UE OXBRIDGE-PREP

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TUMA 2024

Test of Mathematics for University Assessment

WORKBOOK

5th Edition

On-line Practice with Complete Solutions

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We provide application planning consultation, high-quality Oxbridge official summer schools, EPQ/online scientific research projects and personal statement services for outstanding high school students and families applying to Oxbridge and G5 universities.

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ABOUT THE AUTHOR

- Master's Degree, Oxbridge tutor, coach of Maths and Physics Competitions
- 17 years of experience in teaching high school maths and physics
- 8 years of experience in training bilingual teachers
- Independently developed more than 10 courses for oxbridge admissions assessments, maths and physics competitions
- More than 100 students gained offers from ivy league, oxbridge and G5 universities
- Tens of students won top awards in international maths and physics competitions

INTRODUCTION

The test prep course for TMUA has been continuously revised and improved for five years, with the latest 5th edition presented, since the 1st edition was released in 2020. The course content has evolved initially from the pure lecture notes to a comprehensive bundle of learning resources including coursebook, videos, workbook and past questions collections.

Each iteration of the course reflects the latest reform of TMUA and changes to the syllabus. As the difficulty of TMUA grows, more challenging contents are included accordingly. With professional experiences in teaching and researching for nearly 20 years, this systematically designed course allows students of different levels to benefit from it.

So far, this course has helped more than one hundred students achieve outstanding performance in TMUA and gain offers from G5 universities in the UK, including Cambridge, Imperial College and LSE, UCL.

HOW TO USE

First Stage

It is usually recommended to preview each lecture and try as many exemplar questions as possible.

Then watch the on-line lecture videos, which is mainly about key concepts, problem-solving skills. Before watching the videos, you are required to log in by scanning the QR code using WeChat or clicking on the link on the "TMUA Coursebook".

Once finishing studying the video, you should do the practice on the "TMUA Workbook". Detailed solutions will be displayed on-line after you submit your answers after scanning the QR code or clicking on the link. You can correct the answers on your own with the solutions, or your tutor will review along with you and give you feedback.

Second Stage

After the first stage, you can selectively do additional practices (supplements) in the "TMUA Workbook". Detailed solutions can be accessed in the same way as the practice.

Third Stage

In this stage you should do some mock exams from the last 3 years and mainly focus on checking for any weak spots. You may also utilise "TMUA Past Papers" and "TMUA Categorised Questions" to practice the TMUA questions by year or by topic. Detailed solutions of all questions can be accessed on-line.

COURSE STATISTICS

Lecture No.	Number of Exercise Questions	Number of Quiz Questions	Number of Practice Questions	Number of Supplement Questions	Total Number of Questions
01	14	9	15	15	**53**
02	13	12	20	20	**65**
03	16	10	17	15	**58**
04	13	13	16	15	**57**
05	15	14	18	12	**59**
06	17	10	15	12	**54**
07	13	7	17	13	**50**
08	14	10	16	16	**56**
09	5	4	11	12	**32**
10	8	6	15	14	**43**
11	9	9	20	17	**55**
12	14	13	25	24	**76**
13	16	11	25	24	**76**
14	9	8	13	12	**42**
15	15	12	17	14	**58**
16	11	8	18	18	**55**
17	11	11	20	20	**62**
18	14	12	16	12	**54**
19	7	5	8	8	**28**
20	6	4	8	7	**25**
21			20		**20**
22			20		**20**
23			20		**20**
Total	240	188	390	300	**1118**

关于作者

- 硕士，牛剑导师，竞赛金牌教练
- 原北京海淀著名公立国际部首批双语教师
- 17 年国际高中数学、物理课程教学经验
- 8 年双语教师培训经验
- 独立研发 10 余门牛剑笔试、数理竞赛课程
- 超过 100 名学生获得藤校、牛剑 G5 录取
- 数十名学生获得国际数学、物理竞赛顶级奖项

课程简介

TMUA 备考课程自 2020 年推出首版以来，历经五年不断修订和完善，现发展至第 5 版。课程内容由最初的单一讲义演变为一套囊括教材、视频、刷题册和真题集的综合型教学资源。

每一版课程修订都紧扣 TMUA 考试改革和大纲变化。随着 TMUA 试题难度逐渐提升，课程也相应地增加更具挑战性的学习内容。同时，凭借近二十年的教学实践和研发经验，自成一体的系统课程设计让不同层次和水平的学生都能从中受益。

至今，这套课程已经帮助上百位学员在 TMUA 考试中取得优异的成绩，并获得剑桥、帝国理工、伦敦政经和伦敦大学学院等英国 G5 大学的录取。

如何使用

第一轮学习

通常建议先预习每章的内容并尝试做尽可能多的例题。随后在线观看精心录制的讲解视频，视频主要讲解知识点、解题方法和技巧。观看视频需要使用微信扫码或点击《TMUA 教材》(COURSEBOOK)上的链接登录。完成视频学习后，做《TMUA 刷题册》(WORKBOOK)上的课后练习(PRACTICES)，扫码或点击链接提交自己的答案后，可以在线看到详细解析。学生可根据解析自行订正，或由老师批阅并反馈。

第二轮提升

在完成第一轮学习后，可有选择性地完成《TMUA 刷题册》(WORKBOOK)的补充练习(SUPPLEMENTS)。查看详细解析的方式与课后练习一致。

第三轮冲刺

冲刺阶段主要以模拟最近三年真题、知识点查漏补缺为主。若时间相对充裕，可使用配套的《TMUA 历年真题集》和《TMUA 分类真题集》按年份或按专题分类刷真题。所有真题的详细解析可在线查看。

课程统计

章节	例题数量	小测题量	练习题量	补充题量	总题目数
01	14	9	15	15	**53**
02	13	12	20	20	**65**
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關於作者

- 碩士，牛劍導師，競賽金牌教練
- 原北京海淀著名公立國際部首批雙語教師
- 17 年國際高中數學、物理課程教學經驗
- 8 年雙語教師培訓經驗
- 獨立研發 10 餘門牛劍筆試、數理競賽課程
- 超過 100 名學生獲得藤校、牛劍 G5 錄取
- 數十名學生獲得國際數學、物理競賽頂尖獎項

課程簡介

TMUA 備考課程自 2020 年推出首版以來，歷經五年不斷修訂完善，現發展至第 5 版。課程內容由最初的單一講義演變為一套囊括教材、影片、刷題冊和真題集的綜合型教學資源。

每一版課程修訂都緊扣 TMUA 考試改革和大綱變化。隨著 TMUA 試題難度逐漸提升，課程也隨之增加更具挑戰性的學習內容。同時，憑藉著近二十年的教學實踐和研發經驗，自成一體的系統課程設計讓不同層次和程度的學生都能從中受益。

至今，這套課程已經幫助上百位學員在 TMUA 考試中取得優異的成績，並獲得劍橋、帝國理工、倫敦政經和倫敦大學學院等英國 G5 大學的錄取。

如何使用

第一輪學習

通常建議先預習每章的內容並嘗試做盡可能多的例題。隨後在線觀看精心錄製的講解視頻，視頻主要講解知識點、解題方法和技巧。觀看影片需使用微信掃碼或點選《TMUA 教材》(COURSEBOOK)上的連結登入。完成影片學習後，做《TMUA 刷題冊》(WORKBOOK)上的課後練習(PRACTICES)，掃碼或點擊連結提交自己的答案後，可以在線上看到詳細解析。學生可依解析自行訂正，或由老師核准並回饋。

第二輪提升

在完成第一輪學習後，可選擇性地完成《TMUA 刷題冊》(WORKBOOK)的補充練習(SUPPLEMENTS)。查看詳細解析的方式與課後練習一致。

第三輪衝刺

衝刺階段主要以模擬最近三年真題、知識點查漏補缺為主。若時間相對充裕，可使用配對的《TMUA 歷年真題集》及《TMUA 分類真題集》依年份或依專題分類刷真題。所有真題的詳細解析可在線查看。

課程統計

章節	例題數量	小測題量	練習題量	補充題量	總題目數
01	14	9	15	15	**53**
02	13	12	20	20	**65**
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22			20		**20**
23			20		**20**
合計	240	188	390	300	**1118**

TABLE OF CONTENTS

01 Statistics

What's on the Specification?

- Identify possible sources of bias.
- Identify flaws in data collection sheets and questionnaires in an experiment or a survey.
- Group, and understand, discrete and continuous data.
- Extract data from lists and tables.
- Design and use two-way tables.
- Interpret bar charts, pie charts, grouped frequency diagrams, line graphs, and frequency polygons.
- Interpret cumulative frequency tables and graphs, box plots, and histograms (including unequal class width).
- Calculate and interpret mean, median, mode, modal class, range, and inter-quartile range, including the estimated mean of grouped data.
- Calculate average rates when combining samples or events, including solving problems involving average rate calculations (e.g. average survival rates in different wards of different sizes, average speed of a car over a journey where it has travelled at different speeds).
- Interpret scatter diagrams and recognise correlation; using lines of best fit. (The calculation of regression lines is not required.)
- Compare sets of data by using statistical measures or by interpreting graphical representations of their distributions.

Exercises E01

Time Allowed

No limit

Number of Questions

23

Difficulty

★★★☆

Exercises E01

Scan the QR code or click the link above to take the practice online.

01

2

Quiz Pre-1

Four different positive integers are to be chosen so that they have a mean of 2017.

What is the smallest possible range of the chosen integers?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

Quiz Pre-2

Frank's teacher asks him to write down five integers such that the median is one more than the mean, and the mode is one greater than the median. Frank is also told that the median is 10. What is the smallest possible integer that he could include in his list?

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

Quiz Pre-3

The mean, median and mode of the 7 data values 60, 100, x, 40, 50, 200, 90 are all equal to x. What is the value of x?

(A) 50

(B) 60

(C) 75

(D) 90

(E) 100

Ex. 1

The data set $[6, 19, 33, 33, 39, 41, 41, 43, 51, 57]$ has median $Q_2 = 40$, first quartile $Q_1 = 33$, and third quartile $Q_3 = 43$. An outlier in a data set is a value that is more than 1.5 times the interquartile range below the first quartile (Q_1) or more than 1.5 times the interquartile range above the third quartile (Q_3), where the interquartile range is defined as $Q_3 - Q_1$. How many outliers does this data set have?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Ex. 2

A list of five numbers has mean x, median y and range z.

A sixth number is added to the list. This sixth number is greater than x.

Which of the following statements **must** be true?

1 The median of the six numbers cannot be one of the numbers in the list.

2 The mean of the six numbers is greater than x.

3 The range of the six numbers is greater than z.

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Ex. 3

There are two sets of data: the mean of the first set is 15, and the mean of the second set is 20.

One of the pieces of data from the first set is exchanged with one of the pieces of data from the second set.

As a result, the mean of the first set of data increases from 15 to 16, and the mean of the second set of data decreases from 20 to 17.

What is the mean of the set made by combining all the data?

(A) $16\frac{1}{4}$

(B) $16\frac{1}{3}$

(C) $16\frac{1}{2}$

(D) $16\frac{2}{3}$

(E) $16\frac{3}{4}$

Ex. 4

A class of 20 students took a maths test, and their mean mark was 70. The range of these marks was 18.

Five new students joined the class and took the same maths test. When their marks were included, the new mean for the 25 students was 68.

Given only this information, which of the following statements **must** be true?

1 All of the five new students scored 68 marks or less for this test.

2 The mean of the marks for just the five new students was 60.

3 When the marks for the five new students were included, the range of the marks for the class was unchanged.

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Ex. 5

A group of five numbers are such that:

- their mean is 0
- their range is 20

What is the largest possible median of the five numbers?

(A) 0

(B) 4

(C) $4\frac{1}{2}$

(D) $6\frac{1}{2}$

(E) 8

(F) 20

Ex. 6

Melanie computes the mean μ, the median M and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, ..., 12 28s, 11 29s, 11 30s and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$

(B) $M < d < \mu$

(C) $d = M = \mu$

(D) $d < M < \mu$

(E) $d < \mu < M$

Ex. 7

A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

(A) 202

(B) 223

(C) 224

(D) 225

(E) 234

Ex. 8

What is the median of the following list of 4040 numbers?

$$1, 2, 3, \dots, 2020, 1^2, 2^2, 3^2, \dots, 2020^2$$

(A) 1974.5

(B) 1975.5

(C) 1976.5

(D) 1977.5

(E) 1978.5

Ex. 9

The mean, median, unique mode and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

(A) 11

(B) 12

(C) 13

(D) 14

(E) 15

Ex. 10

A list of 11 positive integers has a mean of 10, a median of 9 and a unique mode of 8. What is the largest possible value of an integer in the list?

(A) 24

(B) 30

(C) 31

(D) 33

(E) 35

Ex. 11

When 15 is appended to a list of integers, the mean is increased by 2. When 1 is appended to the enlarged list, the mean of the enlarged list is decreased by 1. How many integers were in the original list?

(A) 4

(B) 5

(C) 6

(D) 7

(E) 8

Ex. 12

Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

(A) 22

(B) 23

(C) 24

(D) 25

(E) 26

Ex. 13

On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points and the rest got 95 points. What is the difference between the mean and the median score on this exam?

(A) 0

(B) 1

(C) 2

(D) 4

(E) 5

Ex. 14

Johann has 64 fair coins. He flips all the coins. Any coin that lands on tails is tossed again. Coins that land on tails on the second toss are tossed a third time. What is the expected number of coins that are now heads?

(A) 32

(B) 40

(C) 48

(D) 56

(E) 64

Quiz 1

The following twelve integers are written in ascending order:

$$1, x, x, x, y, y, y, y, y, 8, 9, 11.$$

The mean of these twelve integers is 7. What is the median?

(A) 6

(B) 7

(C) 7.5

(D) 8

(E) 9

Quiz 2

A list of positive integers has a median of 8, a mode of 9 and a mean of 10.

What is the smallest possible number of integers in the list?

(A) 5

(B) 6

(C) 7

(D) 8

(E) 9

Quiz 3

What is the maximum possible value of the median number of cups of coffee bought per customer on a day when Sundollars Coffee Shop sells 477 cups of coffee to 190 customers, and every customer buys at least one cup of coffee?

(A) 1.5

(B) 2

(C) 2.5

(D) 3

(E) 3.5

Quiz 4

A list of 5 positive integers has mean 5, mode 5, median 5 and range 5.

How many such lists of 5 positive integers are there?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Quiz 5

A set of six **distinct** integers is split into two sets of three.

The first set of three integers has a mean of 10 and a median of 8.

The second set of three integers has a mean of 12 and a median of 9.

What is the smallest possible range of the set of all six integers?

(A) 8

(B) 10

(C) 11

(D) 12

(E) 14

(F) 15

Quiz 6

The table shows statistics relating to the test marks of two groups of students.

	number of students	*mean*	*range*
group X	10	36	16
group Y	20	48	21

The results for the two groups of students are combined.

What can be deduced about the mean and range of the combined results?

(A) mean $= 40$, range ≤ 16

(B) mean $= 40$, $16 <$ range < 21

(C) mean $= 40$, range ≥ 21

(D) mean $= 44$, range ≤ 16

(E) mean $= 44$, $16 <$ range < 21

(F) mean $= 44$, range ≥ 21

Practices P01

Time Allowed

40 min

Number of Questions

15

Difficulty

★★★

Practices P01

Scan the QR code or click the link above to take the practice online.

Q1

The mean of n numbers is p.

The mean of two of these numbers is q.

The mean of the remaining numbers is 10.

Which of the following is a correct expression for n in terms of p and q?

(A) $\frac{2(q-10)}{(p-10)}$

(B) $\frac{2(q-10)}{(10-p)}$

(C) $\frac{2(q-10)}{(p+10)}$

(D) $\frac{2(10-q)}{(p+10)}$

(E) $\frac{2(10+q)}{(p-10)}$

(F) $\frac{2(10+q)}{(10-p)}$

Q2

The expected number of bottles of water sold in a day at a sports ground is directly proportional to the square of the average outside temperature, in degrees Celsius, for that day.

On a day when the average outside temperature is 16 °C, 64 bottles of water, the expected number, are sold.

On a warmer day, when the average outside temperature is T °C, 256 bottles of water are sold, which is 31 bottles more than the expected number for that day.

What is the value of T?

(A) 7.5

(B) $\sqrt{450}$

(C) 30

(D) 32

(E) $\sqrt{1148}$

(F) 56.25

Q3

The mean age of the twenty members of a running club is exactly 28.

The mean age increases by exactly 2 years when two new members join.

What is the mean age of the two new members?

(A) 20 years

(B) 22 years

(C) 30 years

(D) 40 years

(E) 50 years

(F) 52 years

Q4

60% of a sports club's members are women and the remainder are men.

This sports club offers the opportunity to play tennis or cricket. Every member plays exactly one of the two sports.

$\frac{2}{5}$ of the male members of the club play cricket;

$\frac{2}{3}$ of the cricketing members of the club are women.

What is the probability that a member of the club, chosen at random, is a woman who plays tennis?

(A) $\frac{1}{5}$

(B) $\frac{7}{25}$

(C) $\frac{1}{3}$

(D) $\frac{11}{25}$

(E) $\frac{3}{5}$

Q5

Pascal, Newton, Galileo and Fermat all took the same test. The average score of all four candidates was 16; Pascal and Newton had an average of 16, Pascal and Fermat had an average of 13, while Newton and Fermat had an average of 18. What was Galileo's score?

(A) 14

(B) 15

(C) 16

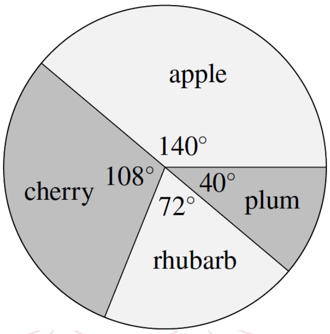
(D) 17

(E) 18

Q6

In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?



(A) 45

(B) 60

(C) 80

(D) 90

(E) 180

Q7

The table shows statistics relating to the test marks of two groups of students.

	number of students	*mean*	*range*
group X	10	36	16
group Y	20	48	21

The results for the two groups of students are combined.

What can be deduced about the mean and range of the combined results?

(A) mean $= 40$, range ≤ 16

(B) mean $= 40$, $16 <$ range < 21

(C) mean $= 40$, range ≥ 21

(D) mean $= 44$, range ≤ 16

(E) mean $= 44$, $16 <$ range < 21

(F) mean $= 44$, range ≥ 21

Q8

A list of n numbers has mean m and a unique mode d .

Two numbers are removed from the list.

The remaining list of numbers also has a unique mode, but this unique mode is not equal to d .

The mean of the remaining $n-2$ numbers is $m+2$.

What was the unique mode, d , of the original list?

(A) $n-m+2$

(B) $n-m-2$

(C) $n+m-2$

(D) $m+n+2$

(E) $m-n+2$

(F) $m-n-2$

Q9

Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon classes mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

(A) 74

(B) 75

(C) 76

(D) 77

(E) 78

Q10

The Dunbar family consists of a mother, a father and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

Q11

A teacher has a list of marks: 17, 13, 5, 10, 14, 9, 12, 16. Which two marks can be removed without changing the mean?

(A) 12 and 17

(B) 5 and 17

(C) 9 and 16

(D) 10 and 12

(E) 10 and 14

Q12

Jane was playing basketball. After a series of 20 shots, Jane had a success rate of 55%. Five shots later, her success rate had increased to 56%. On how many of the last five shots did Jane score?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Q13

Viola has been practising the long jump. At one point, the average distance she had jumped was 3.80 m. Her next jump was 3.99 m and that increased her average to 3.81 m. After the following jump, her average had become 3.82 m. How long was her final jump?

(A) 3.97 m

(B) 4.00 m

(C) 4.01 m

(D) 4.03 m

(E) 4.04 m

Q14

What is the sum of all real numbers x for which the median of the numbers 4, 6, 8, 17 and x is equal to the mean of those five numbers?

(A) -5

(B) 0

(C) 5

(D) $\frac{15}{4}$

(E) $\frac{35}{4}$

Q15

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5

(B) 134

(C) 142

(D) 150.5

(E) 167

Supplements S01

Time Allowed

90 min

Number of Questions

15

Difficulty

★★★★

Supplements S01

Scan the QR code or click the link above to take the practice online.

SQ1

Five positive consecutive integers starting with a have average b. What is the average of 5 consecutive integers that start with b?

(A) $a+3$

(B) $a+4$

(C) $a+5$

(D) $a+6$

(E) $a+7$

SQ2

An *iterative average* of the numbers 1, 2, 3, 4 and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

(A) $\frac{31}{16}$

(B) 2

(C) $\frac{17}{8}$

(D) 3

(E) $\frac{65}{16}$

SQ3

Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82 and 91. What was the last score Mrs. Walter entered?

(A) 71

(B) 76

(C) 80

(D) 82

(E) 91

SQ4

The average value of all the pennies, nickels, dimes and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

SQ5

The average of the numbers 1, 2, 3, ..., 98, 99 and x is $100x$. What is x?

(A) $\frac{49}{101}$

(B) $\frac{50}{101}$

(C) $\frac{1}{2}$

(D) $\frac{51}{101}$

(E) $\frac{50}{99}$

SQ6

The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

(A) 5

(B) 20

(C) 25

(D) 30

(E) 36

SQ7

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?

(A) 85

(B) 88

(C) 93

(D) 94

(E) 98

SQ8

Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer is S is *also* removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

(A) 36.2

(B) 36.4

(C) 36.6

(D) 36.8

(E) 37

SQ9

Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4 and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

(A) 10

(B) 13

(C) 15

(D) 17

(E) 20

SQ10

When the mean, median and mode of the list $10, 2, 5, 2, 4, 2, x$ are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

(A) 3

(B) 6

(C) 9

(D) 17

(E) 20

SQ11

Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

(A) 92

(B) 94

(C) 96

(D) 98

(E) 100

SQ12

For positive integers m and n such that $m + 10 < n + 1$, both the mean and the median of the set $\{m, m + 4, m + 10, n + 1, n + 2, 2n\}$ are equal to n. What is $m + n$?

(A) 20

(B) 21

(C) 22

(D) 23

(E) 24

SQ13

The sum of 49 consecutive integers is 7^5. What is their median?

(A) 7

(B) 7^2

(C) 7^3

(D) 7^4

(E) 7^5

SQ14

The set $\{3, 6, 9, 10\}$ is augmented by a fifth element n, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of n?

(A) 7

(B) 9

(C) 19

(D) 24

(E) 26

SQ15

A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

(A) 55

(B) 56

(C) 57

(D) 58

(E) 59

01

24

02 Counting and Probabilities

What's on the Specification?

- Understand and use the vocabulary of probability and the probability scale.
- Understand and use estimates or measures of probability, including relative frequency and theoretical models.
- List all the outcomes for single and combined events.
- Identify different mutually exclusive outcomes and know that the sum of the probabilities of all these outcomes is 1.
- Construct and use Venn diagrams to solve union and intersection categorization problems and determine probabilities when required. Familiarity with the meaning and use of the terms 'union', 'intersection', and 'complement' is required. The mathematical notation for these ($A \cup B$, $A \cap B$, and A' or A^c) will not be required.
- Know when to add or multiply two probabilities.
- Understand the use of tree diagrams to represent outcomes of combined events:
  - when the probabilities are independent of the previous outcome;
  - when the probabilities are dependent on the previous outcome.
- Compare experimental and theoretical probabilities.
- Understand that if an experiment is repeated, the outcome may be different.

Exercises E02

Time Allowed

No limit

Number of Questions

25

Difficulty

★★★☆

Exercises E02

Scan the QR code or click the link above to take the practice online.

02

26

Quiz Pre-1

Three dice, each showing numbers 1 to 6, are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

(A) 36

(B) 30

(C) 27

(D) 24

(E) 21

Quiz Pre-2

There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

(A) 10

(B) 12

(C) 15

(D) 18

(E) 20

Quiz Pre-3

A bag contains m blue and n yellow marbles. One marble is selected at random from the bag and its colour is noted. It is then returned to the bag along with k other marbles of the same colour. A second marble is now selected at random from the bag. What is the probability that the second marble is blue?

(A) $\frac{m}{m+n}$

(B) $\frac{n}{m+n}$

(C) $\frac{m}{m+n+k}$

(D) $\frac{m+k}{m+n+k}$

(E) $\frac{m+n}{m+n+k}$

Ex. 1

There are 6 gentlemen, A, B, C, D, E, F, and 4 ladies, X, Y, Z, W.

(i) Find the number of different ways when they stand in a line if

(a) there are no restrictions,

(b) all men stand next to each other,

(c) no lady stands next to another,

(d) A must stand next to B,

(e) A must stand next to B, and X must stand next to Y,

(f) A must stand next to B, and X must not stand next to Y,

(g) A must stand next to B, and X must not stand next to B.

(ii) In this part, the 6 gentlemen and 4 ladies are *divided into 2 groups of five*. Find the number of different ways when

(a) there are no other restrictions,

(b) all ladies cannot be in the same group,

(c) numbers of men and ladies are the same in each group,

(d) numbers of men and ladies are the same in each group, A and X must be in the same group, B and Y also must be in the same group,

(e) numbers of men and ladies are the same in each group, A and X must be in the same group, whereas B and Y must not be in the same group.

(iii) If they are divided into 3 groups, find the number of different ways when

(a) no group has a number of persons less than 2,

(b) there is at least one lady and two gentlemen in each group.

(iv) In this part, these 10 people sit down for dinner where they may order one of three types of meals, or order nothing.

(a) How many ways of their orders are possible?

(b) If A orders nothing, three people order the pork meal, three order the chicken meal, and three order the beef meal. A passes the nine meals to the other 9 people in random order. Find the number of ways in which A could pass the meal types to them such that exactly one person receives the type of meal ordered by that person.

Ex. 2

You go into a supermarket to buy two packets of biscuits, which may or may not be of the same variety. The supermarket has 20 different varieties of biscuits and at least two packets of each variety. In how many ways can you choose your two packets?

(A) 400

(B) 210

(C) 200

(D) 190

Ex. 3

Given an unlimited supply of 50p, £1 and £2 coins, in how many different ways is it possible to make a sum of £100?

(A) 1326

(B) 2500

(C) 2601

(D) 5050

(E) 10 000

Ex. 4

In a college there are 100 students taking A level French, German or Spanish. Of these students, 64 are female and the rest are male. There are 50 French students of whom 40 are female and 30 German students of whom 10 are female.

Find the probability that a randomly chosen student

(i) is taking Spanish,

(ii) is male, given that the student is taking Spanish.

College records indicate that 70% of the French students, 80% of the German students and 60% of the Spanish students have applied for University. A student is chosen at random.

(iii) Find the probability that this student has applied for University.

(iv) Given that the student had applied to University, find the probability that the student is studying French.

Ex. 5

Most students in a large college study Mathematics. A teacher chooses three different students at random, one after the other.

Consider these three probabilities:

$R = \mathrm{P}(\text{At least one of the students chosen studies Mathematics})$

$S = \mathrm{P}(\text{The second student chosen studies Mathematics})$

$T = \mathrm{P}(\text{All three of the students chosen study Mathematics})$

Which of the following is true?

(A) $R < S < T$

(B) $R < T < S$

(C) $S < R < T$

(D) $S < T < R$

(E) $T < R < S$

(F) $T < S < R$

Ex. 6

There are only red balls and green balls in a bag.

When I pick a ball from the bag, the probability of picking a red ball is p and the probability of picking a green ball is q, where $q \geq p$.

I pick a ball from the bag and record its colour. I then replace the ball in the bag.

I repeat this procedure once.

Given that

$$\mathrm{P}(\text{the balls are of the same colour}) - \mathrm{P}(\text{the balls are of different colours}) = \frac{1}{4}$$

find the value of

$$\frac{q}{p} - \frac{p}{q}$$

(A) 0

(B) $\frac{3}{2}$

(C) $\frac{5}{6}$

(D) $\frac{8}{3}$

(E) $\frac{247}{48}$

Ex. 7

Tow player take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

(A) $\frac{5}{9}$

(B) $\frac{3}{5}$

(C) $\frac{6}{11}$

(D) $\frac{7}{12}$

Ex. 8

Billie has a die with the numbers 1, 2, 3, 4, 5 and 6 on its six faces.

Niles has a die which has the numbers 4, 4, 4, 5, 5 and 5 on its six faces.

When Billie and Niles roll their dice the one with the larger number wins. If the two numbers are equal it is a draw.

The probability that Niles wins, when written as a fraction in its lowest terms, is $\frac{p}{q}$. What is the value of $7p + 11q$?

Ex. 9

Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition?

(A) $\frac{4}{15}$

(B) $\frac{8}{15}$

(C) $\frac{2}{3}$

(D) $\frac{4}{5}$

(E) $\frac{13}{15}$

Ex. 10

Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

(A) $\frac{1}{4}$

(B) $\frac{5}{16}$

(C) $\frac{3}{8}$

(D) $\frac{7}{16}$

(E) $\frac{1}{2}$

Ex. 11

Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a, b) lies above the parabola $y = ax^2 - bx$?

(A) $\frac{11}{81}$

(B) $\frac{13}{81}$

(C) $\frac{5}{27}$

(D) $\frac{17}{81}$

(E) $\frac{19}{81}$

Ex. 12

A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

(A) $\frac{1}{120}$

(B) $\frac{1}{32}$

(C) $\frac{1}{20}$

(D) $\frac{3}{20}$

(E) $\frac{1}{6}$

Quiz 1

A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100

(B) 2220

(C) 3000

(D) 3120

(E) 3125

Quiz 2

A hockey team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. There are 4 substitutes: 1 goalkeeper, 1 defender, 1 midfielder and 1 forward. A substitute may only replace a player of the same category eg: midfielder for midfielder. Given that a maximum of 3 substitutes may be used and that there are still 11 players on the pitch at the end, how many different teams could finish the game?

(A) 110

(B) 118

(C) 121

(D) 125

(E) 132

Quiz 3

As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours − Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour. In how many ways can he do this?

(A) 32

(B) 48

(C) 72

(D) 108

(E) 162

Quiz 4

In a population, $\frac{3}{5}$ of the adults are overweight. The probability of an overweight adult having Type 2 diabetes is $\frac{9}{50}$; this probability is 6 times the probability of an adult who is not overweight having the disease. An adult is chosen at random from the population.

What is the probability the chosen adult has Type 2 diabetes?

(A) $\frac{27}{250}$

(B) $\frac{3}{25}$

(C) $\frac{63}{500}$

(D) $\frac{37}{250}$

(E) $\frac{39}{50}$

(F) $\frac{21}{100}$

Quiz 5

A bag contains 6 red and 6 green sweets. The sweets are identical apart from their colour. A child takes a sweet at random from the bag. If the sweet is red, the child stops taking sweets. If the sweet is green, it is not replaced and the child takes another sweet. This continues until a red sweet is taken at which point the child stops taking sweets.

What is the probability that the child takes **more** green sweets than red sweets?

(A) $\frac{3}{22}$

(B) $\frac{5}{22}$

(C) $\frac{3}{11}$

(D) $\frac{1}{2}$

(E) $\frac{8}{11}$

(F) $\frac{17}{22}$

Quiz 6

Ivana has two identical dice and on the faces of each are the numbers $-3, -2, -1, 0, 1, 2$. If she throws her dice and multiplies the results, what is the probability that their product is negative?

(A) $\frac{1}{4}$

(B) $\frac{11}{36}$

(C) $\frac{1}{3}$

(D) $\frac{13}{36}$

(E) $\frac{1}{2}$

Quiz 7

A train arriving at Edinburgh has 12 passengers.

The passengers got on the train at three different stations:

5 at Peterborough

4 at Newark

3 at York

The passengers leave the train one at a time in a random order.

What is the probability that the first three to leave did **not** all get on the train at the same station?

(A) $\frac{3}{11}$

(B) $\frac{41}{44}$

(C) $\frac{103}{110}$

(D) $\frac{19}{20}$

(E) $\frac{21}{22}$

(F) $\frac{43}{44}$

(G) $\frac{54}{55}$

(H) $\frac{219}{220}$

Quiz 8

Jerry starts at 0 on the real number line. He tosses a fair coin 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process is $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$? (For example, he succeeds if his sequence of tosses is HTHHHHHH.)

(A) 69

(B) 151

(C) 257

(D) 293

(E) 313

Quiz 9

Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

(A) $\frac{1}{2}$

(B) $\frac{13}{24}$

(C) $\frac{7}{12}$

(D) $\frac{5}{8}$

(E) $\frac{2}{3}$

Ex. 13

Find the sum of those numbers between 1000 and 6000 every one of whose digits is one of the numbers 0, 2, 5 or 7, giving your answer as a product of primes.

Practices P02

Time Allowed

60 min

Number of Questions

20

Difficulty

★★★

Practices P02

Scan the QR code or click the link above to take the practice online.

Q1

An entrance candidate is dealt three cards from a pack of fifty-two playing cards. To one significant figure the probability that he receives exactly one king is:

[There are four kings in a pack of playing cards.]

(A) 0.003

(B) 0.01

(C) 0.2

(D) 0.05

Q2

A pack of cards consists of 52 different cards. A malicious dealer changes one of the cards for a second copy of another card in the pack and he then deals the cards to four players, giving thirteen to each. The probability that one player has two identical cards is

(A) $\frac{3}{13}$

(B) $\frac{12}{51}$

(C) $\frac{1}{4}$

(D) $\frac{13}{51}$

Q3

A child is presented with the following lettered titles: M A M M A L. The number of different "words" he can make using all six tiles is

(A) 6

(B) 30

(C) 60

(D) 120

Q4

Aris, Boris, Clarice and Doris have to decide who will do the washing up. They decide to throw a fair 6-sided die: if it lands showing a 5 or 6, Aris will wash up; otherwise they throw again. The second time, if the result is a 5 or 6, Boris will wash up; otherwise they throw one last time. The final time, if the result is a 5 or 6, Clarice washes up, and otherwise it's Doris. (Of course, this is not a fair procedure!) Of the four, who is *second* most likely to do the washing up?

(A) Aris

(B) Boris

(C) Clarice

(D) Doris

Q5

Tow player take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

(A) $\frac{5}{9}$

(B) $\frac{3}{5}$

(C) $\frac{6}{11}$

(D) $\frac{7}{12}$

Q6

You go into a supermarket to buy two packets of biscuits, which may or may not be of the same variety. The supermarket has 20 different varieties of biscuits and at least two packets of each variety. In how many ways can you choose your two packets?

(A) 400

(B) 210

(C) 200

(D) 190

Q7

Two different faces of a cube are chosen at random. What is the chance of them being opposite one another?

(A) $\frac{1}{3}$

(B) $\frac{1}{4}$

(C) $\frac{1}{5}$

(D) $\frac{1}{6}$

Q8

60% of a sports club's members are women and the remainder are men.

This sports club offers the opportunity to play tennis or cricket. Every member plays exactly one of the two sports.

$\frac{2}{5}$ of the male members of the club play cricket;

$\frac{2}{3}$ of the cricketing members of the club are women.

What is the probability that a member of the club, chosen at random, is a woman who plays tennis?

(A) $\frac{1}{5}$

(B) $\frac{7}{25}$

(C) $\frac{1}{3}$

(D) $\frac{11}{25}$

(E) $\frac{3}{5}$

Q9

A bag contains n red balls, n yellow balls, and n blue balls.

One ball is selected at random and not replaced.

A second ball is then selected at random and not replaced.

Each ball is equally likely to be chosen.

The probability that the two balls are **not** the same colour is

(A) $\frac{n-1}{3n-1}$

(B) $\frac{2n-2}{3n-1}$

(C) $\frac{2n}{3n-1}$

(D) $\frac{(n-1)^3}{27(3n-1)^3}$

(E) $\frac{3(n-1)}{3n-1}$

(F) $\frac{n^3}{27(3n-1)^3}$

Q10

Five runners competed in a race: Fred, George, Hermione, Lavender, and Ron.

Fred beat George.

Hermione beat Lavender.

Lavender beat George.

Ron beat George.

Assuming there were no ties, how many possible finishing orders could there have been, given only this information?

(A) 1

(B) 6

(C) 12

(D) 18

(E) 24

(F) 120

Q11

A group of five numbers are such that:

- their mean is 0
- their range is 20

What is the largest possible median of the five numbers?

(A) 0

(B) 4

(C) $4\frac{1}{2}$

(D) $6\frac{1}{2}$

(E) 8

(F) 20

Q12

With school lunch, students can select tomato sauce, or mayonnaise, or both, or neither.

n students selected both.

$3n + 1$ students selected tomato sauce.

$3n - 1$ students selected **only** mayonnaise.

There were $7n + 5$ students in the group.

The probability of a student, chosen at random, selecting **only** mayonnaise is $\frac{1}{3}$.

By finding n, what is the probability of a student, chosen at random, selecting **only** tomato sauce?

(A) $\frac{3}{11}$

(B) $\frac{7}{26}$

(C) $\frac{13}{33}$

(D) $\frac{3}{8}$

(E) $\frac{7}{13}$

Q13

Box A contains exactly 10 balls, of which 6 are red and 4 are blue.

Box B contains exactly 15 balls, of which 3 are red and 12 are blue.

All the balls are identical in every respect, apart from colour.

One of the two boxes is chosen at random by tossing two fair coins, as follows:

> "If **both** coins show heads, box A is selected. Otherwise box B is selected."

One ball is then randomly taken from the selected box.

What is the probability that a red ball is taken?

(A) $\frac{9}{400}$

(B) $\frac{3}{25}$

(C) $\frac{3}{10}$

(D) $\frac{2}{5}$

(E) $\frac{1}{2}$

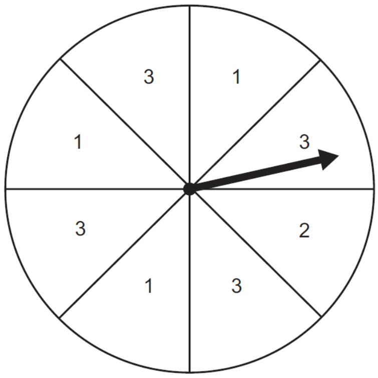
(F) $\frac{4}{5}$

(G) $\frac{323}{400}$

Q14

A fair spinner has eight equal sections.

Each section has one number written on it, as shown.



The spinner is spun twice, and the two numbers scored are added.

What is the probability that the sum of the two numbers is 5?

(A) $\frac{1}{8}$

(B) $\frac{5}{8}$

(C) $\frac{1}{16}$

(D) $\frac{3}{16}$

(E) $\frac{25}{64}$

(F) $\frac{55}{64}$

Q15

Two identical fair six-sided dice each have their faces numbered from 1 to 6, with one number on each face.

Both dice are thrown, and the number on each of the dice is recorded.

They are then both thrown again, and the number on each of the dice is recorded.

What is the probability that at least one of the four recorded numbers is even?

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{9}{16}$

(D) $\frac{3}{4}$

(E) $\frac{15}{16}$

Q16

Two fair six-sided dice are identical except for their colour.

Each of the dice has its faces numbered from 1 to 6, with one number on each face.

One of the dice is red and the other is blue.

The two dice are rolled.

The number shown on the red dice is divided by the number shown on the blue dice to give the score.

What is the probability of a score of 0.5?

(A) 0

(B) $\frac{1}{36}$

(C) $\frac{1}{18}$

(D) $\frac{1}{12}$

(E) $\frac{1}{6}$

Q17

75 pupils in a year group study German or French, or both, or neither.

10 pupils study both languages.

The ratio of those who study both to those that study neither is 5 : 3 respectively.

42 pupils study German.

2 pupils are chosen and each pupil is equally likely to be chosen.

What is the probability that one pupil studies French, and the other pupil studies **only** German?

(A) $\frac{16}{75}$

(B) $\frac{128}{555}$

(C) $\frac{7}{25}$

(D) $\frac{32}{75}$

(E) $\frac{256}{555}$

(F) $\frac{14}{25}$

Q18

During summer activities week 120 students each chose one activity from swimming, archery, and tennis.

46 of the students were girls.

36 of the students chose tennis, and $\frac{2}{3}$ of these were boys; 25 girls chose swimming, and 27 students chose archery.

A boy is picked at random. What is the probability that he chose swimming?

(A) $\frac{3}{20}$

(B) $\frac{9}{37}$

(C) $\frac{4}{15}$

(D) $\frac{16}{37}$

(E) $\frac{32}{57}$

Q19

A pet shop has 4 female rabbits and x male rabbits for sale.

A customer buys 2 of the rabbits, chosen at random, and each rabbit is equally likely to be chosen.

The probability that both the chosen rabbits are male is $\frac{1}{3}$.

What is the value of x?

(A) 2

(B) 4

(C) 6

(D) 8

(E) 9

(F) 11

(G) 12

Q20

There are two red balls and two blue balls in a bag.

Two balls are removed at random without replacement.

Given that at least one of them is red, what is the probability that one of them is blue?

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{4}{5}$

(D) $\frac{5}{6}$

(E) 1

Supplements S02

Time Allowed

90 min

Number of Questions

20

Difficulty

★★★☆

Supplements S02

Scan the QR code or click the link above to take the practice online.

SQ1

Anne, Bert, Clare, Derek and Emily are planning to play a game for which they need to divide themselves into three teams. Each team must have at least one member. The number of different ways they can do this is

(A) 10

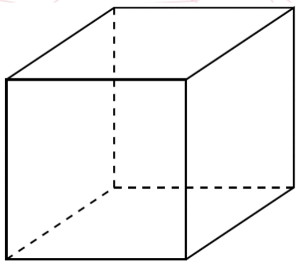
(B) 15

(C) 25

(D) 30

SQ2

The faces of a cube are coloured red or blue. Exactly three are red and three are blue. The number of distinguishable cubes that can be produced (allowing the cube to be turned around) is?



(A) 2

(B) 4

(C) 6

(D) 20

SQ3

A grid of size 3 cm × 5 cm is drawn, ruled at 1 cm intervals. The number of squares that can be drawn using the grid is

(A) 15

(B) 18

(C) 26

(D) 37

SQ4

A cube painted black is cut into 125 identical cubes. How many of them are not painted at all?

(A) 21

(B) 25

(C) 27

(D) 30

SQ5

90 people enter a maze. At each junction a third will go left and two thirds will go right. After three such junctions, what is the most likely combination of turns people will have taken?

(A) Gone right three times

(B) Gone left three times

(C) Gone right twice and once left

(D) Gone twice left and once right

(E) It is impossible to tell

SQ6

A bag contains b blue balls and r red balls. If two balls are picked at random and removed from the bag, what is the probability P that they are different colours?

(A) $\frac{2br}{(b+r)(b+r-1)}$

(B) $\frac{br}{(b+r)(b+r-1)}$

(C) $\frac{br}{(b+r)^2}$

(D) $\frac{2br}{(b+r)^2}$

(E) $2br$

SQ7

We wish to represent integer numbers by using our ten fingers. A finger is assumed to be either stretched out or curled up. How many different integers can we represent with our fingers?

(A) 10

(B) 512

(C) 1000

(D) 20

(E) 1024

SQ8

Ten students need to complete their compulsory practicals for their high school examinations as detailed in the table below:

No. of students	No. of different practicals to complete
2	1
4	2
4	3

The school only has one laboratory in which several different experiments can be set up simultaneously. A maximum of six students are allowed in the school's laboratory for a lesson. Each practical takes one lesson. What is the minimum number of lessons required to complete all the practicals?

(A) 3

(B) 4

(C) 5

(D) 6

(E) 10

SQ9

To get to work, Sylvie first catches a bus and then catches a train.

The probability that the bus is on time is 0.6.

The probability that the bus is late is 0.4.

If the bus is on time, then the probability that she will catch the train is 0.8.

If the bus is late, then the probability that she will catch the train is 0.6.

Given that Sylvie catches the train, what is the probability that the bus was on time?

(A) $\frac{1}{3}$

(B) $\frac{12}{25}$

(C) $\frac{2}{5}$

(D) $\frac{3}{5}$

(E) $\frac{2}{3}$

(F) $\frac{18}{25}$

(G) $\frac{6}{7}$

SQ10

I have two six-sided dice, each with faces numbered from 1 to 6. One of the dice is fair, but the other is not; it will land on numbers 1 to 5 with equal probability, but lands on 6 with a different probability.

When I roll the dice the probability that I get a total of 12 is $\frac{1}{18}$.

What is the probability that I get a total of 2 when I roll the dice?

(A) $\frac{1}{72}$

(B) $\frac{1}{45}$

(C) $\frac{1}{36}$

(D) $\frac{1}{18}$

(E) $\frac{1}{9}$

SQ11

The ratio of the number of boys to the number of girls in a class is 1 : 3

The number of boys in the class is n.

Two students are chosen at random from the class.

The probability that both the students are boys is p.

Which one of the following is a correct expression for n, the number of boys in the class?

(A) $n = \frac{3p-1}{9p-1}$

(B) $n = \frac{3p+1}{9p-1}$

(C) $n = \frac{1}{1-9p}$

(D) $n = \frac{1}{9p-1}$

(E) $n = \frac{4p-1}{16p-1}$

(F) $n = \frac{4p+1}{16p-1}$

(G) $n = \frac{1}{1-16p}$

(H) $n = \frac{1}{16p-1}$

SQ12

A bag contains only n red balls and $2n$ green balls.

One ball is picked and its colour recorded. It is then put back in the bag, and an additional ball of the same colour is added to the bag.

A second ball is then picked.

What is the probability that the two balls picked are **not** the same colour?

(A) $\frac{2n}{3(3n-1)}$

(B) $\frac{4n}{3(3n-1)}$

(C) $\frac{5n}{3(3n-1)}$

(D) $\frac{5n-3}{3(3n-1)}$

(E) $\frac{2n}{3(3n+1)}$

(F) $\frac{4n}{3(3n+1)}$

(G) $\frac{5n}{3(3n+1)}$

(H) $\frac{5n+3}{3(3n+1)}$

SQ13

A bag only contains $2n$ blue balls and n red balls. All the balls are identical apart from colour.

One ball is randomly selected and not replaced. A second ball is then randomly selected.

What is the probability that at least one of the selected balls is red?

(A) $\frac{n-1}{3(3n-1)}$

(B) $\frac{3n-1}{3(3n-1)}$

(C) $\frac{4n-2}{3(3n-1)}$

(D) $\frac{4n}{3(3n-1)}$

(E) $\frac{5n-1}{3(3n-1)}$

(F) $\frac{5n-5}{3(3n-1)}$

SQ14

Two identical fair six-sided dice each have their faces numbered from 1 to 6, with one number on each face.

Both dice are thrown, and the number on each of the dice is recorded.

They are then both thrown again, and the number on each of the dice is recorded.

What is the probability that at least one of the four recorded numbers is even?

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{9}{16}$

(D) $\frac{3}{4}$

(E) $\frac{15}{16}$

SQ15

Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

(A) $\frac{47}{256}$

(B) $\frac{3}{16}$

(C) $\frac{49}{256}$

(D) $\frac{25}{128}$

(E) $\frac{51}{256}$

SQ16

A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that can be selected. What is the remainder when N is divided by 100?

(A) 47

(B) 48

(C) 83

(D) 95

(E) 96

SQ17

How many 15-letter arrangements of 5 A's, 5 B's and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters and no C's in the last 5 letters?

(A) $\sum_{k=0}^{5}\binom{5}{k}^3$

(B) $3^5 \cdot 2^5$

(C) 2^{15}

(D) $\frac{15!}{(5!)^3}$

(E) 3^{15}

SQ18

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

(A) 96

(B) 97

(C) 98

(D) 102

(E) 120

SQ19

A pair of standard 6-sided fair dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

(A) $\frac{1}{36}$

(B) $\frac{1}{12}$

(C) $\frac{1}{6}$

(D) $\frac{1}{4}$

(E) $\frac{5}{18}$

SQ20

Chlo chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chlo's number?

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{3}{4}$

(D) $\frac{5}{6}$

(E) $\frac{7}{8}$

02

56

03 Basis of Logic

What's on the Specification?

- Understand and be able to use mathematical logic in simple situations:
- The terms **true** and **false**;
- The terms **and**, **or** (meaning inclusive or), **not**;
- Statements of the form:
    - if A then B
    - A if B
    - A only if B
    - A if and only if B
- The **converse** of a statement;
- The **contrapositive** of a statement;
- The relationship between the truth of a statement and its converse and its contrapositive.
- **Note:** candidates will not be expected to recognise or use symbolic notation for any of these terms, nor will they be expected to complete formal truth tables.
- Understand and use the terms **necessary** and **sufficient**.
- Understand and use the terms **for all**, **for some** (meaning **for at least one**), and **there exists**.
- Be able to negate statements that use any of the above terms.

Exercises E03

Time Allowed

No limit

Number of Questions

26

Difficulty

★★★

Exercises E03

Scan the QR code or click the link above to take the practice online.

03

58

Quiz Pre-1

What is the greatest number of the following five statements about numbers a, b which can be true at the same time?

$$\frac{1}{a} < \frac{1}{b} \qquad a^2 > b^2 \qquad a < b \qquad a < 0 \qquad b < 0$$

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Quiz Pre-2

The Queen of Hearts had some tarts, but they were eaten. Precisely one of the following statements about the tarts and the Knaves of Clubs, Diamonds and Spades is true. Which one?

(A) None of the three Knaves ate any tarts.

(B) The Knave of Clubs ate some tarts.

(C) Only one of the three Knaves ate any tarts.

(D) At least one of the Knave of Diamonds and the Knave of Spades ate no tarts.

(E) More than one of the three Knaves ate some tarts.

Quiz Pre-3

A magical island is inhabited entirely by knights (who always tell the truth) and knaves (who always tell lies). One day 2014 of the islanders were standing in a long queue. Each person in the queue said, "There are more knaves behind me than knights in front of me".

How many knights were in the queue?

(A) 1

(B) 504

(C) 1007

(D) 1008

(E) 2014

Ex. 1

Draw up the truth table for **(A or B) and (not C)**.

Ex. 2

Draw up the truth tables for

(i) not (A and B).

(ii) (not A) or (not B).

Ex. 3

Draw up the truth tables for

(i) not (A or B).

(ii) (not A) and (not B).

Then show the relationships by means of Venn diagrams.

Ex. 4

Draw up a truth table of **not A or B**? Comparing with the truth table above, what conclusion can you draw from this truth table?

Ex. 5

There are four double-sided cards, and each has a letter on one side and a number on the other. The faces that are up are showing "6", "E", "Q" and "7". They are supposed to obey the following rule: "If there is a vowel on one side, then the card has an even number on the other side."

Question: To see that the rule has been kept, which card(s) must be turned over and checked?

Ex. 6

I make the following statements:

Statement P: if a pig has horns, then it can breathe fire.

Statement Q: if a pig can breathe fire, then it has wings.

Statement R: if a pig has wings, then it has horns.

Each statement is either true or false, but I don't know which.

I then see a pig with wings breathing fire. It has no horns.

Which statements, if any, can I now conclude are definitely true or definitely false?

(A) none of them

(B) P only

(C) Q only

(D) R only

(E) P and Q only

(F) P and R only

(G) Q and R only

(H) P, Q, and R

Ex. 7

Judge whether the following statements are true or false.

(i) ab is even if both a and b are even.

(ii) ab is even only if both a and b are even.

(iii) $a = b$ if $a^2 = b^2$.

(iv) $a = b$ only if $a^2 = b^2$.

(v) a triangle is equilateral if it has three equal sides.

(vi) a triangle is equilateral only if it has three equal sides.

Ex. 8

State whether the following are true or false:

(i) an even number is prime **iff** it is 2.

(ii) an odd number is prime **iff** it is 3.

(iii) $x = 3$ **iff** $x^2 - 9 = 0$.

(iv) a triangle with sides of lengths a, b and c is right-angled **iff** $a^2 + b^2 = c^2$.

(v) a triangle is scalene **iff** no two angles are the same.

For those parts where the statement is false, re-write the statement using "if" or "only if" to make it true.

Ex. 9

A curve has the equation $y = ax^3 + bx^2 + c$. The curve has a maximum stationary point at $x = 0$ and a minimum stationary point in the 4th quadrant (that is, the region where $x > 0$ and $y < 0$).

Which of the following set of conditions is **sufficient** to ensure this?

(A) $a < 0, b < 0, c < 0$.

(B) $a < 0, b < 0, c > 0$.

(C) $a < 0, b > 0, c < 0$.

(D) $a < 0, b > 0, c > 0$.

(E) $a > 0, b < 0, c < 0$.

(F) $a > 0, b < 0, c > 0$.

(G) $a > 0, b > 0, c < 0$.

(H) $a > 0, b > 0, c > 0$.

Ex. 10

Consider the following statement about real numbers a and b:

$$a^2 > b^2 \qquad (*)$$

Which of the following is true?

(A) The condition $a > b$ is necessary but not sufficient for $(*)$ to be true.

(B) The condition $a > b$ is sufficient but not necessary for $(*)$ to be true.

(C) The condition $a > b$ is necessary and sufficient for $(*)$ to be true.

(D) The condition $a > b$ is not necessary and not sufficient for $(*)$ to be true.

Ex. 11

In this question x and y are non-zero real numbers.

Which one of the following is **sufficient** to conclude that $x < y$?

(A) $x^4 < y^4$

(B) $y^4 < x^4$

(C) $x^{-1} < y^{-1}$

(D) $y^{-1} < x^{-1}$

(E) $x^{\frac{3}{5}} < y^{\frac{3}{5}}$

(F) $y^{\frac{3}{5}} < x^{\frac{3}{5}}$

Ex. 12

There are three boxes A, B, and C with two boxes containing marbles and one containing candy. Each box has a note on it.

A: This box does not contain candy.

B: This box contains candy.

C: Box B contains marbles.

Only one of the above statements is true, and the others are false. Which box contains candy?

Ex. 13

Portia has three boxes made from gold, silver and lead. She has placed a prize in one of these boxes and challenges a friend, Bassanio, to find the prize.

She explains that on each box there is a message which may be true or false. On the basis of these messages Bassanio should be able to choose the box with the prize in.

(i) Initially suppose that there are only two boxes, gold and silver, one of which contains the prize. The messages read:

Gold: The prize is not in here.

Silver: Exactly one of these messages is true.

Which box contains the prize? Explain your answer.

(ii) Now suppose that there are all three boxes and that Portia has left the following messages on them:

Gold: The prize is in here.

Silver: The prize is in here.

Lead: At least two of these messages are false.

Which box should Bassanio choose? Explain your answer.

(iii) In this version of the challenge, Portia puts a dagger into one of the boxes. Bassanio must choose a box that does not contain the dagger. The messages on the boxes now read as follows:

Gold: The dagger is in this box.

Silver: The dagger is not in this box.

Lead: At most one of these messages is true.

Which box should Bassanio choose? Explain your answer.

Ex. 14

Five logicians each make a statement, as follows:

Mr. P: Of these five statements, an odd number are true.

Ms. Q: Both statements made by women are true.

Mr. R: My first name is Robert and Mr. P's statement is true.

Ms. S: Exactly one statement made by a man is true.

Mr. T: Neither statement made by a woman is true.

How many of the five statements can be simultaneously true?

(A) none

(B) 1 only

(C) 2 only

(D) 3 only

(E) 4 only

(F) none or 1 only

(G) 1 or 2 only

(H) 2 or 3 only

Quiz 1

a, b and c are real numbers with $a < b < c < 0$.

Which of the following statements must be true?

I $\quad ac < ab < a^2$

II $\quad b(c + a) > 0$

III $\quad \frac{c}{b} > \frac{a}{b}$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Quiz 2

A set of five cards each have a letter printed on their front and a number printed on their back, as follows:

	Card A	Card B	Card C	Card D	Card E
Fronts	A	B	C	D	E
Backs	3	4	1	7	8

Which one of the five cards (**A**, **B**, **C**, **D** or **E**) provides a counterexample to the following statement?

Every card that has a vowel on its front has an even number on its back.

Quiz 3

Three friends make the following statements.

Ben says, "Exactly one of Dan and Cam is telling the truth."

Dan says, "Exactly one of Ben and Cam is telling the truth."

Cam says, "Neither Ben nor Dan is telling the truth."

Which of the three friends is lying?

(A) Just Ben

(B) Just Dan

(C) Just Cam

(D) Each of Ben and Cam

(E) Each of Ben, Cam and Dan

Quiz 4

Some students tried to solve a problem in the Kangaroo competition. The number of boys who solved the problem correctly was equal to the number of girls who did not solve the problem correctly. Which of the following statements is true?

(A) The number of girls is more than the number of students who solved the problem correctly.

(B) The number of girls is less than the number of students who solved the problem correctly.

(C) The number of girls is equal to the number of students who solved the problem correctly.

(D) The situation is impossible.

(E) We need more information to decide on options (A), (B) or (C).

Quiz 5

There are some squares and triangles on the table. Some of them are blue and the rest are red. Some of these shapes are large and the rest are small. We know that

1. If the shape is large, it's a square;
2. If the shape is blue, it's a triangle.

Which of the statements (A)–(E) must be true?

(A) All red figures are squares.

(B) All squares are large.

(C) All small figures are blue.

(D) All triangles are blue.

(E) All blue figures are small.

Quiz 6

Every other day Renate tells the truth for the whole day.

Otherwise she lies for the whole day. Today she made exactly four of the following statements. Which statement could she not have made today?

(A) My name is Renate.

(B) I have a prime number of friends.

(C) I have the same number of girls who are friends as boys.

(D) Three of my friends are older than me.

(E) I always tell the truth.

Quiz 7

Lali and Gregor play a game with five coins, each with Heads on one side and Tails on the other. The coins are placed on a table, with Heads showing. In each round of the game, Lali turns over a coin, and then Gregor turns over a different coin. They play a total of ten rounds. Which of the following statements is then true?

(A) It is impossible for all the coins to show Heads

(B) It is impossible for all the coins to show Tails

(C) It is definite that all the coins show Heads

(D) It is definite that all the coins show Tails

(E) None of the statements (A) to (D) is true

Ex. 15

I have two dice whose faces are all painted different colours. I number the faces of one of them 1, 2, 2, 3, 3, 6 and the other 1, 3, 3, 4, 5, 6. I can now throw a total of 3 in two different ways using the two number 2's on the first die once each. Show that there are seven different ways of throwing a total of 6.

I now renumber the dice (again only using integers in the range 1 to 6) with the results shown in the following table:

Total shown by the two dice	2	3	4	5	6	7	8	9	10	11	12
Different ways of obtaining the total	0	2	1	1	4	3	8	6	5	6	0

Find how I have numbered the dice explaining your reasoning.

[You will only get high marks if the examiner can follow your argument.]

Ex. 16

A Minister and a Bishop were having a cup of tea. There was a knock at the door, and three bell ringers entered the room. After introductions, the Bishop asked the Minister how old the bell ringers were.

> "Well," the Minister said, knowing the Bishop had a penchant for numerical puzzles, "if you multiplied their three ages together, you'd get 2,450. But if you added them, you'd get twice your age."
>
> "Hmm," the Bishop muttered, after several moments' thought. "I haven't enough information to solve that."
>
> "It may help, my dear Bishop," offered the Minister, "to know that I am older than anyone else here in the room."
>
> "Yes, indeed it would," replied the Bishop. "Now I know their ages."

The question is: How old is the Minister?

You may assume that all ages are integers.

Practices P03

Time Allowed

60 min

Number of Questions

17

Difficulty

★★★★

Practices P03

Scan the QR code or click the link above to take the practice online.

Q1

a, b and c are real numbers.

Given that $ab = ac$, which of the following statemnets **must** be true?

I $a = 0$

II $b = 0$ **or** $c = 0$

III $b = c$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Q2

For any real numbers a, b and c where $a \geq b$, consider these three statements:

1. $-b \geq -a$
2. $a^2 + b^2 \geq 2ab$
3. $ac \geq bc$

Which of the statements 1, 2, and 3 **must** be true?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

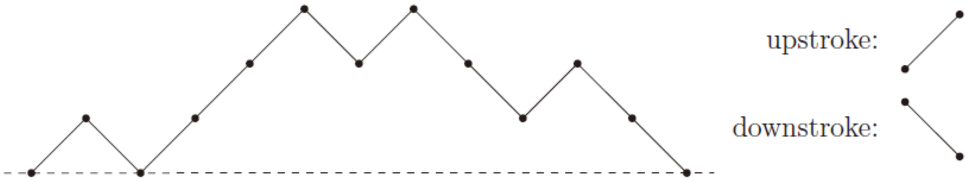
(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Q3 [TMUA, 2018S2Q8]

The diagram shows an example of a *mountain profile*.



This consists of *upstrokes* which go upwards from left to right, and *downstrokes* which go downwards from left to right. The example shown has six upstrokes and six downstrokes. The horizontal line at the bottom is known as *sea level*.

A *mountain profile of order* n consists of n upstrokes and n downstrokes, with the condition that the profile begins and ends at sea level and never goes below sea level (although it might reach sea level at any point). So the example shown is a mountain profile of order 6.

Mountain profiles can be coded by using U to indicate an upstroke and D to indicate a downstroke. The example shown has the code UDUUUDUDDUDD. A sequence of U's and D's obtained from a mountain profile in this way is known as a valid code.

Which of the following statements is/are true?

I If a valid code is written in reverse order, the result is always a valid code.

II If each U in a valid code is replaced by D and each D by U, the result is always a valid code.

III If U is added at the beginning of a valid code and D is added at the end of the code, the result is always a *valid code*.

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

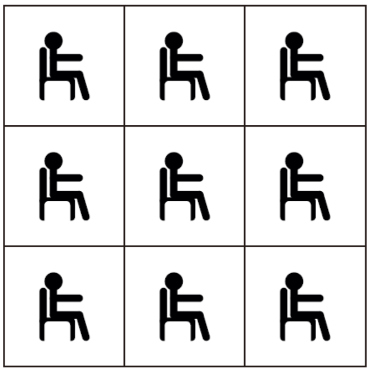
(F) I and III only

(G) II and III only

(H) I, II and III

Q4 [TMUA, 2020S2Q19]

Nine people are sitting in the squares of a 3 by 3 grid, one in each square, as shown. Two people are called *neighbours* if they are sitting in squares that share a side. (People in diagonally adjacent squares, which only have a point in common, are not called neighbours.)



Each of the nine people in the grid is either a truth-teller who **always** tells the truth, or a liar who **always** lies.

Every person in the grid says: 'My neighbours are all liars'.

Given only this information, what are the **smallest** number and the **largest** number of people who could be telling the truth?

(A) **smallest**: 1 **largest**: 4

(B) **smallest**: 2 **largest**: 4

(C) **smallest**: 2 **largest**: 5

(D) **smallest**: 3 **largest**: 4

(E) **smallest**: 3 **largest**: 5

(F) **smallest**: 4 **largest**: 4

(G) **smallest**: 4 **largest**: 5

(H) **smallest**: 5 **largest**: 5

Q5

For the following statements

$$\text{P:}\frac{x(x-2)}{1-x} > 0 \qquad \text{Q: } 1 < x < 2$$

about a real number x,

(A) P implies Q, but Q does not imply P.

(B) Q implies P, but P does not imply Q.

(C) P implies Q, and Q implies P.

(D) P and Q contradict each other.

Q6

Pick a whole number.

Add one.

Square the answer.

Multiply the answer by four.

Subtract three.

Which of the following statements are true regardless of which starting number is chosen?

I The final answer is odd.

II The final answer is one more than a multiple of three.

III The final answer is one more than a multiple of eight.

IV The final answer is not prime.

V The final answer is not one less than a multiple of three.

(A) I, II, and V.

(B) I and IV.

(C) II and V.

(D) I, III, and V.

(E) I and V.

Q7

The fixed positive integers a, b, c, d are such that exactly two of the following four statements are valid:

(i) $a \leq b < c \leq d$

(ii) $a + b = c + d$

(iii) $a = c$ and $b = d$

(iv) $ad = bc$

You are also told that (ii) and (iv) is not the pair of valid statements. Which of the following must be the pair of valid statements?

(A) (i)and (ii)

(B) (i)and (iii)

(C) (i)and (iv)

(D) (iii)and (iv)

Q8

There are real numbers x, y such that precisely one of the statements (a), (b), (c), (d) is true. Which is the true statement?

(A) $x \geq 0$

(B) $x < y$

(C) $x^2 > y^2$

(D) $|x| \leq |y|$

Q9

You are given two whole numbers p and q, and told that three of the following statements concerning p and q are *true* and that the other statement is *false*. Which is the false statement?

(A) pq is even.

(B) $p + q$ is even.

(C) $2p + q^2$ is odd.

(D) $p^2 + 2q$ is odd.

Q10

Three runners, Friedrich, Gottlieb and Hans had a race. Before the race, a commentator said, "Either Friedrich or Gottlieb will win." Another commentator said, "If Gottlieb comes second, then Hans will win." Another said, "If Gottlieb comes third, Friedrich will not win." And another said, "Either Gottlieb or Hans will be second." In the event, it turned out that all the commentators were correct. In what order did the runners finish?

(A) Friedrich, Gottlieb, Hans

(B) Friedrich, Hans, Gottlieb

(C) Hans, Gottlieb, Friedrich

(D) Gottlieb, Friedrich, Hans

(E) Gottlieb, Hans, Friedrich

Q11

Together, Alan and Bill weigh less than Charlie and Dan. Together, Charlie and Edwina weigh less than Bill and Frances. Which of the following is definitely true?

(A) Together, Alan and Edwina weigh less than Dan and Frances.

(B) Together, Dan and Edwina weigh more than Charlie and Frances.

(C) Together, Dan and Frances weigh more than Alan and Charlie.

(D) Together, Alan and Bill weigh less than Charlie and Frances.

(E) Together, Alan, Bill and Charlie weigh the same as Dan, Edwina and Frances.

Q12

A box of fruit contained twice as many apples as pears. Chris and Lily divided them up so that Chris had twice as many pieces of fruit as Lily. Which one of the following statements is always true?

(A) Chris took at least one pear.

(B) Chris took twice as many apples as pears

(C) Chris took twice as many apples as Lily.

(D) Chris took as many apples as Lily took pears.

(E) Chris took as many pears as Lily took apples.

Q13

Pierre said, “Just one of us is telling the truth”.

Qadr said, “What Pierre says is not true”.

Ratna said, “What Qadr says is not true”.

Sven said, “What Ratna says is not true”.

Tanya said, “What Sven says is not true”.

How many of them were telling the truth?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Q14

Isobel: “Josh is innocent”

Genotan: “Tegan is guilty”

Josh: “Genotan is guilty”

Tegan: “Isobel is innocent”

Only the guilty person is lying; all the others are telling the truth. Who is guilty?

(A) Isobel

(B) Josh

(C) Genotan

(D) Tegan

(E) More information required

Q15

The four statements in the box below refer to a mother and her four daughters. One statement is true, three statements are false.

Who is the mother?

Alice is the mother.

Carol and Ella are both daughters.

Beth is the mother.

One of Alice, Diane or Ella is the mother.

(A) Alice

(B) Beth

(C) Carol

(D) Diane

(E) Ella

Q16

Each of the Four Musketeers made a statement about the four of them, as follows.

d'Artagnan: "Exactly one is lying."

Athos: "Exactly two of us are lying."

Porthos: "An odd number of us is lying."

Aramis: "An even number of us is lying."

How many of them were lying (with the others telling the truth)?

(A) one

(B) one or two

(C) two or three

(D) three

(E) four

Q17

The Tour de Clochemerle is not yet as big as the rival Tour de France. This year there were five riders, Arouet, Barthes, Camus, Diderot and Eluard, who took part in five stages. The winner of each stage got 5 points, the runner up 4 points and so on down to the last rider who got 1 point. The total number of points acquired over the five stages was the rider's score. Each rider obtained a different score overall and the riders finished the whole tour in alphabetical order with Arouet gaining a magnificent 24 points. Camus showed consistency by gaining the same position in four of the five stages and Eluard's rather dismal performance was relieved by a third place in the fourth stage and first place in the final stage. Explain why Eluard must have received 11 points in all and find the scores obtained by Barthes, Camus and Diderot.

Where did Barthes come in the final stage?

Supplements S03

Time Allowed

60 min

Number of Questions

15

Difficulty

★★★★

Supplements S03

Scan the QR code or click the link above to take the practice online.

SQ1

A magical island is inhabited by knights (who always tell the truth) and liars (who always lie). A wise man met two people, Chris and Pat, from the island and decided to determine if they were knights or liars. When he asked Chris, "Are you both knights?" he could not be sure of their types. When he then asked Chris, "Are you of the same type?" he could identify their types. What were they?

(A) both liars

(B) both knights

(C) Chris–knight, Pat–liar

(D) Chris–liar, Pat–knight

(E) impossible to specify

SQ2

The island of Nogardia is inhabited by dragons, each of which has either six, seven or eight legs. Dragons with seven legs always lie; dragons with an even number of legs always tell the truth. One day four dragons met.

The blue one said, "We have 28 legs altogether."

The green one said, "We have 27 legs altogether."

The yellow one said, "We have 26 legs altogether."

The red one said, "We have 25 legs altogether."

Which of the following statements is true?

(A) the red dragon definitely has 6 legs

(B) the red dragon definitely has 7 legs

(C) the red dragon definitely has 8 legs

(D) the red dragon has either 6 or 8 legs, but we can't be sure which

(E) the red dragon has 6, 7, or 8 legs, but we can't be sure which

SQ3

The children P, Q, R and S made the following assertions.

P said: Q, R and S are girls

R said: P and Q are lying

Q said: P, R and S are boys

S said: P, Q and R are telling the truth

How many of the children were telling the truth?

(A) 0

(B) 1

(C) 2

(D) 3

(E) It cannot be determined

SQ4

Every second day Charles tells the truth for the whole day, otherwise he lies for the whole day. Today he made exactly four of the following statements. Which statement could he not have made today?

(A) I have a prime number of friends.

(B) I have as many male friends as female friends.

(C) My name is Charles.

(D) I always speak the truth.

(E) Three of my friends are older than me.

SQ5

From noon until midnight Clever Cat sleeps under an oak tree, and from midnight until noon he tells stories. A poster on the oak tree reads:

‘Two hours ago Clever Cat was doing the same thing as he will be doing in one hour's time.’ For how many hours in a day is the statement on the poster true?

(A) 3

(B) 6

(C) 12

(D) 18

(E) 21

SQ6

A lion is hidden in one of three rooms. A note on the door of room 1 reads "The lion is here". A note on the door of room 2 reads "The lion is not here ". A note on the door of room 3 reads "$2 + 3 = 2 \times 3$". Only one of these notes is true. In which room is the lion hidden?

(A) In room 1

(B) In room 2

(C) In room 3

(D) It may be in any room

(E) It may be in either room 1 or room 2

SQ7

One evening, an enclosure contained a number of kangaroos. All of a sudden a kangaroo got up and said: "There are 6 of us here" and jumped out of the enclosure. Then another kangaroo jumped out of the enclosure and said: "Every kangaroo who jumped out before me was lying." After that the rest of the kangaroos jumped out one by one, saying the same thing as the second kangaroo, until there were no kangaroos left in the enclosure. How many kangaroos had told the truth?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

SQ8

Mr Ross always tells the truth on Thursdays and Fridays but always tells lies on Tuesdays.

On the other days of the week he tells the truth or tells lies, at random. For seven consecutive days he was asked what his first name was, and on the first six days he gave the following answers, in order: John, Bob, John, Bob, Pit, Bob. What was his answer on the seventh day?

(A) John

(B) Bob

(C) Pit

(D) Kate

(E) More information is needed

SQ9

Three blackbirds, Isaac, Max and Oscar, are each sitting on their own nest. Isaac says: “I'm more than twice as far away from Max as I am from Oscar”. Max says: “I'm more than twice as far away from Oscar as I am from Isaac”. Oscar says: “I'm more than twice as far away from Max as I am from Isaac”. At least two of them are telling the truth. Who is lying?

(A) Isaac

(B) Max

(C) Oscar

(D) None of them

(E) Impossible to tell

SQ10

The town of Ginkrail is inhabited entirely by knights and liars. Every sentence spoken by a knight is true, and every sentence spoken by a liar is false. One day some inhabitants of Ginkrail were alone in a room and three of them spoke.

The first one said: “There are no more than three of us in the room. All of us are liars.”

The second said: “There are no more than four of us in the room. Not all of us are liars.”

The third said: “There are five of us in the room. Three of us are liars.”

How many people were in the room and how many liars were among them?

(A) 3 people, 1 of whom is a liar

(B) 4 people, 1 of whom is a liar

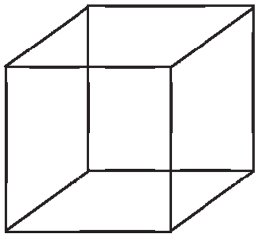
(C) 4 people, 2 of whom are liars

(D) 5 people, 2 of whom are liars

(E) 5 people, 3 of whom are liars

SQ11

At each of the vertices of a cube sits a Bunchkin. Two Bunchkins are said to be adjacent if and only if they sit at either end of one of the cube's edges. Each Bunchkin is either a 'truther', who always tells the truth, or a 'liar', who always lies. All eight Bunchkins say 'I am adjacent to exactly two liars'. What is the maximum number of Bunchkins who are telling the truth?



SQ12

Twenty-five people who always tell the truth or always lie are standing in a queue. The man at the front of the queue says that everyone behind him always lies. Everyone else says that the person immediately in front of them always lies. How many people in the queue always lie?

SQ13

Which is the lowest numbered statement which is true?

Statement 201: "Statement 203 is true".

Statement 202: "Statement 201 is true".

Statement 203: "Statement 206 is false".

Statement 204: "Statement 202 is false".

Statement 205: "None of the statements 201, 202, 203 or 204 are true".

Statement 206: "$1 + 1 = 2$".

SQ14

A box contains seven cards numbered from 301 to 307. Graham picks three cards from the box and then Zoe picks two cards from the remainder. Graham looks at his cards and then says "I know that the sum of the numbers on your cards is even". What is the sum of the numbers on Graham's cards?

SQ15

Sketch, without calculating the stationary points, the graph of the function $f(x)$ given by

$$f(x) = (x-p)(x-q)(x-r),$$

where $p < q < r$. By considering the quadratic equation $f'(x) = 0$, or otherwise, show that

$$(p+q+r)^2 > 3(qr+rp+pq).$$

By considering $(x^2+gx+h)(x-k)$, or otherwise, show that $g^2 > 4h$ is a sufficient condition but not a necessary condition for the inequality

$$(g-k)^2 > 3(h-gk)$$

to hold.

04 Algebraic Manipulations

What's on the Specification?

- Use and manipulation of surds; simplifying expressions that contain surds, including rationalising the denominator; for example, simplifying $\frac{\sqrt{5}}{3+2\sqrt{5}}$, and $\frac{3}{\sqrt{7}-2\sqrt{3}}$.
- Algebraic manipulation of polynomials, including:
  - Expanding brackets and collecting like terms;
  - Factorisation and simple algebraic division (by a linear polynomial, including those of the form $ax + b$, and by quadratics, including those of the form $ax^2 + bx + c$).
- Use of the Factor Theorem and the Remainder Theorem.
- Qualitative understanding that a function is a many-to-one (or sometimes just a one-to-one) mapping. Familiarity with the properties of common functions, including $f(x) = \sqrt{x}$ (which always means the 'positive square root') and $f(x) = |x|$.

Exercises E04

Time Allowed

No limit

Number of Questions

26

Difficulty

★★★☆

Exercises E04

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

It is given that the expansion of $(ax+b)^3$ is $8x^3 - px^2 + 18x - 3\sqrt{3}$, where a, b and p are real constants.

What is the value of p?

(A) $-12\sqrt{3}$

(B) $-6\sqrt{3}$

(C) $-4\sqrt{3}$

(D) $-\sqrt{3}$

(E) $\sqrt{3}$

(F) $4\sqrt{3}$

(G) $6\sqrt{3}$

(H) $12\sqrt{3}$

Quiz Pre-2

The quadratic expression $x^2 - 14x + 9$ factorises as $(x-\alpha)(x-\beta)$, where α and β are positive real numbers.

Which quadratic expression can be factorised as $\left(x-\sqrt{\alpha}\right)\left(x-\sqrt{\beta}\right)$?

(A) $x^2 - \sqrt{10}x + 3$

(B) $x^2 - \sqrt{14}x + 3$

(C) $x^2 - \sqrt{20}x + 3$

(D) $x^2 - 178x + 81$

(E) $x^2 - 176x + 81$

(F) $x^2 + 196x + 81$

Quiz Pre-3

Which of the following is equal to $\sqrt{9-6\sqrt{2}} + \sqrt{9+6\sqrt{2}}$?

(A) $3\sqrt{2}$

(B) $2\sqrt{6}$

(C) $\frac{7\sqrt{2}}{2}$

(D) $3\sqrt{3}$

(E) 6

Ex. 1

(i) Show that $(a+b)(a-b)=a^2-b^2$.

(ii) Show that $(a+b)(a^2-ab+b^2)=a^3+b^3$.

(iii) Factorise (a) x^3-64 and (b) x^6-y^6.

(iv) Find the sum of the following geometric progression:

$$1+t+t^2+t^3+t^4$$

and deduce (by choosing t suitably) that $a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

Write down a factorisation for a^5+b^5.

Ex. 2

Factorise:

(i) $a^2+b^2+c^2+2ab+2bc+2ac$.

(ii) $a^2+b^2+c^2+d^2+2ab+2bc+2cd+2ac+2bd+2ad$.

(iii) $(ac+bd)^2+(ad-bc)^2$.

(iv) $a^2-b^2-c^2+2bc+a+b-c$.

(v) $(a-b)^3+(b-c)^3+(c-a)^3$.

(vi) $abc+ab+bc+ca+a+b+c+1$.

Ex. 3

Find $m^3+\frac{1}{m^3}$ if $m+\frac{1}{m}=2$.

Ex. 4

If both a and b are positive real numbers with $\frac{1}{a}-\frac{1}{b}=\frac{1}{a+b}$, find the value of $\left(\frac{b}{a}\right)^3+\left(\frac{a}{b}\right)^3$.

Ex. 5

Evaluate $\sqrt{4+\sqrt{7}}-\sqrt{4-\sqrt{7}}$.

Ex. 6

Evaluate $(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)(2^{32}+1)$.

Ex. 7

Factorise x^3-2x^2-5x+6.

Ex. 8

The expression $x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 - x^3 - y^3 - z^3 - 2xyz$ factorises as

(A) $(x + y + z)(x - y + z)(-x + y - z)$

(B) $(x + y - z)(x - y - z)(-x + y + z)$

(C) $(x + y - z)(x - y + z)(-x + y + z)$

(D) $(x - y - z)(-x - y + z)(-x + y - z)$

Ex. 9

Show that $x = 15$ is a root of the equation $x^4 - 18x^3 + 35x^2 + 180x - 450 = 0$. Find all the roots of the equation.

Ex. 10

(i) simplify $\sqrt{4 + 2\sqrt{3}}$

(ii) simplify $\sqrt{2 - \sqrt{3}}$

(iii) simplify $\sqrt{17 + 12\sqrt{2}}$

Ex. 11

Evaluate $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$.

Quiz 1

What is the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$?

(A) -7

(B) -5

(C) -4

(D) -1

(E) 4

Quiz 2

Of the following three alleged algebraic identities, at least one is wrong.

(i) $yz(z-y) + zx(x-z) + xy(y-x) = (z-y)(x-z)(y-x)$

(ii) $yz(z-y) + zx(x-z) + xy(y-x) = (z-y)(z-x)(y-x)$

(iii) $yz(z+y) + zx(x+z) + xy(y+x) = (z+y)(z+x)(y+x)$

Which of the following statements is correct?

(A) Only identity (i) is right.

(B) Only identity (ii) is right.

(C) Identities (ii) and (iii) are right.

(D) All these identities are wrong.

Quiz 3

The following equation is true for all a, b and c:

$$a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + kabc$$

What is the value of k?

(A) -6

(B) -3

(C) 0

(D) 3

(E) 6

Quiz 4

If $x^2 - 3x + 1 = 0$, what is the value of $x^2 + \left(\frac{1}{x}\right)^2$?

(A) 7

(B) $\frac{(7-3\sqrt{5})}{2}$

(C) 9

(D) $\frac{(7+3\sqrt{5})}{2}$

(E) 10

Quiz 5

Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

(A) 164

(B) 172

(C) 192

(D) 194

(E) 212

Quiz 6

The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

(A) -1

(B) 0

(C) 1

(D) 2

(E) 8

Quiz 7

Which of the following numbers does *not* have a square root in the form $x + y\sqrt{2}$, where x and y are positive integers?

(A) $17 + 12\sqrt{2}$

(B) $22 + 12\sqrt{2}$

(C) $38 + 12\sqrt{2}$

(D) $54 + 12\sqrt{2}$

(E) $73 + 12\sqrt{2}$

Quiz 8

Find the value of the expression

$$\sqrt{8-4\sqrt{2}+1}+\sqrt{9-12\sqrt{2}+8}$$

(A) $\sqrt{26-16\sqrt{2}}$

(B) $4\sqrt{2}-4$

(C) -2

(D) $4-4\sqrt{2}$

(E) 2

(F) $\sqrt{26}-4\sqrt{2}$

(G) 1

Quiz 9

Which of the following is equal to $\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}}$?

(A) $\sqrt{1003}-\sqrt{1002}$

(B) $\sqrt{1005}-\sqrt{2004}$

(C) $\sqrt{1007}-\sqrt{1005}$

(D) $\sqrt{2005}-\sqrt{2003}$

(E) $\sqrt{2007}-\sqrt{2005}$

Quiz 10

The numbers x, y and z are given by $x=\sqrt{12-3\sqrt{7}}-\sqrt{12+3\sqrt{7}}$, $y=\sqrt{7-4\sqrt{3}}-\sqrt{7+4\sqrt{3}}$ and $z=\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}$.

What is the value of xyz?

(A) 1

(B) – 6

(C) – 8

(D) 18

(E) 12

Ex. 12

(i) Show that $x-3$ is a factor of

$$x^3-5x^2+2x^2y+xy^2-8xy-3y^2+6x+6y. \qquad (*)$$

Express $(*)$ in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

(ii) Factorise $6y^3-y^2-21y+2x^2+12x-4xy+x^2y-5xy^2+10$ into three linear factors.

Ex. 13

Alice plays a game 5 times with her friends Sam and Pam. In each game Alice chooses two integers x and y with $1 \leq x \leq y$. She whispers the sum $x + y$ to Sam, and the product $x \times y$ to Pam, so that neither knows what the other was told. Sam and Pam then have to try to work out what the numbers x and y are. Sam and Pam are well known expert logicians.

(i) In the first game,

Pam says "I know x and y."

What can we deduce about the values of x and y? Explain your answer.

(ii) In the second game,

Pam says "I don't know what x and y are."

Sam then says "I know x and y."

Suppose the sum is 4. What are x and y? Explain your answer.

(iii) In the third game,

Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I now know x and y."

Suppose the product is 4. What are x and y? Explain your answer.

(iv) In the fourth game,

Pam says "I don't know what x and y are."

Sam then says "I already knew that."

Pam then says "I now know x and y."

Suppose the product is 8. What are x and y? Explain your answer.

(v) Finally, in the fifth game,

Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I don't know what x and y are."

Sam then says "I now know x and y."

Suppose the sum is 5. What are x and y? Explain your answer.

Practices P04

Time Allowed

40 min

Number of Questions

16

Difficulty

★★★

Practices P04

Scan the QR code or click the link above to take the practice online.

Q1

Evaluate

$$\left(\sqrt{7}+\sqrt{3}\right)^2 - \left(\sqrt{7}-\sqrt{3}\right)^2$$

(A) 0

(B) $2\sqrt{7}$

(C) $4\sqrt{7}$

(D) $2\sqrt{21}$

(E) 10

(F) $4\sqrt{21}$

(G) 20

Q2

Evaluate

$$\frac{\left(\sqrt{12}+\sqrt{3}\right)^2}{\left(\sqrt{12}-\sqrt{3}\right)^2}$$

(A) 1

(B) 3

(C) $\frac{5}{3}$

(D) $\frac{7}{3}$

(E) $3\sqrt{3}$

(F) 9

Q3

Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

(A) $r + s \leq 0$

(B) $s \leq 0$

(C) $r \leq 0$

(D) $r \geq s$

Q4

The equation gives y in terms of x:

$$y = 3 - 4\left(1 - \frac{x}{2}\right)^2$$

Which one of the following is a rearrangement for x in terms of y?

(A) $x = -2 \pm 2\sqrt{\frac{3-y}{4}}$

(B) $x = -2 \pm 2\sqrt{\frac{4-y}{3}}$

(C) $x = 1 \pm \sqrt{\frac{3-y}{4}}$

(D) $x = 1 \pm 2\sqrt{\frac{3-y}{4}}$

(E) $x = 2 \pm 2\sqrt{\frac{3-y}{4}}$

(F) $x = 2 \pm 2\sqrt{\frac{4-y}{3}}$

(G) $x = 2 \pm 2\sqrt{\frac{3+y}{4}}$

Q5

Which of the following is a simplification of

$$4 - \frac{x(3x+1)}{x^2(3x^2-2x-1)}$$

(A) $\frac{12x^3-8x^2-7x-1}{x(3x-1)(x-1)}$

(B) $\frac{4x^2+4x-1}{x(x+1)}$

(C) $\frac{4x^2+4x+1}{x(x+1)}$

(D) $\frac{4x^2-4x-1}{x(x-1)}$

(E) $\frac{4x^2-4x+1}{x(x-1)}$

(F) $\frac{12x^3-8x^2-x+1}{x(3x-1)(x-1)}$

Q6

When the expression

$$(2x + 3)^2 - (x - 3)^2$$

is written in the form $p(x + q)^2 + r$, where p, q and r are constants, what is the value of r?

(A) -27

(B) -9

(C) 0

(D) 3

(E) 15

Q7

Which one of the following expressions is equivalent to

$$\frac{a}{b/c} - \frac{a/b}{c}$$

(A) 0

(B) $\frac{a(b^2-1)}{bc}$

(C) $\frac{a(b^2-c^2)}{bc}$

(D) $\frac{a^2b^2-c^2}{abc}$

(E) $\frac{a(c^2-1)}{bc}$

(F) $\frac{a^2c^2-b^2}{abc}$

(G) $\frac{b^2-a^2}{abc}$

Q8

Which one of the following is a simplification of

$$1-\left(\frac{3+\sqrt{3}}{6-2\sqrt{3}}\right)^2$$

(A) $-\frac{3}{4}$

(B) $\frac{3}{4}$

(C) $-\frac{3}{4}-\frac{\sqrt{3}}{7}$

(D) $\frac{3}{4}-\frac{\sqrt{3}}{7}$

(E) $-\frac{3}{4}-\sqrt{3}$

(F) $\frac{3}{4}-\sqrt{3}$

(G) $-\frac{\sqrt{3}}{2}$

(H) $\frac{\sqrt{3}}{2}$

Q9

Which one of the following is a simplification of $\frac{x^2-4}{x^2-2x}$ where $x \neq 2$ and $x \neq 0$?

(A) $\frac{x-4}{x-2}$

(B) $\frac{x-2}{x}$

(C) $\frac{2}{x}$

(D) $\frac{x+2}{x}$

(E) $\frac{x+2}{x+1}$

Q10

The equation gives y in terms of x:

$$y = 3 + 2\left(\frac{x}{4} - 1\right)^2$$

Which one of the following is a rearrangement for x in terms of y?

(A) $x = 4\left(1 \pm \sqrt{\frac{y-3}{2}}\right)$

(B) $x = 4\left(1 \pm \sqrt{\frac{y+3}{2}}\right)$

(C) $x = 4\left(-1 \pm \sqrt{\frac{y-3}{2}}\right)$

(D) $x = 4\left(-1 \pm \sqrt{\frac{y+3}{2}}\right)$

(E) $x = 2\left(1 \pm \sqrt{\frac{y-3}{2}}\right)$

(F) $x = 2\left(-1 \pm \sqrt{\frac{y-3}{2}}\right)$

Q11

$(x-1)(x^4+1)(x^2+1)(x+1)$ equals

(A) $x^8 - 1$

(B) $x^8 + x^6 + x^4 + x^2 + 1$

(C) $x^8 + 1$

(D) $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

(E) $x^8 - x^6 + x^4 - 1$

Q12

Let A, M and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$?

(A) 62

(B) 72

(C) 92

(D) 102

(E) 112

Q13

Suppose that real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3.$$

What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$?

(A) 8

(B) $\sqrt{33}+8$

(C) 9

(D) $2\sqrt{10}+4$

(E) 12

Q14

Real numbers x and y satisfy $x+y=4$ and $x\cdot y=-2$. What is the value of

$$x+\frac{x^3}{y^2}+\frac{y^3}{x^2}+y?$$

(A) 360

(B) 400

(C) 420

(D) 440

(E) 480

Q15

If $y+4=(x-2)^2$, $x+4=(y-2)^2$ and $x\neq y$, what is the value of x^2+y^2?

(A) 10

(B) 15

(C) 20

(D) 25

(E) 30

Q16

Let a and b be relatively prime integers with $a > b > 0$ and $\frac{a^3-b^3}{(a-b)^3} = \frac{73}{3}$. What is $a - b$?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Supplements S04

Time Allowed

90 min

Number of Questions

15

Difficulty

★★★☆

Supplements S04

Scan the QR code or click the link above to take the practice online.

SQ1

Which one of the following is a simplification of $(\sqrt{3}-\sqrt{2})^2$?

(A) $1-2\sqrt{3}\sqrt{2}$

(B) $5-2\sqrt{2}\sqrt{3}$

(C) $2\sqrt{3}-2\sqrt{2}$

(D) 1

(E) $5-\sqrt{2}\sqrt{3}$

(F) $13-2\sqrt{2}\sqrt{3}$

(G) $5+2\sqrt{2}\sqrt{3}$

(H) 5

SQ2

The equation gives y in terms of x:

$$y = 3\left(\frac{x}{2}-1\right)^2 - 5$$

Which one of the following is a rearrangement for x in terms of y?

(A) $x = 2 \pm 2\sqrt{\frac{y-5}{3}}$

(B) $x = 2 \pm 2\sqrt{\frac{y+5}{3}}$

(C) $x = 2 \pm 3\sqrt{\frac{y+5}{3}}$

(D) $x = -2 \pm 2\sqrt{\frac{y+5}{3}}$

(E) $x = -2 \pm 3\sqrt{\frac{y+5}{2}}$

(F) $x = 2 + 2\left(\frac{y+5}{3}\right)^2$

(G) $x = -2 + 2\left(\frac{y+5}{3}\right)^2$

SQ3

Which one of the following is a simplification of

$$2-\frac{x^2(9x^2-4)}{x^3(2-3x)}$$

(A) $-1-\frac{2}{x}$

(B) $-1+\frac{2}{x}$

(C) $5-\frac{2}{x}$

(D) $5+\frac{2}{x}$

(E) $5-\frac{3}{x}$

(F) $5+\frac{3}{x}$

SQ4

Make b the subject of the formula:

$$a=\frac{b^2+2}{3b^2-1}$$

(A) $b=\pm\sqrt{\left(\frac{a+2}{3a+1}\right)}$

(B) $b=\pm\sqrt{\left(\frac{a+2}{3a-1}\right)}$

(C) $b=\pm\sqrt{\left(\frac{2-a}{3a+1}\right)}$

(D) $b=\pm\sqrt{\left(\frac{2-a}{3a-1}\right)}$

(E) $b=\pm\sqrt{\left(\frac{3}{3a+1}\right)}$

(F) $b=\pm\sqrt{\left(\frac{3}{3a-1}\right)}$

SQ5

Which one of the following is a simplification of $4+\frac{4-x^2}{x^2-2x}$?

(A) $3-\frac{2}{x}$

(B) $3+\frac{2}{x}$

(C) $4-\frac{2}{x}$

(D) $4+\frac{2}{x}$

(E) $5-\frac{2}{x}$

(F) $5+\frac{2}{x}$

SQ6

Which one of the following expressions is equivalent to $\frac{9^{2n+1}\times 3^{4-3n}}{27^{2-n}}$?

(A) 3^9

(B) 3^{-2n}

(C) 3^{2-2n}

(D) 3^{4n}

(E) 3^{6n-2}

(F) 3^6

SQ7

When simplified, $\frac{1}{(1-\sqrt{2})^3}$ is written in the form $a+b\sqrt{2}$ where a and b are integers.

What is the value of b?

(A) -7

(B) -5

(C) -1

(D) 1

(E) 5

(F) 7

SQ8

The equation gives y in terms of x:

$$y = 3\left(\sqrt{\frac{x+1}{2}}\right) - 1$$

Which one of the following is a rearrangement for x in terms of y?

(A) $x = 2\left(\frac{y+1}{3}\right) - 1$

(B) $x = 2\left(\frac{y+1}{3}\right) + 1$

(C) $x = 2\left(\frac{y+1}{3}\right)^2 - 1$

(D) $x = 2\left(\frac{y+1}{3}\right)^2 + 1$

(E) $x = 3\left(\frac{y+1}{3}\right)^2 - 1$

(F) $x = 3\left(\frac{y+1}{3}\right)^2 + 1$

(G) $x = \left(\frac{2(y+1)}{3}\right)^2 - 1$

SQ9

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

(A) 1

(B) 2

(C) 3

(D) 6

(E) 8

SQ10

What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$

(A) $-\frac{2004}{2003}$

(B) -1

(C) $\frac{2003}{2004}$

(D) 1

(E) $\frac{2004}{2003}$

SQ11

Let a, b, c, d, e, f, g and h be distinct elements in the set

$$\{-7, -5, -3, -2, 2, 4, 6, 13\}.$$

What is the minimum possible value of

$$(a+b+c+d)^2 + (e+f+g+h)^2$$

(A) 30

(B) 32

(C) 34

(D) 40

(E) 50

SQ12

Given that $\left(a+\frac{1}{a}\right)^2 = 6$ and $a^3 + \frac{1}{a^3} = N\sqrt{6}$ and $a > 0$, what is the value of N?

SQ13

The three prime numbers a, b and c are such that $a > b > c$, $a+b+c = 52$ and $a-b-c = 22$.

What is the value of abc?

SQ14

Given that $a+b=5$ and $ab=3$, what is the value of a^4+b^4?

SQ15

The number

$$\sqrt{104\sqrt{6} + 468\sqrt{10} + 144\sqrt{15} + 2006}$$

can be written as $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$, where a, b and c are positive integers. Find $a \cdot b \cdot c$.

05 Equations

What's on the Specification?

- Simultaneous equations: analytical solution by substitution, e.g. of one linear and one quadratic equation.

Exercises E05

Time Allowed

No limit

Number of Questions

29

Difficulty

★★★☆

Exercises E05

Scan the QR code or click the link above to take the practice online.

05

110

Quiz Pre-1

The equation

$$|x| + |x - 1| = 0$$

has

(A) no solutions.

(B) one solution.

(C) two solution.

(D) three solutions.

Quiz Pre-2

For what values of the real number a does the quadratic equation

$$x^2 + ax + a = 1$$

have distinct real roots?

(A) $a \neq 2$

(B) $a > 2$

(C) $a = 2$

(D) all values of a

Quiz Pre-3

There are positive real numbers x and y which solve the equations

$$\begin{aligned} 2x + ky &= 4 \\ x + y &= k \end{aligned}$$

for

(A) all values of k.

(B) no values of k.

(C) $k = 2$ only.

(D) only $k > -2$.

Quiz Pre-4

The numbers x, y and z satisfy the equations

$$x + y + 2z = 850, \qquad x + 2y + z = 950, \qquad 2x + y + z = 1200.$$

What is their mean?

(A) 250

(B) $\frac{1000}{3}$

(C) 750

(D) 1000

(E) More information is needed.

Ex. 1

The number of distinct solutions of the equation $\big|x - |2x+1|\big| = 3$ is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Ex. 2

(i) The function f is defined by $f(x) = |x-a| + |x-b|$, where $a < b$. Sketch the graph of $f(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $g(x)$, where $g(x) = |2x - a - b|$.

What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $\big(b, f(b)\big)$ and $\big(a, f(a)\big)$?

(ii) Show graphically that the equation

$$|x-a| + |x-b| = |x-c|,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x-a| + |x-b| = |x-c| + |x-d|,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

Ex. 3

Given the equation $ax^2 + bx + c = 0$, where a, b and c are real coefficients,

(i) if $a \neq 0$, solve for x by means of completing the square.

(ii) discuss the cases arise according to the number of solutions.

(iii) if the equation has two roots α and β, determine the relationship between the roots and the coefficients, by

(a) the quadratic formula.

(b) comparing the coefficients.

Ex. 4

Depending on the value of the constant d, the equation

$$dx^2 - (d-1)x + d = 0$$

may have two real solutions, one real solution or no real solutions. For how many values of d does it have *just one* real solution?

(A) for one value of d

(B) for two values of d

(C) for three values of d

(D) for infinitely many values of d

Ex. 5

How many positive integer values of a such that the quadratic equation

$$ax^2 + 2(2a-1)x + 4(a-3) = 0$$

has at least one integer solution?

Ex. 6

(i) For what values of the constant k does the quadratic equation

$$x^2 - 2x - 1 = k$$

have:

(a) no real solutions.

(b) one real solution.

(c) two real solutions.

(ii) Showing your working, express $(x^2 - 2x - 1)^2$ as a polynomial of degree 4 in x.

(iii) Show that the quartic equation

$$x^4 - 4x^3 + 2x^2 + 4x + 1 = h$$

has exactly two real solutions if *either* $h = 0$ *or* $h > 4$.

Show that there is no value of h such that the above quartic equation has just one real solution.

Ex. 7

Considering the following general simultaneous equations below:

$$a_1x + b_1y = c_1 \quad (1)$$
$$a_2x + b_2y = c_2 \quad (2)$$

discuss the cases that arise according to the number of solutions.

Ex. 8

Given numbers a, b, c, which of the following statements about the simultaneous equations

$$2x + y = 5$$
$$ax + by = c$$

is true?

(A) There are no solutions when $a = 2b$ and $c = 5b$.

(B) There is a unique solution when $a \neq 2b$ and $c = 5b$.

(C) There are an infinite number of solutions when $a = 2$, $b = 1$, and $c = 0$.

(D) There are no solutions when $a \neq 2b$ and $c = 5b$.

Ex. 9

The simultaneous equations

$$x - 2y + 3z = 1$$
$$2x + 3y - z = 4$$
$$4x - y + 5z = 6$$

have

(A) no solutions.

(B) exactly one solution.

(C) exactly three solutions.

(D) infinitely many solutions.

Ex. 10

(i) Solve the simultaneous equations:

$$x + y - z = 2$$
$$x - y + z = 0$$
$$-x + y + z = 8$$

(ii) Now solve the simultaneous equations:

$$kx + y - z = 2$$
$$x - y + z = 0$$
$$-x + y + z = 8$$

where k is a fixed but unknown number.

Are there any values of k for which the equations have no solution?

Ex. 11

(i) Use the substitution $\sqrt{x} = y$ (where $y \geq 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \frac{1}{2} = 0.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} - 22 = 0$;

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0$.

Quiz 1

p and q are real numbers, and the equation

$$x|x| = px + q$$

has exactly k distinct real solutions for x.

Which one of the following is the complete list of possible values for k?

(A) 0, 1, 2

(B) 0, 1, 2, 3

(C) 0, 1, 2, 3, 4

(D) 0, 2, 4

(E) 1, 2, 3

(F) 1, 2, 3, 4

Quiz 2

For what values of the non-zero real number a does the quadratic equation

$$ax^2 + (a-2)x = 2$$

have distinct real roots?

(A) all values of a

(B) $a = -2$

(C) $a > -2$

(D) $a \neq -2$

(E) no values of a

Quiz 3

Consider the quadratic $f(x) = x^2 - 2px + q$ and the statement:

$(*)$ $\quad f(x) = 0$ has two real roots whose difference is greater than 2 and less than 4.

Which one of the following statements is true **if and only if** $(*)$ is true?

(A) $q < p^2 < q + 4$

(B) $\sqrt{q+1} < p < \sqrt{q+4}$

(C) $q - 3 \leq p^2 - 4 \leq q$

(D) $q < p^2 - 1 < q + 3$

(E) $q - 2 < p^2 - 3 < q + 2$

Quiz 4

The simultaneous equations

$$ax + by = 1$$
$$cx + dy = 0$$

in x and y:

(A) have a solution whatever the values of a, b, c, d may be.

(B) have a unique solution whatever the values of a, b, c, d may be.

(C) have a solution only if $ad \neq bc$.

(D) have a unique solution only if $ad \neq bc$.

Quiz 5

The positive numbers x and y satisfy the equations $x^4 - y^4 = 2009$ and $x^2 + y^2 = 49$. What is the value of y?

(A) 1

(B) 2

(C) 3

(D) 4

(E) more information is needed

Quiz 6

If a, b and c are positive real numbers such that $a(b+c)=152$, $b(c+a)=162$ and $c(a+b)=170$, then abc is

(A) 672

(B) 688

(C) 704

(D) 720

(E) 750

Quiz 7

It is given that the equation $\sqrt{x+p}+\sqrt{x}=p$ has at least one real solution for x, where p is a real constant.

What is the complete set of possible values for p ?

(A) $p=0$ or $p=1$

(B) $p=0$ or $p\geq 1$

(C) $p\geq -x$

(D) $p\geq \sqrt{x}$

(E) $p\geq 0$

(F) $p\geq 1$

Quiz 8

Let a, b and c be real numbers such that

$$a+b+c=2, \text{ and}$$
$$a^2+b^2+c^2=12$$

What is the difference between the maximum and minimum possible values of c?

(A) 2

(B) $\frac{10}{3}$

(C) 4

(D) $\frac{16}{3}$

(E) $\frac{20}{3}$

Quiz 9

How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$
$$\big||x| - |y|\big| = 1$$

(A) 1

(B) 2

(C) 3

(D) 4

(E) 8

Quiz 10

If x, y and z are positive numbers satisfying

$$x + \frac{1}{y} = 4, \qquad y + \frac{1}{z} = 1, \qquad \text{and} \qquad z + \frac{1}{x} = \frac{7}{3},$$

then $xyz =$

(A) $\frac{2}{3}$

(B) 1

(C) $\frac{4}{3}$

(D) 2

(E) $\frac{7}{3}$

Ex. 12

The number of pairs of positive integers x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(A) 0

(B) 2^6

(C) $2^9 - 1$

(D) $2^{10} + 2$

Ex. 13

Find n, the number of all nonnegative integer solutions to the equation

$$xyz + xy + xz + yz + x + y + z = 205.$$

Ex. 14

How many ordered triples (x, y, z) of nonnegative integers are solutions to

$$4x^2 - 4y^2 - 4yz - z^2 = 96?$$

Ex. 15

Find the simultaneous solutions of the three linear equations

$$\begin{aligned} a^2x + ay + z &= a^2 \\ ax + y + bz &= 1 \\ a^2bx + y + bz &= b \end{aligned}$$

for all possible real values of a and b.

Practices P05

Time Allowed

120 min

Number of Questions

18

Difficulty

★★★★

Practices P05V

Scan the QR code or click the link above to take the practice online.

Q1

The sum of the two values of x that satisfy the simultaneous equations $x - 3y + 1 = 0$ and $3x^2 - 7xy = 5$ is

(A) -8.5

(B) -7.5

(C) -1.5

(D) 3.5

(E) 4.5

(F) 5

Q2

Find the complete set of values of the real constant k for which the expression

$$x^2 + kx + 2x + 1 - 2k$$

is positive for all real values of x.

(A) $-12 < k < 0$

(B) $k < -12$ or $k > 0$

(C) $-\sqrt{6} - 3 < k < \sqrt{6} - 3$

(D) $k < -\sqrt{6} - 3$ or $k > \sqrt{6} - 3$

(E) $-2 < k < \frac{1}{2}$

(F) $k < -2$ or $k > \frac{1}{2}$

(G) $0 < k < 4$

(H) $k < 0$ or $k > 4$

Q3

Consider the simultaneous equations

$$3x^2 + 2xy = 4$$
$$x + y = a$$

where a is a real constant.

Find the complete set of values of a for which the equations have two distinct real solutions for x.

(A) There are no values of a

(B) $-2 < a < 2$

(C) $-1 < a < 1$

(D) $a = 0$

(E) $a < -1$ or $a > 1$

(F) $a < -2$ or $a > 2$

(G) All real values of a

Q4

Find the complete set of values of k for which the line $y = x - 2$ crosses or touches the curve $y = x^2 + kx + 2$

(A) $-1 \leq k \leq 3$

(B) $-3 \leq k \leq 5$

(C) $-4 \leq k \leq 4$

(D) $k \leq -1$ or $k \geq 3$

(E) $k \leq -3$ or $k \geq 5$

(F) $k \leq -4$ or $k \geq 4$

Q5

The simultaneous equations

$$ax + by = 1$$
$$cx + dy = 0$$

in x and y:

(A) have a solution whatever the values of a, b, c, d may be.

(B) have a unique solution whatever the values of a, b, c, d may be.

(C) have a solution only if $ad \neq bc$.

(D) have a unique solution only if $ad \neq bc$.

Q6

The equation

$$(2 + x - x^2)^2 = 16$$

has:

(A) no real roots.

(B) one real root.

(C) two real roots.

(D) three real roots.

Q7

The roots of the equation $2x^2 - 11x + c = 0$ differ by 2. The value of c is

(A) $\frac{105}{8}$

(B) $\frac{113}{8}$

(C) $\frac{117}{8}$

(D) $\frac{119}{8}$

Q8

For what values of the non-zero real number a does the quadratic equation $ax^2 + (a - 2)x = 2$ have real distinct roots?

(A) All values of a

(B) $a = -2$

(C) $a > -2$

(D) $a \neq -2$

(E) No values of a

Q9

The simultaneous equations

$$3x^2 - xy = 4$$
$$2x - y = p$$

where p is a real constant, have two distinct real solutions for x.

What is the complete set of values that p can take?

(A) p can take any value

(B) $p < -4,\ p > 4$

(C) $-4 < p < 4$

(D) there are no possible values of p

Q10

The complete set of values of a for which the equation $3x^2 = (a+2)x - 3$ has two real distinct roots is

(A) no values of a

(B) $-4\sqrt{2} < a < 4\sqrt{2}$

(C) $a < -4\sqrt{2}, a > 4\sqrt{2}$

(D) $-4 < a < 8$

(E) $a < -4, a > 8$

(F) $-8 < a < 4$

(G) $a < -8, a > 4$

(H) all values of a

Q11

One of the roots of the equation $2x^2 + 9x - k = 0$, where k is a constant, is 4 more than the other root.

The value of k is

(A) $-\frac{77}{8}$

(B) $-\frac{73}{8}$

(C) $-\frac{65}{8}$

(D) $-\frac{17}{8}$

(E) $\frac{55}{8}$

(F) $\frac{175}{8}$

Q12

Find the sum of the solutions of

$$2\left(\frac{x}{4}+3\right)^2-\left(\frac{x}{4}+3\right)-36=0$$

(A) 2

(B) $\frac{3}{2}$

(C) $\frac{1}{2}$

(D) -4

(E) -13

(F) -22

(G) -26

(H) -34

Q13

The quadratic equation $2x^2-px-4=0$, where p is a positive constant, has two solutions that differ by 6.

What is the value of p?

(A) 2

(B) $4\sqrt{7}$

(C) 12

(D) $4\sqrt{11}$

(E) $4\sqrt{34}$

(F) $6\sqrt{30}$

Q14

p is the greatest solution and q is the least solution of the equation

$$y^4-15y^2+36=0.$$

What is the value of $2p-q$?

(A) $3\sqrt{3}$

(B) $6\sqrt{3}$

(C) $\sqrt{6}$

(D) $3\sqrt{6}$

(E) 6

(F) 21

Q15

Two numbers x and y are such that $x + y = 20$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$. What is the value of $x^2y + xy^2$?

(A) 80

(B) 200

(C) 400

(D) 640

(E) 800

Q16

(i) By setting $y = x + x^{-1}$, find the solutions of

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0.$$

(ii) Solve

$$x^4 + x^3 - 10x^2 - 4x + 16 = 0.$$

Q17

Find all real values of x that satisfy:

(i) $\sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0$;

(ii) $\sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0$;

(iii) $\sqrt{3x^2+1} - 2\sqrt{x} - x + 1 = 0$.

Q18

It is given that x, y, and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \frac{1}{y}$ is either $+1$ or -1.

Supplements S05

Time Allowed

90 min

Number of Questions

12

Difficulty

★★★★

Supplements S05

Scan the QR code or click the link above to take the practice online.

SQ1

The substitution $x = y + t$ transforms the equation $x^3 + ax^2 + bx + c = 0$ into an equation of the form $y^3 + py + q = 0$ when?

(A) $t = \frac{a}{3}$

(B) $t = -\frac{a}{3}$

(C) $t = a$

(D) $t = -a$

SQ2

Let a, b, c and d be real numbers. The two curves $y = ax^2 + c$ and $y = bx^2 + d$ have exactly two points of intersection precisely when

(A) $\frac{a}{b} < 1$

(B) $\frac{a}{b} < \frac{c}{d}$

(C) $a < b$

(D) $c < d$

(E) $(d - c)(a - b) > 0$

SQ3

The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$. What is the mean of x, y and z?

(A) 10

(B) 11

(C) 12

(D) 13

(E) 14

SQ4

There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of these values of a?

(A) -16

(B) -8

(C) 0

(D) 8

(E) 20

SQ5

Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

(A) 0

(B) 1

(C) 2

(D) 3

(E) more than four

SQ6

Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \left(\frac{1}{b}\right)$ and $b + \left(\frac{1}{a}\right)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

(A) $\frac{5}{2}$

(B) $\frac{7}{2}$

(C) 4

(D) $\frac{9}{2}$

(E) 8

SQ7

The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

(A) the coefficient of x^2

(B) the coefficient of x

(C) the y-intercept of the graph of $y = f(x)$

(D) one of the x-intercepts of the graph of $y = f(x)$

(E) the mean of the x-intercepts $f(x)$

SQ8

How many quadratic polynomials with real coefficients are there such that the set of roots equals the set of coefficients? (For clarification: If the polynomial is $ax^2 + bx + c$, $a \neq 0$, and the roots are r and s, then the requirement is that $\{a, b, c\} = \{r, s\}$.)

(A) 3

(B) 4

(C) 5

(D) 6

(E) infinitely many

SQ9

Positive integers a, b and c are chosen so that $< b < c$, and the system of equations

$$2x + y = 2003 \text{ and } y = |x - a| + |x - b| + |x - c|$$

has exactly one solution. What is the minimum value of c?

(A) 668

(B) 669

(C) 1002

(D) 2003

(E) 2004

05 131

SQ10

The numbers x, y and z satisfy the equations $x + y + z = 15$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$. What is the value of $x^2 + y^2 + z^2$?

SQ11

There is a unique real number α that satisfies the equation

$$\alpha^3 + \alpha^2 = 1$$

[You are not asked to prove this.]

(i) Show that $0 < \alpha < 1$.

(ii) Show that

$$\alpha^4 = -1 + \alpha + \alpha^2.$$

(iii) Four functions of α are given in (a) to (d) below. In a similar manner to part (ii), each is equal to a quadratic expression

$$A + B\alpha + C\alpha^2$$

in α; where A, B, C are integers. (So in (ii) we found $A = -1$, $B = 1$, $C = 1$) You may assume in each case that the quadratic expression is unique.

In each case below find the quadratic expression in α.

(a) α^{-1}.

(b) The infinite sum

$$1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots.$$

(c) $(1-\alpha)^{-1}$.

(d) The infinite product

$$(1+\alpha)(1+\alpha^2)(1+\alpha^4)(1+\alpha^8)(1+\alpha^{16})\cdots.$$

SQ12

The first question on an examination paper is:

Solve for x the equation

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}.$$

where (in the question) a and b are given non-zero real numbers. One candidate writes $x = a + b$ as the solution. Show that there are no values of a and b for which this will give the correct answer.

The next question on the examination paper is:

Solve for x the equation

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

where (in the question) a, b and c are given non-zero numbers. The candidate uses the same technique, giving the answer as $x = a + b + c$. Show that the candidate's answer will be correct if and only if a, b and c satisfy at least one of the equations $a + b = 0$, $b + c = 0$ or $c + a = 0$.

06 Polynomials

What's on the Specification?

- Algebraic manipulation of polynomials, including:
    - Expanding brackets and collecting like terms;
    - Factorisation and simple algebraic division (by a linear polynomial, including those of the form $ax + b$, and by quadratics, including those of the form $ax^2 + bx + c$);
    - Use of the Factor Theorem and the Remainder Theorem.

Exercises E06

Time Allowed

No limit

Number of Questions

27

Difficulty

★★★☆

Exercises E06

Scan the QR code or click the link above to take the practice online.

06

134

Quiz Pre-1

When $(3x^2 + 8x - 3)$ is multiplied by $(px - 1)$ and the resulting product is divided by $(x + 1)$, the remainder is 24.

What is the value of p?

(A) -4

(B) 2

(C) 4

(D) $\frac{8}{7}$

(E) $\frac{11}{4}$

Quiz Pre-2

It is given that $x + 2$ is a factor of $x^3 + 4cx^2 + x(c + 1)^2 - 6$.

The sum of the possible values of c is

(A) -10

(B) -6

(C) 0

(D) 6

(E) 10

Quiz Pre-3

The number of solutions of the equation

$$x^3 + ax^2 - x - 2 = 0$$

for which $x > 0$ is

(A) 1

(B) 2

(C) 3

(D) dependent on the value of a

Ex. 1

Find the remainder when $x^{100} + 1$ is divided by $x - 1$.

(A) 100

(B) -100

(C) 10

(D) 2

(E) 2^5

Ex. 2

Find the remainder when $x^{100} - 2x^{51} + 1$ is divided by $x^2 - 1$.

Ex. 3

A polynomial $f(x)$ has remainder c when divided by $x - a$ and remainder d when divided by $x - b$. Find the remainder when $f(x)$ is divided by $(x - a)(x - b)$.

Ex. 4

Find k if $x + 2$ is a factor of $x^3 + kx + 6$.

Ex. 5

Find a and b if $f(x) = x^4 - 5x^3 + 11x^2 + ax + b$ is divisible by $g(x) = (x - 1)^2$.

Ex. 6

The cubic

$$x^3 + ax + b$$

has both $(x - 1)$ and $(x - 2)$ as factors. Then

(A) $a = -7$ and $b = 6$.

(B) $a = -3$ and $b = 2$.

(C) $a = 0$ and $b = -2$.

(D) $a = 5$ and $b = 4$.

Ex. 7

The polynomial

$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor

(A) for no values of n.

(B) for $n = 10$ only.

(C) for $n = 15$ only.

(D) for $n = 10$ and $n = 15$ only.

Ex. 8

Let n be a positive integer. Then $x^2 + 1$ is a factor of

$$(3 + x^4)^n - (x^2 + 3)^n(x^2 - 1)^n$$

for

(A) all n.

(B) even n.

(C) odd n.

(D) $n \geq 3$.

(E) no values of n.

Ex. 9

Let α, β, γ be the solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$.

(i) In terms of a, b, c and d find

(a) $\alpha + \beta + \gamma$

(b) $\beta\gamma + \gamma\alpha + \alpha\beta$

(c) $\alpha\beta\gamma$

(d) $\alpha^2 + \beta^2 + \gamma^2$

(e) $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$

(ii) When $a: b: c: d = 1: 4: 8: 16$, find the cubic equation the solutions of which are α^2, β^2 and γ^2.

Let $\alpha, \beta, \gamma, \delta$ be the solutions of the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$.

(iii) In terms of a, b, c, d and e find similar relationship between the roots and coefficients to part (i).

Ex. 10

For k a positive integer, we define the polynomial $p_k(x)$ as

$$p_k(x) = (1+x)(1+x^2)(1+x^3) \times \cdots \times \left(1+x^k\right) = a_0 + a_1 x + \cdots + a_N x^N,$$

denoting the coefficients of $p_k(x)$ as $a_0, \ldots, a_N$.

(i) Write down the degree N of $p_k(x)$ in terms of k.

(ii) By setting $x = 1$, or otherwise, explain why

$$a_{\max} \geq \frac{2^k}{N+1}$$

where $a_{\max}$ denotes the largest of the coefficients $a_0, \ldots, a_N$.

(iii) Fix $i \geq 0$. Explain why the value of a_i eventually becomes constant as k increases.

A student correctly calculates for $k = 6$ that $p_6(x)$ equals

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 4x^7 + 4x^8 + 5x^9 + 5x^{10} + 5x^{11} + 5x^{12} + 4x^{13} + 4x^{14} + 4x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20} + x^{21}.$$

(iv) On the basis of this calculation, the student guesses that

$$a_i = a_{N-i} \text{ for } 0 \leq i \leq N.$$

By substituting x^{-1} for x, or otherwise, show that the student's guess is correct for all positive integers k.

(v) On the basis of the same calculation, the student guesses that all whole numbers in the range $1, 2, \ldots, a_{\max}$ appear amongst the coefficients $a_0, \ldots, a_N$, for all positive integers k.

Use part (ii) to show that in this case the student's guess is wrong. Justify your answer.

Ex. 11

How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Ex. 12

What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x-2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

(A) $-4 < k < 0$

(B) $-4 < k < 4$

(C) $-4 < k < 16$

(D) $-16 < k < 0$

(E) $-16 < k < 4$

(F) $-16 < k < 16$

Ex. 13

For how many values of a is the equation

$$(x-a)(x^2 - x + a) = 0$$

satisfied by exactly two distinct values of x?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) more than 4

Ex. 14

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d and e are real numbers.

Suppose $f(x) = 1$ has p distinct real solutions, $f(x) = 2$ has q distinct real solutions $f(x) = 3$ has r distinct real solutions, and $f(x) = 4$ has s distinct real solutions.

Which one of the following is **not** possible?

(A) $p = 1, q = 2, r = 4$ and $s = 3$

(B) $p = 1, q = 3, r = 2$ and $s = 4$

(C) $p = 1, q = 4, r = 3$ and $s = 2$

(D) $p = 2, q = 4, r = 3$ and $s = 1$

(E) $p = 4, q = 3, r = 2$ and $s = 1$

Quiz 1

Find the complete set of values of the constant c for which the cubic equation

$$2x^3 - 3x^2 - 12x + c = 0$$

has three distinct real solutions.

(A) $-20 < c < 7$

(B) $-7 < c < 20$

(C) $c > 7$

(D) $c > -7$

(E) $c < 20$

(F) $c < -20$

Quiz 2

When

$$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

Is divided by $x - 1$ the remainder is

(A) 2000

(B) 2500

(C) 3000

(D) 3500

Quiz 3

In this question a and b are real numbers, and a is non-zero.

When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.

The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(A) 1 only

(B) 0 or -2

(C) 1 or 2

(D) 1 or 3

(E) -1 or 2

Quiz 4

$f(x)$ is a polynomial with real coefficients.

The equation $f(x) = 0$ has exactly two real roots, $x = -p$ and $x = p$, where $p > 0$.

Consider the following three statements:

1 $f'(x) = 0$ for exactly one value of x between $-p$ and p.

2 The area between the curve $y = f(x)$, the x-axis and the lines $x = -p$ and $x = p$ is given by $2\int_0^p f(x)\,\mathrm{d}x$.

3 The graph of $y = -f(-x)$ intersects the x-axis at the points $x = -p$ and $x = p$ only.

Which of the above statements **must** be true?

(A) none

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Quiz 5

Which one of the following is a **sufficient** condition for the equation $x^3 - 3x^2 + a = 0$, where a is a constant, to have exactly one real root?

(A) $a > 0$

(B) $a \leq 0$

(C) $a \geq 4$

(D) $a < 4$

(E) $|a| > 4$

(F) $|a| \leq 4$

(G) $a = \frac{9}{4}$

(H) $|a| = \frac{3}{2}$

Quiz 6

The graph of the polynomial

$$P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

has five distinct x-intercepts, one of which is at $(0, 0)$. Which of the following coefficients cannot be zero?

(A) a

(B) b

(C) c

(D) d

(E) e

Quiz 7

Which of the following polynomials has the greatest real root?

(A) $x^{19} + 2018x^{11} + 1$

(B) $x^{17} + 2018x^{11} + 1$

(C) $x^{19} + 2018x^{13} + 1$

(D) $x^{17} + 2018x^{13} + 1$

(E) $2019x + 2018$

Ex. 15

Let

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

where $a_1, a_2, \ldots, a_n$ are given numbers. It is given that $f(x)$ can be written in the form

$$f(x) = (x + k_1)(x + k_2) \cdots (x + k_n).$$

By considering $f(0)$, or otherwise, show that $k_1 k_2 \cdots k_n = a_n$.

Show also that

$$(k_1 + 1)(k_2 + 1) \cdots (k_n + 1) = 1 + a_1 + a_2 + \cdots + a_n$$

and give a corresponding result for $(k_1 - 1)(k_2 - 1) \cdots (k_n - 1)$.

Find the roots of the equation

$$x^4 + 22x^3 + 172x^2 + 552x + 576 = 0,$$

given that they are all integers.

Ex. 16

Sketch the curve

$$f(x) = x^3 + Ax^2 + B$$

first in the case $A > 0$ and $B > 0$, and then in the case $A < 0$ and $B > 0$.

Show that the equation

$$x^3 + ax^2 + b = 0$$

where a and b are real, will have three distinct real roots if

$$27b^2 + 4a^3b < 0,$$

but will have fewer than three if

$$27b^2 + 4a^3b > 0.$$

Ex. 17

For certain real numbers a, b and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(A) -9009

(B) -8008

(C) -7007

(D) -6006

(E) -5005

Practices P06

Time Allowed

40 min

Number of Questions

15

Difficulty

★★★

Practices P06

Scan the QR code or click the link above to take the practice online.

Q1

$(2x+1)$ and $(x-2)$ are factors of $2x^3+px^2+q$

What is the value of $2p+q$?

(A) -10

(B) $-\frac{38}{5}$

(C) $-\frac{22}{3}$

(D) $\frac{22}{3}$

(E) $\frac{38}{5}$

(F) 10

Q2

The expression $3x^3+13x^2+8x+a$, where a is a constant, has $(x+2)$ as a factor.

Which one of the following is a complete factorisation of the expression?

(A) $(x+2)(x-1)(3x-2)$

(B) $(x+2)(x+1)(3x-2)$

(C) $(x+2)(x+1)(3x+2)$

(D) $(x+2)(x-3)(3x+2)$

(E) $(x+2)(x+3)(3x-2)$

(F) $(x+2)(x+3)(3x+2)$

Q3

The polynomial $(x^2-1)(x^2+1)$ is divisible by (that is, has as a factor)

(A) $(x+1)^2$

(B) x^3-x^2+x-1

(C) x^3+x^2-x+1

(D) x^3-1

(E) none of these

Q4

The numbers 10, 11 and -12 are solutions of the cubic equation

(A) $x^3 - 11x^2 - 122x + 1320 = 0$

(B) $x^3 - 9x^2 + 122x - 1320 = 0$

(C) $x^3 - 9x^2 - 142x + 1320 = 0$

(D) $x^3 + 9x^2 - 58x - 1320 = 0$

Q5

The function f is defined by $f(x) = x^3 + ax^2 + bx + c$.

a, b and c take the values 1, 2 and 3 with no two of them being equal and not necessarily in this order.

The remainder when $f(x)$ is divided by $(x + 2)$ is R.

The remainder when $f(x)$ is divided by $(x + 3)$ is S.

What is the largest possible value of $R - S$?

(A) -26

(B) 5

(C) 7

(D) 17

(E) 29

Q6

$ax - 1$ is a factor of $3ax^3 + (6a + 1)x^2 - 4$.

a is a non-zero real number.

What are the possible values of a?

(A) -1 or $-\frac{1}{2}$

(B) $+\frac{1}{2}$ or $+1$

(C) -2 or $+\frac{1}{2}$

(D) $-\frac{1}{2}$ or $+2$

(E) $-\frac{1}{3} - \frac{\sqrt{7}}{3}$ or $-\frac{1}{3} + \frac{\sqrt{7}}{3}$

(F) $+\frac{1}{3} - \frac{\sqrt{7}}{3}$ or $+\frac{1}{3} + \frac{\sqrt{7}}{3}$

Q7

$(x-1)$ and $(x-2)$ are both factors of $x^4 + ax^3 + bx^2 - 12x + 4$.

What are the values of a and b?

(A) $a = -6$ and $b = -23$

(B) $a = -6$ and $b = 13$

(C) $a = 6$ and $b = -11$

(D) $a = 6$ and $b = 1$

Q8

When $x = 2$ is substituted in the expression $x^3 + px^2 + qx + p^2$ the result is 0.

When $x = 1$ is substituted into the same expression, the result is -3.5.

Find all possible value(s) of p.

(A) $p = -1 \pm \frac{\sqrt{6}}{3}$

(B) $p = 1$ or $p = -3$

(C) $p = 1$

(D) $p = 1 \pm \sqrt{7}$

(E) there are no values for p

Q9

How many real roots does the equation $3x^5 - 10x^3 - 120x + 30 = 0$ have?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Q10

How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Q11

The equation

$$x^3 - 300x = 3000$$

has

(A) no real solutions.

(B) exactly one real solution.

(C) exactly two real solutions.

(D) exactly three real solutions.

(E) infinitely many real solutions.

Q12

Given a real constant c, the equation

$$x^4 = (x - c)^2$$

has four real solutions (including possible repeated roots) for

(A) $c \leq \frac{1}{4}$

(B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$

(C) $c \leq -\frac{1}{4}$

(D) all values of c

Q13

It is given that $f(x) = x^3 + 3qx^2 + 2$, where q is a real constant.

The equation $f(x) = 0$ has 3 distinct real roots.

Which of the following statements is/are **necessarily** true?

I The equation $f(x) + 1 = 0$ has 3 distinct real roots.

II The equation $f(x + 1) = 0$ has 3 distinct real roots.

III The equation $f(-x) - 1 = 0$ has 3 distinct real roots.

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Q14

What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x-2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

(A) $-4 < k < 0$

(B) $-4 < k < 4$

(C) $-4 < k < 16$

(D) $-16 < k < 0$

(E) $-16 < k < 4$

(F) $-16 < k < 16$

Q15

The positive real numbers a, b, and c are such that the equation

$$x^3 + ax^2 = bx + c$$

has three real roots, one positive and two negative.

Which one of the following correctly describes the real roots of the equation

$$x^3 + c = ax^2 + bx?$$

(A) It has three real roots, one positive and two negative.

(B) It has three real roots, two positive and one negative.

(C) It has three real roots, but their signs differ depending on a, b, and c.

(D) It has exactly one real root, which is positive.

(E) It has exactly one real root, which is negative.

(F) It has exactly one real root, whose sign differs depending on a, b, and c.

(G) The number of real roots can be one or three, but the number of roots differs depending on a, b, and c.

Supplements S06

Time Allowed

90 min

Number of Questions

12

Difficulty

★★★☆

Supplements S06

Scan the QR code or click the link above to take the practice online.

SQ1

A curve has equation $y = 3x^4 - 4x^3 - 12x^2 + 20$.

What is the complete set of values of the constant k for which the equation

$$3x^4 - 4x^3 - 12x^2 + 20 = k$$

has exactly four distinct real roots?

(A) no values of k

(B) $-12 < k < 15$

(C) $15 < k < 20$

(D) $k > 20$

(E) $7 < k < 20$

(F) all values of k

SQ2

The function f is given by

$$f(x) = 2x^3 + px^2 + qx + 6$$

where p and q are real constants.

When $f(x)$ is divided by $(x + 1)$, the remainder is 12.

When $f(x)$ is divided by $(x - 1)$, the remainder is -6.

Find the remainder when $f(x)$ is divided by $(2x - 1)$.

(A) 0

(B) 4.5

(C) 9

(D) 10.5

(E) 11

SQ3

A cubic polynomial is given by $f(x) = x^3 + bx^2 + cx + d$ where b, c and d are constants.

Two of its factors are $(x - 1)$ and $(x + 1)$

Which of the following statements, taken independently, is/are **necessarily** true?

1 **If** $f(0) = k$ **then** $f(k) = 0$

2 $f(x) = x^3 - x$

3 The graph of $f(x)$ is symmetrical in the y-axis.

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

SQ4

Find the complete set of possible values of the real constant k for which the equation

$$(x-2)^3 - 12(x-2) + k^2 = 0$$

has exactly one real root.

(A) $k < -4$ or $k > 4$

(B) $-4 < k < 4$

(C) $k < 0$ or $k > 4$

(D) $0 < k < 4$

(E) $k < -16$ or $k > 16$

(F) $-16 < k < 16$

(G) $-6 < k < 6$

(H) $k < -6$ or $k > 6$

SQ5

The cubic equation

$$f(x) = p^{\frac{2}{3}}x^3 + px^2 + p^{\frac{1}{3}}x + 3,$$

where p is a positive constant, has exactly one point where $f'(x) = 0$.

What is the value of p?

(A) 0

(B) $\frac{1}{4}$

(C) $\frac{3}{4}$

(D) 1

(E) 3

(F) 6

SQ6

The equation in x

$$3x^4 - 16x^3 + 18x^2 + k = 0$$

has four real solutions

(A) when $-27 < k < 5$

(B) when $5 < k < 27$

(C) when $-27 < k < -5$

(D) when $-5 < k < 0$

SQ7

Let $n \geq 2$ be an integer and $p_n(x)$ be the polynmial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

(A) $\frac{n}{2}$

(B) $\frac{n+1}{2}$

(C) $\frac{n^2+n}{2}$

(D) $-\frac{n}{2}$

SQ8

Given a positive integer n and a real number k, consider the following equation in x,

$$(x-1)(x-2)(x-3)\times\ldots\times(x-n)=k$$

Which of the following statements about this equation is true?

(A) If $n=3$, then the equation has no real solution x for some values of k.

(B) If n is even, then the equation has a real solution x for any given value of k.

(C) If $k\geq 0$ then the equation has (at least) one real solution x.

(D) The equation never has a repeated solution x for any given values of k and n.

SQ9

For some real numbers a and b, the equation

$$8x^3+4ax^2+2bx+a=0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?

(A) -256

(B) -64

(C) -8

(D) 64

(E) 256

SQ10

The polynomial $P(x)=x^3+ax^2+bx+c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y=P(x)$ is 2, what is b?

(A) -11

(B) -10

(C) -9

(D) 1

(E) 5

SQ11

For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(A) -9009

(B) -8008

(C) -7007

(D) -6006

(E) -5005

SQ12

Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0. \qquad (*)$$

(i) Show that if $x = 1$ is a solution of $(*)$ then

$$1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}.$$

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of $(*)$.

(iii) Given that $x = 1$ is a solution, find the value of b for which $(*)$ has a repeated root.

(iv) For this value of b, does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or minimum at its repeated root?

06

156

07 Inequalities

What's on the Specification?

- Solution of linear and quadratic inequalities.

Exercises E07

Time Allowed

No limit

Number of Questions

20

Difficulty

★★★☆

Exercises E07

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

The inequality

$$x^4 < 8x^2 + 9$$

is satisfied precisely when

(A) $-3 < x < 3$

(B) $0 < x < 4$

(C) $1 < x < 3$

(D) $-1 < x < 9$

(E) $-3 < x < -1$

Quiz Pre-2

The inequality

$$\frac{x^2 + 1}{x^2 - 1} < 1$$

is true:

(A) for no values of x.

(B) whenever $-1 < x < 1$.

(C) whenever $x > 1$.

(D) for all values of x.

Quiz Pre-3

x and y satisfy $|2 - x| \leq 6$ and $|y + 2| \leq 4$.

What is the greatest possible value of $|xy|$?

(A) 16

(B) 24

(C) 32

(D) 40

(E) 48

(F) There is no greatest possible value

Ex. 1

Solve the inequalities

(i) $(x+2)(x-3) \geq 0$

(ii) $(x+2)(x+1)x(x-3) \geq 0$

(iii) $(x+2)^3(x+1)x(x-3)^2 \geq 0$

Ex. 2

Solve the inequalities

(i) $\frac{x+2}{x-3} \geq 0$

(ii) $\frac{(x+2)(x+1)}{x(x-3)} \geq 0$

(iii) $\frac{(x+2)^3(x+1)}{x(x-3)^2} < 0$

Ex. 3

Solve $-3 < \frac{1}{x} < 2$.

Ex. 4

Find positive integer m such that $\frac{3x^2+2x+2}{x^2+x+1} > m$ is true for all real values of x.

Ex. 5

Solve $\sqrt{2x+5} > x+1$.

Ex. 6

Find all real numbers x for which $\sqrt{3-x} - \sqrt{x+1} > 1$.

Ex. 7

Solve $|x| - |2x-7| < 1$.

Ex. 8

The graph of $y = |x^2 + 2x - 1|$ lies below the graph of $y = 1$ precisely when

(A) $-1 - \sqrt{3} < x < 1 + \sqrt{3}$

(B) $-1 - \sqrt{3} < x < -2$ or $0 < x < \sqrt{3} - 1$

(C) $-1 - \sqrt{3} < x < -2$ or $0 < x < 1 + \sqrt{3}$

(D) $-2 < x < 0$

(E) none of these

Ex. 9

Positive integers a, b and c are chosen so that $a < b < c$, and the system of equations

$$2x + y = 2003 \text{ and } y = |x - a| + |x - b| + |x - c|$$

has exactly one solution. What is the minimum value of c?

(A) 668

(B) 669

(C) 1002

(D) 2003

(E) 2004

Ex. 10

(i) Sketch the graph of $x+y=1$ in the x-y plane.

Sketch the following subsets of the x-y plane:

(ii) $|x| + |y| \leq 1$.

(iii) $|x - 1| + |y - 1| \leq 1$.

(iv) $|x - 1| - |y + 1| \leq 1$.

(v) $|x||y - 2| \leq 1$.

Quiz 1

If x and n are integers then

$$(1-x)^n(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n}$$

is

(A) negative when $n > 5$ and $x < 5$.

(B) negative when n is odd and $x > 5$.

(C) negative when n is a multiple of 3 and $x > 5$.

(D) negative when n is even and $x < 5$.

Quiz 2

A region R in the (x, y)-plane is defined by the simultaneous inequalities

$$y - x < 3$$

$$y - x^2 < 1$$

Which of the following statements is/are true for **every** point in R?

I $\quad -1 < x < 2$

II $\quad (y-x)(y-x^2) < 3$

III $\quad y < 5$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Quiz 3

What is the minimum value of $f(x) = |x-1| + |2x-1| + |3x-1| + \cdots + |119x-1|$?

(A) 49

(B) 50

(C) 51

(D) 52

(E) 53

Quiz 4

What is the area of the polygon formed by all points (x, y) in the plane satisfying the inequality $\big||x|-2\big|+\big||y|-2\big| \le 4$?

(A) 24

(B) 32

(C) 64

(D) 96

(E) 112

Ex. 11

(i) Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \ge 0.$$

(ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

(iii) On a sketch shade the regions of the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 \ge 0.$$

Ex. 12

(i) Prove that $x^2 + y^2 + z^2 \ge xy + yz + zx$. What can you say about x, y and z when equality holds in this equation?

[**Hint:** you might like to start by multiplying both sides by 2.]

(ii) Write out the following, giving careful explanations of each step (a typical explanation might be 'using $(*)$ with $a = p$ and $b = q$'):

$$\frac{p+q+r+s}{4} \ge \frac{\sqrt{pq}+\sqrt{rs}}{2} \ge \sqrt[4]{pqrs}$$

[Hence AM $\ge$ GM for four numbers.]

(iii) Write out the following, giving careful explanations of each step (you will need to use the result of part (ii) above):

$$\frac{p+q+r}{3} = \frac{p+q+r}{4} + \frac{p+q+r}{12} \ge \sqrt[4]{pqr\left(\frac{p+q+r}{3}\right)}.$$

Deduce that

$$\frac{p+q+r}{3} \ge \sqrt[3]{pqr}.$$

[**Hint:** start by splitting the first fraction into two parts and note that if $A \ge C$ and $B \ge D$ then $A + B \ge C + D$.]

Ex. 13

Here is the set-up for this question. You should justify your answers as briefly as possible.

We have an island with 100 islanders living on it.

Islanders have either blue eyes or brown eyes.

Islanders meet at 9:00 each morning and look into each others' eyes.

Islanders never discuss eye-colour. There are no mirrors or cameras on the island.

Any islander who finds out that he or she has blue eyes must leave the island by the 12:00 boat.

We join the islanders, in spirit anyway, on day one when a ghostly voice at the start of the 9:00 meeting says *'At least one of you has blue eyes'*.

(i) Suppose that no islander leaves the island on the 12:00 boat on day one. What can you conclude?

(ii) Suppose that one islander leaves the island on the 12:00 boat on day one. What can you conclude?

(iii) Suppose instead that no one leaves on the first day, but two islanders leave on the 12:00 boat on the second day. What can you conclude?

(iv) Suppose instead that we don't know if anyone leaves on the first day, but we do know that two islanders leave on the 12:00 boat on the second day. What can you conclude?

(v) Suppose that n islanders leave on the nth day. What can you conclude?

[It might be helpful to do part (ii) before part (i).]

Practices P07

Time Allowed

60 min

Number of Questions

17

Difficulty

★★★

Practices P07

Scan the QR code or click the link above to take the practice online.

Q1

Find the complete set of values of x for which

$$(x+4)(x+3)(1-x) > 0 \text{ and } (x+2)(x-2) < 0$$

(A) $1 < x < 2$

(B) $-2 < x < 1$

(C) $-2 < x < 2$

(D) $x < -2$ or $x > 1$

(E) $x < -4$ or $x > 2$

(F) $x < -4$ or $-3 < x < 1$

(G) $-4 < x < -2$ or $x > 1$

Q2

The complete set of values of x for which $(x^2-1)(x-2) > 0$ is

(A) $x < -1, 1 < x < 2$

(B) $x < -1, x > 2$

(C) $-1 < x < 2$

(D) $x < 1, x > 2$

(E) $-1 < x < 1, x > 2$

Q3

S is the complete set of values of x which satisfy **both** the inequalities

$$x^2 - 8x + 12 < 0 \quad \text{and} \quad 2x + 1 > 9.$$

The set S can also be represented as a single inequality.

Which one of the following single inequalities represents the set S ?

(A) $(x^2 - 8x + 12)(2x + 1) < 0$

(B) $(x^2 - 8x + 12)(2x + 1) > 0$

(C) $x^2 - 10x + 24 < 0$

(D) $x^2 - 10x + 24 > 0$

(E) $x^2 - 6x + 8 < 0$

(F) $x^2 - 6x + 8 > 0$

(G) $x < 2$

(H) $x > 6$

Q4

For which real numbers x does the inequality

$$\frac{x}{x^2+1} \leq \frac{1}{2}$$

hold?

(A) for all real numbers x

(B) for real numbers $x \leq \frac{1}{2}$ and no others

(C) for real numbers $x \leq 1$ and no others

(D) none of the above

Q5

Find the complete set of values of x for which $x - \frac{3}{2} > \frac{1}{x}$

(A) $x < -\frac{1}{2}, x > 2$

(B) $x < -\frac{1}{2}, 0 < x < 2$

(C) $x < -2, 0 < x < \frac{1}{2}$

(D) $x < -2, x > \frac{1}{2}$

(E) $-\frac{1}{2} < x < 0, x > 2$

(F) $-2 < x < 0, x > \frac{1}{2}$

Q6

Find the complete set of values of x for which

$$1 + \frac{x^2+9x+9}{x} > 0$$

(A) $x < -1$ or $x > 0$

(B) $-1 < x < 0$

(C) $x < -9$ or $x > -1$

(D) $-9 < x < -1$

(E) $x < -9$ or $-1 < x < 0$

(F) $-9 < x < -1$ or $x > 0$

Q7

Find the complete set of values of x for which

$$x^3 - 2x^2 - 7x - 4 > 0$$

(A) $x < -1$

(B) $x > -1$

(C) $-1 < x < 4$

(D) $x < -1$ or $x > 4$

(E) $x < 4$

(F) $x > 4$

Q8

The complete set of values of x for which $2x^4 - 9x^2 + 4 > 0$ is

(A) $x < \frac{1}{2}, x > 4$

(B) $\frac{1}{2} < x < 4$

(C) $x < -2, -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, x > 2$

(D) $-2 < x < -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} < x < 2$

(E) $-2 < x < 2$

Q9

Find the complete set of values of x for which

$$\frac{x^3 - 6x^2 + 9x - 4}{x} > 0$$

(A) $x < 0, x > 4$

(B) $0 < x < 4$

(C) $0 < x < 1, x > 4$

(D) $x < 0, 1 < x < 4$

(E) $x < 1, x > 4$

(F) $1 < x < 4$

Q10

Find the set of values of x such that both $x^2 + x - 6 \geq 0$ and $4 + 3x - x^2 \leq 0$

(A) $2 \leq x \leq 4$

(B) $-3 \leq x \leq -1$

(C) $x \leq -3$ or $-1 \leq x \leq 4$

(D) $-3 \leq x \leq -1$ or $2 \leq x \leq 4$

(E) $x \leq -1$ or $x \geq 2$

(F) $x \leq -3$ or $x \geq 4$

Q11

The set of solutions to the inequality $x^2 + bx + c < 0$ is the interval $p < x < q$, where b, c, p and q are real constants with $c < 0$.

In terms of p, q and c, what is the set of solutions to the inequality $x^2 + bcx + c^3 < 0$?

(A) $\frac{p}{c} < x < \frac{q}{c}$

(B) $\frac{q}{c} < x < \frac{p}{c}$

(C) $pc < x < qc$

(D) $qc < x < pc$

(E) $pc^2 < x < qc^2$

(F) $qc^2 < x < pc^2$

Q12

Consider the following inequality:

$$(*): \ a|x| + 1 \leq |x - 2|$$

where a is a real constant.

Which one of the following describes the complete set of values of a such that $(*)$ is **true for all real** x?

(A) $a \leq \frac{3}{2}$

(B) $a \leq 1$

(C) $a \leq \frac{1}{2}$

(D) $a \leq 0$

(E) $a \leq -\frac{1}{2}$

(F) $a \leq -1$

(G) $a \leq -\frac{3}{2}$

(H) There are no such values of a.

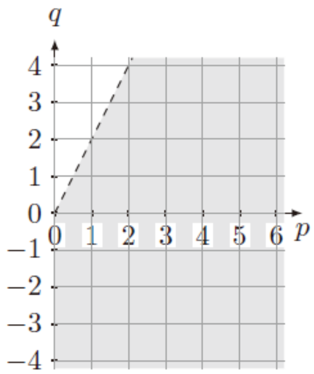
Q13

The graph of the quadratic

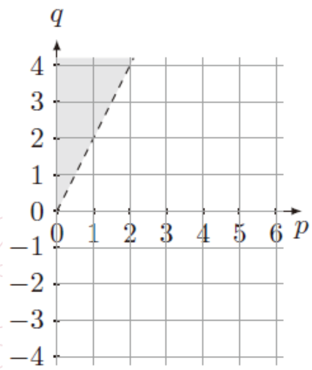
$$y = px^2 + qx + p$$

where $p > 0$, intersects the x-axis at two distinct points.

In which one of the following graphs does the shaded region show the complete set of possible values that p and q could take?



(A)



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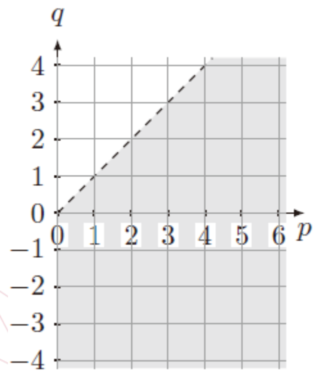

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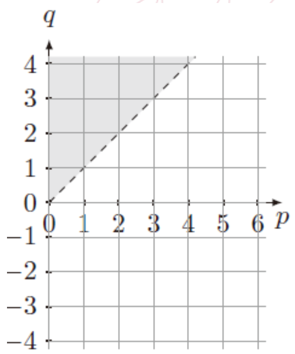

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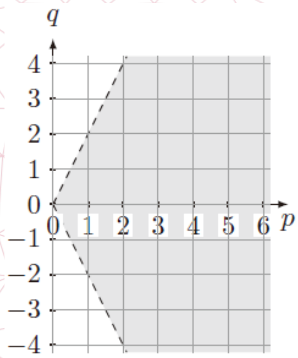

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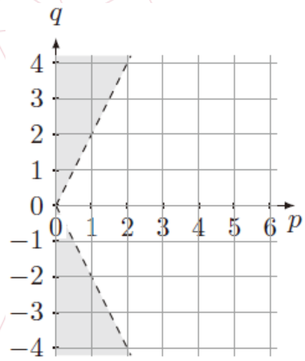

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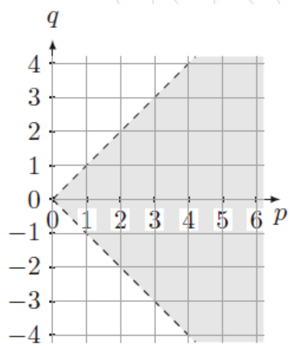
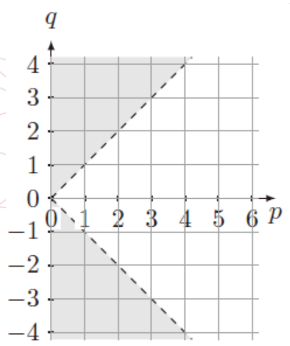


(G)



(H)

Q14

Find the complete set of values of x for which

$$x^4 + 36 < 13x^2$$

(A) $4 < x < 9$

(B) $x < 4,\ x > 9$

(C) $-9 < x < -4,\ 4 < x < 9$

(D) $x < -9, -4 < x < 4,\ x > 9$

(E) $2 < x < 3$

(F) $x < 2,\ x > 3$

(G) $-3 < x < -2,\ \ 2 < x < 3$

(H) $x < -3, -2 < x < 2,\ x > 3$

Q15

The numbers x and y satisfy the following inequalities

$$\begin{aligned} 2x + 3y &\leq 23, \\ x + 2 &\leq 3y, \\ 3y + 1 &\leq 4x. \end{aligned}$$

The largest possible value of x is

(A) 6

(B) 7

(C) 8

(D) 9

Q16

The inequalities $x^2 + 3x + 2 > 0$ and $x^2 + x < 2$, are met by all x in the region:

(A) $x < -2$

(B) $-1 < x < 1$

(C) $x > -1$

(D) $x > -2$

Q17

Show that

$$x^2 - y^2 + x + 3y - 2 = (x - y + 2)(x + y - 1)$$

and hence, or otherwise, indicate by means of a sketch the region of the x-y plane for which

$$x^2 - y^2 + x + 3y > 2.$$

Sketch also the region of the x-y plane for which

$$x^2 - 4y^2 + 3x - 2y < -2.$$

Give the coordinates of a point for which both inequalities are satisfied or explain why no such point exists.

Supplements S07

Time Allowed

90 min

Number of Questions

13

Difficulty

★★★☆

Supplements S07

Scan the QR code or click the link above to take the practice online.

SQ1

Solve fully the inequality

$$2x^2 \geq 15 - x$$

(A) $x \leq -3$

(B) $x \geq 2.5$

(C) $x \leq -1.5, x \geq 5$

(D) $-1.5 \leq x \leq 5$

(E) $x \leq -3, x \geq 2.5$

(F) $-3 \leq x \leq 2.5$

SQ2

Which of the expressions below has the largest value for $0 < x < 1$?

(A) $\frac{1}{x}$

(B) x^2

(C) $\frac{1}{1+x}$

(D) $\frac{1}{\sqrt{x}}$

(E) $\sqrt{x}$

SQ3

How many different integers, n, are there such that the difference between $2\sqrt{n}$ and 7 is less than 1?

(A) 0

(B) 2

(C) 4

(D) 6

(E) 8

SQ4

Let a and b be positive integers such that $a + b = 20$. What is the maximum value that a^2b can take?

(A) 1000

(B) 1152

(C) 1176

(D) 1183

(E) 1196

SQ5

What is the sum of all integer solutions to $1 < (x-2)^2 < 25$?

(A) 10

(B) 12

(C) 15

(D) 19

(E) 25

SQ6

What is the least possible value of $(xy-1)^2 + (x+y)^2$ for real numbers x and y?

(A) 0

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 1

(E) 2

SQ7

The length of the interval of solutions of the inequality $a \leq 2x + 3 \leq b$ is 10. What is $b - a$?

(A) 6

(B) 10

(C) 15

(D) 20

(E) 30

SQ8

What is the area of the region defined by the inequality $|3x - 18| + |2y + 7| \le 3$?

(A) 3

(B) $\frac{7}{2}$

(C) 4

(D) $\frac{9}{2}$

(E) 5

SQ9

What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4) + 2019$$

where x is a real number?

(A) 2017

(B) 2018

(C) 2019

(D) 2020

(E) 2021

SQ10

The set of real numbers x for which

$$\frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011} \ge 1$$

is the union of intervals of the form $a < x \le b$. What is the sum of the lengths of these intervals?

(A) $\frac{1003}{335}$

(B) $\frac{1004}{335}$

(C) 3

(D) $\frac{403}{134}$

(E) $\frac{202}{67}$

SQ11

Suppose that $|x+y|+|x-y|=2$. What is the maximum possible value of x^2-6x+y^2?

(A) 5

(B) 6

(C) 7

(D) 8

(E) 9

SQ12

Prove that, if $|\alpha|<2\sqrt{2}$, then there is no value of x for which

$$x^2-\alpha|x|+2<0 \qquad (*)$$

Find the solution set of $(*)$ for $\alpha=3$.

For $\alpha>2\sqrt{2}$, the sum of the lengths of the intervals in which x satisfies $(*)$ is denoted by S. Find S in terms of α and deduce that $S<2\alpha$.

Sketch the graph of S against α.

SQ13

(i) The numbers m and n satisfy

$$m^3=n^3+n^2+1. \qquad (*)$$

(a) Show that $m>n$. Show also that $m<n+1$ if and only if $2n^2+3n>0$. Deduce that $n<m<n+1$ unless $-\frac{3}{2}\le n\le 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n)=(1,0)$ and $(m,n)=(1,-1)$.

(ii) Find all integer solutions of the equation

$$p^3=q^3+2q^2-1.$$

07

178

08 Functions and Their Graphs

What's on the Specification?

- Quadratic functions and their graphs; the discriminant of a quadratic function; completing the square; solution of quadratic equations.
- Qualitative understanding that a function is a many-to-one (or sometimes just a one-to-one) mapping. Familiarity with the properties of common functions, including $f(x) = \sqrt{x}$ (which always means the 'positive square root') and $f(x) = |x|$.
- Recognise and be able to sketch the graphs of common functions that appear in this specification: these include lines, quadratics, cubics, trigonometric functions, logarithmic functions, exponential functions, square roots, and the modulus function.
- Knowledge of the effect of simple transformations on the graph of $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$, with the value of a positive or negative. Compositions of these transformations.
- Understand how altering the values of a, b and c in $y = a(x + b)^2 + c$ affects the corresponding graph.
- Use differentiation to help determine the shape of the graph of a given function; this might include finding stationary points (excluding inflexions) as well as finding when the function is increasing or decreasing.
- Use algebraic techniques to determine where the graph of a function intersects the coordinate axes; appreciate the possible numbers of real roots a general polynomial can possess.
- Geometric interpretation of algebraic solutions of equations; relationship between the intersections of two graphs and the solutions of the corresponding simultaneous equations.

Exercises E08

Time Allowed

No limit

Number of Questions

24

Difficulty

★★★

Exercises E08

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

The curve S has equation

$$y = px^2 + 6x - q$$

where p and q are constants.

S has a line of symmetry at $x = -\frac{1}{4}$ and touches the x-axis at exactly one point.

What is the value of $p + 8q$?

(A) 6

(B) 18

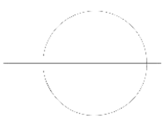
(C) 21

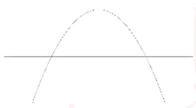
(D) 25

(E) 38

Quiz Pre-2

Which of the following could be part of the graph of the curve $y^2 = x(2 - x)$?


(A)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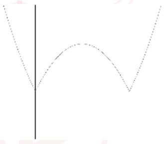
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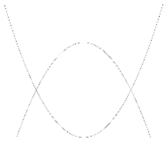
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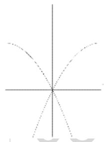

(E)

Quiz Pre-3

The following sequence of transformations is applied to the curve $y = 4x^2$.

1. Translation by $\binom{3}{-5}$
2. Reflection in the x-axis
3. Stretch parallel to the x-axis with scale factor 2

What is the equation of the resulting curve?

(A) $y = -x^2 + 12x - 31$

(B) $y = -x^2 + 12x - 41$

(C) $y = x^2 + 12x + 31$

(D) $y = x^2 + 12x + 41$

(E) $y = -16x^2 + 48x - 31$

(F) $y = -16x^2 + 48x - 41$

(G) $y = 16x^2 - 48x + 31$

(H) $y = 16x^2 - 48x + 41$

Ex. 1

Find a if the following quadratic function is negative for all x.

$$f(x) = ax^2 + (a-1)x + (a-1)$$

Ex. 2

The equation

$$(x^2+1)^{10} = 2x - x^2 - 2$$

(A) has $x = 2$ as a solution.

(B) has no real solutions.

(C) has an odd number of real solutions.

(D) has twenty real solutions.

Ex. 3

The curve with equation

$$x^{17} + x^3 + y^4 + y^{12} = 2$$

has

(A) neither the x-axis nor y-axis as a line of symmetry.

(B) the x-axis but not the y-axis as a line of symmetry.

(C) the y-axis but not the x-axis as a line of symmetry.

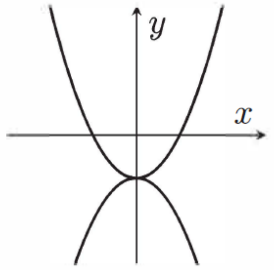
(D) both axes as lines of symmetry.

Ex. 4

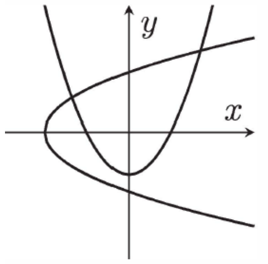
A sketch of the curve

$$(x^8 + 4yx^6 + 6y^2x^4 + 4y^3x^2 + y^4)^2 = 1$$

is given below in



(A)



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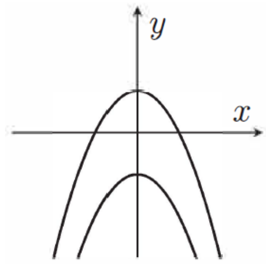

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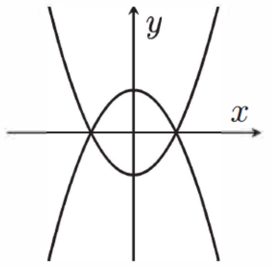

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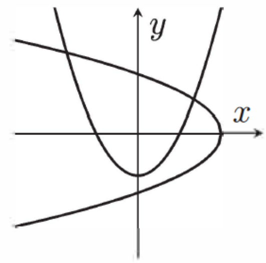


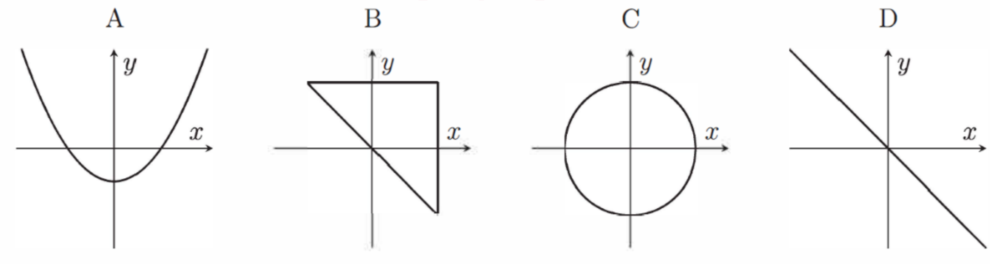
(E)

Ex. 5

Which of the following could be the sketch of a curve

$$p(x) + p(y) = 0$$

for some polynomial p?



(A) A and D, but not B or C

(B) A and B, but not C or D

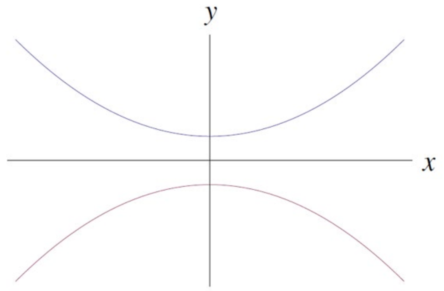
(C) C and D, but not A or B

(D) A, C and D, but not B

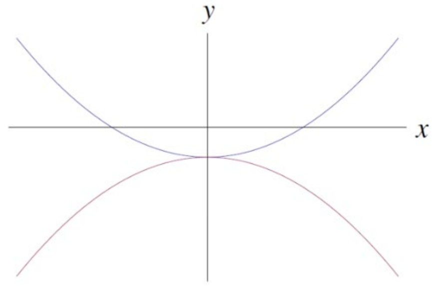
(E) A, B and C, but not D

Ex. 6

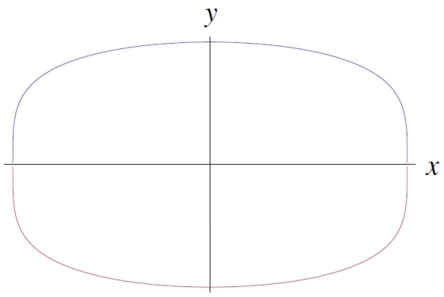
Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



(A)



(B)



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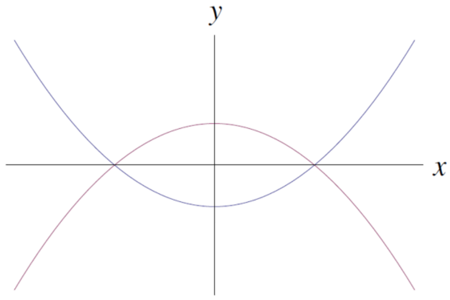


(D)

Ex. 7

The function f satisfies $f(0) = 1$ and

$$f(x - y) = f(x)f(y) - f(a - x)f(a + y)$$

for some fixed number a and all x and y. Without making any further assumptions about the nature of the function show that $f(a) = 0$.

Show that, for all t,

(i) $f(t) = f(-t)$.

(ii) $f(2a) = -1$.

(iii) $f(2a - t) = -f(t)$.

(iv) $f(4a + t) = f(t)$.

Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = \frac{\pi}{2}$. Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = -2$.

Ex. 8

The graph of the function

$$y = 2^{x^2-4x+3}$$

can be obtained from the graph of $y = 2^{x^2}$ by

(A) a stretch parallel to the y-axis followed by a translation parallel to the y-axis.

(B) a translation parallel to the x-axis followed by a stretch parallel to the y-axis.

(C) a translation parallel to the x-axis followed by a stretch parallel to the x-axis.

(D) a translation parallel to the x-axis followed by reflection in the y-axis.

(E) reflection in the y-axis followed by translation parallel to the y-axis.

Ex. 9

If $f(x) = x^2 - 5x + 7$, what are the coordinates of the minimum of $y = f(x-2)$?

(A) $\left(\frac{5}{2}, \frac{3}{4}\right)$

(B) $\left(\frac{9}{2}, \frac{3}{4}\right)$

(C) $\left(\frac{1}{2}, \frac{3}{4}\right)$

(D) $\left(\frac{9}{2}, \frac{-5}{4}\right)$

(E) $\left(\frac{5}{2}, \frac{-5}{4}\right)$

Ex. 10

Find the maximum value of the gradient of the curve with equation

$$y = 2 - 4x + 4x^{\frac{3}{2}} - x^2$$

where $x > 0$.

(A) -4

(B) $-\frac{8}{9}$

(C) $\frac{1}{2}$

(D) 2

(E) 4

Ex. 11

The curve

$$y = x^3 + px^2 + qx + r$$

has a local maximum when $x = -1$ and a local minimum when $x = 3$

What is the value of p?

(A) -9

(B) -3

(C) -1

(D) 1

(E) 3

(F) 9

Ex. 12

The graph of the polynomial function

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

is sketched, where a, b, c, d, e, and f are real constants with $a \neq 0$.

Which one of the following is **not** possible?

(A) The graph has two local minima and two local maxima.

(B) The graph has one local minimum and two local maxima.

(C) The graph has one local minimum and one local maximum.

(D) The graph has no local minima or local maxima.

Ex. 13

$f(x) = x^3 - a^2x$ where a is a positive constant.

Find the complete set of values of x for which $f(x)$ is an increasing function.

(A) $x \leq -a, x \geq a$

(B) $-a \leq x \leq a$

(C) $x \leq -a, 0 \leq x \leq a$

(D) $-a \leq x \leq 0, x \geq a$

(E) $x \leq -\frac{a}{3}, x \geq \frac{a}{3}$

(F) $-\frac{a}{3} \leq a \leq \frac{a}{3}$

(G) $x \leq -\frac{a}{\sqrt{3}}, x \geq \frac{a}{\sqrt{3}}$

(H) $-\frac{a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$

Quiz 1

Which of these describes the graph of $x^2(x+y+1) = y^2(x+y+1)$?

(A) two parallel lines

(B) two intersecting lines

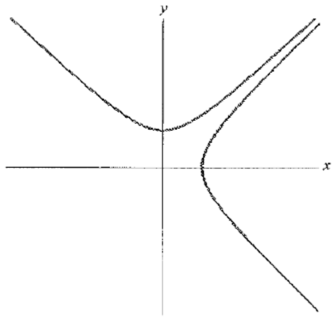
(C) three lines that all pass through a common point

(D) three lines that do not all pass through a common point

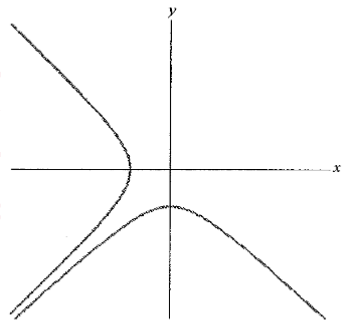
(E) a line and a parabola

Quiz 2

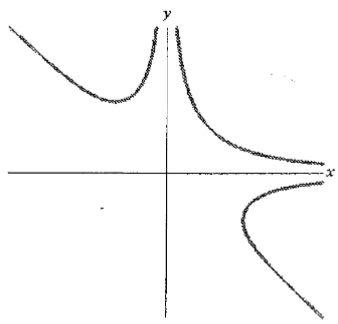
A sketch of the curve with equation $x^2y^2(x+y) = 1$ is drawn in



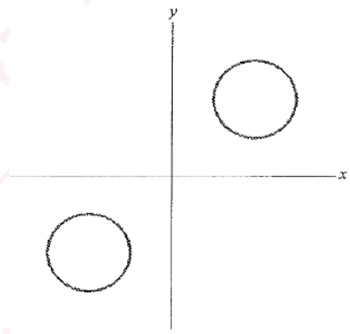
(A)



(B)



(C)



(D)

Quiz 3

The curve C has equation $y = x^2 + bx + 2$, where $b \geq 0$.

Find the value of b that minimises the distance between the origin and the stationary point of the curve C.

(A) $b = 0$

(B) $b = 1$

(C) $b = 2$

(D) $b = \frac{\sqrt{6}}{2}$

(E) $b = \sqrt{2}$

(F) $b = \sqrt{6}$

Quiz 4

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d and e are real numbers.

Suppose $f(x) = 1$ has p distinct real solutions, $f(x) = 2$ has q distinct real solutions $f(x) = 3$ has r distinct real solutions, and $f(x) = 4$ has s distinct real solutions.

Which one of the following is **not** possible?

(A) $p = 1, q = 2, r = 4$ and $s = 3$

(B) $p = 1, q = 3, r = 2$ and $s = 4$

(C) $p = 1, q = 4, r = 3$ and $s = 2$

(D) $p = 2, q = 4, r = 3$ and $s = 1$

(E) $p = 4, q = 3, r = 2$ and $s = 1$

Quiz 5

The graph of the function $y = x^3 + px^2 + qx + 6$, where p and q are real constants, has a local maximum when $x = 2$ and a local minimum when $x = 4$. What are the values of p and q?

(A) $p = -3$ and $q = -8$

(B) $p = -3$ and $q = 8$

(C) $p = 3$ and $q = -8$

(D) $p = -9$ and $q = 24$

(E) $p = 9$ and $q = 24$

(F) $p = 9$ and $q = -24$

Quiz 6

A sequence of translations is applied to the graph of $y = x^3$.

Which of the following graphs could be the result of this sequence of translations?

I $\quad y = x^3 - 3x^2 + 9x - 27$

II $\quad y = x^3 - 9x^2 + 27x - 3$

III $\quad y = 27x^3 - 9x^2 + x - 3$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Quiz 7

The function $f(x)$ is defined for all real numbers.

Consider the following three conditions, where a is a real constant:

I $\quad f(a - x) = f(a + x)$ for all real x.

II $\quad f(2a - x) = f(x)$ for all real x.

III $\quad f(a - x) = f(x)$ for all real x.

Which of these conditions is/are **necessary and sufficient** for the graph of $y = f(x)$ to have reflection symmetry in the line $x = a$?

	Condition I is necessary and sufficient	**Condition II is necessary and sufficient**	**Condition III is necessary and sufficient**
(A)	yes	yes	yes
(B)	yes	yes	no
(C)	yes	no	yes
(D)	yes	no	no
(E)	no	yes	yes
(F)	no	yes	no
(G)	no	no	yes
(H)	no	no	no

Ex. 14

In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $f(x) = \frac{\lfloor x \rfloor}{x}$.

(i) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $f(x) = \frac{7}{12}$.
Solve also the equations $f(x) = \frac{17}{24}$ and $f(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $f(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $f(x) = c$ to have exactly n roots, where $n \geq 1$.

Practices P08

Time Allowed

90 min

Number of Questions

16

Difficulty

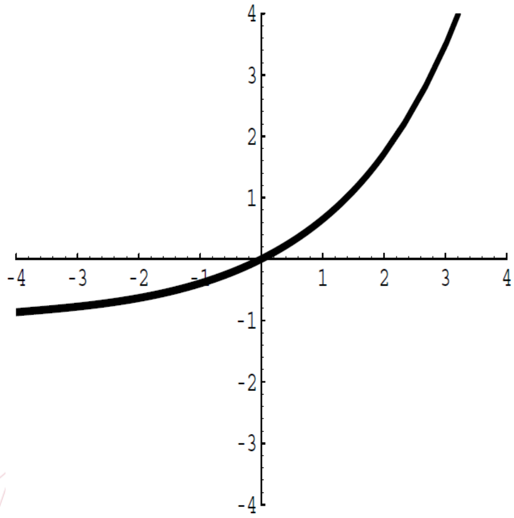
★★★☆

Practices P08

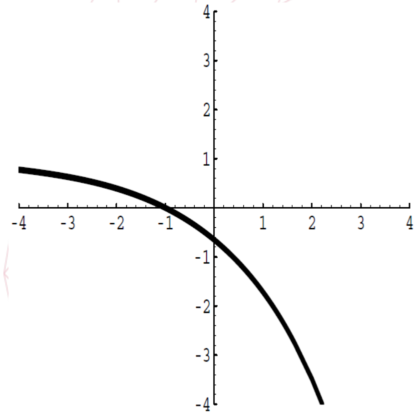
Scan the QR code or click the link above to take the practice online.

Q1

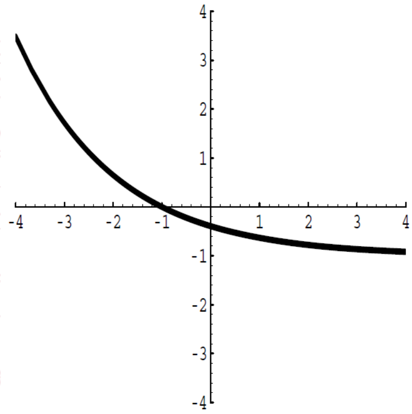
The diagram below shows the graph of the function $y = f(x)$.



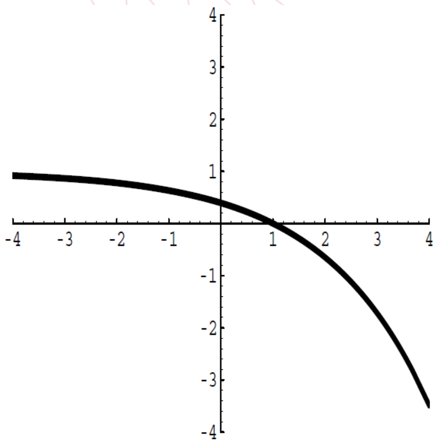
The graph of the function $y = -f(x+1)$ is drawn in which of the following diagrams?



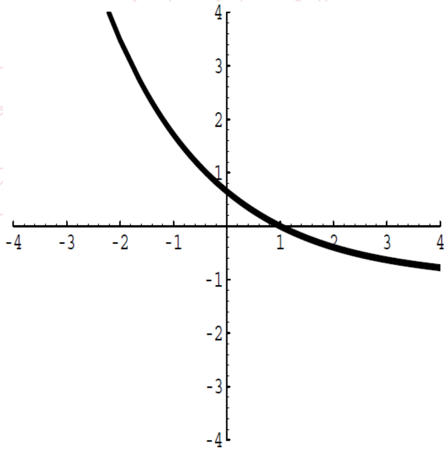
(A)



(B)



(C)



(D)

Q2

The vertices of a rectangle have coordinates:

$$P(4,5) \quad Q(4,8) \quad R(10,8) \quad S(10,5)$$

$PQRS$ is transformed by a clockwise rotation of $90°$ about P followed by a reflection in the x-axis.

What are the coordinates of the final position of R?

(A) $(-8,-10)$

(B) $(-7,-1)$

(C) $(-4,1)$

(D) $(-1,11)$

(E) $(1,-11)$

(F) $(4,-1)$

(G) $(7,1)$

(H) $(8,10)$

Q3

Find the area enclosed by the graph of

$$|x|+|y|=1$$

(A) $\frac{1}{2}$

(B) 1

(C) 2

(D) 4

(E) $\frac{1}{2}\sqrt{2}$

(F) $\sqrt{2}$

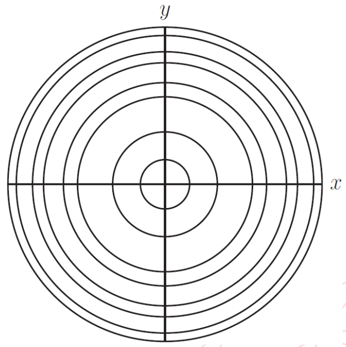
(G) $2\sqrt{2}$

Q4

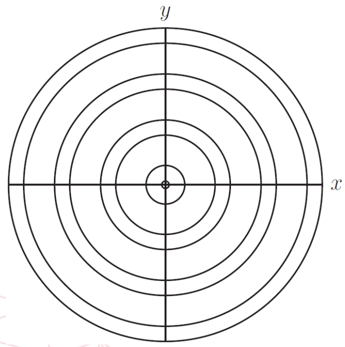
Which of the following sketches shows the graph of

$$\sin(x^2 + y^2) = \frac{1}{2}$$

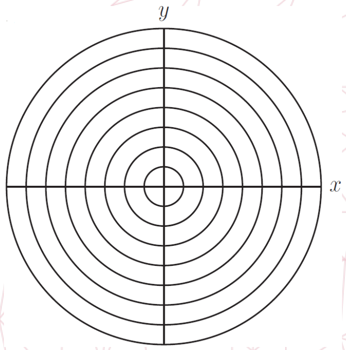
where $x^2 + y^2 \leq 8\pi$?



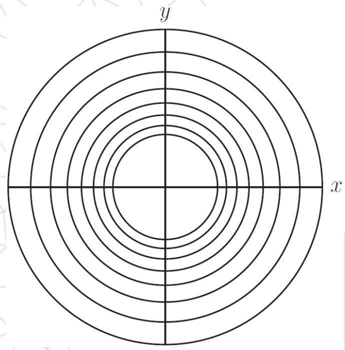
(A)



(B)



(C)



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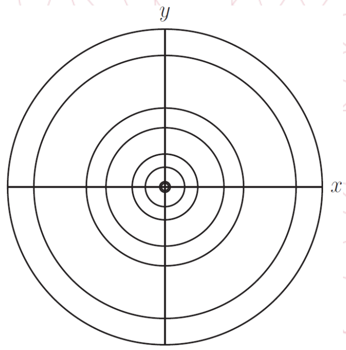


(E)

Q5

What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x-2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

(A) $-4 < k < 0$

(B) $-4 < k < 4$

(C) $-4 < k < 16$

(D) $-16 < k < 0$

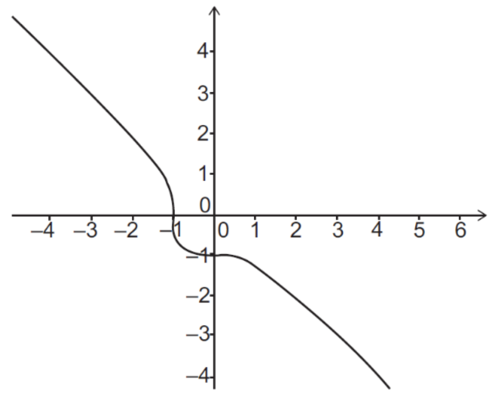
(E) $-16 < k < 4$

(F) $-16 < k < 16$

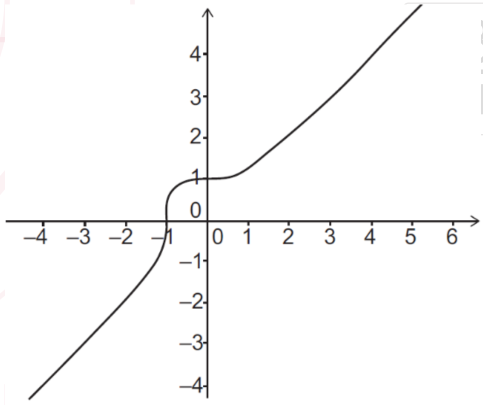
Q6

Which one of the following is a sketch of the graph

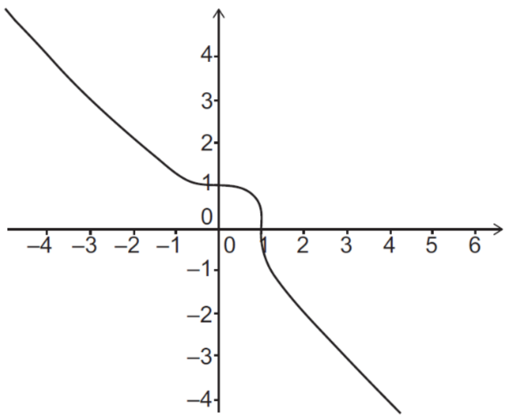
$$(x+y)(x^2 - xy + y^2) = 1?$$



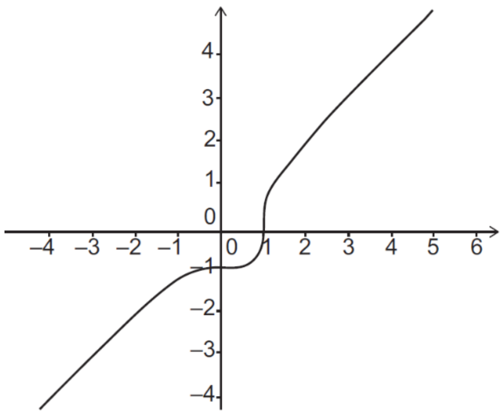
(A)



(B)



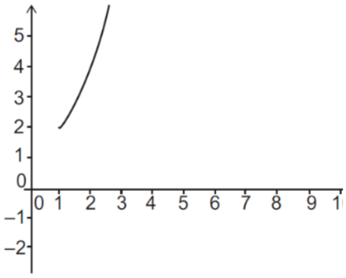
(C)



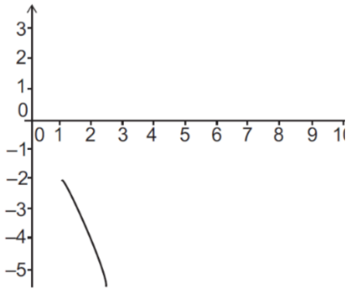
(D)

Q7

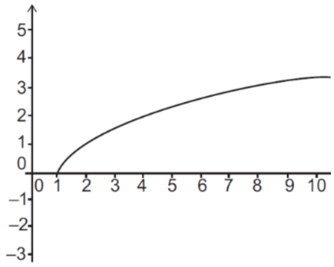
Which one of the following is a sketch of the graph of $y = \log_x 2$ for $x > 1$?



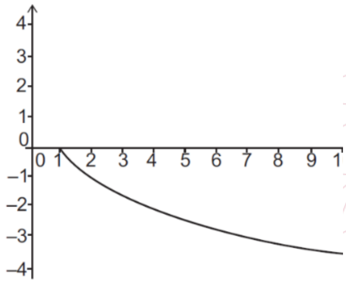
(A)



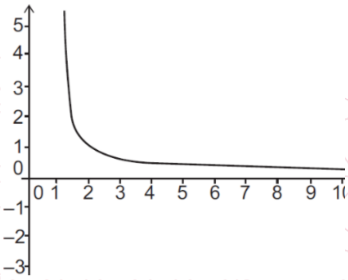
(B)



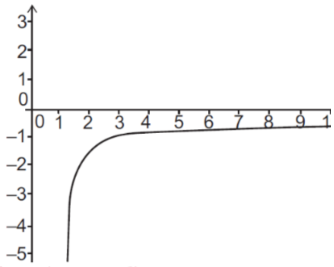
(C)



(D)



(E)



(F)

Q8

The graph of $y = f(x)$ intersects the x-axis at exactly two distinct points.

Consider the graphs of the following:

$$y = f(x) + 2$$
$$y = f(x + 2)$$
$$y = 2f(x)$$
$$y = 2 - f(x)$$
$$y = f(-2x)$$

How many of these graphs **necessarily** intersect the x-axis at exactly two distinct points?

(A) 0

(B) 1

(C) 2

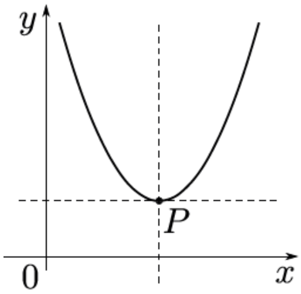
(D) 3

(E) 4

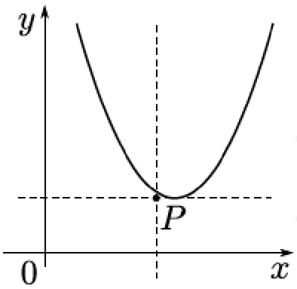
(F) 5

Q9

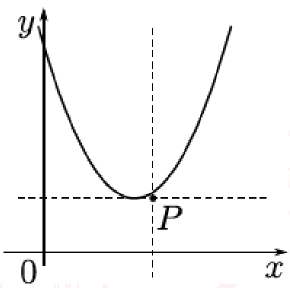
The diagram below shows the graph of $y = x^2 - 2bx + c$. The vertex of this graph is at the point P.



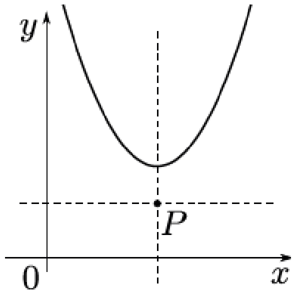
Which one of the following could be the graph of $y = x^2 - 2Bx + c$, where $B > b$?



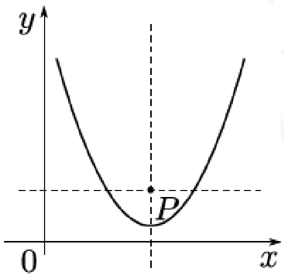
(A)



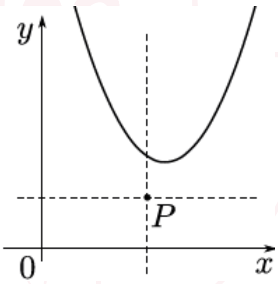
(B)



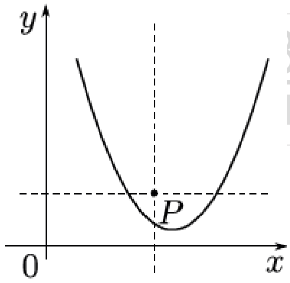
(C)



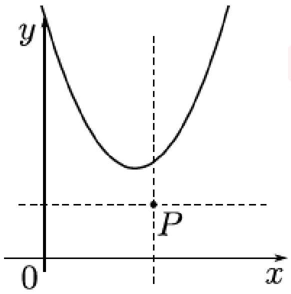
(D)



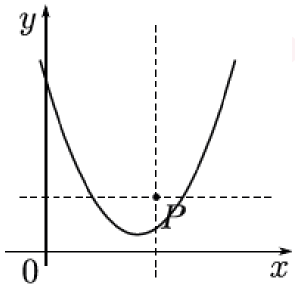
(E)



(F)



(G)



(H)

Q10

When the graph of the function $y = f(x)$, defined on the real numbers, is reflected in the y-axis and then translated by 2 units in the negative x-direction, the result is the graph of the function $y = g(x)$.

When the graph of the same function $y = f(x)$ is translated by 2 units in the negative x-direction and then reflected in the y-axis, the result is the graph of the function $y = h(x)$.

Which one of the following conditions on $y = f(x)$ is **necessary and sufficient** for the functions $g(x)$ and $h(x)$ to be identical?

(A) $f(x) = f(x+2)$ for all x

(B) $f(x) = f(x+4)$ for all x

(C) $f(x) = f(x+8)$ for all x

(D) $f(x) = f(-x)$ for all x

(E) $f(x) = f(2-x)$ for all x

(F) $f(x) = f(4-x)$ for all x

(G) $f(x) = f(8-x)$ for all x

Q11

The function f is defined for whole positive numbers and satisfies $f(1) = 1$ and also the rules

$$f(2n) = f(n),$$
$$f(2n+1) = f(n) + 1,$$

for all values of n. It follows that $f(9)$ equals

(A) 1

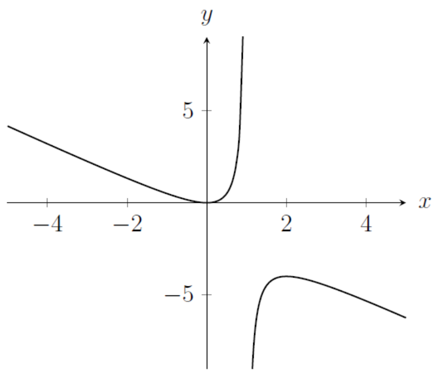
(B) 2

(C) 3

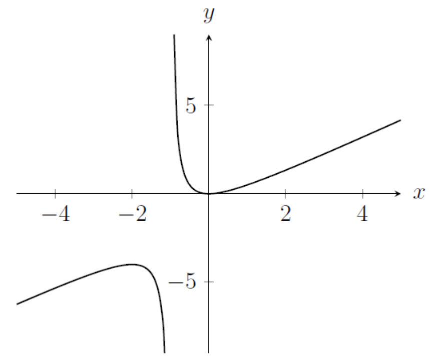
(D) 4

Q12

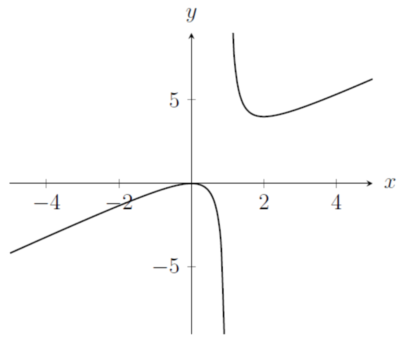
The diagram below shows the graph of $y = f(x)$.



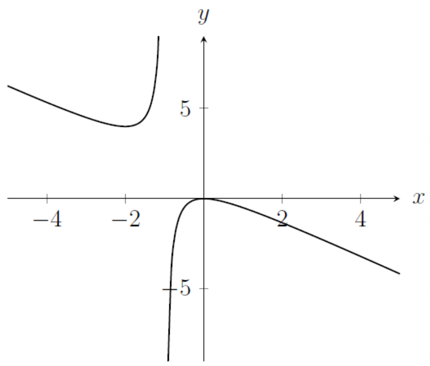
The graph of the function $y = -f(-x)$ is drawn in which of the following diagrams?



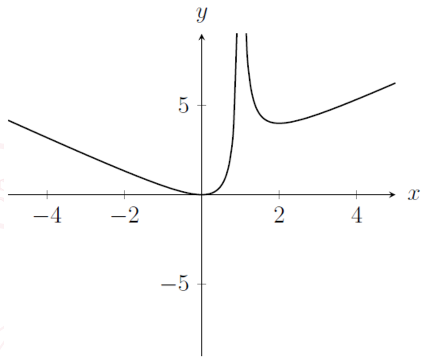
(A)



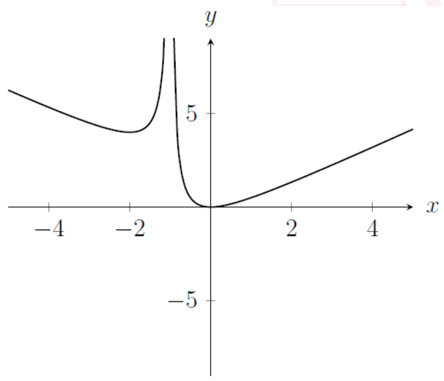
(B)



(C)



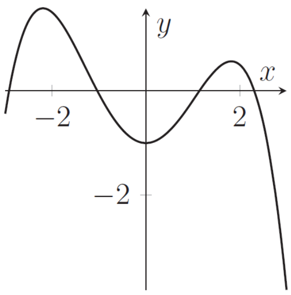
(D)



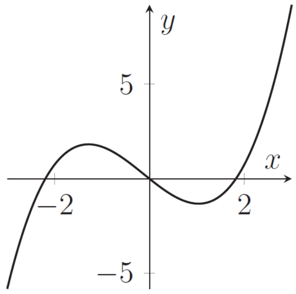
(E)

Q13

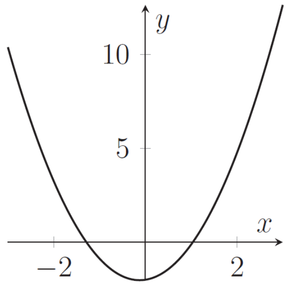
The following five graphs are, in some order, plots of $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $y = \frac{\mathrm{d}g}{\mathrm{d}x}$; that is, three unknown functions and the derivatives of the first two of those functions. Which graph is a plot of $h(x)$?



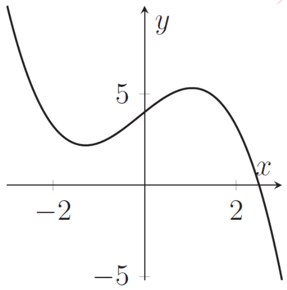
(A)



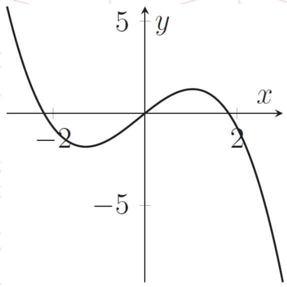
(B)



(C)



(D)



(E)

Q14

Into how many regions is the plane divided when the following equations are graphed, not considering the axes?

$$y = x^3$$
$$y = x^4$$
$$y = x^5$$

(A) 6

(B) 7

(C) 8

(D) 9

(E) 10

Q15

In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that (for example) $\lfloor 2.9 \rfloor = 2$, $\lfloor 2 \rfloor = 2$ and $\lfloor -1.5 \rfloor = -2$.

On separate diagrams draw the graphs, for $-\pi \le x \le \pi$, of:

(i) $y = \lfloor x \rfloor$;

(ii) $y = \sin\lfloor x \rfloor$;

(iii) $y = \lfloor \sin x \rfloor$;

(iv) $y = \lfloor 2\sin x \rfloor$.

In each case, you should indicate clearly the value of y at points where the graph is discontinuous.

Q16

Let

$$f_n(x) = (2 + (-2)^n)x^2 + (n+3)x + n^2$$

where n is a positive integer and x is any real number.

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.

For what values of n does $f_n(x)$ have a maximum value (as x varies)?

[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$.

Find an expression, simplified as much as possible, for

$$f_1(f_1(f_1(\cdots f_1(x))))$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2(f_2(f_2(\cdots f_2(x)))),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

Supplements S08

Time Allowed

90 min

Number of Questions

16

Difficulty

★★★★

Supplements S08

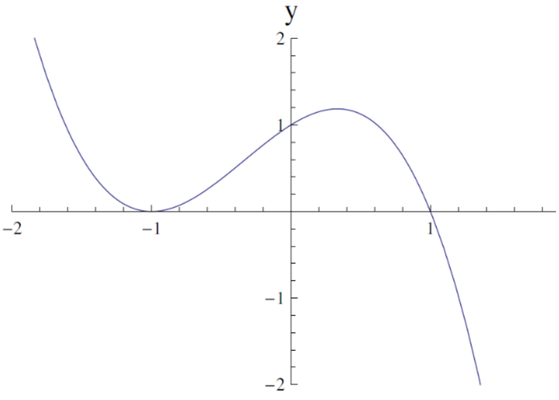
Scan the QR code or click the link above to take the practice online.

08

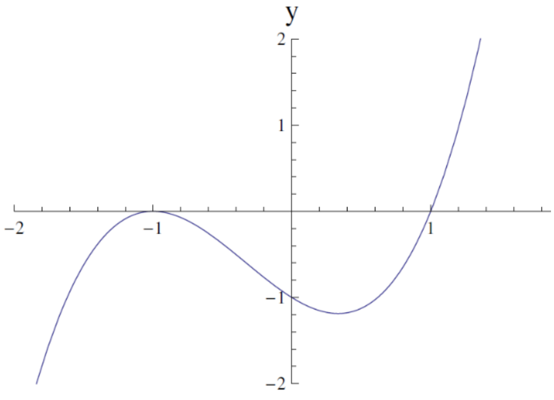
202

SQ1

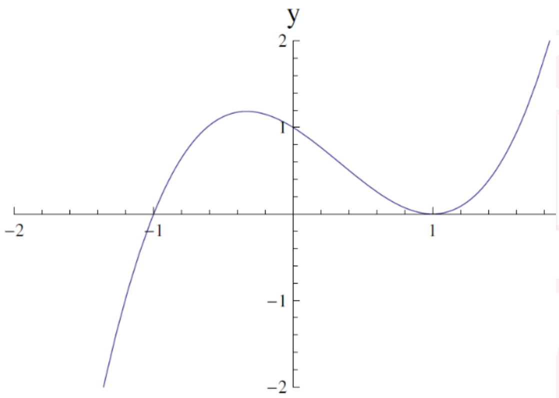
A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



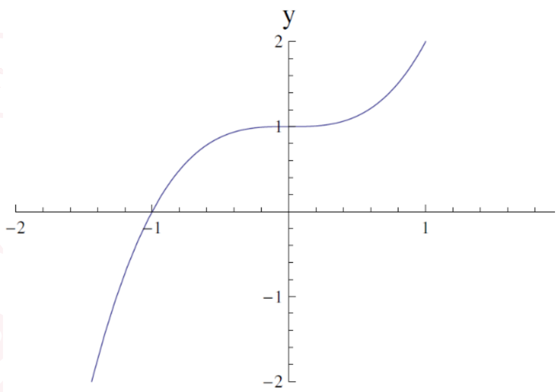
(A)



(B)



(C)



(D)

SQ2

The functions S and T are defined for real numbers by

$$S(x) = x + 1 \text{ and } T(x) = -x.$$

The function S is applied s times and the function T is applied t times, *in some order*, to produce the function

$$F(x) = 8 - x.$$

It is possible to deduce that:

(A) $s = 8$ and $t = 1$

(B) s is odd and t is even

(C) s is even and t is odd

(D) s and t are powers of 2

(E) none of the above

SQ3

Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$
$$y = x^2 - 2x$$
$$y = x^2 + 2x + 2$$

(A) 4

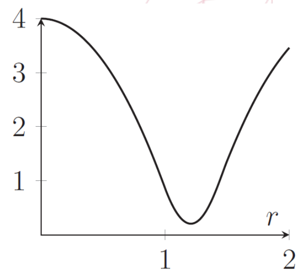
(B) 5

(C) 6

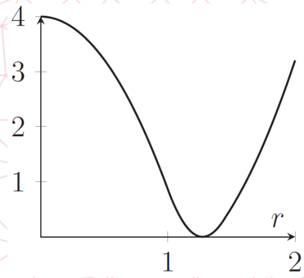
(D) 7

SQ4

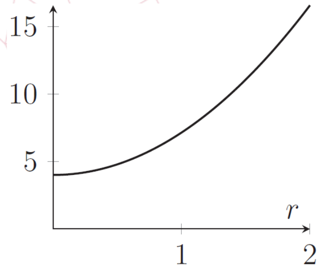
Consider a square with side length 2 and centre $(0, 0)$, and a circle with radius r and centre $(0, 0)$. Let $A(r)$ be the area of the region that is inside the circle but outside the square, and let $B(r)$ be the area of the region that is inside the square but outside the circle. Which of the following is a sketch of $A(r) + B(r)$?



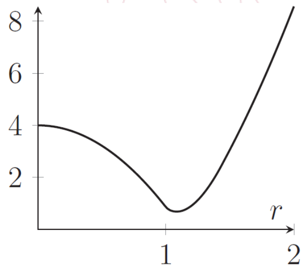
(A)



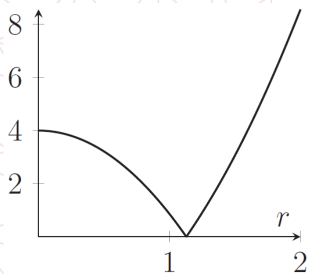
(B)



(C)



(D)



(E)

SQ5

The square $PQRS$ is positioned so that its vertices are at the points with coordinates: $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$.

The square is rotated clockwise through 90° about the origin and then reflected in the line $y = x$.

Which transformation will return the square to its original orientation?

(A) A reflection in the x-axis.

(B) A reflection in the y-axis.

(C) A reflection in the line $y = -x$.

(D) A rotation of 90° clockwise about the origin.

(E) A rotation of 90° anticlockwise about the origin.

SQ6

The curve $y = x^2$ is translated by the vector $\binom{4}{3}$ and then reflected in the line $y = -1$.

Which one of the following is an equation of the resulting curve?

(A) $y = -3 - (x-4)^2$

(B) $y = -3 + (x+4)^2$

(C) $y = 3 - (x+4)^2$

(D) $y = 3 + (x-4)^2$

(E) $y = -5 - (x-4)^2$

(F) $y = -5 + (x+4)^2$

(G) $y = 5 - (x+4)^2$

(H) $y = 5 + (x-4)^2$

SQ7

A line with non-zero gradient m is reflected in the line $y = x$.

What is the gradient of the reflected line?

(A) m

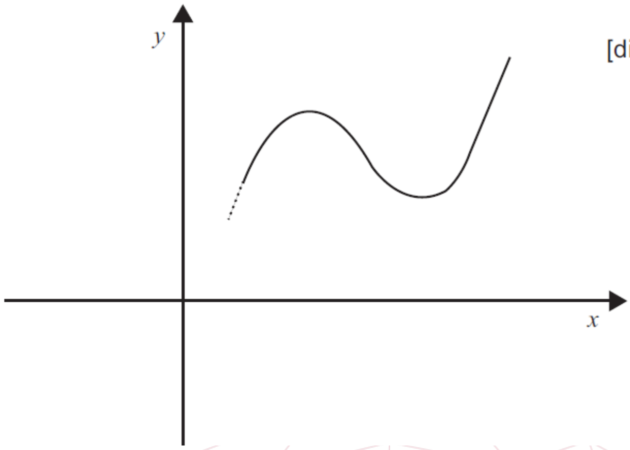
(B) $-m$

(C) $\frac{1}{m}$

(D) $-\frac{1}{m}$

SQ8

The diagram shows a partial **sketch** of the curve $y = 2x^3 - 9x^2 + 12x + p$ where p is constant.



[diagram not to scale]

The curve cuts the x-axis at one point only and does not touch the x-axis at any other point.

The y-coordinates of the turning points of the curve are both positive.

Find the complete range of values of p.

(A) $p > -5$

(B) $p > -4$

(C) $-5 < p < -4$

(D) $4 < p < 5$

(E) $p > 4$

(F) $p > 5$

SQ9

The curve

$$y = x^3 + 3\sqrt{5}px^2 + 3px + 13$$

has two distinct turning points.

What are all the possible values of p?

(A) $p < 0, p > 0.2$

(B) $p \leq 0, p \geq 0.2$

(C) $0 < p < 0.2$

(D) $0 \leq p \leq 0.2$

(E) $p < 0, p > 1.2$

(F) $p \leq 0, p \geq 1.2$

(G) $0 < p < 1.2$

(H) $0 \leq p \leq 1.2$

SQ10

The function f, defined for whole positive numbers, satisfies $f(1) = 1$ and also the rules

$$f(2n) = 2f(n),$$
$$f(2n+1) = 4f(n),$$

for all values of n. How many numbers n satisfy $f(n) = 16$?

(A) 3

(B) 4

(C) 5

(D) 6

SQ11

Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-\frac{1}{3}$

(B) $-\frac{1}{9}$

(C) 0

(D) $\frac{5}{9}$

(E) $\frac{5}{3}$

SQ12

The function f is defined by

$$f(x) = \lfloor |x| \rfloor - |\lfloor x \rfloor|$$

for all real numbers x, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

(A) $\{-1, 0\}$

(B) The set of nonpositive integers

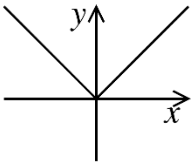
(C) $\{-1, 0, 1\}$

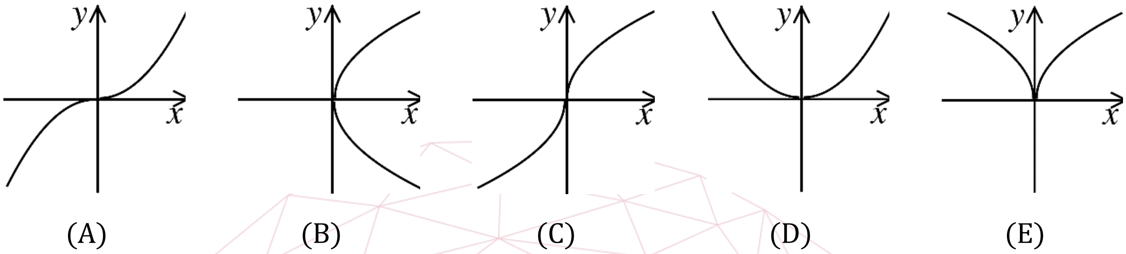
(D) $\{0\}$

(E) The set of nonnegative integers

SQ13

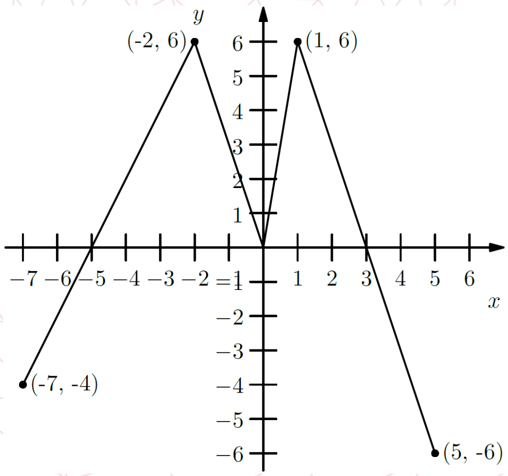
The graph of $y = |x|$ is shown alongside. Which of the following could be a sketch of the graph of $y = x|x|$?





SQ14

The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 6$ have?



(A) 2

(B) 4

(C) 5

(D) 6

(E) 7

SQ15

The function f has the property that for each real number x in its domain, $\frac{1}{x}$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x.$$

What is the largest set of real numbers that can be in the domain of f?

(A) $\{x | x \neq 0\}$

(B) $\{x | x < 0\}$

(C) $\{x | x > 0\}$

(D) $\{x | x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$

(E) $\{-1, 1\}$

SQ16

(i) The functions a, b, c and d are defined by

$$\begin{aligned} a(x) &= x^2 && (-\infty < x < \infty), \\ b(x) &= \ln x && (x > 0), \\ c(x) &= 2x && (-\infty < x < \infty), \\ d(x) &= \sqrt{x} && (x \geq 0). \end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

$$cb, \quad ab, \quad da, \quad ad.$$

(ii) The functions f and g are defined by

$$\begin{aligned} f(x) &= \sqrt{x^2 - 1} && (|x| \geq 1), \\ g(x) &= \sqrt{x^2 + 1} && (-\infty < x < \infty). \end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned} h(x) &= x + \sqrt{x^2 - 1} && (x \geq 1), \\ k(x) &= x - \sqrt{x^2 - 1} && (|x| \geq 1), \end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

08

210

09 Curve Sketching

What's on the Specification?

- Quadratic functions and their graphs; the discriminant of a quadratic function; completing the square; solution of quadratic equations.
- Recognise and be able to sketch the graphs of common functions that appear in this specification: these include lines, quadratics, cubics, trigonometric functions, logarithmic functions, exponential functions, square roots, and the modulus function.
- Knowledge of the effect of simple transformations on the graph of $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$, with the value of a positive or negative. Compositions of these transformations.
- Understand how altering the values of m and c affects the graph of $y = mx + c$.
- Understand how altering the values of a, b and c in $y = a(x + b)^2 + c$ affects the corresponding graph.
- Use differentiation to help determine the shape of the graph of a given function; this might include finding stationary points (excluding inflexions) as well as finding when the function is increasing or decreasing.
- Use algebraic techniques to determine where the graph of a function intersects the coordinate axes; appreciate the possible numbers of real roots a general polynomial can possess.
- Geometric interpretation of algebraic solutions of equations; relationship between the intersections of two graphs and the solutions of the corresponding simultaneous equations.

Exercises E09

Time Allowed

No limit

Number of Questions

9

Difficulty

★★★★

Exercises E09

Scan the QR code or click the link above to take the practice online.

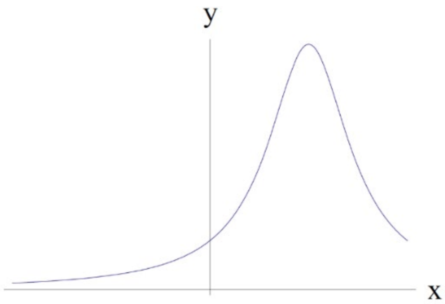
09

212

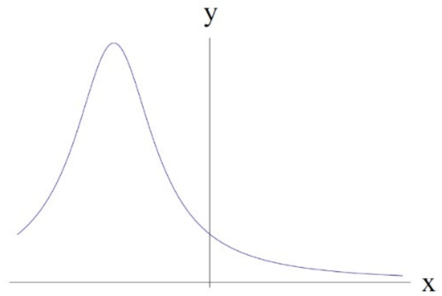
Quiz Pre-1

Which of the graphs below is a sketch of

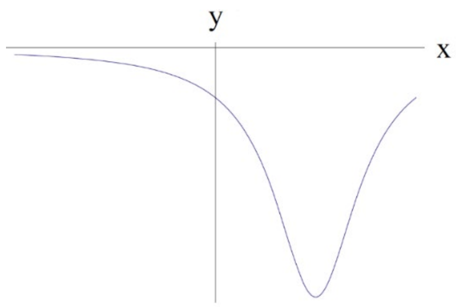
$$y = \frac{1}{4x - x^2 - 5}?$$



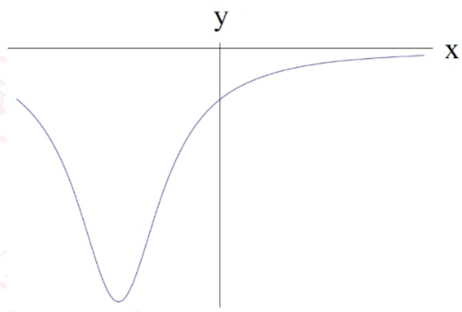
(A)



(B)



(C)



(D)

Quiz Pre-2

Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(A) $-2\sqrt{3} < k < 2\sqrt{3}$

(B) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$

(C) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$

(D) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$

(E) all values of k

Ex. 1

(i) Sketch the curve $y = \frac{x}{x-1}$.

(ii) Sketch the curve $y = \frac{x^2}{x-1}$.

[You may need the derivative of $(x-1)^{-1}$ which is $-(x-1)^{-2}$, by the chain rule.]

(iii) Sketch the curve $y = \frac{1}{x-1} + \frac{1}{x+1}$.

Ex. 2

The curve C has equation

$$y = x(x+1)(x-2)^4.$$

Show that the gradient of C is $(x-2)^3(6x^2+x-2)$ and find the coordinates of all the stationary points. Determine the nature of each stationary point and sketch C.

In separate diagrams draw sketches of the curves whose equations are:

(i) $y^2 = x(x+1)(x-2)^4$;

(ii) $y = x^2(x^2+1)(x^2-2)^4$.

In each case, you should pay particular attention to the points where the curve meets the x axis.

Quiz 1

The function $\frac{1-x}{\sqrt[3]{x^2}}$ is defined for all $x \neq 0$.

The complete set of values of x for which the function is decreasing is

(A) $x \leq -2,\ x > 0$

(B) $-2 \leq x < 0$

(C) $x \leq 1,\ x \neq 0$

(D) $x \geq 1$

(E) $-2 \leq x \leq 1,\ x \neq 0$

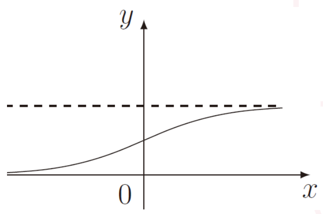
(F) $x \leq -2,\ x \geq 1$

Quiz 2

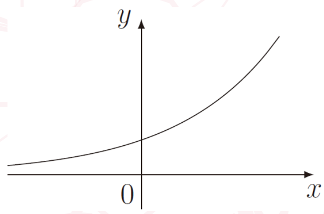
Which one of the following shows the graph of

$$y = \frac{2^x}{1 + 2^x}$$

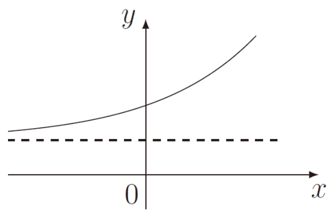
(Dotted lines indicate asymptotes.)



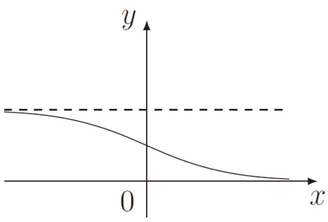
(A)



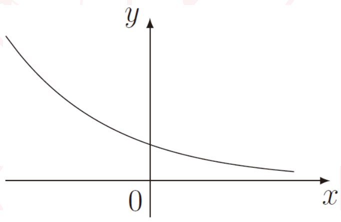
(B)



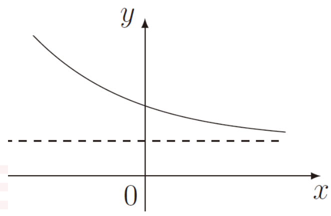
(C)



(D)



(E)



(F)

Ex. 3

(i) Sketch the curve $y = f(x)$, where

$$f(x) = \frac{1}{(x-a)^2 - 1} \qquad (x \neq a \pm 1),$$

and a is a constant.

(ii) The function $g(x)$ is defined by

$$g(x) = \frac{1}{((x-a)^2 - 1)((x-b)^2 - 1)} \quad (x \neq a \pm 1, x \neq b \pm 1),$$

where a and b are constants, and $b > a$. Sketch the curves $y = g(x)$ in the two cases $b > a + 2$ and $b = a + 2$, finding the values of x at the stationary points.

Ex. 4

Sketch the curve with cartesian equation

$$y = \frac{2x(x^2 - 5)}{x^2 - 4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

(i) $3x(x^2 - 5) = (x^2 - 4)(x + 3)$.

(ii) $4x(x^2 - 5) = (x^2 - 4)(5x - 2)$.

(iii) $4x^2(x^2 - 5)^2 = (x^2 - 4)^2(x^2 + 1)$.

Ex. 5

(i) Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0$$

State the values of b, if any, for which (a) n=0; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

[For part (ii) you could sketch $y = x^4 - 6x^2 + ax$ for the particular values of a.]

Practices P09

Time Allowed

120 min

Number of Questions

11

Difficulty

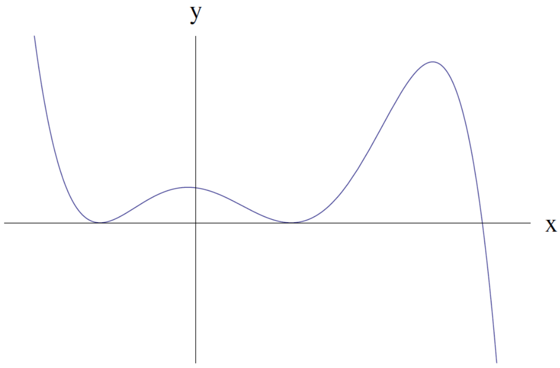
★★★★

Practices P09

Scan the QR code or click the link above to take the practice online.

Q1

Which one of the following equations could possibly have the graph given below?



(A) $y = (3-x)^2(3+x)^2(1-x)$

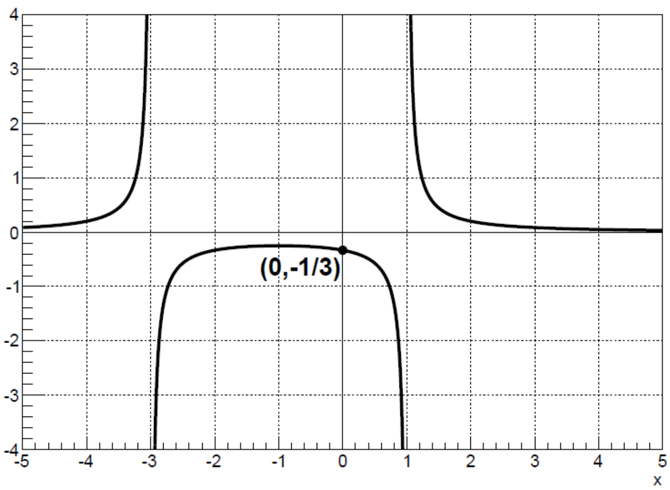
(B) $y = -x^2(x-9)(x^2-3)$

(C) $y = (x-6)(x-2)^2(x+2)^2$

(D) $y = (x^2-1)^2(3-x)$

Q2

The graph below could represent which of the following functions?



(A) $\frac{1}{x-1} + \frac{2}{x+3}$

(B) $\frac{-1}{x^2-2x+3}$

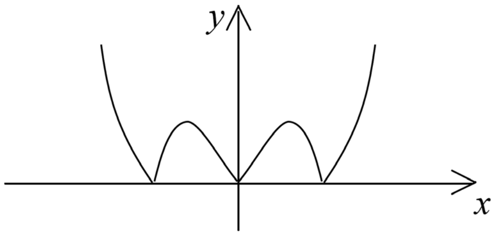
(C) $\frac{1}{x^2+2x-3}$

(D) $\frac{1}{x^2-1} + \frac{2}{x+3}$

(E) $\frac{3}{x^2-9}$

Q3

The graph of $y = |f(x)|$ is shown. Given that the graph of $y = f(x)$ is a continuous curve, how many different possibilities are there for the graph of $y = f(x)$?



(A) 16

(B) 12

(C) 8

(D) 4

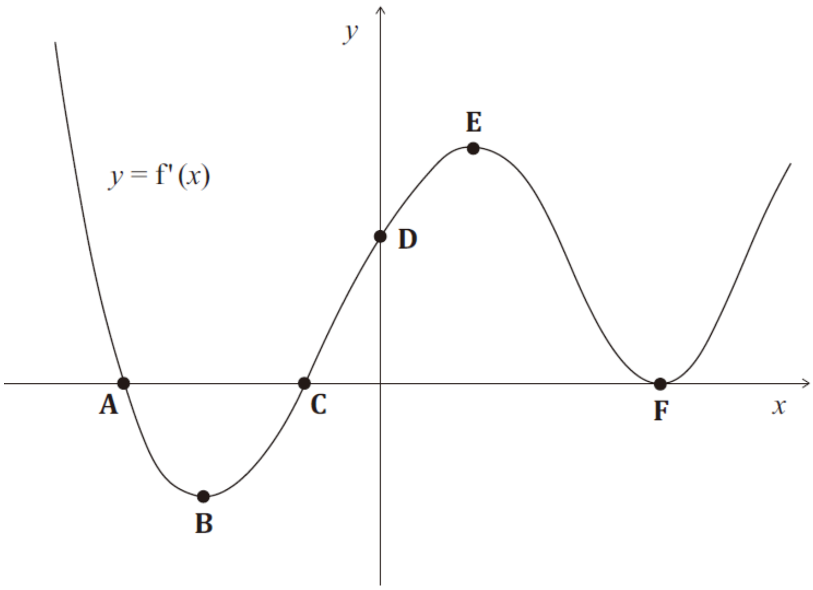
(E) 2

Q4

The function $f(x)$ has derivative $f'(x)$.

The diagram below shows the graph of $y = f'(x)$.

Which point corresponds to a local minimum of $f(x)$?



Q5

The graph of the polynomial function

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

is sketched, where a, b, c, d, e, and f are real constants with $a \neq 0$.

Which one of the following is **not** possible?

(A) The graph has two local minima and two local maxima.

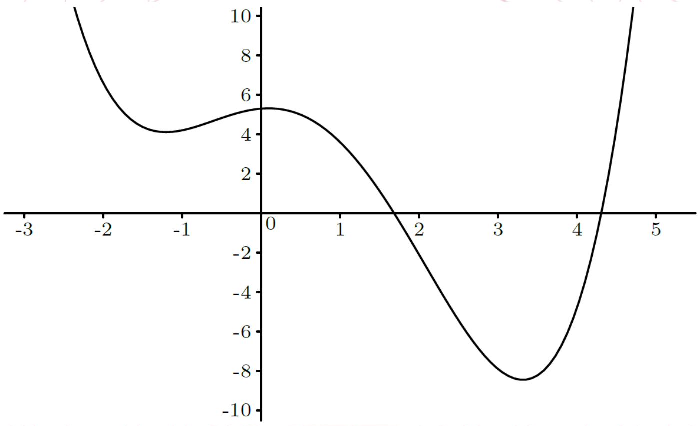
(B) The graph has one local minimum and two local maxima.

(C) The graph has one local minimum and one local maximum.

(D) The graph has no local minima or local maxima.

Q6

The graph below shows a portion of the curve defined by the quartic polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$. Which of the following is the smallest?



(A) $P(-1)$

(B) The product of the zeros of P

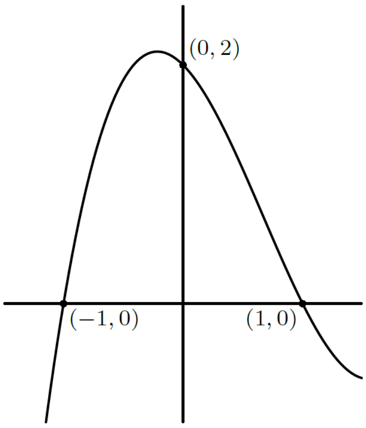
(C) The product of the non-real zeros of P

(D) The sum of the coefficients of P

(E) The sum of the real zeros of P

Q7

Part of the graph of $f(x) = x^3 + bx^2 + cx + d$ is shown. What is b?



(A) -4

(B) -2

(C) 0

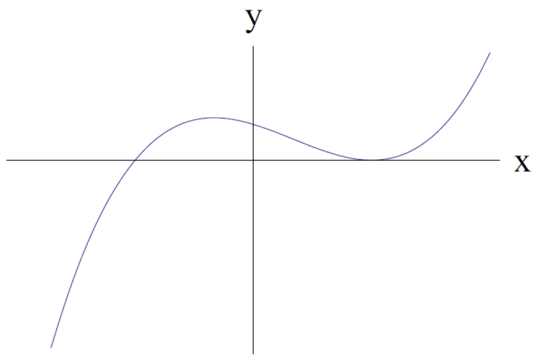
(D) 2

(E) 4

Q8

For applicants in {Math, Math & Statistics, Math & Philosophy, and Math & CS} only. Not for {CS}.

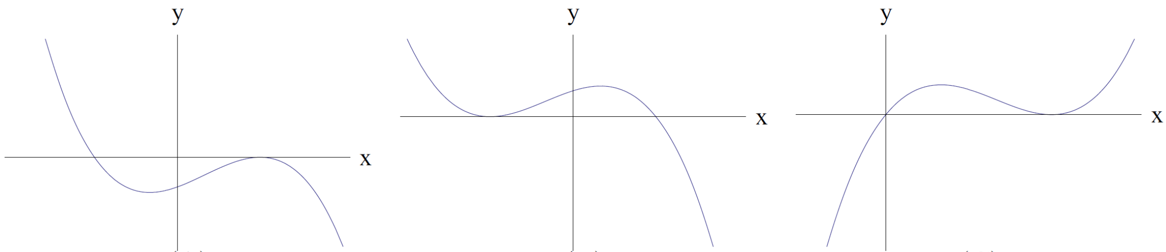
(i) The graph $y = f(x)$ of a certain function has been plotted below.



On the next three pairs of axes (A), (B), (C) are graphs of

$y = f(-x), f(x-1), -f(x)$

in some order. Say which axes correspond to which graphs.



(A) (B) (C)

(ii) Sketch, on the axes opposite, graphs of *both* of the following functions

$$y = 2^{-x^2} \text{ and} y = 2^{2x-x^2}.$$

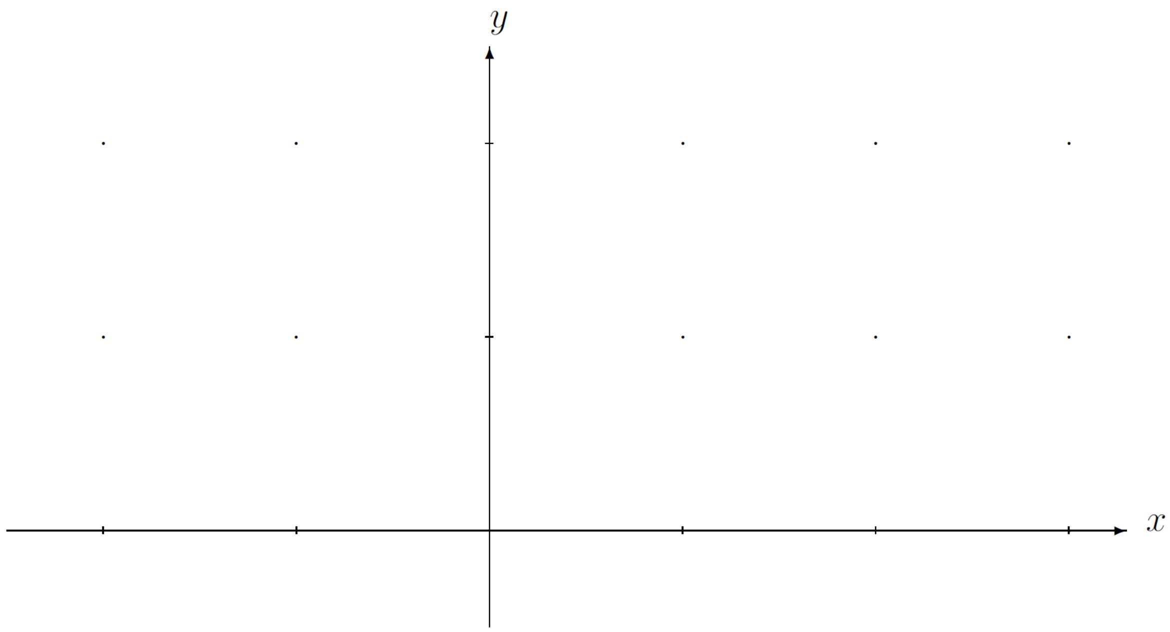
Carefully label any stationary points.

(iii) Let c be a real number and define the following integral

$$I(c) = \int_0^1 2^{-(x-c)^2}\, dx.$$

State the value(s) of c for which $I(c)$ is largest. Briefly explain your reasoning.

[Note you are not being asked to calculate this maximum value.]



y
x

Q9

Let

$$f(x) = \left(c - \frac{1}{c} - x\right)(4 - 3x^2),$$

where c is a positive constant and x varies over the real numbers.

(i) Show that $f(x)$ has one maximum and one minimum.

(ii) Show that the difference between the values of $f(x)$ at its turning points is

$$\frac{4}{9}\left(c + \frac{1}{c}\right)^3.$$

(iii) What is the least value that the difference in (ii) can have for $c > 0$?

Q10

Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

(i) $2b < c\sqrt{3}$;

(ii) $2b = c\sqrt{3} \neq 0$;

(iii) $c\sqrt{3} < 2b < 2c$;

(iv) $b = c \neq 0$;

(v) $b > c > 0$;

(vi) $c = 0,\ b \neq 0$;

(vii) $c = b = 0$.

Sketch also the curves given by

$$y^2 = x^3 - 2bx^2 + c^2x$$

in the cases (i), (v) and (vii).

Q11

The equation of a curve is $y = f(x)$ where

$$f(x) = x - 4 - \frac{16(2x+1)^2}{x^2(x-4)}.$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $f(x) = 0$ has a double root.

(iii) Sketch the curve.

Supplements S09

Time Allowed

120 min

Number of Questions

12

Difficulty

★★★★☆

Supplements S09

Scan the QR code or click the link above to take the practice online.

SQ1

The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of those values?

(A) 4

(B) 5

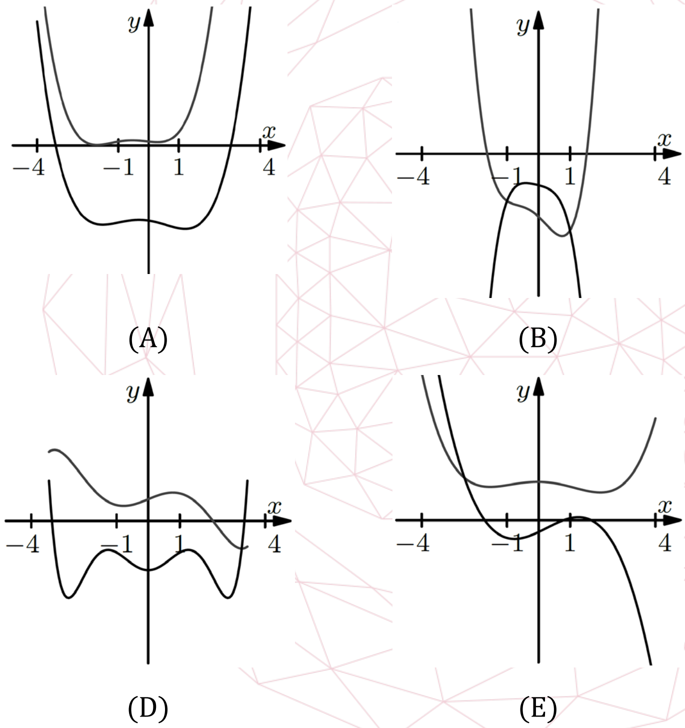
(C) 6

(D) 7

(E) 8

SQ2

The non-zero coefficients of a polynomial P with real coefficients are all replaced by their mean to form a polynomial Q. Which of the following could be a graph of $y = P(x)$ and $y = Q(x)$ over the interval $-4 \le x \le 4$?



SQ3

Sketch the curve $y = e^x(x^2 - 3)$ and find the values of y at the turning points.

SQ4

Sketch the graph of $y = \frac{ax^2}{x^2+1}$, where $a > 0$, and give the equations of any asymptotes.

SQ5

Find the greatest and least values of $e^{-x}(x^2 - 4x + 4)$ when $0 \le x \le 3$.

SQ6

Find any turning points of the function y defined by

$$y = \frac{(x+a)(x+b)}{(x-a)(x-b)}, \qquad a+b \neq 0, \qquad ab \neq 0$$

distinguishing between the cases $ab > 0$ and $ab < 0$.

Plot y in each of the following cases, carefully determining and marking any turning points, asymptotes and zeros:

(i) $0 < a < b$.

(ii) $-b < a < 0 < b$.

SQ7

Find the turning points on the graph of the function

$$y = -3x^3 + 9ax - 2a^2$$

where a is a real number. Sketch the curve (i) for $a < 0$, (ii) $0 < a < 9$ and (iii) for $a > 9$. For each value of a find the maximum value $M(a)$ of the function in the interval $0 \leq x \leq 3$.

SQ8

Let

$$f(x) = x^3 + ax^2 + x + 1$$

where a is a real number. Determine the values of a for which the function f has

(i) two turning points.

(ii) a stationary point of inflection.

(iii) no turning points and no stationary point of inflection.

Find the greatest value and the least value of $f(x)$ for $0 \leq x \leq 1$,

(a) when $a \geq 0$, and

(b) when $a = -2$.

SQ9

Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

SQ10

Let p and q be real numbers. Show that the graph

$$y = x^3 + px + q$$

has turning points if and only if p is negative.

Assume now that p is negative. Find the values of y at the turning points and hence show that the equation

$$x^3 + px + q = 0$$

has three (distinct) real roots if and only if $27q^2 < -4p^3$.

[You may find it helpful to sketch the graph, clearly indicating the turning points.]

SQ11

A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16), \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2, 8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

SQ12

The function $f(x)$ is defined by

$$f(x) = \frac{x(x-2)(x-a)}{x^2 - 1}.$$

Prove algebraically that the line $y = x + c$ intersects the curve $y = f(x)$ if $|a| \geq 1$, but there are values of c for which there are no points of intersection if $|a| < 1$.

Find the equation of the oblique asymptote of the curve $y = f(x)$. Sketch the graph in the two cases (i) $a < -1$, and (ii) $-1 < a < -\frac{1}{2}$. (You need not calculate the turning points.)

10 Trigonometric Functions

What's on the Specification?

- Radian measure, including use for arc length and area of sector and segment.
- The values of sine, cosine, and tangent for the angles 0°, 30°, 45°, 60°, 90°.
- The sine, cosine, and tangent functions; their graphs, symmetries, and periodicity.
- Knowledge and use of $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.

Exercises E10

Time Allowed

No limit

Number of Questions

14

Difficulty

★★★☆

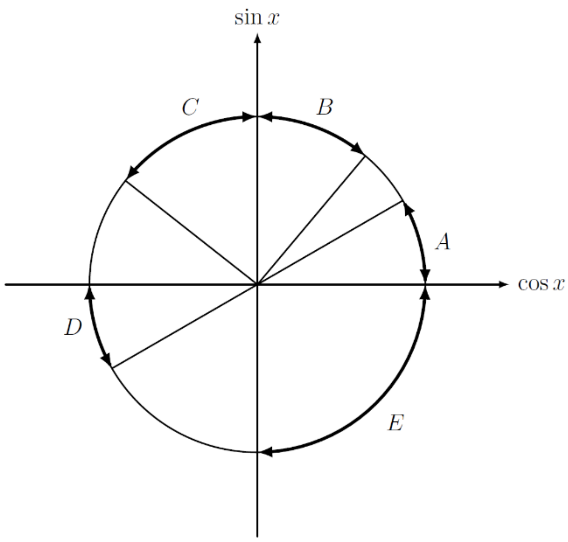
Exercises E10

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some x. Which of the marked arcs corresponds to

$$\tan x < \cos x < \sin x\,?$$



(A) A

(B) B

(C) C

(D) D

(E) E

Quiz Pre-2

Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$?

(A) $\sqrt{\frac{3}{5}} - 1$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

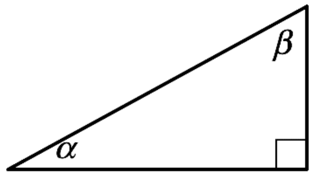
(D) $\frac{2}{3}$

(E) 1

Quiz Pre-3

If $\alpha < \beta$, how many different values are there among the following expressions?

$$\sin\alpha\sin\beta \qquad \sin\alpha\cos\beta \qquad \cos\alpha\sin\beta \qquad \cos\alpha\cos\beta$$



(A) 1

(B) 2

(C) 3

(D) 4

(E) It depends on the value of α

Ex. 1

Find the values of

(i) $\sin 75°$ and $\sin 15°$.

(ii) $\sin\frac{5\pi}{12}\cos\frac{\pi}{12}$.

(iii) $\sin 18°$, hence $\sin 18° \cos 36°$.

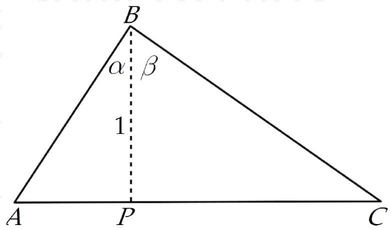
(iv) $\cos 10° \cos 30° \cos 50° \cos 70°$.

(v) $1 - \cos 12° + \cos 24° + \cos 48° - \cos 84°$.

[If you are stuck in these questions, you might proceed to the following section, *Trigonometric identities*, to get some more ideas and learn more methods, then come back to solve them.]

Ex. 2

In the triangle below, BP is perpendicular to AC.



(i) Show by means of area method, or otherwise, that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Use this result, together with properties of sin and cos such as $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, to obtain expressions for $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$.

(ii) By setting $\alpha = \beta$, prove the double-angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Hence, show that (power-reducing formulas)

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}.$$

(iii) By expressing $\sin 3A$ as $\sin(2A + A)$, find an expression for $\sin 3A$ in terms of $\sin A$.

Hence, similarly find an expression for $\cos 3A$ in terms of $\cos A$.

Ex. 3

(i) By choosing proper formulas above, prove the product-to-sum identity

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta).$$

Hence, obtain the other product-to-sum identities for $2\sin\alpha\sin\beta$ and $2\cos\alpha\cos\beta$.

(ii) Prove the sum-to-product identities

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}.$$

Hence, obtain the other sum-to-product identities for $\cos\alpha+\cos\beta$ and $\cos\alpha-\cos\beta$.

(iii) Show that any form of $a\sin\theta + b\cos\theta$ can be expressed in the form of $r\sin(\theta+\varphi)$, that is,

$$a\sin\theta + b\cos\theta \equiv r\sin(\theta+\varphi),$$

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\frac{b}{a}$.

Ex. 4

Just one of the following expressions is equal to $\sin 5\alpha$ for all values of α. Which one is it?

(A) $5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha$

(B) $5\sin\alpha - 20\sin^3\alpha + 14\sin^5\alpha$

(C) $5\sin\alpha - 10\sin^2\alpha + 10\sin^3\alpha - 5\sin^4\alpha + \sin^5\alpha$

(D) $\sin\alpha - 5\sin^2\alpha + 10\sin^3\alpha - 10\sin^4\alpha + 5\sin^5\alpha$

10

234

Ex. 5

(i) Find $\tan(\alpha+\beta)$ if $\sin\alpha+\sin\beta = \frac{1}{4}$, $\cos\alpha+\cos\beta = \frac{1}{3}$.

(ii) Find $\sin\alpha$ if α and β are acute angles with $2\tan\alpha + 3\sin\beta = 7$ and $\tan\alpha - 6\sin\beta = 1$.

(iii) Find the value of $\sin^4\theta + \cos^4\theta$ if $\cos 2\theta = \frac{1}{1+\sqrt{2}}$.

(iv) Find the values of $\sin^3\theta + \cos^3\theta$ if $\sin 2\theta = -\frac{1}{4}$.

Ex. 6

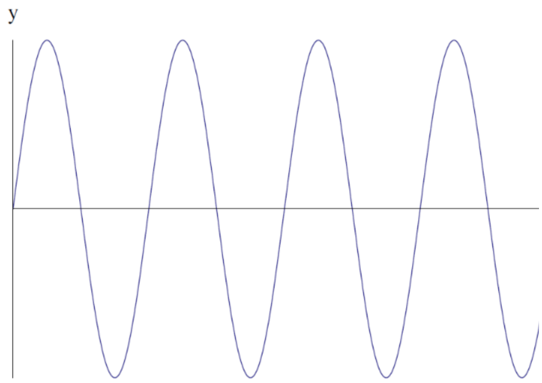
(i) Given that $\sin\theta = \frac{5}{13}$, where $\frac{\pi}{2} < \theta < \pi$, find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

(ii) Find the value of $\tan\alpha$ if $\sin\alpha - \cos\alpha = \frac{1}{5}$, $0 < \alpha < \pi$.

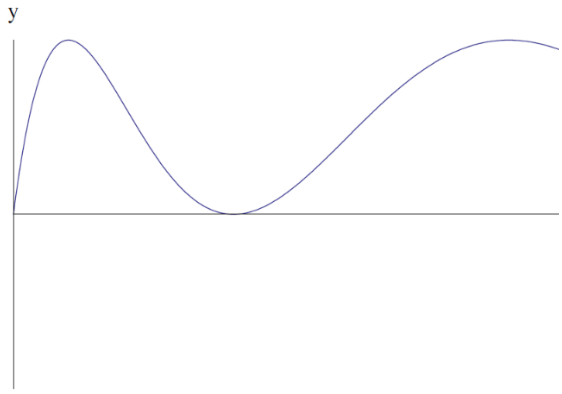
(iii) Find the value of $2\sin x\cos x$ if $\sin x = 5\cos x$.

Ex. 7

The graph of $y = \sin^2 \sqrt{x}$ is drawn in



(A)



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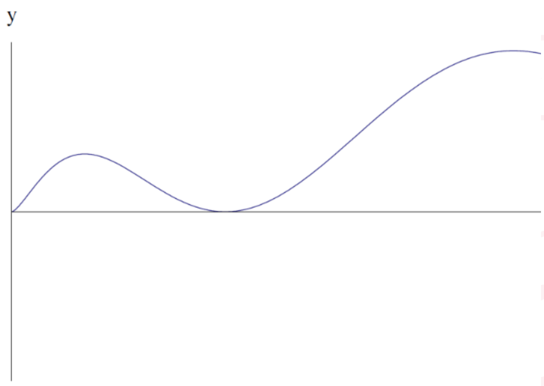
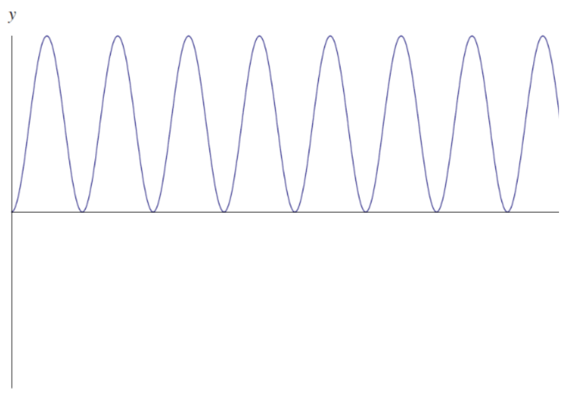


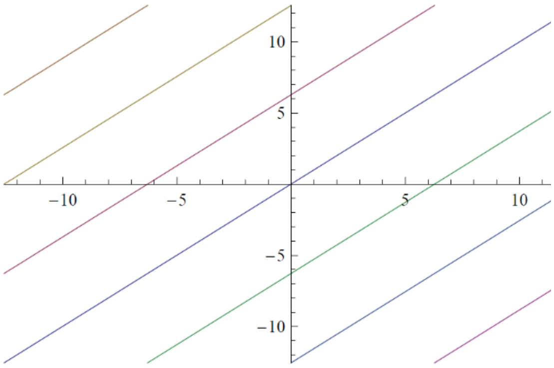
(C)



(D)

Ex. 8

The graph of all those points (x, y) in the x-y plane which satisfy the equation $\sin y = \sin x$ is drawn in



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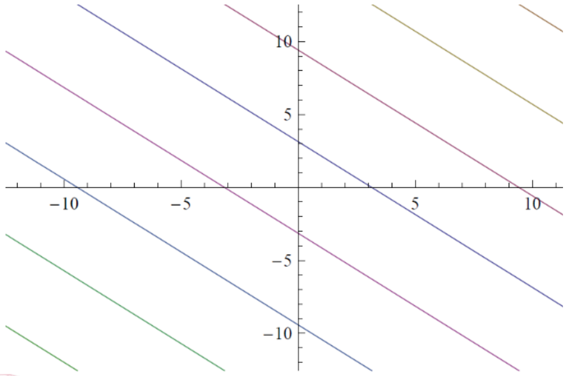
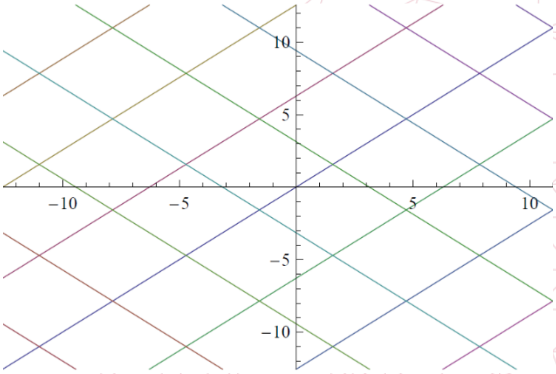


(B)



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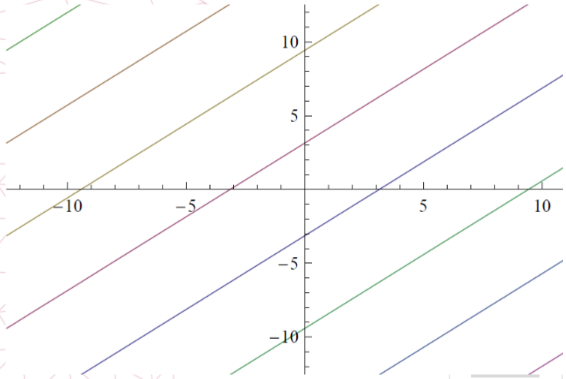
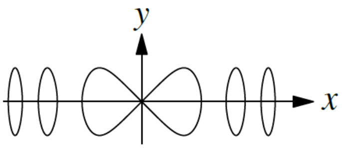
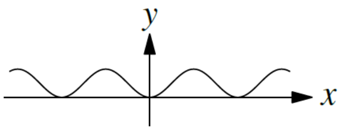


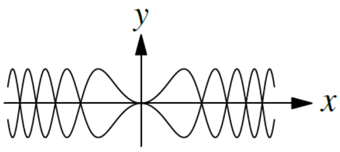
(D)

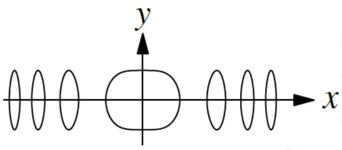
Quiz 1

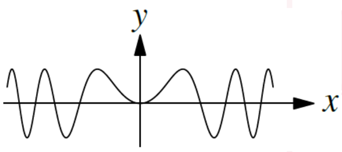
Which of the following could be the graph of $y^2 = \sin(x^2)$?

(A) 

(B) 

(C) 

(D) 

(E) 

Quiz 2

The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

(A) $\frac{\pi}{2}$

(B) π

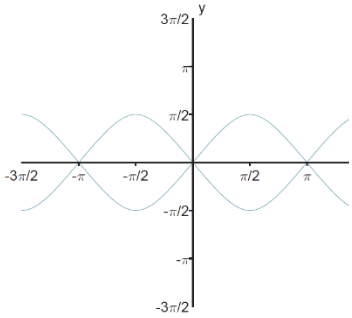
(C) 2π

(D) 4π

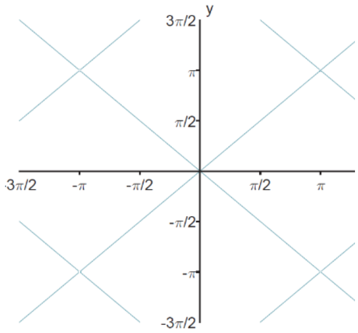
(E) It's not periodic

Quiz 3

The graph of $\cos^2(x) = \cos^2(y)$ is sketched in



(A)



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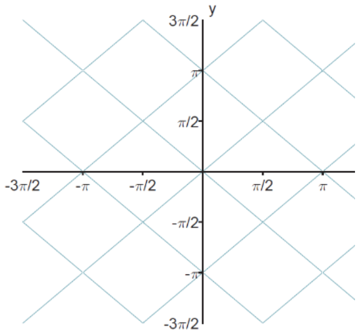
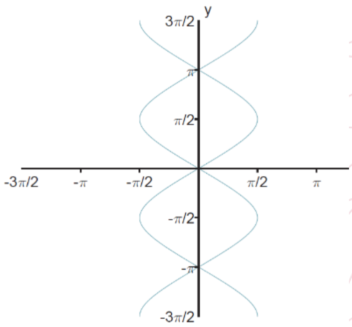
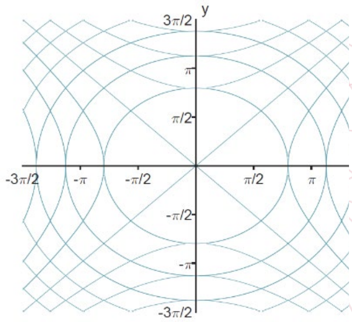


(C)



(D)



(E)

Practices P10

Time Allowed

120 min

Number of Questions

15

Difficulty

★★★★☆

[Practices P10]

Scan the QR code or click the link above to take the practice online.

Q1

The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation

(A) $y = \cos x$

(B) $y = 2 + \sin x$

(C) $y = 4 + \sin x$

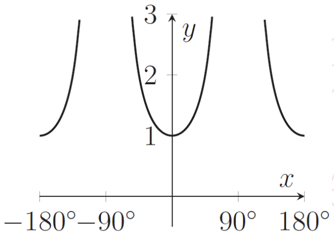
(D) $y = 2 - \cos x$

Q2

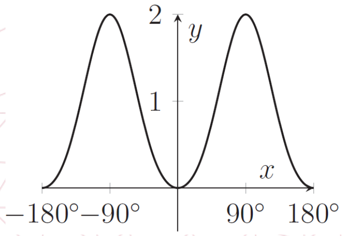
The graph of

$$y = \sin^2 x + \sin^4 x + \sin^6 x + \sin^8 x + \cdots$$

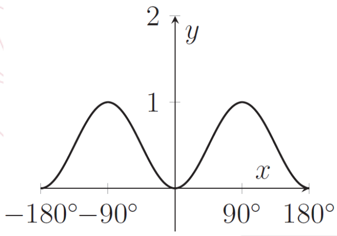
is sketched in



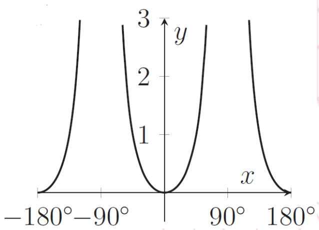
(A)



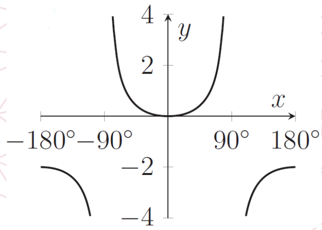
(B)



(C)



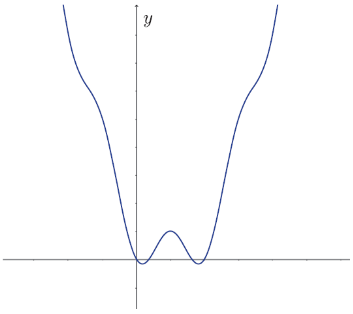
(D)



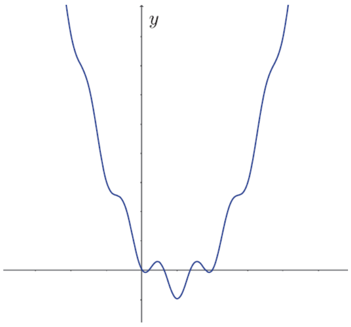
(E)

Q3

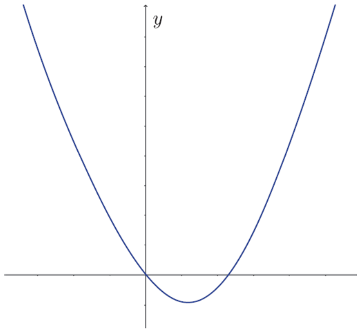
The graph of $y = (x-1)^2 - \cos(\pi x)$ is drawn in



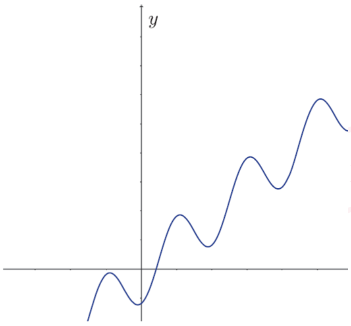
(A)



(B)



(C)



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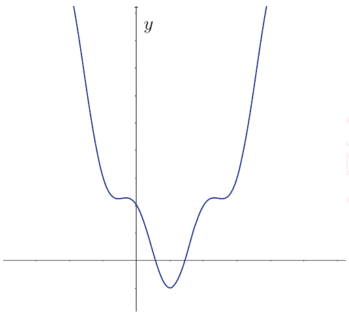


(E)

Q4

The graph of

$$\sin y - \sin x = \cos^2 x - \cos^2 y$$

(A) is empty.

(B) is non-empty but includes no straight lines.

(C) includes precisely one straight line.

(D) includes precisely two straight lines.

(E) includes infinitely many straight lines.

Q5

The curve $y = \cos x$ is reflected in the line $y = 1$ and the resulting curve is then translated by $\frac{\pi}{4}$ units in the positive x-direction. The equation of this new curve is

(A) $y = 2 + \cos\left(x + \frac{\pi}{4}\right)$

(B) $y = 2 + \cos\left(x - \frac{\pi}{4}\right)$

(C) $y = 2 - \cos\left(x + \frac{\pi}{4}\right)$

(D) $y = 2 - \cos\left(x - \frac{\pi}{4}\right)$

Q6

What is the smallest positive value of a for which the line $x = a$ is a line of symmetry of the graph of $y = \sin\left(2x - \frac{4\pi}{3}\right)$?

(A) $\frac{\pi}{12}$

(B) $\frac{5\pi}{12}$

(C) $\frac{7\pi}{12}$

(D) $\frac{11\pi}{12}$

(E) $\frac{19\pi}{12}$

Q7

Find the value of

$$\sin^2 0° + \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 87° + \sin^2 88° + \sin^2 89° + \sin^2 90°.$$

(A) 0.5

(B) 1

(C) 1.5

(D) 45

(E) 45.5

(F) 46

Q8

Given that $\tan\theta = 2$ and $180° < \theta < 360°$, find the value of $\cos\theta$

(A) $\sqrt{3}$

(B) $-\sqrt{3}$

(C) $\frac{\sqrt{3}}{2}$

(D) $-\frac{\sqrt{3}}{2}$

(E) $\frac{\sqrt{5}}{5}$

(F) $-\frac{\sqrt{5}}{5}$

(G) $\frac{2\sqrt{5}}{5}$

(H) $-\frac{2\sqrt{5}}{5}$

Q9

The following question appeared in an examination:

Given that $\tan x = \sqrt{3}$, find the possible values of $\sin 2x$.

A student gave the following answer:

$\tan x = \sqrt{3}$ so $x = 60°$ and $2x = 120°$, therefore $\sin 2x = \frac{\sqrt{3}}{2}$.

Which one of the following statements is correct?

(A) $\frac{\sqrt{3}}{2}$ is the only possible value, and this is fully supported by the reasoning given in the student's answer.

(B) $\frac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\tan x = \sqrt{3}$.

(C) $\frac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\sin 2x = \frac{\sqrt{3}}{2}$.

(D) $\frac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\tan x = \sqrt{3}$.

(E) $\frac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\sin 2x = \frac{\sqrt{3}}{2}$.

Q10

A student wishes to evaluate the function $f(x) = x \sin x$, where x is in radians, but has a calculator that only works in degrees.

What could the student type into their calculator to correctly evaluate $f(4)$?

(A) $(\pi \times 4 \div 180) \times \sin(4)$

(B) $(\pi \times 4 \div 180) \times \sin(\pi \times 4 \div 180)$

(C) $4 \times \sin(\pi \times 4 \div 180)$

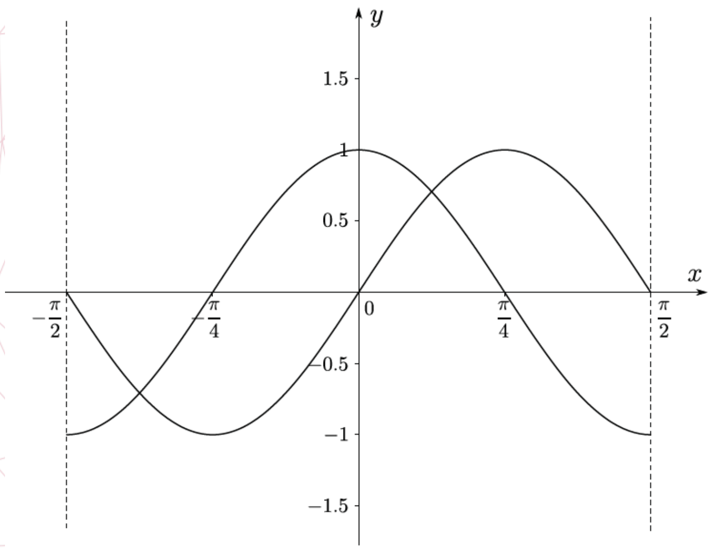
(D) $(180 \times 4 \div \pi) \times \sin(4)$

(E) $(180 \times 4 \div \pi) \times \sin(180 \times 4 \div \pi)$

(F) $4 \times \sin(180 \times 4 \div \pi)$

Q11

The diagram shows the graphs of $y = \sin 2x$ and $y = \cos 2x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



Which one of the following is **not** true?

(A) $\cos 2x < \sin 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(B) $\cos 2x < \tan x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(C) $\sin 2x < \cos 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(D) $\sin 2x < \tan x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(E) $\tan x < \sin 2x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(F) $\tan x < \cos 2x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Q12

The curve $y = \sin x$ is stretched by a scale factor of $\frac{1}{2}$ parallel to the x-axis and then translated by $\frac{\pi}{4}$ in the negative x-direction.

What is the equation of the new curve?

(A) $y = \sin\left(\frac{x}{2} - \frac{\pi}{4}\right)$

(B) $y = \sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$

(C) $y = \sin\left(\frac{x}{2} - \frac{\pi}{8}\right)$

(D) $y = \sin\left(\frac{x}{2} + \frac{\pi}{8}\right)$

(E) $y = \sin\left(2x - \frac{\pi}{4}\right)$

(F) $y = \sin\left(2x + \frac{\pi}{4}\right)$

(G) $y = \sin\left(2x - \frac{\pi}{2}\right)$

(H) $y = \sin\left(2x + \frac{\pi}{2}\right)$

Q13

(i) Given that $\cos\theta = \frac{3}{5}$ and that $\frac{3\pi}{2} \leq \theta \leq 2\pi$, show that $\sin 2\theta = -\frac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \frac{11}{2}$ and that $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

Q14

(i) Find all the values of θ, in the range $0° < \theta < 180°$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18° = \frac{1}{4}\left(\sqrt{5} - 1\right).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0° < \alpha < 90°$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha.$$

Q15

Show that

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)=\frac{\sin\alpha}{4\sin\left(\frac{\alpha}{4}\right)},$$

where $\alpha \neq k\pi$, k an integer.

Prove that, for such α,

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right)=\frac{\sin\alpha}{2^n\sin\left(\frac{\alpha}{2^n}\right)},$$

where n is a positive integer.

Deduce that

$$\alpha=\frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\cdots},$$

and hence that

$$\frac{\pi}{2}=\frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}$$

[Hint: $\lim_{x\to 0}\frac{\sin x}{x}=1$]

Supplements S10

Time Allowed

150 min

Number of Questions

14

Difficulty

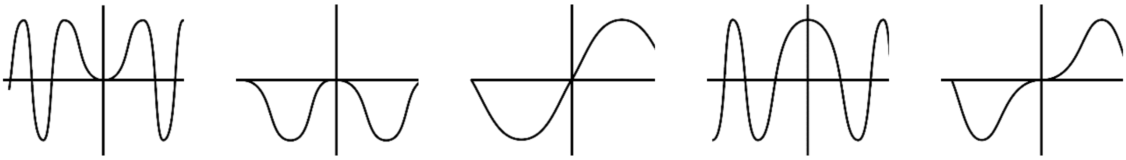
★★★★★

Supplements S10

Scan the QR code or click the link above to take the practice online.

SQ1

Which of the following could be the graph of $y = \sin(x^2)$?



(A) (B) (C) (D) (E)

SQ2

The least and greatest values of $\cos(\cos x)$ in the range $0 \leq x \leq \pi$ are

(A) 0 and 1

(B) $-\cos 1$ and 1

(C) -1 and 1

(D) $\cos 1$ and 1

SQ3

If $\cos\theta = \frac{1}{2}$, which of these cannot equal $\sin 2\theta$?

(A) $\sin\theta$

(B) $\frac{1}{2}$

(C) $-\frac{\sqrt{3}}{2}$

(D) $\frac{\sqrt{3}}{2}$

(E) $2\cos\theta\sin\theta$

SQ4

Which of the following has the greatest value?

(A) $\cos 50°$

(B) $\sin 50°$

(C) $\tan 50°$

(D) $\frac{1}{\sin 50°}$

(E) $\frac{1}{\cos 50°}$

SQ5

The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

(A) $\frac{\pi}{2}$

(B) π

(C) 2π

(D) 4π

(E) It's not periodic

SQ6

If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

(A) $\frac{1}{5}$

(B) $\frac{2}{5}$

(C) $\frac{\sqrt{5}}{5}$

(D) $\frac{3}{5}$

(E) $\frac{4}{5}$

SQ7

The number of x-intercepts on the graph of $y = \sin\left(\frac{1}{x}\right)$ in the interval $(0.0001, 0.001)$ is closest to

(A) 2900

(B) 3000

(C) 3100

(D) 3200

(E) 3300

SQ8

Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$,and $\cos\frac{6\pi}{7}$, where angles are in radians. What is abc?

(A) $-\frac{3}{49}$

(B) $-\frac{1}{28}$

(C) $\frac{3\sqrt{7}}{64}$

(D) $\frac{1}{32}$

(E) $\frac{1}{28}$

SQ9

Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

SQ10

Let $a = \frac{\pi}{2008}$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

SQ11

Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

SQ12

Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

SQ13

Show that if at least one of the four angles $A \pm B \pm C$ is a multiple of π, then

$$\sin^4 A + \sin^4 B + \sin^4 C - 2\sin^2 B \sin^2 C - 2\sin^2 C \sin^2 A - 2\sin^2 A \sin^2 B + 4\sin^2 A \sin^2 B \sin^2 C = 0.$$

SQ14

A curve has the equation $y = f(x)$, where

$$f(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of $f(x)$.

(ii) Determine all values of x in the interval $-\pi \leq x \leq \pi$ for which $f(x) = 0$. Find a value of x in this interval at which the curve touches the x-axis without crossing it.

(iii) Find the value or values of x in the interval $0 \leq x \leq 2\pi$ for which $f(x) = 2$.

10
251

10

252

11 Trigonometric Equations

What's on the Specification?

- Solution of simple trigonometric equations in a given interval; for example: $\tan x = -\frac{1}{\sqrt{3}}$ for $-\pi < x < \pi$; $\sin^2\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $-2\pi < x < 2\pi$; $12\cos^2 x + 6\sin x - 10 = 2$ for $0° < x < 360°$.

Exercises E11

Time Allowed

No limit

Number of Questions

18

Difficulty

★★★☆

Exercises E11

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

How many solutions does the equation

$$\sin 2x = \cos x$$

have in the range $0 \leq x \leq \pi$?

(A) one solution

(B) two solutions

(C) three solutions

(D) four solutions

Quiz Pre-2

The fraction of the interval $0 \leq x \leq 2\pi$, for which one (or both) of the inequalities

$$\sin x \geq \frac{1}{2}, \quad \sin 2x \geq \frac{1}{2}$$

is true, equals

(A) $\frac{1}{3}$

(B) $\frac{13}{24}$

(C) $\frac{7}{12}$

(D) $\frac{5}{8}$

Quiz Pre-3

The non-zero real number c is such that the equation $\cos x = c$ has two solutions for $0 < x < \frac{3}{2}\pi$.

How many solutions of the equation $\cos^2 2x = c^2$ are there in the range $0 < x < \frac{3}{2}\pi$?

(A) 2

(B) 3

(C) 4

(D) 6

(E) 7

(F) 8

Ex. 1

In the range $0 \le x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

(A) has 0 solutions.

(B) has 1 solution.

(C) has 2 solutions.

(D) holds for all values of x.

Ex. 2

Given that x is real, what is the range of $y = 2^{\sin^2 x} + 2^{\cos^2 x}$?

Ex. 3

In the interval $0 \le x \le 2\pi$, the equation

$$\sin(2\cos(2x) + 2) = 0$$

has exactly

(A) 2 solutions.

(B) 3 solutions.

(C) 4 solutions.

(D) 6 solutions.

(E) 8 solutions.

Ex. 4

Find the values of x such that $0 \le x \le 2\pi$ and $\sin 2x = |\sin x|$.

Ex. 5

(i) Find all solutions of the equation $16x^4 - 8x^2 = -1$.

(ii) Show that

$$\sin 3\theta = 4\cos^2\theta \sin\theta - \sin\theta$$

and that

$$\sin 5\theta = 16\cos^4\theta \sin\theta - 12\cos^2\theta \sin\theta + \sin\theta.$$

(iii) Find the general solution of the equation

$$\sin 5\theta + \sin 3\theta + \sin\theta = 0.$$

Ex. 6

Solve the inequality

$$\frac{\sin\theta+1}{\cos\theta}\leq 1.$$

Quiz 1

In the range $0 \leq x < 2\pi$ the equation $\cos(\sin x) = \frac{1}{2}$ has

(A) no solution.

(B) one solution.

(C) two solutions.

(D) three solutions.

Quiz 2

Find the number of solutions and the sum of the solutions of the equation

$$1 - 2\cos^2 x = |\cos x|$$

where $0 \leq x \leq 180°$

(A) Number of solutions $= 2$ Sum of solutions $= 180°$

(B) Number of solutions $= 2$ Sum of solutions $= 240°$

(C) Number of solutions $= 3$ Sum of solutions $= 180°$

(D) Number of solutions $= 3$ Sum of solutions $= 360°$

(E) Number of solutions $= 4$ Sum of solutions $= 240°$

(F) Number of solutions $= 4$ Sum of solutions $= 360°$

Quiz 3

Which of the following are true for all real values of x? All arguments are in radians.

$$\begin{array}{ll} \text{I} & \sin\left(\frac{\pi}{2}+x\right) = \cos\left(\frac{\pi}{2}-x\right) \\ \text{II} & 2 + 2\sin(x) - \cos^2(x) \geq 0 \\ \text{III} & \sin\left(x+\frac{3\pi}{2}\right) = \cos(\pi - x) \\ \text{IV} & \sin(x)\cos(x) \leq \frac{1}{4} \end{array}$$

(A) I and II

(B) I and III

(C) II and III

(D) III and IV

(E) II and IV

Quiz 4

What is the value, in radians, of the largest angle x in the range $0 \le x \le 2\pi$ that satisfies the equation $8\sin^2 x + 4\cos^2 x = 7$?

(A) $\frac{2\pi}{3}$

(B) $\frac{5\pi}{6}$

(C) $\frac{4\pi}{3}$

(D) $\frac{5\pi}{3}$

(E) $\frac{7\pi}{4}$

(F) $\frac{11\pi}{6}$

Quiz 5

How many values of θ in the interval $0 < \theta \le 2\pi$ satisfy

$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

(A) 2

(B) 4

(C) 5

(D) 6

(E) 8

Quiz 6

How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Ex. 7

(i) Show that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ and find a similar expression for $\sin 15°$.

(ii) Show that $\cos\alpha$ is a root of the equation

$$4x^3 - 3x - \cos 3\alpha = 0,$$

and find the other two roots in terms of $\cos\alpha$ and $\sin\alpha$.

(iii) Use parts (i) and (ii) to solve the equation $y^3 - 3y - \sqrt{2} = 0$, giving your answers in surd form.

Ex. 8

Show that $\sin A = \cos B$ if and only if $A = (4n+1)\frac{\pi}{2} \pm B$ for some integers n.

Show also that $|\sin x \pm \cos x| \le \sqrt{2}$ for all values of x and deduce that there are no solutions to the equation $\sin(\sin x) = \cos(\cos x)$.

Sketch, on the same axes, the graphs of $y = \sin(\sin x)$ and $y = \cos(\cos x)$. Sketch, not on the previous axes, the graph of $y = \sin(2\sin x)$.

Ex. 9

In this question, do not consider the special cases in which the denominators of any of your expressions are zero.

Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc.

Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$.

The four real numbers $\theta_1, \theta_2, \theta_3$ and θ_4 lie in the range $0 \le \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

Practices P11

Time Allowed

90 min

Number of Questions

20

Difficulty

★★★★

Practices P11

Scan the QR code or click the link above to take the practice online.

Q1

Given that $7\cos\theta - 3\tan\theta\sin\theta = 1$, which one of the following is true?

(A) $\cos\theta = -\frac{3}{5}$ or $-\frac{1}{2}$

(B) $\cos\theta = -\frac{3}{5}$ or $\frac{1}{2}$

(C) $\cos\theta = \frac{3}{5}$ or $\frac{1}{2}$

(D) $\cos\theta = \frac{3}{5}$ or $-\frac{1}{2}$

Q2

It is given that

$$7\cos x + \tan x \sin x = 5$$

where $0° < x < 90°$

What are the possible values of $\tan x$?

(A) $\frac{1}{2}$ or $\frac{1}{3}$

(B) $\frac{1}{\sqrt{3}}$ or $\frac{1}{2\sqrt{2}}$

(C) $\frac{\sqrt{3}}{2}$ or $\frac{2\sqrt{2}}{3}$

(D) $\sqrt{3}$ or $2\sqrt{2}$

(E) 3 or 2

11
261

Q3

k is the smallest positive value of x which is a solution to **both** the equations $2\sin x + 1 = 0$ **and** $2\cos 2x = 1$.

How many values of x in the range $0 \le x \le k$ are solutions to at least one of these equations?

(A) 0

(B) 2

(C) 3

(D) 4

(E) 8

Q4

How many solutions of the equation $2\sin^3\theta = \sin\theta$ lie in the interval $-\frac{\pi}{2} \le \theta \le \pi$?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

(F) 7

Q5

The three internal angles in a triangle are α, β and θ, and

$$3\tan\alpha - 2\sin\beta = 2$$

$$5\tan\alpha + 6\sin\beta = 8$$

What is the value of θ in degrees?

(A) 15

(B) 45

(C) 75

(D) 105

(E) 135

Q6

The number of solutions in the interval $0 \le \theta \le 4\pi$ of the equation $\sin^2\theta + 3\cos\theta = 3$ is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

Q7

Find the complete set of values of x, with $0 \leq x \leq \pi$, for which

$$(1 - 2\sin x)\cos x \geq 0$$

(A) $0 \leq x \leq \frac{\pi}{6}, \frac{\pi}{2} \leq x \leq \frac{5\pi}{6}$

(B) $0 \leq x \leq \frac{\pi}{6}, \frac{5\pi}{6} \leq x \leq \pi$

(C) $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}, \frac{5\pi}{6} \leq x \leq \pi$

(D) $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

Q8

Find the number of solutions of the equation

$$x \sin 2x = \cos 2x$$

with $0 \leq x \leq 2\pi$.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Q9

Find the maximum angle x in the range $0° \leq x \leq 360°$ which satisfies the equation

$$\cos^2(2x) + \sqrt{3}\sin(2x) - \frac{7}{4} = 0$$

(A) 30°

(B) 60°

(C) 120°

(D) 150°

(E) 210°

(F) 240°

(G) 300°

(H) 330°

Q10

How many solutions does the equation $x \tan x = 1$ have in the interval $-2\pi \leq x \leq 2\pi$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

Q11

How many real solutions are there to the equation

$$3\cos x = \sqrt{x}$$

Where x is in radians?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) infinitely many

Q12

x satisfies the simultaneous equations

$$\sin 2x + \sqrt{3}\cos 2x = -1$$

and

$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{3}$$

where $0° \leq x \leq 360°$.

Find the sum of the possible values of x.

(A) $210°$

(B) $330°$

(C) $390°$

(D) $660°$

(E) $780°$

(F) $930°$

Q13

It is given that

$$y = (1 + 2\cos x)\cos 2x \quad \text{for} \quad 0 < x < \pi.$$

The complete set of values of x for which y is negative is

(A) $0 < x < \frac{\pi}{4}, \ \frac{2\pi}{3} < x < \frac{3\pi}{4}$

(B) $0 < x < \frac{\pi}{4}, \ \frac{3\pi}{4} < x < \pi$

(C) $0 < x < \frac{2\pi}{3}, \ \frac{3\pi}{4} < x < \pi$

(D) $0 < x < \frac{2\pi}{3}, \ \frac{3\pi}{4} < x < \pi$

(E) $\frac{\pi}{4} < x < \frac{2\pi}{3}$

(F) $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Q14

Find the fraction of the interval $0 \leq \theta \leq \pi$ for which the inequality

$$\left(\sin(2\theta) - \frac{1}{2}\right)(\sin\theta - \cos\theta) \geq 0$$

is satisfied.

(A) $\frac{1}{12}$

(B) $\frac{1}{6}$

(C) $\frac{1}{4}$

(D) $\frac{5}{12}$

(E) $\frac{7}{12}$

(F) $\frac{3}{4}$

(G) $\frac{5}{6}$

(H) $\frac{11}{12}$

Q15

The angle x is measured in radians and is such that $0 \leq x \leq \pi$.

The total length of any intervals for which $-1 \leq \tan x \leq 1$ and $\sin 2x \geq 0.5$ is

(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{3}$

(E) $\frac{5\pi}{12}$

(F) $\frac{\pi}{2}$

(G) $\frac{5\pi}{6}$

Q16

Which one of the following is a **necessary and sufficient** condition for

$$\sum_{k=1}^{n} \sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

to be true?

(A) $n = 1$

(B) n is a multiple of 3

(C) n is a multiple of 6

(D) n is 1 more than a multiple of 3

(E) n is 1 more than a multiple of 6

(F) n is 1 more than a multiple of 6 or n is 2 more than a multiple of 6

Q17

Consider these simultaneous equations where c is a constant:

$$y = 3\sin x + 2$$
$$y = x + c$$

Which of the following statements is/are true?

1 For some value of c: there is exactly one solution with $0 \le x \le \pi$ **and** there is at least one solution with $-\pi < x < 0$.

2 For some value of c: there is exactly one solution with $0 \le x \le \pi$ **and** there are no solutions with $-\pi < x < 0$.

3 For some value of c: there is exactly one solution with $0 \le x \le \pi$ **and** there are no solutions with $x > \pi$.

(A) none

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Q18

The angle θ can take any of the values $1°, 2°, 3°, \dots, 359°, 360°$.

For how many of these values of θ is it true that

$$\sin\theta\sqrt{1+\sin\theta}\sqrt{1-\sin\theta}+\cos\theta\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}=0$$

(A) 0

(B) 1

(C) 2

(D) 4

(E) 93

(F) 182

(G) 271

(H) 360

Q19

(i) Show that $2\sin\left(\frac{1}{2}\theta\right)=\sin\theta$ if and only if $\sin\left(\frac{1}{2}\theta\right)=0$.

(ii) Solve the equation $2\tan\left(\frac{1}{2}\theta\right)=\tan\theta$.

(iii) Show that $2\cos\left(\frac{1}{2}\theta\right)=\cos\theta$ if and only if $\theta=(4n+2)\pi\pm 2\varphi$ where φ is defined by $\cos\varphi=\frac{1}{2}(\sqrt{3}-1)$, $0\le\varphi\le\frac{\pi}{2}$, and n is any integer.

Q20

Show that if $\cos(x-\alpha)=\cos\beta$ then either $\tan x=\tan(\alpha+\beta)$ or $\tan x=\tan(\alpha-\beta)$. By choosing suitable values of x, α and β, give an example to show that if $\tan x=\tan(\alpha+\beta)$, then $\cos(x-\alpha)$ need not equal to $\cos\beta$.

Let ω be the acute angle such that $\tan\omega=\frac{4}{3}$.

(i) For $0\le x\le 2\pi$, solve the equation

$$\cos x-7\sin x=5$$

giving both solutions in terms of ω.

(ii) For $0\le x\le 2\pi$, solve the equation

$$2\cos x+11\sin x=10$$

showing that one solution is twice the other and giving both in terms of ω.

Supplements S11

Time Allowed

90 min

Number of Questions

17

Difficulty

★★★☆

Supplements S11

Scan the QR code or click the link above to take the practice online.

SQ1

Find the number of solutions of the equation

$$14\cos^3 x + 10\sin^2 x\cos x = 13\cos x$$

in the range $-2\pi \le x \le 2\pi$.

(A) 4

(B) 6

(C) 8

(D) 10

(E) 12

(F) 14

SQ2

The angle x is measured in radians and is such that $0 \le x \le \pi$.

The total length of any intervals for which $-1 \le \tan x \le 1$ and $\sin 2x \ge 0.5$ is

(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{3}$

(E) $\frac{5\pi}{12}$

(F) $\frac{\pi}{2}$

(G) $\frac{5\pi}{6}$

SQ3

The minimum value achieved by the function

$$f(x) = 9\cos^4 x - 12\cos^2 x + 7$$

Equals

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

SQ4

How many values of x satisfy the equation

$$\sin 2x + \sin^2 x = 1$$

in the range $0 \leq x < 2\pi$?

(A) 2

(B) 4

(C) 6

(D) 8

SQ5

In the range $0 \leq x < 2\pi$, the equation

$$\sin^2 x + 3 \sin x \cos x + 2 \cos^2 x = 0$$

has

(A) 1 solution.

(B) 2 solutions.

(C) 3 solutions.

(D) 4 solutions.

SQ6

The number of solutions x to the equations

$$7 \sin x + 2 \cos^2 x = 5$$

in the range $0 \leq x < 2\pi$, is

(A) 1

(B) 2

(C) 3

(D) 4

SQ7

How many solutions does $\cos^n(x) + \cos^{2n}(x) = 0$ have in the range $0 \leq x \leq 2\pi$ for an integer $n \geq 1$?

(A) 1 for all n

(B) 2 for all n

(C) 3 for all n

(D) 2 for even n and 3 for odd n

(E) 3 for even n and 2 for odd n

SQ8

The simultaneous equations in x, y,

$$(\cos\theta)x - (\sin\theta)y = 2$$
$$(\sin\theta)x + (\cos\theta)y = 1$$

are solvable

(A) for all values of θ in the range $0 \le \theta < 2\pi$.

(B) except for one value of θ in the range $0 \le \theta < 2\pi$.

(C) except for two values of θ in the range $0 \le \theta < 2\pi$.

(D) except for three values of θ in the range $0 \le \theta < 2\pi$.

SQ9

How many values of x satisfy the equation

$$2\cos^2 x + 5\sin x = 4$$

in the range $0 \le x < 2\pi$?

(A) 2

(B) 4

(C) 6

(D) 8

SQ10

In the interval $0 \le x < 360°$, the number of solutions of the equation

$$\sin^3 x + \cos^2 x = 0$$

is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

SQ11

In the range $0 \le x < 2\pi$ the equation $\sin^8 x + \cos^6 x = 1$ has

(A) 3 solutions.

(B) 4 solutions.

(C) 6 solutions.

(D) 8 solutions.

SQ12

In the range $-90° < x < 90°$, how many values of x are there for which the sum to infinity

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots$$

equals $\tan x$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

SQ13

In the range $0 \le x < 2\pi$ the equation $(3 + \cos x)^2 = 4 - 2\sin^8 x$ has

(A) 0 solution.

(B) 1 solution.

(C) 2 solutions.

(D) 3 solutions.

SQ14

How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

Have in the range $0 \le x < 2\pi$?

(A) 0

(B) 1

(C) 2

(D) 3

SQ15

How many solutions does the equation $\tan(2x) = \cos\left(\frac{x}{2}\right)$ have on the interval $[0, 2\pi]$?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

SQ16

Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

SQ17

Let

$$y = \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}.$$

(i) Given that x is real, show that

$$(y\cos\theta - \sin\theta)^2 \geq 4(y-1)^2.$$

Deduce that

$$y^2 + 1 \geq 4(y-1)^2,$$

and hence that

$$\frac{4-\sqrt{7}}{3} \leq y \leq \frac{4+\sqrt{7}}{3}.$$

(ii) In the case $y = \frac{4+\sqrt{7}}{3}$, show that

$$\sqrt{y^2+1} = 2(y-1)$$

and find the corresponding values of x and $\tan\theta$.

12 Exponentials and Logarithms

What's on the Specification?

- Laws of indices for all rational exponents.
- $y = a^x$ and its graph, for simple positive values of a.
- Laws of logarithms:

$$a^b = c \Leftrightarrow b = \log_a c$$
$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$
$$k \log_a x = \log_a (x^k)$$

including the special cases:

$$\log_a \frac{1}{x} = -\log_a x$$
$$\log_a a = 1$$

- Questions requiring knowledge of the change of base formula will not be set.
- The solution of equations of the form $a^x = b$, and equations which can be reduced to this form, including those that need prior algebraic manipulation; for example, $3^{2x} = 4$ and $25^x - 3 \times 5^x + 2 = 0$.

Exercises E12

Time Allowed

No limit

Number of Questions

27

Difficulty

★★★☆

Exercises E12

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

Which is the smallest of these values?

(A) $\log_{10} \pi$

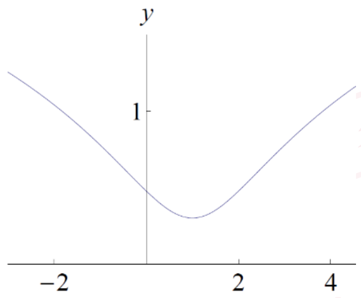
(B) $\sqrt{\log_{10}(\pi^2)}$

(C) $\left(\frac{1}{\log_{10} \pi}\right)^3$

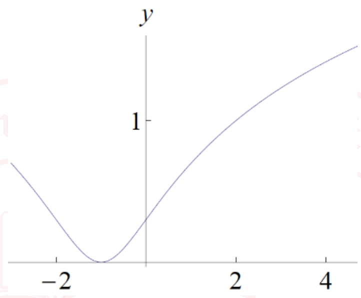
(D) $\frac{1}{\log_{10} \sqrt{\pi}}$

Quiz Pre-2

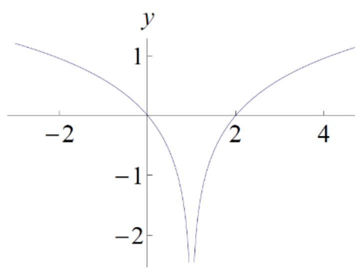
The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



(A)



(B)



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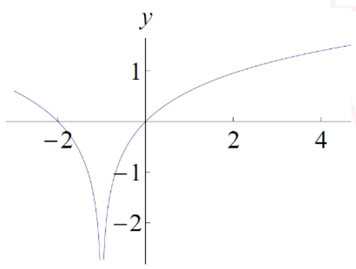
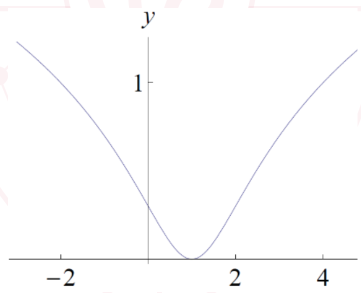


(D)



(E)

Quiz Pre-3

The equation

$$8^x + 4 = 4^x + 2^{x+2}$$

has

(A) no real solutions.

(B) one real solution.

(C) two real solutions.

(D) three real solutions.

Ex. 1

If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

(A) pq

(B) $\frac{3p+q}{5}$

(C) $\frac{1+3pq}{p+q}$

(D) $\frac{3pq}{1+3pq}$

[Tips: use the definition of the logarithm.]

Ex. 2

Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$, the logarithm of 3 to base 2, is

(A) between $1\frac{1}{3}$ and $1\frac{1}{2}$.

(B) between $1\frac{1}{2}$ and $1\frac{2}{3}$.

(C) between $1\frac{2}{3}$ and 2.

(D) none of the above.

Ex. 3

Which pairs are the same?

(A) $y = \log_{10} x^2$, $y = 2\log_{10} x$.

(B) $y = \log_{10} x$, $y = \log_{10} \frac{x^2}{x}$.

(C) $y = 10^{\log_{10} x}$, $y = x$.

(D) $y = \log_{10}(x^2 - 1)$, $y = \log_{10}(x+1) + \log_{10}(x-1)$.

Ex. 4

Given that

$$\log_{10} 2 = 0.3010 \text{ to 4 d.p. and that } 10^{0.2} < 2$$

it is possible to deduce that

(A) 2^{100} begins in a 1 and is 30 digits long.

(B) 2^{100} begins in a 2 and is 30 digits long.

(C) 2^{100} begins in a 1 and is 31 digits long.

(D) 2^{100} begins in a 2 and is 31 digits long.

Ex. 5 (Ex. 1 again)

If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

(A) pq

(B) $\frac{3p+q}{5}$

(C) $\frac{1+3pq}{p+q}$

(D) $\frac{3pq}{1+3pq}$

[Tips: use Formula 1 of the Chang-of-Base Theorem.]

Ex. 6 (Ex. 1 again)

If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

(A) pq

(B) $\frac{3p+q}{5}$

(C) $\frac{1+3pq}{p+q}$

(D) $\frac{3pq}{1+3pq}$

[Tips: use Formula 3 of the Chang-of-Base Theorem.]

Ex. 7

Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

Ex. 8

Let a, b and c be positive integers greater than 1. Show that if $x = \log_a b$, $y = \log_b c$ and $z = \log_c a$, then $xyz = 1$.

Ex. 9

Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers. Then N will definitely be a square number whenever

(A) k is even

(B) $k + n$ is odd

(C) k is odd but $m + n$ is even

(D) $k + n$ is even

[Notes: can you distinguish **whole numbers**, **integers** and **natural numbers**?]

Ex. 10

Given that

$$2^{m+1} + 2^m = 3^{n+2} - 3^n$$

and that m and n are integers, find the values of m and n.

Ex. 11

Find the values of x that satisfy the equation

$$3^{2x} - 34 \times 15^{x-1} + 5^{2x} = 0$$

Ex. 12

The inequality $2^n > n^2$ is true for:

(A) no integers $n \geq 0$

(B) all integers $n \geq 0$

(C) all integers $n > 4$

(D) all integers $n \geq 4$

Ex. 13

Let a, b, c be positive numbers. There are finitely many positive whole numbers x, y which satisfy the inequality

$$a^x > cb^y$$

if

(A) $a > 1$ or $b < 1$

(B) $a < 1$ or $b < 1$

(C) $a < 1$ and $b < 1$

(D) $a < 1$ and $b > 1$

Quiz 1

Three positive numbers a, b, c satisfy

$$\log_b a = 2, \qquad \log_b(c-3) = 3, \qquad \log_a(c+5) = 2$$

This information

(A) specifies a uniquely.

(B) is satisfied by two values of a.

(C) is satisfied by infinitely many values of a.

(D) is contradictory.

Quiz 2

Using the observation that $2^5 \approx 3^3$, it is possible to deduce that $\log_3 2$ is approximately

(A) $\frac{3}{5}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{3}$

(E) $\frac{1}{2}$

(F) 2

Quiz 3

Which is the largest of the following four numbers?

(A) $\log_2 3$

(B) $\log_4 8$

(C) $\log_3 2$

(D) $\log_5 10$

Quiz 4

The numbers a, b and c are each greater than 1.

The following logarithms are all to the same base:

$$\log(ab^2c) = 7$$
$$\log(a^2bc^2) = 11$$
$$\log(a^2b^2c^3) = 15$$

What is this base?

(A) a

(B) b

(C) c

(D) It is possible to determine the base, but the base is not a, b or c.

(E) There is insufficient information given to determine the base.

Quiz 5

Given that c and d are non-zero integers, the expression $\frac{10^{c-2d}\times 20^{2c+d}}{8^c\times 125^{c+d}}$ is an integer if

(A) $c < 0$

(B) $d < 0$

(C) $c < 0$ and $d < 0$

(D) $c < 0$ and $d > 0$

(E) $c > 0$ and $d < 0$

(F) $c > 0$ and $d > 0$

(G) $d > 0$

(H) $c > 0$

Quiz 6

The equation

$$9^x - 3^{x+1} = k$$

Has one or more real solutions precisely when

(A) $k \geq -\frac{9}{4}$

(B) $k > 0$

(C) $k \leq -1$

(D) $k \geq \frac{5}{8}$

Quiz 7

The solution of the simultaneous equations

$$2^x + 3 \times 2^y = 3$$
$$2^{2x} - 9 \times 2^{2y} = 6$$

is $x = p, y = q$.

Find the value of $p - q$.

(A) $\frac{5}{12}$

(B) $\frac{7}{3}$

(C) $\log_2 \frac{5}{12}$

(D) $\log_2 \frac{7}{3}$

(E) $\log_2 9$

(F) $\log_2 15$

Quiz 8

Which one of the following numbers is largest in value?

(All angles are given in radians.)

(A) $\tan\left(\frac{3\pi}{4}\right)$

(B) $\log_{10} 100$

(C) $\sin^{10}\left(\frac{\pi}{2}\right)$

(D) $\log_2 10$

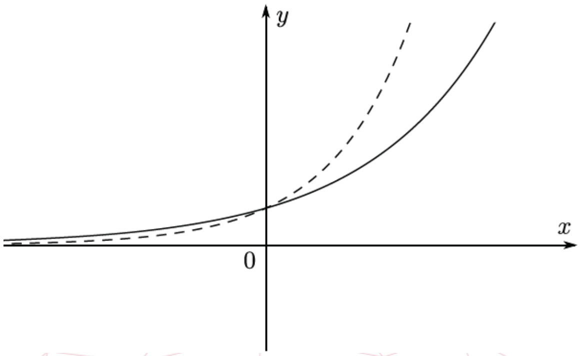
(E) $\left(\sqrt{2} - 1\right)^{10}$

Quiz 9

The graphs of two functions are shown here:

$y = a^x$ is shown with a solid line, where a is a positive real number

$y = f(x)$ is shown with a dashed line



Which of the following statements (1, 2, 3, 4) could be true?

1. $f(x) = b^x$ for some $b > a$
2. $f(x) = b^x$ for some $b < a$
3. $f(x) = a^{kx}$ for some $k > 1$
4. $f(x) = a^{kx}$ for some $k < 1$

(A) 1 only

(B) 2 only

(C) 3 only

(D) 4 only

(E) 1 and 3 only

(F) 2 and 3 only

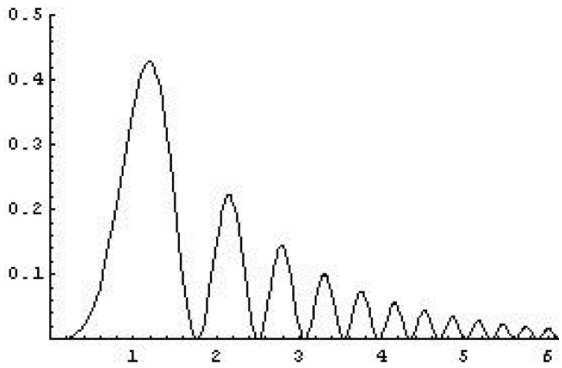
(G) 2 and 3 only

(H) 2 and 4 only

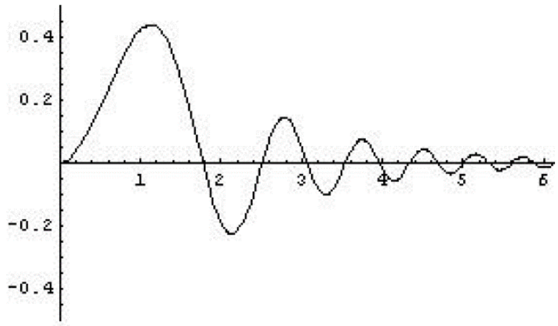
Quiz 10

On which of the axes below is a sketch of the graph

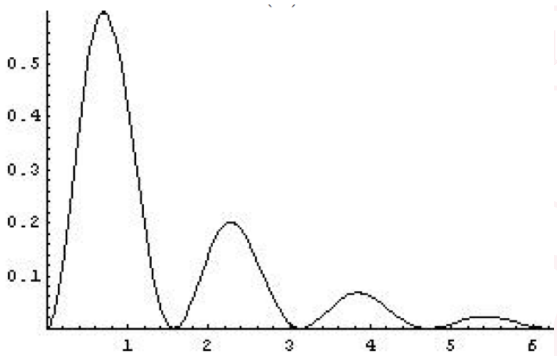
$$y = 2^{-x}\sin^2(x^2)\,?$$



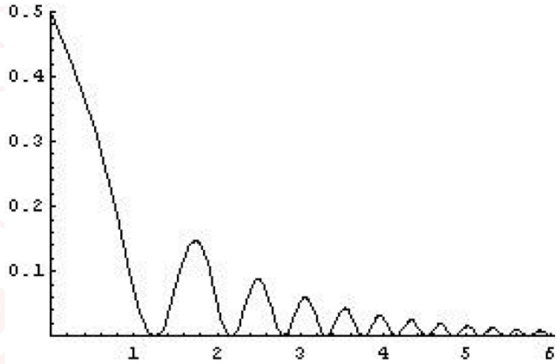
(A)



(B)



(C)



(D)

Ex. 14

(i) Sketch the curve $y = e^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $e^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

[You may assume that $x^n e^x \to 0$ as $x \to -\infty$ for any integer n.]

(ii) Sketch the curve $y = e^{x^2}(2x^4 - 5x^2 + 2)$.

Practices P12

Time Allowed

100 min

Number of Questions

25

Difficulty

★★★☆

[Practices P12]

Scan the QR code or click the link above to take the practice online.

Q1

Given that

$$9^{2x-1} \times \frac{1}{27^x} = 81^x$$

what is the value of x?

(A) $-\frac{2}{3}$

(B) $-\frac{2}{5}$

(C) $-\frac{1}{3}$

(D) $-\frac{1}{4}$

(E) $-\frac{1}{5}$

Q2

Given that

$$5^{x+1} \times 25^{3x} = 125^{19}$$

what is the value of x?

(A) 3

(B) $\frac{9}{2}$

(C) 8

(D) 9

(E) $\frac{19}{4}$

Q3

The positive real numbers a and b satisfy the simultaneous equations:

$$\log_2 4a - \log_2 b = 4$$

$$\log_2 a + \log_2 2b = 3$$

What is the value of $a - 2b$?

(A) 0

(B) $\frac{8}{9}$

(C) 2

(D) $\frac{8}{3}$

(E) 4

(F) 6

Q4

x satisfies the equation

$$\log_3(k + \log_5 x) = 1$$

Which one of the following is an expression for x in terms of k?

(A) $x = 3^{5-k}$

(B) $x = 3^{5+k}$

(C) $x = 5^{3-k}$

(D) $x = 5^{3+k}$

(E) $x = 3^5 - k$

(F) $x = 3^5 + k$

(G) $x = 5^3 - k$

(H) $x = 5^3 + k$

Q5

Given that $a^x b^{2x} c^{3x} = 2$, where a, b and c are positive real numbers, then $x =$

(A) $\log_{10}\left(\frac{2}{a+2b+3c}\right)$

(B) $\frac{\log_{10} 2}{\log_{10}(a+2b+3c)}$

(C) $\frac{2}{\log_{10}(a+2b+3c)}$

(D) $\frac{2}{a+2b+3c}$

(E) $\log_{10}\left(\frac{2}{ab^2c^3}\right)$

(F) $\frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$

(G) $\frac{2}{\log_{10}(ab^2c^3)}$

(H) $\frac{2}{ab^2c^3}$

Q6

Which one of the following numbers is largest in value?

(All angles are given in radians.)

(A) $\tan\left(\frac{3\pi}{4}\right)$

(B) $\log_{10} 100$

(C) $\sin^{10}\left(\frac{\pi}{2}\right)$

(D) $\log_2 10$

(E) $\left(\sqrt{2}-1\right)^{10}$

Q7

The sum of the roots of the equation $2^{2x} - 8 \times 2^x + 15 = 0$ is

(A) 3

(B) 8

(C) $2\log_{10} 2$

(D) $\log_{10}\left(\frac{15}{4}\right)$

(E) $\frac{\log_{10} 15}{\log_{10} 2}$

Q8

What is the value of x that makes the following expression correct?

$$2^{3+2x}4^x8^{-x} = 4\sqrt{2}$$

(A) -2.25

(B) -1.75

(C) -1.5

(D) -0.5

(E) -0.25

Q9

Which one of the following is the real solution of the equation

$$3 \times 5^{2x+1} - 5^x - 2 = 0$$

(A) $x = \log_5\left(\frac{1}{3}\right)$

(B) $x = \log_5\left(\frac{2}{5}\right)$

(C) $x = \log_5\left(\frac{3}{5}\right)$

(D) $x = \log_5\left(\frac{2}{3}\right)$

(E) $x = \log_5\left(\frac{5}{3}\right)$

(F) $x = \log_5\left(\frac{5}{2}\right)$

Q10

Which of the following is a solution to the equation $3^{(2x+1)} - 6(3^x) = 0$?

(A) $\log_2 3$

(B) $\log_3 2$

(C) 2

(D) $\log_{10} 2$

(E) $\frac{2}{3}$

Q11

Given that $y = -\log_{10}(1-x)$ for $x < 1$, find x in terms of y.

(A) $x = -\frac{1}{\log_{10}(1-y)}$

(B) $x = 1 + \log_{10} y$

(C) $x = 1 - \log_{10} y$

(D) $x = 1 - 10^{-y}$

(E) $x = 10^{-y} - 1$

(F) $x = 10^{1-y}$

Q12

Given that $a^x b^{2x} c^{3x} = 2$, where a, b, and c are positive real numbers, then $x =$

(A) $\log_{10}\left(\frac{2}{a+2b+3c}\right)$

(B) $\frac{\log_{10} 2}{\log_{10}(a+2b+3c)}$

(C) $\frac{2}{\log_{10}(a+2b+3c)}$

(D) $\frac{2}{a+2b+3c}$

(E) $\log_{10}\left(\frac{2}{ab^2c^3}\right)$

(F) $\frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$

(G) $\frac{2}{\log_{10}(ab^2c^3)}$

(H) $\frac{2}{ab^2c^3}$

Q13

The sum of the roots of the equation $2^{2x} - 8 \times 2^x + 15 = 0$ is

(A) 3

(B) 8

(C) $2\log_{10} 2$

(D) $\log_{10}\left(\frac{15}{4}\right)$

(E) $\frac{\log_{10} 15}{\log_{10} 2}$

Q14

a, b, x, and y are real and positive.

a and b are constants.

x and y are related.

A graph of $\log y$ against $\log x$ is drawn.

For which one of the following relationships will this graph be a straight line?

(A) $y^b = a^x$

(B) $y = ab^x$

(C) $y^2 = a + x^b$

(D) $y = ax^b$

(E) $y^x = a^b$

Q15

Given that

$$2^{3x} = 8^{(y+3)}$$

and

$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$

What is the value of $x + y$?

(A) -23

(B) -22

(C) -15

(D) -14

(E) -11

(F) -10

Q16

The real roots of the equation $4^{2x} + 12 = 2^{2x+3}$ are p and q, where $p > q$.

The values of $p - q$ can be expressed as

(A) $\frac{3}{4}$

(B) 1

(C) 4

(D) $-\frac{1}{2} + \log_{10}\frac{3}{2}$

(E) $\frac{\log_{10} 3}{\log_{10} 4}$

(F) $\frac{\log_{10} 3}{\log_{10} 2}$

Q17

Find the sum of the real solutions of the equation:

$$3^x - \left(\sqrt{3}\right)^{x+4} + 20 = 0$$

(A) 1

(B) 4

(C) 9

(D) $\log_3 20$

(E) $2\log_3 20$

(F) $4\log_3 20$

Q18

Find the sum of the real values of x that satisfy the simultaneous equations:

$$\log_3(xy^2) = 1$$
$$(\log_3 x)(\log_3 y) = -3$$

(A) $\frac{1}{3}$

(B) 1

(C) 3

(D) $3\frac{1}{9}$

(E) $9\frac{1}{27}$

(F) $9\frac{1}{3}$

(G) 27

(H) $27\frac{1}{9}$

Q19

Find the real non-zero solution to the equation

$$\frac{2^{(9^x)}}{8^{(3^x)}} = \frac{1}{4}$$

(A) $\log_3 2$

(B) $2\log_3 2$

(C) 1

(D) 2

(E) $\log_2 3$

(F) $2\log_2 3$

Q20

Given the simultaneous equations

$$\log_{10} 2 + \log_{10}(y-1) = 2\log_{10} x$$
$$\log_{10}(y + 3 - 3x) = 0$$

the values of y are

(A) $\frac{5}{2} \pm \frac{3\sqrt{5}}{2}$

(B) $3 \pm \sqrt{3}$

(C) $7 \pm 3\sqrt{3}$

(D) $3, 9$

(E) $1, 13$

Q21

Find the positive difference between the two real values of x for which

$$(\log_2 x)^4 + 12\left(\log_2\left(\frac{1}{x}\right)\right)^2 - 2^6 = 0$$

(A) 4

(B) 16

(C) $\frac{15}{4}$

(D) $\frac{17}{4}$

(E) $\frac{255}{16}$

(F) $\frac{257}{16}$

Q22

The graph of $y = \log_{10} x$ is translated in the positive y-direction by 2 units.

This translation is equivalent to a stretch of factor k parallel to the x-axis.

What is the value of k?

(A) 0.01

(B) $\log_{10} 2$

(C) 0.5

(D) 2

(E) $\log_2 10$

(F) 100

Q23

Three **real** numbers x, y and z satisfy $x > y > z > 1$.

Which one of the following statements must be true?

(A) $\frac{2^{z+1}}{2^x} > \frac{2^x+2^z}{2^y}$

(B) $2 > \frac{3^x+3^z}{3^y}$

(C) $\frac{2\times5^x}{5^z} > \frac{5^x+5^z}{5^y}$

(D) $2 < \frac{7^x+7^z}{7^y}$

Q24

To nine decimal places, $\log_{10} 2 = 0.301029996$ and $\log_{10} 3 = 0.477121255$.

(i) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places. By taking logs, or otherwise, show that

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}.$$

Hence write down the first digit of 3^{100}.

(ii) Find the first digit of each of the following numbers: 2^{1000}; $2^{10\,000}$; and $2^{100\,000}$.

Q25

If $x = \log_b c$, express c in terms of b and x and prove that $\frac{\log_a c}{\log_a b} = \log_b c$.

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2.$$

(ii) Given that $\log_2 \frac{\pi}{e} > \frac{1}{5}$ and that $e^2 < 8$, prove that $\ln \pi > \frac{17}{15}$.

(iii) Given that $e^3 > 20$, $\pi^2 < 10$ and $\log_{10} 2 > \frac{3}{10}$, prove that $\ln \pi < \frac{15}{13}$.

Supplements S12

Time Allowed

120 min

Number of Questions

24

Difficulty

★★★★☆

Supplements S12

Scan the QR code or click the link above to take the practice online.

SQ1

It is given that $y = 8^p$ and $z = \left(\frac{1}{2}\right)^{2q}$ where p and q are real numbers.

Which of the following expressions is a simplification of $\log_2\left(\frac{y^3}{z^2}\right)$?

(A) $6p - 4q$

(B) $6p + 4q$

(C) $6p - 8q$

(D) $6p + 8q$

(E) $9p - 4q$

(F) $9p + 4q$

(G) $9p - 8q$

(H) $9p + 8q$

SQ2

Evaluate

$$\log_2\left(\frac{5}{4}\right) + \log_2\left(\frac{6}{5}\right) + \log_2\left(\frac{7}{6}\right) + \cdots + \log_2\left(\frac{64}{63}\right)$$

(A) -2

(B) 3

(C) 4

(D) 6

(E) $\log_2(3!)$

(F) $\log_2 60$

SQ3

Find all possible positive values of x for which $2 + 2\log_5 x = \log_5(24 + 10x)$ is true.

(A) $5 \pm 2\sqrt{6}$

(B) $\frac{6}{5}$

(C) $2\sqrt{6}$

(D) $\frac{4}{5}$

(E) There are no values of x.

SQ4

Find the product of the real roots of the equation

$$(\log_{10} x^2)^2 + \log_{10} x = 3$$

(A) $10^{-\frac{3}{2}}$

(B) 10^{-1}

(C) $10^{-\frac{1}{2}}$

(D) $10^{-\frac{1}{4}}$

(E) $10^{\frac{3}{5}}$

(F) 10^{1}

SQ5

The real solution of the equation

$$5^{2x+1} + 5^x - 4 = 0$$

can be written as

(A) $\frac{2}{\log_2 5} - 1$

(B) $2 \log_2 \frac{2}{5}$

(C) 0

(D) $\frac{2}{\log_2 5}$

(E) $\frac{1}{\log_2 5}$

SQ6

Find the sum of the real roots of the equation

$$49^x - 7^{x+1} + 12 = 0.$$

(A) 1

(B) 7

(C) $\log_{10} 12$

(D) $\log_{10}\left(\frac{12}{49}\right)$

(E) $\frac{\log_{10} 12}{\log_{10} 7}$

(F) $\frac{1}{\log_{10} 7}$

SQ7

It is given that

$$2^x = 3^y$$

and

$$x + y = 2$$

Which one of the following is an expression for x?

(A) $2\log_{10}\frac{1}{2}$

(B) $\log_{10} 3$

(C) $\frac{6}{5}$

(D) $\frac{3}{2}$

(E) $\frac{\log_{10} 9}{\log_{10} 5}$

(F) $\frac{\log_{10} 9}{\log_{10} 6}$

(G) $\frac{\log_{10} 3}{\log_{10} 2}$

SQ8

Which of the following number is largest?

(A) $((2^3)^2)^3$

(B) $(2^3)^{(2^3)}$

(C) $2^{\left((3^2)^3\right)}$

(D) $2^{\left(3^{(2^3)}\right)}$

SQ9

You are given that e^3 is approximately 20 and that 2^{10} is approximately 1000. Using this information, a student can obtain an approximate value for $\ln 2$. Which of the following is it?

(A) $\frac{7}{10}$

(B) $\frac{9}{13}$

(C) $\frac{38}{55}$

(D) $\frac{41}{59}$

SQ10

Which is the smallest of the following numbers?

(A) $\left(\sqrt{3}\right)^3$

(B) $\log_3(9^2)$

(C) $\left(3\sin\frac{\pi}{3}\right)^2$

(D) $\log_2(\log_2(8^5))$

SQ11

Let $f(x)$ be the function $e^{e^{e^x}}$. The value of $f'(x)$ when $x = \ln 3$ is which of the following?

(A) $3e^{e^3}$

(B) $3e^{e^3+3}$

(C) e^{3e+e^3}

(D) $9e^{e^3+1}$

SQ12

Which of the following expressions is equal to $\log_{10}(10 \times 9 \times 8 \times \cdots \times 2 \times 1)$?

(A) $1 + 5\log_{10} 2 + 4\log_{10} 6$

(B) $1 + 4\log_{10} 2 + 2\log_{10} 6 + \log_{10} 7$

(C) $2 + 2\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$

(D) $2 + 6\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$

(E) $2 + 6\log_{10} 2 + 4\log_{10} 6$

SQ13

Which of the following numbers is largest in value? (All angles are given in radians.)

(A) $\tan\left(\frac{5\pi}{4}\right)$

(B) $\sin^2\left(\frac{5\pi}{4}\right)$

(C) $\log_{10}\left(\frac{5\pi}{4}\right)$

(D) $\ln\left(\frac{5\pi}{4}\right)$

SQ14

Let $a, b, c > 0$. The equations

$$\log_a b = c, \qquad \log_b a = c + \frac{3}{2}, \qquad \log_c a = b,$$

(A) specify a, b and c uniquely.

(B) specify c uniquely but have infinitely many solutions for a and b.

(C) specify c and a uniquely but have infinitely many solutions for b.

(D) specify a and b uniquely but have infinitely many solutions for c.

(E) have no solutions for a, b and c.

SQ15

The number of positive values x which satisfy the equation

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

is

(A) 0

(B) 1

(C) 2

(D) 3

SQ16

How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4 - 5x^2 - 6x^3) = 2$$

(A) None.

(B) 1

(C) 2

(D) 4

(E) Infinitely many

SQ17

Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1,$$

then the greatest possible value of a is

(A) $\frac{1}{10}$

(B) 1

(C) $\sqrt{10}$

(D) $10^{\sqrt{2}}$

SQ18

The positive real numbers x and y satisfy $0 < x < y$ and

$$x2^x = y2^y$$

for

(A) no pairs x and y.

(B) exactly one pair x and y.

(C) exactly two pairs x and y.

(D) exactly four pairs x and y.

(E) infinitely many pairs x and y.

SQ19

Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b((b^x)^x) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(A) $b^2 = 4ac$

(B) $b = \frac{1}{a}$

(C) $c = \frac{b}{a}$

(D) $c = \frac{1}{b}$

(E) $a = b = c$

SQ20

Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

(A) 1

(B) $\sqrt{\log_5 6}$

(C) 2

(D) $\sqrt{\log_2 3} + \sqrt{\log_3 2}$

(E) $\sqrt{\log_2 6} + \sqrt{\log_3 6}$

SQ21

The real numbers a and b, where $a > b$, are solutions to the equation $3^{2x} - 10 \times 3^{x+1} + 81 = 0$. What is the value of $20a^2 + 18b^2$?

SQ22

Let x, y and z be real numbers satisfying the system

$$\begin{aligned}\log_2(xyz - 3 + \log_5 x) &= 5\\ \log_3(xyz - 3 + \log_5 y) &= 4\\ \log_4(xyz - 3 + \log_5 z) &= 4\end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

SQ23

Sketch the graph of the function h, where

$$h(x) = \frac{\ln x}{x}, \qquad (x > 0).$$

Hence, or otherwise, find all pairs of distinct positive integers m and n which satisfy the equation

$$n^m = m^n.$$

SQ24

By considering the maximum of $\ln x - x \ln a$, or otherwise, show that the equation $x = a^x$ has no real roots if $a > \mathrm{e}^{\frac{1}{\mathrm{e}}}$.

How many real roots does the equation have if $0 < a < 1$? Justify your answer.

12

304

13 Sequences and Series

What's on the Specification?

- Sequences, including those given by a formula for the nth term and those generated by a simple recurrence relation of the form $x_{n+1} = f(x_n)$.
- Arithmetic series, including the formula for the sum of the first n natural numbers.
- The sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$.
- Binomial expansion of $(1+x)^n$ for positive integer n, and for expressions of the form $\left(a + f(x)\right)^n$ for positive integer n and simple $f(x)$; the notations $n!$ and $\binom{n}{r}$.

Exercises E13

Time Allowed

No limit

Number of Questions

27

Difficulty

★★★☆

Exercises E13

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

Find the coefficient of the x^4 term in the expansion of

$$x^2\left(2x+\frac{1}{x}\right)^6$$

(A) 15

(B) 30

(C) 60

(D) 120

(E) 240

Quiz Pre-2

A sequence (a_n) has the property that

$$a_{n+1}=\frac{a_n}{a_{n-1}}$$

for every $n \geq 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) 2

(E) 3

Quiz Pre-3

The first three terms of an arithmetic progression are p, q and p^2 respectively, where $p < 0$.

The first three terms of a geometric progression are p, p^2 and q respectively.

Find the sum of the first 10 terms of the arithmetic progression.

(A) $\frac{23}{8}$

(B) $\frac{95}{8}$

(C) $\frac{115}{8}$

(D) $\frac{185}{8}$

Ex. 1

Let n be a positive integer. Write down an expression for the expansion of $(1+x)^n$ as a polynomial in x with coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $r = 0,1,\dots,n$.

(i) Show that

$$\binom{n}{r} = \binom{n}{n-r}.$$

(ii) By considering $(1+x)^{n-1}(1+x)$, or otherwise, show that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Ex. 2

Determine the general term of the trinomial $(x+y+z)^n$.

[You may set the general term in the form of $x^a y^b z^c$, and you should state the relationship among a, b, c and n.]

Ex. 3

Let n be a positive integer. The coefficient of x^3y^5 in the expansion of

$$(1+xy+y^2)^n$$

equals

(A) n

(B) 2^n

(C) $\binom{n}{3}\binom{n}{5}$

(D) $4\binom{n}{4}$

(E) $\binom{n}{8}$

Ex. 4

The highest power of x in

$$\{[(2x^6+7)^3+(3x^8-12)^4]^5+[(3x^5-12x^2)^5+(x^7+6)^4]^6\}^3$$

is

(A) x^{424}

(B) x^{450}

(C) x^{500}

(D) x^{504}

Ex. 5

Given an arithmetic progression u_n,

(i) show that if $m+n=p+q$, where m, n, p and q are positive integers, then $u_m+u_n=u_p+u_q$.

(ii) prove the above summation formula of an arithmetic progression.

Ex. 6

Given a geometric progression u_n,

(i) show that if $m+n=p+q$, $u_m u_n=u_p u_q$. And if $p=q$, then $u_m u_n=u_p^2$.

(ii) prove the above summation formula for a geometric progression.

Ex. 7

P and Q are two different geometric progressions.

The 3rd term of each geometric progression is 4.

The 5th term of each geometric progression is 2.

What is the modulus of the difference between the sums to infinity of P and Q?

(A) 0

(B) 8

(C) $8\sqrt{2}$

(D) 16

(E) $16\sqrt{2}$

(F) 32

(G) $32\sqrt{2}$

Ex. 8

The first term of a convergent geometric series is 8.

The fifth term is 2.

The sixth term is real and positive.

What is the sum to infinity of this series?

(The sum to infinity of a convergent geometric series is given by $\frac{a}{1-r}$, where a is the first term and r is the common ratio.)

(A) $8(1+\sqrt{2})$

(B) $8(1-\sqrt{2})$

(C) $8(2+\sqrt{2})$

(D) $8(2-\sqrt{2})$

(E) 16

(F) $\frac{8\sqrt[5]{4}}{\sqrt[5]{4}-1}$

(G) $\frac{8\sqrt[5]{4}}{\sqrt[5]{4}+1}$

Ex. 9

It is given that $\sum_{r=-1}^{n} r^2$ can be written in the form $pn^3 + qn^2 + rn + s$, where p, q, r and s are numbers. By setting $n = -1, 0, 1$ and 2, obtain four equations that must be satisfied by p, q, r and s and hence show that

$$\sum_{r=0}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

Given that $\sum_{r=-2}^{n} r^3$ can be written in the form $an^4 + bn^3 + cn^2 + dn + e$, show similarly that

$$\sum_{r=0}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

Ex. 10

The sequence of numbers $u_1, u_2, u_3, \ldots, u_n, \ldots$ is given by

$$u_1 = 2$$
$$u_{n+1} = pu_n + 3$$

where p is an integer.

The fourth term, u_4, is equal to -7.

What is the value of $u_1 + u_2 + u_3 + u_4$?

(A) -10

(B) -2

(C) -1

(D) 8

(E) 26

Ex. 11

The sequence of functions $f_1(x), f_2(x), f_3(x), \ldots$ is defined as follows:

$$f_1(x) = x^{10}$$
$$f_{n+1}(x) = xf_n'(x) \quad \text{for} \quad n \geq 1$$

where $f_n'(x) = \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}$.

Find the value of

$$\sum_{n=1}^{20} f_n(x).$$

(A) $\frac{x^{10}(x^{20}-1)}{x-1}$

(B) $\frac{x^{10}(x^{21}-1)}{x-1}$

(C) $\left(\frac{10^{20}-1}{9}\right)x^{10}$

(D) $\left(\frac{10^{21}-1}{9}\right)x^{10}$

(E) $\left(\frac{(10x)^{20}-1}{10x-1}\right)x^{10}$

(F) $\left(\frac{(10x)^{21}-1}{10x-1}\right)x^{10}$

(G) $x^{10} + x^9 + x^8 + \cdots + x + 1$

(H) $x^{10} + 10x^9 + (10 \times 9)x^8 + \cdots + (10 \times 9 \times \ldots \times 2)x + (10 \times 9 \times \ldots \times 2 \times 1)$

Ex. 12

The function $F(k)$ is defined for positive integers by $F(1) = 1, F(2) = 1, F(3) = -1$ and by the identities

$$F(2k) = F(k), \qquad F(2k+1) = F(k)$$

for $k \geq 2$. The sum

$$F(1) + F(2) + F(3) + \cdots + F(100)$$

equals

(A) -15

(B) 28

(C) 64

(D) 81

Ex. 13

(i) You are given that

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1},$$

where A and B are constants. Find the values of A and B.

(ii) Simplify

$$\frac{1}{(x-1)^{n+1}(x-2)} - \frac{1}{(x-1)^n(x-2)}.$$

(iii) You are given that

$$\frac{1}{(x-1)^n(x-2)} = \frac{A_0}{x-2} + \sum_{i=1}^{n} \frac{A_i}{(x-1)^i},$$

where $A_0, A_1, A_2, \ldots$ are constants. Using your answers to (i) and (ii), or otherwise, find the values of these constants.

Quiz 1

Sequence 1 is an arithmetic progression with first term 11 and common difference 3.

Sequence 2 is an arithmetic progression with first term 2 and common difference 5.

Some numbers that appear in Sequence 1 also appear in Sequence 2. Let N be the 20th such number.

What is the remainder when N is divided by 7?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

Quiz 2

The terms of an infinite series S are formed by adding together the corresponding terms in two infinite geometric series, T and U.

The first term of T and the first term of U are each 4.

In order, the first three terms of the combined series S are 8, 3, and $\frac{5}{4}$.

What is the sum to infinity of S?

(A) $\frac{32}{5}$

(B) $\frac{20}{3}$

(C) $\frac{64}{5}$

(D) $\frac{40}{3}$

(E) 16

(F) 32

Quiz 3

In the binomial expansion of $(a+2x)^6$, where a is positive, the coefficient of x^4 is 1200.

What is the value of a?

(A) $\frac{\sqrt{2}}{2}$

(B) $\frac{\sqrt{10}}{2}$

(C) $\frac{\sqrt{15}}{2}$

(D) $\sqrt{5}$

(E) $2\sqrt{10}$

Quiz 4

The sequence a_n is given by the rule:

$$a_1 = 2$$
$$a_{n+1} = a_n + (-1)^n \text{ for } n \geq 1$$

What is $\sum_{n=1}^{100} a_n$?

(A) 150

(B) 250

(C) -4750

(D) 5150

(E) $4\left(1-\left(\frac{1}{2}\right)^{100}\right)$

(F) $4\left(\left(\frac{3}{2}\right)^{100}-1\right)$

Quiz 5

You are told that the infinite series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ and $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$ have sums $\frac{\pi^2}{6}$ and $\frac{\pi^2}{8}$ respectively. The infinite series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots+(-1)^{n-1}\frac{1}{n^2}+\cdots$ has sum equal to

(A) $\frac{\pi^2}{9}$

(B) $\frac{\pi^2}{10}$

(C) $\frac{\pi^2}{12}$

(D) $\frac{\pi^2}{16}$

Quiz 6

The inequality

$$(n+1)+(n^4+2)+(n^9+3)+(n^{16}+4)+\cdots+(n^{10000}+100)>k$$

is true for all $n \geq 1$. It follows that

(A) $k < 1300$

(B) $k^2 < 101$

(C) $k \geq 101^{10000}$

(D) $k < 5150$

Quiz 7

The function $f(n)$ is defined for positive integers n according to the rules

$$f(1)=1, \quad f(2n)=f(n), \quad f(2n+1)=\left(f(n)\right)^2-2.$$

The value of $f(1)+f(2)+f(3)+\cdots+f(100)$ is

(A) -86

(B) -31

(C) 23

(D) 58

13
315

Quiz 8

The coefficient of x^3 in the expansion of $(1+2x+3x^2)^6$ is equal to twice the coefficient of x^4 in the expansion of $(1-ax^2)^5$.

Find all possible values of the constant a.

(A) $\pm 2\sqrt{2}$

(B) $\pm\sqrt{17}$

(C) $\pm\sqrt{34}$

(D) $\pm 2\sqrt{17}$

(E) There are no possible values of a.

Ex. 14

By expressing $2\cos 2rt \sin t$ as a difference of sines, prove that

$$1 + 2\sum_{r=1}^{n} \cos 2rt = \frac{\sin(2n+1)t}{\sin t}.$$

Hence find

$$\sum_{r=0}^{n} \cos^2 rt.$$

Ex. 15

A famous problem of the early 18th century, called the *Basel problem*, was to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This problem had baffled many great mathematicians, in particular those of the Bernoulli family who lived in Basel, Switzerland (whence the name attached to the problem). But in the period 1735-1741, or thereabouts, Euler provided no fewer than five different ways of evaluating the sum.

Several ways led to the result

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{8}\pi^2, \qquad (*)$$

(i.e. the sum of the odd terms) from which Euler was able to deduce full result.

(i) Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and let $S_{\text{even}} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$. Write down a relation between S and S_{even}. Hence use Euler's result $(*)$ to evaluate S.

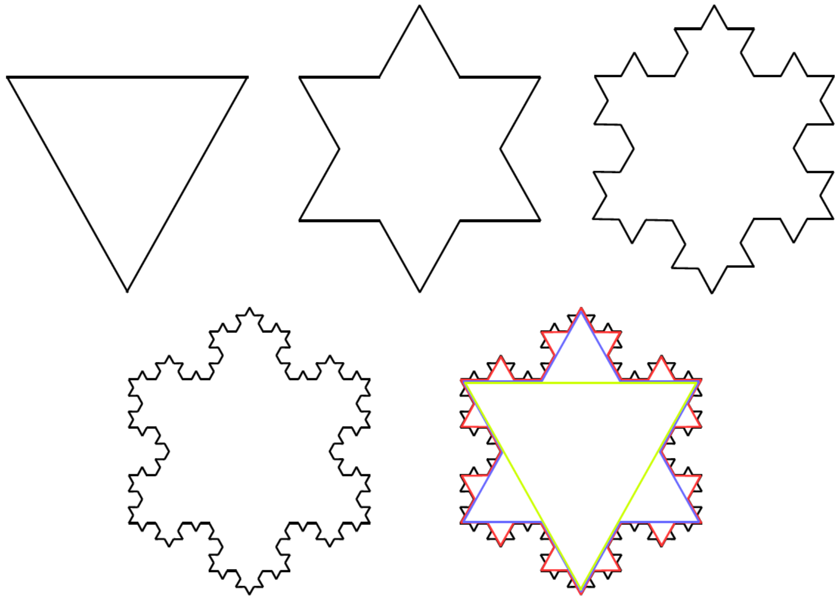
(ii) Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{\cos\frac{1}{2}n\pi}{n^2}$.

(iii) Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{(6n-5)^2} + \frac{1}{(6n-1)^2}\right)$.

Ex. 16

This question is about a *fractal* curve.(Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over.) The diagrams below show how our fractal is created. We start with an equilateral triangle. Then each side is divided into three equal parts and the middle section is replaced by two sides of an equilateral triangle. There are now 12 edges. At the next step each of these edges is divided into three equal parts and again the middle section is replaced by two sides of an equilateral triangle.

The process is then repeated over and over again. The first four diagrams below show the original triangle and the first three iterations. The final diagram just shows these iterations superposed. (This fractal is called the Koch Snowflake.)



(i) How many edges are there in the second iteration? Find an expression for the number of edges in the nth iteration.

(ii) Let each side of the original triangle have length 1. How long is each of the 12 edges in the first iteration? Write down an expression for the length of each edge in the nth iteration.

(iii) Using your answers to parts (i) and (ii) find an expression for the total length of the curve in the nth iteration. What happens to this total length as $n \to \infty$?

(iv) Let the area of the original triangle be A. By considering the areas of each of the 3 small triangles added in the first iteration, find the total area of the first iteration of the curve. Similarly, find an expression for the total area of the second iteration.

Obtain an expression for the total area of the nth iteration as a sum of the form

$$A + \frac{1}{3}A(1 + r + r^2 + \cdots + r^{n-1})$$

for some number r which you should find. What happens to the total area as $n \to \infty$?

Practices P13

Time Allowed

60 min

Number of Questions

25

Difficulty

★★★☆

[Practices P13]

Scan the QR code or click the link above to take the practice online.

Q1

The nth term of a sequence is $2n - 5$.

Which row in the table is correct for this sequence?

	term-to-term rule	*term which has a value of 17*
(A)	subtract 5	11th
(B)	subtract 5	29th
(C)	subtract 2	11th
(D)	subtract 2	29th
(E)	add 5	11th
(F)	add 5	29th
(G)	add 2	11th
(H)	add 2	29th

Q2

An arithmetic progression has first term a and common difference d.

The sum of the first 5 terms is equal to the sum of the first 8 terms.

Which one of the following expresses the relationship between a and d?

(A) $a = -\frac{38}{3}d$

(B) $a = -7d$

(C) $a = -6d$

(D) $a = 6d$

(E) $a = 7d$

(F) $a = \frac{38}{3}d$

Q3

The sequence a_n is defined by the rule:

$$a_n = (-1)^n - (-1)^{n-1} + (-1)^{n+2} \quad \text{for} \quad n \geq 1.$$

Find the value of

$$\sum_{n=1}^{39} a_n \, .$$

(A) -39

(B) -3

(C) -1

(D) 0

(E) 1

(F) 3

(G) 39

Q4

S is a geometric sequence.

The sum of the first 6 terms of S is equal to 9 times the sum of the first 3 terms of S.

The 7^{th} term of S is 360.

Find the 1^{st} term of S.

(A) $\frac{40}{27}$

(B) $\frac{40}{9}$

(C) $\frac{40}{3}$

(D) $\frac{45}{16}$

(E) $\frac{45}{8}$

(F) $\frac{45}{4}$

Q5

The non-zero constant k is chosen so that the coefficients of x^6 in the expansions of $(1+kx^2)^7$ and $(k+x)^{10}$ are equal.

What is the value of k?

(A) $\frac{1}{6}$

(B) 6

(C) $\frac{\sqrt{6}}{6}$

(D) $\sqrt{6}$

(E) $\frac{\sqrt{30}}{30}$

(F) $\sqrt{30}$

Q6

The sum to infinity of a geometric progression is 6.

The sum to infinity of the squares of each term in the progression is 12.

Find the sum to infinity of the cubes of each term in the progression.

(A) 8

(B) 18

(C) 24

(D) $\frac{216}{7}$

(E) 72

(F) 216

Q7

Find the coefficient of x in the expression:

$$(1+x)^0+(1+x)^1+(1+x)^2+(1+x)^3+\cdots+(1+x)^{79}+(1+x)^{80}.$$

(A) 80

(B) 81

(C) 324

(D) 628

(E) 3240

(F) 3321

(G) 6480

(H) 6642

Q8

The 1st, 2nd and 3rd terms of a geometric progression are also the 1st, 4th and 6th terms, respectively, of an arithmetic progression.

The sum to infinity of the geometric progression is 12.

Find the 1st term of the geometric progression.

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

(F) 6

Q9

The sequence x_n is given by:

$$x_1 = 10$$
$$x_{n+1} = \sqrt{x_n} \text{ for } n \geq 1$$

What is the value of x_{100}?

[Note that a^{b^c} means $a^{(b^c)}$]

(A) $10^{2^{99}}$

(B) $10^{2^{100}}$

(C) $10^{2^{-99}}$

(D) $10^{2^{-100}}$

(E) $10^{-2^{99}}$

(F) $10^{-2^{100}}$

(G) $10^{-2^{-99}}$

(H) $10^{-2^{-100}}$

Q10

The non-zero constant k is chosen so that the coefficients of x^6 in the expansions of $(1+kx^2)^7$ and $(k+x)^{10}$ are equal.

What is the value of k?

(A) $\frac{1}{6}$

(B) 6

(C) $\frac{\sqrt{6}}{6}$

(D) $\sqrt{6}$

(E) $\frac{\sqrt{30}}{30}$

(F) $\sqrt{30}$

Q11

The sequence x_n is defined by the rules

$$x_1 = 7$$
$$x_{n+1} = \frac{23x_n - 53}{5x_n + 1}$$

The first three terms in the sequence are 7, 3, 1.

What is the value of x_{100}?

(A) -5

(B) 0

(C) 1

(D) 3

(E) 7

Q12

In the expansion of $(a+bx)^5$ the coefficient of x^4 is 8 times the coefficient of x^2.

Given that a and b are non-zero **positive** integers, what is the smallest possible value of $a+b$?

(A) 3

(B) 4

(C) 5

(D) 9

(E) 13

(F) 17

Q13

Find the coefficient of x^2y^4 in the expansion of $(1 + x + y^2)^7$

(A) 6

(B) 10

(C) 21

(D) 35

(E) 105

(F) 210

Q14

Find the value of

$$\sum_{k=0}^{90} \sin(10 + 90k)^\circ$$

(A) 0

(B) $\sin 10^\circ$

(C) $\sin 100^\circ$

(D) $\sin 190^\circ$

(E) $\sin 280^\circ$

(F) 1

Q15

Find the value of the constant term in the expansion of

$$\left(x^6 - \frac{1}{x^2}\right)^{12}$$

(A) -495

(B) -220

(C) -66

(D) 66

(E) 220

(F) 495

Q16

The first term of a geometric progression is $2\sqrt{3}$ and the fourth term is $\frac{9}{4}$.

What is the sum to infinity of this geometric progression?

(A) $-2(2-\sqrt{3})$

(B) $4(2\sqrt{3}-3)$

(C) $\frac{16(8\sqrt{3}+9)}{37}$

(D) $\frac{4(2\sqrt{3}-3)}{7}$

(E) $\frac{4(2\sqrt{3}+3)}{7}$

(F) $2(2+\sqrt{3})$

(G) $4(2\sqrt{3}+3)$

Q17

The first term of an arithmetic sequence is a and the common difference is d.

The sum of the first n terms is denoted by S_n.

If $S_8 > 3S_6$, what can be deduced about the sign of a and the sign of d?

(A) both a and d are negative

(B) a is positive, d is negative

(C) a is negative, d is positive

(D) a is negative, but the sign of d cannot be deduced

(E) d is negative, but the sign of a cannot be deduced

(F) neither the sign of a nor the sign of d can be deduced

Q18

A geometric series has first term 4 and common ratio r, where $0 < r < 1$.

The first, second, and fourth terms of this geometric series form three successive terms of an arithmetic series.

The sum to infinity of the geometric series is

(A) $\frac{1}{2}(\sqrt{5}-1)$

(B) $2(3-\sqrt{5})$

(C) $2(1+\sqrt{5})$

(D) $2(3+\sqrt{5})$

Q19

The coefficient of x^2 in the expansion of $(4-x^2)[(1+2x+3x^2)^6-(1+4x^3)^5]$ is

(A) 28

(B) 72

(C) 78

(D) 192

(E) 240

(F) 310

(G) 312

Q20

The sequence a_n is given by the rule:

$$a_1 = 2$$
$$a_{n+1} = a_n + (-1)^n \text{ for } n \geq 1.$$

What is

$$\sum_{n=1}^{100} a_n\ ?$$

(A) 150

(B) 250

(C) -4750

(D) 5150

(E) $4\left(1-\left(\frac{1}{2}\right)^{100}\right)$

(F) $4\left(\left(\frac{3}{2}\right)^{100}-1\right)$

Q21

The 1^{st} term of an arithmetic progression is non-zero.

The 5^{th} term of this arithmetic progression is the square of the 1^{st} term.

The 33^{rd} term of this arithmetic progression is 10 times the 3^{rd} term.

What is the 10^{th} term?

(A) 31

(B) 34

(C) 39

(D) 248

(E) 496

Q22

The first three terms of a geometric series are equal to the first, fifth and sixth terms respectively of an arithmetic series.

Given that the terms in the geometric series are all different, find the value of the common ratio.

(A) -1

(B) $-\frac{4}{9}$

(C) $-\frac{1}{4}$

(D) $\frac{1}{4}$

(E) $\frac{4}{9}$

(F) 1

Q23

The first term of a convergent geometric series is 8.

The fifth term is 2.

The sixth term is real and positive.

What is the sum to infinity of this series?

(The sum to infinity of a convergent geometric series is given by $\frac{a}{1-r}$, where a is the first term and r is the common ratio.)

(A) $8(1+\sqrt{2})$

(B) $8(1-\sqrt{2})$

(C) $8(2+\sqrt{2})$

(D) $8(2-\sqrt{2})$

(E) 16

(F) $\frac{8\sqrt[5]{4}}{\sqrt[5]{4}-1}$

(G) $\frac{8\sqrt[5]{4}}{\sqrt[5]{4}+1}$

Q24

The first three terms of a geometric progression, whose terms are all greater than zero, are $(p-2)$, $(2p+2)$ and $(5p+14)$.

What is the fifth term of the progression?

(A) 324

(B) 486

(C) 1250

(D) 1458

(E) 3888

Q25

Show that the coefficient of x^{-12} in the expansion of

$$\left(x^4 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^6$$

is -15, and calculate the coefficient of x^2.

Hence, or otherwise, calculate the coefficients of x^4 and x^{38} in the expansion of

$$(x^2-1)^{11}(x^4+x^2+1)^5.$$

Supplements S13

Time Allowed

120 min

Number of Questions

24

Difficulty

★★★★

Supplements S13

Scan the QR code or click the link above to take the practice online.

SQ1

A sequence (a_n) has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geq 1$. What is the product of the first 15 terms of the sequence?

(A) l^{14}

(B) $15 + l^{14}$

(C) $\frac{1-l^{15}}{1-l}$

(D) l^{105}

(E) $15 + l^{105}$

SQ2

The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \cdots$$

is

(A) $2^n + 1 - 2^{1-n}$

(B) $2^n + 2^{-n}$

(C) $2^{2n} - 2^{3-2n}$

(D) $\frac{2^n - 2^{-n}}{3}$

SQ3

The smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n \geq 100,$$

is

(A) 99

(B) 101

(C) 199

(D) 300

SQ4

The sequence x_n is given by the formula

$$x_n = n^3 - 9n^2 + 631.$$

The largest value of n for which $x_n > x_{n+1}$ is

(A) 5

(B) 7

(C) 11

(D) 17

SQ5

The sum $1 - 4 + 9 - 16 + \cdots + 99^2 - 100^2$ equals

(A) -101

(B) -1000

(C) -1111

(D) -4545

(E) -5050

SQ6

The function $S(n)$ is defined for positive integers n by

$S(n) =$ sum of the digits of n.

For example, $S(723) = 7 + 2 + 3 = 12$. The sum

$$S(1) + S(2) + S(3) + \cdots S(99)$$

equals

(A) 746

(B) 862

(C) 900

(D) 924

SQ7

The power of x which has the greatest coefficient in the expansion of $\left(1 + \frac{1}{2}x\right)^{10}$ is

(A) x^2

(B) x^3

(C) x^5

(D) x^{10}

SQ8

The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \quad \text{for } n \geq 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(A) 1

(B) $\frac{6}{5}$

(C) $\frac{8}{5}$

(D) 3

(E) $\frac{27}{5}$

SQ9

The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide n;

$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n;

$F(n) = F(n-1) + 4$ if 2 and 3 both divide n;

$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

The value of $F(6000)$ equals

(A) 9827

(B) 10121

(C) 11000

(D) 12300

(E) 12352

SQ10

The coefficient of x^3 in the expansion of $\left(2\sqrt{p} + 3x\right)^5$ is 8640.

What is the value of p?

(A) 4

(B) 8

(C) 16

(D) 72

(E) 80

SQ11

S is a geometric progression: $u_1, u_2, u_3, u_4, u_5, u_6, \ldots$

All of the terms in S are positive.

S is split to form two new geometric progressions, O and E.

The terms of O are: $u_1, u_3, u_5, \ldots$

The terms of E are: $u_2, u_4, u_6, \ldots$

The sum to infinity of O is $\frac{8}{9}$, and $u_1 = \frac{2}{3}$

What is the sum to infinity of E?

(A) $\frac{4}{9}$

(B) $\frac{5}{9}$

(C) $\frac{2}{3}$

(D) $\frac{5}{6}$

(E) $\frac{8}{9}$

(F) $\frac{4}{3}$

SQ12

R is the sum to infinity of the geometric progression: $1, \sqrt{0.3}, 0.3, 0.3\sqrt{0.3}, \ldots$

S is the sum to infinity of the geometric progression: $1, -\sqrt{0.3}, 0.3, -0.3\sqrt{0.3}, \ldots$

What is the value of $R - S$?

(A) $-\frac{20}{7}$

(B) $-\frac{2\sqrt{30}}{7}$

(C) $-\frac{2\sqrt{3}}{7}$

(D) $\frac{2\sqrt{3}}{7}$

(E) $\frac{2\sqrt{30}}{7}$

(F) $\frac{20}{7}$

SQ13

The sum of the first 20 terms of an arithmetic progression is 50.

The sum of the next 20 terms of the arithmetic progression is -50.

What is the sum of the first 100 terms of the arithmetic progression?

(A) -750

(B) -350

(C) -50

(D) $-\frac{159}{8}$

(E) $\frac{159}{8}$

(F) 50

(G) 350

(H) 750

SQ14

A sequence is generated by

$$x_{n+1} = -\frac{12}{x_n}$$

n is an integer, where $n \geq 1$.

The 50^{th} term of the sequence is 6.

What is the sum of the first fifteen terms of the sequence?

(A) -2

(B) 6

(C) 10

(D) 22

(E) 26

(F) 34

(G) 58

SQ15

S and T are geometric progressions. For each, the second term is 6 and the sum to infinity is 25.

The first term of S is greater than the first term of T.

What is the fourth term of S?

(A) $\frac{8}{125}$

(B) $\frac{4}{25}$

(C) $\frac{24}{25}$

(D) $\frac{162}{125}$

(E) $\frac{54}{25}$

SQ16

An arithmetic series has first term a, common difference d and n terms. The sum of the series is S.

A second arithmetic series also has common difference d and n terms. The first term of this series is $a + 5$.

What is the sum of the second arithmetic series?

(A) $S + 5$

(B) $S + 5(n - 1)$

(C) $S + 5n$

(D) $S + (a + 5)$

(E) $S + (a + 5)(n - 1)$

(F) $S + (a + 5)n$

SQ17

In a particular arithmetic progression:

1 the 13^{th} term is six times the 1^{st} term

2 the 11^{th} term is 1 less than twice the 5^{th} term

What is the 3^{rd} term of the progression?

(A) -14.5

(B) -11

(C) $\frac{29}{19}$

(D) 3.5

(E) 11

(F) 14.5

SQ18

Given that $y = (2 + 3x)^6$, what is the coefficient of x^3 in $\frac{\mathrm{d}y}{\mathrm{d}x}$?

(A) 240

(B) 4320

(C) 4860

(D) 12 960

(E) 19 440

SQ19

A geometric progression has first term equal to 1 and common ratio $\frac{1}{2}\sin 2x$.

The sum to infinity of the series is $\frac{4}{3}$.

Find the possible values of x in the range $\pi \le x \le 2\pi$.

(A) $\frac{13}{12}\pi, \frac{17}{12}\pi$

(B) $\frac{7}{6}\pi, \frac{4}{3}\pi$

(C) $\frac{7}{6}\pi, \frac{11}{6}\pi$

(D) $\frac{5}{4}\pi, \frac{7}{4}\pi$

(E) there are no values of x in this range

SQ20

What is the coefficient of x^3 in the expansion of $(1-2x)^5(1+2x)^5$?

(A) -6400

(B) -640

(C) -80

(D) 0

(E) 80

(F) 800

(G) 960

SQ21

What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21

(B) $100\log_5 3$

(C) $200\log_3 5$

(D) 2200

(E) 21000

SQ22

Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

SQ23

Let k and n be positive integers such that $n \geq 2k$ and

$$\frac{(n-2)!}{(n-2k)!} = k!\,2^{k-1}$$

(Recall that, for $r \geq 1$, $r!$ is the product $r \cdot (r-1) \cdot (r-2) \cdots 2 \cdot 1$ and that $0!$ is defined to equal 1.)

(i) Suppose that $k = 1$. What are the possible values of n?

(ii) Suppose that $k = 2$. Show that $(n-2)(n-3) = 4$. What are the possible values of n?

(iii) Suppose that $k = 3$. Show that it is impossible that $n \geq 7$.

(iv) Suppose that $k \geq 4$. Show that there are no possible values of n.

SQ24

A list of real numbers $x_1, x_2, x_3, \cdots$ is defined by $x_1 = 1$, $x_2 = 3$ and then for $n \geq 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of x_4 and x_5.

(ii) Find values of real constants A, B, C such that <u>for $n = 1, 2, 3$</u>,

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true <u>for all $n \geq 1$</u>, find the smallest n such that $x_n \geq 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \cdots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n.$$

Find, explaining your reasoning, a formula for y_n which holds for $n \geq 2$.

What is the approximate value of $\frac{x_n}{y_n}$ for large values of n?

14 Differentiation

What's on the Specification?

- The derivative of $f(x)$ as the gradient of the tangent to the graph $y = f(x)$ at a point. In addition:
- Interpretation of a derivative as a rate of change;
- Second-order derivatives;
- Knowledge of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $f'(x)$, and $f''(x)$.
- Differentiation from first principles is excluded.
- Differentiation of x^n for rational n, and related sums and differences. This might require some simplification before differentiating; for example, the ability to differentiate an expression such as $\frac{(3x+2)^2}{x^{\frac{1}{2}}}$ could be required.
- Applications of differentiation to gradients, tangents, normals, stationary points (maxima and minima only), increasing $[f'(x) \geq 0]$ and decreasing $[f'(x) \leq 0]$ functions. Points of inflexion will not be examined, although students are expected to have a qualitative understanding of points of inflexion in the curves of simple polynomial functions.
- An understanding of the Fundamental Theorem of Calculus and its significance to integration. Simple examples of its use may be required in the two forms, $\int_b^a f(x)\,\mathrm{d}x = F(b) - F(a)$, where $F'(x) = f(x)$, and $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$.

Exercises E14

Time Allowed

No limit

Number of Questions

17

Difficulty

★★★

Exercises E14

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

The function

$$y = 2x^3 - 6x^2 + 5x - 7$$

has

(A) no stationary points.

(B) one stationary point.

(C) two stationary points.

(D) three stationary points.

Quiz Pre-2

The function f is given by

$$f(x) = \left(\frac{2}{x} - \frac{1}{2x^2}\right)^2 \quad (x \neq 0)$$

What is the value of $f''(1)$?

(A) -3

(B) -1

(C) 5

(D) 17

(E) 29

(F) 80

Quiz Pre-3

The derivative of $xe^{-x^2}\cos\left(\frac{1}{x}\right)$ is

(A) $-\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right) - 2x^2e^{-x^2}\cos\left(\frac{1}{x}\right) + e^{-x^2}\cos\left(\frac{1}{x}\right)$

(B) $\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right) - 2x^2e^{-x^2}\cos\left(\frac{1}{x}\right) + e^{-x^2}\cos\left(\frac{1}{x}\right)$

(C) $\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right) + 2x^2e^{-x^2}\cos\left(\frac{1}{x}\right) + e^{-x^2}\cos\left(\frac{1}{x}\right)$

(D) $\frac{1}{x}e^{-x^2}\cos\left(\frac{1}{x}\right) - 2x^2e^{-x^2}\cos\left(\frac{1}{x}\right) + e^{-x^2}\cos\left(\frac{1}{x}\right)$

Quiz Pre-4

The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx,$$

as a varies, is

(A) $\frac{3}{20}$

(B) $\frac{4}{45}$

(C) $\frac{7}{13}$

(D) 1

Ex. 1

Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0. \qquad (*)$$

(i) Show that if $x = 1$ is a solution of $(*)$ then

$$1 - \sqrt{2} \le b \le 1 + \sqrt{2}$$

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of $(*)$.

(iii) Given that $x = 1$ is a solution, find the value of b for which $(*)$ has a repeated root.

(iv) For this value of b, does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or minimum at its repeated root?

Ex. 2

As x varies over the real numbers, the largest value taken by the function

$$(4\sin^2 x + 4\cos x + 1)^2$$

equals

(A) $17 + 12\sqrt{2}$

(B) 36

(C) $48\sqrt{2}$

(D) $64 - 12\sqrt{3}$

(E) 81

Ex. 3

The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

(A) $k > 0$

(B) $0 < k < 1$

(C) $k > \frac{1}{2}$

(D) $k < 3$

(E) all values of k

Ex. 4

The functions f, g and h are related by

$$f'(x) = g(x+1), \qquad g'(x) = h(x-1)$$

It follows that $f''(2x)$ equals

(A) $h(2x+1)$

(B) $2h'(2x)$

(C) $h(2x)$

(D) $4h(2x)$

Ex. 5

The expression

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[(2x-1)^4(1-x)^5] + \frac{\mathrm{d}}{\mathrm{d}x}[(2x+1)^4(3x^2-2)^2]$$

is a polynomial of degree

(A) 9

(B) 8

(C) 7

(D) less than 7

14 344

Ex. 6

The derivative of the function

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

is:

(A) $f(x) - f(0)$

(B) $f'(x)$

(C) $f(x)$

(D) $f'(x) - f'(0)$

Ex. 7

For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) \mathrm{d}x.$$

Then $\frac{\mathrm{d}I}{\mathrm{d}a} = 0$ when a equals

(A) $\frac{1+\sqrt{5}}{2}$

(B) $\sqrt{2}$

(C) $\frac{\sqrt{5}-1}{2}$

(D) 1

Quiz 1

The least possible value of the gradient of the curve $y = (2x + a)(x - 2a)^2$ at the point where $x = 1$, as a varies, is

(A) $-\frac{49}{4}$

(B) -8

(C) $-\frac{25}{4}$

(D) $\frac{7}{4}$

(E) $\frac{47}{16}$

Quiz 2

The function $y = \mathrm{e}^{kx}$ satisfies the equation

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y\frac{\mathrm{d}y}{\mathrm{d}x}$$

for

(A) no values of k.

(B) exactly one value of k.

(C) exactly two distinct values of k.

(D) exactly three distinct values of k.

(E) infinitely many distinct values of k.

Quiz 3

The functions f and g are given by $f(x) = 3x^2 + 12x + 4$ and $g(x) = x^3 + 6x^2 + 9x - 8$.

What is the complete set of values of x for which one of the functions is increasing and the other decreasing?

(A) $x \geq -1$

(B) $x \leq -1$

(C) $-3 \leq x \leq -2,\ x \geq -1$

(D) $x \leq -2,\ x \geq -1$

(E) $x \leq -3,\ -2 \leq x \leq -1$

(F) $x \leq -3,\ x \geq -2$

(G) $-2 \leq x \leq -1$

Quiz 4

The smallest possible value of $\int_0^1 (x-a)^2\,\mathrm{d}x$ as a varies is

(A) $\frac{1}{12}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{7}{12}$

(E) 2

Ex. 8

Δ is an operation that takes polynomials in x to polynomials in x; that is, given any polynomial $h(x)$, there is a polynomial called $\Delta h(x)$ which is obtained from $h(x)$ using the rules that define Δ. These rules are as follows:

(i) $\Delta x = 1$.

(ii) $\Delta\big(f(x) + g(x)\big) = \Delta f(x) + \Delta g(x)$ for any polynomials $f(x)$ and $g(x)$.

(iii) $\Delta\big(\lambda f(x)\big) = \lambda \Delta f(x)$ for any constant λ and any polynomial $f(x)$.

(iv) $\Delta\big(f(x)g(x)\big) = f(x)\Delta g(x) + g(x)\Delta f(x)$ for any polynomials $f(x)$ and $g(x)$.

Using these rules show that, if $f(x)$ is a polynomial of degree zero (that is, a constant), then $\Delta f(x) = 0$. Calculate Δx^2 and Δx^3.

Prove that $\Delta h(x) \equiv \frac{\mathrm{d}h(x)}{\mathrm{d}x}$ for any polynomial $h(x)$. You should make it clear whenever you use one of the above rules in your proof.

Ex. 9

(i) Using the fact that, $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x g(t)\,\mathrm{d}t = g(x)$, differentiate with respect to x the following functions of x, when $0 \le x \le \frac{\pi}{2}$.

(a) $\int_0^x \tan^{93} t\,\mathrm{d}t$.

(b) $\int_0^{x^2} \tan^{93} t\,\mathrm{d}t$.

(c) $\int_x^{x^2} \tan^{93} t\,\mathrm{d}t$.

(ii) Work out

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\ln x) \quad \text{for} \quad x > 1.$$

(Note that the function $\ln x$ can be defined as $\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t$, for $x > 0$.) Using integration by parts evaluate

$$\int_2^x \frac{\ln(\ln t)}{t}\,\mathrm{d}t, \quad \text{for} \quad x \ge 2.$$

14
347

Practices P14

Time Allowed

30 min

Number of Questions

13

Difficulty

★★★

Practices P14

Scan the QR code or click the link above to take the practice online.

Q1

Which of the following is an expression for the first derivative with respect to x of

$$\frac{x^3 - 5x^2}{2x\sqrt{x}}$$

(A) $-\frac{\sqrt{x}}{2}$

(B) $\frac{\sqrt{x}}{4}$

(C) $\frac{3x-5}{4\sqrt{x}}$

(D) $\frac{3\sqrt{x}-5}{4\sqrt{x}}$

(E) $\frac{3\sqrt{x}-10}{3\sqrt{x}}$

(F) $\frac{3x^2-10x}{3\sqrt{x}}$

Q2

Find the maximum value of the function

$$f(x) = \frac{1}{5^{2x} - 4(5^x) + 7}$$

(A) $\frac{1}{7}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) 3

(E) 4

(F) 7

Q3

A curve has equation

$$y = (2q - x^2)(2qx + 3)$$

The gradient of the curve at $x = -1$ is a function of q.

Find the value of q which minimises the gradient of the curve at $x = -1$.

(A) -1

(B) $-\frac{3}{4}$

(C) $-\frac{1}{2}$

(D) 0

(E) $\frac{1}{2}$

(F) $\frac{3}{4}$

(G) 1

Q4

The function f is defined for all real x as

$$f(x) = (p - x)(x + 2)$$

Find the complete set of values of p for which the maximum value of $f(x)$ is less than 4.

(A) $-2 - 4\sqrt{2} < p < -2 + 4\sqrt{2}$

(B) $-2 - 2\sqrt{2} < p < -2 + 2\sqrt{2}$

(C) $-2\sqrt{5} < p < 2\sqrt{5}$

(D) $-6 < p < 2$

(E) $-4 < p < 0$

(F) $-2 < p < 2$

Q5

Find the maximum value of

$$4^{\sin x} - 4 \times 2^{\sin x} + \frac{17}{4}$$

for real x.

(A) $\frac{1}{4}$

(B) $\frac{5}{2}$

(C) $\frac{13}{2}$

(D) $\frac{21}{2}$

(E) $\frac{65}{4}$

(F) There is no maximum value.

Q6

Find the lowest positive integer for which $x^2 - 52x - 52$ is positive.

(A) 26

(B) 27

(C) 51

(D) 52

(E) 53

(F) 54

Q7

$$f(x) = \frac{(x^2+5)(2x)}{\sqrt[4]{x^3}}, \quad x > 0$$

Which one of the following is equal to $f'(x)$?

(A) $8x^{\frac{9}{4}} + \frac{40}{3}x^{\frac{1}{4}}$

(B) $\frac{9}{2}x^{\frac{5}{4}} + \frac{5}{2}x^{-\frac{3}{4}}$

(C) $8x^{\frac{9}{4}} + \frac{40}{3}x^{-\frac{1}{4}}$

(D) $\frac{8}{13}x^{\frac{13}{4}} + 8x^{\frac{5}{4}}$

Q8

Given that $y = \frac{(1-3x)^2}{2x^{\frac{3}{2}}}$, which one of the following is a correct expression for $\frac{dy}{dx}$?

(A) $\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

(B) $\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

(C) $\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

(D) $-\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

(E) $-\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

(F) $-\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

Q9

The function f is given, for $x > 0$, by

$$f(x) = \frac{x^3 - 4x}{2\sqrt{x}}$$

Find the values of $f'(4)$.

(A) 3

(B) 9

(C) 9.5

(D) 12

(E) 39.5

(F) 88

Q10

The gradient of the curve $y = \frac{(3x-2)^2}{x\sqrt{x}}$ at the point where $x = 2$ is

(A) $\frac{3}{2}\sqrt{2}$

(B) $3\sqrt{2}$

(C) $4\sqrt{2}$

(D) $\frac{9}{2}\sqrt{2}$

(E) $6\sqrt{2}$

Q11

Consider the following statements about the polynomial $p(x)$, where $a < b$:

I $\quad p(a) \leq p(b)$

II $\quad p'(a) \leq p'(b)$

III $\quad p''(a) \leq p''(b)$

Which of these statements is a **necessary** condition for $p(x)$ to be increasing for $a \leq x \leq b$?

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Q12

The graph of the polynomial function

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

is sketched, where a, b, c, d, e, and f are real constants with $a \neq 0$.

Which one of the following is **not** possible?

(A) The graph has two local minima and two local maxima.

(B) The graph has one local minimum and two local maxima.

(C) The graph has one local minimum and one local maximum.

(D) The graph has no local minima or local maxima.

Q13

Consider the following statement:

$(*)$ **If** $f(x)$ is an integer for every integer x, **then** $f'(x)$ is an integer for every integer x.

Which one of the following is a **counterexample** to $(*)$?

(A) $f(x) = \frac{x^3+x+1}{4}$

(B) $f(x) = \frac{x^4+x^2+x}{2}$

(C) $f(x) = \frac{x^4+x^3+x^2+x}{2}$

(D) $f(x) = \frac{x^4+2x^3+x^2}{4}$

Supplements S14

Time Allowed

90 min

Number of Questions

12

Difficulty

★★★★

Supplements S14

Scan the QR code or click the link above to take the practice online.

14

354

SQ1

The function $y = 2x^3 - 6x^2 + 5x - 7$ has

(A) no stationary points.

(B) one stationary point.

(C) two stationary points.

(D) three stationary points.

SQ2

The smallest value of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

in the range $0 \le x \le 2$ is

(A) 1

(B) 3

(C) 5

(D) 7

SQ3

The greatest value which the function

$$f(x) = (3\sin^2(10x + 11) - 7)^2$$

takes, as x varies over all real values, equals

(A) -9

(B) 16

(C) 49

(D) 100

SQ4

Let

$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n > 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will

(A) always be odd.

(B) always be even.

(C) depend on a but not n.

(D) depend on n but not a.

(E) depend on both a and n.

SQ5

The function

$$y = x^2 \ln x$$

defined for $x > 0$ satisfies

(A) $x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2y + x^2$, (and only this part).

(B) $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all x, (and only this part).

(C) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq 0$ for all x, (and only this part).

(D) all of the above.

SQ6

The largest value achieved by $3\cos^2 x + 2\sin x + 1$ equals

(A) $\frac{11}{5}$

(B) $\frac{13}{3}$

(C) $\frac{12}{5}$

(D) $\frac{14}{9}$

(E) $\frac{12}{7}$

SQ7

The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occur when

(A) $x = 0$

(B) $x = 1 - \frac{1}{\sqrt{3}}$

(C) $x = 1 + \frac{1}{\sqrt{3}}$

(D) $x = 2\frac{1}{5}$

SQ8

A circle of radius 2, centred on the origin, is drawn on a grid of points with integer coordinates. Let n be the number of grid points that lie within or on the circle. What is the smallest amount the radius needs to increase by for there to be $2n - 5$ grid points within or on the circle?

(A) $\sqrt{5} - 2$

(B) $\sqrt{6} - 2$

(C) $\sqrt{8} - 2$

(D) 1

(E) $\sqrt{8}$

SQ9

For which of the following do we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \ln y\,?$$

(A) $y = e^{e^{2x}}$

(B) $y = e^{2e^{x}}$

(C) $y = e^{e^{x^2}}$

(D) $y = 2e^{e^{x}}$

SQ10

A cubic has equation $y = x^3 + ax^2 + bx + c$ and has turning points at $(1, 2)$ and $(3, d)$ for some d. What is the value of d?

(A) -4

(B) -2

(C) 0

(D) 2

(E) 4

SQ11

The functions f and g are related (for all real x) by

$$g(x) = f(x) + \frac{1}{f(x)}.$$

Express $g'(x)$ and $g''(x)$ in terms of $f(x)$ and its derivatives.

If $f(x) = 4 + \cos 2x + 2\sin x$, find the stationary points of g for $0 \le x \le 2\pi$, and determine which are maxima and which are minima.

SQ12

An operator D is defined, for any function f, by

$$\mathrm{D}f(x) = x\frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

The notation D_n means that D is applied n times; for example

$$\mathrm{D}^2 f(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right).$$

Show that, for any constant a, $\mathrm{D}^2 x^a = a^2 x^a$.

(i) Show that if $P(x)$ is a polynomial of degree r (where $r \ge 1$) then, for any positive integer n, $\mathrm{D}^n P(x)$ is also a polynomial of degree r.

(ii) Show that if n and m are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n = 0.$$

15 Integration

What's on the Specification?

- Definite integration as related to the 'area between a curve and an axis.' Candidates are expected to understand the difference between finding a definite integral and finding the area between a curve and an axis. Integrals could be given with respect to x or with respect to y.
- Finding definite and indefinite integrals of x^n for n rational, $n \neq -1$, and related sums and differences, including expressions which require simplification prior to integrating; for example, $\int (x+2)^2 \, \mathrm{d}x$, and $\int \frac{(3x-5)^2}{x^{\frac{1}{2}}} \, \mathrm{d}x$.
- Combining integrals with either equal or contiguous ranges; for example, $\int_2^5 f(x) \, \mathrm{d}x + \int_2^5 g(x) \, \mathrm{d}x = \int_2^5 [f(x) + g(x)] \, \mathrm{d}x$, and $\int_2^4 f(x) \, \mathrm{d}x + \int_4^3 f(x) \, \mathrm{d}x = \int_3^2 f(x) \, \mathrm{d}x$.
- Approximation of the area under a curve using the trapezium rule; determination of whether this constitutes an overestimate or an underestimate.
- Solving differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.

Exercises E15

Time Allowed

No limit

Number of Questions

27

Difficulty

★★★☆

Exercises E15

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

Find the value of

$$\int_1^2 \left(x^2 - \frac{4}{x^2}\right)^2 \mathrm{d}x$$

(A) $\frac{43}{15}$

(B) 3

(C) $\frac{97}{15}$

(D) $\frac{103}{15}$

(E) $\frac{163}{15}$

(F) 18

Quiz Pre-2

Which of these integrals has the largest value? You are not expected to calculate the exact value of any of these.

(A) $\int_0^2 (x^2 - 4)\sin^8(\pi x)\,\mathrm{d}x$

(B) $\int_0^{2\pi} (2 + \cos x)^3\,\mathrm{d}x$

(C) $\int_0^{\pi} \sin^{100} x\,\mathrm{d}x$

(D) $\int_0^{\pi} (3 - \sin x)^6\,\mathrm{d}x$

(E) $\int_0^{8\pi} 108(\sin^3 x - 1)\,\mathrm{d}x$

Quiz Pre-3

The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis equals

(A) 2

(B) $\frac{5}{2}$

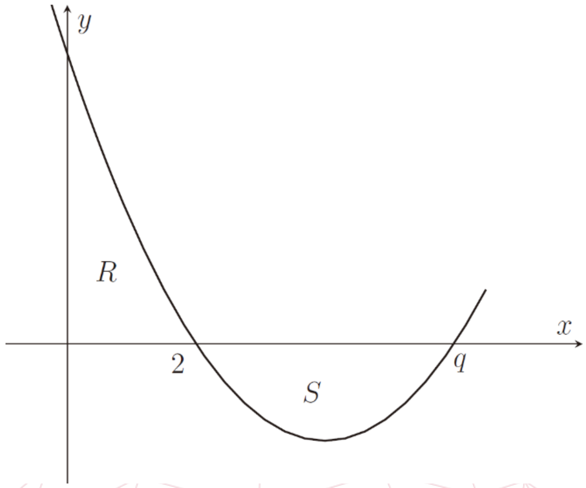
(C) 3

(D) $\frac{10}{3}$

(E) $\frac{16}{3}$

Quiz Pre-4

The quadratic function shown passes through $(2,0)$ and $(q,0)$, where $q > 2$.



What is the value of q such that the area of region R equals the area of region S?

(A) $\sqrt{6}$

(B) 3

(C) $\frac{18}{5}$

(D) 4

(E) 6

(F) $\frac{33}{5}$

Ex. 1

For a particular function $f(x)$, it is given that:

$$\int_{-2}^{2} 2f(x)\,\mathrm{d}x + \int_{2}^{4} f(x)\,\mathrm{d}x = 4$$

and also:

$$\int_{-2}^{2} 5f(x)\,\mathrm{d}x - \int_{-2}^{4} f(x)\,\mathrm{d}x = 7$$

Find the value of $\int_2^4 f(x)\,\mathrm{d}x$.

(A) $\frac{1}{3}$

(B) $\frac{11}{7}$

(C) $\frac{11}{6}$

(D) $\frac{13}{6}$

(E) $\frac{13}{3}$

Ex. 2

Evaluate

$$\int_{-1}^{3} |x|(1-x)\,\mathrm{d}x$$

(A) $\frac{17}{3}$

(B) $-\frac{17}{3}$

(C) $\frac{16}{3}$

(D) $-\frac{16}{3}$

(E) $\frac{11}{3}$

(F) $-\frac{11}{3}$

Ex. 3

The square bracket notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor \sqrt{24} \rfloor = 4$ and $\lfloor 5 \rfloor = 5$.

(i) Sketch the graph of $y = \sqrt{\lfloor x \rfloor}$ and show that

$$\int_0^a \sqrt{\lfloor x \rfloor}\,\mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\int_0^a 2^{\lfloor x \rfloor}\,\mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\int_0^a 2^{\lfloor x \rfloor}\,\mathrm{d}x$ when a is positive but not an integer.

Ex. 4

Let

$$T = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,\mathrm{d}x\right) \times \left(\int_{\pi}^{2\pi} \sin x\,\mathrm{d}x\right) \times \left(\int_0^{\frac{\pi}{8}} \frac{\mathrm{d}x}{\cos 3x}\right).$$

Which of the following is true?

(A) $T = 0$

(B) $T < 0$

(C) $T > 0$

(D) T is not defined

Ex. 5

In the region $0 < x \leq 2\pi$, the equation

$$\int_0^x \sin(\sin t)\,\mathrm{d}t = 0$$

has

(A) no solution.

(B) one solution.

(C) two solutions.

(D) three solutions.

Ex. 6

The function f is defined by

$$f(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geq 0,$$

and the function g is the inverse function to f, so that $g\big(f(x)\big) = x$. Sketch $f(x)$ and $g(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x + \int_0^k g(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}} + 1)},$$

where $k = \frac{1}{\sqrt{\mathrm{e}}+1}$, and explain this result by means of a diagram.

Ex. 7

The function $f(x)$ is increasing and $f(0) = 0$.

The positive constants a and b are such that $a < b$.

The area of the region enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is denoted by R.

The function $g(x)$ is defined by $g(x) = f(x) + 2f(b)$.

Which of the following is an expression for the area enclosed by the curve $y = g(x)$, the x-axis and the lines $x = a$ and $x = b$?

(A) $R + (b - a)f(b)$

(B) $R + 2(b - a)f(b)$

(C) $R + 2f(b) - f(a)$

(D) $R + 2f(b)$

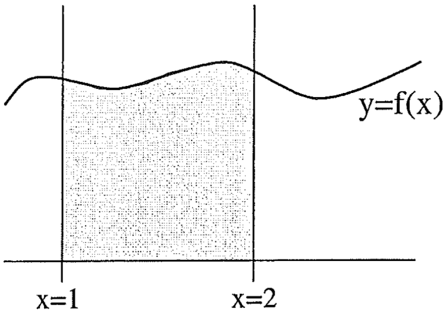
(E) $R + \big(f(b)\big)^2$

(F) $R + \big(f(b)\big)^2 - \big(f(a)\big)^2$

(G) $R + 2\big(f(b) - f(a)\big)f(b)$

Ex. 8

The (shaded) area under the graph of $y = f(x)$ between $x = 1$ and $x = 2$ is given to be 1.



The area under the graph of $y = 2f(3 - x)$ between $x = 1$ and $x = 2$ is therefore

(A) 1

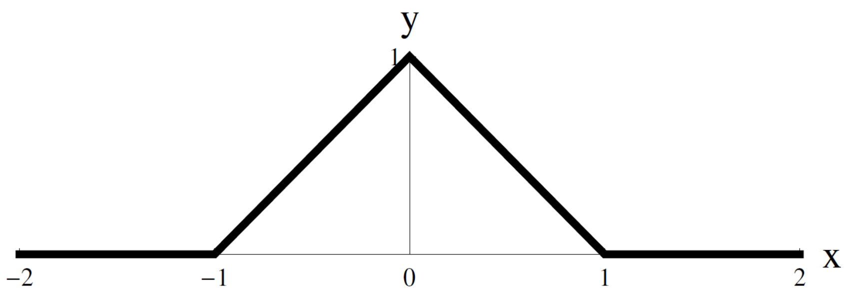
(B) 2

(C) 3

(D) 6

Ex. 9

A graph of the function $y = f(x)$ is sketched on the axes below:



The value of $\int_{-1}^{1} f(x^2 - 1)\,dx$ equals

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{3}{5}$

(D) $\frac{2}{3}$

Ex. 10

The area bounded by the graphs

$$y = \sqrt{2 - x^2} \quad \text{and} \quad x + (\sqrt{2} - 1)y = \sqrt{2}$$

equals

(A) $\frac{\sin\sqrt{2}}{\sqrt{2}}$

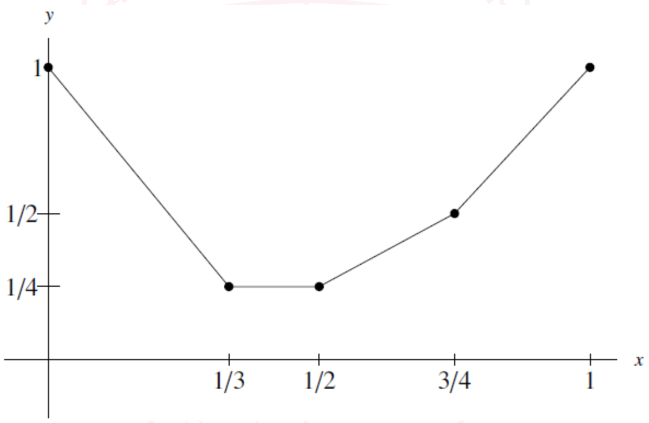
(B) $\frac{\pi}{4} - \frac{1}{\sqrt{2}}$

(C) $\frac{\pi}{2\sqrt{2}}$

(D) $\frac{\pi^2}{6}$

Ex. 11

The graph $y = f(x)$ of a function is drawn below for $0 \leq x \leq 1$.



The trapezium rule is then used to estimate

$$\int_0^1 f(x)\,\mathrm{d}x$$

By dividing $0 \leq x \leq 1$ into n equal intervals. The estimate calculated will equal the actual integral when

(A) n is a multiple of 4.

(B) n is a multiple of 6.

(C) n is a multiple of 8.

(D) n is a multiple of 12.

Ex. 12

When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x \, \mathrm{d}x$$

by dividing the interval $0 \leq x \leq 1$ into N subintervals the answer achieved is

(A) $\frac{1}{2N}\left\{1+\frac{1}{2^{\frac{1}{N}}+1}\right\}$

(B) $\frac{1}{2N}\left\{1+\frac{2}{2^{\frac{1}{N}}-1}\right\}$

(C) $\frac{1}{N}\left\{1-\frac{1}{2^{\frac{1}{N}}-1}\right\}$

(D) $\frac{1}{2N}\left\{\frac{5}{2^{\frac{1}{N}}+1}-1\right\}$

Ex. 13

It is given that

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{24\pi(t-1)}{(1+\sqrt{t})} \quad \text{for} \quad t \geq 1$$

and $V = 7$ when $t = 1$.

Find the value of V when $t = 9$.

(A) $208\pi + 7$

(B) $216\pi + 7$

(C) $224\pi + 7$

(D) $416\pi + 7$

(E) $608\pi + 7$

(F) $744\pi + 7$

Quiz 1

Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - \frac{2-3x}{x^3}, \quad x \neq 0$$

and $y = 5$ when $x = 1$, find y in terms of x.

(A) $y = \frac{1}{3}x^3 + x^{-2} - 3x^{-1} + 6\frac{2}{3}$

(B) $y = x^3 + \frac{1}{2}x^{-2} - 3x^{-1} + 6\frac{1}{2}$

(C) $y = x^3 + x^{-2} - 3x^{-1} + 6$

(D) $y = x^3 + x^{-2} - x^{-1} + 4$

(E) $y = x^3 + 2x^{-2} - x^{-1} + 3$

(F) $y = 3x^3 + x^{-2} - x^{-1} + 2$

Quiz 2

Given that

$$2\int_0^1 f(x)\,\mathrm{d}x + 5\int_1^2 f(x)\,\mathrm{d}x = 14$$

and

$$\int_0^1 f(x+1)\,\mathrm{d}x = 6$$

find the value of

$$\int_0^2 f(x)\,\mathrm{d}x$$

(A) -8

(B) -4

(C) -2

(D) 2

(E) 4

(F) $\frac{29}{5}$

(G) $\frac{32}{5}$

(H) 14

Quiz 3

The two functions $F(n)$ and $G(n)$ are defined as follows for positive integers n:

$$F(n) = \frac{1}{n}\int_0^n (n-x)\,\mathrm{d}x$$

$$G(n) = \sum_{r=1}^{n} F(r)$$

What is the smallest positive integer n such that $G(n) > 150$?

(A) 22

(B) 23

(C) 24

(D) 25

(E) 26

Quiz 4

$f(x)$ is a function defined for all real values of x.

Which one of the following is a **sufficient** condition for $\int_1^3 f(x)\,\mathrm{d}x = 0$?

(A) $f(2) = 0$

(B) $f(1) = f(3) = 0$

(C) $f(-x) = -f(x)$ for all x

(D) $f(x+2) = -f(2-x)$ for all x

(E) $f(x-2) = -f(2-x)$ for all x

Quiz 5

Find the total area enclosed between the curve $y = x(x+a)(x-2a)$, the x-axis and the lines $x = -a$ and $x = a$, where a is a positive constant.

(A) $\frac{2}{3}a^4$

(B) $\frac{5}{6}a^4$

(C) $\frac{4}{3}a^4$

(D) $\frac{3}{2}a^4$

(E) $\frac{7}{2}a^4$

Quiz 6

The function f is such that $0 < f(x) < 1$ for $0 \le x \le 1$.

The trapezium rule with n equal intervals is used to estimate $\int_0^1 f(x)\,\mathrm{d}x$ and produces an underestimate.

Using the same number of equal intervals, for which one of the following does the trapezium rule produce an overestimate?

(A) $\int_0^1 (f(x) + 1)\,\mathrm{d}x$

(B) $\int_0^1 2f(x)\,\mathrm{d}x$

(C) $\int_{-1}^0 f(x+1)\,\mathrm{d}x$

(D) $\int_{-1}^0 f(-x)\,\mathrm{d}x$

(E) $\int_0^1 \big(1 - f(x)\big)\,\mathrm{d}x$

Quiz 7

If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,\mathrm{d}x,$$

by splitting the interval $0 \le x \le 1$ into 10 intervals then an overestimate of the integral is produced. It follows that

(A) the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\,\mathrm{d}x$.

(B) the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x) - 1)\,\mathrm{d}x$.

(C) the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\,\mathrm{d}x$.

(D) the trapezium rule with 10 intervals underestimates $\int_0^1 \big(1 - f(x)\big)\,\mathrm{d}x$.

Quiz 8

For a real number x we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x. Let

$$f(x) = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor$$

The smallest number of equal width strips for which the trapezium rule produces an overestimate for the integral

$$\int_0^5 f(x)\,\mathrm{d}x$$

is

(A) 2

(B) 3

(C) 4

(D) 5

(E) it never produces an overestimate

Ex. 14

For all real numbers x, the function $f(x)$ satisfies

$$6 + f(x) = 2f(-x) + 3x^2\left(\int_{-1}^{1} f(t)\,\mathrm{d}t\right).$$

It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(A) 4

(B) 6

(C) 11

(D) $\frac{27}{2}$

(E) 23

Ex. 15

A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q - p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q - p)^4$.

Practices P15

Time Allowed

60 min

Number of Questions

17

Difficulty

★★★☆

Practices P15

Scan the QR code or click the link above to take the practice online.

Q1

Find the value of

$$\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x.$$

(A) $-\frac{13}{2}$

(B) $-\frac{85}{16}$

(C) $-\frac{13}{8}$

(D) -1

(E) $-\frac{1}{4}$

(F) $\frac{7}{4}$

(G) 7

Q2

What is the total area enclosed between the curve $y = x^2 - 1$, the x-axis and the lines $x = -2$ and $x = 2$?

(A) $\frac{4}{3}$

(B) $\frac{8}{3}$

(C) 4

(D) $\frac{16}{3}$

(E) 12

(F) 16

Q3

p is a positive constant.

Find the area enclosed between the curves $y = p\sqrt{x}$ and $x = p\sqrt{y}$.

(A) $\frac{2}{3}p^{\frac{5}{2}} - \frac{1}{2}p^2$

(B) $\frac{4}{3}p^{\frac{2}{5}} - p^2$

(C) $\frac{p^4}{6}$

(D) $\frac{p^4}{3}$

(E) $\frac{2}{3}p^3 - \frac{1}{2}p^4$

(F) $\frac{4}{3}p^3 - p^4$

(G) $2p^4$

Q4

The polynomial function $f(x)$ is such that $f(x) > 0$ for all value of x.

Given $\int_2^4 f(x)\,\mathrm{d}x = A$, which one of the following statements **must** be correct?

(A) $\int_0^2 [f(x+2)+1]\,\mathrm{d}x = A+1$

(B) $\int_0^2 [f(x+2)+1]\,\mathrm{d}x = A+2$

(C) $\int_2^4 [f(x+2)+1]\,\mathrm{d}x = A+1$

(D) $\int_2^4 [f(x+2)+1]\,\mathrm{d}x = A+2$

(E) $\int_4^6 [f(x+2)+1]\,\mathrm{d}x = A+1$

(F) $\int_4^6 [f(x+2)+1]\,\mathrm{d}x = A+2$

Q5

A curve has equation $y = f(x)$, where

$$f(x) = x(x-p)(x-q)(r-x)$$

with $0 < p < q < r$.

You are given that:

$$\int_0^r f(x)\,\mathrm{d}x = 0$$
$$\int_0^q f(x)\,\mathrm{d}x = -2$$
$$\int_p^r f(x)\,\mathrm{d}x = -3$$

What is the total area enclosed by the curve and the x-axis for $0 \le x \le r$?

(A) 0

(B) 1

(C) 4

(D) 5

(E) 6

(F) 10

Q6

The area enclosed between the line $y = mx$ and the curve $y = x^3$ is 6.

What is the value of m?

(A) 2

(B) 4

(C) $\sqrt{3}$

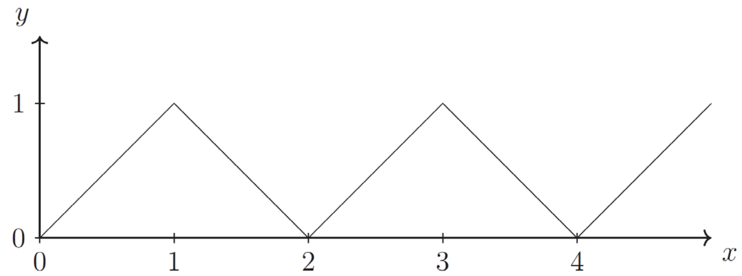
(D) $\sqrt{6}$

(E) $2\sqrt{3}$

(F) $2\sqrt{6}$

Q7

The diagram shows the graph of $y = f(x)$.



The graph consists of alternating straight-line segments of gradient 1 and -1 and continues in this way for all values of x.

The function g is defined as

$$g(x) = \sum_{r=1}^{10} f(2^{r-1}x).$$

Find the value of

$$\int_0^1 g(x)\,\mathrm{d}x.$$

(A) $\frac{1023}{1024}$

(B) $\frac{1023}{512}$

(C) 5

(D) 10

(E) $\frac{55}{2}$

(F) 55

Q8

Curve C has equation $y = 9 - x^2$.

Line L has equation $y = 5$.

What is the area enclosed between C and L?

(A) $\frac{32}{3}$

(B) $\frac{62}{3}$

(C) $\frac{92}{3}$

(D) $\frac{122}{3}$

(E) $\frac{152}{3}$

Q9

Find $\int_1^2 \left(3x + \frac{1}{x}\right)^2 \mathrm{d}x$.

(A) 13.5

(B) 14.75

(C) 21.5

(D) 26.5

(E) 27.5

(F) 28.75

Q10

What is the area of the region enclosed between the curve $y = \frac{1}{2}x^2$, the line $y = -x$, and the lines $x = 1$ and $x = 3$?

(A) $\frac{1}{3}$

(B) 2

(C) 4

(D) 6

(E) $\frac{25}{3}$

(F) $\frac{28}{3}$

Q11

What is the area enclosed by the line $x = 7$ and the curve $x = 3(y-1)^2 + 4$?

(A) 4

(B) 8

(C) 10

(D) 11

(E) 14

(F) 20

Q12

P is the area of the region bounded by the curve $y = 5x - x^2$ and the x-axis.

Q is the area of the region between the curve $y = 5x - x^2$ and the line $y = 2x$.

What is the ratio $Q : P$?

(A) $27 : 125$

(B) $54 : 125$

(C) $81 : 125$

(D) $27 : 250$

(E) $81 : 250$

Q13

The tangents to the curve $y = x^2 - 9$ are drawn at the points where the curve meets the x-axis.

What is the area of the closed region bounded by the curve and the two tangents?

(A) 9

(B) 18

(C) 54

(D) 72

(E) 90

Q14

The two functions f and g satisfy

$$f'(x) = ax + g(x)$$

where a is a constant.

Given that

$$\int_2^4 g(x)\,\mathrm{d}x = 12$$

and

$$f(4) = 18 + f(2)$$

what is the value of a?

(A) 1

(B) 3

(C) 5

(D) 6

(E) 15

Q15

Which one of the following is the area enclosed by the curves $y = -x^2 + 5x - 4$ and $y = x^2 - x$?

(A) $\frac{1}{3}$

(B) $\frac{5}{3}$

(C) 2

(D) $\frac{8}{3}$

(E) 5

Q16

Given that

$$\int_0^2 x^m \, \mathrm{d}x = \frac{16\sqrt{2}}{7}$$

And

$$\int_0^2 x^{m+1} \, \mathrm{d}x = \frac{32\sqrt{2}}{9}$$

what is the value of m?

(A) $-\frac{11}{2}$

(B) $-\frac{9}{2}$

(C) $-\frac{22}{29}$

(D) $\frac{7}{22}$

(E) $\frac{5}{2}$

(F) $\frac{7}{2}$

Q17

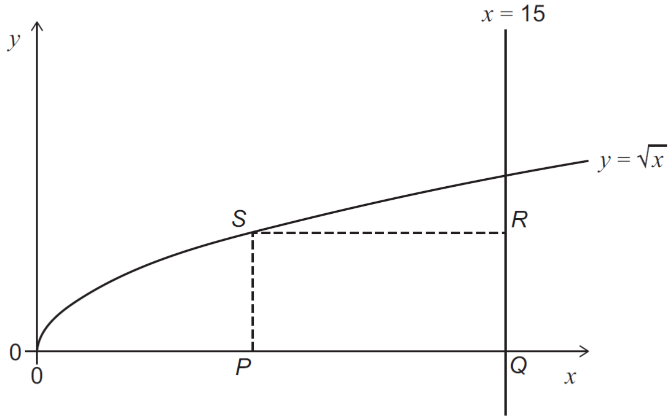
$PQRS$ is a rectangle.

P and Q lie on the x-axis.

Q and R lie on the line $x = 15$

S lies on the curve $y = \sqrt{x}$

What is the maximum possible area of the rectangle?



(A) $5\sqrt{5}$

(B) $10\sqrt{5}$

(C) 50

(D) $25\sqrt{5}$

(E) 100

(F) 125

Supplements S15

Time Allowed

120 min

Number of Questions

14

Difficulty

★★★★

Supplements S15

Scan the QR code or click the link above to take the practice online.

15

382

SQ1

The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

(A) $\frac{9}{2}$

(B) $\frac{7}{3}$

(C) $\frac{7}{2}$

(D) $\frac{11}{2}$

SQ2

What is the value of $\int_0^1 (\mathrm{e}^x - x)(\mathrm{e}^x + x)\,\mathrm{d}x$?

(A) $\frac{3e^2-2}{6}$

(B) $\frac{3e^2+2}{6}$

(C) $\frac{2e^2-3}{6}$

(D) $\frac{3e^2-5}{6}$

(E) $\frac{e^2+3}{6}$

SQ3

What is the value of the definite integral

$$\int_1^2 \frac{\mathrm{d}x}{x + x^3}?$$

(A) $\ln 2 - \frac{\pi}{6}$

(B) $2\ln 2 - \ln 5$

(C) $\frac{1}{2}\ln\frac{8}{5}$

(D) None of the above

SQ4

Let

$$f(x) = \int_0^1 (xt)^2 \, \mathrm{d}t, \text{ and } g(x) = \int_0^x t^2 \, \mathrm{d}t.$$

Let $A > 0$. Which of the following statements is true?

(A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$.

(B) $f\big(g(A)\big)$ is always bigger than $g\big(f(A)\big)$.

(C) They are always equal.

(D) $f\big(g(A)\big)$ is bigger if $A < 1$, and $g\big(f(A)\big)$ is bigger if $A > 1$.

(E) $g\big(f(A)\big)$ is bigger if $A < 1$ and $f\big(g(A)\big)$ is bigger if $A > 1$.

SQ5

The area between the parabolas with equations $y = x^2 + 2ax + a$ and $y = a - x^2$ equals 9. The possible values of a are

(A) $a = 1$

(B) $a = -3$ or $a = 3$

(C) $a = -3$

(D) $a = -1$ or $a = 1$

(E) $a = 1$ or $a = 3$

15

384

SQ6

Which of the following integrals has the greatest value?

(A) $\int_0^{\pi/2} \sin^2 x \cos x \, \mathrm{d}x$

(B) $\int_0^{\pi} \sin^2 x \cos x \, \mathrm{d}x$

(C) $\int_0^{\pi/2} \sin x \cos^2 x \, \mathrm{d}x$

(D) $\int_0^{\pi/2} \sin 2x \cos x \, \mathrm{d}x$

SQ7

A line is tangent to the parabola $y = x^2$ at the point (a, a^2) where $a > 0$. The area of the region bounded by the parabola, the tangent line, and the x-axis equals

(A) $\frac{a^2}{3}$

(B) $\frac{2a^2}{3}$

(C) $\frac{a^3}{12}$

(D) $\frac{5a^3}{6}$

(E) $\frac{a^4}{10}$

SQ8

Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,\mathrm{d}x + \int_1^2 2f(x)\,\mathrm{d}x = 7,$$
$$\int_0^2 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x = 1.$$

It follows that $\int_0^2 f(x)\,\mathrm{d}x$ equals

(A) -1

(B) 0

(C) $\frac{1}{2}$

(D) 2

SQ9

Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a.$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(A) all values of a

(B) $a > 1$

(C) $a > \frac{6}{5}$

(D) $a > \left(\frac{4}{3}\right)^{\frac{3}{2}}$

(E) $a > \left(\frac{6}{5}\right)^4$

SQ10

For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x]\,\mathrm{d}x$$

equals

($k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.)

(A) $\log_2\big((2^n - 1)!\big)$

(B) $n2^n - \log_2\big((2^n)!\big)$

(C) $n2^n$

(D) $\log_2\big((2^n)!\big)$

SQ11

Determine the area inside the circle defined by:

$$x^2 + y^2 - 8x + 4y + 4 = 0$$

but outside the triangle bounded by the three lines below.

$$y = x - 7$$
$$y = \frac{1}{5}(2x - 29)$$
$$x = 7$$

SQ12

(a) On the same axes, sketch the functions $y = x^2 + 1$, $y = 2/x$, and $y = 3x + 1$. [3]

(b) Determine the exact x coordinates at which any of the graphs intersect and mark these on the x-axis. [3]

(c) Find the exact area enclosed between $y = 3x + 1$ and $y = x^2 + 1$ that is also below the curve $y = 2/x$. [4]

SQ13

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS}.

Let

$$f(x) = x^3 - 3x^2 + 2x.$$

(i) On the axes below, sketch the curve $y = f(x)$ for the range $-1 < x < 3$, carefully labelling any turning points.

(ii) The equation $f(x) = k$ has exactly one positive solution and exactly one negative solution. Find k.

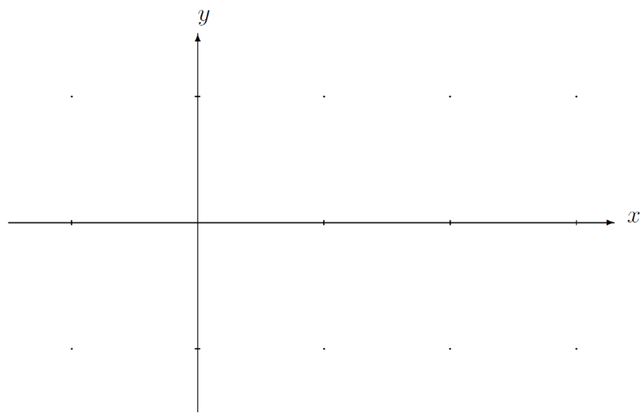
For x in the range $0 \le x \le 2$ the functions $g(x)$ and $h(x)$ are defined by

$$g(x) = \int_0^x f(t)\,\mathrm{d}t,$$
$$h(x) = \int_0^x |f(t)|\,\mathrm{d}t.$$

(iii) Find the value X_1 of x in the range $0 \le x \le 2$ for which $g(x)$ is greatest. Calculate $g(X_1)$.

(iv) Find the value X_2 of x in the range $0 \le x \le 2$ for which $h(x)$ is greatest.

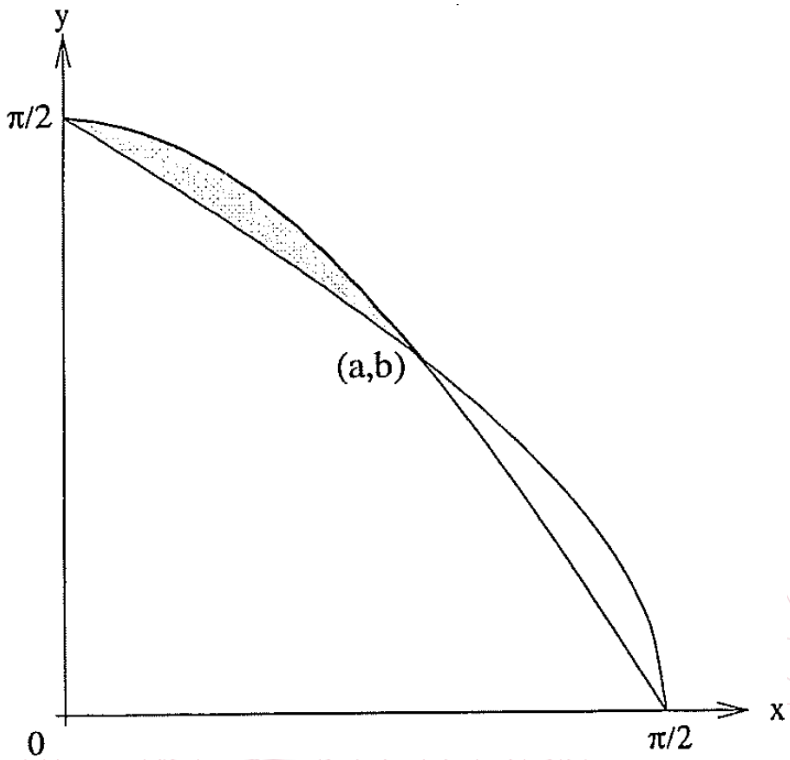
[You are not asked to calculate $h(X_2)$.]



15
387

SQ14

The curves $y = \frac{1}{2}\pi\cos x$ and $x = \frac{1}{2}\pi\cos y$ intersect at the three points $\left(0, \frac{1}{2}\pi\right)$, (a, b), $\left(\frac{1}{2}\pi, 0\right)$, as shown in the figure below.



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388

(i) Explain why $a = b = \frac{1}{2}\pi\cos b$.

(ii) Show that $\pi\sin b = \sqrt{\pi^2 - 4b^2}$.

(iii) Show that the area of the shaded region is

$$\sqrt{\pi^2 - 4b^2} - \frac{\pi}{2} - b^2.$$

16 Plane Geometry

What's on the Specification?

- The sine and cosine rules, and the area of a triangle in the form $\frac{1}{2}ab\sin C$. The sine rule includes an understanding of the 'ambiguous' case (angle-side-side). Problems might be set in 2- or 3-dimensions.
- Use of the following circle properties:
- The perpendicular from the centre to a chord bisects the chord;
- The tangent at any point on a circle is perpendicular to the radius at that point;
- The angle subtended by an arc at the centre of a circle is twice the angle subtended by the arc at any point on the circumference;
- The angle in a semicircle is a right angle;
- Angles in the same segment are equal;
- The opposite angles in a cyclic quadrilateral add to 180°;
- The angle between the tangent and chord at the point of contact is equal to the angle in the alternate segment.

Exercises E16

Time Allowed

No limit

Number of Questions

19

Difficulty

★★★☆

Exercises E16

Scan the QR code or click the link above to take the practice online.

16

390

Quiz Pre-1

In the triangle PQR, $PR = 2$, $QR = p$ and $\angle RPQ = 30°$.

What is the set of **all** the values of p for which this information uniquely determines the length of PQ?

(A) $p = 1$

(B) $p = \sqrt{3}$

(C) $1 \leq p < 2$

(D) $\sqrt{3} \leq p < 2$

(E) $p = 1$ or $p \geq 2$

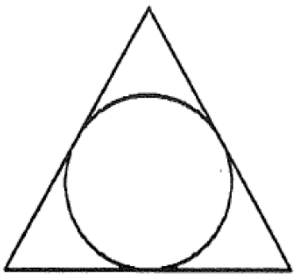
(F) $p = \sqrt{3}$ or $p \geq 2$

(G) $p < 2$

(H) $p \geq 2$

Quiz Pre-2

A circle is inscribed in an equilateral triangle as shown in the diagram. The area of the circle, as a percentage of the triangle's, is approximately



(A) 40%

(B) 50%

(C) 60%

(D) 70%

Quiz Pre-3

If you look at a clock and the time is 9:45, what is the angle between the hour and the minute hands?

(A) 0°

(B) 7.5°

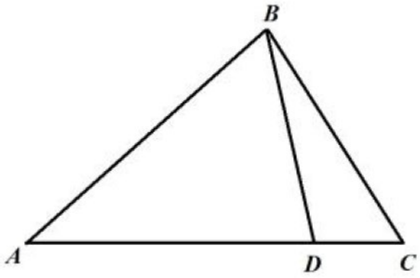
(C) 15°

(D) 22.5°

(E) 30°

Ex. 1

In ΔABC, $AB = 5$, $BC = 7$, $AC = 9$ and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD:DC$.



Ex. 2

The points O, A, B and C have coordinates $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Ex. 3

In triangle PQR, $PQ = 4x$ cm, $QR = (8-3x)$ cm, $\angle PQR = 60°$.

What is the maximum value of the area, in cm^2, of triangle PQR?

(A) $\frac{8\sqrt{3}}{3}$

(B) $\frac{16}{3}$

(C) $\frac{69\sqrt{3}}{16}$

(D) $\frac{16\sqrt{3}}{3}$

(E) $\frac{32}{3}$

(F) $\frac{32\sqrt{3}}{3}$

Ex. 4

The triangle PQR has a right angle at R.

The length of PQ is 4 cm, correct to the nearest centimetre.

The length of PR is 2 cm, correct to the nearest centimetre.

Find the minimum possible length, in centimetres, of QR.

(A) $\sqrt{6}-\frac{1}{2}$

(B) $2\sqrt{3}-\frac{1}{2}$

(C) $2\sqrt{5}-\frac{1}{2}$

(D) $2\sqrt{5}$

(E) $2\sqrt{3}$

(F) $\sqrt{6}$

Ex. 5

A triangle ABC is to be drawn with $AB = 10$ cm, $BC = 7$ cm and the angle at A equal to θ, where θ is a certain specified angle.

Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one.

What is the value of $\cos\theta$?

(A) $\frac{5}{7}$

(B) $\frac{151}{200}$

(C) $\frac{2\sqrt{2}}{5}$

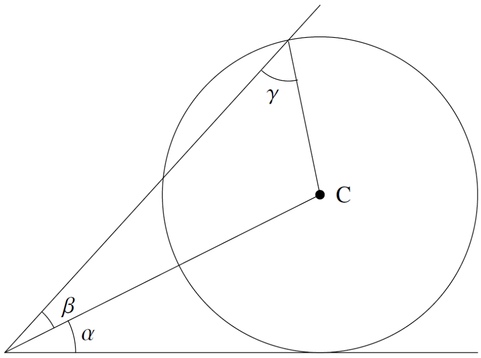
(D) $\frac{\sqrt{17}}{5}$

(E) $\frac{\sqrt{51}}{8}$

(F) $\frac{\sqrt{34}}{8}$

Ex. 6

The circle in the diagram has centre C. Three angles α, β, γ are also indicated.



The angles α, β, γ are related by the equation:

(A) $\cos\alpha = \sin(\beta + \gamma)$

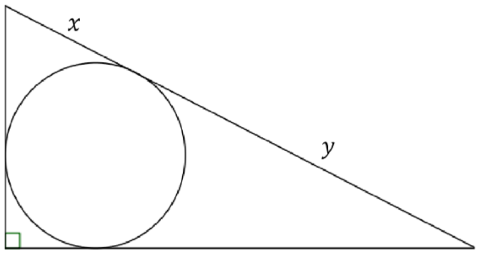
(B) $\sin\beta = \sin\alpha \sin\gamma$

(C) $\sin\beta\,(1 - \cos\alpha) = \sin\gamma$

(D) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$

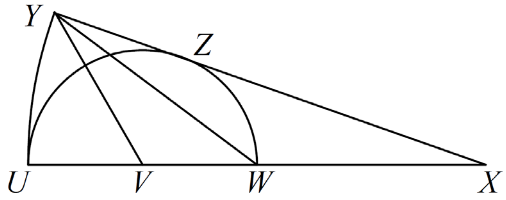
Ex. 7

In the diagram below, the circle touches all three sides of the triangle. Find the area of the triangle in terms of x and y.



Ex. 8

A semicircle of radius r is drawn with centre V and diameter UW. The line UW is then extended to the point X, such that UW and WX are of equal length. An arc of the circle with centre X and radius $4r$ is then drawn so that the line XY is a tangent to the semicircle at Z, as shown. What, in terms of r, is the area of triangle YVW?



(A) $\frac{4r^2}{9}$

(B) $\frac{2r^2}{3}$

(C) r^2

(D) $\frac{4r^2}{3}$

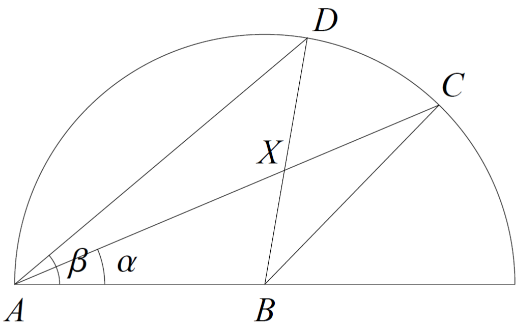
(E) $2r^2$

Ex. 9

For applicants in {Math, Math & Statistics, Math & Philosophy} only. Not for {Math & CS, CS and CS & Philosophy}.

In the diagram below is sketched a semicircle with centre B and radius 1. Three points A, C, D lie on the semicircle as shown with α denoting angle CAB and β denoting angle DAB. The triangles ABC and ABD intersect in a triangle ABX.

Throughout the question we shall consider the value of α fixed. Assume for now that $0 < \alpha \leq \beta \leq \frac{\pi}{2}$.



(i) Show that the area of the triangle ABC equals

$$\frac{1}{2}\sin(2\alpha).$$

(ii) Let

$$F = \frac{\text{area of triangle } ABX}{\text{area of triangle } ABC}.$$

Without calculation, explain why, for every k in the range $0 \leq k \leq 1$, there is a unique value of β such that $F = k$.

(iii) Find the value of β such that $F = \frac{1}{2}$.

(iv) Show that

$$F = \frac{\sin(2\beta)\sin\alpha}{\sin(2\beta - \alpha)\sin(2\alpha)}.$$

(v) Suppose now that $0 < \beta < \alpha \leq \frac{\pi}{2}$. Write down, without further calculation, an expression for the area of ABX and hence a formula for F.

Quiz 1

The hands of a 12-hour analogue clock move continuously. When the time on the clock is 4:00, the angle between the minute hand and the hour hand is 120°.

What is the angle between the two hands at 4:40?

(A) 80°

(B) 100°

(C) 110°

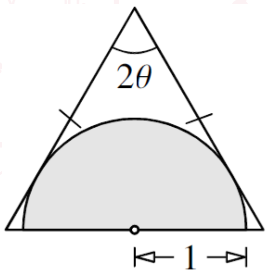
(D) 120°

(E) 140°

Quiz 2

The diagram shows a semicircle of radius 1 inside an isosceles triangle. The diameter of the semicircle lies along the 'base' of the triangle, and the angle of the triangle opposite the 'base' is equal to 2θ. Each of the two equal sides of the triangle is tangent to the semicircle.

What is the area of the triangle?



(A) $\frac{1}{2}\tan 2\theta$

(B) $\sin\theta\cos\theta$

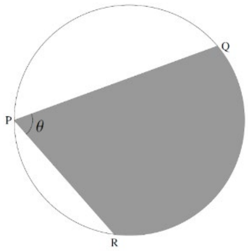
(C) $\cos\theta + \sin\theta$

(D) $\frac{1}{2}\cos 2\theta$

(E) $\frac{1}{\sin\theta\cos\theta}$

Quiz 3

If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,



then the largest possible area of the shaded region RPQ is

(A) $\theta\left(1+\cos\left(\frac{\theta}{2}\right)\right)$

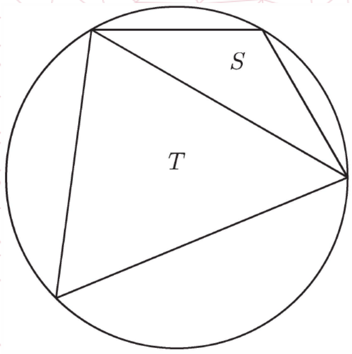
(B) $\theta+\sin\theta$

(C) $\frac{\pi}{2}(1-\cos\theta)$

(D) θ

Quiz 4

Two triangles S and T are inscribed in a circle, as shown in the diagram below.



The triangles have respective areas s and t and S is the smaller triangle so that $s<t$.

The smallest value that

$$\frac{4s^2+t^2}{5st}$$

can equal is

(A) $\frac{2}{5}$

(B) $\frac{3}{5}$

(C) $\frac{4}{5}$

(D) 1

(E) $\frac{3}{2}$

Quiz 5

The lengths of the sides QR, RP and PQ in triangle PQR are a, $a+d$ and $a+2d$ respectively, where a and d are positive and such that $3d > 2a$.

What is the full range, in degrees, of possible values for angle PRQ?

(A) $0 <$ angle $PRQ < 60$

(B) $0 <$ angle $PRQ < 120$

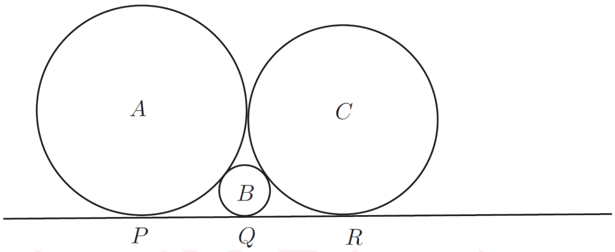
(C) $60 <$ angle $PRQ < 120$

(D) $60 <$ angle $PRQ < 180$

(E) $120 <$ angle $PRQ < 180$

Ex. 10

(i)



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

Ex. 11

The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1),$$

where $q = \frac{b+c}{a}$. Deduce that

$$R \leq \frac{1}{\pi(\pi - 1)}.$$

Practices P16

Time Allowed

90 min

Number of Questions

18

Difficulty

★★★★

Practices P16

Scan the QR code or click the link above to take the practice online.

Q1

The equilateral triangle PQR has sides of length 8 cm.

A circle, centre O, passes through each of the vertices of the triangle.

Find an expression for the circumference of the circle, in cm.

(A) $\frac{\sin 60°}{8\pi}$

(B) $\frac{8\pi}{\sin 60°}$

(C) $\frac{\cos 60°}{8\pi}$

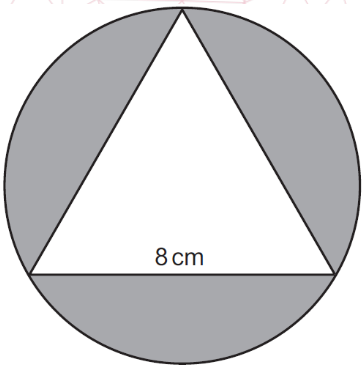
(D) $\frac{8\pi}{\cos 60°}$

(E) $\frac{\tan 60°}{8\pi}$

(F) $\frac{8\pi}{\tan 60°}$

Q2

An equilateral triangle of side 8 cm is drawn so that its vertices lie on the circumference of a circle, as shown in the diagram.



What is the total of the three areas shaded in the diagram, in cm^2?

(A) $8(2\pi - 3)$

(B) $24(\pi - \sqrt{3})$

(C) $48(4\pi - \sqrt{3})$

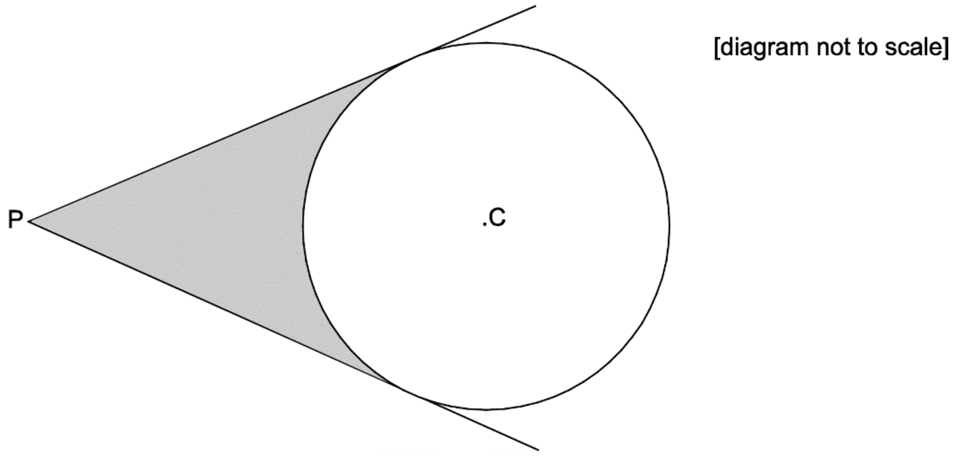
(D) $\frac{16}{3}(4\pi - 6 - 3\sqrt{3})$

(E) $\frac{16}{3}(4\pi - 3\sqrt{3})$

Q3

Tangents are drawn from a point P to a circle of radius 10 cm.

The centre of the circle is C and the distance PC is 20 cm.



Which one of the following is an expression for the shaded area in square centimetres?

(A) $\frac{100}{3}(3\sqrt{3}-\pi)$

(B) $\frac{100}{3}(3\sqrt{5}-\pi)$

(C) $\frac{50}{3}(6\sqrt{3}-\pi)$

(D) $\frac{50}{3}(6\sqrt{5}-\pi)$

(E) $\frac{50}{3}(\sqrt{3}-2\pi)$

(F) $\frac{50}{3}(2\pi-\sqrt{3})$

Q4

Two circles have centres P and Q.

The radius of each circle is 1 cm.

The distance PQ is 1 cm.

What is the area of overlap, in cm^2, of the two circles?

(A) $\frac{\pi}{3}-\frac{1}{4}$

(B) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

(C) $\frac{2\pi}{3}-\frac{1}{2}$

(D) $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

(E) $\frac{4\pi}{3}-\frac{1}{4}$

(F) $\frac{4\pi}{3}-\frac{\sqrt{3}}{2}$

Q5

The triangle PQR has a right angle at R.

The length of PQ is 4 cm, correct to the nearest centimetre.

The length of PR is 2 cm, correct to the nearest centimetre.

Find the minimum possible length, in centimetres, of QR.

(A) $\sqrt{6} - \frac{1}{2}$

(B) $2\sqrt{3} - \frac{1}{2}$

(C) $2\sqrt{5} - \frac{1}{2}$

(D) $2\sqrt{5}$

(E) $2\sqrt{3}$

(F) $\sqrt{6}$

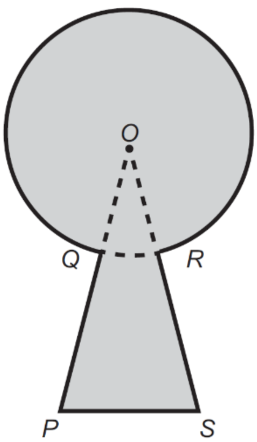
Q6

The diagram shows the outline of a keyhole consisting of three straight sides and an arc from a circle.

The sides PQ and RS are both 18 mm in length and when extended meet at the centre of the circle O forming an angle of $\frac{\pi}{6}$ radians.

The longer arc from Q to R has length 22π mm.

[diagram not to scale]



What is the area, in mm², of the keyhole as shaded in the diagram?

(A) $121\pi + \frac{841}{4}$

(B) $121\pi + \frac{841\sqrt{3}}{4}$

(C) $132\pi + 225$

(D) $132\pi + 225\sqrt{3}$

(E) $144\pi + 225$

(F) $144\pi + 225\sqrt{3}$

Q7

A triangle ABC is to be drawn with $AB = 10$ cm, $BC = 7$ cm and the angle at A equal to θ, where θ is a certain specified angle.

Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one.

What is the value of $\cos\theta$?

(A) $\frac{5}{7}$

(B) $\frac{151}{200}$

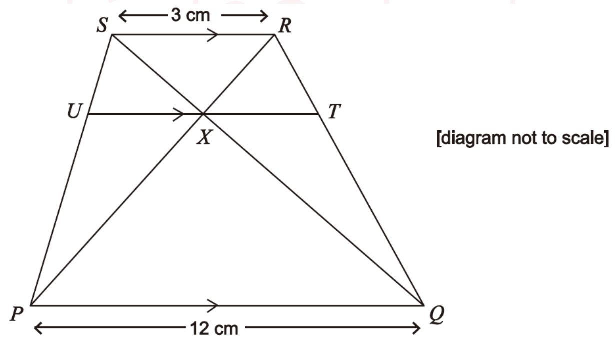
(C) $\frac{2\sqrt{2}}{5}$

(D) $\frac{\sqrt{17}}{5}$

(E) $\frac{\sqrt{51}}{8}$

(F) $\frac{\sqrt{34}}{8}$

Q8



In the figure, $PQRS$ is a trapezium with PQ parallel to SR.

The diagonals of the trapezium meet at X.

U lies on SP and T lies on RQ such that UT is a line segment through X parallel to PQ.

The length of PQ is 12 cm and the length of SR is 3 cm.

What, in centimetres is the length of UT?

(A) 4.2

(B) 4.5

(C) 4.8

(D) 5.25

(E) 6

Q9

Each interior angle of a regular polygon with n sides is $\frac{3}{4}$ of each interior angle of a second regular polygon with m sides.

How many pairs of positive integers n and m are there for which this statement is true?

(A) none

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

(H) infinitely many

Q10

A rectangle has perimeter P and area A. The values P and A must satisfy

(A) $P^3 > A$

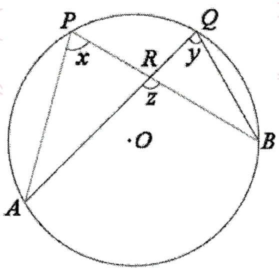
(B) $A^2 > 2P + 1$

(C) $P^2 \geq 16A$

(D) $PA \geq A + P$

Q11

In the figure above, A, B, P and Q are arbitrary distinct points on a circle centred at O. The chords AQ and BP meet at R, and the angles $\angle APB$, $\angle AQB$ and $\angle ARB$ are equal to x, y and z respectively. Which of the following statements is true?



(A) $x + y = \frac{\pi}{2}$

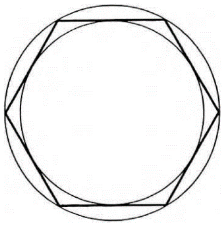
(B) $x + y = z$

(C) $x + y = 2x$

(D) $x + y = \pi$

Q12

The figure shows a regular hexagon with its circumscribed and inscribed circles. What is the ratio of the area of the two circles?



(A) $4:3$

(B) $6:5$

(C) $7:5$

(D) $\sqrt{3}:2$

Q13

The triangle ABC is right-angled at B and the side lengths are positive numbers in geometric progression. It follows that $\tan \angle BAC$ is either

(A) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1-\sqrt{5}}{2}}$

(B) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{\sqrt{3}-1}{2}}$

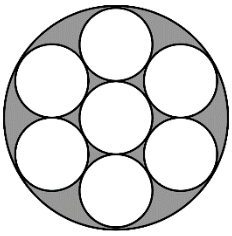
(C) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{\sqrt{5}-1}{2}}$

(D) $-\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1+\sqrt{5}}{2}}$

(E) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{1-\sqrt{3}}{2}}$

Q14

The diagram shows seven circles of equal radius which fit snugly in the larger circle. What is the ratio of the unshaded area to the shaded area?



(A) $7 : 1$

(B) $7 : 2$

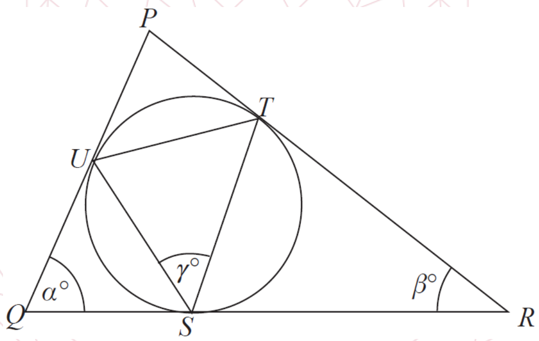
(C) $2\sqrt{3} : 1$

(D) $9 : 2$

(E) $1 : 1$

Q15

A circle touches the sides of triangle PQR at the points S, T and U as shown. Also $\angle PQR = \alpha°$, $\angle PRQ = \beta°$ and $\angle TSU = \gamma°$. Which of the following gives γ in terms of α and β?



(A) $\frac{1}{2}(\alpha + \beta)$

(B) $180 - \frac{1}{2}(\alpha + \beta)$

(C) $180 - (\alpha + \beta)$

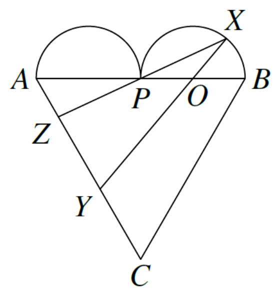
(D) $\alpha + \beta$

(E) $\frac{1}{3}(\alpha + \beta)$

Q16

The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.

What is $\angle ZXY$?



(A) $20°$

(B) $25°$

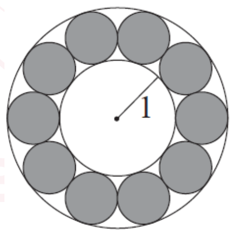
(C) $30°$

(D) $40°$

(E) $45°$

Q17

The diagram shows ten equal discs that lie between two concentric circles-an inner circle and an outer circle. Each disc touches two neighbouring discs and both circles. The inner circle has radius 1. What is the radius of the *outer* circle?



(A) $2\tan 36°$

(B) $\frac{\sin 36°}{1-\sin 36°}$

(C) $\frac{1+\sin 18°}{1-\sin 18°}$

(D) $\frac{2}{\cos 18°}$

(E) $\frac{9}{5}$

Q18

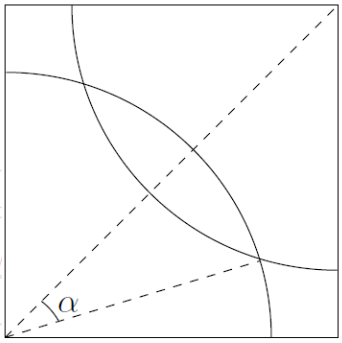
For applicants in {Math, Math & Statistics and Math &Philosophy} only.

A horse is attached by a rope to the corner of a square field of side length 1.

(i) What length of rope allows the horse to reach precisely half the area of the field?

Another horse is placed in the field, attached to the corner diagonally opposite from the first horse. Each horse has a length of rope such that each can reach half the field.

(ii) Explain why the area that both can reach is the same as the area neither can reach.



(iii) The angle α is marked in the diagram above. Show that $\alpha = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$ and hence show that the area neither can reach is $\frac{4}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$. Note that $\cos^{-1}$ can also be written as arccos.

A third horse is placed in the field, and the three horses are rearranged. One horse is now attached to the midpoint of the bottom side of the field, and another horse is now attached to the midpoint of the left side of the field. The third horse is attached to the upper right corner.

(iv) Given each horse can access an equal area of the field and that none of the areas overlap, what length of rope must each horse have to minimise the area that no horse can reach?

The horses on the bottom and left midpoints of the field are each replaced by a goat; each goat is attached by a rope of length g to the same midpoint as in part (iii). The remaining horse is attached to the upper right corner with rope length h.

(v) Given that $0 \leq h \leq 1$, and that none of the animals' areas can overlap, show that $\frac{\sqrt{5}-2}{2} \leq g \leq \frac{1}{2\sqrt{2}}$ holds if the area that the animals can reach is maximised.

Supplements S16

Time Allowed

90 min

Number of Questions

18

Difficulty

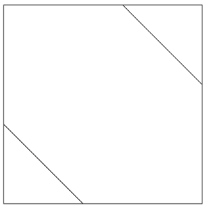
★★★★

Supplements S16

Scan the QR code or click the link above to take the practice online.

SQ1

An irregular hexagon with all sides of equal length is placed inside a square of side length 1, as shown below (not to scale). What is the length of one of the hexagon sides?



(A) $\sqrt{2}-1$

(B) $2-\sqrt{2}$

(C) 1

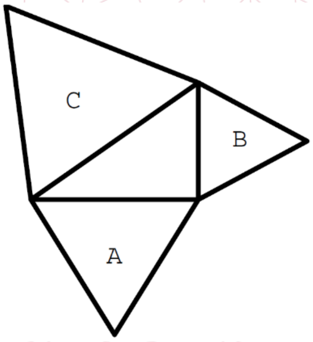
(D) $\frac{\sqrt{2}}{2}$

(E) $2+\sqrt{2}$

SQ2

Three equilateral triangles with areas A, B, C are drawn on the sides of a right-angled triangle as in the diagram.

These areas are related by the equation:



(A) $A^{1/2}+B^{1/2}=C^{1/2}$

(B) $A+B=C$

(C) $A^{3/2}+B^{3/2}=C^{3/2}$

(D) $A^2+B^2=C^2$

SQ3

The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which of the following is true?

(A) $\frac{A}{P} = \frac{5}{4\pi}$

(B) $P < A$

(C) $\frac{P}{A} = \frac{10}{3\pi}$

(D) P^2 is rational

SQ4

An equilateral triangle has centre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. The minimum possible length of PQ is

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

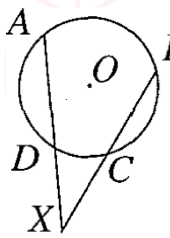
(C) $\frac{\sqrt{3}}{3}$

(D) $\frac{2}{3}$

(E) $\frac{\sqrt{3}}{2}$

SQ5

In the diagram, O is the centre of the circle, $\angle AOB = \alpha$ and $\angle COD = \beta$. What is the size of $\angle AXB$ in terms of α and β?



(A) $\frac{1}{2}\alpha - \frac{1}{2}\beta$

(B) $90° - \frac{1}{2}\alpha - \frac{1}{2}\beta$

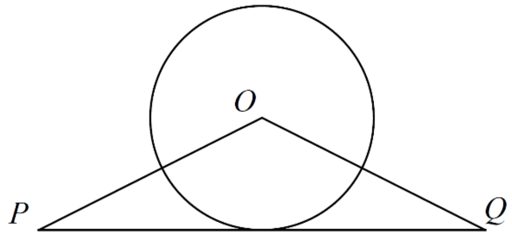
(C) $\alpha - \beta$

(D) $180° - \alpha - \beta$

(E) more information needed

SQ6

The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. The area of the circle is equal to the area of the triangle. What is the ratio of the length of PQ to the circumference of the circle?



(A) 1 : 1

(B) 2 : 3

(C) 2 : π

(D) 3 : 2

(E) π : 2

SQ7

An equilateral triangle of side length 4 cm is divided into smaller equilateral triangles, all of which have side length equal to a whole number of centimetres. Which of the following cannot be the number of smaller triangles obtained?

(A) 4

(B) 8

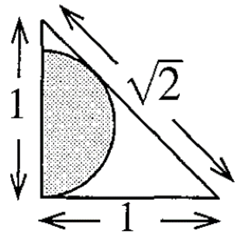
(C) 12

(D) 13

(E) 16

SQ8

What is the radius of the shaded semicircle?



(A) $\sqrt{2}-1$

(B) $\frac{1}{\sqrt{2}}$

(C) $3-2\sqrt{2}$

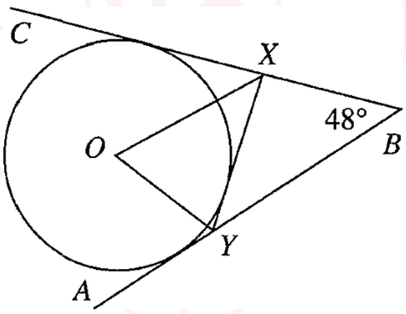
(D) $\frac{1}{2}$

(E) $2-\sqrt{2}$

SQ9

In the diagram, AB, CB and XY are tangents to the circle with centre O and $\angle ABC = 48°$.

What is the size of $\angle XOY$?



(A) 42°

(B) 69°

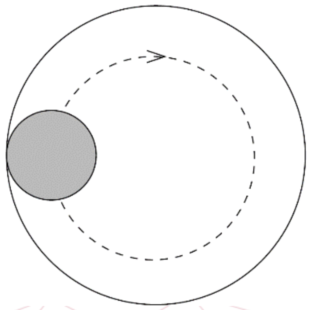
(C) 66°

(D) 48°

(E) 84°

SQ10

A circular disc of diameter d rolls without slipping around the inside of a ring of internal diameter $3d$, as shown in the diagram. By the time that the centre of the inner disc returns to its original position for the first time, how many times will the inner disc have turned about its centre?



(A) 1

(B) π

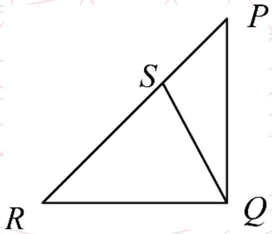
(C) 3

(D) 2π

(E) 2

SQ11

Triangle PQR has a right angle at Q and $PQ = QR$. The line through Q which divides the angle PQR in the ratio $1 : 2$ meets PR at S. What is the ratio $RS : SP$?



(A) $\sqrt{2} : 1$

(B) $\sqrt{3} : 1$

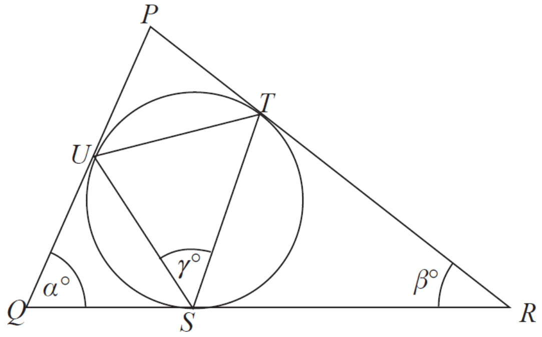
(C) $2 : 1$

(D) $\sqrt{5} : 1$

(E) $3 : 1$

SQ12

A circle touches the sides of triangle PQR at the points S, T and U as shown. Also $\angle PQR = \alpha°$, $\angle PRQ = \beta°$ and $\angle TSU = \gamma°$. Which of the following gives γ in terms of α and β?



(A) $\frac{1}{2}(\alpha + \beta)$

(B) $180 - \frac{1}{2}(\alpha + \beta)$

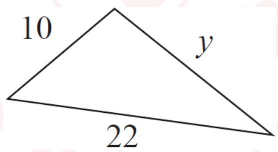
(C) $180 - (\alpha + \beta)$

(D) $\alpha + \beta$

(E) $\frac{1}{3}(\alpha + \beta)$

SQ13

The triangle shown has an area of 88 square units. What is the value of y?



(A) 17.6

(B) $2\sqrt{46}$

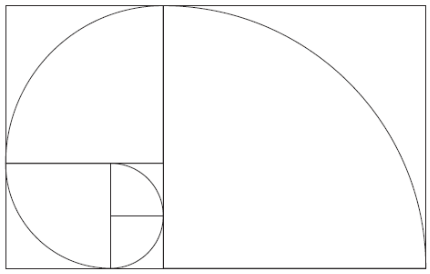
(C) $6\sqrt{10}$

(D) $13\sqrt{2}$

(E) $8\sqrt{5}$

SQ14

Five square tiles are put together side by side. A quarter circle is drawn on each tile to make a continuous curve as shown. Each of the smallest squares has side-length 1. What is the total length of the curve?



(A) 6π

(B) 6.5π

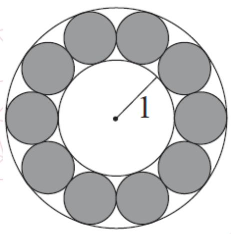
(C) 7π

(D) 7.5π

(E) 8π

SQ15

The diagram shows ten equal discs that lie between two concentric circles-an inner circle and an outer circle. Each disc touches two neighbouring discs and both circles. The inner circle has radius 1. What is the radius of the *outer* circle?



(A) $2\tan 36°$

(B) $\frac{\sin 36°}{1-\sin 36°}$

(C) $\frac{1+\sin 18°}{1-\sin 18°}$

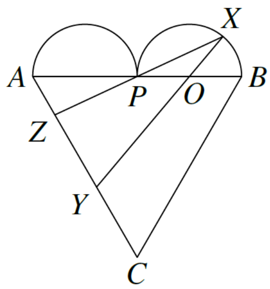
(D) $\frac{2}{\cos 18°}$

(E) $\frac{9}{5}$

SQ16

The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.

What is $\angle ZXY$?



(A) 20°

(B) 25°

(C) 30°

(D) 40°

(E) 45°

SQ17

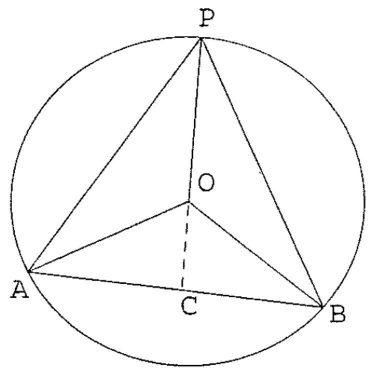
In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2 \alpha)x$.

The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

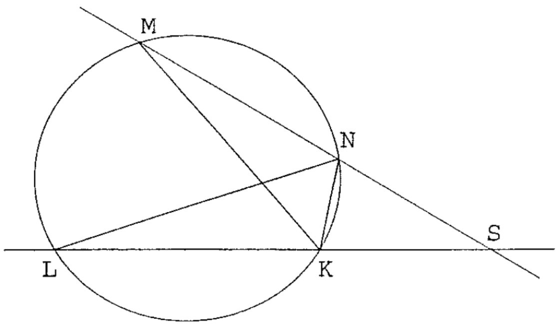
SQ18

(i) Three points P, A, B lie on a circle which has centre O. The point C is where PO extends to meet AB as shown in the diagram below.



Show that $\angle AOC = 2\angle APC$ and $\angle BOC = 2\angle BPC$. Why does this mean that $\angle APB$ is independent of the choice of the point P?

(ii) Four points K, L, M, N lie on a circle and the lines LK and MN meet outside the circle at a point S, as shown in the diagram below.

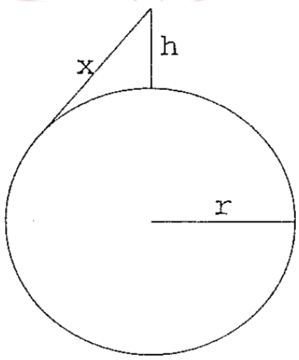


Using part (i) and the Sine Rule show that

$$\frac{KS}{NS} = \frac{SM}{SL}.$$

[You may also assume that part (ii) holds true in the special case when $M = N$ in which case the line SM is the tangent to the circle at M.]

(iii) A tower has height h. Assuming the earth to be a perfect sphere of radius r, determine the greatest distance x from the top of the tower at which an observer can still see it.



17 Coordinate Geometry

What's on the Specification?

- Equation of a straight line, including $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$; conditions for two straight lines to be parallel or perpendicular to each other; finding equations of straight lines given information in various forms.
- Coordinate geometry of the circle: using the equation of a circle in the forms $(x-a)^2 + (y-b)^2 = r^2$, and $x^2 + y^2 + cx + dy + e = 0$.

Exercises E17

Time Allowed

No limit

Number of Questions

22

Difficulty

★★★☆

Exercises E17

Scan the QR code or click the link above to take the practice online.

Quiz Pre-1

The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

(A) $k \leq 0$

(B) $k \geq 4$

(C) $k \geq 0$ or $k \leq -4$

(D) $-4 \leq k \leq 0$

Quiz Pre-2

The point on the circle

$$x^2 + y^2 + 6x + 8y = 75,$$

which is closest to the origin, is at what distance from the origin?

(A) 3

(B) 4

(C) 5

(D) 10

Quiz Pre-3

$PQRS$ is a rectangle.

The coordinates of P and Q are $(0, 6)$ and $(1, 8)$ respectively.

The perpendicular to PQ at Q meets the x-axis at R.

What is the area of $PQRS$?

(A) $\frac{5}{2}$

(B) $4\sqrt{10}$

(C) 20

(D) $8\sqrt{10}$

(E) 40

Quiz Pre-4

A curve has equation $y = 3x^2 + 2$ and a line has equation $y = 5x - 6$.

What is the shortest distance parallel to the y-axis between the curve and the line?

(A) $\frac{5}{6}$

(B) $\frac{11}{6}$

(C) $\frac{49}{12}$

(D) $\frac{71}{12}$

(E) $\frac{73}{12}$

Ex. 1

The shortest distance from the origin to the line $3x + 4y = 25$ is

(A) 3

(B) 4

(C) 5

(D) 6

Ex. 2

The point lying between

$P(2,3)$ and $Q(8,-3)$ which divides the line PQ in the ratio $1:2$ has coordinates

(A) $(4,-1)$

(B) $(6,-2)$

(C) $\left(\frac{14}{3},2\right)$

(D) $(4,1)$

(E) none of these

Ex. 3

The reflection of the point $(1,0)$ in the line $y = mx$ has coordinates

(A) $\left(\frac{m^2+1}{m^2-1},\frac{m}{m^2-1}\right)$

(B) $(1,m)$

(C) (m,m)

(D) $\left(\frac{1-m^2}{1+m^2},\frac{2m}{1+m^2}\right)$

(E) $(1-m^2,m)$

Ex. 4

The equations of two straight lines are $y = 3 + (2p^2 - p)x$ and $y = 7 + (p - 2)x$, where p is a real constant.

For certain values of p, the two lines are perpendicular.

Which of the following numbers is closest to the greatest such value of p?

(A) 2.00

(B) 1.75

(C) 1.50

(D) 1.00

(E) -0.25

(F) -0.50

Ex. 5

$PQRS$ is a rectangle.

The coordinates of P and Q are $(0, 6)$ and $(1, 8)$ respectively.

The perpendicular to PQ at Q meets the x-axis at R.

What is the area of $PQRS$?

(A) $\frac{5}{2}$

(B) $4\sqrt{10}$

(C) 20

(D) $8\sqrt{10}$

(E) 40

Ex. 6

Find the coordinates of the point(s) at which the line $y = m(3x - 2)$ is tangent to the curve $y = 9x^2 + 6x - 7$. In the above m is a real constant.

Ex. 7

The curve C has equation $y = x^3 - 7x - 6$.

Find the equation of the tangent to C at the point where C cuts the positive x-axis.

(A) $y = 4 - 4x$

(B) $y = 2x - 6$

(C) $y = 5x - 10$

(D) $y = 9x - 27$

(E) $y = 20x - 60$

Ex. 8

Given θ in the range $0 \leq \theta < \pi$, the equation

$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

represents a circle for

(A) $0 < \theta < \frac{\pi}{3}$

(B) $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$

(C) $0 < \theta < \frac{\pi}{2}$

(D) all values of θ

Ex. 9

The point on the circle

$$(x-5)^2 + (y-4)^2 = 4$$

which is closest to the circle

$$(x-1)^2 + (y-1)^2 = 1$$

is

(A) (3.4,2.8)

(B) (3,4)

(C) (5,2)

(D) (3.8,2.4)

Ex. 10

Let a and b be positive real numbers. If $x^2 + y^2 \leq 1$ then the largest that $ax + by$ can equal is

(A) $\frac{1}{a} + \frac{1}{b}$

(B) $\max(a, b)$

(C) $\sqrt{a^2 + b^2}$

(D) $a + b$

(E) $a^2 + ab + b^2$

Quiz 1

The perpendicular bisector of the line segment joining the points $(2, -6)$ and $(5, 4)$ cuts the x-axis at the point with x-coordinate

(A) $\frac{1}{20}$

(B) $\frac{1}{6}$

(C) $\frac{1}{3}$

(D) $\frac{19}{5}$

(E) $\frac{41}{6}$

Quiz 2

The line joining the points with coordinates $(p, p-1)$ and $(1-p, 2p)$ is parallel to the line with equation $2x + 3y + 1 = 0$.

What is the value of p?

(A) -1

(B) $-\frac{1}{7}$

(C) $\frac{1}{9}$

(D) $\frac{1}{8}$

(E) 1

(F) $\frac{5}{4}$

(G) 2

(H) 5

Quiz 3

The parallelogram $OPQR$, labelled clockwise, is in the first quadrant ($x \geq 0, y \geq 0$) with O at the origin. The point R has coordinates $\left(\frac{3a}{2}, 0\right)$ and the point Q has coordinates $(2a, a+1)$. The area of $OPQR$ is 9 square units.

What are the coordinates of point P?

(A) $\left(\frac{\sqrt{3}}{2}, 1+\sqrt{3}\right)$

(B) $(1, 3)$

(C) $(1.5, 4)$

(D) $(2, 3)$

(E) $(3, 4)$

(F) $\left(2\sqrt{3}, 1+\sqrt{3}\right)$

Quiz 4

Find the shortest distance between the curve $y = x^2 + 4$ and the line $y = 2x - 2$.

(A) 2

(B) $\sqrt{5}$

(C) $\frac{6\sqrt{5}}{5}$

(D) 3

(E) $\frac{5\sqrt{5}}{3}$

(F) 5

(G) 6

Quiz 5

The two circles with equations

$$x^2 + y^2 = 1, \qquad (x-a)^2 + (y-b)^2 = r^2$$

(where $r > 0$) do *not* intersect if

(A) $\sqrt{a^2 + b^2} + r < 1$, (and only this part).

(B) $\sqrt{a^2 + b^2} + 1 < r$, (and only this part).

(C) $\sqrt{a^2 + b^2} - r > 1$, (and only this part).

(D) all of the above.

Quiz 6

A circle has equation $x^2 + y^2 - 18x - 22y + 178 = 0$.

A regular hexagon is drawn inside this circle so that the vertices of the hexagon touch the circle.

What is the area of the hexagon?

(A) 6

(B) $6\sqrt{3}$

(C) 18

(D) $18\sqrt{3}$

(E) 36

(F) $36\sqrt{3}$

(G) 48

(H) $48\sqrt{3}$

Quiz 7

The circle C_1 has equation $(x + 2)^2 + (y - 1)^2 = 3$.

The circle C_2 has equation $(x - 4)^2 + (y - 1)^2 = 3$.

The straight line l is a tangent to both C_1 and C_2 and has positive gradient.

The acute angle between l and the x-axis is θ.

Find the value of $\tan\theta$.

(A) $\frac{1}{2}$

(B) 2

(C) $\frac{\sqrt{2}}{2}$

(D) $\sqrt{2}$

(E) $\frac{\sqrt{6}}{2}$

(F) $\frac{\sqrt{6}}{3}$

(G) $\frac{\sqrt{3}}{3}$

(H) $\sqrt{3}$

Ex. 11

(i) Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geq 0$, is $c(m^2 + 1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C.

Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2(1 + (y')^2). \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1 + m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

Practices P17

Time Allowed

60 min

Number of Questions

20

Difficulty

★★★☆

Practices P17

Scan the QR code or click the link above to take the practice online.

17

432

Q1

Consider the four lines with the following equations.

1 $2x + 6y = 3$

2 $9y = 3x - 4$

3 $2y = 6x + 3$

4 $4x + 6y - 9 = 0$

Which two lines are perpendicular?

(A) 1 and 2

(B) 1 and 3

(C) 1 and 4

(D) 2 and 3

(E) 2 and 4

(F) 3 and 4

Q2

The straight lines

$$5x + 2y = 20$$
$$y = 3x - 23$$
$$x = 0$$

enclose a region with area K square units.

What is the value of K?

(A) 39

(B) 78

(C) 99

(D) 129

(E) 198

(F) 258

Q3

The straight line P has equation $3y - 2x = 12$ and intercepts the y-axis at the point $(0, p)$.

The straight line Q is parallel to P, passes through the point $(6, -1)$ and intercepts the y-axis at the point $(0, q)$.

What is the value of $p - q$?

(A) -9

(B) -7

(C) 1

(D) 9

(E) 14

(F) 17

Q4

A straight line is drawn joining the points with coordinates $(7, 1 - p)$ and $(2p + 1, -1)$, where p is a constant.

What is the complete set of values of p for which the gradient of this line is finite and greater than zero?

(A) $p < -4, p > 0$

(B) $-4 < p < 0$

(C) $p < 0$

(D) $p < 2$

(E) $2 < p < 3$

(F) $p < 2, p > 3$

Q5

Find the area of the shape bounded by the four lines:

$$2y + x = 4$$
$$x = -6$$
$$x = 0$$
$$y = 0$$

(A) 4

(B) 12

(C) 21

(D) 25

(E) 27

(F) 30

Q6

The equation of a curve C is given by

$$y = \frac{x^2 - 2}{\sqrt{x}}$$

Find the gradient of C at the point $(2, \sqrt{2})$.

(A) $\sqrt{2}$

(B) $\frac{5}{4}\sqrt{2}$

(C) $\frac{7}{4}\sqrt{2}$

(D) $\frac{7}{2}\sqrt{2}$

(E) $4\sqrt{2}$

(F) $\frac{9}{2}\sqrt{2}$

(G) $8\sqrt{2}$

Q7

The line with equation

$$(1 + \sqrt{3})y = px + 5$$

is perpendicular to the line with equation

$$y = (2 - \sqrt{3})x + 8$$

What is the value of p?

(A) $-5 - 3\sqrt{3}$

(B) $-5 + 3\sqrt{3}$

(C) $5 - 3\sqrt{3}$

(D) $5 + 3\sqrt{3}$

Q8

The radius of the circle $2x^2 + 2y^2 - 8x + 12y + 15 = 0$ is

(A) $\sqrt{\frac{5}{2}}$

(B) $\sqrt{\frac{11}{2}}$

(C) $\sqrt{\frac{41}{2}}$

(D) $\sqrt{37}$

(E) $\sqrt{67}$

Q9

The line L with equation $y = mx + c$, where $m > 0$ and $c \geq 0$, passes through the point $(2, 4)$.

A line is drawn through the point $(2, 4)$ perpendicular to L.

The triangle enclosed between the two lines and the y-axis has area 5 square units.

What is the **larger** of the two possible values of m?

(A) -0.5

(B) 0.5

(C) 1.25

(D) 2

(E) 5

Q10

$f(x)$ is a quadratic function in x.

The graph of $y = f(x)$ passes through the point $(1, -1)$ and has a turning point at $(-1, 3)$.

Find an expression for $f(x)$.

(A) $-x^2 - 2x + 2$

(B) $-x^2 + 2x + 3$

(C) $x^2 - 2x$

(D) $x^2 + 2x - 4$

(E) $2x^2 + 4x + 1$

(F) $-2x^2 - 4x + 5$

Q11

Find the shortest distance between the two circles with equations:

$$(x + 2)^2 + (y - 3)^2 = 18$$
$$(x - 7)^2 + (y + 6)^2 = 2$$

(A) 0

(B) 4

(C) 16

(D) $2\sqrt{2}$

(E) $5\sqrt{2}$

Q12

A line is drawn normal to the curve $y = \frac{2}{x^2}$ at the point on the curve where $x = 1$.

This line cuts the x-axis at P and the y-axis at Q.

The length of PQ is

(A) $\frac{3\sqrt{5}}{2}$

(B) $\frac{3\sqrt{17}}{4}$

(C) $\frac{7\sqrt{17}}{4}$

(D) $\frac{35}{4}$

(E) $\frac{35\sqrt{5}}{2}$

(F) $\frac{3\sqrt{17}}{2}$

Q13

A line l has equation $y = 6 - 2x$.

A second line is perpendicular to l and passes through the point $(-6, 0)$.

Find the area of the region enclosed by the two lines and the x-axis.

(A) $16\frac{1}{5}$

(B) 18

(C) $21\frac{3}{5}$

(D) 27

(E) $40\frac{1}{2}$

Q14

A tangent to the circle $x^2 + y^2 = 144$ passes through the point $(20, 0)$ and crosses the positive y-axis.

What is the value of y at the point where the tangent meets the y-axis?

(A) 12

(B) 15

(C) $\frac{49}{3}$

(D) 20

(E) $\frac{64}{3}$

(F) $\frac{80}{3}$

Q15

The circles with equations

$$(x+4)^2 + (y+1)^2 = 64 \quad \text{and}$$
$$(x-8)^2 + (y-4)^2 = r^2 \quad \text{where } r > 0$$

have exactly one point in common.

Find the difference between the two possible values of r.

(A) 4

(B) 10

(C) 16

(D) 26

(E) 50

Q16

The line segment joining the points $(3, 3)$ and $(7, 5)$ is a diameter of a circle.

This circle is translated by 3 units in the negative x-direction, then reflected in the x-axis, and then enlarged by a scale factor of 4 about the centre of the resulting circle.

The equation of the final circle is

(A) $(x-2)^2 + (y-4)^2 = 320$

(B) $(x-2)^2 + (y+4)^2 = 320$

(C) $(x-2)^2 + (y-4)^2 = 80$

(D) $(x-2)^2 + (y+4)^2 = 80$

(E) $(x-2)^2 + (y-4)^2 = 20$

(F) $(x-2)^2 + (y+4)^2 = 20$

Q17

The line $y = mx + 4$ passes through the points $(3, \log_2 p)$ and $(\log_2 p, 4)$.

What are the possible values of p?

(A) $p = 1$ and $p = 4$

(B) $p = 1$ and $p = 16$

(C) $p = \frac{1}{4}$ and $p = 4$

(D) $p = \frac{1}{4}$ and $p = 64$

(E) $p = \frac{1}{64}$ and $p = 4$

(F) $p = \frac{1}{64}$ and $p = 16$

Q18

A curve C has equation $y = f(x)$ where

$$f(x) = p^3 - 6p^2x + 3px^2 - x^3$$

and p is real.

The gradient of the normal to the curve C at the point where $x = -1$ is M.

What is the greatest possible value of M as p varies?

(A) $-\frac{3}{2}$

(B) $-\frac{2}{3}$

(C) $-\frac{1}{2}$

(D) $\frac{1}{4}$

(E) $\frac{2}{3}$

(F) $\frac{3}{2}$

Q19

The line $y = mx + 5$, where $m > 0$, is normal to the curve $y = 10 - x^2$ at the point (p, q).

What is the value of p ?

(A) $\frac{\sqrt{2}}{6}$

(B) $-\frac{\sqrt{2}}{6}$

(C) $\frac{3\sqrt{2}}{2}$

(D) $-\frac{3\sqrt{2}}{2}$

(E) $\sqrt{5}$

(F) $-\sqrt{5}$

Q20

Find the complete set of values of m in terms of c such that the graphs of $y = mx + c$ and $y = \sqrt{x}$ have two points of intersection.

(A) $0 < m < \frac{1}{4c}$

(B) $0 < m < 4c^2$

(C) $m > \frac{1}{4c}$

(D) $m < \frac{1}{4c}$

(E) $m > 4c^2$

(F) $m < 4c^2$

Supplements S17

Time Allowed

90 min

Number of Questions

20

Difficulty

★★★★

Supplements S17

Scan the QR code or click the link above to take the practice online.

SQ1

The equation of a curve is $y = px^2 + qx$ where p and q are constants.

The curve passes through the points $(2, 6)$ and $(4, -4)$.

What is the value of $q - p$?

(A) 1

(B) 2

(C) 5

(D) 6

(E) 9

(F) 16

SQ2

The graph of $y = x^2 + ax + b$ meets the straight line $y = x + 1$ when $x = 2$ and $x = 4$.

Find a and b.

(A) $a = -5, b = 9$

(B) $a = 5, b = 9$

(C) $a = -5, b = 11$

(D) $a = 5, b = 11$

(E) $a = -6, b = 11$

(F) $a = 6, b = 11$

(G) $a = -6, b = 13$

(H) $a = 6, b = 13$

SQ3

Find the y-coordinate of the points on the curve $y = x^2$ that are closest to the point $\left(0, \frac{9}{2}\right)$.

(A) 0

(B) $\frac{1}{4}$

(C) $\frac{4}{3}$

(D) 4

(E) $\frac{9}{2}$

SQ4

The straight line with equation $y = mx + 3$, where $m > 0, m \neq 1$, is perpendicular to the line with equation $y = px + 2$.

The lines cut the x-axis at the points L and M respectively. The length of LM is 5 units.

What is the value of $m + p$ given that $m > 1$?

(A) $-\frac{8}{3}$

(B) $-\frac{13}{6}$

(C) $-\frac{5}{6}$

(D) $\frac{5}{6}$

(E) $\frac{13}{6}$

(F) $\frac{8}{3}$

SQ5

A square $PQRS$ is drawn above the x-axis with the side PQ on the x-axis.

P is the point $(-5, 0)$ and Q is the point $(1, 0)$.

A circle is drawn inside the square with diameter equal in length to the side of the square.

Which one of the following is an equation of the circle?

(A) $x^2 + y^2 - 4x + 6y + 4 = 0$

(B) $x^2 + y^2 - 4x + 6y + 9 = 0$

(C) $x^2 + y^2 + 4x - 6y + 4 = 0$

(D) $x^2 + y^2 + 4x - 6y + 9 = 0$

(E) $x^2 + y^2 - 6x - 4y + 9 = 0$

(F) $x^2 + y^2 - 6x + 4y + 4 = 0$

(G) $x^2 + y^2 + 6x - 4y + 4 = 0$

(H) $x^2 + y^2 + 6x + 4y + 9 = 0$

SQ6

The line $y = x + k$, where k is a constant, is a tangent to the curve $y = 3x^2 - 2x + 1$.

What is the value of k?

(A) -2

(B) -1

(C) $\frac{1}{4}$

(D) $\frac{1}{3}$

(E) $\frac{1}{2}$

(F) $\frac{3}{4}$

(G) 1

(H) 2

SQ7

The point lying between $P(2, 3)$ and $Q(8, -3)$ which divides the line PQ in the ratio $1 : 2$ has co-ordinates

(A) $(4, -1)$

(B) $(6, -2)$

(C) $\left(\frac{14}{3}, 2\right)$

(D) $(4, 1)$

SQ8

Which of the following lines is a tangent to the circle with equation

$$x^2 + y^2 = 4?$$

(A) $x + y = 2$

(B) $y = x - 2\sqrt{2}$

(C) $x = \sqrt{2}$

(D) $y = \sqrt{2} - x$

SQ9

A square has centre $(3, 4)$ and one corner at $(1, 5)$. Another corner is at

(A) $(1, 3)$

(B) $(5, 5)$

(C) $(4, 2)$

(D) $(2, 2)$

(E) $(5, 2)$

SQ10

The turning point of the parabola

$$y = x^2 - 2ax + 1$$

is closest to the origin when

(A) $a = 0$

(B) $a = \pm 1$

(C) $a = \pm\frac{1}{\sqrt{2}}$ or $a = 0$

(D) $a = \pm\frac{1}{\sqrt{2}}$

SQ11

The origin lies inside the circle with equation

$$x^2 + ax + y^2 + by = c$$

precisely when

(A) $c > 0$

(B) $a^2 + b^2 > c$

(C) $a^2 + b^2 < c$

(D) $a^2 + b^2 > 4c$

(E) $a^2 + b^2 < 4c$

SQ12

What is the reflection of the point (3,4) in the line $3x + 4y = 50$?

(A) (9,12)

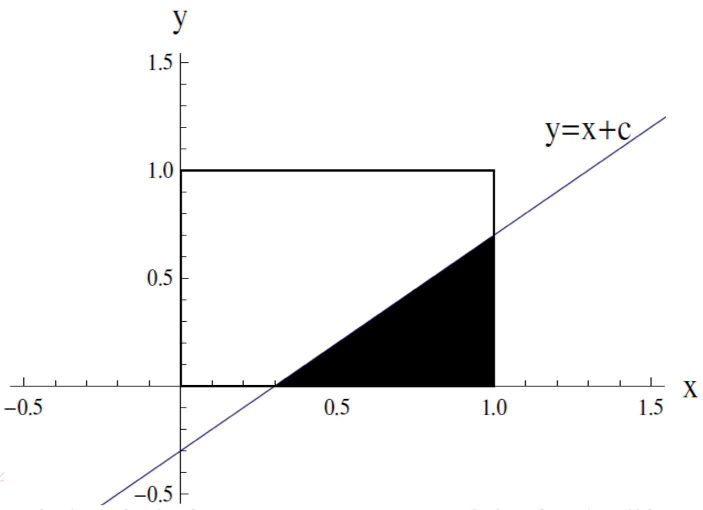
(B) (6,8)

(C) (12,16)

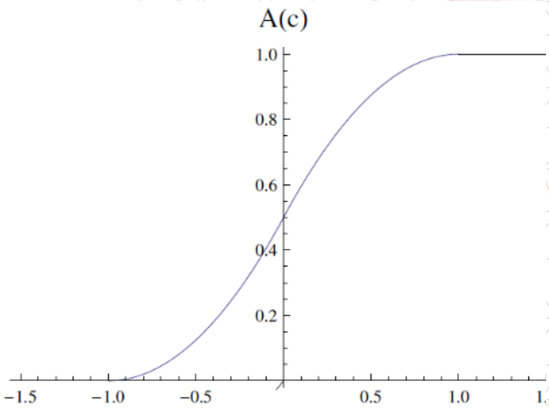
(D) (16,12)

SQ13

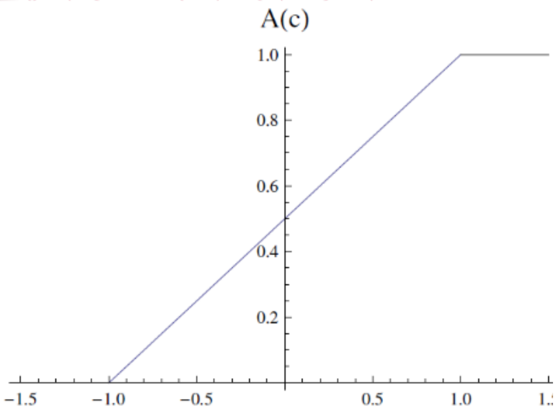
Shown below is a diagram of the square with vertices (0,0), (0,1), (1,1), (1,0) and the line $y = x + c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.



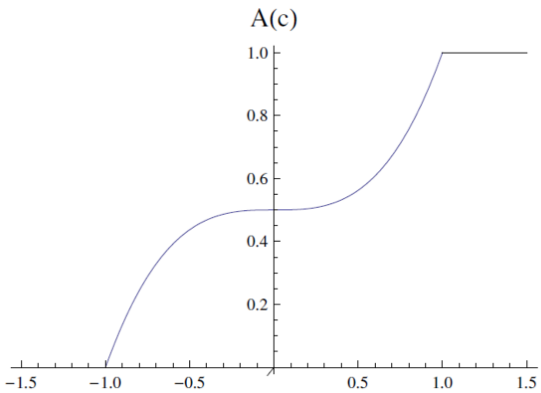
Which of the following graphs shows $A(c)$ as c varies?



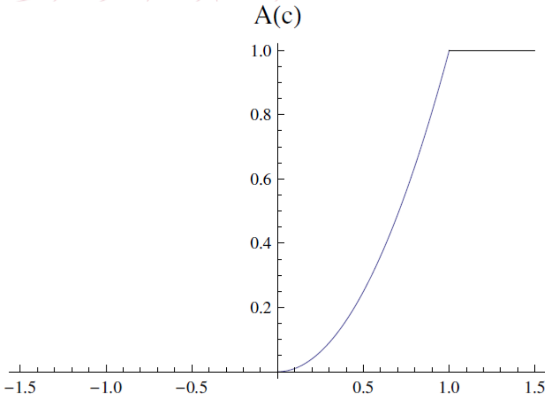
(A)



(B)



(C)



(D)

SQ14

A particle moves in the xy-plane, starting at the origin $(0, 0)$. At each turn, the particle may move in one of two ways:

- it may move two to the right and one up, that is, it may be translated by the vector $(2, 1)$, or
- it may move one to the right and two up, that is, it may be translated by the vector $(1, 2)$.

What is the closest the particle may come to the point $(25, 75)$?

(A) 0

(B) $5\sqrt{5}$

(C) $2\sqrt{53}$

(D) 25

(E) 35

SQ15

For all θ in the range $0 \leq \theta < 2\pi$ the line

$$(y - 1)\cos\theta = (x + 1)\sin\theta$$

divides the disc $x^2 + y^2 \leq 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(A) one value of θ.

(B) two values of θ.

(C) three values of θ.

(D) four values of θ.

(E) all values of θ.

SQ16

The parabolas with equations $y = x^2 + c$ and $y^2 = x$ touch (that is, meet tangentially) at a single point. It follows that c equals

(A) $\frac{1}{2\sqrt{3}}$

(B) $\frac{3}{4\sqrt[3]{4}}$

(C) $-\frac{1}{2}$

(D) $\sqrt{5} - \sqrt{3}$

(E) $\sqrt{\frac{2}{3}}$

SQ17

The numbers x and y satisfy

$$(x-1)^2 + y^2 \leq 1.$$

The largest that $x + y$ can be is

(A) 2

(B) $1+\sqrt{2}$

(C) 3

(D) $2+\sqrt{2}$

SQ18

Points $A = (3,9)$, $B = (1,1)$, $C = (5,3)$ and $D = (a,b)$ lie in the first quadrant and are the vertices of quadrilateral $ABCD$. The quadrilateral formed by joining the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ is a square. What is the sum of the coordinates of point D?

(A) 7

(B) 9

(C) 10

(D) 12

(E) 16

SQ19

Points $A(6,13)$ and $B(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$

(B) $\frac{21\pi}{2}$

(C) $\frac{85\pi}{8}$

(D) $\frac{43\pi}{4}$

(E) $\frac{87\pi}{8}$

SQ20

The curve $x^2 + y^2 = 25$ is drawn. Points on the curve whose x-coordinate and y-coordinate are both integers are marked with crosses. All of those crosses are joined in turn to create a convex polygon P. What is the area of P?

18 Solid Figures

What's on the Specification?

- Use Pythagoras' theorem in 2-dimensions and 3-dimensions.
- Use 2-dimensional representations of 3-dimensional shapes.

Exercises E18

Time Allowed

No limit

Number of Questions

26

Difficulty

★★★☆

Exercises E18

Scan the QR code or click the link above to take the practice online.

18

450

Quiz Pre-1

The total surface area of a cylinder, measured in square centimetres, is numerically the same as its volume, measured in cubic centimetres.

The radius of the cylinder is r cm, the height is h cm.

Express h in terms of r.

(A) $h = \frac{2r}{r-2}$

(B) $h = \frac{2r}{r+2}$

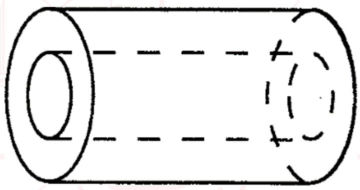
(C) $h = r + 2$

(D) $h = r - 2$

(E) $h = 2r(r - 2)$

Quiz Pre-2

A cylindrical hole of radius r and of length $4r$ is bored symmetrically through a solid cylinder of radius $2r$ and length $4r$. What is the total surface area of the resulting solid?



(A) $22\pi r^2$

(B) $24\pi r^2$

(C) $28\pi r^2$

(D) $30\pi r^2$

(E) $36\pi r^2$

Quiz Pre-3

A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

(A) 3

(B) 6

(C) 8

(D) 9

(E) 12

1. Examples

Ex. 1

A cone has a height equal to the diameter of a sphere. If the volumes of the two objects are equal, and the radius of the sphere is r, what is the radius of the base of the cone?

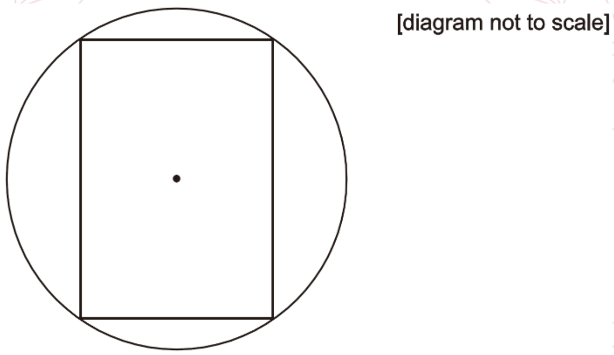
Ex. 2

You want to make a snowman out of modelling clay. The snowman consists of 2 spheres, where one sphere has a radius r, the other has a radius $2r$. The modelling clay comes in the form of a cylinder with radius $r/2$. What length of modelling clay is required to make the snowman?

Ex. 3

A right circular cylinder is contained within a sphere of radius 5 cm in such a way that the whole of the circumferences of both ends of the cylinder are in contact with the sphere.

The diagram shows a planar cross section through the centre of the sphere and cylinder.



Find, in cubic centimetres, the maximum possible volume of the cylinder.

(A) 250π

(B) 500π

(C) 1000π

(D) $\frac{250\sqrt{3}}{3}\pi$

(E) $\frac{500\sqrt{3}}{9}\pi$

(F) $\frac{1000\sqrt{3}}{9}\pi$

Ex. 4

The cross-section of a triangular prism is an equilateral triangle with side $2x$ cm.

The length of the prism is d cm.

Let the total surface area of the prism be T cm². Given that the volume of the prism is T cm³, which one of the following is an expression for d in terms of x?

(A) $\frac{x}{2x-3}$

(B) $\frac{3x}{3x-2\sqrt{3}}$

(C) $\frac{2x}{x-4\sqrt{3}}$

(D) $\frac{2x}{x-2\sqrt{3}}$

(E) $\frac{2x}{x-\sqrt{3}}$

Ex. 5

A solid cone has a base radius x cm.

The ratio of the perpendicular height of the cone to the radius of the cone is 5 : 2.

A solid hemisphere of radius $\frac{y}{2}$ cm is made from the same material as the cone.

Which one of the following is a correct expression for

$$\frac{\text{volume of the cone}}{\text{volume of the hemisphere}}$$

(Volume of a cone $= \frac{1}{3}\pi r^2 h$ where r is the radius and h is the perpendicular height.)

(Volume of a sphere $= \frac{4}{3}\pi r^3$ where r is the radius.)

(A) $\frac{5x^3}{y^3}$

(B) $\frac{5x^3}{4y^3}$

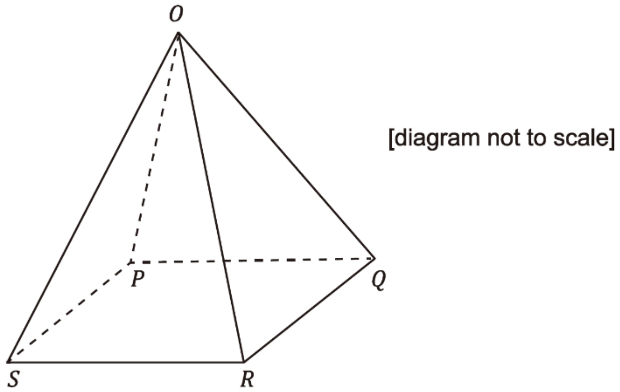
(C) $\frac{8x^3}{5y^3}$

(D) $\frac{10x^3}{y^3}$

(E) $\frac{14x^3}{y^3}$

Ex. 6

The diagram shows a square-based pyramid with base $PQRS$ and vertex O. All the edges of the pyramid are of length 20 metres.



Find the shortest distance, in metres, along the outer surface of the pyramid from P to the midpoint of OR.

(A) $10\sqrt{5-2\sqrt{3}}$

(B) $10\sqrt{3}$

(C) $10\sqrt{5}$

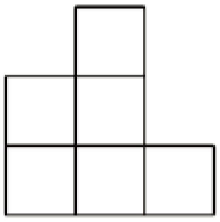
(D) $10\sqrt{7}$

(E) $10\sqrt{5+2\sqrt{3}}$

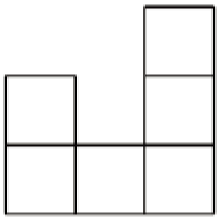
Ex. 7

Some identical unit cubes are used to construct a three-dimensional object by gluing them together face to face.

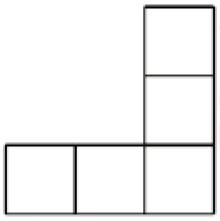
Sketches of this object are made by looking at it from the right-hand side from the front and from above. These sketches are called the side elevation the front elevation and the plan view respectively.



This is the side elevation of the object.



This is the front elevation of the object.



This is the plan view of the object.

How many cubes were used to construct the object?

(A) exactly 6

(B) either 6 or 7

(C) exactly 7

(D) either 7 or 8

(E) exactly 8

(F) either 8 or 9

(G) exactly 9

Ex. 8

The dimensions of a solid cuboid, in cm, are x, $2x$ and y.

The volume of the cuboid is 576 cm^3.

At this volume, the surface area of the cuboid has its maximum value.

What is the area, in cm^2, of the face that has the largest area?

(A) $2(288)^{\frac{2}{3}}$

(B) 72

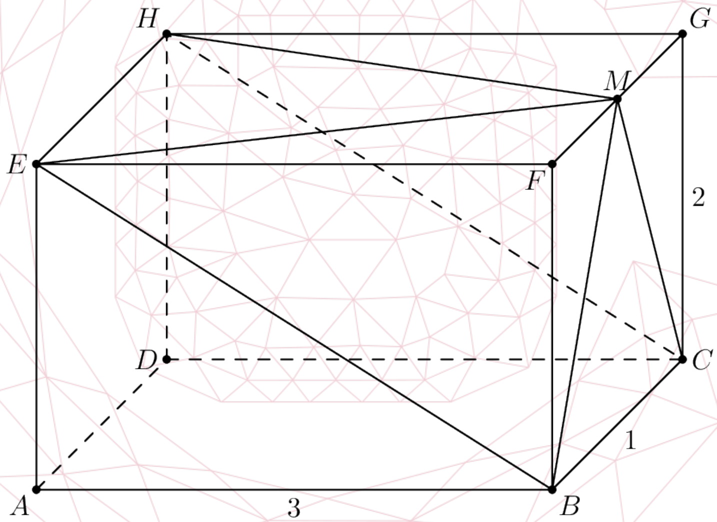
(C) 96

(D) 432

(E) $4(144)^{\frac{2}{3}}$

Ex. 9

In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$ and $CG = 2$. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



(A) 1

(B) $\frac{4}{3}$

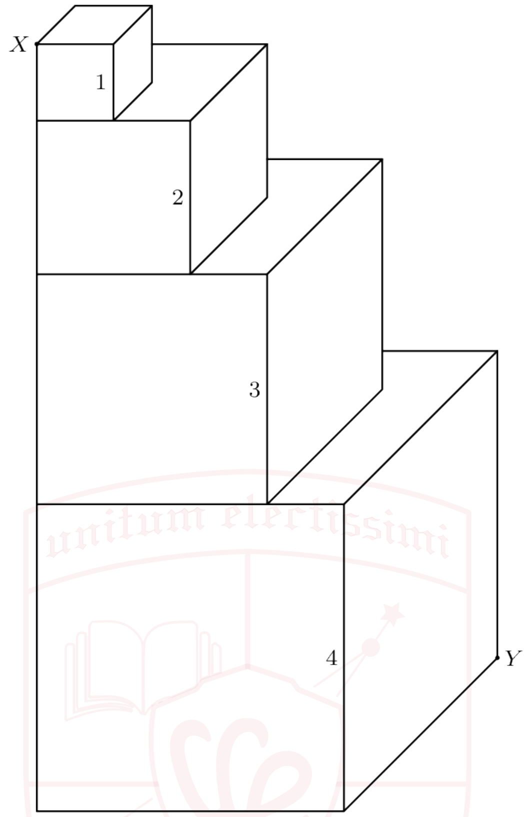
(C) $\frac{3}{2}$

(D) $\frac{5}{3}$

(E) 2

Ex.10

Four cubes with edge lengths 1, 2, 3 and 4 are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length 3?



(A) $\frac{3\sqrt{33}}{5}$

(B) $2\sqrt{3}$

(C) $\frac{2\sqrt{33}}{3}$

(D) 4

(E) $3\sqrt{2}$

Ex. 11

What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$ and $CD = 5$?

(A) 3

(B) $2\sqrt{3}$

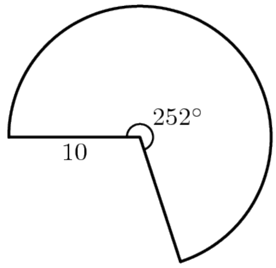
(C) 4

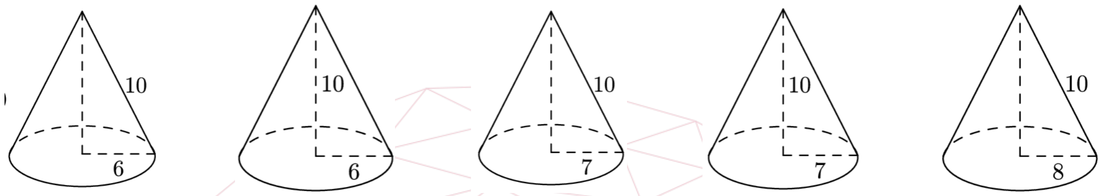
(D) $3\sqrt{3}$

(E) 6

Ex. 12

Which of the cones listed below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?





(A) (B) (C) (D) (E)

Ex. 13

An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cones height to its radius?

(A) 2 : 1

(B) 3 : 1

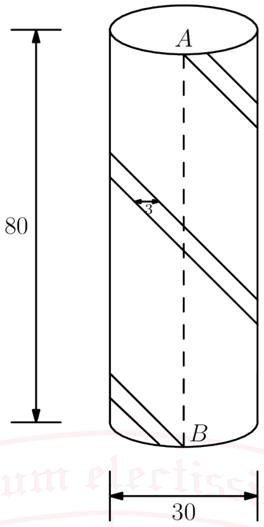
(C) 4 : 1

(D) 16 : 3

(E) 6 : 1

Ex. 14

A white cylindrical silo has a diameter of 30 feet and a height of 80 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?



(A) 120

(B) 180

(C) 240

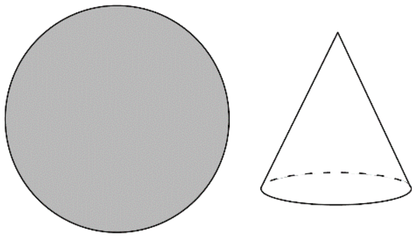
(D) 360

(E) 480

Quiz 1

Coco is making clown hats from a circular piece of cardboard. The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard.

What is the maximum number of hats that Coco can make from the piece of cardboard?



(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

Quiz 2

A cuboid has sides of lengths 22, 2 and 10. It is contained within a sphere of the smallest possible radius. What is the side-length of the largest cube that will fit inside the same sphere?

(A) 10

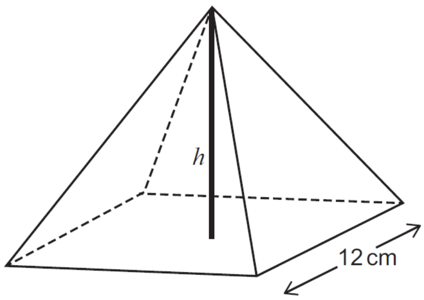
(B) 11

(C) 12

(D) 13

(E) 14

Quiz 3



A solid pyramid has a square base of side length 12 cm and a vertical height of h cm.

The volume of the pyramid, in cm^3, is equal to the total surface area of the pyramid, in cm^2.

What is the value of h?

(volume of pyramid $= \frac{1}{3} \times$ area of base $\times$ vertical height)

(A) $\frac{72}{35}$

(B) $2\sqrt{3}$

(C) 6

(D) $\frac{144}{23}$

(E) 8

(F) $2\sqrt{21}$

Quiz 4

An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

(A) 8π

(B) $\frac{28\pi}{3}$

(C) 12π

(D) 14π

(E) $\frac{44\pi}{3}$

Quiz 5

A solid cube is divided into two pieces by a single rectangular cut. As a result, the total surface area increases by a fraction f of the surface area of the original cube. What is the greatest possible value of f?

(A) $\frac{1}{3}$

(B) $\frac{\sqrt{3}}{4}$

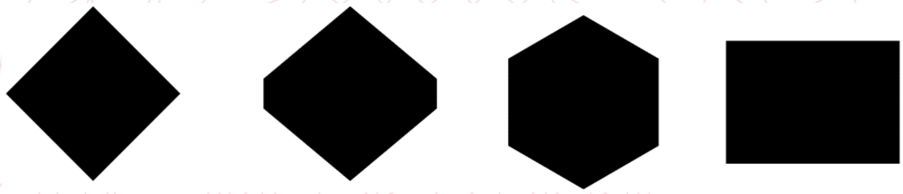
(C) $\frac{\sqrt{2}}{3}$

(D) $\frac{1}{2}$

(E) $\frac{1}{\sqrt{3}}$

Quiz 6

When a solid cube is held up to the light, how many of the following shapes could its shadow have?



(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Quiz 7

The region consisting of all points in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π. What is the length AB?

(A) 6

(B) 12

(C) 18

(D) 20

(E) 24

Quiz 8

Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}$, $\overline{MC}$ and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

(A) $24\sqrt{5}$

(B) 60

(C) $28\sqrt{5}$

(D) 66

(E) $8\sqrt{70}$

Quiz 9

A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have?

(A) 36

(B) 60

(C) 72

(D) 84

(E) 108

Practices P18

Time Allowed

40 min

Number of Questions

16

Difficulty

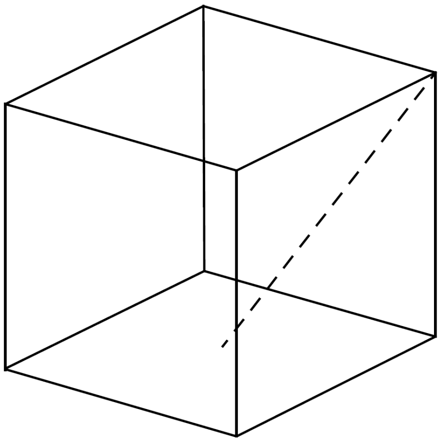
★★★☆

Practices P18

Scan the QR code or click the link above to take the practice online.

Q1

A cube has sides of unit length. What is the length of a line joining a vertex to the midpoint of one of the opposite faces (the dashed line in the diagram below)?



[diagram not to scale]

(A) $\sqrt{\frac{2}{3}}$

(B) $\sqrt{2}$

(C) $\sqrt{\frac{2}{5}}$

(D) $\sqrt{3}$

(E) $\sqrt{5}$

Q2

A cuboid has sides of length x, $\sqrt{2}x$ and $2x$, measured in cm.

The volume, in cm^3, of the cuboid is numerically equal to twice the total surface area, in cm^2, of the cuboid.

What is the value of x?

(A) 10

(B) $6 + 2\sqrt{2}$

(C) 5

(D) $3 + \sqrt{2}$

(E) $\frac{5}{2}$

(F) $\frac{3}{2} + \frac{1}{2}\sqrt{2}$

Q3

A solid sphere of radius r fits inside a hollow cylinder. The cylinder has the same internal diameter and length as the diameter of the sphere.

The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere.

What fraction of the space inside the cylinder is taken up by the sphere?

(A) $\frac{1}{4}$

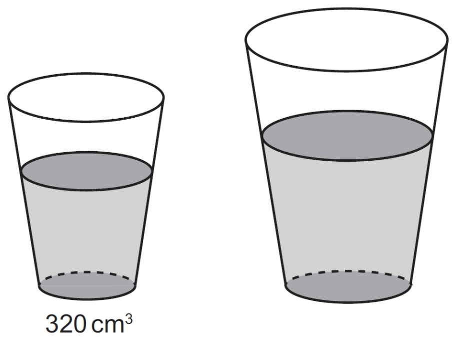
(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

(E) $\frac{3}{4}$

Q4



[diagram not to scale]

At a cinema, drinks are sold in regular and large sizes.

The cups for these are mathematically similar.

The ratio of the heights of the cups and the ratio of the depths of the drinks are both 4 : 5.

The volume of drink in a regular size cup is 320 cm^3.

What is the volume, in cm^3, of drink in a large size cup?

(A) 384

(B) 400

(C) 500

(D) 576

(E) 625

(F) 640

Q5

The ball for a garden game is a solid sphere of volume 192 cm^3.

For the children's version of the game the ball is a solid sphere made of the same material, but the radius is reduced by 25%.

What is the volume, in cm^3, of the children's ball?

(A) 48

(B) 81

(C) 96

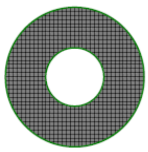
(D) 108

(E) 144

Q6

The diagram shows the cross section of a cylindrical roll of paper towels which are used in domestic kitchens.

[diagram not to scale]



The roll consists of 64 towels, each of which is a square of side 250 mm, and they are tightly wrapped around an inner cardboard tube of diameter 5 cm. The outer diameter of the roll is 11 cm and the length of the cylinder is 25 cm.

The thickness of an individual towel, in millimetres, is estimated from this information.

Which one of these estimates is the best?

(A) 0.01 mm

(B) 0.05 mm

(C) 0.1 mm

(D) 0.5 mm

(E) 1 mm

(F) 2 mm

Q7

A scale model of a cylindrical pillar is to be made.

The full-sized pillar has a volume of 12π m^3.

The model will use a length scale of 1 : 40.

The model is to be a solid cylinder made of a plastic which has a density of $\frac{4}{3}$ g cm^{-3}.

What is the mass of the model in grams?

(A) $\frac{9}{640}\pi$

(B) $\frac{1}{40}\pi$

(C) 40π

(D) $\frac{1125}{8}\pi$

(E) 250π

(F) $10\,000\pi$

(G) $225\,000\pi$

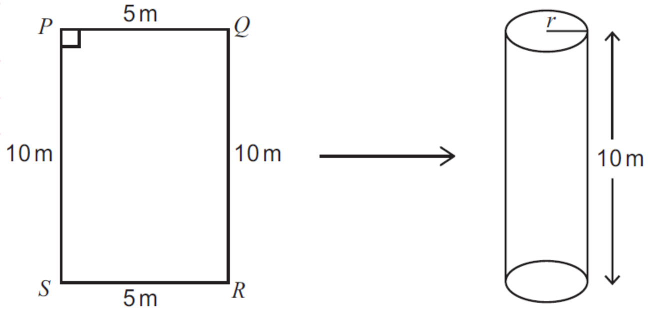
(H) $400\,000\pi$

Q8

A thin rectangular sheet of metal 10 m by 5 m is made into an open ended cylinder by joining the edges PS and QR.

The height of the cylinder is 10 m.

What is the volume, in cubic metres, enclosed by this cylinder?



(A) $\frac{5}{2\pi}$

(B) $\frac{25}{4\pi}$

(C) $\frac{125}{2\pi}$

(D) 62.5π

(E) $\frac{125}{\pi}$

(F) 250π

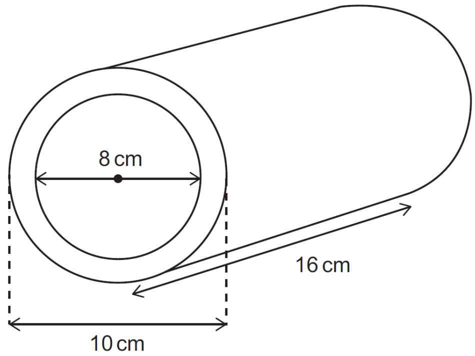
Q9

A cylindrical hollow metal pipe is 16 cm long.

It has an external diameter of 10 cm and an internal diameter of 8 cm.

The density of the metal from which the pipe is made is 8 grams per cm^3.

[diagram not to scale]



What is the mass of the pipe in grams?

(A) 8π

(B) 16π

(C) 18π

(D) 72π

(E) 128π

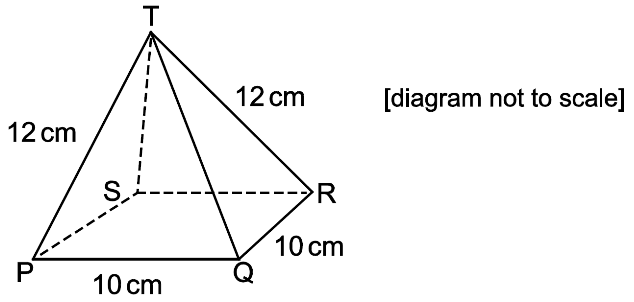
(F) 512π

(G) 1152π

(H) 4608π

Q10

A box is a hollow pyramid. The base of the box is a square with sides 10 cm and all the slant edges of the box are 12 cm long.



What is the angle made by the slant edge TP with the base $PQRS$?

(A) $\sin^{-1}\frac{2\sqrt{5}}{12}$

(B) $\sin^{-1}\frac{5}{12}$

(C) $\sin^{-1}\frac{5\sqrt{2}}{12}$

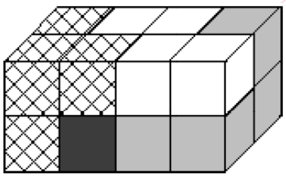
(D) $\cos^{-1}\frac{2\sqrt{5}}{12}$

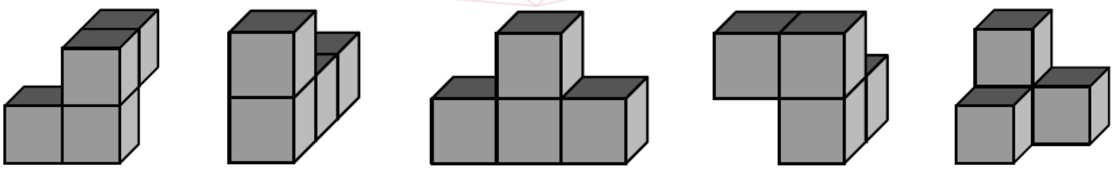
(E) $\cos^{-1}\frac{5}{12}$

(F) $\cos^{-1}\frac{5\sqrt{2}}{12}$

Q11

The cuboid shown has been built using four shapes, each made from four small cubes. Three of the shapes can be completely seen, but the dark one is only partly visible. Which of the following shapes could be the dark one?





(A) (B) (C) (D) (E)

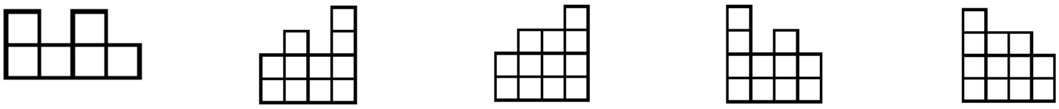
Q12

John has made a building of unit cubes standing on a 4×4 grid. The diagram shows the number of cubes standing on each cell. When John looks horizontally at the building from behind, what does he see?

BEHIND

4	2	3	2
3	3	1	2
2	1	3	1
1	2	1	2

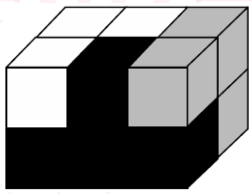
FRONT

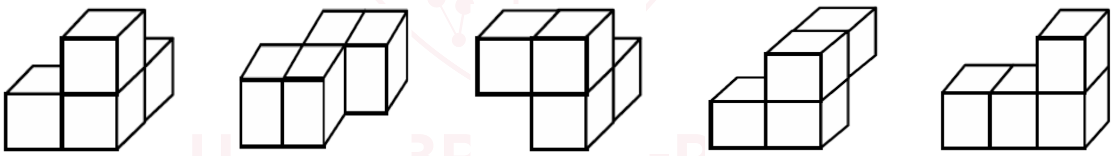


(A) (B) (C) (D) (E)

Q13

A cuboid has been built using 3 shapes (not necessarily different) each made from 4 little cubes as shown. The shape shaded black is completely visible, but both of the others are only partially visible. Which of the following shapes is the unshaded one?





(A) (B) (C) (D) (E)

Q14

A cube is inscribed in a sphere of diameter 1 m. What is the surface area of the cube?

(A) 2 m^2

(B) 3 m^2

(C) 4 m^2

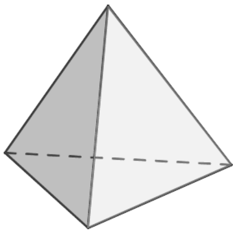
(D) 5 m^2

(E) 6 m^2

Q15

A regular tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle, as shown. A solid regular tetrahedron is cut into two pieces by a single plane cut.

Which of the following could *not* be the shape of the section formed by the cut?



(A) a pentagon

(B) a square

(C) a rectangle that is not a square

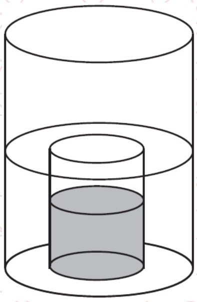
(D) a trapezium

(E) a triangle that is not equilateral

Q16

Two vases are cylindrical in shape. The larger vase has diameter 20 cm. The smaller vase has diameter 10 cm and height 16 cm. The larger vase is partially filled with water. Then the empty smaller vase, with the open end at the top, is slowly pushed down into the water, which flows over its rim. When the smaller vase is pushed right down, it is half full of water.

What was the original depth of the water in the larger vase?



(A) 10 cm

(B) 12 cm

(C) 14 cm

(D) 16 cm

(E) 18 cm

Supplements S18

Time Allowed

60 min

Number of Questions

12

Difficulty

★★★★

Supplements S18

Scan the QR code or click the link above to take the practice online.

SQ1

The trunk of a monkey-puzzle tree has diameter 40 cm. As a protection from fire, the trunk of the tree has a bark which makes up 19% of its volume. On average, roughly how thick is the bark of the trunk?

(A) 0.4 cm

(B) 1.2 cm

(C) 2 cm

(D) 2.8 cm

(E) 4 cm

SQ2

A roll of adhesive tape is wound round a central cylindrical core of radius 3 cm. The outer radius of a roll containing 20 m of tape is 4 cm. Approximately, what is the outer radius of a roll containing 80 m of tape?

(A) 5 cm

(B) 5.5 cm

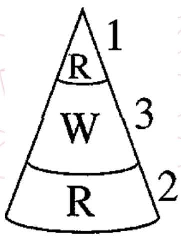
(C) 6 cm

(D) 7 cm

(E) 12 cm

SQ3

A traffic cone is painted with red (R) and white (W) bands of paint as shown. The sloping heights of the bands are in the ratio 1 : 3 : 2. What is the ratio of the area painted white to the area painted red?



(A) 5 : 9

(B) 5 : 7

(C) 1 : 1

(D) 7 : 5

(E) 9 : 5

SQ4

A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length of 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

(A) 14.0

(B) 16.0

(C) 20.0

(D) 33.3

(E) 55.6

SQ5

An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cones height to its radius?

(A) 2 : 1

(B) 3 : 1

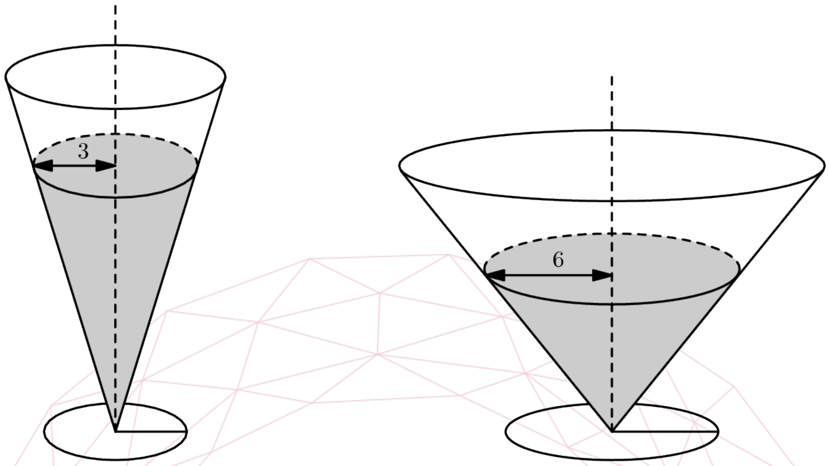
(C) 4 : 1

(D) 16 : 3

(E) 6 : 1

SQ6

Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into ach cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



(A) 1 : 1

(B) 47 : 43

(C) 2 : 1

(D) 40 : 13

(E) 4 : 1

18
476

SQ7

A rectangular box has a total surface area of 94 square inches. The sum of the lengths of all its edges is 48 inches. What is the sum of the lengths in inches of all of its interior diagonals?

(A) $8\sqrt{3}$

(B) $10\sqrt{2}$

(C) $16\sqrt{3}$

(D) $20\sqrt{2}$

(E) $40\sqrt{2}$

SQ8

An inverted cone with base radius 12 cm and height 18 cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24 cm. What is the height in centimeters of the water in the cylinder?

(A) 1.5

(B) 3

(C) 4

(D) 4.5

(E) 6

SQ9

One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

(A) 8

(B) 27

(C) 64

(D) 125

(E) 216

SQ10

A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

(A) 7

(B) 8

(C) 10

(D) 12

(E) 15

SQ11

A cylindrical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet. What is the volume of the water, in cubic feet?

(A) $24\pi - 36\sqrt{2}$

(B) $24\pi - 24\sqrt{3}$

(C) $36\pi - 36\sqrt{3}$

(D) $36\pi - 24\sqrt{2}$

(E) $48\pi - 36\sqrt{3}$

SQ12

A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Find the radius of the cylinder.

(A) $\frac{8}{3}$

(B) $\frac{30}{11}$

(C) 3

(D) $\frac{25}{8}$

(E) $\frac{7}{2}$

19 Mathematical Proofs

What's on the Specification?

- Follow a proof of the following types, and in simple cases know how to construct such a proof:
- Direct deductive proof ('Since A, therefore B, therefore C, ⋯, therefore Z, which is what we wanted to prove.');
- Proof by cases (for example, by considering even and odd cases separately);
- Proof by contradiction;
- Disproof by counterexample.
- Deduce implications from given statements.
- Make conjectures based on small cases, and then justify these conjectures.
- Rearrange a sequence of statements into the correct order to give a proof for a statement.
- Problems requiring a sophisticated chain of reasoning to solve.
- Identifying errors in purported proofs.
- Be aware of common mathematical errors in purported proofs; for example, claiming 'if $ab = ac$, then $b = c$' or assuming 'if $\sin A = \sin B$, then $A = B$' neither of which are valid deductions.

Exercises E19

Time Allowed

No limit

Number of Questions

12

Difficulty

★★★★

Exercises E19

Scan the QR code or click the link above to take the practice online.

19

480

Quiz Pre-1

Consider the following statement:

A car journey consists of two parts. In the first part, the average speed is u km/h. In the second part, the average speed is v km/h. Hence the average speed for the whole journey is $\frac{1}{2}(u+v)$ km/h.

Which of the following examples of car journeys provide(s) a **counterexample** to the statement?

I In the first part of the journey, the car travels at a constant speed of 50 km/h for 100 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

II In the first part of the journey, the car travels at a constant speed of 50 km/h for one hour. In the second part of the journey, the car travels at a constant speed of 40 km/h for one hour.

III In the first part of the journey, the car travels at a constant speed of 50 km/h for 80 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Quiz Pre-2

A student makes the following claim:

For all integers n, the expression $4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)$ is divisible by 3.

Here is the student's argument:

$$
\begin{aligned}
4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right) &= 2\left(2\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)\right) && (1)\\
&= 2(9n+1-3n-1) && (2)\\
&= 2(6n) && (3)\\
&= 12n && (4)\\
&= 3(4n) && (5)
\end{aligned}
$$

which is always a multiple of 3. (6)

So the expansion $4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)$ is always divisible by 3.

Which one of the following is true?

(A) The argument is correct.

(B) The argument is incorrect, and the first error occurs on line (1).

(C) The argument is incorrect, and the first error occurs on line (2).

(D) The argument is incorrect, and the first error occurs on line (3).

(E) The argument is incorrect, and the first error occurs on line (4).

(F) The argument is incorrect, and the first error occurs on line (5).

(G) The argument is incorrect, and the first error occurs on line (6).

Ex. 1

Five runners competed in a race: Fred, George, Hermione, Lavender, and Ron.

Fred beat George.

Hermione beat Lavender.

Lavender beat George.

Ron beat George.

Assuming there were no ties, how many possible finishing orders could there have been, given only this information?

(A) 1

(B) 6

(C) 12

(D) 18

(E) 24

(F) 120

Ex. 2

Consider the statement:

($*$) A whole number n is prime if it is 1 less or 5 less than a multiple of 6.

How many counterexamples to ($*$) are there in the range $0 < n < 50$?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

Ex. 3

47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

Ex. 4

For any real numbers a, b and c where $a \geq b$, consider these three statements:

1. $-b \geq -a$
2. $a^2 + b^2 \geq 2ab$
3. $ac \geq bc$

Which of the statements 1, 2, and 3 **must** be true?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Ex. 5

Show that, if n is an integer such that

$$(n-3)^3 + n^3 = (n+3)^3, \qquad (*)$$

then n is even and n^2 is a factor of 54. Deduce that there is no integer n which satisfies the equation $(*)$.

Show that, if n is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \qquad (**)$$

then n is even. Deduce that there is no integer n which satisfies the equation $(**)$.

Ex. 6

If $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ are sequences of positive numbers, we write

$$(s_n) \leq (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leq t_n$ whenever $n \geq m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m; in the case of a false statement, you should give a counter-example.

(i) $(1000n) \leq (n^2)$.

(ii) If it is not the case that $(s_n) \leq (t_n)$, then it is the case that $(t_n) \leq (s_n)$.

(iii) If $(s_n) \leq (t_n)$ and $(t_n) \leq (u_n)$, then $(s_n) \leq (u_n)$.

(iv) $(n^2) \leq (2^n)$.

Ex. 7

In this question a, b and c are positive integers.

The following is an attempted proof of the false statement:

If a divides bc, then a divides b or a divides c.

['a divides bc' means 'a is a factor of bc']

Which line contains the error in this proof?

1. The statement is equivalent to 'if a does not divide b and a does not divide c then a does not divide bc'.
2. Suppose a does not divide b and a does not divide c. Then the remainder when dividing b by a is r, where $0 < r < a$, and the remainder when dividing c by a is s, where $0 < s < a$.
3. So $b = ax + r$ and $c = ay + s$ for some integers x and y.
4. Thus $bc = a(axy + xs + yr) + rs$.
5. So the remainder when dividing bc by a is rs.
6. Since $r > 0$ and $s > 0$, it follows that $rs > 0$.
7. Hence a does not divide bc.

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

(E) Line 5

(F) Line 6

Quiz 1

The real numbers a, b and c are non-zero and $a \leq b$.

Consider these three statements:

1 $\frac{1}{a} \geq \frac{1}{b}$

2 $2^a \leq 2^b$

3 $ac \leq bc$

Which of the above statements **must** be true?

(A) none

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Quiz 2

Consider the following three statements:

1 $10p^2 + 1$ and $10p^2 - 1$ are both prime when p is an odd prime.

2 Every prime greater than 5 is of the form $6n + 1$ for some integer n.

3 No multiple of 7 greater than 7 is prime.

The result $91 = 7 \times 13$ can be used to provide a **counterexample** to which of the above statements?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Quiz 3

Consider the following attempt to prove this true theorem:

Theorem: $a^3 + b^3 = c^3$ has no solutions with a, b and c positive integers.

Attempted proof:

Suppose that there are positive integers a, b and c such that $a^3 + b^3 = c^3$.

I We have $a^3 = c^3 - b^3$.

II Hence $a^3 = (c-b)(c^2 + cb + b^2)$.

III It follows that $a = c - b$ and $a^2 = c^2 + cb + b^2$, since $a \leq a^2$ and $c - b \leq c^2 + cb + b^2$.

IV Eliminating a, we have $(c-b)^2 = c^2 + cb + b^2$.

V Multiplying out, we have $c^2 - 2cb + b^2 = c^2 + cb + b^2$.

VI Hence $3cb = 0$ so one of b and c is zero.

But this is a contradiction to the original assumption that all of a, b and c are positive. It follows that the equation has no solutions.

Comment on this proof by choosing one of the following options:

(A) The proof is correct.

(B) The proof is incorrect and the first mistake occurs on line I.

(C) The proof is incorrect and the first mistake occurs on line II.

(D) The proof is incorrect and the first mistake occurs on line III.

(E) The proof is incorrect and the first mistake occurs on line IV.

(F) The proof is incorrect and the first mistake occurs on line V.

(G) The proof is incorrect and the first mistake occurs on line VI.

Practices P19

Time Allowed

20 min

Number of Questions

8

Difficulty

★★★★

Practices P19

Scan the QR code or click the link above to take the practice online.

19

488

Q1

The four real numbers a, b, c and d are all greater than 1.

Suppose that they satisfy the equation $\log_c d = (\log_a b)^2$.

Use some of the lines given to construct a proof that, in this case, it follows that

$$(*) \log_b d = (\log_a b)(\log_a c).$$

(1) Let $x = \log_a b$ and $y = \log_a c$

(2) $d = (c^x)^2$

(3) $d = c^{(x^2)}$

(4) $d = b^{xy}$

(5) $d = (a^y)^{(x^2)}$

(6) $d = \left((a^y)x\right)^2$

(7) $d = (a^x)^{xy}$

(8) $d = a^{(y^{2x})}$

(9) $d = a^{(x^2 y)}$

(A) (1). Then (2), so (6), so (8), so (7), and therefore (4), hence $(*)$ as required.

(B) (1). Then (2), so (7), so (8), so (6), and therefore (4), hence $(*)$ as required.

(C) (1). Then (3), so (5), so (9), so (7), and therefore (4), hence $(*)$ as required.

(D) (1). Then (3), so (7), so (9), so (5), and therefore (4), hence $(*)$ as required.

(E) (1). Then (4), so (5), so (9), so (7), and therefore (3), hence $(*)$ as required.

(F) (1). Then (4), so (6), so (8), so (7), and therefore (2), hence $(*)$ as required.

(G) (1). Then (4), so (7), so (8), so (6), and therefore (2), hence $(*)$ as required.

(H) (1). Then (4), so (7), so (9), so (5), and therefore (3), hence $(*)$ as required.

Q2

Which one of the following functions provides a **counterexample** to the statement:

if $f'(x) > 0$ for all real x, **then** $f(x) > 0$ for all real x.

(A) $f(x) = x^2 + 1$

(B) $f(x) = x^2 - 1$

(C) $f(x) = x^3 + x + 1$

(D) $f(x) = 1 - x$

(E) $f(x) = 2^x$

Q3

Consider the following conjecture:

If N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1, **then** N is a prime number.

Here are three numbers:

I $N = 101$ (which is a prime number)

II $N = 1001$ (which equals $7 \times 11 \times 13$)

III $N = 10001$ (which equals 73×137)

Which of these provide(s) a counterexample to the conjecture?

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Q4

Consider the following statement:

Every positive integer N that is greater than 6 can be written as the sum of two non-prime integers that are greater than 1.

Which of the following is/are **counterexample**(s) to this statement?

I $N = 5$

II $N = 7$

III $N = 9$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

Q5

A student attempts to solve the equation

$$\cos x + \sin x \tan x = 2\sin x - 1$$

in the range $0 \le x \le 2\pi$.

The student's attempt is as follows:

$\cos x + \sin x \tan x = 2\sin x - 1$

So $\cos x - \sin x + \sin x \tan x - \sin x = -1$ (I)

So $(\sin x - \cos x)(\tan x - 1) = -1$ (II)

So $\sin x - \cos x = -1$ or $\tan x - 1 = -1$ (III)

So $(\sin x - \cos x)^2 = 1$ or $\tan x = 0$ (IV)

So $2\sin x \cos x = 0$ or $\tan x = 0$ (V)

So $x = 0,\ \frac{\pi}{2}, \pi,\ \frac{3}{2}\pi, 2\pi$ (VI)

Which of the following best describes this attempt?

(A) It is completely correct

(B) It is incorrect, and the first error occurs on line (I)

(C) It is incorrect, and the first error occurs on line (II)

(D) It is incorrect, and the first error occurs on line (III)

(E) It is incorrect, and the first error is that extra solutions were introduced on line (IV)

(F) It is incorrect, and the first error is that extra solutions were introduced on line (V)

(G) It is incorrect, and the first error is not eliminating the values where $\tan x$ is undefined on line (VI)

Q6

Consider the following attempt to solve an equation. The steps have been numbered for reference.

$$\begin{array}{ll} \sqrt{x+5}=x+3 & \\ & \downarrow (1) \\ x+5=x^2+6x+9 & \\ & \downarrow (2) \\ x^2+5x+4=0 & \\ & \downarrow (3) \\ (x+4)(x+1)=0 & \end{array}$$

$$x=-4 \text{ or } x=-1$$

Which one of the following statements is true?

(A) Both -4 and -1 are solutions of the equation.

(B) Neither -4 nor -1 are solutions of the equation.

(C) One solution is correct and the incorrect solution arises as a result of step (1).

(D) One solution is correct and the incorrect solution arises as a result of step (2).

(E) One solution is correct and the incorrect solution arises as a result of step (3).

Q7

For any real numbers a, b and c where $a \geq b$, consider these three statements:

1 $\quad -b \geq -a$

2 $\quad a^2 + b^2 \geq 2ab$

3 $\quad ac \geq bc$

Which of the above statements must be true?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

Q8

Consider the following problem:

Solve the inequality $\left(\frac{1}{4}\right)^n < \left(\frac{1}{32}\right)^{10}$, where n is a positive integer.

A student produces the following argument:

$$\left(\frac{1}{4}\right)^n < \left(\frac{1}{32}\right)^{10}$$

$$\Big\downarrow \text{(I)}$$

$$\log_{\frac{1}{2}}\left(\frac{1}{4}\right)^n < \log_{\frac{1}{2}}\left(\frac{1}{32}\right)^{10}$$

$$\Big\downarrow \text{(II)}$$

$$n\log_{\frac{1}{2}}\left(\frac{1}{4}\right) < 10\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$$

$$\Big\downarrow \text{(III)}$$

$$n < \frac{10\log_{\frac{1}{2}}(\frac{1}{32})}{\log_{\frac{1}{2}}(\frac{1}{4})}$$

$$\Big\downarrow \text{(IV)}$$

$$n < \frac{10 \times 5}{2} = 25$$

$$\Big\downarrow \text{(V)}$$

$$1 \leqslant n \leqslant 24$$

Which step (if any) in the argument is invalid?

(A) There are no invalid steps; the argument is correct.

(B) Only step (I) is invalid; the rest are correct

(C) Only step (II) is invalid; the rest are correct

(D) Only step (III) is invalid; the rest are correct

(E) Only step (IV) is invalid; the rest are correct

(F) Only step (V) is invalid; the rest are correct

Supplements S19

Time Allowed

20 min

Number of Questions

8

Difficulty

★★★★

Supplements S19

Scan the QR code or click the link above to take the practice online.

SQ1

Consider the following attempt to solve the equation $4x\sqrt{2x-1} = 10x - 5$:

$$4x\sqrt{2x-1} = 10x - 5$$

(I)

$$4x\sqrt{2x-1} = 5(2x-1)$$

(II)

$$16x^2(2x-1) = 25(2x-1)^2$$

(III)

$$16x^2 = 25(2x-1)$$

(IV)

$$16x^2 - 50x + 25 = 0$$

(V)

$$(8x-5)(2x-5) = 0$$

(VI)

The solutions of the original equation are $x = \frac{5}{8}$ and $x = \frac{5}{2}$.

Which one of the following is true?

(A) The solution is correct.

(B) Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (II).

(C) Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (III).

(D) Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (IV).

(E) There is another value of x that satisfies the original equation and the error arises as a result of step (II).

(F) There is another value of x that satisfies the original equation and the error arises as a result of step (III).

(G) There is another value of x that satisfies the original equation and the error arises as a result of step (IV).

SQ2

The following is an attempted proof of the conjecture:

if $\tan\theta > 0$, **then** $\sin\theta + \cos\theta > 1$.

Suppose $\tan\theta > 0$, so in particular $\cos\theta \neq 0$.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\sin\theta\cos\theta = \tan\theta\cos^2\theta > 0$. (I)

It follows that $1 + 2\sin\theta\cos\theta > 1$. (II)

Therefore $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta > 1$, (III)

which factorises to give $(\sin\theta + \cos\theta)^2 > 1$. (IV)

Therefore $\sin\theta + \cos\theta > 1$. (V)

Which one of the following is the case?

(A) The proof is correct.

(B) The proof is incorrect, and the first error occurs in line (I).

(C) The proof is incorrect, and the first error occurs in line (II).

(D) The proof is incorrect, and the first error occurs in line (III).

(E) The proof is incorrect, and the first error occurs in line (IV).

(F) The proof is incorrect, and the first error occurs in line (V).

SQ3

For which one of the following statements can the fact that $12^2 + 16^2 = 20^2$ be used to produce a **counterexample**?

(A) If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, and the three numbers have no common divisor, then two of them are odd and the other is even.

(B) The equation $a^4 + b^2 = c^2$ has no solutions for which a, b and c are positive integers.

(C) The equation $a^4 + b^2 = c^2$ has no solutions for which a, b and c are positive integers.

(D) If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, then one is the arithmetic mean of the other two.

SQ4

A student is asked to prove whether the following statement $(*)$ is true or false:

$(*)$ For all real numbers a and b, $|a+b| < |a| + |b|$

The student's proof is as follows:

Statement $(*)$ is **false**. A counterexample is $a = 3$, $b = 4$, as $|3+4| = 7$ and $|3| + |4| = 7$, but $7 < 7$ is false.

Which of the following best describes the student's proof?

(A) The statement $(*)$ is true, and the student's proof is not correct.

(B) The statement $(*)$ is false, but the student's proof is not correct: the counterexample is not valid.

(C) The statement $(*)$ is false, but the student's proof is not correct: the student needs to give all the values of a and b where $|a+b| < |a| + |b|$ is false.

(D) The statement $(*)$ is false, but the student's proof is not correct: the student should have instead stated that for all real numbers a and b, $|a+b| \leq |a| + |b|$.

(E) The statement $(*)$ is false, and the student's proof is fully correct.

SQ5

The Fundamental Theorem of Calculus (FTC) tells us that for any polynomial f:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x f(t)\,\mathrm{d}t\right) = f(x)$$

A student calculates $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right)$ as follows:

(I) $\int_x^{2x} t^2\,\mathrm{d}t = \int_0^{2x} t^2\,\mathrm{d}t - \int_0^x t^2\,\mathrm{d}t$

(II) By FTC, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x t^2\,\mathrm{d}t\right) = x^2$

(III) By FTC, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) = (2x)^2 = 4x^2$

(IV) So $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 4x^2 - x^2$

(V) giving $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 3x^2$

Which of the following best describes the student's calculation?

(A) The calculation is completely correct.

(B) The calculation is incorrect, and the first error occurs on line (I).

(C) The calculation is incorrect, and the first error occurs on line (II).

(D) The calculation is incorrect, and the first error occurs on line (III).

(E) The calculation is incorrect, and the first error occurs on line (IV).

(F) The calculation is incorrect, and the first error occurs on line (V).

SQ6

In this question, x and y are non-zero real numbers.

Consider the three statements:

1 $x > y$ **if** $\frac{x}{y} > 1$

2 $\frac{x}{y} > 1$ **if and only if** $\frac{y}{x} < 1$

3 **If** $xy < 1$ **then** both $x < 1$ and $y < 1$

Which of these statements, taken independently, is/are true?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

SQ7

Consider the following three statements:

1 $x^2 - 3x + 2 = 0$ **if** $x = 1$

2 $x^2 - 3x + 2 = 0$ **only if** $x = 1$

3 $x^2 - 3x + 2 = 0$ **if and only if** $x = 1$

Which of the statements is/are true?

(A) none of them

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

SQ8

Which of the following values of n is a counterexample to the statement, 'If n is a prime number, then exactly one of $n-2$ and $n+2$ is prime'?

(A) 11

(B) 19

(C) 21

(D) 29

(E) 37

19

20 Logic of Arguments

What's on the Specification?

- The **converse** of a statement;
- The **contrapositive** of a statement;
- The relationship between the truth of a statement and its converse and its contrapositive.
- **Note:** candidates will not be expected to recognise or use symbolic notation for any of these terms, nor will they be expected to complete formal truth tables.
- Understand and use the terms **necessary** and **sufficient**.
- Understand and use the terms **for all**, **for some** (meaning **for at least one**), and **there exists**.
- Be able to negate statements that use any of the above terms.

Exercises E20

Time Allowed

No limit

Number of Questions

10

Difficulty

★★★

Exercises E20

Scan the QR code or click the link above to take the practice online.

20

502

Quiz Pre-1

Consider the statement:

$f(x) > x$ for all real values of $x > 1$

Which one of the following is a negation of this statement?

(A) $f(x) \leq x$ for all real values of $x \leq 1$

(B) $f(x) \leq x$ for all real values of $x > 1$

(C) $f(x) \leq x$ for at least one real value of $x \leq 1$

(D) $f(x) \leq x$ for at least one real value of $x > 1$

(E) $f(x) > x$ for at least one real value of $x \leq 1$

(F) $f(x) > x$ for at least one real value of $x > 1$

(G) $f(x) > x$ for no real values of $x \leq 1$

(H) $f(x) \leq x$ for no real values of $x > 1$

Quiz Pre-2

Consider the following statement about the positive integer n:

Statement $(*)$: The sum of the four consecutive integers, the smallest of which is n, is a multiple of 6.

Which one of the following is true?

(A) Statement $(*)$ is true for all values of n.

(B) Statement $(*)$ is true for all values of n which are odd, but not for any other values of n.

(C) Statement $(*)$ is true for all values of n which are multiples of 3, but not for any other values of n.

(D) Statement $(*)$ is true for all values of n which are multiples of 6, but not for any other values of n.

(E) Statement $(*)$ is not true for any value of n.

Ex. 1

An electronic circuit contains three light bulbs, X, Y and Z, which are each either on or off at any particular time. It is known that if bulb X is off or bulb Y is on, then bulb Z is on.

Which one of these statements **necessarily** follows from this?

(A) If bulb Z is on, then bulb X is off **or** bulb Y is on.

(B) If bulb Z is on, then bulb X is on **and** bulb Y is off.

(C) If bulb Z is on, then bulb X is on **or** bulb Y is on.

(D) If bulb Z is off, then bulb X is off **and** bulb Y is off.

(E) If bulb Z is off, then bulb X is on **or** bulb Y is off.

(F) If bulb Z is off, then bulb X is on **and** bulb Y is on.

Ex. 2

A set S of whole numbers is called *stapled* **if and only if** for every whole number a which is in S there exists a prime factor of a which divides at least one other number in S.

Let T be a set of whole numbers. Which of the following is true **if and only if** T is **not** stapled?

(A) For every number a which is in T, there is no prime factor of a which divides every other number in T.

(B) For every number a which is in T, there is no prime factor of a which divides at least one other number in T.

(C) For every number a which is in T, there is a prime factor of a which does not divide any other number in T.

(D) For every number a which is in T, there is a prime factor of a which does not divide at least one other number in T.

(E) There exists a number a which is in T such that there is no prime factor of a which divides every other number in T.

(F) There exists a number a which is in T such that there is no prime factor of a which divides at least one other number in T.

(G) There exists a number a which is in T such that there is a prime factor of a which does not divide any other number in T.

(H) There exists a number a which is in T such that there is a prime factor of a which does not divide at least one other number in T.

Ex. 3

Consider the following statement:

For any positive integer N there is a positive integer K such that $N(Km+1)-1$ is not prime for any positive integer m.

Which one of the following is the negation of this statement?

(A) For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is prime.

(B) For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is not prime.

(C) For any positive integer N there is a positive integer K such that for any positive integer m, $N(Km+1)-1$ is not prime.

(D) For any positive integer N, any positive integer K and any positive integer m, $N(Km+1)-1$ is not prime.

(E) There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is not prime.

(F) There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is prime.

(G) There is a positive integer N such that for any positive integer K and any positive integer m, $N(Km+1)-1$ is prime.

(H) There is a positive integer N and a positive integer K for which there is no positive integer m for which $N(Km+1)-1$ is prime.

20 505

Ex. 4

Given a statement,

A square is a rectangle.

Write down:

(i) the converse,

(ii) the inverse,

(iii) the contrapositive

of the statement and judge whether they are true. Find a counterexample if it is false.

Ex. 5

Given a statement,

All four-sided plane figures are rectangles.

Write down:

(i) the converse,

(ii) the inverse,

(iii) the contrapositive

of the statement and judge whether they are true. Find a counterexample if it is false.

Ex. 6

If $n > 2$, then $n^2 > 4$.

Find the converse, inverse, and contrapositive. Determine if each resulting statement is true or false. If it is false, find a counterexample.

Quiz 1

Consider the statement about Fred:

($*$) Every day next week, Fred will do at least one maths problem.

If statement ($*$) is **not** true, which of the following is **certainly** true?

(A) Every day next week, Fred will do more than one maths problem.

(B) Some day next week, Fred will do more than one maths problem.

(C) On no day next week will Fred do more than one maths problem.

(D) Every day next week, Fred will do no maths problems.

(E) Some day next week, Fred will do no maths problems.

(F) On no day next week will Fred do no maths problems.

Quiz 2

An arithmetic sequence T has first term a and common difference d, where a and d are non-zero integers.

Property P is:

For some positive integer m, the sum of the first m terms of the sequence is equal to the sum of the first $2m$ terms of the sequence.

For example, when $a = 11$ and $d = -2$, the sequence T has property P, because

$$11 + 9 + 7 + 5 = 11 + 9 + 7 + 5 + 3 + 1 + (-1) + (-3)$$

i.e. the sum of the first 4 terms equals the sum of the first 8 terms.

Which of the following statements is/are **true**?

I For T to have property P, it is **sufficient** that $ad < 0$.

II For T to have property P, it is **necessary** that d is even.

(A) neither of them

(B) I only

(C) II only

(D) I and II

Practices P20

Time Allowed

30 min

Number of Questions

8

Difficulty

★★★

[Practices P20]

Scan the QR code or click the link above to take the practice online.

20

508

Q1

Five sealed urns, labelled P, Q, R, S, and T, each contain the **same** (non-zero) number of balls. The following statements are attached to the urns.

Urn P This urn contains one or four balls.

Urn Q This urn contains two or four balls.

Urn R This urn contains more than two balls and fewer than five balls.

Urn S This urn contains one or two balls.

Urn T This urn contains fewer than three balls.

Exactly one of the urns has a true statement attached to it.

Which urn is it?

(A) Urn P

(B) Urn Q

(C) Urn R

(D) Urn S

(E) Urn T

Q2

Considering the following statement about the positive integers a, b and n:

$(*)$: ab is divisible by n

The condition 'either a or b is divisible by n' is:

(A) necessary but not sufficient for $(*)$

(B) sufficient but not necessary for $(*)$

(C) necessary and sufficient for $(*)$

(D) not necessary and not sufficient for $(*)$

Q3

Consider the following conditions on a **parallelogram** $PQRS$, labelled anticlockwise:

I length of PQ = length of QR

II The diagonal PR intersects the diagonal QS at right angles

III $\angle PQR = \angle QRS$

Which of these conditions is/are individually **sufficient** for the parallelogram $PQRS$ to be a square?

(A) **Condition I is sufficient**: yes **Condition II is sufficient**: yes **Condition III is sufficient**: yes

(B) **Condition I is sufficient**: yes **Condition II is sufficient**: yes **Condition III is sufficient**: no

(C) **Condition I is sufficient**: yes **Condition II is sufficient**: no **Condition III is sufficient**: yes

(D) **Condition I is sufficient**: yes **Condition II is sufficient**: no **Condition III is sufficient**: no

(E) **Condition I is sufficient**: no **Condition II is sufficient**: yes **Condition III is sufficient**: yes

(F) **Condition I is sufficient**: no **Condition II is sufficient**: yes **Condition III is sufficient**: no

(G) **Condition I is sufficient**: no **Condition II is sufficient**: no **Condition III is sufficient**: yes

(H) **Condition I is sufficient**: no **Condition II is sufficient**: no **Condition III is sufficient**: no

Q4

$PQRS$ is a quadrilateral, labelled anticlockwise.

Which one of the following is a **necessary** but **not sufficient** condition for $PQRS$ to be a parallelogram?

(A) $PQ = SR$ and PS is parallel to QR

(B) $PQ = SR$ and PQ is parallel to SR

(C) $PQ = QR = SR = PS$

(D) $PR = QS$

Q5

A multiple-choice test question offered the following four options relating to a certain statement:

Given that exactly one of these options was correct, which one was it?

(A) The statement is true **if and only if** $x > 1$

(B) The statement is true **if** $x > 1$

(C) The statement is true **if and only if** $x > 2$

(D) The statement is true **if** $x > 2$

Q6

Let S be a set of positive integers, for example S could consist of 3, 4, and 8.

A positive integer n is called an S-number **if and only if** for every factor m of n with $m > 1$, the number m is a multiple of some number in S.

So in the above example, 9 is an S-number; this is because the factors of 9 greater than 1 are 3 and 9, and each of these is a multiple of 3.

Positive integer n is therefore **not** an S-number **if and only if**

(A) for **every** (positive) factor m of n with $m > 1$, there is a number in S which is not a factor of m.

(B) for **every** (positive) factor m of n with $m > 1$, there is no number in S which is a factor of m.

(C) for **every** (positive) factor m of n with $m > 1$, every number in S is a factor of m.

(D) for **some** (positive) factor m of n with $m > 1$, there is a number in S which is not a factor of m.

(E) for **some** (positive) factor m of n with $m > 1$, there is no number in S which is a factor of m.

(F) for **some** (positive) factor m of n with $m > 1$, every number in S is a factor of m.

Q7

x is a real number and f is a function.

Given that **exactly one** of the following statements is true, which one is it?

(A) $x \geq 0$ **only if** $f(x) < 0$

(B) $x < 0$ **if** $f(x) \geq 0$

(C) $x \geq 0$ **only if** $f(x) \geq 0$

(D) $f(x) < 0$ **if** $x < 0$

(E) $f(x) \geq 0$ **only if** $x \geq 0$

(F) $f(x) \geq 0$ **if and only if** $x < 0$

Q8

The expression $(ax + b)(cx + d)$ is expanded, where a, b, c and d are real numbers.

Consider the following three conditions:

condition 1 $b = -d$

condition 2 $ad = -bc$

condition 3 $a = c = 1$ and $b = -d$

Which of these conditions is/are **sufficient** for the expansion of $(ax + b)(cx + d)$ to equal $px^2 + q$, for all x, where p and q are real numbers?

	condition 1	condition 2	condition 3
(A)	not sufficient	not sufficient	not sufficient
(B)	not sufficient	not sufficient	sufficient
(C)	not sufficient	sufficient	not sufficient
(D)	not sufficient	sufficient	sufficient
(E)	sufficient	not sufficient	not sufficient
(F)	sufficient	not sufficient	sufficient
(G)	sufficient	sufficient	not sufficient
(H)	sufficient	sufficient	sufficient

Supplements S20

Time Allowed

20 min

Number of Questions

7

Difficulty

★★★

Supplements S20

Scan the QR code or click the link above to take the practice online.

SQ1

A region is defined by the inequalities $x + y > 6$ and $x - y > -4$.

Consider the three statements:

1. $x > 1$
2. $y > 5$
3. $(x + y)(x - y) > -24$

Which of the above statements is/are true for **every** point in the region?

(A) none

(B) 1 only

(C) 2 only

(D) 3 only

(E) 1 and 2 only

(F) 1 and 3 only

(G) 2 and 3 only

(H) 1, 2 and 3

SQ2

An **arithmetic** series has n terms, all of which are **integers**.

The sum of the series is 20.

Which of the following statements **must** be true?

I The first term of the series is even.

II n is even.

III The common difference is even.

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

SQ3

The real numbers a, b, c and d satisfy both

$$0 < a + b < c + d$$

and

$$0 < a + c < b + d$$

Which of the following inequalities **must** be true?

I $\quad a < d$

II $\quad b < c$

III $\quad a + b + c + d > 0$

(A) none of them

(B) I only

(C) II only

(D) III only

(E) I and II only

(F) I and III only

(G) II and III only

(H) I, II and III

SQ4

A positive integer is called a *squaresum* **if and only if** it can be written as the sum of the squares of two integers. For example, 61 and 9 are both squaresums since $61 = 5^2 + 6^2$ and $9 = 3^2 + 0^2$.

A prime number is called *awkward* **if and only if** it has a remainder of 3 when divided by 4. For example, 23 is awkward since $23 = 5 \times 4 + 3$.

A (true) theorem due to Fermat states that:

A positive integer is a squaresum **if and only if** each of its awkward prime factors occurs to an even power in its prime factorisation.

It follows that 5×23^2 is a squaresum, since 23 occurs to the power 2, but 5×23^3 is not, since 23 occurs to the power 3.

Which one of the following statements is **not** true?

(A) Every square number is a squaresum.

(B) If N and M are squaresums, then so is NM.

(C) If NM is a squaresum, then N and M are squaresums.

(D) If N is not a squaresum, then kN is a squaresum for some number k which is a product of awkward primes.

SQ5

Triangles ABC and XYZ have the **same area**.

Which of these extra conditions, taken independently, would **imply** that they are congruent?

(1) $AB = XY$ **and** $BC = YZ$

(2) $AB = XY$ **and** $\angle ABC = \angle XYZ$

(3) $\angle ABC = \angle XYZ$ **and** $\angle BCA = \angle YZX$

(A) Condition (1): Does not imply congruent; Condition (2): Does not imply congruent; Condition (3): Does not imply congruent.

(B) Condition (1): Does not imply congruent; Condition (2): Does not imply congruent; Condition (3): Implies congruent.

(C) Condition (1): Does not imply congruent; Condition (2): Implies congruent; Condition (3): Does not imply congruent.

(D) Condition (1): Does not imply congruent; Condition (2): Implies congruent; Condition (3): Implies congruent.

(E) Condition (1): Implies congruent; Condition (2): Does not imply congruent; Condition (3): Does not imply congruent.

(F) Condition (1): Implies congruent; Condition (2): Does not imply congruent; Condition (3): Implies congruent.

(G) Condition (1): Implies congruent; Condition (2): Implies congruent; Condition (3): Does not imply congruent.

(H) Condition (1): Implies congruent; Condition (2): Implies congruent; Condition (3): Implies congruent.

SQ6

The positive real numbers $a \times 10^{-1}$, $b \times 10^{-2}$ and $c \times 10^{-1}$ are each in standard form, and

$$(a \times 10^{-3}) + (b \times 10^{-2}) = (c \times 10^{-1}).$$

Which of the following statements (I, II, III, IV) **must** be true?

I $\quad a > 9$

II $\quad b > 9$

III $\quad a < c$

IV $\quad b < c$

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I and IV only

(F) II and III only

(G) II and IV only

(H) I, II, III and IV

SQ7

I have forgotten my 5-character computer password, but I know that it consists of the letters **a**, **b**, **c**, **d**, **e** in some order. When I enter a potential password into the computer, it tells me exactly how many of the letters are in the correct position.

When I enter **abcde**, it tells me that none of the letters are in the correct position. The same happens when I enter **cdbea** and **eadbc**.

Using the best strategy, how many **further** attempts must I make in order to **guarantee** that I can **deduce** the correct password?

(A) None: I can deduce it immediately

(B) One

(C) Two

(D) Three

(E) More than three

20

518

21 UEIE TMUA Mock 2022

Time Allowed

75 min

Number of Questions

20

Difficulty

★★★★☆

TMUA Mock 2022

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22 UEIE TMUA Mock 2023

Time Allowed

75 min

Number of Questions

20

Difficulty

★★★★☆

TMUA Mock 2023

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22

522

23 UEIE TMUA Mock 2024

THIS MOCK TEST IS **UNAVAILABLE** BEFORE 2024.10.10.

Time Allowed

75 min

Number of Questions

20

Difficulty

★★★★☆

TMUA Mock 2024

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