Chapter 2

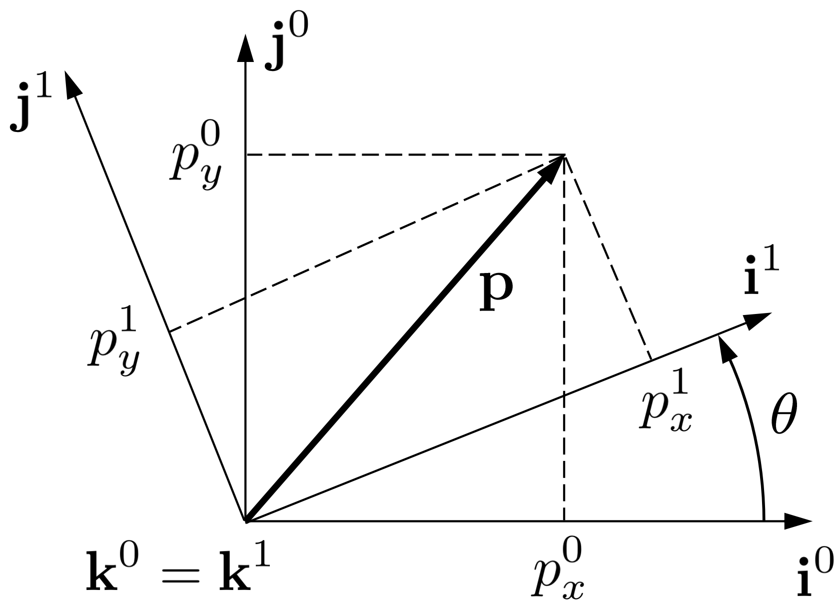
Coordinate Frames

Reference Frames

- In guidance and control of aircraft, reference frames are an essential idea
- Describe relative position and orientation of objects
  - Aircraft relative to direction of wind
  - Camera relative to aircraft
  - Aircraft relative to inertial frame
- Quantities most easily calculated or described in a specific reference frames:
  - Newton's law
  - Aircraft attitude
  - Aerodynamic forces/torques
  - Accelerometers, rate gyros
  - GPS
  - Mission requirements

Must know how to transform between different reference frames

Rotation of Reference Frame



$$\mathbf{p} = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

$$\mathbf{p} = p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1$$

$$p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1 = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

Take dot product of both sides – first with $\mathbf{i}^1$, then $\mathbf{j}^1$, then $\mathbf{k}^1$

$$\mathbf{p}^1 \triangleq \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \mathbf{i}^1 \cdot \mathbf{i}^0 & \mathbf{i}^1 \cdot \mathbf{j}^0 & \mathbf{i}^1 \cdot \mathbf{k}^0 \\ \mathbf{j}^1 \cdot \mathbf{i}^0 & \mathbf{j}^1 \cdot \mathbf{j}^0 & \mathbf{j}^1 \cdot \mathbf{k}^0 \\ \mathbf{k}^1 \cdot \mathbf{i}^0 & \mathbf{k}^1 \cdot \mathbf{j}^0 & \mathbf{k}^1 \cdot \mathbf{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

$$\mathbf{p}^1 = \mathcal{R}_0^1 \mathbf{p}^0 \quad \text{where} \quad \mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(rotation about $\mathbf{k}$ axis)

Rotation of Reference Frame

Right-handed rotation about $\mathbf{j}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Right-handed rotation about $\mathbf{i}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Orthonormal matrix properties:

P.1. $(\mathcal{R}_a^b)^{-1} = (\mathcal{R}_a^b)^\top = \mathcal{R}_b^a$

P.2. $\mathcal{R}_b^c \mathcal{R}_a^b = \mathcal{R}_a^c$

P.3. $\det\big(\mathcal{R}_a^b\big) = 1$

Rotation of Reference Frame: Alternate View

Note that

$$\mathbf{i}_0^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{j}_0^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{k}_0^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and that

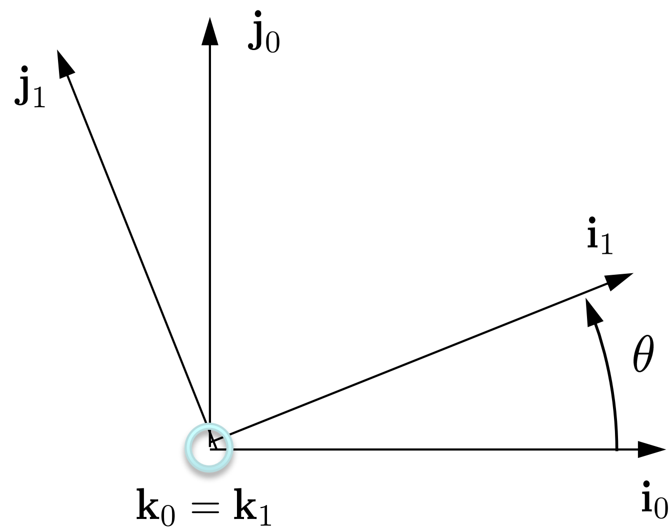
$$\mathbf{i}_1^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{j}_1^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{k}_1^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now express $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ in the 1-frame:

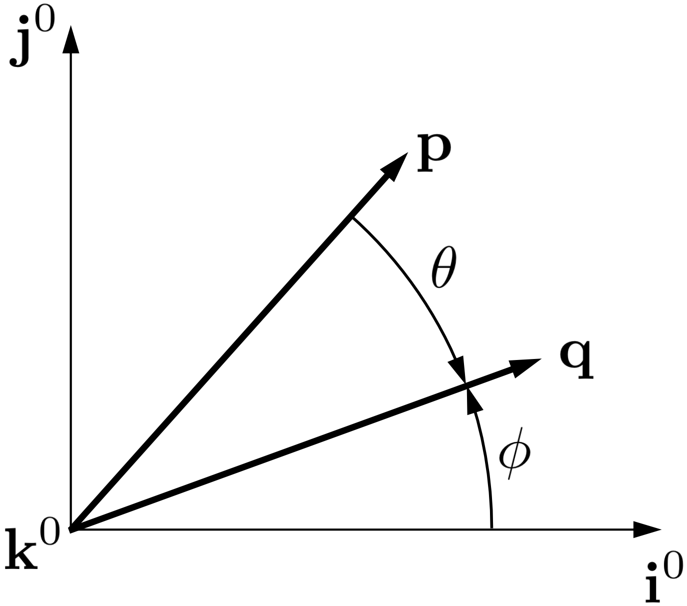
$$\mathbf{i}_0^1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}, \qquad \mathbf{j}_0^1 = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \qquad \mathbf{k}_0^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Therefore

$$\mathcal{R}_0^1 = \begin{pmatrix} \mathbf{i}_0^1 & \mathbf{j}_0^1 & \mathbf{k}_0^1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotation of a Vector



$\mathcal{R}_0^1$ can be interpreted as left-handed rotation of vector by angle θ

Let $p \triangleq |\mathbf{p}| = q \triangleq |\mathbf{q}|$, then

$$\mathbf{p} = \begin{pmatrix} p\cos(\theta+\phi) \\ p\sin(\theta+\phi) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} p\cos\theta\cos\phi - p\sin\theta\sin\phi \\ p\sin\theta\cos\phi + p\cos\theta\sin\phi \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p\cos\phi \\ p\sin\phi \\ 0 \end{pmatrix}$$

Define

$$\mathbf{q} = \begin{pmatrix} p\cos\phi \\ p\sin\phi \\ 0 \end{pmatrix}$$

then

$$\mathbf{p} = (\mathcal{R}_0^1)^\top \mathbf{q} \qquad \Longrightarrow \qquad \mathbf{q} = \mathcal{R}_0^1 \mathbf{p}$$

Passive vs. Active Rotation

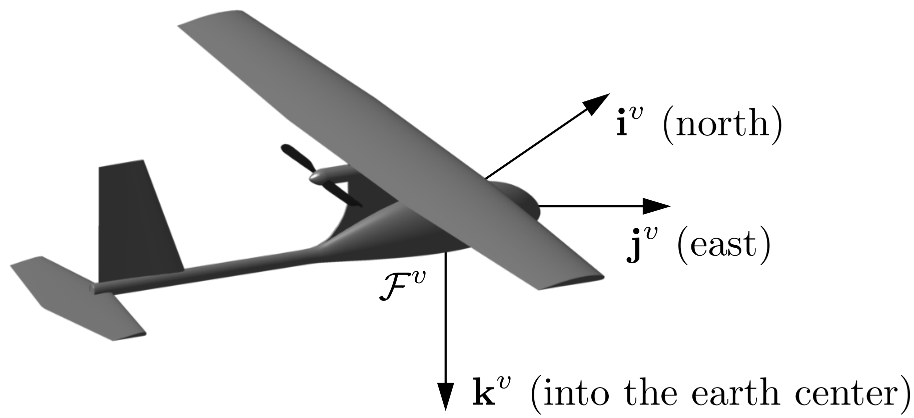
Passive Rotation/Transformation: The vector $\mathbf{p}$ is stationary and the coordinate frame changes

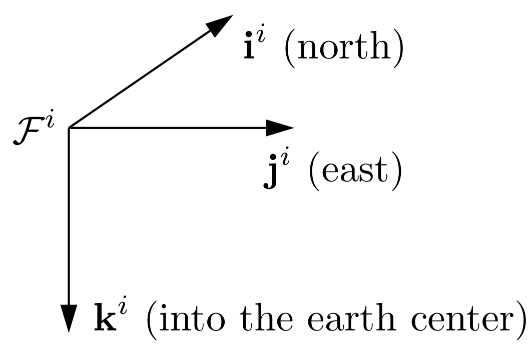
Active Rotation/Transformation: The coordinate frame remains fixed, but the vector is rotated

R_0^1 as defined earlier, represents a right-handed passive transformation, or a left-handed active transformation

If we stick with right-handed rotations, then R_0^1 represents a right-handed passive rotation, and $R_0^{1\top}$ represents a right-handed active rotation

Inertial Frame and Vehicle Frame





- Vehicle frame has same orientation as inertial frame
- Vehicle frame is fixed at cm of aircraft
- Inertial and vehicle frames are referred to as NED frames
- $N \longrightarrow x$, $E \longrightarrow y$, $D \longrightarrow z$

Euler Angles

- Need way to describe attitude of aircraft
- Common approach: Euler angles

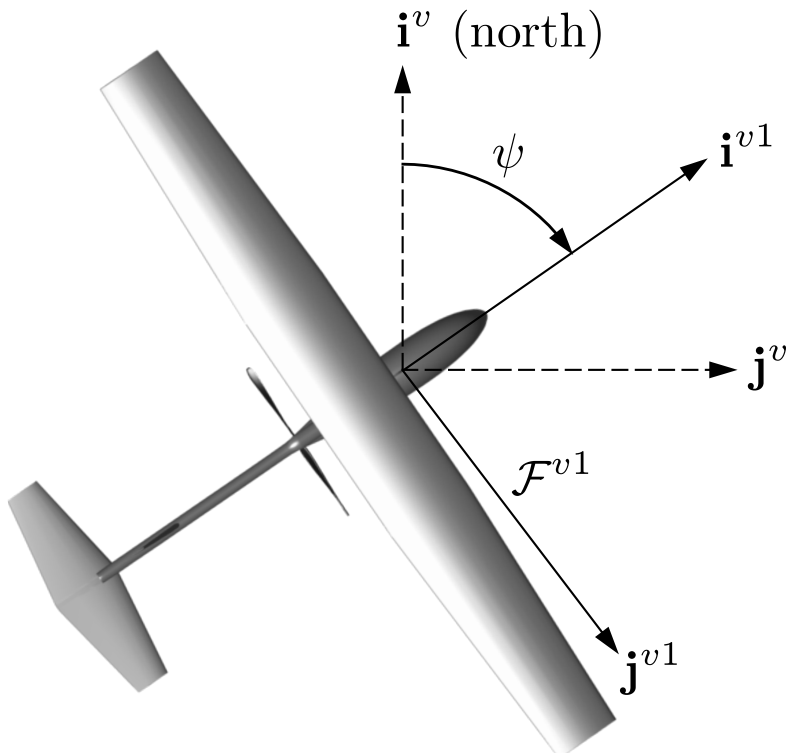
ψ: heading (yaw)

θ: elevation (pitch)

ϕ: bank (roll)

- Pro: Intuitive
- Con: Mathematical singularity
  - Quaternions are alternative for overcoming singularity

Vehicle-1 Frame



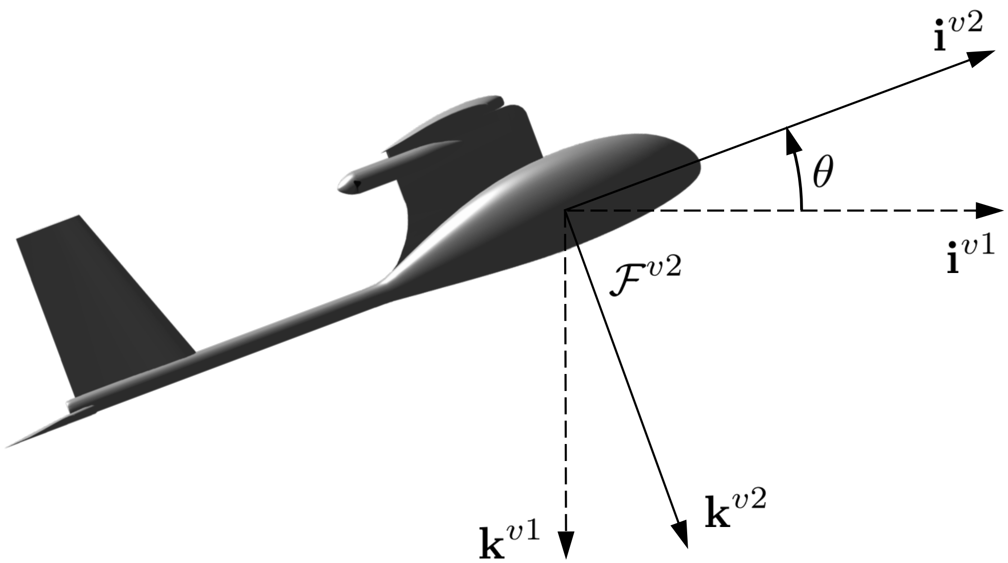
$$\mathcal{R}_v^{v1}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives

$$\mathbf{p}^{v1} = \mathcal{R}_v^{v1}(\psi)\mathbf{p}^v$$

where ψ is the heading

Vehicle-2 Frame



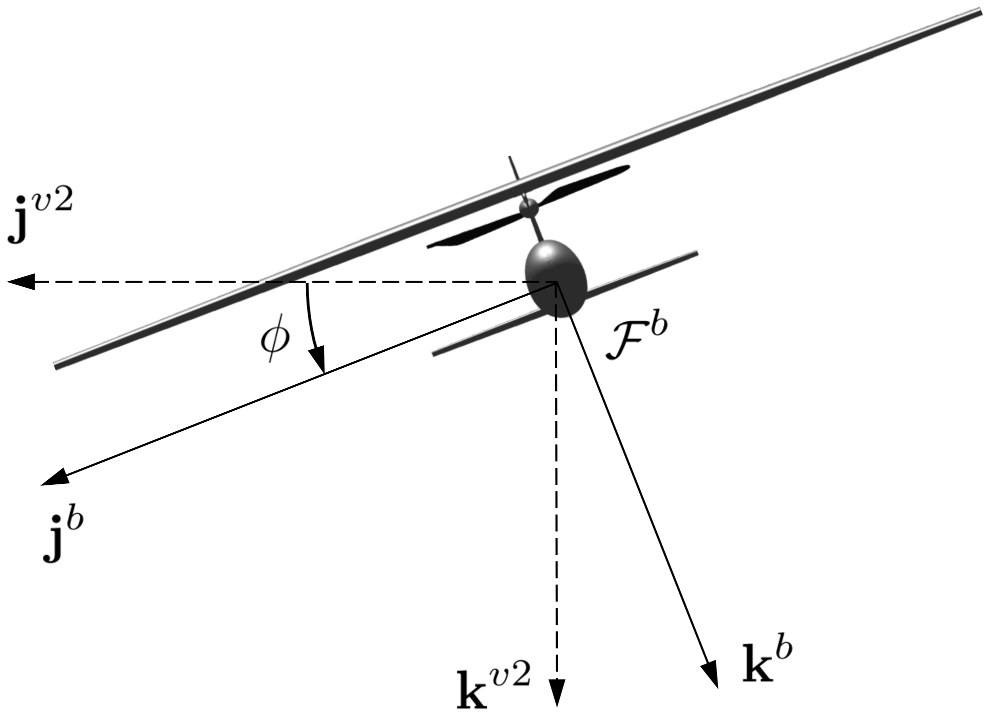
$$\mathcal{R}_{v1}^{v2}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

gives

$$\mathbf{p}^{v2} = \mathcal{R}_{v1}^{v2}(\theta)\mathbf{p}^{v1}$$

where θ is the pitch angle

Body Frame



$$\mathcal{R}_{v2}^{b}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

gives

$$\mathbf{p}^b = \mathcal{R}_{v2}^{b}(\phi)\mathbf{p}^{v2}$$

where ϕ is the roll (bank) angle

Inertial Frame to Body Frame Transformation

Define

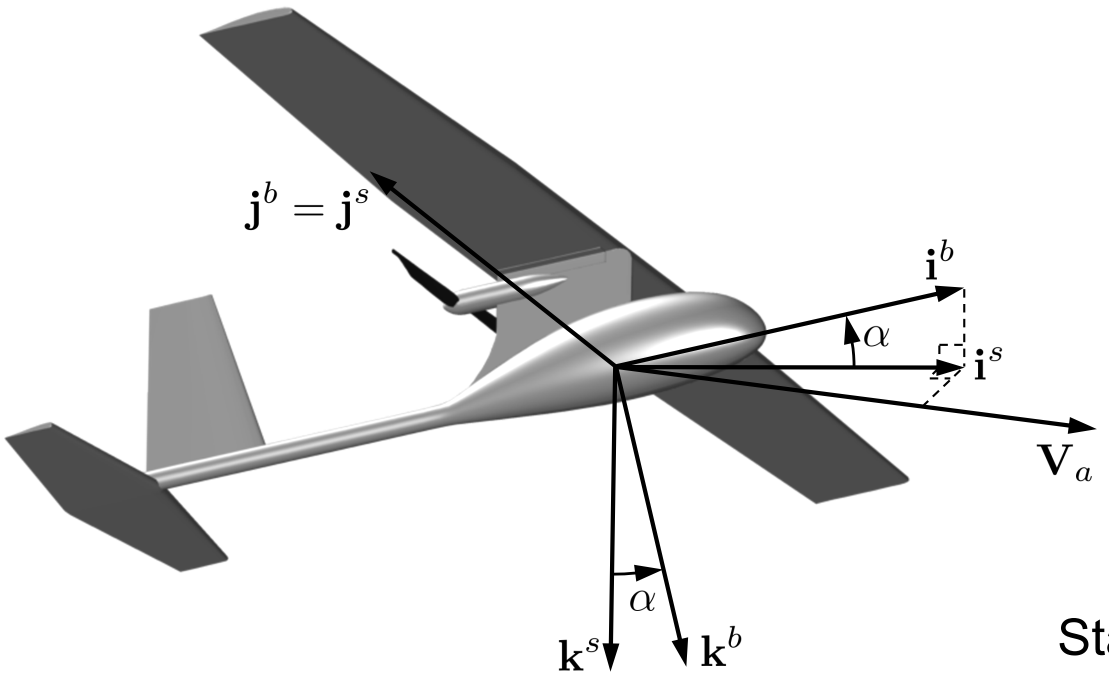
$$
\begin{aligned}
\mathcal{R}_v^b(\phi,\theta,\psi) &= \mathcal{R}_{v2}^b(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_v^{v1}(\psi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}
\end{aligned}
$$

to give

$$\mathbf{p}^b = \mathcal{R}_v^b(\Theta)\mathbf{p}^v$$

where, $\Theta = (\phi,\theta,\psi)^\top$

Stability Frame



Right-handed rotation

$$\mathcal{R}_s^b(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

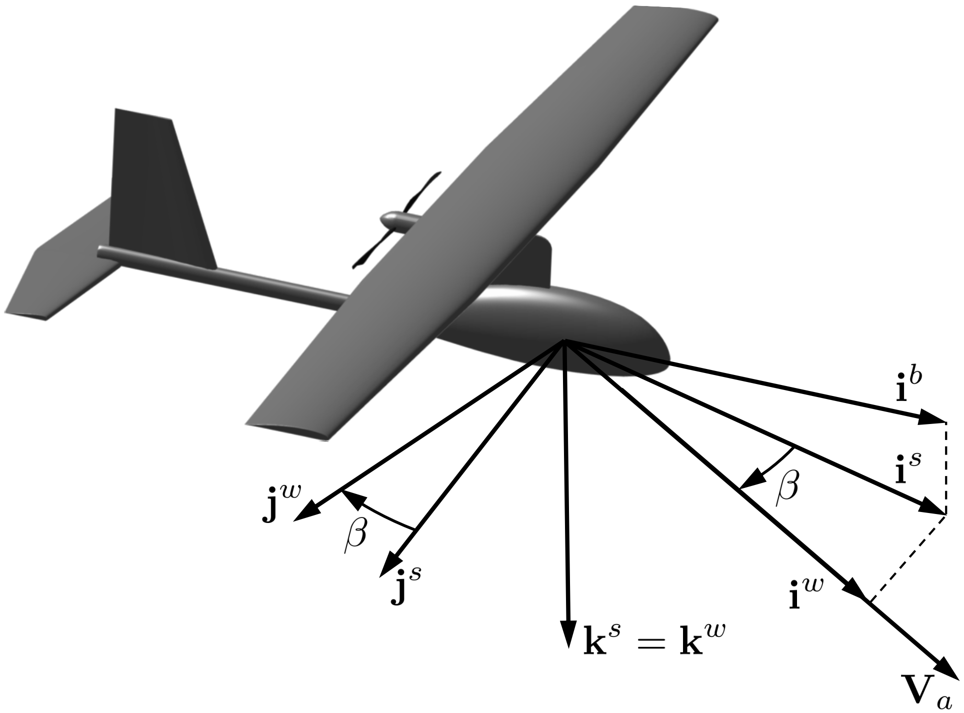
$$\mathbf{p}^b = \mathcal{R}_s^b(\alpha)\mathbf{p}^s$$

$$\mathbf{p}^s = \mathcal{R}_b^s(\alpha)\mathbf{p}^b = \mathcal{R}_s^b(\alpha)^\top \mathbf{p}^b$$

Stability frame helps us rigorously define angle of attack and is useful for analyzing stability of aircraft

Angle of attack defined as a positive RH rotation from stability to body frame

Wind Frame

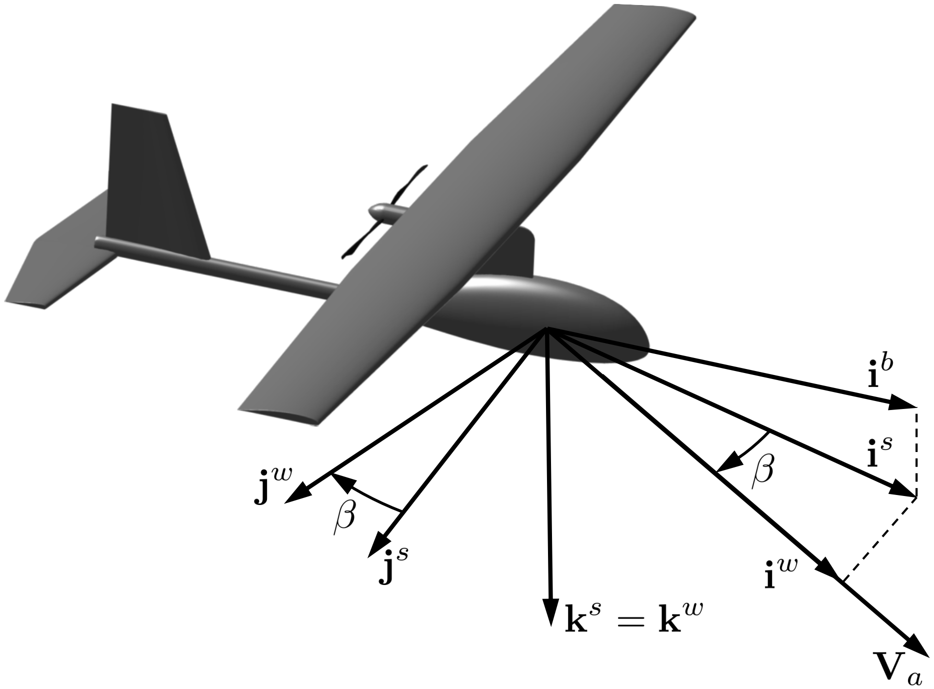


$$\mathcal{R}_s^w(\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}^w = \mathcal{R}_s^w(\beta)\mathbf{p}^s$$

$$\begin{aligned} \mathcal{R}_b^w(\alpha,\beta) &= \mathcal{R}_s^w(\beta)\mathcal{R}_b^s(\alpha) \\ &= \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta\cos\alpha & \sin\beta & \cos\beta\sin\alpha \\ -\sin\beta\cos\alpha & \cos\beta & -\sin\beta\sin\alpha \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \end{aligned}$$

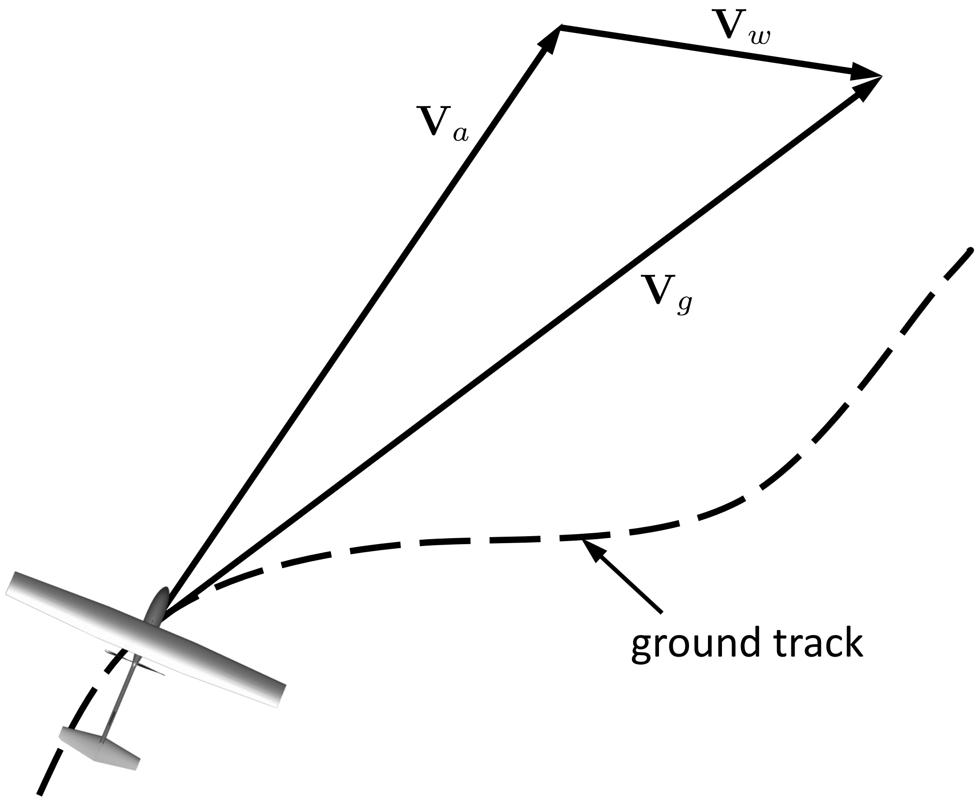
Wind Frame (continued)



- Wind frame helps us rigorously define side-slip angle
- Forces and moments are most naturally defined in wind frame
- Side-slip angle is nominally zero for tailed aircraft

$$\mathcal{R}_w^b(\alpha,\beta) = (\mathcal{R}_b^w)^\top(\alpha,\beta) = \begin{pmatrix} \cos\beta\cos\alpha & -\sin\beta\cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \cos\beta\sin\alpha & -\sin\beta\sin\alpha & \cos\alpha \end{pmatrix}$$

Airspeed, Wind Speed, Ground Speed



$$\mathbf{V}_a = \mathbf{V}_g - \mathbf{V}_w$$

$$\mathbf{V}_g^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

a/c wrt to inertial frame expressed in body frame

wind wrt to inertial frame expressed in body frame

$$\mathbf{V}_w^b = \begin{pmatrix} u_w \\ v_w \\ w_w \end{pmatrix} = \mathcal{R}_v^b(\phi, \theta, \psi) \begin{pmatrix} w_n \\ w_e \\ w_d \end{pmatrix}$$

$$\mathbf{V}_a^w = \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

a/c wrt to surrounding air expressed in body frame

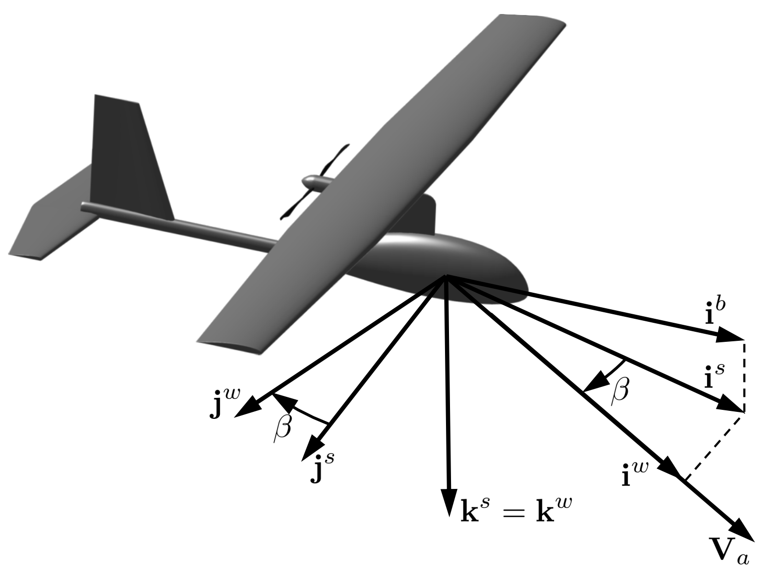
$$\mathbf{V}_a^b = \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \begin{pmatrix} u - u_w \\ v - v_w \\ w - w_w \end{pmatrix}$$

Airspeed, Angle of Attack, Sideslip Angle

$$\mathbf{V}_a^b = \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \mathcal{R}_w^b \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha & -\sin\beta\cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta & -\sin\beta\sin\alpha \\ \cos\beta\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = V_a \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{pmatrix}$$



$$V_a = \sqrt{u_r^2 + v_r^2 + w_r^2}$$

$$\alpha = \tan^{-1}\left(\frac{w_r}{u_r}\right)$$

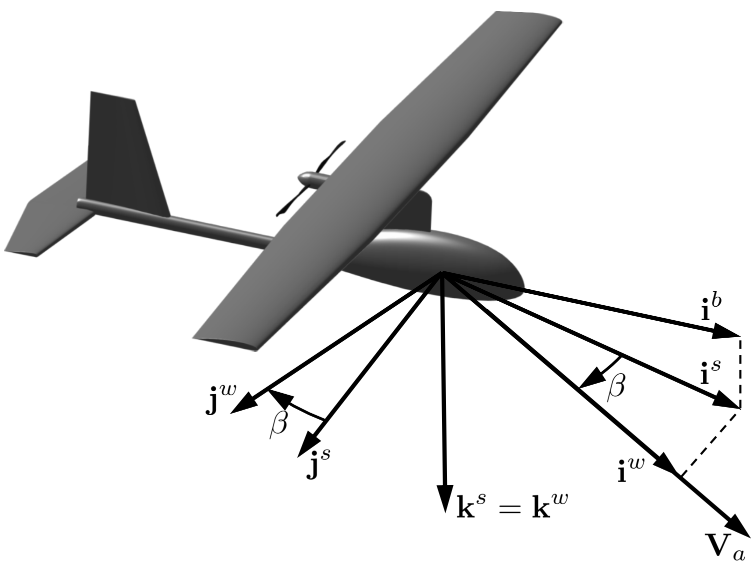
$$\beta = \sin^{-1}\left(\frac{v_r}{\sqrt{u_r^2 + v_r^2 + w_r^2}}\right)$$

Airspeed, Angle of Attack, Sideslip Angle

$$\mathbf{V}_a^b = \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \mathcal{R}_w^b \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha & -\sin\beta\cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta & -\sin\beta\sin\alpha \\ \cos\beta\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = V_a \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{pmatrix}$$

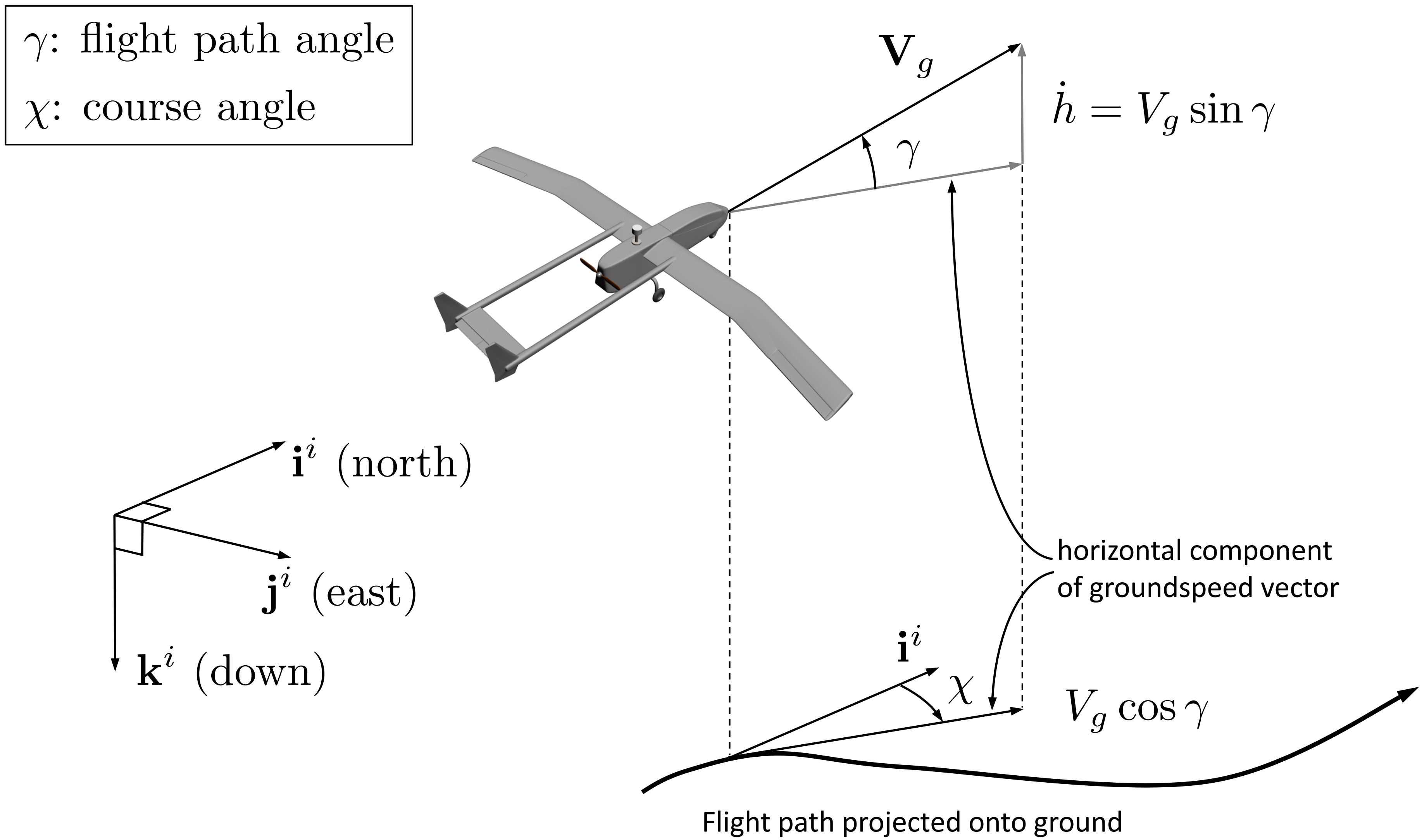


$$V_a = \sqrt{u_r^2 + v_r^2 + w_r^2}$$

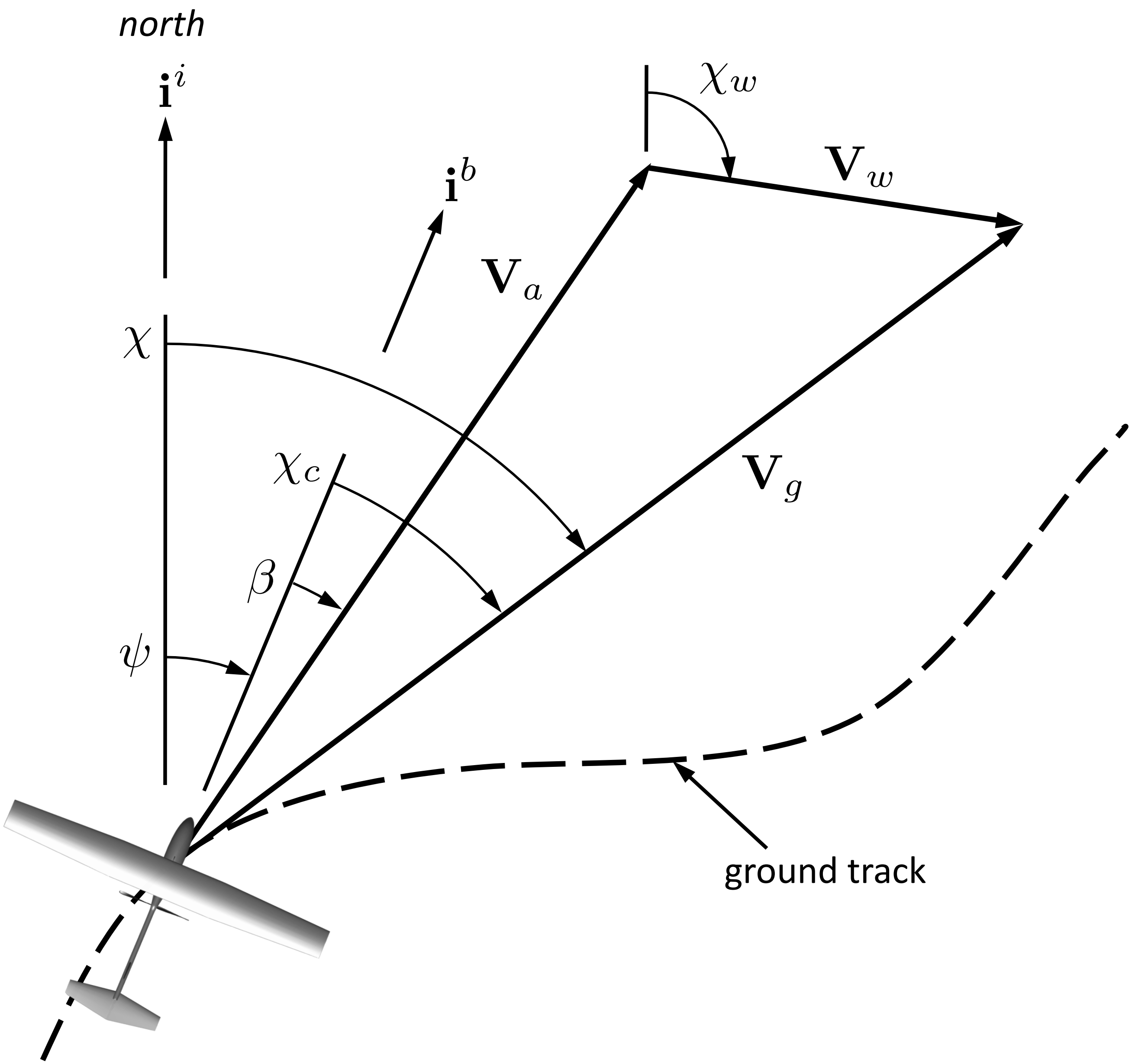
$$\alpha = \tan^{-1}\left(\frac{w_r}{u_r}\right)$$

OR $$\beta = \tan^{-1}\left(\frac{v_r}{\sqrt{u_r^2 + w_r^2}}\right)$$

Course and Flight Path Angles



Wind Triangle



χ: course angle

ψ: heading angle

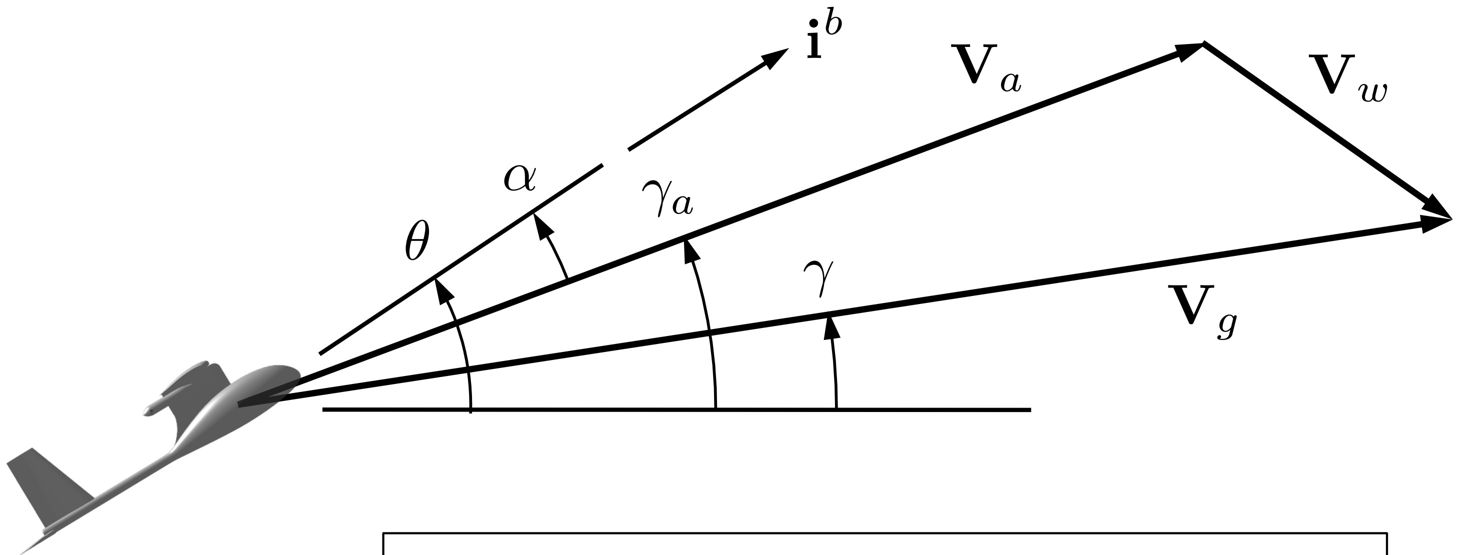
β: side slip angle

χ_c: crab angle

χ_w: wind direction

$$\chi_c \triangleq \chi - \psi$$

Wind Triangle



θ: pitch angle

α: angle of attack

γ: flight path angle

γ_a: air-mass-relative flight path angle

$$\gamma_a = \theta - \alpha$$

When wind speed and sideslip are zero…

If both the windspeed and the sideslip angles are zero, i.e.,

$$
\begin{aligned}
V_w &= 0 \\
\beta &= 0
\end{aligned}
$$

then we have the following simplifications

$$
\begin{aligned}
V_a &= V_g \quad \text{Airspeed equals groundspeed} \\
u &= u_r \quad \text{Velocity equals velocity relative to the air mass} \\
v &= v_r \\
w &= w_r \\
\psi &= \chi \quad \text{Heading equals course} \\
\gamma &= \gamma_a \quad \text{Flight path angle equals air-mass-referenced flight path angle}
\end{aligned}
$$

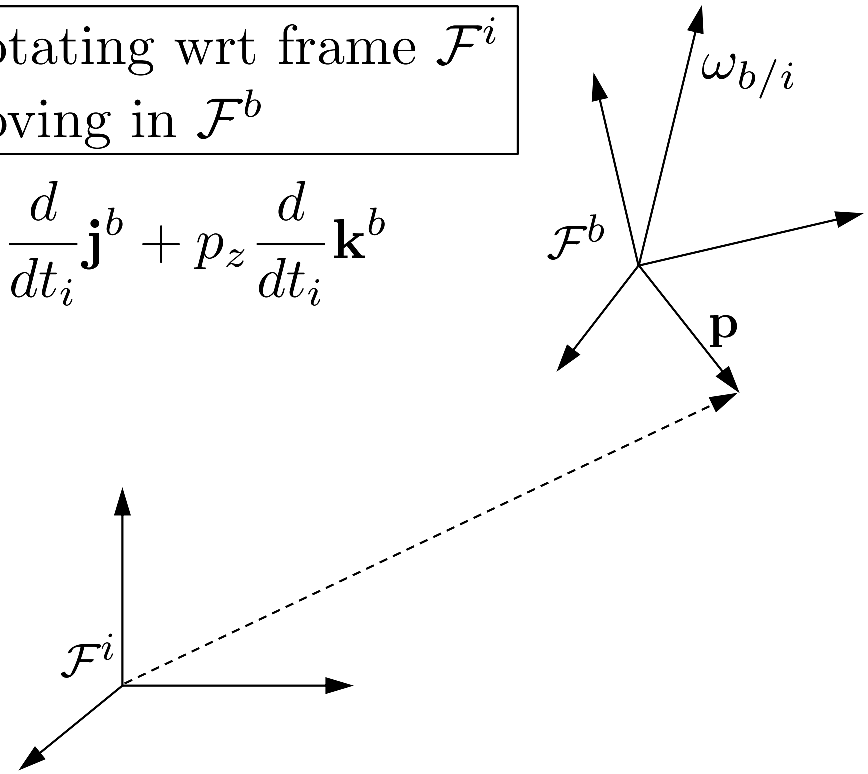
Differentiation of a Vector

Frame $\mathcal{F}^b$ rotating wrt frame $\mathcal{F}^i$
Vector $\mathbf{p}$ moving in $\mathcal{F}^b$

$$\mathbf{p} = p_x \mathbf{i}^b + p_y \mathbf{j}^b + p_z \mathbf{k}^b$$

$$\frac{d}{dt_i}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b + p_x \frac{d}{dt_i}\mathbf{i}^b + p_y \frac{d}{dt_i}\mathbf{j}^b + p_z \frac{d}{dt_i}\mathbf{k}^b$$

$$\frac{d}{dt_b}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b$$



$$\dot{\mathbf{i}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{i}^b$$

$$\dot{\mathbf{j}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{j}^b$$

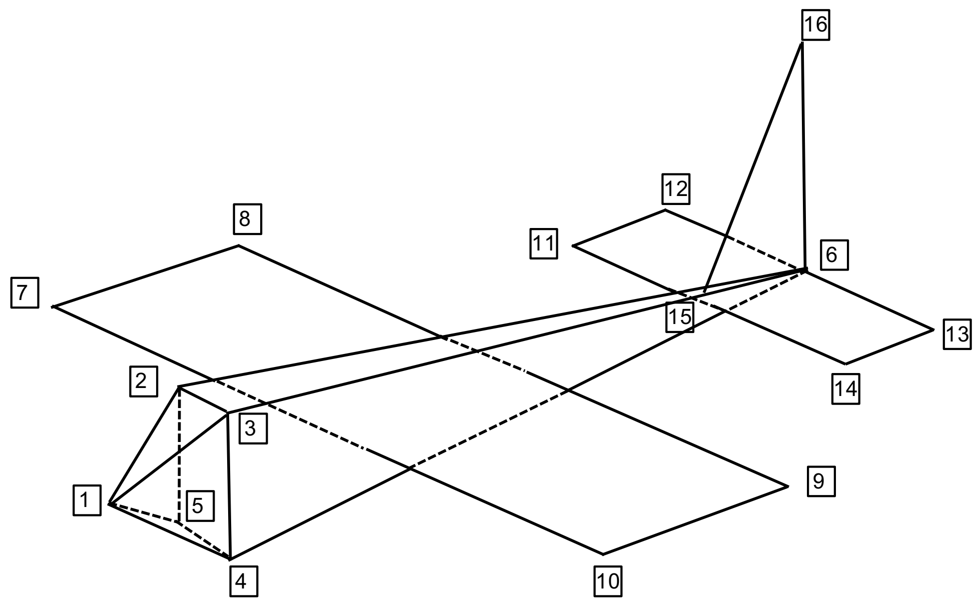
$$\dot{\mathbf{k}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{k}^b$$

$$\begin{aligned} p_x \dot{\mathbf{i}}^b + p_y \dot{\mathbf{j}}^b + p_z \dot{\mathbf{k}}^b &= p_x(\boldsymbol{\omega}_{b/i} \times \mathbf{i}^b) + p_y(\boldsymbol{\omega}_{b/i} \times \mathbf{j}^b) + p_z(\boldsymbol{\omega}_{b/i} \times \mathbf{k}^b) \\ &= \boldsymbol{\omega}_{b/i} \times \mathbf{p} \end{aligned}$$

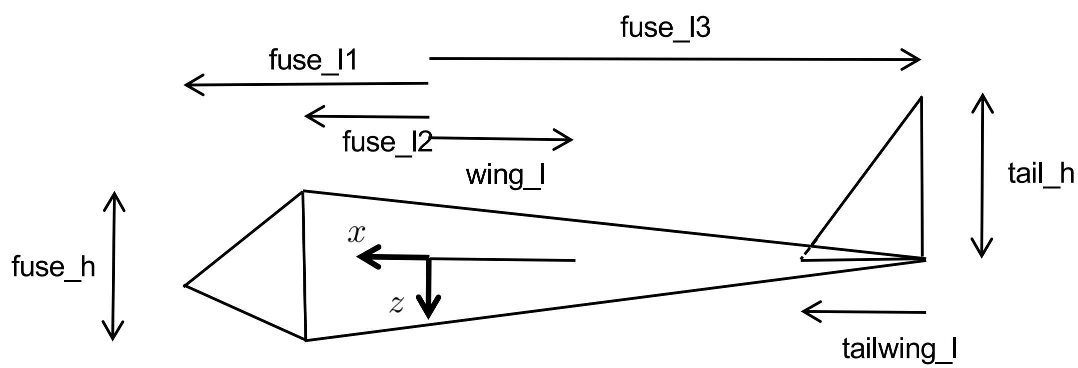
$$\frac{d}{dt_i}\mathbf{p} = \frac{d}{dt_b}\mathbf{p} + \boldsymbol{\omega}_{b/i} \times \mathbf{p}$$

Note: Frame $\mathcal{F}^b$ does not translate wrt frame $\mathcal{F}^i$

Project Aircraft



Project Aircraft



Project Aircraft