Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 13 October 2021 – Afternoon

Time 2 hours

Paper reference **9MA0/02**

Mathematics

Advanced

PAPER 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1. In an arithmetic series

- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.

(2)

(b) Hence find the sum of the first 500 terms of the series.

(2)

a) $u_1 = a = 16$ $\quad u_{21} = a + 20d = 24$ 1 $\leftarrow$ $u_n = a + (n-1)d$

$16 + 20d = 24$

$20d = 8$

$d = \frac{8}{20} = 0.4$ 1

b) $S_{500} = \frac{500}{2}\left[2(16) + (500-1)0.4\right]$ 1

$= 57,900$ 1 $\quad$ $S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$

Question 1 continued

(Total for Question 1 is 4 marks)

2. The functions f and g are defined by

$$f(x) = 7 - 2x^2 \qquad x \in \mathbb{R}$$

$$g(x) = \frac{3x}{5x-1} \qquad x \in \mathbb{R} \qquad x \neq \frac{1}{5}$$

(a) State the range of f

(1)

(b) Find gf(1.8)

(2)

(c) Find $g^{-1}(x)$

(2)

a) $f(x) = 7 - 2x^2$

$-2x^2 \leq 0 \ \forall x$

so max $f(x)$ is when $-2x^2 = 0$

$f(x) \leq 7$ ①

b) $gf(1.8)$

$f(1.8) = 7 - 2(1.8)^2 = 0.52$

$g(0.52) = \frac{3(0.52)}{5(0.52)-1}$ ① $= 0.975$ ①

c) $g(x) = \frac{3x}{5x-1}$

swap x and y, rearrange for y

$x = \frac{3y}{5y-1}$

$x(5y-1) = 3y$

$5xy - x = 3y$

$5xy - 3y = x$

$y(5x-3) = x$ ①

$y = \frac{x}{5x-3}$

$g^{-1}(x) = \frac{x}{5x-3}$ ①

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Question 2 continued

(Total for Question 2 is 5 marks)

3. Using the laws of logarithms, solve the equation

$$\log_3(12y + 5) - \log_3(1 - 3y) = 2$$

(3)

$$\log_3(12y+5) - \log_3(1-3y) = 2$$

$\log_a b - \log_a c = \log_a \frac{b}{c}$

$$\log_3\left(\frac{12y+5}{1-3y}\right) = 2 \quad 1$$

$$\frac{12y+5}{1-3y} = 3^2$$

$$12y+5 = 9(1-3y)$$

$$12y+5 = 9-27y \quad 1$$

$$39y = 4$$

$$y = \frac{4}{39} \quad 1$$

make sure to substitute $y = \frac{4}{39}$ into $12y+5$ and $1-3y$ to ensure they are positive and so the logarithms are valid.

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Question 3 continued

(Total for Question 3 is 3 marks)

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

small angle approximations: $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\sin\theta \approx \theta$

$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx 4\left(\frac{\theta}{2}\right) + 3\left(1 - \frac{\theta^2}{2}\right)^2$ 1

$= 2\theta + 3\left(1 - \theta^2 + \frac{\theta^4}{4}\right)$

$= 2\theta + 3 - 3\theta^2 + \frac{3}{4}\theta^4$ 1

ignore θ^4 term because θ is small

$= 3 + 2\theta - 3\theta^2$ 1

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Question 4 continued

(Total for Question 4 is 3 marks)

5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) (i) $y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$

$\dfrac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ (1) (1)

(ii) $\dfrac{d^2y}{dx^2} = 60x^2 - 144x + 84$ (1)

b) (i) If C has a stationary point at $x=1$, then $\left.\dfrac{dy}{dx}\right|_{x=1} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 20(1)^3 - 72(1)^2 + 84(1) - 32$

$= 20 - 72 + 84 - 32 = 0$ ✓ (1)

so there is a stationary point at $x = 1$ (1)

(ii) $\left.\dfrac{d^2y}{dx^2}\right|_{x=0.8} = 7.2 > 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=1.2} = -2.4 < 0$ (1)

Since there is a change in sign, $x=1$ is a point of inflection. (1)

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Question 5 continued

(Total for Question 5 is 7 marks)

6.

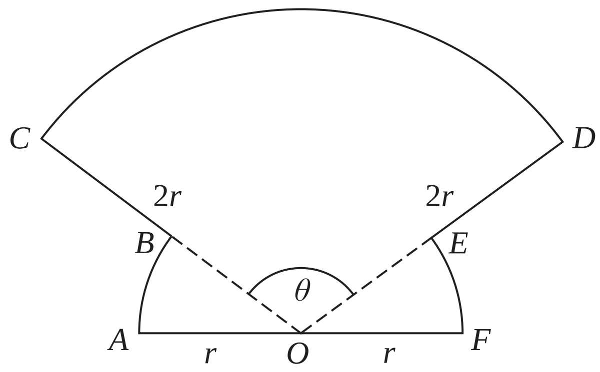


Figure 1

The shape $OABCDEFO$ shown in Figure 1 is a design for a logo.

In the design
- OAB is a sector of a circle centre O and radius r
- sector OFE is congruent to sector OAB
- ODC is a sector of a circle centre O and radius $2r$
- AOF is a straight line

Given that the size of angle COD is θ radians,

(a) write down, in terms of θ, the size of angle AOB

(1)

(b) Show that the area of the logo is

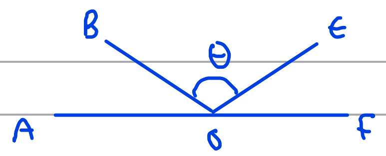
$$\frac{1}{2}r^2(3\theta + \pi)$$

(2)

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π.

(2)

a)



$\angle AOF = \pi$ and $\angle AOB = \angle FOE$

so $\angle AOB = \dfrac{\pi - \theta}{2}$ (1)

$= 180° = \pi^c$

b) Area $= 2 \times$ +

Area of a sector $= \dfrac{1}{2}\theta r^2$

θ in radians

$= 2 \times \left(\frac{1}{2}r^2\left(\frac{\pi - \theta}{2}\right)\right) + \frac{1}{2}\theta(2r)^2$ (1)

$= \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta + 2r^2\theta = \frac{3}{2}r^2\theta + \frac{1}{2}r^2\pi = \frac{1}{2}r^2(3\theta + \pi)$ (1)

expanded brackets

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Question 6 continued

c) arc length $= r\theta$, θ in radians

arc length $CD = 2r\theta$

arc length AB and $EF = \left(\frac{\pi - \theta}{2}\right)r$

Perimeter $= 4r + 2r\left(\frac{\pi - \theta}{2}\right) + 2r\theta$ (1)

$= 4r + r\theta + r\pi$

$= r(4 + \pi + \theta)$ (1)

Question 6 continued

Question 6 continued

(Total for Question 6 is 5 marks)

7. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

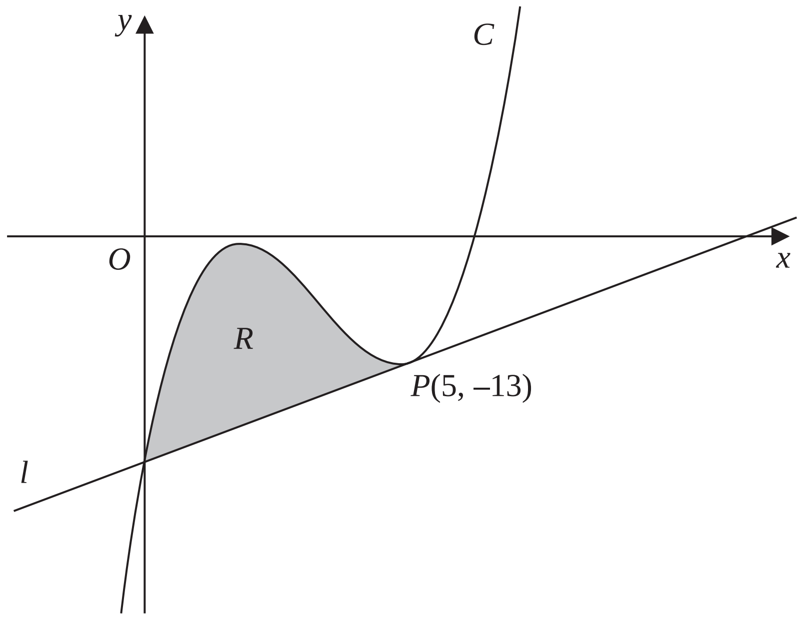


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

a) $y = x^3 - 10x^2 + 27x - 23$

$\frac{dy}{dx} = 3x^2 - 20x + 27$ 1

$\left.\frac{dy}{dx}\right|_{x=5} = 3(5)^2 - 20(5) + 27 = 2$ 1

$\therefore$ l has gradient 2 and goes through $(5, -13)$

$\Rightarrow$ $y + 13 = 2(x - 5)$ 1

$y + 13 = 2x - 10$

$y = 2x - 23$ 1

P 6 8 7 3 2 A 0 1 6 4 8

Question 7 continued

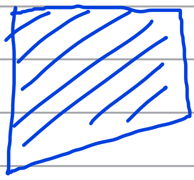
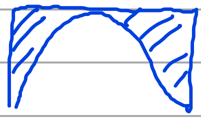
b) on the y-axis, $x=0$

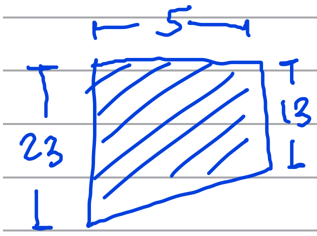
C: $y = 0^3 - 10(0)^2 + 27(0) - 23 = -23$

L: $y = 2\times 0 - 23 = -23$

Both C and l pass through $(0,-23)$

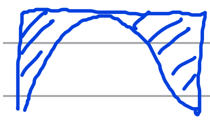
So C meets l again on the y-axis ①

c) Area R =  − 



$$\text{Area} = \frac{13+23}{2} \times 5 = 90 \quad ①$$

– at the front since this area is below the x-axis, it will be negative. We are interested in the positive area.



$$\text{Area} = -\int_0^5 (x^3 - 10x^2 + 27x - 23)\,dx$$

$$= -\left[\frac{1}{4}x^4 - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x\right]_0^5 \quad ①$$

$$= \frac{455}{12} \quad ①$$

$$R = 90 - \frac{455}{12} = \frac{625}{12} \quad ①$$

Question 7 continued

Question 7 continued

(Total for Question 7 is 9 marks)

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

a) $px^3 + qxy + 3y^2 = 26$

$\frac{d}{dx}(qxy) = qy + qx\frac{dy}{dx}$ (1)

$3px^2 + qy + qx\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ (1)

$3px^2 + qy + \frac{dy}{dx}(qx + 6y) = 0$

$\frac{dy}{dx}(qx + 6y) = -3px^2 - qy$ (1)

$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$ (1)

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Question 8 continued

b) $P(-1,-4)$ lies on C:

$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$ (1)

$-p \quad + 4q \quad + 48 \quad = 26$

$-p + 4q = -22$ (1)

Normal to C at P has equation $19x + 26y + 123 = 0$

$\Rightarrow y = -\frac{19}{26}x - \frac{123}{26} \quad \therefore m = -\frac{19}{26}$ (1)

$m_{normal} = \frac{-1}{m_{tangent}}$

$\left.\frac{dy}{dx}\right|_{x=-1, y=-4} = -\frac{1}{-\frac{19}{26}} = \frac{26}{19}$

$\Rightarrow \frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = \frac{26}{19}$ (1)

$\frac{-3p + 4q}{-q - 24} = \frac{26}{19}$

$19(-3p + 4q) = 26(-q - 24)$

$-57p + 76q = -26q - 624$

$624 = 57p - 102q$ (2)

(1)

solve (1) and (2) simultaneously using calculator:

$p = 2 \quad q = -5$

(1)

Question 8 continued

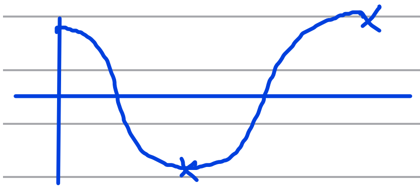
Question 8 continued

(Total for Question 8 is 9 marks)

9. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28}$$

(3)



$\cos(180n)^\circ = (-1)^n$

so $\displaystyle\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \sum_{n=2}^{\infty}\left(-\frac{3}{4}\right)^n$

$\left(\frac{3}{4}\right)^n(-1)^n = \left(-\frac{3}{4}\right)^n$

geometric series with

$a = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}$ 1

$r = -\frac{3}{4}$

1

$S_\infty = \dfrac{a}{1-r} = \dfrac{\frac{9}{16}}{1+\frac{3}{4}} = \dfrac{9}{28}$ 1

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Question 9 continued

(Total for Question 9 is 3 marks)

10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b$$

where l metres is the length of the pendulum and a and b are constants.

(a) Show that this relationship can be written in the form

$$\log_{10} T = b\log_{10} l + \log_{10} a$$

(2)

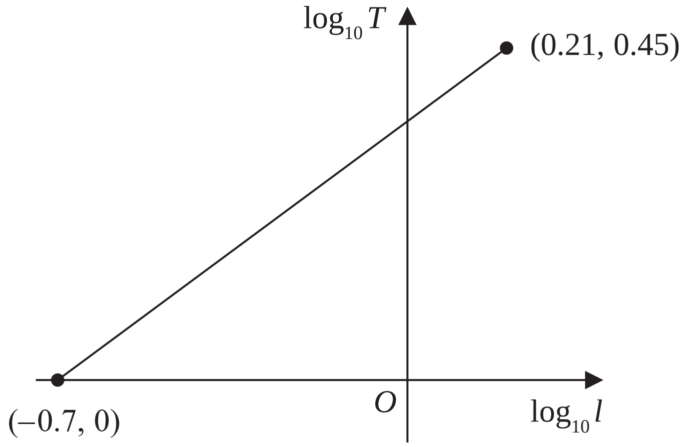


Figure 3

A student carried out an experiment to find the values of the constants a and b.

The student recorded the value of T for different values of l.

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data. The straight line passes through the points (–0.7, 0) and (0.21, 0.45)

Using this information,

(b) find a complete equation for the model in the form

$$T = al^b$$

giving the value of a and the value of b, each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant a.

(1)

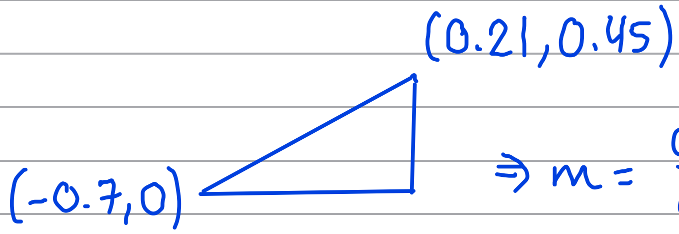
a) $T = al^b \Rightarrow \log_{10} T = \log_{10} al^b$

$\log_{10} T = \log_{10} a + \log_{10} l^b$ 1

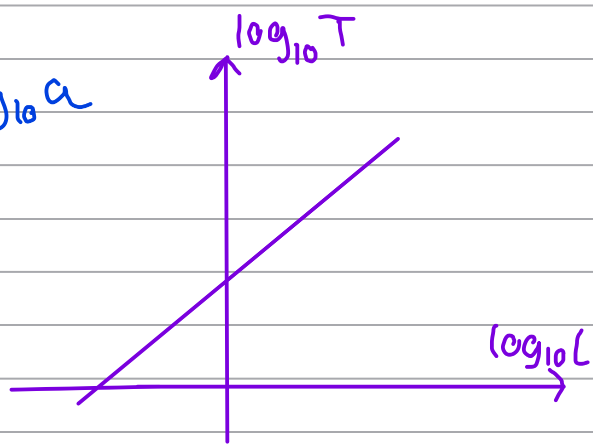
$\log_{10} T = \log_{10} a + b\log_{10} l$ 1

Question 10 continued

b)



$\Rightarrow m = \dfrac{0.45-0}{0.21+0.7} = \dfrac{45}{91}$



we have $\log_{10} T = b\log_{10} L + \log_{10} a$

$\Rightarrow$ gradient $= b$

$\therefore b = \dfrac{45}{91}$ (1)

sub in $(-0.7, 0)$:

$0 = \dfrac{45}{91}(-0.7) + \log_{10} a$ (1)

$\log_{10} a = 0.346\ldots$

$a = 10^{0.346\ldots} = 2.218\ldots$

$T = 2.22L^{0.495}$ (1)

c) a is the time taken for a pendulum of length 1m to complete one full swing. (1)

Question 10 continued

Question 10 continued

(Total for Question 10 is 6 marks)

11.

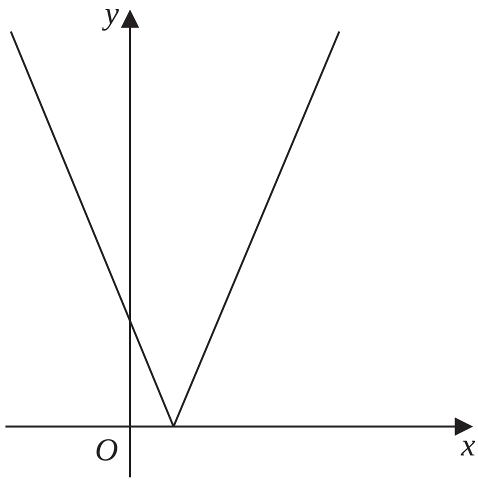


Figure 4

Figure 4 shows a sketch of the graph with equation

$$y = |2x - 3k|$$

where k is a positive constant.

(a) Sketch the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = k - |2x - 3k|$$

stating

- the coordinates of the maximum point
- the coordinates of any points where the graph cuts the coordinate axes

(4)

(b) Find, in terms of k, the set of values of x for which

$$k - |2x - 3k| > x - k$$

giving your answer in set notation.

(4)

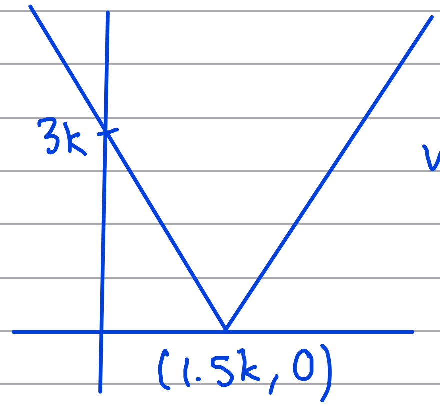
(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation

$$y = 3 - 5\mathrm{f}\left(\frac{1}{2}x\right)$$

(2)

Question 11 continued

a)



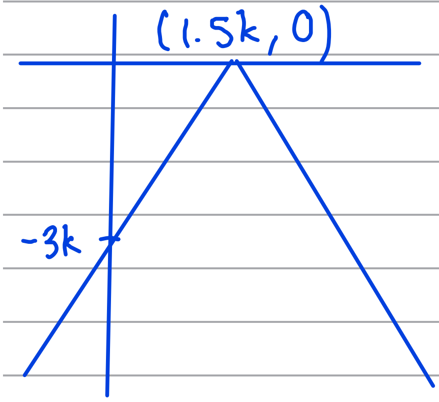
$y = |2x - 3k| \quad k > 0$

vertex: $|2x - 3k| = 0$
$2x - 3k = 0$
$2x = 3k$
$x = 1.5k$

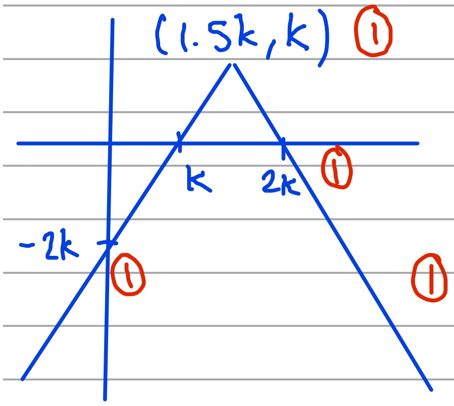
y-intercept:
$y = |0 - 3k|$
$= |-3k|$
$= 3k$

want to sketch $y = k - |2x - 3k|$.

Method: first sketch $y = -|2x - 3k|$, then add k.



$y = -|2x - 3k|$ is a reflection of the first graph in the x-axis.



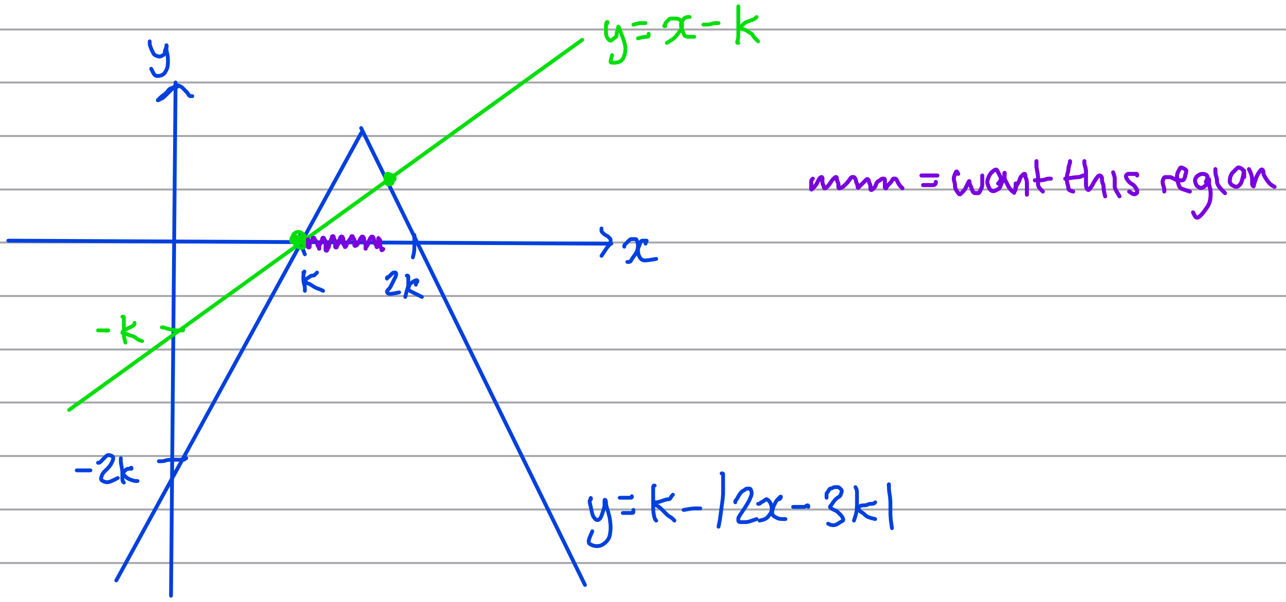
$y = k - |2x - 3k|$ is a translation of $\begin{pmatrix} 0 \\ k \end{pmatrix}$ of the previous graph.

find roots: $k - |2x - 3k| = 0$
$|2x - 3k| = k$

$2x - 3k = k \Rightarrow x = 2k$ $\qquad$ $2x - 3k = -k \Rightarrow x = k$

Question 11 continued

b) $k - |2x - 3k| > x - k$



first critical value is $x = k$ (1)

$\therefore \quad x > k$

find second critical value:

$k - |2x - 3k| = x - k$ (1)

$|2x - 3k| = -x + 2k$

$2x - 3k = -x + 2k$
$3x = 5k$
$x = \frac{5k}{3}$ (1)

$2x - 3k = x - 2k$
$x = k$
found already

$\therefore \quad x < \frac{5k}{3}$

set notation: $\left\{x : x < \frac{5k}{3}\right\} \cap \left\{x : x > k\right\}$ (1)

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Question 11 continued

$y = 3 - 5f\left(\frac{1}{2}x\right)$

$f(x)$ has maximum point
$(1.5k, k)$

$f\left(\frac{1}{2}x\right)$ has maximum point
$(3k, k)$

$-5f\left(\frac{1}{2}x\right)$ has minimum point
$(3k, -5k)$

$3 - 5f\left(\frac{1}{2}x\right)$ has minimum point

$(3k, 3 - 5k)$
1 1

(Total for Question 11 is 10 marks)

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

a) $\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx$

let $u = 1+\sqrt{x}$

when $x=0$, $u=1$
when $x=16$, $u=5$

$u-1 = \sqrt{x}$

$u = 1+\sqrt{0} = 1$
$u = 1+\sqrt{16} = 1+4 = 5$

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$ 1

$\Rightarrow dx = 2(u-1)du$

$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_1^5 \frac{(u-1)^2}{u} \times 2(u-1)\,du$ 1 $= \int_1^5 \frac{2(u-1)^3}{u}\,du$ 1

Question 12 continued

b) $\int_1^5 \frac{2(u-1)^3}{u}\,du = 2\int_1^5 \frac{(u-1)^3}{u}\,du$

$= 2\int_1^5 \frac{u^3 - 3u^2 + 3u - 1}{u}\,du$

$= 2\int_1^5 u^2 - 3u + 3 - \frac{1}{u}\,du$ (1)

$= 2\left[\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right]_1^5$ (1)

$= 2\left(\frac{1}{3}(5)^3 - \frac{3}{2}(5)^2 + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1\right)\right)$ (1)

$= \frac{104}{3} - 2\ln 5$ (1)

Question 12 continued

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Question 12 continued

(Total for Question 12 is 7 marks)

13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ
(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$
(3)

a) $y = \operatorname{cosec}^3\theta$

$\frac{dy}{d\theta} = 3\operatorname{cosec}^2\theta \times -\operatorname{cosec}\theta \cot\theta$

$= -3\operatorname{cosec}^3\theta \cot\theta$ (1)

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ (1)

$x = \sin 2\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta$

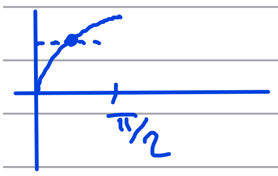
$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ (1)

b) find θ when $y = 8$

$8 = \operatorname{cosec}^3\theta$

$\operatorname{cosec}\theta = 2$

$\sin\theta = \frac{1}{2}$ (1)



$0 < \theta < \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\frac{\pi}{6}\cot\frac{\pi}{6}}{2\cos\frac{2\pi}{6}}$ (1)

$\frac{dy}{dx} = -24\sqrt{3}$ (1)

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Question 13 continued

(Total for Question 13 is 6 marks)

14.

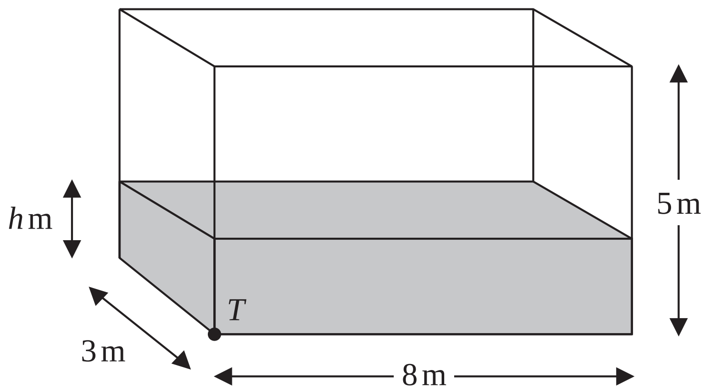

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

Question 14 continued

a)

let V= volume of water in the tank at time t

$V = 3 \times 8 \times h = 24h$

$\frac{dV}{dh} = 24$ 1

given that water moves in at $0.48m^3$ per minute
and water moves out at $0.1h\ m^3$ per minute

$\therefore \frac{dV}{dt} = 0.48 - 0.1h$ 1

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{24} \times (0.48 - 0.1h)$ 1

$\frac{dh}{dt} = \frac{24}{1200} - \frac{5h}{1200}$

$1200 \frac{dh}{dt} = 24 - 5h$ 1

b) when $t=0$, $h=2$

$1200 \frac{dh}{dt} = 24 - 5h$ separate variables

$\Rightarrow \int \frac{1200}{24-5h} dh = \int dt$ 1

if $y = \ln(24-5h)$

$\frac{dy}{dh} = \frac{-5}{24-5h}$

$-240\ln|24-5h| = t + c$ 1

we need numerator $=1200$, so multiply by -240

Question 14 continued

$$-240\ln|24-5h| = t + c$$

sub in $t=0$, $h=2$

$$0 + c = -240\ln|24-10|$$
$$c = -240\ln 14 \quad ①$$

$$\therefore \quad -240\ln|24-5h| = t - 240\ln 14$$

$$t = 240\ln 14 - 240\ln|24-5h| \quad ①$$

$$\frac{t}{240} = \ln\left(\frac{14}{24-5h}\right)$$

$$\Rightarrow e^{\frac{t}{240}} = \frac{14}{24-5h}$$

$$24e^{\frac{t}{240}} - 5he^{\frac{t}{240}} = 14$$

$$5he^{\frac{t}{240}} = 24e^{\frac{t}{240}} - 14 \quad ①$$

$$h = \frac{24e^{t/240} - 14}{5e^{t/240}}$$

$$h = 4.8 - 2.8e^{-\frac{t}{240}} \quad ①$$

c) as $t \to \infty$, $e^{-\frac{t}{240}} \to 0$ so $h \to 4.8$ ①

The tank is 5m high, and the limit for h is 4.8m, ①
so the tank will never become full.

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Question 14 continued

(Total for Question 14 is 12 marks)

15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

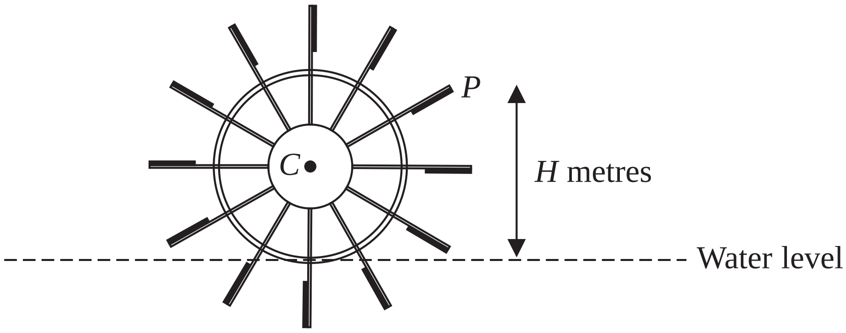

Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

Question 15 continued

a) $2\cos\theta - \sin\theta = R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

comparing coefficients:

$R\cos\alpha = 2$

$R\sin\alpha = 1$

$R = \sqrt{1^2+2^2} = \sqrt{5}$ ①

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2} \Rightarrow \alpha = 0.464$ (3d.p) ① ①

b) $H = 3+4\cos(0.5t) - 2\sin(0.5t)$

(i) max H

$H = 3+2(2\cos(0.5t) - \sin(0.5t))$

$= 3+2\sqrt{5}\cos(0.5t+0.464)$

max H happens when $\cos(0.5t+0.464) = 1$, i.e. $H = 3+2\sqrt{5}$ ①

(ii) $\Rightarrow$ $0.5t + 0.464 = \cos^{-1}(1)$ ①

$= 2\pi$

$0.5t = 2\pi - 0.464$

$t = 11.6$ (1dp) ①

Question 15 continued

c) $H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

T seconds when $H<0$. Find T.

Method: find two consecutive times when $t=0$, then find the difference between them

$3 + 2\sqrt{5}\cos(0.5t + 0.464) = 0$ (1)

$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$

$0.5t + 0.464 = \cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$

$t = 2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right) - 0.464\right)$ (1)

(1)

time required is e.g. $2(3.977... - 0.464) - 2(2.306.. - 0.464)$

$= 3.34$ (3sf) (1)

d) the 3 would need to be variable. (1)

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Question 15 continued

Question 15 continued

(Total for Question 15 is 11 marks)

TOTAL FOR PAPER IS 100 MARKS