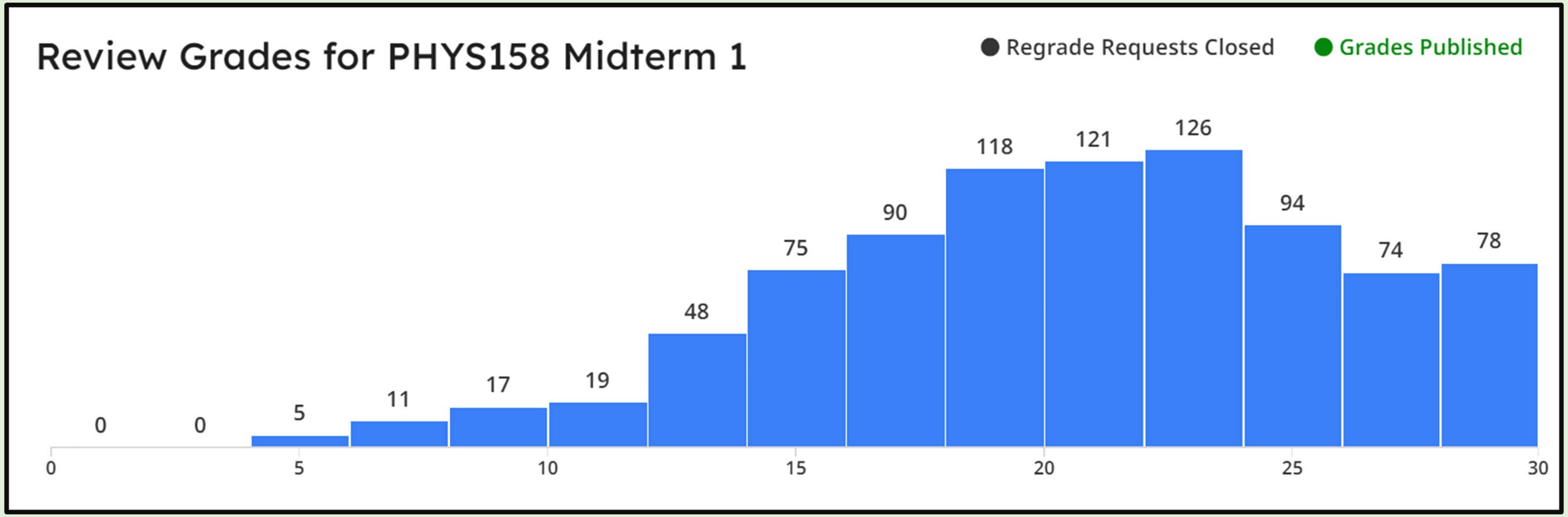
Lecture 36.

Maxwell’s equations: Let’s solve them!
Electromagnetic waves at a glance.
Concluding remarks.


Review Grades for PHYS158 Midterm 1
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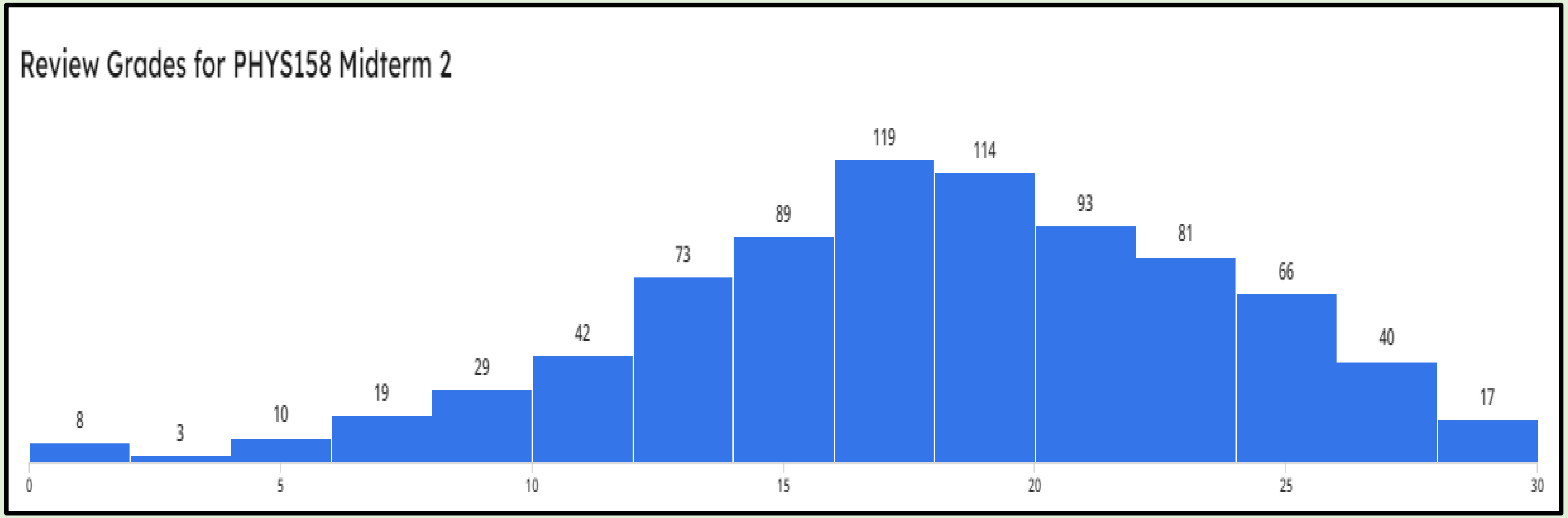
Review Grades for PHYS158 Midterm 2
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Our plans:

- We are aiming at finishing PHYS 158 material today.

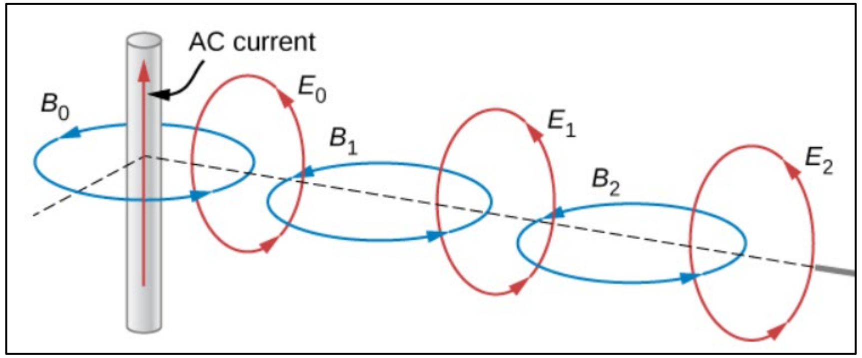
- Friday: optional (but highly recommended) in-class quiz: **NOT** for marks
  - ~40 mins, on Canvas, with password, can be done only from class
  - You will review the course content in a test-like atmosphere: self-evaluation
  - This will help us to gauge our teaching this term and compare the results with earlier years

- My office hours:
  - Wed (today) 13:00-14:00, Hebb 112
  - Fri (Apr 12th) 10:00-11:30, Hebb 112

- Help sessions:
  - Apr. 15 2-4 pm: Henn 302 & Henn 304
  - Apr. 17 10 am-12 pm: Henn 304 & Henn 318
  - Apr. 17 2-4 pm: Henn 302 & Henn 304

Law	Integral form	Differential form	Meaning
Gauss's law	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}$	$\text{div}\,\vec{E} = \frac{\rho}{\varepsilon_0}$	Electric charges produce electric field.
Gauss's law for magnetism	$\oint \vec{B} \cdot d\vec{A} = 0$	$\text{div}\,\vec{B} = 0$	There are no magnetic monopoles ("magnetic charges").
Faraday's law	$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$	$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Electric field can be also produced by changing magnetic field.
Ampère-Maxwell law	$\oint \vec{B} \cdot d\vec{l} =$ $= \mu_0 I_{ext} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$	$\text{curl}\,\vec{B} =$ $= \mu_0 \left(J + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$	Magnetic field is produced by electric currents and changing electric field.

Maxwell's Equations: Solutions



- How do the solutions of these equations look like, in the simplest case?

- Assumptions:
  - ➢ No external charges and currents
  - ➢ Assume that the fields depend on:
    - ❖ time, t
    - ❖ only one coordinate, x
  - ➢ Let us show that in this case the solution of this system of equations is what is called a harmonic plane wave!

Gauss's law	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0} \oint \vec{E} \cdot d\vec{A} = \cancel{\frac{Q_{\text{in}}}{\varepsilon_0}}$
Gauss's law (magnetism)	$\oint \vec{B} \cdot d\vec{A} = 0$
Faraday's law	$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampère-Maxwell law	$\oint \vec{B} \cdot d\vec{l} = \cancel{\mu_0 I_{ext}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$

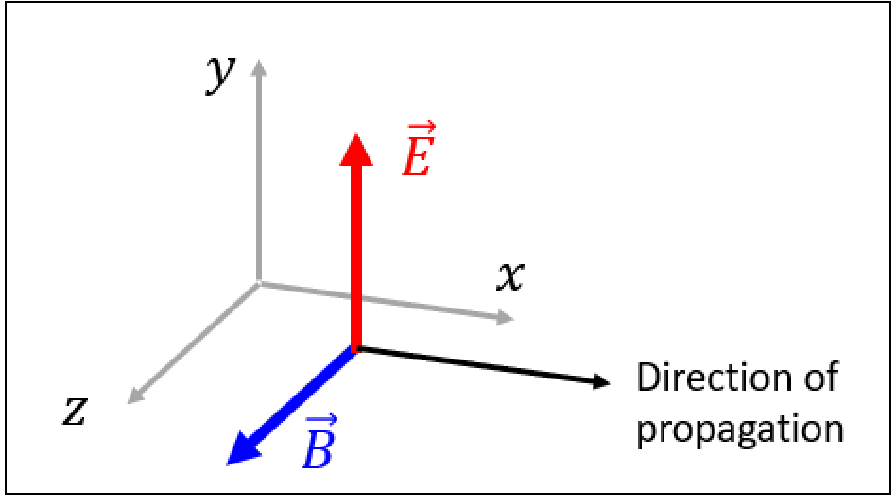
Maxwell's Equations: Solutions

- We are going to check that the solution of these equations look like this:
  - $\vec{E} \perp \vec{B} \perp$ propagation direction (x)
  - $\vec{E} = (0, E_y, 0)$, and $E_y = E_y(x,t)$
  - $\vec{B} = (0, 0, B_z)$, and $B_z = B_z(x,t)$

- …and they satisfy the same harmonic wave equation:

$$\boxed{\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \frac{\partial^2 E_y}{\partial x^2}}$$

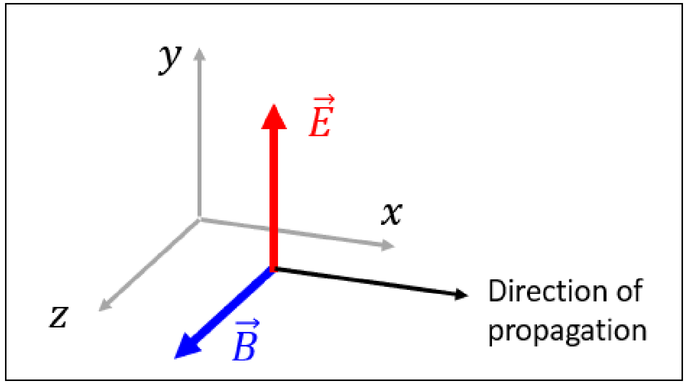
$$\boxed{\frac{\partial^2 B_z}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \frac{\partial^2 B_z}{\partial x^2}}$$



Gauss's law	$\oint \vec{E} \cdot d\vec{A} = 0$
Gauss's law (magnetism)	$\oint \vec{B} \cdot d\vec{A} = 0$
Faraday's law	$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampère-Maxwell law	$\oint \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$

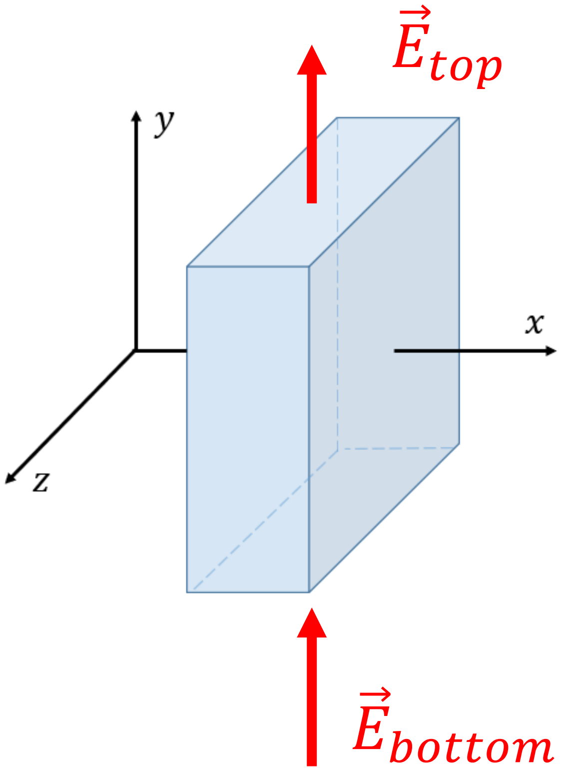
1. Gauss's law (E-field)

$$\oint \vec{E} \cdot d\vec{A} = \Phi_E = 0$$



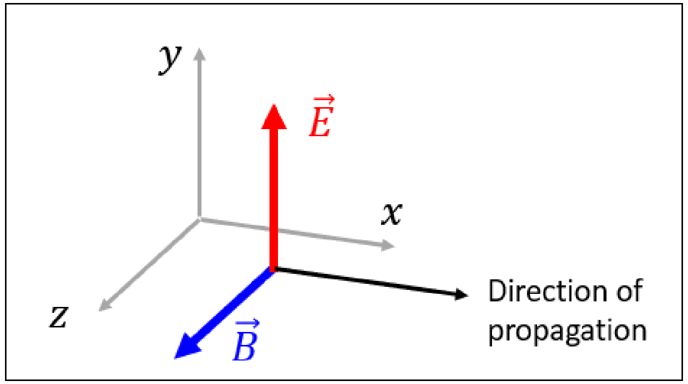
- Flux trough this Gaussian surface:

  - ➢ Only y-component of E-field
  - ➢ $E_{top} = E_{bottom}$ since $E_y = E_y(x)$ does not depend on y
  - ➢ $\Phi_E = 0$ => it works!



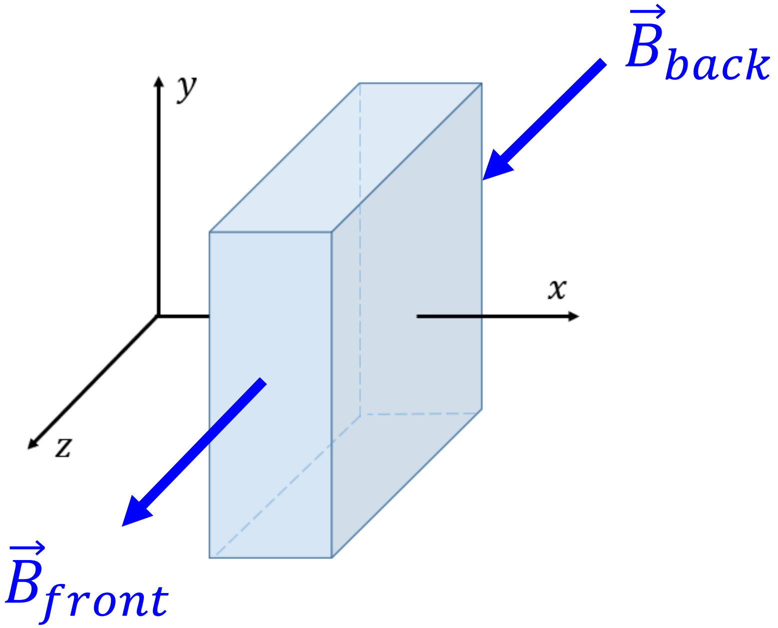
2. Gauss's law (B-field)

$$\oint \vec{B} \cdot d\vec{A} = \Phi_B = 0$$



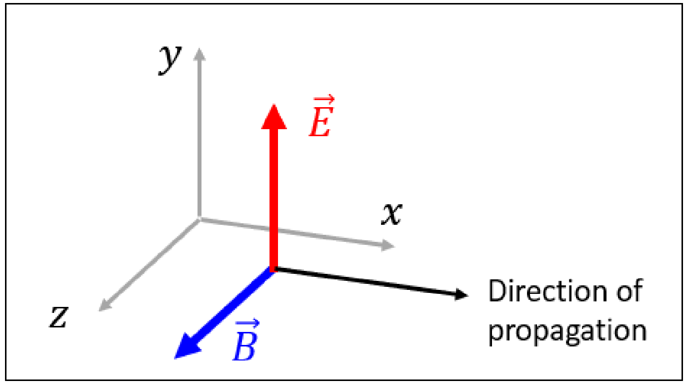
- Flux trough this Gaussian surface:

  - Only z-component of B-field
  - $B_{front} = B_{back}$ since $B_z = B_z(x)$ does not depend on z
  - $\Phi_B = 0$ => it works!



3. Faraday's law

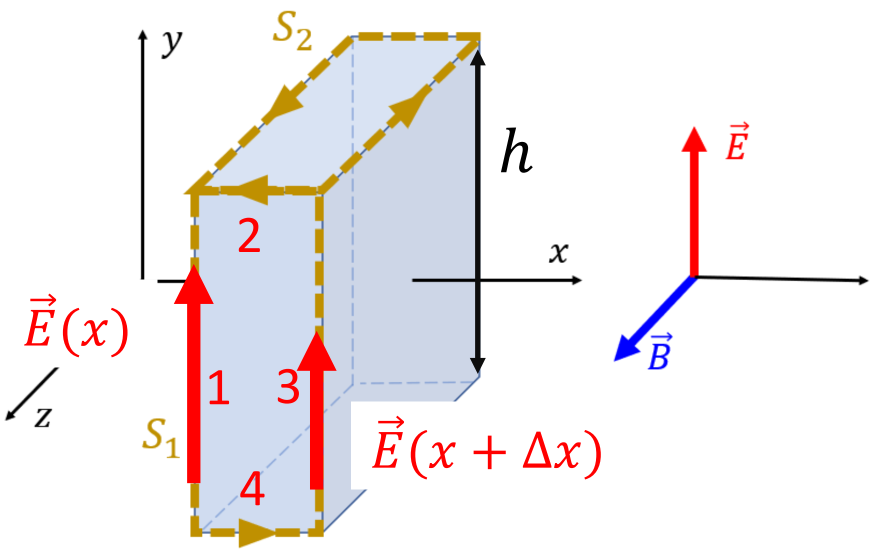
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$



- Using path S_1:

$$\oint \vec{E}\cdot d\vec{l} = \int_1 \vec{E}\cdot d\vec{l} + \underbrace{\cancelto{0}{\int_2 \vec{E}\cdot d\vec{l}}}_{\vec{E}\perp d\vec{l}} + \int_3 \vec{E}\cdot d\vec{l} + \underbrace{\cancelto{0}{\int_4 \vec{E}\cdot d\vec{l}}}_{\vec{E}\perp d\vec{l}} =$$

$$= \underbrace{\int_1 \vec{E}\cdot d\vec{l}}_{\text{const}} + \underbrace{\int_3 \vec{E}\cdot d\vec{l}}_{\text{const}} = -E_y(x)h + E_y(x+\Delta x)h$$



- Now: $$\frac{d\Phi_B}{dt} = \frac{d}{dt}(h\Delta x\, B_z)$$

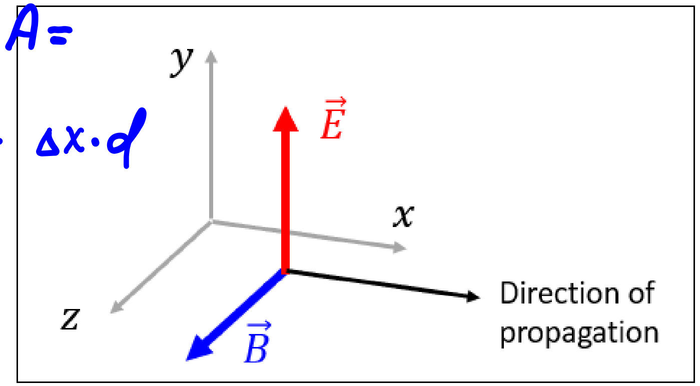
- Hence: $$h\frac{\left(E_y(x+\Delta x) - E_y(x)\right)}{\Delta x} = -h\cancel{\Delta x}\, B_z \qquad \lim_{\Delta x\to 0}$$

$$\boxed{\frac{dE_y(x)}{dx} = -\frac{dB_z}{dt}}$$

4. Ampere-Maxwell's law

$$\oint \vec{B}\cdot d\vec{l} = +\varepsilon_0\mu_0 \frac{d\Phi_E}{dt}$$

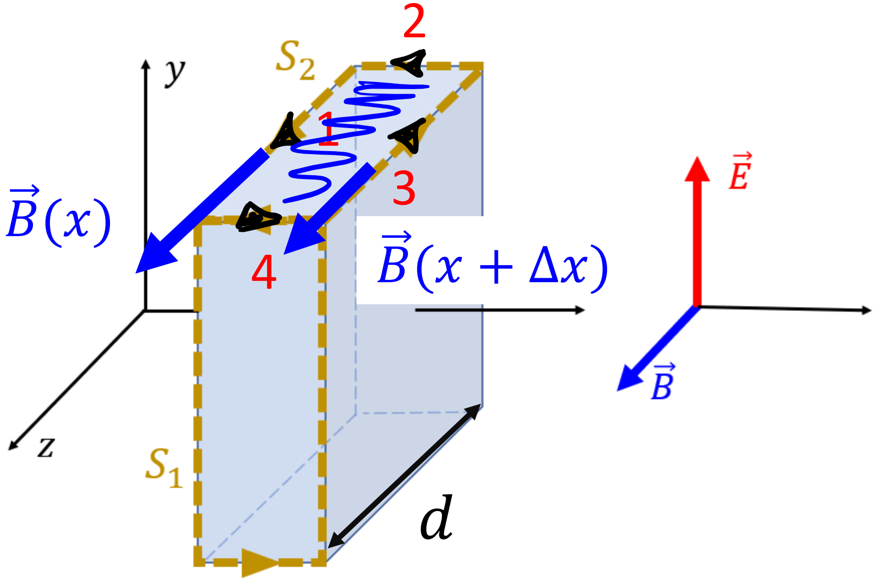
$\Phi_E = \vec{E}\cdot\vec{A} = E\cdot \Delta x \cdot d$



- Using path S_2:

$$\oint \vec{B}\cdot d\vec{l} = \underbrace{\int_1 \vec{B}\cdot d\vec{l}}_{B\,dl} + \underbrace{\cancel{\int_2 \vec{B}\cdot d\vec{l}}}_{\vec{B}\perp d\vec{l}\;\to\;0} + \underbrace{\int_3 \vec{B}\cdot d\vec{l}}_{-B\,dl} + \underbrace{\cancel{\int_4 \vec{B}\cdot d\vec{l}}}_{\vec{B}\perp d\vec{l}\;\to\;0} =$$

$$= \underbrace{\int_1 \vec{B}\cdot d\vec{l}}_{\text{const}} + \underbrace{\int_3 \vec{B}\cdot d\vec{l}}_{\text{const}} = +B_z(x)d - B_z(x+\Delta x)d$$



- Now: $\dfrac{d\Phi_E}{dt} = \dfrac{d}{dt}\left(d\Delta x\, E_y\right)$

$\lim \Delta x \to 0$

- Hence: $-d\underbrace{\left(B_z(x+\Delta x) - B_z(x)\right)}_{\Delta x} = \varepsilon_0\mu_0\, d\cancel{\Delta} x\, E_y$ $\Rightarrow$

$$\boxed{\frac{dB_z(x)}{dx} = -\varepsilon_0\mu_0 \frac{dE_y}{dt}}$$

5. Let's put it together:

$$\boxed{\frac{dE_y(x)}{dx} = -\frac{dB_z}{dt}} \quad (1)$$

$$\boxed{\frac{dB_z(x)}{dx} = -\varepsilon_0\mu_0\frac{dE_y}{dt}} \quad (2)$$

- Let's compute $\frac{d}{dx}$ of (1):

$$\frac{d^2E_y(x)}{dx^2} = -\frac{d}{dx}\left(\frac{dB_z}{dt}\right) = -\frac{d}{dt}\left(\frac{dB_z}{dx}\right)$$

- Using (2):

$$\frac{d^2E_y(x)}{dx^2} = +\frac{d}{dt}\left(\varepsilon_0\mu_0\frac{dE_y}{dt}\right) = \varepsilon_0\mu_0\frac{d^2E_y}{dt^2}$$

- Let's compute $\frac{d}{dx}$ of (2):

$$\frac{d^2B_z(x)}{dx^2} = -\frac{d}{dx}\left(\frac{dE_y}{dt}\right) = -\frac{d}{dt}\left(\frac{dE_y}{dx}\right)$$

- Using (1):

$$\frac{d^2B_z(x)}{dx^2} = +\frac{d}{dt}\left(\varepsilon_0\mu_0\frac{dB_z}{dt}\right) = \varepsilon_0\mu_0\frac{d^2B_z}{dt^2}$$

Differential operator nabla: $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Just fyi:

The beauty of differential operators

$\operatorname{div}\vec{E} = 0$	$\nabla \cdot \vec{E} = 0$
$\operatorname{div}\vec{B} = 0$	$\nabla \cdot \vec{B} = 0$
$\operatorname{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\operatorname{curl}\vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$	$\nabla \times \vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$

$$\nabla \cdot \vec{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$-\nabla^2\vec{B} = \nabla(\nabla\cdot\vec{B}) - \nabla^2\vec{B} = \nabla\times\nabla\times\vec{B} = \mu_0\varepsilon_0\left(\nabla\times\frac{\partial\vec{E}}{\partial t}\right) = \mu_0\varepsilon_0\frac{\partial}{\partial t}(\nabla\times\vec{E}) = \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

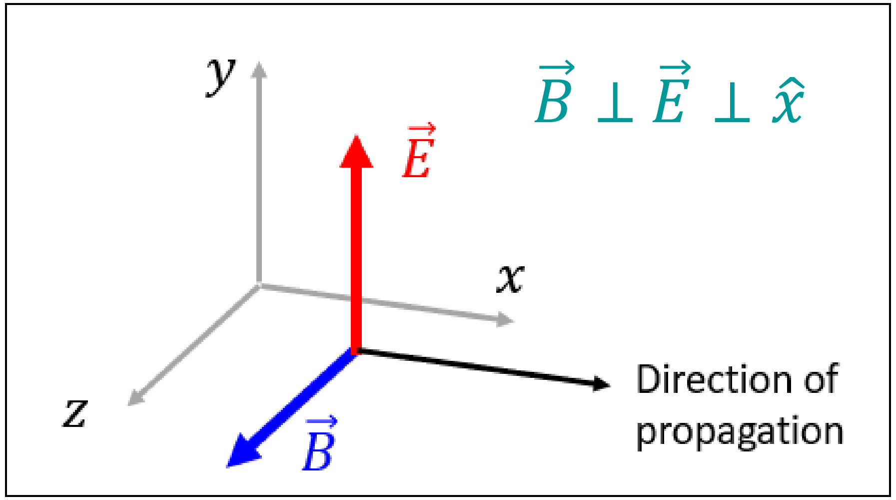
$$\boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}$$

wave equation for B-field, with ∇^2 expressing its second derivatives with respect to coordinates

Answer:

- Electromagnetic wave propagating in x-direction (rearranged for d/dt^2):

$$\frac{\partial^2 E_y(x)}{\partial t^2} = \overbrace{\left(\frac{1}{\varepsilon_0 \mu_0}\right)}^{c^2} \frac{\partial^2 E_y}{\partial x^2} \qquad \frac{\partial^2 B_z(x)}{\partial t^2} = \overbrace{\left(\frac{1}{\varepsilon_0 \mu_0}\right)}^{c^2} \frac{\partial^2 B_z}{\partial x^2}$$



Q: Look at the factor $\frac{1}{\varepsilon_0 \mu_0}$ appearing in the wave equation. Calculate its numerical value, and find out its units. Does it remind you of something?

$$\varepsilon_0 = \frac{1}{4\pi K} = \frac{1}{4\pi \cdot 9 \cdot 10^9} \frac{C^2}{N \cdot m^2}, \qquad \mu_0 = 4\pi \cdot 10^{-7} \frac{T \cdot m}{A} \qquad [T] = \frac{[N]}{[A][m]}$$

$$\left[\frac{1}{\varepsilon_0 \mu_0}\right] = \frac{N \cdot m^2}{C^2} \cdot \frac{A}{T \cdot m} = \frac{\cancel{N} m^2}{C^2} \cdot \frac{A \cdot \cancel{m} \cdot A^2}{\cancel{N} \cdot \cancel{m}} = \frac{m^2}{s^2}$$

$$[C] = [A][s]$$

$$\frac{1}{\varepsilon_0 \mu_0} = 9 \cdot 10^{16} \frac{m^2}{s^2} = (3 \cdot 10^8) \frac{m}{s}$$

-- famous speed of light!

Q: does that mean the electromagnetic waves is what we call “light”?
What, in fact, are they?

A. Yes, “light” and “electromagnetic waves” are synonyms.
B. Electromagnetic waves are NOT light! It’s a coincidence.
C. Light is just one example of electromagnetic waves. Sound waves is another example.
D. Light is just one example of electromagnetic waves. Thermal radiation is another example.
E. None of the above

Electromagnetic waves: a flashlight and beyond

$f\lambda = c$

- There is a whole spectrum of electromagnetic waves. They differ from each other by frequency and wavelength.
- All electromagnetic waves have the same properties and show the same effects.
- They all travel at the speed of light, c.

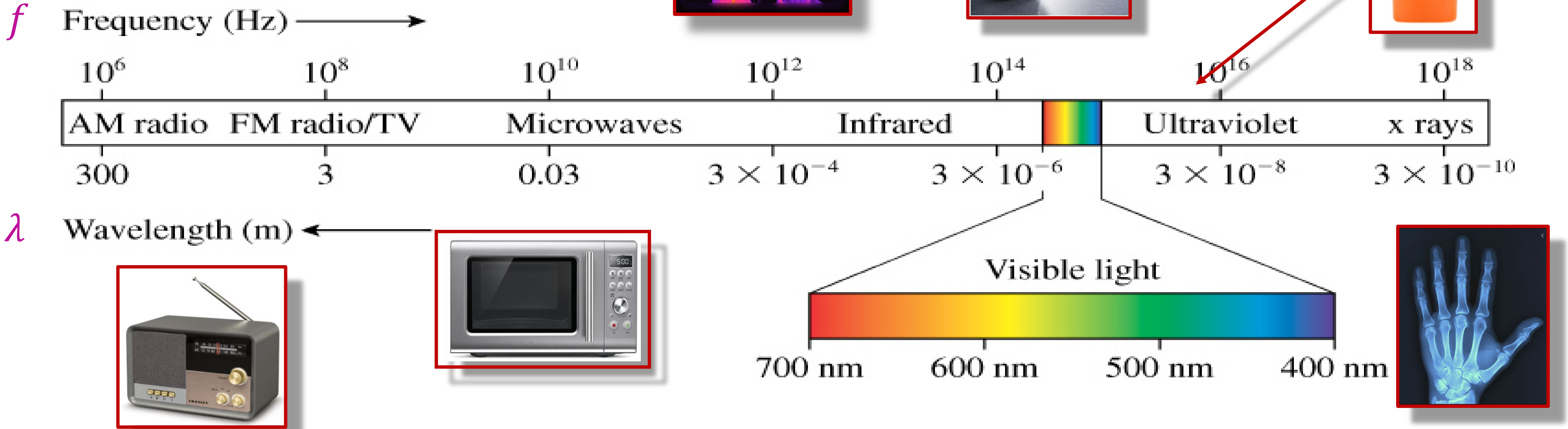
("ultra" = beyond)

ultraviolet

visible light

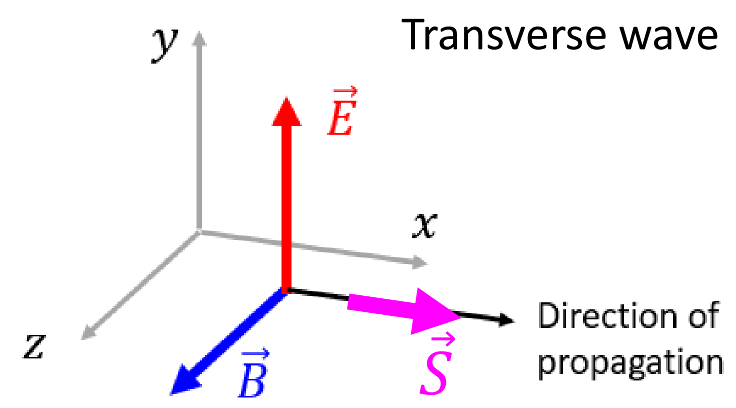
("infra" = below)

infrared



How electromagnetic waves look like:

$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2}$$



- The solution of the wave equation above is a harmonic wave (prove this by substitution):

$$E_y = E_0 \sin(kx - \omega t)$$

$$B_z = B_0 \sin(kx - \omega t)$$

with $\omega = ck$.

- "Right-handed light":

  The direction of propagation is parallel to $\vec{E} \times \vec{B}$

  True for light in vacuum, always!

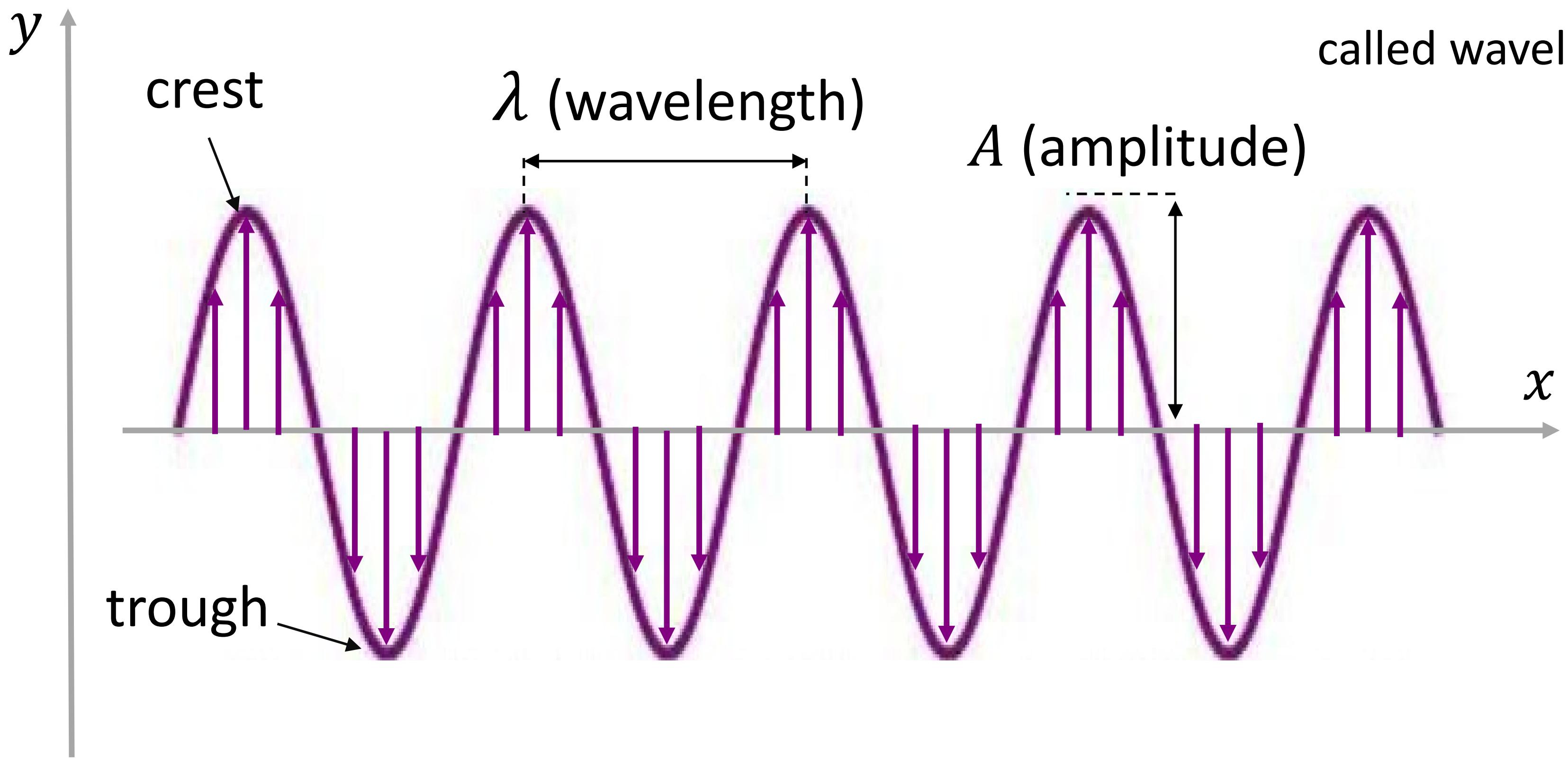
  "Left-handed light" exists in metamaterials (difficult to achieve)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

- Poynting vector (energy flow in EM wave)

Reminder (just in case)

- Each point on x-axis moves up and down, making a full cycle in time T (wave period).
- The distance between two adjacent crests is called wavelength λ



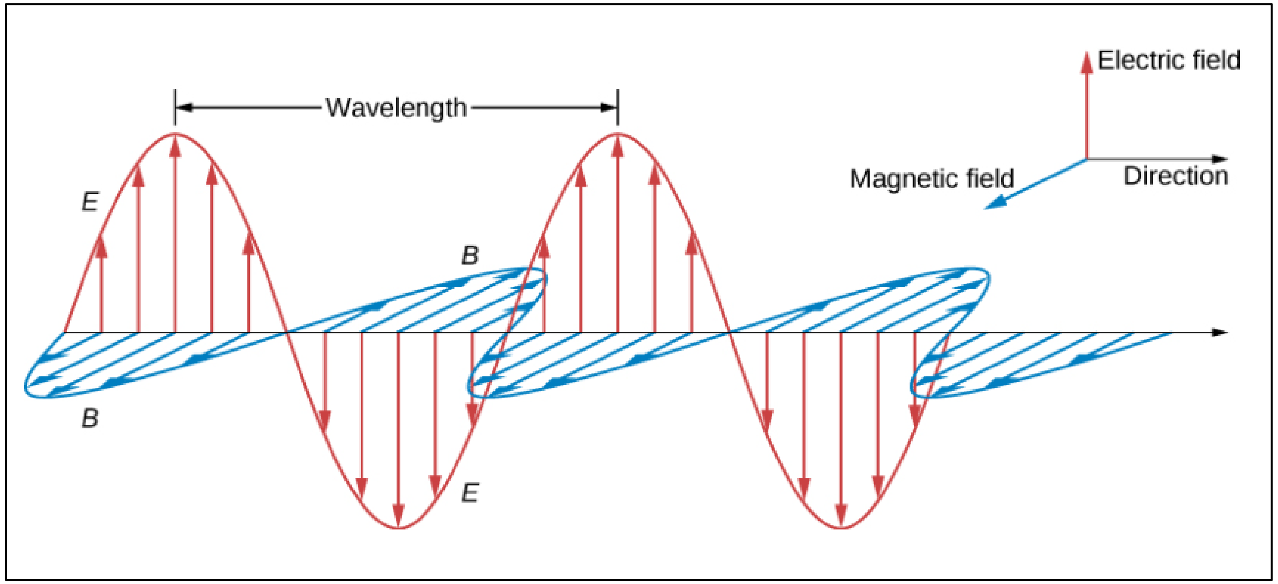
- Period T relates to frequency f and angular frequency ω: $f = \frac{1}{T} = \frac{\omega}{2\pi}$
- Wavelength λ relates to wave number k: $k = \frac{2\pi}{\lambda}$
- Vertical displacement: $y(x,t) = A\sin(kx - \omega t)$

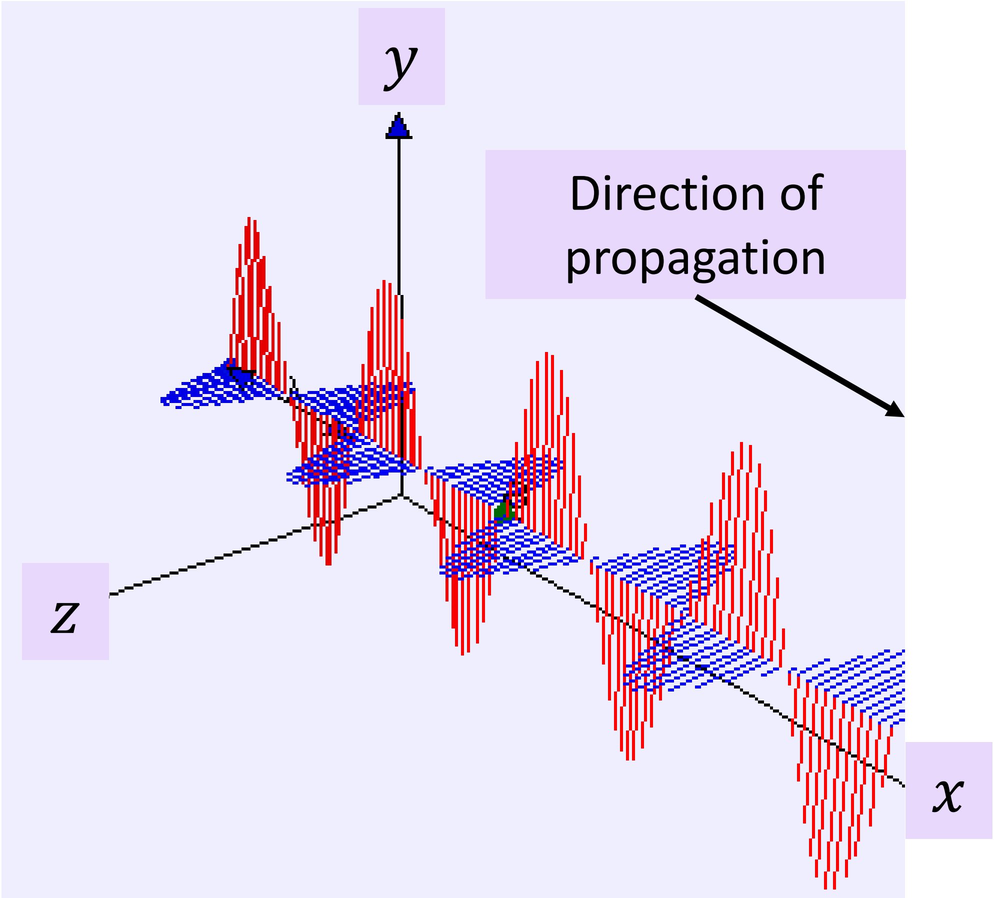
- Speed of propagation:

$$c = f\lambda = \frac{\omega}{k}$$

Electromagnetic wave

https://en.wikipedia.org/wiki/Electromagnetic_radiation#/media/File:Electromagneticwave3D.gif

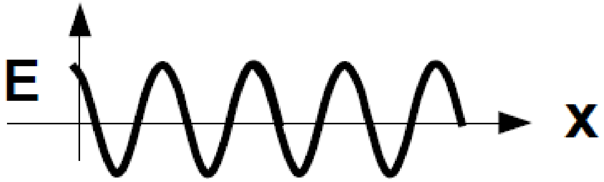


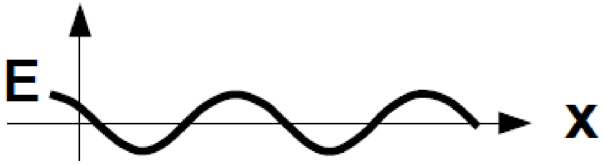


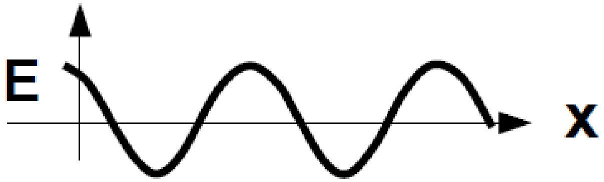
- Done! That's how a plane electromagnetic wave looks like…

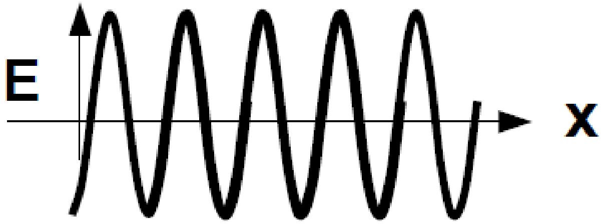
- The direction of propagation is given by the cross product $\vec{E} \times \vec{B}$

Q: The snapshots of four E&M waves travelling in vacuum are shown below. Which wave travels the fastest?

A. 

B. 

C. 

D. 

E. All the same

Q: In which direction each of the two waves is travelling?

A. Wave A in +x, Wave B in –x.

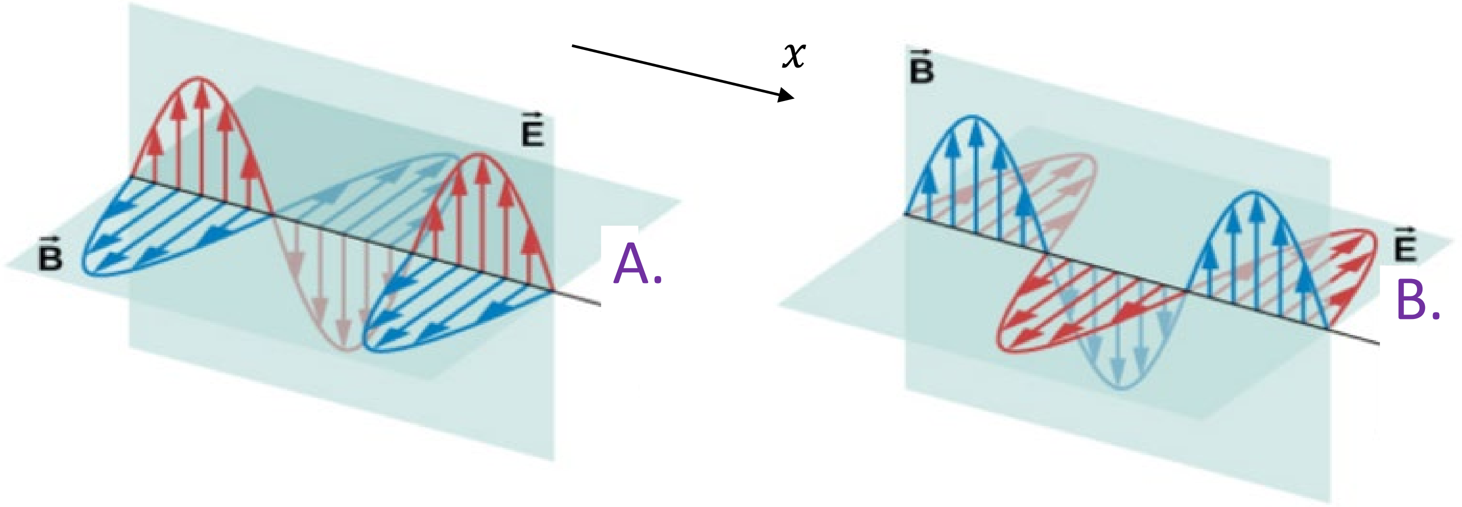
B. Wave A in –x, Wave B in +x.

C. Wave A in +x, Wave B in +x.

D. Wave A in –x, Wave B in –x.

E. None of the above

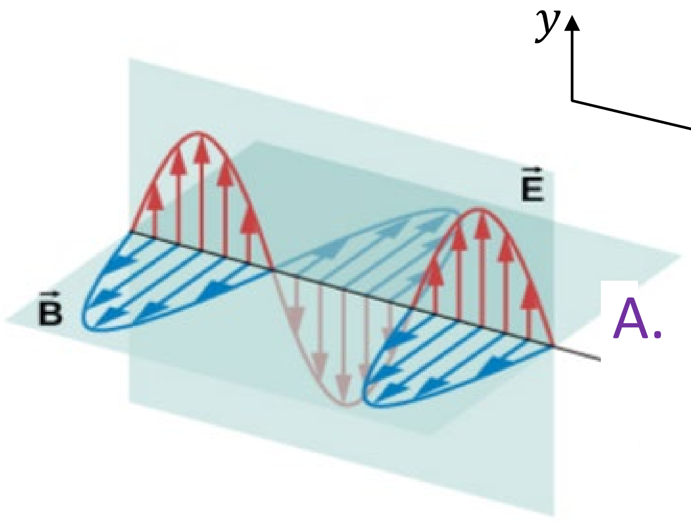
$\hat{x} : \quad \vec{E} \times \vec{B}$

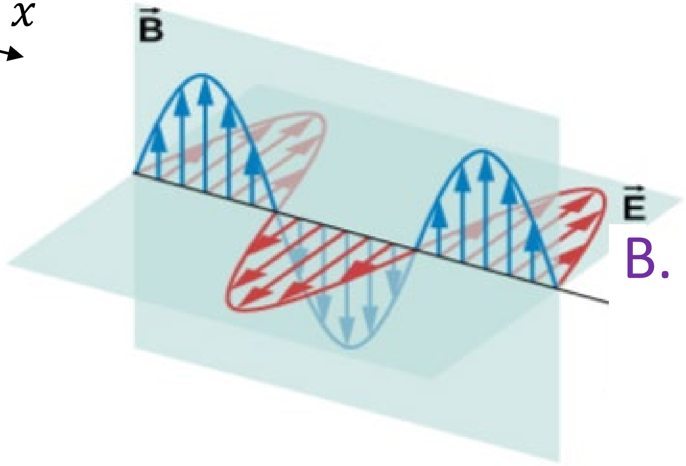


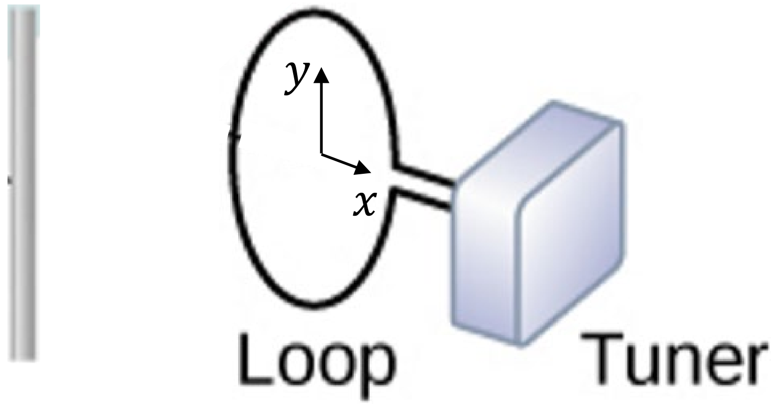
Q: Which of these two waves will excite current in a vertical wire? A.

Q: Which of these two waves will excite current in a vertical loop? A.

A. Only Wave A
B. Only Wave B
C. Both
D. Neither







Comments:

- Q: So we solved these equations. Are we done?
- A: Nope. That's actually where the fun starts...

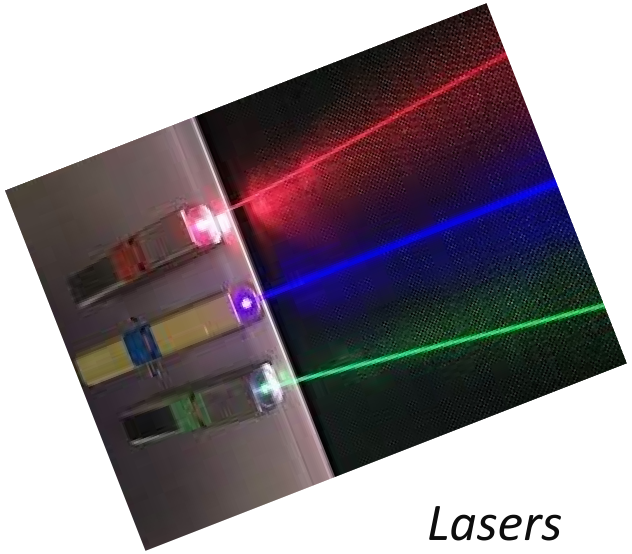
- Q: How many equations, how many unknowns?
- A: It depends!
  - $\vec{E}$ and $\vec{B}$ are vectors => $E_x, E_y, E_z, B_x, B_y, B_z$
  - Maxwell's equations are <u>differential equations</u> => you will need initial conditions and boundary conditions
  - Geometry plays a huge role (if you have a surface, or interface, or a sharp tip.....)
  - Material plays a huge role (polarizability, magnetization, symmetry of the crystal...)

Gauss's law	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0} \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}$
Gauss's law (magnetism)	$\oint \vec{B} \cdot d\vec{A} = 0$
Faraday's law	$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampère-Maxwell law	$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{ext} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$

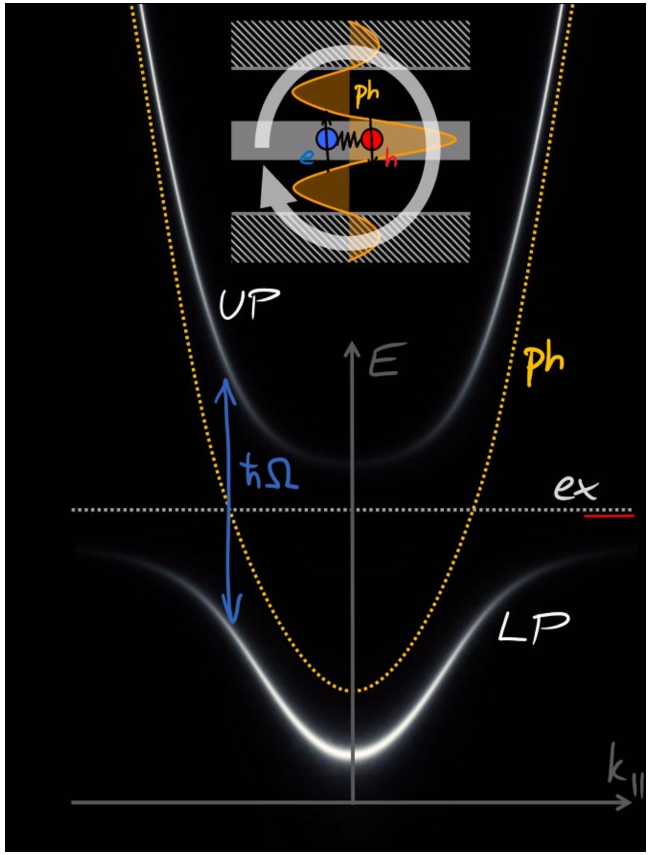
You can spend your whole life solving Maxwell's equations. (That's what happened to me)

All these objects (and many others) are described by simply solving Maxwell's equations.

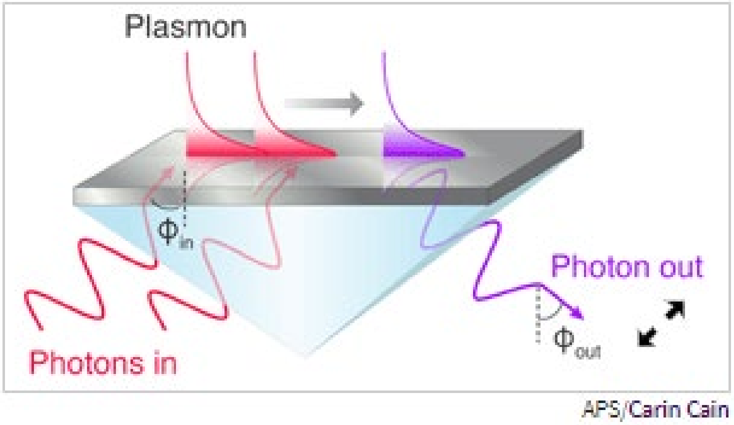
(With addition of quantum mechanics, to be honest)



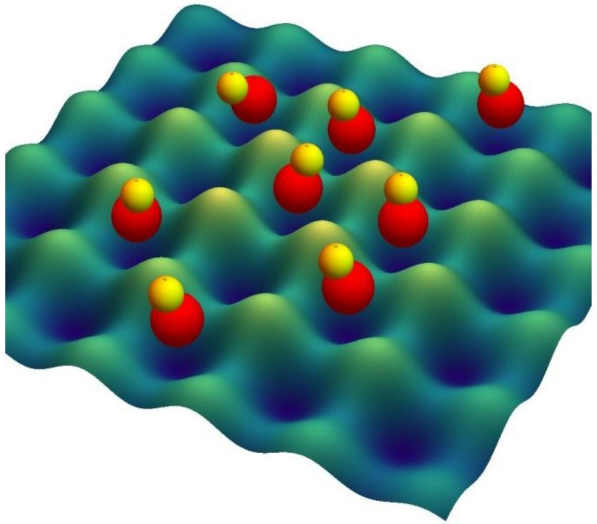
Lasers



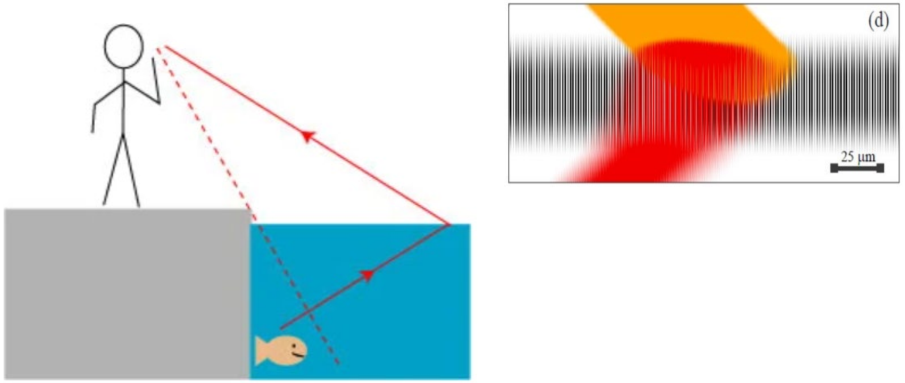
Polaritons (light hybridized with atomic transitions in a crystal)



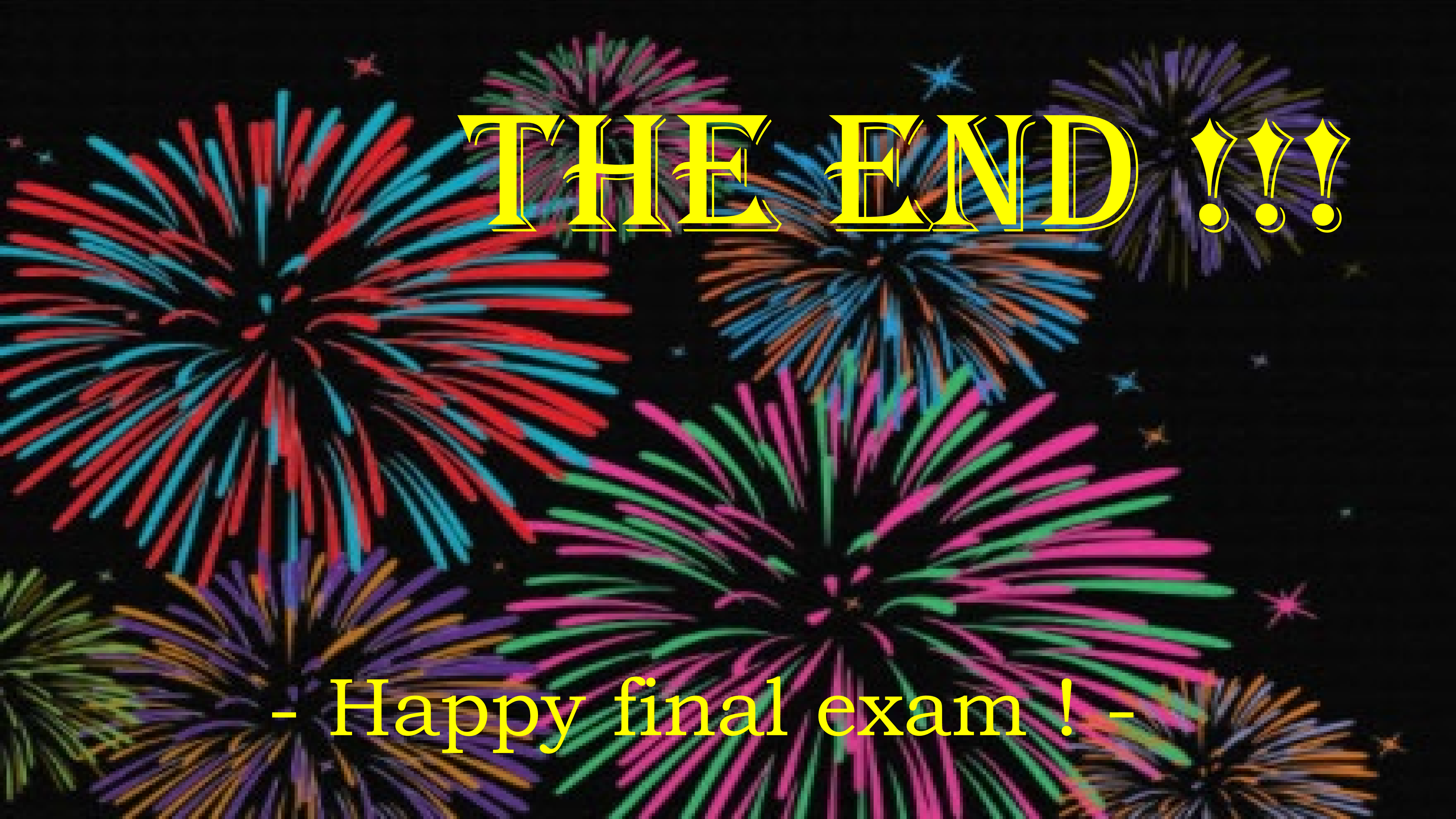
Light interacting with the waves in the "sea of electrons"



Polar molecules in an optical lattice



Negative refraction (light that bends "wrongly")


THE END !!!
- Happy final exam ! -