MATH 101A — ASSIGNMENT 1

Learning goals

- Compute the average value of a given function on a given interval.
- Calculate moving averages for a given time-varying function.
- Interpret integrals involving flow rates in practical terms.
- Confront the Keeling curve and its consequences.

Contributors

On the first page of your submission, list the student numbers and full names (with the last name in bold*) of all team members. Indicate members who have not contributed using the comment "(non-contributing)".*

Assignment questions

The questions in this section contribute to your assignment grade*. Stars indicate the difficulty of the questions, as described on Canvas.*
Parts (a), (b), and (h) of Question 2 require plots. We will accept either careful handmade sketches or computer-generated graphics, as produced by software like Desmos.

The graph below may be the most important one you study all year. It shows the concentration of CO_2 in the atmosphere as a function of time, which has a strong influence on the global climate.

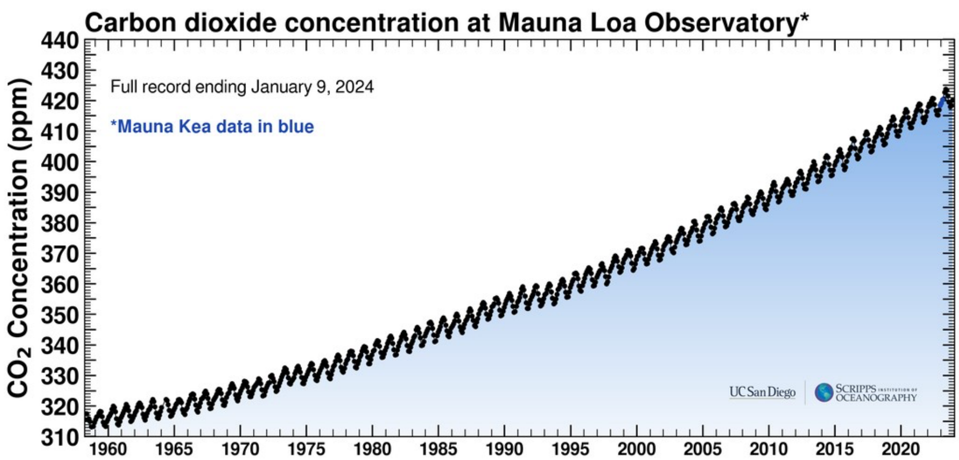


Fig. 1: CO_2 concentrations in recent years

Today's CO_2 levels have never been seen before in human history, and they will cause enormous global changes in the next few decades. Understanding this graph in detail will be an important step in preparing for, perhaps even influencing, the immediate future of humanity and indeed all life on Earth.

2. (12 marks) There is a clear increasing trend in Figure 1, but there are also short-term ripples. To systematically remove the ripples and extract the trend, we can use a *moving average*.

   For any integrable function f and interval $[a, b]$, the *average value of f on* $[a, b]$ is the number f_{AV} defined by

$$f_{\mathrm{AV}} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Rearranging this definition reveals a natural interpretation:

$$\int_a^b f(t)\,dt = (b-a)f_{\text{AV}} = \int_a^b f_{\text{AV}}\,dt.$$

The number f_{AV} is *the unique constant* that produces the same integral as f over $[a, b]$.

(a) ★☆☆☆ Calculate the average value of the function $f(t) = \sin(t)$ on the interval $[0, \pi]$. Then plot the graphs of $y = f(t)$ and $y = f_{\text{AV}}$ on the same set of axes, restricting the sketch to $0 \le t \le \pi$.

Solution:

First, we plug in the values a = 0, b = π and $f(t) = \sin(t)$ into the function for obtaining a moving average,

$$f_{AV} = \frac{1}{\pi-0}\int_0^{\pi} \sin(t)\,dt\ .$$

We then evaluate the integral of $\sin(t)$ with respect to t, and we know that the antiderivative of $\sin(t)$ is $-cos(t)$, this becomes

$$f_{AV} = \frac{1}{\pi}[-cos(t)]_0^{\pi}.$$

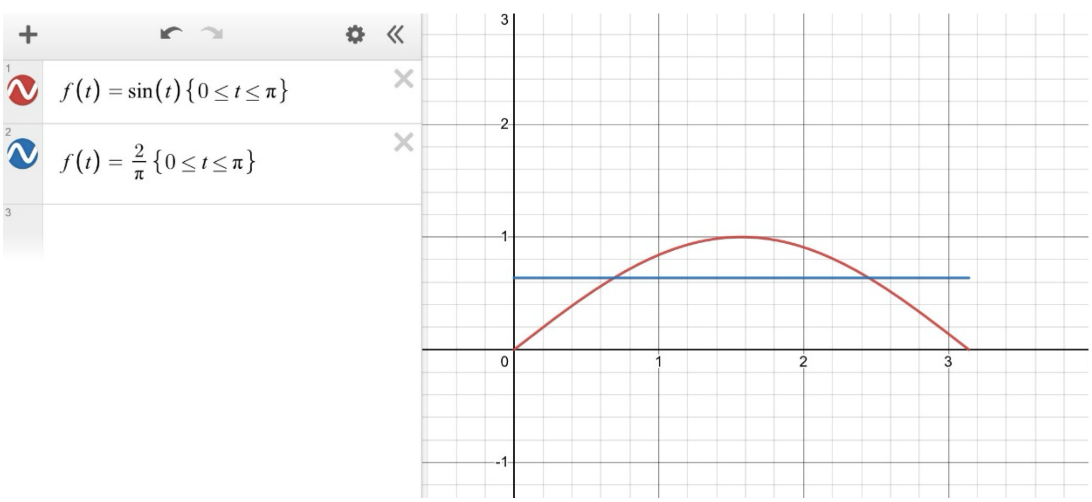
Simplifying further, we get

$$f_{AV} = \frac{1}{\pi}[-cos(\pi) - (-cos(0))],$$

$$f_{AV} = \frac{1}{\pi}[1+1],$$

$$f_{AV} = \frac{2}{\pi}.$$

With the obtained functions, we can plot the graph below.



(b) ★☆☆☆ Calculate the average value of the function $f(t) = t^2$ on the interval [1,2]. Then plot the graphs of $y = f(t)$ and $y = f_{AV}$ on the same set of axes, restricting the sketch to $1 \le t \le 2$.

Solution:

First, we plug in the values a = 1, b = 2 and $f(t) = t^2$ into the function for obtaining a moving average,

$$f_{AV} = \frac{1}{2-1}\int_1^2 t^2\, dt\ .$$

We then evaluate the integral of t^2 with respect to t, and we know that the antiderivative of t^2 is $\frac{1}{3}t^3$, this becomes

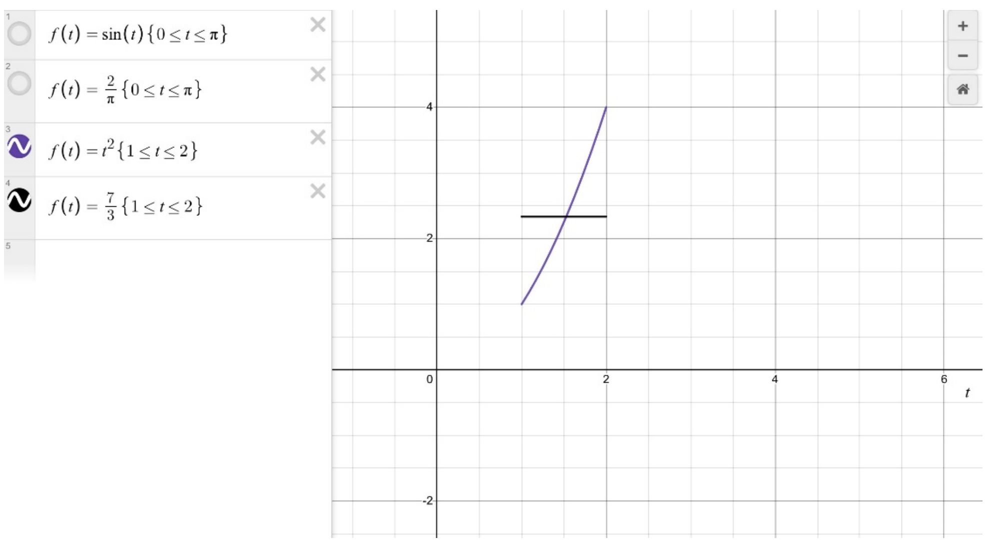
$$f_{AV} = [\frac{1}{3}t^3]_1^2.$$

Simplifying further, we get

$$f_{AV} = \frac{1}{3}[2^3 - 1^3],$$

$$f_{AV} = \frac{7}{3}.$$

With the obtained functions, we can plot the graph below.



Now imagine that some time-varying function f is given. If we pick some number $r > 0$, we can imagine the interval $[t - r, t]$ as a moving window of width r and final time t. Calculating the average value of f over this window will produce a result that depends on t, thus defining a new function $\bar{f}$ as follows:

$$\bar{f}(t) = \frac{1}{r}\int_{t-r}^{t} f(x)\, dx.$$

Let's investigate the relationship between f and $\bar{f}$ in various situations.

(c) ★☆☆☆ Let $f_0(t)= 1$, $f_1(t)= t$, and $f_2(t)= t^2$. Find $\bar{f}_0(t)$, $\bar{f}_1(t)$, and $\bar{f}_2(t)$. (The results may depend on r.)

Solution:

For $f_0(t)$= 1, the corresponding $\bar{f}_0(t)$ will be,

$$\bar{f}_0(t) = \frac{1}{r}\int_{t-r}^{t} 1 \; dx.$$

We also know that the antiderivative of 1 is x, so

$$\bar{f}_0(t) = \frac{1}{r}[x]_{t-r}^{t}.$$

Simplifying the expression, we get

$$\bar{f}_0(t) = \frac{1}{r}[\text{t} - (\text{t-r})],$$

$$\bar{f}_0(t) = 1.$$

For $f_1(t)= t$, the corresponding $\bar{f}_1(t)$ will be,

$$\bar{f}_1(t) = \frac{1}{r}\int_{t-r}^{t} x \; dx.$$

We also know that the antiderivative of x is $\frac{1}{2}x^2$, so

$$\bar{f}_1(t) = \frac{1}{r}[\frac{1}{2}x^2]_{t-r}^{t}.$$

Simplifying the expression, we get

$$\bar{f}_1(t) = \frac{1}{2r}[t^2 - (t-r)^2],$$

$$\bar{f}_1(t)= \frac{1}{2r}[2tr - r^2],$$

$$\bar{f}_1(t)= t - \frac{r}{2}.$$

For $f_2(t)= t^2$, the corresponding $\bar{f}_2(t)$ will be,

$$\bar{f}_1(t) = \frac{1}{r}\int_{t-r}^{t} x^2 \, dx.$$

We also know that the antiderivative of x^2 is $\frac{1}{3}x^3$, so

$$\bar{f}_1(t) = \frac{1}{r}[\frac{1}{3}x^3]_{t-r}^{t}.$$

Simplifying the expression, we get

$$\bar{f}_2(t) = \frac{1}{3r}[t^3 - (t-r)^3],$$

$$\bar{f}_2(t) = \frac{1}{3r}[-(3\text{t}r^2 - 3t^2r - r^3)],$$

$$\bar{f}_2(t) = t^2 - tr + \frac{r^2}{3}.$$

(d) ★☆☆☆ Express $\bar{f}(t)$ in terms of r for the general quadratic $f(t)= at^2+ bt + c$, where *a, b, c* are real constants.

Solution:

Writing $\bar{f}(t)$ in terms of r for the general quadratic $f(t)= at^2+ bt + c$,

$$\bar{f}(t) = \frac{1}{r}\int_{t-r}^{t} ax^2 + bx + c \; dx.$$

We know that the antiderivative of $ax^2 + bx + c$ is $\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$, this gives

$$\bar{f}(t) = \frac{1}{r}[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx]_{t-r}^{t}.$$

Simplifying the expression, we get

$$\bar{f}(t) = \frac{1}{r}\left[\left(\frac{a}{3}t^3 + \frac{b}{2}t^2 + ct\right) - \left(\frac{a}{3}(t-r)^3 + \frac{b}{2}(t-r)^2 + c(t-r)\right)\right].$$

We then further expand and simplify the expression,

$$\bar{f}(t) = \frac{1}{r}\left[-\left(\frac{a}{3}(3tr^2 - 3t^2r - r^3) + \frac{b}{2}(-2tr + r^2) + cr\right)\right].$$

Rearranging this, we arrive at,

$$\bar{f}(t) = a(t^2 - tr + \frac{r^2}{3}) + b(t - \frac{r}{2}) + c.$$

(e) ★☆☆☆ Given a constant $\omega > 0$, let $g(t)= \cos(\omega t)$. Find $\bar{g}(t)$ in terms of r (and, of course, ω).

Solution:

Consider $g(t)= \cos(\omega t)$, the corresponding $\bar{g}(t)$ is expressed as,

$$\bar{g}(t)= \frac{1}{r}\int_{t-r}^{t} cos(\omega x)\ dx.$$

We know that the antiderivative of $\cos(\omega x)$ is $\sin(\omega x)$, *so*

$$\bar{g}(t)= \frac{1}{r}[sin(\omega x)]_{t-r}^{t}.$$

Simplifying the expression, we get

$$\bar{g}(t)= \frac{1}{\omega r}[sin(\omega t) - sin(\omega t - \omega r)].$$

(f) ★★★☆ It's reasonable to expect that a moving average with a very short window should produce a new function very close to the original one, provided the original function is continuous. Test this expectation on the functions from parts (d) and (e) above by taking the limit as $r \to 0^+$ in your expressions for $\bar{f}(t)$ and $\bar{g}(t)$.

Solution:

When taking the limit, you will get the original function since the formula mirrors the fundamental theorem of calculus. This is shown here,

$$\bar{f}(t) = at^2 - atr + \frac{ar^2}{3} + bt - \frac{br}{2} + c,$$

$$\lim_{r\to 0^+} \bar{f}(t) = at^2 + bt + c.$$

Next, we can carry this operation out with

$$\bar{g}(t)= \frac{1}{\omega r}[sin(\omega t) - sin(\omega t - \omega r)].$$

$$\lim_{r\to 0^+} \bar{g}(t) = \lim_{r\to 0^+}\left[\frac{\frac{d}{dr}sin(\omega t) - sin(\omega t - \omega r)}{\frac{d}{dr}(\omega r)}\right],$$

$$\lim_{r\to 0^+} \bar{g}(t) = \lim_{r\to 0^+}\left[\frac{\omega\cos(\omega(t-r))}{\omega}\right],$$

$$\lim_{r\to 0^+} \bar{g}(t) = \cos(\omega t).$$

(g) ★★☆☆ For a generic continuous function h, find an algebraic formula involving r for $\bar{h}'(t)$.

Solution:

Let the generic continuous function be h(x) and plug it into the general function for calculating its average value, we get

$$\bar{h}(t) = \frac{1}{r}\int_{t-r}^{t} h(x)dx.$$

Differentiating the expression with respect to t and simplifying it, we arrive at

$$\bar{h}'(t) = \frac{1}{r}[h(t)\cdot t' - h(t-r)\cdot(t-r)'],$$

$$\bar{h}'(t) = \frac{1}{r}[h(t) - h(t-r)].$$

(h) ★☆☆☆ For this part only, define $g(t)$= $\cos(2\pi t)$ and use $r = \frac{1}{2}$ to define $\bar{g}(t)$. Plot both $y = g(t)$ and $y = \bar{g}(t)$ for $0 \le t \le 4$ on the same axes.

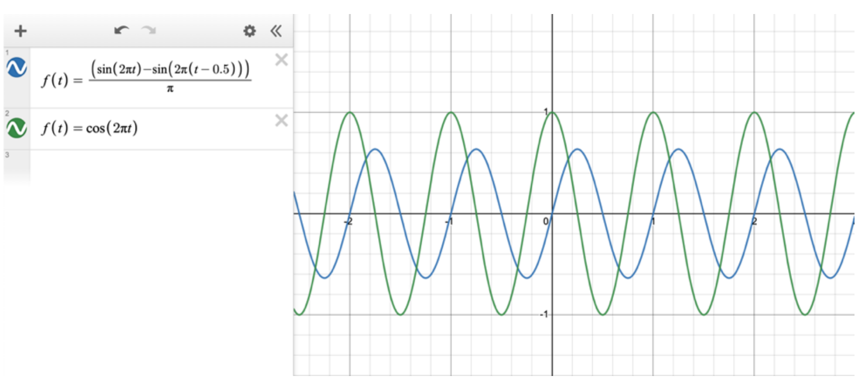
When $g(t) = \cos(2\pi t)$ and if $r = \frac{1}{2}$, we get

$$\bar{g}(t) = \frac{1}{0.5}\int_{t-0.5}^{t} \cos(2\pi t)dt\,.$$

Integrating this expression and simplifying it gives us

$$\bar{g}(t) = \frac{[\sin(2\pi t) - \sin(2\pi(t-0.5))]}{\pi}.$$

From this, we can plot the following graph.

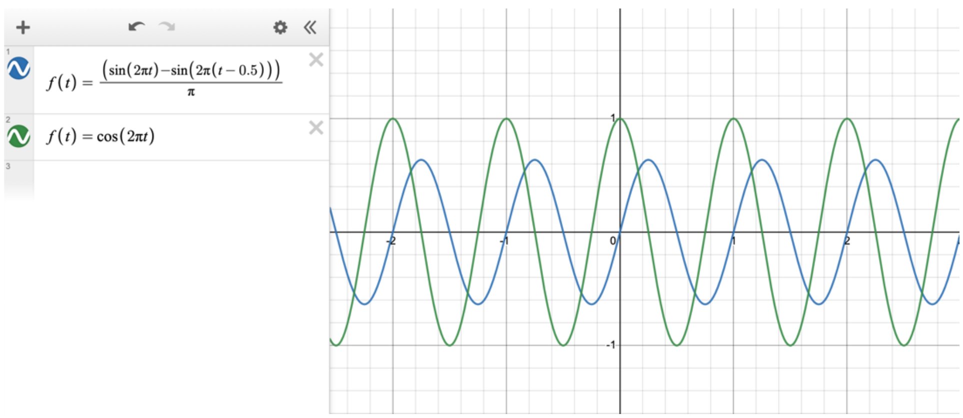


(i) ★★☆☆ Look at the maximum and minimum values on your plot in part (h). Notice that:

1. the extreme values of $\bar{g}(t)$ *occur later* than the extreme values of $g(t)$, and
2. the extreme values of $\bar{g}(t)$ *are smaller in magnitude* than the extreme values of $g(t)$.

Write brief intuitive explanations of these two facts. (*Note*: "That's what the calculations produce" is simultaneously (i) logically correct, and (ii) a dreadful intuitive explanation.)

Solution:



This graph shows the blue graph, denoted as $\bar{g}(t)$, calculates the mean value from the interval between t and t-r, and the green graph, which is g(t). The peak of $g(t)$ is larger than the peak of $\bar{g}(t)$, which logically makes sense, since the average value encompasses the peak at point t to point t-r. Overall, the graph of $\bar{g}(t)$ still has the same period as g(t), but the maximums and minimums are shifted to the left because of its sinusoidal nature. (compared to the cosine nature of g(t), which is the sin graph shifted to the left by half a period).

(j) ★★★☆ Extend your observations in part (f) above to explain why taking the limit as $r \to 0^+$ in the definition of $\bar{h}(t)$ will recover the original value of $h(t)$, for any given continuous function h. (*Hint*: This question is *fundamental*. You have confronted it before.)

Solution:

Let's say that $h(t)$ is the original and let's say that $H(t)$ is the antiderivative of this function and $\bar{h}(t) = \frac{1}{r}\int_{t-r}^{t} h(x)dx$ this is essentially the fundamental theorem of calculus.

$$\lim_{r\to 0^+} \bar{h}(t) = \lim_{r\to 0^+} \frac{d}{dr}\left(\frac{\int_{t-r}^{t} h(t)dt}{r}\right),$$

$$= \lim_{r\to 0^+} \frac{\frac{d}{dr}(H(t)-H(t-r))}{\frac{d}{dr}r},$$

$$= \lim_{r\to 0^+} -h(t-r)(-1),$$

$$= h(t).$$

(k) ★★☆☆ When $g(t) = \cos(\omega t)$ as above, some special values of $r > 0$ are extremely effective at removing ripples. Find all $r > 0$ for which $\bar{g}(t)$ is a constant function. (Expect the answers to involve ω.)

Solution:

For the function $g(t) = \cos(\omega t)$ which simplifies to $\bar{g}(t) = \frac{[\sin(\omega t) - \sin(\omega(t-r))]}{r\omega}$.

For this function this also by definition means that $\bar{g}'(t) = 0$. Taking this, we can differentiate $\bar{g}(t)$ to solve for r.

$$\bar{g}'(t) = \frac{[\omega\cos(\omega t) - \omega\cos(\omega(t-r))]}{r\omega} = \frac{[\cos(\omega t) - \cos(\omega(t-r))]}{r} = 0.$$

For the expression to be equal to zero,

$$\cos(\omega t) - \cos(\omega(t-r)) = 0.$$

Using trigonometric identities to simplify the expression, we get

$$2\sin(\frac{\omega r}{2})\sin(\omega t - \omega t + \frac{\omega r}{2}) = 0.$$

$$2\sin(\frac{\omega r}{2})\sin(\frac{\omega r}{2}) = 0$$

For the expression to be true, $\sin(\frac{\omega r}{2})$ must be equal to 0.

This implies that, for all positive real numbers k,

$$\frac{\omega r}{2} = \frac{k\pi}{2}$$

From this, we can get

$$r = \frac{2k\pi}{\omega} \text{ (where } k \in \mathrm{R}).$$

3. (5 marks) Figure 1 shows that the CO_2 concentration in the atmosphere varies from season to season. Photosynthesis helps explain this: there is more land, and therefore more plant material, in Earth's northern hemisphere than in the south. When the northern hemisphere is having summer, plants absorb CO_2 and turn it into tissue. When it is winter in the north, the plants slow down their absorption and the humans turn up their emissions. It's summer in the south, but there aren't enough plants there to compensate for all this and keep the absorption rate steady.

Suppose $M(t)$ denotes the total mass of CO_2 in the atmosphere at time t, measuring time t in years from the usual Year 0, and measuring mass in kg. Then we can write

$$\frac{dM}{dt} = i(t) - o(t),$$

where $i(t)$ is the rate at which CO_2 is added to the atmosphere due to respiration and emissions ["influx"] and $o(t)$ is the rate at which CO_2 leaves the atmosphere due to photosynthesis and other effects ["outflux"].

(a) ★☆☆☆ Let $t_0 \geq 0$ and $t_1 > t_0$ represent two different points in time. Write a brief description, including units, for the physical meaning of the integrals shown below:

- $\int_{t0}^{t1} i(t)dt$,
- $\int_{t0}^{t1} (i(t) - o(t))\, dt$.

Solution:

The integral $\int_{t0}^{t1} i(t)dt$ calculates the total amount of CO_2 added to the atmosphere (influx) between two points in time, t_0 and t_1. The units would be in kilograms of CO_2, as $i(t)$ represents the rate at which CO2 is added per unit time.

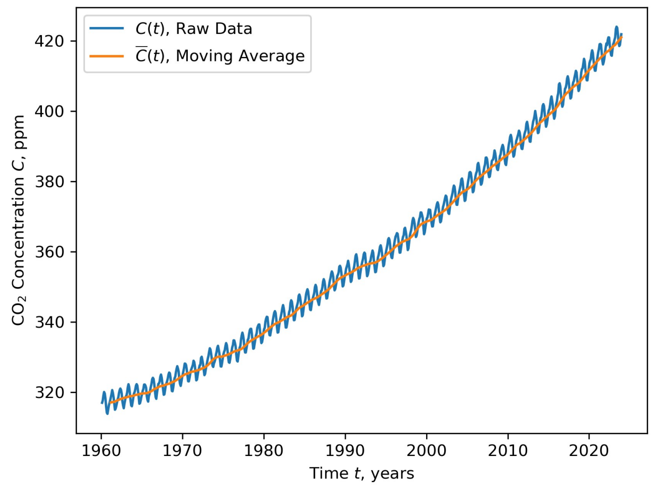
The integral $\int_{t0}^{t1} (i(t) - o(t))\, dt$ gives the net change in atmospheric CO_2 mass over the time interval [t0, t1]. It accounts for both the influx $i(t)$ and the outflux $o(t)$ of CO_2, providing the cumulative effect on CO2 mass for that period. The units remain in kilograms.

The total mass of the atmosphere changes slowly enough to be treated as a constant, M_0. Define $C(t) = 10^6 M(t)/M_0$: since both $M(t)$ and M_0 are measured in kg, their ratio is a pure number with no units. Multiplying that ratio by 10^6 produces numbers of convenient sizes and explains the description of $C(t)$ as a concentration in "parts per million (ppm)". This is the function plotted in Figure 1.

If the ripples in Figure 1 are really caused by the seasons, their influence might have a form like $\cos(\omega t)$, with ω chosen so that a full cycle takes exactly one year. This calls for $\omega = 2\pi$ (with units y^{-1}). With the results of problem 2(k) above in mind, we choose $r = 2\pi/\omega = 1$ (units: y) and introduce

$$\overline{C}(t) = \frac{1}{r}\int_{t-r}^{t} C(x)\,dx = \int_{t-1}^{t} C(x)\,dx.$$

The accurate measurements plotted in Fig. 1 are freely available online. They are tabulated monthly, so, anyone can use the Trapezoidal Rule with 12 subintervals to calculate a good approximation for $\bar{C}(t)$ at each measurement time t. The next plot shows the graphs of the measured function $C(t)$ and the computed values of $\bar{C}$(t).



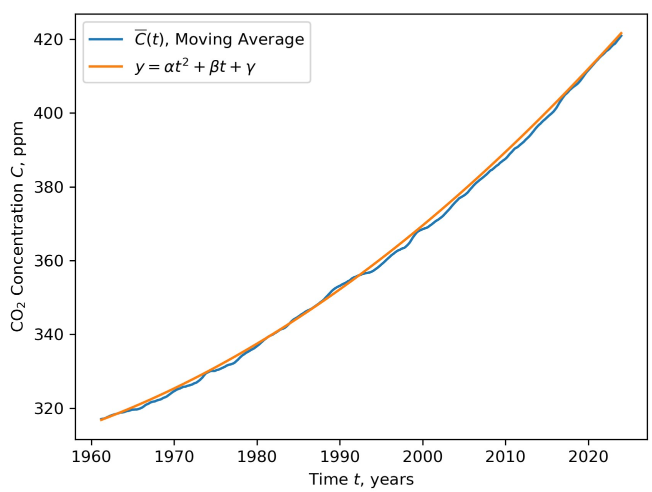
The curve $y = \bar{C}(t)$ looks not quite linear, so we look for a quadratic approximation

$$\bar{C}(t) \approx y(t), \qquad \text{for } y = \alpha t^2 + \beta t + \gamma.$$

Standard methods out of scope for this assignment provide the constants

$$\alpha = 0.012997, \qquad \beta = -50.125, \qquad \gamma = 48631.$$

Here is a plot showing the target curve and approximating parabola defined above:



(b) ★★☆☆ Consider an idealized function $C_{101}(t)$ with the form below, where all the parameters except t are constants:

$$C_{101}(t) = at^2 + bt + c + A\cos(2\pi t + \phi).$$

We want to choose these constants so that the moving average $\bar{C}_{101}(t)$ is identical to the approximating parabola $y = \alpha t^2 + \beta t + \gamma$ plotted above. Use this criterion to find a, b, and c to 5-digit accuracy.

(Hint: Our choice of $r = 2\pi/\omega = 1$ makes the function $\bar{C}_{101}(t)$ independent of the cosine term. This is a small extension of Problem 2(k): you don't have to prove it. So, for this part only, it's valid to calculate as if $A = 0$ in $C_{101}(t)$.)

Solution:

Rewriting the function $C_{101}(t)$ by taking $A = 0$ in $C_{101}(t)$ gives

$$C_{101}(t) = at^2 + bt + c.$$

From the information above, we can establish the following equation,

$$\bar{C}_{101}(t) = \frac{1}{1} \cdot \int_{t-1}^{t} ax^2 + bx + c \;\, dx = \alpha t^2 + \beta t + \gamma.$$

We can first evaluate the integral. Referencing the answer from 2d,

$$\frac{1}{r}\int_{t-r}^{t} ax^2 + bx + c \; dx = a\left(t^2 - tr + \frac{r^2}{3}\right) + b\left(t - \frac{r}{2}\right) + c,$$

where r, in this case, is 1. So, the above integral can be simplified to,

$$a\left(t^2 - t + \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right) + c.$$

Rearranging it to match the form on the right side, we get

$$at^2 + (b-a)t + \left(\frac{a}{3} - \frac{b}{2} + c\right).$$

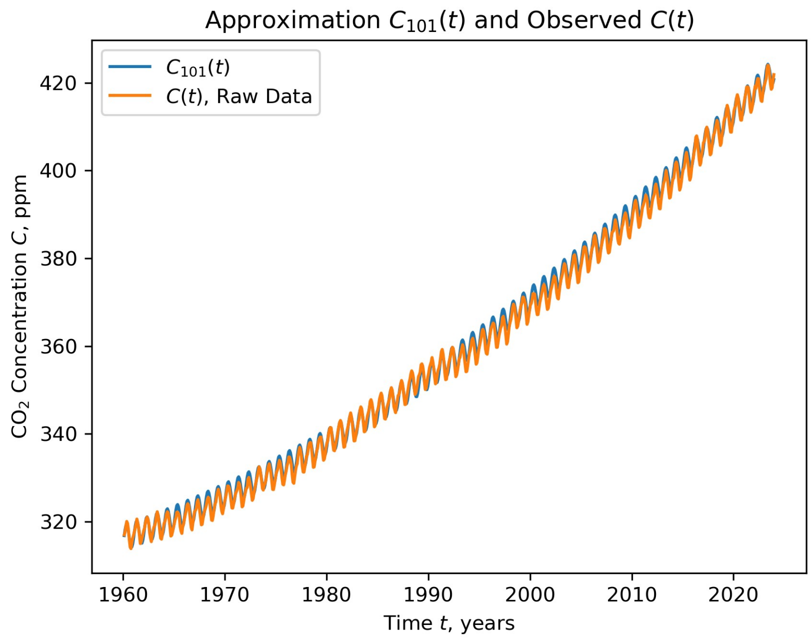
Comparing the expressions on both sides, we can form the following relationships,

$$\begin{cases} \alpha = a \\ \beta = b - a \\ \gamma = \frac{a}{3} - \frac{b}{2} + c \end{cases}.$$

Substituting $\alpha = 0.012997$, $\beta = -50.125$ and $\gamma = 48631$ into the equations, we find that

$$\begin{cases} a = 0.012997 \\ b = -50.112 \\ c = 48606 \end{cases}.$$

Using the values of a,b,c found in part (b) and choosing $A = 2.8$ and $\phi = 0.9\pi$ completes the definition of our approximating function $C_{101}(t)$. The sketch below shows that the graph of C_{101} tracks the actual measurements reasonably well.



(c) ★☆☆☆ In 2015, the nations of the world met in Paris and agreed to limit the average global temperature to 1.5°C above its pre-industrial average. Then, in 2016, scientists advising the IPCC said that we need to maintain $C(t) \leq 430$ to achieve this goal. If current trends continue, in what year will $\bar{C}_{101}(t)$= 430?

Solution:

$$\text{Given} \begin{cases} a = \ 0.012997 \\ b = \ -50.112 \\ c = 48606 \end{cases}, \text{A=2.8}, \ \bar{C}_{101}(t) = 430, \text{ and } \emptyset = 0.9\pi,$$

$$\bar{C}_{101}(t) = \ \int_{t-1}^{t} at^2 + tx + c + A\cos(2\pi t \ + \ \phi) \ dx.$$

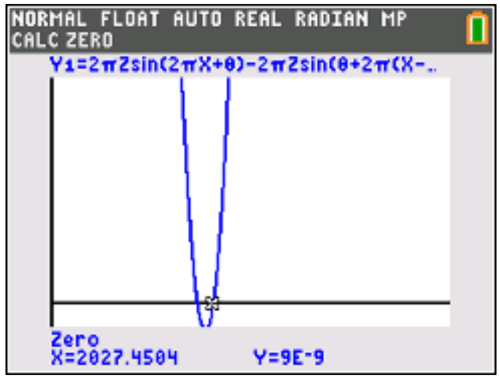
We can evaluate this integral to find the following,

$$(\tfrac{a}{3}t^3 + \tfrac{b}{2}t^2 + ct + \tfrac{1}{2\pi}Asin(2\pi t + \theta)) - (\tfrac{a}{3}(t-1)^3 + \tfrac{b}{2}(t-1)^2 + c(t-1) + \tfrac{1}{2\pi}Asin(2\pi(t-1) + \theta)).$$

We can equate this to 430,

$$430 = (\tfrac{a}{3}t^3 + \tfrac{b}{2}t^2 + ct + \tfrac{1}{2\pi}Asin(2\pi t + \theta)) - (\tfrac{a}{3}(t-1)^3 + \tfrac{b}{2}(t-1)^2 + c(t-1) + \tfrac{1}{2\pi}Asin(2\pi(t-1) + \theta)).$$

After simplifying this expression and graphing it we are able to find zeros at two locations,



The two zeros are,

$$x = 1829.2086 \; and \; 2027.4504.$$

Since scientists recommended the proposal 2015 this disregards the first solution and only leaves $x = 2027.4504$.

All in all, this means that at current trends we will reach $\bar{C}_{101}(t)$= 430 in the year 2027.

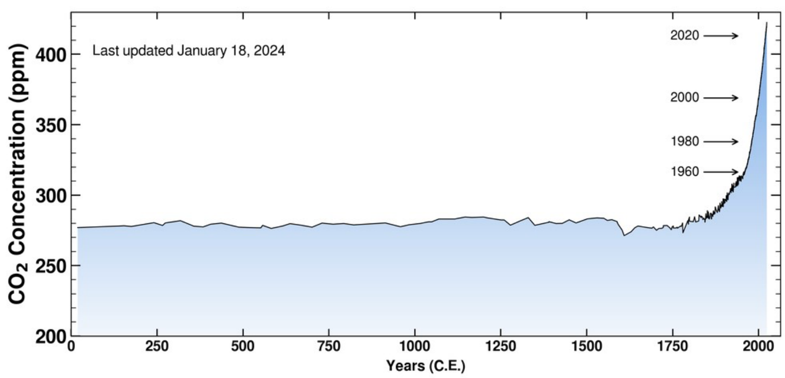
Notes and References: Statements in the text above are corroborated by many online sources. Representative samples are given below.

- The monthly data used to produce the figures here comes from the NOAA, a government agency in the USA, at this URL: https://gml.noaa.gov/ccgg/trends/data.html

- The Paris Agreement was announced on 12 December 2015 at the conclusion of the COP21 climate conference sponsored by the United Nations. The UN page describing highlights of the agreement is here: https://unfccc.int/process-and-meetings/the-paris-agreement

- The graph in Fig. 1 is called the Keeling Curve. The sketch itself comes from the Scripps Institution of Oceanography at the University of California San Diego, on the "Full Record" tab: see https: //keelingcurve.ucsd.edu/

- On the page just cited, the "2K years" tab extends the *t*-axis in Figure 1 back to Year 0 in our current numbering scheme. The world's current situation is truly exceptional.



- Various credible online sources provide the 430 ppm target for atmospheric CO_2, including Reuters:

"Science advisers on the Intergovernmental Panel on Climate Change have estimated the limits imply an atmospheric CO2 concentration of no more than 450 parts per million (for 2 degrees) or 430 ppm (for 1.5 degrees)."

The same group of scientists is cited on the MIT Climate Portal:

"In 2016, a worldwide body of climate scientists said that a CO_2 level of 430 ppm would push the world past its target for avoiding dangerous climate change."

- Confronting the facts explored in this assignment can be emotionally difficult. UBC has gathered some resources to help with this in the online STEM and Climate Wellbeing Toolkit.