

Test of Mathematics for University Admission, 2019 Paper 2

Worked Solutions

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Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be ‘definitive’ solutions.

The questions themselves are available on the ‘Preparing for the test’ section on the Admissions Testing website.

Question 1

We need the coefficient of x^2 in the bracketed term to get the term in x^4 in the whole expansion.
We expand this using the binomial theorem to get

$$x^2 \left(2x + \frac{1}{x}\right)^6 = x^2 \left((2x)^6 + \binom{6}{1}(2x)^5 \left(\frac{1}{x}\right) + \binom{6}{2}(2x)^4 \left(\frac{1}{x}\right)^2 + \dots\right).$$

We stop here as the power of x in this last term is $x^{4-2} = x^2$, as we require. This term is then

$$\begin{aligned} x^2 \binom{6}{2} (2x)^4 \left(\frac{1}{x}\right)^2 &= 15x^2 \cdot 16x^4 \cdot x^{-2} \\ &= 240x^4 \end{aligned}$$

so the answer is E.

Question 2

Since $2x + 1 = 2(x + \frac{1}{2})$ is a factor of $f(x) = 2x^3 + px^2 + q$, it follows that $f(-\frac{1}{2}) = 0$. Similarly, $f(2) = 0$.

Substituting these values in to $f(x)$ gives

$$\begin{aligned}2(-\frac{1}{2})^3 + p(-\frac{1}{2})^2 + q &= 0 \\2(2)^3 + p(2^2) + q &= 0\end{aligned}$$

so

$$\begin{aligned}-\frac{1}{4} + \frac{1}{4}p + q &= 0 \\16 + 4p + q &= 0\end{aligned}$$

Multiplying the first equation by 16 gives

$$-4 + 4p + 16q = 0.$$

Subtracting $16 + 4p + q = 0$ from this gives $-20 + 15q = 0$, so $q = \frac{4}{3}$, hence $4p = -16 - q = -\frac{52}{3}$. Therefore $2p + q = -\frac{26}{3} + \frac{4}{3} = -\frac{22}{3}$, which is option C.

Question 3

It might be tempting to divide by a to get $b = c$, but a might be zero, and we cannot divide by zero. Instead, we can rearrange and factorise to get $ab - ac = 0$, so $a(b - c) = 0$. From this, we can deduce that $a = 0$ or $b - c = 0$ (or both).

- I This might not be true; we could have $b = c$ instead.
- II This might not be true; we could have $b = c = 1$, for example.
- III This might not be true; we could have $a = 0$ instead.

So the correct answer is A.

Question 4

A counterexample will have the antecedent (“ N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1”) true, and the consequent (“ N is a prime number”) false. We can now check each of the three offered values of N :

- I The form of N is correct: there is 1 zero, and 1 is odd, so the antecedent is true. However, the consequent is also true, so this is not a counterexample.
- II The form of N is not correct, as there are 2 zeros, so the antecedent is false. Hence this is not a counterexample.
- III The form of N is correct, as there are 3 zeros, and the consequent is false, as N is not prime. Hence this is a counterexample.

The correct solution is D.

Question 5

Saying that ‘either a or b is divisible by n ’ is **necessary** for $(*)$ is the same as saying that **if** ab is divisible by n , **then** either a or b is divisible by n . We can look for a counterexample or proof for this assertion.

On the surface of it, it seems surprising: when a product is divisible by a number, why should one of the two factors also be? We know something similar to this for a product which equals zero, and we may know that if a product is divisible by a prime, then one of the factors must be. But there is no mention of zero or primes here. So let us try a simple example where n is not prime, such as $n = 4$ (the smallest non-prime other than 1). Then the statement becomes: **if** ab is divisible by 4, **then** either a or b is divisible by 4. A small amount of exploration finds counterexamples, such as $a = 2$, $b = 2$ or $a = 6$, $b = 2$. So the condition is **not necessary** for $(*)$.

What about sufficiency? Saying that ‘either a or b is divisible by n ’ is **sufficient** for $(*)$ is the same as saying that **if** either a or b is divisible by n , **then** ab is divisible by n . We can again look for a counterexample or proof for this assertion.

This time, though, a proof is clear: if a is divisible by n , then a is a multiple of n , so ab is also a multiple of n (as b is an integer). This means that ab is divisible by n . The same is the case if b is divisible by n , so the condition is **sufficient**.

The correct answer is therefore B.

Question 6

The following is just one approach to checking an argument.

We start by checking the final solutions on line (VI). $x = 0$ makes the LHS (left hand side) equal 1 but the RHS (right hand side) equal -1 , so the attempt cannot be correct. Furthermore, this is not a value of x for which $\tan x$ is undefined, so G is not the correct answer. We must then identify where the error arose.

We can now test the incorrect solution $x = 0$ at various stages of the argument to see where things first go wrong. Rather than working line-by-line, we start about half way through to see whether the mistake happens in the first half of the argument or the second half; this is a little more efficient.

Let us try the incorrect solution $x = 0$ in line (III) (which is midway between (I) and (V)): this gives $-1 = -1$ or $-1 = -1$, so $x = 0$ is valid at this point. This tells us that the error must occur on line (I), (II) or (III).

On line (I), $x = 0$ gives $1 = -1$, while on line (II), $x = 0$ gives $(-1) \times (-1) = -1$. So it is most likely that the first error occurs on line (III).

We check the algebra on line (I): all that has happened is that $2 \sin x$ has been subtracted from both sides, so this is correct.

We likewise check the algebra on line (II): by multiplying out the brackets, we get the equation on line (I), so this is correct. (We might be concerned about the case $\cos x = 0$ here, but this is not possible on either line (I) or line (II) because of the $\tan x$ terms on each line.)

Finally, we are left with line (III). The implication here is that because two numbers multiply to give -1 , one of them must be -1 . This would be true if we were dealing with integers, but we are not, so this is an invalid implication.

The correct solution is therefore D: the first error occurs on line (III).

Question 7

We work through these in turn:

- A A counterexample here is a case for which the antecedent (the ‘if’ part) is true but the consequent (the ‘then’ part) is false. In this case, taking $a = 12$, $b = 16$, $c = 20$ to match $12^2 + 16^2 = 20^2$ does not satisfy the antecedent, as they do have a common divisor of 4. Even if we were to divide all three by 4 to get $3^2 + 4^2 = 5^2$, we would still not have a counterexample, as two of them are odd and the other is even. (The statement is actually true, so there are no counterexamples.)
- B A counterexample in this case is an example to show that there is a solution. Since $16 = 4^2$, we can rewrite $12^2 + 16^2 = 20^2$ as $4^4 + 12^2 = 20^2$, so this does provide a counterexample.
- C Since neither 12 nor 20 is a perfect square, we cannot use this fact to obtain a counterexample. (And that is quite a relief, since Fermat proved that this statement is true! It is the simplest case of Fermat’s Last Theorem.)
- D We note that $12^2 + 16^2 = 20^2$ satisfies the antecedent, but as 16 is the arithmetic mean of 12 and 20, this does not provide a counterexample. (The statement is false, and a counterexample is $5^2 + 12^2 = 13^2$.)

The correct answer is therefore B.

Question 8

When thinking about inequalities involving negative numbers, we must take care to reverse the sign of any inequality when multiplying or dividing by a negative number. In this case, all three of a , b and c are negative.

I We start with $a < b < c$ and multiply by $a < 0$ to obtain

$$a^2 > ab > ac$$

so this statement must be true.

II Since $a < 0$ and $c < 0$, $a + c < 0$. Since $b < 0$, it follows that $b(a + c) > 0$. So this statement must be true.

III We start with $c > a$ (since $a < c$). Dividing by $b < 0$ then gives

$$\frac{c}{b} < \frac{a}{b}$$

and so this statement must be false.

The correct answer is E.

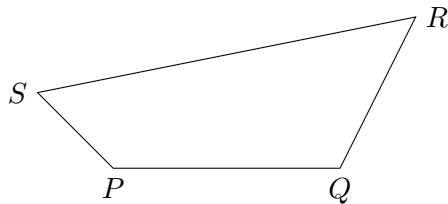
Question 9

If the next person **must** sit next to someone, there cannot be three spaces between one person and the next. But if there are only two spaces between them, someone sitting down in one of those spaces must sit next to someone. So the smallest number of people will be achieved when all of the gaps are exactly two spaces, except perhaps for the final person to the first person.

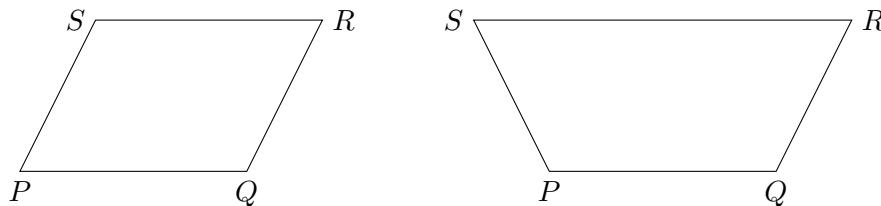
So counting the seats around the table as 1, 2, 3, ..., we place people in seats 1, 4, 7, 10, ..., 37, 40. (The final person is sitting immediately next to the first person, but that does not matter.) There are 14 seats in this list, so the answer is D.

Question 10

We sketch a quadrilateral for ease of reference.



- A** The sides PQ and SR are equal in length, which is certainly necessary if $PQRS$ is a parallelogram, and PS being parallel to QR is also necessary. If these are both the case, then on first glance it may seem that $PQRS$ must be a parallelogram. However, if we draw two parallel lines and ask how we can join them with two equal-length sides. We discover there are two ways:



So this condition is necessary but not sufficient.

- B** As before, $PQ = SR$ is necessary, and PQ being parallel to SR is obviously necessary, so the whole condition is necessary.

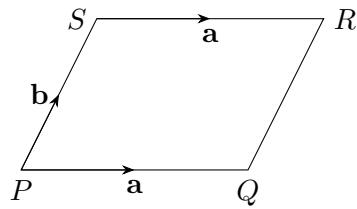
Unlike in the previous part, though, this is also sufficient: if two sides of the quadrilateral are both equal and parallel, the remaining two sides must also be equal and parallel. (This may seem obvious, and it can be proven using geometric or vector methods; see below for details.)

- C** This condition is not necessary, as not all four sides of a parallelogram need to be the same length. It does mean that the parallelogram is a rhombus, so it is a sufficient condition.
- D** The diagonals of a parallelogram need not be equal in length, so this is not a necessary condition. It is also not sufficient.

The correct answer is therefore A.

We now give two proofs that condition B is sufficient, that is, if $PQ = SR$ and PQ is parallel to SR , then $PQRS$ is a parallelogram.

We first give a vector proof. Since $PQ = SR$ and they are parallel, $\overrightarrow{PQ} = \overrightarrow{SR}$; we call this vector \mathbf{a} . Let $\overrightarrow{PS} = \mathbf{b}$, as shown in this diagram:

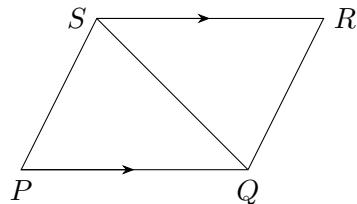


Then

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR} = (-\mathbf{a}) + \mathbf{b} + \mathbf{a} = \mathbf{b}$$

so $\overrightarrow{QR} = \overrightarrow{PS}$, showing that PS and QR are equal and parallel, and thus $PQRS$ is a parallelogram.

And here is a geometric approach. We are given that $PQ = RS$ and that they are parallel. Draw in the diagonal QS , as shown.



Then $\angle PQS = \angle RSQ$ as they are alternate angles.

Therefore triangles PQS and RSQ are congruent by side-angle-side, as $PQ = RS$ (given), $QS = SQ$ (common side) and $\angle PQS = \angle RSQ$.

Thus $\angle QSP = \angle SQR$, so SP is parallel to QR (by alternate angles) and $PS = RQ$.

Hence $PQRS$ is a parallelogram.

Question 11

Let the first term of the series be a and the common difference be d , as usual. Therefore we have

$$\frac{1}{2}n(2a + (n - 1)d) = 20.$$

Multiplying by 2 gives

$$n(2a + (n - 1)d) = 40,$$

so we need to find integers n , a and d which satisfy this. We list values of n which are factors of 40 and the corresponding value of $2a + (n - 1)d$.

n	$2a + (n - 1)d$
1	$40 = 2a$
2	$20 = 2a + d$
4	$10 = 2a + 3d$
5	$8 = 2a + 4d$
8	$5 = 2a + 7d$
10	$4 = 2a + 9d$
20	$2 = 2a + 19d$
40	$1 = 2a + 39d$

With this table to hand, we consider the possibilities for each of the statements.

I $n = 1$ gives $a = 20$, which is even.

$n = 2$ gives $2a + d = 20$. If we take $d = 2$, this gives $a = 9$, and indeed $9 + 11 = 20$, so this statement need not be true.

II The table shows that we could have $n = 1$ and $a = 20$, or $n = 5$, $a = 2$, $d = 1$, giving $2 + 3 + 4 + 5 + 6 = 20$. So this statement need not be true.

III The example in the previous part, $2 + 3 + 4 + 5 + 6 = 20$ shows that this is not necessarily true.

Therefore the correct answer is A.

Question 12

One way to answer this question is to draw a tree diagram and to work out the probabilities of the three events.

Another way is to reason as follows.

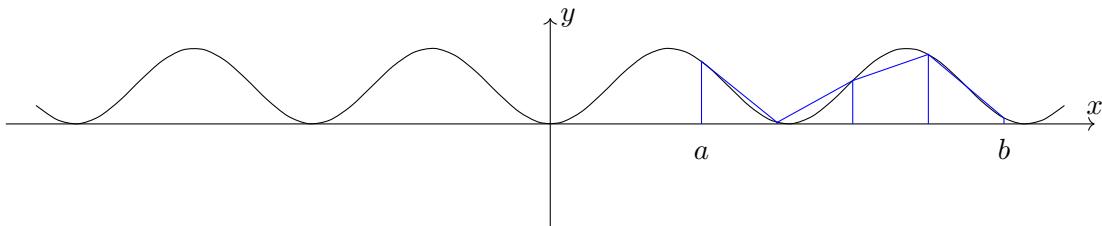
If all three students study Mathematics, then certainly the second student chosen studies Mathematics and at least one of the students chosen studies Mathematics. Therefore $T < S$ and $T < R$.

Likewise, if the second student chosen studies Mathematics, certainly at least one of the students chosen studies Mathematics. Therefore $S < R$.

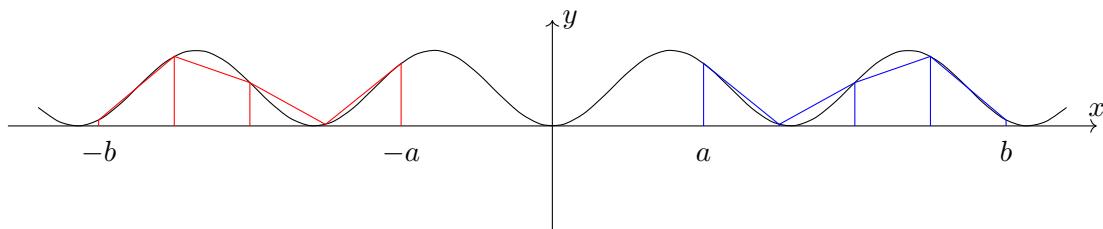
Putting these together gives $T < S < R$, and the answer is F.

Question 13

It will help us to draw a sketch of this situation, remembering that we do not know that values of a and b . The exact shape of the graph does not matter; the only thing that is important for this question is that it is symmetrical about the y -axis, since $\sin^2(-x) = (-\sin x)^2 = \sin^2 x$.

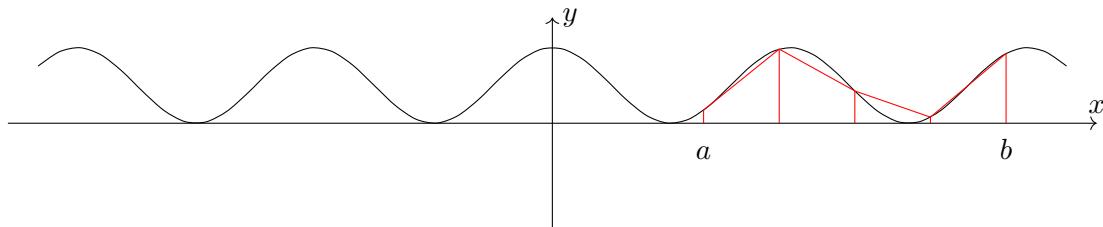


- I We can now add to our sketch the trapezium rule calculation for the integral $\int_{-b}^{-a} \sin^2 x \, dx$:

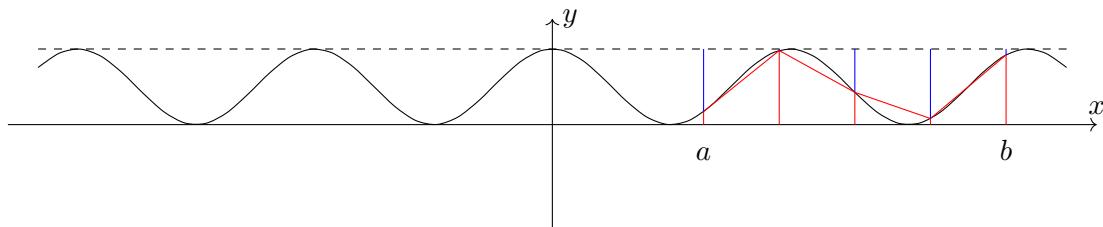


We see that the strips are just a reflection of the original 4 strips. Since the original approximation was an overestimate, this approximation is also an overestimate.

- II We note that $\cos^2 x$ is related to $\sin^2 x$ by the identity $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$. Thus the graph of $y = \cos^2 x$ is a reflection of the graph of $y = \sin^2 x$ in the line $y = \frac{1}{2}$ (why?). Therefore the sketch of the graph and trapezia for $\int_a^b \cos^2 x \, dx$ is:



Since $\cos^2 x = 1 - \sin^2 x$, we can also draw on the original trapezia, coming down from the line $y = 1$:



We can see from this sketch that the blue and red trapezia exactly sum to $\int_a^b \sin^2 x \, dx + \int_a^b \cos^2 x \, dx$; since the blue trapezia give an overestimate, the red ones must give an underestimate.

Hence this statement is also necessarily true.

Therefore the correct answer is D.

Question 14

A necessary condition is one which must follow. To show that a condition is not necessary, we need to exhibit a counterexample, while to show that a condition is necessary, we need to give a justification.

- I If $p(x)$ is increasing for $a \leq x \leq b$, the value of the polynomial is not decreasing in this interval, so we must have $p(a) \leq p(b)$, and this is therefore a necessary condition.
- II A quadratic such as $p(x) = 9 - x^2$ provides a counterexample: it is increasing for $-3 \leq x \leq 0$, but $p(-3) > 0$ while $p(0) = 0$. Hence this is not a necessary condition.
- III It is not clear why this should be the case, and we need to look beyond quadratics (as $p''(x)$ is constant for every quadratic). Let us see if we can construct a counterexample. Let us take $p''(x) = -x$, so that it is decreasing. Then $p'(x) = -\frac{1}{2}x^2 + c$, giving $p(x) = -\frac{1}{6}x^3 + cx$. (There is a further constant of integration here, but we can choose it to be zero, as it has no impact on where $p(x)$ is increasing.)

If we now choose some $c > 1$, then $p'(x)$ will be positive for $0 \leq x \leq 1$, so $p(x)$ will be increasing there, yet $p''(0) < p''(1)$.

Therefore this condition is not necessary.

Another way of thinking about this is: to say that $p(x)$ is increasing for $a \leq x \leq b$ only tells us about the function values; it does not tell us anything about the derivatives, so neither II nor III could possibly be necessary conditions.

Either way, the correct answer is B.

Question 15

We expand the expressions using the logarithm rules, namely $\log(ab) = \log a + \log b$ and $\log a^n = n \log a$ when $a > 1$. This gives

$$\log a + 2\log b + \log c = 7$$

$$2\log a + \log b + 2\log c = 11$$

$$2\log a + 2\log b + 3\log c = 15$$

This is a set of simultaneous equations! If we write $A = \log a$, $B = \log b$ and $C = \log c$, they become

$$A + 2B + C = 7 \quad (1)$$

$$2A + B + 2C = 11 \quad (2)$$

$$2A + 2B + 3C = 15 \quad (3)$$

Subtracting (3) – (2) and (3) – 2 × (1) to eliminate A gives

$$B + C = 4 \quad (4)$$

$$-2B + C = 1 \quad (5)$$

Subtracting these then gives

$$3B = 3$$

so $B = 1$ (and thus $C = 3$, $A = 2$).

Therefore $\log b = 1$, so the base of logarithms is b (and also $a = b^2$, $c = b^3$). The correct answer is option B.

Question 16

For the quadratic to intersect the x -axis at two distinct points, we require the discriminant to be positive. In this case, this gives

$$q^2 - 4p^2 > 0.$$

We can either deduce the correct graph by solving this inequality, or decide between the given options by checking various points. We show both approaches here.

First, we can write the inequality as $(q - 2p)(q + 2p) > 0$. Therefore, this inequality holds when either $q - 2p > 0$ and $q + 2p > 0$, or when $q - 2p < 0$ and $q + 2p < 0$. The first case gives $q > 2p$ and $q > -2p$. As $p > 0$, this simplifies to $q > 2p$. The second case gives $q < 2p$ and $q < -2p$, which simplifies to $q < -2p$. So the solution is $q > 2p$ or $q < -2p$, and this is shown in graph F.

Alternatively, we can check values of p and q . When $p = 1$, $q = 0$, $q^2 - 4p^2 = -4 < 0$, so this point cannot lie in the region. This eliminates options A, C, E and G. Next, we test a point with $q < 0$, say $p = 1$, $q = -4$. This gives $q^2 - 4p^2 = 12 > 0$, so this point is in the region. This eliminates options B and D, so we are left with F or H. Finally, we can check a point which distinguishes the two, say $p = 2$, $q = 3$, which gives $q^2 - 4p^2 = -7 < 0$, so this point is not in the region. Therefore the correct answer is graph F.

Question 17

As exactly one of the options was correct, if option X is implied by option Y, then option Y cannot be correct (as then option X would also be correct).

Option A implies option B: if A is correct, then the statement is true **if and only if** $x > 1$, which means the statement is true **if** $x > 1$ and the statement is true **only if** $x > 1$, so option B is also correct. Therefore option A cannot be the correct option.

For the same reason, option C cannot be correct, as it implies option D.

We must therefore consider options B and D. Let us assume that option B is correct and ask whether option D follows from it. Option B tells us that whenever $x > 1$, the statement is true, so certainly whenever $x > 2$, the statement is true. Therefore option D follows from option B, and option B cannot be the correct option. The opposite direction does not follow, though: the statement could be true for all $x > 2$, but be false for $x = \frac{3}{2}$, for example.

Therefore the correct option must have been option D.

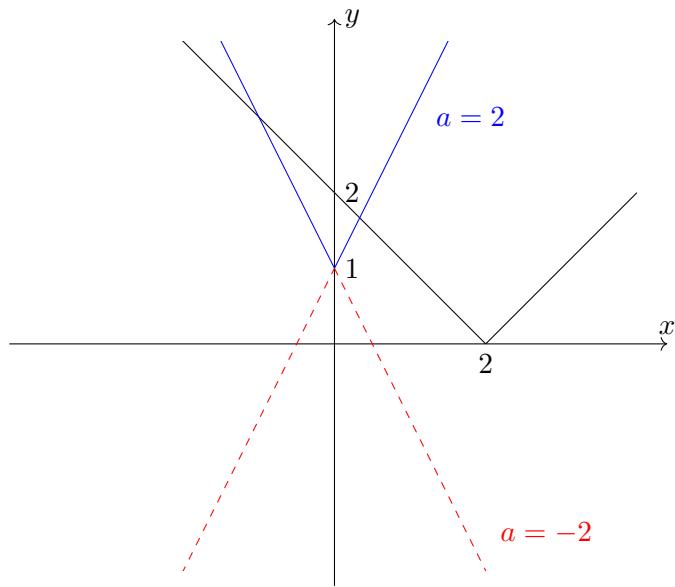
Question 18

We would do well to sketch a graph for this question! There are two functions involved, $y = a|x| + 1$ and $y = |x - 2|$, so we sketch these for different values of a .

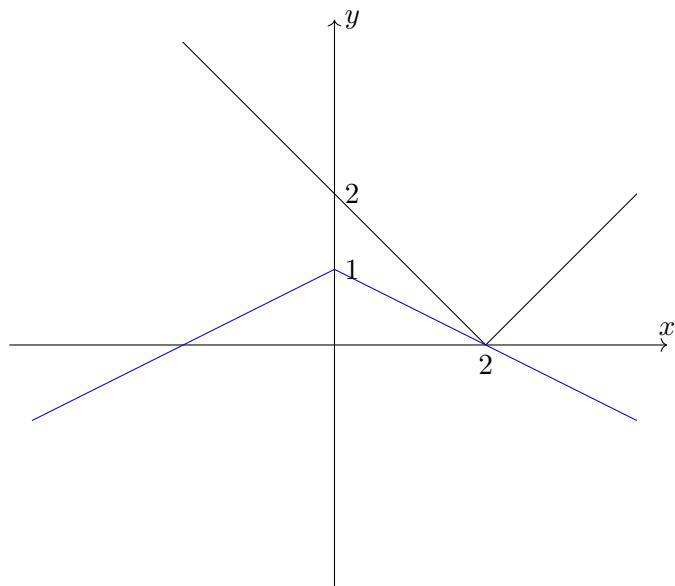
The graph of $y = a|x| + 1$ is the graph of $y = |x|$ first stretched by a factor of a in the y -direction, and then translated by 1 in the positive y -direction.

The graph of $y = |x - 2|$ is the graph of $y = |x|$ translated by 2 in the positive x -direction.

This gives us the following sketch:



What we can see from this sketch is that $y = a|x| + 1$ will be below $y = |x - 2|$ when a is sufficiently negative; when a is positive, it will certainly intersect the graph of $y = |x - 2|$. We can find the value of a at which the behaviour changes by sketching the graph at this point and then doing some algebra:



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We can work out the value of a by noting that the graph passes through $(2, 0)$:

$$a|2| + 1 = 0$$

therefore $a = -\frac{1}{2}$, and hence the correct answer is $a \leq -\frac{1}{2}$, which is option E.

Question 19

This is a rather ugly looking expression! Let us consider the first square root to begin with:

$$\sqrt{8 - 4\sqrt{2} + 1}$$

We could simplify this to $\sqrt{9 - 4\sqrt{2}}$, but it seems likely that the examiners have written it in this way to help us.

One way to approach this question is to square the given expression, see what comes out, and then compare the answer with the squares of the given options. We will try this approach below.

Another approach is to see if we can write the inside of the square root as the square of something. We have $8 = (2\sqrt{2})^2$ and $1 = 1^2$, which suggests that we try $2\sqrt{2} + 1$. But as the middle term has a minus sign, we should actually try $2\sqrt{2} - 1$. Squaring this gives

$$(2\sqrt{2} - 1)^2 = 8 - 4\sqrt{2} + 1$$

and so

$$\sqrt{8 - 4\sqrt{2} + 1} = 2\sqrt{2} - 1.$$

It is important that we take the positive square root; since $2\sqrt{2} > 1$, this is the positive square root.

Similarly,

$$\sqrt{9 - 12\sqrt{2} + 8} = 3 - 2\sqrt{2},$$

where again we note that $3 > 2\sqrt{2}$, so this is the positive square root.

These sum to 2, and hence the answer is E.

The alternative approach was to square the given expression. Writing it as

$$\sqrt{9 - 4\sqrt{2}} + \sqrt{17 - 12\sqrt{2}}$$

we can square it to get

$$\begin{aligned} 9 - 4\sqrt{2} + 17 - 12\sqrt{2} + 2\sqrt{(9 - 4\sqrt{2})(17 - 12\sqrt{2})} \\ = 26 - 16\sqrt{2} + 2\sqrt{153 - 68\sqrt{2} - 108\sqrt{2} + 96} \\ = 26 - 16\sqrt{2} + 2\sqrt{249 - 176\sqrt{2}} \end{aligned}$$

Unfortunately, we are now left in an even worse situation than we started with, as we have to find some way of simplifying this square root; can we find a and b with $(a - b\sqrt{2})^2 = 249 - 176\sqrt{2}$? It turns out that $11 - 8\sqrt{2}$ does the job; this, however, is negative (as $11^2 < (8\sqrt{2})^2$), so the positive square root is $8\sqrt{2} - 11$. Thus the whole expression simplifies to $26 - 16\sqrt{2} + 2(8\sqrt{2} - 11) = 4$, and so the correct solution is $\sqrt{4} = 2$, which is option E.

The first approach was certainly simpler.

Question 20

Commentary: It could be helpful for this question to take a piece of paper and physically move it around following the given instructions to get a feel for what's going on. This suggests that we may be looking for a periodic function.

We could then try to see what happens with a sine or cosine function, but these turn out to have too much symmetry, and could lead us to the wrong answer. A tan graph is periodic, but does not have line symmetry, so is likely to be better for our needs.

With all of this, though, we might still get ourselves in quite a muddle, so beginning with an algebraic approach is likely to be more successful.

When we reflect the graph of $y = f(x)$ in the y -axis, we obtain the graph of $y = f(-x)$. When we translate this by 2 units in the negative x -direction, we obtain the graph of $y = f(-(x + 2))$, so $g(x) = f(-(x + 2)) = f(-x - 2)$.

Similarly, when we translate the graph of $y = f(x)$ by 2 units in the negative x -direction, we obtain the graph of $y = f(x + 2)$. When we reflect this in the y -axis, we obtain the graph of $y = f(-x + 2)$, so $h(x) = f(-x + 2)$.

If the functions $g(x)$ and $h(x)$ are identical, then we require

$$f(-x - 2) = f(-x + 2) \text{ for all } x.$$

This is quite close to what we want, so let us let $u = -x - 2$, so that $-x + 2 = u + 4$. Then the above becomes

$$f(u) = f(u + 4) \text{ for all } u,$$

which is option B, just with x replaced by u .

This means that the function is periodic with a period of 4 (though it may have a smaller period such as 2 or $\frac{1}{7}$, as long as it repeats every 4).