Exercise 1D.1

A bag contains 3 red balls and 2 blue balls. Define the events:

$A =$ get a red on the first draw

$B =$ get a blue on the second draw assuming the first ball is removed (but you don't know the color)

What is $P(B)$?

Hint: Use the Law of Total Probability

ANSWER: 0.4

$P(A) = 3/5 = 0.6$
$P(A^C) = 1 - P(A) = 1 - 0.6 = 0.4$
$P(B|A) = P(\textit{blue on 2nd draw}|\textit{first was red}) = \frac{\textit{2 blue left}}{\textit{4 balls left}} = 1/2 = 0.5$
$P(B|A^C) = P(\textit{blue on 2nd draw}|\textit{first was blue}) = \frac{\textit{1 blue left}}{\textit{4 balls left}} = 1/4 = 0.25$
$P(B) = P(B|A)P(A) + P(B|A^C)P(A^C) = 0.5 \times 0.6 + 0.25 \times 0.4 = 0.4$

Exercise 1D.2

There are three candidates in the election for governor of a certain state, Mr. Bluth, Ms. Fünke, and Mr. Loblaw. The biggest campaign issue is that of lowering property taxes. If Mr. Bluth is elected there is an 80% chance that he will lower taxes, if Ms. Fünke is elected there is a 10% chance she will lower taxes, and if Mr. Loblaw is elected there is a 55% chance he will lower taxes. The last election poll showed that there was a 60% chance that Mr. Bluth would win the election, a 20% chance that Ms. Fünke will win, and a 20% chance that Mr. Loblaw will win. What is the probability that property taxes will be lowered after the election?

Hint: Use the Law of Total Probability

ANSWER: 0.61

$P(\textit{lower taxes}|\textit{Bluth elected}) = 0.8$
$P(\textit{lower taxes}|\textit{Funke elected}) = 0.1$
$P(\textit{lower taxes}|\textit{Loblaw elected}) = 0.55$
$P(\textit{Bluth elected}) = 0.6$
$P(\textit{Funke elected}) = 0.2$
$P(\textit{Loblaw elected}) = 0.2$
By Law of Total Probability:
$P(\textit{lower taxes})$
$= P(\textit{lower}|\textit{Bluth}) \times P(\textit{Bluth}) + P(\textit{lower}|\textit{Funke}) \times P(\textit{Funke}) + P(\textit{lower}|\textit{Loblaw}) \times P(\textit{Loblaw})$
$= 0.8 \times 0.6 + 0.1 \times 0.2 + 0.55 \times 0.2$
$= 0.61$

Exercise 1D.3

A pregnancy test claims that it is "99% accurate" – if 100 pregnant woman took the test, 99 of them would get a positive result. If 100 non-pregnant women took the test, 10 would (falsely) test positive. Jane takes a pregnancy test and it comes up positive.

(a) Assume that the overall proportion of pregnant women in the population is 5%. What is the probability that Jane really is pregnant?

ANSWER: 0.34

Let A = *pregnant,* B = *positive test*
$P(A) = 0.05 \rightarrow P(A^C) = 1 - 0.05 = 0.95$
$P(B|A) = 99/100 = 0.99$
$P(B|A^C) = 10/100 = 0.1$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)} = \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.1 \times 0.95} = 0.34$$

(b) Assume that the overall proportion of pregnant women in the population is 50%. What is the probability that Jane really is pregnant?

ANSWER: 0.91

$P(A) = 0.5 \rightarrow P(A^C) = 1 - 0.5 = 0.5$ *(different from before)*
$P(B|A) = 99/100 = 0.99$ *(same as before)*
$P(B|A^C) = 10/100 = 0.1$ *(same as before)*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)} = \frac{0.99 \times 0.5}{0.99 \times 0.5 + 0.1 \times 0.5} = 0.91$$

Exercise 1D.4

Suppose on a particular multiple choice exam, a student has a 1 in 4 chance of getting the question right if she does not know the correct answer.

(a) If the student guesses on 15% of the questions, what percentage of all the questions on the test do you expect the student to answer correctly?

ANSWER: 0.8875

$P(\textit{correct}|\textit{guess}) = 1/4 = 0.25$
$P(\textit{correct}|\textit{not guess}) = 1$ *(knows the answer, assume correct)*
$P(\textit{guess}) = 0.15 \rightarrow P(\textit{not guess}) = 1 - 0.15 = 0.85$
Law of Total Probability:

$$\begin{aligned} P(\textit{correct}) &= P(\textit{correct}|\textit{guess}) \times P(\textit{guess}) + P(\textit{correct}|\textit{not guess}) \times P(\textit{not guess}) \\ &= 0.25 \times 0.15 + 1 \times 0.85 = 0.8875 \end{aligned}$$

(b) Given that the student got a particular answer correct, what is the probability that she actually knew the answer?

ANSWER: 0.96

Want $P(\textit{not guess}|\textit{correct}) \rightarrow$ *need Bayes' Rule to flip conditioning*

$$\begin{aligned} P(\textit{not guess}|\textit{correct}) &= \frac{P(\textit{correct}|\textit{not guess}) \times P(\textit{not guess})}{P(\textit{correct}|\textit{guess}) \times P(\textit{guess}) + P(\textit{correct}|\textit{not guess}) \times P(\textit{not guess})} \\ &= \frac{1 \times 0.85}{0.25 \times 0.15 + 1 \times 0.85} = \frac{0.85}{0.8875} = 0.96 \end{aligned}$$

(note denominator is answer from part (a))

Exercise 1D.5

Patients with newly diagnosed heart failure can be treated at three hospitals: A, B, and C. Suppose that for a particular year, 500 patients went to A, 200 went to B, and 300 went to C (no patient went to more than one hospital). The proportions of patients who survived for at least 6 months were 50%, 80%, and 75%, at hospitals A, B, and C, respectively.

(a) What is the probability that a randomly selected patient was treated at hospital A or B?

ANSWER: 0.7

$P(A) = 500/1000 = 0.5 \quad P(B) = 200/1000 = 0.2$
$P(A \cup B) = P(A) + P(B) = 0.5 + 0.2 = 0.7$ *(no intersection to subtract, A and B are disjoint)*

(b) What is the probability that a randomly selected patient was treated at hospitals A and B?

ANSWER: 0

$P(A \cap B) = 0$ *since A and B are disjoint*

(c) What is the probability that a patient who was treated at hospital B survived at least 6 months?

ANSWER: 0.8

$P(\textit{survive}|B) = 0.8$, *given in text of problem*

(d) What is the probability that a randomly selected patient was treated at hospital B and survived at least 6 months?

ANSWER: 0.16

$$P(B \cap \textit{survive}) = P(\textit{survive}|B) \times P(B) = 0.8 \times 0.2 = 0.16$$

(e) Sue and Joe both went to hospital B. What is the probability they both survived at least 6 months?

ANSWER: 0.64

$$\begin{aligned} P(\textit{Sue survived} \cap \textit{Joe survived}|B) &= P(\textit{Sue survived}|B) \times P(\textit{Joe survived}|B) \textit{ (independence)} \\ &= 0.8 \times 0.8 = 0.64 \end{aligned}$$

(f) Given that Danielle survived at least 6 months, what is the probability that she went to hospital B?

ANSWER: 0.252

$$\begin{aligned} P(B|\textit{survived}) &= \frac{P(\textit{survived}|B)P(B)}{P(\textit{survived})} \\ &= \frac{P(\textit{survived}|B)P(B)}{P(\textit{survived}|A)P(A) + P(\textit{survived}|B)P(B) + P(\textit{survived}|C)P(C)} \\ &= \frac{0.8 \times 0.2}{0.5 \times 0.5 + 0.8 \times 0.2 + 0.75 \times 0.3} = 0.252 \end{aligned}$$