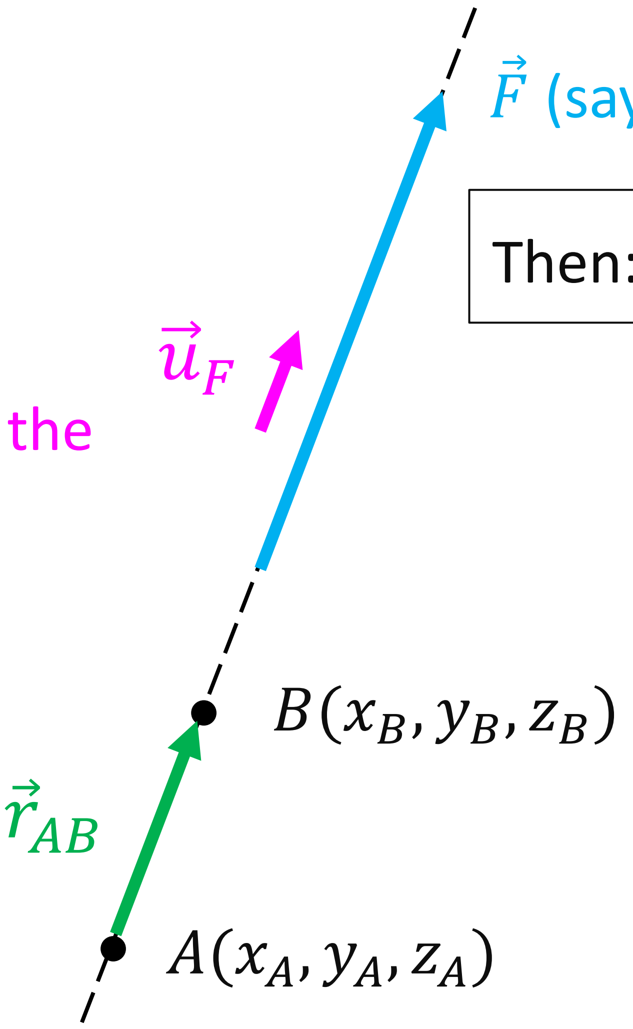
Dot product

Text: 2.9

Content:

- Dot product of two vectors: definition and properties
- Using dot product to find the angle between two vectors
- Dot product and projecting vectors
- Practice (W2-5 – W2-6)

Last Time:



$$\boxed{\text{Then: } \quad \vec{F} = F\,\vec{u}_F}$$ ✓

$$\boxed{\vec{F} = F\,\frac{\vec{r}_{AB}}{r_{AB}}}$$

- $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A = \vec{i}\,(x_B - x_A) + \vec{j}\,(y_B - y_A) + \vec{k}(z_B - z_A)$,
- $r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

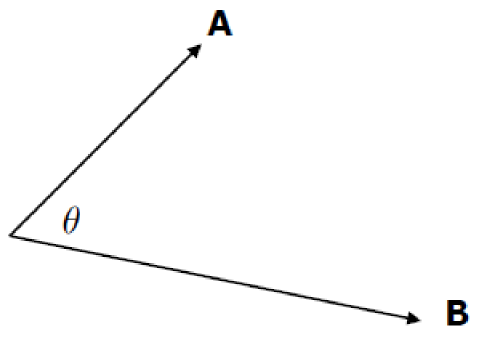
DOT PRODUCT

- Dot product: an operation that takes **two vectors** and produces a **scalar**:

$$\boxed{\vec{A} \cdot \vec{B} = A\,B\cos\theta}$$

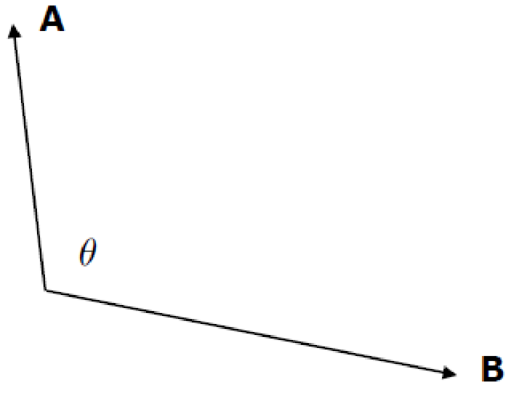
$$0^o < \theta < 180^o$$

angle between $\vec{A}, \vec{B}$



$$0 < \theta < 90^o:$$

dot product is positive



$$90^o < \theta < 180^o:$$

dot product is negative

- Example of a situation when we may want to produce a scalar out of two vectors: Work is a scalar, but it depends on two vectors (force and displacement)

DOT PRODUCT: Properties

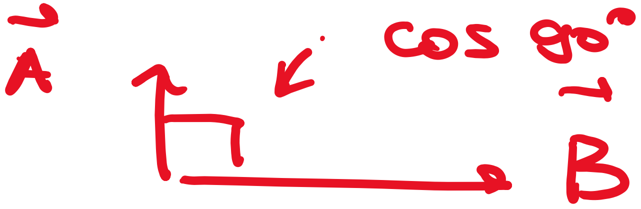
- It has the following properties similar to regular multiplication:

  - $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ *(commutative)*
  - $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ *(distributive)*
  - $\vec{A} \cdot (a\vec{B}) = a\,(\vec{A} \cdot \vec{B})$ *(scalar multiplication)*

- The units are the product of the units of the vectors being multiplied.

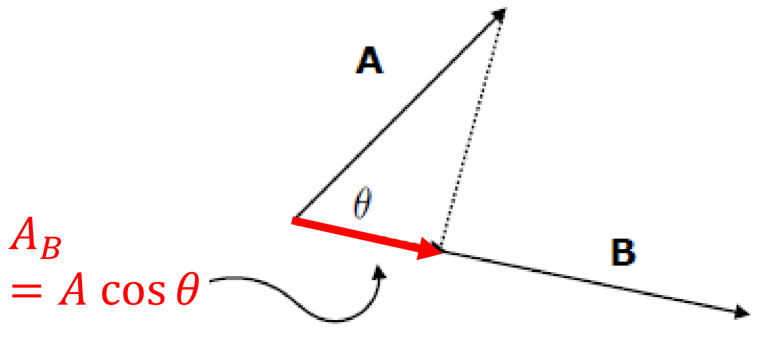
Unlike regular multiplication it is possible that $\vec{A} \cdot \vec{B} = 0$ when neither A nor B are zero.

$\cos 90^\circ \Rightarrow 0$



- Q: What is the angle between $\vec{A}$ and $\vec{B}$ in this case?

DOT PRODUCT: Connection to projections

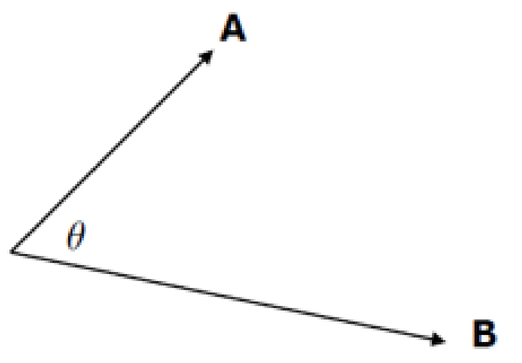


projection of vector $\vec{A}$ onto
the direction of vector $\vec{B}$

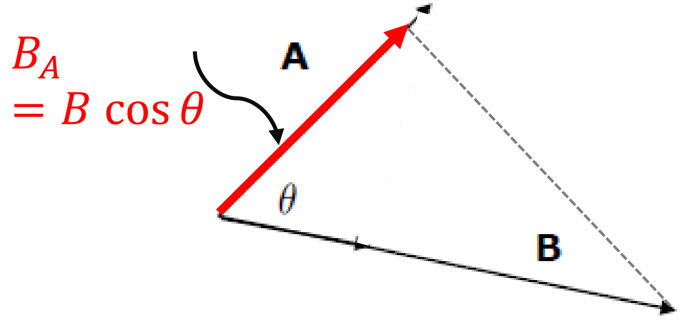
$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$= A_B B$$

$$= AB_A$$



projection of vector $\vec{B}$ onto
the direction of vector $\vec{A}$

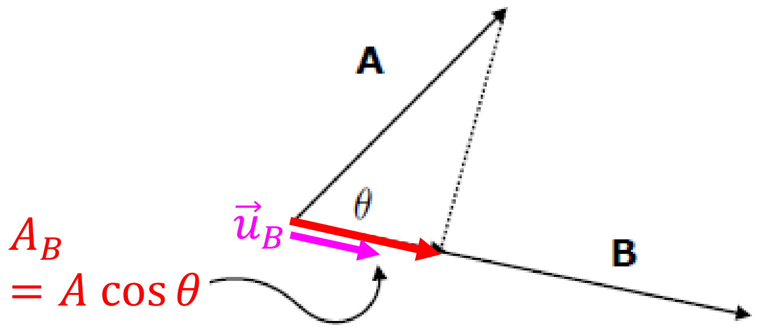


Take the length of $\vec{A}$ that points in the direction of $\vec{B}$ and multiply it by B

or

Take the length of $\vec{B}$ that points in the direction of $\vec{A}$ and multiply it by A

DOT PRODUCT: Connection to projections



projection of vector $\vec{A}$ onto the direction of vector $\vec{B}$

- We found:

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_B B$$

(here A_B is the projection of $\vec{A}$ on the direction of $\vec{B}$)

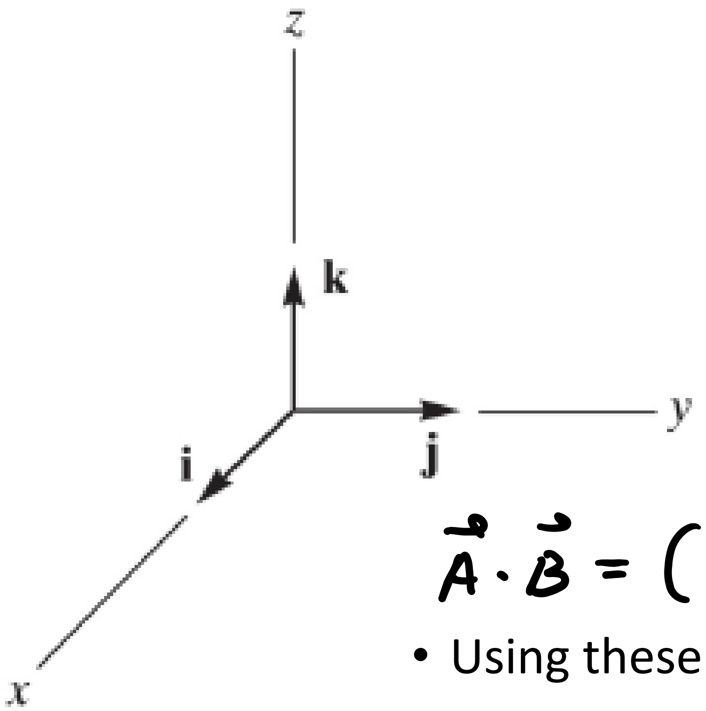
- What will we get if we project $\vec{A}$ onto the unit vector in the direction of $\vec{B}$?

- We will simply get A_B!

$$\vec{A}\cdot\vec{u}_B = A\,u_B\cos\theta = A\cos\theta = A_B$$

DOT PRODUCT: in Cartesian components

$$\vec{i}\cdot\vec{i} = \underbrace{|\vec{i}|}_{1}\underbrace{|\vec{i}|}_{1}\underbrace{\cos 0^\circ}_{1} = 1$$



- Since the angle between a vector and its copy is 0:

$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$$

$$\vec{i}\cdot\vec{j} = |\vec{i}||\vec{j}|\cos 90^\circ = 1\cdot 1\cdot 0 = 0$$

- Since the angle between $\vec{i}$ and $\vec{j}$, etc., is 90°:

$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{i}\cdot\vec{k} = \vec{k}\cdot\vec{i} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{j} = 0$$

$$\vec{A}\cdot\vec{B} = (\vec{i}A_x + \vec{j}A_y + \vec{k}A_z)\cdot(\vec{i}B_x + \vec{j}B_y + \vec{k}B_z)$$

- Using these equalities, we can get an important formula for two arbitrary vectors $\vec{A} = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z$ and $\vec{B} = \vec{i}B_x + \vec{j}B_y + \vec{k}B_z$:

W2-5: Prove this equation.

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

DOT PRODUCT: Application #1

Practice: Calculate the angle between these vectors: $\vec{A} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{B} = 2\vec{i} + 3\vec{j} - \vec{k}$

A. Between 70º and 80º
B. Between 80º and 90º
C. Between 90º and 100º
D. Between 100º and 110º
E. Not enough information

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = A \cdot B \cdot \cos\theta$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{A \cdot B} =$$

$$A = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$B = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$= \frac{(1)(2) + (-1)(3) + (1)(-1)}{\sqrt{3} \cdot \sqrt{14}}$$

$$\cos\theta = \cdots\cdots \longrightarrow \theta = \cdots\cdots$$

$$\theta = 108°$$

DOT PRODUCT: Application #1 (Summary)

- On one hand: $\vec{A} \cdot \vec{B} = A\,B\cos\theta$

- On the other hand: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

- Using these two identities, you can connect the Cartesian components of two vectors, (A_x, A_y, A_z) and (B_x, B_y, B_z), with the angle θ between these two vectors!

$$\boxed{\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{A\,B}}$$

…and the magnitudes of the vectors of course are $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$, $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$.

DOT PRODUCT: Application #2

- **Application #2** allows us to resolve one vector ($\vec{A}$) into components that are parallel and perpendicular to another vector ($\vec{B}$).

  - Reminder: if you know $\vec{B}$, you also know $\vec{u}_B$. What is $\vec{u}_B$?

$$\vec{u}_B = \frac{\vec{B}}{B} = \vec{i}\left(\frac{B_x}{B}\right) + \vec{j}\left(\frac{B_y}{B}\right) + \vec{k}\left(\frac{B_z}{B}\right), \quad B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

  - What is A_B?

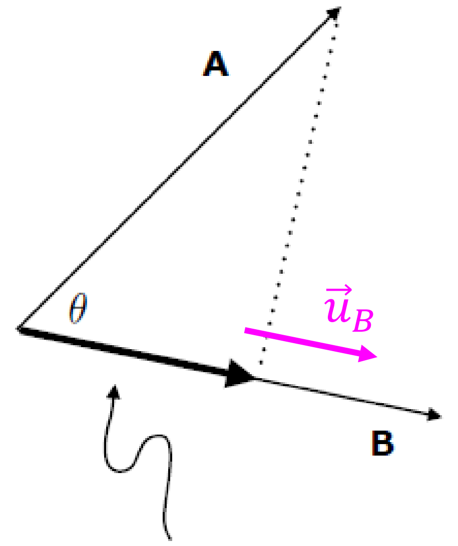
$$A_B = \vec{A} \cdot \vec{u}_B = A_x \cdot \frac{B_x}{B} + A_y \frac{B_y}{B} + A_z \frac{B_z}{B}$$

  - What is $\vec{A}_{\parallel \text{ to } \vec{B}}$?

$$\vec{A}_{\parallel \text{ to } B} = A_B \cdot \vec{u}_B = \overbrace{(\vec{A} \cdot \vec{u}_B)}^{\text{scalar}} \cdot \vec{u}_B$$

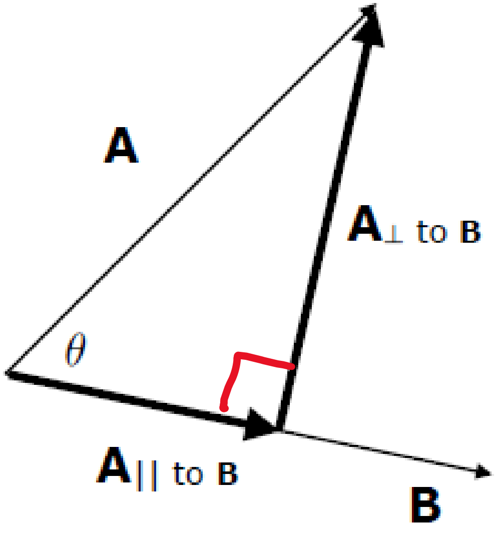
  - What is $\vec{A}_{\perp \text{to } \vec{B}}$?

$$\vec{A}_{\perp \text{ to } \vec{B}} = \vec{A} - \vec{A}_{\parallel \text{ to } \vec{B}}$$



The projection of **A** onto **B**.

$$\vec{A} = \vec{A}_{\parallel \text{ to } \vec{B}} + \vec{A}_{\perp \text{ to } \vec{B}}$$

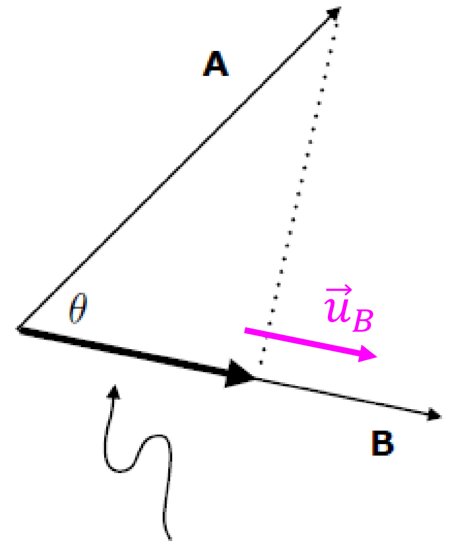


DOT PRODUCT: Application #2 (Summary)

- **Application #2** allows us to resolve one vector ($\vec{A}$) into components that are parallel and perpendicular to another vector **($\vec{B}$).** To do that, let us define projection of one vector onto another vector:

Here $A_B = A\cos\theta = \vec{A}\cdot\vec{u}_B$, where $\vec{u}_B$ is a unit vector along $\vec{B}$.

This definition of projection means that it is a scalar
(other definitions are also possible)



The projection of **A** onto **B**.

- Now we can split vector $\vec{A}$ into the component parallel to $\vec{B}$ and the component perpendicular to $\vec{B}$:

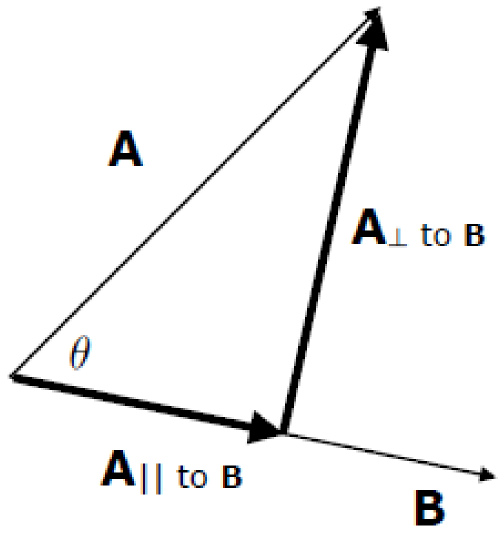
$$\vec{A} = \vec{A}_{\parallel \text{ to } \vec{B}} + \vec{A}_{\perp \text{ to } \vec{B}}$$

$$\boxed{\vec{A}_{\parallel \text{ to } \vec{B}} = A_B\,\vec{u}_B = \left(\vec{A}\cdot\vec{u}_B\right)\vec{u}_B}$$

$$\boxed{\vec{A}_{\perp \text{ to } \vec{B}} = \vec{A} - \vec{A}_{\parallel \text{ to } \vec{B}} = \vec{A} - \left(\vec{A}\cdot\vec{u}_B\right)\vec{u}_B}$$

- Pythagoras theorem:

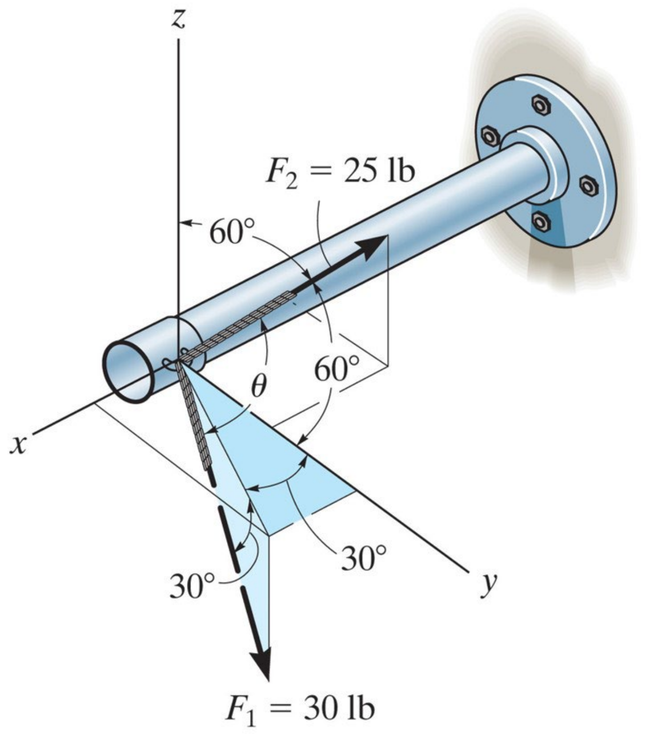
$$A^2_{\parallel \text{ to } \vec{B}} + A^2_{\perp \text{to } \vec{B}} = A^2$$



W2-6. Two cables exert forces on the pipe as shown.

a) Determine the projected component of $\vec{F}_1$ along the line of action of $\vec{F}_2$.

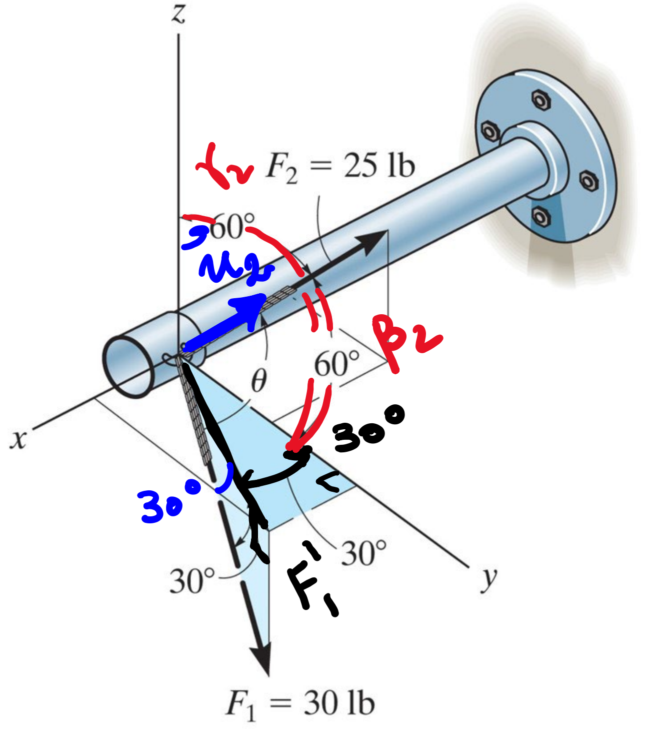
b) Determine the angle between the two cables.



W2-6. Two cables exert forces on the pipe as shown.

a) Determine the projected component of $\vec{F}_1$ along the line of action of $\vec{F}_2$.

b) Determine the angle between the two cables.



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- What is projection of F_1 on F_2?

θ - angle between $\vec{F}_1$ and $\vec{F}_2$

$$F_{1\,on\,2} = F_1 \cos\theta = \vec{F}_1 \cdot \vec{u}_2$$

i) $\vec{F}_1 = \vec{i}\,(F_1 \cos 30° \cdot \sin 30°) + \vec{j}\,(F_1 \cos 30° \cdot \cos 30°) + \vec{k}\,(-F_1 \sin 30°)$

$$F_1' = F_1 \cos 30°$$

$$F_{1,x}' = F_1' \cdot \sin 30°$$

$$F_{1,y}' = F_1' \cos 30°$$

$$\vec{u}_2 = \vec{i} \cos\alpha_2 + \vec{j} \cos\beta_2 + \vec{k} \cos\gamma_2 =$$

$$= \vec{i} \cos\alpha_2 + \vec{j} \cos 60° + \vec{k} \cos 60°$$

$$-1/\sqrt{2} \qquad 1/2 \qquad 1/2$$

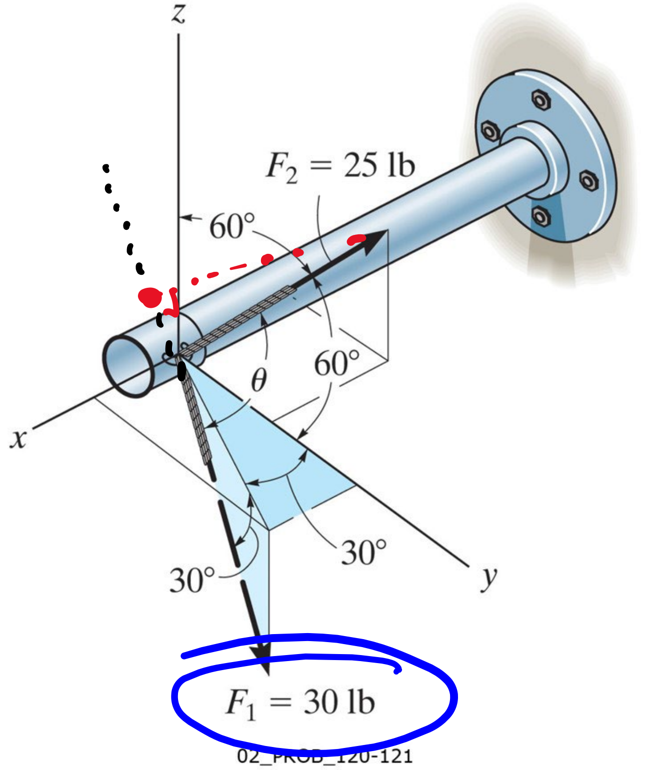
$$\cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2 = 1 \qquad \cos^2\alpha_2 = \frac{1}{2}$$

$$\cos\alpha_2 = {+1/\sqrt{2} \atop -1/\sqrt{2}} \qquad \alpha_2 \begin{cases} 45° \\ 135° \end{cases}$$

W2-6. Two cables exert forces on the pipe as shown.

a) Determine the projected component of $\vec{F}_1$ along the line of action of $\vec{F}_2$.

b) Determine the angle between the two cables.



- What is the projection of $\vec{F}_1$ on $\vec{F}_2$?

$F_1 = 30$

$$F_{1on2} = F_1 \cos\theta = \vec{F}_1 \cdot \vec{u}_2$$

$$\vec{F}_1 = \vec{i}\,(30\cos 30^o \sin 30^o) + \vec{j}\,(30\cos 30^o \cos 30^o) + \vec{k}(-30\sin 30^o)$$

$$\vec{u}_2 = \vec{i}\left(-\frac{1}{\sqrt{2}}\right) + \vec{j}\left(\frac{1}{2}\right) + \vec{k}\left(\frac{1}{2}\right)$$

$$F_{1on2} = F_{1x}\cdot u_{2x} + F_{1y}\cdot u_{2y} + F_{1z}\cdot u_{2z} = -5.4356 \longrightarrow$$

dir: opposite to $\vec{F}_2$

magn: $\boxed{5.44 \text{ lb}}$

- What is θ?

$$\cos\theta = \frac{F_{1on2}}{F_1} = \frac{-5.436}{30} \longrightarrow \boxed{\theta = \arccos(\dots) = 100^\circ}$$

IMPORTANT EQUATIONS: Summary

- Unit vector expressed in terms of direction angles: $\vec{u}_A = \vec{i}\ \cos\alpha + \vec{j}\ \cos\beta + \vec{k}\ \cos\gamma = \vec{i}\ \dfrac{A_x}{A} + \vec{j}\ \dfrac{A_y}{A} + \vec{k}\ \dfrac{A_z}{A}$

  Since $\vec{u}_A$ is a unit vector, we have: $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$

- Displacement vector pointing **from** point A **to** point B: $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A = \vec{i}\ (x_B - x_A) + \vec{j}\ (y_B - y_A) + \vec{k}\ (z_B - z_A)$

  $r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ -- its magnitude

- Force pointing in the direction **from** point A **to** point B: $\vec{F} = F\ \vec{u}_{AB} = F\ \dfrac{\vec{r}_{AB}}{r_{AB}} = F\ \dfrac{\vec{i}\,(x_B - x_A)\ + \vec{j}\,(y_B - y_A) + \vec{k}\,(z_B - z_A)}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$

- Four forms of the dot product: $\vec{A} \cdot \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z = A_B B = A B_A$

- Resolving vector $\vec{A}$ into components parallel and perpendicular to vector $\vec{B}$:

$$\vec{A}_{\parallel \text{ to } \vec{B}} = A_B\ \vec{u}_B = \left(\vec{A} \cdot \vec{u}_B\right) \vec{u}_B \qquad \vec{A}_{\perp \text{ to } \vec{B}} = \vec{A} - \vec{A}_{\parallel \text{ to } \vec{B}} = \vec{A} - \left(\vec{A} \cdot \vec{u}_B\right) \vec{u}_B \qquad \vec{A} = \vec{A}_{\parallel \text{ to } \vec{B}} + \vec{A}_{\perp \text{ to } \vec{B}}$$