PC1101 Cheatsheet, by randomwish

https://github.com/randomwish/schoolNotes

2. Monochromatic Waves

Monochromatic: single color $\rightarrow$ wave has a single frequency.

- Frequency v: units in Hz.
- Phase φ: only matters if there are two or more waves (relative phase).
- Pulsation $\omega = 2\pi v$: units in rad/s.
- Time dependence: $\cos(2\pi vt + \varphi) \equiv \cos(\omega t + \varphi)$.
- Conversion: $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$.

Space Dependence

1D Monochromatic Waves

We represent space as $x - ct$, where t represents time coordinates. Space and time dependence of a 1D monochromatic wave:

$$\cos(kx - \omega t + \varphi),$$

where k is the wave number:

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

The minus sign indicates that the wave moves to the right (positive direction).

3D Monochromatic Waves

- **Plane wave**: $\cos(\vec{k} \cdot \vec{x} - \omega t + \varphi)$.
  - Wavefronts are planes; propagates eternally from $-\infty$ to ∞ in the direction $\hat{e_k}$.
  - The space vibrates in unison in the perpendicular direction to $\hat{e_k}$.
- **Spherical wave**:

  $$\frac{1}{r(\vec{x})} \cos(kr(\vec{x}) - \omega t + \varphi),$$

  where k is the wave number and $r(\vec{x})$ is the distance of $\vec{x}$ from the center $\vec{x_0}$ where the wave originates.

  $$r(\vec{x}) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

- wave fronts are surfaces of constant r, or spheres centered at $\vec{x_0}$

Electric Field Representation

To represent light as an electromagnetic wave, we need to add:
1. Amplitude E_0: Gives the wave units of electric field (V/m) and describes how large the field is.
2. Polarization: Direction of oscillation of the electric field vector.
For a monochromatic plane wave propagating along $\hat{e_z}$:

Linear Polarization

$$\vec{E}(\vec{x}, t) = E_0 \hat{e}_\theta \cos(kz - \omega t)$$

Where $\hat{e}_\theta \equiv \cos\theta \hat{e}_x + \sin\theta \hat{e}_y$

Circular Polarization

$$\vec{E}(\vec{x}, t) = E_0[\cos(kz - \omega t)\hat{e}_\theta \pm \sin(kz - \omega t)\hat{e}_{\theta+\pi/2}]$$

+ sign: right-handed polarization - sign: left-handed polarization

Elliptical Polarization

Most general form:

$$\vec{E}(\vec{x}, t) = E_0[\cos(kz - \omega t)\hat{e}_\theta + \cos(kz - \omega t + \phi)\hat{e}_{\theta+\pi/2}]$$

Intensity

Intensity at position $\vec{x}$ and time t:

$$I(\vec{x}, t) = \vec{E}(\vec{x}, t) \cdot \vec{E}(\vec{x}, t) \equiv ||\vec{E}(\vec{x}, t)||^2$$

Energy density in vacuum:

$$u(\vec{x}, t) = \frac{1}{2}\epsilon_0(||\vec{E}(\vec{x}, t)||^2 + c^2||\vec{B}(\vec{x}, t)||^2) \equiv \epsilon_0 I(\vec{x}, t)$$

Complex Notation

Physicists often use complex numbers for calculations:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Advantages: - Adding phase is multiplication:
$e^{i(kx-\omega t)} \rightarrow e^{i\phi}e^{i(kx-\omega t)}$ - Simplifies many calculations

Note: Take real part before computing intensities, since $|e^{i\theta}|^2 = 1$

Superposition

Linear Superposition

For two sources producing fields $\vec{E}_1(\vec{x}, t)$ and $\vec{E}_2(\vec{x}, t)$:

$$\vec{E}(\vec{x}, t) = \vec{E}_1(\vec{x}, t) + \vec{E}_2(\vec{x}, t)$$

This is valid in normal media far from saturation.

Huygens' Principle

Each point on a wavefront is a source of secondary waves Secondary waves propagate in all directions with the same speed as the primary wave The new wavefront is the tangent surface to all secondary wavelets

Diffraction by an Aperture

Diffraction occurs when waves encounter an obstacle or opening. It's the apparent bending of waves around small obstacles or spreading out past small openings.

Fraunhofer Diffraction (2D)

Consider a monochromatic plane wave impinging on a slit of width 2a along the x-direction. Setup:

Wave propagates in z-direction Slit width: 2a Observation point: $\vec{x} = (R\sin\theta, R\cos\theta)$

Using Huygens' principle, we sum the contributions from all points in the slit:

$$A(\vec{x}, t) = \alpha \int_{-a}^{a} dx_0 \frac{1}{((\vec{x} - \vec{x_0})^2)^{1/4}} \cos(k\sqrt{(\vec{x} - \vec{x_0})^2} - \omega t)]$$

Where α is the infinitesimal amplitude of each Huygens wave.

Far-field Approximation

In the far-field (Fraunhofer) limit where $R \gg a$:

$$A(\vec{x}, t) \approx \frac{\alpha}{\sqrt{R}} \int_{-a}^{a} dx_0 \cos(kR - kx_0\sin\theta - \omega t)$$

Solving this integral leads to:

$$A(\vec{x}, t) = \frac{2a\alpha}{\sqrt{R}} \cos(kR - \omega t)\mathrm{sinc}(ka\sin\theta)$$

Where $\mathrm{sinc}(u) = \frac{\sin(u)}{u}$

Interpretation

For $ka \ll 1$ (slit much smaller than wavelength):

$\mathrm{sinc}(ka\sin\theta) \approx 1$ for all θ Behaves like a point source

For $ka \gg 1$ (slit much larger than wavelength):

Intensity is significant only when $ka\sin\theta \ll \pi$ Light propagates mostly in the forward direction

Extension to 3D

For a circular aperture in 3D, the diffraction pattern is described by a Bessel function:

$$A(\vec{x}, t) \propto J_1(ka\sin\theta)$$

Where J_1 is the Bessel function of the first kind of order 1.

Resolution and Rayleigh Criterion

The Rayleigh criterion for resolution states that two point sources are just resolvable when the central maximum of one diffraction pattern coincides with the first minimum of the other. For a circular aperture:

$$\theta_{min} = 1.22\frac{\lambda}{D}$$

Where D is the diameter of the aperture. This criterion sets the fundamental limit for the angular resolution of optical instruments like telescopes and microscopes.

3. Propagation of light in media

Basic idea: light slows down when propagating in dense media

$$c \to v = \frac{c}{m}$$

, where c is speed of light in vacuum, v is the speed of light in the medium and n is the refractive index. Ides is because light impinges on atoms, makes electrons vibrate, vibration of the charges in turn emits light, wavelength becomes shorter as there is no change to the frequency

3.1 Dense Media: Refractive Index

The Model: Driven Damped Oscillator

For our simple model, we consider that the electrons are "attached" to the atom (the wave will not ionize them away) and model this with a spring of fixed resonance frequency ω_0. We neglect the charged character and focus on the spring's reaction to a sinusoidal drive.

Setting the Problem

We use Newton's second law $ma = F$, where the total force F consists of three forces:

- The proper force of an oscillator (spring):

  $$F_{\text{osc}} = -k_0 x = -m\omega_0^2 x$$

  where k_0 is the spring constant, and $\omega_0 = \sqrt{\frac{k_0}{m}}$ is the natural pulsation.

- A damping term, representing a negative force proportional to speed:

  $$F_{\text{damp}} = -m\gamma v = -m\gamma \frac{dx}{dt}$$

  where $\gamma = \frac{1}{\tau}$ is the relaxation rate, and τ is the relaxation time.

- The driving force due to the electric field at the position of the dipole:

  $$F_{\text{drive}} = qE(t) = F\cos(\omega t)$$

Newton's equation is:

$$m\frac{d^2x(t)}{dt^2} = -m\omega_0^2 x(t) - m\gamma\frac{dx(t)}{dt} + F\cos(\omega t) \tag{3.3}$$

Calculations

To solve this, we switch to complex numbers. We aim to solve the equation:

$$\frac{d^2}{dt^2}z(t) + \gamma\frac{d}{dt}z(t) + \omega_0^2 z(t) = F' e^{-i\omega t} \tag{3.4}$$

where $F' = \frac{F}{m}$ and $z(t) = A(\omega)e^{i(-\omega t+\phi(\omega))}$ is our Ansatz. Using the following identities:

$$\frac{d}{dt}z(t) = -i\omega z(t) \quad \text{and} \quad \frac{d^2}{dt^2}z(t) = -\omega^2 z(t)$$

we simplify the differential equation into:

$$A(\omega)e^{i\phi(\omega)}\left[\omega_0^2 - \omega^2 - i\gamma\omega\right] = F' \tag{3.6}$$

The amplitude $A(\omega)$ is given by:

$$A(\omega) = \frac{F}{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \tag{3.8}$$

The phase shift $\phi(\omega)$ is:

$$\tan\phi(\omega) = \frac{\gamma\omega}{\omega_0^2-\omega^2} \tag{3.9}$$

From One Atom to the Medium

In a dense medium, the total phase accumulated when a wave travels a distance x is:

$\phi(\omega, x) = \phi(\omega)\cdot\frac{x}{a}$ where a is the typical distance between atoms.

The refractive index can be expressed as:

$$n(\omega) - 1 = \frac{c}{a\omega}\phi(\omega) \tag{3.10}$$

Given that $n(\omega) \approx 1-3$ for typical media, we approximate the delay per atom as:

$$\phi(\omega) \approx \frac{\gamma\omega}{\omega_0^2-\omega^2} \tag{3.11}$$

Substituting into (3.10), we get:

$$n(\omega) \approx 1 + \frac{a}{c}\frac{\omega_0^2}{\gamma}\left(1+\frac{\omega^2}{\omega_0^2}\right) \tag{3.12}$$

Final Comparison: Sellmeier's Equation

For multiple resonances, the refractive index follows Sellmeier's equation:

$$n^2(\omega) = 1 + \sum_i \frac{B_i}{1-(\omega/\omega_i)^2} \tag{3.13}$$

For $\omega \ll \omega_1$, we approximate:

$$n^2(\omega) \approx 1 + B_1 + B_2 \quad \text{(plus small corrections in } \omega\text{)}.$$

For $\omega \gg \omega_2$, we find $n^2(\omega) \to 1$, meaning the medium becomes transparent for high frequencies.

To Go Further

In some media, the refractive index is complex, described as:

$$n(\omega) = n_r(\omega) + in_i(\omega) \tag{3.14}$$

where $n_i(\omega) \geq 0$ accounts for attenuation.

4.1.1 Physical Mechanisms at a Glance

There are two possible mechanisms for the emission of light: Motion of a charge and Change of internal energy levels.

4.1.2 Blackbody Radiation

A blackbody absorbs all incident electromagnetic radiation and emits radiation at thermal equilibrium with the characteristic shape:

$$u_\lambda(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right)-1} \tag{4.1}$$

where λ is the wavelength. The energy density per frequency is:

$$u_\nu(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right)-1} \tag{4.2}$$

The peak wavelength of emission follows Wien's law:

$$\lambda_{\text{peak}} \approx \frac{2898\ \mu\text{m}\cdot\text{K}}{T\ [K]} \tag{4.3}$$

4.1.3 Lasers

A laser beam, propagating along the z-axis, can be written as:

$$f(\mathbf{x}, t) \propto G(x,y)\cos(kz - \omega t + \phi(t)) \tag{4.4}$$

with the beam confinement along the transverse directions (x, y) typically represented as a Gaussian profile:

$$G(x,y) \propto \exp\left(-\frac{x^2+y^2}{4a^2}\right) \tag{4.6}$$

4.2 How Light is Detected

4.2.1 Mechanisms

Most light detectors, whether biological or artificial, are based on the photoelectric effect, which transforms light into an electric current. Detectors can operate in two regimes: single-photon, where we detect single photons, and proportional regime, where the photocurrent is proportional to the light intensity impinging on the detector.

4.2.2 Proportional Detectors and Time Resolution

The signal of a proportional detector is proportional to the intensity of the light and averaged over a time interval Δt, called the time resolution of the detector. The time resolution is much longer than the period of the light wave:

$$\Delta t \gg \frac{2\pi}{\omega} \tag{4.8}$$

Thus, the detected signal averages the wave over many oscillations. For a wave $\psi(x,t) = \alpha e^{i(kx-\omega t)}$, the detected signal is:

$$P_{\text{det}}(t) \propto |\alpha|^2 \tag{4.9}$$

4.3 Manipulating Light Beams: Linear Devices

4.3.1 Delays, Lenses, and Prisms

To introduce a phase delay on a beam, let it pass through a material with refractive index $n > 1$. The phase shift is given by:

$$\phi(\omega, L) = 2\pi\frac{(n(\omega)-1)L}{\lambda} \tag{4.12}$$

where L is the thickness of the material. This principle is used in the design of lenses and prisms, which manipulate light based on the frequency dependence of n.

4.3.2 Beam-Splitters

Beam-splitters are symmetric devices that split incoming light into transmitted and reflected components. If T is the transmittivity and $R = 1 - T$ is the reflectivity, we have:

$$t = \sqrt{T}, \quad r = \sqrt{R}$$

The real version of a beam-splitter transformation is:

$$\psi_x \to t\psi_x + r\psi_y, \quad \psi_y \to t\psi_y - r\psi_x \tag{4.13}$$

This conserves the total intensity of the beam.

4.3.3 Birefringent Devices: Manipulating Polarisation

Birefringent materials have different refractive indices for different polarizations, transforming the polarization of a beam as:

$$\psi_x[\alpha\hat{e}_o + \beta\hat{e}_e] \to \psi_x\left[\alpha e^{i\phi_o}\hat{e}_o + \beta e^{i\phi_e}\hat{e}_e\right] \tag{4.15}$$

where ϕ_o and ϕ_e are the phase shifts for the ordinary and extraordinary axes, respectively.
Common birefringent devices include:

- **Half-wave plate ($\lambda/2$ plate):** Rotates the polarization by shifting the phase of one component by π:
$$\psi_x[\alpha\hat{e}_o + \beta\hat{e}_e] \to \psi_x[\alpha\hat{e}_o - \beta\hat{e}_e] \tag{4.18}$$
- **Quarter-wave plate ($\lambda/4$ plate):** Converts linearly polarized light to circularly polarized light by introducing a phase shift of $\frac{\pi}{2}$:
$$\psi_x[\alpha\hat{e}_o + \beta\hat{e}_e] \to \psi_x[\alpha\hat{e}_o - i\beta\hat{e}_e] \tag{4.19}$$

Interferometry

As seen in history, interference is what eventually convinced people that light is a wave. It has not ceased to play a crucial role since, both for fundamental physics and for applications. We shall see some examples here and in the later parts of this course.

5.1 Wavefront-Splitting Interferometry: Young's Double Slit

The setup used by Young is a milestone in the history of science, worth knowing well. A plane wave hits a screen with two apertures of size $2a$ at a distance $2D$ (distance between centers). We observe the resulting wave at a distance R, larger than the size and separation of the slits, and of course larger than the wavelength.

5.1.1 Two Point-Like Apertures

Using Huygens' principle, each aperture creates a circular wave. At position $\mathbf{x} = (R\sin\theta, R\cos\theta)$, the total amplitude of the wave at time t is:

$$A(\mathbf{x}, t) \propto \frac{1}{\sqrt{2\pi R}} e^{i(kR-\omega t)}[e^{-ikD\sin\theta} + e^{ikD\sin\theta}]$$

which simplifies to:

$$A(\mathbf{x}, t) \propto \frac{\sqrt{2}}{\sqrt{2\pi R}} e^{i(kR-\omega t)}\cos(kD\sin\theta)$$

Constructive interference occurs when $kD\sin\theta = m\pi$, while destructive interference occurs when $kD\sin\theta = \frac{\pi}{2} + m\pi$, where $m \in \mathbb{Z}$.

5.1.2 Two Finite Slits

When we account for the finite aperture $2a$, we integrate over each slit. The total amplitude becomes:

$$A(\mathbf{x}, t) = \frac{4a\alpha}{\sqrt{2\pi R}} e^{i(kR-\omega t)}\,\mathrm{sinc}(ka\sin\theta)\cos(kD\sin\theta)$$

For comparison:

$$\text{One slit: } 2a\alpha\ \mathrm{sinc}(ka\sin\theta)$$
$$\text{Two slits: } 4a\alpha\ \mathrm{sinc}(ka\sin\theta)\cos(kD\sin\theta)$$

The interference term $\cos(kD\sin\theta)$ is multiplied by the diffraction factor $\mathrm{sinc}(ka\sin\theta)$ from each slit.

5.2 Amplitude-Splitting Interferometry

In amplitude-splitting interferometers, the beam of light is split and recombined at different times. These devices measure relative phases.

5.2.1 The Mach-Zehnder Interferometer

In the Mach-Zehnder interferometer, the phase delay ϕ due to the difference in optical path Δl or time delay $\Delta\tau$ is given by:

$$\phi = 2\pi\frac{\Delta l}{\lambda} = \omega\Delta\tau$$

For two balanced beam splitters ($t = r = \frac{1}{\sqrt{2}}$), we compute the result:

$$\psi_x \to \frac{1}{\sqrt{2}}[\psi_x + i\psi_y], \quad \psi_y \to \frac{1}{\sqrt{2}}[e^{i\phi}\psi_x + i\psi_y]$$

The final outputs at the detectors are:

$$P_{\mathrm{det},x} \propto \cos^2\frac{\phi}{2}, \quad P_{\mathrm{det},y} \propto \sin^2\frac{\phi}{2}$$

This result encodes the time delay and allows one to retrieve the wavelength using the relation for Δl.

5.2.2 The Michelson Interferometer

The Michelson interferometer operates similarly to the Mach-Zehnder interferometer, but the delay is created by moving a mirror. The additional optical length is given by:

$$\Delta l = 2\delta x$$

Michelson's interferometer has been used in several historical experiments, including:

- The Michelson-Morley experiment, proving the constancy of the speed of light.
- LIGO (Laser Interferometer Gravitational-Wave Observatory), which detected gravitational waves in 2016.

5.2.3 Interferometric Precision

Once the wavelength of a source is calibrated using known delays Δl (etalons), interferometry can measure distances Δl that are fractions of the wavelength. This is the principle behind interferometric precision, used in the Michelson-Morley experiment and LIGO measurements.

More about Waves

So far, we have worked with monochromatic waves and superposed such waves at the same frequency coming from different directions. In this last chapter on waves, we have a look at other waves and a glimpse of them all. This allows us to introduce important notions: different velocities, coherence length, and time. Let us get started with examples.

Examples of Non-monochromatic 1D Waves

Superposing Two Frequencies

Consider the superposition of two monochromatic plane waves (in 1D) at different frequencies. For simplicity, assume they have the same amplitude:

$$\psi(x,t) = A\cos(k_1 x - \omega_1 t + \phi_1) + A\cos(k_2 x - \omega_2 t + \phi_2)$$

This can be rewritten as:

$$\psi(x,t) = 2A\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t + \frac{\phi_1+\phi_2}{2}\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega}{2}\right.$$

where we used the identity $\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$.

This expression describes a wave at the frequency $\frac{\omega_1+\omega_2}{2}$, modulated by a wave of lower frequency $\frac{\omega_1-\omega_2}{2}$. This is known as frequency modulation or beat.

Gaussian Wave Packet in Vacuum

Next, consider a Gaussian wave packet in 1D. The equation, pending normalization, is:

$$\psi(x,t) \propto e^{-\kappa^2(x-ct)^2/2}\cos(k_0 x - \omega_0 t) \quad [\text{for } |k_0| \gg \kappa]$$

with $\omega_0 = c|k_0|$. Sometimes it's more convenient to use the complex version:

$$\psi(x,t) \propto e^{-\kappa^2(x-ct)^2/2}e^{i(k_0 x - \omega_0 t)} \quad [\text{for } |k_0| \gg \kappa]$$

Wave Packets in Media

Consider how the Gaussian packet propagates in a medium with a refractive index $n(\omega)$. Since n is a function of ω, its effect on monochromatic waves is clear from:

$$\psi(x,t) = \int_{-\infty}^{\infty}\tilde{\psi}(k)e^{i(n(\omega)kx - \omega t)}\,dk$$

If the frequency spectrum is narrow, you can approximate $n(\omega) \approx n_0$ over the spectrum, and the packet moves at speed $\frac{c}{n_0}$, for example:

$$\psi(x,t) \propto e^{-(n_0\kappa)^2(x-(c/n_0)t)^2/2}e^{i(n_0 k_0 x - \omega_0 t)}$$

On Velocities

The dispersion relation describes the relation between frequency ω and wave-vector $\mathbf{k}$. For light, it is given by:

$$|\mathbf{k}(\omega)| = \frac{n(\omega)\omega}{c}$$

Based on this relation, two velocities can be defined:

Phase Velocity The phase velocity is defined for each k as:

$$v_\phi(k) = \frac{\omega}{k}$$

This is the velocity at which a fixed point of the cosine wave moves.

Group Velocity

The group velocity is defined as:

$$v_g(k) = \frac{d\omega(k)}{dk}$$

This is the velocity of the envelope of the wave packet.

Notion of Coherence

Coherence captures the stability of the relationship between two points on a wave over time. We can define two types of coherence:

Longitudinal Coherence

Longitudinal coherence refers to the relation between $\psi(x,t)$ and $\psi(x,t+\Delta\tau)$.

Transverse Coherence

Transverse coherence refers to the relation between $\psi(x_1,t)$ and $\psi(x_2,t)$, as seen in Young's double-slit experiment.

Longitudinal Coherence of a Wave Packet

For a wave packet, coherence is measured by its coherence length l_c, or equivalently, the coherence time $\tau_c = \frac{l_c}{c}$. For interference effects, the input waves from each direction must arrive at the same time at the beam splitter.

The output signal difference is given by:

$$\Delta P \propto \int_t^{t+\Delta t} \mathrm{Re}\left(\Psi(t)^*\Psi(t-\Delta\tau)\right) dt$$

The coherence time τ_c can be understood as the value of $\Delta\tau$ for which the interference term disappears.

Longitudinal Coherence of a Laser

For a laser, the temporal dependence is modeled as:

$$\Psi(t) = e^{-i\omega_0 t + \phi(t)}$$

where $\phi(t)$ is a slowly varying phase. The detected power is given by:

$$P_{\mathrm{det},x}(t) \propto 2 + 2\mathrm{Re}\left(e^{-i\omega_0\Delta\tau}\int_t^{t+\Delta t} e^{i(\phi(t-\Delta\tau)-\phi(t))}dt\right)$$

Transverse Coherence

Transverse coherence is related to setups like Young's double-slit experiment. If the wavefront that impinges on the slits has transverse phase fluctuations, the interference term $\cos(kD\sin\theta)$ will no longer be visible, allowing us to assess the wavefront's stability.

$$A(\mathbf{x},t) \propto \frac{1}{\sqrt{2\pi R}} e^{i(kR-\omega t)}\left(e^{-i(kD\sin\theta+\phi(D,t))} + e^{i(kD\sin\theta+\phi(-D,t))}\right)$$

Interference disappears when the phase fluctuation is large enough that the coherence length is exceeded.

Identities and Notation

Trigonometric Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x\cos y}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x\cos y}$$

Notation and Meaning

- $\sin x, \cos x, \tan x$: The sine, cosine, and tangent of angle x (similar for y). - These identities are used to express sums and differences of trigonometric functions in terms of products, simplifying calculations in trigonometry and calculus.

Euler's Formula and Complex Conjugates

$$(e^{i\theta})^* = e^{-i\theta}$$
$$e^{ix} = \cos x + i\sin x$$
$$|z|^2 = zz^*$$
$$i = e^{i\frac{\pi}{2}} \quad \left(\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1\right)$$

Notation and Meaning

- $e^{i\theta}$: Euler's formula relates complex exponentials to trigonometric functions. - z^*: The complex conjugate of z. For a complex number $z = x+iy$, the conjugate is $x-iy$. - i: The imaginary unit, defined as $i^2 = -1$, can also be expressed as $e^{i\pi/2}$ due to Euler's identity. - $|z|^2$: The modulus (magnitude) of a complex number, equal to x^2+y^2 for $z = x+iy$.

Transformation of Random Variables

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dy}{dx}\right|^{-1}$$

Notation and Meaning

- $f_X(x)$, $f_Y(y)$: The probability density functions (PDFs) of random variables X and Y. - This is the formula used to find the PDF of a transformed random variable, where $Y = g(X)$. The formula accounts for how the change of variables affects the distribution.

Sound Intensity and Decibel Levels

$$\frac{P'}{P_0} = 10^{-\alpha d/10}$$

Notation and Meaning

- P' and P_0: Sound intensity at two points. - α: The attenuation coefficient of the medium. - d: Distance through the medium. - This equation relates the sound intensity at two different points, considering the decay in intensity due to the distance traveled.

Taylor Expansion

The Taylor expansion of $\frac{1}{1-\delta}$, neglecting δ^2 and higher-order terms, is $1+\delta$.

Notation and Meaning

- δ: A small perturbation or variable. - This is a first-order approximation (linearization) of the function $\frac{1}{1-\delta}$, commonly used in physics and engineering to simplify expressions when δ is small.