Moment of a Force (Torque) About a Point

Text: 4.1 – 4.4

Content:

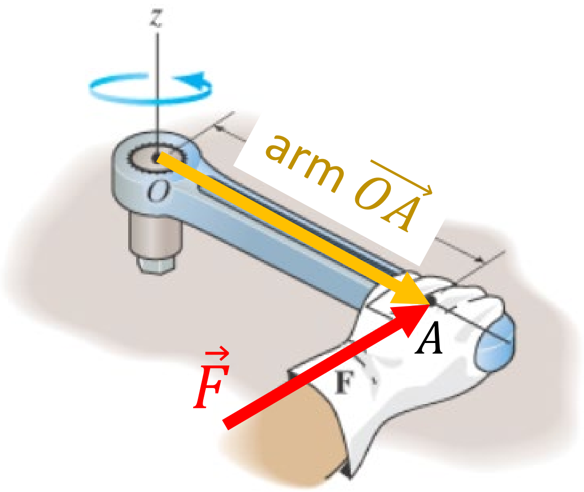
- Moment about a point: scalar definition (using moment arm)
- Moment as a vector
- Right-Hand Rule (RHR)
- Cross product of two vectors
- Moment about a point as a scalar product

MOMENT OF A FORCE (= TORQUE)

- Moment of a force: a quantity that describes the **amount of rotation** caused by a force applied to a body.

What does the rotation effect depend on?

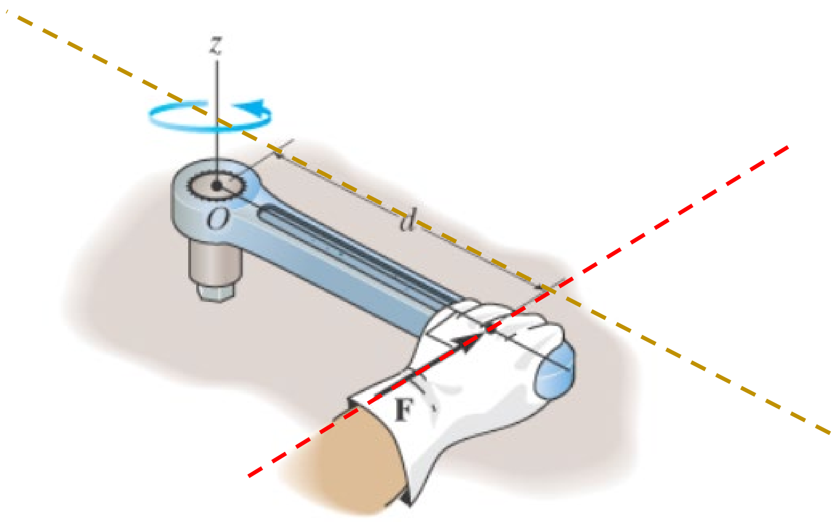
- Applied force magnitude, F: larger force => larger rotation
- Distance, d, from the point O to the line of action of the force (moment arm): larger distance => larger rotation (that is why the doorknobs are at the maximum distance from the hinges!)



- Angle between the force and the moment arm, $\overrightarrow{OA}$
  - Maximum rotation when $\vec{F} \perp \overrightarrow{OA}$
  - No rotation when $\vec{F} \parallel \overrightarrow{OA}$
- Direction: rotations can be clockwise (cw) and counterclockwise (ccw)

MOMENT OF A FORCE (= TORQUE): Scalar definition $\neq 1$

- Moment of a force: a quantity that describes the **amount of rotation** caused by a force applied to a body.



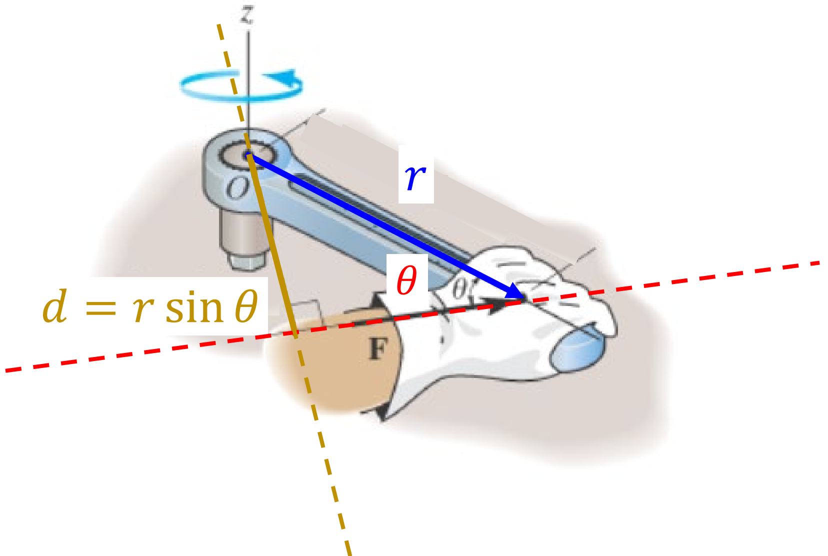
The scalar definition of a moment **about a point $\boldsymbol{O}$**:

$$\boxed{M_O = \pm F d} = \pm F \cdot r \cdot \sin\theta$$

$\widehat{\vec{F}, \vec{r}}$

O → arbitrary point on line of action

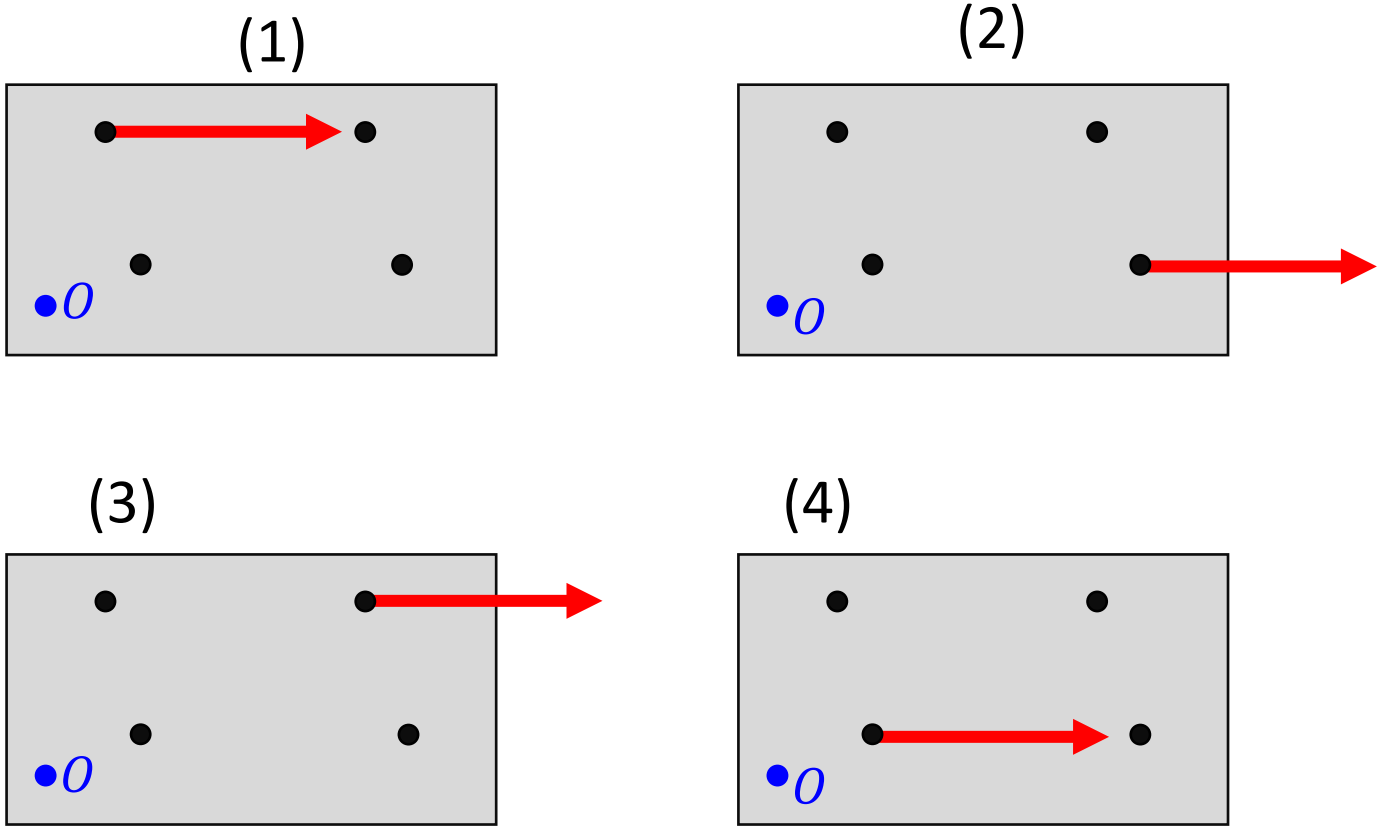
- ➢ F = the magnitude of the applied force
- ➢ d = moment arm = perpendicular distance between O and the line of action of the force $\vec{F}$
- ➢ + for CCW rotation, − for CW rotation



- Note that you can connect O with an arbitrary point at the line of action of the force $\vec{F}$ by a vector $\vec{r}$, and use the angle between $\vec{F}$ and $\vec{r}$ to find d:

$$d = r\sin\theta \quad \Rightarrow \quad M_O = \pm F r \sin\theta$$

Q: Consider rotation of these 2D rectangles about point O. The red arrow is a force vector. Which of these rectangles has the largest moment about point O?

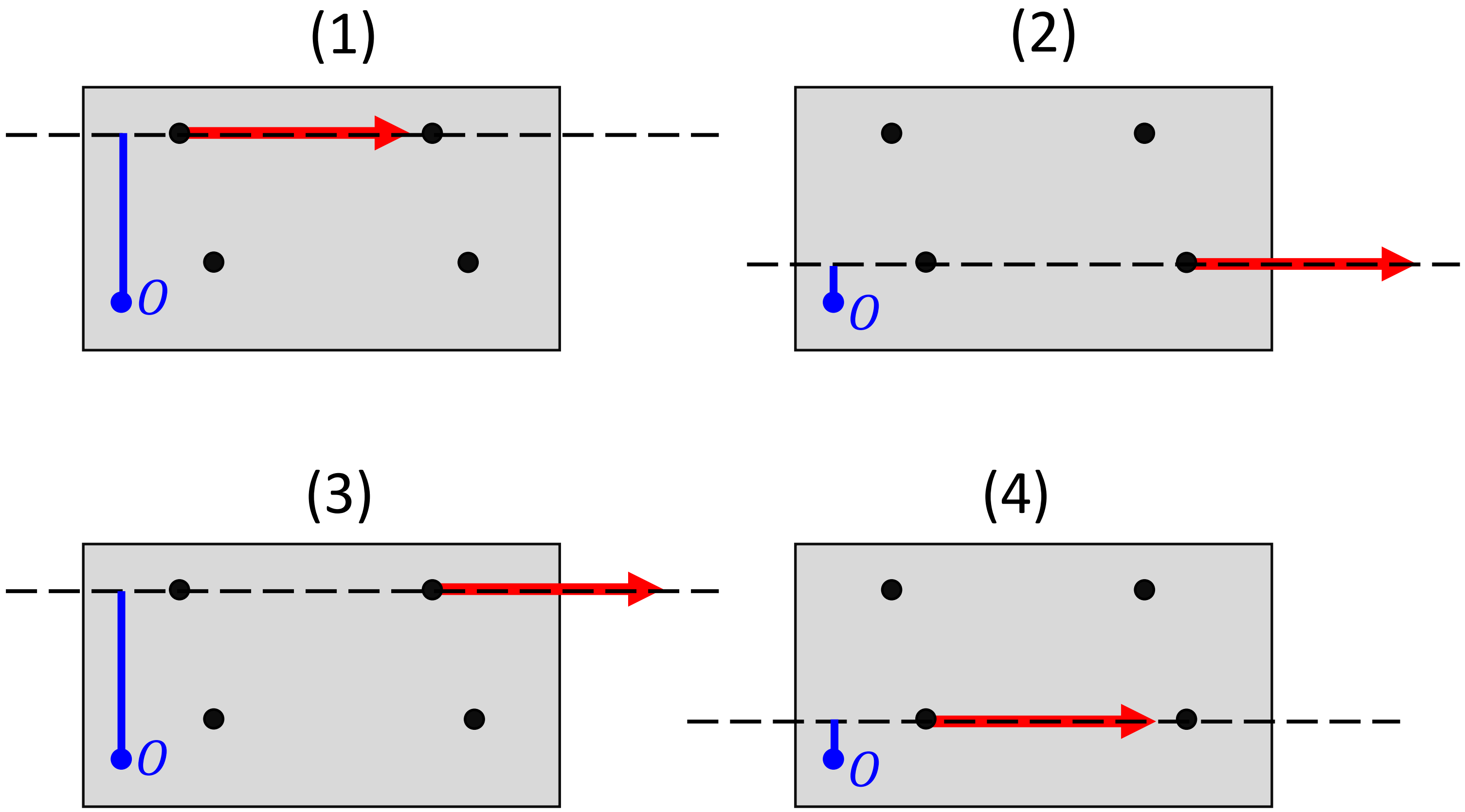


A. Only (1)
B. Only (2)
C. Only (3)
D. Only (4)
E. (1) and (3)

$$M_o = \pm F \cdot d$$

Q: Consider rotation of these 2D rectangles about point O. The red arrow is a force vector. Which of these rectangles has the largest moment about point O?

$$M_O = -Fd$$

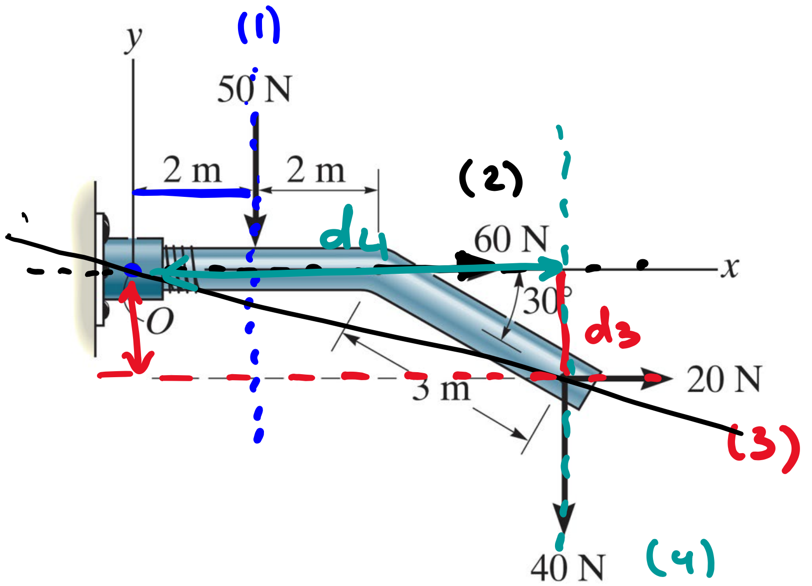


A. Only (1)
B. Only (2)
C. Only (3)
D. Only (4)
E. (1) and (3)

The dashed line shows the line of action of the force, and the blue segment is the arm of the force moment in each case. We see that the arm remains the same if we shift the force anywhere along its line of action. We say it this way: force is a sliding vector.

To figure out a moment arm, it is often convenient to <u>extend the line of action of the force</u> (dotted lines)

Practice: Determine the resultant moment of the four forces acting on the rod about point *O*.



$$M_O^{(1)} = -(50\,N)(2\,m)$$
$$+$$
$$M_O^{(2)} = 0 \qquad (d=0)$$
$$+$$
$$M_O^{(3)} = +(20)(3\sin 20^\circ)$$
$$+$$
$$M_O^{(4)} = -(40)(4 + 3\cos 30^\circ)$$

$$M_O = -334\ N\cdot m,$$

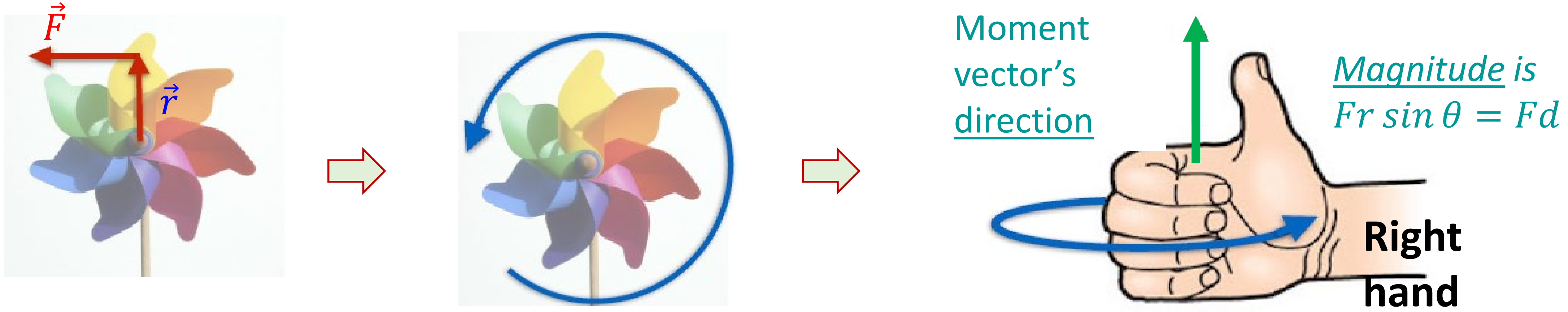
$$\boxed{334\ N\cdot m \text{ clockwise}}$$

$$d_4 = 2 + 2 + 3\cos 30^\circ$$

A. 394 N m
B. -394 N m
C. 334 N m
D. -334 N m
E. Something else

MOMENT OF A FORCE (= TORQUE): Vector definition #2

- It is actually much more convenient and useful to define a moment of a force about a point as a vector:
  - Its magnitude is consistent with our scalar definition: $|\vec{M}_O| = Fr\sin\theta = Fd$
  - Its direction is given by the "curled fingers" **Right-Hand Rule** (RHR) and captures CCW or CW rotation

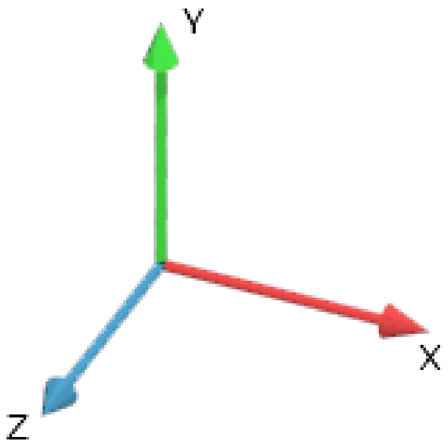


- Curl fingers of your **right** hand so that they follow the sense of expected rotation
- Thumb then points in the direction of the moment
- Note that the moment is perpendicular to the plane containing $\vec{d}$ and $\vec{F}$

- For the picture in the middle, the direction of the moment is **out of the page.**

- Associating a vector with a rotation gives us a great tool in visualizing rotations: If you know that a certain rotation is described by a moment $\vec{M}$, by applying curled-fingers RHR you can immediately figure out: (i) the plane of rotation, and (ii) the direction of rotations!

Q: A moment $\vec{M}$ is directed along positive x. What can you say about the corresponding rotation?



A. Plane XZ, CW (from X to Z)
B. Plane XZ, CCW (from Z to X)
C. Plane YZ, CW (from Z to Y)
D. Plane YZ, CCW (from Y to Z)
E. Correct option is not shown

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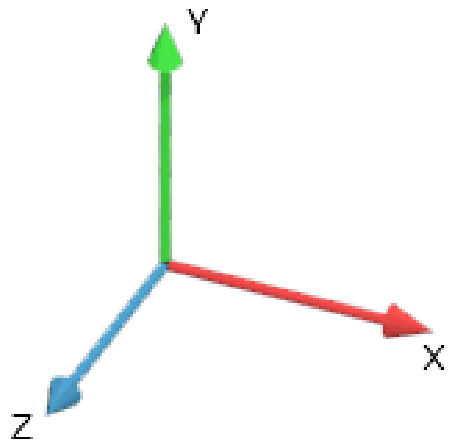


A. Plane XZ, CW (from X to Z)

B. Plane XZ, CCW (from Z to X)

C. Plane YZ, CW (from Z to Y)

D. Plane YZ, CCW (from Y to Z)

E. Correct option is not shown

The rotation is in the plane perpendicular to $\vec{M}$ => it is (y,z)-plane. Direction: place your right thumb along positive x, then your curled fingers will show CCW rotation (assuming that you are looking from positive-x direction).

- Associating a vector with a rotation gives us a great tool in visualizing rotations: If you know that a certain rotation is described by a moment $\vec{M}$, by applying curled-fingers RHR you can immediately figure out: (i) the plane of rotation, and (ii) the direction of rotations!

Q: What's the direction of moment $\vec{M}$ associated with my rotation?

A. Up
B. Down
C. Towards the class
D. Away from the class
E. Left
F. Right

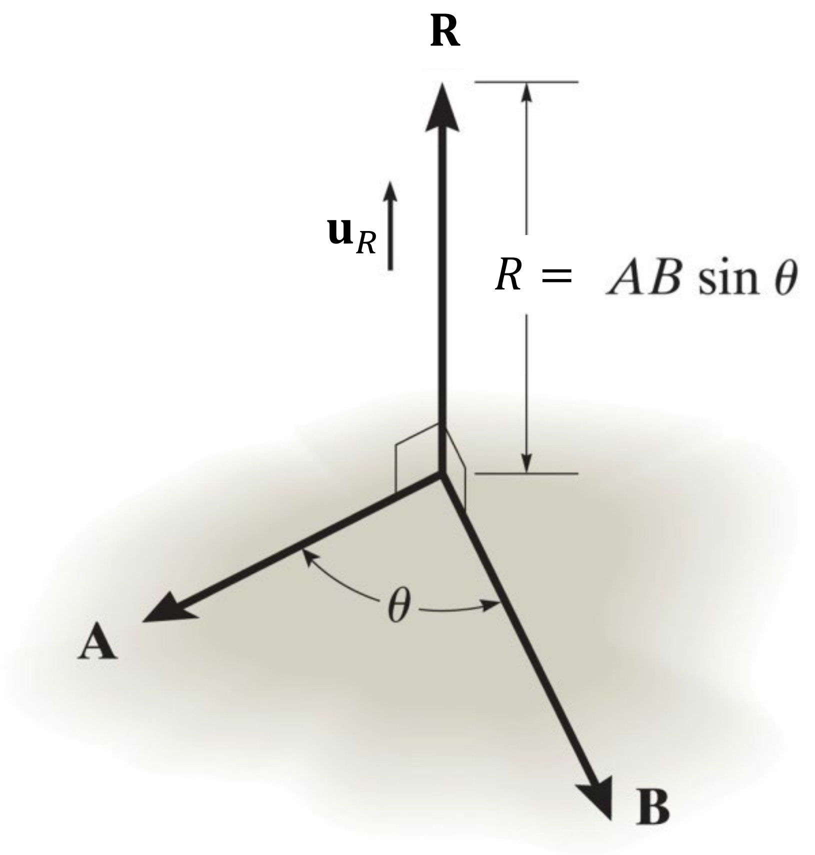
If F is your choice, don't submit your answer, but make a wish instead. It will come true if it's the correct choice.

CROSS PRODUCT OF TWO VECTORS: Math digression

- **(!!!)** In a vector product, each of the three vectors has its own role. Here by $\vec{A}$ we denote the first vector, by $\vec{B}$ the second vector, by $\vec{R}$ the resulting vector.

$$\boxed{\vec{A} \times \vec{B} = \vec{R}}$$

First Second Result



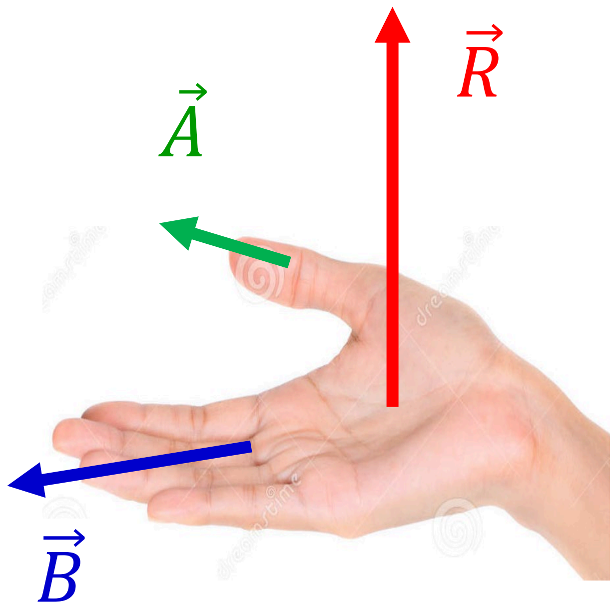
- Notation: $\boxed{\vec{A} \times \vec{B} = \vec{R}}$
- Magnitude of $\vec{R}$: $R = AB\sin\theta$
- Direction of $\vec{R}$:
  - Perpendicular to the plane containing $\vec{A}$ and $\vec{B}$
  - Given by yet another **Right-Hand Rule:**
  - Fingers of right hand rotating $\vec{A}$ into $\vec{B}$**,** right thumb extends in direction of $\vec{R}$

- All the properties of the moment are captures by a quantity called **cross product** of vectors $\vec{r}$ and $\vec{F}$!!!

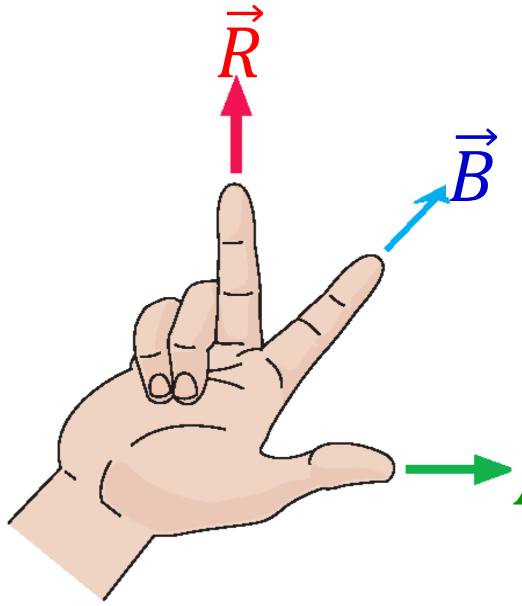
CROSS PRODUCT OF TWO VECTORS: mnemonic rules

$$\boxed{\vec{A} \times \vec{B} = \vec{R}}$$

First Second Result

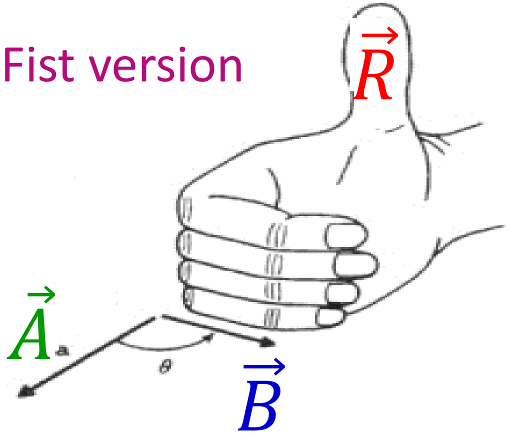


Open palm version (my favorite)



Fingers version

Fist version



Use your <u>right</u> hand for these configurations!

Corkscrew / screwdriver version (of these two objects, choose the one you are more familiar with!)

Rotate a screwdriver or a corkscrew from the 1st vector to the 2nd – it will move in the direction of $\vec{R}$

(…not your pencil-free hand!)

CROSS PRODUCT OF TWO VECTORS: properties

- Important: vector product is NOT commutative, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$. Instead,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

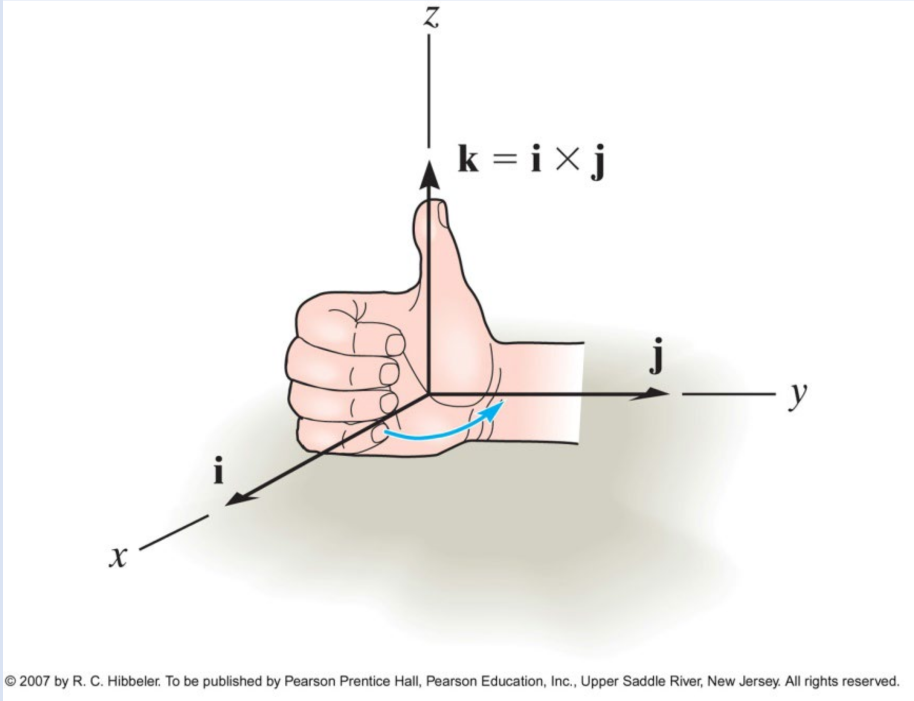
- Multiplication by a scalar:

$$a(\vec{A} \times \vec{B}) = (a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B})$$

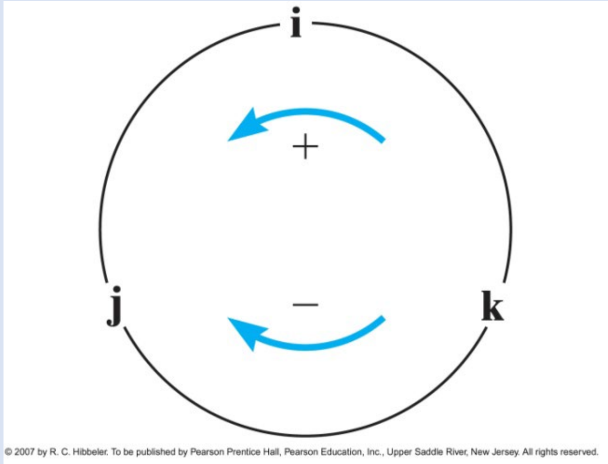
- Distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

CROSS PRODUCT OF VECTORS $\vec{i}, \vec{j}, \vec{k}$



- What is $\vec{i} \times \vec{i}$?
  - Magnitude: $|\vec{i}||\vec{i}| \sin 0^o = 1 \cdot 1 \cdot 0 = 0$
- What is $\vec{i} \times \vec{j}$?
  - Magnitude: $|\vec{i}||\vec{j}| \sin 90^o = 1 \cdot 1 \cdot 1 = 1$
  - Direction: RHR => along positive z
- Hence, $\vec{i} \times \vec{j} = \vec{k}$, and then $\vec{j} \times \vec{i} = -\vec{k}$.



- In general:

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{i} \times \vec{k} = -\vec{j} \qquad \vec{i} \times \vec{i} = 0$$
$$\vec{j} \times \vec{k} = \vec{i} \qquad \vec{j} \times \vec{i} = -\vec{k} \qquad \vec{j} \times \vec{j} = 0$$
$$\vec{k} \times \vec{i} = \vec{j} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{k} \times \vec{k} = 0$$

Meet determinants

Q: How familiar are you with determinants?

A. Determinants? Never heard of them.

B. Heard of them but prefer to not think about them.

C. Heard of them & know where to look them up.

D. Know what they are & can easily calculate a determinant of a 2 x 2 matrix.

E. Know what they are & can easily calculate a determinant of a 3 x 3 matrix.

DETERMINANT OF A MATRIX: Reminder

- 2 × 2 matrix:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \qquad \det X = \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} = X_{11}\, X_{22} - X_{12} X_{21}$$

- 3 × 3 matrix: $\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$ Its determinant can be found as a combination of three 2x2 determinants:

For element **i**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_y B_z - A_z B_y)$

For element **j**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_x B_z - A_z B_x)$ note the "−"

For element **k**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_x B_y - A_y B_x)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = +\vec{i}\left(A_y B_z - A_z B_y\right) - \vec{j}\left(A_x B_z - A_z B_x\right) + \vec{k}\left(A_x B_y - A_y B_x\right)$$

Extra Practice: Using the cross products of unit vectors, prove that:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Extra Practice: Using the cross products of unit vectors, prove that:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

To prove that, start with $\vec{A} \times \vec{B} = \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right) \times \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)$, apply the distributive law and the relations for the products $\vec{i} \times \vec{i} = 0$, etc:

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_xB_x(\mathbf{i} \times \mathbf{i}) + A_xB_y(\mathbf{i} \times \mathbf{j}) + A_xB_z(\mathbf{i} \times \mathbf{k}) \\
&+ A_yB_x(\mathbf{j} \times \mathbf{i}) + A_yB_y(\mathbf{j} \times \mathbf{j}) + A_yB_z(\mathbf{j} \times \mathbf{k}) \\
&+ A_zB_x(\mathbf{k} \times \mathbf{i}) + A_zB_y(\mathbf{k} \times \mathbf{j}) + A_zB_z(\mathbf{k} \times \mathbf{k}) \\
&= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \\
&= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}
\end{aligned}$$

Now calculate the determinant of the matrix $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ and convince yourself that you get the same expression.

W3-3. a) Compute $\vec{A}\times\vec{B}$ for $\vec{A} = -\vec{i} + 5\vec{j} + 3\vec{k}$ and $\vec{B} = 10\,\vec{i} - 20\vec{j} + 5\vec{k}$

b) Show that $(\vec{A}\times\vec{B})\cdot\vec{A} = (\vec{A}\times\vec{B})\cdot\vec{B} = 0$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & 3 \\ 10 & -20 & 5 \end{vmatrix} =$$

$$= \vec{i}\,[(5)(5) - (3)(-20)] - \vec{j}\,[(-1)(5) - (3)(10)] +$$

$$+ \vec{k}\,[(-1)(-20) - (5)(10)] =$$

$$= \vec{i}\,(85) + \vec{j}\,(35) + \vec{k}\,(-30)$$

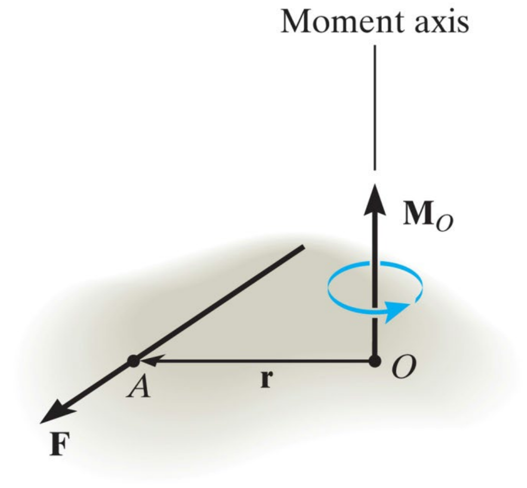
MOMENT AS A CROSS PRODUCT

- What does it all have to do with the moment of a force?

- The moment of a force $\vec{F}$ about a point O is a cross product of the position vector drawn from O to <u>any point on the line of action</u> of this force, and the force $\vec{F}$ (i.e. <u>we don't need to always use a perpendicular arm</u>):

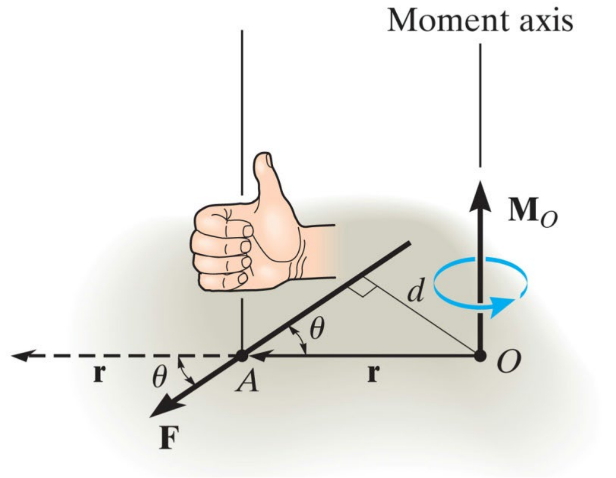
$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

- Magnitude: $M_O = r\,F \sin\theta$
- Direction: Right Hand Rule



- Resultant moment of many forces:

vectorial sum of their moments, $\boxed{\vec{M}_O = \sum_i (\vec{r}_i \times \vec{F}_i)}$



- In other words: Rotations add up as vectors!

Writhing solutions up

Show your work!

- For each cross-product, set up the 3 x 3 determinant
- Show the transformations that you make while calculating it
- The more work you show, the greater the chance that you get partial marks if something goes wrong