2. Finding Inverse of a Matrix

A. Activities

1. If $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find X by using the following activity

by $R_2 + R_1$

and $R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ \boxed{0} & \boxed{3} & 5 \\ 0 & 0 & \boxed{1} \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \text{ By } \left(\frac{1}{3}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ \boxed{0} & \boxed{1} & 5/3 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ \boxed{1} \\ 2 \end{bmatrix}, \text{ By } R_1 - 2R_2$$

$$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{-1/3} \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} \boxed{-1} \\ 1 \\ 2 \end{bmatrix}, \text{ By } R_1 + \frac{1}{3} R_3 ,\ R_2 - \frac{5}{3} R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \boxed{0} & 1 & \boxed{0} \\ 0 & \boxed{0} & 1 \end{bmatrix} x = \begin{bmatrix} \boxed{-1/3} \\ \boxed{-7/3} \\ 2 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} -1/3 \\ \boxed{-7/3} \\ 2 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, then show that $A^{-1} = \frac{1}{6}(A - 5I)$ by completing the following activity.

Consider $AA^{-1} = I$,

$$\therefore \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \frac{1}{4} R_1 \rightarrow \begin{bmatrix} 1 & 5/4 \\ 2 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \boxed{1/4} & \boxed{0} \\ 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow \begin{bmatrix} 1 & 5/4 \\ \boxed{0} & \boxed{-3/2} \end{bmatrix} A^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\frac{-2}{3} R_2 \rightarrow \begin{bmatrix} 1 & 5/4 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1/4 & 0 \\ \boxed{1/3} & \boxed{-2/3} \end{bmatrix}$$

$R_1 - \frac{5}{4}R_2$, $\begin{bmatrix} \boxed{1} & \boxed{0} \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \boxed{\frac{-1}{6}} & \boxed{\frac{5}{6}} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{6}\begin{bmatrix} -1 & \boxed{5} \\ \boxed{2} & -4 \end{bmatrix}$ ....(i)

Now $\frac{1}{6}(A - 5I) = \frac{1}{6}\left\langle \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$

$= \frac{1}{6}\begin{bmatrix} \boxed{-1} & 5 \\ 2 & \boxed{-4} \end{bmatrix}$ ....(ii)

$\therefore$ From (i) and (ii) $A^{-1} = \boxed{\frac{1}{6}(A-5I)}$

3. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by completing the following activity.

Consider $A.A^{-1} = I$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 - R_1$ and $R_3 - 2R_1$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{-1} & \boxed{2} \\ \boxed{} & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & \boxed{1} & \boxed{0} \\ -2 & 0 & \boxed{} \end{bmatrix}$, By $-R_2$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{1} & \boxed{-2} \\ \boxed{} & 0 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \boxed{-1} & \boxed{0} \\ -2 & 0 & \boxed{} \end{bmatrix}$, $R_1 - 2R_2$

$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{7} \\ 0 & +1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \boxed{-1} & \boxed{2} & \boxed{0} \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$R_1 - 7R_3$ and $R_2 + 2R_3$

$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ \boxed{-3} & \boxed{-1} & \boxed{2} \\ -2 & 0 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \boxed{13} & \boxed{2} & \boxed{-7} \\ \boxed{-3} & \boxed{-1} & \boxed{2} \\ \boxed{-2} & \boxed{0} & \boxed{1} \end{bmatrix}$

4. Carry out the following activity

If $A=\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

M_{11} = ...-3.......

M_{12} = ...12.......

M_{13} = ...6.......

M_{21} = ...1.......

M_{22} = ...3.......

M_{23} = ...-2.......

M_{31} = ...-11.......

M_{32} = ...9.......

M_{33} = ...1.......

$$A_{11} = (-1)^{1+1} M_{11} = \boxed{-3}$$

$$A_{12} = (-1)^{1+2} M_{12} = \boxed{-12}$$

$$A_{13} = (-1)^{1+3} \boxed{6} = 6$$

$$A_{21} = (-1)^{2+1} (1) = \boxed{-1}$$

$$A_{22} = (-1)^{2+2} \boxed{3} = 3$$

$$A_{23} = (-1)^{2+3} \boxed{-2} = \boxed{2}$$

$$A_{31} = (-1)^{3+1} \boxed{-11} = -11$$

$$A_{32} = (-1)^{3+2} \boxed{9} = -9$$

$$A_{33} = (-1)^{3+3} \boxed{1} = \boxed{1}$$

$$\text{Cofactor Matrix} = \begin{bmatrix} -3 & \boxed{-12} & \boxed{6} \\ \boxed{-1} & 3 & \boxed{2} \\ \boxed{-11} & \boxed{-9} & 1 \end{bmatrix}$$

$$\text{adjoint of } A = \begin{bmatrix} -3 & \boxed{-1} & -11 \\ \boxed{-12} & 3 & \boxed{-9} \\ 6 & \boxed{2} & 1 \end{bmatrix}$$

B. Solve the Following

Q.1. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by the elementary row transformations.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

$\therefore |A| = 1(7-20) - 2(7-10) + 3(4-2) = -13+6+6 = -1 \neq 0$

$\therefore A^{-1}$ exists.

Consider $AA^{-1} = I$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $R_2 \rightarrow (-1)R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $R_1 \rightarrow R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $R_1 \rightarrow R_1 - 7R_3$ and $R_2 \rightarrow R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

Q.2. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ using adjoint method.

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$\therefore |A| = 1(0-2) - 2(3-0) + 1(6-0) = -2-6+6 = -2 \neq 0$

$\therefore A^{-1}$ exists.

$A_{11} = (-1)^{1+1} M_{11} = (1)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0-2 = -2$

$A_{12} = (-1)^{1+2} M_{12} = (-1)\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = -(3-0) = -3$

$A_{13} = (-1)^{1+3} M_{13} = (1)\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6-0 = 6$

$A_{21} = (-1)^{2+1} M_{21} = (-1)\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = -(2-2) = 0$

$A_{22} = (-1)^{2+2} M_{22} = (1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (1-0) = 1$

$A_{23} = (-1)^{2+3} M_{23} = (-1)\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2$

$A_{31} = (-1)^{3+1} M_{31} = (1)\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2-0 = 2$

$A_{32} = (-1)^{3+2} M_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -(1-3) = 2$

$A_{33} = (-1)^{3+3} M_{33} = (1)\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = (0-6) = -6$

$\therefore$ Cofactor matrix $= \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & -2 \\ 2 & 2 & -6 \end{bmatrix}$

$\therefore$ adj $A = [\text{Cofactor matrix}]^T = \begin{bmatrix} -2 & 0 & 2 \\ -3 & 1 & 2 \\ 6 & -2 & -6 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{-2}\begin{bmatrix} -2 & 0 & 2 \\ -3 & 1 & 2 \\ 6 & -2 & -6 \end{bmatrix}$

Q.3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, Show that $A^2 - 5A + 7I = 0$. Hence find A^{-1}.

Let $A^2 - 5A + 7I = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\therefore \boxed{A^2 - 5A + 7I = 0}$ —— (1)

Now $|A| = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 6-(-1) = 7 \neq 0$

$\therefore A^{-1}$ exists.

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \cdot 2 = 2$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \cdot (-1) = 1$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \cdot 1 = -1$

$A_{22} = (-1)^{2+2} M_{22} = 1 \cdot (3) = 3$

$\therefore$ Cofactor matrix $= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

$\therefore \text{Adj}(A) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{Adj}(A) = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Sign of Teacher :