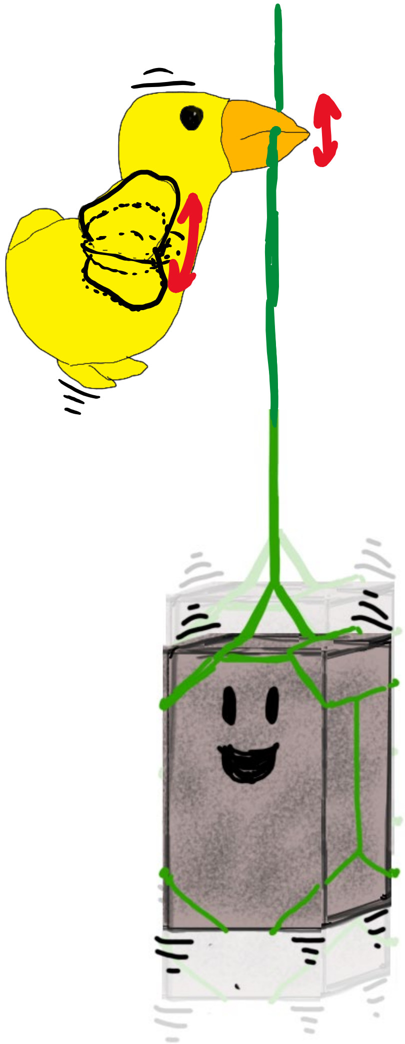


Lecture 30.

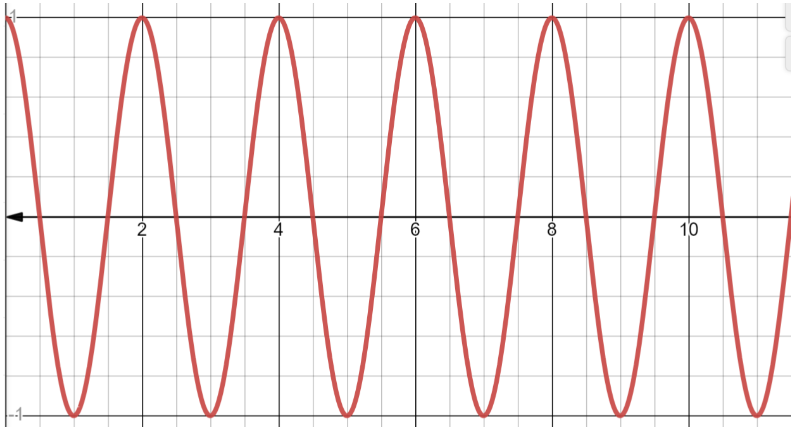
Damped oscillators.

Resonance (if time permits).

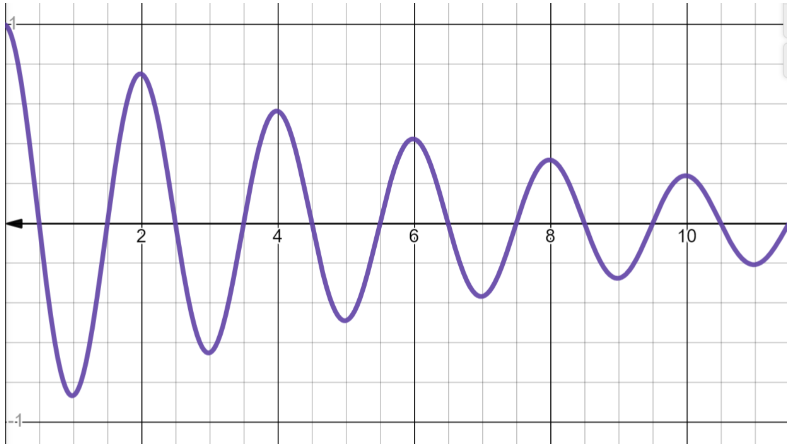
Damping

very low damping

Q: Assume that a mass oscillates on a spring, and there is a friction force between the mass and the table. Which of these graphs can represent the coordinate of the mass as function of time?

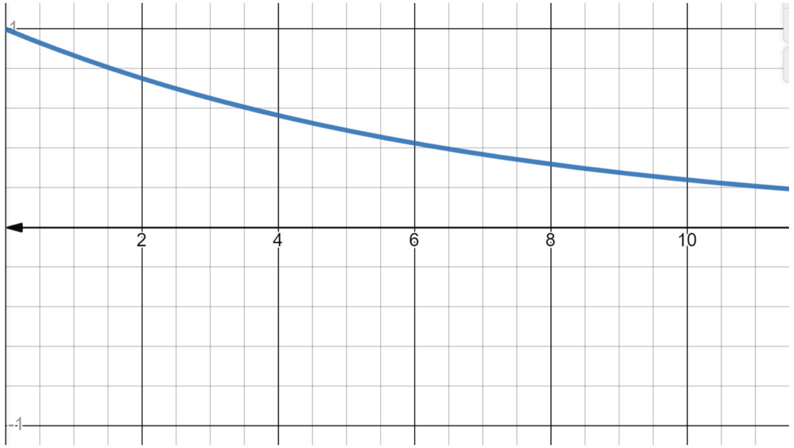


(1)



(2)

overdamped



(3)

A. (1)
B. (2)
C. (3)
D. All of them
E. None of them

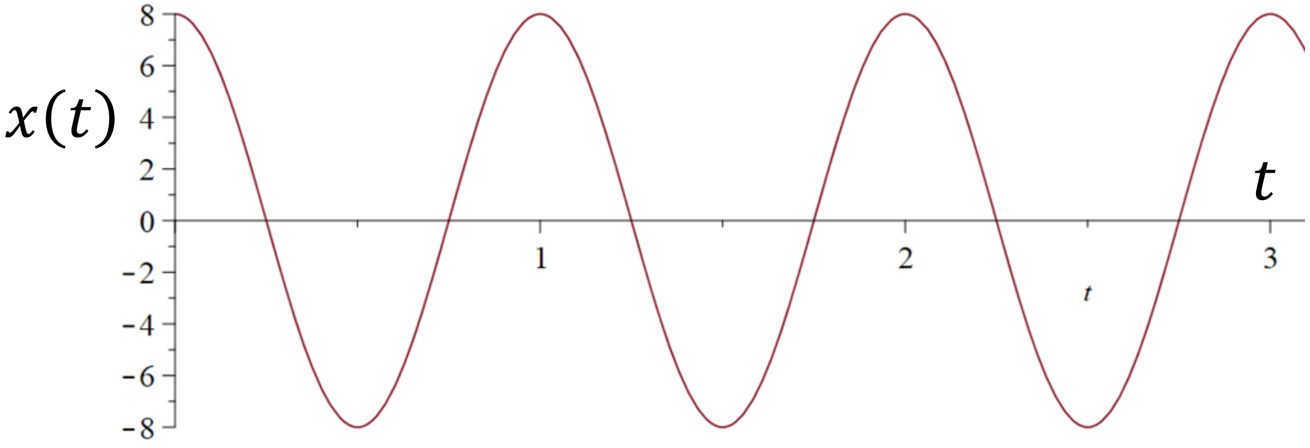
Demo: Mass on spring with position sensor, with and without damping

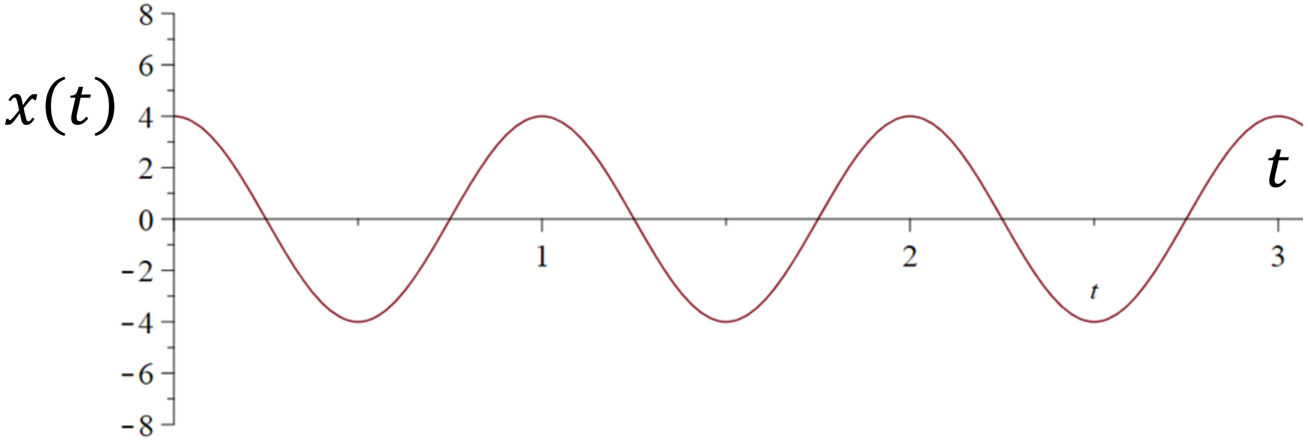
☹

Q: The two graphs show different oscillations for the same system. Compared with the first case, the total potential plus kinetic energy in the second case is

$$E_{tot} = KE + PE = \frac{kA^2}{2}$$

$$E_{tot} \propto A^2$$



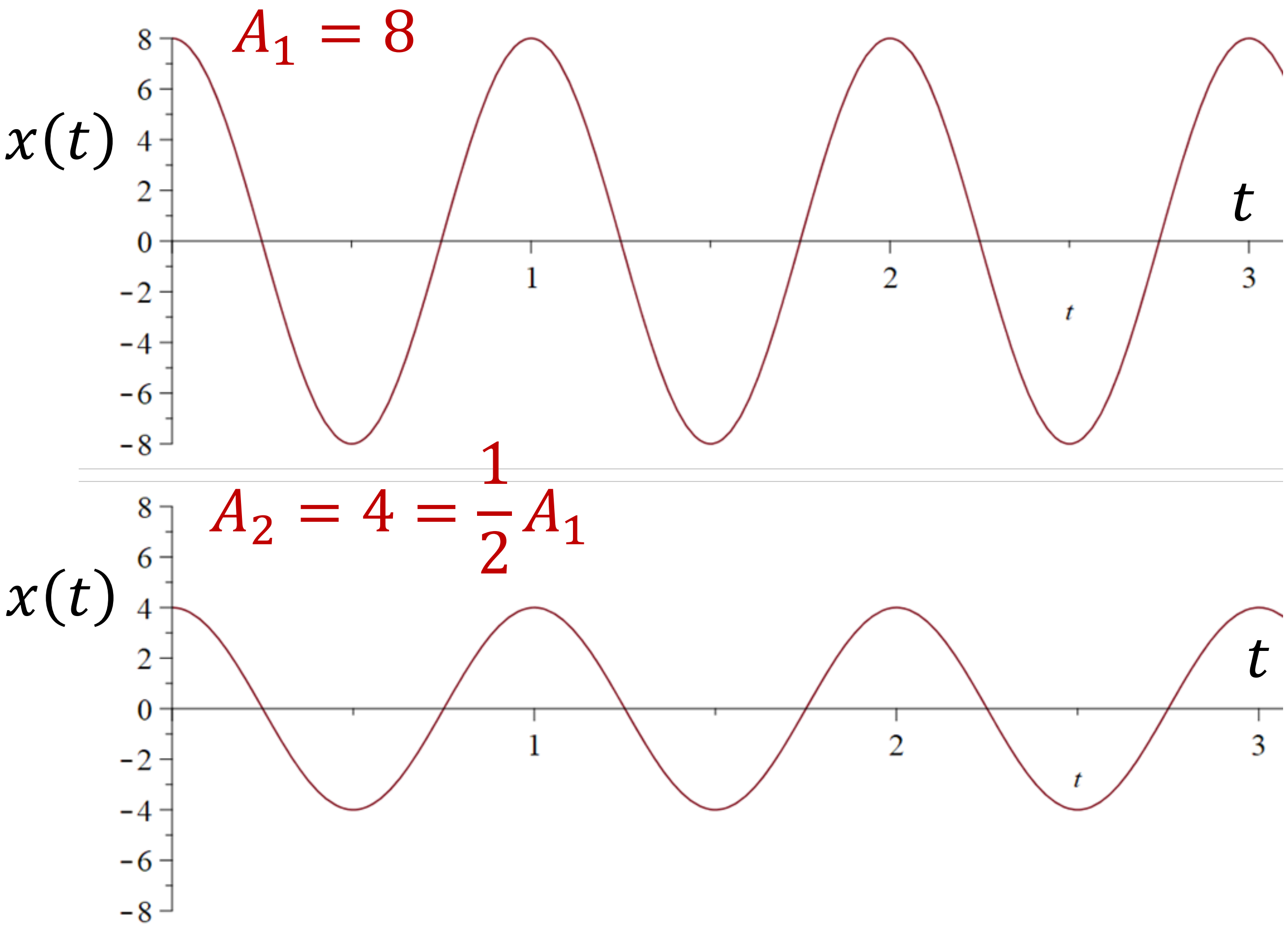


A. The same
B. Twice as big
C. Half as big
D. One quarter as big
E. One 16th as big

Q: The two graphs show different oscillations for the same system. Compared with the first case, the total potential plus kinetic energy in the second case is

- Energy is same at all times
- $E_{tot} = \frac{1}{2} k A^2$
- So:

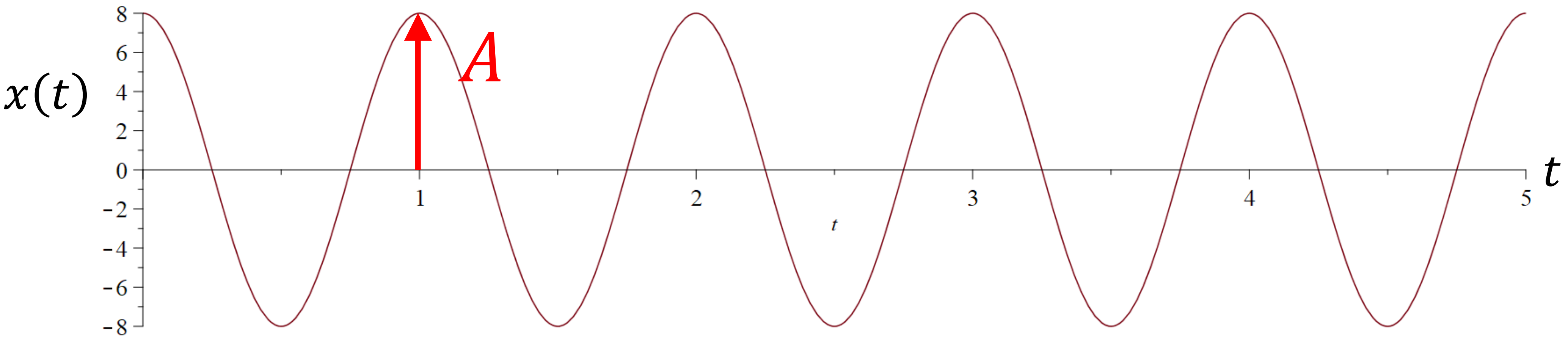
$$\frac{E_{2,tot}}{E_{1,tot}} = \frac{A_2^2}{A_1^2} = \frac{1}{4}$$



A. The same
B. Twice as big
C. Half as big
D. One quarter as big ✓
E. One 16th as big

- Key fact about oscillating systems:

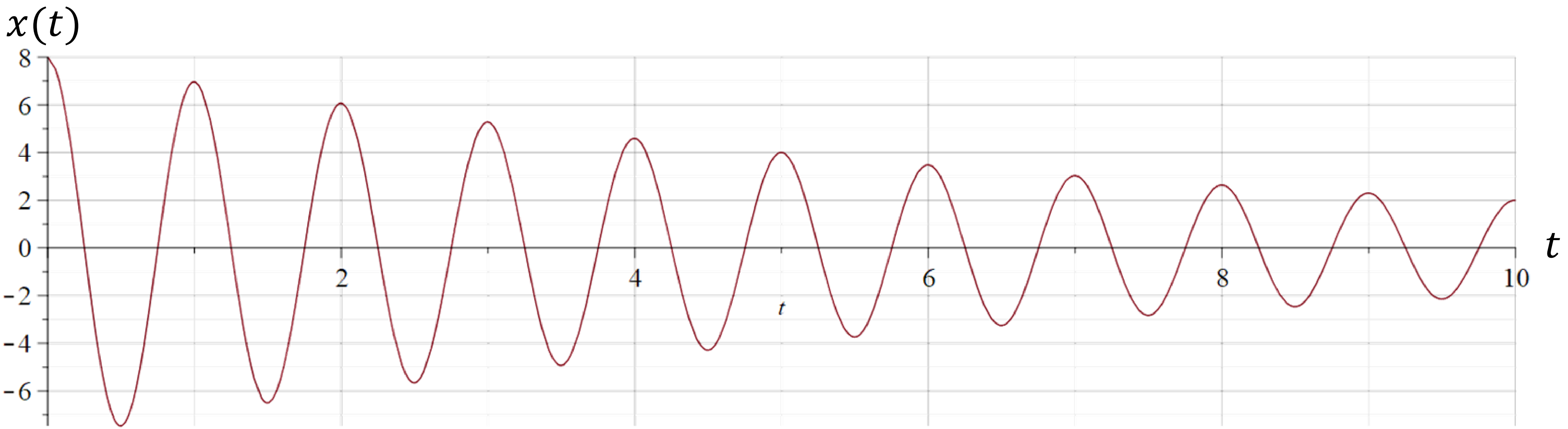
  - Their energy is proportional to amplitude squared



$$\boxed{E \propto A^2}$$

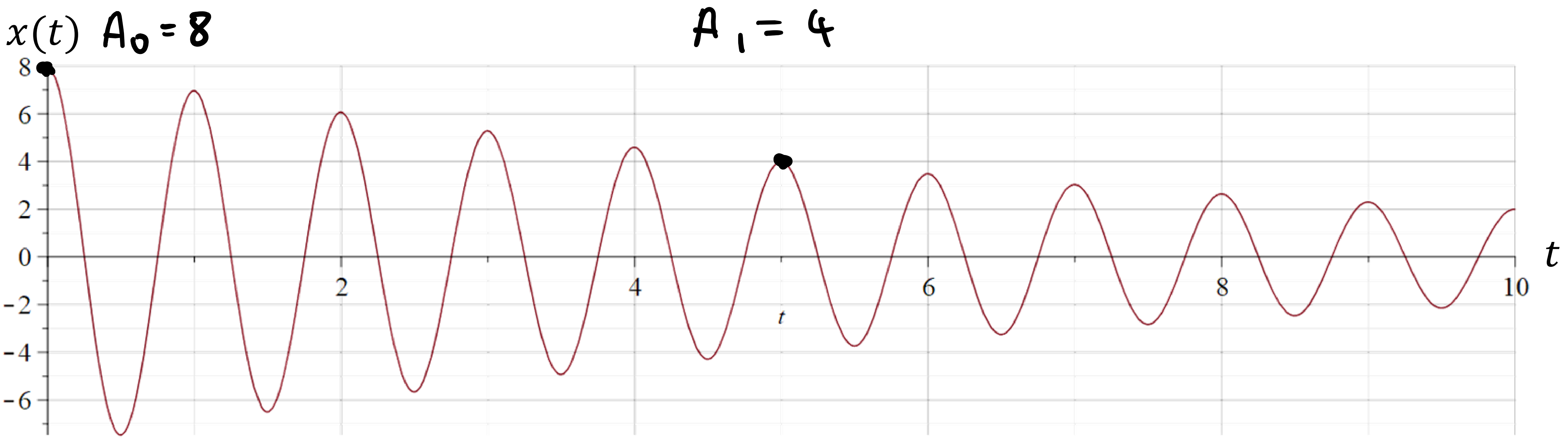
Real oscillators:

- Energy is lost (friction, air/fluid drag, heating of system…)



- Amplitude decreases with time
- Energy decreases as square of the amplitude

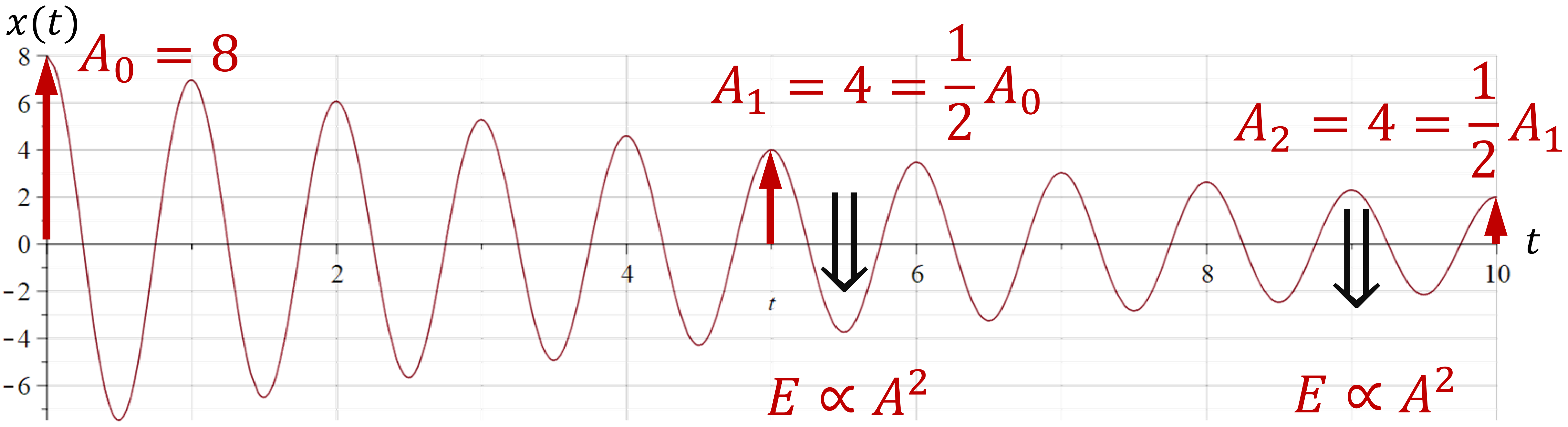
Q1: What fraction of the original kinetic + potential energy remains in the oscillator at $t = 5\ s$?



A. All of it
B. Half of it
C. One quarter of it
D. $1/\sqrt{2}$ of it

Q2: What fraction of the energy at $t = 5\ s$ remains at $t = 10\ s$?

Q1: What fraction of the original kinetic + potential energy remains in the oscillator at $t = 5\ s$?



$$E_1 = \frac{1}{4}E_0$$

$$E_2 = \frac{1}{4}E_1$$

A. All of it
B. Half of it
C. One quarter of it ✓
D. $1/\sqrt{2}$ of it

→ Q2: What fraction of the energy at $t = 5\ s$ remains at $t = 10\ s$? $1/4$

Common situation: amplitude decreases by same fraction each full oscillation

Example:

$$t = 0 \quad \longrightarrow \quad A = A_0$$

$$t = T \quad \longrightarrow \quad A = A_0 \cdot r$$

fraction

$$t = 2T \quad \longrightarrow \quad A = A_0 \cdot r^2$$

$$t = 3T \quad \longrightarrow \quad A = A_0 \cdot r^3$$

- We assume that $r < 1$, so A is decreasing at each iteration

General:

$$t \quad \longrightarrow \quad A = A_0 \cdot r^{t/T}$$

t/T = number of periods (integer)

Exponential decay

- Show that: $A(t) = A_0 \cdot r^{\,t/T}$ is equivalent to $A(t) = A_0 e^{-t/t_0}$ with $t_0 = -\dfrac{T}{\ln(r)}$

$$A(t) = A_0 e^{+\frac{t}{T}\ln r} = A_0 e^{\ln r \cdot \frac{t}{T}} =$$

$$= A_0 \left(e^{\ln r}\right)^{t/T} = A_0 r^{t/T}$$

$$\boxed{t_0 = -\frac{T}{\ln(r)}}$$

➢ Time constant

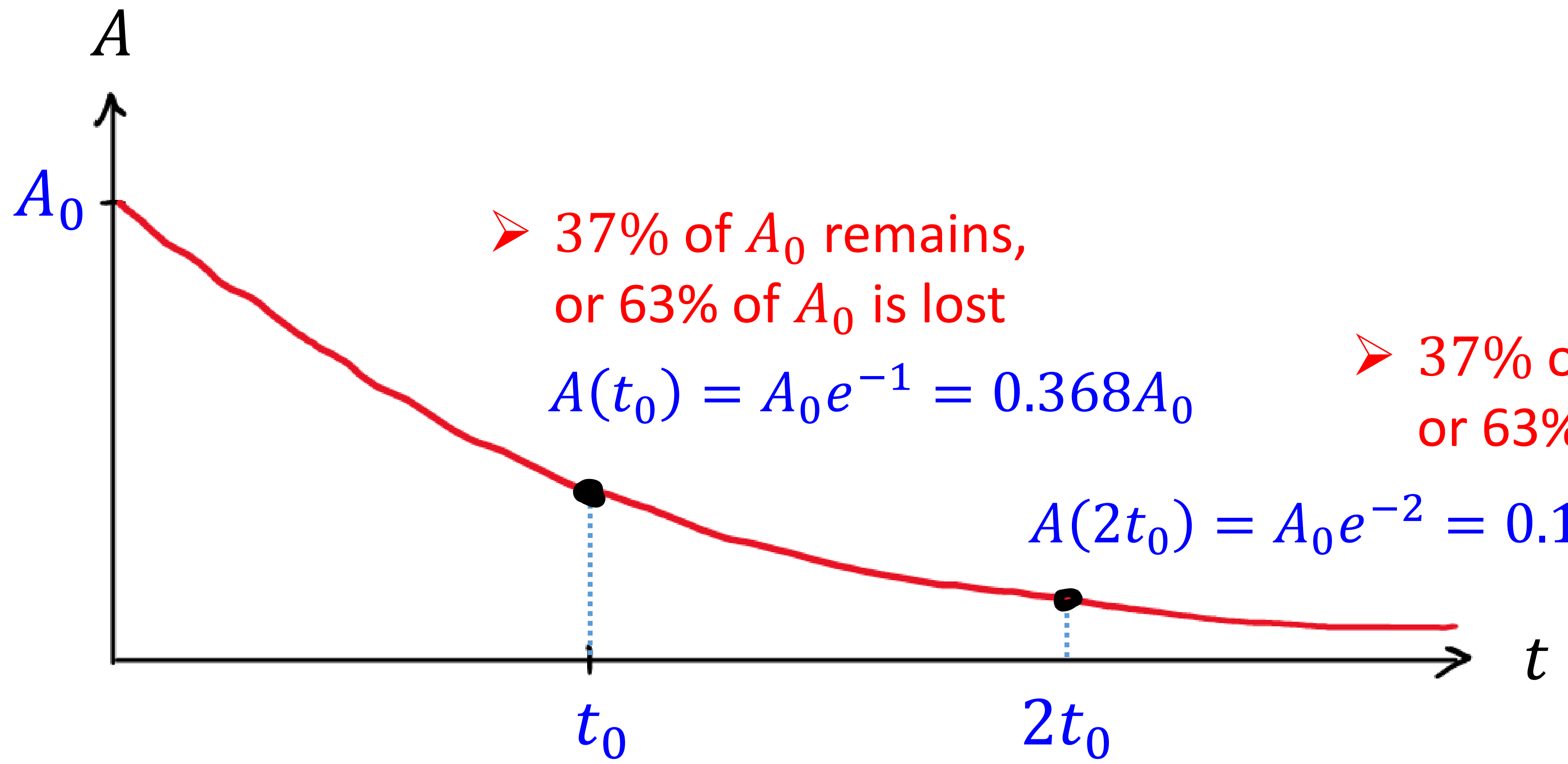
Exponential decay

- Show that: $A(t) = A_0 \cdot r^{\,t/T}$ is equivalent to $A(t) = A_0 e^{-t/t_0}$ with $t_0 = -\dfrac{T}{\ln(r)}$

$$A(t) = A_0 e^{-t/t_0} = A_0 e^{\ln(r)\cdot t/T} = A_0 \left(e^{\ln(r)}\right)^{t/T} = A_0\, r^{t/T}$$

$$\boxed{t_0 = -\frac{T}{\ln(r)}}$$

- Time constant



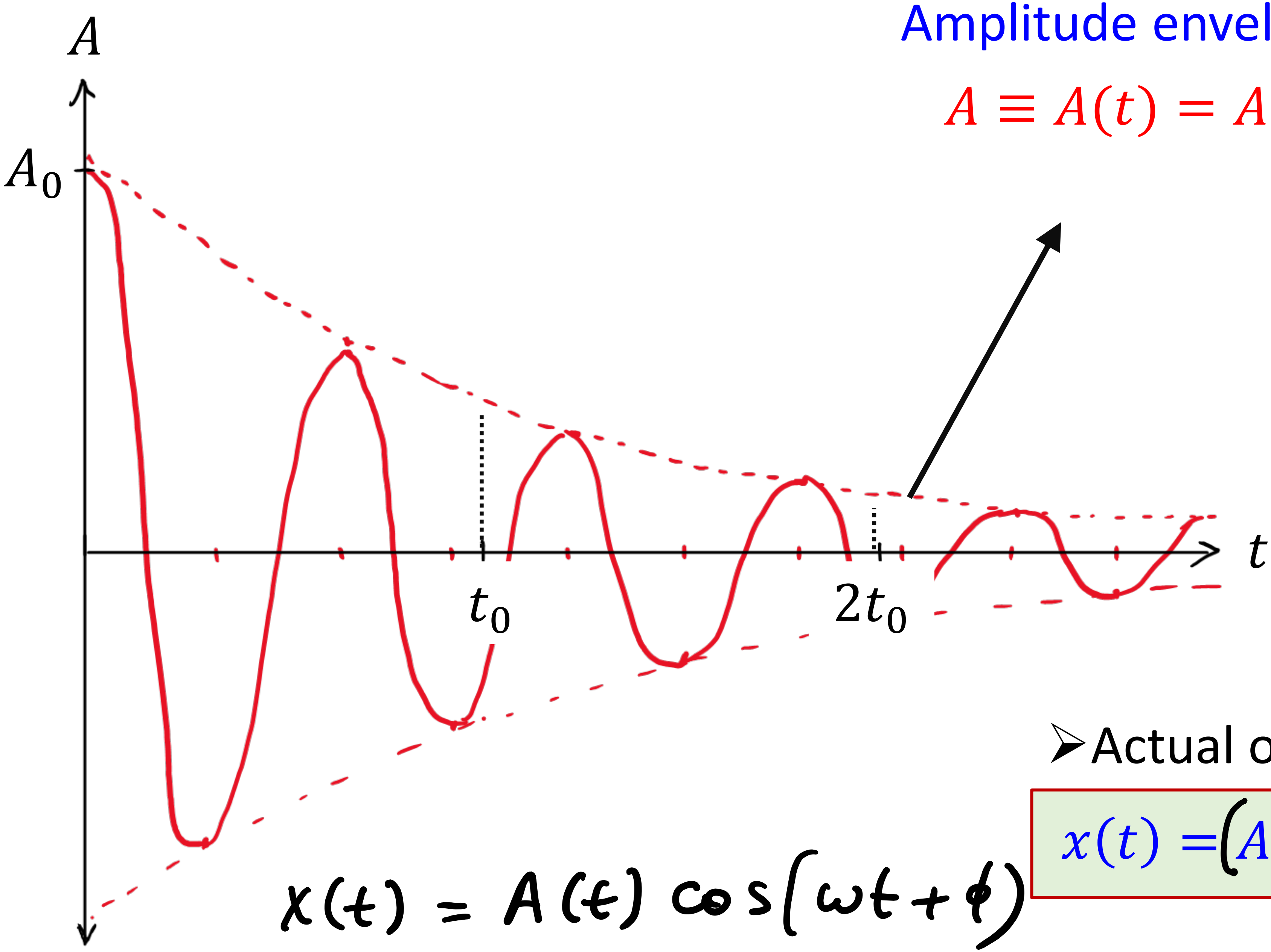
- 37% of A_0 remains, or 63% of A_0 is lost

$A(t_0) = A_0 e^{-1} = 0.368 A_0$

- 37% of $A(t_0)$ remains, or 63% of $A(t_0)$ is lost

$A(2t_0) = A_0 e^{-2} = 0.135 A_0$

Damped Oscillations



Amplitude envelope:

$$A \equiv A(t) = A_0 e^{-t/t_0}$$

multiplies cosine!

➢Actual oscillation

$$x(t) = \underbrace{\left(A_0 e^{-t/t_0}\right)}_{A(t)} \cos(\omega t + \phi)$$

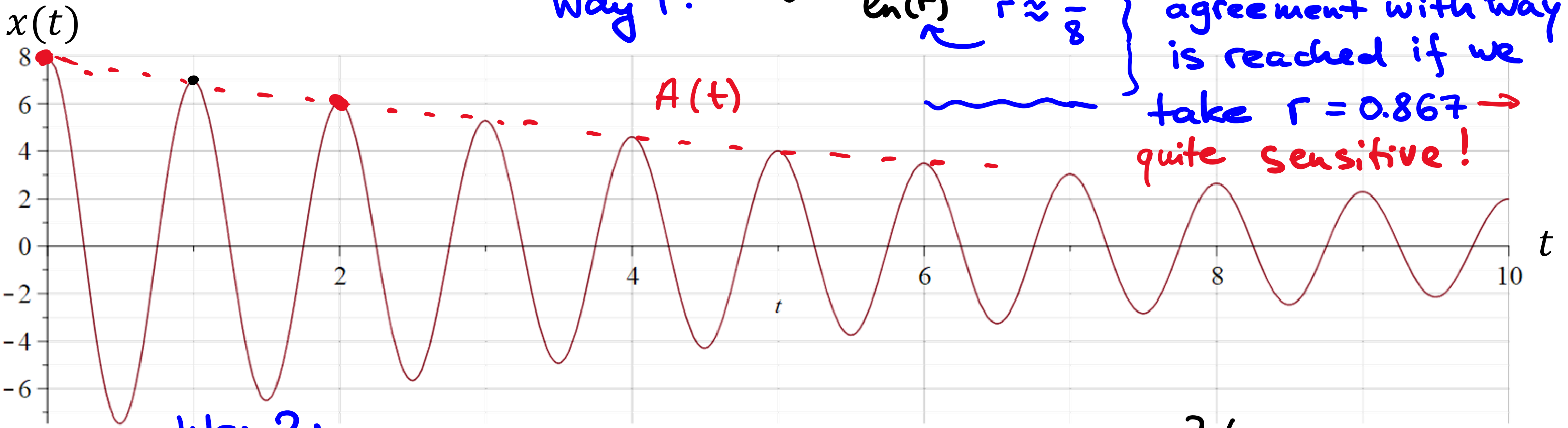
$$x(t) = A(t) \cos(\omega t + \phi)$$

Q: The graph shows displacement vs time for a damped oscillation. The time constant t_0 in this case is nearest to

Way 1: $t_0 = -\dfrac{T}{\ln(r)}$ ← 1s $r \approx \dfrac{7}{8}$

$7/8 = 0.875$ gives $t_0 = 7.5 \approx 7$. Perfect agreement with way 2 is reached if we take $r = 0.867$ →

quite sensitive!



Way 2:

$$A(t) = A_0 e^{-t/t_0}$$

$$x(t=2s) = A_0 e^{-\frac{2s}{t_0}}$$

$$6 = 8 e^{-\frac{2s}{t_0}}$$

$$\frac{6}{8} = e^{-2/t_0}$$

$$\ln(6/8) = -\frac{2}{t_0}$$

$$t_0 = \frac{2}{[-\ln(6/8)]} > 0$$

A. 1 s

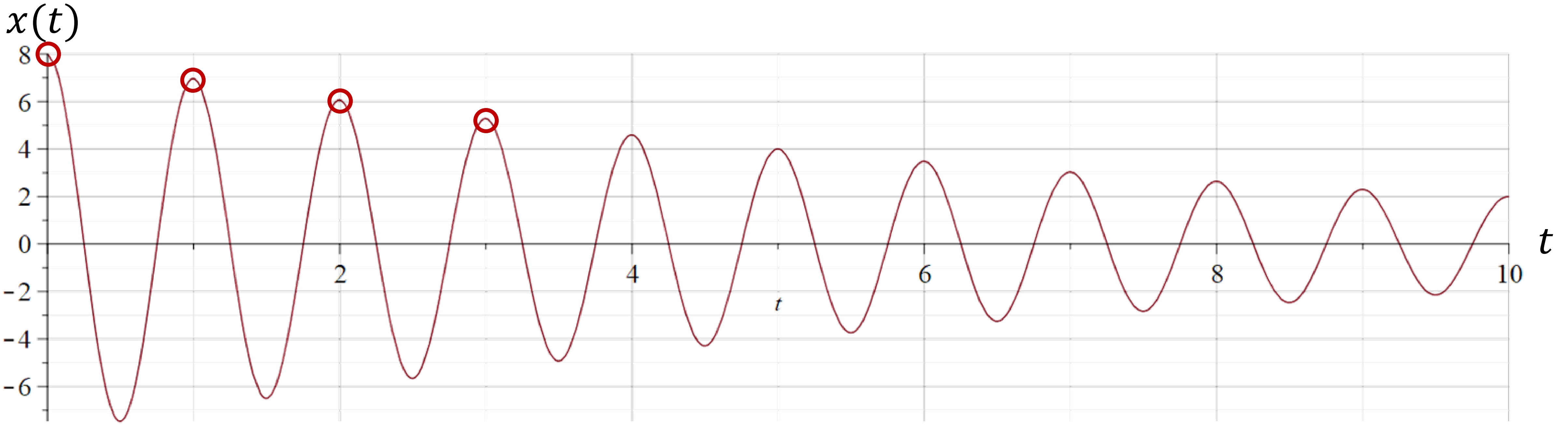
B. 3 s

C. 5 s

D. 7 s

E. 9 s

Q: The graph shows displacement vs time for a damped oscillation. The time constant t_0 in this case is nearest to



A. 1 s

B. 3 s

C. 5 s

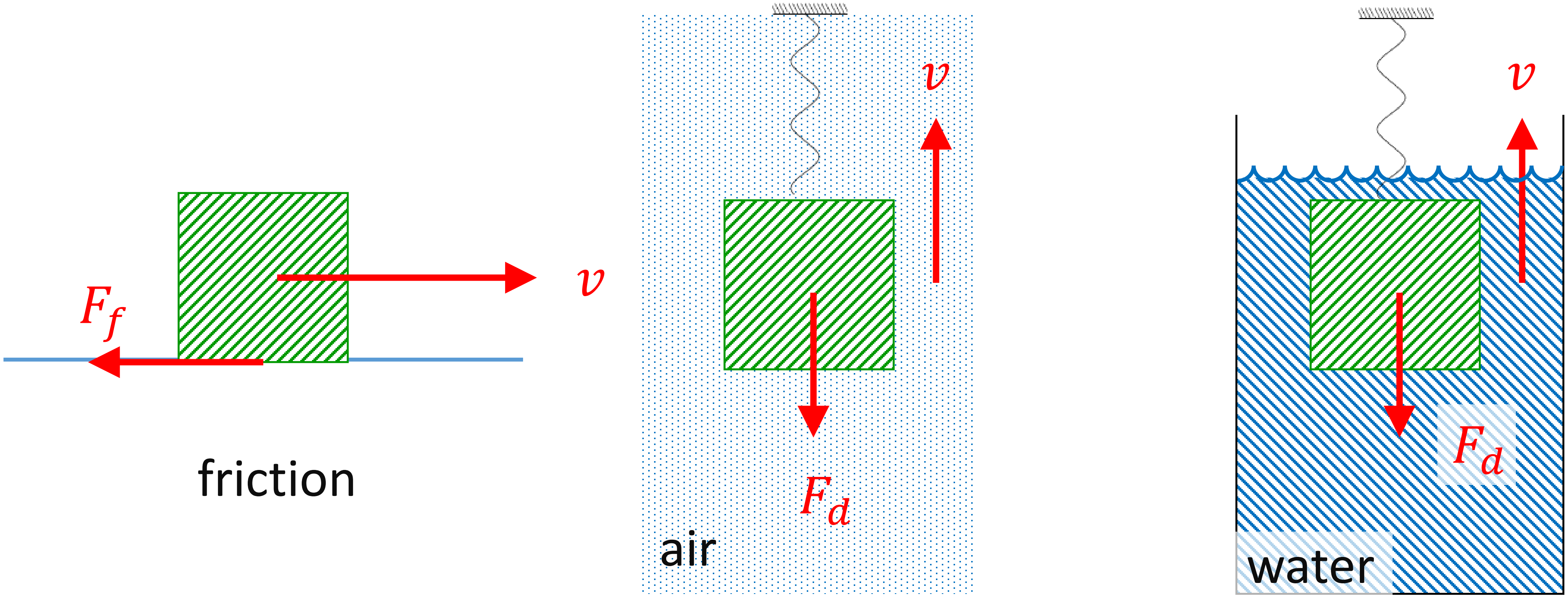
D. 7 s

E. 9 s

- At circled points, cos $=1$ so $x(t)=A_0e^{-t/t_0}$
- At $t=0$, $x=8\ cm$. At $t=2\ s$, $x=6\ cm$.
- $6\ cm=8\ cm\cdot e^{-2/t_0}\Longrightarrow e^{-2/t_0}=0.75$

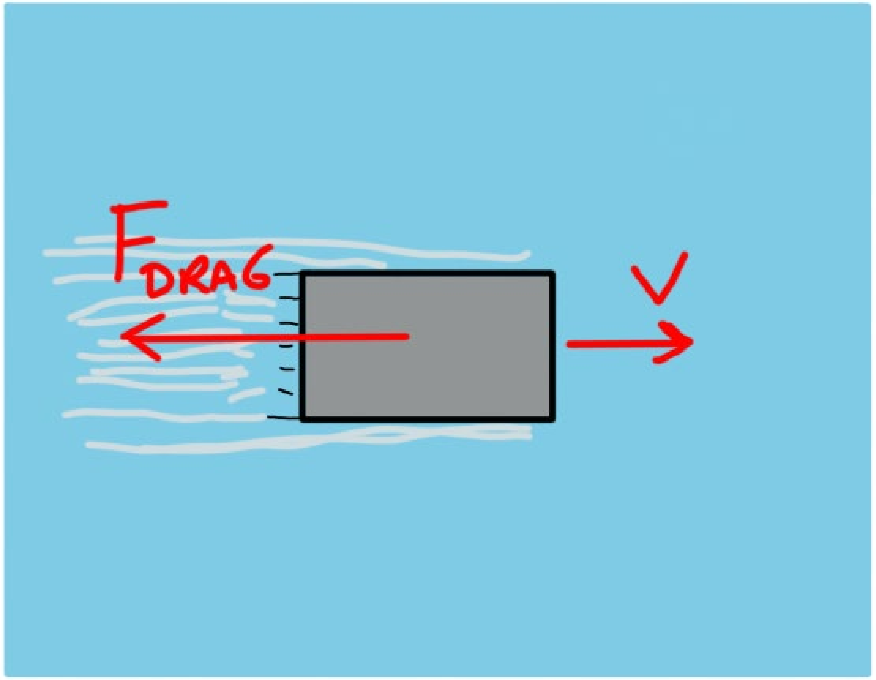
$$\Longrightarrow -2/t_0=\ln(0.75)\quad\Longrightarrow\quad t_0=7\ s$$

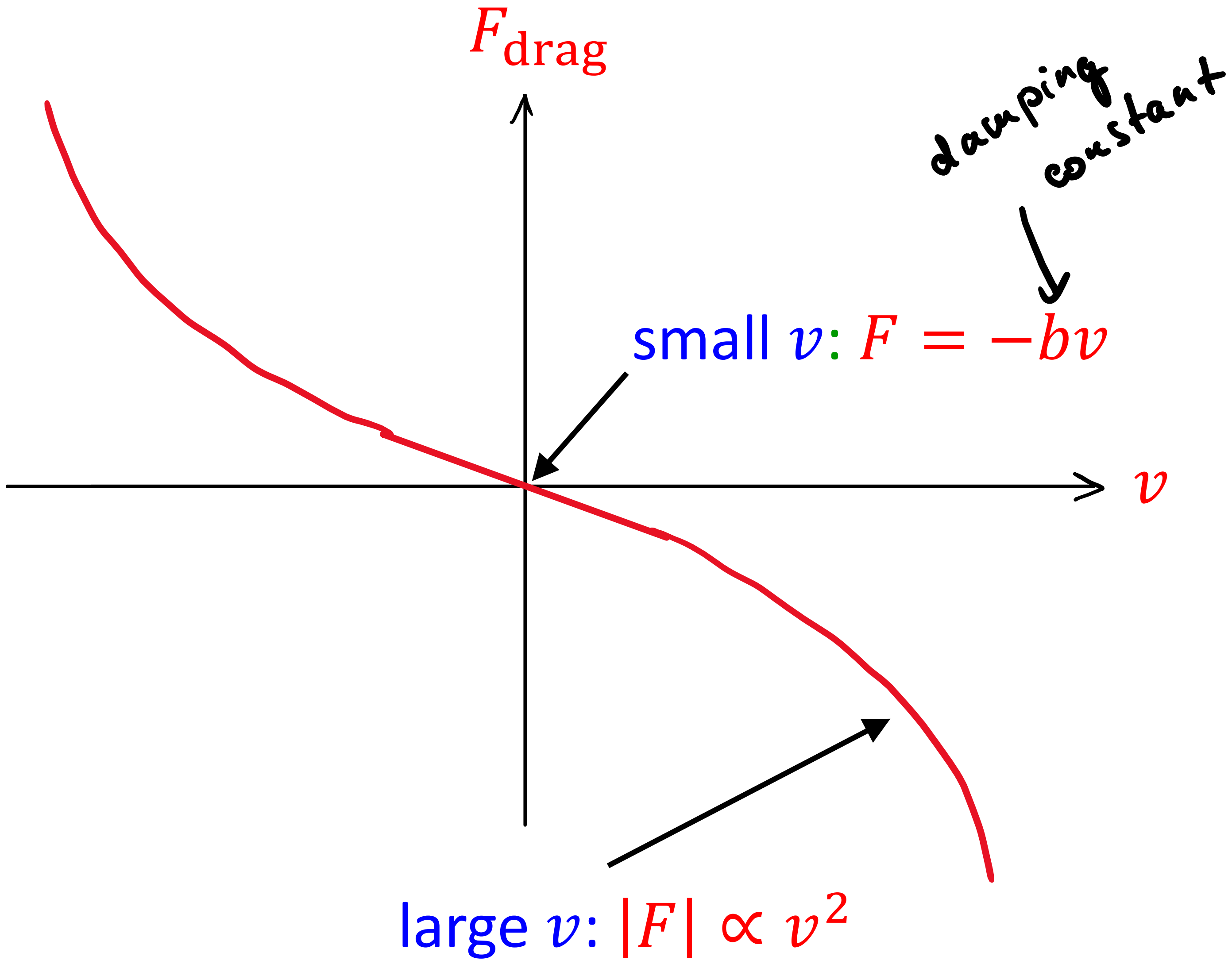
Forces that lead to damping are velocity-dependent
& opposite in direction to velocity



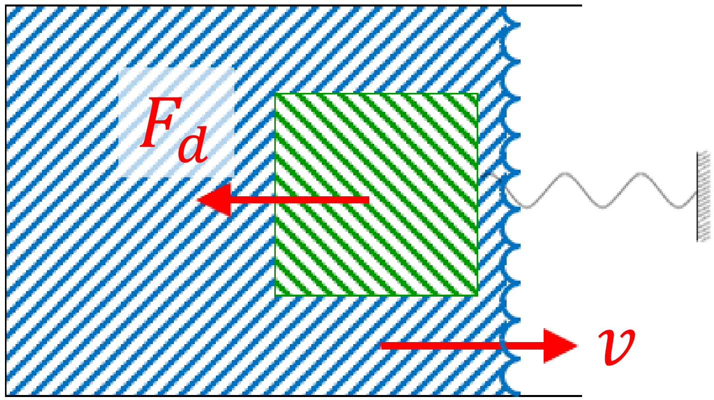
drag forces in fluids (air or liquid)

Example: drag forces from air/liquid





Example: viscous fluid drag



damping constant

- Forces:
  - $F_d = -bv$
  - $F_{net} = -kx - bv$

SHM

with damping

- Equations of motion:
  - $\dfrac{dx}{dt} = v$
  - $\dfrac{dv}{dt} = a = \dfrac{F_{net}}{m} = -\dfrac{k}{m}x - \dfrac{b}{m}v$

- Use these to predict how x and v change with time

Example: viscous fluid drag

- Equations of motion:
  - $\frac{dx}{dt} = v$
  - $\frac{dv}{dt} = -\frac{k}{m}x - \frac{b}{m}v$
- The solution is:
  - $x(t) = A_0 e^{-t/t_0}\cos(\omega t + \phi)$

    with $t_0 = \frac{2m}{b}$ and $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
  - Valid for $b < 2\sqrt{km}$

Check it: calculate $v = \frac{dx}{dt}$, and then verify 2nd equation

$$0 = \sin(\omega t)\underbrace{(\cdots\cdots)}_{=0} + \cos(\omega t)\underbrace{(\cdots\cdots)}_{=0}$$

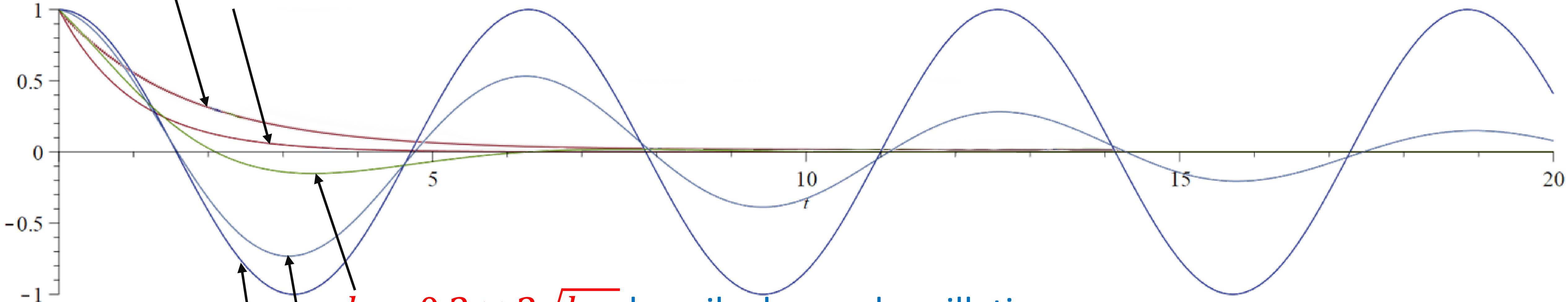
Classifying damping

$$x(t) = A_0 e^{-t/t_0} \cos(\omega t + \phi) \quad \text{with} \quad t_0 = \frac{2m}{b} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Accurate to just use $\omega = \sqrt{k/m}$ unless very highly damped

$b > 2\sqrt{km}$ overdamped, even slower to reach equilibrium

$b = 2\sqrt{km} \Longrightarrow \omega = 0$ critical damping (pure decay, no oscillations)



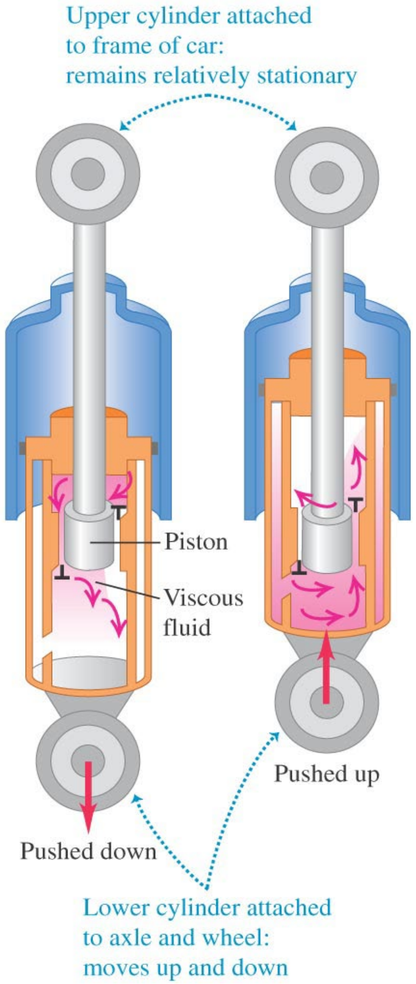
$b = 0.2 \times 2\sqrt{km}$ heavily damped oscillations

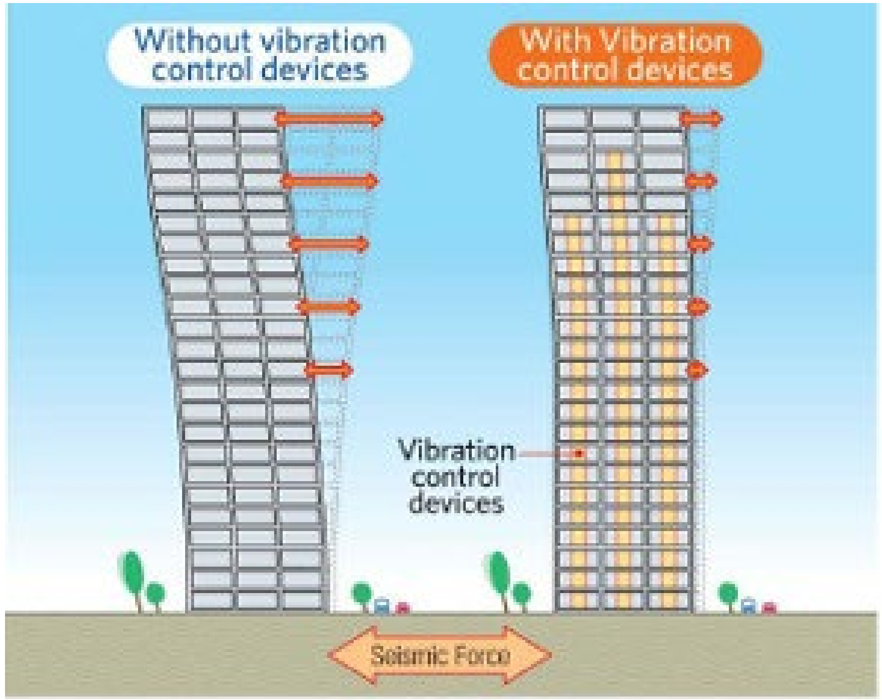
$b = 0.1 \times 2\sqrt{km}$ lightly damped oscillations ($\omega \approx \omega_{b=o} = \sqrt{k/m}$)

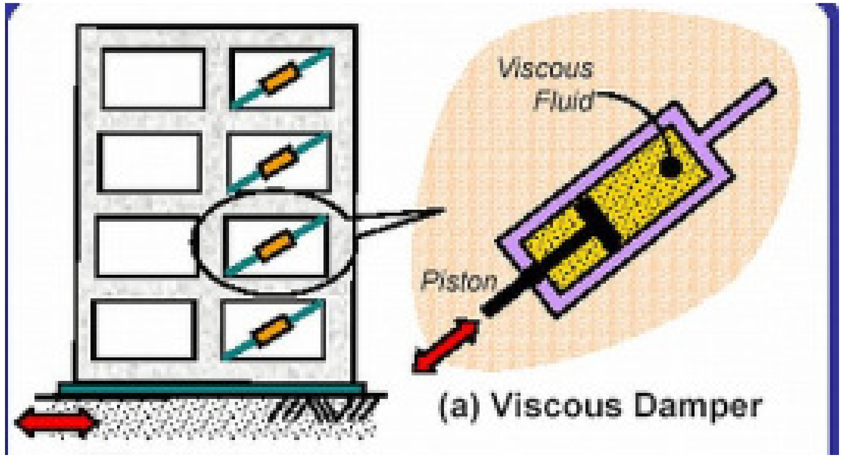
$b = 0$ no damping (no decay, pure oscillations)

Damped oscillations – applications

- Shock absorbers: cars, bikes, doors, buildings…



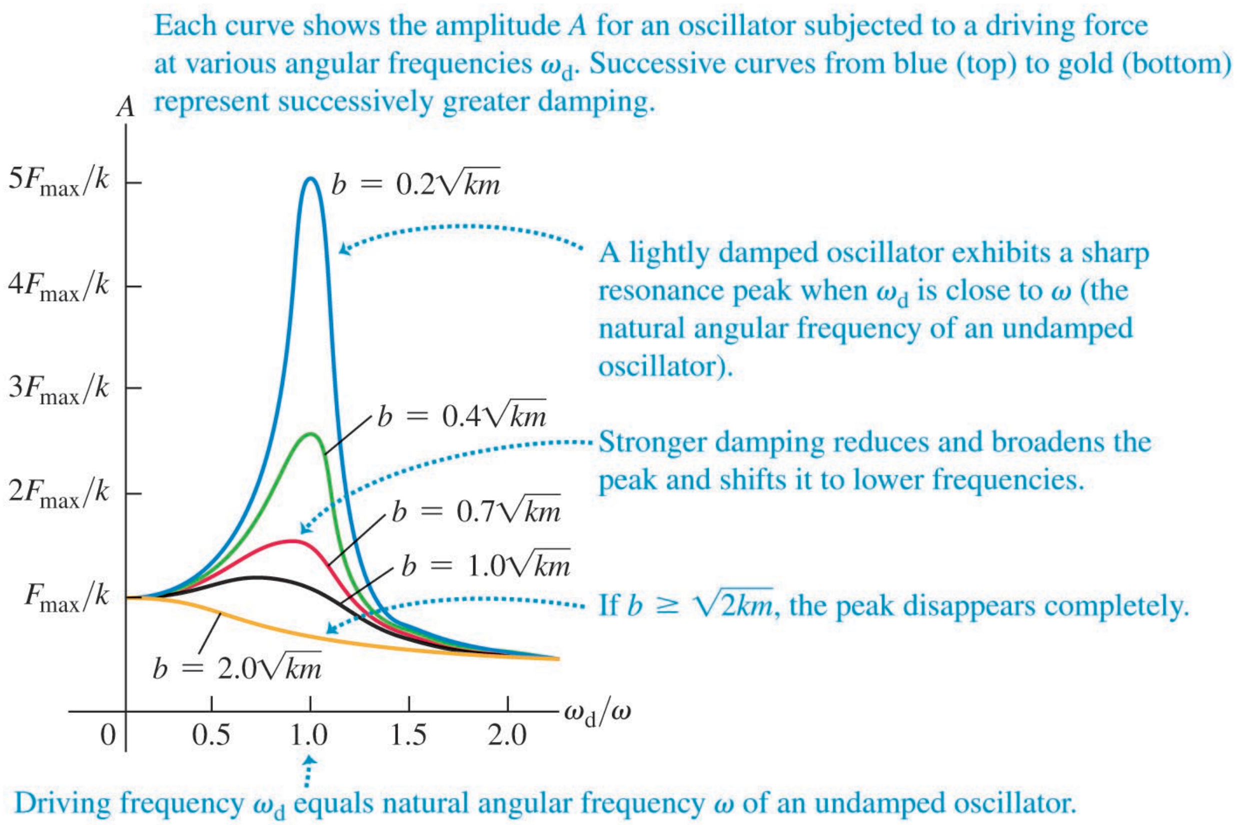




Forced Oscillations and Resonance
(extra – you are not responsible for this material)

- A damped oscillator left to itself will eventually stop moving
- But we can maintain a constant-amplitude oscillation by applying a periodic driving force
- Applying a driving force F with angular frequency ω_d to a damped harmonic oscillator results in a forced oscillation
- The forced oscillation amplitude is:

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}$$



Q: What happens if ω_d is close to the natural angular frequency of the oscillator, $\omega = \sqrt{k/m}$?

Breaking a wine glass: High-speed video