore \quad \tan\theta = x, \qquad x \in R \; and -\frac{\pi}{2} < -\theta < \frac{\pi}{2}$

$\therefore -\theta = \tan^{-1} x$

$-\tan^{-1} x = \tan^{-1} x$

$\therefore \tan^{-1}(-x) = -\tan^{-1} x$

Ex. (4) Prove that : $\tan^{-1}\left[\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right] = \frac{\pi}{4} + \theta$ if $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution : $L.H.S. = \tan^{-1}\left[\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right]$

$$= \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\tan\left(\frac{\pi}{4}\right) + \tan\theta}{1 - \tan\left(\frac{\pi}{4}\right)\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right] \quad \text{------------------} \quad (1)$$

We have, $\tan^{-1}(\tan\theta) = \theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. ---------------- (2)

Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$\therefore -\frac{\pi}{4} + \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{\pi}{4} + \frac{\pi}{4}$

$\therefore \; 0 < \theta + \frac{\pi}{4} < \frac{\pi}{2}$

From equation (1) we get,

$L.H.S. = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$

$= \frac{\pi}{4} + \theta$ From equation (2).

$= \text{R. H. S}$

Thus, $\tan^{-1}\left[\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right]=\dfrac{\pi}{4}+\theta$ for $\theta\in\left(-\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$

Ex. (5) If $\sin^{-1}(1-x)-2\sin^{-1}x=\dfrac{\pi}{2}$ then find the value of x.

Solution :

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

Put $x=\sin y$

$\sin^{-1}(1-\sin y)-2\sin^{-1}(\sin y)=\frac{\pi}{2}$

$\therefore \sin^{-1}(1-\sin y)=\frac{\pi}{2}+2y$

$\therefore 1-\sin y=\sin\left(\frac{\pi}{2}+2y\right)$

$1-\sin y=\cos 2y$

$1-\sin y=1-2\sin^2 y$

$1-\sin y-1+2\sin^2 y=0$

$2\sin^2 y-\sin y=0$

$\sin y\,(2\sin y-1)=0$

$\sin y=0$ or $2\sin y-1=0$

$\sin y=0$ or $\sin y=\frac{1}{2}$

$\therefore x=0$ or $x=\frac{1}{2}$ $\because x=\sin y$

Let $x=\frac{1}{2}$

$\text{LHS}=\sin^{-1}(1-x)-2\sin^{-1}x$

$=\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{6}-2\times\frac{\pi}{6}$

$=-\frac{\pi}{6}$

$\therefore \text{LHS}\neq\text{RHS}$

$\therefore x=\frac{1}{2}$ is not sol^n

$\boxed{\therefore x=0}$

Ex. (6) Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=-2\pi+2\cos^{-1}x$ *if* $-1\le x\le-\dfrac{1}{\sqrt{2}}$

Solution :

we know that

$\sin^{-1}(\sin x)=x,\ -\frac{\pi}{2}<x\le\frac{\pi}{2}$

$\text{LHS}=\sin^{-1}(2x\sqrt{1-x^2}$ —(I)

put $x=\sin\theta\Rightarrow\theta=\sin^{-1}x$

$-1\le x\le 1,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\text{LHS}=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$=\sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$=\sin^{-1}(2\sin\theta\cdot\cos\theta)$

$=\sin^{-1}(\sin 2\theta)$

where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

multiplying by 2 b.s

$-\pi\le 2\theta\le\pi$

Here $x=2\theta$ does not

Satisfy $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

we have to find the

value of x for which

$-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

$\therefore \text{LHS}=\sin^{-1}(\sin 2\theta)$

$=\sin^{-1}\left[-\sin(\pi+2\theta)\right]$

$=\sin^{-1}\left[\sin[-(\pi+2\theta)]\right]$

$=\sin^{-1}\left[\sin(-\pi-2\theta)\right]$

$-1\le x\le-\frac{1}{2}$

$-\sin\frac{\pi}{2}\le\sin\theta\le-\sin\frac{\pi}{4}$

$\therefore -\frac{\pi}{2}\le\theta\le-\frac{\pi}{4}$

$-\pi\le 2\theta\le-\frac{\pi}{2}$

$\frac{\pi}{2}\le-2\theta\le\pi$

$$-\pi + \frac{\pi}{2} \le -\pi - 2\theta \le -\pi + \pi$$

$$-\frac{\pi}{2} \le -\pi - 2\theta \le 0$$

$\therefore$ LHS $= \sin^{-1}(\sin(-\pi - 2\theta))$

where $-\frac{\pi}{2} \le -\pi - 2\theta \le 0$

LHS $= -\pi - 2\theta$ from (I)

LHS $= -\pi - 2\sin^{-1}x$

$= -\pi - 2\left(\frac{\pi}{2} - \cos^{-1}x\right)$

$\left(\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right)$

LHS $= -\pi - \pi + 2\cos^{-1}x$

$= -2\pi + 2\cos^{-1}x$

$\therefore$ LHS $=$ RHS

Ex. (7) Prove that : $\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\frac{\theta}{2}$, if $\theta \in (-\pi, 0)$

Solution :

LHS $= \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)$

$= \tan^{-1}\left(\sqrt{\tan^2\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(\pm \tan\frac{\theta}{2}\right)$

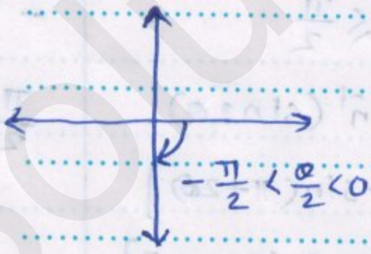
Since $\theta \in (-\pi, 0)$

$\therefore -\pi < \theta < 0$

dividing by 2 on b.s

$-\frac{\pi}{2} < \frac{\theta}{2} < 0$

$\frac{\theta}{2}$ is in IV quadrant



$\therefore$ LHS $= \tan^{-1}\left(-\tan\frac{\theta}{2}\right)$

$= -\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\therefore \tan^{-1}(-x) = -\tan^{-1}x$

LHS $= -\frac{\theta}{2}$

$\therefore$ LHS $=$ RHS

Sign of Teacher :

Q. 26. A solenoid of length π m and 5 cm in diameter has a winding of 1000 turns and carries a current of 5A. Calculate the magnetic field at its centre along the axis.

SECTION – D

Attempt any THREE questions of the following : [12]

Q. 27. What is Ferromagnetism? Explain it on the basis of domain theory.

Q. 28. Obtain an expression for average power dissipated in a series LCR circuit.

Q. 29. Distinguish between interference and diffraction of light.

A double slit arrangement produces interference fringes for sodium light of wavelength 589 nm, that are 0.20 degree apart. What is the angular fringe separation if the entire arrangement is immersed in water?

(R.I. of water = 1.33)

Q. 30. State Einstein's photoelectric equation and mention physical significance of each term involved in it.

The wavelength of incident light is 4000Å. Calculate the energy of incident photon.

Q. 31. State any four uses of Van de Graaff generator.

In a parallel plate air capacitor, intensity of electric field is changing at the rate of 2×10^{11} V/ms. If area of each plate is 20 cm^2, calculate the displacement current.

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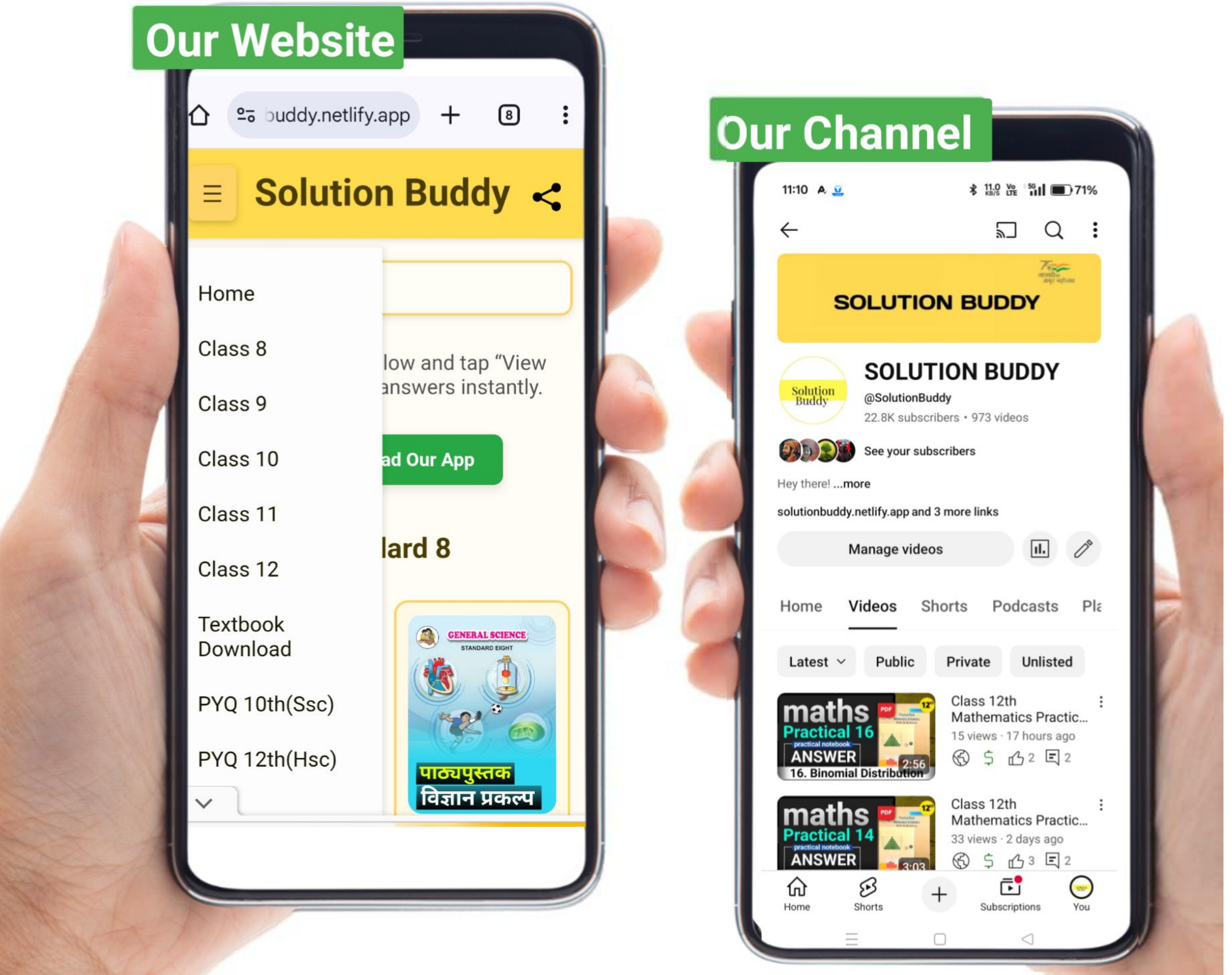

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