4. DEFINITE INTEGRATION

Let us Study

- Definite integral as limit of sum.
- Fundamental theorem of integral calculus.
- Methods of evaluation and properties of definite integral.

4. 1 Definite integral as limit of sum :

In the last chapter, we studied various methods of finding the primitives or indefinite integrals of given function. We shall now interprete the definite integrals denoted by $\int_a^b f(x)\, dx$, read as the integral from a to b of the function $f(x)$ with respect to x. Here $a < b$, are real numbers and $f(x)$ is definited on $[a, b]$. At present, we assume that $f(x) \geq 0$ on $[a, b]$ and $f(x)$ is continuous.

$\int_a^b f(x)\, dx$ is defined as the area of the region bounded by $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. If $g(x)$ is the primitive of $f(x)$ then the area is $g(b) - g(a)$.

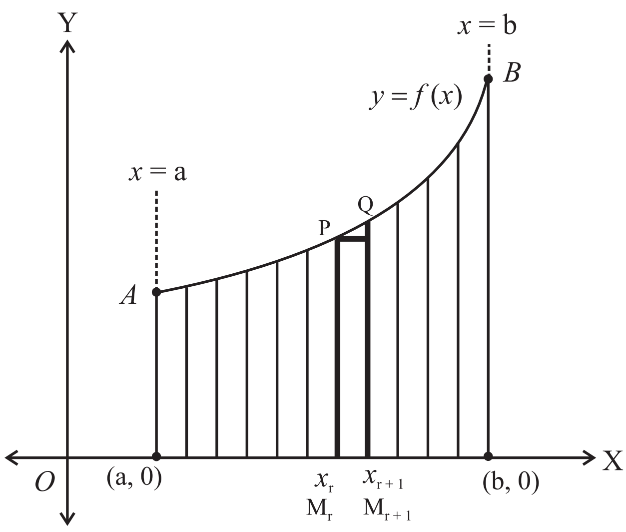

Fig. 4.1

The reason of the above definition will be clear from the figure 4.1. and the discussion that follows here. We are using the mean value theorem learnt earlier. Divide the interval $[a, b]$ into a equal parts by

$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$

Draw the curve $y = f(x)$ in $[a, b]$ and divide the interval $[a, b]$ into n equal parts by

$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$

Divide the region whose area is measured into their strips as above.

Note that, the area of each strip can be approximated by the area of a rectangle $M_r\, M_{r+1}$ QP as shown in the figure 4.1, which is $(x_{r-1} - x_r) \times f(\mathrm{T})$ where T is a point on the curve $y = f(x)$ between P and Q.

The mean value theorem states that if $g(x)$ is the primitive of $f(x)$,

$$g(x_{r+1}) - g(x_r) = (x_{r+1} - x_r) \cdot f(t_r) \qquad \text{where } x_r < t_r < x_{r+1}.$$

Now we can replace $f(\mathrm{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaded region as $\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(\mathrm{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaed region as

$$\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n=1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$$

Thus taking limit as $n \to \infty$

$$\begin{aligned} g(b) - g(a) &= \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r) \\ &= \lim_{n\to\infty} \mathrm{S}_n \\ &= \int_a^b f(x)\, dx \end{aligned}$$

The word 'to integrate' means 'to find the sum of'. The technique of integration is very useful in finding plane areas, length of arcs, volume of solid revolution etc...

SOLVED EXAMPLES

Ex. 1 : $\int_1^2 (2x + 5)\, dx$

Solution : Given, $\int_1^2 (2x + 5)\, dx = \int_a^b f(x)\, dx$

$f(x) = 2x + 5 \qquad a = 1 \,;\, b = 2$

$$\begin{aligned} \Rightarrow \quad f(a + rh) &= f(1 + rh) \\ &= 2(1 + rh) + 5 \\ &= 2 + 2rh + 5 \\ &= 7 + 2rh \end{aligned}$$

and $h = \dfrac{b - a}{n}$

$h = \dfrac{2 - 1}{n}$

$\therefore \quad nh = 1$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a + rh)$

$$\therefore \quad \int_1^2 (2x+5)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\cdot(7+2rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (7h + 2rh^2)$$

$$= \lim_{n\to\infty} \left(7h\sum_{r=1}^{n} 1 + 2h^2\sum_{r=1}^{n} r\right)$$

$$= \lim_{n\to\infty} \left[7h\cdot(n) + 2h^2\left(\frac{n\,(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[7nh + h^2n^2\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[7\,(1) + (1)^2\left(1+\frac{1}{n}\right)\right]$$

$$= 7 + 1\,(1+0) = 8$$

Ex. 2 : $\int_2^3 7^x \;\; dx$

Solution : Given, $\int_2^3 7^x \;\; dx = \int_a^b f(x)\,dx$

$f(x) = 7^x \qquad a = 2\,;\, b = 3$

$\Rightarrow \quad f(a+rh) = f(1+rh) \qquad$ and $\qquad h = \dfrac{b-a}{n}$

$= 7^{2+rh} \qquad h = \dfrac{3-2}{n} \qquad \therefore \qquad nh = 1$

$= 7^2\cdot 7^{rh}$

We know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\cdot f(a+rh)$

$$\therefore \quad \int_1^3 7^x \;\; dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\cdot\left(7^2\cdot 7^{r\cdot h}\right)$$

$$= \lim_{n\to\infty} 7^2\cdot \sum_{r=1}^{n} h\cdot 7^{r\cdot h}$$

$$= \lim_{n\to\infty} 7^2\cdot h\cdot\left[7^h + 7^{2h} + 7^{3h} + 7^{4h} + \ldots + 7^{nh}\right]$$

$$= \lim_{n\to\infty} 7^2\cdot h\cdot\left(\frac{7^h\left[(7^h)^n - 1\right]}{7^h - 1}\right) = \lim_{n\to\infty} 7^2\cdot\left(\frac{7^h\,(7^{nh}-1)}{\frac{7^h-1}{h}}\right)$$

$$= \lim_{n\to\infty} 7^2\cdot\left(\frac{7^h\,(7^{(1)}-1)}{\frac{7^h-1}{h}}\right)$$

$$= \frac{7^2\cdot\, 7^0\cdot\,(7-1)}{\log 7} = \frac{(49)(1)(6)}{\log 7} = \frac{294}{\log 7}$$

Ex. 3 : $\int_0^4 (x - x^2) \; dx$

Solution : $\int_0^4 (x - x^2) \; dx = \int_a^b f(x)\, dx$

$f(x) = x - x^2$ $\qquad a = 0 \; ; \; b = 4$

$\Rightarrow \quad f(a + rh) = f(0 + rh)$ $\qquad$ and $\qquad h = \dfrac{b-a}{n}$

$= f(rh)$ $\qquad h = \dfrac{4-0}{n}$

$= (rh) - (rh)^2$

$= rh - r^2h^2$ $\qquad \therefore \qquad nh = 4$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \qquad \int_0^4 (x - x^2)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \left(rh - r^2h^2\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(rh^2 - r^2h^3\right)$$

$$= \lim_{n\to\infty} \left(h^2 \cdot \sum_{r=1}^{n} r - h^3 \cdot \sum_{r=1}^{n} r^2\right)$$

$$= \lim_{n\to\infty} \left[h^2\left(\frac{n(n+1)}{2}\right) - h^3\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{h^2 \cdot n \cdot n\left(1 + \frac{1}{n}\right)}{2} - \frac{h^3 \cdot n \cdot n\left(1 + \frac{1}{n}\right) n\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(nh)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(nh)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(4)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(4)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \frac{(4)^2 \cdot (1 + 0)}{2} - \frac{(4)^3(1 + 0)(2 + 0)}{6}$$

$$= 8 - \frac{(64)(2)}{6}$$

$$= -\frac{40}{3}$$

Ex. 4 : $\int_0^{\pi/2} \sin x \; dx$

Solution : $\int_0^{\pi/2} \sin x \; dx = \int_0^{\pi/2} f(x)\, dx$

$$f(x) = \sin x \qquad a = 0 \;;\; b = \frac{\pi}{2}$$

$$\Rightarrow \quad f(a + rh) = \sin(a + rh)$$

$$= \sin(0 + rh) \qquad \text{and} \qquad h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{n}$$

$$= \sin rh \qquad \therefore \quad nh = \frac{\pi}{2}$$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \quad \int_0^{\pi/2} \sin x \; dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \sum_{r=1}^{n} \sin rh$$

$$= \lim_{n\to\infty} h \cdot [\sin h + \sin 2h + \sin 3h + \ldots + \sin nh] \qquad \ldots \text{(I)}$$

Consider,

$$\sum_{r=1}^{n} \sin rh = \sin h + \sin 2h + \sin 3h + \ldots + \sin nh$$

$$= 2\sin\frac{h}{2} \cdot \sin h + 2\sin\frac{h}{2} \cdot \sin 2h + 2\sin\frac{h}{2} \cdot \sin 3h + \ldots + 2\sin\frac{h}{2} \cdot \sin nh$$

$$\because \quad 2\sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin\frac{h}{2} \cdot \sum_{r=1}^{n} \sin rh = \left[\left(\cos\frac{h}{2} - \cos\frac{3h}{2}\right) + \left(\cos\frac{3h}{2} - \cos\frac{5h}{2}\right) + \left(\cos\frac{5h}{2} - \cos\frac{7h}{2}\right) + \ldots\right.$$

$$\left. + \ldots + \left(\cos\left(\frac{2n-1}{2}\right)h - \left(\cos\left(\frac{2n+1}{2}\right)h\right)\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2n+1}{2}\right)h\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2nh}{2} + \frac{h}{2}\right)\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{\pi}{2} + \frac{h}{2}\right)\right] \qquad \because \quad nh = \frac{\pi}{2}$$

$$= \left(\cos\frac{h}{2} + \sin\frac{h}{2}\right)$$

$$\therefore \quad \sum_{r=1}^{n} \sin rh = \frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}$$

Now from I,

$$\int_0^{\pi/2} \sin x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}\right]$$

$$\because \quad nh = \frac{\pi}{4} \text{ as } n\to\infty \Rightarrow h\to 0 \left(\frac{1}{n} \to 0\right)$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{\frac{2\cdot\sin\frac{h}{2}}{h}}\right]$$

$$= \frac{\cos 0 + \sin 0}{\left(\frac{1}{2}\right)}$$

$$= \frac{1+0}{2\cdot\frac{1}{2}} = 1$$

$$\therefore \quad \int_0^{\pi/2} \sin x \quad dx = 1$$

EXERCISE 4.1

I. Evaluate the following integrals as limit of sum.

(1) $\int_1^3 (3x - 4)\, dx$ (2) $\int_0^4 x^2\, dx$ (3) $\int_0^2 e^x\, dx$

(4) $\int_0^2 (3x^2 - 1)\, dx$ (5) $\int_1^3 x^3\, dx$

4.2 Fundamental theorem of integral calculus :

Let f be the continuous function defined on $[a, b]$ and if $\int f(x)\,dx = g(x) + c$

then
$$\int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$$
$$= [(g(b) + c) - (g(a) + c)]$$
$$= g(b) + c - g(a) - c$$
$$= g(b) - g(a)$$

Thus $\int_a^b f(x)\,dx = g(b) - g(a)$

Ex. :
$$\int_2^5 (x^2 - x)\,dx = \left[\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_2^5$$
$$= \left[\left(\frac{5^3}{3} - \frac{5^2}{2}\right) - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\right]$$
$$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2}$$
$$= \frac{117}{3} - \frac{21}{2} = \frac{234 - 83}{6}$$
$$\therefore \int_2^5 (x^2 - x)\,dx = \frac{151}{3}$$

In $\int_a^b f(x)\,dx$ a is called as a lower limit and b is called as an upper limit.

Now let us discuss some fundamental properties of definite integration.

These properties are very useful in evaluation of the definite integral.

4.2.1

Property I : $\int_a^a f(x)\,dx = 0$

Let $\int f(x)\,dx = g(x) + c$
$$\therefore \int_a^a f(x)\,dx = \left[g(x) + c\right]_a^a$$
$$= [(g(a) + c) - (g(a) + c)]$$
$$= 0$$

Property II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Let $\int f(x)\,dx = g(x) + c$
$$\therefore \int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$$
$$= [(g(b) + c) - (g(a) + c)]$$
$$= g(b) - g(a)$$
$$= -[g(a) - g(b)]$$
$$= -\int_b^a f(x)\,dx$$

Thus $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex.
$$\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3$$
$$= \frac{3^2}{2} - \frac{1^2}{2} = \frac{9}{2} - \frac{1}{2} = 4$$

Ex.
$$\int_3^1 x\,dx = \left[\frac{x^2}{2}\right]_3^1$$
$$= \frac{1^2}{2} - \frac{3^2}{2} = \frac{1}{2} - \frac{9}{2} = -4$$

Property III : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Let $\int f(x)\,dx = g(x) + c$

L.H.S. : $\int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) - g(a)$ . . . . . (i)

R.H.S. : $\int_a^b f(t)\,dt = \left[g(t) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) - g(a)$ . . . . . (ii)

from (i) and (ii)

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

i.e. definite integration is independent of the variable.

Ex. $\int_{\pi/6}^{\pi/3} \cos x \;\; dx = \left[\sin x\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2}$

Ex. $\int_{\pi/6}^{\pi/3} \cos t \;\; dt = \left[\sin t\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2}$

Property IV : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ i.e. $c \in [a, b]$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= \left[g(x) + c\right]_a^c + \left[g(x) + c\right]_c^b$

$= [(g(c) + c) - (g(a) + c)] + [(g(b) + c) - (g(c) + c)]$

$= g(c) + c - g(a) - c + g(b) + c - g(c) - c$

$= g(b) - g(a)$

$= \left[g(x) + c\right]_a^b$

$= \int_a^b f(x)\,dx$: L.H.S.

Thus $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Ex. : $\int_{-1}^{5}(2x+3)\,dx = \int_{-1}^{3}(2x+3)\,dx + \int_{3}^{5}(2x+3)\,dx$

L.H.S. : $\int_{-1}^{5}(2x+3)\,dx$

$$= \left[2\frac{x^2}{2} + 3x\right]_{-1}^{5}$$

$$= \left[x^2 + 3x\right]_{-1}^{5}$$

$$= [(5)^2 + 3\,(5)] - [(-1)^2 + 3\,(-1)]$$

$$= (25 + 15) - (1 - 3)$$

$$= 40 + 2 = 42$$

R.H.S. : $\int_{-1}^{3}(2x+3)\,dx + \int_{3}^{5}(2x+3)\,dx$

$$= \left[x^2 + 3x\right]_{-1}^{3} + \left[x^2 + 3x\right]_{3}^{5}$$

$$= [((3)^2 + 3\,(3)) - ((-1)^2 + 3\,(-1))] + [((5)^2 + 3\,(5)) - ((3)^2 + 3\,(3))]$$

$$= [(9+9)-(1-3)]+[(25+15)-(9-9)]$$

$$= 18 + 2 + 40 - 18$$

$$= 42$$

Property V : $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_{a}^{b} f(a+b-x)\,dx$

put $a+b-x = t$ i.e. $x = a+b-t$

$\therefore$ $-dx = dt \Rightarrow dx = -dt$

As $x \to a \Rightarrow t \to b$ and $x \to b \Rightarrow t \to a$

therefore $= \int_{b}^{a} f(t)\,(-dt)$

$$= -\int_{b}^{a} f(t)\,dt$$

$$= \int_{a}^{b} f(t)\,dt \ldots \left(\because \int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx\right)$$

$= \int_{a}^{b} f(x)\,dx$ . . . as definite integration is independent of the variable.

= L. H. S.

Thus $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

Ex. :

$\int_{\pi/6}^{\pi/3} \sin^2 x \;\; dx$

$$I = \int_{\pi/6}^{\pi/3} \sin^2 x \;\; dx \quad \ldots \text{(i)}$$

$$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right) dx$$

$$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_{\pi/6}^{\pi/3} \cos^2 x \;\; dx \quad \ldots \text{(ii)}$$

adding (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} \sin^2 x \;\; dx + \int_{\pi/6}^{\pi/3} \cos^2 x \;\; dx$$

$$2I = \int_{\pi/6}^{\pi/3} (\sin^2 x + \cos^2 x) \;\; dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1 \;\; dx = \left[\, x \,\right]_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \qquad \therefore \quad I = \frac{\pi}{12}$$

$$\int_{\pi/6}^{\pi/3} \sin^2 x \;\; dx = \frac{\pi}{12}$$

Property VI : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_0^a f(a-x)\,dx$

put $a - x = t$ i.e. $x = a - t$

$\therefore\ -dx = dt \Rightarrow dx = -dt$

As x varies from 0 to a, t varies from a to 0

therefore I $= \int_a^0 f(t)\,(-dt)$

$= -\int_a^0 f(t)\,dt$

$= \int_0^a f(t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$

$= \int_0^a f(x)\,dx$ ... as definite integration is independent of the variable.

$=$ L. H. S.

Thus

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Ex. : $\int_0^{\pi/4} \log(1+\tan x)\,dx$

Let $\int_0^{\pi/4} \log(1+\tan x)\,dx$ ... (i)

$$\mathrm{I} = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\,dx$$

$$= \int_0^{\pi/4} (\log 2)\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$\mathrm{I} = (\log 2)\int_0^{\pi/4} 1\,dx - \mathrm{I} \quad \text{...by eq. (i)}$$

$$\mathrm{I}+\mathrm{I} = (\log 2)\Big[x\Big]_0^{\pi/4}$$

$$2\mathrm{I} = (\log 2)\left[\frac{\pi}{4}-0\right]$$

$$\therefore\ \mathrm{I} = \frac{\pi}{8}(\log 2)$$

Thus

$$\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}(\log 2)$$

Property VII :

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

R.H.S. : $\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$$= I_1 + I_2 \qquad \ldots \text{(i)}$$

Consider $I_2 = \int_0^a f(2a-x)\,dx$

put $2a - x = t$ i.e. $x = 2a - t$

$\therefore\ -1\,dx = 1\,dt \Rightarrow dx = -\,dt$

As x varies from 0 to $2a$, t varies from $2a$ to 0

$$I = \int_{2a}^{a} f(t)\,(-\,dt)$$

$$= -\int_{2a}^{a} f(t)\,dt$$

$$= \int_0^{2a} f(t)\,dt \ldots\left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^{2a} f(x)\,dx \ldots\left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

$$\therefore \int_0^a f(x)\,dx = \int_0^{2a} f(x)\,dx$$

from eq. (i)

$$\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(x)\,dx$$

$$= \int_0^{2a} f(x)\,dx : \text{L.H.S}$$

Thus,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Property VIII :

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \qquad \text{, if } f(x) \text{ is odd function}$$

$f(x)$ even function if $f(-x) = f(x)$

and $f(x)$ odd function if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \ldots \text{(i)}$$

Consider $\int_{-a}^{0} f(x)\,dx$

put $x = -t$ $\therefore$ $dx = -dt$

As x varies from $-a$ to 0, t varies from a to 0

$$I = \int_a^0 f(-t)\,(-dt) = -\int_a^0 f(-t)\,dt$$

$$= \int_0^a f(-t)\,dt \ldots\left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^a f(-x)\,dx \ldots\left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

Equation (i) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a [f(-x) + f(x)]\,dx$$

If $f(x)$ is odd function then $f(-x) = -f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 0$$

If $f(x)$ is even function then $f(-x) = f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$$

Hence :

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \qquad \text{, if } f(x) \text{ is odd function}$$

Ex. :

1. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx$

Let $f(x) = x^3 \sin^4 x$

$f(-x) = (-x)^3 [\sin(-x)]^4 = -x^3 [-\sin x]^4 = -x^3 \sin^4 x$

$= -f(x)$

$f(x)$ is odd function.

$\therefore \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx = 0$

2. $\int_{-1}^{1} \frac{x^2}{1+x^2} dx$

Let $f(x) = \frac{x^2}{1+x^2}$

$f(-x) = \frac{(-x)^2}{1+(-x)^2}$

$= \frac{x^2}{1+x^2}$

$= f(x)$

$f(x)$ is even function.

$$\int_{-1}^{1} \frac{x^2}{1+x^2} dx = 2\int_{0}^{1} \frac{x^2}{1+x^2} dx$$

$$= 2\int_{0}^{1} \frac{1+x^2-1}{1+x^2} dx$$

$$= 2\int_{0}^{1} \left[1 - \frac{1}{1+x^2}\right] dx$$

$$= 2\left[x - \tan^{-1}x\right]_0^1$$

$$= 2\left\{(1-\tan^{-1}x) - (0-\tan^{-1}x)\right\}$$

$$= 2\left\{1 - \frac{\pi}{4} - 0\right\}$$

$$= 2\left(1-\frac{\pi}{4}\right) = \left(\frac{4-\pi}{2}\right)$$

$$\therefore \int_{-1}^{1} \frac{x^2}{1+x^2} dx = \frac{4-\pi}{2}$$

SOLVED EXAMPLES

Ex. 1 : $\int_{1}^{3} \frac{1}{\sqrt{2+x}+\sqrt{x}} dx$

Solution : $= \int_{1}^{3} \left(\frac{1}{\sqrt{2+x}+\sqrt{x}}\right)\left(\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}\right) dx$

$$= \int_{1}^{3} \left(\frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}\right) dx$$

$$= \frac{1}{2}\int_{1}^{3} (\sqrt{2+x}-\sqrt{x})\, dx$$

$$= \frac{1}{2}\left[\frac{(2+x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3}\left[(2+x)^{\frac{3}{2}} - (x)^{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3}\left\{\left[(2+3)^{\frac{3}{2}} - (3)^{\frac{3}{2}}\right] - \left[(2+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 3^{\frac{3}{2}} - 3^{\frac{3}{2}} + 1^{\frac{3}{2}}\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right\}$$

$$\therefore \int_{1}^{3} \frac{1}{\sqrt{2+x}+\sqrt{x}} dx = \frac{1}{3}\left[5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right]$$

Ex. 2 : $\int_0^{\pi/2} \sqrt{1-\cos 4x}\; dx$

Solution : Let $I = \int_0^{\pi/2} \sqrt{1-\cos 4x}\; dx$

$$I = \int_0^{\pi/2} \sqrt{2\sin^2 2x}\; . dx$$

$$\left(\because 1-\cos A = 2\sin^2 \frac{A}{2}\right)$$

$$= \sqrt{2}\int_0^{\pi/2} \sin 2x\; dx$$

$$= \sqrt{2}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{2}\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{\sqrt{2}}{2}\; [\cos \pi - \cos 0]$$

$$= -\frac{\sqrt{2}}{2}\; (-1-1) = \sqrt{2}$$

$$\therefore \quad \int_0^{\pi/2} \sqrt{1-\cos 4x}\; dx = \sqrt{2}$$

Ex. 3 : $\int_0^{\pi/2} \cos^3 x\; dx$

Solution : Let $I = \int_0^{\pi/2} \cos^3 x\; dx$

$$= \int_0^{\pi/2} \frac{1}{4}\,[\cos 3x + 3\cos x]\; dx$$

$$= \frac{1}{4}\left[\sin 3x \cdot \frac{1}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{1}{3}\sin 3\frac{\pi}{2} + 3\sin\frac{\pi}{2}\right) - \left(\frac{1}{3}\sin 3\,(0) + 3\sin (0)\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}\sin\frac{3\pi}{2} + 3\sin\frac{\pi}{2} - \frac{1}{3}\sin 0 + 3\sin 0\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}(-1) + 3\,(1) \; - 0\right]$$

$$= \frac{1}{4}\left[-\frac{1}{3} + 3\right] = \frac{1}{4}\left[\frac{8}{3}\right] = \frac{2}{3}$$

$$\therefore \quad \int_0^{\pi/2} \cos^3 x\; dx = \frac{2}{3}$$

Ex. 4 : $\int_0^{\pi/4} \frac{\sec^2 x}{2\tan^2 x + 5\tan x + 1}\; dx$

Solution : Let $I = \int_0^{\pi/4} \frac{\sec^2 x}{2\tan^2 x + 5\tan x + 1}\; dx$

put $\tan x = t \qquad \therefore\; \sec^2 x\; dx = 1\; dt$

As x varies from 0 to $\frac{\pi}{4}$

t varies from 0 to 1

$$= \int_0^1 \frac{1}{2t^2 + 4t + 1}\; dt$$

$$= \frac{1}{2}\int_0^1 \frac{1}{t^2 + 2t + \frac{1}{2}}\; dt$$

$$= \frac{1}{2}\int_0^1 \frac{1}{t^2 + 2t + 1 - 1 + \frac{1}{2}}\; dt$$

$$= \frac{1}{2}\int_0^1 \frac{1}{(t+1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\; dt$$

$$= \frac{1}{2}\,\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\left[\log\left[\frac{(t+1) - \frac{1}{\sqrt{2}}}{(t+1) + \frac{1}{\sqrt{2}}}\right]\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\log\left[\left(\frac{\sqrt{2}\,t + \sqrt{2} - 1}{\sqrt{2}\,t + \sqrt{2} + 1}\right)\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{\sqrt{2}\,(1) + \sqrt{2} - 1}{\sqrt{2}\,(1) + \sqrt{2} + 1}\right) - \log\left(\frac{\sqrt{2}\,(0) + \sqrt{2} - 1}{\sqrt{2}\,(0) + \sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1}\right) - \log\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\log\left[\left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1}\right) \div \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\log\left[\frac{3 + \sqrt{2}}{3 - \sqrt{2}}\right]$$

Ex. 5 : $\int_1^2 \frac{\log x}{x^2}\, dx$

Solution : Let $I = \int_1^2 (\log x)\left(\frac{1}{x^2}\right) dx$

$$= \left[(\log x)\int \frac{1}{x^2}\, dx\right]_1^2 - \int_1^2 \frac{d}{dx}\log x\left(\int \frac{1}{x^2}\, dx\right) dx$$

$$= \left[(\log x)\left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x}\left(-\frac{1}{x}\right) dx$$

$$= \left[-\frac{1}{x}\log x\right]_1^2 + \int_1^2 \frac{1}{x^2}\, dx$$

$$= \left[-\frac{1}{x}\log x\right]_1^2 + \left[-\frac{1}{x}\right]_1^2$$

$$= \left[\left(-\frac{1}{2}\log 2\right) - \left(-\frac{1}{1}\log 1\right)\right] + \left[\left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)\right]$$

$$= -\frac{1}{2}\log 2 - 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2}\log 2 \qquad \because \log 1 = 0$$

$$\therefore \int_1^2 \frac{\log x}{x^2}\, dx = \frac{1}{2}\left(1 - \log 2\right)$$

Ex. 6 : $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx$

Solution : Let $I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx$

$$= \int_0^{\pi/2} \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)}\, dx$$

$$= \int_0^{\pi/2} \frac{\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]}{2\left[\cos\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right] dx = \int_0^{\pi/2} \left[1 - \tan\left(\frac{x}{2}\right)\right] dx$$

$$= \frac{1}{2}\left[x - \log\left(\sec\frac{x}{2}\right)\frac{1}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 2\log\left(\sec\frac{\pi}{4}\right) - (0 - 2\log\sec 0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 2\log\sqrt{2} - 0 + 2(0)\right] \qquad = \frac{1}{2}\left[\frac{\pi}{2} - 2\log\sqrt{2}\right] \qquad = \frac{\pi}{4} - \log\sqrt{2}$$

$$\therefore \qquad \int_0^{\pi/2} \frac{\sec^2 x}{1 + \cos x + \sin x}\, dx = \frac{\pi}{4} - \log\sqrt{2}$$

Ex. 7 : $\displaystyle\int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}}\, dx$

Solution : Let $\displaystyle I = \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}}\, dx$

put $x = \sin\theta \quad \therefore \quad 1\, dx = \cos\theta\, d\theta$

As x varies from 0 to $\frac{1}{2}$, θ varies from 0 to $\frac{\pi}{6}$

$$= \int_0^{\pi/6} \frac{\cos\theta}{(1 - 2\sin^2\theta)\sqrt{1 - \sin^2\theta}}\, d\theta = \int_0^{\pi/6} \frac{\cos\theta}{(\cos 2\theta)\sqrt{\cos^2\theta}}\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos 2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \sec 2\theta\, d\theta$$

$$= \left[\log(\sec 2\theta + \tan 2\theta)\ \frac{1}{2}\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\log\left(\sec 2\left(\frac{\pi}{6}\right) + \tan 2\left(\frac{\pi}{6}\right)\right) - \log(\sec 0 + \tan 0)\right]$$

$$= \frac{1}{2}\left[\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log(1 + 0)\right] \qquad \because \log 1 = 0$$

$$= \frac{1}{2}\left[\log(2 + \sqrt{3}) - 0\right]$$

$$= \frac{1}{2}\log(2 + \sqrt{3})$$

$$\therefore \quad \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}}\, dx = \frac{1}{2}\log(2 + \sqrt{3})$$

Ex. 8 : $\int_0^2 \frac{2^x}{2^x(1+4^x)}\, dx$

Solution : Let $I = \int_0^2 \frac{2^x}{2^x(1+4^x)}\, dx$

put $2^x = t$ $\quad \therefore \quad 2^x \cdot \log 2 \; dx = 1\, dt$

As x varies from 0 to 2, t varies from 1 to 4

$$= \int_1^4 \frac{\frac{1}{\log 2}}{t(1+t^2)}\, dt$$

$$= \frac{1}{\log 2} \int_1^4 \frac{1}{t(1+t^2)}\, dt$$

$$= \frac{1}{\log 2} \int_1^4 \frac{1+t^2-t^2}{t(1+t^2)}\, dt$$

may be solved by method of partial fraction

$$= \frac{1}{\log 2} \int_1^4 \left[\frac{1+t^2}{t(1+t^2)} - \frac{t^2}{t(1+t^2)}\right] dt$$

$$= \frac{1}{\log 2} \int_1^4 \left[\frac{1}{t} - \frac{t}{1+t^2}\right] dt$$

$$= \frac{1}{\log 2} \left[\int_1^4 \frac{1}{t}\, dt - \frac{1}{2}\int_1^4 \frac{2t}{1+t^2}\, dt\right]$$

$$= \frac{1}{\log 2} \left[\log(t) - \frac{1}{2}\log(1+t^2)\right]_1^4$$

$$= \frac{1}{\log 2} \left[\left(\log 4 - \frac{1}{2}\log 17\right) - \left(\log 1 - \frac{1}{2}\log 2\right)\right]$$

$$= \frac{1}{\log 2} \left[\log 4 - \frac{1}{2}\log 17 + \frac{1}{2}\log 2\right]$$

$\because \log 1 = 0$

$$= \frac{1}{\log 2} \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$\therefore \int_0^2 \frac{2^x}{2^x(1+4^x)}\, dx = \frac{1}{(\log 2)} \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$= \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right)$$

Ex. 9 : $\int_{-1}^1 |5x-3|\, dx$

Solution : Let $I = \int_{-1}^1 |5x-3|\, dx$

$$|5x-3| = -(5x-3) \text{ for } (5x-3) < 0 \text{ i.e. } x < \frac{3}{5}$$

$$= (5x-3) \text{ for } (5x-3) > 0 \text{ i.e. } x > \frac{3}{5}$$

$$= \int_{-1}^{3/5} |5x-3|\, dx + \int_{3/5}^1 |5x-3|\, dx \quad = \int_{-1}^{3/5} -(5x-3)\, dx + \int_{3/5}^1 (5x-3)\, dx$$

$$= \left[-\left(5\frac{x^2}{2} - 3x\right)\right]_{-1}^{3/5} + \left[\left(5\frac{x^2}{2} - 3x\right)\right]_{3/5}^1 \quad = \left[3x - \frac{5}{2}x^2\right]_{-1}^{3/5} + \left[\frac{5}{2}x^2 - 3x\right]_{3/5}^1$$

$$= \left[\left(3\left(\frac{3}{5}\right) - \frac{5}{2}\left(\frac{3}{5}\right)^2\right) - \left(3(-1) - \frac{5}{2}(-1)^2\right)\right] + \left[\left(\frac{5}{2}(1)^2 - 3(1)\right) - \left(\frac{5}{2}\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right)\right)\right]$$

$$= \left[\left(\frac{9}{5}-\frac{9}{10}\right)-\left(-3-\frac{5}{2}\right)\right]+\left[\left(\frac{5}{2}-3\right)-\left(\frac{9}{10}-\frac{9}{5}\right)\right]$$

$$= \frac{9}{5}-\frac{9}{10}+3+\frac{5}{2}+\frac{5}{2}-3-\frac{9}{10}+\frac{9}{5} = 2\left(\frac{9}{5}-\frac{9}{10}+\frac{5}{2}\right) = 2\left(\frac{18-9+25}{5}\right) = \frac{34}{5}$$

$$\therefore \int_{-1}^{1} |5x-3|\, dx = \frac{34}{5}$$

Ex. 10 : $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt[3]{\tan x}}\, dx$

Solution : Let $\displaystyle I = \int_0^{\pi/2} \frac{1}{1+\sqrt[3]{\tan x}}\, dx$

$$= \int_0^{\pi/2} \left[\frac{1}{1+\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\, dx \qquad \ldots \text{(i)}$$

By property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt[3]{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt[3]{\sin\left(\frac{\pi}{2}-x\right)}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\, dx \qquad \ldots \text{(ii)}$$

adding (i) and (ii)

$$I + I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\, dx + \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\, dx$$

$$2I = \int_0^{\pi/2} 1\, dx$$

$$I = \frac{1}{2}\Big[\,x\,\Big]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2} \frac{1}{1+\sqrt[3]{\tan x}}\, dx = \frac{\pi}{4}$$

with the help of the above solved/ illustrative example verify whether the following examples evaluates their definite integrate to be equal to / as $\frac{\pi}{4}$

$\displaystyle\int_0^{\pi/2} \frac{1}{1+\cot^3 x}\, dx$; $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx$; $\displaystyle\int_0^{\pi/2} \frac{\sec x}{\sec x+\operatorname{cosec} x}\, dx$;

$\displaystyle\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\, dx$; $\displaystyle\int_0^{\pi/2} \frac{\operatorname{cosec}^{\frac{5}{2}} x}{\operatorname{cosec}^{\frac{5}{2}} x+\sec^{\frac{5}{2}} x}\, dx$

Ex. 11 : $\int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2}\, dx$

Solution : Let $I = \int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2}\, dx \quad \ldots (i)$

By property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_3^8 \frac{[11-(8+3-x)]^2}{[8+3-x]^2+[11-(8+3-x)]^2}\, dx = \int_3^8 \frac{[11-(11-x)]^2}{(11-x)^2+[11-(11-x)]^2}\, dx$$

$$= \int_3^8 \frac{x^2}{(11-x)^2+x^2}\, dx \quad \ldots (ii)$$

adding (i) and (ii)

$$I + I = \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2}\, dx + \int_3^8 \frac{x^2}{(11-x)^2+x^2}\, dx$$

$$2I = \int_3^8 \frac{(11-x)^2+x^2}{x^2+(11-x)^2}\, dx$$

$$I = \frac{1}{2}\int_3^8 1 \; dx$$

$$I = \frac{1}{2}\Big[\, x \,\Big]_3^8 = \frac{1}{2}[8-3] = \frac{5}{2}$$

$$\therefore \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2}\, dx = \frac{5}{2}$$

Note that : In general $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx = \frac{1}{2}(b-a)$

verify the generalisation for the following examples :

$\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\, dx$; $\qquad \int_2^7 \frac{x^3}{(9-x)^3+x^3}\, dx$;

$\int_4^9 \frac{x^{\frac{1}{4}}}{(13-x)^{\frac{1}{4}}+x^{\frac{1}{4}}}\, dx \qquad \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\, dx$

$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\operatorname{cosec} x}}\, dx$

Ex. 12 : $\int_0^{\pi} x \ \sin^2 x \ dx$

Solution :

Consider, $\quad I = \int_0^{\pi} x \ \sin^2 x \ dx \ldots$ (i)

$$I = \int_0^{\pi} (\pi - x) \ [\sin(\pi - x)]^2 x \ dx$$

$$I = \int_0^{\pi} (\pi - x) \ \sin^2 x \ dx$$

$$I = \int_0^{\pi} \pi \ \sin^2 x \ dx - \int_0^{\pi} x \ \sin^2 x \ dx$$

$$I = \pi \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \cdot dx - I \ \ldots \text{by (i)}$$

$$I + I = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \ dx$$

$$2I = \frac{\pi}{2}\left[x - \sin 2x \ \frac{1}{2}\right]_0^{\pi}$$

$$I = \frac{\pi}{4}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}[\pi] \qquad \because \sin 0 = 0; \sin 2\pi = 0$$

$$= \frac{\pi^2}{4}$$

$$\therefore \int_0^{\pi} x^2 \cdot \sin^2 x \ dx = \frac{\pi^2}{4}$$

Ex. 13 : Evaluate the integral $\int_0^{\pi} \cos^2 x \ dx$ using the result/ property.

Solution :

$$\int_0^{2a} f(x) \ dx = \int_0^{a} f(x) \ dx + \int_0^{a} f(2a - x) \ dx$$

Let, $I = \int_0^{\pi} \cos^2 x \ dx$

$$= \int_0^{2\left(\frac{\pi}{2}\right)} \cos^2 x \ dx$$

$$= \int_0^{\pi/2} \cos^2 x \ dx + \int_0^{\pi/2} \left[\cos\left(2\frac{\pi}{2} - x\right)\right]^2 dx$$

$$= \int_0^{\pi/2} \cos^2 x \ dx + \int_0^{\pi/2} \cos^2 x \ dx$$

$\because \ \cos(\pi - x) = -\cos x$

$$= 2 \cdot \int_0^{\pi/2} \cos^2 x \ dx$$

$$= \int_0^{\pi/2} (1 + \cos 2x) \ dx$$

$$= \left[x + \sin 2x \cdot \frac{1}{2}\right]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$

$$= \frac{\pi}{2} + 0 \qquad \because \sin 0 = 0; \sin \pi = 0$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \cos^2 x \ dx = \frac{\pi}{2}$$

Ex. 14 : $\displaystyle\int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x}\,dx$

Solution : Let $\displaystyle I = \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x}\,dx$

$$= \left[\left(\int_{-\pi}^{\pi} \frac{x}{1+\cos^2 x}\,dx\right) + \left(\int_{-\pi}^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx\right)\right]$$

The function $\dfrac{x}{1+\cos^2 x}$ is odd function and the function $\dfrac{x\ \sin x}{1+\cos^2 x}$ is even function.

$\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if $f(x)$ even function

$= 0$, if $f(x)$ is odd function

$$\therefore\ I = 0 + 2\int_{0}^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx$$

$$\therefore\ I = 2\int_{0}^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx \qquad \ldots (i)$$

$$= 2\int_{0}^{\pi} \frac{(\pi - x)\ \sin(\pi - x)}{1+[\cos(\pi - x)]^2}\,dx$$

$$= 2\int_{0}^{\pi} \frac{(\pi - x)\ \sin x}{1+(-\cos x)^2}\,dx$$

$$= 2\pi\int_{0}^{\pi} \frac{\pi\ \sin x - x\ \sin x}{1+\cos^2 x}\,dx$$

$$= 2\pi\int_{0}^{\pi} \frac{\sin x\ dx}{1+\cos^2 x} - 2\int_{0}^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\,dx$$

$$I = 2\pi\int_{0}^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I \ldots \text{by eq.(i)}$$

$$I + I = 2\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx \qquad \ldots (ii)$$

put $\cos x = t$ $\quad\therefore\quad -\sin x\ dx = +\,dt$

As varies from 0 to π, t varies from 1 to -1

$$2I = 2\pi\int_{-1}^{1} \frac{-1}{1+t^2}\,dt$$

$$I = \pi\ 2\int_{0}^{1} \frac{1}{1+t^2}\,dt \qquad \left(\text{where } \frac{1}{1+t^2} \text{ is even function.}\right)$$

$$I = 2\pi \left[\tan^{-1} t \right]_0^1$$

$$= 2\pi \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 2\pi \left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{2}$$

$$\therefore \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x}\, dx = \frac{\pi^2}{2}$$

Ex. 15 : $\int_0^3 x\,[\,x\,]\; dx$, where $[\,x\,]$ denote greatest integrate function not greater than x.

Solution : Let $I = \int_0^3 x\,[\,x\,]\; dx$

$$I = \int_0^1 x\,[\,x\,]\; dx + \int_1^2 x\,[\,x\,]\; dx + \int_2^3 x\,[\,x\,]\; dx$$

$$= \int_0^1 x\,(0)\; dx + \int_1^2 x\,(1)\; dx + \int_2^3 x\,(2)\; dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2 + \left[x^2\right]_2^3$$

$$= 0 + \left(\frac{4}{2} - \frac{1}{2}\right) + (9-4)$$

$$= \frac{3}{2} + 5 = \frac{13}{2}$$

$$\therefore \int_0^3 x\,[\,x\,]\; dx = \frac{13}{2}$$

EXERCISE 4.2

I. Evaluate :

(1) $\int_1^9 \frac{x+1}{\sqrt{x}}\, dx$

(2) $\int_2^3 \frac{1}{x^2+5x+6}\, dx$

(3) $\int_0^{\pi/4} \cot^2 x$

(4) $\int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x}\, dx$

(5) $\int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}}\, dx$

(6) $\int_0^1 \frac{x^2-2}{x^2+1}\, dx$

(7) $\int_0^{\pi/4} \sin 4x \sin 3x\, dx$

(8) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$

(9) $\int_0^{\pi/4} \sin^4 x\, dx$

(10) $\int_{-4}^{2} \frac{1}{x^2+4x+13}\, dx$

(11) $\int_0^4 \frac{1}{\sqrt{4x-x^2}}\, dx$

(12) $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}}\, dx$

(13) $\int_0^{\pi/2} x \sin x\, dx$

(14) $\int_0^1 x \tan^{-1} x\, dx$

(15) $\int_0^{\infty} x e^{-x}\, dx$

II. Evaluate :

(1) $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$

(2) $\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x + 4\tan x + 1} dx$

(3) $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

(4) $\int_0^{\pi/2} \sqrt{\cos x}\ \sin^3 x\ dx$

(5) $\int_0^{\pi/2} \frac{1}{5 + 4\cos x} dx$

(6) $\int_0^{\pi/4} \frac{\cos x}{4 - \sin^2 x} dx$

(7) $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$

(8) $\int_{-1}^{1} \frac{1}{a^2 e^x + b^2 e^{-x}} dx$

(9) $\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx$

(10) $\int_0^{\pi/4} \sec^4 x\ dx$

(11) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

(12) $\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)(1+\cos x)^2\ dx$

(13) $\int_0^{\pi/2} \sin 2x\ \tan^{-1}(\sin x)\ dx$

(14) $\int_{\frac{1}{\sqrt{2}}}^{1} \frac{(e^{\cos^{-1} x})(\sin^{-1} x)}{\sqrt{1-x^2}} dx$

(15) $\int_2^3 \frac{\cos(\log x)}{x} \cdot dx$

III. Evaluate :

(1) $\int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} dx$

(2) $\int_0^{\pi/2} \log \tan x\ dx$

(3) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

(4) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} dx$

(5) $\int_0^3 x^2 (3-x)^{\frac{5}{2}} dx$

(6) $\int_{-3}^{3} \frac{x^3}{9 - x^2} dx$

(7) $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2+\sin x}{2-\sin x}\right) dx$

(8) $\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$

(9) $\int_{-\pi/4}^{\pi/4} x^3\ \sin^4 x\ dx$

(10) $\int_0^1 \frac{\log(x+1)}{x^2+1} dx$

(11) $\int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2+4}} dx$

(12) $\int_{-a}^{a} \frac{x + x^3}{16 - x^2} dx$

(13) $\int_0^1 t^2 \sqrt{1-t}\ dt$

(14) $\int_0^{\pi} x\ \sin x\ \cos^2 x\ dx$

(15) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Note that :

To evaluate the integrals of the type $\int_0^{\pi/2} \sin^n x \ \ dx$ and $\int_0^{\pi/2} \cos^n x \ \ dx$, the results used are known as **'reduction formulae'** which are stated as follows :

$$\int_0^{\pi/2} \sin^n x \ \ dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5}\,\frac{2}{3}, \qquad \text{if } n \text{ is odd.}$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4}\,\frac{1}{2} \cdot \frac{\pi}{2}, \qquad \text{if } n \text{ is even.}$$

$$\int_0^{\pi/2} \cos^n x \ \ dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^n dx \qquad \text{. . . by property}$$

$$= \int_0^{\pi/2} [\sin x]^n \ dx$$

$$= \int_0^{\pi/2} \sin^n x \ \ dx$$

$$\int_0^{\pi/2} \sin^7 x \ \ dx = \frac{(7-1)}{7} \cdot \frac{(7-3)}{(7-2)} \cdot \frac{(7-5)}{(7-4)}$$

$$= \frac{(7-1)\cdot(7-3)\cdot(7-5)}{7\cdot(7-2)\cdot(7-4)}$$

$$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3} = \frac{16}{35}$$

$$\int_0^{\pi/2} \cos^8 x \ \ dx = \frac{(8-1)}{8} \cdot \frac{(8-3)}{(8-2)} \cdot \frac{(8-5)}{(8-4)} \cdot \frac{(8-7)}{(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{(8-1)\cdot(8-3)\cdot(8-5)\cdot(8-7)}{8\cdot(8-2)\cdot(8-4)\cdot(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

Let us Remember

- $\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n=1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$

  Thus taking limit as $n \to \infty$

  $$g(b) - g(a) = \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r) = \lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$$

- **Fundamental theorem of integral calculus :** $\int_a^b f(x)\,dx = g(b) - g(a)$

  Property I : $\int_a^a f(x)\,dx = 0$

  Property II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

  Property III : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

  Property IV : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ i.e. $c \in [a, b]$

  Property V : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

  Property VI : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

  Property VII : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

  Property VIII : $\int_{-a}^a f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx$, if $f(x)$ even function

  $= 0$, if $f(x)$ is odd function

  $f(x)$ even function if $f(-x) = f(x)$ and $f(x)$ odd function if $f(-x) = -f(x)$

- **'Reduction formulae'** which are stated as follows :

  $$\int_0^{\pi/2} \sin^n x \; dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5}\,\frac{2}{3}, \quad \text{if } n \text{ is odd.}$$

  $$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4}\,\frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even.}$$

  $$\int_0^{\pi/2} \cos^n x \; dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - 0\right)\right]^n dx = \int_0^{\pi/2} [\sin x]^n \; dx = \int_0^{\pi/2} \sin^n x \; dx$$

MISCELLANEOUS EXERCISE 4

(I) Choose the correct option from the given alternatives :

(1) $\int_2^3 \frac{dx}{x(x^3-1)} =$

(A) $\frac{1}{3}\log\left(\frac{208}{189}\right)$ (B) $\frac{1}{3}\log\left(\frac{189}{208}\right)$ (C) $\log\left(\frac{208}{189}\right)$ (D) $\log\left(\frac{189}{208}\right)$

(2) $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(1+\cos x)^2} =$

(A) $\frac{4-\pi}{2}$ (B) $\frac{\pi-4}{2}$ (C) $4-\frac{\pi}{2}$ (D) $\frac{4+\pi}{2}$

(3) $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\, dx =$

(A) $3+2\pi$ (B) $4-\pi$ (C) $2+\pi$ (D) $4+\pi$

(4) $\int_0^{\pi/2} \sin^6 x \cos^2 x\, dx =$

(A) $\frac{7\pi}{256}$ (B) $\frac{3\pi}{256}$ (C) $\frac{5\pi}{256}$ (D) $\frac{-5\pi}{256}$

(5) If $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \frac{k}{3}$, then k is equal to

(A) $\sqrt{2}\,(2\sqrt{2}-2)$ (B) $\frac{\sqrt{2}}{3}\left(2-2\sqrt{2}\right)$ (C) $\frac{2\sqrt{2}-2}{3}$ (D) $4\sqrt{2}$

(6) $\int_1^2 \frac{1}{x^2} e^{\frac{1}{x}}\, dx =$

(A) $\sqrt{e}+1$ (B) $\sqrt{e}-1$ (C) $\sqrt{e}\left(\sqrt{e}-1\right)$ (D) $\frac{\sqrt{e}-1}{e}$

(7) If $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx = a + \frac{b}{\log 2}$, then

(A) $a = e, b = -2$ (B) $a = e, b = 2$ (C) $a = -e, b = 2$ (D) $a = -e, b = -2$

(8) Let $I_1 = \int_e^{e^2} \frac{dx}{\log x}$ and $I_2 = \int_1^2 \frac{e^x}{x}\, dx$, then

(A) $I_1 = \frac{1}{3} I_2$ (B) $I_1 + I_2 = 0$ (C) $I_1 = 2I_2$ (D) $I_1 = I_2$

(9) $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\, dx =$

(A) 9 (B) $\frac{9}{2}$ (C) 0 (D) 1

(10) The value of $\int_{-\pi/4}^{\pi/4} \log\left(\frac{2+\sin\theta}{2-\sin\theta}\right) d\theta$ is

(A) 0 (B) 1 (C) 2 (D) π

(II) Evaluate the following :

(1) $\int_0^{\pi/2} \frac{\cos x}{3\cos x+\sin x}\, dx$

(2) $\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^3}\, d\theta$

(3) $\int_0^1 \frac{1}{1+\sqrt{x}}\, dx$

(4) $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x}\, dx$

(5) $\int_0^1 t^5\sqrt{1-t^2}\, dt$

(6) $\int_0^1 (\cos^{-1} x)^2\, dx$

(7) $\int_{-1}^1 \frac{1+x^3}{9-x^2}\, dx$

(8) $\int_0^{\pi} x\sin x\cos^4 x\, dx$

(9) $\int_0^{\pi} \frac{x}{1+\sin^2 x}\, dx$

(10) $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}\, dx$

(III) Evaluate :

(1) $\int_0^1 \left(\frac{1}{1+x^2}\right)\sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

(2) $\int_0^{\pi/2} \frac{1}{6-\cos x}\, dx$

(3) $\int_0^a \frac{1}{a^2+ax-x^2}\, dx$

(4) $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x+\cos x}\, dx$

(5) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

(6) $\int_0^{\pi/4} \frac{\cos 2x}{1+\cos 2x+\sin 2x}\, dx$

(7) $\int_0^{\pi/2} \left(2\log\sin x-\log\sin 2x\right)\, dx$

(8) $\int_0^{\pi} \left(\sin^{-1} x+\cos^{-1} x\right)^3 \sin^3 x\, dx$

(9) $\int_0^4 \left[\sqrt{x^2+2x+3}\right]^{-1} dx$

(10) $\int_{-2}^3 \left|x-2\right|\, dx$

(IV) Evaluate the following :

(1) If $\int_0^a \sqrt{x}\, dx = 2a \int_0^{\pi/2} \sin^3 x\, dx$ then find the value of $\int_a^{a+1} x\, dx$

(2) If $\int_0^k \frac{1}{2 + 8x^2}\, dx = \frac{\pi}{16}$ Find k.

(3) If $f(x) = a + bx + cx^2$, show that $\int_0^1 f(x)\, dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$.

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