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TOOLBOX

Abstract. This is a collection of probability inequalities and other useful results. First some very basic inequalities, then results connected to regular variation and finally the techniques used to prove the results for our particular matrix structure.

1. Basic Inequalities

- For any X, Y and $\epsilon > 0$:

$$\mathbb{P}(X+Y>\epsilon) \leq \mathbb{P}(X>\epsilon/2)+\mathbb{P}(Y>\epsilon/2).$$

- $\mathbb{P}(X\mathbb{1}_A > \epsilon) \leq \mathbb{P}(A)$.
- For $X \geq 0$ we have $\mathbb{P}(\sum_i X_i \mathbb{1}_{A_i} > \epsilon) \leq \mathbb{P}(\bigcup_i A_i) \leq \sum_i \mathbb{P}(A_i)$.
- for $\mathbb{1}_{\{\sum Z^2 \leq a_{np}^2\}}$ use Markov inequality
- Markov inequality. X nonnegative integrable rv.

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

- Chebyschev inequality. X nonnegative integrable rv.

$$\mathbb{P}(|X-\mathbb{E}[X]| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2}.$$

- Kolmogorov inequality. $X_1, \ldots, X_n$ independent, zero mean, finite variances. S_k partial sum.

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right) \leq \frac{1}{\lambda^2}\sum_{k=1}^{n} \mathrm{Var}(X_k).$$

- Etemadi inequality. $X_1, \ldots, X_n$ independent real valued. $\lambda \geq 0$.

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq 3\lambda\right) \leq 3 \max_{1\leq k\leq n} \mathbb{P}\left(|S_k| \geq \lambda\right).$$

- Bernstein inequality. $X_1, \ldots, X_n$ independent real valued, zero mean. Suppose $|X_i| \leq M, t > 0$.

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > t\right) \leq \exp\left(-\frac{t^2}{2(\sum \mathbb{E}[X_i^2] + Mt/3)}\right).$$

- Hölder inequality. Assume that p and q are in the open interval $(1,\infty)$ with $1/p+1/q=1$. We have

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q}$$

- Lyapunov inequality: Let $1 \leq r \leq s$ and $E[|X|^s] < \infty$. Then

$$E[|X|^r]^{1/r} \leq E[|X|^s]^{1/s}.$$

- Minkowski. $r \geq 1$.

$$E[|X+Y|^r]^{1/r} \leq E[|X|^r]^{1/r} + E[|Y|^r]^{1/r}$$

- Rosenthal. If (X_i) is a sequence of independent rvs with expectation zero, then for any $r \geq 2$,
$$\mathbb{E}\Big[\Big|\sum_{i=1}^n X_i\Big|^r\Big] \leq C_r\Big(\Big(\sum_{i=1}^n \mathbb{E}[X_i^2]\Big)^{r/2} + \sum_{i=1}^n E[|X_i|^r]\Big).$$

2. Matrix norms

Let A be an $m \times n$ matrix, and $p, q \geq 1$. The $L_{p,q}$ norm is defined as
$$\|A\|_{p,q} = \Big[\sum_{j=1}^n \Big(\sum_{i=1}^m |a_{ij}|^p\Big)^{q/p}\Big]^{1/q}.$$

3. Other Formulas

Assume $X \geq 0$ and g increasing differentiable with $g(0) = 0$, then:
$$\mathbb{E}[g(x)] = [-g(t)\bar{F}(t)]_0^\infty - \int_0^\infty \bar{F}(t)d(-g(t)) = \int_0^\infty g'(t)\bar{F}(t)dt.$$

4. Regular Variation

- Let ℓ be slowly varying at infinity. Then for all $\epsilon > 0$,
$$\lim_{x\to\infty} x^\epsilon \ell(x) = \infty,$$
$$\lim_{x\to\infty} x^{-\epsilon} \ell(x) = 0,$$
$$\lim_{x\to\infty} \frac{\log \ell(x)}{\log x} = 0.$$
- **Potter's bounds.** $f \in \mathrm{RV}_\infty(\alpha)$. For any $\epsilon > 0$ and $C > 1$, there esxists x_0 such that for all $y \geq x \geq x_0$,
$$C^{-1}(y/x)^{\alpha-\epsilon} \leq \frac{f(y)}{f(x)} \leq C(y/x)^{\alpha+\epsilon}$$
- $\mathbb{P}(\max_{1\leq i\leq k} X_i) \sim k\bar{F}(x), \quad x \to \infty.$
- Maxima and sums of non-negative rvs:
  $M_n/S_n \to 1$ in probability iff $\bar{F}$ is slowly varying at infinity.
  M_n/S_n has a non degenerate distribution iff $\bar{F}$ is regularly varying with $\alpha \in (0, 1)$.
- If $\bar{F}$ is regularly varying, then for all $\epsilon > 0$, there exists a constant C such that
$$1 - F^{*n}(x) \leq C(1 + \epsilon)^n \bar{F}(x).$$
- **Truncated Moments.** Assume X is regularly varying with index α. Then
$$\mathbb{E}[X^\beta \mathbb{1}_{\{X\leq x\}}] \sim \frac{\alpha}{\beta - \alpha} x^\beta \bar{F}(x), \quad \beta > \alpha,$$
$$\mathbb{E}[X^\beta \mathbb{1}_{\{X>x\}}] \sim \frac{\alpha}{\alpha - \beta} x^\beta \bar{F}(x), \quad \beta < \alpha.$$
- Lemma for Breiman. Assume X is regularly varying with index α. Then there exists a constant C such that for any $x, y > 0$,
$$\frac{\mathbb{P}(yX > x)}{\mathbb{P}(X > x)} \leq C(1 \vee y)^{\alpha+\epsilon}.$$
- **Breiman.** $\mathbb{P}(AB > x) \sim \mathbb{E}[A^\alpha]\mathbb{P}(B > x)$.
- By Karamata: Sequence $s_n \to \infty$, then
$$\mathbb{E}[s_n^{-1} Z^2 \mathbb{1}_{\{Z^2\leq s_n\}}] \leq c\mathbb{P}(Z^2 > s_n).$$

Theorem 4.1 (Classical Karamata). *L slowly varying, locally bounded for in $[x_0, \infty)$ for some $x_0 \geq 0$. Then*

(1) *for $\rho > -1$,*
$$\int_{x_0}^{x} t^{\rho} L(t)dt \sim \frac{1}{\rho+1} x^{\rho+1} L(x), \quad x \to \infty,$$

(2) *for $\rho < -1$,*
$$\int_{x}^{\infty} t^{\rho} L(t)dt \sim -\frac{1}{\rho+1} x^{\rho+1} L(x), \quad x \to \infty.$$

Theorem 4.2 (Modified Karamata with $\alpha > 0$.). *$\bar{F}(x) = x^{-\alpha} L(x)$. Then*

(1) *for $\alpha \in (0,1)$,*
$$\int_{x_0}^{x} \bar{F}(t)dt \sim \frac{1}{1-\alpha} x\bar{F}(x), \quad x \to \infty,$$

(2) *for $\alpha > 1$,*
$$\int_{x}^{\infty} \bar{F}(t)dt \sim \frac{1}{\alpha-1} x\bar{F}(x), \quad x \to \infty.$$

We collect some useful results about regularly varying distributions. Let $\{Z_i\}$ be an iid sequence of regularly varying random variables, i.e.

$$\mathbb{P}(|Z_1| > x) = x^{-\alpha} L(x)$$
$$\frac{\mathbb{P}(Z_1 > x)}{\mathbb{P}(|Z_1| > x)} \to p_+.$$

Assume $\mathbb{E}[|Z_1|^{\alpha}] = \infty$ and define

$$a_n = \inf\{x : \mathbb{P}(|Z_1| > x) \leq \frac{1}{n}\},$$
$$b_n = \inf\{x : \mathbb{P}(|Z_1 Z_2| > x) \leq \frac{1}{n}\}.$$

Then we get

$$\mathbb{P}(|Z_1 Z_2| > x) = x^{-\alpha} \tilde{L}(x),$$
$$\frac{\mathbb{P}(Z_1 Z_2 > x)}{\mathbb{P}(|Z_1 Z_2| > x)} \to p_+^2 + (1-p_+)^2,$$
$$\lim_{x \to \infty} \frac{\mathbb{P}(|Z_1 Z_2| > x)}{\mathbb{P}(|Z_1| > x)} = \infty,$$
$$\lim_{n \to \infty} \frac{b_n}{a_n} = \infty.$$

Example 4.3. *If $\mathbb{P}(|Z_1| > x) \sim x^{-\alpha}$, then*

$$\mathbb{P}(|Z_1 Z_2| > x) = x^{-\alpha}(1 + \alpha \log x),$$
$$a_n \sim n^{1/\alpha},$$
$$b_n \sim (n \log n)^{1/\alpha}.$$

In particular $\frac{a_n^2}{b_n} \to \infty$. This shows that if we normalize $Z_1 Z_2$ by a_n^2 it will be negligible in probability.

Lemma 4.4. *$\alpha = 1$ and Z is regularly varying with index α. Assume $\mathbb{E}[|Z|] = \infty$. The function*

$$L(x) = \frac{\mathbb{E}[|x^{-1} Z| \mathbb{1}_{\{|Z| \leq x\}}]}{\mathbb{P}(|Z| > x)}, \quad x > 0,$$

is slowly varying and converges to infinity, but

$$\mathbb{E}[|x^{-1}Z|\mathbb{1}_{\{|Z|\le x\}}] + \mathbb{P}(|Z| > x), \quad x > 0,$$

decreases and is regularly varying with index -1.

5. Point Processes

Remark 5.1. *Let* $\{Y_{ni}\}_{i,n=1,\ldots,\infty}$ *be an array of dependent identically distributed random variables. A standard set of sufficient conditions to guarantee that*

$$\sum_{i=1}^{n} \varepsilon_{(i/n, Y_{ni})} \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{(t_i, j_i)}$$

for a PRM($\mathrm{d}t \times \mathrm{d}\nu$*) is*

- *a mixing condition, f.i.* D^**; or* m*-dependence.*
- *a mean value condition*
$$n\mathbb{P}(Y_{n1} \in \cdot) \xrightarrow{v} \nu.$$
- *And a condition to control the error in the small blocks large blocks approximation of the Laplace functional; for example function* $g \le 1$ *and*
$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{i=2}^{[\frac{n}{k}]} \mathbb{E}[g(Y_{n1})g(Y_{ni})] = 0.$$