TEST OF MATHEMATICS   D513/12
FOR UNIVERSITY ADMISSION

PAPER 2

Wednesday 8 November 2017   Time: 75 minutes

Additional Materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the second of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt all 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators and dictionaries must **NOT** be used.
There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2

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1 Given that $y = \dfrac{(1-3x)^2}{2x^{\frac{3}{2}}}$, which one of the following is a correct expression for $\dfrac{dy}{dx}$?

A $\quad \frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

B $\quad \frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

C $\quad \frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

D $\quad -\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

E $\quad -\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

F $\quad -\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

2 $PQRS$ is a rectangle.

The coordinates of P and Q are $(0,6)$ and $(1,8)$ respectively.

The perpendicular to PQ at Q meets the x-axis at R.

What is the area of $PQRS$?

A $\frac{5}{2}$

B $4\sqrt{10}$

C 20

D $8\sqrt{10}$

E 40

3 The first term of a geometric progression is $2\sqrt{3}$ and the fourth term is $\frac{9}{4}$

What is the sum to infinity of this geometric progression?

A $\quad -2(2 - \sqrt{3})$

B $\quad 4(2\sqrt{3} - 3)$

C $\quad \dfrac{16(8\sqrt{3} + 9)}{37}$

D $\quad \dfrac{4(2\sqrt{3} - 3)}{7}$

E $\quad \dfrac{4(2\sqrt{3} + 3)}{7}$

F $\quad 2(2 + \sqrt{3})$

G $\quad 4(2\sqrt{3} + 3)$

4 The following question appeared in an examination:

Given that $\tan x = \sqrt{3}$, find the possible values of $\sin 2x$.

A student gave the following answer:

$\tan x = \sqrt{3}$ so $x = 60°$ and $2x = 120°$,
therefore $\sin 2x = \dfrac{\sqrt{3}}{2}$.

Which one of the following statements is correct?

A $\dfrac{\sqrt{3}}{2}$ is the only possible value, and this is fully supported by the reasoning given in the student's answer.

B $\dfrac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\tan x = \sqrt{3}$.

C $\dfrac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\sin 2x = \dfrac{\sqrt{3}}{2}$.

D $\dfrac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\tan x = \sqrt{3}$.

E $\dfrac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\sin 2x = \dfrac{\sqrt{3}}{2}$.

5   Consider the following three statements:

    1 $10p^2 + 1$ and $10p^2 - 1$ are both prime when p is an odd prime.

    2   Every prime greater than 5 is of the form $6n + 1$ for some integer n.

    3   No multiple of 7 greater than 7 is prime.

The result $91 = 7 \times 13$ can be used to provide a counterexample to which of the above statements?

A none of them

B **1** only

C **2** only

D **3** only

E **1** and **2** only

F **1** and **3** only

G **2** and **3** only

H **1**, **2** and **3**

6 A sequence u_0, u_1, u_2, ... is defined as follows:

$$u_0 = 1$$
$$u_n = \int_0^1 4x u_{n-1}\, dx \quad \text{for } n \geqslant 1$$

What is the value of u_{1000}?

A $\quad 2^{1000}$

B $\quad 4^{1000}$

C $\quad \dfrac{4}{1000!}$

D $\quad \dfrac{4}{1001!}$

E $\quad \dfrac{2^{1000}}{1000!}$
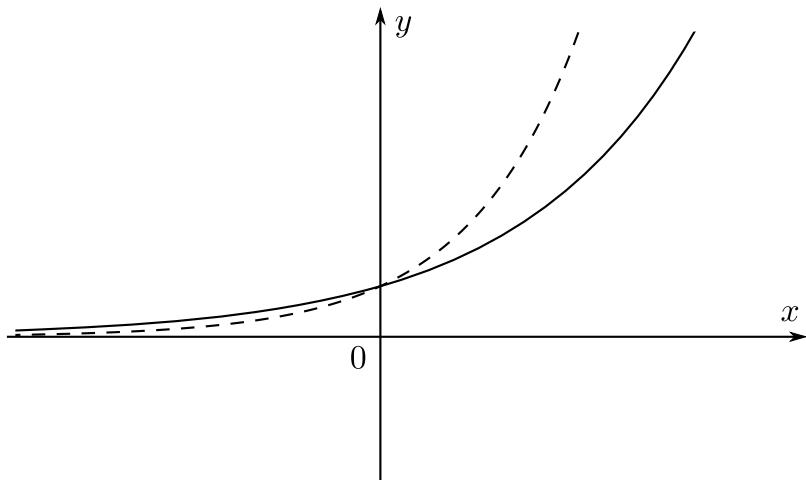
F $\quad \dfrac{4^{1000}}{1000!}$

G $\quad \dfrac{2^{1000}}{1001!}$

H $\quad \dfrac{4^{1000}}{1001!}$

7 The graphs of two functions are shown here:

- $y = a^x$ is shown with a solid line, where a is a positive real number
- $y = f(x)$ is shown with a dashed line

Which of the following statements (**1**, **2**, **3**, **4**) could be true?

1 $f(x) = b^x$ for some $b > a$

2 $f(x) = b^x$ for some $b < a$

3 $f(x) = a^{kx}$ for some $k > 1$

4 $f(x) = a^{kx}$ for some $k < 1$

A   **1** only

B   **2** only

C   **3** only

D   **4** only

E   **1** and **3** only

F   **1** and **4** only

G   **2** and **3** only

H   **2** and **4** only

8 Which one of the following numbers is smallest in value?

A $\log_2 7$

B $(2^{-3} + 2^{-2})^{-1}$

C $2^{(\pi/3)}$

D $\dfrac{1}{4(\sqrt{2}-1)^3}$

E $4\sin^2\left(\dfrac{\pi}{4}\right)$

9 Consider the following attempt to prove this true theorem:

Theorem: $a^3 + b^3 = c^3$ has no solutions with a, b and c positive integers.

Attempted proof:
Suppose that there are positive integers a, b and c such that $a^3 + b^3 = c^3$.

I   We have $a^3 = c^3 - b^3$.

II   Hence $a^3 = (c - b)(c^2 + cb + b^2)$.

III   It follows that $a = c - b$ and $a^2 = c^2 + cb + b^2$, since $a \leqslant a^2$ and $c - b \leqslant c^2 + cb + b^2$.

IV   Eliminating a, we have $(c - b)^2 = c^2 + cb + b^2$.

V   Multiplying out, we have $c^2 - 2cb + b^2 = c^2 + cb + b^2$.

VI   Hence $3cb = 0$ so one of b and c is zero.

But this is a contradiction to the original assumption that all of a, b and c are positive. It follows that the equation has no solutions.

Comment on this proof by choosing one of the following options:

A The proof is correct

B The proof is incorrect and the first mistake occurs on line I.

C The proof is incorrect and the first mistake occurs on line II.

D The proof is incorrect and the first mistake occurs on line III.

E The proof is incorrect and the first mistake occurs on line IV.

F The proof is incorrect and the first mistake occurs on line V.

G The proof is incorrect and the first mistake occurs on line VI.

10 $f(x)$ is a function defined for all real values of x.

Which one of the following is a **sufficient** condition for $\int_1^3 f(x)\,dx = 0$?

A $f(2) = 0$

B $f(1) = f(3) = 0$

C $f(-x) = -f(x)$ for all x

D $f(x+2) = -f(2-x)$ for all x

E $f(x-2) = -f(2-x)$ for all x

11 The function $f(x)$ is increasing and $f(0) = 0$.

The positive constants a and b are such that $a < b$.
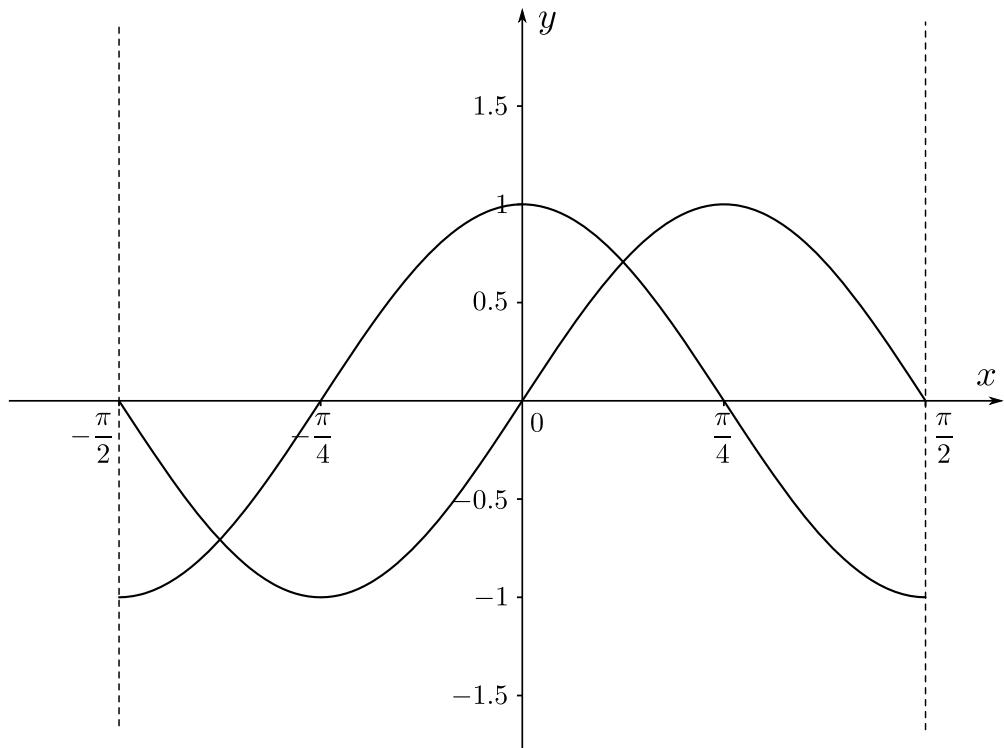
The area of the region enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is denoted by R.

The function $g(x)$ is defined by $g(x) = f(x) + 2f(b)$.

Which of the following is an expression for the area enclosed by the curve $y = g(x)$, the x-axis and the lines $x = a$ and $x = b$?

A $R + (b - a)f(b)$

B $R + 2(b - a)f(b)$

C $R + 2f(b) - f(a)$

D $R + 2f(b)$

E $R + (f(b))^2$

F $R + (f(b))^2 - (f(a))^2$

G $R + 2(f(b) - f(a))f(b)$

12   The diagram shows the graphs of $y = \sin 2x$ and $y = \cos 2x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Which one of the following is **not** true?

A $\cos 2x < \sin 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

B $\cos 2x < \tan x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

C $\sin 2x < \cos 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

D $\sin 2x < \tan x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

E $\tan x < \sin 2x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

F $\tan x < \cos 2x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
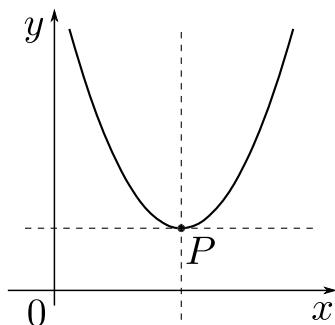
13 The positive real numbers $a \times 10^{-3}$, $b \times 10^{-2}$ and $c \times 10^{-1}$ are each in standard form, and

$$(a \times 10^{-3}) + (b \times 10^{-2}) = (c \times 10^{-1}).$$
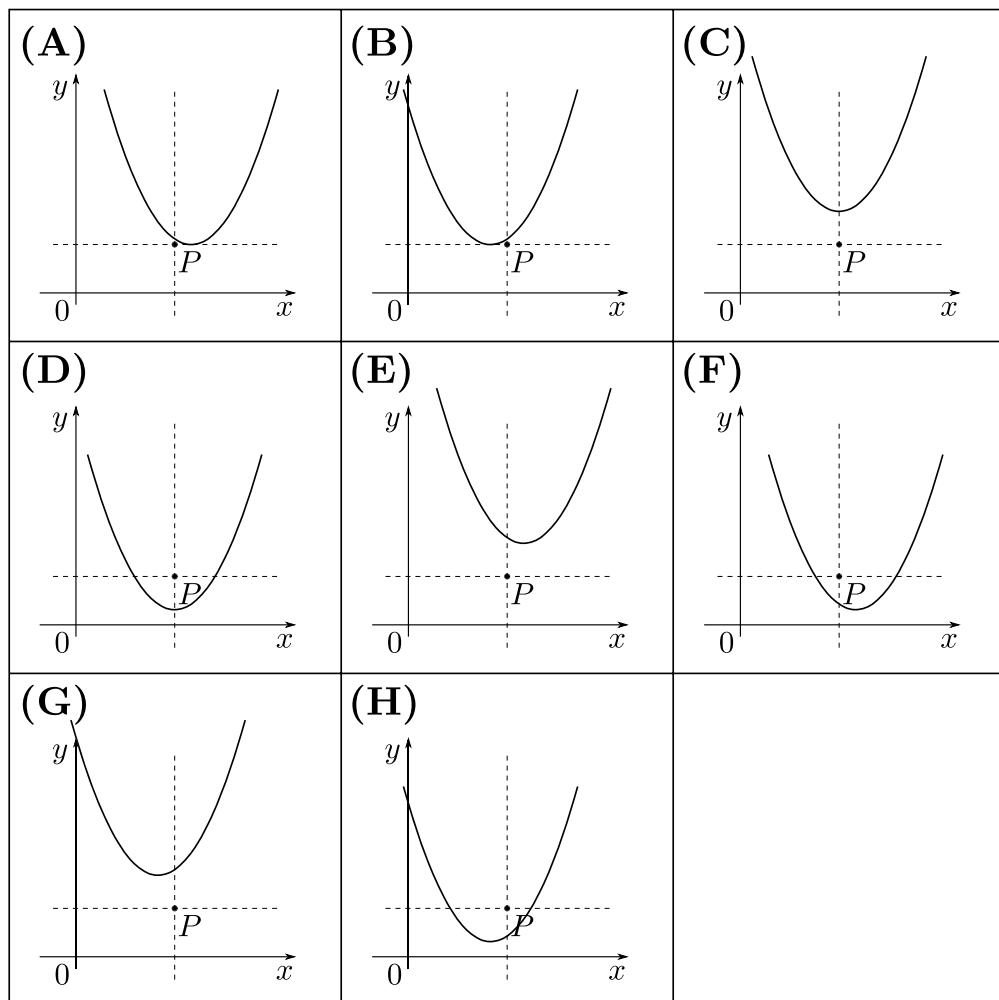
Which of the following statements (I, II, III, IV) **must** be true?

    I $a > 9$

    II $b > 9$

    III $a < c$

    IV $b < c$

A I only

B II only

C I and II only

**D** I and III only

**E** I and IV only

F II and III only

G II and IV only

H I, II, III and IV

14   The diagram below shows the graph of $y = x^2 - 2bx + c$. The vertex of this graph is at the point P.

Which one of the following could be the graph of $y = x^2 - 2Bx + c$, where $B > b$?

15 The function f is defined on the positive integers as follows:

$$f(1) = 5, \text{ and for } n \geqslant 1: \quad \begin{array}{ll} f(n+1) = 3f(n) + 1 & \text{if } f(n) \text{ is odd} \\ f(n+1) = \frac{1}{2}f(n) & \text{if } f(n) \text{ is even} \end{array}$$

The function g is defined on the positive integers as follows:

$$g(1) = 3, \text{ and for } n \geqslant 1: \quad \begin{array}{ll} g(n+1) = g(n) + 5 & \text{if } g(n) \text{ is odd} \\ g(n+1) = \frac{1}{2}g(n) & \text{if } g(n) \text{ is even} \end{array}$$

What is the value of $f(1000) - g(1000)$?

A -6

B -5

C 1

D 2

E 4

F 8

16   Consider the following statement:

>   (∗) **If** $f(x)$ is an integer for every integer x, **then** $f'(x)$ is an integer for every integer x.

Which one of the following is a **counterexample** to (∗)?

A $f(x) = \dfrac{x^3 + x + 1}{4}$

B $f(x) = \dfrac{x^4 + x^2 + x}{2}$

C $f(x) = \dfrac{x^4 + x^3 + x^2 + x}{2}$

D $f(x) = \dfrac{x^4 + 2x^3 + x^2}{4}$

17 A set S of whole numbers is called *stapled* **if and only if** for every whole number a which is in S there exists a prime factor of a which divides at least one other number in S.

Let T be a set of whole numbers. Which of the following is true **if and only if** T is **not** stapled?

A For every number a which is in T, there is no prime factor of a which divides every other number in T.

B For every number a which is in T, there is no prime factor of a which divides at least one other number in T.

C For every number a which is in T, there is a prime factor of a which does not divide any other number in T.

D For every number a which is in T, there is a prime factor of a which does not divide at least one other number in T.

E There exists a number a which is in T such that there is no prime factor of a which divides every other number in T.

F There exists a number a which is in T such that there is no prime factor of a which divides at least one other number in T.

G There exists a number a which is in T such that there is a prime factor of a which does not divide any other number in T.

H There exists a number a which is in T such that there is a prime factor of a which does not divide at least one other number in T.

18 Consider the following problem:

Solve the inequality $\left(\dfrac{1}{4}\right)^{n} < \left(\dfrac{1}{32}\right)^{10}$, where n is a positive integer.

A student produces the following argument:

$$\left(\frac{1}{4}\right)^{n} < \left(\frac{1}{32}\right)^{10}$$

$\}$ (I)

$$\log_{\frac{1}{2}}\left(\frac{1}{4}\right)^{n} < \log_{\frac{1}{2}}\left(\frac{1}{32}\right)^{10}$$

$\}$ (II)

$$n\log_{\frac{1}{2}}\left(\frac{1}{4}\right) < 10\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$$

$\}$ (III)

$$n < \frac{10\log_{\frac{1}{2}}\left(\frac{1}{32}\right)}{\log_{\frac{1}{2}}\left(\frac{1}{4}\right)}$$

$\}$ (IV)

$$n < \frac{10 \times 5}{2} = 25$$

$\}$ (V)

$$1 \leqslant n \leqslant 24$$

Which step (if any) in the argument is invalid?

A There are no invalid steps; the argument is correct

B Only step (I) is invalid; the rest are correct

C Only step (II) is invalid; the rest are correct

D Only step (III) is invalid; the rest are correct

E Only step (IV) is invalid; the rest are correct

F Only step (V) is invalid; the rest are correct

19 Which one of the following is a **sufficient** condition for the equation
$x^3 - 3x^2 + a = 0$, where a is a constant, to have exactly one real root?

A $a > 0$

B $a \leqslant 0$

C $a \geqslant 4$

D $a < 4$

E $|a| > 4$

F $|a| \leqslant 4$

G $a = \frac{9}{4}$

H $|a| = \frac{3}{2}$

20 I have forgotten my 5-character computer password, but I know that it consists of the letters `a`, `b`, `c`, `d`, `e` in some order. When I enter a potential password into the computer, it tells me exactly how many of the letters are in the correct position.

When I enter `abcde`, it tells me that none of the letters are in the correct position. The same happens when I enter `cdbea` and `eadbc`.

Using the best strategy, how many **further** attempts must I make in order to **guarantee** that I can **deduce** the correct password?

A None: I can deduce it immediately

B One

C Two

D Three

E More than three

END OF TEST

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