UMass CMPSCI 383 (AI) HW6: Chapters 18 & 20

YOUR NAME HERE

Assigned: Nov 27 2017; Due: Dec 11 2017 @ 11:55 PM EST

Abstract

Submit a (.zip) file to both Moodle and Gradescope containing your latex (.tex) file and rendered pdf. All written HW responses should be done in latex (use sharelatex.com or overleaf.com). All code should be written in Python and should run on the edlab machines ([yourusername]@elnux[1,2,...].cs.umass.edu). You may NOT use `sklearn` for this assignment.

1 Regression (20 pts)

L2-regularized Linear Regression, also called "Ridge" Regression, adds the L2 norm of the weights to the loss function: $||w||^2 = \sum_d w_d^2$. In this problem, you will train a "Ridge" Regression model using gradient descent on a dataset of (X,y) pairs we have provided . We have provided you with template code for loading the dataset, training the model, and plotting the resulting fit (see `linearRegression.py`). The relevant loss and gradient for training the model are below.

$$L(y, f(x; w)) = \frac{1}{2}\sum_i (w^T x_i - y_i)^2 + \alpha||w||^2$$

$$\nabla_w L(y, f(x; w)) = \sum_i (w^T x_i - y_i)x_i + 2\alpha w = (X^T X + 2\alpha I)w - X^T y$$

$$\nabla_w L(y, f(x; w)) = 0 \Rightarrow w = (X^T X + 2\alpha I)^{-1} X^T y$$

In the previous homework, you used the pseudoinverse to exactly learn the utility of a given policy in Policy Iteration. As a reminder, the pseudoinverse is defined as

$$A^+ = (A^T A)^{-1} A^T = \texttt{np.linalg.pinv(A)}$$

In this problem, you'll need to compute the L2-regularized pseudoinverse yourself.

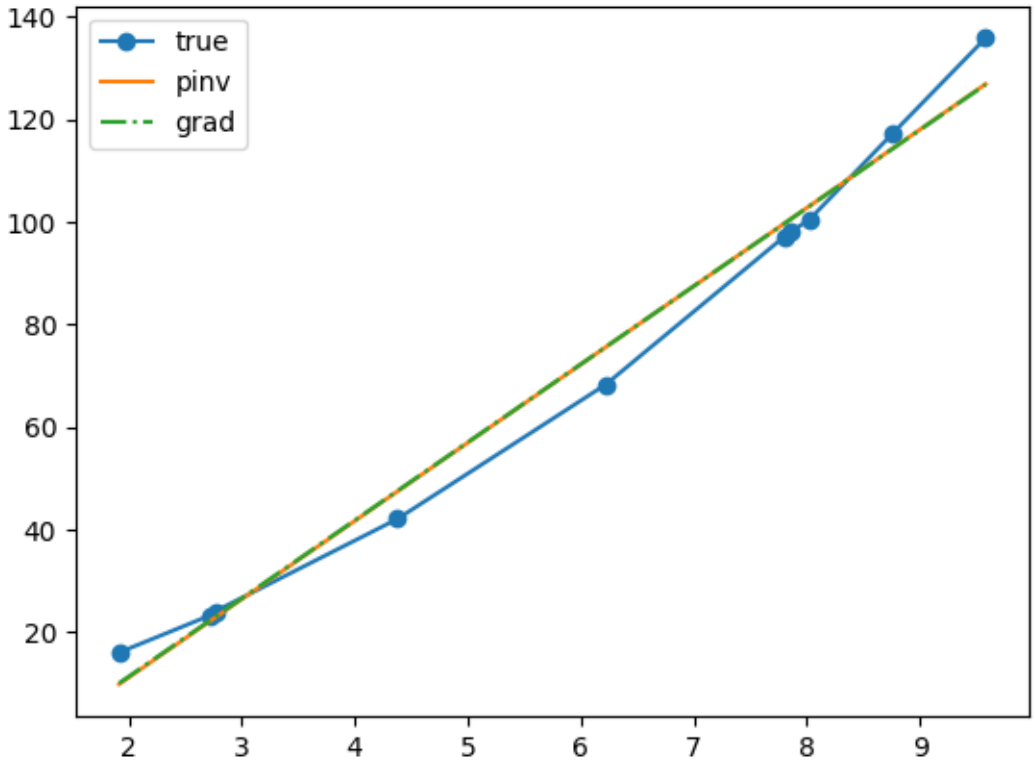
$$\tilde{A}_\alpha = A^T A + 2\alpha I = \texttt{np.dot(A.T,A) + 2*}\alpha\texttt{*np.eye(A.shape[1])}$$

$$A^+_\alpha = \tilde{A}^{-1}_\alpha A^T = \texttt{np.linalg.inv(}\tilde{A}_\alpha\texttt{).dot(A.T)}$$

Initialize the weights to be zero and use $\alpha = 0.01$. Run gradient descent for 10,000 steps with a step size of 0.001.

1. Show that gradient descent returns the same weights as the exact solution using the pseudoinverse (to within 3 digits). Report the weights of both solutions below. (10 pts)
   Pseudoinverse: [15.2696443 -19.3703566]
   Gradient Descent: [15.23341758 -19.11470538]

2. Plot your fitted line with the data and display the plot below. Are you over-fitting, under-fitting, or fitting the data just right? Explain your thought process. (10 pts)
   The data is being over-fitted because the loss function is convex by definition. In order to fit the true loss function, which is increasing with an increasing slope, the regression fits underneath the loss function. For the bulk of the function, the fit line is overestimating the value of the loss function. Towards the end values of the function, the regression underestimates the value again. Since the majority of the estimates are above the loss function, this is over-fitting the



model.

2 Classification (30 pts)

2.1 Logistic Regression

In this problem, you will learn to classify using Logistic Regression. Logistic Regression outputs the probability of x belonging to the positive class $y = T$, $P(y = T|x, w)$, where w is the weight vector. You can simply use $1 - P(y = T|x, w)$ to obtain the probability for the negative class. Note that you can add an L2-regularizer to Logistic Regression too, but we will not in this problem. We have provided you with template code for loading the dataset, training the model, and plotting the decision boundary (see `logisticRegression.py`). Logistic Regression is typically trained with a cross entropy loss. As before, we are providing you with the cross entropy loss for logistic regression along with its gradient (derived below).

$$P(y|x, w) = f(x; w) = \sigma(z) \text{ where } z = w^T x, \quad \sigma(z) = \frac{1}{1 + e^{-z}}, \quad \frac{\partial \sigma}{\partial z} = \sigma(z)(1 - \sigma(z))$$

$$l(y, \sigma(z)) = -y \log(\sigma(z)) - (1 - y) \log(1 - \sigma(z))$$

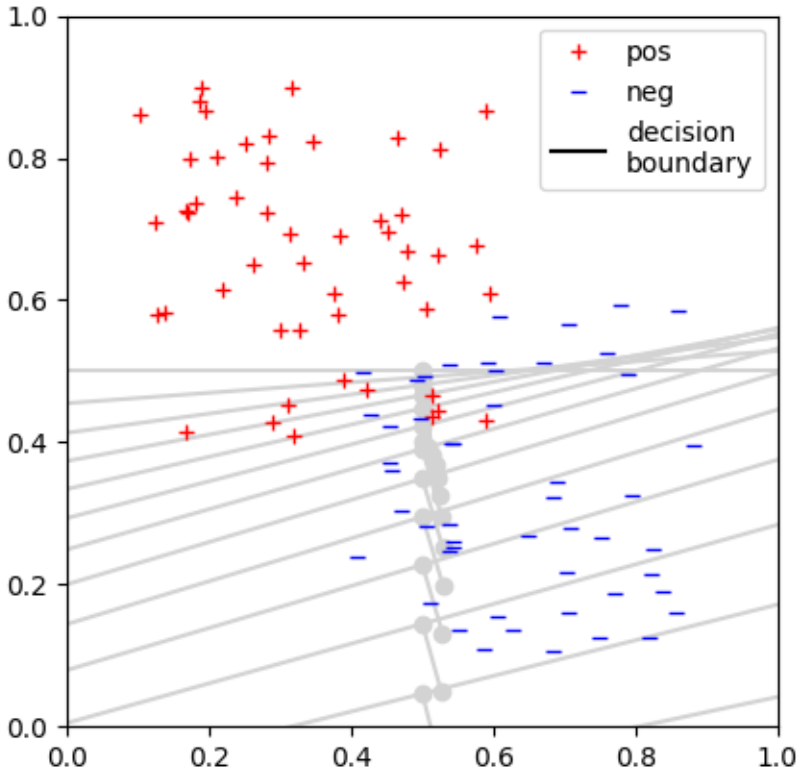
$$\frac{\partial l}{\partial w} = \frac{\partial l}{\partial \sigma} \frac{\partial \sigma}{\partial z} \frac{\partial z}{\partial w} = \Big(\frac{-y}{\sigma(z)} + \frac{1 - y}{1 - \sigma(z)} \Big) \sigma(z)(1 - \sigma(z)) x = \Big(\sigma(z) - y \Big) x$$

$$L(\mathbf{y}, \sigma(X)) = \frac{1}{N} \sum_i l(y_i, \sigma(z_i)) \text{ where } N = \text{ \# of samples} = \sum_i 1$$

$$\nabla_w L = \Big(\sigma(Z) - \mathbf{y} \Big)^T X \text{ where } Z = X^T w$$

Initialize the weights to $[0, -1, 0.5]$. Run gradient descent for 10,000 steps with a step size of 0.0001.

1. Train a Logistic Regression model with gradient descent on the entire data set (X,y) and plot the decision boundary. Report the loss, L. (10 pts)



3585.74558167

2. Train LR on the training set only (`X_train,y_train`) and then compute and report the loss on the test set (`X_test,y_test`). In your own words, explain why, in general, the loss on a held out test set gives a better estimate for how the model will perform in the "real world". (10 pts)
   The test set can be generated to fit more closely to the expected value of the regression, allowing for fewer steps to be taken before convergence and lower loss between steps.

2.2 Decision Trees

In this problem, we are revisiting the Decision Tree problem from the book (p. 697-704). We will be considering the same dataset as well (Figure 18.3).

1. Assume we are building our decision tree, starting at the root. Section 18.3.4 explains the reasoning for choosing to split on the *Patrons* attribute over the *Type* attribute. Instead, calculate the expected entropy of the *Estimated Wait Time* attribute (EST). Is the expected entropy higher or lower than Patrons ($\mathbb{E}_{v=\{None,Some,Full\}}[H(v)] = 0.459$)? Of those two, which attribute should you choose to split on when learning a decision tree and why? (10 pts)

2. (Bonus) Implement a decision tree and train it on Stuart Russel's restaurant preferences. We have provided you with template code for loading the dataset, training the model, and printing the resulting tree (see `decisionTree.py`). Draw the tree that your model learns (using Google Drive drawings) and display it below. (10 pts)

3 Clustering (20 pts)

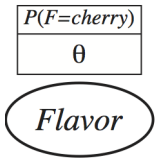
In this problem, you will learn to cluster using K-Means. We have provided you with template code for loading the dataset, training the model, and plotting the clusters (see `kmeans.py`).

1. Implement k-means clustering with k=3 using randomly initialized centers. Does k-means return the same result every time? If not, why not? (10 pts)

2. Run k-means using randomly initialized centers and k=2,3,5. Provide a single plot of the clusters for each k. Which k is best? (10 pts)

4 Bayesian Learning (30 pts)

In class (Lecture 20, slides 49-51), we derived the MAP result for the cherry-lime candy bag (a BN with one node) given the number of limes (l) and cherries (c) seen so far and a $beta[a, b]$ prior: $\theta_{\text{MAP}} = (c + a - 1)/(c + l + a + b - 2)$. In this problem,



we will assume the candy bag contains only limes.

1. Consider the five hypotheses with their priors listed at the bottom of page 802.

   (a) h_1: 100% cherry, $P(h_1) = 0.1$
   (b) h_2: 75% cherry, 25% lime, $P(h_2) = 0.2$
   (c) h_3: 50% cherry, 50% lime, $P(h_3) = 0.4$
   (d) h_4: 25% cherry, 75% lime, $P(h_4) = 0.2$
   (e) h_5: 100% lime, $P(h_5) = 0.1$

   Use full Bayesian learning (last Eqn of slide 9) to derive a formula for the probability that the $(N + 1)$th candy is lime using all these hypotheses:

   $$P(X_{N+1}|X_{1:N}) = \alpha \sum_{i=1}^{5} P(X_{N+1}|h_i)P(h_i) \prod_{j=1}^{N} P(X_j|h_i)$$

   where X_j represents the flavor of the jth candy drawn from the bag (with replacement). Use the priors listed above. You can check your result by plotting it and comparing it to Figure 20.1(b). You can check your intermediate results for $P(h_i|d)$ by comparing to Figure 20.1(a). (10 pts)

   $$P(X_{N+1} = lime|X_{1:N}) = ?$$

2. Consider an infinite number of hypotheses with a hypothesis prior that is a beta distribution, beta[a,b]. Use full Bayesian learning to obtain a formula for the posterior as a function of N:

   $$P(X_{N+1}|X_{1:N}) = \alpha \int P(X_{N+1}|\theta)P(\theta) \prod_{j=1}^{N} P(X_j|\theta)d\theta$$

   You will also need to know the partition function of a beta:

   $$Z = \int_{\theta} \theta^{a-1}(1-\theta)^{b-1}d\theta = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

   You will need to install the scipy package to import the gamma function (`conda install scipy`). Report the formula below. (10 pts)

   $$P(X_{N+1} = lime|X_{1:N}) = ?$$

3. Plot the three different posteriors against each other: MAP vs finite vs infinite as a function of N ($N = 0$ to 10). Use $a = b = 2$ for the beta prior. (10 pts)