

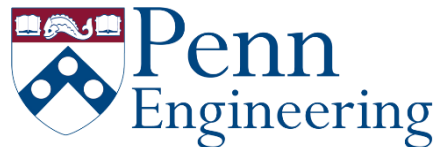
Robotics

Estimation and Learning
with Dan Lee

Week 1.

Gaussian Model Learning

1.3.2 MLE of Multivariate Gaussian



Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- Objective

$$\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}} = \arg \max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} p(\{\mathbf{x}_i\} | \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- Objective

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \max_{\boldsymbol{\mu}, \Sigma} p(\{\mathbf{x}_i\} | \boldsymbol{\mu}, \Sigma)$$



Assuming independence of observations,

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^N p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^N p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

(1) Take the log!

$$\arg \max \textit{likelihood} \leftrightarrow \arg \max \ln(\textit{likelihood})$$

$$\log(x_1 \times x_2 \times \cdots \times x_k) = \log(x_1) + \log(x_2) + \cdots + \log(x_k)$$

$$\arg \max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^N p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma) \quad \Rightarrow \quad \arg \max_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^N \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \max_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^N \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

(2) Gaussian!

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

$$\ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma) \longrightarrow \left\{ -\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) - \frac{1}{2} \ln |\Sigma| + c \right\}$$

$$c = -D/2 \ln(2\pi)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \max_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^N \left\{ -\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) - \frac{1}{2} \ln |\Sigma| + c \right\}$$



$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \min_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^N \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln |\Sigma| \right\}$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg \min_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^N \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln |\Sigma| \right\}$$

$J(\boldsymbol{\mu}, \Sigma)$

- At optimum,

$$\textcircled{1} \quad \frac{\partial J}{\partial \boldsymbol{\mu}} = \mathbf{0} \longrightarrow \hat{\boldsymbol{\mu}}$$

$$\textcircled{2} \quad \frac{\partial J(\hat{\boldsymbol{\mu}}, \Sigma)}{\partial \Sigma} = 0 \longrightarrow \hat{\Sigma}$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

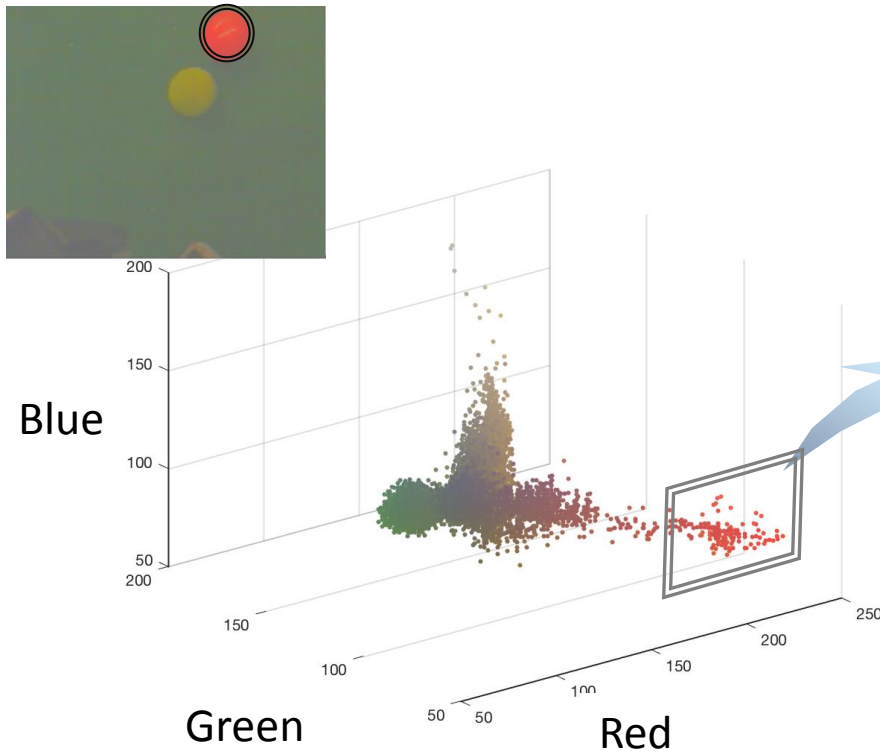
- In summary, we have

$$\hat{\boldsymbol{\mu}} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{i=1}^N (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^\top$$

Multi-dimension Distribution: Example

- Ball color in multi-channels



$$\hat{\mu} = [227 \quad 74]$$

$$\hat{\Sigma} = \begin{bmatrix} 150 & -48 \\ -48 & 46 \end{bmatrix}$$

