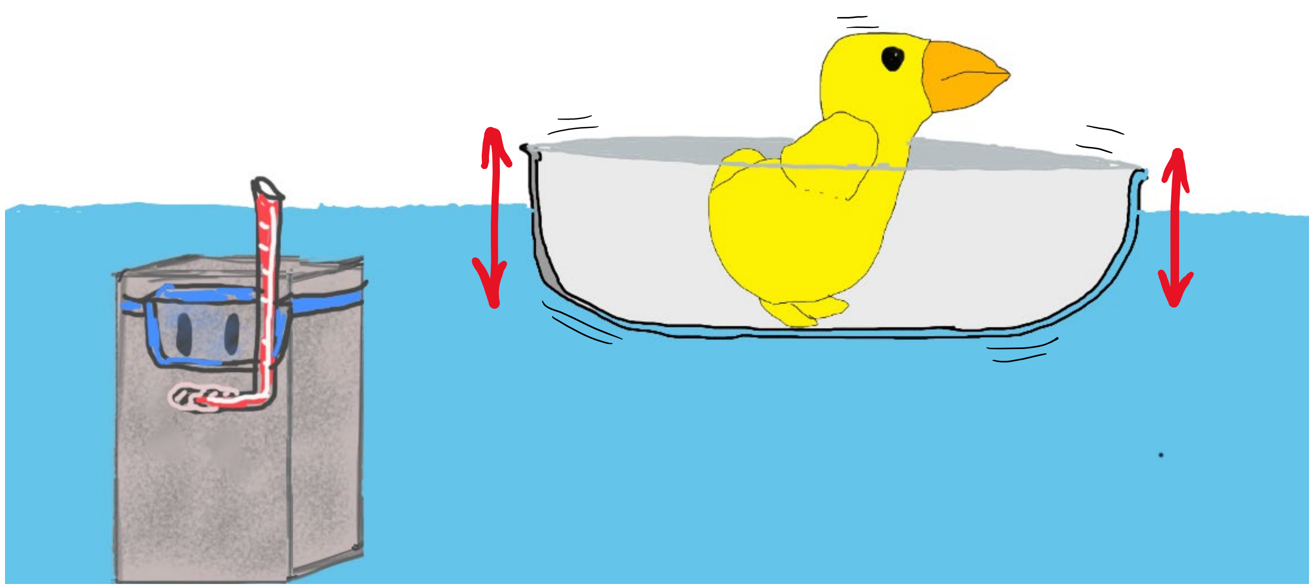
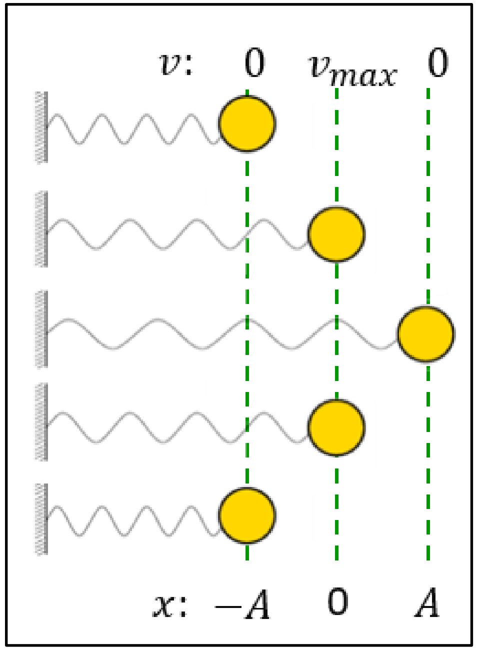
Lecture 29.

SHM: examples.

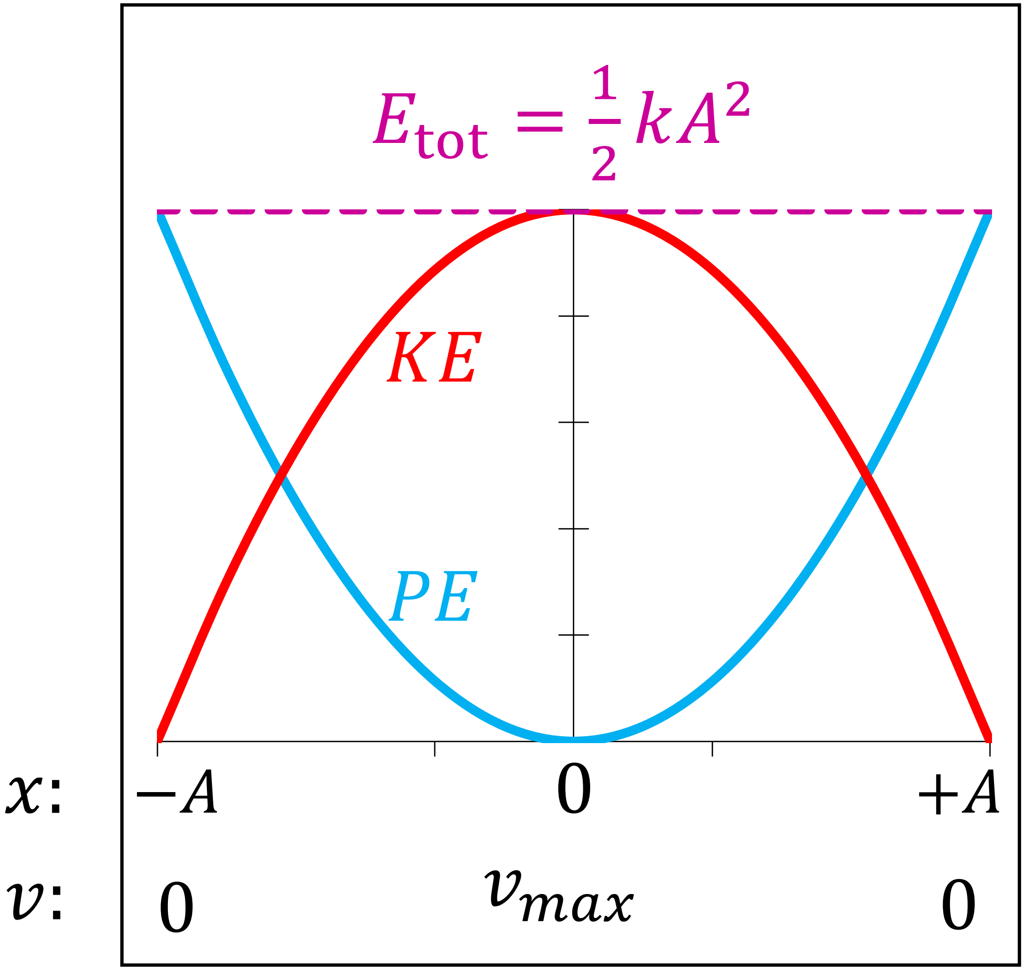
Natural angular frequency of oscillations.



Last Time



Energy conservation in SHM



$$E_{\text{tot}} = KE + PE$$

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$

m k

Q: A 0.5 kg mass is attached to a horizontal spring of spring constant 200 N/m.

If the spring is initially compressed by 0.1 m, and the mass is then released, $\longrightarrow$ SHM

what is the speed of the block when the spring is at its equilibrium length?

$v = ?$ $x = 0$ 0

A

$m, T \rightarrow \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

$x(t) = A\cos(\omega t + \phi)$ $\phi = 0$

m, k

$PE = 0$

$v \leftrightarrow KE = \frac{mv^2}{2}$

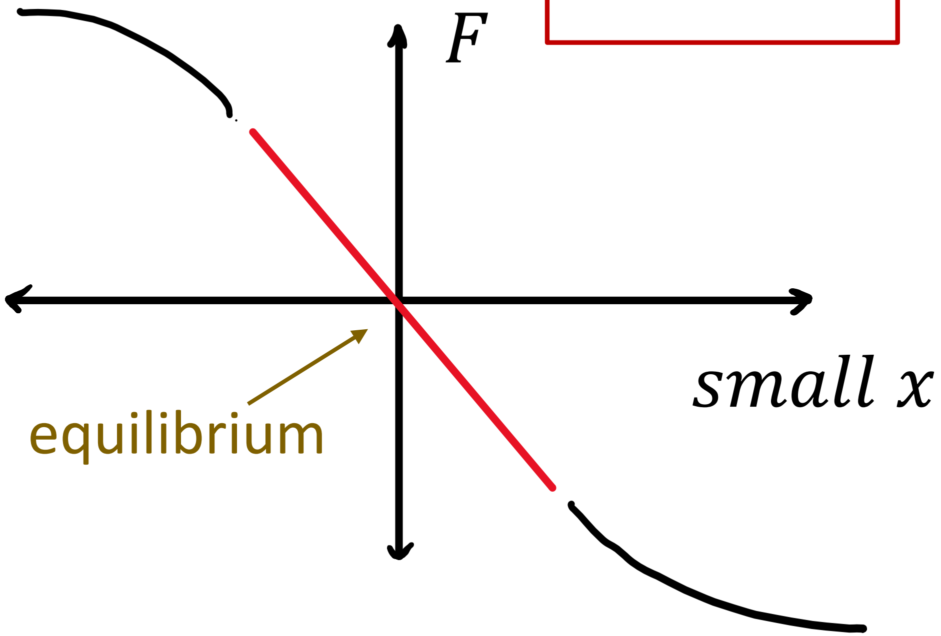
$KE = E_{tot} \rightarrow \frac{mv^2}{2} = \frac{kA^2}{2}$

A. 1 m/s
B. 2 m/s
C. 3 m/s
D. 4 m/s
E. 5 m/s

Week 10

Hooke's Law

$$\boxed{F = -kx}$$



- F is the restoring force
- x is the displacement from the equilibrium position
- "$-$" captures the restoring character of the force
- k is the slope

- It is the slope k who determines the angular oscillation frequency ω in SHM:

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$

Hooke's law   Newton's 2nd law

- $F = -kx$ and $F = ma$

  gives $-kx = ma$.

- Now: $a = \frac{d^2x}{dt^2}$. Then:

$$\frac{d^2x}{dt^2} = -\boxed{\frac{k}{m}}x \equiv -\omega^2 x$$

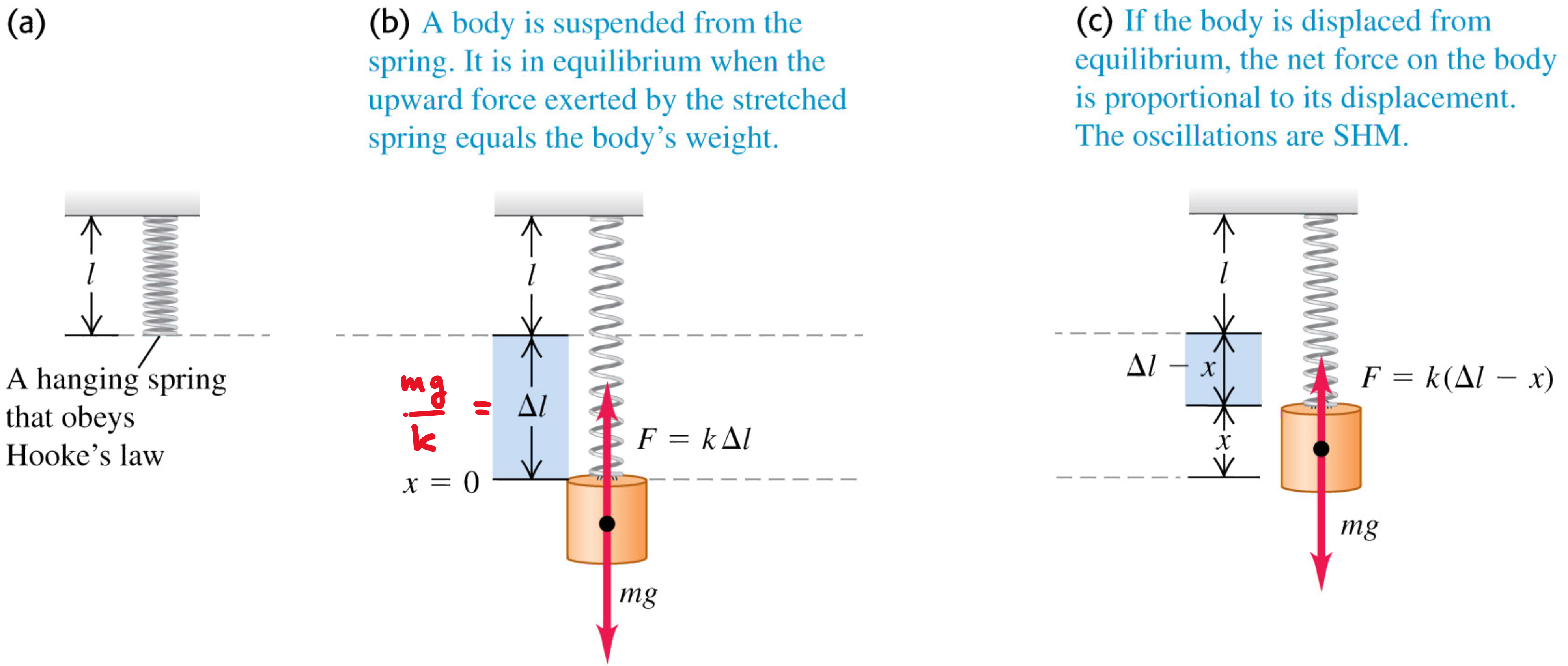
$$a = -\omega^2 x \quad (\text{SHM})$$

where $\omega = \sqrt{k/m}$.

$\frac{k}{m} = \omega^2$

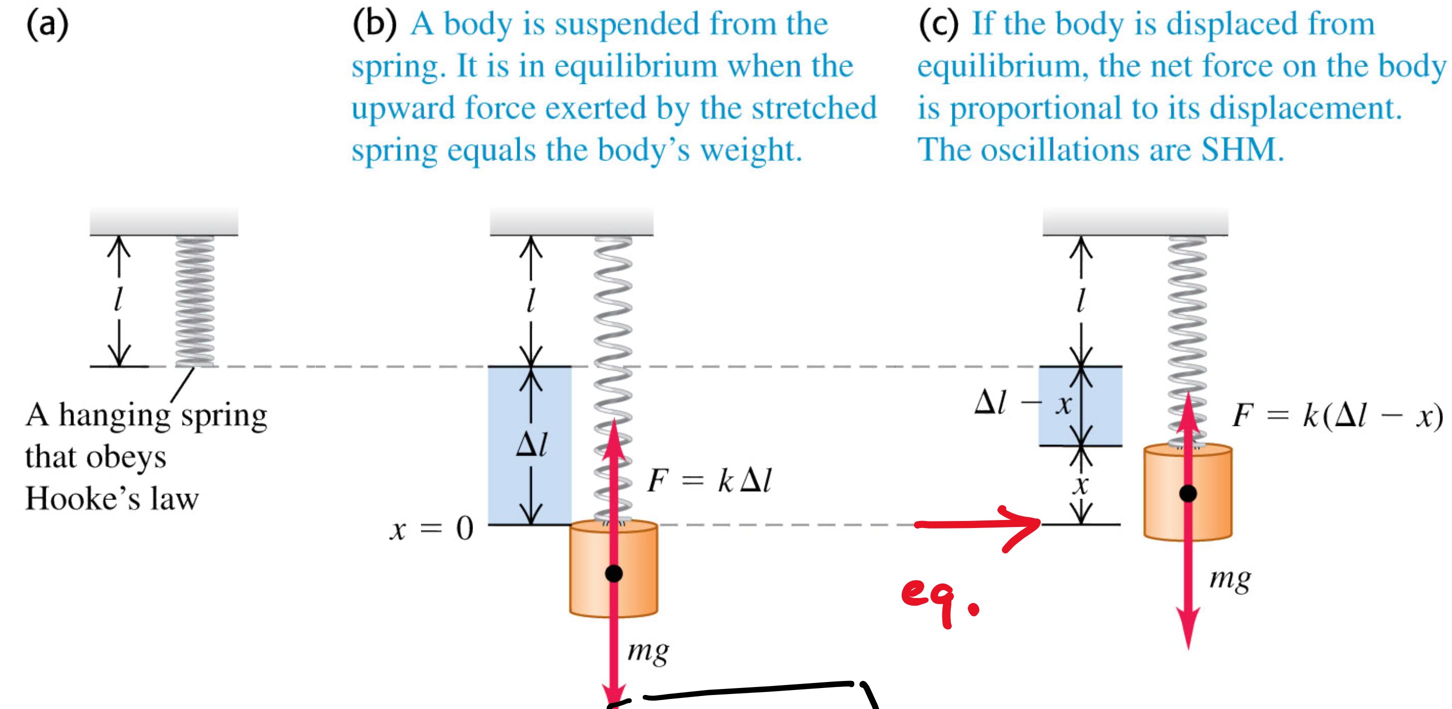
$\omega = \sqrt{\frac{k}{m}}$

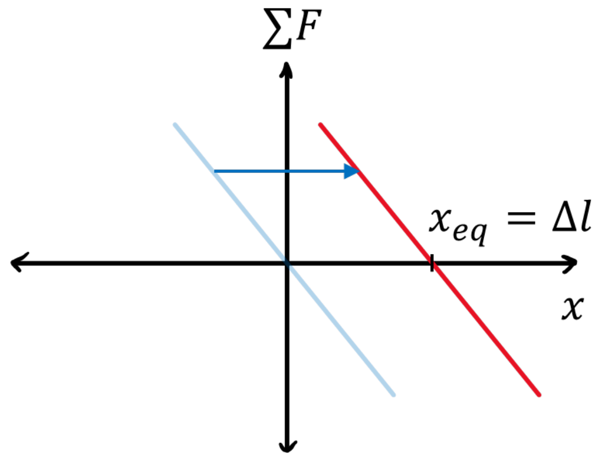
Applications – Vertical SHM



- If we attach mass m, it stretches the spring until $\sum \vec{F} = \vec{F}_{\text{elas}} + \vec{F}_{\text{grav}} = 0$
- $F_{\text{elas}} = k\Delta l$ and $F_{\text{grav}} = mg$, so $mg = k\Delta l$ and equilibrium position is $\Delta l = \frac{mg}{k}$.
- If we displace it from this new equilibrium by x, will it still execute SHM? With same ω?

Applications – Vertical SHM





What matters for SHM is the SLOPE of $\sum F$ vs x at x_{eq} and that hasn't changed

- Equilibrium position: $\boxed{\Delta l = \frac{mg}{k}}$. $mg = k\Delta l$
- Elastic force (magnitude) when displaced from this new equilibrium by x: $F_{\text{elas}} = k(\Delta l - x)$
- Net force: $\sum F = F_{\text{elas}} - F_{\text{grav}} = k(\Delta l - x) - mg$
- Newton's 2nd law: $(k\Delta l - kx) - \underline{k\Delta l} = ma$. Now:

$\Rightarrow ma = (\cancel{k\Delta l} - kx) - \cancel{k\Delta l}$ ...or $a = -(k/m)x$ $\Rightarrow$ Same SHM with same ω!

Different systems, different ω's

$F = -kx \rightarrow -\frac{dF}{dx} = k$

- You will always have SHM whenever the force is:
  1. restoring back to an equilibrium position
  2. proportional to the displacement from equilibrium

  i.e. $\boxed{F_{\text{SHM}} = -\text{const}\left(x - x_{\text{equil}}\right)}$

- This always gives $a = -\omega^2\, x$, with solution $x(t) = A\cos(\omega t + \phi)$

- For a mass-spring system where the force is given by Hooke's Law we have:
  - $F = ma = -kx$, so $a = -\left(\frac{k}{m}\right)x = -\omega^2\, x$ and $\omega = \sqrt{\frac{k}{m}}$
  - NOTE that $k = -\frac{dF}{dx}$

- For other systems, the dependence of the force on displacement can be different and this leads to different expressions for ω…
  - But you can find an "effective spring constant" from $k = -\frac{dF}{dx}$ and still use $\omega = \sqrt{\frac{k}{m}}$

General approach for finding ω

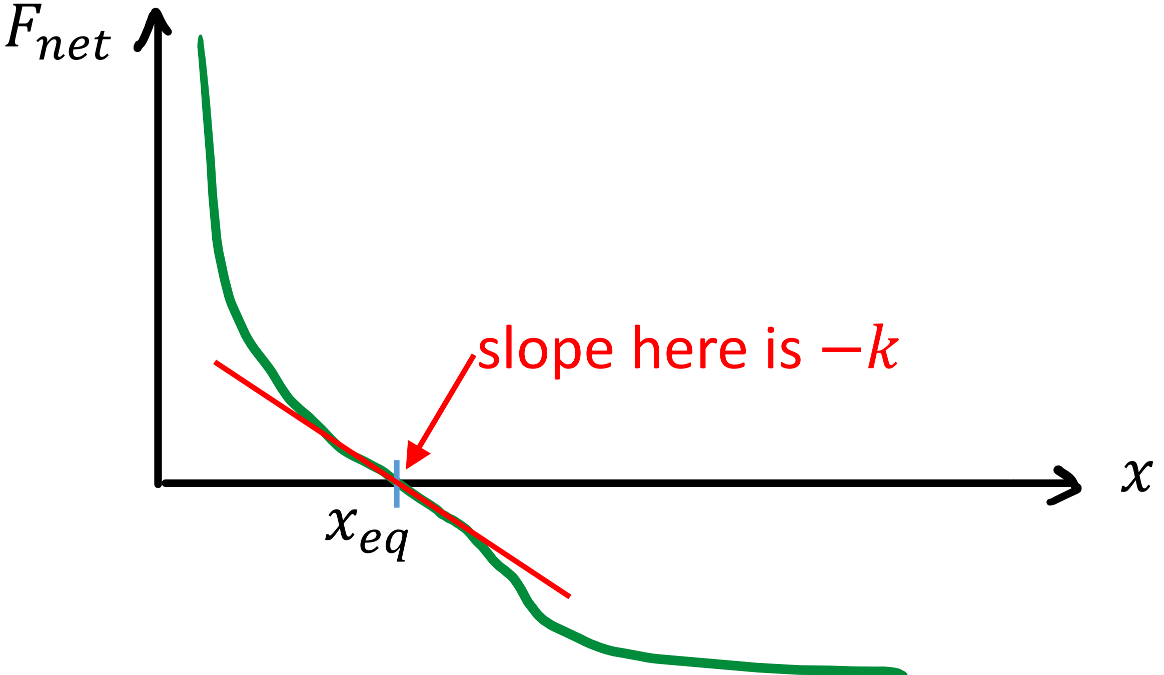
1) Find $F_{net} = \sum F$ as a function of position x

2) Find equilibrium position x_{eq} by solving $F_{net}(x_{eq}) = 0$

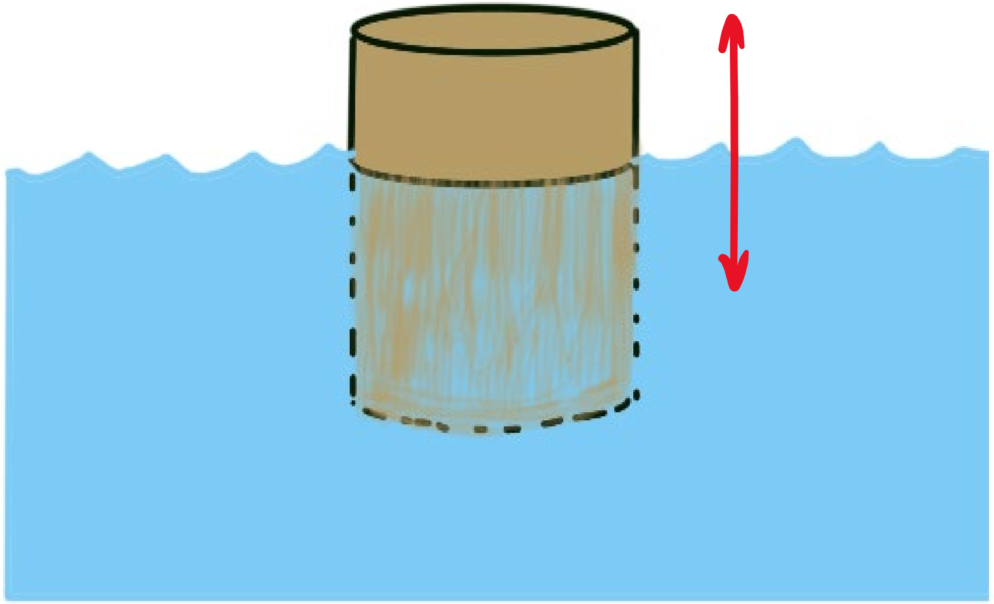
3) Find "effective spring constant" at $x = x_{eq}$:

$$k = -\frac{dF_{net}(x_{eq})}{dx}$$

- Then $\omega = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the oscillator



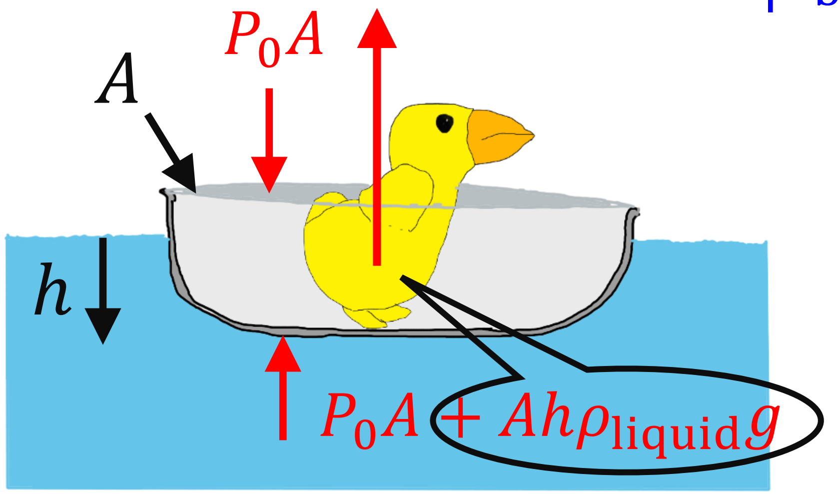
Example: Vertical bobbing

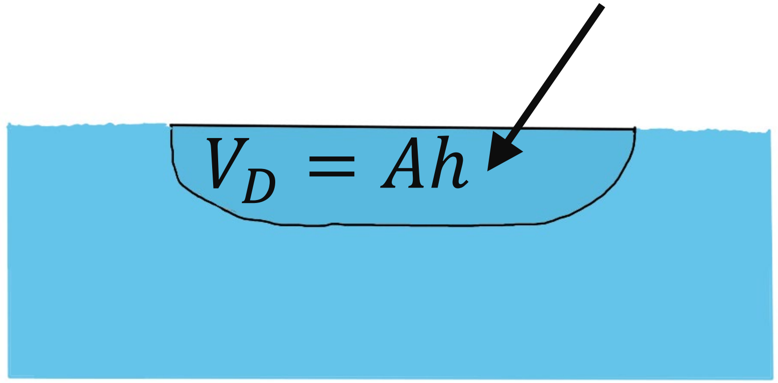


- Is it SHM?
- If yes, what does the frequency depend on?

The buoyant force

$$\left|F_{\text{buoyant}}\right| = V_D \cdot \rho_{\text{liquid}} \cdot g$$





- If an object floats, the buoyant force acting on it compensates its weight
- Appears as a result of pressure difference

- V_D = volume of liquid displaced
- ρ_{liquid} = density of liquid
- $V_D \cdot \rho_{\text{liquid}}$ = mass of liquid displaced

F_{buoyant}: its magnitude is equal to the weight of liquid displaced

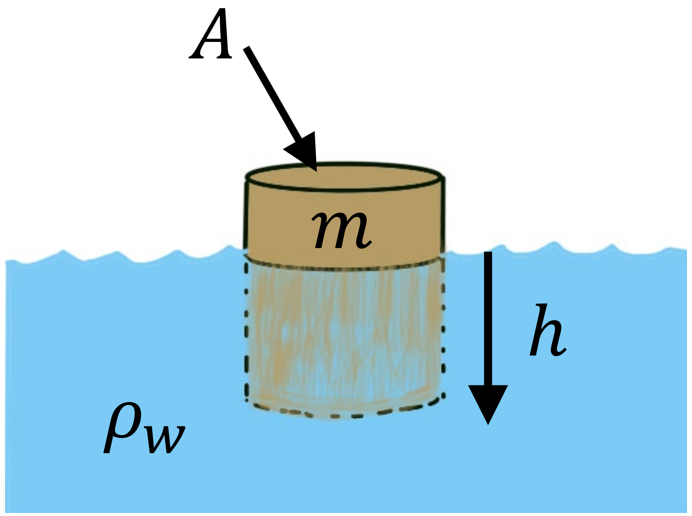
Example: Vertical bobbing

A cylindrical object of mass m and cross-sectional area A is placed in water.

a) Draw a free-body diagram for the object showing the vertical forces.

b) Calculate the magnitude of the net downwards force on the object as a function of the depth h that the object is in the water.

c) What is the oscillation frequency in terms of h, A, m, g, and ρ_{water}?



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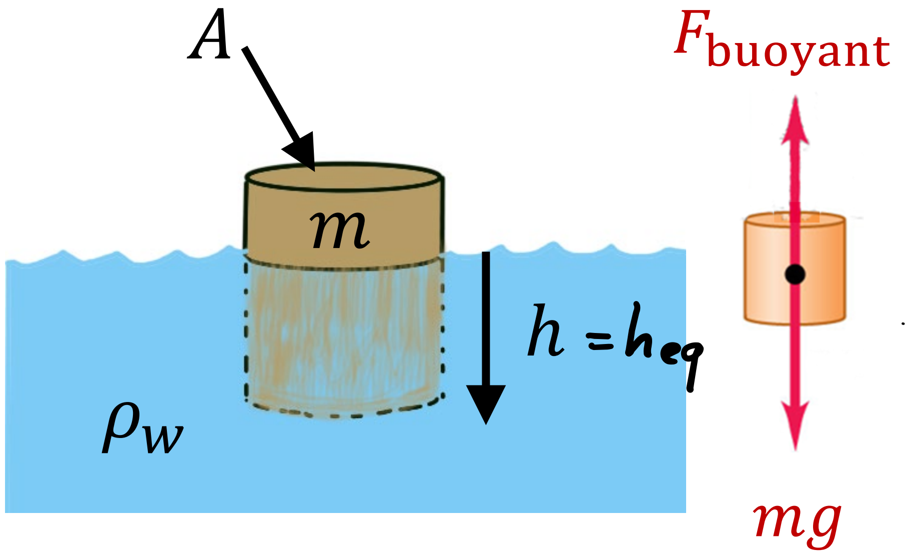

1) Net force:

$$F_{\text{buoyant}} = g(\text{mass of water displaced}) = g\rho_w A h$$

$$F_{net} = \sum F = mg - g\rho_w A h$$

$F_{net}(h)$

2) Find equilibrium position from $F_{net} = 0$:

$$h_{eq} = \frac{m}{\rho_w A}$$

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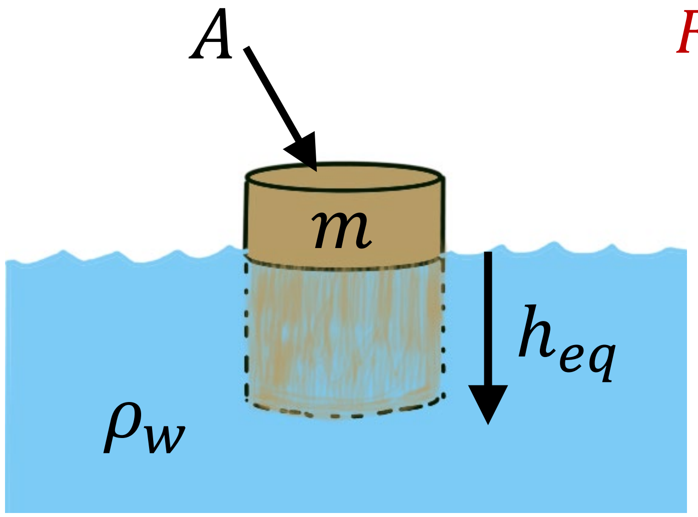


$$h_{eq} = \frac{m}{\rho_w A} \qquad F_{net} = mg - g\rho_w A h$$

3) Find "effective spring constant" k at $h = h_{eq}$:

$$k = -\frac{dF_{net}(h_{eq})}{dh} = g\rho_w A$$

$$\omega = \sqrt{k/m} = \sqrt{g\rho_w A / m}$$

Smaller frequency for larger m, smaller A

Step 3: $k = -\frac{dF_{net}(h_{eq})}{dh}$

$$\boxed{h_{eq} = \frac{m}{\rho_w A} \qquad F_{net} = mg - g\rho_w A h} = F_{net}$$

$$k = -\frac{d}{dh}\Big(\underbrace{\cancel{mg}}_{=0} - g\rho_w A \cdot h\Big)\Big|_{h=h_{eq}} =$$

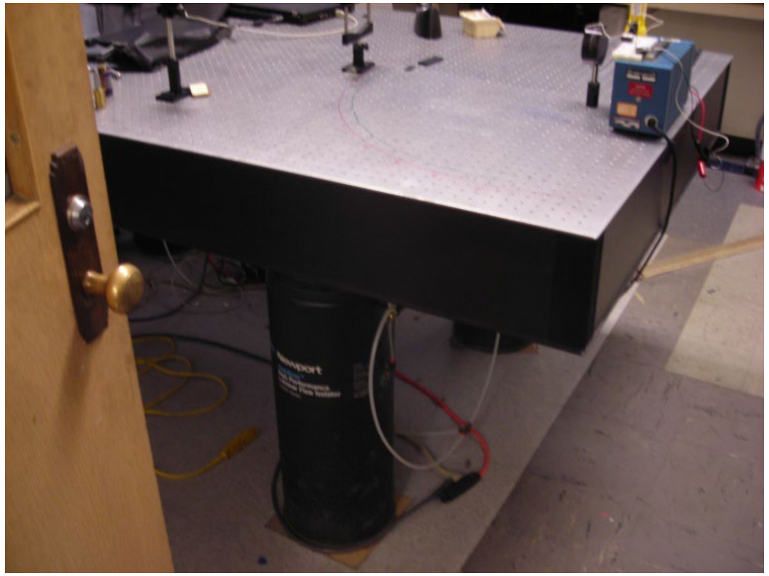
equal to 1

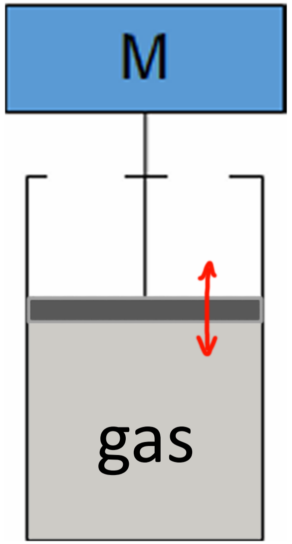
k !

$$+\frac{d}{dh}\big((g\rho_w A)h\big)\Big|_{h=h_{eq}} = g\rho_w A \cdot \left(\frac{dh}{dh}\right)_{h=h_{eq}} = g\rho_w A$$

Example: Air Leg

➢ used to isolate sensitive equipment from vibrations





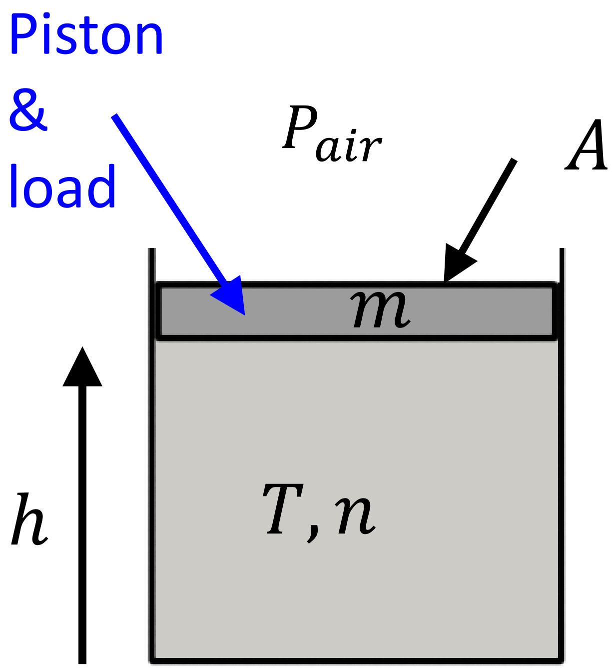
• Assumption: any motion of piston is slow so compression/expansion is isothermal

Example: Air Leg

a) Draw a free body diagram for the object of mass m showing the vertical forces.

b) Calculate the magnitude of the net upwards force on the object as a function of the height h of the piston.

Answer in terms of $\boldsymbol{h, n, T, A, m, g}$, and $\boldsymbol{P_{air}}$



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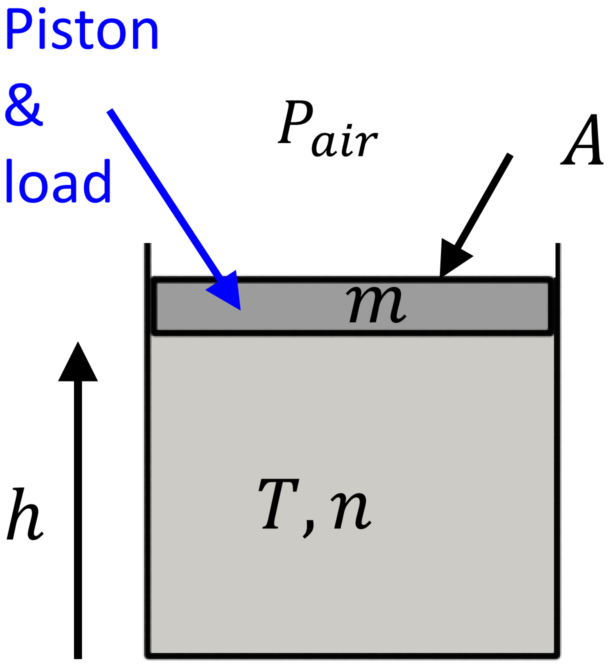
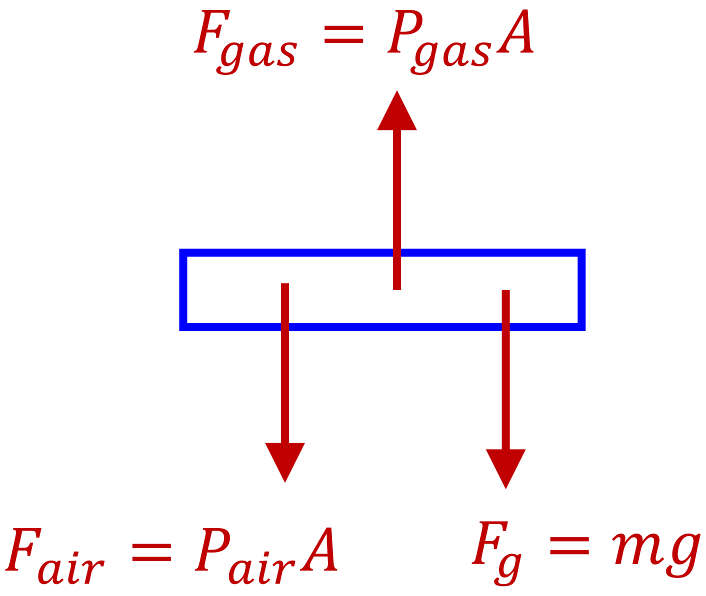




Step 1:

Have: $P_{gas} = \frac{nRT}{V} = \frac{nRT}{Ah}$

So, $F_{gas} = P_{gas}A = \frac{nRT}{h}$

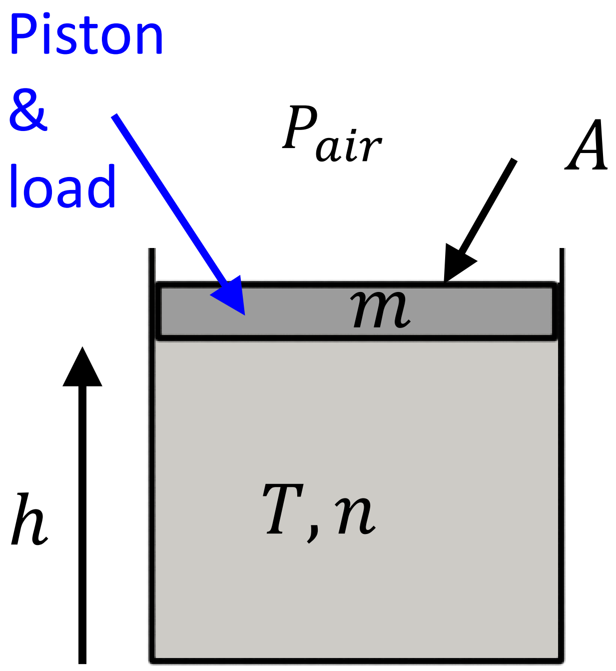
$$\Longrightarrow F_{net} = \frac{nRT}{h} - \underbrace{P_{air}A - mg}_{\text{const}}$$

$$F_{net} = nRT\frac{1}{h} - (P_{air}A + mg)$$

Example: Air Leg

c) Graph this upward force as a function of h, for positive values of h up to the height of the object.

d) What is the equilibrium height of the piston?

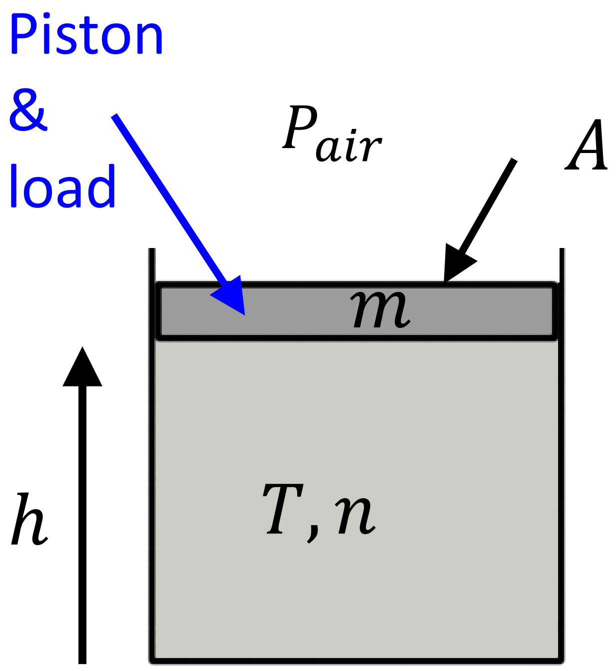


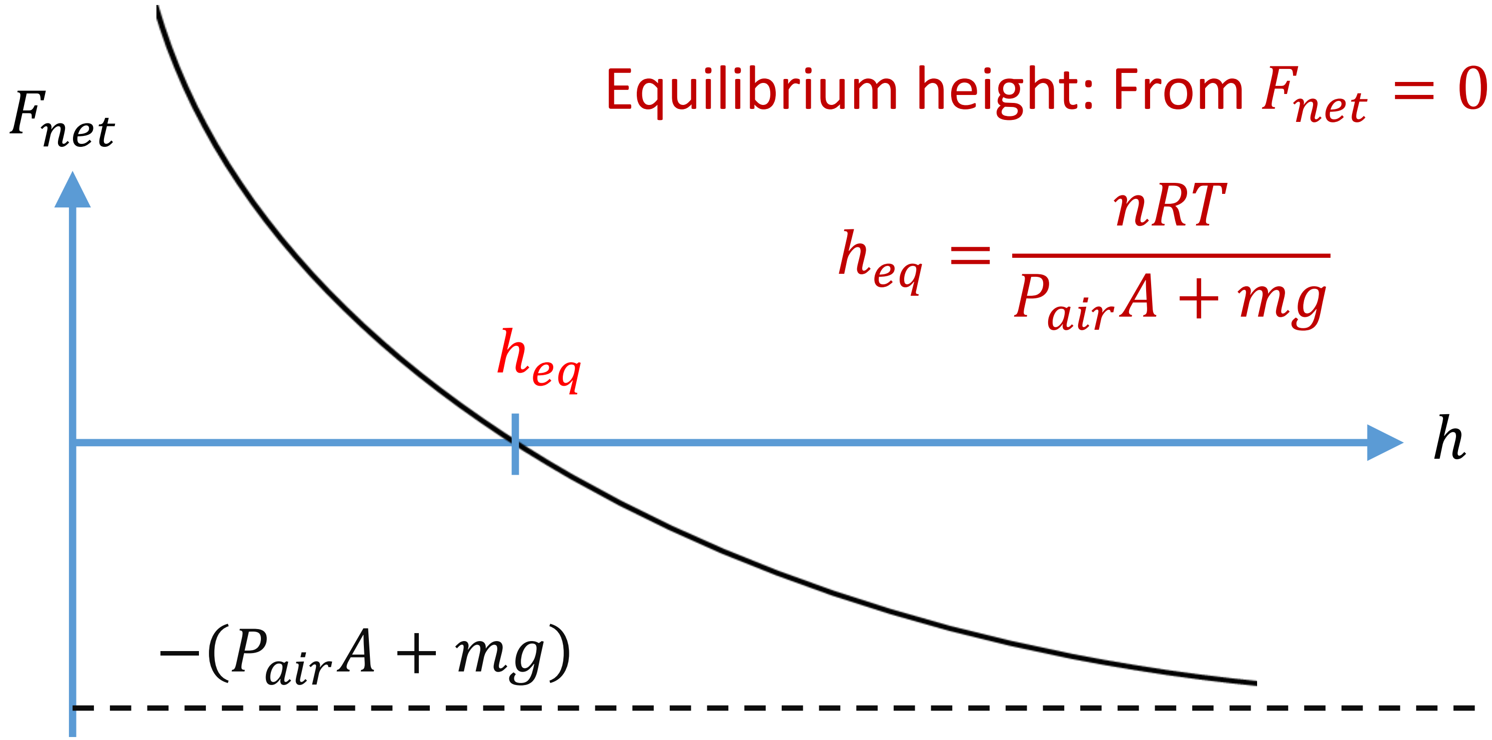
Example: Air Leg

c) Graph this upward force as a function of h, for positive values of h up to the height of the object.

d) What is the equilibrium height of the piston?

$$\boxed{F_{net} = nRT\frac{1}{h} - (P_{air}A + mg)}$$





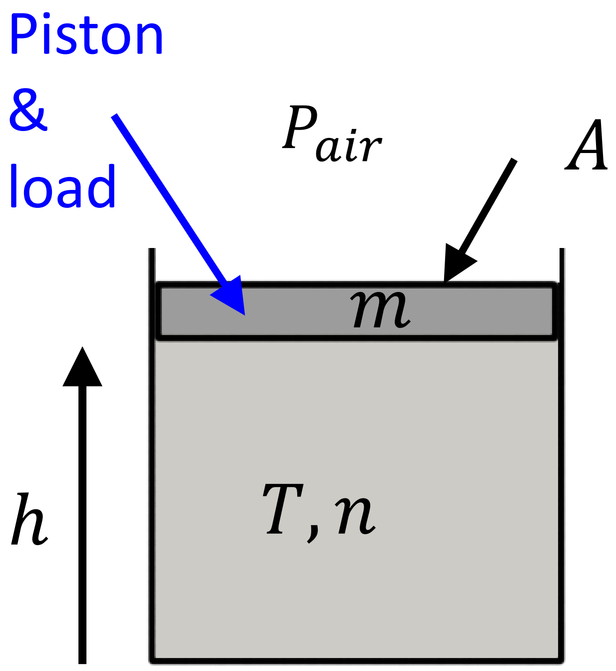
Step 2:

Equilibrium height: From $F_{net} = 0$

$$h_{eq} = \frac{nRT}{P_{air}A + mg}$$

Example: Air Leg

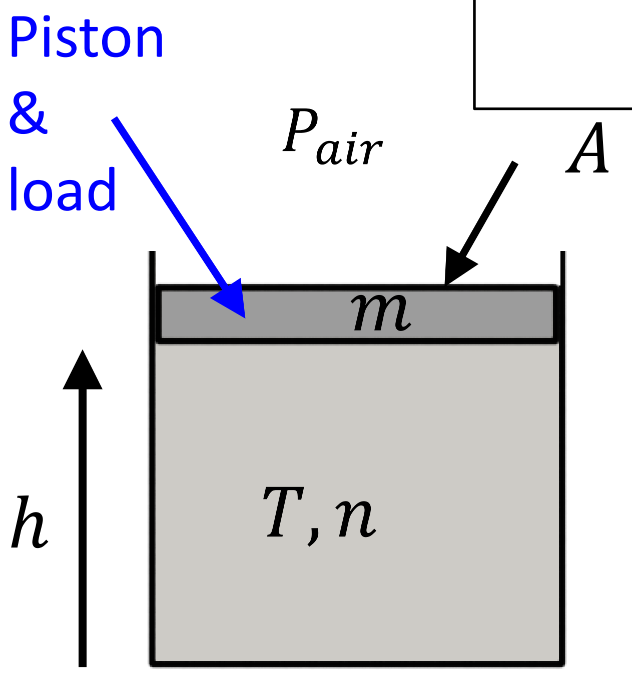
e) What is the oscillation frequency?

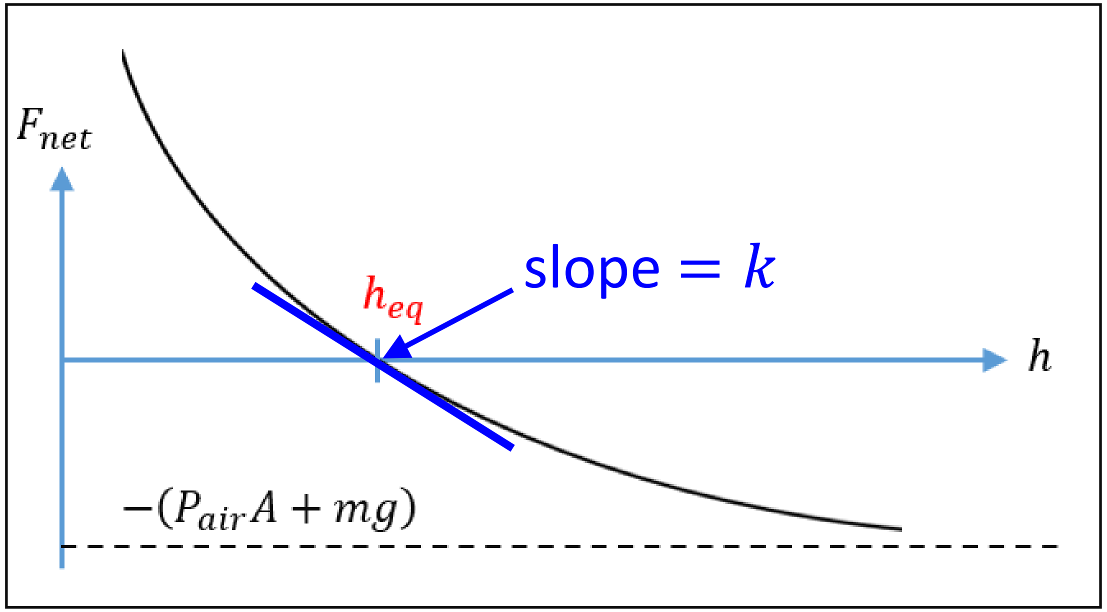


Example: Air Leg

e) What is the oscillation frequency?

$$F_{net} = nRT\frac{1}{h} - (P_{air}A + mg)$$
$$h_{eq} = \frac{nRT}{P_{air}A + mg}$$





Step 3: $k = -\frac{dF_{net}(h_{eq})}{dh}$

$$k = -\frac{dF_{net}(h_{eq})}{dh} = \frac{nRT}{h_{eq}^2} = \frac{(P_{air}A + mg)^2}{nRT}$$

Angular frequency: $\omega = \sqrt{k/m} = \frac{P_{air}A+mg}{\sqrt{nRTm}}$

Step 3: $k = -\frac{dF_{net}(h_{eq})}{dh}$

$$F_{net} = nRT\frac{1}{h} - (P_{air}A + mg)$$

$$h_{eq} = \frac{nRT}{P_{air}A + mg}$$

$$k = -\frac{d}{dh}\left(nRT \cdot \frac{1}{h} - \cancel{const}^{\,0}\right)\Big|_{h=h_{eq}} =$$

$$= -\frac{d}{dh}\left((nRT)\frac{1}{h}\right)\Big|_{h=h_{eq}} = -nRT \cdot \frac{d}{dh}\left(\frac{1}{h}\right)\Big|_{h=h_{eq}} = -nRT \cdot \left(-\frac{1}{h^2}\right)_{h=h_{eq}} =$$

$$= +\frac{nRT}{h_{eq}^2} = \frac{\cancel{(nRT)}\,(P_{air}A+mg)^2}{(nRT)^{\cancel{2}}} = \frac{(P_{air}A + mg)^2}{nRT} = k\,!$$