# Lecture 10.

## AC circuits and how to deal with them.
## Phasors. Impedance.

# AC circuits

Text: 31.1-6

- Ch 30.1: Sinusoidal alternating $v(t)$ and $i(t)$, Phasors, RMS values
- Ch 31.2: AC circuits with $R, L, C$
- Ch 31.3: AC L-R-C series circuit:
- Ch 31.4: Power in AC circuits:
- Ch 31.5: Resonance in AC circuits

# AC circuits

- The electrical voltage/current delivered to our houses oscillates at 60Hz
- In much of the rest of the world, electrical AC power is delivered at 50Hz

AC power for home or industrial use

NOTE: AC circuits are not limited to 60Hz (or 50Hz) !!

**Examples:** AM/FM radio stations (AM~500kHz, FM~100MHz);

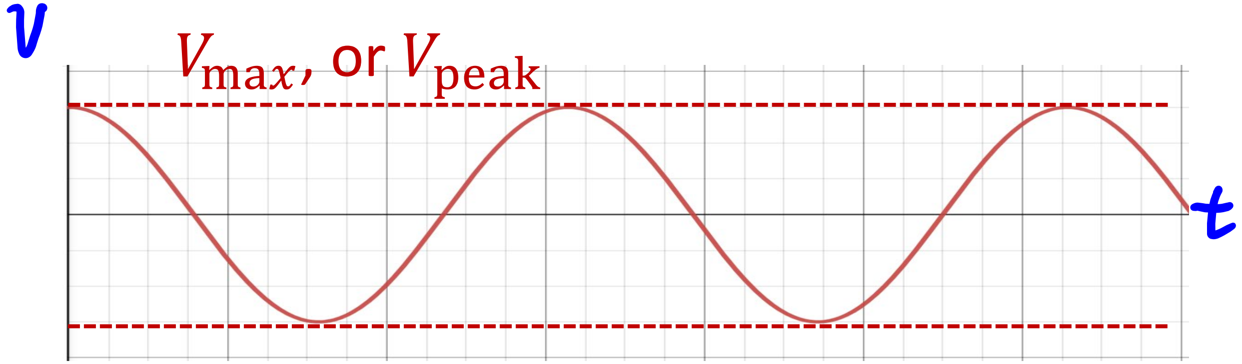
Audio, Television, and all other telecommunications range from 100 Hz to $10^9$ Hz

# Voltage in an AC circuits: Root-Mean-Square average

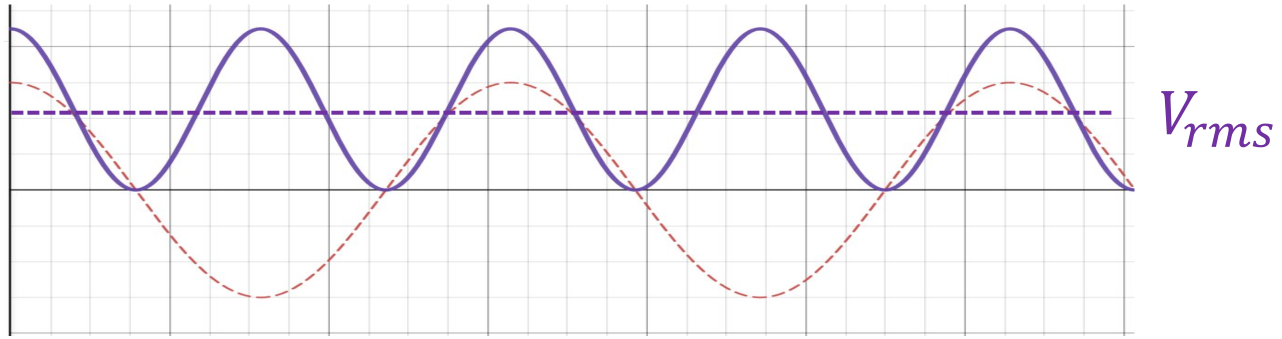
- AC source produces alternating voltage:

$$v(t) = V_{\max}\cos(\omega t)$$

- It's time average value is zero ☹




- The time average of $v^2(t) = V_{\max}^2 \cos^2(\omega t)$ is not zero! ☺

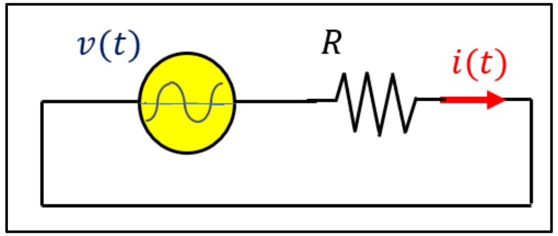



- $\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$
- $V_{rms}^2 = \frac{V_{max}^2}{T}\int_0^T \cos^2(\omega t) = \frac{V_{max}^2}{T}\cdot\frac{T}{2} = \frac{V_{max}^2}{2}$

- Physical meaning: in AC circuits, ammeter and voltmeter read RMS, not instantaneous or peak values

$$\boxed{V_{rms} = \sqrt{\langle v^2(t)\rangle_t} = \frac{V_{\max}}{\sqrt{2}}}$$

Q: The figure shows the source voltage and current in an AC $R$-circuit. What is the frequency of the source EMF in Hz? (Pick the closest answer)




A. 1000 Hz

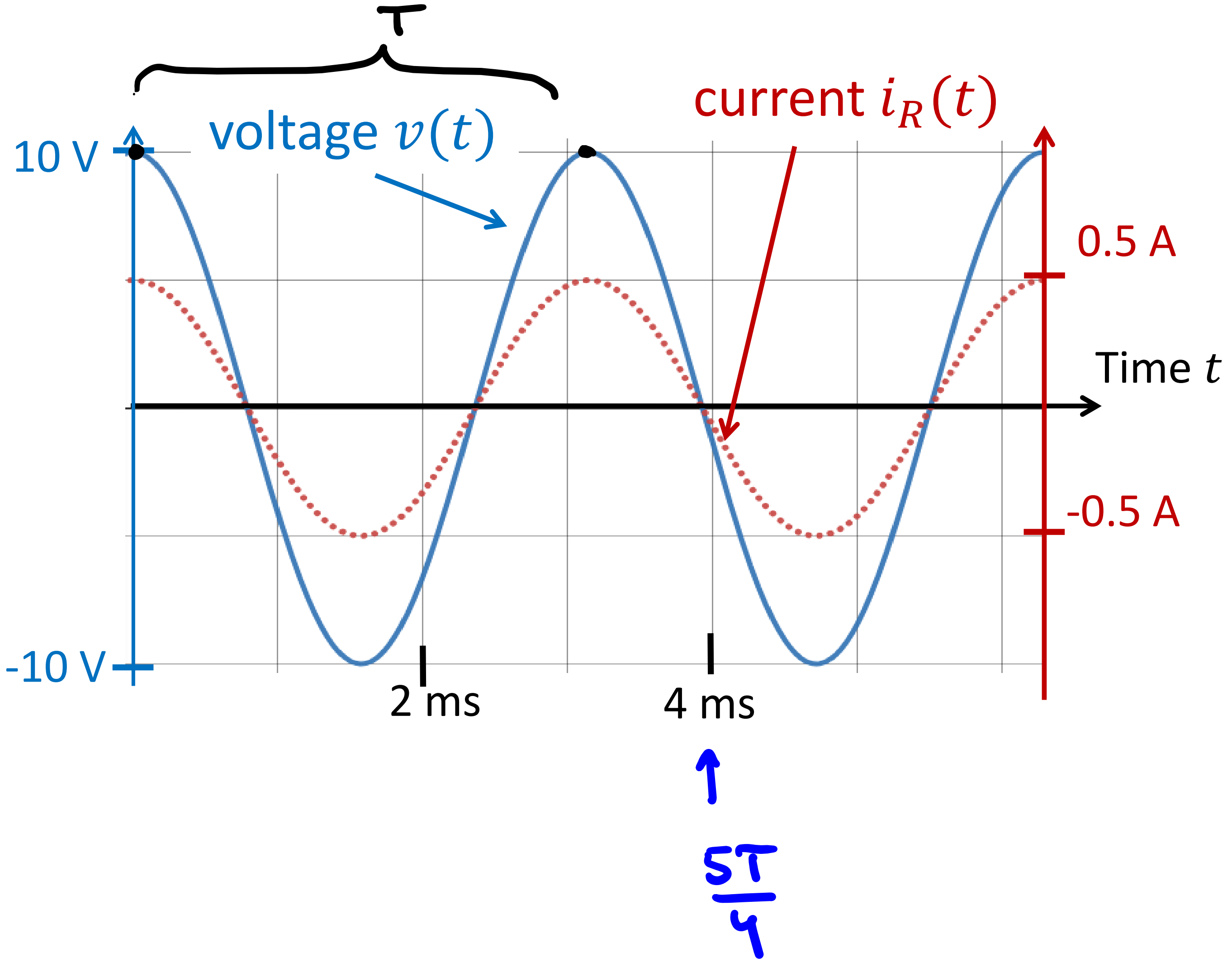
B. 320 Hz

C. 250 Hz

D. 100 Hz

E. 60 Hz

$$f = \frac{1}{T}$$



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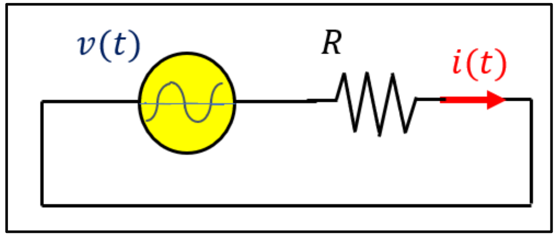


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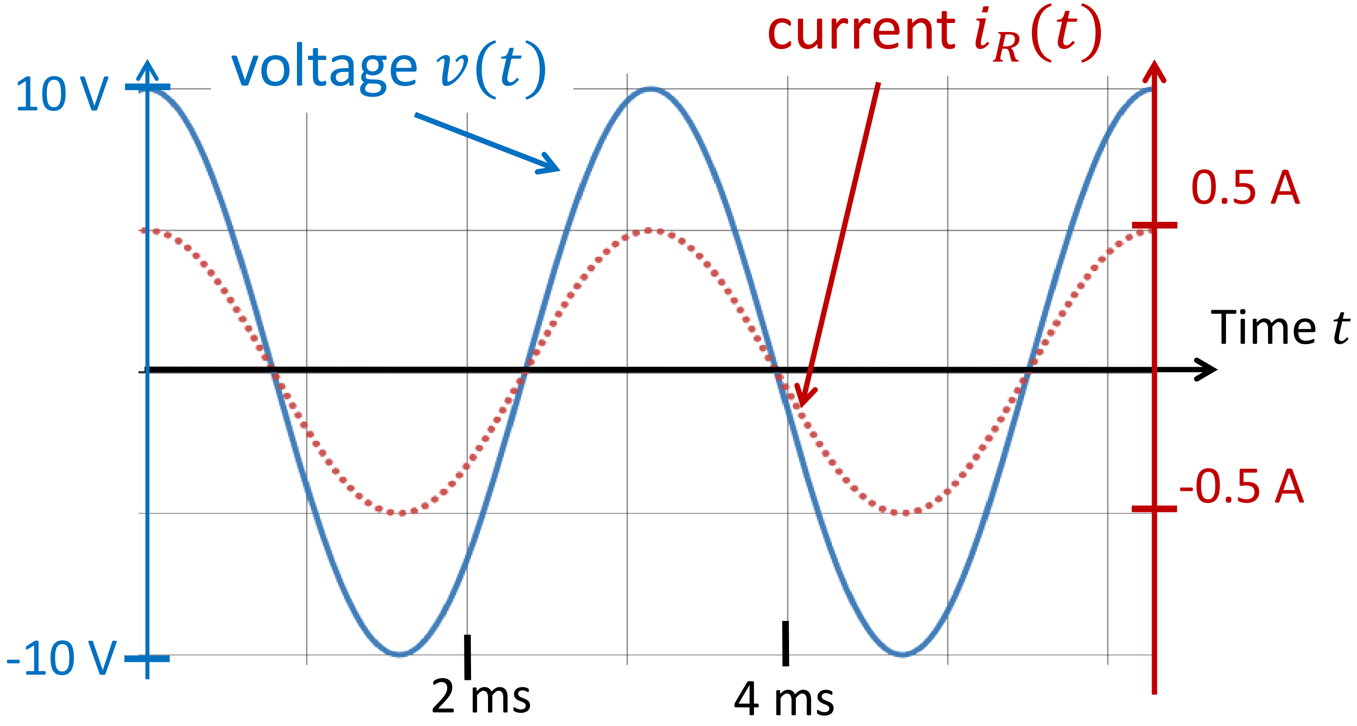
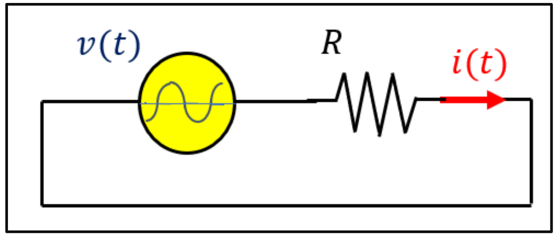


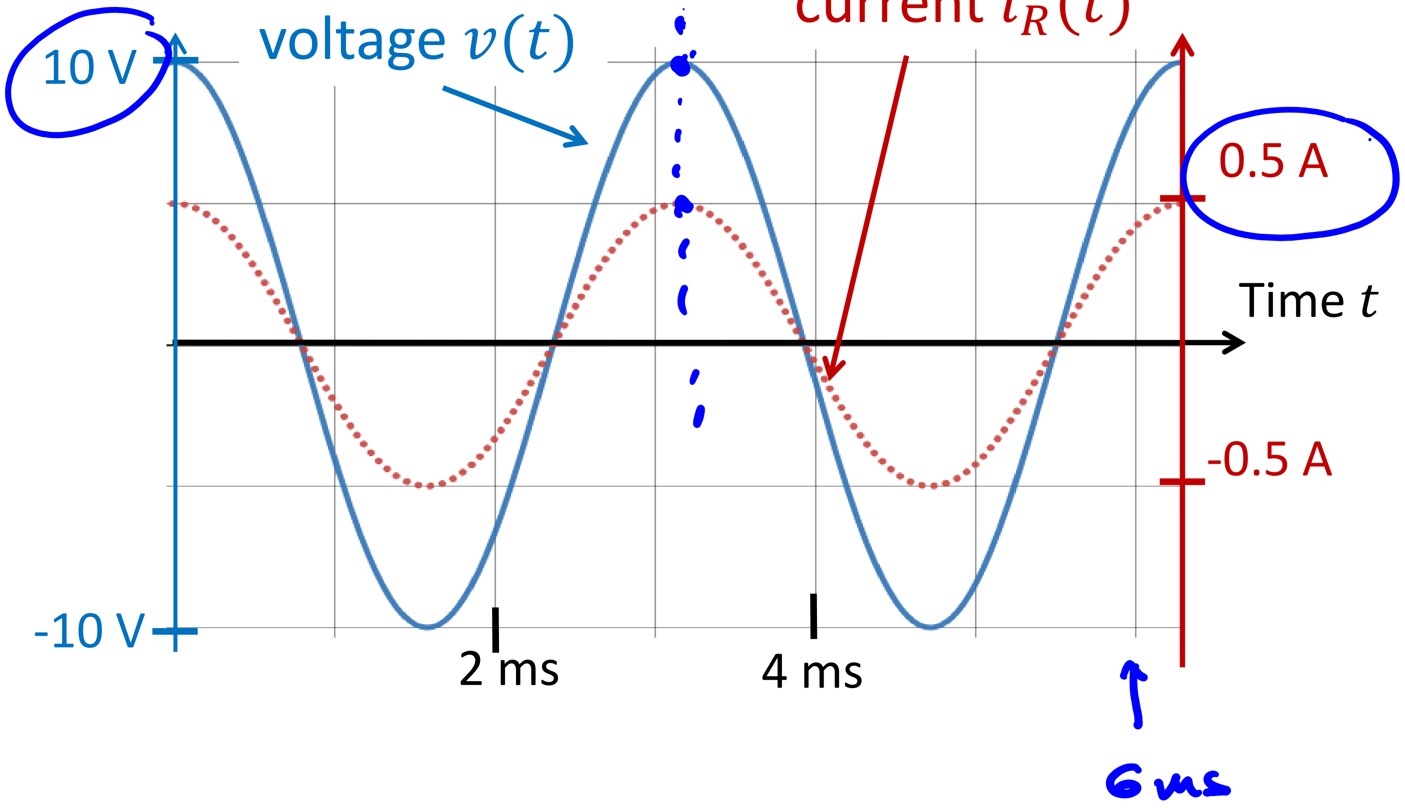

$$\frac{5T}{4} \approx 4\,ms \Rightarrow \quad T \approx 3.2\,ms$$

$$f = \frac{1}{T} \approx \frac{1}{3.1ms} \approx 313\,Hz$$

Q: The figure shows the source voltage and current in an AC $R$-circuit. What is the value of resistance $R$ at $t = 6\ ms$? (Pick the closest answer)

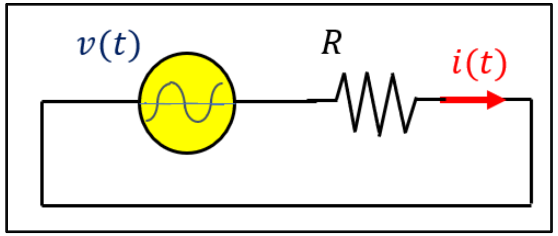



A. Instantaneously, at $t = 6\ ms$, $R$ is undefined
B. 0 Ω
C. 5 Ω
D. 10 Ω
E. 20 Ω




$\Delta V_R = IR$

Q: The figure shows the source voltage and current in an AC $R$-circuit. What is the value of resistance $R$ at $t = 6\ ms$? (Pick the closest answer)

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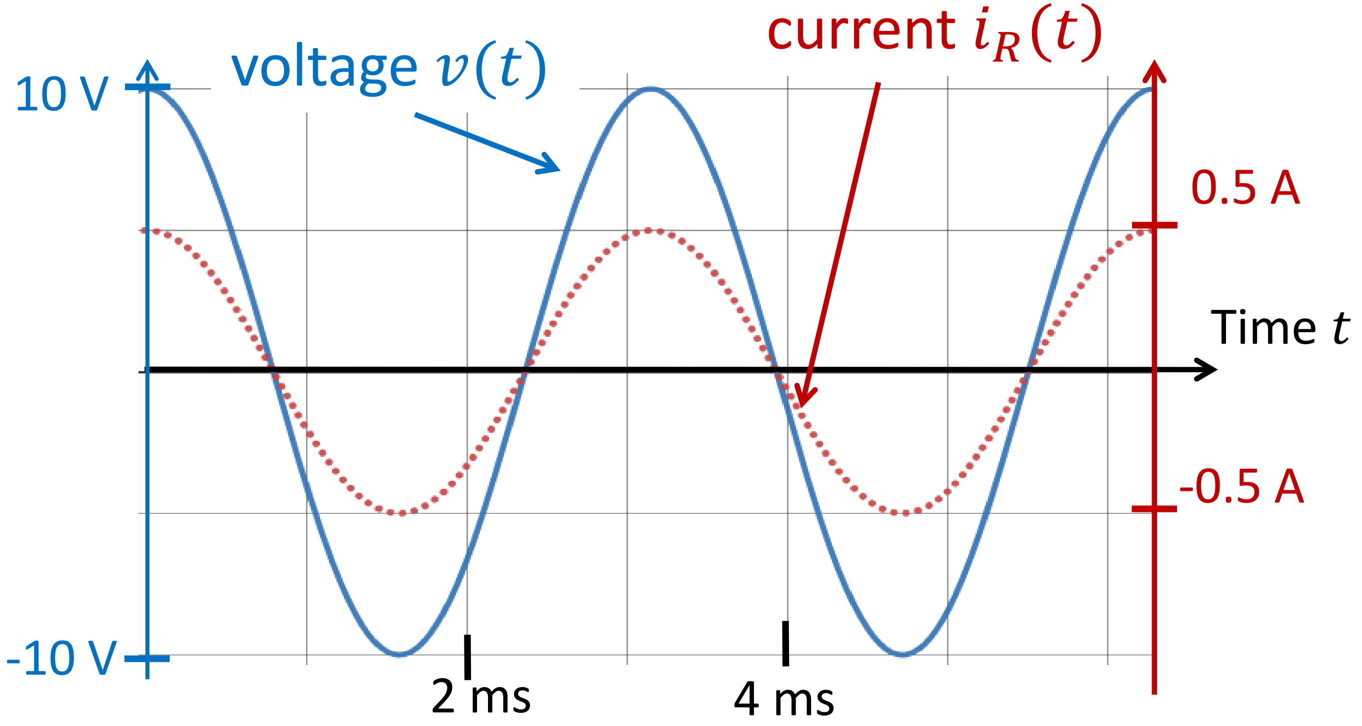
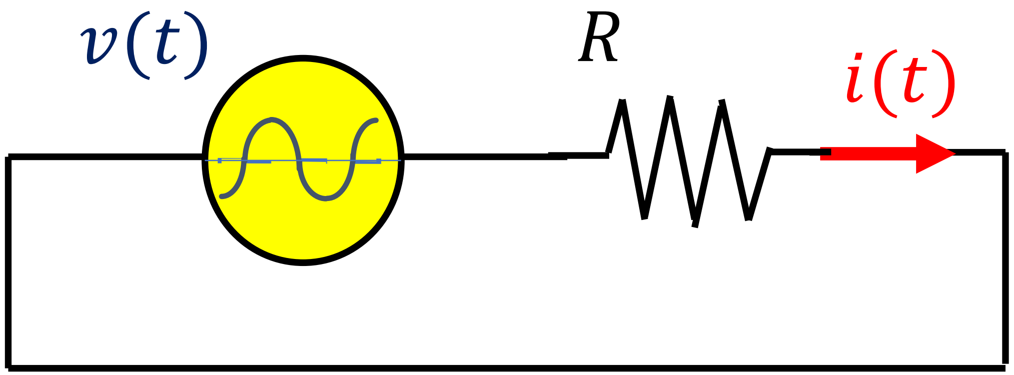



- No matter which moment of time we analyze, the resistance is always $R = V/I$.
- At max, we have: $R\ = 10\ V/0.5\ A = 20\ \Omega$

# AC circuit: purely "R" = Resistive




$$v(t) = V_{max}\cos(\omega t)$$

$I_{max}$

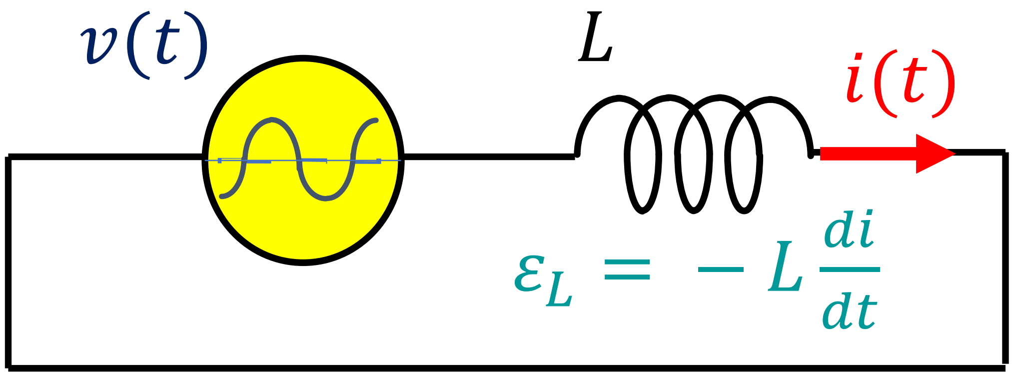
- Circuit analysis:

$$v(t) - i(t)R = 0$$

$$i(t) = \frac{v(t)}{R} = \left(\frac{V_{max}}{R}\right)\cos(\omega t)$$

- The current through the Resistor is "in phase" with the oscillating Voltage

# AC circuit: purely "L" = Inductive




$$v(t) = V_{max}\cos(\omega t)$$

$$\sin\alpha = \cos(\alpha - \pi/2)$$

- Circuit analysis:

$$v(t) - L\,di(t)/dt = 0$$

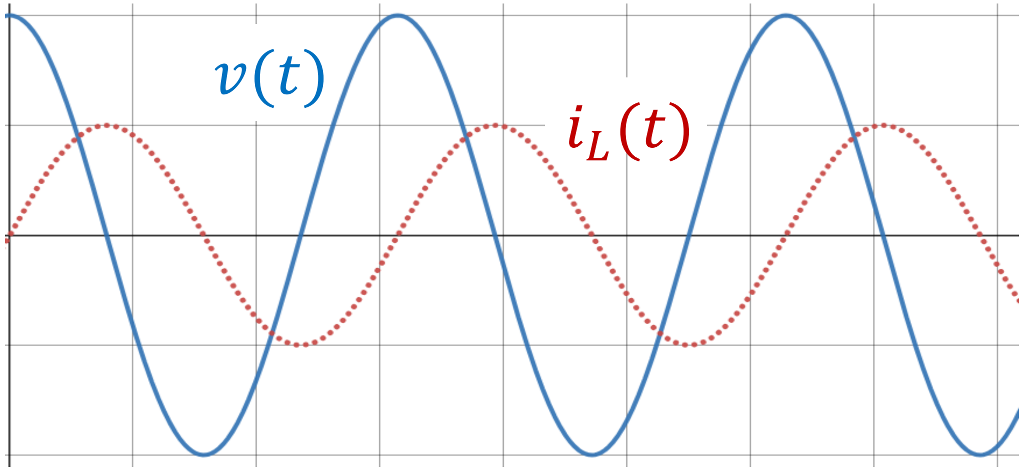
$$V_{max}\cos(\omega t) - L\,di(t)/dt = 0$$

$$di(t)/dt = (V_{max}/L)\cos(\omega t)$$

$$i(t) = \left(\frac{V_{max}}{\omega L}\right)\sin(\omega t) = \left(\frac{V_{max}}{\omega L}\right)\cos\left(\omega t - \frac{\pi}{2}\right)$$

- The current through the inductor is "by $\frac{\pi}{2}$ behind" the oscillating voltage

# AC circuit: purely "L" = Inductive




$$\boxed{v(t) = V_{max}\cos(\omega t)}$$

$$\boxed{i(t) = \left(\frac{V_{max}}{X_L}\right)\cos\left(\omega t - \frac{\pi}{2}\right)}$$

$$X_L = \omega L$$

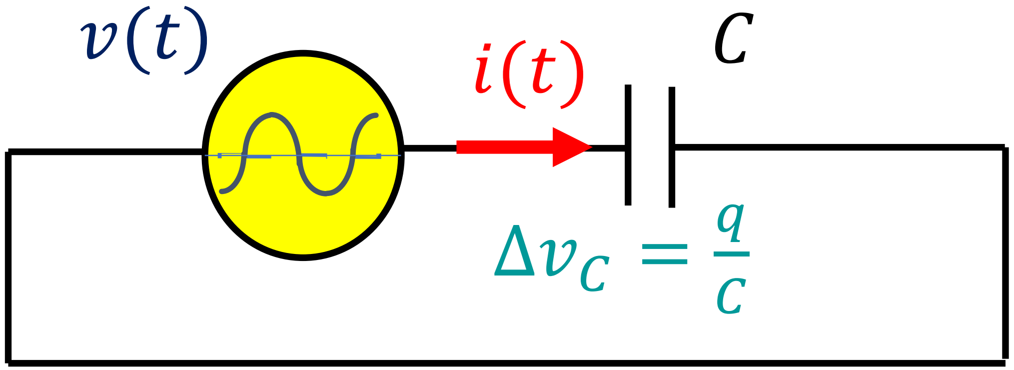
"Inductive resistance"

- Units: Ohm

- We say: the current $i(t)$ lags behind the voltage, or "the voltage leads the current".

- Note: $V_{max}$ and $I_{max}$ occur at different times!

# AC circuit: purely "C" = Capacitive




$$v(t) = V_{max}\cos(\omega t)$$

$-\sin\alpha = \cos(\alpha + \pi/2)$

- Circuit analysis:

$$v(t) - q(t)/C = 0$$

$$V_{max}\cos(\omega t) - q(t)/C = 0$$

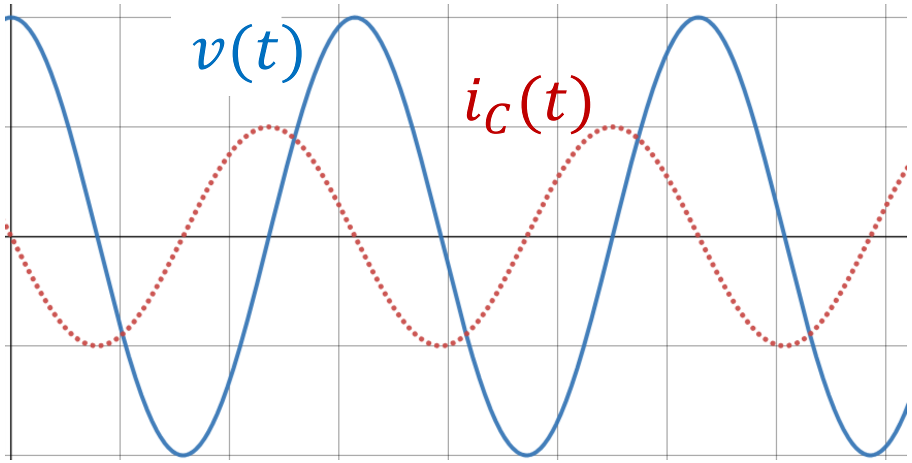
$$i(t) = \frac{dq(t)}{dt} = -(\omega C V_{max})\sin(\omega t)$$

$$i(t) = -(\omega C\, V_{max})\sin(\omega t) = (\omega C\, V_{max})\cos\left(\omega t + \frac{\pi}{2}\right)$$

- The current through the capacitor is "by $\frac{\pi}{2}$ ahead" the oscillating voltage

# AC circuit: purely "C" = Capacitive

$I_{max} = \omega C V_{max}$




$$V(t) = V_{max}\cos(\omega t)$$

$$i(t) = \left(\frac{V_{max}}{X_C}\right)\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$X_C = \frac{1}{\omega C}$$

"Capacitive resistance"

- Units: Ohm

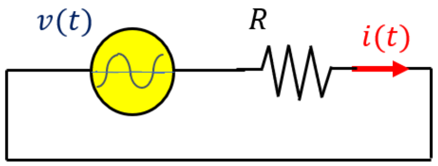
leads

- We say: the current $i(t)$ ~~lags behind~~ the voltage, or "the voltage ~~lead~~s the current".

lags behind

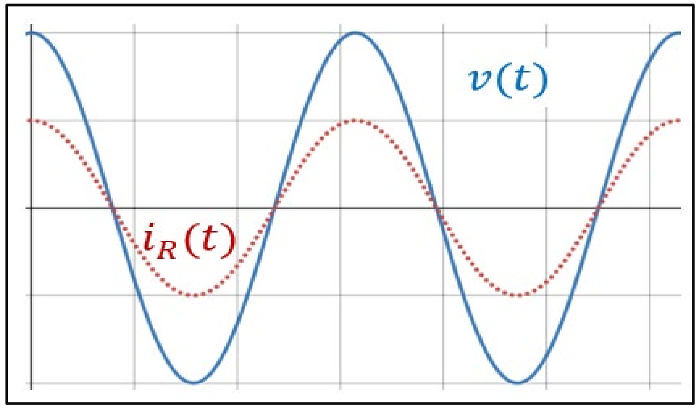
- Note: $V_{max}$ and $I_{max}$ occur at different times!

## R-circuit



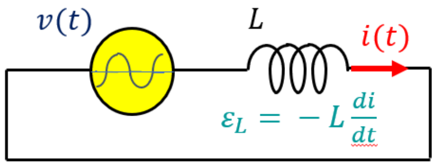

$$i(t) = \left(\frac{V_{max}}{X_R}\right)\cos(\omega t)$$

$$X_L = R$$



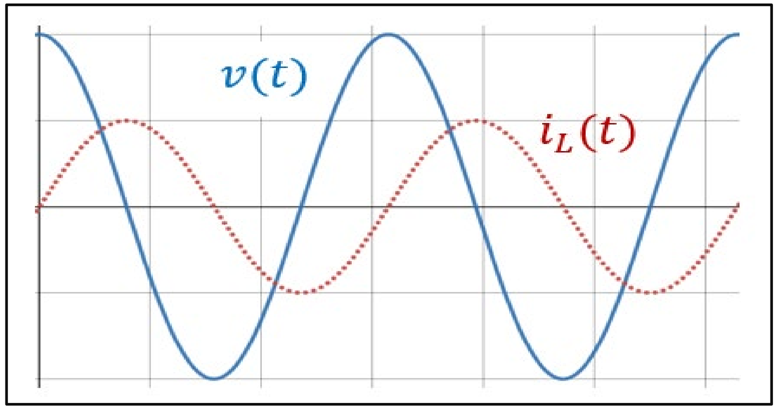

- Current in phase with voltage

## L-circuit



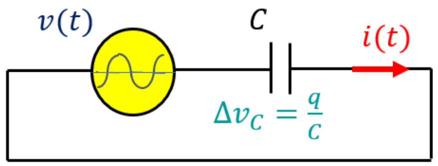

$$i(t) = \left(\frac{V_{max}}{X_L}\right)\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$X_L = \omega L$$



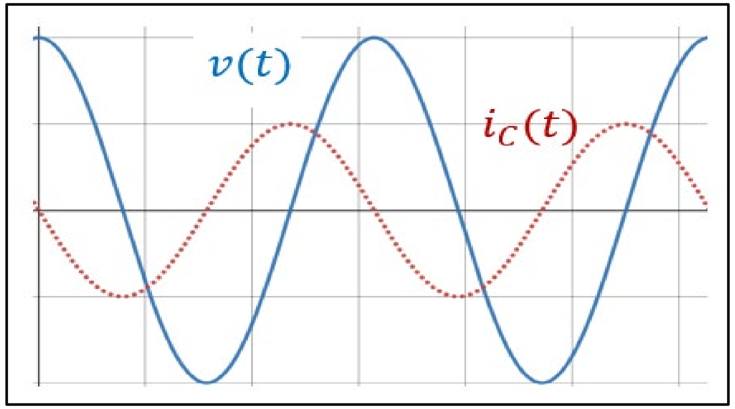

- Current lags behind voltage by $\frac{\pi}{2}$ (voltage leads current)

## C-circuit



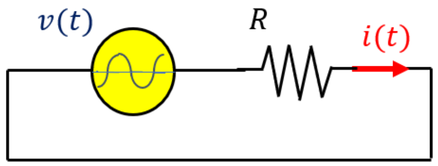

$$i(t) = \left(\frac{V_{max}}{X_C}\right)\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$X_C = \frac{1}{\omega C}$$



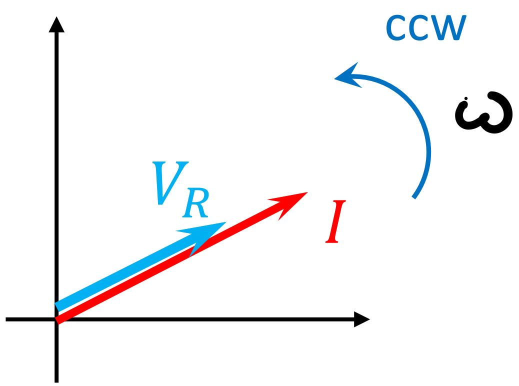

- Voltage lags behind current by $\frac{\pi}{2}$ (current leads voltage)

## R-circuit



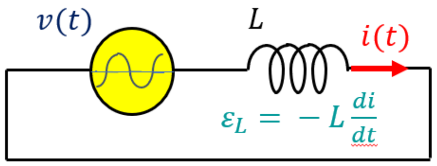

$$i(t) = \left(\frac{V_{max}}{X_R}\right)\cos(\omega t)$$

$$X_L = R$$



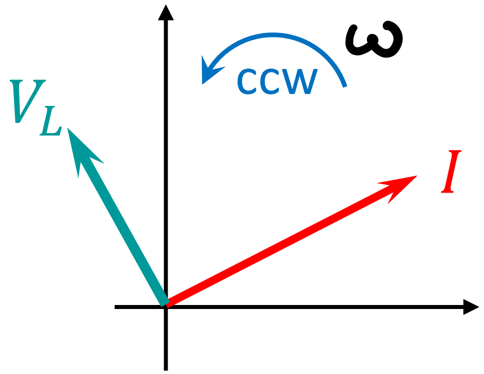

- Current is in phase with voltage

## L-circuit



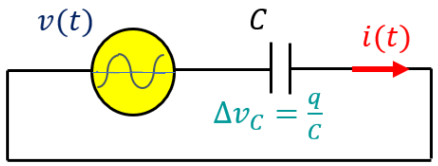

$$i(t) = \left(\frac{V_{max}}{X_L}\right)\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$X_L = \omega L$$



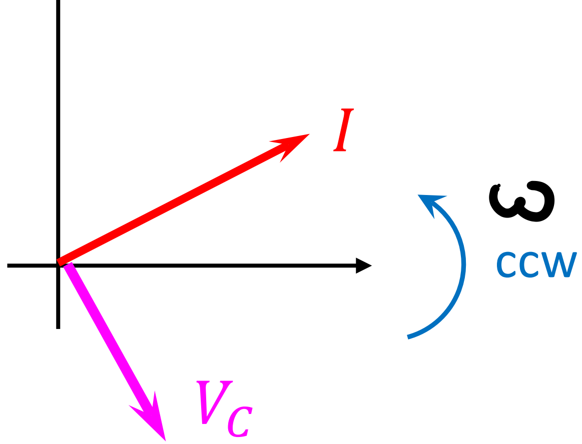

- Current lags behind voltage by $\frac{\pi}{2}$ (voltage leads current)

## C-circuit




$$i(t) = \left(\frac{V_{max}}{X_C}\right)\cos\left(\omega t + \frac{\pi}{2}\right)$$

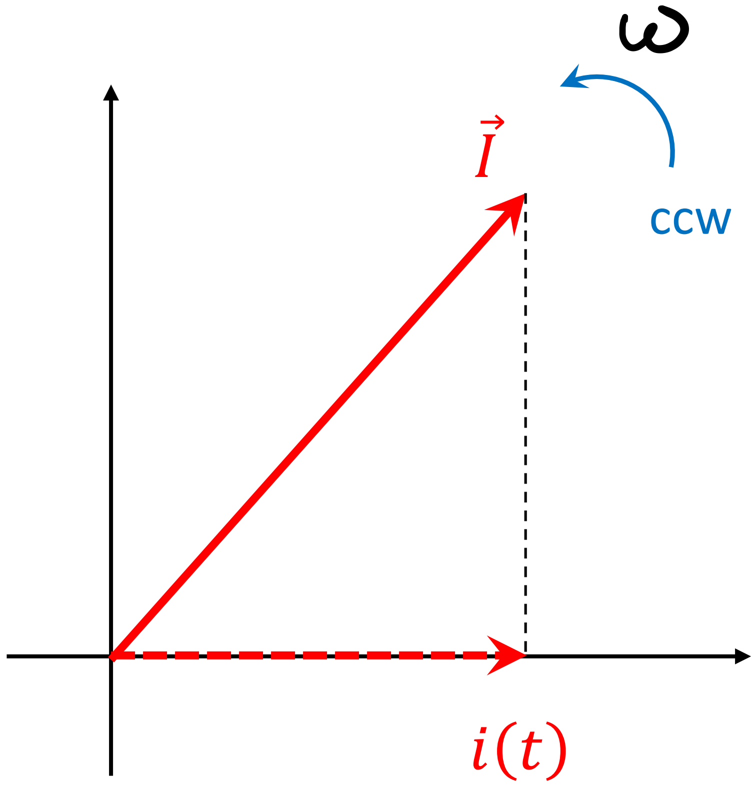
$$X_C = \frac{1}{\omega C}$$

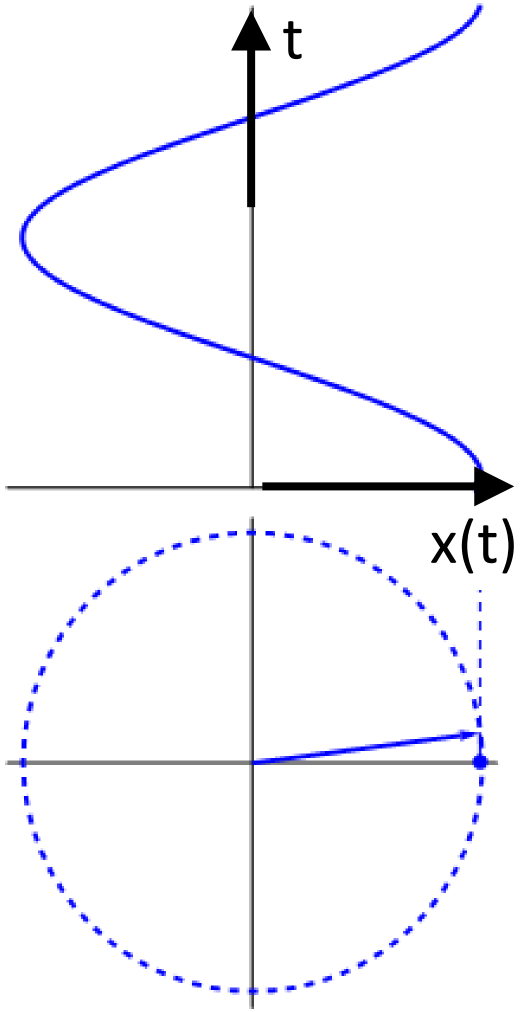


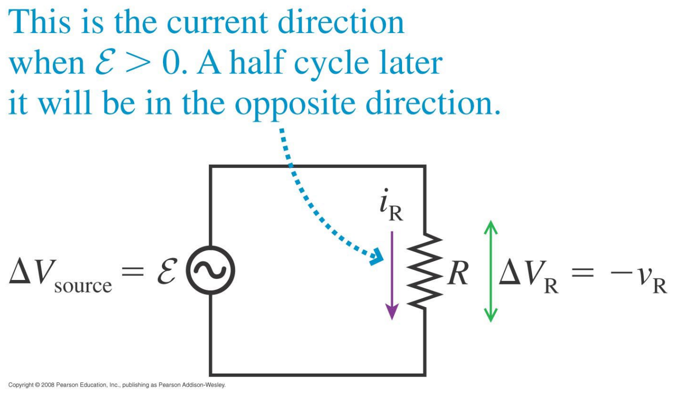

- Voltage lags behind current by $\frac{\pi}{2}$ (current leads voltage)

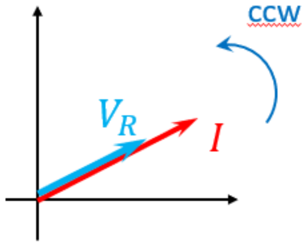
# Phasors

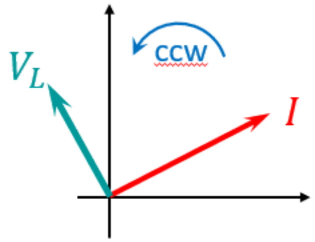
- AC current is represented by a vector performing CCW rotation $i(t) = I_{max} \cos \omega t$

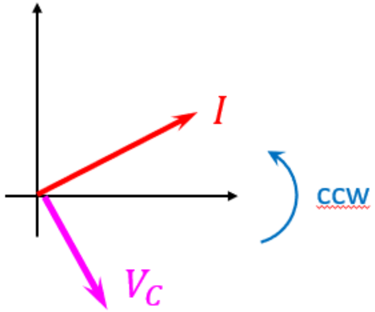


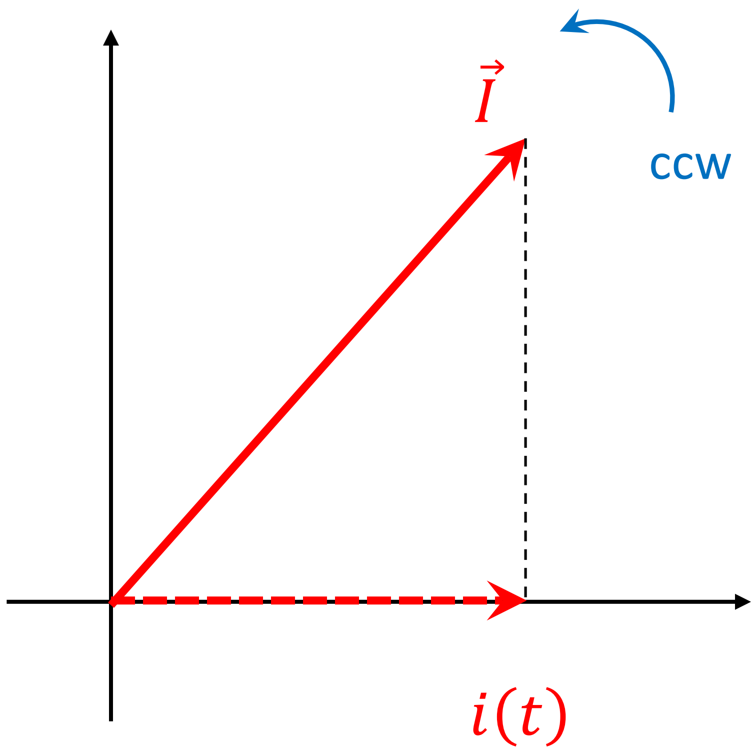



# Phasors

- Here comes the idea of phasors – abstract vectors, that help us to account for the phases (= delays) for voltages and currents in AC circuits.

- AC current is represented by a vector performing CCW rotation

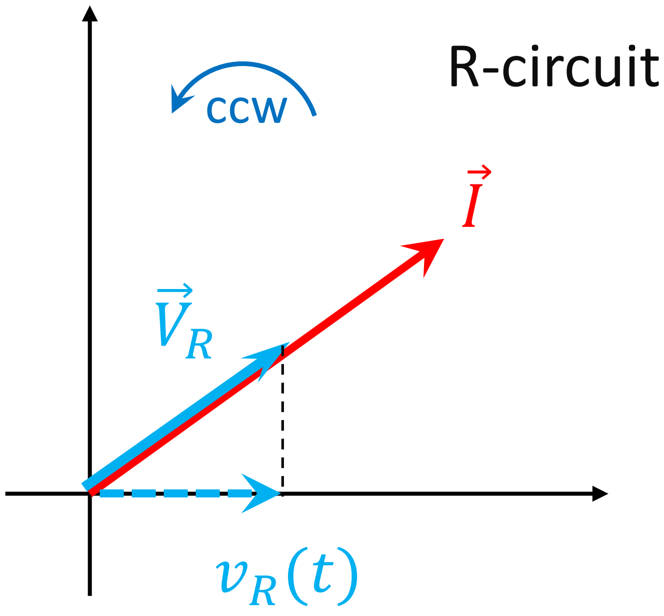


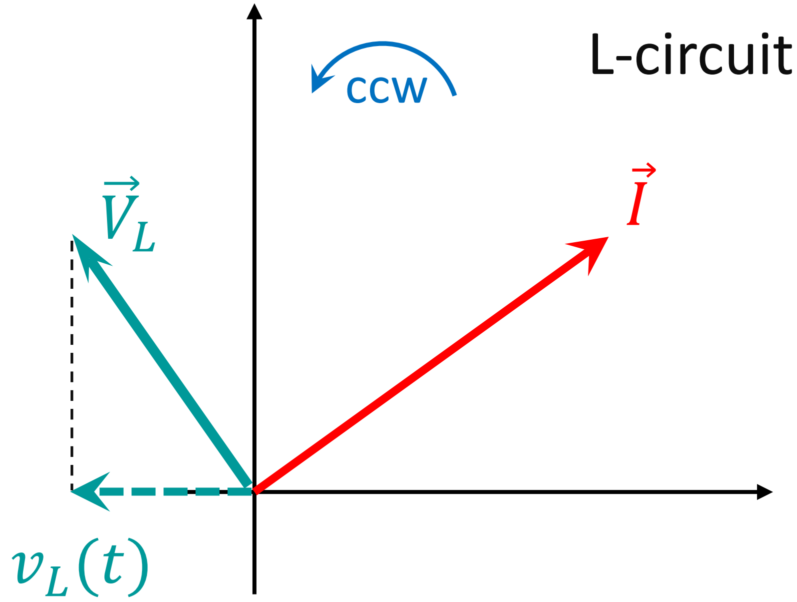

- ➢ Its length is equal to maximum current, $I_{max}$
- ➢ It rotates with the angular frequency $\omega$ of the source voltage
- ➢ Its projection onto the horizontal axis at time $t$ is equal to the current at that instant:

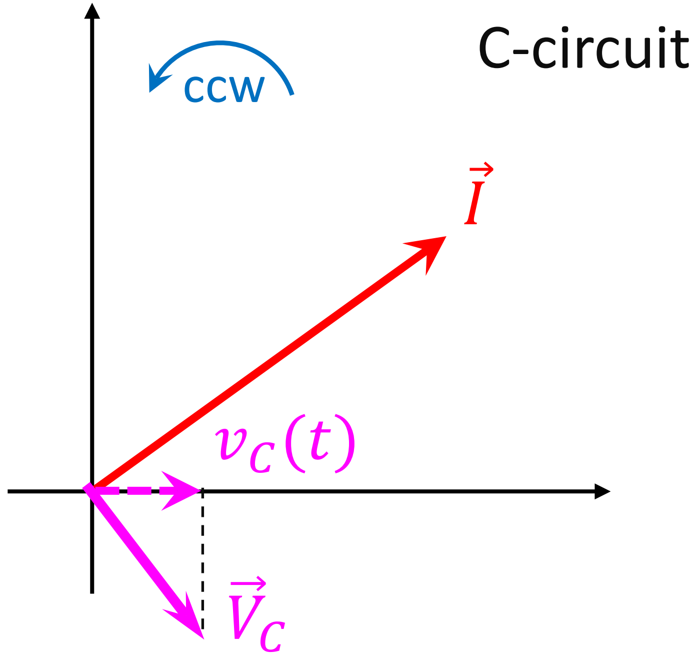
$$i(t) = I_{max} \cos \omega t$$

# Phasors

- We can also invent phasors for the voltages: $\vec{V}_R, \vec{V}_L, \vec{V}_C$
- Their projections on the horizontal axis will represent instantaneous voltages across R, L, and C in the direction opposite to the current.



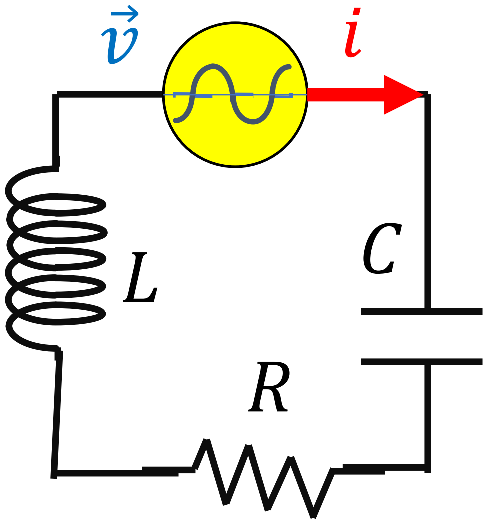




- Resistor voltage phasor:
  - In phase with $I_R$
  - $V_R = I_R R$
- Inductor voltage phasor:
  - $\pi/2$ ahead of $I_L$
  - $V_L = I_L X_L \quad (X_L = \omega L)$
- Capacitor voltage phasor:
  - $\pi/2$ behind $I_C$
  - $V_C = I_C X_C \quad (X_C = 1/\omega C)$

# AC RLC series circuit

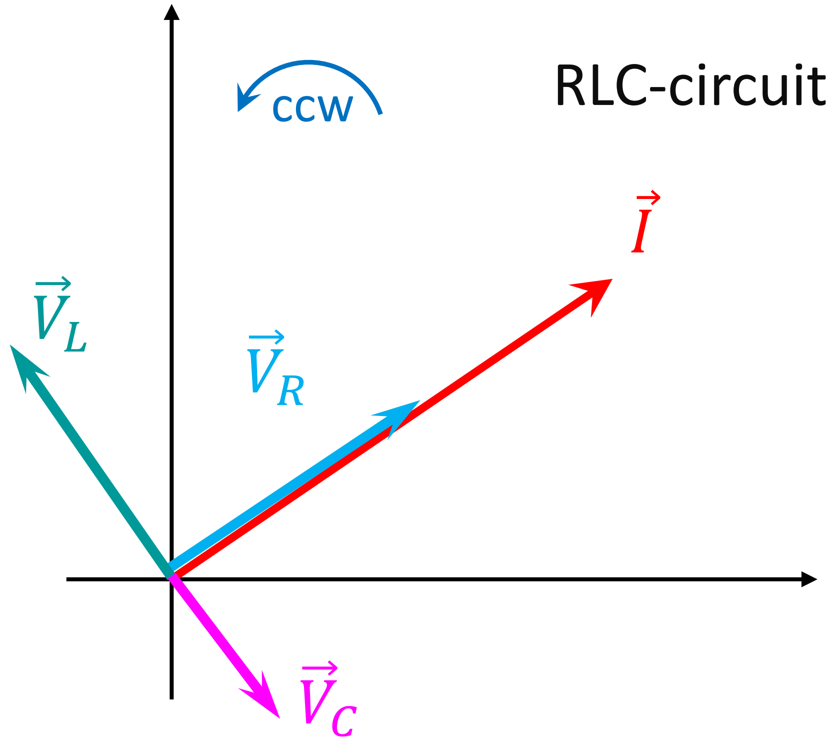
- Phasors help us to add up voltages when combining different elements in an AC circuit




➢ Q: What is common for all these elements?

➢ A: They have the same current!

- Let's combine all the voltage phasors in one diagram:




➢ Q: How can we find the relationship between source voltage and the current?

➢ A: $\vec{V} = \vec{V}_L + \vec{V}_C + \vec{V}_R$ ☺

Q: What is the magnitude of the source voltage phasor, $\vec{V} = \vec{V}_L + \vec{V}_C + \vec{V}_R$?

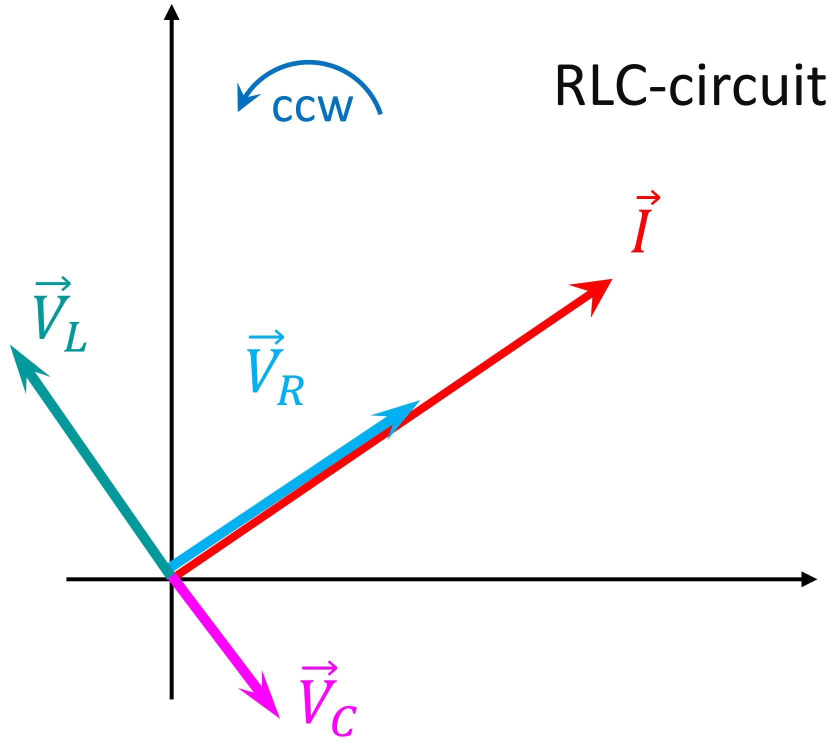
A. $V = V_R + V_L + V_C$

B. $V = V_R + (V_L - V_C)$

C. $V = \sqrt{V_R^2 + (V_L + V_C)^2}$

D. $V = \sqrt{V_R^2 + (V_L - V_C)^2}$

E. ☹



# AC RLC series circuit: Impedance

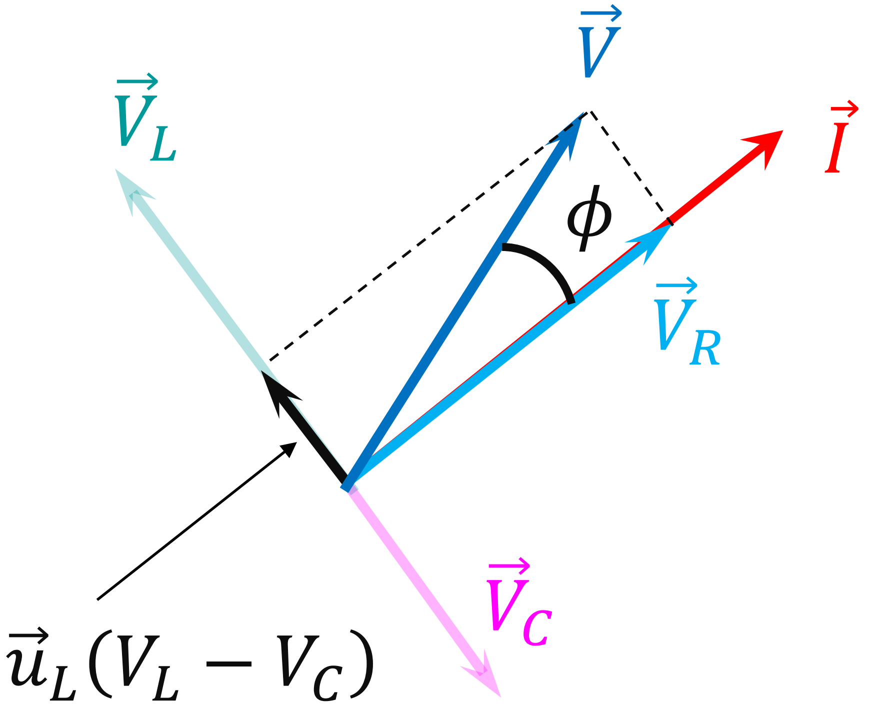
$$\vec{V} = \vec{V}_L + \vec{V}_C + \vec{V}_R$$

R

$V_R = X_R I_{max}$

$V_L = X_L I_{max}$

$V_C = X_C I_{max}$



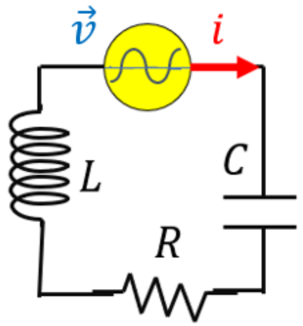

Unit vector in the direction of $\vec{V}_L$

$$V_{max} = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$= \sqrt{(X_R I_{max})^2 + (X_L I_{max} - X_C I_{max})^2}$$

$$= I_{max}\sqrt{X_R^2 + (X_L - X_C)^2} \equiv I_{max} Z$$

- $Z = \sqrt{X_R^2 + (X_L - X_C)^2}$ is called the impedance.
- $\tan(\phi) = \frac{X_L - X_C}{X_R}$ is the phase between the current and the source voltage




- So: if $i(t) = I_{max}\cos\omega t$, then $v(t) = (I_{max}Z)\cos(\omega t + \phi)$.