Chapter 5

Linear Design Models

Equations of Motion

$$\dot{p}_n = (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w$$

$$\dot{p}_e = (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w$$

$$\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{T_p(\delta_t, V_a)}{\mathrm{m}}$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right]$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = q\sin\phi\sec\theta + r\cos\phi\sec\theta$$

$$\dot{p} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] + \Gamma_3 Q_p(\delta_t, V_a)$$

$$\dot{q} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right]$$

$$\dot{r} = \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right] + \Gamma_4 Q_p(\delta_t, V_a)$$

Equations of Motion

$$
\begin{aligned}
C_{p_0} &= \Gamma_3 C_{l_0} + \Gamma_4 C_{n_0} \\
C_{p_\beta} &= \Gamma_3 C_{l_\beta} + \Gamma_4 C_{n_\beta} \\
C_{p_p} &= \Gamma_3 C_{l_p} + \Gamma_4 C_{n_p} \\
C_{p_r} &= \Gamma_3 C_{l_r} + \Gamma_4 C_{n_r} \\
C_{p_{\delta_a}} &= \Gamma_3 C_{l_{\delta_a}} + \Gamma_4 C_{n_{\delta_a}} \\
C_{p_{\delta_r}} &= \Gamma_3 C_{l_{\delta_r}} + \Gamma_4 C_{n_{\delta_r}} \\
C_{r_0} &= \Gamma_4 C_{l_0} + \Gamma_8 C_{n_0} \\
C_{r_\beta} &= \Gamma_4 C_{l_\beta} + \Gamma_8 C_{n_\beta} \\
C_{r_p} &= \Gamma_4 C_{l_p} + \Gamma_8 C_{n_p} \\
C_{r_r} &= \Gamma_4 C_{l_r} + \Gamma_8 C_{n_r} \\
C_{r_{\delta_a}} &= \Gamma_4 C_{l_{\delta_a}} + \Gamma_8 C_{n_{\delta_a}} \\
C_{r_{\delta_r}} &= \Gamma_4 C_{l_{\delta_r}} + \Gamma_8 C_{n_{\delta_r}}
\end{aligned}
$$

$$
\begin{aligned}
C_X(\alpha) &\triangleq -C_D(\alpha)\cos\alpha + C_L(\alpha)\sin\alpha \\
C_{X_q}(\alpha) &\triangleq -C_{D_q}\cos\alpha + C_{L_q}\sin\alpha \\
C_{X_{\delta_e}}(\alpha) &\triangleq -C_{D_{\delta_e}}\cos\alpha + C_{L_{\delta_e}}\sin\alpha \\
C_Z(\alpha) &\triangleq -C_D(\alpha)\sin\alpha - C_L(\alpha)\cos\alpha \\
C_{Z_q}(\alpha) &\triangleq -C_{D_q}\sin\alpha - C_{L_q}\cos\alpha \\
C_{Z_{\delta_e}}(\alpha) &\triangleq -C_{D_{\delta_e}}\sin\alpha - C_{L_{\delta_e}}\cos\alpha
\end{aligned}
$$

Lift and Drag Models

Nonlinear model with stall effects

$$C_L(\alpha) = (1 - \sigma(\alpha))\left[C_{L_0} + C_{L_\alpha}\alpha\right] + \sigma(\alpha)\left[2\,\mathrm{sign}(\alpha)\sin^2\alpha\cos\alpha\right]$$

$$\sigma(\alpha) = \frac{1 + e^{-M(\alpha-\alpha_0)} + e^{M(\alpha+\alpha_0)}}{\left(1 + e^{-M(\alpha-\alpha_0)}\right)\left(1 + e^{M(\alpha+\alpha_0)}\right)}$$

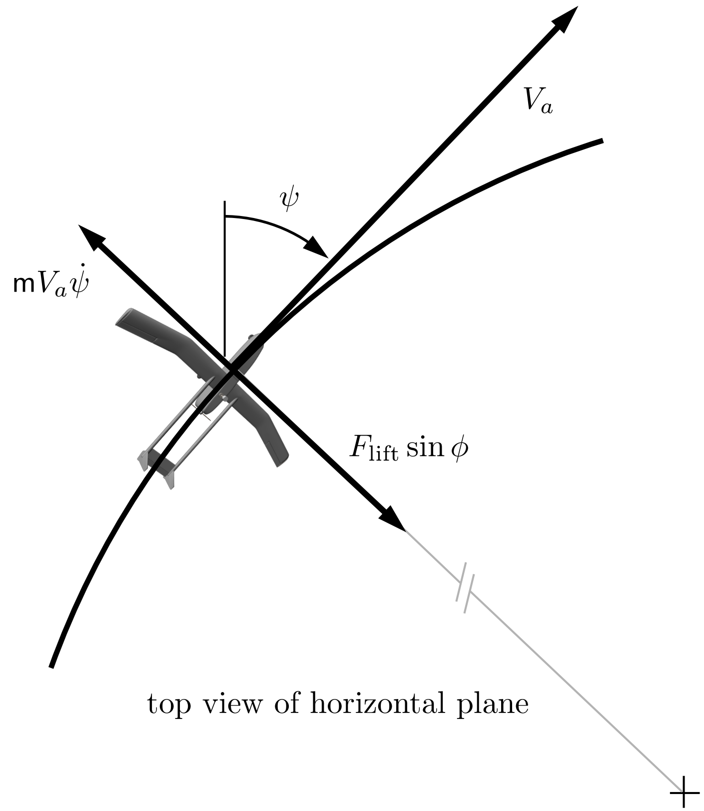
$$C_D(\alpha) = C_{D_p} + \frac{(C_{L_0} + C_{L_\alpha}\alpha)^2}{\pi e AR}$$

Linear model

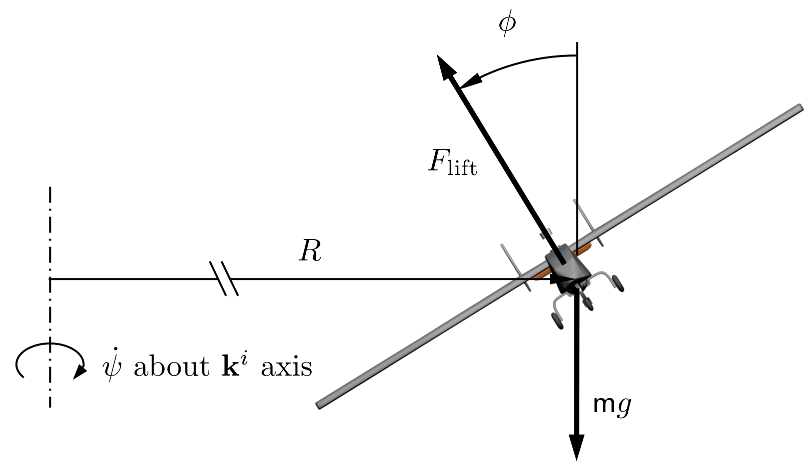
$$C_L(\alpha) = C_{L_0} + C_{L_\alpha}\alpha$$

$$C_D(\alpha) = C_{D_0} + C_{D_\alpha}\alpha$$

Coordinated Turn – No Wind



top view of horizontal plane



view in direction of $-\mathbf{i}^b$ axis
forces shown in $\mathbf{j}^b$-$\mathbf{k}^b$ plane

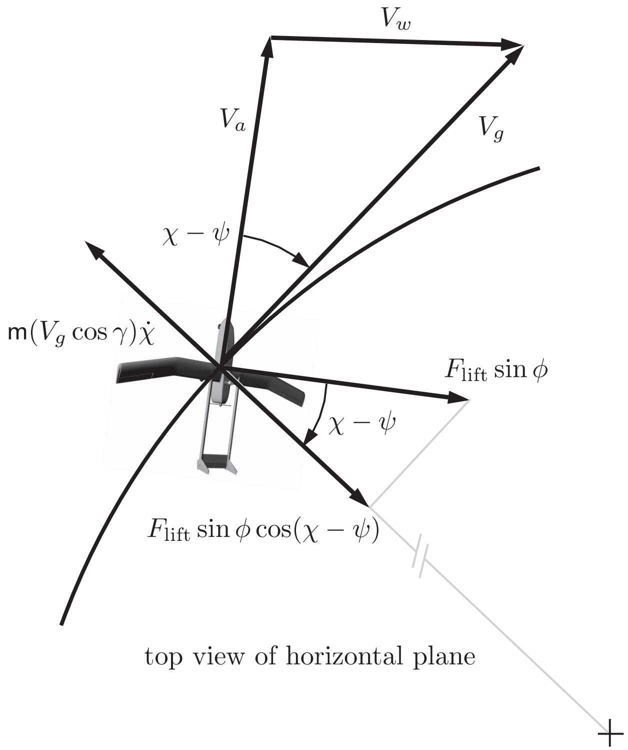
$$F_{\text{lift}}\cos\phi = \mathsf{m}g$$

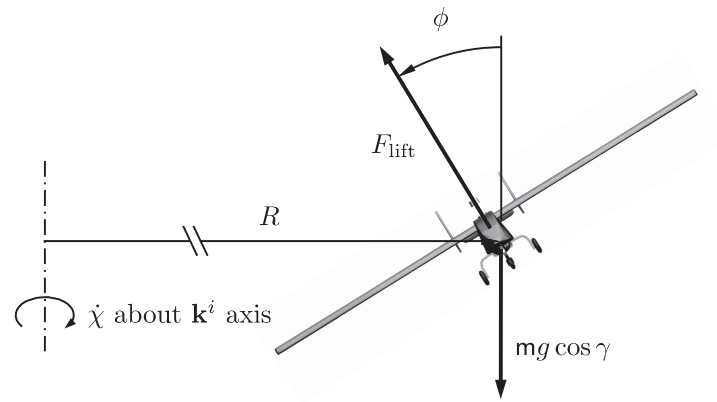
$$\begin{aligned} F_{\text{lift}}\sin\phi &= \mathsf{m}\frac{v^2}{R} \\ &= \mathsf{m}v\omega \\ &= \mathsf{m}V_a\dot{\psi} \end{aligned}$$

Dividing gives $\tan\phi = \dfrac{V_a\dot{\psi}}{g}$

and $\dot{\psi} = \dfrac{g}{V_a}\tan\phi$

Coordinated Turn





$$
\begin{aligned}
F_{\text{lift}} \sin\phi \cos(\chi - \psi) &= \mathsf{m}\frac{v^2}{R} \\
&= \mathsf{m} v \omega \\
&= \mathsf{m}(V_g \cos\gamma)\dot{\chi}
\end{aligned}
$$

$$
F_{\text{lift}} \cos\phi = \mathsf{m} g \cos\gamma
$$

Coordinated Turn

Dividing the two expressions gives

$$\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi)$$

which is the coordinated turn condition in wind

In the absence of wind, we have $V_a = V_g$ and $\psi = \chi$ which gives

$$\dot{\psi} = \frac{g}{V_a} \tan\phi$$

which is the expression commonly seen in the literature

The turning radius is given by

$$R = \frac{V_g \cos\gamma}{\dot{\chi}} = \frac{V_g^2 \cos\gamma}{g \tan\phi \cos(\chi - \psi)}$$

For level flight in the absence of wind we have the standard formula

$$R = \frac{V_a^2}{g \tan\phi}$$

Trim Conditions

Given the nonlinear system

$$\dot{x} = f(x, u)$$

The equilibria (x^*, u^*) are defined by

$$f(x^*, u^*) = 0$$

An aircraft in equilibrium is in a *trim condition*. In general, trim conditions may include states that are not constant. Therefore, trim conditions are given by

$$\dot{x}^* = f(x^*, u^*)$$

Calculating Trim

Objective is to compute trim states and inputs when aircraft simultaneously satisfies three conditions:

- Traveling at constant speed V_a^*
- Climbing at constant flight path angle γ^*
- In constant orbit of radius R^*

V_a^*, γ^*, and R^*, are inputs to the trim calculations

$$\text{States:} \qquad x \triangleq (p_n, p_e, p_d, u, v, w, \phi, \theta, \psi, p, q, r)^\top$$

$$\text{Inputs:} \qquad u \triangleq (\delta_e, \delta_a, \delta_r, \delta_t)^\top$$

Calculating Trim

For constant-climb orbit:
$\rightarrow$ speed of aircraft not changing $\rightarrow \dot{u}^* = \dot{v}^* = \dot{w}^* = 0$
$\rightarrow$ roll and pitch angles constant $\rightarrow \dot{\phi}^* = \dot{\theta}^* = \dot{p}^* = \dot{q}^* = 0$

Turn rate constant and given by

$$\dot{\psi}^* = \frac{V_a^*}{R^*}\cos\gamma^* \quad \rightarrow \quad \dot{r}^* = 0$$

Climb rate constant, and given by

$$\dot{h}^* = V_a^* \sin\gamma^*$$

Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = (\dot{p}_n^* \ \ \dot{p}_e^* \ \ \dot{h}^* \ \ \dot{u}^* \ \ \dot{v}^* \ \ \dot{w}^* \ \ \dot{\phi}^* \ \ \dot{\theta}^* \ \ \dot{\psi}^* \ \ \dot{p}^* \ \ \dot{q}^* \ \ \dot{r}^* \ )^\top$$

(continued...)

Calculating Trim

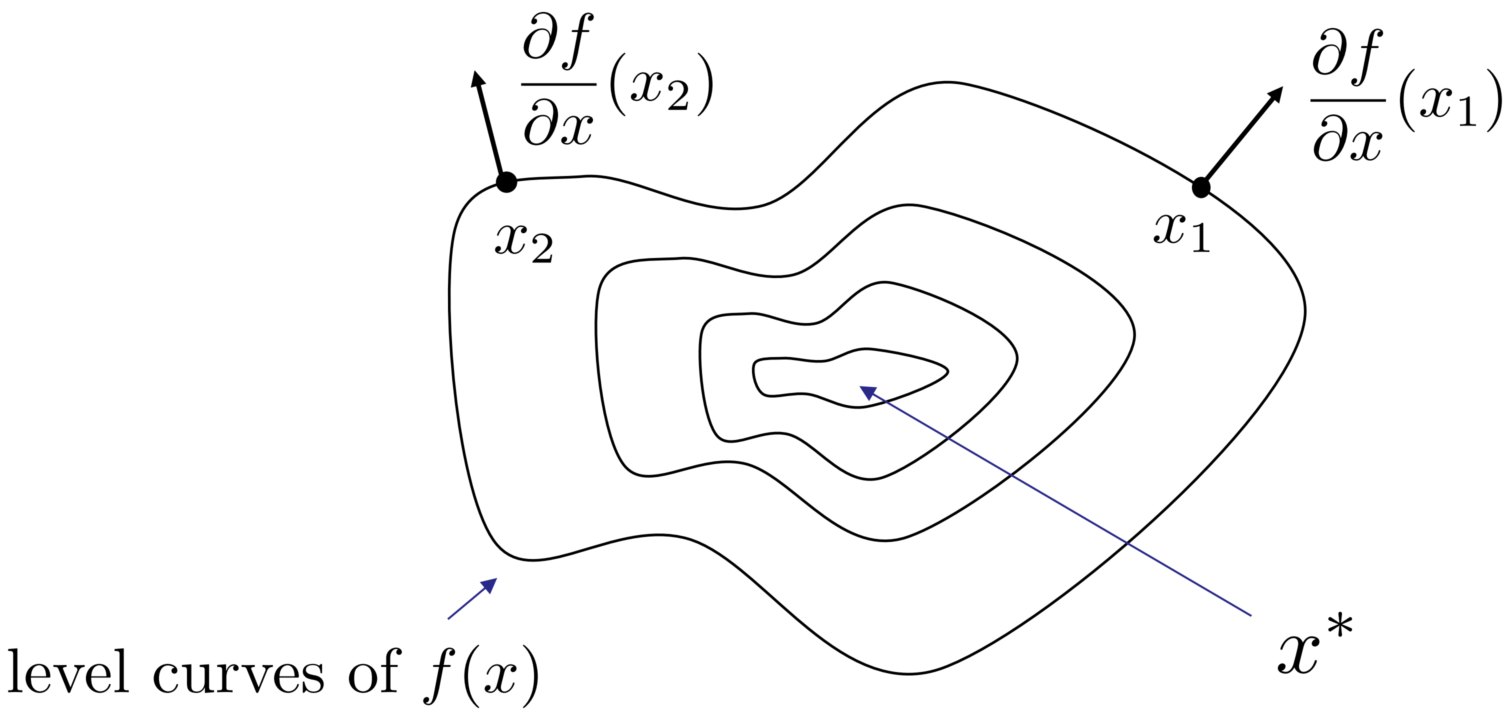
Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = \begin{pmatrix} \dot{p}_n^* \\ \dot{p}_e^* \\ \dot{h}^* \\ \dot{u}^* \\ \dot{v}^* \\ \dot{w}^* \\ \dot{\phi}^* \\ \dot{\theta}^* \\ \dot{\psi}^* \\ \dot{p}^* \\ \dot{q}^* \\ \dot{r}^* \end{pmatrix} = \begin{pmatrix} \text{[don't care]} \\ \text{[don't care]} \\ V_a^* \sin\gamma^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{V_a^*}{R^*}\cos\gamma^* \\ 0 \\ 0 \\ 0 \end{pmatrix} = f(x^*, u^*)$$

Problem of finding x^* and u^* such that $\dot{x}^* = f(x^*, u^*)$, reduces to solving nonlinear algebraic systems of equations

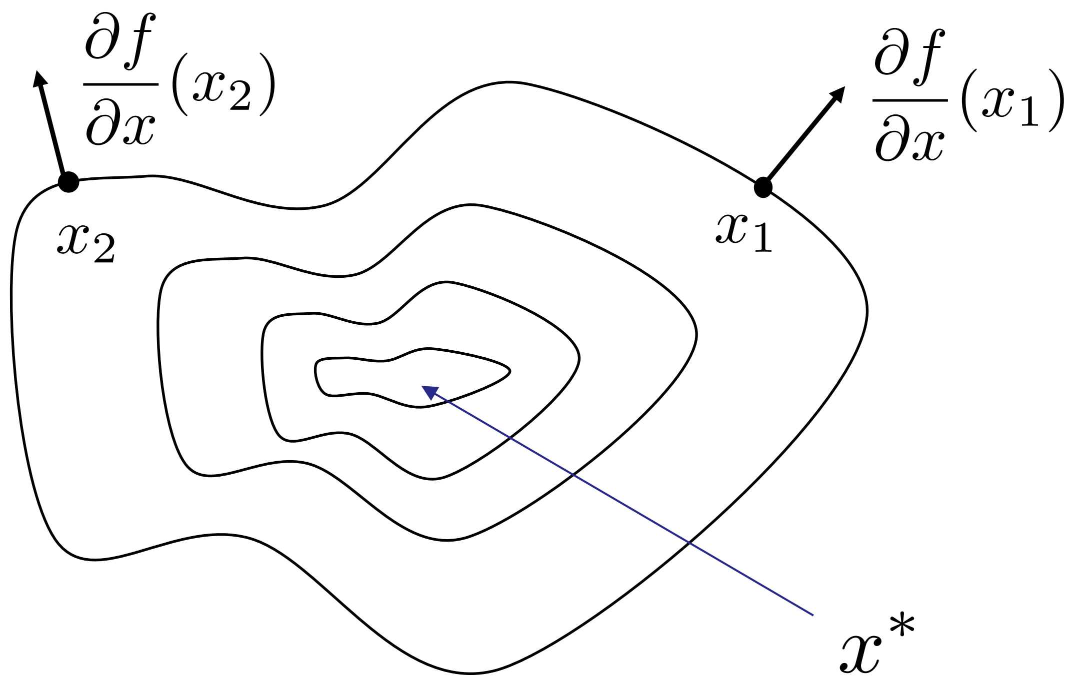
Can use Matlab trim command – see Appendix F

Unconstrained Optimization



$$x^* = \arg\min_{x \in \mathbb{R}^n} f(x)$$

Unconstrained Optimization



Gradient descent algorithm:

$$x^{[k+1]} = x^{[k]} - \alpha_k \frac{\partial f}{\partial x}(x^{[k]})$$

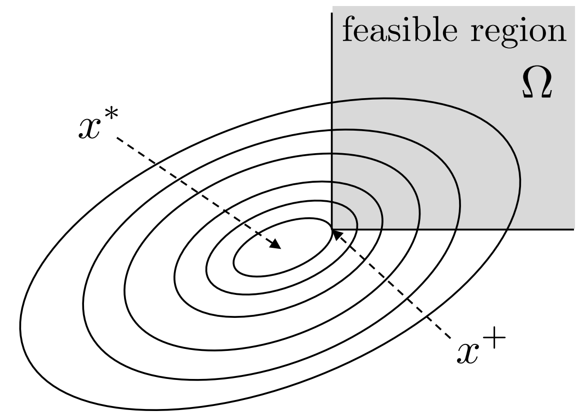
Constrained Optimization

Can generally pose as:

$$\min_{x \in \Omega} \quad f(x)$$

Usually more managable to pose as:

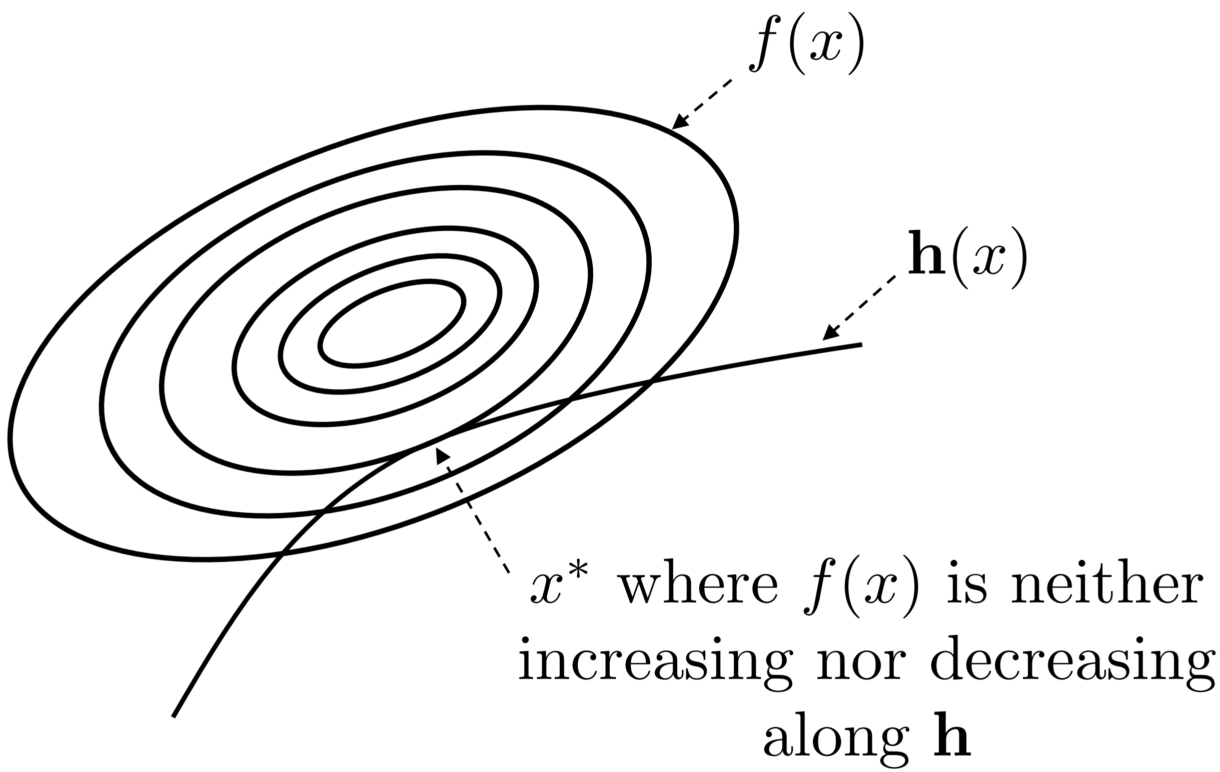
$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = 0, \\
& \vdots, \\
& h_m(x) = 0, \\
& g_1(x) \leq 0, \\
& \vdots, \\
& g_p(x) \leq 0
\end{aligned}$$



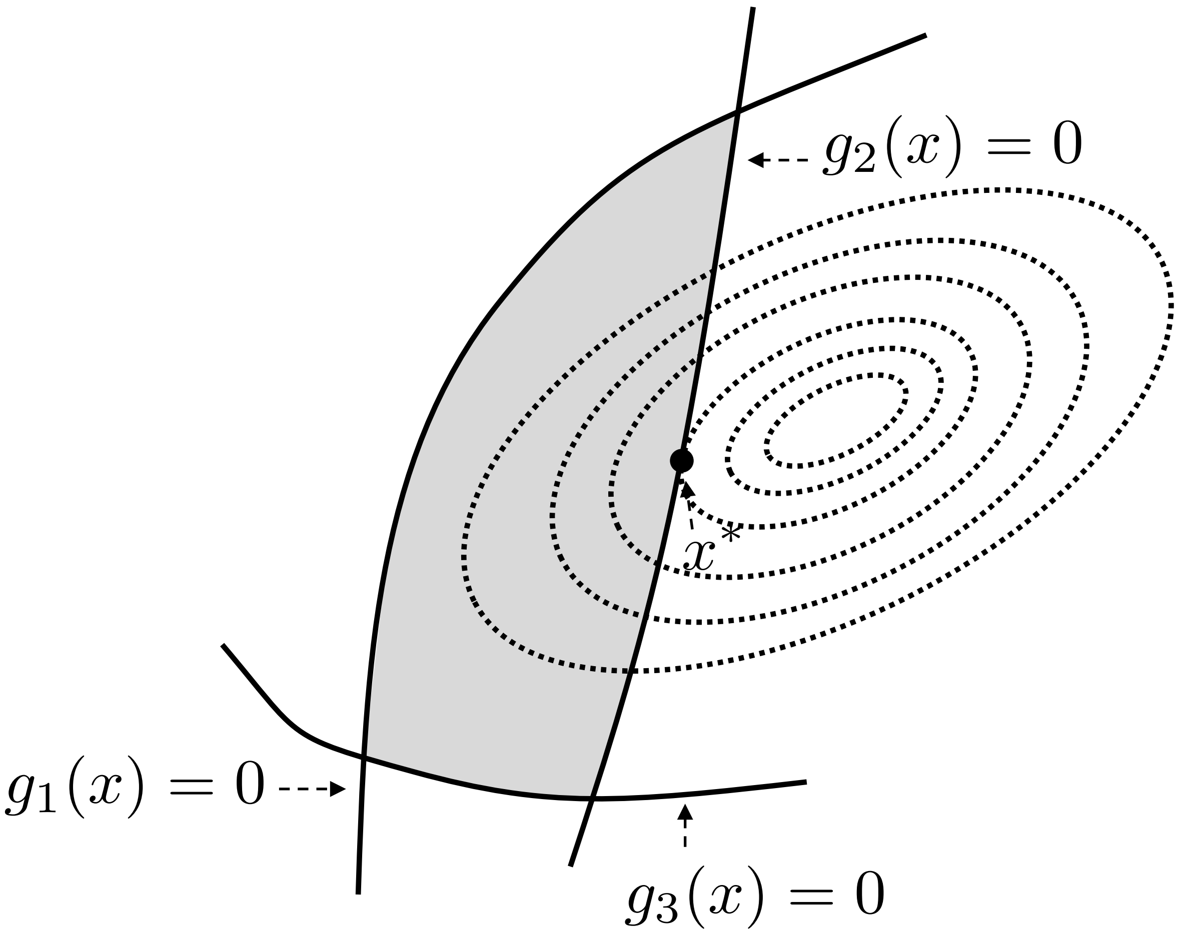
Note that the constrained optimum x^+ does not equal the unconstrained optimum x^*.

$h_i(x) = 0$ are called "equality constraints"
$g_i(x) \leq 0$ are called "inequality constraints"

Equality Constraints



Inequality Constraints



Optimization in Python

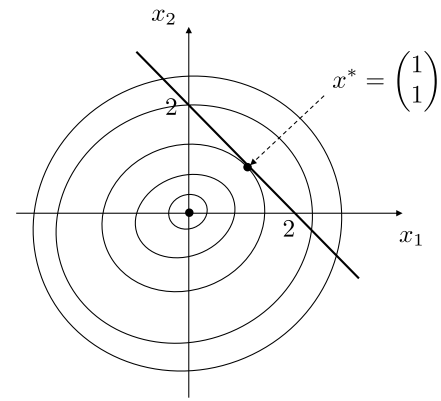
```
from scipy.optimize import minimize

minimize(fun, # function to be minimized
         x0, # initial guess for gradient descent
         args=(), # additional arguments for fun
         method=None, # optimization method
         jac=None, # function that defines Jacobian of f
         hess=None, # function that defines Hessian of f
         hessp=None, # Hession in a specific direction
         bounds=None, # inequality bounds on x
         constraints=(), # definition of constraints
         tol=None, # tolerance for termination
         callback=None, # function called after each iteration
         options=None) # other algorithm options
```

Example: Optimization in Python

$$\min \quad x_1^2 + x_2^2$$

$$\text{s.t.} \quad x_1 + x_2 = 2$$




```
from scipy.optimize import minimize
import numpy as np

def objective(x):
      return(x[0]**2 + x[1]**2)

x0 = np.array([[5], [2]]) # initial condition
cons = ({'type': 'eq',
        'fun': lambda x: np.array([
            x[0]+x[1]-2, # x1+x2=2
            ]),
       })

res = minimize(objective, x0, method='SLSQP', constraints=cons)
print("xstar =", res.x)
```

Objective Function for Trim

```
def trim_objective_fun(x, mav, Va, gamma):
      state = x[0:13]
      delta = MsgDelta(elevator=x.item(13),
      aileron=x.item(14),
      rudder=x.item(15),
      throttle=x.item(16))
      desired_trim_state_dot
            = np.array([[0., 0., -Va*np.sin(gamma), 0.,
                        0., 0., 0., 0., 0., 0., 0., 0.,
                        0.]]).T
      mav._state = state
      mav._update_velocity_data()
      forces_moments = mav._forces_moments(delta)
      f = mav._derivatives(state, forces_moments)
      tmp = desired_trim_state_dot - f
      J = np.linalg.norm(tmp[2:13])**2.0
      return J
```

Constraints for Trim

```
cons = ({'type': 'eq',
         'fun': lambda x: np.array([
                      x[3]**2 + x[4]**2 + x[5]**2 - Va**2, # magnitude of velocity
                                                          # vector is Va
                     x[4], # v=0, force side velocity to be zero
                     x[6]**2 + x[7]**2 + x[8]**2 + x[9]**2 - 1., # quaternion is unit length
                     x[7], # e1=0 - forcing e1=e3=0 ensures zero roll and zero yaw in trim
                     x[9], # e3=0
                     x[10], # p=0 - angular rates should all be zero
                     x[11], # q=0
                     x[12], # r=0
                     ]),
         'jac': lambda x: np.array([
         [0., 0., 0., 2*x[3], 2*x[4], 2*x[5], 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 2*x[6], 2*x[7], 2*x[8], 2*x[9], 0., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0.],
         [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0.],
         ])
         })
```

Lateral Transfer Functions - Roll

Objective: Find simple transfer function relationship between ϕ and δ_a

Start with: $\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$

Assuming that θ is small and defining the linearization error (can be considered as a control disturbance) to be $d_{\phi_1} \stackrel{\triangle}{=} q\sin\phi\tan\theta + r\cos\phi\tan\theta$ gives

$$\dot{\phi} = p + d_{\phi_1}$$

Differentiating and substituting for $\dot{\phi}$ gives

$$\begin{aligned}
\ddot{\phi} &= \dot{p} + \dot{d}_{\phi_1} \\
&= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1} \\
&= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{b}{2V_a}(\dot{\phi} - d_{\phi_1}) + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1} \\
&= \left(\frac{1}{2}\rho V_a^2 SbC_{p_p}\frac{b}{2V_a}\right)\dot{\phi} + \left(\frac{1}{2}\rho V_a^2 SbC_{p_{\delta_a}}\right)\delta_a \\
&\qquad + \left\{\Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta - C_{p_p}\frac{b}{2V_a}(d_{\phi_1}) + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1}\right\} \\
&= -a_{\phi_1}\dot{\phi} + a_{\phi_2}\delta_a + d_{\phi_2}
\end{aligned}$$

Equations of Motion

$$\dot{p}_n = (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w$$

$$\dot{p}_e = (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w$$

$$\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right]$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right]$$

$$\boxed{\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = q\sin\phi\sec\theta + r\cos\phi\sec\theta$$

$$\boxed{\dot{p} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right]}$$

$$\dot{q} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right]$$

$$\dot{r} = \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]$$

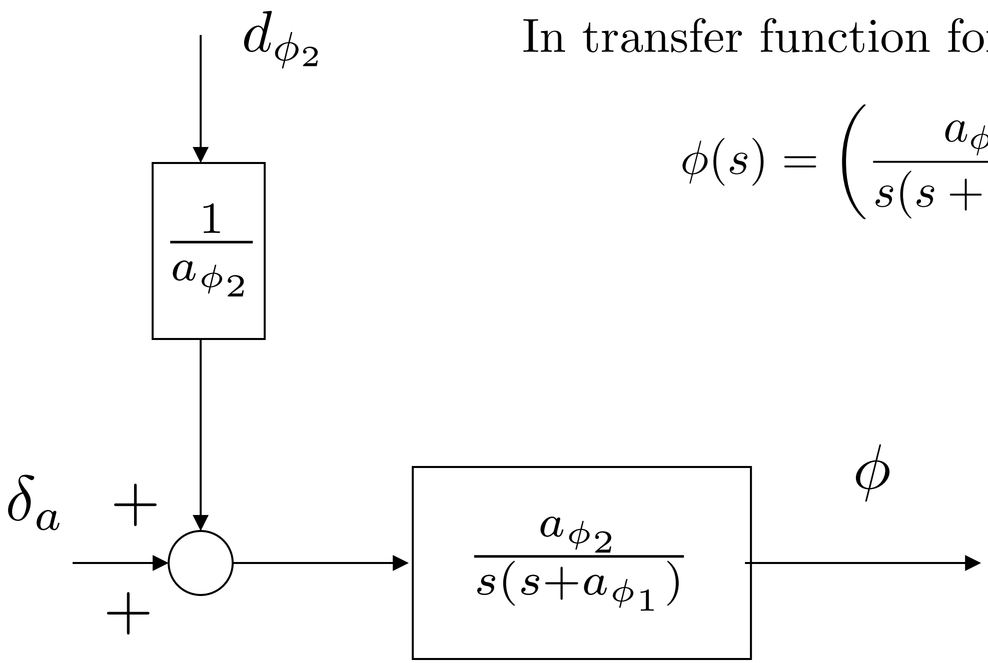
Roll Dynamics

Linearization leads to second-order ordinary differential equation

$$\ddot{\phi} = -a_{\phi_1}\dot{\phi} + a_{\phi_2}\delta_a + d_{\phi_2}$$

In transfer function form we have

$$\phi(s) = \left(\frac{a_{\phi_2}}{s(s+a_{\phi_1})}\right)\left(\delta_a(s) + \frac{1}{a_{\phi_2}}d_{\phi_2}(s)\right)$$



Lateral Transfer Functions - Course

To derive transfer function from roll ϕ to course χ, start with no-wind coordinated turn condition

$$\dot{\chi} = \frac{g}{V_g} \tan\phi$$

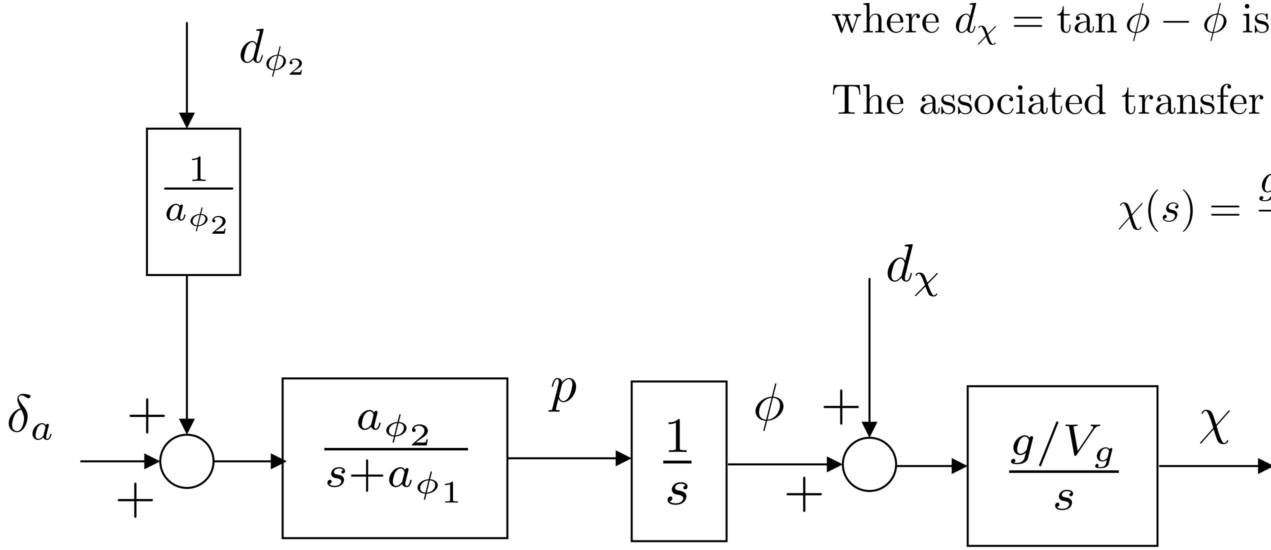
Add and subtract $\frac{g}{V_g}\phi$ to get

$$\begin{aligned}\dot{\chi} &= \frac{g}{V_g}\phi + \frac{g}{V_g}(\tan\phi - \phi) \\ &= \frac{g}{V_g}\phi + \frac{g}{V_g} d_\chi\end{aligned}$$

where $d_\chi = \tan\phi - \phi$ is an input disturbance

The associated transfer function is

$$\chi(s) = \frac{g/V_g}{s}\left(\phi(s) + d_\chi(s)\right)$$



Lateral Transfer Functions - Sideslip

No-wind conditions: $v = V_a \sin\beta$

Constant airspeed: $\dot{v} = (V_a \cos\beta)\dot{\beta}$

Substituting for $\dot{v}$ from the dynamics gives

$$(V_a \cos\beta)\dot{\beta} = pw - ru + g\cos\theta\sin\phi$$
$$+ \frac{\rho V_a^2 S}{2\mathsf{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

Solving for $\dot{\beta}$ and grouping terms gives

$$\dot{\beta} = -a_{\beta_1}\beta + a_{\beta_2}\delta_r + d_\beta$$

where

$$a_{\beta_1} = -\frac{\rho V_a S}{2\mathsf{m}\cos\beta}C_{Y_\beta} \qquad a_{\beta_2} = \frac{\rho V_a S}{2\mathsf{m}\cos\beta}C_{Y_{\delta_r}}$$

$$d_\beta = \frac{1}{V_a\cos\beta}(pw - ru + g\cos\theta\sin\phi) + \frac{\rho V_a S}{2\mathsf{m}\cos\beta}\left[C_{Y_0} + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a\right]$$

Equations of Motion

$$\dot{p}_n = (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w$$

$$\dot{p}_e = (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w$$

$$\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right]$$

$$\boxed{\dot{v} = pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]}$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right]$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = q\sin\phi\sec\theta + r\cos\phi\sec\theta$$

$$\dot{p} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right]$$

$$\dot{q} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right]$$

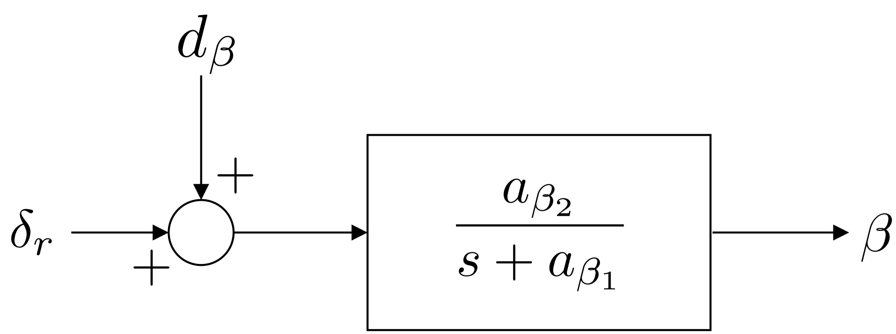
$$\dot{r} = \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]$$

Sideslip Dynamics

The differential equation

$$\dot{\beta} = -a_{\beta_1}\beta + a_{\beta_2}\delta_r + d_\beta$$

leads to the block diagram



$$a_{\beta_1} = -\frac{\rho V_a S}{2\mathrm{m}} C_{Y_\beta}$$

$$a_{\beta_2} = \frac{\rho V_a S}{2\mathrm{m}} C_{Y_{\delta_r}}$$

$$d_\beta = \frac{1}{V_a}(pw - ru + g\cos\theta\sin\phi) + \frac{\rho V_a S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a\right]$$

Longitudinal Transfer Functions - Pitch

To derive the transfer function from elevator to pitch angle

$$\begin{aligned}
\dot{\theta} &= q\cos\phi - r\sin\phi \\
&= q + q(\cos\phi - 1) - r\sin\phi \\
&\stackrel{\triangle}{=} q + d_{\theta_1}
\end{aligned}$$

Differentiating and substituting $\dot{q}$ from the dynamics gives

$$\begin{aligned}
\ddot{\theta} &= \Gamma_6(r^2 - p^2) + \Gamma_5 pr + \frac{\rho V_a^2 cS}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right] + \dot{d}_{\theta_1} \\
&= \Gamma_6(r^2 - p^2) + \Gamma_5 pr + \frac{\rho V_a^2 cS}{2J_y}\left[C_{m_0} + C_{m_\alpha}(\theta - \gamma) + C_{m_q}\frac{c}{2V_a}(\dot{\theta} - d_{\theta 1}) + C_{m_{\delta_e}}\delta_e\right] + \dot{d}_{\theta_1} \\
&= \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_q}\frac{c}{2V_a}\right)\dot{\theta} + \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_\alpha}\right)\theta + \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_{\delta_e}}\right)\delta_e + \Big\{\Gamma_6(r^2 - p^2) \\
&\qquad + \Gamma_5 pr + \frac{\rho V_a^2 cS}{J_y}\left[C_{m_0} - C_{m_\alpha}\gamma - C_{m_q}\frac{c}{2V_a}d_{\theta_1}\right] + \dot{d}_{\theta_1}\Big\} \\
&= -a_{\theta_1}\dot{\theta} - a_{\theta_2}\theta + a_{\theta_3}\delta_e + d_{\theta_2}
\end{aligned}$$

Equations of Motion

$$\dot{p}_n = (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w$$

$$\dot{p}_e = (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w$$

$$\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\text{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\text{prop}} C_{\text{prop}}}{2\text{m}}\left[(k_{\text{motor}}\delta_t)^2 - V_a^2\right]$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\text{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\text{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right]$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$

$$\boxed{\dot{\theta} = q\cos\phi - r\sin\phi}$$

$$\dot{\psi} = q\sin\phi\sec\theta + r\cos\phi\sec\theta$$

$$\dot{p} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right]$$

$$\boxed{\dot{q} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right]}$$

$$\dot{r} = \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]$$

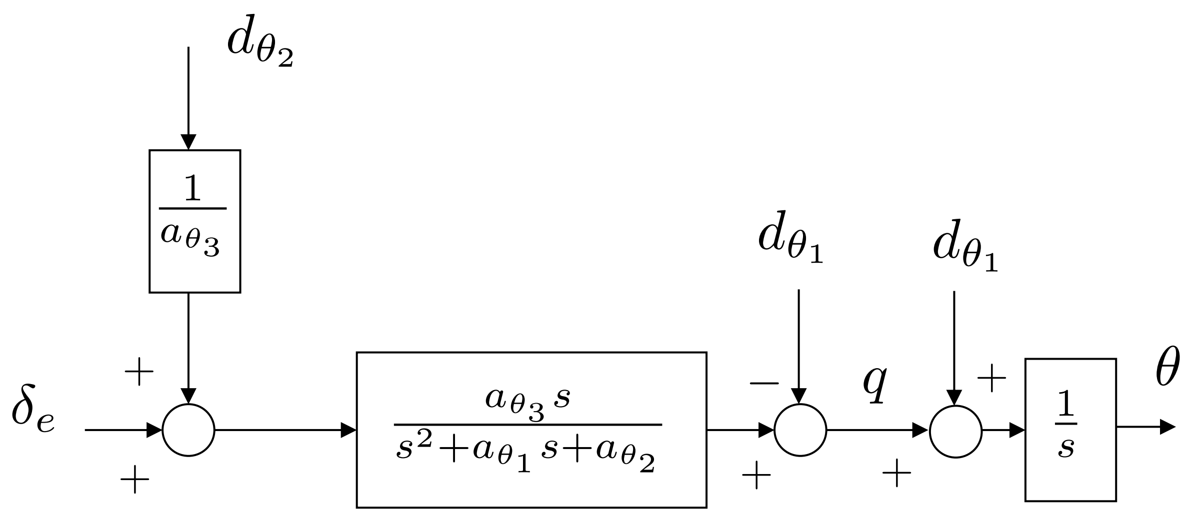
Longitudinal Transfer Functions - Pitch

The differential equation model is

$$\ddot{\theta} = -a_{\theta_1}\dot{\theta} - a_{\theta_2}\theta + a_{\theta_3}\delta_e + d_{\theta_2}$$

with associated transfer function

$$\theta(s) = \left(\frac{a_{\theta_3}}{s^2 + a_{\theta_1}s + a_{\theta_2}}\right)\left(\delta_e(s) + \frac{1}{a_{\theta_3}}d_{\theta_2}(s)\right)$$



Longitudinal TF - Altitude from Pitch

To derive the transfer function from pitch to altitude (assuming constant airspeed)

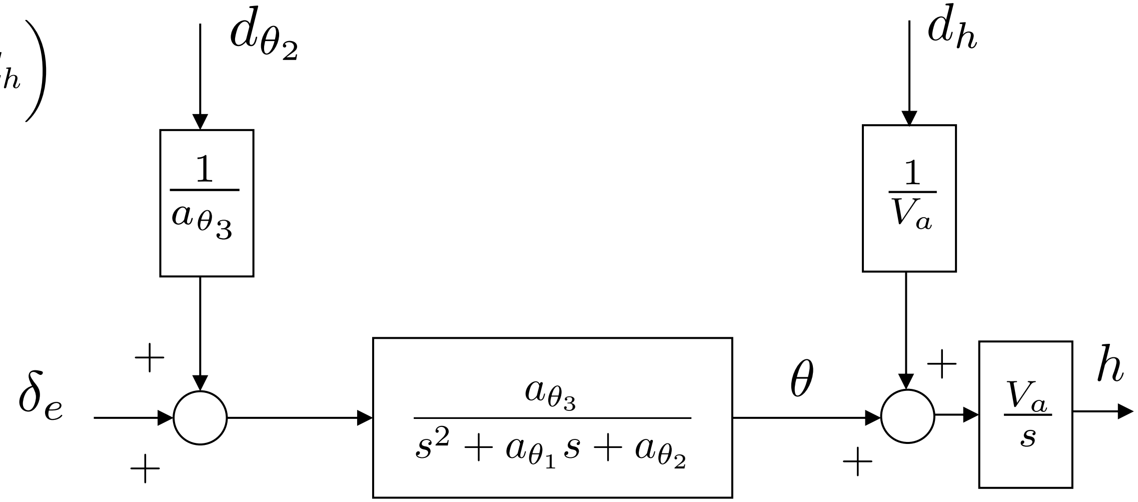
$$\begin{aligned}\dot{h} &= u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\ &= V_a\theta + (u\sin\theta - V_a\theta) - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\ &= V_a\theta + d_h\end{aligned}$$

where

$$d_h \stackrel{\triangle}{=} (u\sin\theta - V_a\theta) - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

The associated transfer function is

$$h(s) = \frac{V_a}{s}\left(\theta + \frac{1}{V_a}d_h\right)$$



Equations of Motion

$$
\begin{aligned}
\dot{p}_n &= (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w \\
\dot{p}_e &= (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w \\
&\boxed{\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta} \\
\dot{u} &= rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right] \\
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{w} &= qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right] \\
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= q\sin\phi\sec\theta + r\cos\phi\sec\theta \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{q} &= \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]
\end{aligned}
$$

Longitudinal Transfer Functions - Airspeed

Both pitch angle and throttle strongly affect airspeed. We wish to derive the relationship.

If there is no wind, then

$$V_a = \sqrt{u^2 + v^2 + w^2}$$

Differentiating gives

$$\dot{V}_a = \dot{u}\cos\alpha\cos\beta + \dot{v}\sin\beta + \dot{w}\sin\alpha\cos\beta = \dot{u}\cos\alpha + \dot{w}\sin\alpha + d_{V_1}$$

where

$$d_{V_1} = -\dot{u}(1-\cos\beta)\cos\alpha - \dot{w}(1-\cos\beta)\sin\alpha + \dot{v}\sin\beta$$

Substituting from the dynamics gives

$$\begin{aligned}
\dot{V}_a &= rV_a\cos\alpha\sin\beta - pV_a\sin\alpha\sin\beta - g\cos\alpha\sin\theta + g\sin\alpha\cos\theta\cos\phi \\
&\quad + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_D(\alpha) - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_1} \\
&= (rV_a\cos\alpha - pV_a\sin\alpha)\sin\beta - g\sin(\theta-\alpha) - g\sin\alpha\cos\theta(1-\cos\phi) \\
&\quad + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_1} \\
&= -g\sin\gamma + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_2}
\end{aligned}$$

Longitudinal Transfer Functions - Airspeed

$$\dot{V}_a = -g\sin\gamma + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_2}$$

Since this equation is nonlinear in V_a, we will linearize about trim: α^*, V_a^*, θ^*, δ_e^*, δ_t^*, and let

$$\bar{V}_a = V_a - V_a^*, \qquad \bar{\theta} = \theta - \theta^*, \qquad \bar{\delta}_e = \delta_e - \delta_e^*, \qquad \bar{\delta}_t = \delta_e - \delta_t^*$$

to get

$$\begin{aligned}
\dot{\bar{V}}_a &= -g\cos(\theta^* - \alpha^*)\bar{\theta} + \left\{\frac{\rho V_a^* S}{\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha^* - C_{D_{\delta_e}}\delta_e^*\right] + \frac{1}{\mathsf{m}}\frac{\partial T_p}{\partial V_a}(\delta_t^*, V_a^*)\right\}\bar{V}_a \\
&\qquad + \frac{1}{\mathsf{m}}\frac{\partial T_p}{\partial \delta_t}(\delta_t^*, V_a^*)\bar{\delta}_t + d_V \\
&= -a_{V_1}\bar{V}_a + a_{V_2}\bar{\delta}_t - a_{V_3}\bar{\theta} + d_V
\end{aligned}$$

where

$$\begin{aligned}
a_{V_1} &= \frac{\rho V_a^* S}{\mathsf{m}}\left[C_{D_0} + C_{D_\alpha}\alpha^* + C_{D_{\delta_e}}\delta_e^*\right] - \frac{1}{\mathsf{m}}\frac{\partial T_p}{\partial V_a}(\delta_t^*, V_a^*) \\
a_{V_2} &= \frac{1}{\mathsf{m}}\frac{\partial T_p}{\partial \delta_t}(\delta_t^*, V_a^*) \\
a_{V_3} &= g\cos(\theta^* - \alpha^*)
\end{aligned}$$

Longitudinal Transfer Functions - Airspeed

The associated transfer function is

$$\bar{V}_a(s) = \frac{1}{s + a_{V_1}} \left(a_{V_2}\bar{\delta}_t(s) - a_{V_3}\bar{\theta}(s) + d_V(s) \right)$$

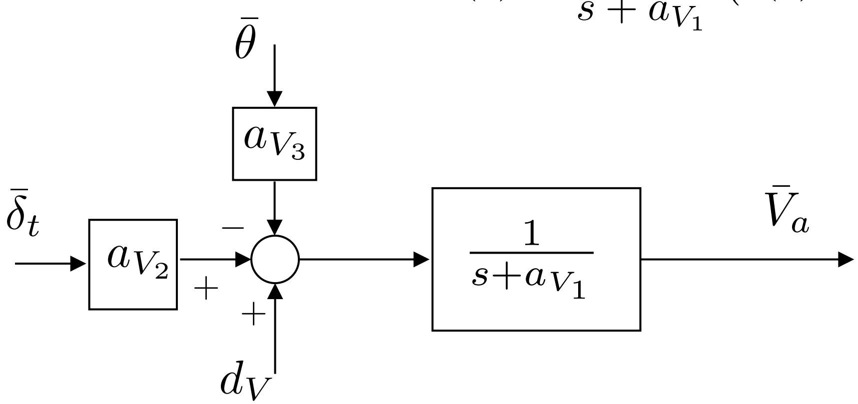
Key inputs are pitch angle and throttle command

For throttle we have

$$\bar{V}_a(s) = \frac{a_{V_2}}{s + a_{V_1}} \left(\bar{\delta}_t(s) + d_V(s) \right)$$

and for pitch angle we have

$$\bar{V}_a(s) = \frac{-a_{V_3}}{s + a_{V_1}} \left(\bar{\theta}(s) + d_V(s) \right)$$



Linear State-space Models

Nonlinear state equations: $\dot{x} = f(x, u)$

Trim condition: $\dot{x}^* = f(x^*, u^*)$

Deviation from trim: $\bar{x} = x - x^*$

Rewriting the state equation in terms of deviation from trim gives

$$\begin{aligned}
\dot{\bar{x}} &= \dot{x} - \dot{x}^* \\
&= f(x, u) - f(x^*, u^*) \\
&= f(x + x^* - x^*, u + u^* - u^*) - f(x^*, u^*) \\
&= f(x^* + \bar{x}, u^* + \bar{u}) - f(x^*, u^*)
\end{aligned}$$

Using a Taylor series expansion around trim to linearize gives

$$\begin{aligned}
\dot{\bar{x}} &= f(x^*, u^*) + \frac{\partial f(x^*, u^*)}{\partial x}\bar{x} + \frac{\partial f(x^*, u^*)}{\partial u}\bar{u} + H.O.T - f(x^*, u^*) \\
&\approx \frac{\partial f(x^*, u^*)}{\partial x}\bar{x} + \frac{\partial f(x^*, u^*)}{\partial u}\bar{u} \\
&\triangleq A\bar{x} + B\bar{u}
\end{aligned}$$

Lateral State-space Equations

The lateral states and inputs are defined as

$$\dot{x}_{\text{lat}} \stackrel{\triangle}{=} (v, p, r, \phi, \psi)^\top \qquad u_{\text{lat}} \stackrel{\triangle}{=} (\delta_a, \delta_r)^\top$$

The nonlinear equations of motion are

$$\begin{aligned}
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2\mathrm{m}}\frac{b}{2}\left[C_{Y_p}p + C_{Y_r}r\right] \\
&\quad + \frac{\rho(u^2+v^2+w^2)S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2}\frac{b^2}{2}\left[C_{p_p}p + C_{p_r}r\right] + \Gamma_3 Q_p(\delta_t, V_a) \\
&\quad + \frac{1}{2}\rho(u^2+v^2+w^2)Sb\left[C_{p_0} + C_{p_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2}\frac{b^2}{2}\left[C_{r_p}p + C_{r_r}r\right] + \Gamma_4 Q_p(\delta_t, V_a) \\
&\quad + \frac{1}{2}\rho(u^2+v^2+w^2)Sb\left[C_{r_0} + C_{r_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right] \\
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\psi} &= q\sin\phi\sec\theta + r\cos\phi\sec\theta
\end{aligned}$$

where we have used $\beta = \tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right)$ and $V_a = \sqrt{u^2+v^2+w^2}$

Jacobian Matrices

$$A_{\rm lat} = \frac{\partial f_{\rm lat}}{\partial x_{\rm lat}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial v} & \frac{\partial \dot{v}}{\partial p} & \frac{\partial \dot{v}}{\partial r} & \frac{\partial \dot{v}}{\partial \phi} & \frac{\partial \dot{v}}{\partial \psi} \\ \frac{\partial \dot{p}}{\partial v} & \frac{\partial \dot{p}}{\partial p} & \frac{\partial \dot{p}}{\partial r} & \frac{\partial \dot{p}}{\partial \phi} & \frac{\partial \dot{p}}{\partial \psi} \\ \frac{\partial \dot{r}}{\partial v} & \frac{\partial \dot{r}}{\partial p} & \frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial \phi} & \frac{\partial \dot{r}}{\partial \psi} \\ \frac{\partial \dot{\phi}}{\partial v} & \frac{\partial \dot{\phi}}{\partial p} & \frac{\partial \dot{\phi}}{\partial r} & \frac{\partial \dot{\phi}}{\partial \phi} & \frac{\partial \dot{\phi}}{\partial \psi} \\ \frac{\partial \dot{\psi}}{\partial v} & \frac{\partial \dot{\psi}}{\partial p} & \frac{\partial \dot{\psi}}{\partial r} & \frac{\partial \dot{\psi}}{\partial \phi} & \frac{\partial \dot{\psi}}{\partial \psi} \end{pmatrix}$$

$$B_{\rm lat} = \frac{\partial f_{\rm lat}}{\partial u_{\rm lat}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial \delta_a} & \frac{\partial \dot{v}}{\partial \delta_r} \\ \frac{\partial \dot{p}}{\partial \delta_a} & \frac{\partial \dot{p}}{\partial \delta_r} \\ \frac{\partial \dot{r}}{\partial \delta_a} & \frac{\partial \dot{r}}{\partial \delta_r} \\ \frac{\partial \dot{\phi}}{\partial \delta_a} & \frac{\partial \dot{\phi}}{\partial \delta_r} \\ \frac{\partial \dot{\psi}}{\partial \delta_a} & \frac{\partial \dot{\psi}}{\partial \delta_r} \end{pmatrix}$$

Take partial derivatives and evaluate them at trim state and trim input

Lateral State-space Equations

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{w}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w & X_q & -g\cos\theta^* & 0 \\ Z_u & Z_w & Z_q & -g\sin\theta^* & 0 \\ M_u & M_w & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^* & -\cos\theta^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}$$

Lateral	Formula
Y_v	$\frac{\rho S b v^*}{4mV_a^*}\left[C_{Y_p}p^* + C_{Y_r}r^*\right] + \frac{\rho S v^*}{\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta^* + C_{Y_{\delta_a}}\delta_a^* + C_{Y_{\delta_r}}\delta_r^*\right] + \frac{\rho S C_{Y_\beta}}{2\mathrm{m}}\sqrt{u^{*2} + w^*}$
Y_p	[illegible]
Y_r	[illegible]
Y_{δ_a}	$\frac{\rho V_a^{*2} S}{2\mathrm{m}} C_{Y_{\delta_a}}$
Y_{δ_r}	$\frac{\rho V_a^{*2} S}{2\mathrm{m}} C_{Y_{\delta_r}}$
L_v	$\frac{\rho S b^2 v^*}{4V_a^*}\left[C_{p_p}p^* + C_{p_r}r^*\right] + \rho S b v^*\left[C_{p_0} + C_{p_\beta}\beta^* + C_{p_{\delta_a}}\delta_a^* + C_{p_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{p_\beta}}{2}\sqrt{u^{*2} + w}$
L_p	$\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_p}$
L_r	$-\Gamma_2 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_r}$
L_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_a}}$
L_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_r}}$
N_v	$\frac{\rho S b^2 v^*}{4V_a^*}\left[C_{r_p}p^* + C_{r_r}r^*\right] + \rho S b v^*\left[C_{r_0} + C_{r_\beta}\beta^* + C_{r_{\delta_a}}\delta_a^* + C_{r_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{r_\beta}}{2}\sqrt{u^{*2} + w}$
N_p	$\Gamma_7 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_p}$
N_r	$-\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_r}$
N_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_a}}$
N_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_r}}$

dimensional stability derivatives

Alternative Form - Lateral

The lateral dynamics are often expressed using β instead of v. Recall that

$$v = V_a \sin\beta$$

Therefore

$$\bar{v} = V_a^* \cos\beta^* \bar{\beta}$$

Differentiating gives

$$\dot{\bar{\beta}} = \frac{1}{V_a^* \cos\beta^*}\dot{\bar{v}}$$

Utilizing side-slip angle instead of lateral velocity

The associated state space equations are

$$\begin{pmatrix} \dot{\bar{\beta}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & \frac{Y_p}{V_a^* \cos\beta^*} & \frac{Y_r}{V_a^* \cos\beta^*} & \frac{g\cos\theta^*\cos\phi^*}{V_a^*\cos\beta^*} & 0 \\ L_v V_a^* \cos\beta^* & L_p & L_r & 0 & 0 \\ N_v V_a^* \cos\beta^* & N_p & N_r & 0 & 0 \\ 0 & 1 & \cos\phi^*\tan\theta^* & \begin{matrix} q^*\cos\phi^*\tan\theta^* \\ -r^*\sin\phi^*\tan\theta^* \end{matrix} & 0 \\ 0 & 0 & \cos\phi^*\sec\theta^* & \begin{matrix} p^*\cos\phi^*\sec\theta^* \\ -r^*\sin\phi^*\sec\theta^* \end{matrix} & 0 \end{pmatrix} \begin{pmatrix} \bar{\beta} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} \frac{Y_{\delta_a}}{V_a^*\cos\beta^*} & \frac{Y_{\delta_r}}{V_a^*\cos\beta^*} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}$$

Longitudinal State-space Equations

The longitudinal states and inputs are defined as

$$\dot{x}_{\text{lon}} \triangleq (u, w, q, \theta, h)^\top \qquad u_{\text{lon}} \triangleq (\delta_e, \delta_t)^\top$$

The nonlinear equations of motion are

$$\begin{aligned}
\dot{u} &= -qw - g\sin\theta + \frac{\rho(u^2+w^2)S}{2\text{m}}\left[C_{X_0} + C_{X_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{X_{\delta_e}}\delta_e\right] \\
&\qquad + \frac{\rho\sqrt{u^2+w^2}S}{4\text{m}}C_{X_q}cq + \frac{\rho S_{\text{prop}}}{2\text{m}}C_{\text{prop}}\left[(k\delta_t)^2 - (u^2+w^2)\right] \\
\dot{w} &= qu + g\cos\theta + \frac{\rho(u^2+w^2)S}{2\text{m}}\left[C_{Z_0} + C_{Z_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{Z_{\delta_e}}\delta_e\right] \\
&\qquad + \frac{\rho\sqrt{u^2+w^2}S}{4\text{m}}C_{Z_q}cq \\
\dot{q} &= \frac{1}{2J_y}\rho(u^2+w^2)cS\left[C_{m_0} + C_{m_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{m_{\delta_e}}\delta_e\right] + \frac{1}{4J_y}\rho\sqrt{u^2+w^2}SC_{m_q}c^2q \\
\dot{\theta} &= q \\
\dot{h} &= u\sin\theta - w\cos\theta
\end{aligned}$$

where we have used $\alpha = \tan^{-1}\left(\frac{w}{u}\right)$ and $V_a = \sqrt{u^2+w^2}$

Jacobian Matrices

$$A_{\text{lon}} = \frac{\partial f_{\text{lon}}}{\partial x_{\text{lon}}} = \begin{pmatrix} \frac{\partial \dot{u}}{\partial u} & \frac{\partial \dot{u}}{\partial w} & \frac{\partial \dot{u}}{\partial q} & \frac{\partial \dot{u}}{\partial \theta} & \frac{\partial \dot{u}}{\partial h} \\ \frac{\partial \dot{w}}{\partial u} & \frac{\partial \dot{w}}{\partial w} & \frac{\partial \dot{w}}{\partial q} & \frac{\partial \dot{w}}{\partial \theta} & \frac{\partial \dot{w}}{\partial h} \\ \frac{\partial \dot{q}}{\partial u} & \frac{\partial \dot{q}}{\partial w} & \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial \theta} & \frac{\partial \dot{q}}{\partial h} \\ \frac{\partial \dot{\theta}}{\partial u} & \frac{\partial \dot{\theta}}{\partial w} & \frac{\partial \dot{\theta}}{\partial q} & \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial h} \\ \frac{\partial \dot{h}}{\partial u} & \frac{\partial \dot{h}}{\partial w} & \frac{\partial \dot{h}}{\partial q} & \frac{\partial \dot{h}}{\partial \theta} & \frac{\partial \dot{h}}{\partial h} \end{pmatrix}$$

$$B_{\text{lon}} = \frac{\partial f_{\text{lon}}}{\partial u_{\text{lon}}} = \begin{pmatrix} \frac{\partial \dot{u}}{\partial \delta_e} & \frac{\partial \dot{u}}{\partial \delta_t} \\ \frac{\partial \dot{w}}{\partial \delta_e} & \frac{\partial \dot{w}}{\partial \delta_t} \\ \frac{\partial \dot{q}}{\partial \delta_e} & \frac{\partial \dot{q}}{\partial \delta_t} \\ \frac{\partial \dot{\theta}}{\partial \delta_e} & \frac{\partial \dot{\theta}}{\partial \delta_t} \\ \frac{\partial \dot{h}}{\partial \delta_e} & \frac{\partial \dot{h}}{\partial \delta_t} \end{pmatrix}$$

Take partial derivatives and evaluate them at trim state and trim input

Longitudinal State-space Equations

$$
\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{w}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w & X_q & -g\cos\theta^* & 0 \\ Z_u & Z_w & Z_q & -g\sin\theta^* & 0 \\ M_u & M_w & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^* & -\cos\theta^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}
$$

Longitudinal	Formula
X_u	$\frac{u^*\rho S}{\mathsf{m}}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta_e}}\delta_e^*\right] - \frac{\rho S w^* C_{X_\alpha}}{2\mathsf{m}} + \frac{\rho S c C_{X_q} u^* q^*}{4\mathsf{m}V_a^*} - \frac{\rho S_{\text{prop}} C_{\text{prop}} u^*}{\mathsf{m}}$
X_w	$-q^* + \frac{w^*\rho S}{\mathsf{m}}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta_e}}\delta_e^*\right] + \frac{\rho S c C_{X_q} w^* q^*}{4\mathsf{m}V_a^*} + \frac{\rho S C_{X_\alpha} u^*}{2\mathsf{m}} - \frac{\rho S_{\text{prop}} C_{\text{prop}} w^*}{\mathsf{m}}$
X_q	$-w^* + \frac{\rho V_a^* S C_{X_q} c}{4\mathsf{m}}$
X_{δ_e}	$\frac{\rho V_a^{*2} S C_{X_{\delta_e}}}{2\mathsf{m}}$
X_{δ_t}	[illegible]
Z_u	[illegible] $\frac{\ldots w^*}{2\mathsf{m}} + \frac{u^*\rho S C_{Z_q} c q^*}{4\mathsf{m}V_a^*}$
Z_w	$\frac{w^*\rho S}{\mathsf{m}}\left[C_{Z_0} + C_{Z_\alpha}\alpha^* + C_{Z_{\delta_e}}\delta_e^*\right] + \frac{\rho S C_{Z_\alpha} u^*}{2\mathsf{m}} + \frac{\rho w^* S c C_{Z_q} q^*}{4\mathsf{m}V_a^*}$
Z_q	$u^* + \frac{\rho V_a^* S C_{Z_q} c}{4\mathsf{m}}$
Z_{δ_e}	$\frac{\rho V_a^{*2} S C_{Z_{\delta_e}}}{2\mathsf{m}}$
M_u	$\frac{u^*\rho S c}{J_y}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta_e}}\delta_e^*\right] - \frac{\rho S c C_{m_\alpha} w^*}{2J_y} + \frac{\rho S c^2 C_{m_q} q^* u^*}{4J_y V_a^*}$
M_w	$\frac{w^*\rho S c}{J_y}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta_e}}\delta_e^*\right] + \frac{\rho S c C_{m_\alpha} u^*}{2J_y} + \frac{\rho S c^2 C_{m_q} q^* w^*}{4J_y V_a^*}$
M_q	$\frac{\rho V_a^* S c^2 C_{m_q}}{4J_y}$
M_{δ_e}	$\frac{\rho V_a^{*2} S c C_{m_{\delta_e}}}{2J_y}$

dimensional stability derivatives

Alternative Form – Longitudinal

The longitudinal dynamics are often expressed using α instead of w. Recall that

$$w = V_a \sin\alpha \cos\beta.$$

At trim, when $\beta = 0$ we have

$$\bar{w} = V_a^* \cos\alpha^* \bar{\alpha}.$$

Utilizing angle of attack instead of downward velocity

Differentiating gives

$$\dot{\bar{\alpha}} = \frac{1}{V_a^* \cos\alpha^*} \dot{\bar{w}}.$$

The associated state space equations are

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{\alpha}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w V_a^* \cos\alpha^* & X_q & -g\cos\theta^* & 0 \\ \frac{Z_u}{V_a^* \cos\alpha^*} & Z_w & \frac{Z_q}{V_a^* \cos\alpha^*} & \frac{-g\sin\theta^*}{V_a^* \cos\alpha^*} & 0 \\ M_u & M_w V_a^* \cos\alpha^* & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^* & -V_a^* \cos\theta^* \cos\alpha^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{\alpha} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ \frac{Z_{\delta_e}}{V_a^* \cos\alpha^*} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}$$

Numerical Computation of State Space Equations

Let $w = f(z)$ where $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, i.e.,

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

The Jacobian of f with respect to x is defined as

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \ldots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Numerical Computation of State Space Equations

Let

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}, \qquad \text{where} \qquad \frac{\partial f}{\partial x_i} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \frac{\partial f_2}{\partial x_i} \\ \vdots \\ \frac{\partial f_n}{\partial x_i} \end{pmatrix}.$$

If ϵ is a small fixed number, then

$$\frac{\partial f}{\partial x_i}(x) \approx \frac{f(x + \epsilon e_i) - f(x)}{\epsilon},$$

which can be computed numerically, with two calls of $f(\cdot)$.

Note that computing $\frac{\partial f}{\partial x}(x)$ will require $n+1$ functions calls to $f(\cdot)$.

Numerical Computation of State Space Equations

```
import numpy as np
def df_dx(x, z):
    # take partial of f(x, z) with respect to x
    eps = 0.001  # deviation
    A = np.zeros((m, n))  # Jacobian of f wrt x
    f_at_x = f(x,z)
    for i in range(0, n):
        x_eps = np.copy(x)
        x_eps[i][0] += eps  # add eps to ith state
        f_at_x_eps = f(x_eps, z)
        df_dxi = (f_at_x_eps - f_at_x) / eps
        A[:,i] = df_dxi[:,0]
    return A
```

Numerical Computation of State Space Equations

Define:

$$\text{The Euler state: } x_e = (p_n, p_e, p_d, u, v, w, \phi, \theta, \psi, p, q, r)^\top \in \mathbb{R}^{12}$$
$$\text{The Quaternion state: } x_q = (p_n, p_e, p_d, u, v, w, e_0, e_x, e_y, e_z, p, q, r)^\top \in \mathbb{R}^{13}$$

Let

$$\begin{aligned}
\mathbf{p} &= (p_n, p_e, p_d)^\top \\
\mathbf{v} &= (u, v, w)^\top \\
\boldsymbol{\vartheta} &= (\phi, \theta, \psi)^\top \\
\mathbf{q} &= (e_0, e_1, e_2, e_3)^\top \\
\boldsymbol{\omega} &= (p, q, r)^\top,
\end{aligned}$$

Then

$$\begin{aligned}
x_e &= (\mathbf{p}^\top, \mathbf{v}^\top, \boldsymbol{\vartheta}^\top, \boldsymbol{\omega}^\top)^\top \\
x_q &= (\mathbf{p}^\top, \mathbf{v}^\top, \mathbf{q}^\top, \boldsymbol{\omega}^\top)^\top.
\end{aligned}$$

Numerical Computation of State Space Equations

Let $\boldsymbol{\vartheta} = \Theta(\mathbf{q})$ convert quaternions to Euler angles, and $\mathbf{q} = Q(\boldsymbol{\vartheta})$ converts Euler angles to quaternions.
Define

$$T_e \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \\ \mathbf{q} \\ \boldsymbol{\omega} \end{pmatrix} \triangleq \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \\ \Theta(\mathbf{q}) \\ \boldsymbol{\omega} \end{pmatrix}$$

$$T_q \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \\ \boldsymbol{\vartheta} \\ \boldsymbol{\omega} \end{pmatrix} \triangleq \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \\ Q(\boldsymbol{\vartheta}) \\ \boldsymbol{\omega} \end{pmatrix},$$

where

$$\frac{\partial T_e}{\partial x_q}(x_q) = \begin{pmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 4} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 4} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \frac{\partial \Theta}{\partial \mathbf{q}}(x_q) & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 4} & I_{3\times 3} \end{pmatrix}.$$

Numerical Computation of State Space Equations

The Python dynamics file returns:

$$\dot{x}_q = f_q(x_q, u).$$

In terms of Euler states we have:

$$\begin{aligned}\dot{x}_e &= \frac{d}{dt} T_e(x_q) = \frac{\partial T_e}{\partial x_q}(x_q)\dot{x}_q = \frac{\partial T_e}{\partial x_q}(x_q) f_q(x_q, u) \\ &= \frac{\partial T_e}{\partial x_q}(T_q(x_e)) f_q(T_q(x_e), u)).\end{aligned}$$

If $\tilde{x}_e = x_e - x_e^*$, $\tilde{u} = u - u^*$, the state space equations are

$$\dot{\tilde{x}}_e = A\tilde{x}_e + B\tilde{u},$$

where

$$\begin{aligned}A &= \frac{\partial}{\partial x_e}\left(\frac{\partial T_e}{\partial x_q}(T_q(x_e)) f_q(T_q(x_e), u))\right) \\ B &= \frac{\partial}{\partial u}\left(\frac{\partial T_e}{\partial x_q}(T_q(x_e)) f_q(T_q(x_e), u))\right).\end{aligned}$$

Longitudinal State Space Equations

The longitudinal states and inputs are

$$x_{lon} = \begin{pmatrix} u \\ w \\ q \\ \theta \\ h \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_e \\ p_d \\ u \\ v \\ w \\ \phi \\ \theta \\ \psi \\ p \\ q \\ r \end{pmatrix} \triangleq E_1 x_e$$

$$u_{lon} = \begin{pmatrix} \delta_e \\ \delta_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_t \\ \delta_e \\ \delta_a \\ \delta_r \end{pmatrix} \triangleq E_2 u.$$

Therefore the longitudinal state space equations are

$$\dot{x}_{lon} = A_{lon} x_{lon} + B_{lon} u_{lon}$$

where

$$A_{lon} = E_1 A E_1^\top$$
$$B_{lon} = E_1 B E_2^\top.$$

Lateral State Space Equations

The lateral states and inputs are

$$x_{lat} = \begin{pmatrix} v \\ p \\ r \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_e \\ p_d \\ u \\ v \\ w \\ \phi \\ \theta \\ \psi \\ p \\ q \\ r \end{pmatrix} \triangleq E_3 x_e$$

$$u_{lat} = \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_t \\ \delta_e \\ \delta_a \\ \delta_r \end{pmatrix} \triangleq E_4 u,$$

Therefore the lateral state space equations are

$$\dot{x}_{lat} = A_{lat} x_{lat} + B_{lat} u_{lat}$$

where

$$A_{lat} = E_3 A E_3^\top$$
$$B_{lat} = E_3 B E_4^\top.$$

Reduced Order Modes

<u>Longitudinal Modes</u>

- Short-period Mode
    - Short-period mode is the fast mode seen in pitch rate q and pitch angle
- Phugoid Mode
    - Phugoid mode is slow mode seen in pitch angle and u
- Induced by impulse on elevator

<u>Lateral Modes</u>

- Roll Mode
    - First order fast mode between aileron and roll rate
- Spiral-divergence Mode
    - First order (really) slow mode between aileron and yaw/course
- Dutch-roll Mode
    - Second order mode: coupling between roll, yaw, and side slip. Like a duck wagging its tail
- Induced by doublet on aileron or rudder

Simulation Project

- Find trim states and inputs. For the aerosonde model use $V_a = 25$ m/s, $\gamma = 0$, and $R = \infty$ m.
- Find the transfer functions derived in this chapter.
- Find the state space models derived in this chapter.
- Compute the eigenvalues of A_{lon} and A_{lat} and associate them with the phugoid mode, the short period mode, the roll mode, the dutch roll mode, and the spiral mode.