Distinguishability of Qudit Hyperentangled States with Linear Evolution and Local Measurement

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May, 2014

Abstract

Entanglement is a property of quantum systems that is central to many applications such as quantum teleportation, quantum secret sharing, quantum dense coding, and quantum cryptography. All of these applications require measurement of entangled particles in the "Bell state basis." The goal of this kind of measurement is to distinguish qudit hyperentangled states, or states of two particles simultaneously entangled in multiple variables, where each variable takes an arbitrary number d of distinct values in a particular basis. Such measurements are difficult in general, and so measurements that use only linear evolution and local measurement (LELM) are of particular interest. However, it is known that LELM measurements cannot reliably distinguish the full set of Bell states from each other. We show a theoretical bound on the number of qudit hyperentangled Bell states that can be distinguished by LELM devices, setting limitations on various quantum information protocols.

Acknowledgments

To the professors of the physics and math departments at Harvey Mudd College, I extend my thanks for their assistance and advice throughout the past four years. I would especially like to thank my thesis advisor, Professor Theresa Lynn, for many insightful and interesting discussions about physics, and for her help throughout this thesis. I would also like to acknowledge Professor Michael Orrison, who has helped me solidify much of the math that appears in this thesis, as well as Neal Pisenti, Lucas Brady, Julien Devin, and Philipp Gaebler, whose previous work laid the path for mine. Finally, I am greatly indebted to my friends and family, who have supported me in all my endeavors.

Contents

Chapter 1

Introduction

In this thesis, we develop fundamental bounds on quantum state measurements by a certain class of measurement apparatus. Bounds on quantum measurements can inform the development of practical quantum communication protocols and other protocols that rely on quantum mechanics. In particular, we discuss measurement apparatuses that are constrained to *linear evolution, local measurement* (LELM). Apparatuses of this type are of particular interest both because they are relatively simple to implement in the lab and because they perform measurements more reliably than apparatuses that are not so constrained. However, it is well-known that LELM devices have limitations when it comes to the measurements they can perform [15, 19, 20, 23, 25].

In particular, we are interested in using LELM devices to distinguish Bell states in a multi-particle (*multipartite*) system. The *Bell states* are a set of basis states that are useful when discussing entanglement between two particles (for a discussion of entanglement, see Chapter 2). Unfortunately, LELM devices cannot be used to distinguish all of the Bell states. The most well-known result regarding these limitations shows that, for systems of two particles entangled in a single two-state variable, an LELM device can only return three distinct results, despite there being four different Bell states [15]. One of these three results actually indicates a class of two Bell states, between which the apparatus can not distinguish. Previous work has generalized this result to quantify the limitations of LELM apparatuses when measuring two-particle systems entangled in an arbitrary number of two-state variables [19, 20].

Limitations on the measurement of Bell states have significant implications for many quantum information protocols, such as quantum teleportation, entanglement swapping, and dense coding, all of which rely on Bell state measurements [6, 7, 16, 18]. In quantum teleportation, for example, the goal is to take a particle in an unknown state and reproduce it in another, possibly distant, location (the original state is destroyed in the process). Due to the nature of quantum mechanics, attempts to measure the state of the particle directly would destroy the state without yielding enough information to fully recreate it after the fact. However, we can exploit the properties of quantum entanglement to teleport the state of the particle without ever measuring the state directly. This approach, however, requires that we be able to measure the particle to be teleported and another helper particle unambiguously in the Bell basis; a measurement that yields an ambiguous result destroys the original state and causes a failed teleportation.

Physicists have found methods to circumvent some of the limitations on LELM measurement by using entanglement in additional variables to assist unambiguous measurement of the Bell states for the variable of interest [25]. Development of methods such as these, as well as extension of existing methods, requires an understanding of the theory underlying the limitations on LELM devices. In particular, this thesis seeks to generalize results that have been found previously for entanglement of two particles in an arbitrary number of two-state variables [19, 20] to the case of entanglement in an arbitrary number of variables with any finite number of states.

The remaining chapters are divided as follows: Chapter 2 gives the necessary theoretical background for understanding hyperentanglement, and introduces much of the terminology used throughout the document. Chapter 3 more formally states the problem and explains notation and concepts specific to this thesis. Chapter 4 discusses previous work done on theoretical limits for Bell state measurements. Chapter 5 proves two lower bounds on the size of Bell state classes distinguished by an LELM device, and Chapter 6 proves a maximum distinguishability bound for the case when these Bell states classes are disjoint; these two chapters constitute the main work of the thesis. Chapter 7 concludes the document and indicates some possible avenues for further research. Finally, Appendix A relates progress made on the problem of proving that LELM devices with potentially overlapping detection signatures cannot distinguish a greater number of Bell states than an optimal LELM apparatus with disjoint detection signatures.

Chapter 2

Background

2.1 Introduction

One common way to think of classical information is in terms of bits, or variables that can take on one of two values (which we commonly label as 0 and 1). One example is the state of a light switch, which can either be off (0) or on (1). When we consider quantum information, we use a quantum bit, or *qubit*. Similar to the classical case, the qubit has two possible values, labeled $|0\rangle$ and $|1\rangle$. However, the qubit is not restricted to only these states, as is the classical bit, but it can rather take on any superposition of the two. In general, a single-qubit state $|q\rangle$ can be expressed in the form

$$|q\rangle = c_0 |0\rangle + c_1 |1\rangle , \tag{2.1}$$

where c_0 and c_1 are complex constants with which we can control the relative magnitude and phase of each state. We normalize this state by requiring that $|c_0|^2 + |c_1|^2 = 1$.

Although this may seem unnecessarily abstract at first glance, qubits are quite common in the world of quantum mechanics. For example, if we consider a single photon propagating along the z-axis, we can measure its linear polarization in the $\{x, y\}$ basis. Such a measurement will always yield one of two values: polarized along the x-axis, which we denote $|x\rangle$, or polarized along the y-axis, denoted $|y\rangle$. Although these are the only two values that a measurement of this kind can yield, the state of the photon prior to measurement can be any complex superposition of the two possible values:

$$|q\rangle = c_x |x\rangle + c_y |y\rangle .$$

This is precisely the qubit we saw in Eq. (2.1), with the state $|0\rangle$ and $|1\rangle$ relabeled as $|x\rangle$ and $|y\rangle$. If we were to measure the linear polarization of this photon in the $\{x, y\}$ basis we would *collapse* the system in to one state or the other; the probability that the measurement would yield $|x\rangle$ is $|c_x|^2$, and the probability that it would yield $|y\rangle$ is $|c_y|^2$. We can see then why we require that $|c_x|^2 + |c_y|^2 = 1$: so that the total probability is always 1. For this reason, the constants c_0 and c_1 in Eq. (2.1), and c_x and c_y in this example, are referred to as *probability amplitudes*.

The polarization of a photon can also be expressed as a superposition of left and right circularly polarized states ($|L\rangle$ and $|R\rangle$, respectively). This is an example of a change of basis (as we will see later, the states of a qubit live in a vector space). As you might expect, we can express these two new basis vectors in terms of our original basis:

$$
\begin{aligned}
|R\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \\
|L\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle).
\end{aligned}
$$

In general, there are an infinite number of bases we could choose to express the state for any qubit; however, carefully choosing the basis we use will often lead to insights about the system in question.

There are many other examples of qubits. The spin angular momentum state of an electron can take on any superposition of the states $|\uparrow\rangle$ (corresponding to spin angular momentum $+\hbar/2$ along a chosen axis) and $|\downarrow\rangle$ (corresponding to spin angular momentum $-\hbar/2$ along that axis). We could also consider current in a small superconducting loop, which is quantized and can take on any superposition of 'clockwise' and 'counterclockwise.' Indeed, any quantum system in which measurements are restricted to two possible outcomes can be represented as a qubit. In general, we will label the states of a qubit as $|0\rangle$ and $|1\rangle$, and be aware that these two states could represent the two possible states of any qubit that we consider.

Many quantum systems can take on superpositions of more than two distinct states. For example, the energy of a hydrogen atom can take on an infinite number of discrete values, as can the orbital angular momentum of a photon. A system with three states is sometimes called a *qutrit* and in general we refer to a system with d distinct states as a *qudit*, and label these states $|0\rangle, |1\rangle, \ldots, |d-1\rangle$.

2.2 Mathematical Foundation

Before we continue to discuss quantum states, we must develop a mathematical formalism with which to manipulate them. As we saw in Section 2.1, a qubit can be expressed in the form given by Eq. (2.1). In actual fact, the set of all states $|q\rangle$ of the qubit system in question forms a Hilbert space V of dimension 2, and the set $\{|0\rangle, |1\rangle\}$ is a basis for this space (a Hilbert space is simply a complete inner product space). We see then that Eq. (2.1) is simply a statement that any vector in V may be expressed as a superposition of the basis vectors $|0\rangle$ and $|1\rangle$, as should be the case. When we incorporate the restriction that $|c_1|^2 + |c_2|^2 = 1$, we simply restrict ourselves to considering only those elements of V with magnitude 1.

A vector in the Hilbert space is referred to as a *ket*, and is always written in the form $|\cdot\rangle$. The reason for this name and notation will soon become apparent. We also consider the dual space V^* to the Hilbert space. A vector in the dual space is referred to as a *bra*, and is always written in the form $\langle\cdot|$. Every element $|q\rangle \in V$ has a corresponding vector in the dual space, which we denote $\langle q|$. By definition, any bra in the dual space V^* is a linear functional on the Hilbert space V, and so we can consider its action on vectors in V. We denote the action of a bra $\langle r|$ on a ket $|q\rangle$ by $\langle r|q\rangle$. Note that this corresponds exactly to the inner product of $|r\rangle$ and $|q\rangle$ in V, and we will often abuse this relationship by referring to the action of $\langle r|$ on $|q\rangle$ as an inner product. Now the reason for the naming scheme becomes clear: when we take the inner product of a *bra* $\langle r|$ and a *ket* $|q\rangle$, we form the *bra*c*ket* $\langle r|q\rangle$.

We now must extend our formalism to include single-particle systems consisting of multiple variables as well as multi-particle systems. For example, we could simultaneously consider the spin angular momentum and the linear momentum of an electron. The set of all spin states of the electron forms a Hilbert space V_1, and the set of all linear momentum states of the electron forms a second Hilbert space V_2. The total Hilbert space for the system is then the tensor product space $V = V_1 \otimes V_2$. If the spin state of the electron is $|q_1\rangle \in V_1$ and the linear momentum state is $|q_2\rangle \in V_2$, then the state of the whole system is the tensor product $|q_1\rangle \otimes |q_2\rangle \in V_1 \otimes V_2$ (we will often denote tensor products of kets implicitly, so that $|q_1\rangle \otimes |q_2\rangle$ becomes $|q_1\rangle |q_2\rangle$). Considering multi-particle systems is much the same. The Hilbert space of a system of multiple particles is the tensor product space of the Hilbert spaces of each particle, and the total state of the system is the tensor product of the states of the constituent particles.

2.3 Entanglement

2.3.1 Single-Variable Entanglement

We are now prepared to discuss quantum entanglement, a rather bizarre result in quantum mechanics. Let's consider a system of two particles, in which we consider the same qubit variable for each particle. At first, it might seem that a general state in this system could always be expressed in the form

$$|q\rangle = |q_1\rangle |q_2\rangle = (a_0 |0\rangle + a_1 |1\rangle)(b_0 |0\rangle + b_1 |1\rangle). \quad \text{(wrong!)} \tag{2.4}$$

However, this is not the case. Consider, for example, the state

$$|q\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle). \tag{2.5}$$

Now, this state is definitely an element of the Hilbert space for the two-particle system, and yet we cannot express it in the form given in Eq. (2.4). The state given in Eq. (2.5) is what is known as an *entangled state*, and the two particles in the system are referred to as *entangled*. Although neither particle in the system has a definite state, there is a clear relationship between the particles. For example, the first particle has a 50% chance to take on either possible value if measured in the $\{|0\rangle, |1\rangle\}$ basis, but once it is measured, the entire wave function collapses and the value of the second particle is decided as well.

To emphasize how bizarre this really is, let's compare the correlation in Eq. (2.5) with a similar classically correlated state; in particular, let's consider a pair of qubits that, when both measured in the $\{|0\rangle, |1\rangle\}$ basis, have a 50% chance of yielding $|0\rangle |0\rangle$ and a 50% chance of yield $|1\rangle |1\rangle$ (this is called a *mixed state* in the context of quantum mechanics). It may seem at first glance that these two systems will behave in exactly the same way. After all, if we measure both qubits of the state given in Eq. (2.5) in the $\{|0\rangle, |1\rangle\}$ basis, we will see the same results: 50% of the time the measurement will yield $|0\rangle |0\rangle$ and 50% of the time it will yield $|1\rangle |1\rangle$. However, let us see what happens when we consider measurement in a different basis. Namely, let's look at the basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

We can express the states $|0\rangle$ and $|1\rangle$ in this basis as

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Let's first consider what happens when we measure the particles of the classically correlated state in this basis. We know that 50% of the time, the classically correlated system is in the state

$$\begin{aligned}|0\rangle|0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ &= \frac{1}{2}(|+\rangle|+\rangle + |+\rangle|-\rangle + |-\rangle|+\rangle + |-\rangle|-\rangle),\end{aligned}$$

and 50% of the time, it is in the state

$$\begin{aligned}|1\rangle|1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{1}{2}(|+\rangle|+\rangle - |+\rangle|-\rangle - |-\rangle|+\rangle + |-\rangle|-\rangle).\end{aligned}$$

We see that in either case, measuring both particles in the $\{|+\rangle, |-\rangle\}$ basis can yield any of the four possible results $|+\rangle|+\rangle$, $|+\rangle|-\rangle$, $|-\rangle|+\rangle$, $|-\rangle|-\rangle$; that is, although the mixed state exhibits correlation in the $\{|0\rangle, |1\rangle\}$ basis, it does not exhibit correlation in the $\{|+\rangle, |-\rangle\}$ basis. However, if we rewrite the state of Eq. (2.5) in this new basis, we find

$$\begin{aligned}|q\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \\ &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(|+\rangle + |-\rangle)(|+\rangle + |-\rangle) + \frac{1}{2}(|+\rangle - |-\rangle)(|+\rangle - |-\rangle)\right] \\ &= \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle).\end{aligned}$$

Thus, if we measure both of the particles of the state $|q\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis, the only results we can get are $|+\rangle|+\rangle$ and $|-\rangle|-\rangle$; the entangled state $|q\rangle$ exhibits correlation in both bases! In fact, we can show that this entangled state will exhibit correlation (or anticorrelation) in any basis we could choose to measure in. This is a fundamental difference between classical correlation and entanglement.

Entanglement has another rather spooky consequence: the wave function collapse from measuring one of the entangled particles occurs instantaneously regardless of the separation between the two particles, meaning that a measurement of one entangled particle on Earth would result in an instantaneous change in the state of its entangled partner on Alpha Centauri. This result bothered many prominent physicists for quite some time. In fact, Albert Einstein himself, along with Podolski and Rosen, claimed that this result was an obvious indicator of some flaw in quantum mechanics [12]. This has become known as the Einstein–Podolski–Rosen (EPR) paradox. Einstein proposed an alternative theory, in which all particles have a definite state (in contrast with quantum mechanics), but that some variables affecting this state were "hidden" from the observer. In this theory, the observed behavior that quantum mechanics strove to explain (how identically-prepared particles could give various different results upon applying the same measurement) was instead explained by claiming that particles that *seemed* identical to the observer in fact differed in one of the hidden variables, and that this difference is what would determine the outcome of the measurement. However, experiments have since been devised that can differentiate between local hidden variable theories and quantum mechanics, and the results overwhelmingly confirm quantum mechanics [1–5, 11].

A convenient concept when thinking about entanglement is that of the *Schmidt rank* of a given state. The Schmidt rank of a state $|q\rangle$ is the minimum number of tensor product terms that must be summed to express the state. For example, the state given in Eq. (2.5) clearly has Schmidt rank 2, because it cannot be factored into a single tensor product, but is already expressed as the sum of two tensor product terms. Contrast this with the state

$$|q\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |0\rangle\,|1\rangle)$$

which has Schmidt rank 1 because it can be rewritten in the form

$$|q\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

The reason that this quantity is called the Schmidt *rank* is as follows: Let $\{|b_1\rangle\,, |b_2\rangle\,, \ldots, |b_d\rangle\}$ be a basis for the Hilbert space of each particle, so that the state of the two-particle system can be expressed in the form

$$|q\rangle = \sum_{i=0}^{d}\sum_{j=0}^{d} c_{ij}|b_i\rangle|b_j\rangle$$

for some constants $c_{ij} \in \mathbb{C}$. If $\mathbf{Q}$ is the coordinate matrix of the state q, with (i, j) entry $Q_{ij} = c_{ij}$, then the Schmidt rank of the state $|q\rangle$ is the matrix rank of $\mathbf{Q}$.

2.3.2 Hyperentanglement

Two particles are said to be *hyperentangled* if they are simultaneously entangled in multiple variables. For example, in the process of spontaneous parametric down-conversion, the two photons created are entangled with each other in polarization, orbital angular momentum, linear momentum, and energy.

2.4 Bases

In this thesis, we seek to find bounds on the ability of LELM devices to distinguish between different entangled states (called Bell states) of a pair of particles. Such a device takes as its input two particles, and the goal is to measure them in a basis composed of these entangled states—the Bell basis. However, due to limitations on the device from the LELM restriction, a measurement result may not yield a particular Bell state, but rather a superposition of Bell states among which the device cannot distinguish. Our task is to determine the minimum number of Bell states in such a superposition, and correspondingly, the maximum number of Bell states such a device can distinguish.

In discussing the problem at hand, there are multiple Hilbert spaces we must consider. Each of the two particles has its own *single-particle Hilbert space*, which contains the state of that particle alone. We also consider the *joint-particle Hilbert space* that contains the state of both particles together.

For the single-particle Hilbert spaces, we will always use the *single-particle basis*, which is the basis we have been discussing thus far. For a single d-state variable, the single-particle basis is $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$. When considering multiple variables, the single-particle basis consists of every possible tensor product of states from the single-particle basis for each variable. For ease of notation, we can condense such a state into a single ket:

$$|a_1, a_2, a_3, \ldots, a_n\rangle := |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_n\rangle .$$

We order this basis lexicographically; for example, if we consider a qubit variable and a qutrit variable, our basis would be ordered as

$$\{|0,0\rangle, |0,1\rangle, |0,2\rangle, |1,0\rangle, |1,1\rangle, |1,2\rangle\}.$$

The most natural basis for the joint-particle Hilbert space is what we will call the *joint-particle basis*. This basis consists of every possible tensor product of states from the single-particle bases for each particle. We again order this basis lexicographically: for a two-particle system entangled in a single qubit variable, the joint-particle basis is

$$\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}.$$

To avoid confusion, note that throughout this thesis we will refer to states of the joint-particle basis as *joint-particle states* or *joint-particle kets*.

Another basis for the joint-particle Hilbert space, which is particularly convenient when considering entanglement, is the *Bell basis*. This basis consists of maximally entangled states—entangled states for which the state of each individual particle is completely undefined, and thus all information is stored in correlation—and so is inherently useful when working with entanglement. For two particles entangled in a single qubit variable, the four *Bell states* that form the Bell basis are

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \qquad (2.18)\\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)\\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)\\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle).
\end{aligned}$$

These states are characterized by two properties: correlation and relative phase. The $|\Phi\rangle$ states are perfectly correlated in the $\{|0\rangle, |1\rangle\}$ basis—though neither particle has a definite state, if a measurement is performed in this basis on one particle and then the other, the result will be the same. The $|\Psi\rangle$ states are perfectly anticorrelated in the $\{|0\rangle, |1\rangle\}$ basis—measurements of each particle in this basis will always yield opposite results. The superscript sign indicates the relative phase between the two tensor product kets that make up the state.

For two particles entangled in a single d-state variable, the Bell basis is given by

$$\left\{ |\Phi_c^p\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i2\pi pj/d} |j\rangle |j + c \pmod d\rangle \,\middle|\, c, p \in \{0, 1, \ldots, d-1\} \right\}.$$

The index c specifies the correlation of the state, while the index p specifies the relative phases. As we have seen, when $d = 2$, there are two types of correlation and two choices of relative phase. (In our more general notation, the Bell states given in Eq. (2.18) are $|\Phi^+\rangle = |\Phi_0^0\rangle$, $|\Phi^-\rangle = |\Phi_0^1\rangle$, $|\Psi^+\rangle = |\Phi_1^0\rangle$, and $|\Psi^+\rangle = |\Phi_1^1\rangle$.) In general, for a d-state variable there are d choices for each of correlation and relative phase. Thus, the number of Bell states for entanglement in a single d-state variable is d^2 (note that this agrees with the number of states in the joint-particle basis, as it should). The Bell state $|\Phi_c^p\rangle$ is an equal superposition of all joint-particle kets $|a\rangle\,|b\rangle$ such that $b - a \equiv c \pmod{d}$.

As an example, let's explicitly write out the Bell states for the qutrit, i.e., the $d = 3$ case:

$$
\begin{aligned}
\left|\Phi_0^0\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle + |2\rangle\,|2\rangle\right) \\
\left|\Phi_0^1\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|0\rangle + e^{i2\pi/3}\,|1\rangle\,|1\rangle + e^{i4\pi/3}\,|2\rangle\,|2\rangle\right) \\
\left|\Phi_0^2\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|0\rangle + e^{i4\pi/3}\,|1\rangle\,|1\rangle + e^{i8\pi/3}\,|2\rangle\,|2\rangle\right) \\
\left|\Phi_1^0\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|1\rangle + |1\rangle\,|2\rangle + |2\rangle\,|0\rangle\right) \\
\left|\Phi_1^1\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|1\rangle + e^{i2\pi/3}\,|1\rangle\,|2\rangle + e^{i4\pi/3}\,|2\rangle\,|0\rangle\right) \\
\left|\Phi_1^2\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|1\rangle + e^{i4\pi/3}\,|1\rangle\,|2\rangle + e^{i8\pi/3}\,|2\rangle\,|0\rangle\right) \\
\left|\Phi_2^0\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|2\rangle + |1\rangle\,|0\rangle + |2\rangle\,|1\rangle\right) \\
\left|\Phi_2^1\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|2\rangle + e^{i2\pi/3}\,|1\rangle\,|0\rangle + e^{i4\pi/3}\,|2\rangle\,|1\rangle\right) \\
\left|\Phi_2^2\right\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle\,|2\rangle + e^{i4\pi/3}\,|1\rangle\,|0\rangle + e^{i8\pi/3}\,|2\rangle\,|1\rangle\right).
\end{aligned}
$$

Looking at this example, we notice that these states are pairwise orthogonal. In fact, this is a general property of the Bell basis: the Bell states for any multipartite system are always pairwise orthogonal, and they span the entire joint-particle Hilbert space (meaning that even an unentangled state can be written as a superposition of Bell states).

We can define a $d \times d$ unitary transformation matrix $\mathbf{M}_d$, which is simply a change-of-basis matrix from the joint-particle basis to the Bell basis.

When encoding the transformation as a matrix, we order the Bell basis as

$$\left\{ \left|\Phi_0^0\right\rangle, \left|\Phi_0^1\right\rangle, \ldots, \left|\Phi_0^{d-1}\right\rangle, \left|\Phi_1^0\right\rangle, \ldots, \left|\Phi_1^{d-1}\right\rangle, \ldots, \left|\Phi_{d-1}^0\right\rangle, \ldots, \left|\Phi_{d-1}^{d-1}\right\rangle \right\}.$$

For the case of entanglement in a single qubit, we see that

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

For entanglement in n variables of varying state number, the complete two-particle hyperentangled Bell state is a tensor product of Bell states for each degree of freedom. The Hilbert space for such a system has dimension $d_1^2 \cdot d_2^2 \cdots d_n^2$, where the ith variable has d_i orthogonal states, and the transformation from the joint-particle basis to the Bell basis is represented by the matrix $\mathbf{M} = \mathbf{M}_{d_1} \otimes \mathbf{M}_{d_2} \otimes \cdots \otimes \mathbf{M}_{d_n}$, where here the symbol $\otimes$ denotes the Kronecker product. For notational convenience, we let $D = d_1 d_2 \cdots d_n$. Thus, the number of Bell states in the hyperentangled Bell basis is D^2.

2.5 Correlation Classes

The hyperentangled Bell states and the joint-particle basis states can be partitioned into equivalence classes based on the correlation type within each variable. Two hyperentangled Bell states

$$\left|\Phi_{c_1}^{p_1}\right\rangle \left|\Phi_{c_2}^{p_2}\right\rangle \cdots \left|\Phi_{c_n}^{p_n}\right\rangle \quad \text{and} \quad \left|\Phi_{c_1'}^{p_1'}\right\rangle \left|\Phi_{c_2'}^{p_2'}\right\rangle \cdots \left|\Phi_{c_n'}^{p_n'}\right\rangle$$

are in the same equivalence class if and only if $c_i = c_i'$ for all i, while two joint-particle states

$$\left|a_1, a_2, \ldots, a_n\right\rangle \left|b_1, b_2, \ldots, b_n\right\rangle \quad \text{and} \quad \left|a_1', a_2', \ldots, a_n'\right\rangle \left|b_1', b_2', \ldots, b_n'\right\rangle$$

are in the same equivalence class if and only if $b_i - a_i \equiv b_i' - a_i' \pmod{d_i}$ for all i. We refer to these equivalence classes as *correlation classes*, and we indicate a correlation class by the ordered multiset $\{c_1, c_2, \ldots, c_n\}$. In principle, it is straightforward to determine experimentally what correlation class a state is in: one simply has to perform a projective measurement of each particle into the single-particle basis and then compare the values obtained in each measurement for every variable. The number of distinct correlation classes is then D, which is the number of distinct ordered multisets $\{c_1, c_2, \ldots, c_n\}$.

Chapter 3

Formalism

3.1 Introduction

For a number of practical reasons, we are concerned with measurement apparatuses constrained to *linear evolution and local measurement* (LELM). In particular, we are interested in linear evolution because linear processes have much higher cross-sections than non-linear ones, and are therefore more reliable. That is, the probability of achieving a successful measurement via linear processes is much higher than that for non-linear processes, which often rely on low-probability spatial coincidences and weak interactions between separate particles.

The second restriction—local measurement—is simply a necessary addition due to the manner in which measurements of microscopic entities are typically carried out. We measure a particle by registering a click in some detector; this is an annihilation event that projects the particle into a localized state, because it must have entered the detector to register the click. The result of our measurement is given by the state of a detector after registering a click.

Apparatuses constrained to LELM are also of particular interest because they are simple to construct. Consider an optical system. It has been shown that all LELM devices can be constructed using a series of mirrors, beam splitters, and wave plates [21]. The proof involves decomposing the mathematical representation of an arbitrary LELM device into components that behave like these optical components. As a result, LELM apparatuses are relatively easy to realize experimentally, which partly explains the emphasis on their use in real-world quantum measurements.

Although LELM devices are convenient in many ways, it is well known that they have limitations when used to measure entangled states, which are inherently nonlocal [15, 19, 20, 23, 25]. In this thesis, we address how well we can quantify these limits and explain the physics that gives rise to them.

3.2 Measurement with an LELM Apparatus

In general, an LELM apparatus can be realized as a unitary operator that maps single-particle input states to a set of single-particle output states. A particle is measured by being annihilated in a detector at the output of the apparatus. We consider annihilation in the Fock state basis, so that we can register multiple "clicks" in a single detector. The ability to detect both particles in the same output channel will be important later in our analysis of the maximum distinguishability problem.

The input to our LELM apparatus consists of two separate spatial channels, labeled *L*eft and *R*ight (or L and R for short). It is through these channels that the two particles we wish to measure will enter our device—one through L and one through R. The evolution of a particle through the apparatus can depend on which channel it entered through, and so we must also keep track of this. For this reason, our single-particle input states are tensored with the appropriate ket $|L\rangle$ or $|R\rangle$. For example, if we are considering two particles entangled in two different qubit variables, a particle in state $|0,0\rangle$ entering the left channel would correspond to the input state $|0,0\rangle\,|L\rangle$. As a result, the Hilbert space of single-particle input states to the apparatus has size $2D$. A diagram of an LELM apparatus is given in Fig. 3.1.

The following notation is inherited from Neal Pisenti [19, 20]. We will index all possible single-particle input states by $|\phi_j\rangle$, where i runs from 1 to $2D$. Here, even indices i are given to right channel states and odd indices are given to left channel states. In particular, if the single-particle states (without the additional $|L\rangle$ or $|R\rangle$) are ordered lexicographically as $\{|\chi_1\rangle, |\chi_2\rangle, \ldots, |\chi_D\rangle\}$, then

$$|\phi_{2j-1}\rangle = |\chi_j\rangle\,|L\rangle \quad \text{and} \quad |\phi_{2j}\rangle = |\chi_j\rangle\,|R\rangle\,.$$

The space of joint-particle input states to the apparatus (considering both input particles) is then spanned by the set of states $|\phi_j\rangle|\phi_k\rangle$, where we have the restriction $j \not\equiv k \pmod 2$, because we only accept states involving one particle in each of the spatial input channels. If we are considering indistinguishable particles, then the input states $|\phi_j\rangle$ also carry with them

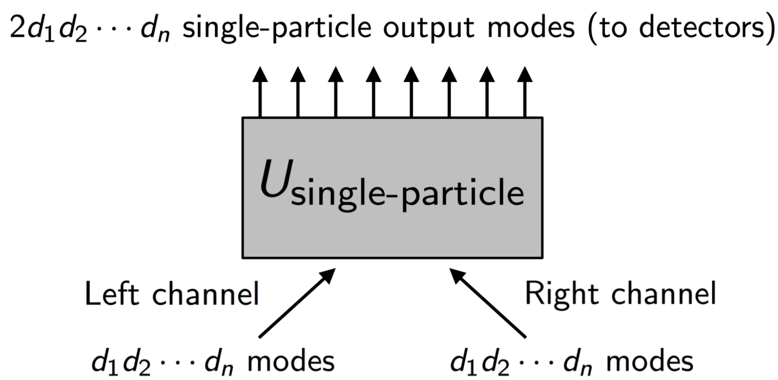

Figure 3.1 *Schematic of an LELM device*: Two particles enter the apparatus via separate spatial channels, labeled L and R. Each particle evolves independently of the other (the linear evolution restriction), and so single-particle input modes evolve unitarily to the set of single-particle output modes. Local measurement corresponds to registering two clicks in the detectors situated at each of the output modes, which projects the system into the tensor product state of two of the output modes (potentially the same mode twice).

an implicit (anti)symmetrization, so that the state designated by $|\phi_j\rangle|\phi_k\rangle$ is in fact

$$\frac{1}{\sqrt{2}}\left(|\phi_j\rangle_1|\phi_k\rangle_2 \pm |\phi_k\rangle_1|\phi_j\rangle_2\right).$$

Whether this state is symmetrized or antisymmetrized depends on whether the particles in question are bosons or fermions.

Because the apparatus is simply a unitary transformation over the single-particle input states (this is what it means for the apparatus to be restricted to linear evolution), the single-particle output space of our device must have the same dimension as the input space, and hence the maximum number of detectors that can be useful to us is $2D$. This assumes that we use no extra particles as ancillas, but it has been shown previously that for projective measurements with linear evolution, distinguishability between input states cannot be improved by the use of auxiliary modes if the input states are of definite particle number [10, 24], as is the case in our setup. Our device then has $2D$ single-particle detectors at its output. We assume that these detectors span the Hilbert space of output states (if this were not the case, then the device could destroy input particles without returning a result, which is clearly not useful if our goal is to measure the input state in the Bell basis).

Let $\hat{U}$ be the single-particle unitary operator carried out by our apparatus, with corresponding matrix $\mathbf{U}$ encoded in the single-particle basis. In general, any detector mode can be expressed in the form

$$|i\rangle = \sum_{j=1}^{2D} U_{ij} \left|\phi_j\right\rangle , \tag{3.1}$$

as an arbitrary superposition of all single-particle input states $\left|\phi_j\right\rangle$. The detectors are numbered 1 through $2D$ so that we may easily refer to them by a detector number.

A detection event consists of projecting the joint-particle input state onto a state of the form $|i\rangle \otimes |j\rangle$. As with the joint-particle input states, the detector states carry the implicit (anti)symmetrization

$$|i\rangle\, |j\rangle \to \frac{1}{\sqrt{2}} \left(|i\rangle_1 \,|j\rangle_2 \pm |j\rangle_1 \,|i\rangle_2\right)$$

for systems of indistinguishable particles. Because we require that our joint-particle input states consist of one particle in each of the spatial input channels, the actual relevant detector state is the projection of $|i\rangle\,|j\rangle$ onto the subspace of joint-particle input states that meet this criterion. We refer to this projection as the *detection signature*, and denote it $\hat{P}_{LR}\,|i\rangle\,|j\rangle$, where $\hat{P}_{LR}$ is the projection operator onto the aforementioned subspace.

A detection composed of two 'click' events realizes a projective measurement, projecting the state onto

$$|i\rangle\, |j\rangle = \sum_{k=1}^{2D} \sum_{l=1}^{2D} U_{ik} U_{jl}\, |\phi_k\rangle\, |\phi_l\rangle\, .$$

The corresponding detection signature is then of the form

$$P_{LR}\,|i\rangle\, |j\rangle = \sum_{k=1}^{2D} \sum_{l=1}^{2D} a_{kl}^{ij}\, |\phi_k\rangle\, |\phi_l\rangle\, ,$$

where the coefficient a_{kl}^{ij} are appropriately renormalized constants related to $U_{ik}U_{jl}$ and satisfying the propery $a_{kl}^{ij} = 0$ if $k \equiv l \pmod 2$. The form of the above detection signature is convenient because we can easily transform the state into the Bell basis via the matrix $\mathbf{M}$. This makes it easy to understand which Bell states have an amplitude to trigger a given detection event based on their presence in its representation, and hence which Bell states are indistinguishable under that detection outcome.

We express the coefficients $U_{ik}U_{jl}$ of the detection signature $P_{LR}\left|i\right\rangle\left|j\right\rangle$ in a matrix $\boldsymbol{\tau}^{ij}$, where

$$\left(\boldsymbol{\tau}^{ij}\right)_{kl} = a^{ij}_{kl} \tag{3.2}$$

Note that $\boldsymbol{\tau}^{ij}$ is the coordinate matrix for the detection signature $P_{LR}\left|i\right\rangle\left|j\right\rangle$, and so the rank of $\boldsymbol{\tau}^{ij}$ is the Schmidt rank for this state.

Because the physically significant state is the properly (anti)symmetrized state $\frac{1}{\sqrt{2}}\left(\left|\phi_k\right\rangle\left|\phi_l\right\rangle \pm \left|\phi_l\right\rangle\left|\phi_k\right\rangle\right)$, we also define

$$\mathbf{T}^{ij} = \frac{1}{\sqrt{2}}\left[\boldsymbol{\tau}^{ij} \pm \left(\boldsymbol{\tau}^{ij}\right)^{\mathsf{T}}\right]. \tag{3.3}$$

Consequently, the entry $T^{ij}_{kl} = \frac{1}{\sqrt{2}}\left(a^{ij}_{kl} \pm a^{ij}_{lk}\right)$ is the amplitude that our system existed in a given joint-particle state $\frac{1}{\sqrt{2}}\left(\left|\phi_k\right\rangle\left|\phi_l\right\rangle \pm \left|\phi_l\right\rangle\left|\phi_k\right\rangle\right)$ when detectors (i, j) are triggered.

A single-particle detector $\left|i\right\rangle$ can be expressed as a superposition of two separate kets that represent arbitrary superpositions of only L-channel or R-channel states:

$$\left|i\right\rangle = \alpha_i\left|\ell_i\right\rangle + \beta_i\left|r_i\right\rangle.$$

The detection signature $P_{LR}\left|i\right\rangle\left|j\right\rangle$ then becomes

$$\begin{aligned} P_{LR}|i\rangle|j\rangle &= P_{LR}(\alpha_i|\ell_i\rangle + \beta_i|r_i\rangle)(\alpha_j|\ell_j\rangle + \beta_j|r_j\rangle) \\ &= P_{LR}(\alpha_i\alpha_j|\ell_i\rangle|\ell_j\rangle + \alpha_i\beta_j|\ell_i\rangle|r_j\rangle + \beta_i\alpha_j|r_i\rangle|\ell_j\rangle + \beta_i\beta_j|r_i\rangle|r_j\rangle) \\ &= \frac{\alpha_i\beta_j}{\sqrt{|\alpha_i|^2|\beta_j|^2 + |\alpha_j|^2|\beta_i|^2}}|\ell_i\rangle|r_j\rangle + \frac{\alpha_j\beta_i}{\sqrt{|\alpha_i|^2|\beta_j|^2 + |\alpha_j|^2|\beta_i|^2}}|r_i\rangle|\ell_j\rangle. \end{aligned} \tag{3.4}$$

We see then that the Schmidt rank of a detection signature must always be at most two (without accounting for implicit (anti)symmetrization), and thus $\boldsymbol{\tau}^{ij}$ has rank at most two. (A more general proof is given in [9], which shows that any apparatus restricted to LELM cannot achieve a Positive Operator-Valued Measure with a Schmidt number greater than the number of input particles.) This fact will be useful later.

Let us consider an example to illustrate how detection signatures are found. Consider two particles entangled in a single qubit variable, and detector states $\left|i\right\rangle$ and $\left|j\right\rangle$ given by

$$\begin{aligned} \left|i\right\rangle &= \frac{1}{\sqrt{2}}(\left|0, L\right\rangle - \left|0, R\right\rangle) \\ \left|j\right\rangle &= \frac{1}{\sqrt{2}}(\left|1, L\right\rangle + \left|1, R\right\rangle). \end{aligned}$$

The joint-detector state is then

$$
\begin{aligned}
|i\rangle\,|j\rangle &= \frac{1}{2}(|0,L\rangle - |0,R\rangle)(|1,L\rangle + |1,R\rangle) \\
&= \frac{1}{2}(|0,L\rangle\,|1,L\rangle + |0,L\rangle\,|1,R\rangle - |0,R\rangle\,|1,L\rangle - |0,R\rangle\,|1,R\rangle),
\end{aligned}
$$

with corresponding detection signature

$$
P_{LR}\,|i\rangle\,|j\rangle = \frac{1}{\sqrt{2}}(|0,L\rangle\,|1,R\rangle - |0,R\rangle\,|1,L\rangle) = \left|\Psi^-\right\rangle.
$$

In this case, we see that the detection event (i, j) corresponds to a measurement of the Bell state $|\Psi^-\rangle$.

Chapter 4

Previous Work

4.1 Theoretical Results

This chapter serves simply to discuss previous work that has been done regarding the maximal distinguishability problem. It is shown in Section 6.1 that Bell state measurements using an LELM apparatus are constrained by a naïve upper bound to distinguishing at most $2D$ Bell state classes. Previous work has shown, however, that this is not the least upper bound for the case of qubit variables. In particular, algebraic and computational approaches have previously shown that for systems entangled in $n = 1$ or $n = 2$ qubit variables, the actual maximum number of distinguishable Bell state classes is one less than the naïve upper bound, or $2^{n+1} - 1$ [15, 23, 25]. In the case of entanglement in a single qubit, this means that only three classes of Bell states are distinguishable, though there are four Bell states in total. Neal Pisenti (HMC '11) generalized these results to two-particle systems entangled in an arbitrary number n of qubit variables, proving that such systems are always restricted to distinguishing at most $2^{n+1} - 1$ Bell state classes [19, 20]. Philipp Gaebler (HMC '09) showed by construction that one can *always* distinguish $2^{n+1} - 1$ different Bell state classes using LELM devices [13], proving that this bound is tight.

Pisenti's proof of the $2^{n+1} - 1$ bound in the general qubit case relied on physical arguments regarding symmetry constraints on indistinguishable particles, in contrast to the approaches given in Chapter 5 and Chapter 6 of this thesis, which are primarily algebraic. An algebraic proof of the $2^{n+1} - 1$ bound for entanglement in a single qubit (n = 1) variable was first published in 1999 [15]. The maximum number of distinguishable Bell states for hyperentanglement in two qubit variables was discovered computationally in 2007 by testing all $\binom{16}{8} = 12,870$ combinations of $2^{n+1} = 8$ states to show that no such set was distinguishable [25]. A similar computational proof for entanglement in three qubit variables would involve testing $\binom{64}{16} \approx 4.9 \times 10^{14}$ cases; clearly, this problem quickly becomes computationally intractable.

We will now briefly discuss the proofs given in [19, 20] for the general (arbitrary n) qubit bound. This material is reviewed here for two reasons: first, because much of my work is an extension of that done by Pisenti in his thesis and is based on some of his methods, and second, because I wish to present a small correction to his proof. The argument begins as follows. First, we note that we can assume, without loss of generality, that a particular detector fires first (say, detector number i). Then, we have limited the possible number of outcomes to 2^{n+1}, corresponding to the number of detectors which could register the second click and realize a full detection event. We then note that if and only if all of these 2^{n+1} detection events distinguish unique Bell state classes, we achieve the naïve bound of 2^{n+1} (for a full argument for why these facts are true, see Section 6.1).

We then consider fermions and bosons separately. The fermion case is the simpler of the two: if our system is composed of fermionic particles, then the detection event $P_{LR} |i\rangle |i\rangle$ cannot occur, because it is inherently symmetric. This establishes the $2^{n+1} - 1$ bound for fermionic systems.

In the boson case, there is a small oversight in the original thesis [20]. First, we note that if either α_i or β_i of Eq. (3.1) is zero (meaning that detector i can detect particles from only one of the two input channels), then the detection event $P_{LR} |i\rangle |i\rangle$ cannot occur, and we establish the bound as in the fermionic case. If $|i\rangle$ is a nontrivial superposition $|i\rangle = \alpha_i |\ell_i\rangle + \beta_i |r_i\rangle$ of left- and right-channel kets, then there must be some linear combination of output states satisfying

$$|X\rangle = \sum_j \epsilon_j |j\rangle = \alpha_i |\ell_i\rangle - \beta_i |r_i\rangle ,$$

because we assume that the output states span the Hilbert space. We see that the hypothetical detection signature $P_{LR} |i\rangle |X\rangle$ is zero, because it is inherently antisymmetric, and thus cannot occur in a bosonic system. This means that

$$\sum_j \epsilon_j P_{LR} |i\rangle |j\rangle = 0. \tag{4.1}$$

Now consider any j with $\epsilon_j \neq 0$. Any Bell state in the Bell basis representation of $P_{LR} |i\rangle |j\rangle$ must also appear in another detection signature $P_{LR} |i\rangle |l\rangle$, for some l with $\epsilon_l \neq 0$, or else Eq. (4.1) could not sum to zero in the Bell basis. This is where the oversight is made. At this point, the original thesis claims that this establishes the desired bound because the class of Bell states associated with the detection event $P_{LR} |i\rangle |j\rangle$ cannot be distinguished from the class of Bell states associated with the detection event $P_{LR} |i\rangle |l\rangle$. Because there must be at least one Bell state in both classes, it is certainly true that one would not be able to distinguish between any random choice of Bell states from the two classes; however, there are potentially many choices of individual Bell states from the two Bell state classes that could be distinguished using these two detection events. In this way, the result given in the original thesis for the bosonic case is weaker than that for the fermionic case; the proof given for the fermionic case establishes that there does not exist any set of 2^{n+1} Bell states that can all be distinguished from one another by an LELM device.

Fortunately, the same holds for the bosonic case, but the argument given in [20] must be tweaked slightly. The argument for why a given Bell state appearing in the Bell basis representation of $P_{LR} |i\rangle |j\rangle$ must also appear in another detection signature $P_{LR} |i\rangle |l\rangle$ must hold for each Bell state appearing in the Bell basis representation of $P_{LR} |i\rangle |j\rangle$, and this must further hold for every detection signature $P_{LR} |i\rangle |k\rangle$ for k with $\epsilon_k \neq 0$. This creates at least one closed cycle of detection signatures that overlap in the Bell basis. When such a cycle exists, it is impossible to choose a set of Bell states, one from each detection signature in the cycle, and be able to distinguish between every pair of Bell states in this set using the LELM apparatus. This establishes the stronger version of the bound: for bosonic systems, there does not exist any set of 2^{n+1} Bell states that can all be distinguished from one another by an LELM device.

For a discussion of the optimal qubit LELM apparatus given by Gaebler, see [13, 20].

4.2 Experimental Results

There is also experimental work being done to implement quantum communication protocols and detection schemes using LELM devices. For example, the Kwiat research group has explored the use of *embedded Bell state measurements*, in which hyperentanglement in additional variables is used to achieve complete Bell state distinguishability in a variable of interest [14]. The well-known limit for entanglement in a single qubit—that LELM devices can distinguish only three of the four Bell states—has already constrained a number of quantum information experiments [7, 16, 18]. Additionally, hyperentanglement of two particles in up to three qubit variables has been demonstrated experimentally.

Theoretical bounds on hyperentangled Bell state distinguishability can be used to constrain and inform future experiments conducted using LELM devices. The remainder of this thesis generalizes previously-known bounds to the case of hyperentanglement in arbitrary qudit variables.

Chapter 5

Minimum Class Size

5.1 Introduction

We now move on to addressing the problem of minimum class size. We have seen previously that an apparatus restricted to LELM will not be able to fully distinguish individual Bell states of the joint-particle input state [15, 19, 20, 23, 25]. Our goal is to determine, for two particles entangled in n qudit variables, the maximum number of Bell states among which such an apparatus can deterministically distinguish. Formally, we say that two Bell states are distinguishable by the apparatus if the two Bell states never appear together in the Bell basis representation of any individual detection signature. A set of Bell states is distinguishable if they are pairwise distinguishable.

In this chapter, we take a large step toward quantifying the maximum distinguishability of an LELM apparatus by addressing the problem of minimum class size. Here, we consider a single detection signature, and ask how many Bell states must, of necessity, be able to trigger the corresponding detection event. Two Bell states that can both trigger the same detection event are clearly not distinguishable by the apparatus. Formally speaking, we are interested in determining, for any given detection signature, the minimum number of Bell states that must appear when that detection signature is expressed in the Bell basis. This is the smallest possible number of Bell states among which that detection signature imposes indistinguishability.

Much of the approach used in this chapter was adapted from the approach taken by Neal Pisenti [19, 20] for qubit variables.

Below are the two minimum class size results that this chapter will prove in the general case of n qudit variables:

Theorem 1 If a detector fires twice, we can distinguish a class of no fewer than D states. □

Theorem 2 If two distinct detectors fire, we can distinguish a class of no fewer than $\lceil D/2 \rceil$ states. □

Recall that the ceiling function $\lceil \cdot \rceil$, used in Theorem 2, returns the least integer greater than its argument. Before proving these two results, we will first discuss the relationship between superpositions of joint-particle kets and the number of states in the corresponding Bell basis representations. This chapter will be rather mathematical in nature, and gaining a thorough understanding of its contents will rely on some previous knowledge of linear algebra and group theory. For those who don't want to fully wade through the math, a summary of the key observations is included immediately before the proofs at the end of the chapter.

5.2 Bell States and Joint-Particle Input States

5.2.1 Bell Basis Representation of Joint-Particle States

As we have seen, a detection signature represented in the joint-particle input basis is a superposition of kets of the form $|\phi_j\rangle |\phi_k\rangle$. Because we are interested in find which states are in the Bell basis representation of the detection signature, we must understand how the joint-particle kets individually relate to the Bell states. Given a joint-particle input state $|\phi_j\rangle |\phi_k\rangle$, we wish to know which states are in its Bell-basis representation. To gain some intuition, let's consider two particles entangled in a single qubit. Looking at the Bell states given in Eq. (2.18), we see that

$$
\begin{aligned}
|0\rangle\,|0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Phi^{+}\right\rangle + \left|\Phi^{-}\right\rangle\right)\\
|0\rangle\,|1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Phi^{+}\right\rangle - \left|\Phi^{-}\right\rangle\right)\\
|1\rangle\,|0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Psi^{+}\right\rangle + \left|\Psi^{-}\right\rangle\right)\\
|1\rangle\,|1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\Psi^{+}\right\rangle - \left|\Psi^{-}\right\rangle\right).
\end{aligned}
$$

We see that the two correlated states, $|0\rangle\,|0\rangle$ and $|1\rangle\,|1\rangle$, are equal superpositions of all the Bell states that exhibit perfect correlation, while the two anticorrelated states, $|0\rangle\,|1\rangle$ and $|1\rangle\,|0\rangle$, are equal superpositions of all the Bell states that exhibit perfect anticorrelation. This is true in general. From the state of $\left|\phi_j\right\rangle|\phi_k\rangle$ in each variable, we can determine its correlation class $\{c_1, c_2, \ldots, c_n\}$. The state $\left|\phi_j\right\rangle|\phi_k\rangle$ will be an equal superposition of all D of the Bell states in this correlation class. We can see then that two states $\left|\phi_j\right\rangle|\phi_k\rangle$ and $|\phi_l\rangle\,|\phi_m\rangle$ that belong to the same correlation class will have identical Bell states in their representations (with different relative phases).

However, two states which belong to *different* correlation classes will necessarily have disjoint Bell states in their representations, as they will always have a different single-variable Bell state in the Bell state tensor product for variables where they differ in correlation. Therefore, superpositions of joint-particle kets that belong to the same correlation class have the chance to add destructively in their Bell basis representations, whereas the disjoint nature of states belonging to different correlation classes will require that they always add to increase the number of states in the Bell basis.

For example, imagine we have a system of two particles entangled in two qubit variables. Consider the two joint-particle kets $|0,0\rangle\,|0,0\rangle$ and $|0,1\rangle\,|0,1\rangle$, which are both in the correlation class $\{0,0\}$. Each of these states has $D = 2 \cdot 2 = 4$ Bell states in its Bell basis representation, as we can see below:

$$\begin{aligned}
|0,0\rangle\,|0,0\rangle &= \left[\frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)\right]_{v_1} \otimes \left[\frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)\right]_{v_2} \\
&= \frac{1}{2}\left(\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle + \left|\Phi^+\right\rangle\left|\Phi^-\right\rangle + \left|\Phi^-\right\rangle\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\left|\Phi^-\right\rangle\right) \\
|0,1\rangle\,|0,1\rangle &= \left[\frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)\right]_{v_1} \otimes \left[\frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)\right]_{v_2} \\
&= \frac{1}{2}\left(\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle - \left|\Phi^+\right\rangle\left|\Phi^-\right\rangle + \left|\Phi^-\right\rangle\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\left|\Phi^-\right\rangle\right).
\end{aligned} \tag{5.5}$$

Above, we have use the subscripts v_1 and v_2 to indicate the variable associated to each single-variable Bell state superposition. Note that when we create an equal superposition of these two joint-particle kets, the Bell basis representation has a reduced size of $2 = D/2$:

$$\frac{1}{\sqrt{2}}\left(|0,0\rangle\,|0,0\rangle + |0,1\rangle\,|0,1\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\left|\Phi^+\right\rangle\right).$$

On the other hand, let's consider superposing the state $|0,0\rangle\,|0,0\rangle$ with the state $|0,0\rangle\,|0,1\rangle$, which belongs to a different correlation class. The latter state has a Bell basis representation

$$\begin{aligned}
|0,0\rangle\,|0,1\rangle &= \left[\frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)\right]_{1} \otimes \left[\frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)\right]_{2} \\
&= \frac{1}{2}\left(\left|\Phi^+\right\rangle\left|\Psi^+\right\rangle + \left|\Phi^+\right\rangle\left|\Psi^-\right\rangle + \left|\Phi^-\right\rangle\left|\Psi^+\right\rangle + \left|\Phi^-\right\rangle\left|\Psi^-\right\rangle\right).
\end{aligned} \tag{5.6}$$

We see that the Bell basis representation of the state $|0,0\rangle\,|0,1\rangle$ has no Bell states in common with the Bell basis representation of the state $|0,0\rangle\,|0,0\rangle$, and so no matter what relative magnitude or relative phase we choose when superposing these states, the resulting superposition will have eight Bell states in its Bell basis representation.

Because the theorems above depend on proving that certain detection events cannot represent *fewer* than a given number of Bell states, we want to find ways to minimize the number of states in the Bell basis representation. Thus, we will only concern ourselves with superpositions of joint-particle kets belonging to the same correlation class, as only these will produce a reduction in the Bell state basis.

5.2.2 Single-Particle Kets and Correlation Classes

Devices constrained to LELM act on single-particle kets, and so we must also understand the relationship between single-particle states and correlation classes. In particle, we can show that two joint-particle states belonging to the same correlation class cannot share any n-variable single-particle ket. That is to say, if two states $|\phi_j\rangle\,|\phi_k\rangle$ and $|\phi_l\rangle\,|\phi_m\rangle$ are in the same correlation class, then $j \neq l \neq k \neq m$. To see this, consider two different joint-particle input states $|\phi_j\rangle\,|\phi_k\rangle$ and $|\phi_j\rangle\,|\phi_m\rangle$ that share the single-particle ket $|\phi_j\rangle$. Since $|\phi_j\rangle\,|\phi_k\rangle$ and $|\phi_j\rangle\,|\phi_m\rangle$ are distinct states, $|\phi_k\rangle$ and $|\phi_m\rangle$ must differ in at least one variable. In that variable, the joint-particle kets $|\phi_j\rangle\,|\phi_k\rangle$ and $|\phi_j\rangle\,|\phi_l\rangle$ will have different correlations, and hence cannot belong to the same correlation class.

For example, consider again the system of two particles entangled in two qubit variables, and the three states discussed in the example given in Section 5.2.1. We can see that the states $|0,0\rangle\,|0,0\rangle$ and $|0,1\rangle\,|0,1\rangle$, which share no single-particle kets, are both in the correlation class $\{0,0\}$. On the other hand, the state $|0,0\rangle\,|0,1\rangle$, which shares a single-particle ket with each of the other states, is in the correlation class $\{0,1\}$. Examining the Bell basis representations of these states given in Eq. (5.5) and Eq. (5.6), we see that indeed the Bell basis representations of the states $|0,0\rangle\,|0,0\rangle$ and $|0,1\rangle\,|0,1\rangle$ are both superpositions of perfectly correlated Bell states, while the Bell basis representation of the state $|0,0\rangle\,|0,1\rangle$ consists of Bell states that are perfectly correlated in the first variable and perfectly anticorrelated in the second variable.

Recalling from Section 5.2.1 that only joint-particle kets from the same correlation class can add destructively in the Bell basis, we conclude that any superpositions of joint-particle states that directly reduce the size of the Bell basis representation cannot share any single particle states $|\phi_j\rangle$.

5.2.3 Bell State Reduction Efficiency

We have now established that reduction in the Bell basis can only be achieved by superposing joint-particle kets from the same correlation class, and that such joint-particle kets do not share single-particle kets. We must now establish how efficiently we can use such superpositions to reduce the size of the Bell basis representation. In Section 5.2.1, we saw that a single joint-particle ket $|\phi_j\rangle\,|\phi_k\rangle$ contains D Bell states in its Bell basis representation. We are interested in finding the minimum number r of joint-particle kets from the same correlation class that must be superposed to reduce the Bell basis representation of a detection signature to a size $q \leq D$.

In order to determine the relationship between r and q, we introduce a group of linear symmetry operators on the space of Bell states that alter the relative phase of a Bell state in a given variable. In particular, let g_s be an operator that increments (mod d_s) the relative phase index p_s of the sth variable in a Bell state. For example, if we have a pair of particles entangled in three qutrit variables and we consider the Bell state $|\Phi_0^0\rangle\,|\Phi_2^1\rangle\,|\Phi_2^2\rangle$, then the operators g_1, g_2, and g_3 act as shown below:

$$\begin{aligned} g_1|\Phi_0^0\rangle|\Phi_2^1\rangle|\Phi_2^2\rangle &= |\Phi_0^1\rangle|\Phi_2^1\rangle|\Phi_2^2\rangle \\ g_2|\Phi_0^0\rangle|\Phi_2^1\rangle|\Phi_2^2\rangle &= |\Phi_0^0\rangle|\Phi_2^2\rangle|\Phi_2^2\rangle \\ g_3|\Phi_0^0\rangle|\Phi_2^1\rangle|\Phi_2^2\rangle &= |\Phi_0^0\rangle|\Phi_2^1\rangle|\Phi_2^0\rangle. \end{aligned}$$

Note that these operators do not affect the correlation indices of any of the single-variable Bell states. These operators generate a group $G = \langle g_1, g_2, \ldots, g_n\rangle$. These generators commute with one another, and the subgroup $H_s = \langle g_s\rangle$ generated by the phase-shifting operator on the sth variable is cyclic of order d_s for all s, so we see that

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}.$$

We see then that $|G| = D$. Thus, given any Bell state in a particular correlation class, all D Bell states in that correlation class can be generated by applying elements of G, i.e., if we view the elements of any particular correlation class as a G-module, then the action of G on this module is transitive.

Let us then look at how the elements of G act on kets in the joint-particle basis. By inspecting the action of g_s on the Bell states, we see that if we have a joint-particle ket

$$|\phi_j\rangle\,|\phi_k\rangle = |a_1, a_2, \ldots, a_s, \ldots, a_n, L\rangle\,|b_1, b_2, \ldots, b_n, R\rangle\,,$$

then

$$g_s\,|\phi_j\rangle\,|\phi_k\rangle = e^{i2\pi a_s/d_s}\,|a_1, a_2, \ldots, a_s, \ldots, a_n, L\rangle\,|b_1, b_2, \ldots, b_n, R\rangle\,.$$

Note that although the elements of G permute the Bell states, they only change the overall phase of joint-particle kets. Thus, the elements of G transform one Bell state to another by changing the relative phases between states in the joint-particle representation, but they do not change which joint-particle kets appear in that representation.

Now, consider a superposition of q Bell states formed by superposing r joint-particle kets from the correlation class $\{c_1, c_2, \ldots, c_n\}$. Applying any element $g_s \in G$ to this state will not change the r joint-particle kets in the superposition, but if we apply all D elements of G to the superposition, then we have created D potentially different superpositions of q Bell states each. Every Bell state in the correlation class $\{c_1, c_2, \ldots, c_n\}$ must appear in at least one of the superpositions generated in this way, due to the transitivity of the group action. Let S be the set of superpositions created in this process. We know then that r must be at least as large as the maximum number of linearly independent vectors in S, i.e., $r \geq \dim\langle S\rangle$. This is simply because every element of S is a superposition of the same r joint-particle kets, and so S can, at most, span the corresponding r-dimensional subspace of the Hilbert space. We note then that the minimum possible value of $\dim\langle S\rangle$ over all superpositions of q Bell states from the correlation class $\{c_1, c_2, \ldots, c_n\}$ is a lower bound for r. The corresponding minimal spanning set T will consist of superpositions that are maximally disjoint; however, we note that each Bell state in the correlation class $\{c_1, c_2, \ldots, c_n\}$ must still appear in at least one superposition in T. If this were not the case, then any superposition in S containing the "missing" Bell state (we know that at least one exists from the transitivity of the action of G) could be added to T to create a larger set of vectors that was still linearly independent, contradicting the notion that T spans $\langle S\rangle$. Thus, T must contain at least enough vectors to cover all D Bell states in the correlation class $\{c_1, c_2, \ldots, c_n\}$. This condition requires that $|T| \geq D/q$, and so we see that $r \geq D/q$. That is, we must superpose at least D/q joint-particle kets from the same correlation class in order to reduce the number of states in the Bell basis representation of the superposition to q.

5.2.4 Summary

In preparing for the proofs at the end of this chapter, we have made several key observations, all of which are summarized below.

1. The Bell basis representation of a joint-particle ket is an equal superposition of all D Bell states in the correlation class associated with that joint-particle ket.

2. Only joint-particle kets from within the same correlation class can add destructively in the Bell basis. That is, superposing two joint-particle kets can reduce the total number of Bell states to a number less than D if and only if both joint-particle kets are in the same correlation class.

3. Two joint-particle kets belonging to the same correlation class cannot have any single-particle kets in common. That is, if two states $|\phi_j\rangle\,|\phi_k\rangle$ and $|\phi_l\rangle\,|\phi_m\rangle$ are in the same correlation class, then $j \neq l \neq k \neq m$.

4. In order to contain only $q \leq D$ Bell states from a particular correlation class in its Bell basis representation, a superposition of joint-particle kets must contain $r \geq D/q$ joint-particle kets from that correlation class.

We now move on to proving the theorems presented at the beginning of the chapter.

5.3 Proof of Minimal Bell State Class Size

The two theorems we wish to prove are restated below:

Theorem 1 If a detector fires twice, we can distinguish a class of no fewer than D states. □

Theorem 2 If two distinct detectors fire, we can distinguish a class of no fewer than $\lceil D/2 \rceil$ states. □

Throughout these two proofs, we will be careful to differentiate between Schmidt ranks that do and do not take into account the implicit (anti)symmetrization of the joint-particle kets. Recall that the rank of the matrix $\boldsymbol{\tau}^{ij}$ has rank equal to the *unsymmetrized* Schmidt rank of the corresponding state.

PROOF (THEOREM 1) In the case that a single detector fires twice, the detection signature is given by $P_{LR} |i\rangle |i\rangle$. We see, referring back to Eq. (3.4) that

$$|i\rangle |i\rangle = \frac{1}{\sqrt{2}}(|\ell_i\rangle |r_i\rangle + |r_i\rangle |\ell_i\rangle),$$

which clearly has Schmidt rank two (without considering symmetrization or antisymmetrization). Note that this detection signature is inherently symmetric, and so a detection event in which the same detector fires twice cannot occur in a fermionic system (notice that if we antisymmetrize the state, it vanishes); we then need only consider the bosonic case. Note that if our system obeys boson statistics, then the implicitly symmetrized states $|\ell_i\rangle\,|r_i\rangle$ and $|r_i\rangle\,|\ell_i\rangle$ are, in fact, the same state. In this case, $|i\rangle\,|i\rangle = |\ell_i\rangle\,|r_i\rangle$ in our implicit symmetrization notation, and the implicit symmetrization does not increase the Schmidt rank of the state. From here on out, we will consider the bosonic state

$$|i\rangle\,|i\rangle = |\ell_i\rangle\,|r_i\rangle\,,$$

which is implicitly symmetrized and has *symmetrized* Schmidt rank two.

We now play the game of trying to design such a detection signature with a minimal Bell state class size. If we imagine starting out with an entire correlation class of D Bell states—say by having the detection signature initially equal to a single joint-particle ket from this correlation class—then the only way we can attempt to reduce the size of this class is to superpose additional joint-particle states from the same correlation class. However, joint-particle states from the same correlation class do not share single-particle kets, as we found in Section 5.2.2, and so a superposition of any number r of (implicitly symmetrized) joint-particle states all from the same correlation class will have symmetrized Schmidt rank $2r$. Because the repeated-detector detection signature is restricted to have symmetrized Schmidt rank two, we see that we cannot superpose even two joint-particle states from the same correlation class. We could try to solve this problem by also superposing joint-particle states from multiple correlation classes, in an attempt to reduce the Schmidt rank back to two, but each time we superpose a joint-particle state from a new correlation class, we introduce D new Bell states to the Bell state class. To reduce the total number of Bell states in the detection signature below D would require a reduction within each correlation class that is not achievable while satisfying the Schmidt rank two condition. Thus, the detection signature must have at least D Bell states in its representation.

To be more formal, we consider the matrix $\boldsymbol{\tau}^{ij}$ given in Eq. (3.2). In this case, we see that

$$\left(\boldsymbol{\tau}^{ii}\right)_{kl} = a^{ii}_{kl} \propto \begin{cases} U_{ik}U_{il} & k \not\equiv l \pmod 2 \\ 0 & k \equiv l \pmod 2 \end{cases}$$

We see that $\boldsymbol{\tau}^{ii}$ is symmetric. Thus, $\mathbf{T}^{ii} = \sqrt{2}\boldsymbol{\tau}^{ii}$ for bosonic systems and $\mathbf{T}^{ii} = 0$ for fermionic systems (this indicates the fact that the same detector cannot fire twice in a fermionic system). Let us then consider the bosonic case. As discussed in Section 3.2, the rank of $\boldsymbol{\tau}^{ii}$ is the Schmidt rank of of

$P_{LR}\,|i\rangle\,|i\rangle$. Thus, we see that $\boldsymbol{\tau}^{ii}$, and therefore $\mathbf{T}^{ii}$, must have rank two. The fact that $\boldsymbol{\tau}^{ii}$ and $\mathbf{T}^{ii}$ have the same rank is the mathematical realization of our earlier observation that in the bosonic case, symmetrization does not change the Schmidt rank of the state. We can then decompose $\mathbf{T}^{ii}$ as

$$\mathbf{T}^{ii} = \mathbf{u}_1\mathbf{v}_1^{\mathsf{T}} + \mathbf{u}_2\mathbf{v}_2^{\mathsf{T}},$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are vectors in $\mathbb{C}^{2D}$. This is the singular value decomposition of $\mathbf{T}^{ii}$. We know because $\left(\boldsymbol{\tau}^{ii}\right)_{kl} = a^{ii}_{kl}$ that the matrix entries of $\mathbf{T}^{ii}$ that have row and column index of the same parity must be zero. Because of this, we see that $\mathbf{u}_1$ and $\mathbf{v}_1$ cannot have nonzero entries in positions of the same parity, and similarly for $\mathbf{u}_2$ and $\mathbf{v}_2$. Without loss of generality, let us assume that $\mathbf{u}_1$ and $\mathbf{v}_2$ have nonzero entries only in even-index positions, while $\mathbf{u}_2$ and $\mathbf{v}_1$ have nonzero entries only in odd-index positions. Because $\mathbf{T}^{ii}$ is symmetric, this leads to the restriction that $\mathbf{u}_1 = \mathbf{v}_2$ and $\mathbf{u}_2 = \mathbf{v}_1$, and so we have

$$\mathbf{T}^{ii} = \mathbf{u}\mathbf{v}^{\mathsf{T}} + \mathbf{v}\mathbf{u}^{\mathsf{T}}.$$

Let us then consider building such a detection signature out of joint-particle kets. Say we begin with the joint-particle ket $|\phi_j\rangle\,|\phi_k\rangle$ (without loss of generality, assume that j is even and k is odd). If this state is to be detected by $P_{LR}\,|i\rangle\,|i\rangle$, we must set u_j and v_k to nonzero values. As discussed in Section 5.2.1, this introduces D Bell states to the detection signature. The only way we could potentially reduce this number is by adding in another joint-particle state from the same correlation class. As we concluded in Section 5.2.2, this means introducing a new state $|\phi_l\rangle\,|\phi_m\rangle$ with $l \neq j$ and $m \neq k$ (again, assume without loss of generality that l is even and m is odd). Thus, we set u_l and v_m to nonzero values. However, we see that this also introduces the new joint-particle states $|\phi_j\rangle\,|\phi_m\rangle$ and $|\phi_l\rangle\,|\phi_k\rangle$, neither of which are in the same correlation class as the original two states (these unintentionally added states may or may not be from the same correlation class as one another). Thus, in attempting to reduce the size of the Bell state class associated to the original correlation class, we have introduced at least one new correlation class of Bell states to the mix. If our goal is to reduce the number of Bell states from the original correlation class to q, then we must superpose $r \geq D/q$ joint-particle states from this correlation class, which introduces states from at least r total correlation classes. Each of these correlation classes has at most r of its joint-particle kets represented in the superposition, and so the associated Bell state class has size at least q. Thus, the number of Bell states in the superposition is at least $r \cdot q \geq D/q \cdot q = D$. This completes the proof. ∎

PROOF (THEOREM 2) In the case that two distinct detectors fire, the detection signature is given by Eq. (3.4) and has Schmidt rank at most two (considering the implicit symmetrization, the state has Schmidt rank of at most four). From the discussion in the previous proof, we see that such a detection signature cannot represent more than two (implicitly (anti)symmetrized) joint-particle kets from a single correlation class without also representing joint-particle kets from other correlation classes. If both kets $|\ell_i\rangle\,|r_j\rangle$ and $|r_i\rangle\,|\ell_j\rangle$ are from the same correlation class, then we can potentially get a reduction to $\lceil D/2 \rceil$ Bell states, but superposing any additional joint-particle kets from the same correlation class in an attempt to reduce further will necessitate superposing joint-particle kets from other correlation classes and therefore introducing new Bell states, as discussed in the previous proof. We conclude that a detection signature associated with two distinct detectors firing cannot distinguish between fewer than $\lceil D/2 \rceil$ Bell states.

Once again, we seek to formalize this argument. In this case,

$$\left(\boldsymbol{\tau}^{ij}\right)_{kl} = a_{kl}^{ij} \propto \begin{cases} U_{ik}U_{jl} & k \not\equiv l \pmod 2 \\ 0 & k \equiv l \pmod 2 \end{cases}$$

As discussed in Section 3.2, the rank of $\boldsymbol{\tau}^{ij}$ is at most two, and so the rank of the matrix $\mathbf{T}^{ij} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\tau}^{ij} \pm \left(\boldsymbol{\tau}^{ij}\right)^{\mathsf{T}}\right)$, as given in Eq. (3.3), is at most four.

Consider k Bell states represented by $r \geq D/q$ joint-particle states all from the same correlation class $\{c_1, c_2, \ldots, c_n\}$ that are nonzero in the detection signature $P_{LR}\,|i\rangle\,|j\rangle$. None of these joint-particle kets share any single-particle ket. Each joint-particle ket $|\phi_k\rangle\,|\phi_l\rangle$ (not symmetrized or antisymmetrized) corresponds to a unique entry in $\mathbf{T}^{ij}$, and these entries are distinct for different values of q and l. Therefore, because $\mathbf{T}^{ij}$ is symmetric, each of the r joint-particle kets from the correlation class $\{c_1, c_2, \ldots, c_n\}$ corresponds to a different pair of rows of $\mathbf{T}^{ij}$ that must be nonzero. Thus, $\mathbf{T}^{ij}$ will have at least $2r$ nonzero rows. Because $\mathbf{T}^{ij}$ has rank at most four, it can have at most four linearly independent rows, and so this implies that at least one row of the matrix must have at least $(2r)/4 = r/2$ nonzero entries. These entries correspond to at least $r/2$ joint-particle kets that all share a single-particle ket, and therefore all belong to distinct correlation classes. As before, each of the corresponding Bell state classes could be reduced, at best, to size q, and so the detection signature must contain at least $q \cdot r/2 \geq q \cdot (D/q)/2 = D/2$ Bell states. Because D is not guaranteed to be even, this ensures that the detection signature contains at least $\lceil D/2 \rceil$ Bell states, completing the proof. ∎

Finally, it is worth mentioning that, as noted in [19, 20], the optimal schemes that have previously published for systems entangled in $n = 1$ and $n = 2$ qubit variables all have exactly the class-size structure given by these two theorems [8, 14, 15, 17, 22, 23, 25, 26]. When any single detector fires twice, they distinguish a class of 2^n Bell states, and all other detector pairings that can fire in the apparatus distinguish classes of 2^{n-1} Bell states each. Additionally, there are always some detector pairings that cannot fire together within the apparatus due to symmetry restrictions.

Chapter 6

Maximal Distinguishability

6.1 Introduction

We are interested in how many of the Bell states can be reliably distinguished by an LELM device, because this bound limits the performance of some Bell-measurement based quantum communication protocols. The exact distinguishability question of interest depends somewhat on the protocol. For some protocols, such as quantum teleportation, the Bell measurement projects a previously unentangled pair into one of the Bell states at random. In this case, performance depends on the number of classes of Bell states into which the LELM apparatus can partition the full set of D^2 Bell states. Other protocols, however, allow the user to select a subset of the Bell states and then discriminate only among those; then performance depends on how many individual Bell states can be reliably distinguished, even if these must be carefully chosen out of their detection signatures, and even if those detection signatures are individually larger than necessary and overlap with one other in some Bell states. We can answer the former question, for the case of teleportation, quite handily. We can also address the latter question if we can show that an apparatus with disjoint detection signatures is sufficient to achieve optimal distinguishability. While we have not been able to prove this yet, there are a (perhaps surprisingly) large number of situations in which the detection signatures are disjoint, which we will discuss in this chapter.

Before moving on to discuss bounds on maximal distinguishability in the case of disjoint detection signatures, it is worth noting that it is relatively simply to prove a naïve bound of $2D$ on the maximum number of Bell states distinguishable by the apparatus. The basic insight is that the detection of a single particle does not give any information about the entangled joint-particle state in the Bell basis. Because the Bell states are all maximally entangled, they each contain every single-particle ket somewhere in their joint-particle basis representation. In particular, if either particle is considered individually, then it is found to be in an equal mixture of every basis vector, regardless of the basis chosen. Thus, if a particular detector in the apparatus can fire at all, it must have nonzero amplitude to fire for any Bell state input, because the apparatus acts over the space of single-particle kets rather than joint-particle kets. (This fact can be shown more formally for general qudits with exactly the same method as given for qubits in [19].) Therefore, no information can be gained from the first single-detector event. Distinguishability must then come from one of the $2D$ possibilities for the second single-detector event. This is what provides us with the naïve upper bound of $2D$ for maximal distinguishability: there simply aren't enough free parameters in the detection space for any LELM device to achieve a better detection resolution.

Considering the ideas found in the proof of the naïve bound given above, we see that we can assume, without loss of generality, that detector i fires first (rather than any other particular detector). This will be useful in future discussions, as it reduces the total number of possible outcomes for a measurement to $2D$.

6.2 Assuming Disjoint Detection Signatures

For the time being, let us assume that the detection signatures of our apparatus are pairwise disjoint (two detection signatures either have no Bell states in common between their Bell basis representations, or they distinguish exactly the same class of Bell states), and see what kinds of bounds we can find for the maximal distinguishability of the apparatus.

If we assume that the detection signatures of the apparatus are disjoint, then the associated classes of Bell states must partition the set of all Bell states. We can then utilize the minimum class size results from Chapter 5 to determine the maximum possible number of Bell state classes that the apparatus can distinguish among. Call this maximum number x. If our system is bosonic, then we know that one repeated-detector event is possible (namely $P_{LR} |i\rangle |i\rangle$), and it has an associated class of Bell states of size at least D. The other $x - 1$ Bell state classes will be associated with two-detector events, and so will have associated Bell state classes of size at least $\lceil D/2 \rceil$. The total number of Bell states in all these classes cannot exceed

D^2, the total number of Bell states, and the classes are disjoint. Thus, we have

$$D + (x-1)\lceil D/2 \rceil \leq D^2. \tag{6.1}$$

We then go into casework. If $D \equiv 2 \pmod 2$, then $\lceil D/2 \rceil = D/2$, and Eq. (6.1) simplifies to

$$x \leq 2D - 1.$$

This is our maximal distinguishability result for this case.

If $D \not\equiv 2 \pmod 2$, then $\lceil D/2 \rceil = (D+1)/2$, and things become slightly more complicated. In this case, we can rearrange Eq. (6.1) to reach

$$x \leq 2D - \frac{3D-1}{D+1}.$$

The smallest odd value of D we can consider is $D = 3$; for this value of D, $\lceil \frac{3D-1}{D+1} \rceil = 2$. For all larger odd values of D, $\lceil \frac{3D-1}{D+1} \rceil = 3$. Thus, if $D = 3$, we have

$$x \leq 2D - 2 = 4,$$

and if D is odd and $D > 3$, then we have

$$x \leq 2D - 3.$$

We now consider the fermionic case. Here, no repeated-detector event is possible, as discussed in Section 5.3, and so we have

$$x\lceil D/2 \rceil \leq D^2. \tag{6.2}$$

In the case that $D \equiv 2 \pmod 2$, this simplifies to $x \leq 2D$. However, we can actually use a symmetry argument to give a stricter bound in this case. The naïve bound of $2D$ includes the possibility of a repeated-detector event; however, we know that such a detection event cannot occur in a fermionic system, and so we in fact have

$$x \leq 2D - 1.$$

This is precisely the symmetry argument given in [19, 20] for the fermion maximal distinguishability case.

In the case that $D \not\equiv 2 \pmod 2$, then Eq. (6.2) becomes

$$x \leq 2D - \frac{2D}{D+1}.$$

The smallest odd value of D we can consider is $D = 3$; for all $D \geq 3$, $\lceil \frac{2D}{D+1} \rceil = 2$. Thus, we have

$$x \leq 2D - 2$$

in this case.

The results of this section are shown in Table 6.1. Note that this agrees with the results of [19, 20]; for a system hyperentangled in n qubit variables, $D = 2^n$, and so our result indicate an upper bound of $2^{n+1} - 1$ for the maximum number of distinguishable Bell states.

Case	Max # of Distinguishable Bell States
$D \equiv 0 \pmod 2$	$2D - 1$
$D \equiv 1 \pmod 2$, fermions	$2D - 2$
$D \equiv 1 \pmod 2$, bosons	$2D - 3$
$D = 3$, bosons	4

Table 6.1 Maximal distinguishability results, assuming disjoint detection signatures

It is not known in general if these upper bounds are achievable (tight), as they are for hyperentanglement in only qubit variables.

Chapter 7

Conclusion

This thesis extends the analysis of the maximal distinguishability problem for LELM-constrained devices to general hyperentangled states. For a system of two particles entangled in n arbitrary discrete variables, we show that a detection event by an LELM device in which the same detector fires twice distinguishes a class of no fewer than D Bell states, while a detection event in which two distinct detectors fire distinguishes a class of no fewer than $\lceil D/2 \rceil$ Bell states, where D is the dimension of the single-particle Hilbert space. These results are then used to discuss maximal distinguishability under the assumption that the detection signatures of the apparatus are all pairwise orthogonal. In this case, precise numerical bounds are found for the maximal number of Bell states distinguishable by the device, which agree with previous results [15, 19, 20, 23, 25]. The reasonability of such an assumption is discussed in Appendix A.

This thesis also provides a new approach through which to analyze maximal distinguishability. Previously, algebraic approaches have proved fruitful in very small cases [15], while computational methods have been used for slightly larger systems [25]. In his thesis, Neal Pisenti (HMC '11) was able to prove a general qudit result using physical symmetry arguments which do not easily generalize to the case of qudits [19, 20]. This thesis returns to a more formal algebraic approach that is applicable to the general case, based off of work done previously by Pisenti.

The most apparent future research direction for this project is showing, if possible, that LELM apparatuses with disjoint detection signatures are sufficient to achieve the optimal maximal distinguishability, when considering the use of projective measurements to deterministically distinguish Bell states. Anther possible extension could involve seeking bounds on distinguishability for states that are not maximally entangled. Additionally, future research could investigate whether the bounds presented here are actually achievable, or if the least upper bounds are lower for as yet unknown reasons.

The results contained herein will hopefully be useful in guiding future experimental work in quantum communication, and provide a formal approach for further analysis of the maximal distinguishability problem.

Appendix A

Possible Approaches to a Proof of Disjoint Detection Signature Sufficiency

A.1 Introduction

In this appendix, we present progress made in proving the sufficiency of LELM devices with disjoint detection signatures to perform optimally. That is, our ultimate goal is to prove that LELM devices with potentially overlapping detection signatures cannot distinguish a greater number of Bell states than an optimal LELM apparatus with disjoint detection signatures. If we were able to prove this, then the results presented in Sec. 6.2 would hold in all cases. The results presented here can be used to aid future attempts toward such a proof.

A.2 When are detection signatures orthogonal?

First, we will discuss situations in which detection signatures are orthogonal. It should be noted that detection signature orthogonality *does not* imply disjointness. Consider, as a simple example, the two states

$$
\begin{aligned}
|q_1\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right) \\
|q_2\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right),
\end{aligned}
$$

which are orthogonal but contain exactly the same Bell states in their Bell basis representation. However, orthogonality is a related condition (disjointness *does* imply orthogonality), and so it may be useful to find conditions under which we know that two detection signatures will be orthogonal. The algebra that follows is somewhat tedious to work through, but the insights it leads to are worthwhile, and so it is included for completeness.

As discussed in Section 6.1, we will assume without loss of generality that detector $|i\rangle$ fires first. Let detectors $|i\rangle$, $|j\rangle$, and $|k\rangle$ be given by

$$\begin{aligned}
|i\rangle &= \alpha_i \,|\ell_i\rangle + \beta_i \,|r_i\rangle \\
|j\rangle &= \alpha_j \,|\ell_j\rangle + \beta_j \,|r_j\rangle \\
|k\rangle &= \alpha_k \,|\ell_k\rangle + \beta_i \,|r_k\rangle \,,
\end{aligned}$$

as defined in Eq. (3.1). Throughout this section, we will explicitly show all (anti)symmetrization, and we will use subscripts 1 and 2 for particles 1 and 2 to avoid confusion (note that the subscript corresponding to a bra appears to the *left* of the bra). We then consider the two detection events corresponding to the (anti)symmetrized product states

$$\begin{aligned}
|i\rangle|j\rangle \to \frac{1}{\sqrt{2}} &\left(|i\rangle_1|j\rangle_2 \pm |j\rangle_1|i\rangle_2\right) \\
&= \frac{1}{\sqrt{2}} \Big[(\alpha_i|\ell_i\rangle_1 + \beta_i|r_i\rangle_1)(\alpha_j|\ell_j\rangle_2 + \beta_j|r_j\rangle_2) \\
&\qquad \pm(\alpha_j|\ell_j\rangle_1 + \beta_j|r_j\rangle_1)(\alpha_i|\ell_i\rangle_2 + \beta_i|r_i\rangle_2)\Big] \\
&= \frac{1}{\sqrt{2}} \Big[\alpha_i\alpha_j|\ell_i\rangle_1|\ell_j\rangle_2 + \alpha_i\beta_j|\ell_i\rangle_1|r_j\rangle_2 \\
&\qquad + \beta_i\alpha_j|r_i\rangle_1|\ell_j\rangle_2 + \beta_i\beta_j|r_i\rangle_1|r_j\rangle_2 \\
&\qquad \pm \alpha_j\alpha_i|\ell_j\rangle_1|\ell_i\rangle_2 \pm \alpha_j\beta_i|\ell_j\rangle_1|r_i\rangle_2 \\
&\qquad \pm\beta_j\alpha_i|r_j\rangle_1|\ell_i\rangle_2 \pm \beta_j\beta_i|r_j\rangle_1|r_i\rangle_2\Big]
\end{aligned}$$

and (omitting repeated work)

$$\begin{aligned}
|i\rangle|k\rangle \to \frac{1}{\sqrt{2}} &\Big[\alpha_i\alpha_k|\ell_i\rangle_1|\ell_k\rangle_2 + \alpha_i\beta_k|\ell_i\rangle_1|r_k\rangle_2 \\
&\quad + \beta_i\alpha_k|r_i\rangle_1|\ell_k\rangle_2 + \beta_i\beta_k|r_i\rangle_1|r_k\rangle_2 \\
&\quad \pm \alpha_k\alpha_i|\ell_k\rangle_1|\ell_i\rangle_2 \pm \alpha_k\beta_i|\ell_k\rangle_1|r_i\rangle_2 \\
&\quad \pm\beta_k\alpha_i|r_k\rangle_1|\ell_i\rangle_2 \pm \beta_k\beta_i|r_k\rangle_1|r_i\rangle_2\Big] \,.
\end{aligned}$$

We see then that

$$P_{LR}|i\rangle|j\rangle \to \frac{\alpha_i\beta_j|\ell_i\rangle_1|r_j\rangle_2 + \beta_i\alpha_j|r_i\rangle_1|\ell_j\rangle_2 \pm \alpha_j\beta_i|\ell_j\rangle_1|r_i\rangle_2 \pm \beta_j\alpha_i|r_j\rangle_1|\ell_i\rangle_2}{\sqrt{2\left(|\alpha_i|^2|\beta_j|^2 + |\alpha_j|^2|\beta_i|^2\right)}}$$

and

$$P_{LR}|i\rangle|k\rangle \to \frac{\alpha_i\beta_k|\ell_i\rangle_1|r_k\rangle_2 + \beta_i\alpha_k|r_i\rangle_1|\ell_k\rangle_2 \pm \alpha_k\beta_i|\ell_k\rangle_1|r_i\rangle_2 \pm \beta_k\alpha_i|r_k\rangle_1|\ell_i\rangle_2}{\sqrt{2\left(|\alpha_i|^2|\beta_k|^2 + |\alpha_k|^2|\beta_i|^2\right)}}.$$

Let

$$N = 2\sqrt{\left(|\alpha_i|^2|\beta_j|^2 + |\alpha_j|^2|\beta_i|^2\right)\left(|\alpha_i|^2|\beta_k|^2 + |\alpha_k|^2|\beta_i|^2\right)}.$$

We see then that

$$\begin{aligned}
\langle k|\langle i|P_{LR}^\dagger P_{LR}|i\rangle|j\rangle \to \frac{1}{N}\, &\big[\big(\alpha_i^*\beta_k^*{}_2\langle r_k|_1\langle \ell_i| + \alpha_k^*\beta_i^*{}_2\langle \ell_k|_1\langle r_i| \pm \alpha_k^*\beta_i^*{}_2\langle r_i|_1\langle \ell_k| \pm \alpha_i^*\beta_k^*{}_2\langle \ell_i|_1\langle r_k|\big) \\
&\Big(\alpha_i\beta_j|\ell_i\rangle_1|r_j\rangle_2 + \alpha_j\beta_i|r_i\rangle_1|\ell_j\rangle_2 \pm \alpha_j\beta_i|\ell_j\rangle_1|r_i\rangle_2 \pm \alpha_i\beta_j|r_j\rangle_1|\ell_i\rangle_2\Big)\Big] \\
= \frac{1}{N}\Big[&|\alpha_i|^2\beta_j\beta_k^*\langle r_k|r_j\rangle \pm \alpha_i^*\alpha_j\beta_i\beta_k^*\langle \ell_i|\ell_j\rangle\langle r_k|r_i\rangle \\
&+ \alpha_j\alpha_k^*|\beta_i|^2\langle \ell_k|\ell_j\rangle \pm \alpha_i\alpha_k^*\beta_i^*\beta_j\langle \ell_k|\ell_i\rangle\langle r_i|r_j\rangle \\
&\pm \alpha_i\alpha_k^*\beta_i^*\beta_j\langle \ell_k|\ell_i\rangle\langle r_i|r_j\rangle \mp \alpha_j\alpha_k^*|\beta_i|^2\langle \ell_k|\ell_j\rangle \\
&\pm \alpha_i^*\alpha_j\beta_i\beta_k^*\langle \ell_i|\ell_j\rangle\langle r_k|r_i\rangle \mp |\alpha_i|^2\beta_j\beta_k^*\langle r_k|r_j\rangle\Big] \\
= \frac{1}{N}\Big[&|\alpha_i|^2\beta_j\beta_k^*\langle r_k|r_j\rangle(1\mp 1) + \alpha_j\alpha_k^*|\beta_i|^2\langle \ell_k|\ell_j\rangle(1\mp 1) \\
&\pm 2\alpha_i^*\alpha_j\beta_i\beta_k^*\langle \ell_i|\ell_j\rangle\langle r_k|r_i\rangle \pm 2\alpha_i\alpha_k^*\beta_i^*\beta_j\langle \ell_k|\ell_i\rangle\langle r_i|r_j\rangle\Big].
\end{aligned}$$

For ease of referencing back to this equation without having to look at the clutter above, we restate the result below:

$$\begin{aligned}
\langle k|\langle i|P_{LR}^\dagger P_{LR}|i\rangle|j\rangle \to \frac{1}{N}\Big[&|\alpha_i|^2\beta_j\beta_k^*\langle r_k|r_j\rangle(1\mp 1) + \alpha_j\alpha_k^*|\beta_i|^2\langle \ell_k|\ell_j\rangle(1\mp 1) \\
&\pm 2\alpha_i^*\alpha_j\beta_i\beta_k^*\langle \ell_i|\ell_j\rangle\langle r_k|r_i\rangle \pm 2\alpha_i\alpha_k^*\beta_i^*\beta_j\langle \ell_k|\ell_i\rangle\langle r_i|r_j\rangle\Big]
\end{aligned} \tag{A.1}$$

Now, we assume that the single-particle detectors are orthogonal to one another. This is not a strong assumption; we have already assumed that the single-particle detector states span the output Hilbert space. If we had an LELM device whose single-particle detector states were not orthogonal, we could orthogonalize it using the Gram-Schmidt process to create an LELM device with orthogonal single-particle detector states that could distinguish at least as many Bell states as the original. Now, this orthogonality condition is given by

$$0 = \langle i|j\rangle = \alpha_i^* \alpha_j \left\langle \ell_i \middle| \ell_j \right\rangle + \beta_i^* \beta_j \left\langle r_i \middle| r_j \right\rangle .$$

This equation and the corresponding equations found when considering $\langle k|i\rangle$ and $\langle k|j\rangle$ give the conditions

$$\begin{aligned} \alpha_i^* \alpha_j \left\langle \ell_i \middle| \ell_j \right\rangle &= -\beta_i^* \beta_j \left\langle r_i \middle| r_j \right\rangle \\ \alpha_i^* \alpha_k \left\langle \ell_k \middle| \ell_i \right\rangle &= -\beta_i^* \beta_k \left\langle r_k \middle| r_i \right\rangle \\ \alpha_j^* \alpha_k \left\langle \ell_k \middle| \ell_j \right\rangle &= -\beta_j^* \beta_k \left\langle r_k \middle| r_j \right\rangle . \end{aligned}$$

Using these, we can rewrite Eq. (A.1) as

$$\frac{1}{N} \left[\left(|\alpha_i|^2 - |\beta_i|^2 \right) \beta_j \beta_k^* \langle r_k | r_j \rangle (1 \mp 1) \pm 4\alpha_i^* \alpha_j \beta_i \beta_k^* \langle \ell_i | \ell_j \rangle \langle r_k | r_i \rangle \right] \tag{A.2}$$

We can see that the first term in Eq. (A.2) vanishes if *any* of the following conditions are satisfied:

- $|\alpha_i|^2 = |\beta_i|^2$, i.e., the detector state $|i\rangle$ has equal left and right weight,
- $\left\langle r_k \middle| r_j \right\rangle = 0$ (or equivalently, $\left\langle \ell_k \middle| \ell_j \right\rangle = 0$), i.e., the right (or left) parts of the detector states $|j\rangle$ and $|k\rangle$ are orthogonal to one another,
- the system in question is bosonic.

Similarly, the second term in Eq. (A.2) vanishes if *any* of the following conditions are satisfied:

- $\left\langle \ell_i \middle| \ell_j \right\rangle = 0$ (or equivalently, $\left\langle r_i \middle| r_j \right\rangle = 0$), i.e., the right (or left) parts of the detector states $|i\rangle$ and $|j\rangle$ are orthogonal to one another,
- $\langle \ell_k | \ell_i \rangle = 0$ (or equivalently, $\langle r_k | r_i \rangle = 0$), i.e., the right (or left) parts of the detector states $|k\rangle$ and $|i\rangle$ are orthogonal to one another.

Thus, if at least one condition from each of these two lists is satisfied, then the detection signatures $P_{LR} |i\rangle |j\rangle$ and $P_{LR} |i\rangle |k\rangle$ will be orthogonal.

A.2.1 When these conditions are not satisfied

We will now make a small observation about a pair of detection signatures that satisfy neither of the conditions on the second of the two lists given above. From Eq. (3.4), we know that $P_{LR}\left|i\right\rangle\left|j\right\rangle$ and $P_{LR}\left|i\right\rangle\left|k\right\rangle$ have the form

$$\begin{aligned}
P_{LR}\left|i\right\rangle\left|j\right\rangle &= \frac{\alpha_i\beta_j}{\sqrt{|\alpha_i|^2|\beta_j|^2+|\alpha_j|^2|\beta_i|^2}}\left|\ell_i\right\rangle\left|r_j\right\rangle + \frac{\alpha_j\beta_i}{\sqrt{|\alpha_i|^2|\beta_j|^2+|\alpha_j|^2|\beta_i|^2}}\left|r_i\right\rangle\left|\ell_j\right\rangle \\
P_{LR}\left|i\right\rangle\left|k\right\rangle &= \frac{\alpha_i\beta_k}{\sqrt{|\alpha_i|^2|\beta_k|^2+|\alpha_k|^2|\beta_i|^2}}\left|\ell_i\right\rangle\left|r_k\right\rangle + \frac{\alpha_k\beta_i}{\sqrt{|\alpha_i|^2|\beta_k|^2+|\alpha_k|^2|\beta_i|^2}}\left|r_i\right\rangle\left|\ell_k\right\rangle.
\end{aligned}$$

Let the states $\left|x_1\right\rangle, \left|x_2\right\rangle, \ldots, \left|x_a\right\rangle$ and $\left|y_1\right\rangle, \left|y_2\right\rangle, \ldots, \left|y_b\right\rangle$ be such that

$$\begin{aligned}
\left|\ell_i\right\rangle &= \frac{1}{\sqrt{a}}(\left|x_1\right\rangle + \left|x_2\right\rangle + \cdots + \left|x_a\right\rangle) \\
\left|r_i\right\rangle &= \frac{1}{\sqrt{b}}(\left|y_1\right\rangle + \left|y_2\right\rangle + \cdots + \left|y_b\right\rangle).
\end{aligned}$$

If we assume that

$$\begin{aligned}
\left\langle \ell_i \middle| \ell_j \right\rangle &\neq 0 \\
\left\langle r_i \middle| r_j \right\rangle &\neq 0 \\
\left\langle \ell_k \middle| \ell_i \right\rangle &\neq 0 \\
\left\langle r_k \middle| r_i \right\rangle &\neq 0,
\end{aligned}$$

then it must be the case that there exist indices $1 \leq m_1, m_2 \leq a$ and $1 \leq m_3, m_4 \leq b$ such that

$$\begin{aligned}
\left\langle x_{m_1} \middle| \ell_j \right\rangle &\neq 0 \\
\left\langle x_{m_2} \middle| \ell_k \right\rangle &\neq 0 \\
\left\langle y_{m_3} \middle| r_j \right\rangle &\neq 0 \\
\left\langle y_{m_4} \middle| r_k \right\rangle &\neq 0.
\end{aligned}$$

Then, we see that the detection signature $P_{LR}\left|i\right\rangle\left|j\right\rangle$ contains the states

$$\left|x_{m_2}\right\rangle\left|y_{m_3}\right\rangle \quad \text{and} \quad \left|y_{m_4}\right\rangle\left|x_{m_1}\right\rangle$$

and the detection signature $P_{LR}\left|i\right\rangle\left|k\right\rangle$ contains the states

$$\left|x_{m_1}\right\rangle\left|y_{m_4}\right\rangle \quad \text{and} \quad \left|y_{m_3}\right\rangle\left|x_{m_2}\right\rangle.$$

Upon implicit (anti)symmetrization, the detection signatures $P_{LR}\left|i\right\rangle\left|j\right\rangle$ and $P_{LR}\left|i\right\rangle\left|k\right\rangle$ will therefore share (at least) the four states

$$\left|x_{m_2}\right\rangle\left|y_{m_3}\right\rangle, \quad \left|y_{m_3}\right\rangle\left|x_{m_2}\right\rangle, \quad \left|y_{m_4}\right\rangle\left|x_{m_1}\right\rangle, \quad \left|x_{m_1}\right\rangle\left|y_{m_4}\right\rangle.$$

Note that the two states may share fewer states if some of the indices m_1, m_2, m_3, m_4 are the same.

A.3 Determining if a detection signature is useful

In attempting to prove (or disprove) the conjecture that disjoint detection signature LELM devices can perform optimally, we might search for a counter-example. If we manage to find a counter-example, then we have disproved the conjecture, and even if we don't, we might learn something about why we could not.

A counter-example would come in the form of a set of detectors for a system with the properties that

1. the detection signatures defined by these detectors have nontrivial overlap, and

2. the resulting LELM apparatus can distinguish a number Bell states *strictly* exceeding the bounds given in Chapter 6.

When looking for such an example, it is useful to have a number of methods that can quickly determine if a detector (or detectors) will *not* be useful in finding a valid counter-example. For a given two-particle system entangled in n qudit variables, let B be the corresponding bound given in Chapter 6. For example, if we consider a system of two particles entangled in one qubit variable and one qutrit variable ($n = 2$, $d_1 = 2$, $d_2 = 3$), then we have $D = 6$, so $B = 2D - 1 = 11$. In this case, a single detection signature is not useful if its Bell basis representation contains *more than* $D^2 - B$ Bell states. The reason for this is simple: only one Bell state from a particular detection signature can be distinguished by a set of detection events that include that detection signature. Thus, if such a detector were included, then only one Bell state from its Bell basis representation could be distinguished, and the number of Bell states remaining to be distinguished by other detection events would be strictly less than $D^2 - \left(D^2 - B\right) = B$. Thus, such a detection signature could not be used to exceed the bound B, and therefore could not be useful in finding a counter-example.

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