MATH 101A — ASSIGNMENT 4

Learning goals

- Consolidate ideas of convergence and numerical approximation for series.
- Use Fourier methods to find some exact sums.
- Apply concepts from geometric series in support of larger problems.
- Be amazed by the ingenuity and power of Feymann's Integral Trick.

Contributors

On the first page of your submission, list the student numbers and full names (with the last name in bold) of all team members. Indicate members who have not contributed using the comment "(non-contributing)".

Assignment questions

The questions in this section contribute to your assignment grade. Stars indicate the difficulty of the questions, as described on Canvas.

Evaluating Series by Fourier Methods

In Assignment 3, we considered Fourier Sine Series, i.e., functions defined in terms of real constants $b_1, b_2, b_3, \ldots$, as follows:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \; = \; b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots, \qquad 0 < x < \pi. \tag{1}$$

Assuming that a given function f has this form, we derived the formulas

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx, \qquad n = 1, 2, 3, \ldots. \tag{2}$$

It's a remarkable fact that this works in reverse. That is, for *any* function f that is continuous on $[0,\pi]$ and differentiable on $(0,\pi)$, calculating the coefficients from (2) and using them to form the series in line (1) produces a valid identity in (1). (Note that line (1) makes no promises at the endpoints of the interval $(0,\pi)$, because the right side will have the value 0 at these points, no matter what the values of $f(0)$ and $f(1)$ may be.) We saw examples and supporting plots in Assignment 3, Question 5.

2. Complete the following, noting that the coefficient sequences $b_1, b_2, \ldots$ may be different in each part. In parts (a) and (b), you may quote results derived on Assignment 3.
   (a) (1 mark) ★☆☆☆ Find the constants b_n for which

$$1 = b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots, \qquad 0 < x < \pi. \tag{3}$$

   Integrate both sides of this identity on $[0,\pi]$ to find the exact value for the series

$$T = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Solution:

To find these values of b_n we must use the provided formula and the fact that $f(x) = 1$ to evaluate the following equation:

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\, dx,$$

$$= \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^\pi,$$

$$= \frac{2-2\cos(\pi n)}{\pi n}.$$

Now by taking a closer look at this result, we can figure out the constants b_n. For n being an integer, $\cos(\pi n)$ will alternate between the values of 1 and -1 depending on the parity of the number. Therefore,

$$b_n = \frac{4}{\pi n} \text{ for odd n}$$

And

$$b_n = 0 \text{ for even n.}$$

This means for our series for $f(x) = 1$ will only have nonzero b_n for all odd n will simplify to:

$$1 = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x),$$

Integrating both sides of (3) on $[0, \pi]$, we get,

$$\int_0^\pi 1\, dx = \int_0^\pi \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x)\, dx,$$

$$\pi = \sum_{k=1}^{\infty} \left[\frac{4(-\cos((2k-1)x))}{\pi(2k-1)^2}\right]_0^\pi,$$

$$\pi = \sum_{k=1}^{\infty} \frac{4\cdot 2}{\pi(2k-1)^2},$$

$$1 = \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

Given the expression we found for b_n and the fact that these coefficients directly correspond to the inverse squares of the odd numbers as shown in T. The exact value for this series is then equal to

$$T = \frac{\pi^2}{8}.$$

(b) (1 mark) ★☆☆☆ Find the constants b_n for which:

$$x = b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + ..., \qquad 0 < x < \pi. \tag{4}$$

Multiply both sides of this identity by x and integrate the result on $[0, \pi]$ to solve the famous "Basel Problem", i.e., to find the exact value for the series:

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution:
To find these values of b_n we must use the provided formula and the fact that $f(x) = x$ to evaluate the following equation:

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx$$

After calculating b_n we can multiply both sides of the identity by x and integrate the result on $[0, \pi]$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx$$

$$= \frac{2}{\pi} [\frac{xcos(nx)}{n}]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \cos(nx)\, dx$$

$$= \frac{2(\sin(\pi n) - \pi ncos(\pi n))}{\pi n^2}$$

Since $\sin(\pi n) = 0$ for all integer values of n, and $\cos(\pi n) = (-1)^n$, we can simplify the expression to:

$$b_n = \frac{2(-1)^{n+1}\pi n}{\pi n^2} = \frac{2(-1)^{n+1}}{n}.$$

Now with this formula for b_n and multiplying both sides of (4) by x and integrating the result on $[0, \pi]$ and integrating it, we get,

$$\int_0^{\pi} x^2 dx = \int_0^{\pi} \sum_{k=1}^{\infty} \frac{2(-1)^{n+1}}{n^2} \cdot xsin(nx)\, dx,$$

$$\frac{\pi^3}{3} = \sum_{k=1}^{\infty} [\frac{2(-1)^{n+1}}{n} \cdot \frac{\left(\sin(\pi n) - \pi ncos(\pi n)\right)}{n}]$$

$$\frac{\pi^3}{3} = \sum_{k=1}^{\infty} [\frac{2(-1)^{n+1}}{n} \cdot \frac{-\pi(-1)^n}{n}]$$

$$\frac{\pi^3}{3} = \sum_{k=1}^{\infty} [\frac{2\pi}{n^2}]$$

It directly relates to the terms of the Basel Problem, and we know that this series sums to x over the interval $0 < x < \pi$ we can directly find the sum of the series:

$$S = \frac{\pi^2}{6}$$

(c) (1 mark) ★☆☆☆ Suppose f is a well-behaved function defined on $[0, \pi]$ with $f(0) = 0 = f(\pi)$, and each b_n is defined by (2). Derive the formula:

$$b_n = -\frac{2}{n^2\pi}\int_0^\pi f''(x)\sin(nx)\,dx, \qquad n = 1, 2, 3, \ldots.$$

Solution:

To start, we integrate by parts twice to relate b_n to $f''(x)$, the second derivative of f(x).

$$\int udv = uv - \int vdu$$

For our case, we will set $u = f(x)$ and $dv = \sin(nx)$, this makes $du = f'(x)dx$ and $v = \frac{-1}{n}\cos(nx)$. Applying this formula we get,

$$\int_0^\pi f(x)\sin(nx)\,dx = \left[-\frac{f(x)\cos(nx)}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi f'(x)\cos(nx)\,dx.$$

Then integrating by parts again with the terms $u = f'(x)$ and $dv = \cos(nx)\,dx$ we find,

$$\int_0^\pi f'(x)\sin(nx)\,dx = \left[\frac{f'(x)\sin(nx)}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi f''(x)\sin(nx)\,dx.$$

Since $f(0) = 0 = f(\pi)$, the boundary terms disappear leading to,

$$b_n = \frac{2}{\pi}\left(-\frac{1}{n^2}\int_0^\pi f''(x)\sin(nx)\,dx\right).$$

We can simplify to obtain,

$$b_n = -\frac{2}{\pi n^2}\int_0^\pi f''(x)\sin(nx)\,dx.$$

(d) (1 mark) ★☆☆☆ Find the constants b_n in the identity below:

$$\pi^2 x - x^3 = b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \ldots, \qquad 0 < x < \pi. \tag{5}$$

Solutions:

First we can calculate $f''(x)$ for $f(x) = \pi^2 x - x^3$. This gives us,

$$f''(x) = -6x$$

Now we can plug this into the formula for b_n.

$$b_n = -\frac{2}{\pi n^2}\int_0^{\pi} -6x\sin(nx)\,dx.$$

Calculating this integral we get,

$$b_n = \frac{12(\sin(\pi n) - \pi n cos(\pi n))}{\pi n^4}.$$

Since $\sin(\pi n) = 0$ for all integer values of n, and $\cos(\pi n) = (-1)^n$, we can simplify the expression to,

$$b_n = \frac{12(-1)^{n+1}}{n^3}.$$

For $n = 1, b_1 = 12$

For $n = 2, b_2 = -\frac{3}{2}$

For $n = 3, b_3 = \frac{4}{9}$

For $n = 4, b_4 = -\frac{3}{16}$

For $n = 5, b_5 = \frac{12}{125}$

And so on with this general formula.

(e) (2 marks) ★★★☆ Find the exact value of the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Solution:

Multiplying x on both sides of (5) and integrating it over $[0, \pi]$, we get,

$$\int_0^{\pi} x(\pi^2 x - x^3)dx = \int_0^{\pi}\sum_{k=1}^{\infty}\frac{12(-1)^{n+1}}{n^3}\cdot xsin(nx)\,dx,$$

$$[\frac{x^3\pi^2}{3} - \frac{x^4}{4}]_0^{\pi} = \sum_{k=1}^{\infty}\left[\frac{12(-1)^{n+1}}{n^3}\cdot\frac{-\pi(-1)^n}{n}\right],$$

$$\frac{\pi^5}{3} - \frac{\pi^5}{5} = \sum_{k=1}^{\infty}\frac{12(-1)^n}{n^4}\cdot\pi(-1)^n,$$

$$\frac{2\pi^5}{15} = \sum_{k=1}^{\infty}\frac{12}{n^4}\cdot\pi,$$

$$\sum_{k=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Enhancing Convergence for Numerical Approximations

Suppose a given series $S = \sum_{n=1}^{\infty} a_n$ is known to converge. For any number m, the exact value of the sum can be split into a combination of the partial sum of m terms and the corresponding "tail": in symbols,

$$S = S_m + T_m, \qquad \text{where} \quad S_m = \sum_{n=1}^{m} a_n \quad \text{and} \quad T_m = \sum_{n=m+1}^{\infty} a_n .$$

By definition, the sequence S_m converges to S, so it's correct to say $S \approx S_m$ for large m. Actually, using this approximation is equivalent to applying the approximation $T_m \approx 0$, an approach described as "truncating the series after index m". Now saying $T_m \approx 0$ is not wrong, but using a better approximation for T_m can lead to an estimate for S that is much more accurate than simple truncation for essentially no extra effort. The next question explores one of the many approaches to this.

3. (a) (1 mark) ★☆☆☆ Suppose $G = A_0 + A_1 + A_2 + \cdots$ is a convergent geometric series. Derive a formula that expresses the value of G in terms of just the first two terms, A_0 and A_1. (Assume $A_0 \neq A_1$.)

Solution:

Since G is a geometric series, we can express G with just $A_0 + A_1$ by:

$$G = A_0 + A_1 + A_1 \mathrm{r} + A_1 r^2 \ldots \ldots,$$

where r is the common ratio of the series.

Since the series is convergent, $|r| < 1$. The formula for the infinite series is,

$$G = \frac{A_0}{1 - r}$$

We can express r with A_0 and A_1 with,

$$r = \frac{A_1}{A_0}$$

Thus,

$$G = \frac{A_0}{1 - \frac{A_1}{A_0}},$$

$$G = \frac{{A_0}^2}{A_0 - A_1}.$$

For a general series S as discussed in the preamble, suppose we have computed the partial sums up to some index N, so we know the numbers S_{N-2}, S_{N-1}, and S_N. The Big Idea is to split

$$S = S_{N-2} + T_{N-2}$$

and use S_{N-2} with the additional values S_{N-1} and S_N to approximate T_{N-2}.

(b) (1 mark) ★★☆☆ Let $\hat{T}_{N-2}$ be the geometric series whose first two terms match the first two terms in the original definition of T_{N-2}. Assuming that $\widehat{T}_{N-2}$ converges, express the exact value of $\widehat{T}_{N-2}$ in terms of the partial sums S_{N-2}, S_{N-1}, and S_N.

Solution:
The first two terms of $\hat{T}_{N-2}$ are a_{N-1} and a_N.
Since $\hat{T}_{N-2}$ is a convergent geometric series, based on part a), it can be expressed as,

$$\hat{T}_{N-2} = \frac{(a_{N-1})^2}{a_{N-1} - a_N}.$$

Based on the definition of the partial sum,

$$S_N - S_{N-1} = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n,$$

$$= a_N.$$

$$S_{N-1} - S_{N-2} = \sum_{n=1}^{N-1} a_n - \sum_{n=1}^{N-2} a_n,$$

$$= a_{N-1}.$$

Thus, $\hat{T}_{N-2}$ can be expressed as,

$$\hat{T}_{N-2} = \frac{(S_{N-1} - S_{N-2})^2}{(S_{N-1} - S_{N-2}) - (S_N - S_{N-1})},$$

$$\hat{T}_{N-2} = \frac{(S_{N-1} - S_{N-2})^2}{2S_{N-1} - S_{N-2} - S_N}.$$

(c) (1 mark) ★☆☆☆ The number $\widehat{T}_{N-2}$ is often a surprisingly good approximation for T_{N-2} even when the true tail series is not geometric. This suggests the following approximation to the original sum:

$$S = S_{N-2} + T_{N-2} \approx S_{N-2} + \hat{T}_{N-2}.$$

Show the algebra needed to rearrange this approximation into a form that could be described as taking the most recent partial sum and adding a correction term, like this:

$$S \approx S_N - \frac{(S_N - S_{N-1})^2}{\boxed{??}}.$$

Here the denominator expression should involve only the quantities S_{N-2}, S_{N-1}, and S_N. For later developments, let $\widehat{S}_N$ denote the full expression on the right here, so $S \approx \hat{S}_N$.

Solution:

From the expression obtained from b), we get,

$$S \approx S_{N-2} + \hat{T}_{N-2} \approx S_{N-2} + \frac{(S_{N-1} - S_{N-2})^2}{2S_{N-1} - S_{N-2} - S_N},$$

Replacing $S_{N-2} + \frac{(S_{N-1}-S_{N-2})^2}{2S_{N-1}-S_{N-2}-S_N}$ with $S_N - x$,

$$x = S_N - S_{N-2} - \frac{(S_{N-1} - S_{N-2})^2}{2S_{N-1} - S_{N-2} - S_N},$$

$$= \frac{(S_N - S_{N-2})(2S_{N-1} - S_{N-2} - S_N) - (S_{N-1} - S_{N-2})^2}{2S_{N-1} - S_{N-2} - S_N},$$

By expansion and simplification, we arrive at,

$$x = -\frac{(S_N - S_{N-1})^2}{2S_{N-1} - S_{N-2} - S_N}.$$

Therefore,

$$S \approx S_N - \frac{(S_N - S_{N-1})^2}{S_{N-2} + S_N - 2S_{N-1}}.$$

(d) (1 mark) ★☆☆☆ Try this idea on the series $S = \sum_{n=1}^{\infty} \frac{n3^n}{4^n}$, whose exact sum is known to be 12. Using computer assistance as appropriate, calculate the following values for each *N* = 5,6,...,25:

- the partial sums S_N and their errors ($12 - S_N$),
- the improved approximations $\widehat{S}_N$ and their errors ($12 - \widehat{S}_N$), and
- the "accuracy improvement factors" $|12 - S_N| \,/\, \left|12 - \widehat{S}_N\right|$ (round these to integers).

(For presentation, a table showing the requested values would be ideal. Presenting a screenshot of a spreadsheet would be one way to achieve this.)

Solution:

	A	B	C	D	E	F	G	H	I
1	N	Partial sum SN	True value	Error		N	Improved approximations	True value	Error
2	5	5.592773438	12	6.407226563		5	23.390625	12	-11.390625
3	6	6.660644531	12	5.339355469		6	16.27148438	12	-4.27148438
4	7	7.595031738	12	4.404968262		7	14.13574219	12	-2.13574219
5	8	8.395935059	12	3.604064941		8	13.20135498	12	-1.20135498
6	9	9.071697235	12	2.928302765		9	12.72081299	12	-0.72081299
7	10	9.634832382	12	2.365167618		10	12.45050812	12	-0.45050812
8	11	10.09941888	12	1.900581121		11	12.28961236	12	-0.28961236
9	12	10.4795351	12	1.520464897		12	12.19005811	12	-0.19005811
10	13	10.78837954	12	1.211620465		13	12.12670541	12	-0.12670541
11	14	11.03783081	12	0.962169193		14	12.08552615	12	-0.08552615
12	15	11.23828272	12	0.761717278		15	12.05831328	12	-0.05831328
13	16	11.39864425	12	0.601355745		16	12.04009038	12	-0.04009038
14	17	11.52643235	12	0.47356765		17	12.02775488	12	-0.02775488
15	18	11.62791113	12	0.372088868		18	12.01932929	12	-0.01932929
16	19	11.7082485	12	0.291751498		19	12.0135305	12	-0.0135305
17	20	11.77167274	12	0.22832726		20	12.00951364	12	-0.00951364
18	21	11.82161933	12	0.178380672		21	12.00671551	12	-0.00671551
19	22	11.86086308	12	0.139136924		22	12.00475682	12	-0.00475682
20	23	11.89163374	12	0.108366258		23	12.00337984	12	-0.00337984
21	24	11.91571513	12	0.084284867		24	12.00240814	12	-0.00240814
22	25	11.93452872	12	0.065471281		25	12.0017201	12	-0.0017201

Improper Integrals, Geometric Series, and Exact Values

The *Laplace Transform* is an operator that takes in a function *f* defined on the real interval $[0,+\infty)$ and returns a new function defined on some interval $(a,+\infty)$ that depends on *f*. Typically the new function is named *F*, and the operation is denoted $F = L\,\{f\}$. The letter *t* is often used for inputs to *f*, and the letter *s* is the standard choice for inputs to *F*. In these symbols, the defining relationship is

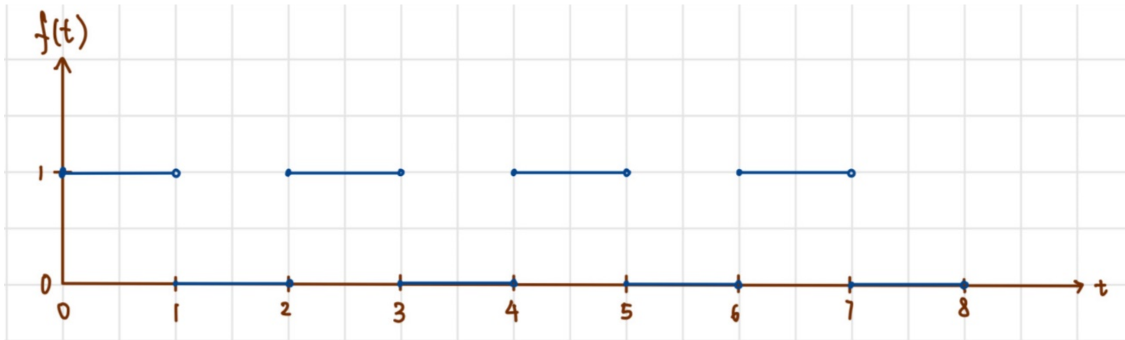
$$F = L\,\{f\} \qquad \Longleftrightarrow \qquad F(s) = \int_0^{\infty} e^{-st} f(t)\,dt, \quad s > a.$$

Students may remember finding $L\,\{f\}$ for some simple functions *f* on WeBWorK 08, Questions 23–24.

4. Consider the function $f(t)$ defined below:

$$f(t) = \begin{cases} 1, & \text{if } 2n \le t < 2n+1 \text{ for some integer } n, \\ 0, & \text{if } 2n+1 \le t < 2n+2 \text{ for some integer } n. \end{cases}$$

(a) (1 mark) Make a reasonable sketch of the graph of *f* on an interval containing [0,8]. (Computer plots are welcome.)



(b) (2 marks) Working directly from the definition, explain why $F = L\{f\}$ is given by

$$F(s) = \frac{1}{s\,(1+e^{-s})}, \qquad s > 0.$$

Suggestion: Start by writing $F(s)$ as a series of simple integrals.

Solution:

The Laplace Transformation can be derived by integrating over the intervals where the function is equal to 1 because there is no contribution to the area when the function is equal to 0. Given the alternating pattern of this function the transformation can be written as an infinite series of integrals of e^{-st} over the intervals where the function equals 1.

$$F(s) = \sum_{n=0}^{\infty} \int_{2n}^{2n+1} e^{-st} dt,$$

$$\int_{2n}^{2n+1} e^{-st} dt = \left[-\frac{1}{s} e^{-st}\right]_{2n}^{2n+1}.$$

This simplifies to,

$$-\frac{1}{s}\left(e^{-s(2n+1)} - e^{-2ns}\right),$$

And further to,

$$\frac{e^{-2ns}}{s} - \frac{e^{-s(2n+1)}}{s}.$$

We can consider these two terms separately over all intervals. For the e^{-2ns} terms,

$$\sum_{n=0}^{\infty} e^{-2ns} = \frac{1}{1-e^{-2s}}$$

And for -$e^{-s(2n+1)}$ terms:

$$-\sum_{n=0}^{\infty} e^{-s(2n+1)} = \frac{-e^{-s}}{1-e^{-2s}}$$

Finally, we can combine both of these parts to get,

$$F(s) = \frac{1}{s}(\frac{1}{1-e^{-2s}} - \frac{e^{-s}}{1-e^{-2s}})$$

$$F(s) = \frac{1}{s}(\frac{1-e^{-s}}{1-e^{-2s}})$$

Now we can notice that,

$$1 - e^{-2s} = (1 + e^{-s})(1 - e^{-s})$$

So, then we get this,

$$F(s) = \frac{1}{s(1+e^{-s})}.$$

This is the expected result and shows the overall nature of the Laplace Transform.

5. The following improper integral is known to converge:

$$I = \int_{-\infty}^{\infty} \frac{\sin(t)}{t}\,dt.$$

(Note that the singularity at $t = 0$ is removable, because $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$.) Find the exact value of I by following the steps below:

(a) (1 mark) ★★☆☆ Find the Laplace Transform of $f(t) = \sin(t)$. Call it $F(s)$; assume $s > 0$.

Solution:

Given that the definition of the Laplace Transformation is,

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt, \quad s > a.$$
,

We can substitute f(t) = sin(t) into the question,

$$F(s) = \int_0^{\infty} e^{-st}\sin(t)\,dt.$$

By integration,

$$F(s) = [-\frac{e^{-st}}{s}\sin(t)]_0^{\infty} + \int_0^{\infty} \frac{e^{-st}}{s}\cos(t)\,dt,$$

$$= [-\frac{e^{-st}}{s}\sin(t)]_0^\infty - [\frac{e^{-st}}{s^2}\cos(t)]_0^\infty + \frac{1}{s^2}\int_0^\infty e^{-st}\sin(t)\,dt,$$

$$= [-\frac{e^{-st}\,(\mathrm{s}\cdot\sin(t)+\cos(t))}{s^2+1}]_0^\infty,$$

The Laplace Transformation of $f(t) = \sin(t)$ is a standard result:

$$F(s) = \frac{1}{s^2+1}.$$

Now, inspired by the form of I, we invent the function

$$G(s) = \int_0^\infty e^{-st}\left(\frac{\sin(t)}{t}\right)\,dt.$$

This is the key step, with two essential properties:

1. When $s = 0$, the factor e^{-st} turns into 1 and makes $G(0)$ very similar to the integral we seek; and
2. the derivative of $G(s)$ has a simple form.

(b) (1 mark) Express $G'(s)$ as a rational function of s in the region where $s > 0$.
Suggestion: Use without proof the natural extension of the *Leibniz integral rule* to this situation.

Solution:

Using the Leibniz integral rule, we can write:

$$G'(s) = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}\frac{sint}{t})\,dt$$

Applying the product rule with respect to s, we get:

$$G'(s) = -\int_0^\infty (e^{-st}\sin(t))$$

Thus,

$$G'(s) = \frac{d}{ds}\int_0^\infty (e^{-st}\frac{sint}{t})\,dt = -\int_0^\infty e^{-st}\sin(t)\,dt$$

Evaluating the definite integral of G'(s), we get:

$$G'(s) = -\frac{1}{s^2+1}$$

Since the Laplace Transform of $\sin(t)$ was defined above as $\mathrm{F(s)} = \frac{1}{s^2+1}$, we can clearly see that G'(s) is the negative of the Laplace transformation.

(c) (1 mark) Use the result above to find $\lim_{R\to\infty}[G(R) - G(0)]$.

Solution:

We can solve this part by understanding that $G(0)$ is essentially the integral I from the bounds of 0 to ∞ because when s = 0, the entire term e^{-st} becomes 1.

Since we have found that,

$$G'(s) = -\frac{1}{s^2 + 1}$$

And we want to find $\lim_{R\to\infty}[G(R) - G(0)]$. Given the Laplace Transform $F(s)$ of $\sin(t)$ is $\frac{1}{s^2+1}$, and since $G'(s) = -F(s)$, we can integrate $G'(s)$ to find $G(s)$:

$$G(s) = -\int F(s)ds$$

$$G(s) = -\int \frac{1}{s^2 + 1} ds$$

$$= -\arctan(s) + C.$$

The integral of this with respect s is $-\arctan(s)$. So as R approaches infinity, $\arctan(R)$ approaches $\frac{\pi}{2}$, and thus the limit is:

$$\lim_{R\to\infty}[G(R) - G(0)] = \lim_{R\to\infty}[-\arctan(R)] = -\frac{\pi}{2}$$

(d) (1 mark) Determine the exact value of I.

Solution:
We can approach the evaluation of the integral of I as follows,

$$I = \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt = \int_{-\infty}^{0} \frac{\sin(t)}{t} dt + \int_{0}^{\infty} \frac{\sin(t)}{t} dt.$$

Given that $\frac{\sin(t)}{t}$ is an even function, which satisfies $\frac{\sin(-t)}{-t} = \frac{\sin(t)}{t}$, we can rewrite the above expression as,

$$I = 2\int_{0}^{\infty} \frac{\sin(t)}{t} dt.$$

In part c) we have shown that,

$$\int e^{-st} \frac{\sin(t)}{t} dt = -\arctan(s) + C.$$

Thus, we can infer that,

$$I = 2\lim_{n\to\infty}[\arctan(R)] = 2 \times \frac{\pi}{2} = \pi.$$

Notes and Comments

The Fourier theory is used throughout applied mathematics, from the theory of waves, heat flow, and probability, to quantum mechanics.

The idea used in Question 3 to accelerate convergence of partial sums of a given series can be applied to any convergent sequence. It is known as *the Shanks Transformation*, in spite of having been invented and published decades before Daniel Shanks's paper of 1955. It is one of a large number of methods for accurately predicting the limit of a slowly-converging sequence. (To learn more about such methods, see, e.g., Weniger, E. J., *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series,* https://arxiv.org/pdf/math/0306302.pdf.)

The idea in Question 5 is popularly known as Feynman's Integral Trick: it has many fascinating applications. There are nice discussions online at *Cantor's Paradise* and *a personal site of Zaharia Burghelea*. The function $\sin(t)/t$ (extended with the value 1 when $t = 0$) comes up often enough to deserve a special name: it's known as the *sinc function*.