MATH 101A — ASSIGNMENT 2

Learning goals

- Calculate volumes of revolution by the disk method (review).
- Calculate volumes of revolution by the method of cylindrical shells.
- Calculate solid volumes by slicing.
- Evaluate the Gaussian Integral, $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Contributors

On the first page of your submission, list the student numbers and full names (with the last name in **bold***) of all team members. Indicate members who have not contributed using the comment "(non-contributing)".*

Assignment questions: Three Volumes, Three Ways

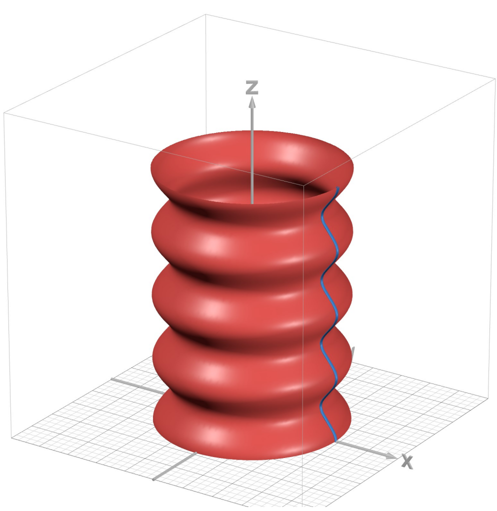
The questions in this section contribute to your assignment grade*. Stars indicate the difficulty of the questions, as described on Canvas.*

Before starting work on this assignment, please read through all the questions and take note of the special instructions in the section headed "Notes and Comments" below.

For any positive-valued function r and interval $[z_0,z_1]$, the equation

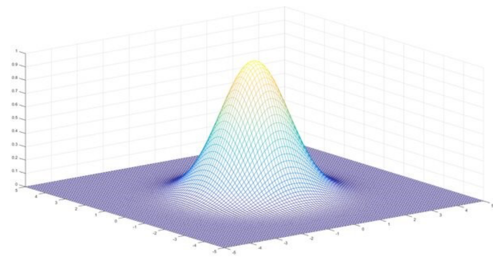
$$x^2 + y^2 = r(z)^2, \qquad z_0 \le z \le z_1$$

describes a 3D surface for which the horizontal slice at level $z = c$ consists of points (x,y,z) where $x^2 + y^2 = r(c)^2$ and, of course, $z = c$. (The informal phrase "flying circle" could be used.) Textbooks call shapes like this "surfaces of revolution", because they can be produced by revolving a curve in the xz-plane (where $y = 0$) around the z-axis. The curve is defined by the pair of equations $y = 0$ and $x = r(z)$. Here is an example, in which $[z_0,z_1] = [0, \pi]$ and $r(z) = \sqrt{(2 + \cos^2(2z))/3}$.



This assignment considers 3 idealized shapes whose sides are surfaces of revolution resembling real objects: the cone, the skep, and the bump.





We can make these by spinning curves with the special form $z = f(x)$, $y = 0$, where $r_0 \le x \le r_1$ and $r_0 \ge 0$. Spinning any point $(x,0,z)$ around the z-axis will produce a flying circle at height z with radius x. Algebraically, replacing x with $r = \sqrt{x^2 + y^2}$ upgrades the curve just mentioned into the following 3D equation defining the surface of revolution:

$$z = f(r), \qquad r_0 \le r \le r_1; \qquad r = \sqrt{x^2 + y^2}.$$

(Notice how substituting $y = 0$ recovers $r = x$ and produces the original curve.)
For each of these solids, we will complete and compare three approaches to finding the volume:

- the pile-of-disks interpretation discussed in Small Class 4;
- the method of cylindrical shells, explained in the CLP-2 textbook, Example 1.6.9;
- the slicing idea illustrated in CLP-2 Example 1.6.6, in which the volume of any solid is given by
$$V = \int_{x_0}^{x_1} A(x)\,dx,$$
where $A(x)$ is the area of the cross-section of the solid in the plane perpendicular to the x-axis at the named point x, and the limits of integration correspond to the exact interval $x_0 < x < x_1$ in which $A(x) > 0$.

(The pile-of-disks approach is actually a special case of the slicing method, built using slices perpendicular to the z-axis. Question 2 shows the profound difference that a wise choice of axis can make to the intensity of calculations required for the same result.)

2. (9 marks) Consider the solid circular cone with base radius $a > 0$ and height $h > 0$ formed by joining each point $(x,y,0)$ where $x^2 + y^2 \le a^2$ to the vertex $(0,0,h)$. The volume of this solid is well known: $V = \pi a^2 h/3$. Prove this in three different ways, as follows.

    (a)★☆☆☆Define a function f and an interval $[r_0, r_1]$ with $r_0 \ge 0$ for which the cone's lateral surface in 3D has the form $z = f(r)$, $r_0 \le r \le r_1$; $r = \sqrt{x^2 + y^2}$.

    Solution:
    First, we must consider the cross section on the x-z plane and write an equation to define the slant height of the cone cone is given by,

    $$z - h = -\frac{h}{a}(x - 0),$$

    in the interval [0, a].

Replace x with

$$r = \sqrt{x^2 + y^2},$$

$$z = f(r) = -\frac{h}{a}(r) + h.$$

(b)★☆☆☆ Use the pile-of-disks interpretation to set up and evaluate an integral for *V*.

Solution:

The pile-of-disks interpretation of the volume of the cone can be expressed as,

$$V = \lim_{\Delta z \to 0} \pi \sum_{i=1}^{n} \Delta z \cdot [r(z_i)]^2$$

$$V = \pi \int_0^h (a(1 - \tfrac{z}{h})^2 \, dz,$$

$$V = a^2\pi \int_0^h 1 - \tfrac{2z}{h} + \tfrac{z^2}{h^2} \, dz,$$

$$V = a^2\pi \, [z - \tfrac{z^2}{h} + \tfrac{z^3}{3h^2}]_0^h,$$

$$V = a^2\pi \left[h - h + \tfrac{h}{3}\right],$$

$$V = \pi a^2 \tfrac{h}{3}.$$

(c) ★★☆☆ Use the method of cylindrical shells to set up and evaluate an integral for *V*.

Solution:

Using the method of cylindrical shells, the volume of the cone can be expressed as,

$$V = 2\pi \int_0^a x \cdot (-\tfrac{h}{a} + h)dx,$$

$$V = 2h\pi \int_0^a x - \tfrac{x^2}{a} dx,$$

$$V = 2h\pi[\tfrac{x^2}{2} - \tfrac{x^3}{3a}]_0^a,$$

$$V = 2h\pi \left[\tfrac{a^2}{2} - \tfrac{a^2}{3}\right],$$

$$V = \pi a^2 \tfrac{h}{3}.$$

Our third approach to finding the same volume involves slicing up the cone with planes parallel to the *z*-axis. This calls for some heavy calculations, broken into parts (d)–(i) below. To simplify this development, assume $h = 1$ in all that follows.

For each fixed x_0, with $|x_0| \le a$, imagine slicing the solid cone of interest with a plane perpendicular to the *x*-axis at the location x_0. Let $A(x_0)$ denote the area of the slice obtained.

(a)★☆☆☆ Find $A(0)$.

Solution:

At $x_0 = 0$, slicing the cone reveals a triangle with height h (=1) and base 2a. So,

$$A(0) = \tfrac{1}{2} \cdot 2a \cdot h,$$

$$A(0) = a.$$

(b)★★★☆ For each fixed x_0, with $0 < x_0 < a$, set up a definite integral whose value equals $A(x_0)$.

Solution:

The limits of integration for this are set by the positive and negative Y intercepts.

$$Yint^2 = a^2 - x_0^2,$$

$$Yint = \pm\sqrt{a^2 - x_0^2}.$$

These are the bounds of integration now applying this to the equation found in part a and substituting h = 1 and r = $\sqrt{x^2 + y^2}$. Putting these together we get,

$$A(x_0, y) = \int_{-\sqrt{a^2-x_0^2}}^{\sqrt{a^2-x_0^2}} 1 - \frac{\sqrt{x_0^2+y^2}}{a}\,dy.$$

As the area of the slices is the same on both sides of the cone, this can be simplified into,

$$A(x_0, y) = 2\int_0^{\sqrt{a^2-x_0^2}} 1 - \frac{\sqrt{x_0^2+y^2}}{a}\,dy.$$

(c)★★★☆ Show how to evaluate the integral for $A(x_0)$ in part (e). After you finish, change x_0 to x throughout the bottom line to obtain the slice-area function $A = A(x)$ for all x with $0 < |x| \leq a$.

Solution:

$$A(x_0, y) = 2\int_0^{\sqrt{a^2-x_0^2}} 1 - \frac{\sqrt{x_0^2+y^2}}{a}\,dy$$

Evaluating the integral we can solve in two parts,

$$A(x) = 2\left[\int_0^{\sqrt{a^2-x_0^2}} dy - \frac{1}{a}\int_0^{\sqrt{a^2-x_0^2}} \sqrt{x_0^2+y^2}\,dy\right].$$

Evaluating the first part,

$$\int_0^{\sqrt{a^2-x_0^2}} dy = [y]_0^{\sqrt{a^2-x_0^2}},$$

$$= \sqrt{a^2 - x_0^2}.$$

Evaluating the second part,

$$\frac{1}{a}\int_0^{\sqrt{a^2-x_0{}^2}}\sqrt{x_0{}^2+y^2}dy = \frac{x_0}{a}\int_0^{\sqrt{a^2-x_0{}^2}}\sqrt{1+(\frac{y}{x_0})^2}\,dy,$$

Substituting $\frac{y}{x_0} = \tan\theta$, and dy = $x_0\,sec^2\,\theta\,d\theta$, the new integral will also have a upper limit, b, of arctan$\left(\frac{\sqrt{a^2-x_0{}^2}}{x_0}\right)$ and a lower limit, a, of 0.

$$\frac{x_0}{a}\int_0^{\sqrt{a^2-x_0{}^2}}\sqrt{1+(\frac{y}{x_0})^2}\,dy = \frac{x_0}{a}\int_a^b \sec\theta \cdot x_0\,sec^2\,\theta\,d\theta,$$

$$= \frac{x_0{}^2}{2a}[\tan\theta\cdot\sec\theta]_a^b + \frac{x_0{}^2}{2a}\int_a^b \sec\theta\,d\theta\,,$$

$$= \frac{\sqrt{a^2-x_0{}^2}}{2} + \frac{x_0{}^2}{2a}[ln|\tan\theta+\sec\theta|]_a^b\,,$$

$$= \frac{\sqrt{a^2-x_0{}^2}}{2} + \frac{x_0{}^2}{2a}\,ln\left|\frac{a+\sqrt{a^2-x_0{}^2}}{x_0}\right|.$$

Combining the two parts and substituting $x_0 = x$, we get

$$A(x) = \sqrt{a^2-x^2} - \frac{x^2}{a}\,ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|.$$

(d) ★☆☆☆ Taken together, parts (d) and (f) above define $A(x)$ for each x where $|x| \le a$. Show that the function A is continuous at 0.

Solution:

The function is continuous if $\lim_{x\to\pm 0} A(x) = A(0) = a$,

$$\lim_{x\to\pm 0} A(x) = \lim_{x\to\pm 0}\sqrt{a^2-x^2} - \frac{x^2}{a}\,ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$$
$$= a\,.$$

Therefore, this function is continuous.

(h) ★★★☆ Verify that the following function is an antiderivative for $A(x)$:

$$F(x) = \frac{a^2}{3}\sin^{-1}\left(\frac{x}{a}\right) + \frac{2x}{3}\sqrt{a^2-x^2} + \frac{x^3}{3a}\log|x| - \frac{x^3}{3a}\log\left|a+\sqrt{a^2-x^2}\right|.$$

(*Hint*: Finding $\int A(x)dx$ directly is much harder than showing that $F(x) + C$ is correct.)

Solution:

If F(x) is an antiderivative for A(x), then

$$A(x) = \frac{dF}{dx}.$$

Now to verify both sides of this equation,

$$\text{LHS} = \sqrt{a^2 - x^2} - \frac{x^2}{a} ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$$

RHS =

$$\frac{a}{3\sqrt{1-\left(\frac{x}{a}\right)^2}} + \frac{2}{3}\sqrt{a^2 - x^2} - \frac{2x^2}{3\sqrt{a^2-x^2}} + \frac{x^2}{3a} + \frac{x^2}{a}\ln|x| - \frac{x^2}{a}\ln\left|a + \sqrt{a^2 - x^2}\right| + \frac{x^4}{3a(a+\sqrt{a^2-x^2})(\sqrt{a^2-x^2})}.$$

Simplifying both sides,

$$= \frac{a^2-2x^2+2(a^2-x^2)}{3\sqrt{a^2-x^2}} + \frac{x^2}{3a} + \frac{x^2}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + \frac{x^4}{3a(\sqrt{a^2-x^2})} \cdot \frac{1}{(a+\sqrt{a^2-x^2})}.$$

$$= \frac{a^2-2x^2+2(a^2-x^2)}{3\sqrt{a^2-x^2}} + \frac{x^2}{3a} + \frac{x^2}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + \frac{x^4}{3a(\sqrt{a^2-x^2})} \cdot \frac{a-\sqrt{a^2-x^2}}{x^2},$$

$$= \frac{a^2-2x^2+2(a^2-x^2)}{3\sqrt{a^2-x^2}} + \frac{x^2}{3a} + \frac{x^2}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + \frac{x^2(a-\sqrt{a^2-x^2})}{3a(\sqrt{a^2-x^2})},$$

$$= \frac{a^3-2ax^2+2a(a^2-x^2)+x^2(a-\sqrt{a^2-x^2})+x^2\sqrt{a^2-x^2}}{3a\sqrt{a^2-x^2}} + \frac{x^2}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right|,$$

$$=\sqrt{a^2 - x^2} + \frac{x^2}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right|,$$

$$= \sqrt{a^2 - x^2} - \frac{x^2}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|.$$

LHS = RHS which proves that F(x) was the antiderivative of A(x).

(i) ★☆☆☆ Use the results from parts (f) and (h) to evaluate the cone's volume by slicing, i.e., find $\int_{-a}^{a} A(x)dx$.

Solution:

Using fundamental theorem of calculus, we can take this expression and say it is equivalent to

$$= F(a) - F(-a).$$

Evaluating this we get,

$$\frac{a^2}{3}\arcsin(1) + \frac{a^3}{3a}\ln(a) - \frac{a^3}{3a}\ln|a| - \left[\frac{a^3}{3}\arcsin(-1) - \frac{a^3}{3a}\ln(a) + \frac{a^3}{3a}\ln|-a|\right].$$

Simplifying this we get,

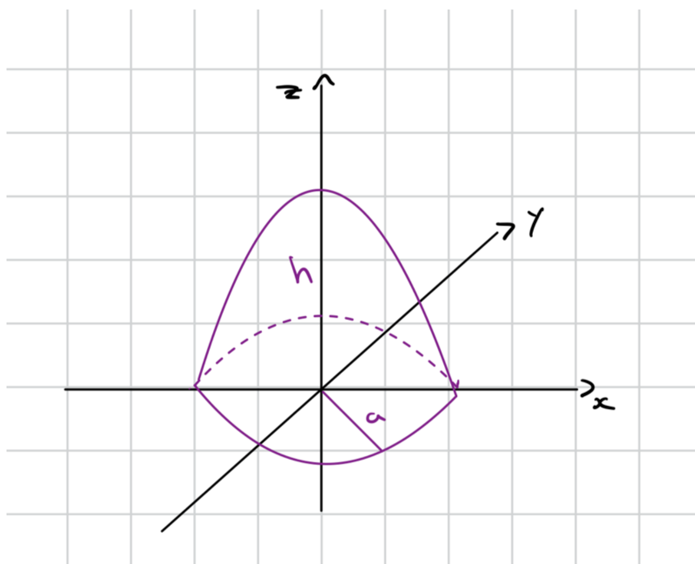
$$= \frac{\pi a^2}{6} + \frac{\pi a^2}{6},$$

$$A(x) = \frac{1}{3}\pi a^2.$$

3. (4 marks) Let V denote the volume of the 3D solid where $0 \le z \le h(a^2 - x^2 - y^2)/a^2$, taking a and h as given parameters. Find V in three different ways, as follows:

(a)★☆☆☆Describe and sketch the solid object of interest.

Solution:



In this question the solid object is a paraboloid that opens downwards where a is the radius of the object and h is the max height.

(b)★☆☆☆ Find V using the pile-of-disks interpretation.

Solution:

Since we are given the expression

$$z = \frac{h(a^2-x^2-y^2)}{a^2},$$

and by basic geometric principles,

$$r^2 = x^2 + y^2,$$

we can substitute and rearrange our equation to get,

$$r^2 = a^2(1 - \frac{z}{h}).$$

And once more to get,

$$r(z) = a\sqrt{1 - \frac{z}{h}}.$$

The area of a disk's cross-section is represented by $A(z) = \pi[r(z)]^2$. To calculate the volume of the object, we integrate this area over z. Given that z ranges from 0 to h, with the condition $0 \le z \le h(a^2 - x^2 - y^2)/a^2$ translating to $0 \le z \le h(a^2 - r^2)/a^2$ we can deduce this range applies when r approaches zero, acknowledging that r is always non-negative, thus setting the bounds for z from 0 to h. Therefore, the integration would go as follows.

$$V = \int_0^h \pi[r(z)]^2\, dz,$$

$$V = \pi \int_0^h a^2(1 - \frac{z}{h})\, dz.$$

Evaluating this we get,

$$V = \pi a^2[z - \frac{z^2}{h}]_0^h,$$

$$V = \pi a^2(h - \frac{h}{2}).$$

$$V = \frac{\pi a^2 h}{2}.$$

(c) ★★☆☆ Find V using the method of cylindrical shells.

Solution:

Given that the height of this shell at any point is given by:

$$z = h \cdot \frac{a^2 - x^2}{a^2}$$

So, using this we can express volume as

$$V = \int_0^a 2\pi x \cdot h \cdot \frac{a^2 - x^2}{a^2}\,\mathrm{dx}.$$

Evaluating this integral we get,

$$V = 2\pi h \int_0^a x \cdot \left(1 - \frac{x^2}{a^2}\right)\, dx,$$

$$V = 2\pi h \int_0^a \left(x - \frac{x^3}{a^2}\right) dx,$$

$$V = 2\pi h \left(\frac{a^2}{2} - \frac{a^2}{4}\right).$$

Simplifying this, we get,

$$V = \frac{\pi a^2 h}{2}.$$

(d) ★★☆☆ Find V by considering slices perpendicular to the x-axis.

Solution:

First, we must consider the cross section on the y-z plane and write an equation to define this area.

$$z = \frac{h(a^2 - x^2 - y^2)}{a^2},$$

$$z = h - \frac{h}{a^2}x^2 - \frac{h}{a^2}y^2,$$

$$y^2 = a^2 - x^2 - \frac{a^2}{h}z.$$

From this we can say that after setting $z = 0$, we arrive on the upper and lower bounds to first find the function $A(x)$,

$$y = \pm\sqrt{a^2 - x^2}.$$

Putting this into an integral,

$$A(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z(y)dy.$$

Simplifying and evaluating this integral we get,

$$A(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h - \frac{h}{a^2}x^2 - \frac{h}{a^2}y^2 dy,$$

$$A(x) = 2[hy - \frac{h}{a^2}x^2y - \frac{h}{3a^2}y^3]_0^{\sqrt{a^2-x^2}},$$

$$A(x) = 2(h\sqrt{a^2-x^2} - \frac{h}{a^2}x^2\sqrt{a^2-x^2} - \frac{h}{3a^2}(a^2-x^2)^{\frac{3}{2}}).$$

Finally, to calculate for volume we have the expression:

$$V = \int_{-a}^{a} A(x)\, dx.$$

$$V = \int_{-a}^{a} 2(h\sqrt{a^2-x^2} - \frac{h}{a^2}x^2\sqrt{a^2-x^2} - \frac{h}{3a^2}(a^2-x^2)^{\frac{3}{2}})dx,$$

$$V = 2h\int_{-a}^{a} \sqrt{a^2-x^2} - \frac{1}{a^2}x^2\sqrt{a^2-x^2} - \frac{1}{3a^2}(a^2-x^2)^{\frac{3}{2}})dx,$$

$$V = 2h\int_{-a}^{a} a\sqrt{1-\left(\frac{x}{a}\right)^2} - \frac{a}{a^2}x^2\sqrt{1-\left(\frac{x}{a}\right)^2} - \frac{a}{3a^2}(a^2-x^2)\sqrt{1-\left(\frac{x}{a}\right)^2}dx,$$

$$V = 2ah\int_{-a}^{a} \sqrt{1-\left(\frac{x}{a}\right)^2} - \frac{1}{a}x^2\sqrt{1-\left(\frac{x}{a}\right)^2} - \frac{1}{3a}(a^2-x^2)\sqrt{1-\left(\frac{x}{a}\right)^2}\mathrm{dx},$$

Let $\frac{x}{a} = sin\theta, dx = acos\theta \mathrm{d}\theta$,

$$V = 4ah\int_0^{\frac{\pi}{2}}(cos\theta - asin^2\theta cos\theta - \frac{1}{3a}a^2cos^3\theta)\ acos\theta\mathrm{d}\theta,$$

$$V = 4a^2h\int_0^{\frac{\pi}{2}} cos^2\theta - asin^2\theta cos^2\theta - \frac{1}{3a}a^2cos^4\theta \; \mathrm{d}\theta,$$

$$V = 4a^2h\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+cos2\theta) - \frac{a}{4}sin^2 2\theta - \frac{a}{12}(1+cos2\theta)^2 \; \mathrm{d}\theta,$$

$$V = 4a^2h\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+cos2\theta) - \frac{a}{8}(1-cos4\theta) - \frac{a}{12}(1+2cos2\theta+cos^2 2\theta) \; \mathrm{d}\theta,$$

$$V = 4a^2h\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+cos2\theta) - \frac{a}{8}(1-cos4\theta) - \frac{a}{12}(1+2cos2\theta+\frac{1}{2}(1+cos4\theta)) \; \mathrm{d}\theta,$$

Integrating on these bounds we come to:

$$V = \frac{\pi a^2 h}{2}.$$

Some of the integrals in the next question are "improper". We will study these more thoroughly later in the course. For an operational preview, simply rely on the equations below whenever the symbol on the left does not fit our usual situations:

$$\int_a^b f(x)\,dx = \lim_{\alpha\searrow a}\int_\alpha^b f(x)\,dx, \qquad \int_a^b f(x)\,dx = \lim_{\beta\nearrow b}\int_a^\beta f(x)\,dx.$$

Whenever f happens to be integrable on $[a,b]$, these equations are valid but uninformative. But when $a = -\infty$ or $b = +\infty$ or the integrand is unbounded near one of a or b, we can use the expressions on the right as *definitions* for the symbols on the left—provided the indicated limits exist. Identifying situations when all this works will be a key topic later in the course. In this exercise, however, all the necessary limits can be handled by methods from a prerequisite course.

4. (4 marks) Let V denote the volume of the 3D solid where $0 \le z \le e^{-(x^2+y^2)}$. This is the infinite object above the plane $z = 0$ and below the bump sketched in the introduction.

(a) ★★☆☆ Find V using the pile-of-disks interpretation. (Consider only disks with $z > 0$; use an improper integral as suggested above.)

Solution:

We know from the graph of $z = e^{-(x^2+y^2)}$ that this is a "bump" like shape that spans the entire plane of x and y, with a bump at the origin of x, y, and z (positive z values). To use the pile of disks interpretation, if we visualize the graph with the z axis being the height of the graph, then we can slice horizontally. That gives us the width in terms of the z axis, and the area of each slice in terms of the x and y axis. Since each slice is of a circular shape, we can compute the surface area using the equation:

$$A = \pi r^2$$

And the radius can be represented as:

$$r^2 = x^2 + y^2$$

Since we have $z = e^{-(x^2+y^2)}$, we can substitute r^2 for $x^2 + y^2$:

$$z = e^{-r^2}$$

Now we can solve for the radius.

$$\ln(\text{z}) = \ln\,(e^{-r^2})$$

$$\ln\,(\text{z}) = -r^2\ln\,(e)$$

$$\ln(z) = -r^2$$

$$-\ln(z) = r^2$$

$$r^2 = -\ln(z)$$

Using the pile of disks interpretation, we know that $A(z) = \pi(r(z))^2$, and each width of the disk to be dz. Therefore, we can get the total volume by integrating the function:

$$V = \int_0^1 \pi(-\ln(z))dz.$$

Evaluating this integral we get

$$V = \pi.$$

(b) ★☆☆☆ Find V using the method of cylindrical shells.

Solution:

We know that the height of the shell at any point is given by:

$$h = e^{-r^2}.$$

To find the total value of V we integrate dV over the radius of the solid from the bounds:

$$r = 0\ to\ r = \infty.$$

This is because the solid extends indefinitely in the x and y direction, but the height z approaches 0 as r increases.

$$V = \int_0^\infty 2\pi r e^{-r^2}\,dr,$$

Let $u = r^2$, so $du = 2rdr$,

$$V = \pi\int_0^\infty e^{-u}\,du.$$

Evaluating this integral we get,

$$V = \pi[-e^{-u}]_0^\infty,$$

As $e^{-\infty} \approx 0$, we arrive at,

$$V = \pi.$$

(c)★★★☆Define $I = \int_{-\infty}^{\infty} e^{-u^2} du$. This is a positive number whose exact value we intend to discover. Use slices perpendicular to the *x*-axis to find an equation relating *V* to *I*.

Solution:

$$I = \int_{-\infty}^{\infty} e^{-u^2} du$$

The function e^{-u^2} is symmetric about the y-axis, meaning it has the same shape and value for both positive and negative values of u. This symmetry implies that to capture the entire area under the curve the integration limits must extend from $-\infty \; to + \infty$,

$$V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy.$$

The inner integral represents the area under the curve of $e^{-(x^2+y^2)}$ for a fixed y over the range x. The outer integral sums these areas over all possible values of y, effectively stacking all these slices to form the volume of the entire solid. When we inspect this closer, we get,

$$V = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy.$$

Both integrals are equivalent to the function I which yields us with the equation,

$$V = I^2.$$

(d)★☆☆☆ Determine the exact value of *I*. (This is an important constant worth remembering.)

Solution:

Since $V = \pi$ we can solve for I,

$$\pi = I^2,$$

$$I = \sqrt{\pi}.$$

Notes and Comments

For any questions requiring plots or sketches, we will accept either careful handmade sketches or computer generated graphics, as produced by software like Desmos.

Questions 2, 3, and 4 tackle the shapes of interest in order of increasing complexity. The algebraic complexity of the work requested above follows the opposite progression. The requested calculations are easiest in Question 4 (though somewhat conceptual), intermediate in Question 3, and challenging in Question 2.

To help with Question 2, you may simply quote the following famous antiderivative without deriving it (see CLP-2 Example 1.8.22 for that):

$$\int (\sec\theta)^3\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + C.$$

Question 2(h) provides a convenient check on the result in Question 2(f), but Q2(f) calls for an independent calculation. (This accounts for the phrase, “Show how to ...,” in the statement.) Some approaches to Q2(h) may benefit from the difference-of-squares trick:

$$\frac{1}{p+\sqrt{q}} = \frac{1}{p+\sqrt{q}}\left(\frac{p-\sqrt{q}}{p-\sqrt{q}}\right) = \frac{p-\sqrt{q}}{p^2-q}.$$

Credits

- The 3D plot in the introduction was made with Desmos.
- The cone image came from Amazon.ca. This particular one has stock number ASIN B0757QY9B7.
- The skep image also came from Amazon.ca. Its product number is ASIN B07MRGFZH7.
- The Gaussian bump illustration was copied from a recent open-source publication by Fankui Hu, Haibing Chen, Xiaofei Wang: *An Intuitionistic Kernel-Based Fuzzy C-Means Clustering Algorithm With Local Information for Power Equipment Image Segmentation*. For details, use the unique Digital Object Identifier (DOI) 10.1109/ACCESS.2019.2963444.