# Maximal LELM Distinguishability of Qubit and Qutrit Bell States using Projective and Non-Projective Measurements

**Nathaniel Leslie**

Theresa Lynn, Advisor

Francis Su, Reader

HARVEY
MUDD
COLLEGE

**Department of Physics**

May, 2017

Copyright © 2017 Nathaniel Leslie.

The author grants Harvey Mudd College and the Claremont Colleges Library the nonexclusive right to make this work available for noncommercial, educational purposes, provided that this copyright statement appears on the reproduced materials and notice is given that the copying is by permission of the author. To disseminate otherwise or to republish requires written permission from the author.

# Abstract

Many quantum information tasks require measurements to distinguish between different quantum-mechanically entangled states (Bell states) of a particle pair. In practice, measurements are often limited to linear evolution and local measurement (LELM) of the particles. We investigate LELM distinguishability of the Bell states of two qubits (two-state particles) and qutrits (three-state particles), via standard projective measurement and via generalized measurement, which allows detection channels beyond the number of orthogonal single-particle states. Projective LELM can only distinguish 3 of 4 qubit Bell states; we show that generalized measurement does no better. We show that projective LELM can distinguish only 3 of 9 qutrit Bell states for bosons that generalized LELM allows at most 5 of 9 for any particle statistics. We have also made progress on distinguishing qubit × qutrit hyperentangled Bell states, which are made up of tensor products of the qubit Bell states and the qutrit Bell states, showing that the maximum number distinguishable with projective LELM measurements is between 9 and 11.

# Contents

**Abstract** iii

**Acknowledgments** ix

**1 Background** 1

1.1 Qubits, Qutrits and Qudits . . . . . 1
1.2 Bipartite Systems, Entanglement and Bell States . . . . . 3
1.3 General LELM Apparatus . . . . . 6
1.4 Projective LELM Apparatus and Detection Modes . . . . . 7
1.5 Particle Statistics . . . . . 8
1.6 Detection Signatures . . . . . 8
1.7 Creation and Annihilation Operators . . . . . 9
1.8 Necessary Conditions for Distinguishability . . . . . 10

**2 Maximum Distinguishability of Qutrit Bell States with Projective Measurement** 13

2.1 Overview . . . . . 13
2.2 Applications of Bell State Measurements . . . . . 13
2.3 The Qubit Bell States are not Completely Distinguishable . . 14
2.4 3 Qutrit Bell States are Distinguishable, but 5 are not . . . . . 14
2.5 Equivalence Classes of 4 Qutrit Bell States . . . . . 15
2.6 Putting Further Limits on Detection Modes . . . . . 20
2.7 Detector Modes in the Bell Basis and in the Joint-Particle Basis 20
2.8 Ruling out 4-Ket Detection Modes . . . . . 21
2.9 Ruling out Single-Channel Detection Modes . . . . . 22
2.10 Forcing a 6-Ket Mode in a Specific Tic-Tac-Toe Winner . . . . 25
2.11 Forcing No Solution to the Necessary Conditions . . . . . 27

**3 Distinguishability of Qubit and Qutrit Bell States with POVMs 31**
3.1 Overview . . . . . . . . . . 31
3.2 Introduction to POVMs . . . . . . . . . . 31
3.3 General Limits of Bell State Discrimination . . . . . . . . . . 32
3.4 POVMs cannot distinguish all 4 qubit Bell states . . . . . . . 34
3.5 POVMs cannot distinguish 6 qutrit Bell states . . . . . . . . . 34

**4 Maximum Distinguishability of Qubit × Qutrit Bell States with Projective Measurement 41**
4.1 Overview . . . . . . . . . . 41
4.2 Hyperentanglement . . . . . . . . . . 41
4.3 Hyperentangled Bell States . . . . . . . . . . 42
4.4 9 Qubit × Qutrit Hyperentangled Bell States are Distinguishable, but 12 are not . . . . . . . . . . 43
4.5 Distinguishability Classes of Qubit × Qutrit Hyperentangled Bell States . . . . . . . . . . 44

**5 Conclusion 49**

**Bibliography 51**

# List of Figures

1.1 This general two-particle LELM apparatus takes two particles in separate input channels and performs single-particle unitary operations that may mix the channels, but do not involve conditional evolution of one particle based on the state of the second. Then the outputs from the channels are detected by detectors which comprise a POVM, which may be either projective or non-projective. . . . . . . . . . . . . . . . 6
1.2 The projective LELM apparatus simply performs single-particle unitary transformations before sending the particles to an orthogonal set of projective detectors. . . . . . . . . . . . . . . . . 8

2.1 The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^1_0\rangle, |\Psi^1_2\rangle, |\Psi^2_2\rangle\}$. . 16
2.2 The tic-tac-toe diagram of the set $\{|\Psi^2_0\rangle, |\Psi^1_2\rangle, |\Psi^2_1\rangle, |\Psi^0_2\rangle\}$. This belongs to the tic-tac-toe winners class because it has 3 in a row diagonally. . . . . . . . . . . . . . . . . . . . . . . . . 16
2.3 The tic-tac-toe diagram of the set $\{|\Psi^1_0\rangle, |\Psi^0_1\rangle, |\Psi^1_1\rangle, |\Psi^2_2\rangle\}$. This belongs to the tic-tac-toe winners class because it has 3 in a row diagonally when we permute the columns or allow wrap-around boundaries. . . . . . . . . . . . . . . . . . . . . 17
2.4 The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^1_2\rangle\}$. Even with column permutation or wrap-around boundaries, we cannot get 3 in a row, so it is a tic-tac-toe loser. . . . . . . 17
2.5 The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^1_0\rangle, |\Psi^1_1\rangle, |\Psi^2_2\rangle\}$. We will show that single-channel detectors cannot be part of an apparatus that distinguishes these states. . . . . . . . . . . 23
2.6 The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^1_0\rangle, |\Psi^2_0\rangle, |\Psi^0_1\rangle\}$. We will show that any apparatus that distinguishes these states must have a detector with all 6 joint-particle kets. . . . 25

3.1 The tic-tac-toe diagram of the set $\{|\Psi_0^0\rangle, |\Psi_0^2\rangle, |\Psi_1^1\rangle, |\Psi_1^2\rangle, |\Psi_2^0\rangle, |\Psi_2^1\rangle\}$. This belongs to the tic-tac-toe anti-winners class because with wrap-around boundary conditions or column permutation, the missing states win at tic-tac-toe. . . . . . . . . . . . . . . . . 37
3.2 The tic-tac-toe diagram of the set $\{|\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle, |\Psi_2^0\rangle, |\Psi_2^1\rangle, |\Psi_2^2\rangle\}$. This belongs to the tic-tac-toe anti-losers class because with even with wrap-around boundary conditions or column permutation, the missing states do not win at tic-tac-toe. . . . . 38
3.3 The subset $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle\}$ is shown with orange Xs. The tic-tac-toe anti-winner adds $|\Psi_1^2\rangle$, shown as a maroon X with a large circle. The tic-tac-toe anti-loser adds $|\Psi_2^0\rangle$, shown as a blue X with a small rectangle. By showing that the subset cannot remain orthogonal after one detection, we will show that both representatives from the two classes cannot remain orthogonal after one detection. . . . . . . . . 39

# Acknowledgments

I'd like to thank my kind and caring professors for all that they have taught me and all of the ways that they have supported and encouraged me, especially Prof. Benjamin, Prof. Donnelly, Prof. Esin, Prof. Gallicchio, Prof. Gerbode, Prof. Lynn, Prof. Saeta, Prof. Su, Prof. Sullivan and Prof. Yong. I'd also like to thank my parents and my sister Danielle for their incredibly constant support and assistance. I would not be where I am without them. Lastly, I want to thank the many amazing friends that I have made here at Mudd. They have taught me so much and supported me through a lot. My conversations with them have made me a much better person. I owe so much of my success to them.

# Chapter 1

# Background

## 1.1 Qubits, Qutrits and Qudits

Classically, the simplest form of information is the bit, a logical construct that can either take the value 0 or 1. Computers store information and perform computation using very large numbers of these bits. In quantum information, the analogous construct is the qubit. A qubit is a quantum mechanical system whose state can be expressed as a linear combination or *superposition* of two distinguishable states $|0\rangle$ and $|1\rangle$, which we call the standard basis states. The $|\rangle$ symbol is notation used to describe a quantum state, and it is called a ket. Generally, the description of the state is put inside the ket. The general form of a ket that represents a qubit looks like

$$|\psi_{\text{qubit}}\rangle = a\,|0\rangle + b\,|1\rangle \tag{1.1}$$

where $a$ and $b$ are complex numbers that satisfy the normalization condition

$$|a|^2 + |b|^2 = 1. \tag{1.2}$$

One physical example of a qubit would be the z-projection of the spin of a spin-$\frac{1}{2}$ particle, which has spin-up and spin-down as its basis states. Another would be the polarization of a photon, whose basis states are vertical and horizontal polarization. We encode these physical states as the standard basis states $|0\rangle$ and $|1\rangle$.

Such a state gives us information when it is measured. The simplest way to measure such a state is to make a projective measurement, which projects the state into one of a set of orthogonal basis states with a probability that depends on the state and the basis in which it is measured. If we were to make a simple projective measurement of the qubit in Equation 1.1 in the standard basis ($\{|0\rangle, |1\rangle\}$), we would get the result 0 and collapse the state into $|0\rangle$ with probability $|a|^2$ and we would get the result 1 and collapse the state into $|1\rangle$ with probability $|b|^2$. Probability conservation yields Equation 1.2.

Kets that represent qubits can also be treated as vectors in a two-dimensional complex vector space. As this vector space is complete and has an inner product (which we will describe soon), we call it a Hilbert space. When we choose a basis, we can represent kets that represent qubits as column vectors in $\mathbb{C}^2$. For example, if we choose to represent the state from Equation 1.1 in the $\{|0\rangle, |1\rangle\}$ basis, we get

$$|\psi_{\text{qubit}}\rangle \to \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.3}$$

For each ket $|\psi\rangle$, there exists a corresponding object $\langle\psi|$, called a bra, which lives in the dual space to the Hilbert space and can be represented as the adjoint of the ket. For example, the corresponding bra to $|\psi_{\text{qubit}}\rangle$ can be represented as

$$\langle\psi_{\text{qubit}}| \to \begin{pmatrix} a^* & b^* \end{pmatrix}, \tag{1.4}$$

where $a^*$ is the complex conjugate of $a$. Because this is a Hilbert space, it has an inner product. Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $|\phi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$. To calculate the inner product between these two states, we must put one in bra form and then multiply the corresponding vectors. The inner product between these two states is

$$\langle\phi|\psi\rangle = \begin{pmatrix} \beta_0^* & \beta_1^* \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0\beta_0^* + \alpha_1\beta_1^*, \tag{1.5}$$

and the result is called the probability amplitude to measure $|\psi\rangle$ in the state $|\phi\rangle$. This turns out to be related to the probability of measuring $|\psi\rangle$ in the state $|\phi\rangle$ in an appropriate projective measurement. To form a projective measurement, we need an orthonormal basis. Because qubits live in a two-dimensional Hilbert space, we only need one other orthogonal state to $|\phi\rangle$, which might be $|\phi^\perp\rangle = \beta_1^* |0\rangle - \beta_0^* |1\rangle$. We can verify that the states are orthogonal:

$$\langle\phi|\phi^\perp\rangle = \begin{pmatrix} \beta_0^* & \beta_1^* \end{pmatrix} \begin{pmatrix} \beta_1^* \\ -\beta_0^* \end{pmatrix} = \beta_0^*\beta_1^* - \beta_1^*\beta_0^* = 0. \tag{1.6}$$

If a projective measurement is made of $|\psi\rangle$ in the $\{|\phi\rangle, |\phi^{\perp}\rangle\}$ basis, the probability of measuring $|\phi\rangle$ is the square of the probability amplitude:

$$p = \left|\langle\phi|\psi\rangle\right|^2 = \left|\alpha_0\beta_0^* + \alpha_1\beta_1^*\right|^2. \tag{1.7}$$

We can also define operators that act on kets to transform them. When a projective measurement is made on a state, a projection operator acts on its ket. The projection operator that projects a state onto $|\phi\rangle$ is

$$|\phi\rangle\langle\phi| \rightarrow \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\begin{pmatrix}\beta_0^* & \beta_1^*\end{pmatrix} = \begin{pmatrix}|\beta_0|^2 & \beta_0\beta_1^*\\ \beta_0^*\beta_1 & |\beta_1|^2\end{pmatrix}. \tag{1.8}$$

After a projection operator acts on a ket, it must be renormalized by multiplying it by a scalar so that it satisfies Equation 1.2.

This concept of a qubit can be generalized to higher-dimensional Hilbert spaces. A quantum state in the three-dimensional Hilbert space with the standard basis $\{|0\rangle, |1\rangle, |2\rangle\}$ is called a qutrit. In general, a quantum state in a $d$-dimensional Hilbert state with the standard basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ is called a qudit.

When a state is not being measured, it evolves in time according to the Schrödinger Equation, shown below:

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle, \tag{1.9}$$

where $\hat{H}$ is an operator called the Hamiltonian, which is related to the energy of a particle in a system. As the Schrödinger Equation is a linear differential equation, the evolution of independent systems in time is linear. Along with probability conservation, this condition tells us that evolution of independent systems when not being measured are described by a unitary operator $\hat{U}$. Unitary means that when written as a matrix, $U$ satisfies $U^{\dagger}U = I$, where $U^{\dagger}$ is the adjoint of $U$. So the relationship between the state of a system at one point in time ($t = t_1$) and the system at a later point in time ($t = t_2$) can always be described by a unitary operator like so:

$$|\psi(t_2)\rangle = U|\psi(t_1)\rangle. \tag{1.10}$$

## 1.2 Bipartite Systems, Entanglement and Bell States

A bipartite state is the combined state of two quantum mechanical systems together. A system of an $n$-state variable and an $m$-state variable will live in a $mn$-dimensional Hilbert space spanned by all possible tensor products of the basis states of the $n$-state and $m$-state systems. For example, a general two qubit state will look like

$$|\Psi_{\text{two qubit}}\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle\,, \tag{1.11}$$

where $|ab\rangle$ is a shorthand for $|a\rangle\,|b\rangle$, which is a shorthand for $|a\rangle \otimes |b\rangle$ and for normalization,

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1. \tag{1.12}$$

Again, let $|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$ and $|\phi\rangle = \beta_0\,|0\rangle + \beta_1\,|1\rangle$. Then one could write the bipartite system of these two qubits as

$$|\psi\rangle\,|\phi\rangle = \alpha_0\beta_0\,|00\rangle + \alpha_0\beta_1\,|01\rangle + \alpha_1\beta_0\,|10\rangle + \alpha_1\beta_1\,|11\rangle\,. \tag{1.13}$$

We see that the state in Equation 1.11 is more general than that in Equation 1.13. If a bipartite system cannot be factored as two independent single-particle states, then it is in an *entangled* state. If it can, like the state in Equation 1.13, the state is *unentangled*.

Entangled states have information stored in the correlation of the two particles with each other, beyond what is stored in the individual particle states. For example, the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) \tag{1.14}$$

is fully entangled, and it gives absolutely no information about the state of the first or second system, but it tells exactly how the two systems are correlated: if they were measured in this basis, they would either both be in the state $|0\rangle$ or both be in the state $|1\rangle$. One could not measure one in the state $|0\rangle$ and the other in the state $|1\rangle$ because the coefficients of $|01\rangle$ and $|10\rangle$ are zero, which means that their probability amplitudes are zero, so the probabilities for such measurements are also zero.

In Equation 1.11, we expressed the state of a two-qubit bipartite system in what we call the joint-particle basis, which is comprised of the joint-particle kets: $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$. Each of these basis states are obviously unentangled as they are tensor products of the single-particle basis states. It is also possible to construct a basis for the same Hilbert space using entangled states. One very useful such basis is the Bell basis (we will detail some applications of these states in Section 2.2), comprising the Bell states. For qubits, the Bell states are

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) \tag{1.15a}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) \tag{1.15b}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big) \tag{1.15c}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big) \tag{1.15d}$$

The qubit Bell states are labelled with $\Psi$ and $\Phi$ based on the correlation between the particles and with a + or − based on the relative phase between the terms.

General two-qudit bipartite systems also have Bell bases of their own. For qudits, the $\Psi/\Phi$ distinction is generalized to a *correlation class*, in which all of the joint-particle kets that make up the Bell state have the same difference between the variable values (mod $d$). The $+/-$ distinction is generalized to a *phase class*, in which all Bell states have consecutive terms in the joint-particle ket representation differing by a constant complex phase. The general qudit Bell state in correlation class $c$ and phase class $p$ is shown below in Equation 1.16.

$$|\Psi_c^p\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i2\pi p j/d}\,|j\rangle\,|j + c\ (\mathrm{mod}\ d)\rangle \tag{1.16}$$

Because much of the work in this thesis will be done with qutrit Bell states, I list the qutrit Bell states below.

$$|\Psi_0^0\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + |11\rangle + |22\rangle\,\big) \tag{1.17a}$$

$$|\Psi_0^1\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + e^{i\frac{2\pi}{3}}\,|11\rangle + e^{i\frac{4\pi}{3}}\,|22\rangle\,\big) \tag{1.17b}$$

$$|\Psi_0^2\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + e^{i\frac{4\pi}{3}}\,|11\rangle + e^{i\frac{2\pi}{3}}\,|22\rangle\,\big) \tag{1.17c}$$

$$|\Psi_1^0\rangle = \frac{1}{\sqrt{3}}\big(\,|01\rangle + |12\rangle + |20\rangle\,\big) \tag{1.17d}$$

$$|\Psi_1^1\rangle = \frac{1}{\sqrt{3}}\big(\,|01\rangle + e^{i\frac{2\pi}{3}}\,|12\rangle + e^{i\frac{4\pi}{3}}\,|20\rangle\,\big) \tag{1.17e}$$

$$|\Psi_1^2\rangle = \frac{1}{\sqrt{3}}\big(\,|01\rangle + e^{i\frac{4\pi}{3}}\,|12\rangle + e^{i\frac{2\pi}{3}}\,|20\rangle\,\big) \tag{1.17f}$$

$$|\Psi_2^0\rangle = \frac{1}{\sqrt{3}}\big( |02\rangle + |10\rangle + |21\rangle \big) \tag{1.17g}$$

$$|\Psi_2^1\rangle = \frac{1}{\sqrt{3}}\big( |02\rangle + e^{i\frac{2\pi}{3}} |10\rangle + e^{i\frac{4\pi}{3}} |21\rangle \big) \tag{1.17h}$$

$$|\Psi_2^2\rangle = \frac{1}{\sqrt{3}}\big( |02\rangle + e^{i\frac{4\pi}{3}} |10\rangle + e^{i\frac{2\pi}{3}} |21\rangle \big) \tag{1.17i}$$

## 1.3 General LELM Apparatus

The goal of this thesis is to better understand how to reliably distinguish these Bell states from each other. To distinguish the Bell states, we will consider what is called an LELM (Linear Evolution and Local Measurement) apparatus. We will define exactly what is meant by Linear Evolution and Local Measurement and why we add these restrictions below. A general two-particle LELM apparatus is shown in Figure 1.1 below.

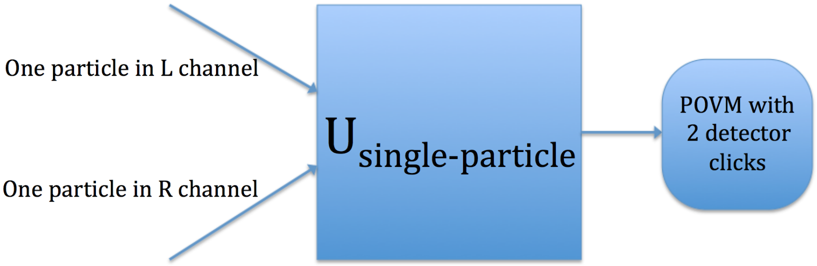
One particle in L channel
One particle in R channel
U single-particle
POVM with 2 detector clicks

**Figure 1.1** This general two-particle LELM apparatus takes two particles in separate input channels and performs single-particle unitary operations that may mix the channels, but do not involve conditional evolution of one particle based on the state of the second. Then the outputs from the channels are detected by detectors which comprise a POVM, which may be either projective or non-projective.

The first restriction on the LELM apparatus, Linear Evolution, refers to each particle individually and requires that each particle must evolve (change) independently from the other particle and evolve according to Equation 1.10, although it may mix the particles between the two channels. It is always true that the whole system must evolve linearly before measurement, but this restriction further requires that both particles must individually evolve according to some unitary operator $U_{\text{single-particle}}$. Notably, this disallows operations that impact one qubit that are conditional on the state of the other qubit. We consider this restriction because linear or non-conditional evolution can be performed more reliably in practice.

The second restriction, Local Measurement, requires that each measurement must act on single particles at a time, so there will be two distinct detection events, or detector clicks. Each detector click heralds the presence of one particle in a certain output mode of the apparatus; because of the channel mixing allowed in the linear evolution step, a detector click might not unambigiously identify which channel the detected particle originated in. This restriction simply describes the physical reality of measurement.

In this LELM device, because there are two particles in distinct channels, we add a channel variable to the state of each particle. Thus there are $2d$ basis states for single-particle states:

$$\{\left|0, L\right\rangle, \left|0, R\right\rangle, \ldots, \left|d-1, L\right\rangle, \left|d-1, R\right\rangle\} \tag{1.18}$$

Any input state into this system will be acted upon by the single-particle unitary operator to transform it before it is detected. A general detection is described by a Positive-Operator-Valued-Measure (POVM), which we will describe in Section 3.2. Until then, we will consider a more specific case of measurement, called projective measurement, which allows us to make some nice simplifications.

## 1.4 Projective LELM Apparatus and Detection Modes

If we limit our LELM device to be projective, then all of the detectors make projective measurements, like the measurements described in Section 1.1. Then the apparatus is much simpler. After the single-particle unitary transformations operate, particles in any of the $2d$ orthogonal channels are simply detected in output detectors. Now the output detectors will be projection operators of the form $\left|i\right\rangle\left\langle i\right|$, where $\left|i\right\rangle$ is what we will call a detector mode. The detector modes will be defined by the unitary transformation $U_{\text{single-particle}}$ as follows:

$$\left|2k+1\right\rangle = U_{\text{single-particle}}\left|k, L\right\rangle, \quad \left|2k+2\right\rangle = U_{\text{single-particle}}\left|k, R\right\rangle. \tag{1.19}$$

Because the transformation is unitary and the input modes (the $2d$ basis kets from Equation 1.18) are orthogonal, the output modes must also be orthogonal. This apparatus is shown in Figure 1.2.

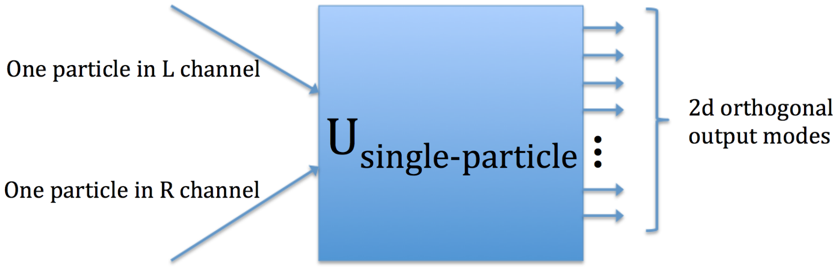
One particle in L channel
One particle in R channel
$U_{\text{single-particle}}$
2d orthogonal output modes

**Figure 1.2** The projective LELM apparatus simply performs single-particle unitary transformations before sending the particles to an orthogonal set of projective detectors.

## 1.5 Particle Statistics

Dealing with two-particle states in the LELM apparatus requires us to consider the statistics of the particles. When particles are distinguishable, we can simply write the states of the particles separately. But when particles are indistinguishable, we must deal with two cases: fermions, which must be antisymmetric under exchange and bosons, which must be symmetric under exchange. We will deal with this implicitly by letting

$$|i\rangle\,|j\rangle = \frac{1}{\sqrt{2}}\big(\,|i\rangle_1\,|j\rangle_2 \pm |j\rangle_1\,|i\rangle_2\,\big), \tag{1.20}$$

where the subscripts 1 and 2 denote particles 1 and 2 and the sign is a plus for bosons and a minus for fermions.

## 1.6 Detection Signatures

Because Bell states are two-particle states, we will be getting 2 detector modes to click. This pair of detections is called a detection signature. In order to determine which Bell states trigger which detectors, we will determine the form of these detection signatures in terms of the input states. We have already dealt with indistinguishability of the particles in the last section, so the order of the tensor product of the detection modes does not matter. We can't simply write the detection signatures as a raw tensor product of two detection modes $|i\rangle \otimes |j\rangle$, because that would contain inputs of two particles from the left and two particles on the right. So we introduce the projection operator $P_{LR}$, which projects these detection signatures onto the subspace of two particle states where one comes from the left side and one comes from the right side (and then renormalizes). So the detection signature resulting from a detection in detector $i$ and a detection in detector $j$ is

$$|i\rangle\,|j\rangle = P_{LR}\,|i\rangle \otimes |j\rangle\,. \tag{1.21}$$

To make this process clearer, let's consider an example. Say that we have two detection modes:

$$|i\rangle = \frac{1}{\sqrt{2}}\big(\,|0,L\rangle + |0,R\rangle\,\big) \quad |j\rangle = \frac{1}{\sqrt{3}}\big(\,|0,L\rangle + |1,L\rangle + |2,R\rangle\,\big) \tag{1.22}$$

Then the tensor product of the two is

$$|i\rangle\otimes|j\rangle = \frac{1}{\sqrt{6}}\big(\,|0,L\rangle\,|0,L\rangle+|0,L\rangle\,|1,L\rangle+|0,L\rangle\,|2,R\rangle+|0,R\rangle\,|0,L\rangle+|0,R\rangle\,|1,L\rangle+|0,R\rangle\,|2,R\rangle\,\big). \tag{1.23}$$

After we apply the projection operator, $P_{LR}$, we get the detection signature:

$$|i\rangle\,|j\rangle = P_{LR}\,|i\rangle \otimes |j\rangle = \frac{1}{\sqrt{3}}\big(\,|0,L\rangle\,|2,R\rangle + |0,L\rangle\,|0,R\rangle + |1,L\rangle\,|0,R\rangle\,\big). \tag{1.24}$$

From now on, we will simply use the notation $|i\rangle\,|j\rangle$ to denote a detection signature after $P_{LR}$ acts. When these detectors are used to distinguish between Bell states, it is helpful to express these detection signatures in the Bell basis. Then the two detectors that make up the signature can trigger if and only if any Bell state in the Bell-basis representation of the signature is fed into the apparatus. So an apparatus cannot distinguish between two Bell states that appear in one detection signature.

## 1.7 Creation and Annihilation Operators

Another useful formalism for representing states and detectors involves creation operators ($\hat{a}_i^\dagger$) and annihilation operators ($\hat{a}_i$). These operators essentially create or remove particles from certain states. We can express states by having the creation operators act on the vacuum (no-particle) state $|\mathbf{0}\rangle$ as follows:

$$|k,L\rangle = \hat{a}_{2k}^\dagger\,|\mathbf{0}\rangle\,, \quad |k,R\rangle = \hat{a}_{2k+1}^\dagger\,|\mathbf{0}\rangle \tag{1.25}$$

We can also use multiple creation operators to represent multiple particles in certain states. For example, we can write

$$|0,L\rangle\,|1,R\rangle = \hat{a}_0^\dagger \hat{a}_3^\dagger\,|\mathbf{0}\rangle\,. \tag{1.26}$$

Now annihilation operators can act as projective detectors and remove particles from a system after they are detected. If we apply the projection operator $|0, L\rangle \langle 0, L|$ to the state in Equation 1.26, we could act on it with $\hat{a}_0$ to get

$$\hat{a}_0 \hat{a}_0^\dagger \hat{a}_3^\dagger |\mathbf{0}\rangle = \hat{a}_3^\dagger |\mathbf{0}\rangle = |1, R\rangle . \tag{1.27}$$

However, if instead we applied the projector $|0, R\rangle \langle 0, R|$, we should get nothing, because both particle states are orthogonal to $|0, R\rangle$. So when we act with $\hat{a}_1$, we get

$$\hat{a}_1 \hat{a}_0^\dagger \hat{a}_3^\dagger |\mathbf{0}\rangle = 0. \tag{1.28}$$

Now we can write any detection mode as a superposition of these annihilation operators. If we have a general detection mode that comes from $U_{\text{single-particle}}$, it will look like

$$|c\rangle = \nu_0 |0, L\rangle + \nu_1 |0, R\rangle + \ldots + \nu_{2d-1} |d-1, R\rangle , \tag{1.29}$$

where the $\nu$s are entries in the $c$th column of $U_{\text{single-particle}}$ and the annihilation operator corresponding to that mode is

$$\hat{c} = \nu_0 \hat{a}_0 + \nu_1 \hat{a}_1 + \ldots + \nu_{2d-1} \hat{a}_{2d-1}. \tag{1.30}$$

## 1.8 Necessary Conditions for Distinguishability

One method of determining whether a set of states is distinguishable using an LELM apparatus is detailed in van Loock and Lütkenhaus (2004). The idea is that all states must remain orthogonal after one particle is detected in some detector. From Equation 1.30, we can write the annihilation operator for a detection mode as

$$\hat{c} = \nu_0 \hat{a}_0 + \nu_1 \hat{a}_1 + \ldots + \nu_{2d-1} \hat{a}_{2d-1}.$$

This means that if a state $|\psi\rangle$ has one particle detected in this detector, the remaining state is

$$\hat{c} |\psi\rangle . \tag{1.31}$$

So for a set of states $\{|\psi_i\rangle\}$ to be distinguishable, these remaining states must be orthogonal, so we must have

$$\langle \psi_k | \hat{c}^\dagger \hat{c} | \psi_l \rangle = 0 \qquad \forall k \neq l. \tag{1.32}$$

These are only necessary conditions on one detector; they are definitely not sufficient for showing that a set of states is distinguishable. For a set of states to be distinguishable, after any one detector fires, all of the states must trigger different second detectors. This implies that the states after one detection must be orthogonal, but the states could still be orthogonal but trigger the same second detector. Still, these necessary conditions are still very useful for ruling out sets of states as indistinguishable.

# Chapter 2

# Maximum Distinguishability of Qutrit Bell States with Projective Measurement

## 2.1 Overview

In this chapter, we will establish the maximum size of a set of bosonic qutrit Bell states that can be distinguished from each other reliably with a projective LELM device. We begin by motivating Bell measurement and we summarize results in maximal Bell state distinguishability from before my work. It has already been established that only 3 out of 4 qubit Bell states can be distinguished with projective LELM devices and that 3 or 4 is the maximum number of qutrit Bell states that can be distinguished with projective LELM devices. The criteria from Equation 1.32 were not sufficient to determine whether a set of 4 qutrit Bell states could be distinguished, so in this chapter, I detail additional arguments that were made to rule out such a possibility for bosonic qutrits.

## 2.2 Applications of Bell State Measurements

Making unambiguous and deterministic measurements in the Bell basis is required in many applications. If one wants to send a quantum state over some distance, they may want to use a quantum teleportation protocol, which requires a Bell measurement. The famous qubit teleportation protocol was introduced in Bennett et al. (1993). If this distance is sufficiently long, one might want to use quantum repeaters, which can extend the distance that one would send a quantum state through a noisy channel. These devices, introduced in Briegel et al. (1998), also require Bell measurements to be made. If one wants to send many bits of classical information with relatively few qubits, they may use quantum dense coding protocols. The first of these was introduced in Mattle et al. (1996), and it requires a Bell measurement. Additionally, using quantum states for computation can allow one to build quantum computers, which theoretically offer an exponential speedup over classical computers. In order to make these computers fault tolerant, one needs to use quantum error correction protocols, which also make use of Bell measurements. Some of these protocols were developed in Gottesman and Chuang (1999).

## 2.3 The Qubit Bell States are not Completely Distinguishable

The qubit Bell state no-go theorem establishes that projective measurements can only distinguish 3 out of 4 qubit Bell states. To show this, we must show both that 3 out of 4 qubit Bell states can be distinguished and that all 4 cannot be distinguished. By applying the condition in Equation 1.32 to all 4 qubit Bell states, we can show that they cannot all be distinguished from each other. In Lütkenhaus et al. (1999), an apparatus that can distinguish 3 of the qubit Bell states was detailed. So at most 3 qubit Bell states can be distinguished by projective measurement. The next objective was to determine maximum distinguishability of the qutrit Bell states using projective measurement.

## 2.4 3 Qutrit Bell States are Distinguishable, but 5 are not

Like what was done for qubits, to show that $n$ is the maximum number of qutrit Bell states distinguishable using an LELM apparatus, we need to show both that $n$ qutrit Bell states are distinguishable and that $n + 1$ qutrit Bell states are not distinguishable.

First, we will show by a simple construction that 3 qutrit Bell states are distinguishable. If an LELM apparatus simply measures both particles in the standard basis, it can distinguish between the 3 correlation classes by measuring the correlation between the variable values. So we can pick 3 Bell states from different correlation classes and distinguish them using this method, which means that at least 3 are distinguishable.

To show that 3 was the maximum number of qutrit Bell states distinguishable with an LELM device, we would need to show that 4 were not distinguishable. Unfortunately, the necessary criteria from Equation 1.32 were not sufficient to do that. However, they are sufficient to show that 5 were not distinguishable. Julien Devin (HMC '11), a previous student in this group used those criteria to show that 5 qutrit Bell states were not distinguishable. He wrote a Mathematica program that would generate the system of equations that come from the criteria above for each of the 126 sets of 5 qutrit Bell states. He added a normalization condition for the detector and then ran the FindInstance command on all of the systems of equations and all gave no solutions. From that point, the only question was whether or not 4 qutrit Bell states were distinguishable.

## 2.5 Equivalence Classes of 4 Qutrit Bell States

Julien also used that program to generate the criteria from Equation 1.32 for the 126 sets of 4 qutrit Bell states. 54 of these had no solutions and 72 of these either gave solutions or did not complete. He was not able to determine whether or not the solutions could make valid sets of 6 orthogonal detectors; that proved to be too computationally difficult. The class of 54 that had no solutions do not meet the necessary conditions for distinguishability; no detector exists that would leave those sets orthogonal to each other after one particle is detected, so they are indistinguishable using an LELM apparatus. The class of 72 that gave solutions or did not complete were inconclusive, so we could not determine simply from these criteria whether or not they were distinguishable.

First, to understand the distinction between these two groups of sets of Bell states, it is useful to look at sets of qutrit Bell states (Equation 1.17) in a $3 \times 3$ grid where the $c$ (correlation) index corresponds to rows and the $p$ (phase) index corresponds to columns; we will call these grids tic-tac-toe diagrams. For example, the set $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_2^1\rangle, |\Psi_2^2\rangle\}$ can be expressed by Figure 2.1 below.

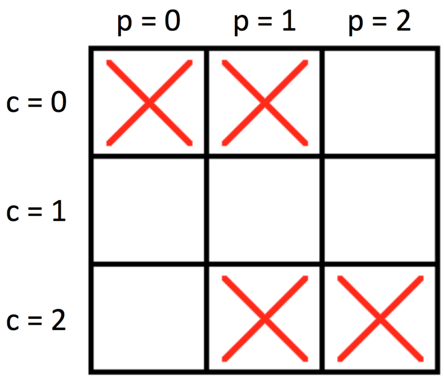

**Figure 2.1** The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^1_0\rangle, |\Psi^1_2\rangle, |\Psi^2_2\rangle\}$.

We call the 72 sets of Bell states that were inconclusive with the necessary distinguishability conditions the "tic-tac-toe winners" because they form boards that win at tic-tac-toe (they have 3 Xs in a row vertically, horizontally or diagonally) when we allow for column permutation or allow wrap-around boundaries. For example, the set $\{|\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^2_1\rangle, |\Psi^0_2\rangle\}$ shown in Figure 2.2 below is a tic-tac-toe winner.

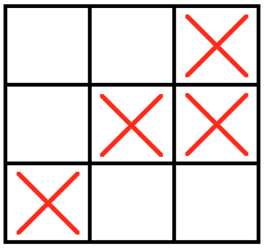

**Figure 2.2** The tic-tac-toe diagram of the set $\{|\Psi^2_0\rangle, |\Psi^1_2\rangle, |\Psi^2_1\rangle, |\Psi^0_2\rangle\}$. This belongs to the tic-tac-toe winners class because it has 3 in a row diagonally.

The set $\{|\Psi^1_0\rangle, |\Psi^0_1\rangle, |\Psi^1_1\rangle, |\Psi^2_2\rangle\}$ shown in Figure 2.3 below is also a tic-tac-toe winner. But unlike $\{|\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^2_1\rangle, |\Psi^0_2\rangle\}$ in Figure 2.2 above, $\{|\Psi^1_0\rangle, |\Psi^0_1\rangle, |\Psi^1_1\rangle, |\Psi^2_2\rangle\}$ requires either row permutation or wrap-around boundaries to give it 3 in a row.

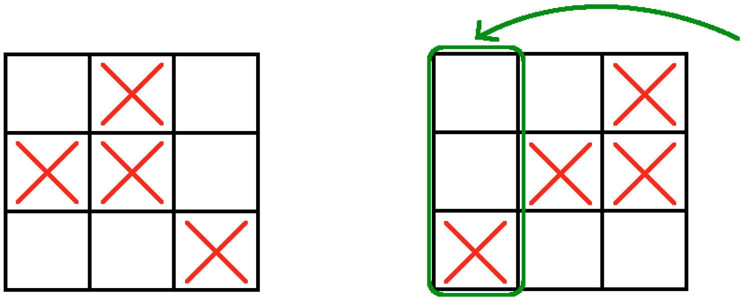

**Figure 2.3** The tic-tac-toe diagram of the set $\{|\Psi^1_0\rangle, |\Psi^0_1\rangle, |\Psi^1_1\rangle, |\Psi^2_2\rangle\}$. This belongs to the tic-tac-toe winners class because it has 3 in a row diagonally when we permute the columns or allow wrap-around boundaries.

We call the 54 sets of Bell states that were already determined to be indistinguishable the "tic-tac-toe losers", because they do not form boards that win at tic-tac-toe, even allowing column permutation or wrap-around boundaries. For example, the set $\{|\Psi^0_0\rangle, |\Psi^1_0\rangle, |\Psi^1_2\rangle, |\Psi^2_2\rangle\}$ in Figure 2.1 above and the set $\{|\Psi^0_0\rangle, |\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^1_2\rangle\}$ in Figure 2.4 below are both tic-tac-toe losers. Now we will show that all of the sets of Bell states in these two classes must share distinguishability or indistinguishability. We will argue that any set of qutrit Bell states in one of these classes can be transformed into any other set in the same class using the following four transformations:

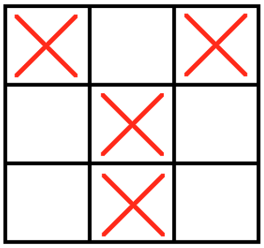

**Figure 2.4** The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^1_2\rangle\}$. Even with column permutation or wrap-around boundaries, we cannot get 3 in a row, so it is a tic-tac-toe loser.

1. Cycle the variable value in the right channel:

$$|0, R\rangle \to |1, R\rangle \to |2, R\rangle\,. \tag{2.1}$$

   This increases the c index of all of the Bell states.

2. Add phases to kets in the left channel:

$$|1,L\rangle \to e^{\frac{2\pi i}{3}}\,|1,L\rangle\,, \quad |2,L\rangle \to e^{\frac{4\pi i}{3}}\,|2,L\rangle\,. \tag{2.2}$$

This increases the p index of all of the Bell states.

3. Add phases to kets with variable value 0:

$$|0,L\rangle \to e^{\frac{2\pi i}{3}}\,|0,L\rangle\,, \quad |0,R\rangle \to e^{\frac{4\pi i}{3}}\,|0,R\rangle\,. \tag{2.3}$$

This increases the p index of the Bell states by their c index.

4. Change basis to

$$|0'\rangle = \frac{1}{\sqrt{3}}\big(\,|0\rangle + |1\rangle + |2\rangle\,\big) \tag{2.4a}$$

$$|1'\rangle = \frac{1}{\sqrt{3}}\big(\,|0\rangle + e^{\frac{2\pi i}{3}}\,|1\rangle + e^{\frac{4\pi i}{3}}\,|2\rangle\,\big) \tag{2.4b}$$

$$|2'\rangle = \frac{1}{\sqrt{3}}\big(\,|0\rangle + e^{\frac{4\pi i}{3}}\,|1\rangle + e^{\frac{2\pi i}{3}}\,|2\rangle\,\big) \tag{2.4c}$$

add phases to kets in the new basis with variable value 0′ as in transformation 3, and transform back:

$$|0,L\rangle \to \frac{1}{\sqrt{3}}\big(e^{i\frac{\pi}{6}}\,|0,L\rangle + e^{i\frac{5\pi}{6}}\,|1,L\rangle + e^{i\frac{5\pi}{6}}\,|2,L\rangle\,\big), \tag{2.5a}$$

$$|1,L\rangle \to \frac{1}{\sqrt{3}}\big(e^{i\frac{5\pi}{6}}\,|0,L\rangle + e^{i\frac{\pi}{6}}\,|1,L\rangle + e^{i\frac{5\pi}{6}}\,|2,L\rangle\,\big), \tag{2.5b}$$

$$|2,L\rangle \to \frac{1}{\sqrt{3}}\big(e^{i\frac{5\pi}{6}}\,|0,L\rangle + e^{i\frac{5\pi}{6}}\,|1,L\rangle + e^{i\frac{\pi}{6}}\,|2,L\rangle\,\big), \tag{2.5c}$$

$$|0,R\rangle \to \frac{1}{\sqrt{3}}\big(e^{-i\frac{\pi}{6}}\,|0,R\rangle + e^{-i\frac{5\pi}{6}}\,|1,R\rangle + e^{-i\frac{5\pi}{6}}\,|2,R\rangle\,\big), \tag{2.5d}$$

$$|1,R\rangle \to \frac{1}{\sqrt{3}}\big(e^{-i\frac{5\pi}{6}}\,|0,R\rangle + e^{-i\frac{\pi}{6}}\,|1,R\rangle + e^{-i\frac{5\pi}{6}}\,|2,R\rangle\,\big), \tag{2.5e}$$

$$|2,R\rangle \to \frac{1}{\sqrt{3}}\big(e^{-i\frac{5\pi}{6}}\,|0,R\rangle + e^{-i\frac{5\pi}{6}}\,|1,R\rangle + e^{-i\frac{\pi}{6}}\,|2,R\rangle\,\big). \tag{2.5f}$$

This increases the c index of the Bell states by their p index (and adds a phase of $\frac{4\pi}{3}$ if the p index is nonzero).

Originally, Julien showed that these transformations transformed all sets in each class into each other in Mathematica. It can also be verified by hand by noting that the transformations do the following to the tic-tac-toe diagrams.

1. Cycle all rows down.
2. Cycle all columns right.
3. Cycle the rows differently. Don't cycle the top row, cycle the middle row right and cycle the bottom row left.
4. Cycle the columns differently. Don't cycle the left column, cycle the middle column down and cycle the right column up.

One can use these transformations to transform any specific tic-tac-toe winner into all of the other tic-tac-toe winners or any specific tic-tac-toe loser into all of the other tic-tac-toe losers. We will leave this as an exercise to the reader, but we will show that is all that is required to show that any set can be transformed into any other in the same class. First, we note that we can perform the inverse of any sequence of transformations. Because each of these transformations can be undone by applying itself twice again, the inverse of any sequence of transformations can be performed by performing each transformation twice in reverse. Now if we let the representative tic-tac-toe winner set be $W$ and the representative tic-tac-toe loser set be $L$, the sequence of transformations to get from any set $A$ to any other set $B$ in the same class can be described simply. This sequence is just the inverse of the sequence from $W$ or $L$ to $A$ followed by the sequence from $W$ or $L$ to $B$.

Because these transformations are all unitary operations on each channel, they can be realized in an LELM apparatus. If any set in one of the classes is distinguishable, then any other set in that class could be transformed into that set with an LELM apparatus and be distinguished that way. Thus if one set in a class is distinguishable then all classes in the set are distinguishable. Similarly, if one set in one of the classes is indistinguishable, it can be transformed into any other set in its class using an LELM apparatus and must remain indistinguishable. Therefore, if any set in a class is indistinguishable, then all sets in that class must also be indistinguishable. So all of the sets of Bell states in both classes must share distinguishability or indistinguishability. We know the tic-tac-toe losers are all indistinguishable. Thus we only need to consider the distinguishability of one tic-tac-toe winner to determine whether or not all of them are distinguishable.

## 2.6 Putting Further Limits on Detection Modes

The necessary distinguishability criteria of Equation 1.32 were not enough to show that the tic-tac-toe winners were indistinguishable. To investigate further, I chose to search for limits on what the detectors could look like in order to either determine the form of detectors that could distinguish the tic-tac-toe winners or to add additional conditions that, along with the equations from the necessary distinguishability criteria, would be able to rule out distinguishability of the tic-tac-toe winners.

The goal was to put as many restrictions as possible on the number of kets (input basis states from Equation 1.18) that could be in a detection mode. Because all of the tic-tac-toe winners share distinguishability, we only needed to do this for a single set of Bell states in the tic-tac-toe winners class.

We were able to frequently take advantage of the possibility of a double click in a single detector to rule out detection modes, but the corresponding detections signatures are only nonzero for bosonic qutrits, so we will be considering bosonic statistics and assume the plus sign from Equation 1.20 for the remainder of this chapter.

## 2.7 Detector Modes in the Bell Basis and in the Joint-Particle Basis

In order to put restrictions on the number of kets that can be in a detection mode, I need to know the minimum and maximum number of Bell states that are in detection signatures involving those modes. First, we will establish two restrictions on detection modes and detection signatures.

If I can show that a detection signature must necessarily contain multiple Bell states that I am trying to distinguish, then I can say that that detection signature cannot be part of an apparatus that distinguishes those Bell states.

We note that the state of one of the particles in a Bell state is completely random, so we can not get any meaningful information from our first detection. We would have to discriminate between the Bell states solely based on information from the second particle. For a more rigorous treatment of this argument, see Pisenti et al. (2011). Therefore, if I can show that no detection signatures involving a mode contain a certain Bell state that I am trying to distinguish, then I can say that that mode cannot be part of an apparatus that distinguishes that Bell state from any others.

It will be much easier to construct detection signatures from modes in the joint-particle basis, so next, we want to know how many Bell states can be in a signature based on its joint-particle representation. We see that both joint-particle-kets and Bell states are grouped by correlation class, so we can separate the joint-particle kets into correlation classes that contain different numbers of Bell states in the same correlation class.

Next, we will establish that for each correlation class, there are a few restrictions on the number of Bell states that can be in a signature based on the number of joint-particle kets in that signature. First, the only way for a detection signature to have a single Bell state in a correlation class is for the joint-particle kets in that correlation class to form a scalar multiple of the Bell state, which requires all 3 joint-particle kets in that correlation class. Second, a single joint-particle ket in a correlation class consists of all Bell states in its correlation class. Lastly, it is possible for a state made up of 2 Bell states in a correlation class to be made up of only 2 joint-particle kets.

## 2.8 Ruling out 4-Ket Detection Modes

First, for any set of tic-tac-toe winners, we were able to rule out 4-ket detection modes using the following argument. A 4-ket detection mode either has 1 ket in one channel and 3 in the other or 2 in both channels. If a 4-ket mode has 1 ket in one channel and 3 in the other, then without loss of generality, let it have 1 ket in the left channel and three in the right. Let $a$, $b$ and $c$ represent 0,1 and 2 in an arbitrary permutation and let $\alpha$, $\beta$, $\gamma$ and $\delta$ be arbitrary nonzero coefficients. Then this detection mode would look like

$$|\text{4-ket}_1\rangle = \alpha\,|a, L\rangle + \beta\,|a, R\rangle + \gamma\,|b, R\rangle + \delta\,|c, R\rangle\,. \tag{2.6}$$

Then the detection signature corresponding to two clicks in this detector would be

$$|\text{4-ket}_1\rangle\,|\text{4-ket}_1\rangle = \alpha\beta\,|aa\rangle + \alpha\gamma\,|ab\rangle + \alpha\delta\,|ac\rangle\,. \tag{2.7}$$

Here, as each joint-particle ket must contain a left-channel ket and a right-channel ket, we can leave off the $L$ and $R$, and put the left-channel variable value first. We notice that this is a superposition of 3 joint-particle kets from different correlation classes. Because 1 joint-particle ket in a correlation class must be made of a superposition of all 3 Bell states in that class, this detection signature will be a superposition of all 9 Bell states, so it would not be able to distinguish any Bell states. So to distinguish 4 of them, we cannot have 4-ket modes with 1 ket in one channel and 3 kets in the other. That leaves 2 kets in each channel.

If there are 2 kets in each channel, there are 2 possibilities. Either the kets in both channels have the same variable values or they only share one variable value. If they have the same variable values, they look like

$$|4\text{-ket}_2\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|a, R\rangle + \delta\,|b, R\rangle \tag{2.8}$$

and the detection signature corresponding to two clicks in this detector would be

$$|4\text{-ket}_2\rangle\,|4\text{-ket}_2\rangle = \alpha\gamma\,|aa\rangle + \beta\delta\,|bb\rangle + \alpha\delta\,|ab\rangle + \beta\gamma\,|ba\rangle\,. \tag{2.9}$$

Because $a$ and $b$ are different, $|ab\rangle$ and $|ba\rangle$ are single joint-particle kets in the $c = 1$ and $c = 2$ correlation classes. Again, these must each be made up of all 3 Bell states in their correlation classes. $|aa\rangle$ and $|bb\rangle$ are both from the $c = 0$ correlation class, and 2 joint-particle kets are made up of at least 2 Bell states, so this must have at least 2 Bell states in the $c = 0$ correlation class. In total, this signature must be a superposition of at least 8 out of the 9 Bell states, so it could not be present in an apparatus that distinguishes 4 of them.

If there are 2 kets in each channel and the kets in both channels only share one variable value, then the mode must look like

$$|4\text{-ket}_3\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|a, R\rangle + \delta\,|c, R\rangle \tag{2.10}$$

and the detection signature corresponding to two clicks in this detector would be

$$|4\text{-ket}_3\rangle\,|4\text{-ket}_3\rangle = \alpha\gamma\,|aa\rangle + \beta\delta\,|bc\rangle + \alpha\delta\,|ac\rangle + \beta\gamma\,|ba\rangle\,. \tag{2.11}$$

Here, $|aa\rangle$ and $|bc\rangle$ are the only joint-particle kets in their correlation classes, so again we get all 6 Bell states from both of those correlation classes. $|ac\rangle$ and $|ba\rangle$ are both in the other correlation class, which gives at least 2 more Bell states. So again, we get at least 8 or 9 Bell states, which is not allowed. So we cannot have 2 kets in each channel either, which means 4-ket modes cannot be present in an apparatus to distinguish 4 of 9 qutrit Bell states.

## 2.9 Ruling out Single-Channel Detection Modes

In Section 2.5, we listed out transformations that could be realized as part of an LELM device that allowed us to transform any set of 4 Bell states into another set in the same class. Because these transformations act separately on each channel (i.e. they do not send particles from one channel to the other), they transform single-channel detectors into single-channel detectors and transform multi-channel detectors into multi-channel detectors. Thus, if we want to show that single-channel detectors are not allowed to distinguish any set of tic-tac-toe winners, then we only need to show that single-channel detectors cannot distinguish one set of tic-tac-toe winners. If a single-channel detector could be used to distinguish one set, then any other set could also be distinguished using a transformed single-channel detector.

We will demonstrate that single-channel detectors cannot be in an apparatus that can distinguish $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_1^1\rangle, |\Psi_2^2\rangle\}$, shown below in Figure 2.5, to show that single-channel detectors cannot be in an apparatus that can distinguish any set of tic-tac-toe winners.

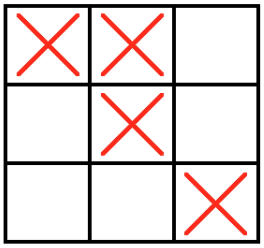

**Figure 2.5** The tic-tac-toe diagram of the set $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_1^1\rangle, |\Psi_2^2\rangle\}$. We will show that single-channel detectors cannot be part of an apparatus that distinguishes these states.

Without loss of generality, we will let the single-channel modes be in the left channel. Single-channel modes can have either 1, 2 or 3 kets in them. If a single-channel mode were to have 1 ket, it would look like

$$|S_1\rangle = |a, L\rangle\,. \tag{2.12}$$

Then we can write an arbitrary mode as

$$|i\rangle = x\,|a, R\rangle + y\,|b, R\rangle + z\,|c, R\rangle + |L\rangle\,, \tag{2.13}$$

where $x$, $y$ and $z$ could be zero and $|L\rangle$ is a superposition of kets in the left channel. We can then write all detection signatures involving a click in the 1-ket single-channel detector as

$$|S_1\rangle\,|i\rangle = x\,|aa\rangle + y\,|ab\rangle + z\,|ac\rangle\,. \tag{2.14}$$

If $x = 0$, then there are no joint-particle kets in the $c = 0$ correlation class. If $x \neq 0$, then there is only one joint-particle ket in the $c = 0$ correlation class, which means that there are all 3 Bell states in the $c = 0$ correlation class. Neither of these possibilities allow $|\Psi_0^0\rangle$ and $|\Psi_0^1\rangle$ to be distinguished, so there can't be any 1-ket single-channel detector modes. If a single-channel mode were to have 2 kets, it would look like

$$|S_2\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle\,. \tag{2.15}$$

Using the arbitrary mode in Equation 2.13, we can write all detection signatures involving a click in the 2-ket single-channel detector as

$$|S_2\rangle\,|i\rangle = x\alpha\,|aa\rangle + y\beta\,|bb\rangle + y\alpha\,|ab\rangle + z\beta\,|bc\rangle + x\beta\,|ba\rangle + z\alpha\,|ac\rangle\,. \tag{2.16}$$

If $x$, $y$ and $z$ are all 0, there is no detection signature. If one or two of them is zero, then 2 correlation classes only have 1 joint-particle ket, which means that those two correlation classes must have all 6 Bell states in them. As we can see in Figure 2.5, each correlation class (row) has at least one Bell state (X) in it, so this detection signature would render at least two of those Bell states indistinguishable. So after $|S_2\rangle$, all of the remaining detector modes must have either no right-channel kets or all of them.

Those that have all of the right-channel kets would create detection signatures with $|S_2\rangle$ that would have 2 joint-particle kets in each correlation class. That means that they would have to have at least 2 Bell states in each correlation class. This could distinguish $|\Psi_0^0\rangle$ or $|\Psi_0^1\rangle$ from the rest, but the signature would have to contain one of those two, so it could not distinguish $|\Psi_1^1\rangle$ or $|\Psi_2^2\rangle$. So there can be no 2-ket single-channel detector modes.

If a single-channel mode were to have 3 kets, it would look like

$$|S_3\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|c, L\rangle\,. \tag{2.17}$$

Using the arbitrary mode in Equation 2.13, we can write all detection signatures involving a click in the 3-ket single-channel detector as

$$\begin{aligned}|S_3\rangle\,|i\rangle = x\alpha\,|aa\rangle + y\beta\,|bb\rangle + z\gamma\,|cc\rangle + y\alpha\,|ab\rangle \\ + z\beta\,|bc\rangle + x\gamma\,|ca\rangle + z\alpha\,|ac\rangle + x\beta\,|ba\rangle + y\gamma\,|cb\rangle\,.\end{aligned} \tag{2.18}$$

As we just showed at the end of the previous example, if the $c = 0$ correlation class has 2 or more Bell states, this signature cannot distinguish $|\Psi_1^1\rangle$ or $|\Psi_2^2\rangle$. So there must be at least 2 detection signatures that detect $|\Psi_1^1\rangle$ and $|\Psi_2^2\rangle$

and both must only have $|\Psi_0^2\rangle$ as the only Bell state in the $c = 0$ correlation class. This fixes $x$, $y$ and $z$ up to an overall phase, which allows there to be only one signature that has $|\Psi_0^2\rangle$ as the only Bell state in the $c = 0$ correlation class, where we needed two such signatures. So there can be no 3-ket single-channel signatures.

So we cannot have any single-channel detector modes in an apparatus that can distinguish $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_1^1\rangle, |\Psi_2^2\rangle\}$. Again, because transformations between the tic-tac-toe winners preserve single-channel detectors, we cannot have single-channel modes in any tic-tac-toe winner. With this piece of information, we are now ready to focus on our final tic-tac-toe winner.

## 2.10 Forcing a 6-Ket Mode in a Specific Tic-Tac-Toe Winner

Now that we have established multiple restrictions on the general form of a tic-tac-toe winner, we can start to restrict a specific tic-tac-toe winner to determine distinguishability of the entire class. The set that we will look at is $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle\}$, shown below in Figure 2.6.

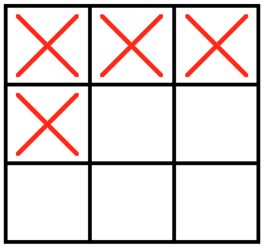

**Figure 2.6** The tic-tac-toe diagram of the set $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle\}$. We will show that any apparatus that distinguishes these states must have a detector with all 6 joint-particle kets.

The advantage of using this set is that it contains all 3 Bell states in the $c = 0$ correlation class. Thus, detection signatures would have to have only one of the Bell states in the $c = 0$ correlation class, so they must have either 0 or 1 $c = 0$ joint-particle kets.

In any apparatus that could distinguish this set, we can show that there cannot be any 5-ket modes. Without loss of generality, such a mode could be missing one ket in the right channel. It would then look like

$$|\text{5-ket}\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|c, L\rangle + \delta\,|a, R\rangle + \epsilon\,|b, R\rangle\,. \tag{2.19}$$

Then the detection signature corresponding to two clicks in this detector is

$$|\text{5-ket}\rangle\,|\text{5-ket}\rangle = \alpha\delta\,|aa\rangle + \beta\epsilon\,|bb\rangle + \alpha\epsilon\,|ab\rangle + \gamma\delta\,|ca\rangle + \beta\delta\,|ba\rangle + \gamma\epsilon\,|cb\rangle\,. \tag{2.20}$$

We see that this has 2 joint-particle kets from the $c = 0$ correlation class, so it is not allowed. So there cannot be any 5-ket modes.

There clearly cannot be any 1-ket modes, because those are necessarily single-channel. Next, we will show that there cannot be any 2-ket modes. We know such a mode can't be single-channel, so we will consider only multi-channel 2-ket modes. One of these detectors could either have kets of the same variable value or different variable values. If the detector had kets with the same variable value, it would look like

$$|\text{2-ket}_1\rangle = \alpha\,|a, L\rangle + \beta\,|a, R\rangle \tag{2.21}$$

and the detection signature corresponding to two clicks in this detector would be

$$|\text{2-ket}_1\rangle\,|\text{2-ket}_1\rangle = |aa\rangle\,. \tag{2.22}$$

This has only one joint-particle ket in the $c = 0$ class, so it is not allowed. If the detector had kets with different variable values, it would look like

$$|\text{2-ket}_2\rangle = \alpha\,|a, L\rangle + \beta\,|b, R\rangle\,. \tag{2.23}$$

In order to have 6 orthogonal detectors, one other detector would have to contain the $|a, R\rangle$ ket. So this detector would look like

$$|i\rangle = v\,|a, R\rangle + w\,|b, R\rangle + x\,|c, R\rangle + \eta\,|a, R\rangle + y\,|b, R\rangle + z\,|c, R\rangle\,, \tag{2.24}$$

where $v, w, x, y$ and $z$ may be zero, but $\eta$ is nonzero. The detection signature corresponding to clicks in these two detectors would be

$$|\text{2-ket}_2\rangle\,|i\rangle = \alpha\eta\,|aa\rangle + w\beta\,|bb\rangle + (y\alpha + v\beta)\,|ab\rangle + z\alpha\,|ac\rangle + x\beta\,|cb\rangle\,. \tag{2.25}$$

We see that this has either 1 or 2 joint-particle kets in the $c = 0$ correlation class (based on whether w is 0 or not). This is also not allowed. So there can be no 2-ket modes.

Now all that is left are multi-channel 3-ket modes and 6-ket modes. We will show that there must be a 6-ket mode by showing that this apparatus cannot consist of six 3-ket modes.

If we have 1 multi-channel 3-ket mode, it must have 2 kets in one channel and 1 ket in the other. Without loss of generality, we will let it have 2 kets in the left channel. Then it will look like

$$|3\text{-ket}_1\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|d, R\rangle\,, \tag{2.26}$$

where $d = a$, $b$ or $c$. Then the detection signature corresponding to two clicks in this detector will be

$$|3\text{-ket}_1\rangle\,|3\text{-ket}_1\rangle = \alpha\gamma\,|ad\rangle + \beta\gamma\,|bd\rangle\,. \tag{2.27}$$

If $d = a$ or $b$, then this signature will have 1 joint-particle ket in the $c = 0$ correlation class, so we must have $d = c$. So

$$|3\text{-ket}_1\rangle = \alpha\,|a, L\rangle + \beta\,|b, L\rangle + \gamma\,|c, R\rangle\,. \tag{2.28}$$

Another detector would have to have $|a, R\rangle$, $|b, R\rangle$ or $|c, L\rangle$. If it did not have all 3, the detection signature $|3\text{-ket}_1\rangle\,|3\text{-ket}_2\rangle$ would have fewer than 3 joint-particle kets in the $c = 0$ correlation class. So we would have to have

$$|3\text{-ket}_2\rangle = \delta\,|c, L\rangle + \epsilon\,|a, R\rangle + \eta\,|b, R\rangle\,. \tag{2.29}$$

Then the detection signature $|3\text{-ket}_1\rangle\,|3\text{-ket}_2\rangle$ is

$$|3\text{-ket}_1\rangle\,|3\text{-ket}_2\rangle = \alpha\epsilon\,|aa\rangle + \beta\eta\,|bb\rangle + \gamma\delta\,|cc\rangle + \alpha\eta\,|ab\rangle + \beta\epsilon\,|ba\rangle\,. \tag{2.30}$$

This has all 3 joint-particle kets in the $c = 0$ correlation class, but it also has only 1 joint-particle ket in each of the other correlation classes. So it would have to have at least 7 Bell states, which would have to contain 2 of the 4 Bell states that we are trying to distinguish. This is also not allowed, so we would have to have at least 1 detector mode with all 6 kets.

## 2.11 Forcing No Solution to the Necessary Conditions

Now that we have the requirement that at least one detector must have all 6 kets, we look back at the necessary distinguishability criteria from Section 1.8. For the set $\{|\Psi^0_0\rangle\,, |\Psi^1_0\rangle\,, |\Psi^2_0\rangle\,, |\Psi^0_1\rangle\}$, shown in Figure 2.6, we will use the following subset of the conditions in Equation 1.32:

$$\langle\Psi^0_0|\,\hat{c}^\dagger\hat{c}\,|\Psi^0_1\rangle = 0 \tag{2.31a}$$

$$\langle\Psi^1_0|\,\hat{c}^\dagger\hat{c}\,|\Psi^0_1\rangle = 0 \tag{2.31b}$$

$$\langle\Psi_0^2|\,\hat{c}^\dagger\hat{c}\,|\Psi_1^0\rangle = 0 \tag{2.31c}$$

If we now consider the detector mode that must have 6 kets:

$$|\text{6-ket}\rangle = v_0^*\,|0,L\rangle + v_1^*\,|1,L\rangle + v_2^*\,|2,L\rangle + v_3^*\,|0,R\rangle + v_4^*\,|1,R\rangle + v_5^*\,|2,R\rangle\,, \tag{2.32}$$

the associated annihilation operator is

$$\hat{c} = v_0\hat{a}_0 + v_1\hat{a}_1 + v_2\hat{a}_2 + v_3\hat{a}_3 + v_4\hat{a}_4 + v_5\hat{a}_5. \tag{2.33}$$

Plugging this into the conditions above in Equations 2.31 gives Equations 2.34 below.

$$\langle\Psi_0^0|\,\hat{c}^\dagger\hat{c}\,|\Psi_1^0\rangle = (v_4v_3^* + v_2v_0^*) + (v_5v_4^* + v_0v_1^*) + (v_3v_5^* + v_1v_2^*) = 0 \tag{2.34a}$$

$$\langle\Psi_0^1|\,\hat{c}^\dagger\hat{c}\,|\Psi_1^0\rangle = (v_4v_3^*+v_2v_0^*)+e^{i\frac{4\pi}{3}}(v_5v_4^*+v_0v_1^*)+e^{i\frac{2\pi}{3}}(v_3v_5^*+v_1v_2^*) = 0 \tag{2.34b}$$

$$\langle\Psi_0^2|\,\hat{c}^\dagger\hat{c}\,|\Psi_1^0\rangle = (v_4v_3^*+v_2v_0^*)+e^{i\frac{2\pi}{3}}(v_5v_4^*+v_0v_1^*)+e^{i\frac{4\pi}{3}}(v_3v_5^*+v_1v_2^*) = 0 \tag{2.34c}$$

Using various linear combinations of these, we can get

$$v_4v_3^* + v_2v_0^* = 0 \tag{2.35a}$$

$$v_5v_4^* + v_0v_1^* = 0 \tag{2.35b}$$

$$v_3v_5^* + v_1v_2^* = 0. \tag{2.35c}$$

We can then rewrite these as

$$v_4v_3^* = -v_2v_0^* \tag{2.36a}$$

$$v_5v_4^* = -v_0v_1^* \tag{2.36b}$$

$$v_3v_5^* = -v_1v_2^*. \tag{2.36c}$$

If we multiply all of these together, we get

$$|v_0|^2|v_1|^2|v_2|^2 = -|v_3|^2|v_4|^2|v_5|^2. \tag{2.37}$$

Because all of these magnitudes are non-negative, both sides would have to be zero. But because $|\text{6-ket}\rangle$ must have all 6 kets, none of the $v$s can be 0. So these conditions cannot be satisfied.

This means that, for bosons, the set $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle\}$ cannot be distinguished with an LELM apparatus. Because the set $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle\}$

is a tic-tac-toe winner, all of the tic-tac-toe winners must be indistinguishable with an LELM apparatus if the particles are bosons. Lastly, because for bosons, both the tic-tac-toe winners and the tic-tac-toe losers are indistinguishable, it is not possible for an LELM apparatus to distinguish 4 out of 9 bosonic qutrit Bell states.

So the maximum number of bosonic qutrit Bell states that can be distinguished with a projective LELM device is 3. This is a notably restrictive upper limit on distinguishable states, since it is realized simply by an apparatus that measures each particle separately in the standard basis.

# Chapter 3

# Distinguishability of Qubit and Qutrit Bell States with POVMs

## 3.1 Overview

In this chapter, we will establish LELM distinguishability bounds for qubit and qutrit Bell states when the requirement of projective measurement is removed. This allows for a more general form of measurement called a POVM. We will offer a mathematical description of a POVM and then present arguments that show that all 4 qubit Bell states cannot be distinguished even with non-projective LELM devices and that no more than 5 qutrit Bell states can be distinguished with general LELM devices.

## 3.2 Introduction to POVMs

So far, we have been considering distinguishability of Bell states in the special case of projective measurements. A more general quantum measurement is called a POVM, a positive operator-valued measurement. A POVM is made up of a set of *Kraus operators* $\hat{E}_i$ and *POVM elements* $\hat{\Pi}_i = \hat{E}_i^{\dagger}\hat{E}_i$, which are positive operators that satisfy

$$\sum_i \hat{\Pi}_i = \hat{I},$$

where $\hat{I}$ is the identity operator in the Hilbert space of the particle(s) being measured. Each index value $i$ corresponds to a measurement that transforms a pure state according to

$$|\psi\rangle \rightarrow \frac{\hat{E}_i |\psi\rangle}{\sqrt{\langle\psi| \hat{E}_i{}^{\dagger} \hat{E}_i |\psi\rangle}} \tag{3.1}$$

with probability

$$p_i = \langle\psi| \hat{E}_i{}^{\dagger} \hat{E}_i |\psi\rangle . \tag{3.2}$$

If all of the Kraus operators are projection operators (then so are the POVM elements), the POVM is a projective measurement, which is what we have been using all along. Otherwise, it is a non-projective measurement. Non-projective POVMs are useful in many distinguishability applications. POVMs can be used for minimum error discrimination and unambiguous state discrimination, and both schemes are able to distinguish between non-orthogonal states in ways that projective measurements cannot. In minimum error discrimination, a measurement returns a result which may be incorrect, but the probability of an incorrect result is minimized. In unambiguous state discrimination, the resulting measurement is never wrong, but there is a probability that the resulting measurement may be inconclusive. These protocols have uses, but we are interested in potential use of non-projective POVMs to discriminate Bell states perfectly, with no theoretical probability of failure. But POVMs show potential for this as well. POVMs have the useful property that the number of measurement outcomes is not limited by the dimension of the Hilbert space. In a projective qudit ($d$-state variable) Bell state discrimination scheme, we are limited to $2d$ detectors that can project onto orthogonal states. In a general POVM scheme, we do not generally get that restriction.

## 3.3 General Limits of Bell State Discrimination

In this thesis, we will develop restrictions on a device that maximally distinguishes Bell states with LELM measurements. We will show that a general POVM cannot distinguish more than $2d$ Bell states in the general case. We will also restrict the form of a maximally distinguishing Kraus operator in a POVM.

For distinguishable particles, we have to measure the two particles separately. Again, from Section 2.7, because the state of an individual particle in a Bell state is random, only a measurement of the second particle can give meaningful information. We know that even in the general POVM case, only orthogonal states can be reliably distinguished (See Preskill (1998)). The second particle only has a $d$-state variable, so only $d$ orthogonal states could exist in its Hilbert space. Therefore, only $d$ Bell states of distinguishable particles could be distinguished.

For indistinguishable particles, we can mix the two particles in two channels, which gives an extra channel variable, doubling the dimensions of the Hilbert spaces for both particles. So again, because the state of the first particle is completely random, we can only distinguish orthogonal states of the second particle. As the second particle has both $d$-state variable and a 2-state channel variable, there are a maximum of $2d$ orthogonal states of the second particle that can be used to distinguish the Bell states. So a general POVM can not distinguish more than $2d$ Bell states.

Now we will restrict the form of a maximally distinguishing Kraus operator in a POVM. Based on whether the particle is a boson or a fermion, we explicitly symmetrize or antisymmetrize the Bell states. Then the Bell states look like

$$|\Psi_p^c\rangle = \eta_1^{c,p}|0,L\rangle|\chi_1^{c,p}\rangle + \eta_2^{c,p}|0,R\rangle|\chi_2^{c,p}\rangle + \ldots + \eta_{2d}^{c,p}|d,R\rangle|\chi_{2d}^{c,p}\rangle, \tag{3.3}$$

where all of the $\eta_i^{c,p}$s all have magnitude $\frac{1}{\sqrt{2d}}$ and $|\chi_1^{c,p}\rangle, \ldots, |\chi_{2d}^{c,p}\rangle$ are some permutation of all of the basis states. After the first measurement, the first particle will be transformed according to Equation 3.1. To maximize distinguishability, we would like the remaining states to be orthogonal. If any remaining states are mixed, then it is impossible to have $2d-1$ states that are orthogonal to every state in the ensemble, so it is not possible to reliably distinguish them. To be able to distinguish $2d$ qudit Bell states, the state remaining after the first measurement would have to be unentangled.

Because the state in Equation 3.3 has each basis state for the first particle and each basis state for the second particle, the state that each Kraus operator takes each basis state to must be a scalar multiple of some fixed state that can be factored out of the resulting 2-particle state. This means that each Kraus operator must be rank 1.

Because of this, I will write a general Kraus operator as

$$\hat{E}_i = \begin{bmatrix} \alpha_1 n_1 & \alpha_2 n_1 & \ldots & \alpha_{2d} n_1 \\ \alpha_1 n_2 & \alpha_2 n_2 & \ldots & \alpha_{2d} n_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 n_{2d} & \alpha_2 n_{2d} & \ldots & \alpha_{2d} n_{2d} \end{bmatrix} \tag{3.4}$$

## 3.4 POVMs cannot distinguish all 4 qubit Bell states

Now we will apply Equation 3.4 to the qubit case to show that POVMs cannot distinguish all 4 qubit Bell states. When we symmetrize or antisymmetrize the qubit Bell states, we get Equation 3.5.

$$|\Phi^+\rangle = \frac{1}{2}(|0,L\rangle\,|0,R\rangle \pm |0,R\rangle\,|0,L\rangle + |1,L\rangle\,|1,R\rangle \pm |1,R\rangle\,|1,L\rangle) \tag{3.5a}$$

$$|\Phi^-\rangle = \frac{1}{2}(|0,L\rangle\,|0,R\rangle \pm |0,R\rangle\,|0,L\rangle - |1,L\rangle\,|1,R\rangle \mp |1,R\rangle\,|1,L\rangle) \tag{3.5b}$$

$$|\Psi^+\rangle = \frac{1}{2}(|0,L\rangle\,|1,R\rangle \pm |0,R\rangle\,|1,L\rangle + |1,L\rangle\,|0,R\rangle \pm |1,R\rangle\,|0,L\rangle) \tag{3.5c}$$

$$|\Psi^-\rangle = \frac{1}{2}(|0,L\rangle\,|1,R\rangle \mp |0,R\rangle\,|1,L\rangle - |1,L\rangle\,|0,R\rangle \pm |1,R\rangle\,|0,L\rangle) \tag{3.5d}$$

Now we can use the 4 × 4 version of Equation 3.4 on each of these Bell states, which leave the second particles in the states in Equation 3.6.

$$|\Phi^+\rangle_{i,2} = \frac{1}{2}(\pm\alpha_2\,|0,L\rangle + \alpha_1\,|0,R\rangle \pm \alpha_4\,|1,L\rangle + \alpha_3\,|1,R\rangle) \tag{3.6a}$$

$$|\Phi^-\rangle_{i,2} = \frac{1}{2}(\pm\alpha_2\,|0,L\rangle + \alpha_1\,|0,R\rangle \mp \alpha_4\,|1,L\rangle - \alpha_3\,|1,R\rangle) \tag{3.6b}$$

$$|\Psi^+\rangle_{i,2} = \frac{1}{2}(\pm\alpha_4\,|0,L\rangle + \alpha_3\,|0,R\rangle \pm \alpha_2\,|1,L\rangle + \alpha_1\,|1,R\rangle) \tag{3.6c}$$

$$|\Psi^-\rangle_{i,2} = \frac{1}{2}(\pm\alpha_4\,|0,L\rangle - \alpha_3\,|0,R\rangle \mp \alpha_2\,|1,L\rangle + \alpha_1\,|1,R\rangle) \tag{3.6d}$$

All of these must be orthogonal to be distinguishable. In both the boson and fermion cases, the 6 pairwise orthogonality conditions are not satisfiable. So there is no way for such an apparatus to distinguish all four qubit Bell states.

## 3.5 POVMs cannot distinguish 6 qutrit Bell states

Next we will apply Equation 3.4 to the qutrit case to show that POVMs cannot distinguish 6 qutrit Bell states. When we symmetrize or antisymmetrize the qutrit Bell states, we get Equations 3.7.

$$\begin{aligned}|\Psi^0_0\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle\,|0,R\rangle \pm |0,R\rangle\,|0,L\rangle + |1,L\rangle\,|1,R\rangle \\ &\pm |1,R\rangle\,|1,L\rangle + |2,L\rangle\,|2,R\rangle \pm |2,R\rangle\,|2,L\rangle)\end{aligned} \tag{3.7a}$$

$$\begin{aligned}|\Psi_0^1\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|0,R\rangle \pm |0,R\rangle|0,L\rangle + e^{i\frac{2\pi}{3}}|1,L\rangle|1,R\rangle \\ &\pm e^{i\frac{2\pi}{3}}|1,R\rangle|1,L\rangle + e^{i\frac{4\pi}{3}}|2,L\rangle|2,R\rangle \pm e^{i\frac{4\pi}{3}}|2,R\rangle|2,L\rangle)\end{aligned}\tag{3.7b}$$

$$\begin{aligned}|\Psi_0^2\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|0,R\rangle \pm |0,R\rangle|0,L\rangle + e^{i\frac{4\pi}{3}}|1,L\rangle|1,R\rangle \\ &\pm e^{i\frac{4\pi}{3}}|1,R\rangle|1,L\rangle + e^{i\frac{2\pi}{3}}|2,L\rangle|2,R\rangle \pm e^{i\frac{2\pi}{3}}|2,R\rangle|2,L\rangle)\end{aligned}\tag{3.7c}$$

$$\begin{aligned}|\Psi_1^0\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|1,R\rangle \pm |1,R\rangle|0,L\rangle + |1,L\rangle|2,R\rangle \\ &\pm |2,R\rangle|1,L\rangle + |2,L\rangle|0,R\rangle \pm |0,R\rangle|2,L\rangle)\end{aligned}\tag{3.7d}$$

$$\begin{aligned}|\Psi_1^1\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|1,R\rangle \pm |1,R\rangle|0,L\rangle + e^{i\frac{2\pi}{3}}|1,L\rangle|2,R\rangle \\ &\pm e^{i\frac{2\pi}{3}}|2,R\rangle|1,L\rangle + e^{i\frac{4\pi}{3}}|2,L\rangle|0,R\rangle \pm e^{i\frac{4\pi}{3}}|0,R\rangle|2,L\rangle)\end{aligned}\tag{3.7e}$$

$$\begin{aligned}|\Psi_1^2\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|1,R\rangle \pm |1,R\rangle|0,L\rangle + e^{i\frac{4\pi}{3}}|1,L\rangle|2,R\rangle \\ &\pm e^{i\frac{4\pi}{3}}|2,R\rangle|1,L\rangle + e^{i\frac{2\pi}{3}}|2,L\rangle|0,R\rangle \pm e^{i\frac{2\pi}{3}}|0,R\rangle|2,L\rangle)\end{aligned}\tag{3.7f}$$

$$\begin{aligned}|\Psi_2^0\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|2,R\rangle \pm |2,R\rangle|0,L\rangle + |1,L\rangle|0,R\rangle \\ &\pm |0,R\rangle|1,L\rangle + |2,L\rangle|1,R\rangle \pm |1,R\rangle|2,L\rangle)\end{aligned}\tag{3.7g}$$

$$\begin{aligned}|\Psi_2^1\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|2,R\rangle \pm |2,R\rangle|0,L\rangle + e^{i\frac{2\pi}{3}}|1,L\rangle|0,R\rangle \\ &\pm e^{i\frac{2\pi}{3}}|0,R\rangle|1,L\rangle + e^{i\frac{4\pi}{3}}|2,L\rangle|1,R\rangle \pm e^{i\frac{4\pi}{3}}|1,R\rangle|2,L\rangle)\end{aligned}\tag{3.7h}$$

$$\begin{aligned}|\Psi_2^2\rangle = \frac{1}{\sqrt{6}}(&|0,L\rangle|2,R\rangle \pm |2,R\rangle|0,L\rangle + e^{i\frac{4\pi}{3}}|1,L\rangle|0,R\rangle \\ &\pm e^{i\frac{4\pi}{3}}|0,R\rangle|1,L\rangle + e^{i\frac{2\pi}{3}}|2,L\rangle|1,R\rangle \pm e^{i\frac{2\pi}{3}}|1,R\rangle|2,L\rangle)\end{aligned}\tag{3.7i}$$

Now we can use the $6 \times 6$ version of Equation 3.4 on each of these Bell states, which leave the second particles in the states in Equation 3.8.

$$\begin{aligned}|\Psi_0^0\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_2\,|0,L\rangle + \alpha_1\,|0,R\rangle \pm \alpha_4\,|1,L\rangle \\ &+ \alpha_3\,|1,R\rangle \pm \alpha_6\,|2,L\rangle + \alpha_5\,|2,R\rangle)\end{aligned} \tag{3.8a}$$

$$\begin{aligned}|\Psi_0^1\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_2\,|0,L\rangle + \alpha_1\,|0,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_4\,|1,L\rangle \\ &+ e^{i\frac{2\pi}{3}}\alpha_3\,|1,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_6\,|2,L\rangle + e^{i\frac{4\pi}{3}}\alpha_5\,|2,R\rangle)\end{aligned} \tag{3.8b}$$

$$\begin{aligned}|\Psi_0^2\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_2\,|0,L\rangle + \alpha_1\,|0,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_4\,|1,L\rangle \\ &+ e^{i\frac{4\pi}{3}}\alpha_3\,|1,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_6\,|2,L\rangle + e^{i\frac{2\pi}{3}}\alpha_5\,|2,R\rangle)\end{aligned} \tag{3.8c}$$

$$\begin{aligned}|\Psi_1^0\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_4\,|0,L\rangle + \alpha_5\,|0,R\rangle \pm \alpha_6\,|1,L\rangle \\ &+ \alpha_1\,|1,R\rangle \pm \alpha_2\,|2,L\rangle + \alpha_3\,|2,R\rangle)\end{aligned} \tag{3.8d}$$

$$\begin{aligned}|\Psi_1^1\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_4\,|0,L\rangle + e^{i\frac{4\pi}{3}}\alpha_5\,|0,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_5\,|1,L\rangle \\ &+ \alpha_1\,|1,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_2\,|2,L\rangle + e^{i\frac{2\pi}{3}}\alpha_3\,|2,R\rangle)\end{aligned} \tag{3.8e}$$

$$\begin{aligned}|\Psi_1^2\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_4\,|0,L\rangle + e^{i\frac{2\pi}{3}}\alpha_5\,|0,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_6\,|1,L\rangle \\ &+ \alpha_1\,|1,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_2\,|2,L\rangle + e^{i\frac{4\pi}{3}}\alpha_3\,|2,R\rangle)\end{aligned} \tag{3.8f}$$

$$\begin{aligned}|\Psi_2^0\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_6\,|0,L\rangle + \alpha_3\,|0,R\rangle \pm \alpha_2\,|1,L\rangle \\ &+ \alpha_5\,|1,R\rangle \pm \alpha_4\,|2,L\rangle + \alpha_1\,|2,R\rangle)\end{aligned} \tag{3.8g}$$

$$\begin{aligned}|\Psi_2^1\rangle_{i,2} = \frac{1}{\sqrt{6}}(&\pm\alpha_6\,|0,L\rangle + e^{i\frac{2\pi}{3}}\alpha_3\,|0,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_2\,|1,L\rangle \\ &+ e^{i\frac{4\pi}{3}}\alpha_5\,|1,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_4\,|2,L\rangle + \alpha_1\,|2,R\rangle)\end{aligned} \tag{3.8h}$$

$$|\Psi^2_2\rangle_{i,2} = \frac{1}{\sqrt{6}}(\pm\alpha_6\,|0,L\rangle + e^{i\frac{4\pi}{3}}\alpha_3\,|0,R\rangle \pm e^{i\frac{4\pi}{3}}\alpha_2\,|1,L\rangle + e^{i\frac{2\pi}{3}}\alpha_5\,|1,R\rangle \pm e^{i\frac{2\pi}{3}}\alpha_4\,|2,L\rangle + \alpha_1\,|2,R\rangle) \quad (3.8i)$$

Now, we are left with $\binom{9}{6} = 84$ sets of 6 Bell states to determine distinguishability for. We can simplify this greatly by considering equivalence classes of 6-state sets that we can generate using the operations from Section 2.5.

It can be verified by hand using the transformations on a tic-tac-toe diagram that these operations establish 2 classes, which we will call the tic-tac-toe anti-winners and the tic-tac-toe anti-losers. Any set of 6 states leaves out 3 of the qutrit Bell states. The location of those states is what determines which class a set belongs to.

If the missing states in a set win at tic-tac-toe with wrap-around boundary conditions or column permutation, then that set is a tic-tac-toe anti-winner. An example is shown in Figure 3.1.

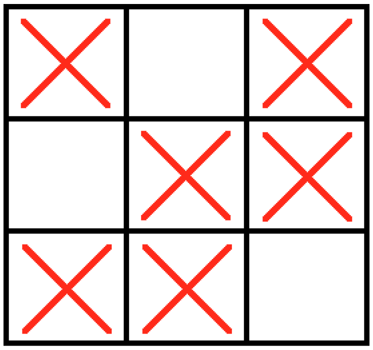

**Figure 3.1** The tic-tac-toe diagram of the set $\{|\Psi^0_0\rangle, |\Psi^2_0\rangle, |\Psi^1_1\rangle, |\Psi^2_1\rangle, |\Psi^0_2\rangle, |\Psi^1_2\rangle\}$. This belongs to the tic-tac-toe anti-winners class because with wrap-around boundary conditions or column permutation, the missing states win at tic-tac-toe.

If the missing states in a set do not win at tic-tac-toe with wrap-around boundary conditions or column permutation, then that set is a tic-tac-toe anti-loser. An example is shown in Figure 3.2.

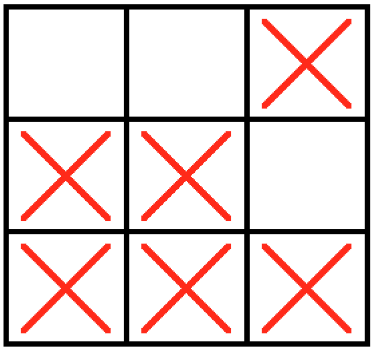

**Figure 3.2** The tic-tac-toe diagram of the set $\{|\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle, |\Psi_2^0\rangle, |\Psi_2^1\rangle, |\Psi_2^2\rangle\}$. This belongs to the tic-tac-toe anti-losers class because with even with wrap-around boundary conditions or column permutation, the missing states do not win at tic-tac-toe.

We can simultaneously show that both a representative of the tic-tac-toe anti-winners and a representative of the tic-tac-toe anti-losers cannot be distinguishable by showing that a subset of both representatives cannot be orthogonal after 1 detection. This subset is $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle\}$ and it is a subset of the tic-tac-toe anti winner $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle, |\Psi_1^2\rangle\}$ and the tic-tac-toe anti-loser $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle, |\Psi_2^0\rangle\}$. All of these sets are shown in Figure 3.3.

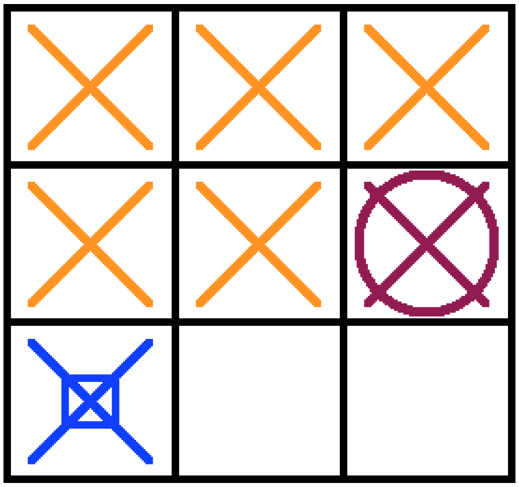

**Figure 3.3** The subset $\{|\Psi_0^0\rangle, |\Psi_0^1\rangle, |\Psi_0^2\rangle, |\Psi_1^0\rangle, |\Psi_1^1\rangle\}$ is shown with orange Xs. The tic-tac-toe anti-winner adds $|\Psi_1^2\rangle$, shown as a maroon X with a large circle. The tic-tac-toe anti-loser adds $|\Psi_2^0\rangle$, shown as a blue X with a small rectangle. By showing that the subset cannot remain orthogonal after one detection, we will show that both representatives from the two classes cannot remain orthogonal after one detection.

To show that the subset cannot remain orthogonal after one detection, we will use the general forms of the second particles in Equations 3.8a-3.8e. Imposing $\langle\Psi_0^0|_{i,2}\,|\Psi_0^1\rangle_{i,2} = 0$ and $\langle\Psi_1^0|_{i,2}\,|\Psi_1^1\rangle_{i,2} = 0$ (in either the fermion or boson case) gives the conditions

$$|\alpha_1| = |\alpha_3| = |\alpha_5| \quad \text{and} \quad |\alpha_2| = |\alpha_4| = |\alpha_6|.$$

Imposing $\langle\Psi_0^0|_{i,2}\,|\Psi_1^0\rangle_{i,2} = 0$, $\langle\Psi_0^1|_{i,2}\,|\Psi_1^0\rangle_{i,2} = 0$ and $\langle\Psi_0^2|_{i,2}\,|\Psi_1^0\rangle_{i,2} = 0$ (again in either the fermion or boson case) gives the conditions

$$\alpha_2\alpha_4^* = -\alpha_1\alpha_5^*$$
$$\alpha_4\alpha_6^* = -\alpha_3\alpha_1^*$$
$$\alpha_6\alpha_2^* = -\alpha_5\alpha_3^*.$$

If we multiply all of these together, we get

$$|\alpha_2|^2|\alpha_4|^2|\alpha_6|^2 = -|\alpha_1|^2|\alpha_3|^2|\alpha_5|^2.$$

The only way that all of these conditions can be satisfied is if all of the coefficients are zero, which obviously cannot happen. So no set of 6 qutrit Bell states is distinguishable with any LELM apparatus.

# Chapter 4

# Maximum Distinguishability of Qubit × Qutrit Bell States with Projective Measurement

## 4.1 Overview

In this chapter, we will detail progress toward determining maximal distinguishability of a different type of Bell states with projective LELM devices. Here, we consider hyperentangled Bell states, which are entangled in more than one property, specifically qubit × qutrit Bell states. We will begin by introducing the concept of hyperentanglement and hyperentangled Bell states. Then we will present some minor results and progress toward determining exactly how many of these qubit × qutrit Bell states can be distinguished with a projective LELM device.

## 4.2 Hyperentanglement

So far in this thesis, we have considered the distinguishability of Bell states of two particles entangled in one $d$-dimensional variable. When two particles are entangled in more than one variable, they are called *hyperentangled*. For example, a pair of photons could be entangled in both spin and momentum if their momenta and spins were not individually defined, but their correlations were. Hyperentanglement is an important phenomenon, and hyperentangled states have many applications. Earlier in this thesis, we showed that the qubit Bell states could not be distinguished deterministically by any LELM device. But if we allow entanglement in additional variables, the 4 qubit Bell states *can* be distinguished with an LELM apparatus. This protocol is detailed in Walborn et al. (2003).

Hyperentangled states can be written as tensor products of the states of the individual variables. For example, if I have a two-particle system that is entangled in the $|\Phi^+\rangle$ state in qubit variable $A$ and entangled in the $|\Phi^-\rangle$ state in qubit variable $B$, we could write this state as

$$\begin{aligned} |\Psi\rangle = |\Phi^+\rangle_A \otimes |\Phi^-\rangle_B &= \left( |00\rangle_A + |11\rangle_A \right) \otimes \left( |00\rangle_B - |11\rangle_B \right) \\ &= \frac{1}{2}\left( |00\rangle_A |00\rangle_B - |00\rangle_A |11\rangle_B + |11\rangle_A |00\rangle_B - |11\rangle_A |11\rangle \right). \end{aligned} \tag{4.1}$$

## 4.3 Hyperentangled Bell States

As the section title suggests, one can also make hyperentangled Bell states, which form an entangled basis for Hilbert spaces of two-particle multiple-variable states. The state shown above in Equation 4.1 is one of the qubit × qubit Bell states. One can make a complete set of hyperentangled Bell states for a set of variables of a specific dimension by tensoring all of the possible Bell states for each variable together. For example, the qubit × qubit Bell states are made up of all 16 possible tensor products of qubit Bell states. In general, the Bell states for a hyperentangled system with variables of dimension $d_1, d_2, \ldots d_n$ are all of the $d_1^2 d_2^2 \ldots d_n^2$ combinations of the Bell states for those variables.

Neal Pisenti (HMC '11), another previous student in this group, considered distinguishability of hyperentangled $n$-qubit Bell states (particles entangled in $n$ qubit variables) in his thesis. He showed that $2^{n+1} - 1$ is the maximum number of hyperentangled $n$-qubit Bell states distinguishable with projective LELM measurements (Pisenti (2011)). Unfortunately, his argument did not fully generalize to other sets of hyperentangled Bell states, so we had to start almost from scratch.

Next, we consider the simplest hyperentangled case that has not yet been solved: qubit × qutrit Bell states. We have since made some progress on determining the maximum number of qubit × qutrit Bell states that can be distinguished deterministically with projective LELM measurements.

## 4.4 9 Qubit × Qutrit Hyperentangled Bell States are Distinguishable, but 12 are not

One might hope that because 3 qubit Bell states can be distinguished with LELM measurements that 9 qubit × qubit Bell states could be distinguished. But by substituting $n = 2$ into the maximum that Pisenti proved, we see that only 7 can be distinguished, as originally shown in Wei et al. (2007). For maximum qubit Bell state discrimination (detailed in Lütkenhaus et al. (1999)), the particles must be mixed together in a way that can't be undone for measurement on the second qubit variable. But in the qubit × qutrit case, we don't have that problem because the maximally distinguishing protocol on the qutrit variable does not mix the particles; it just requires separate measurements of each particle in the standard basis. So to distinguish 9 qubit × qutrit Bell states, one can simply measure the particles in the standard basis of the qutrit variable and then perform the maximally distinguishing qubit measurement from Lütkenhaus et al. (1999). This means that at least 9 qubit × qutrit Bell states are distinguishable.

If we consider the form of the projective LELM device for measuring qubit × qutrit Bell states, we see that we only have 12 detectors, 3 for the qutrit variable times 2 for the qubit variable times 2 for the channel variable. As in the non-hyperentangled Bell states, the first detection gives no information because all Bell states can trigger all detectors. So distinguishability again depends on one measurement, giving us a maximum of 12 distinguishable qubit × qutrit Bell states.

Neal Pisenti argues in his thesis that the corresponding maximum of $2^{n+1}$ cannot be achieved for $n$-qubit hyperentangled Bell states. This argument can be extended to this case to show that 12 qubit × qutrit Bell states cannot be distinguished. The argument considers three possible cases for particle statistics: distinguishable particles, fermions and bosons.

Distinguishable particles do not have the channel variable because there is no possibility of mixing the channels, so after the first particle is detected, there can only be 6 orthogonal detectors to detect the second, so 12 can definitely not be distinguished.

Fermions must be antisymmetric under exchange, so if detector $|i\rangle$ fires for the first particle, it cannot fire again, because that would yield the symmetric state $|i\rangle |i\rangle$ for the pair, which is not allowed, so only 11 of the 12 detectors could fire second.

In the case of bosons, we will consider a general detector $|i\rangle$ which we will let fire first. If $|i\rangle$ is a single-channel detection mode, then we can't have a second click in that detector, because both particles can't be detected in the same channel. In other words, the detection signature $P_{LR}\,|i\rangle\,|i\rangle = 0$. This allows only 11 possibilities to distinguish qubit × qutrit Bell states. Otherwise, $|i\rangle$ would be a nontrivial superposition of states in the left channel and states in the right channel. We can express this state as shown below in Equation 4.2.

$$|i\rangle = |L\rangle + |R\rangle \tag{4.2}$$

Here we require that $|L\rangle$ is only made of left-channel states, $|R\rangle$ is only made of right-channel states, $|L\rangle \neq 0$, and $|R\rangle \neq 0$. Now we consider the hypothetical state

$$|X\rangle = \sum_j \epsilon_j\,|j\rangle = |L\rangle - |R\rangle\,, \tag{4.3}$$

where the $|j\rangle$s are the detectors and the $\epsilon_j$s are chosen for the second equality to hold. Now we see that

$$P_{LR}\,|i\rangle\,|X\rangle = P_{LR}(|L\rangle + |R\rangle)(|L\rangle - |R\rangle) = P_{LR}(|L\rangle\,|L\rangle - |R\rangle\,|R\rangle) = 0. \tag{4.4}$$

So by rewriting $|X\rangle$ differently as shown in Equation 4.3, we get that

$$\sum_j \epsilon_j P_{LR}\,|i\rangle\,|j\rangle = 0. \tag{4.5}$$

We know that at least one of the $\epsilon_j$s must be nonzero. So there is a nonzero term $\epsilon_j P_{LR}\,|i\rangle\,|j\rangle$ in that expression that must have at least one of the qubit × qutrit Bell states in it, specifically one that we are trying to distinguish from others by getting the detection $(i, j)$. In order for Equation 4.5 to sum to zero in the Bell basis, one of the other detection signatures $P_{LR}\,|i\rangle\,|k\rangle$ must have that same Bell state in it, which does not allow this state to be distinguished. So not all 12 qubit × qutrit Bell states are distinguishable. As this is true for all 3 types of particle statistics, it is true in general.

## 4.5 Distinguishability Classes of Qubit × Qutrit Hyperentangled Bell States

Future investigations into whether sets of 10 or 11 qubit × qutrit Bell states are distinguishable would have to consider a ridiculously large number of

Bell state sets. The easiest case is 10 qubit × qutrit Bell states, and there are $\binom{36}{10} = 254,186,856$ of them. Because we do not want to investigate over a quarter of a billion cases individually, we want to simplify the search using equivalence classes. In order to generate some distinguishability classes for qubit × qutrit Bell states like those used in Section 2.5, I extended what was known from the equivalence classes for qutrit Bell states and qubit Bell states, specifically the single-particle operations that permuted the Bell states.

First, conveniently, the qubit Bell states are arbitrarily permutable! We will show that every element in $S_4$ that acts on the Bell states is obtainable from unitary operations on the individual qubits. To do this, we begin by labelling the qutrit Bell states as shown below:

$$1 : |\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle_A\,|0\rangle_B + |1\rangle_A\,|1\rangle_B\,\big) \tag{4.6a}$$

$$2 : |\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle_A\,|0\rangle_B - |1\rangle_A\,|1\rangle_B\,\big) \tag{4.6b}$$

$$3 : |\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle_A\,|1\rangle_B + |1\rangle_A\,|0\rangle_B\,\big) \tag{4.6c}$$

$$4 : |\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle_A\,|1\rangle_B - |1\rangle_A\,|0\rangle_B\,\big) \tag{4.6d}$$

We know that $S_4$ is generated by the set $\{(12), (23), (34)\}$, so we will show that (12), (23) and (34) can be performed. The operation (12) is performed by performing the same conditional phase operation on both qubits:

$$(12) : |1\rangle_A \rightarrow i\,|1\rangle_A\,, |1\rangle_B \rightarrow i\,|1\rangle_B \tag{4.7}$$

The operation (23) can be done with a general basis change performed on both qubits:

$$(23) : |0\rangle \rightarrow \frac{1}{2}(|0\rangle + |1\rangle\,, |1\rangle) \rightarrow \frac{1}{2}(|0\rangle - |1\rangle) \tag{4.8}$$

The last operation, (34), can be performed by different conditional phase operations on the qubits:

$$(34) : |1\rangle_A \rightarrow i\,|1\rangle_A\,, |0\rangle_B \rightarrow i\,|0\rangle_B \tag{4.9}$$

So because the qubit × qutrit Bell states all have one of the four qubit Bell states for their qubit variable, performing these operations on them allow many permutations on the whole set. Specifically, sets of Bell states are equivalent if they have the same qutrit Bell states that share qubit Bell states, and the specific identities of the qubit Bell states do not matter. So far, our best option is to consider this freedom alongside the operations on the qutrit Bell states from Section 2.5. One can see from operations on the tic tac toe diagram, that the operations on the qutrit Bell states split the 9 qutrit Bell states into 1 class for sets of $1, 2, 7, 8$, and 9 Bell states and 2 classes for sets of $3, 4, 5$, and 6 Bell states. So we can count every equivalence class of $n$ qubit × qutrit Bell states by partitioning $n$ into 4 integers between 1 and 9, counting how many qutrit Bell states are associated with each qubit Bell state and for each partition, choosing which set of qutrit Bell states reduces the total number of classes the most by performing the qutrit permuting operations to reduce it to the one or two qutrit Bell state classes.

For example, if we are looking for equivalence classes for sets of $n = 10$ qubit × qutrit Bell states, we may choose the partition $(6, 2, 1, 1)$, which corresponds to one qubit Bell state having 6 qutrit Bell states that are tensored with it, another having 2 and the last two having 1 each. Without qutrit operations, we would have $\binom{9}{6}\binom{9}{2}\binom{9}{1}\binom{9}{1} = 84 \times 36 \times 9 \times 9 = 244{,}944$ equivalence classes, but we choose to use the qutrit Bell state permutation operations to reduce the $\binom{9}{6}$ down to 2. So we only get $2\binom{9}{2}\binom{9}{1}\binom{9}{1} = 2 \times 36 \times 9 \times 9 = 5{,}832$ equivalence classes in this case.

We list all of the valid partitions and the number of equivalence classes in each partition below, so that we can calculate a total number of equivalence classes:

| Partitions of 10 Qubit × Qutrit Bell States | Number of Equivalence Classes |
|---|---|
| (9,1,0,0) | 1 |
| (8,2,0,0) | 9 |
| (8,1,1,0) | 81 |
| (7,3,0,0) | 72 |
| (7,2,1,0) | 324 |
| (7,1,1,1) | 729 |
| (6,4,0,0) | 168 |
| (6,3,1,0) | 1,512 |
| (6,2,2,0) | 2,592 |
| (6,2,1,1) | 5,832 |
| (5,5,0,0) | 252 |
| (5,4,1,0) | 2,268 |
| (5,3,2,0) | 6,048 |
| (5,3,1,1) | 13,608 |
| (5,2,2,1) | 23,328 |
| (4,4,2,0) | 9,072 |
| (4,4,1,1) | 20,412 |
| (4,3,3,0) | 14,112 |
| (4,3,2,1) | 54,432 |
| (4,2,2,2) | 93,312 |
| (3,3,3,1) | 127,008 |
| (3,3,2,2) | 217,728 |
| Total Equivalence Classes: | 592,900 |

As you can see, we have not yet reached an easy computation, but we have reduced the number of cases to check by about a factor of 500. With further Bell state permutation operations that could be expressed as operations on single qubits, we may be able to get this down to something that is computationally feasible. In Wei et al. (2007),12,870 cases for sets of 8 qubit × qubit Bell states were investigated in Mathematica. Because we need to solve a larger system of equations for each case, we may reach computational feasibility at around 1,000 cases.

# Chapter 5

# Conclusion

In this thesis, we have established multiple distinguishability limits for Bell states with LELM devices, with and without the restriction of projective measurement. For two qubits, it has been known for quite a while that LELM cannot perform a complete Bell measurement, but at least it is possible to distinguish 3 out of 4 Bell states, where measuring the two particles separately only allows us to distinguish 2 Bell states. If we consider bosons, not even this much can be said for qutrit Bell states. We have shown that the most effective projective LELM scheme for distinguishing bosonic qutrit Bell states is a simple measurement of each particle separately in the standard basis. As this only allows 3 Bell states to be distinguished, the same number as for qubit Bell states, the viability of qutrit Bell states in quantum information protocols looks less promising, at least for bosons. Future work might entail making similar arguments to restrict detector format for dectecting fermions in order to potentially extend this result to both statistics of indistinguishable particles, but this may be slightly more difficult as double detections cannot happen.

We have also shown that general non-projective POVMs do not allow for better distinguishability for qubit Bell states with an LELM apparatus; only 3 can be distinguished with a general LELM apparatus. We also establish that no more than 5 qutrit Bell states can be distinguished with a general LELM apparatus. Both results suggest that non-projective measurement may not improve the outlook for deterministic unambiguous Bell state discrimination, but we have not yet ruled it out for the qutrit case or in general. This is consistent with my intuition, as non-projective measurement tends to introduce an additional element of chance in the measurement outcomes, so it can increase a probability of successful state discrimination in some applications, but it does not work reliably.

Finally, we have made some progress on distinguishability of qubit × qutrit Bell states with projective LELM devices. We know that the maximum number distinguishable is between 9 and 11 and we have made progress in simplifying the investigation of these cases by considering the equivalence classes that arise from the operations that permute the non-hyperentangled qubit and qutrit Bell states. Future work on this project would involve consideration of new operations that might permute the hyperentangled qubit × qutrit Bell states, which could further reduce the number of equivalence classes down to around 1,000, which may be computationally feasible. Alternatively, one might be able to make arguments about the detector modes similar to those that I made in Chapter 2, which may circumvent the difficulty of considering the system from Equation 1.32 for every equivalence class.

# Bibliography

Bennett, Charles H., Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. 1993. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys Rev Lett* 70:1895–1899. doi:10.1103/PhysRevLett.70.1895. URL https://link.aps.org/doi/10.1103/PhysRevLett.70.1895.

Briegel, H.-J., W. Dür, J. I. Cirac, and P. Zoller. 1998. Quantum repeaters: The role of imperfect local operations in quantum communication. *Phys Rev Lett* 81(26):5932–5935. doi:10.1103/PhysRevLett.81.5932.

Gottesman, Daniel, and Isaac L. Chuang. 1999. Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations. *Nature* 402(6760):390–393. URL http://dx.doi.org/10.1038/46503.

Lütkenhaus, N., J. Calsamiglia, and K.-A. Suominen. 1999. Bell measurements for teleportation. *Phys Rev A* 59(5):3295–3300. doi:10.1103/PhysRevA.59.3295.

Mattle, Klaus, Harald Weinfurter, Paul G. Kwiat, and Anton Zeilinger. 1996. Dense coding in experimental quantum communication. *Phys Rev Lett* 76(25):4656–4659. doi:10.1103/PhysRevLett.76.4656.

Pisenti, N., C. P. E. Gaebler, and T. W. Lynn. 2011. Distinguishability of hyperentangled bell states by linear evolution and local projective measurement. *Phys Rev A* 84:022,340. doi:10.1103/PhysRevA.84.022340. URL http://link.aps.org/doi/10.1103/PhysRevA.84.022340.

Pisenti, Neal C. 2011. Distinguishability of hyper-entangled bell states with linear devices. Senior Thesis, Harvey Mudd College.

Preskill, John. 1998. Lecture notes for physics 229: Quantum information and computation .

van Loock, Peter, and Norbert Lütkenhaus. 2004. Simple criteria for the implementation of projective measurements with linear optics. *Phys Rev A* 69(1):012,302. doi:10.1103/PhysRevA.69.012302.

Walborn, S. P., S. Pádua, and C. H. Monken. 2003. Hyperentanglement-assisted bell-state analysis. *Phys Rev A* 68:042,313. doi:10.1103/PhysRevA.68.042313. URL https://link.aps.org/doi/10.1103/PhysRevA.68.042313.

Wei, Tzu-Chieh, Julio T. Barreiro, and Paul G. Kwiat. 2007. Hyperentangled bell-state analysis. *Phys Rev A* 75(6):060,305. doi:10.1103/PhysRevA.75.060305.