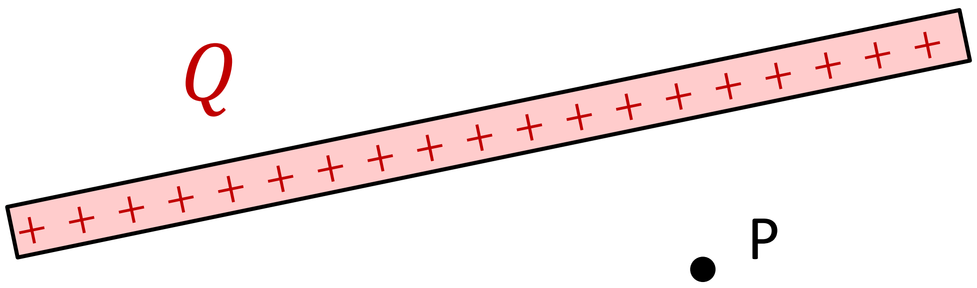
Lecture 15.

E-field of a charged ring.
E-field of a charged rod.
Electric field lines.
Force and torque on electric dipole (if time permits).

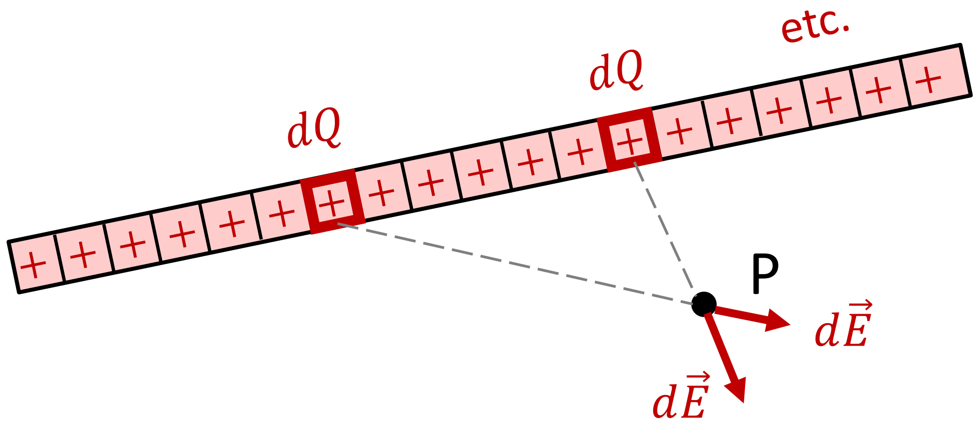
Last Time Electric field due to a continuous charge distribution

- What we know so far is the electric field produced by a point charge: $\vec{E} = \pm\frac{kq}{r^2}\,\vec{u}_r$

- How can we calculate the field produced by a continuous charge distribution?



➢ Say, we know that total charge Q is distributed evenly over this rod, and we want to know E-field at point P



➢ Big idea: let's cut the object into tiny almost-point-like charges dQ

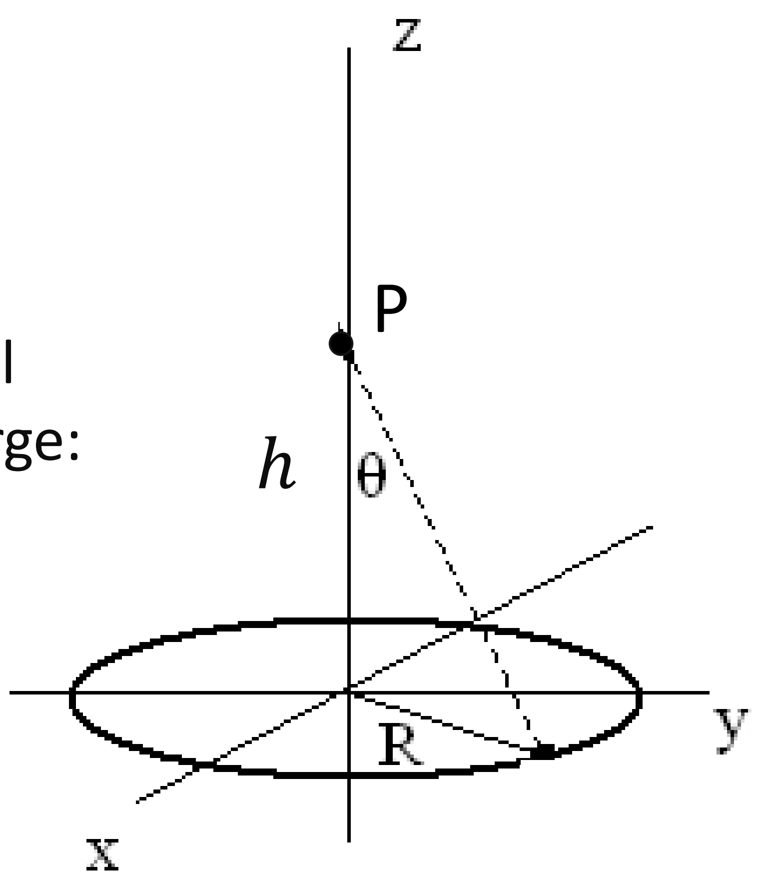
➢ Then find the field created by each of them, and sum these fields up!

Example #1: Thin Ring of Charge - 1

- Consider a thin ring-shaped conductor with radius R that has a total charge $+Q$ uniformly distributed around it.

- What is the electric field at the point P located at a position on the z-axis a distance h above the ring's centre?

Total charge: Q



Example #1: Thin Ring of Charge - 2

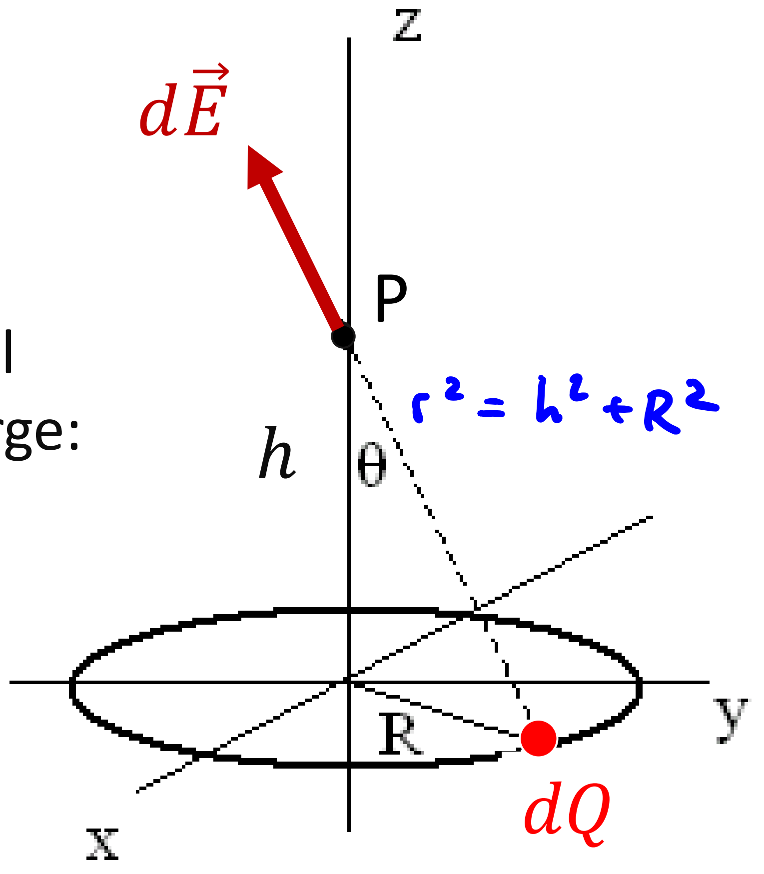
Why a ring? It's a good starter. It's a highly symmetric object, and we will be looking for a field at a point with high symmetry (otherwise it will be a nightmare)

1) Menatlly cut the object into infinitesimal charges dQ.

2) Calculate $d\vec{E}$ at P due to a point charge dQ.

- Magnitude: $dE = k\frac{dQ}{R^2+h^2}$
- Direction: away from dQ, at an angle (shown in the picture)

Total charge: Q



- Next step would be to add all tiny fields $d\vec{E}$ up.
- We need to remember though that they add as vectors (in components)

Example #1: Thin Ring of Charge - 3

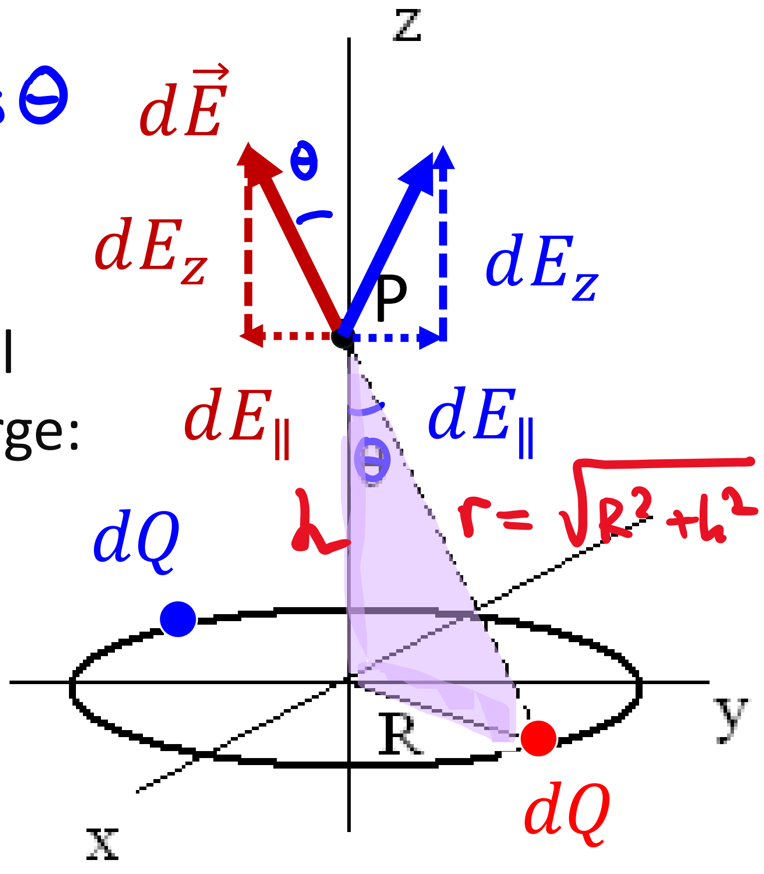
Why a ring? It's a good starter. It's a highly symmetric object, and we will be looking for a field at a point with high symmetry (otherwise it will be a nightmare)

3) Find the components of $d\vec{E}$. Do we actually need all of them?

$dE_z = dE \cdot \cos\theta$

- Due to the symmetry of the ring, all the components of E-filed in x and y directions will cancel out. And the components in z-direction will add up!
- We expect: $\vec{E} = \{0, 0, E_z\}$
- We have:

Total charge: Q



$$dE_z = dE\cos\theta \qquad dE = \frac{k\ dQ}{R^2 + h^2} \qquad \cos\theta = \frac{h}{\sqrt{R^2 + h^2}}$$

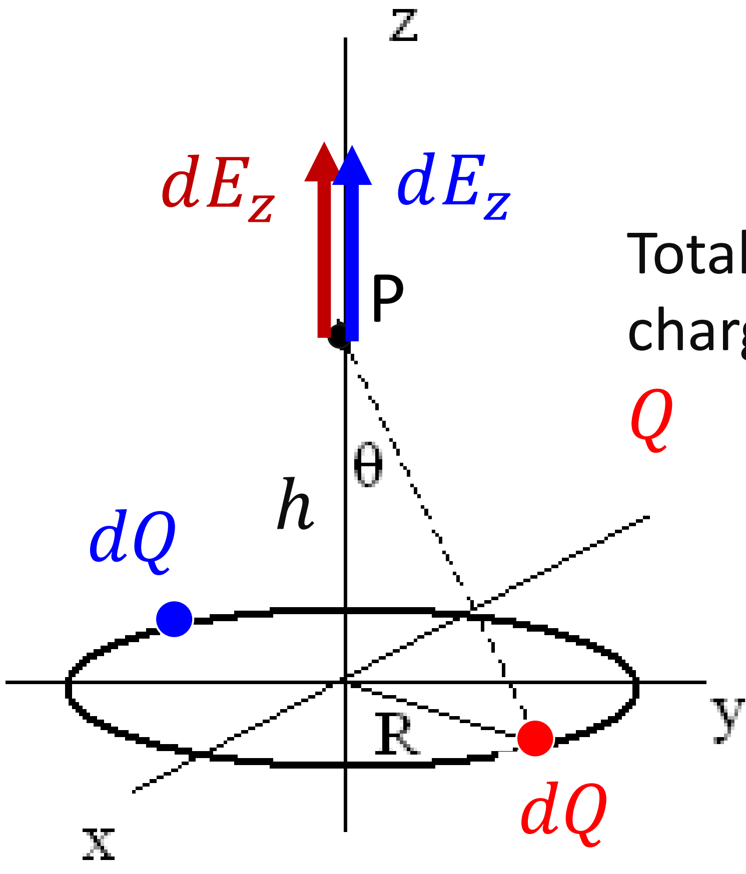
Example #1: Thin Ring of Charge - 4

4) Find $E_x = \int dE_x$, $E_y = \int dE_y$, $\boxed{E_z = \int dE_z.}$ Basically, you are just adding up the fileds produced by all tiny charges (superposition principle)

$$dE_z = \frac{k\ dQ}{R^2 + h^2} \cdot \frac{h}{\sqrt{R^2 + h^2}} = \frac{kh\ dQ}{(R^2 + h^2)^{3/2}}$$

$$E_z = \int dE_z = \int_{ring} \frac{kh\ dQ}{(R^2 + h^2)^{\frac{3}{2}}} = \frac{kh}{(R^2 + h^2)^{\frac{3}{2}}} \underbrace{\int_{ring} dQ}_{\text{Total charge!}}$$

$$\boxed{E_z = \frac{kh\ Q}{(R^2 + h^2)^{\frac{3}{2}}}}$$



Total charge: Q

Example #1: Thin Ring of Charge - 5

*) Sanity check:

$$\vec{E} = \frac{kh\,Q}{(\cancel{R}^2 + \underline{\underline{h^2}})^{\frac{3}{2}}}\,\vec{u}_z$$

- Limiting cases:

- Far away from the ring: $h \gg R$

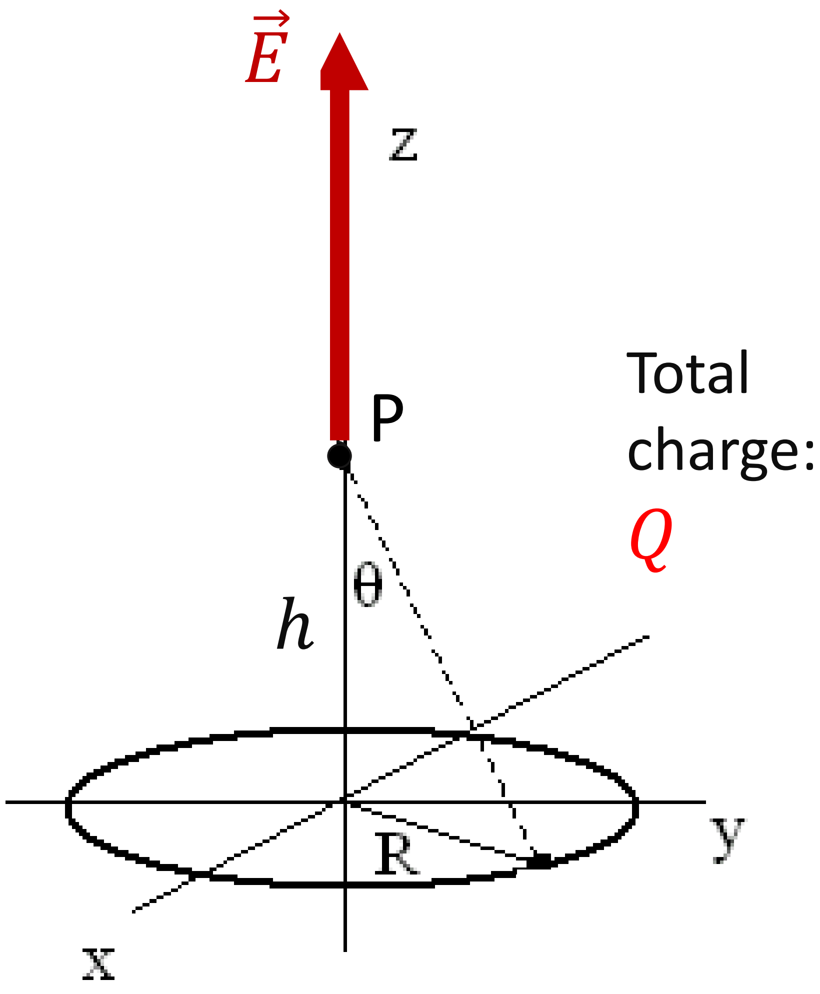
$\frac{khQ}{h^3}$

$$\vec{E} = \frac{k\,Q}{h^2}\,\vec{u}_z \qquad \text{-- point charge!}$$

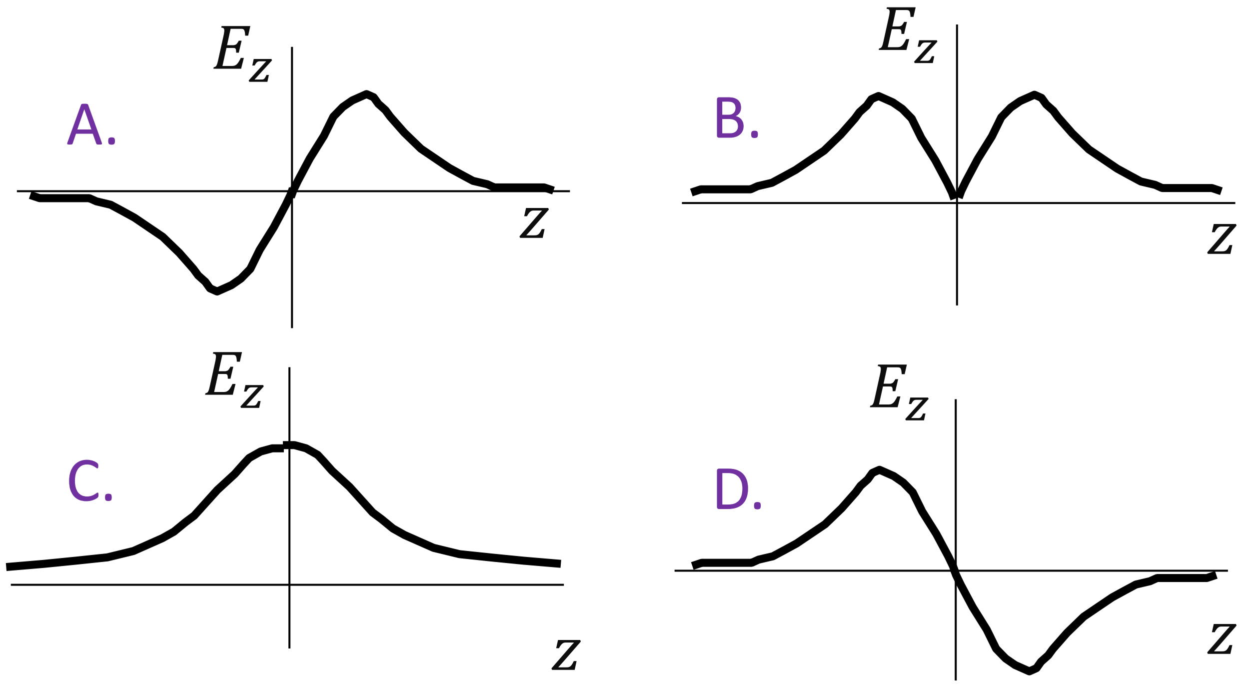
- Center of the ring: $h = 0$

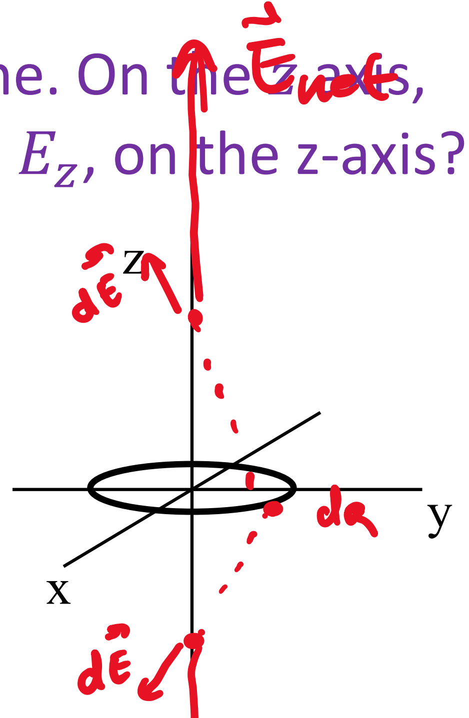
$E(h=0) = 0$

$$\vec{E} = 0 \qquad \text{-- by symmetry}$$



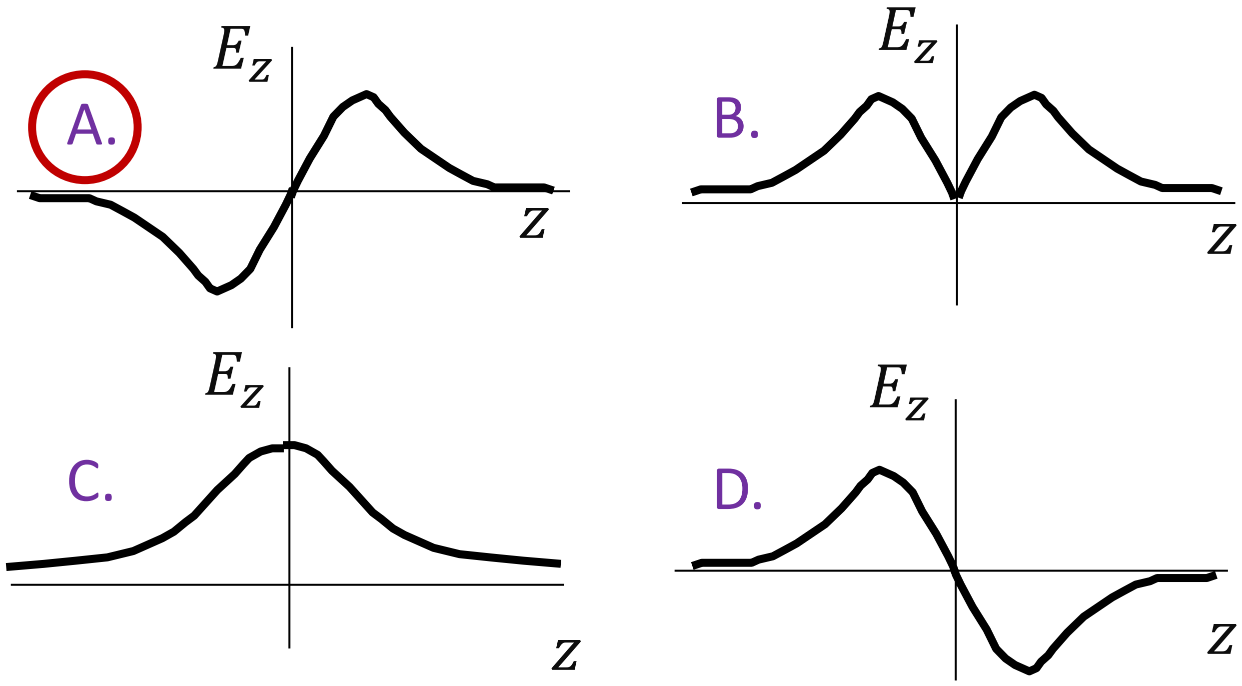
Q: A circular ring uniformly charged (charge $+Q$) is shown in the (x,y) plane. On the z-axis, $\vec{E} = E_z \vec{k}$. Which graph correctly represents the electric field component, E_z, on the z-axis?

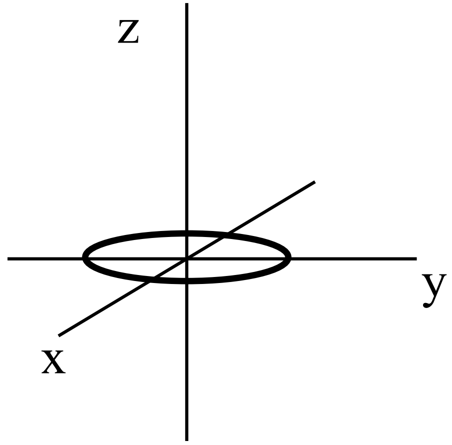




Application: loop antenna

Q: A circular ring uniformly charged (charge $+Q$) is shown in the (x,y) plane. On the z axis, $\vec{E} = E_z \vec{k}$. Which graph correctly represents the electric field component, E_z, on the z-axis?

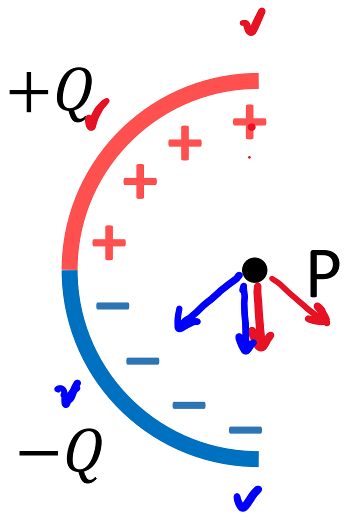




$$E_z = \left(\frac{kzQ}{(R^2+z^2)^{3/2}}\right) \begin{array}{l} = 0 \text{ for } z = 0 \text{ (centre of ring)} \\ = \text{ max for } z = \pm\frac{R}{\sqrt{2}} \end{array}$$

Application: loop antenna

Q: A positive charge $+Q$ is uniformly distributed on the upper half of a semicircular rod and a negative charge $-Q$ is uniformly distributed on the lower half. What is the direction of the electric field at the point P, the center of the semicircle?

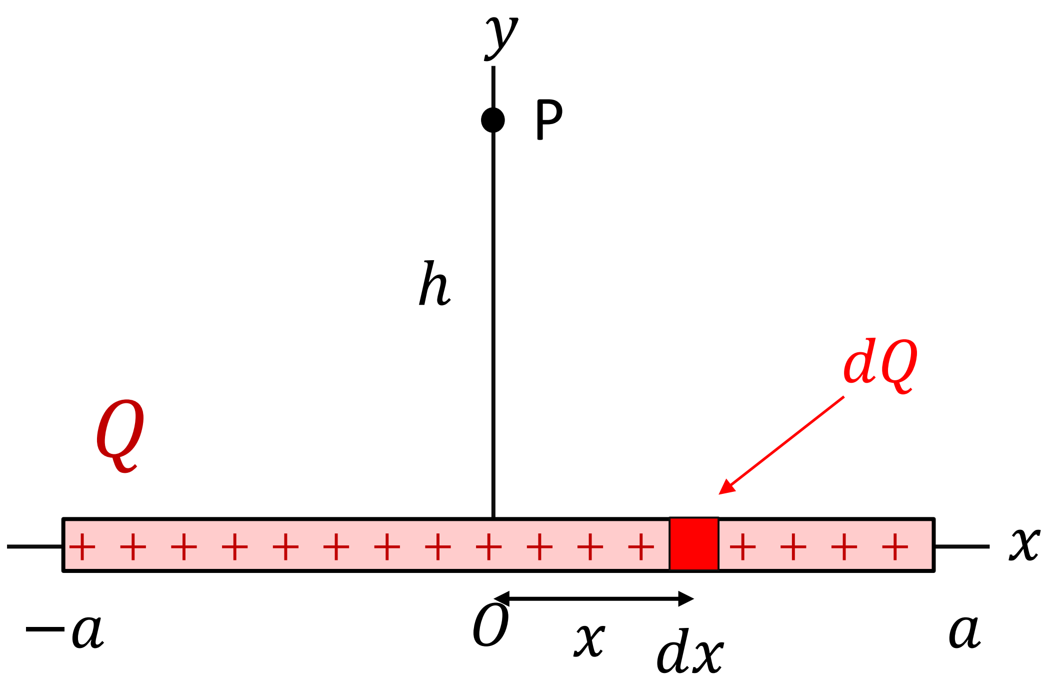


A. Upward
B. Downward
C. Left
D. Right
E. 45 degree north of east

Example #2: E-field of a Finite Line of Charge - 1

- Consider charge $+Q$ distributed uniformly along a line $L = 2a$.

- Find the electric field $\vec{E}$ at point P at the symmetry axis at distance h from the rod.



➢ Consider the line to be made up of infinitesimal segments, dx, at the position x

➢ Find the superposition of tiny fields, $d\vec{E}$, produced by these charged at point P
(!) E-fields add up in components!

Example #2: E-field of a Finite Line of Charge - 2

1) Menatlly cut the object into infinitesimal charges dQ.

Q: How much charge, dQ, is sitting in a segment of length dx?

A. $\frac{Q}{2a}$

B. $\frac{Q}{2a}dx$

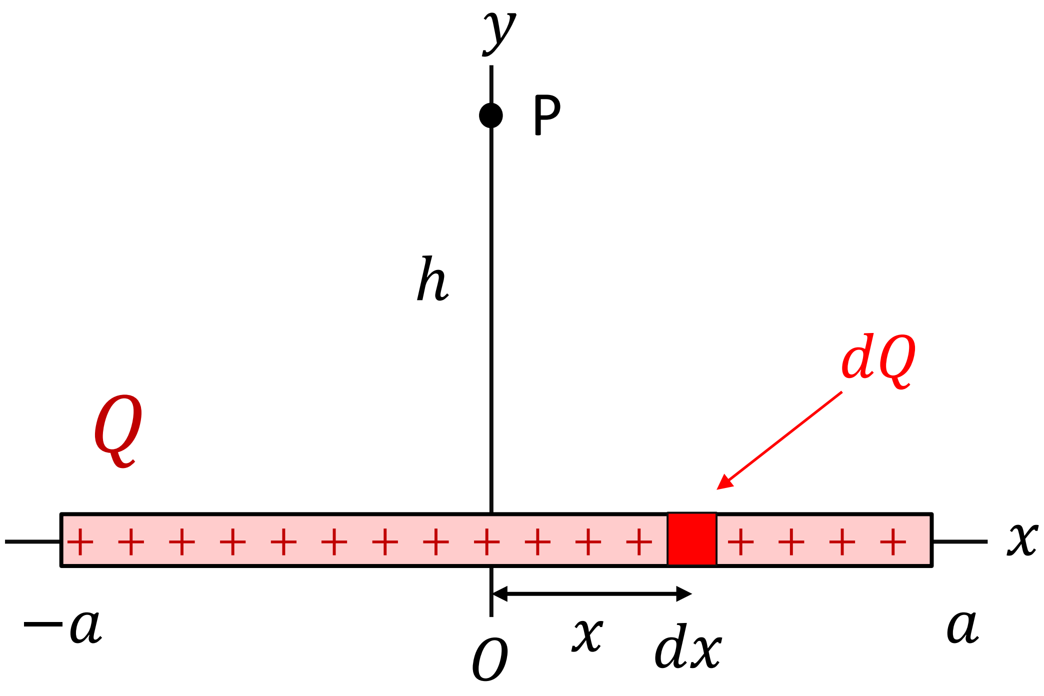
C. $Q\ dx$

D. $Q\frac{2a}{dx}$

E. Something else

$$\frac{dQ}{Q} = \frac{dx}{L} = \frac{dx}{2a}$$

$$dQ = \frac{Q}{2a}dx$$



Example #2: E-field of a Finite Line of Charge - 3

2) Calculate $d\vec{E}$ at P due to a point charge dQ.

Q: What is the magnitude of the electric field $d\vec{E}$ created by a tiny charge dQ highlighted in the picture?

A. $dE = k\dfrac{dQ}{h^2}$

B. $dE = k\dfrac{dQ}{x^2}$

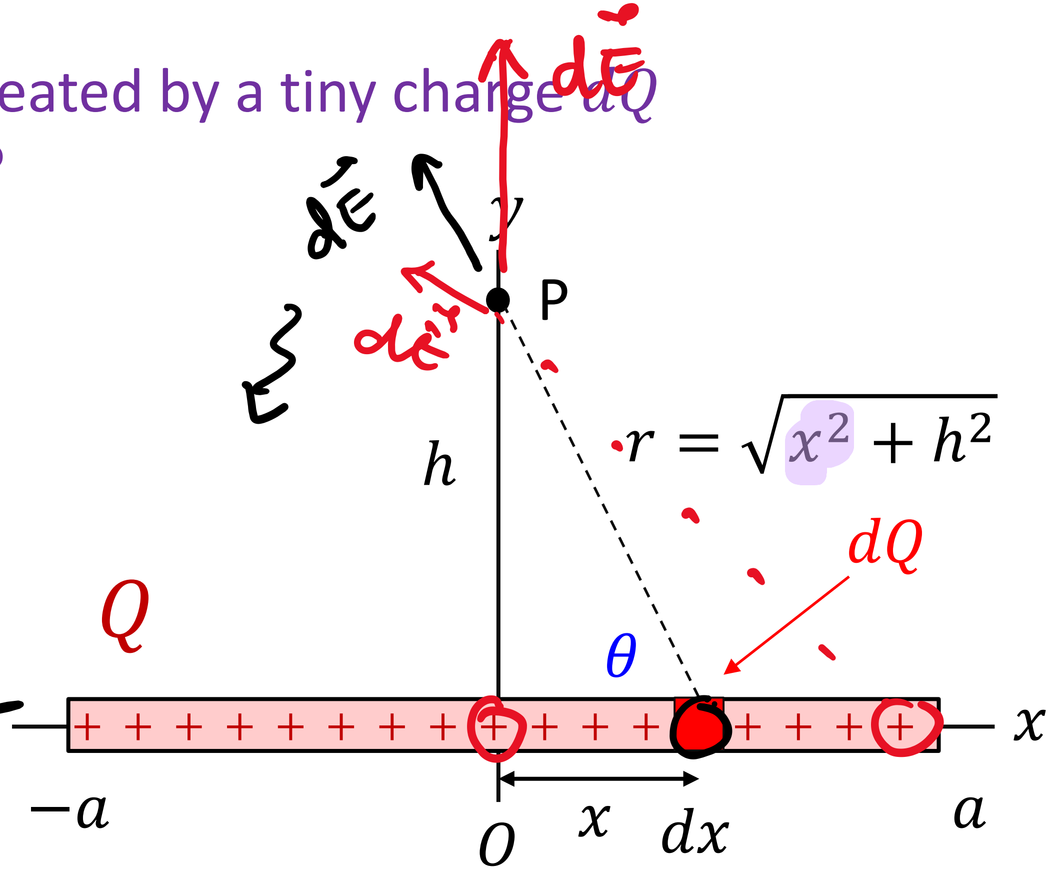
C. $dE = k\dfrac{dQ}{x^2+h^2}$

D. $dE = k\dfrac{dQ}{x^2+h^2}\cos\theta$

E. $dE = k\dfrac{dQ}{x^2+h^2}\sin\theta$

$E_{p.ch.} = k\dfrac{q}{r^2}$

$dE = k\dfrac{dQ}{r^2}$



Q: What is its direction? Show in the picture.

Example #2: E-field of a Finite Line of Charge - 4

$$dQ = \frac{Q}{2a}dx$$

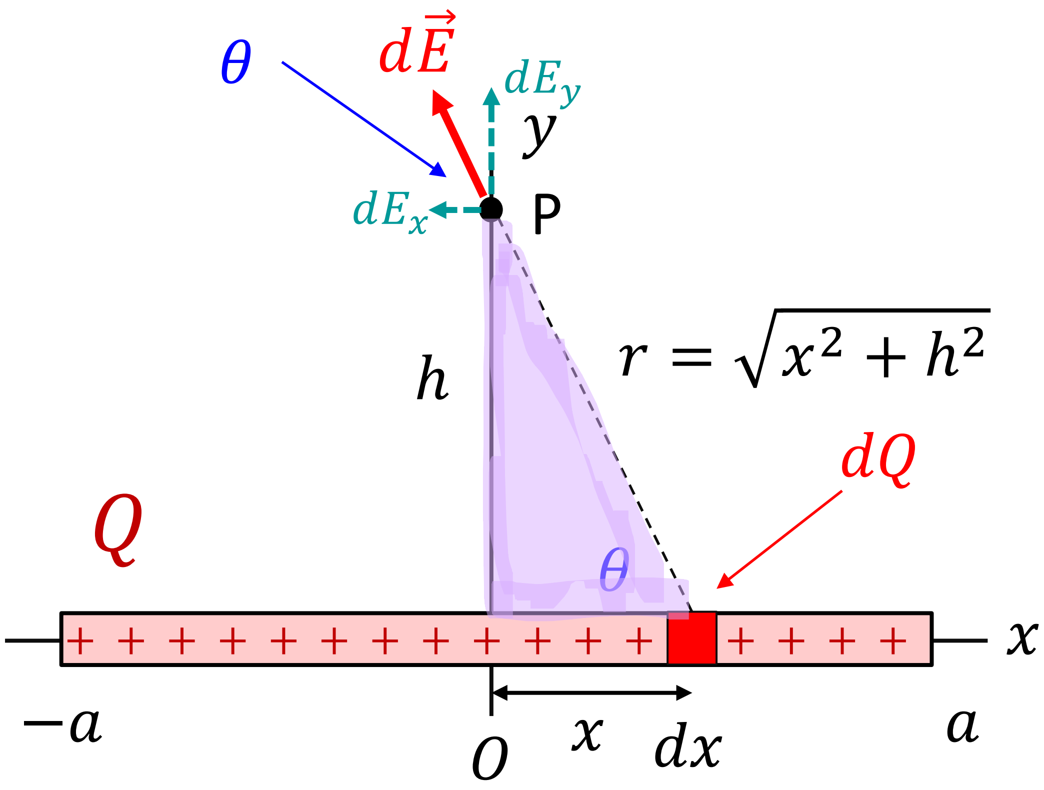
3) Find the components of $d\vec{E}$. Do we actually need all of them?

$$dE = \frac{kQ}{2a}\frac{dx}{x^2+h^2}$$

$$\boxed{dE = k\frac{dQ}{x^2+h^2}}$$

$$dE_x = -dE\cos\theta = -\left(\frac{kQ}{2a}\frac{dx}{x^2+h^2}\right)\frac{x}{\sqrt{x^2+h^2}}$$

$$dE_y = +dE\sin\theta = +\left(\frac{kQ}{2a}\frac{dx}{x^2+h^2}\right)\frac{h}{\sqrt{x^2+h^2}}$$



- Next: $E_x = \int_{rod} dE_x\,,\quad E_y = \int_{rod} dE_y$

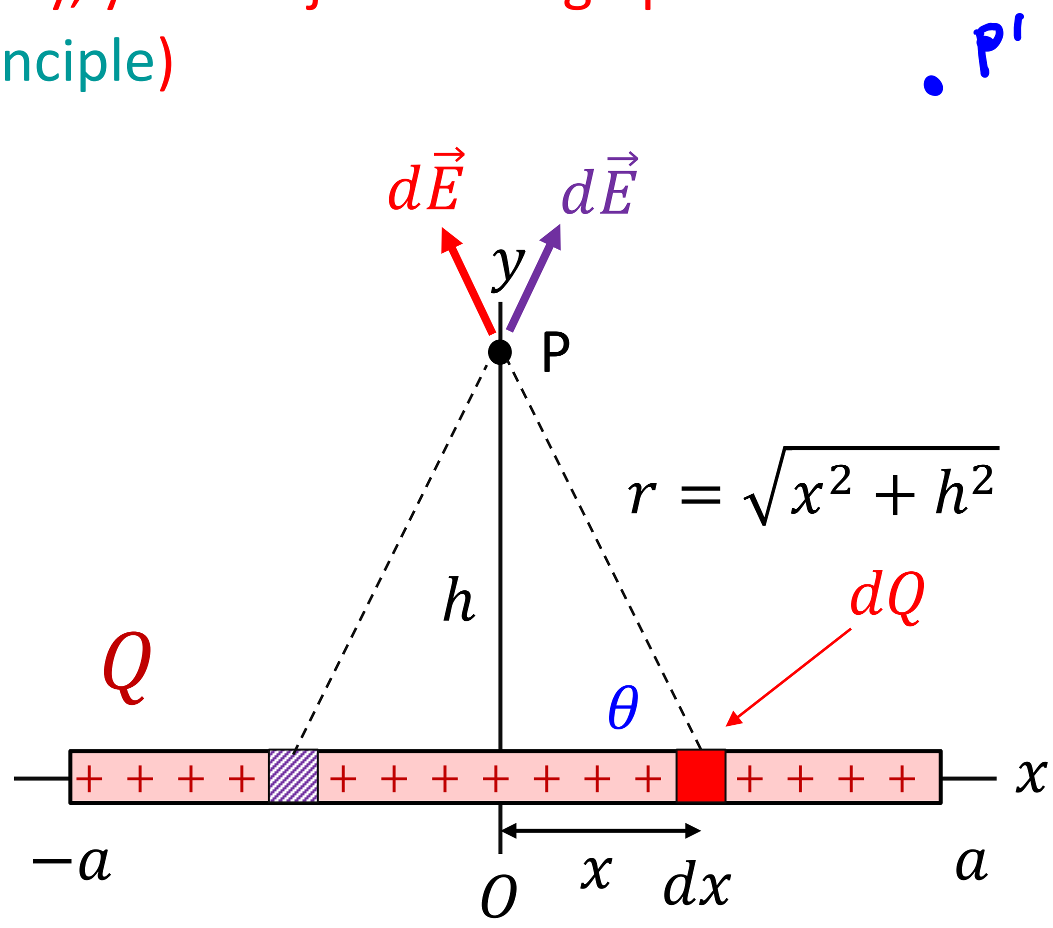
$$\int_{rod}(\ldots) = \int_{x=-a}^{x=+a}(\ldots)$$

Example #2: E-field of a Finite Line of Charge - 5

4) Find $E_x = \int dE_x$, $E_y = \int dE_y$, $E_z = \int dE_z$. Basically, you are just adding up the fields produced by all tiny charges (superposition principle)

$$E_x = \frac{kQ}{2a}\int_{-a}^{+a} \frac{-x\,dx}{(x^2+h^2)^{3/2}} = 0 \quad \text{by symmetry}$$

$$E_y = \frac{kQ}{2a}\int_{-a}^{+a} \frac{h\,dx}{(x^2+h^2)^{3/2}} = \frac{kQ}{h\sqrt{a^2+h^2}}$$



- Use this standard integral:

$$\int \frac{dx}{(x^2+c^2)^{3/2}} = \frac{x}{c^2\sqrt{x^2+c^2}}$$

Example #2: E-field of a Finite Line of Charge - 6

$$E_y(h) = \frac{kQ}{h\sqrt{\cancel{a^2} + h^2}}$$

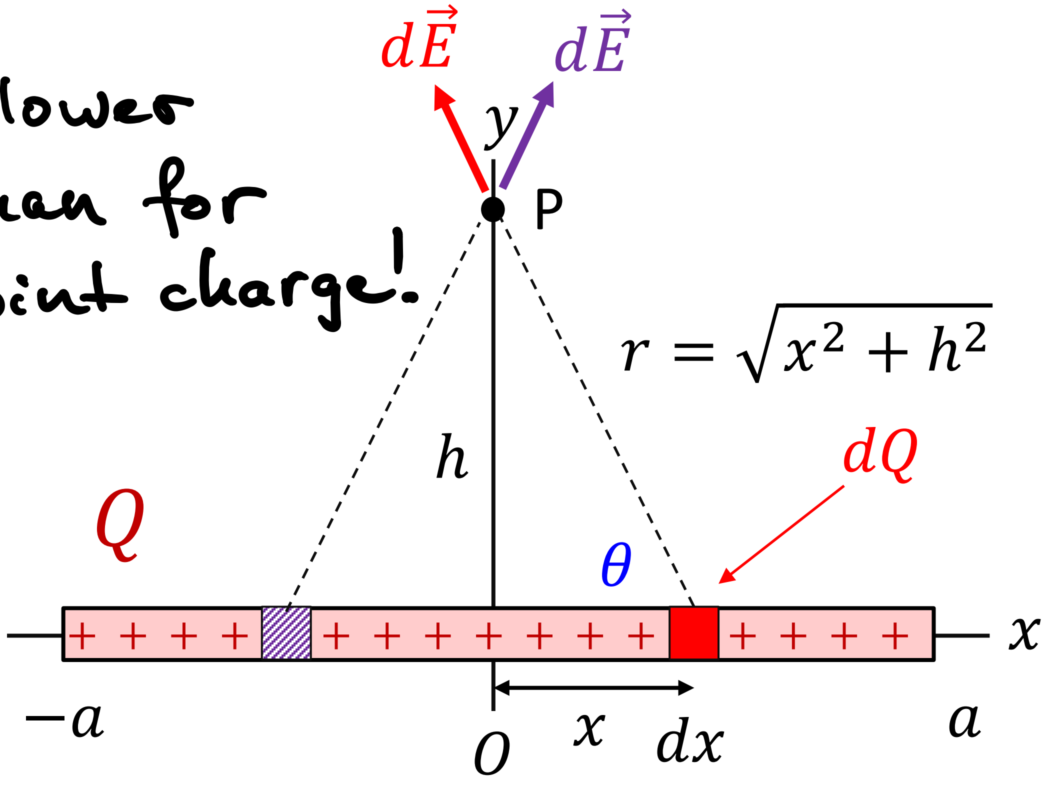
*) Limiting cases:

- $a \gg h$ (very long rod) ←

$$E_y \to \frac{kQ}{h\,a} = 2k\left(\frac{Q}{2a}\right)\frac{1}{h} = \frac{2k\lambda}{h}$$ ←

slower than for point charge!

  - $\lambda = Q/(2a)$: linear charge density (charge per unit length)
  - Electric field decays as 1/(distance from the rod)



- $a \ll h$ (very short rod)

$$E_y \to \frac{kQ}{h^2}$$ = E-field of a point charge Q – faster than rod!

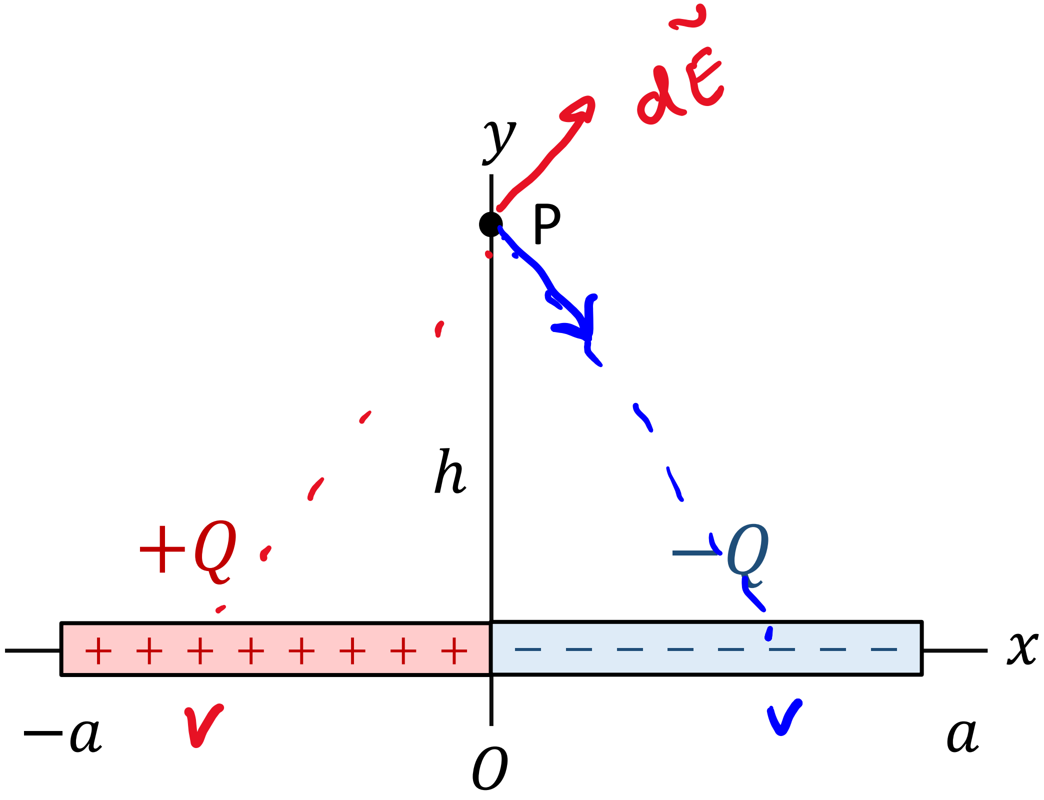
Q: The same line as before now has charges uniformly distributed as:

$+Q$ between $x = -a$ and $x = 0$

$-Q$ between $x = 0$ and $x = +a$

$$E_y(h) = \frac{kQ}{h\sqrt{a^2 + h^2}}$$

In this situation, the electric field at point P is:



A. in the positive x direction
B. in the negative x direction
C. in the positive y direction
D. in the negative y direction
E. zero

Calculations with continuous charge distribution: Summary

Step 0: Try to reduce the problem in hand to what you already know

Consider symmetry

1. Mentally cut the object into infinitesimal segments = tiny charges dq.

2. Write the electric field, $d\vec{E}$, at the point P due to each tiny charge dq.

3. Write the *components* of $d\vec{E}$ (dE_x, dE_y, dE_z) at the point P due to a segment dq.

4. Add up (as vectors, *componentwise*!) the fields produced by all the small segments = integrate the projections dE_x, dE_y, dE_z:

$$E_x = \int dE_x, \qquad E_y = \int dE_y, \qquad E_z = \int dE_z$$

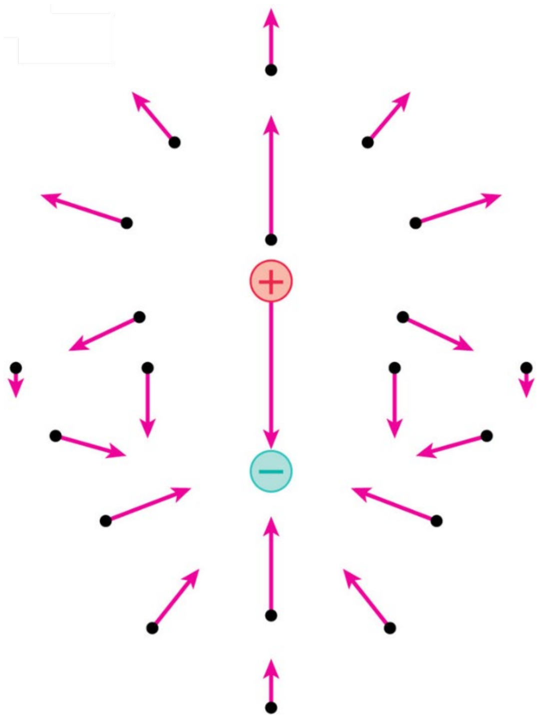
5. Calculate the magnitude of the electric field from its components:

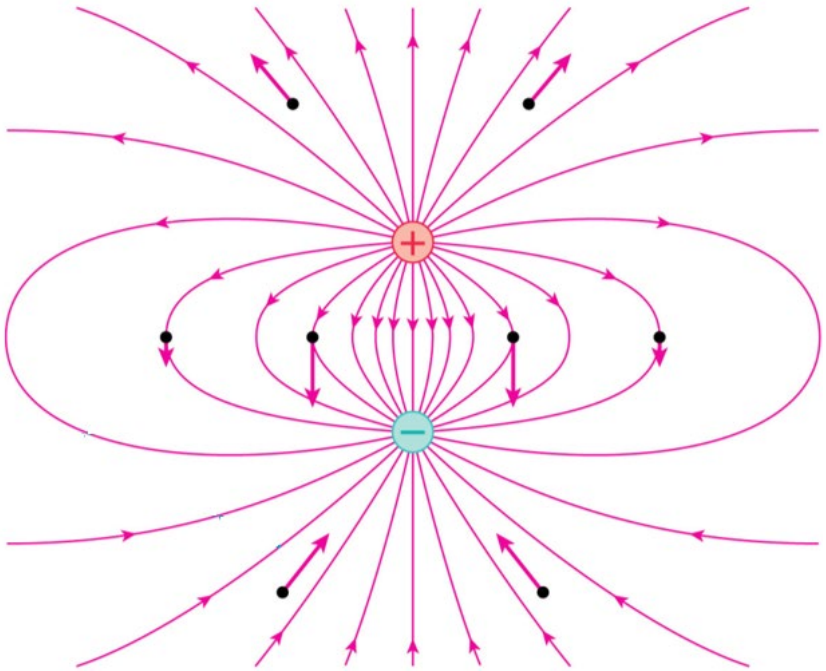
$$E = \sqrt{E_x^2 + E_y^2 + E_z^2}$$

Remember what is your integration variable, and what is constant (i.e. does not change)

Sanity-check your answer: look at limiting cases (field far away and near the charged object)

Field Lines



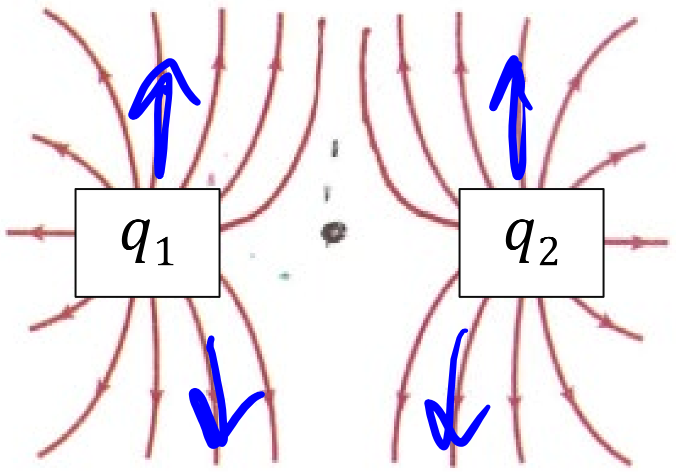


Field Lines:
show you in which direction a positive test charge would move if placed at this point.

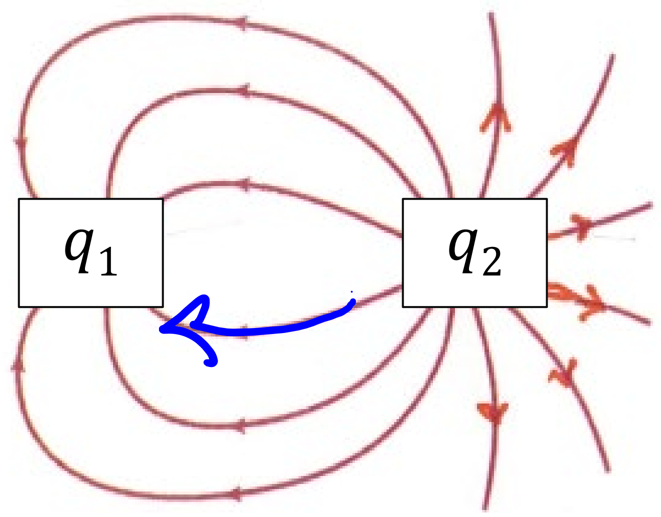
1. The field vector is always tangent to the field line at any point.
2. Field lines start on positive charges and end on negative charges.
3. Field strength: density of field lines. Closer spacing = stronger field.
4. Field lines never cross.

Q: What can you say about charges q_1 and q_2?

(a)



(b)



A. (a) + & +, (b) + & -

B. (a) + & +, (b) − & +

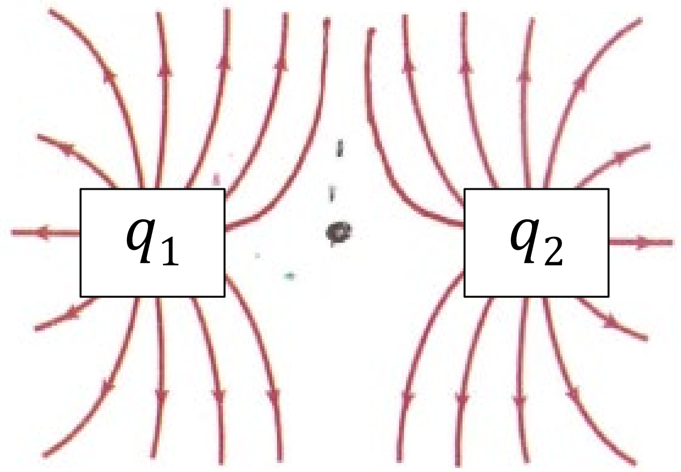
C. (a) − & −, (b) + & −

D. (a) − & −, (b) − & +

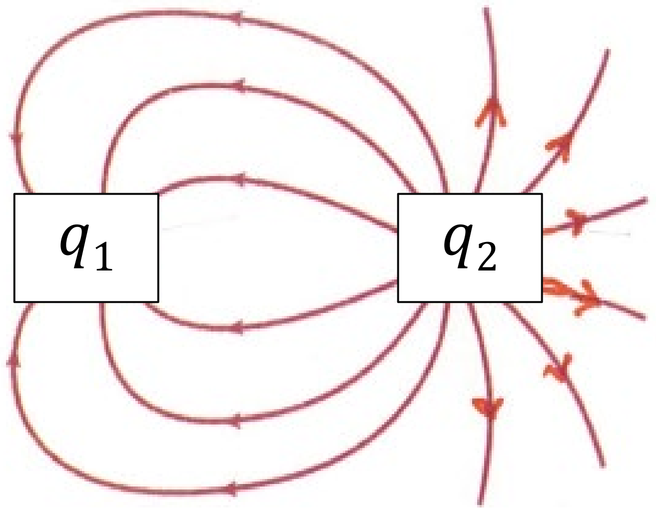
E. Other

Q: What can you say about the magnitudes of these charges?

(a)



(b)

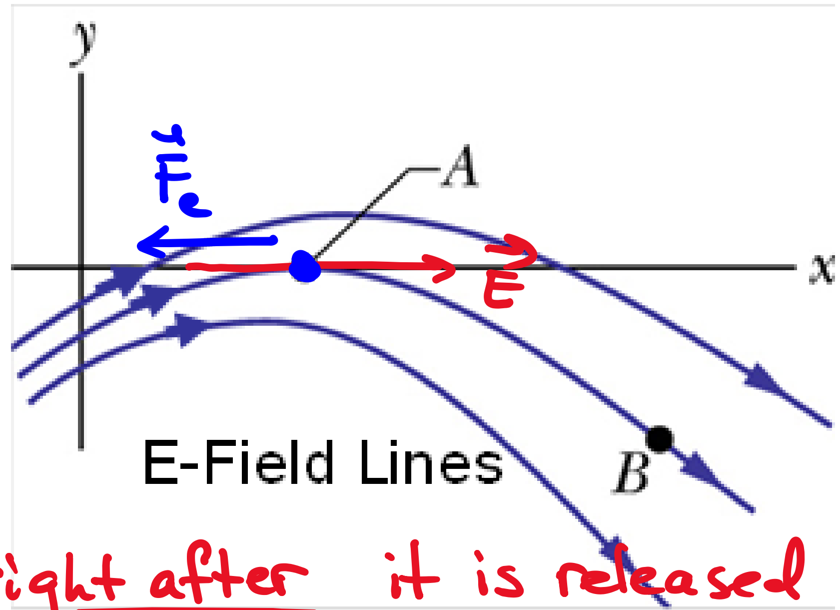


A. (a) q1 = q2, (b) q1 = q2
B. (a) q1 = q2, (b) q1 > q2
C. (a) q1 = q2, (b) q1 < q2
D. (a) q1 > q2, (b) q1 = q2
E. (a) q1 < q2, (b) q1 = q2

density of E-field lines
proportional to the E-field strength

Q: If a negative charge is released from rest at A, in which direction will it initially travel?

$\vec{F} = q_{\pm}\vec{E}$



3) When we know the direction of the force acting on the charge, we also the direction in which it accelerates right after it is released from rest.

A. To the left
B. To the right
C. Toward B along the field line
D. Away from B along the field line
E. Other

1) $\vec{E}$-field is tangent to the field lines → we know the direction of $\vec{E}$

2) Force on a charge $q_{\pm}$ in electric field $\vec{E}$:

$\vec{F} = q_{\pm}\vec{E}$ → $q>0$: $\vec{F} \uparrow\uparrow \vec{E}$; $q<0$: $\vec{F} \uparrow\downarrow \vec{E}$