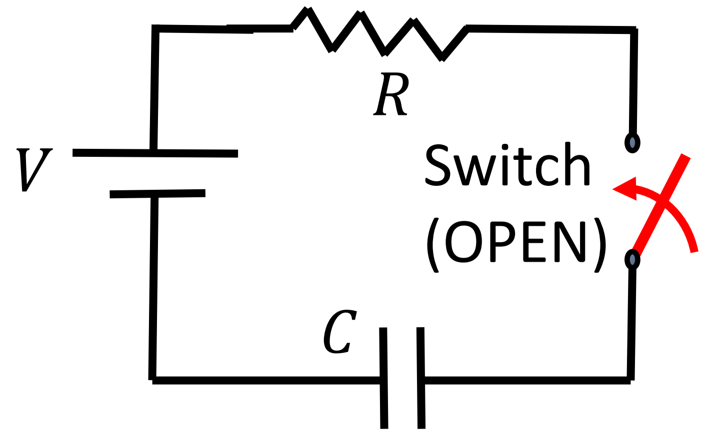
Lecture 6.

RC-circuits: circuits with dynamics

Last Time



initially uncharged

- Limiting cases:
  - Immediately after the switch is closed:
    - Path created => current starts flowing
    - At t $= 0$: $q = 0 \quad \Rightarrow \quad \Delta V_C = 0 \quad \Rightarrow$

      capacitor acts as ideal wire
  - After a very long time:
    - Capacitor is fully charged => does not accept more charge => no current
    - At t $= \infty$:

      capacitor acts as an open switch
- What happens in between??

CASE 1: Charging a capacitor - 3

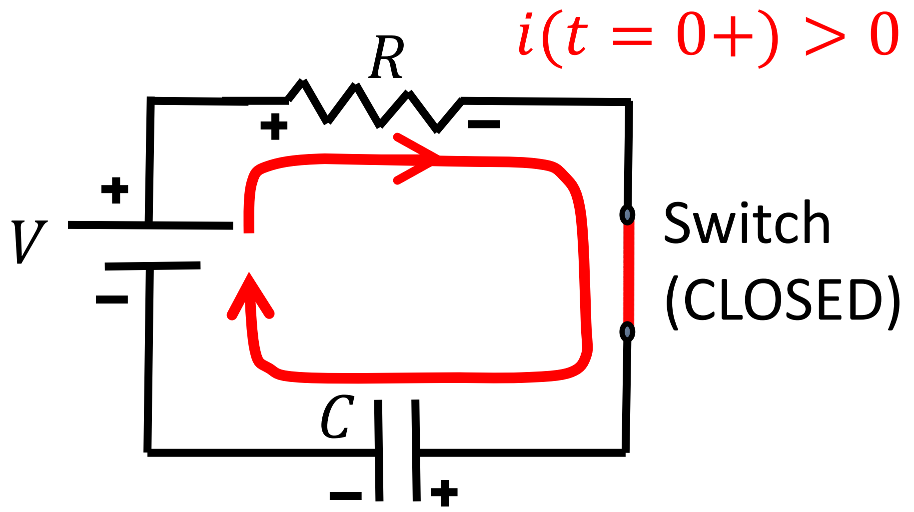
Initial charge $q_c(t=0) = 0$
Switch is suddenly closed at $t = 0+$

- Kirchhoff loop law (travel CW):

$$V - iR - \frac{q}{C} = 0$$

- Let's take time derivative:

$$0 - R\frac{di}{dt} - \frac{1}{C}\frac{dq}{dt} = 0$$



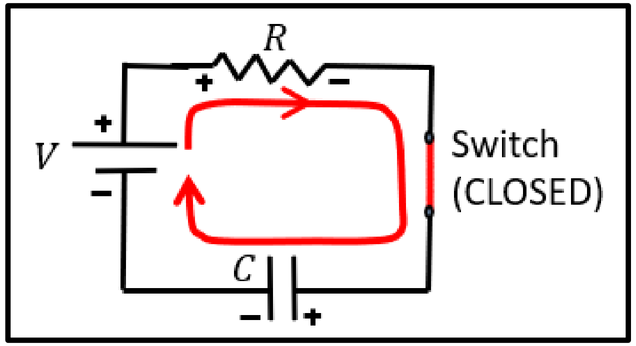
- Note that: $\boxed{\frac{dq}{dt} = i,}$ hence: $\boxed{-R\frac{di}{dt} - \frac{i}{C} = 0.}$

- Here we take $\frac{dq}{dt} = +i$ since current i brings + charge to the positive plate => charge on the plates increases => $\frac{dq}{dt} > 0.$

- We have got a differential equation for $i(t)$, and a connection between $q(t)$ and $i(t)$

CASE 1: Charging a capacitor - 4

$$\boxed{\frac{dq}{dt} = i,} \quad (1) \qquad \boxed{-R\frac{di}{dt} - \frac{i}{C} = 0.} \quad (2)$$



- We need to find their solutions, $i(t)$ and $q(t)$, that satisfy initial conditions:
  - For current: $i(t=0) = I_0 = V/R$
  - For charge: $q(t=0) = Q_0 = 0$ & $q(t=\infty) = CV$

$$\boxed{i(t) = \frac{V}{R} e^{-\left(\frac{t}{RC}\right)}}$$

initial current

$t \to \infty$

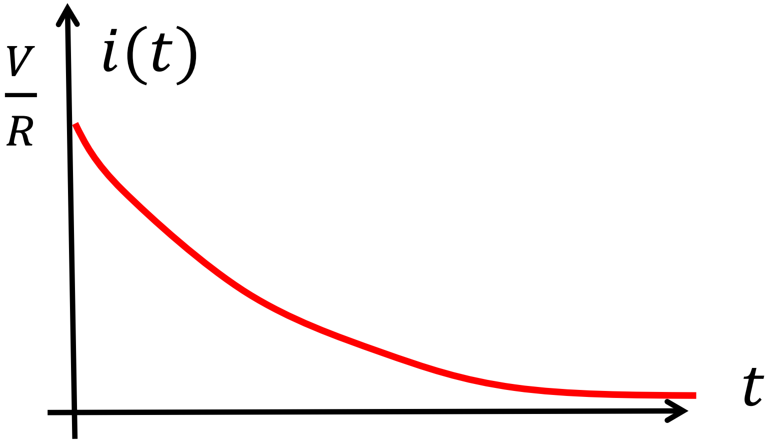
$$\boxed{q(t) = CV\left(1 - e^{-\left(\frac{t}{RC}\right)}\right)}$$

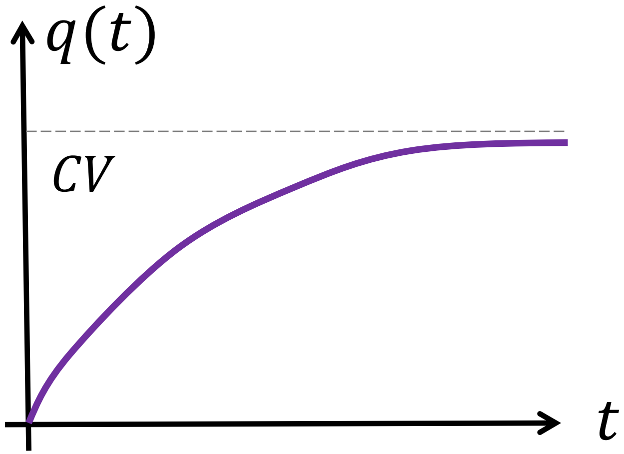
$q(t=\infty) = CV$

final charge

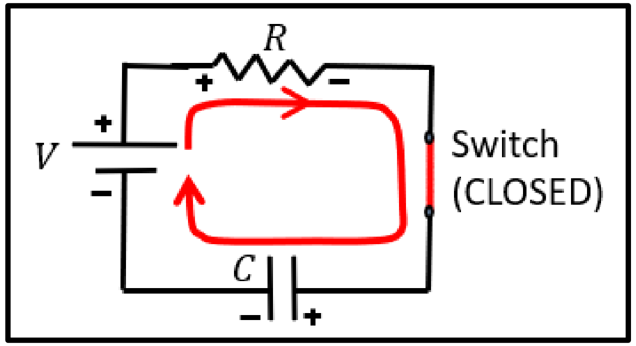
$$\boxed{\tau = RC}$$

time constant





CASE 1: Charging a capacitor - 5 Math details



Loop law $$V - i(t)R - \frac{q(t)}{C} = 0$$

Take the derivative $$0 - R\frac{di}{dt} - \frac{1}{C}\frac{dq}{dt} = 0, \qquad i = \frac{dq}{dt}$$

Separate variables $$\frac{di}{dt} = -\frac{1}{RC}dt = 0$$

$$\int \frac{di}{i} = -\frac{1}{RC}\int dt$$

using $i(0) = V/R$

$$\ln(i) = -\frac{t}{RC} + \text{const}$$

$$i(t) = \text{const}'\, e^{-t/RC}$$

$$\boxed{i(t) = I_0 e^{-t/RC} = \frac{V}{R}e^{-t/RC}}$$

$RC = \tau =$ time constant

$$i(t) = \frac{dq(t)}{dt} \quad \Rightarrow \quad q(t) = \int i(t)dt$$

$$q(t) = \frac{V}{R}\int e^{-\frac{t}{RC}}\,dt$$

$$q(t) = -RC\frac{V}{R}e^{-\frac{t}{RC}} + \text{const}$$

$$q(t) = -CV\, e^{-\frac{t}{RC}} + \text{const}$$

- Use $q(0) = 0$: $\quad 0 = -CV + \text{const}$

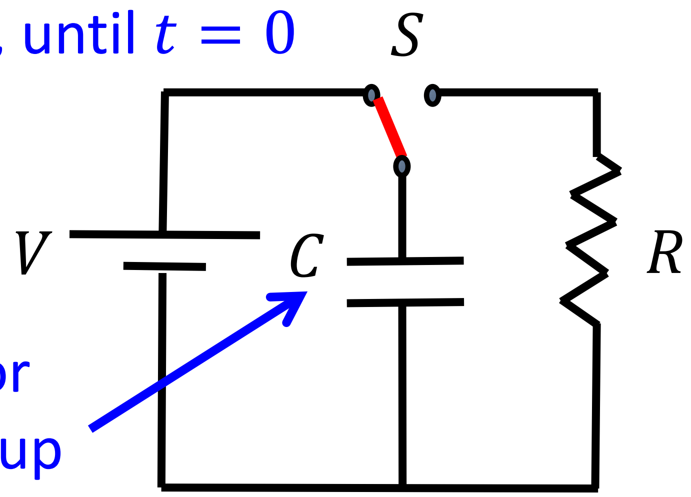
$$\Rightarrow \quad \text{const} = CV$$

Note that:
$$q(\infty) = CV = Q_f$$

$$\boxed{q(t) = CV\left(1 - e^{-t/RC}\right)}$$

CASE 2: Discharging a capacitor - 1

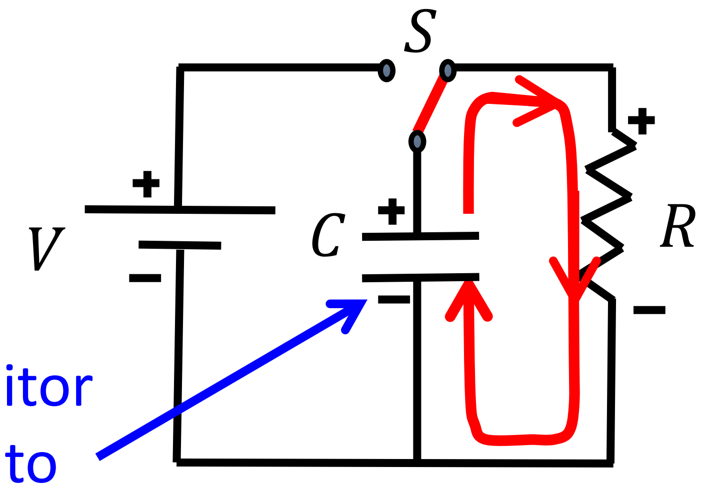
Switch is at the left position for a very long time, until $t = 0$



Capacitor charges up to voltage V

Initial charge: $Q_0 = CV$

Switch is suddenly flipped to the right at $t = 0+$



Capacitor starts to discharge through R

- Now the capacitor is discharging: positive charges from the $+$ plate flow through the resistor and recombine (meet and annihilate) with negative charges on the – plate.

- Once the switch is flipped to right, Kirchhoff's loop law gives: $\dfrac{q(t)}{C} - iR = 0$

$i(t)$

CASE 2: Discharging a capacitor - 2   Math details

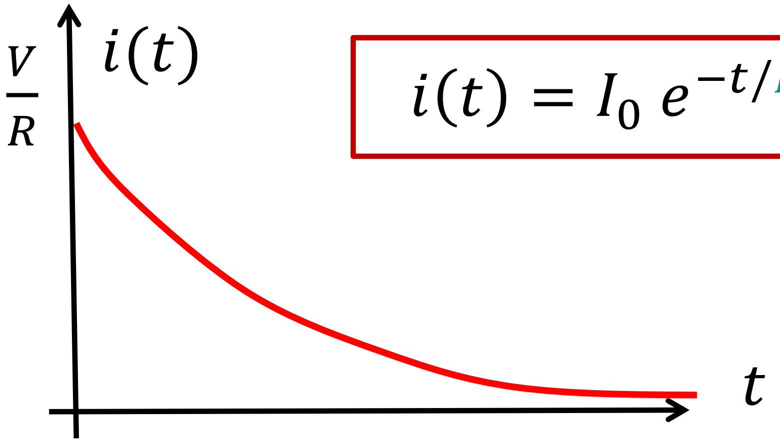
$$\frac{q(t)}{C} - i(t)R = 0$$

Note that here $i = -\frac{dq}{dt}$ since q is decreasing.

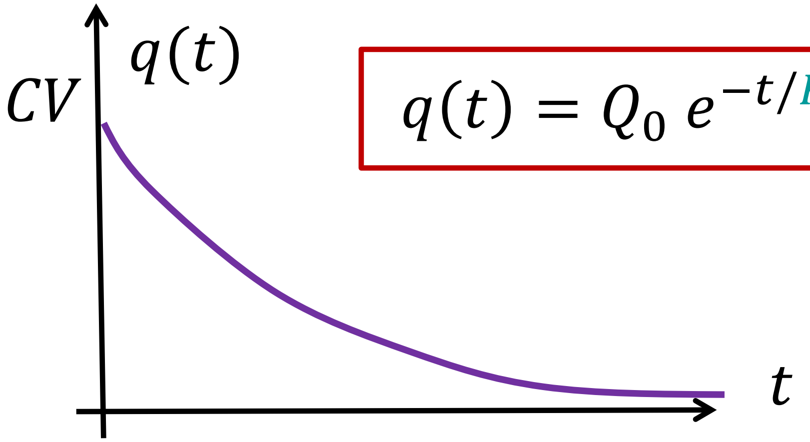
$$\frac{q}{C} + R\frac{dq}{dt} = 0 \quad \Rightarrow \quad \frac{dq}{q} = -\frac{1}{RC}dt \quad \Rightarrow \quad q(t) = \text{const}\, e^{-t/RC}$$

- Initial condition: $q(0) = CV \quad \Rightarrow \quad q(t) = Q_0\, e^{-t/RC}$, with $Q_0 = CV$

- Current: $i = -\frac{dq}{dt} = -Q_0\left(-\frac{1}{RC}\right)e^{-t/RC} = I_0 e^{-t/RC}$, with $I_0 = \frac{Q_0}{RC} = \frac{CV}{RC} = \frac{V}{R}$



$$\boxed{i(t) = I_0\, e^{-t/RC}}$$



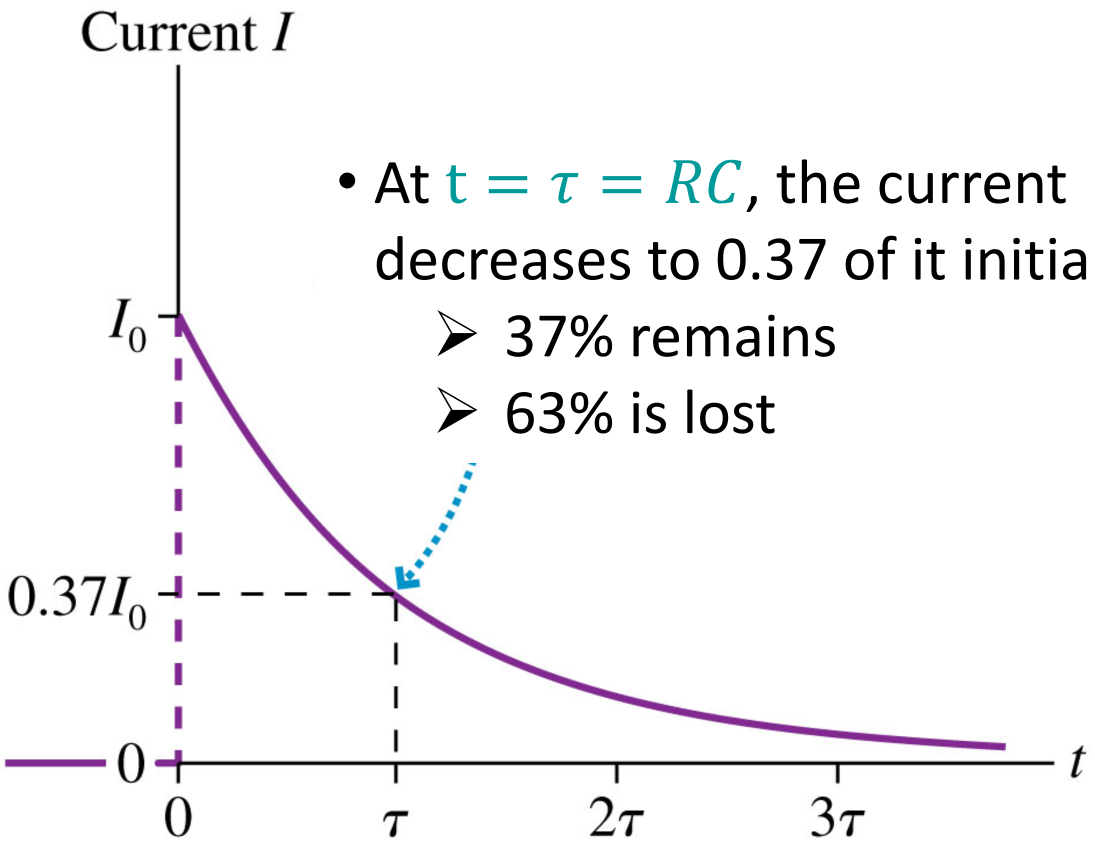
$$\boxed{q(t) = Q_0\, e^{-t/RC}}$$

$$\boxed{\tau = RC}$$

time constant

How long is *very long*?

- The relevant time scale is $\tau = RC$:



- At $t = \tau = RC$, the current decreases to 0.37 of it initial value:
  - ➢ 37% remains
  - ➢ 63% is lost

$$I(t) = I_0 e^{-\frac{t}{\tau}}, \quad \text{with} \quad I_0 = V/R$$

$$I(t = \tau) = I_0 e^{-1} = \frac{I_0}{e} = \frac{I_0}{2.71} = 0.37\, I_0$$

$$I(t = 2\tau) = 0.37 I(t = \tau) = 0.14\, I_0$$

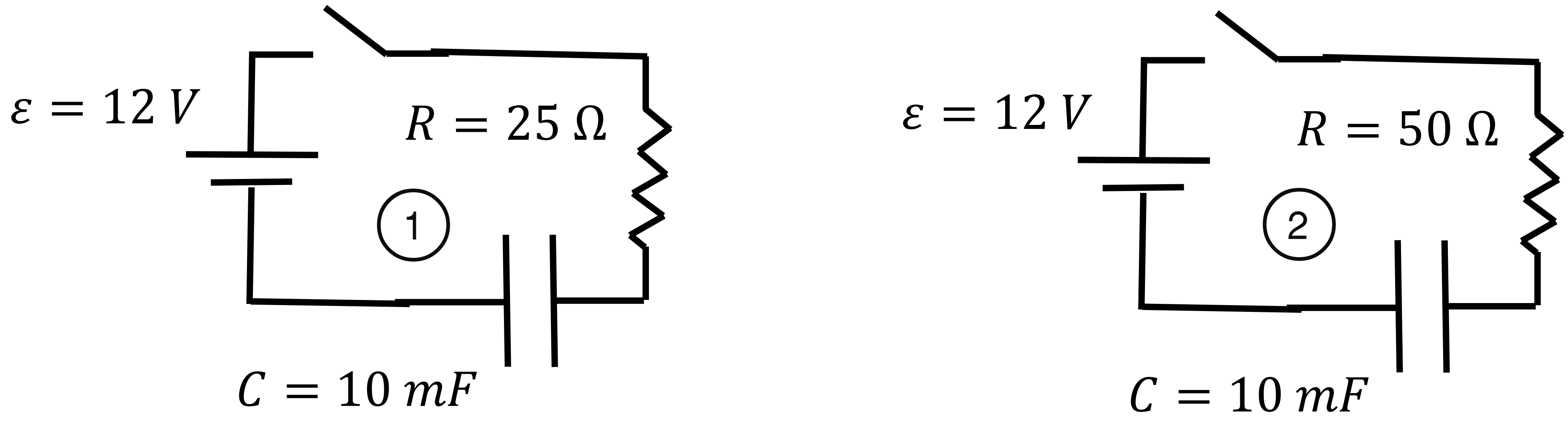
$$I(t = 3\tau) = 0.37 I(t = 2\tau) = 0.05\, I_0$$

…

$$I(t = 6\tau) = 0.37 I(t = 5\tau) = 0.002\, I_0$$

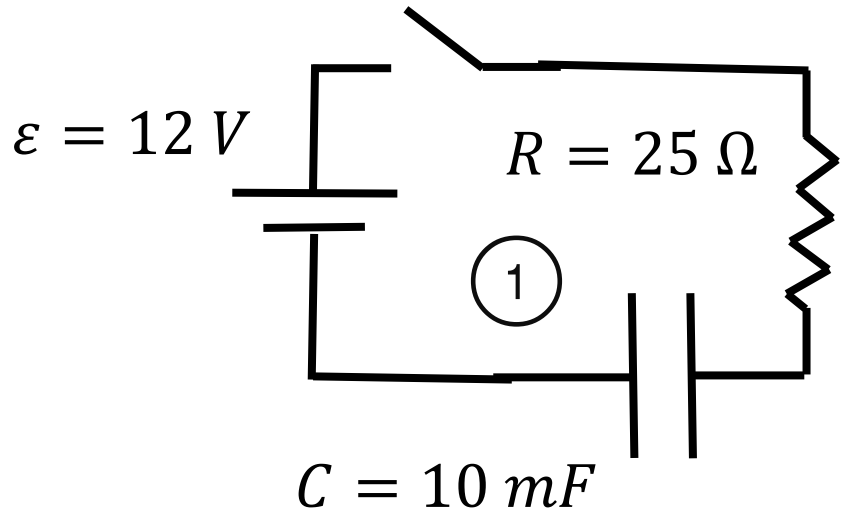
“Very long” means $t \gg \tau = RC$

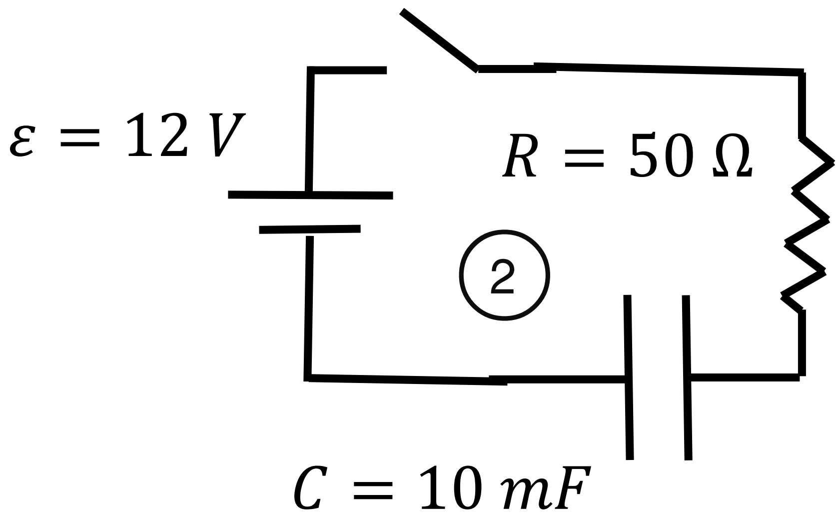
Q: In which of these RC-circuits the charge decays faster?



A. Faster in 1
B. Faster in 2
C. Decay at the same speed
D. Not enough information

Q: In which of these RC-circuits the charge decays faster?





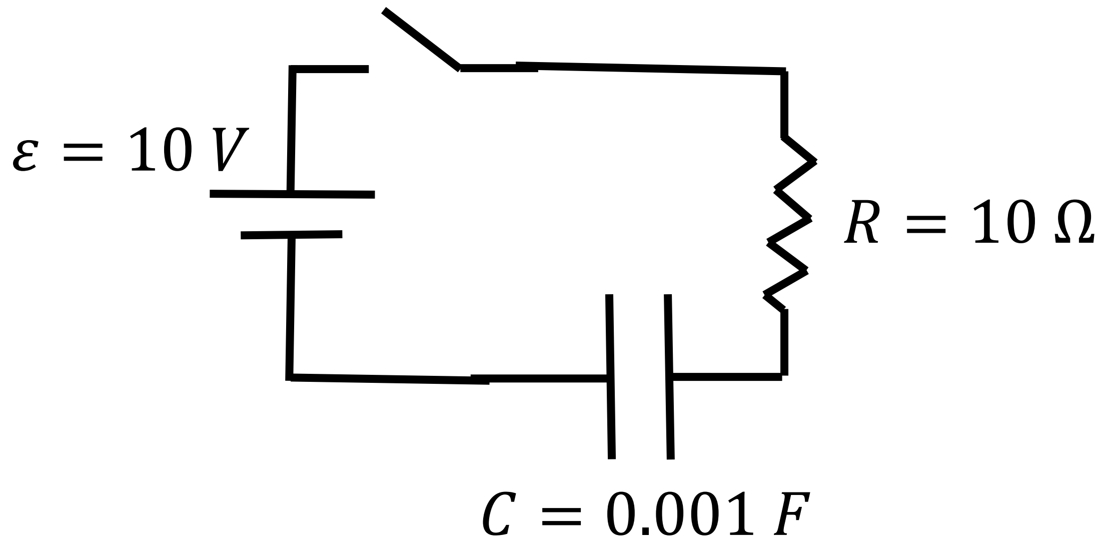
- $\tau_1 = 250\ ms, \quad \tau_2 = 500\ ms$
- For circuit 2 it takes twice longer to lose 63% of charge =>
- Charge in circuit 1 decays faster.

A. Faster in 1
B. Faster in 2
C. Decay at the same speed
D. Not enough information

Q: $q_C(0-) = 0$. At time $t = 0$, the switch is closed.

What is the voltage across the capacitor immediately after the switch is closed ($t = 0+$) ?

A. 0 V
B. 10 V
C. 5 V
D. None of these.



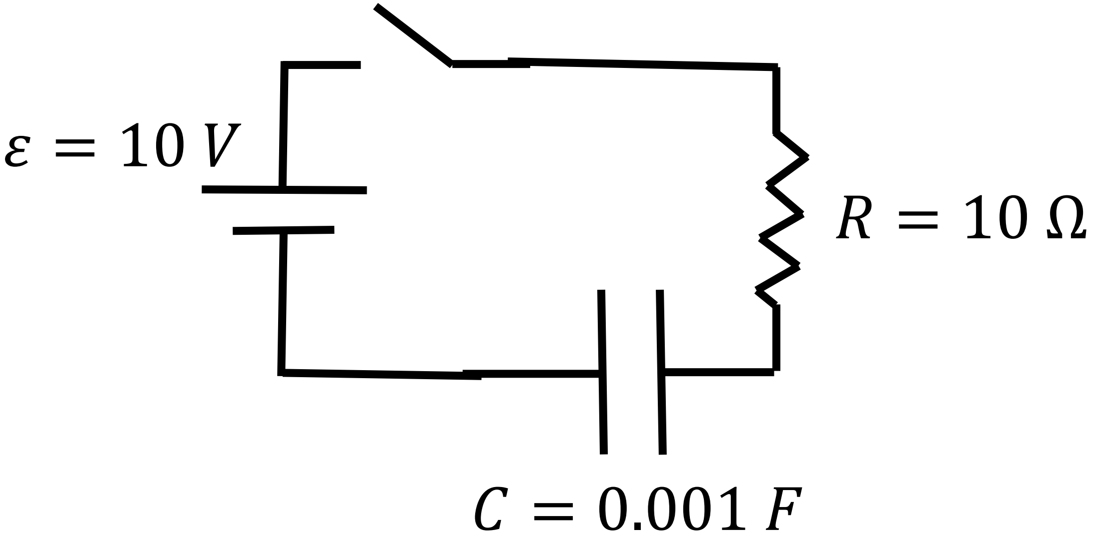
Q: $q_C(0-) = 0$. At time $t = 0$, the switch is closed.

What is the voltage across the capacitor immediately after the switch is closed ($t = 0+$) ?

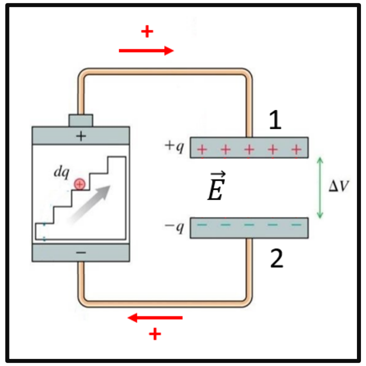
A. 0 V
B. 10 V
C. 5 V
D. None of these.

$= 0$

$$\underbrace{\Delta V_c}_{} = \frac{Q^{\,=0}}{C}$$

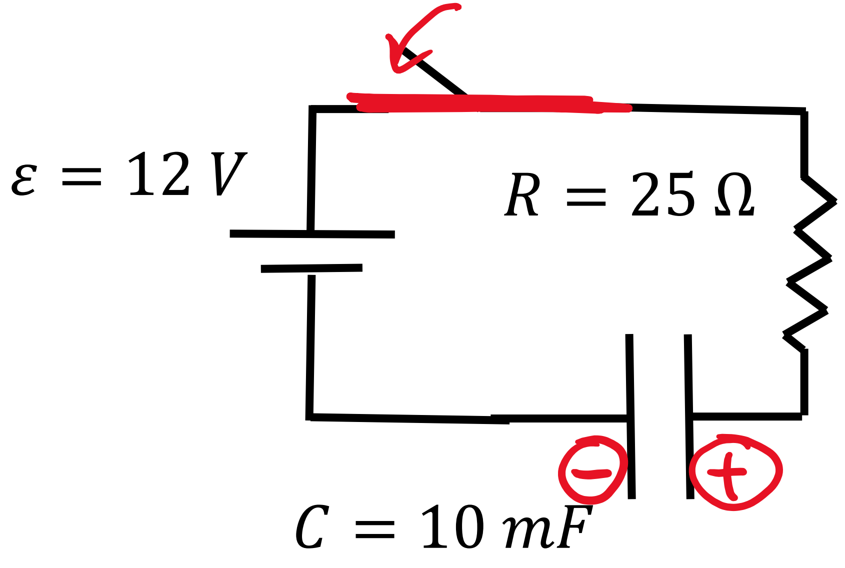


- Since $q(0) = 0$, $\Delta V_C(0) = 0$
- Since $\Delta V_C(0) = 0$, immediately after the switch is closed it acts as an ideal wire (there is no charge flow through it, but there is a charge flow around it as if it was an ideal wire)



Q: This capacitor is fully charged. What is the magnitude of the voltage drop across the resistor?

$I = 0$

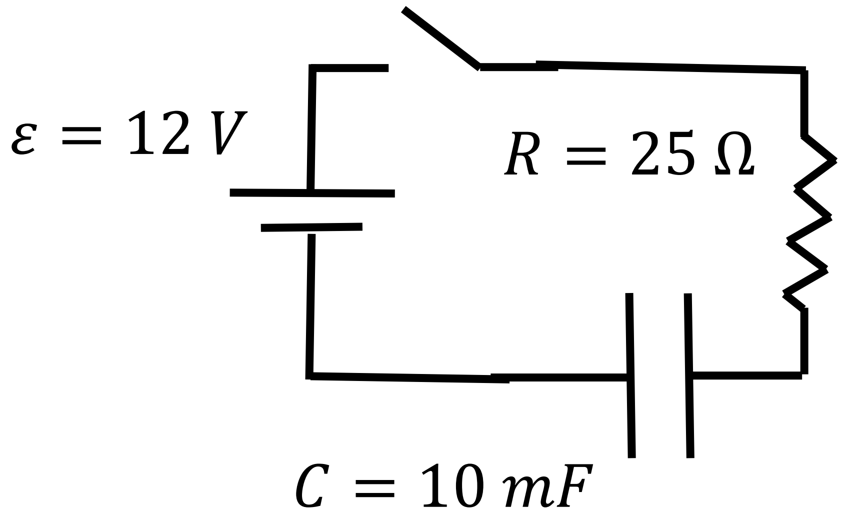


$$|\Delta V_R| + |\Delta V_C| = \varepsilon$$

$$|\Delta V_R| = \underset{0}{I}R = 0$$

A. 0
B. 4 V
C. 6 V
D. 8 V
E. 12 V

Q: This capacitor is fully charged. What is the magnitude of the voltage drop across the resistor?



- When the capacitor is fully charged, there is no more current in the circuit (the capacitor does not accept more charge)

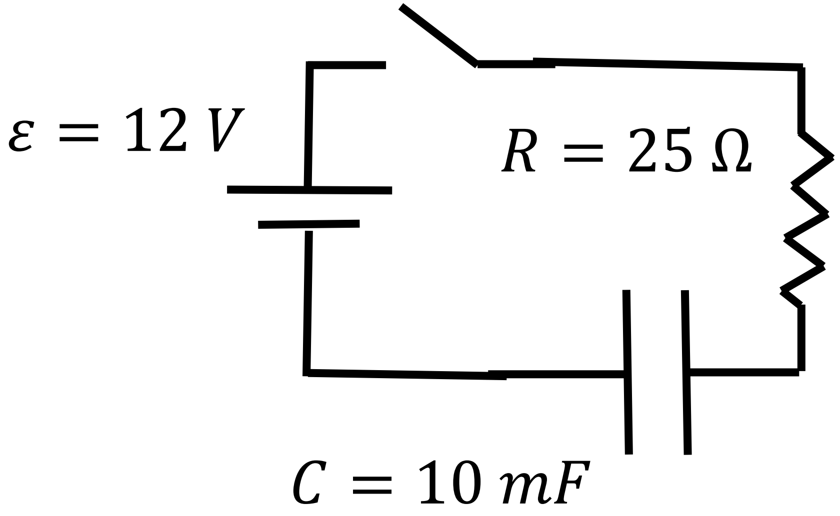
A. 0
B. $4\ V$
C. $6\ V$
D. $8\ V$
E. $12\ V$

- The voltage drop across the resistor is: $\Delta V_R = IR = 0$ since $I = 0$

Q: Before we close the switch, $q_C = 0$.

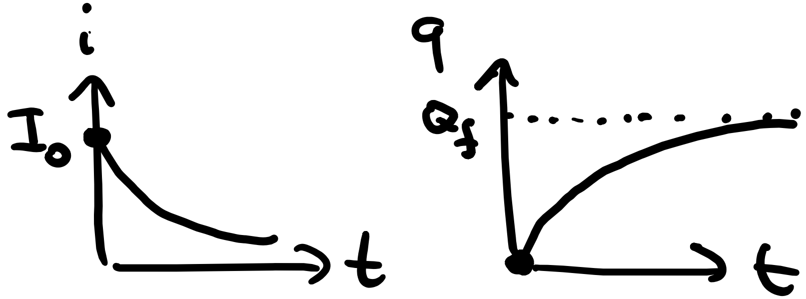
1) Compute the current $i(t)$ as a function of time

2) Compute $q_C(t_1 = 250\ ms)$



a) $i(t) = I_0 e^{-\frac{t}{\tau}}$ $\qquad$ $q(t) = Q_f \left[1 - e^{-\frac{t}{\tau}} \right]$

$\tau = R \cdot C$ $\qquad$ $I_0 = \frac{\varepsilon}{R}$ (no cap) $\qquad$ $Q_f = C\varepsilon$ (no resistor)



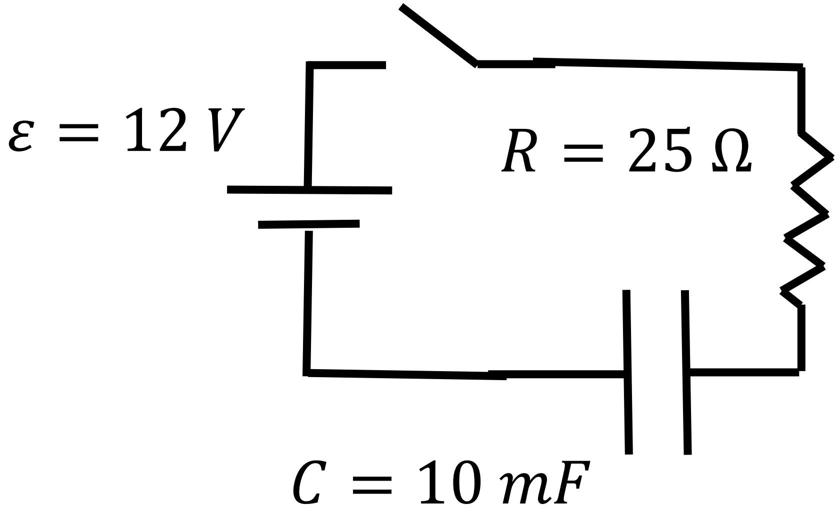
b) $q(t = 250\ ms) = Q_f \left[1 - e^{-\frac{250 ms}{\tau}} \right] = \dots$

Remember the graphs

Q: Before we close the switch, $q_C = 0$.

1) Compute the current $i(t)$ as a function of time

2) Compute $q_C(t_1 = 250\ ms)$



- Here the capacitor is charging =>

$$\boxed{i(t) = I_0 e^{-t/RC}, \quad q(t) = Q_f\left(1 - e^{-t/RC}\right)}$$

ms

- $\underline{\tau = RC = 0.25}\ ms$
- $I_0 = \varepsilon / R = 12/2.5 = 0.48\ A$
- $Q_f = \varepsilon C = 12 \cdot 10^{-2} C$

$i(t) = 0.48\ e^{-t/0.25} = 0.48\ e^{-4t}$ Amper $= 0.48\ e^{-t/0.25}$

$q(t = 0.25\ s) = 12 \cdot 10^{-2}\left(1 - e^{-\frac{0.25}{0.25}}\right) = 0.076$ Coulomb

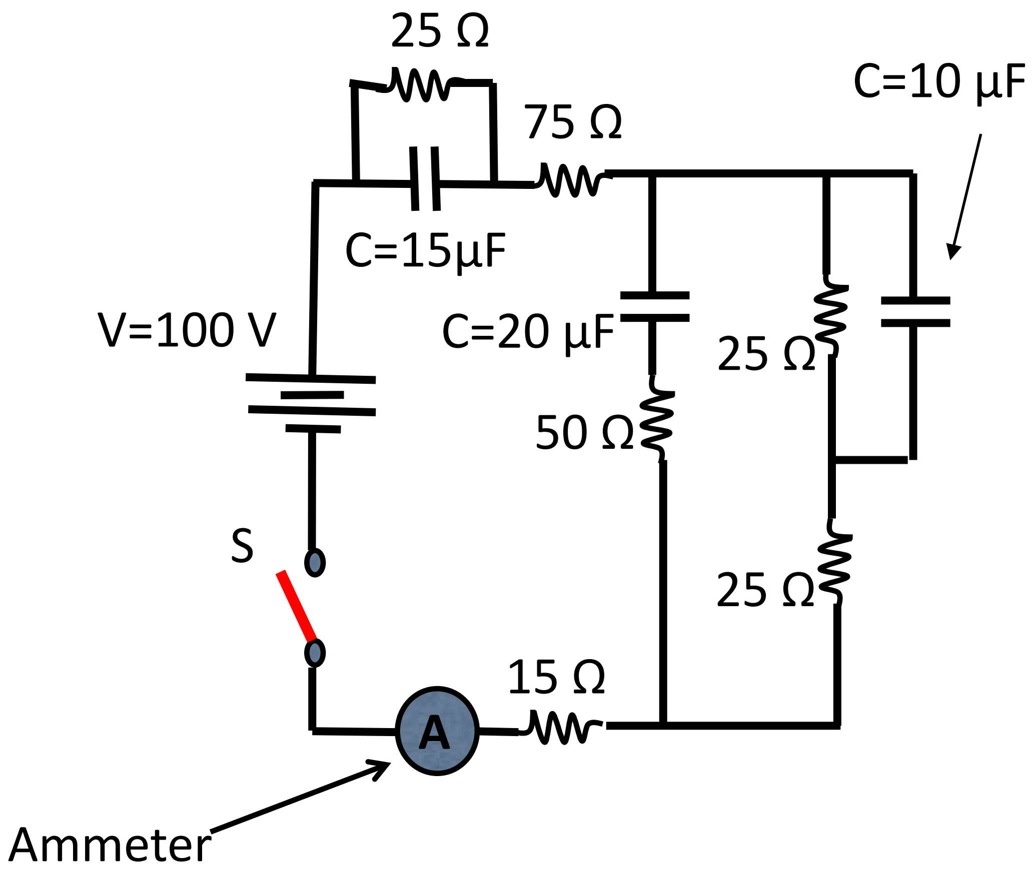
Remember the graphs

Q: All the capacitors in the circuit below are initially uncharged.

a) Find the current through the ammeter, $I_A(0+)$, just after the switch S is closed.
b) Find $I_A(\infty)$ a long time after the switch S has been closed.

HINT--

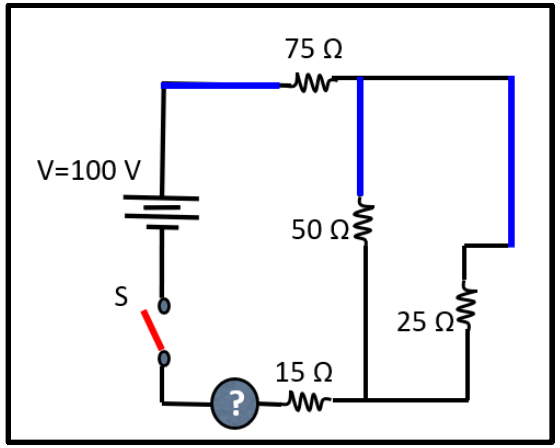
a) Redraw the circuit at $t = 0$
b) Redraw the circuit at $t = \infty$

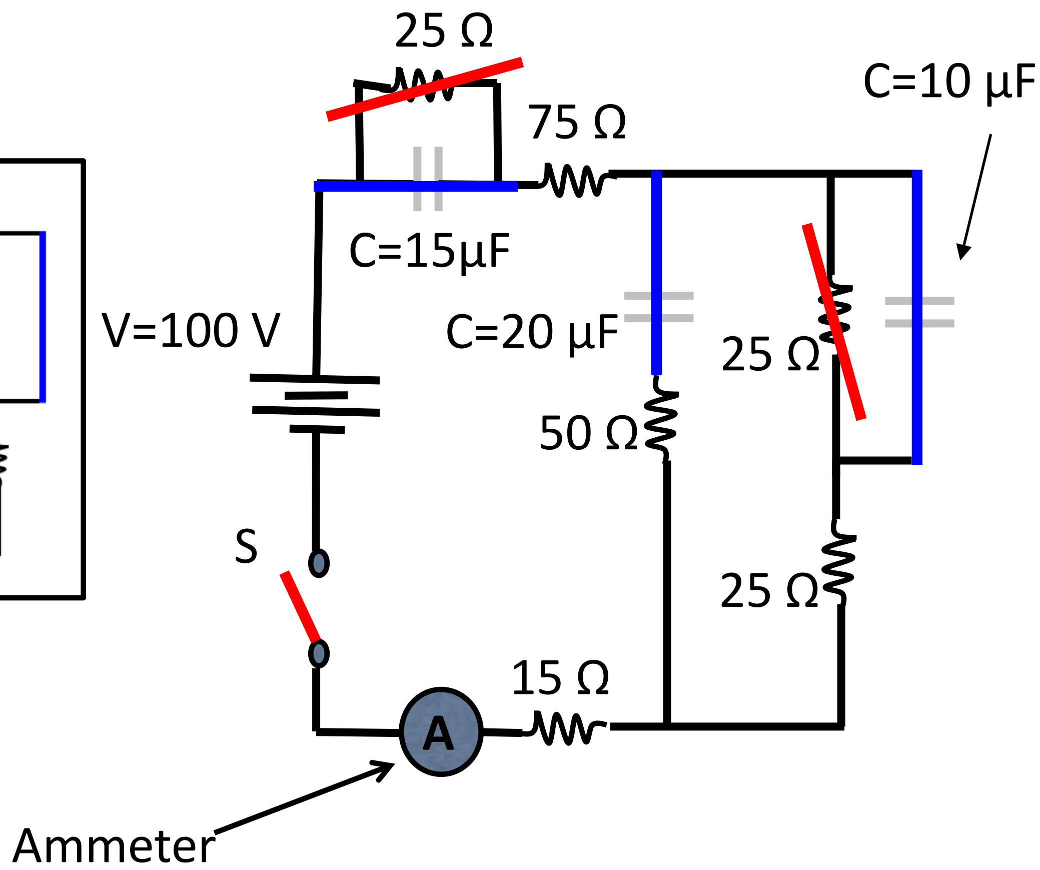


Q: All the capacitors in the circuit below are initially uncharged.

a) Find the current through the ammeter, $I_A(0+)$, just after the switch S is closed.
b) Find $I_A(\infty)$ a long time after the switch S has been closed.

- Right after the switch is closed, the capacitors act as a wire!
- Junction between an ideal wire and resistor => the resistor is short-cut





$$R_{eq} = 75 + \frac{50 \cdot 25}{50 + 25} + 15$$
$$= 106.7\ \Omega$$

$$I_{tot}(0) = I_A = \frac{V}{R_{eq}} = 0.94\ A$$

Q: All the capacitors in the circuit below are initially uncharged.

a) Find the current through the ammeter, $I_A(0+)$, just after the switch S is closed.

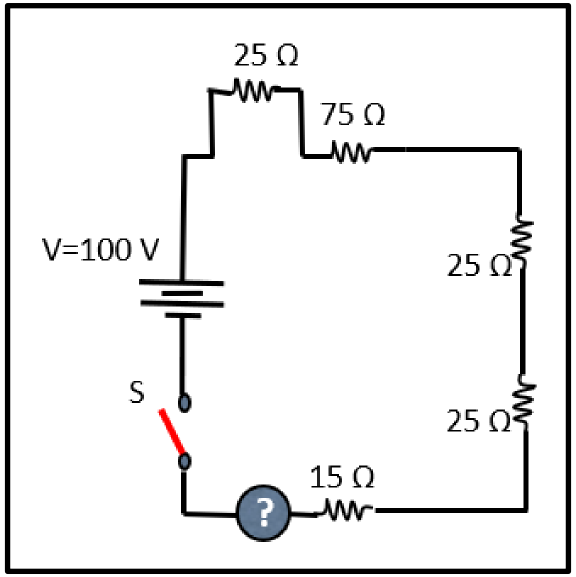
b) Find $I_A(\infty)$ a long time after the switch S has been closed.

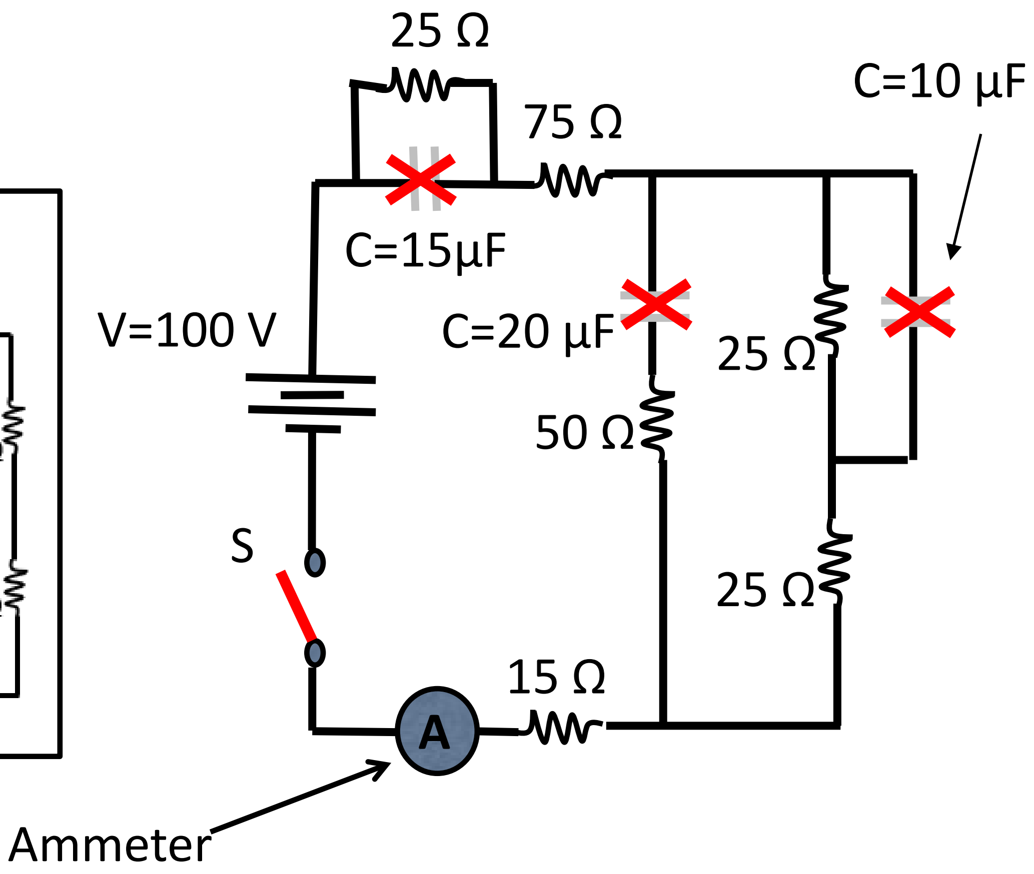
- When a capacitor is fully charged, it does not let any charge to move around => it acts as an open switch!

$$R_{eq} = 25 + 75 + 25 + 25 + 15$$

$$= 165\ \Omega$$

$$I_{tot}(\infty) = I_A = \frac{V}{R_{eq}} = 0.61\ A$$





Summary

- Capacitors are circuit elements that store electric charge and electric energy. However, in circuits with dynamics they can "act as other circuit elements":
  - Immediately after an empty capacitor is connected to a circuit, it acts as an ideal wire (there is no voltage drop across it)
  - After a capacitor is fully charged, it acts as an open switch (it does not accept more charge => no current flow around it)

- There is no voltage drop across a resistor if there is no current flowing through it (in a sense, it acts as an ideal wire)