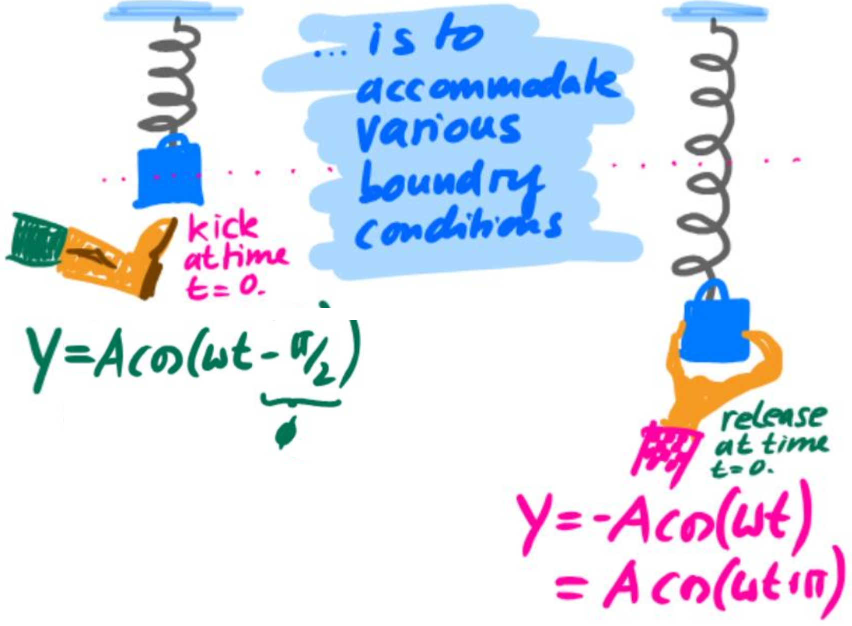
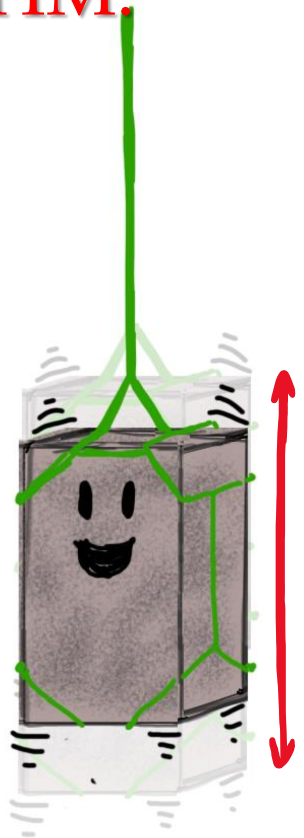
Lecture 27.

Phase. Displacement, velocity and acceleration in SHM.



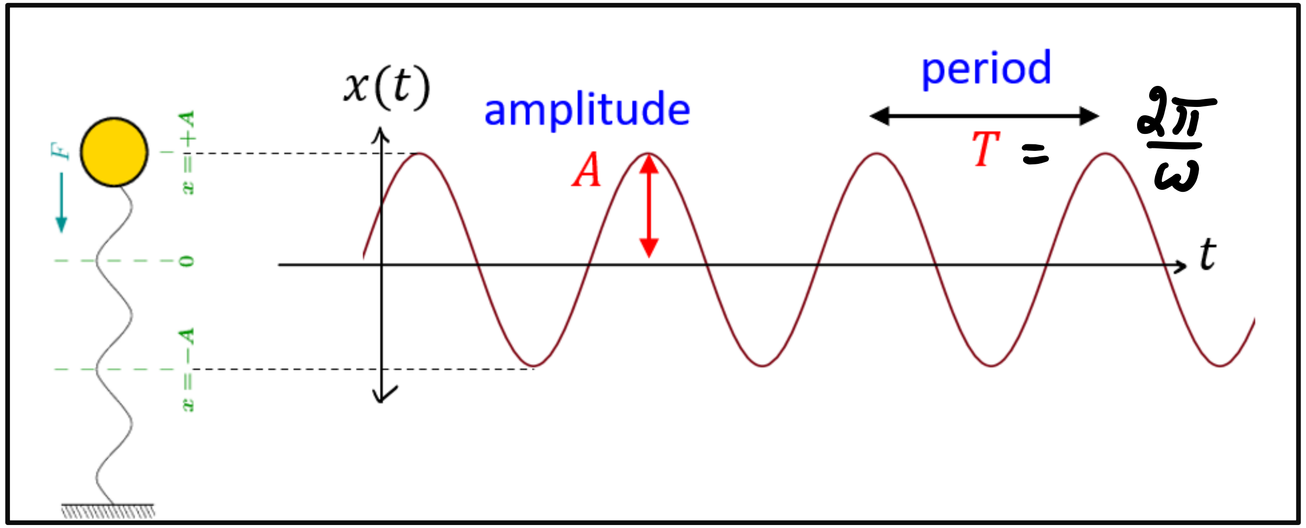


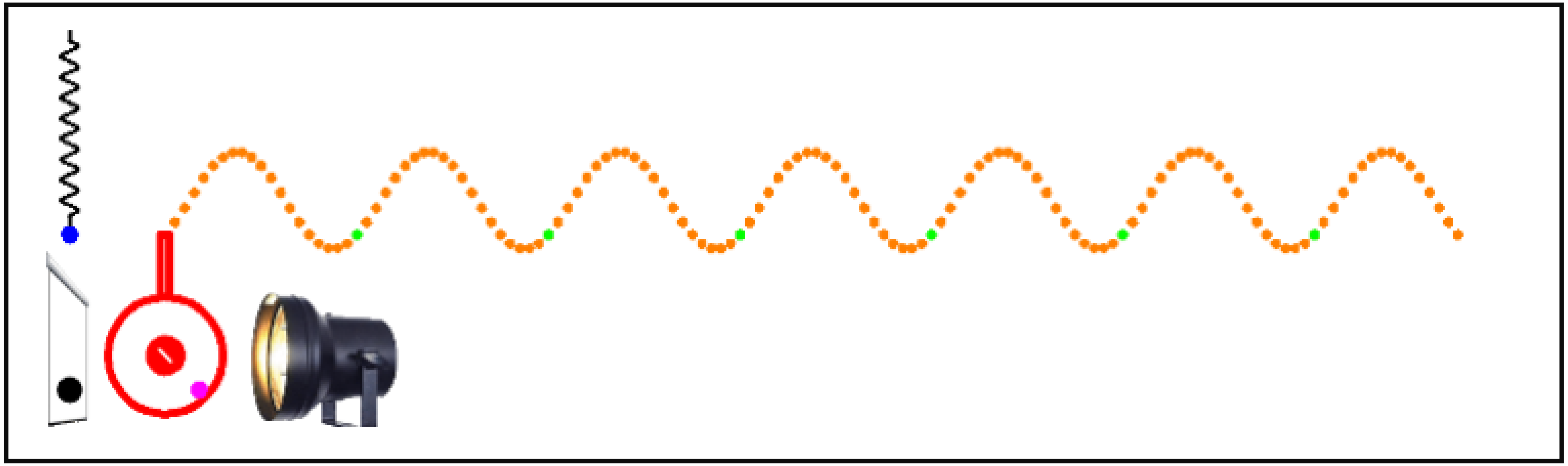
Last Time

- Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi)$$

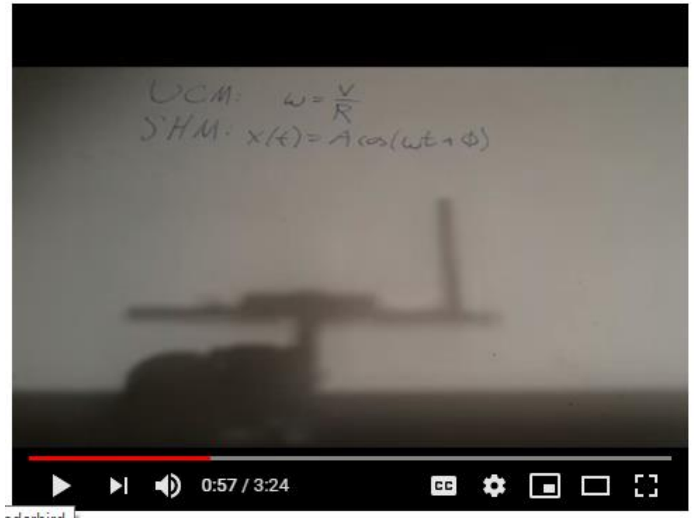
- A = amplitude
- ω = angular frequency
- ϕ = phase = ?

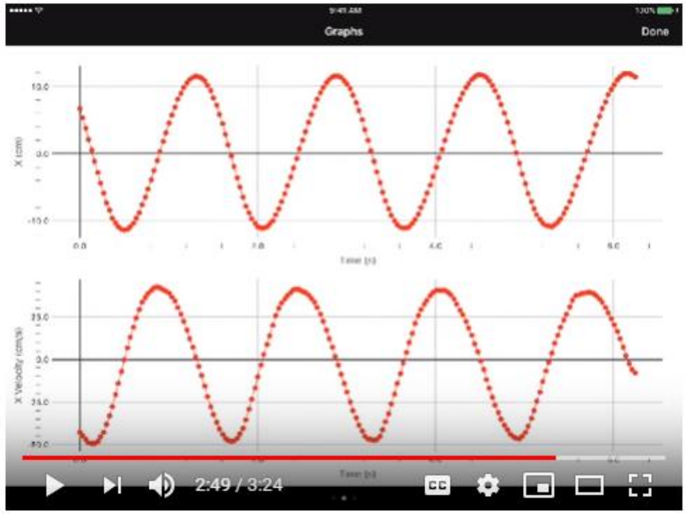




Bicycle wheel and harmonic motion

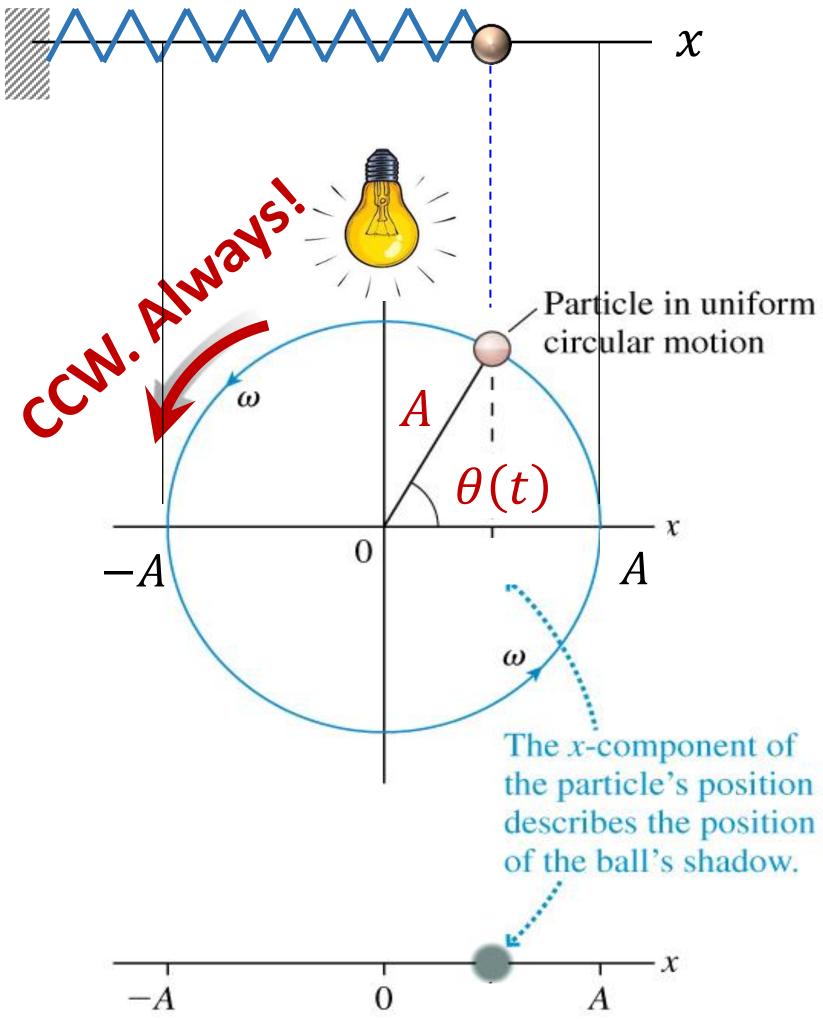
https://www.youtube.com/watch?v=nZYbHygjyTc





$x(t)$

$y(t)$



- Assume the ball is illuminated from behind, and we are looking at the projection of its motion onto a screen perpendicular to the plane of its rotation.

$$x(t) = ??$$

- x coordinate: $x(t) = A\cos\theta(t)$

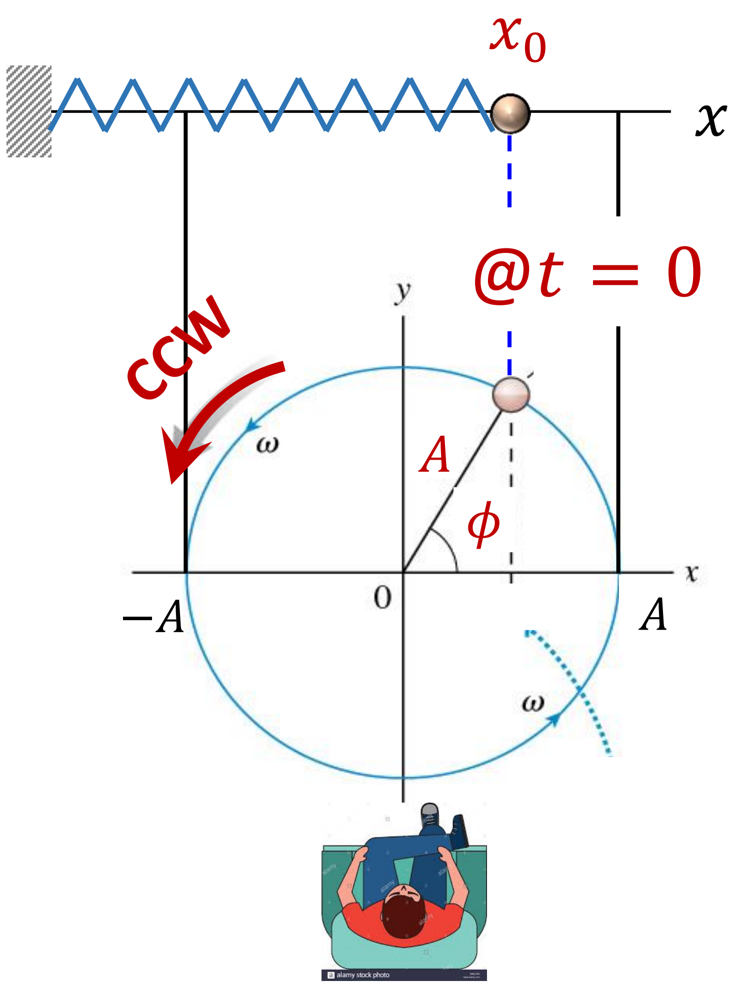
- Uniform circular motion: $\theta(t) = \omega t$ is the angle swept out in a time t

- We get: $\boxed{x(t) = A\cos(\omega t)}$

Q: But what is the physical meaning of the phase, ϕ, in $x(t) = A\cos(\omega t + \phi)$?

Phase ϕ

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$



- At $t = 0$:

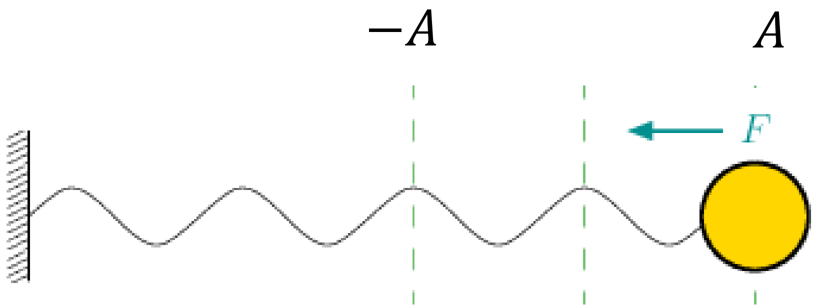
$$x_0 = x(t = 0) = A\cos(\phi)$$

- Hence, phase determines the position of the mass at $t = 0$. In other words, phase is the initial condition.

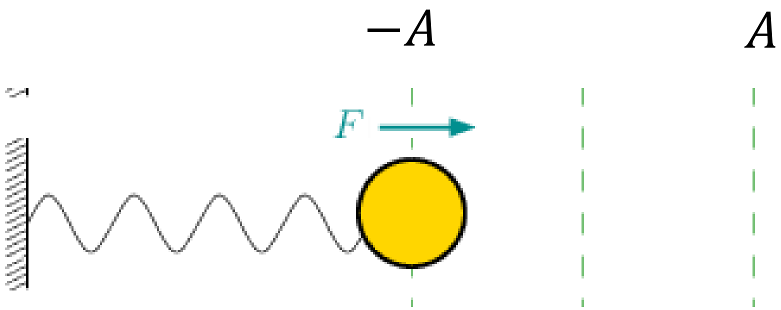
- How to find: $$\boxed{\phi = \cos^{-1}\left(\frac{x_0}{A}\right)}$$

Q: What is the phase, if the position of the mass at $t = 0$ is as shown? $x(t) = A\cos(\omega t + \phi)$

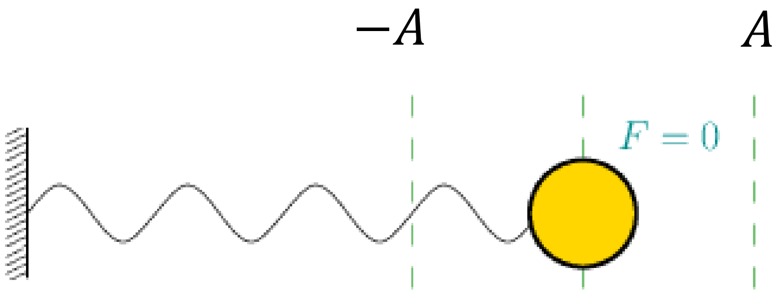
(a)



(b)



(c)



$\phi = ?$

↓

A. 0

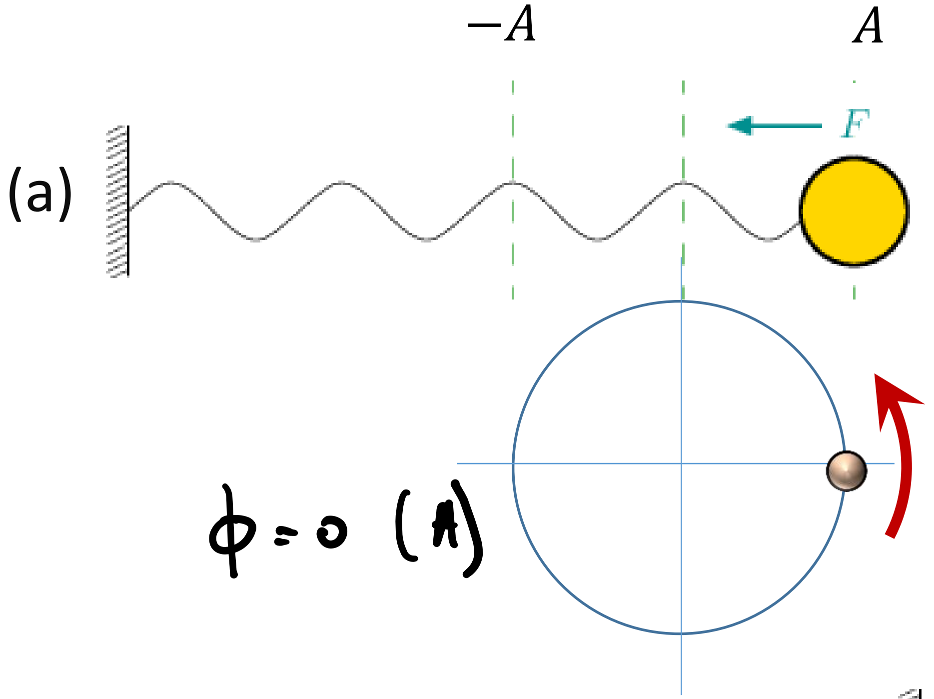
B. $-\frac{\pi}{2}$

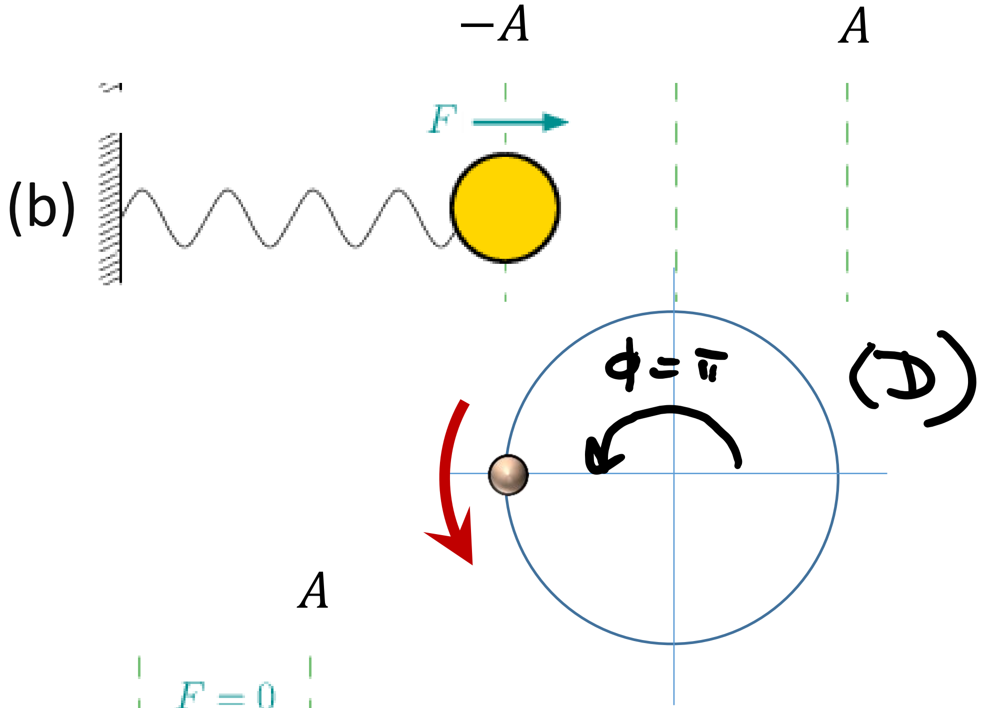
C. $\frac{\pi}{2}$

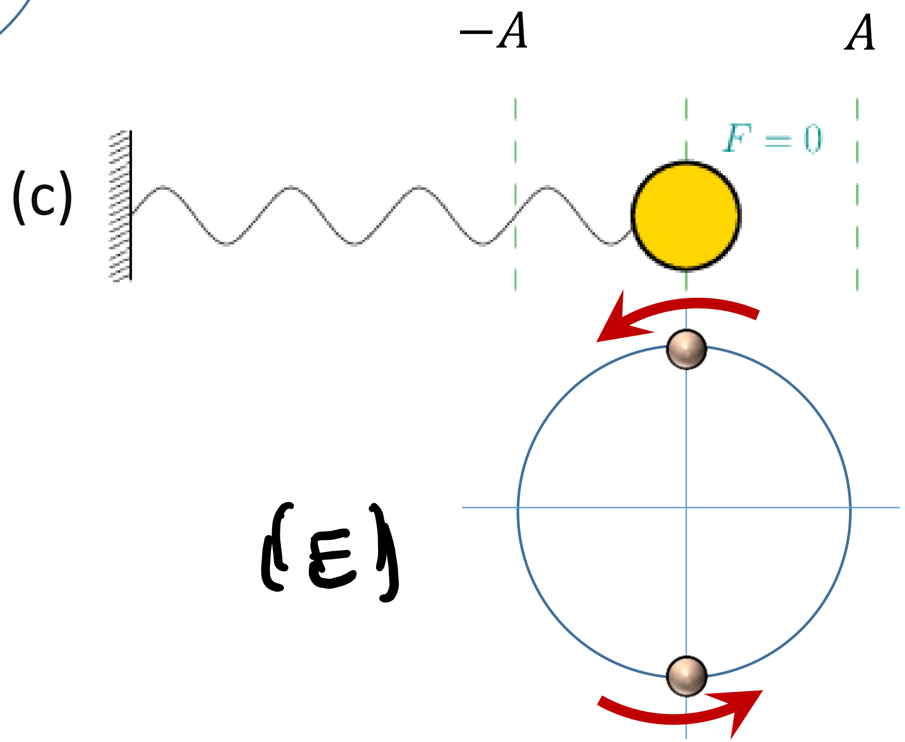
D. π

E. Not enough information

Q: What is the phase, if the position of the mass at $t = 0$ is as shown?







A. 0

B. $-\frac{\pi}{2}$

C. $\frac{\pi}{2}$

D. π

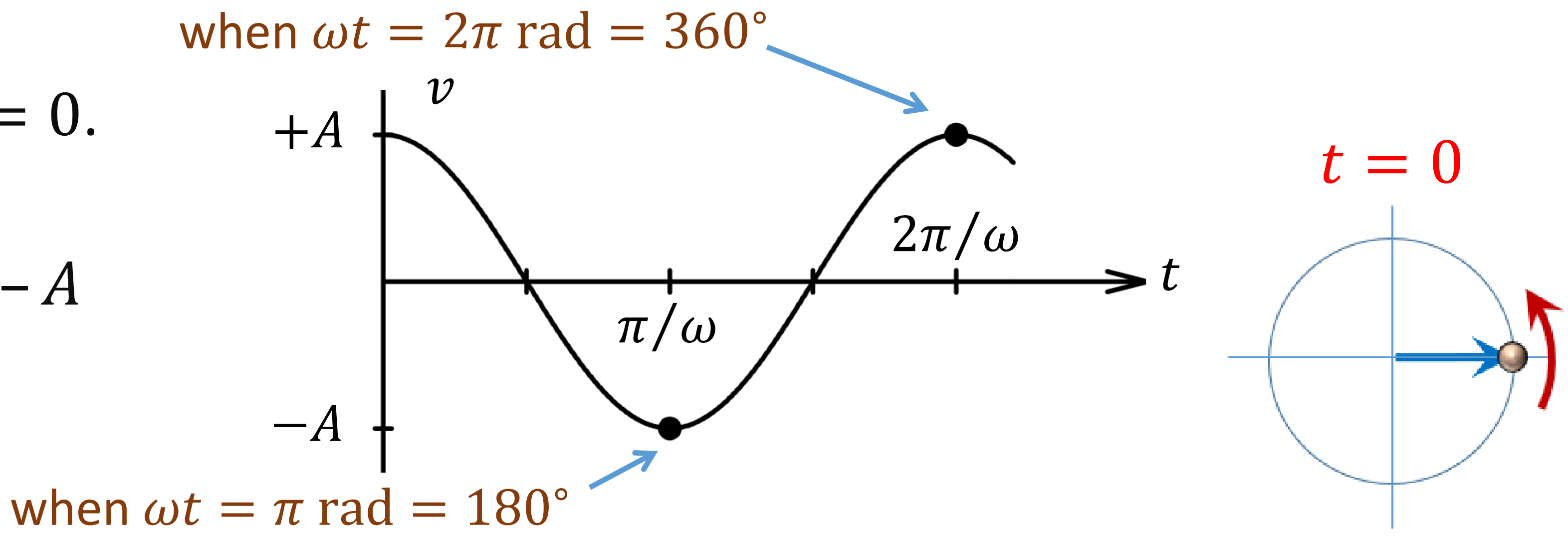
E. Not enough information

We need to know in which direction the particle on the spring is moving at the moment shown!

Phase as an offset

$x(t) = A\cos(\omega t)$

- This graph shows oscillations of a particle starting from $x = A$ at $t = 0$.
- It reaches $x = A$ for the first time when $t = T = 2\pi/\omega$, and is at $-A$ when $t = T/2 = \pi/\omega$.
- Shift of 2π is the whole period.

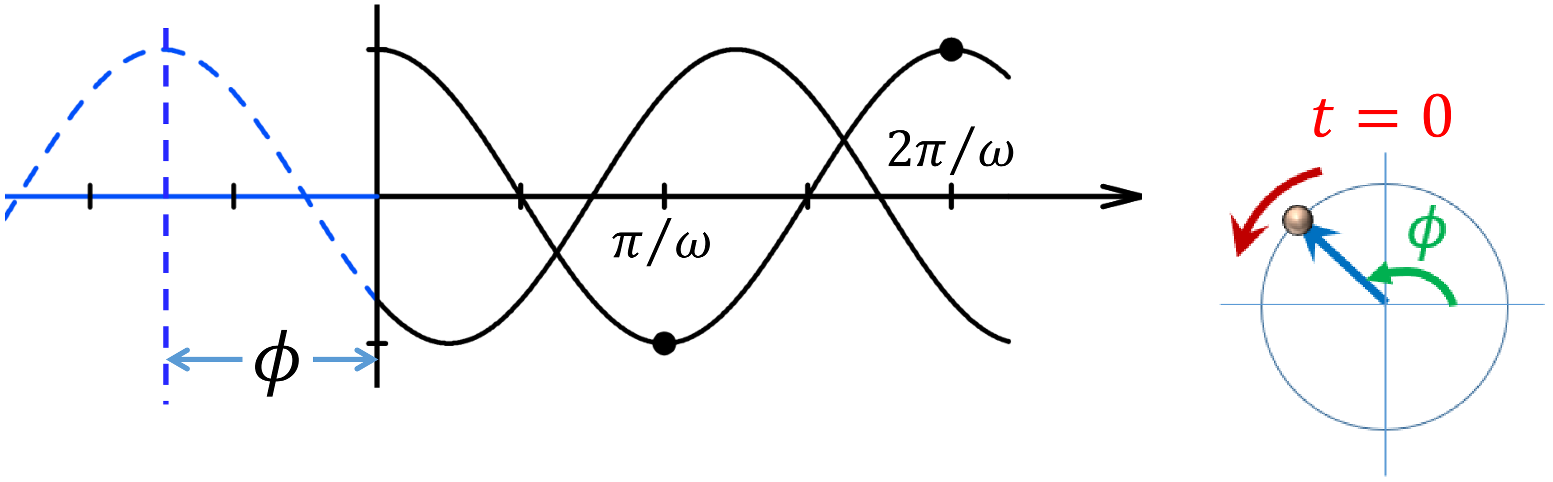


$x(t) = A\cos(\omega t + \phi)$

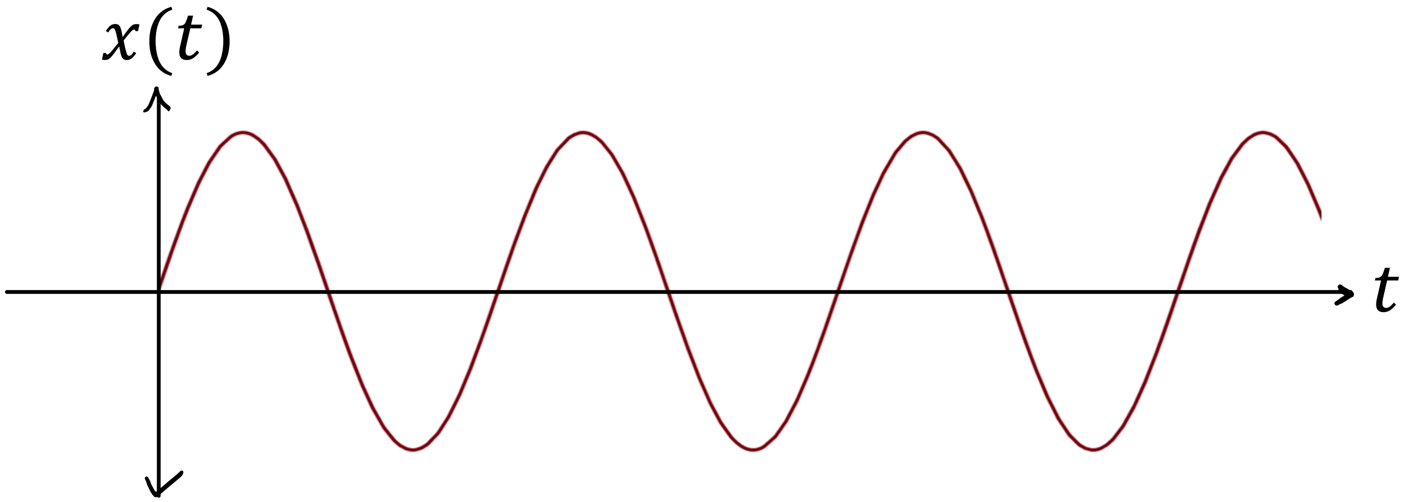
- Including an offset, ϕ:
- Now $x = A$ is reached for the first time at $\omega t + \phi = 0$ =>

  i.e. at $t = -\phi/\omega$ =>

  the graph shifts to the left.

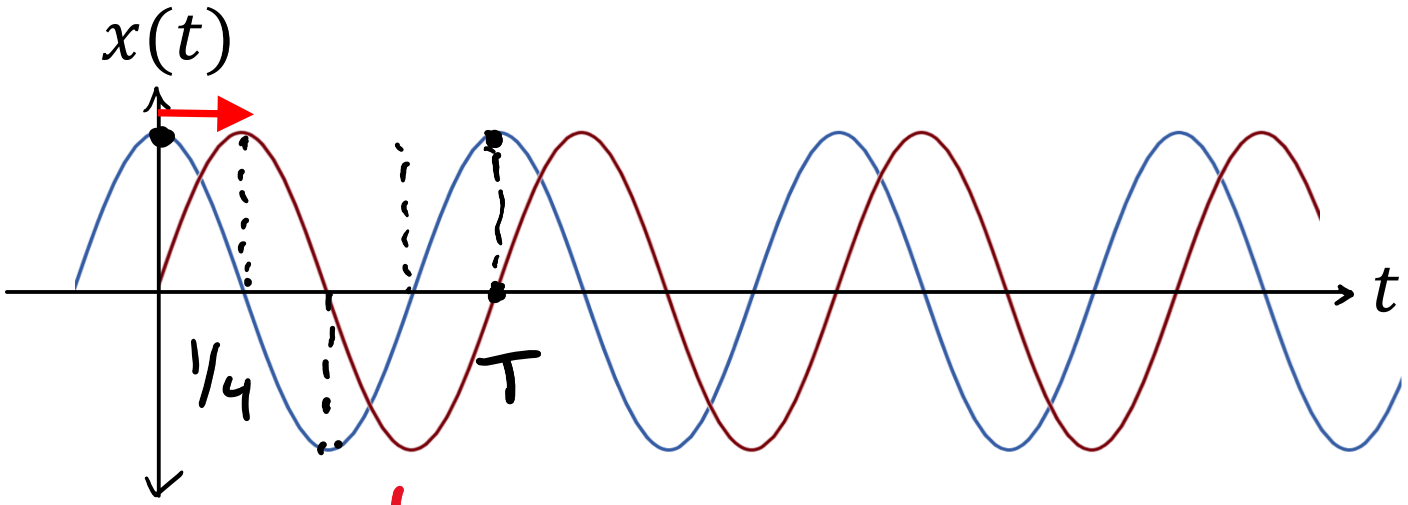


Q: For this displacement graph, what is the phase ϕ? Assume $x(t) = A\cos(\omega t + \phi)$.

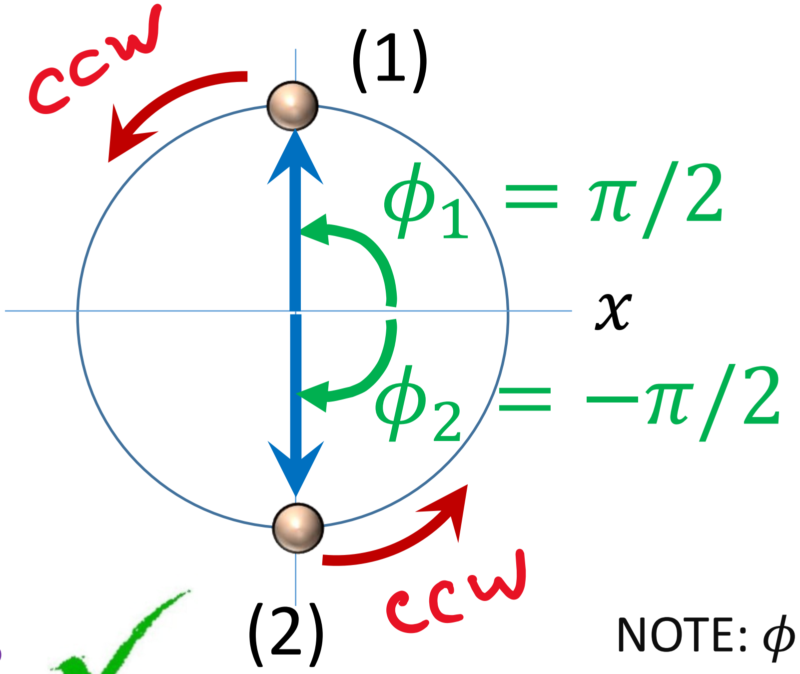


A. 0
B. $\pi/2$
C. π
D. $-\pi/2$

Q: For this displacement graph, what is the phase ϕ? Assume $x(t) = A\cos(\omega t + \phi)$.



- Way 1: positive / negative shifts:
  - Shifts to the right by ¼ period, so $\phi = -\frac{2\pi}{4} = -\frac{\pi}{2}$



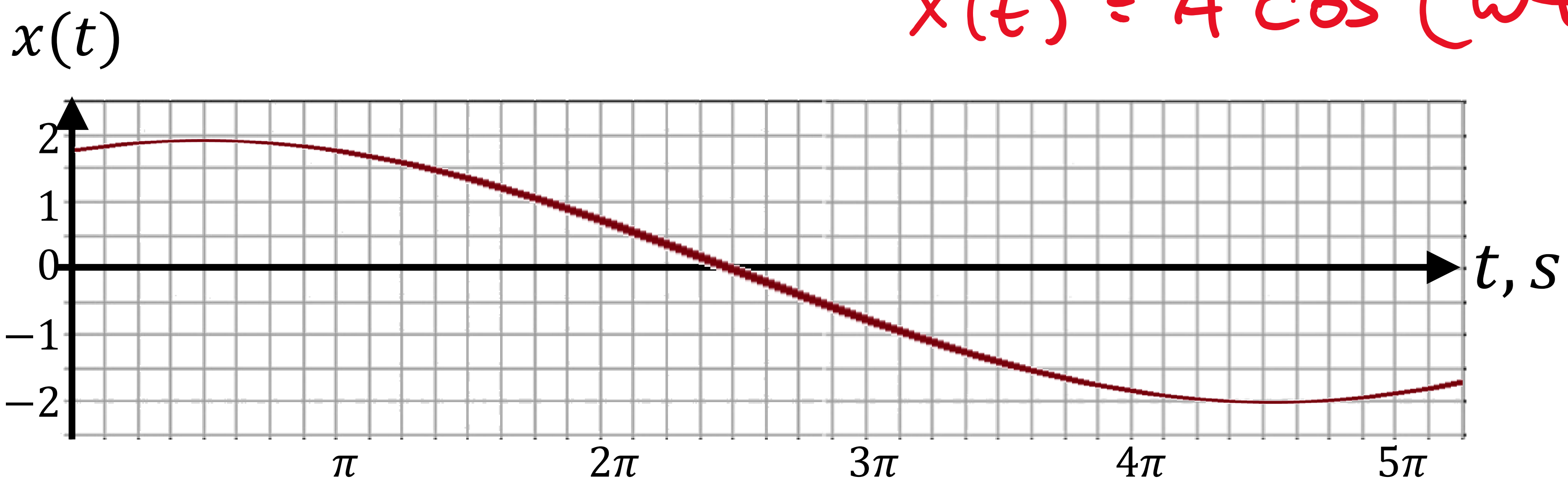
- Way 2: looking at the wheel:
  - At $t = 0$ the particle is at $x = 0$ => two potential locations, (1) and (2)
  - Graph shows: at $t = 0$ the particle moves towards $+x$ => only (2) would work => $\phi = -\frac{\pi}{2}$

A. 0
B. $\pi/2$
C. π
D. $-\pi/2$

NOTE: $\phi = -\frac{\pi}{2} \pm n \times 2\pi$ would also be correct, since adding or subtracting an integer multiple of 2π gives the same displacement vs time

Q: For this displacement graph shown, what is the phase ϕ?

$x(t) = A\cos(\omega t + \phi)$



A. $-\pi/8$

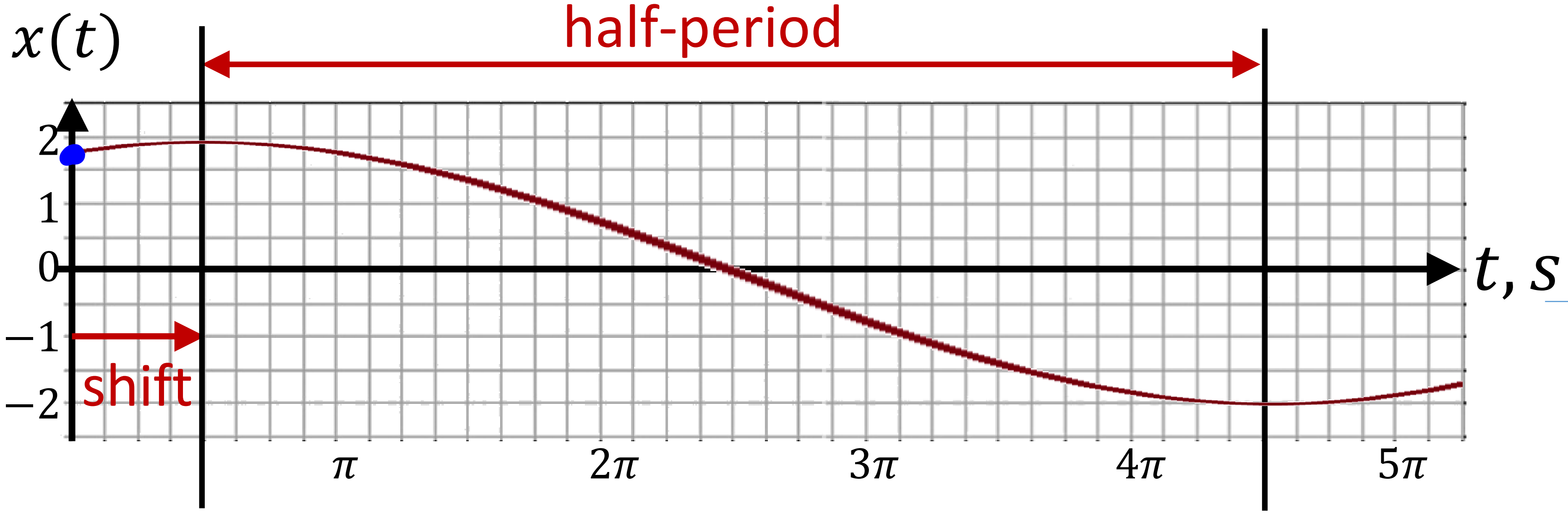
B. $-\pi/4$

C. $-\pi/2$

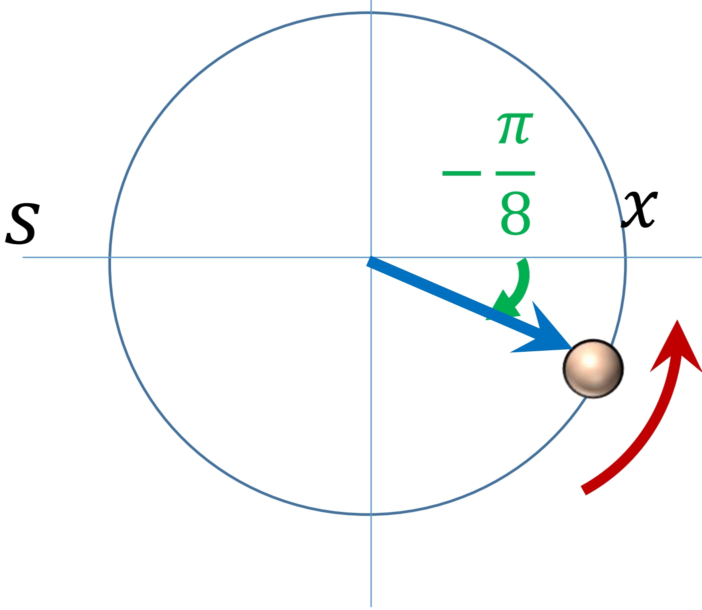
D. $+\pi/4$

E. $+\pi/8$

Q: For this displacement graph shown, what is the phase ϕ?



Check-up:



- Positive /negative shifts

A. $-\pi/8$

B. $-\pi/4$

C. $-\pi/2$

D. $+\pi/4$

E. $+\pi/8$

shift to the right, so ϕ is $-$smth

$x(t) = A\cos(\omega t + \phi)$ $\qquad \phi(rad) = \pm 2\pi \cdot \dfrac{\text{shift (sec)}}{\text{period (sec)}}$

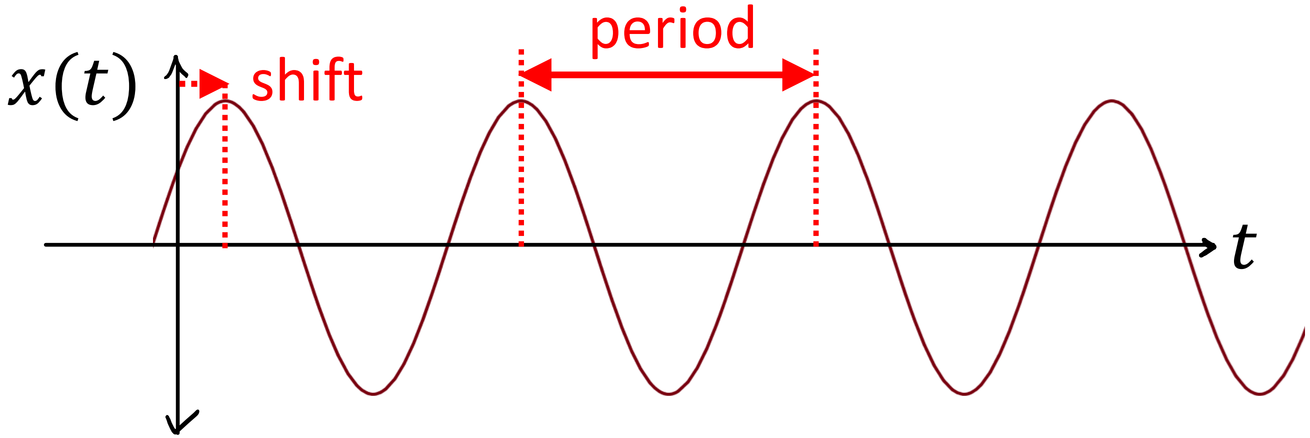
$\dfrac{\phi}{2\pi} = \dfrac{\text{shift(s)}}{\text{period(s)}}$

Period is $T = 2 \cdot \left(\frac{9\pi}{2} - \frac{\pi}{2}\right) = 8\pi$ sec and shift is $\frac{\pi}{2}$ sec

So phase is $\phi = -2\pi \cdot \frac{\text{shift}}{\text{period}} = -2\pi \cdot \frac{\pi/2}{8\pi} = -\frac{\pi}{8}$ rad

How to find ϕ?

- Way 1: positive /negative shifts



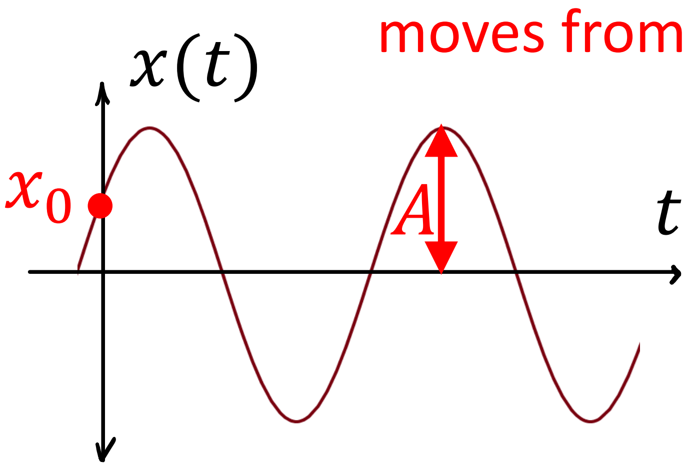
$$\phi(\text{rad}) = \pm 2\pi \cdot \frac{\text{shift (sec)}}{\text{period (sec)}}$$

+ to the left

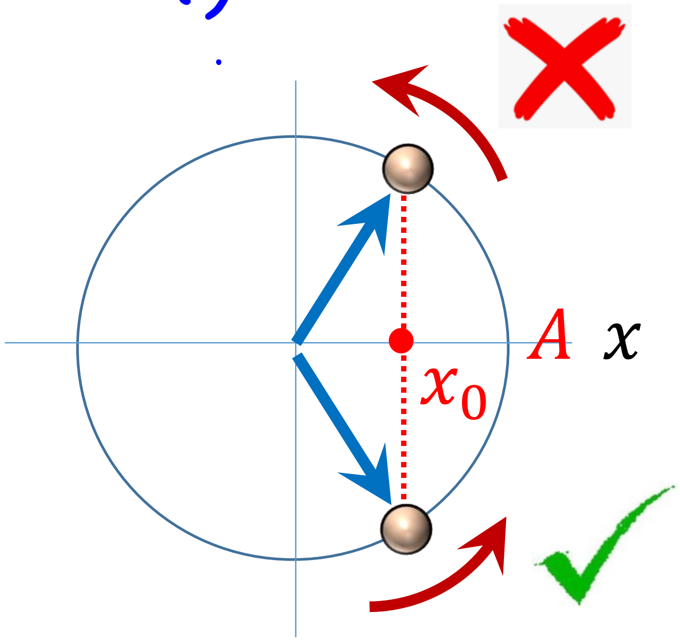
− to the right

$x = A\cos(\omega t + \phi)$

- Way 2: from initial condition:

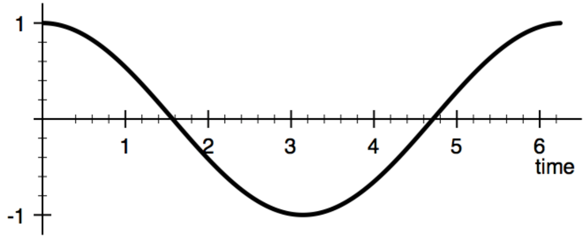


$$\cos\phi = \frac{x_0}{A}$$



How to find $x(t)$, $v(t)$, $a(t)$

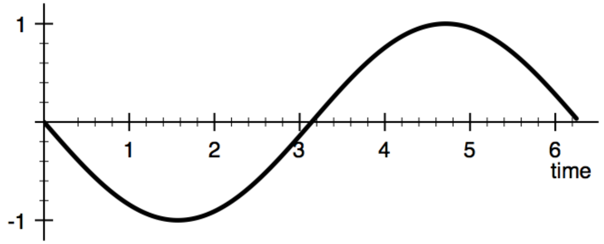
- Position



$$x(t) = A\cos(\omega t + \phi)$$

$\dfrac{d}{dt}$ (slope)

- Velocity

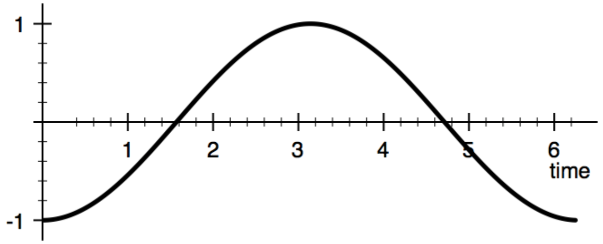


$$v(t) = -A\,\omega\,\sin(\omega t + \phi)$$

chain rule!

$\dfrac{d}{dt}$ (slope)

- Acceleration



$$a(t) = -A\,\omega^2\cos(\omega t + \phi)$$

chain rule!

➢ Note that acceleration depends on time!

Position, velocity, acceleration

- Position:

$$x(t) = \underbrace{A}\cos(\omega t + \phi)$$

- Velocity (time derivative of position):

$$v(t) = -\underbrace{\omega A}_{v_{max}}\sin(\omega t + \phi)$$

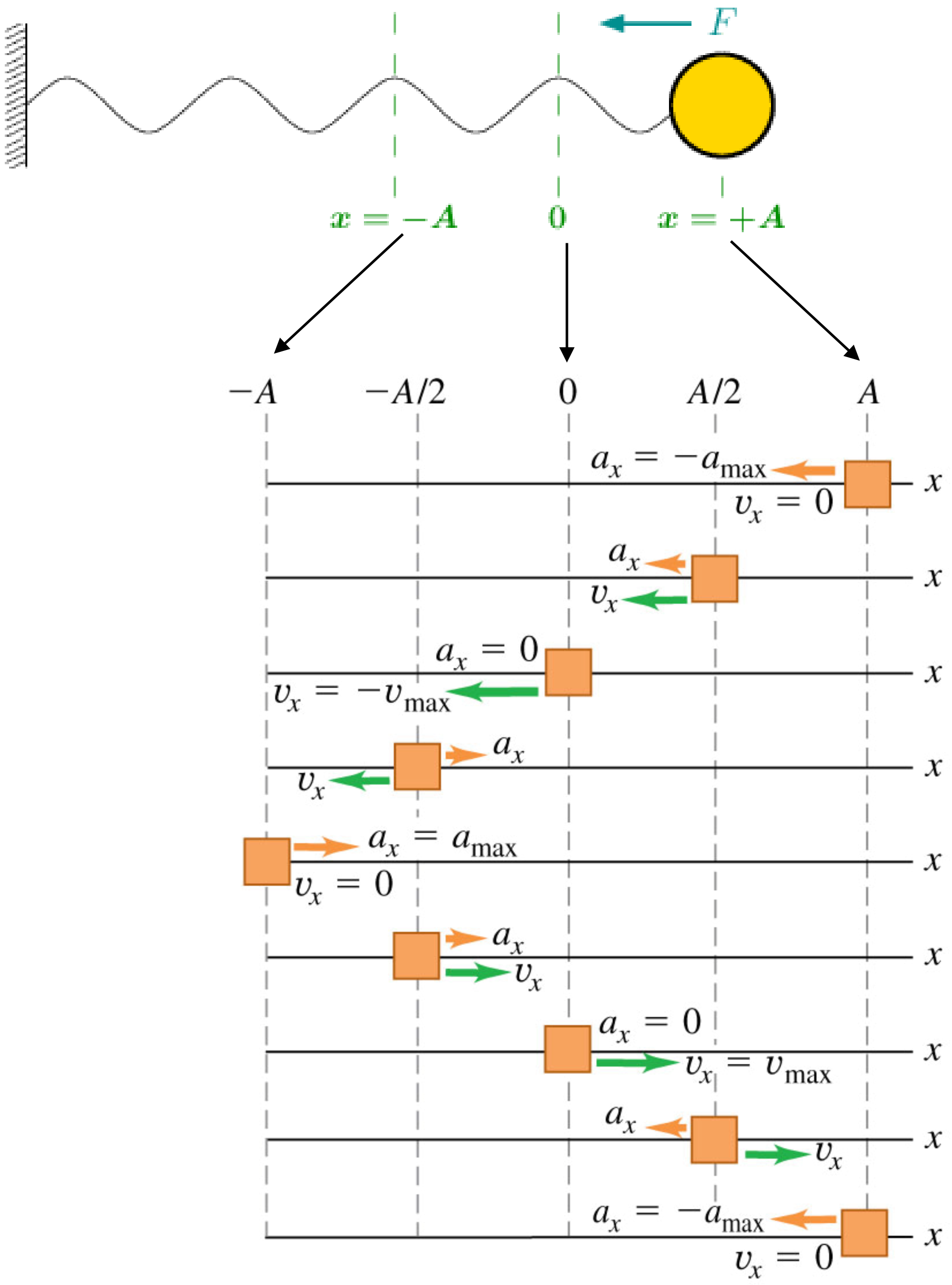
- Acceleration (time derivative of velocity)

$$a(t) = -\underbrace{\omega^2 A}_{a_{max}}\cos(\omega t + \phi)$$

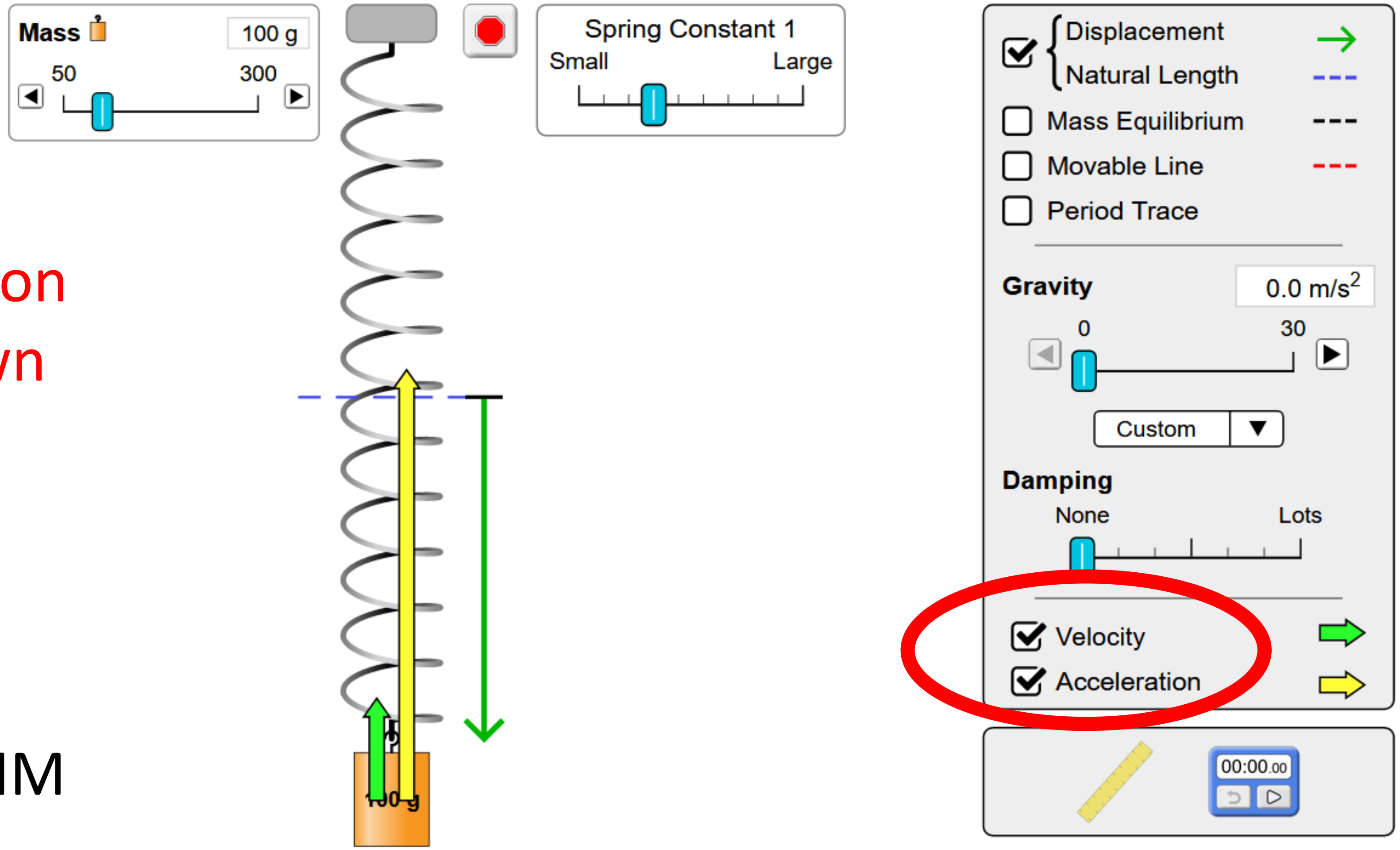
!

➢ Note that $\boxed{a(t) = -\omega^2 x(t)}$.

➢ This is an alternative definition of SHM.



PHET Simulation: Weight on a spring

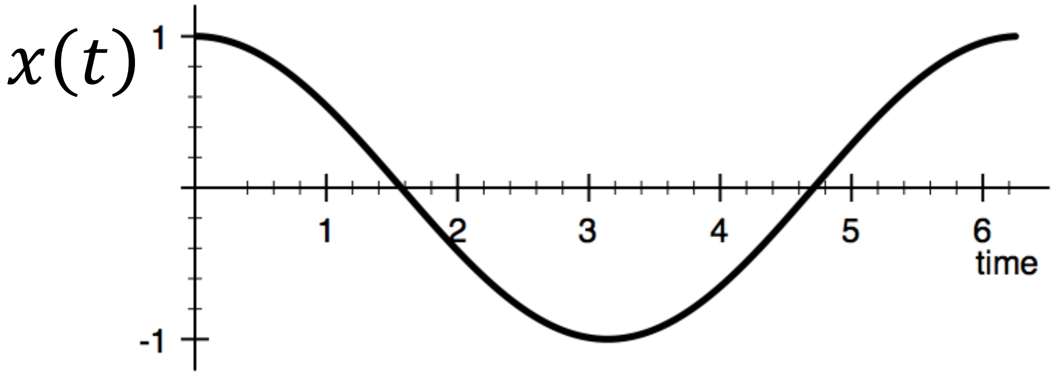


Check it on your own

Vertical SHM

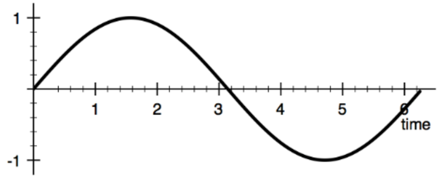
https://phet.colorado.edu/sims/html/masses-and-springs/latest/masses-and-springs_en.html

Q: A plot of displacement as a function of time is shown to the left below. Which of the diagrams to the right describes the velocity as a function of time for the same motion?

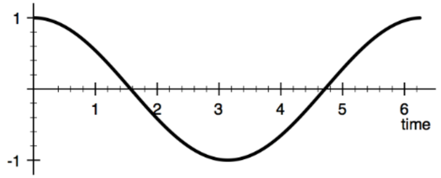


$v(t)$

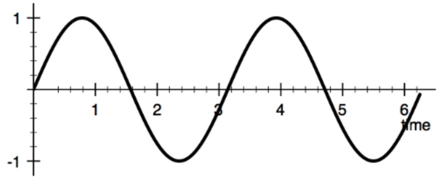
A.



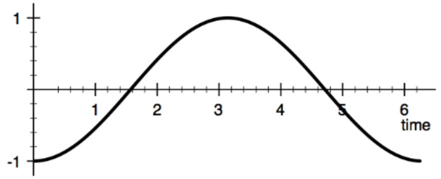
B.



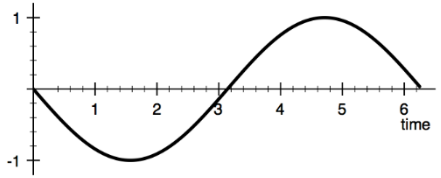
C.



D.



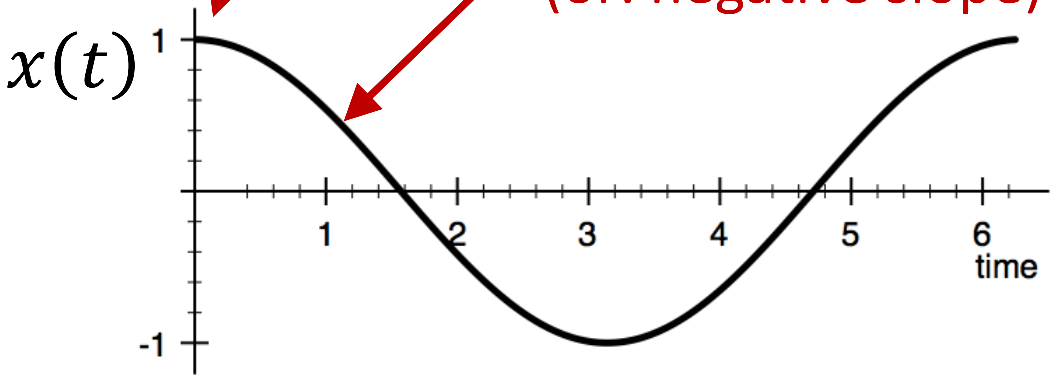
E.



Q: A plot of displacement as a function of time is shown to the left below. Which of the diagrams to the right describes the velocity as a function of time for the same motion?

U-turn, so $v = 0$ (or: zero slope)

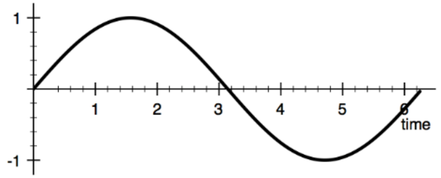
Object moves in $-x$ direction, so $v < 0$ (or: negative slope)



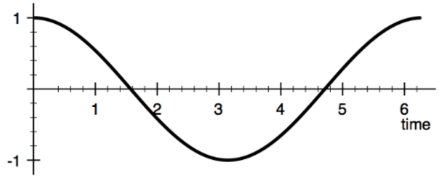
$$\boxed{v = \frac{dx}{dt} = \text{slope of graph}}$$

$v(t)$

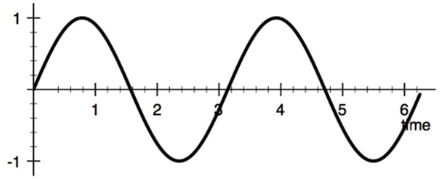
A.



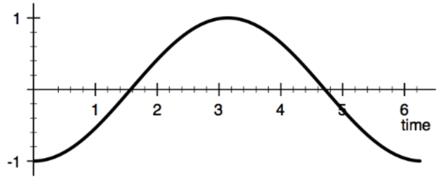
B.



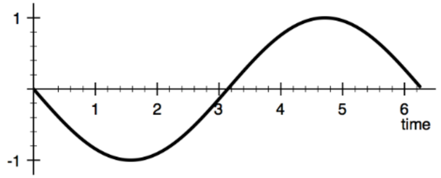
C.



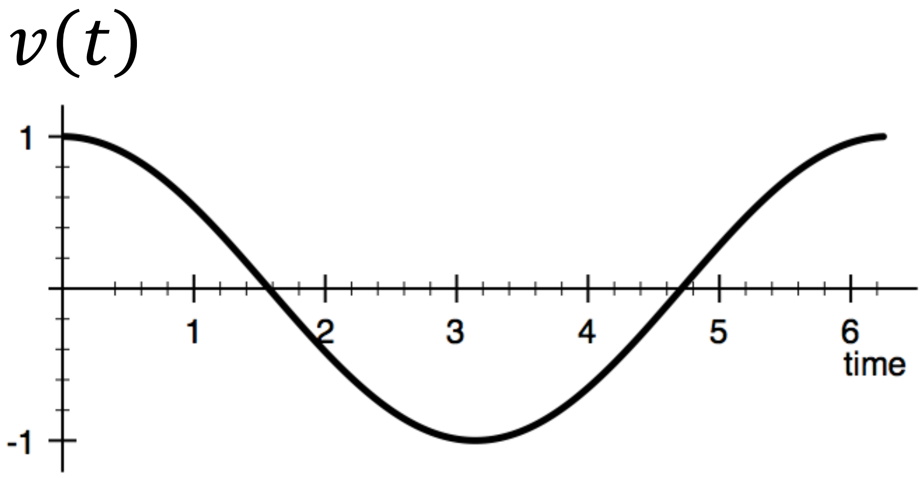
D.

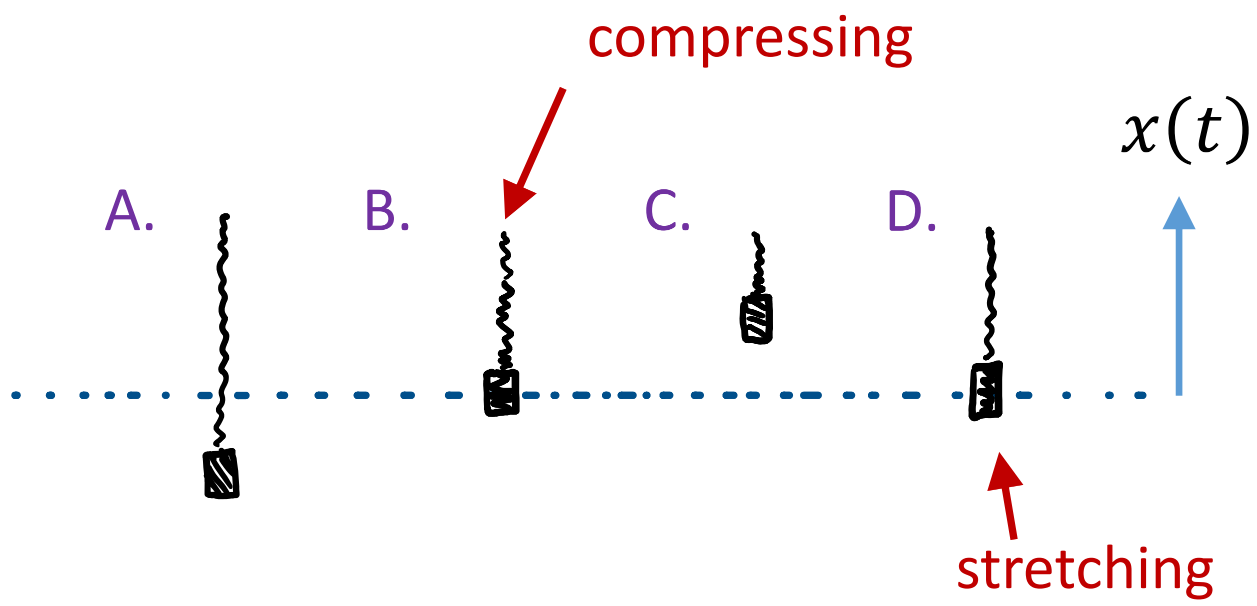


E.



Q: A plot of upward **velocity** (in cm/s) as a function of time (in s) is shown below for a mass hanging from a spring. Which of the pictures best represents the situation at $t = 1.6$ s?



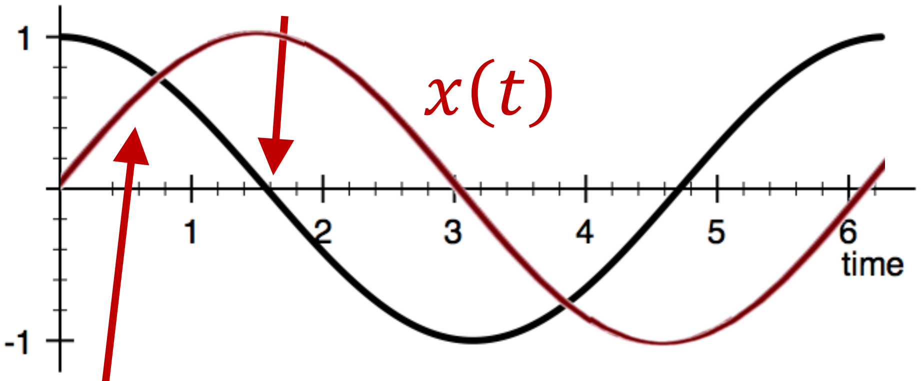


Q: What does $x(t)$ look like?

Q: A plot of upward **velocity** (in cm/s) as a function of time (in s) is shown below for a mass hanging from a spring. Which of the pictures best represents the situation at $t = 1.6$ s?

$v = 0$ here, so must be at highest or lowest point

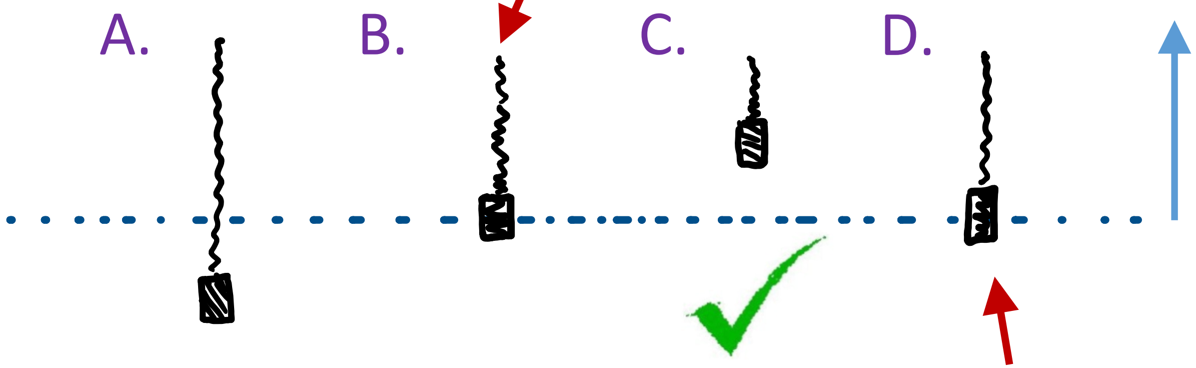
$v(t)$



$x(t)$

compressing

$x(t)$



stretching

Prior to $t = 1.6$, v was positive, so object moving up;

After $t = 1.6$, v will be negative, so object will be moving down

Q: What does $x(t)$ look like?