Problem E4.2(☆☆): Two conducting spheres with radii R_1 and R_2 are placed far apart (distance $d \gg R_1, R_2$) from each other and are connected by a thin conducting wire.

(a) If the total charge on the spheres is Q (positive), find the equilibrium charges Q_1 and Q_2 on each sphere.

(b) Now the wire has been discarded, leaving the charges in place. Find the magnitude of the electric field E at the surface of each sphere.
Hint: use the potential from (a) and ignore the E-field from the other sphere.

(c) Sketch the potential V as a function of the position x along the line connecting the centres of the two spheres.

(a) The key realization is that $V_1 = V_2$ in equilibrium, else charges would flow.

The potential for each sphere is $V = \frac{kQ}{R}$, so:

$$\frac{kQ_1}{R_1} = \frac{kQ_2}{R_2} \quad \text{also} \quad Q_1 + Q_2 = Q$$

$$\therefore Q_1 = \frac{R_1 Q}{R_1 + R_2} \qquad Q_2 = \frac{R_2 Q}{R_1 + R_2}$$

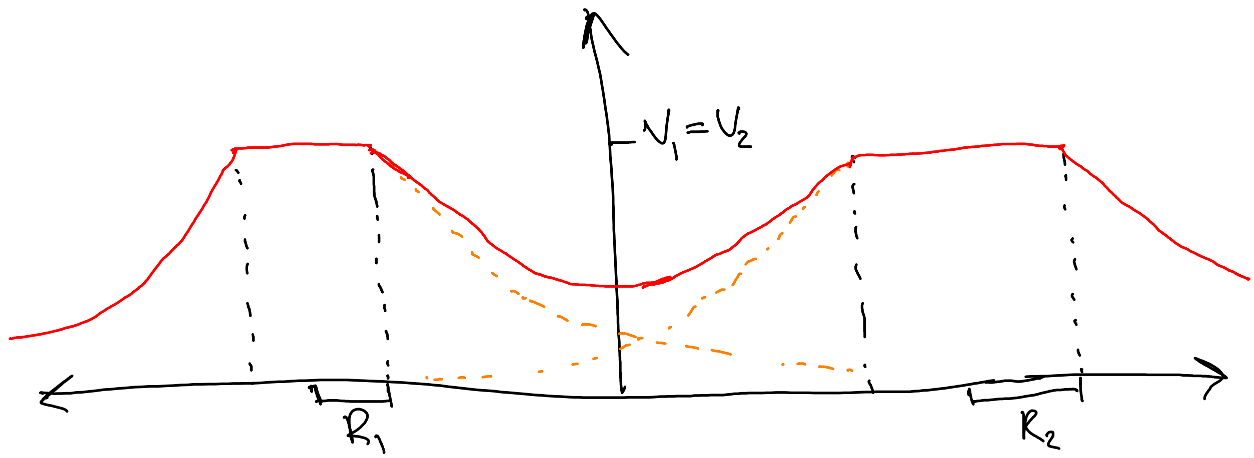
(b) We can take the derivative to get $E(r)$

$$E_1(r) = -\frac{dV}{dr}\bigg|_{r=R_1} = -\frac{d}{dr}\left(\frac{kQ_1}{r}\right)\bigg|_{r=R_1}$$

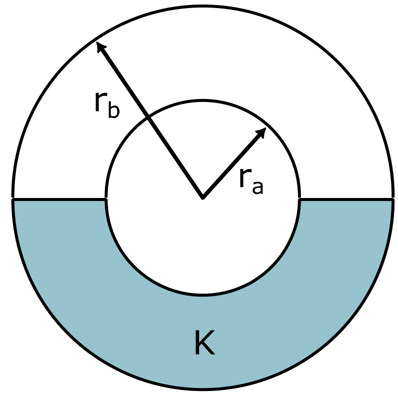
$$= \frac{kQ_1}{R_1^2} = \frac{kQ}{R_1(R_1+R_2)}$$

similarly: $E_2(r) = \dfrac{kQ}{R_2(R_1+R_2)}$

(c)



Problem E4.3(☆☆): An isolated spherical capacitor has charge $+Q$ on its inner conductor (radius r_a) and charge $-Q$ on its outer conductor (radius r_b). Half of the volume between the two conductors is filled with a liquid dielectric of constant K, as shown right.



(a) Find the capacitance of the half-filled capacitor.
Hint: find the capacitance of a spherical capacitor filled with air first, then apply the dielectric.

(b) Find the magnitude of E in the volume between the two conductors as a function of the distance r from the center of the capacitor. Give answers for the upper and lower halves of this volume.

(c) Find the surface density of free charge on the upper and lower halves of the inner and outer conductors.

(a) The capacitor equation is $Q=CV$. We can determine the potential difference by integrating the electric field:

$$\Delta V = kQ\int_{r_a}^{r_b}\frac{dr}{r^2} = KQ\left(\frac{1}{r_b}-\frac{1}{r_a}\right) = kQ\left(\frac{r_a-r_b}{r_a r_b}\right)$$

$$\therefore C_0 = \frac{Q}{V} = \frac{4\pi\varepsilon_0 r_a r_b}{r_a - r_b}$$

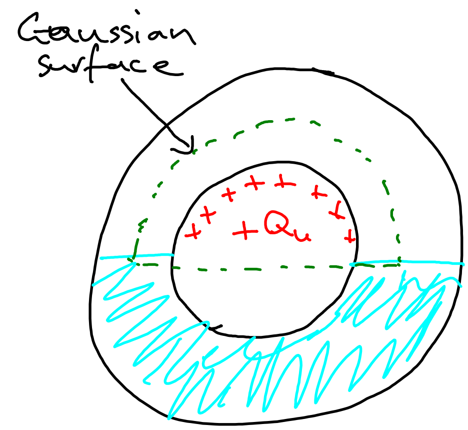
We switched to $k=\frac{1}{4\pi\varepsilon_0}$ to avoid confusion w/ dielectric K.

Lets consider the upper and lower halves separately. With half the plate area of a normal capacitor:

$$C_u = \frac{C_0}{2} \qquad C_L = \frac{KC_0}{2}$$

$$\therefore C_{tot} = \frac{(1+K)}{2}C_0 = 2\pi\varepsilon_0(1+K)\frac{r_a r_b}{r_a - r_b}$$

(b) It is tempting just to say $E' = \frac{E_0}{K}$, but because the sphere is only half-filled, the distribution of charges may have changed and will no longer be symmetrical.



Instead we can take a Gaussian surface around each respective hemisphere:

$$E_u \frac{4\pi r^2}{2} = \frac{Q_u}{\varepsilon_0} \Rightarrow E_u = \frac{Q_u}{2\pi\varepsilon_0 r^2}$$

$$E_L \frac{4\pi r^2}{2} = \frac{Q_L}{K\varepsilon_0} \Rightarrow E_L = \frac{Q_L}{2\pi\varepsilon_0 K r^2}$$

But we know from the capacitor equation that

$$V = \frac{Q_u}{C_u} = \frac{Q_L}{C_L} = \frac{2Q_u}{C_0} = \frac{2Q_L}{KC_0} \quad \text{so} \quad Q_u = \frac{Q_L}{K}$$

and $Q_u + Q_L = Q$ so $Q_u = \frac{Q}{1+K}$ & $Q_L = \frac{KQ}{1+K}$

$$\therefore \vec{E}_u = \frac{2}{1+K} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \vec{E}_L$$

Thus we find the E-fields are equal, which is good because the capacitor plates are equipotential and $\Delta V = \int_{r_a}^{r_b} \vec{E}\cdot d\vec{r}$ implies that $\vec{E}_u = \vec{E}_L$

(c) The free charge densities are just $\frac{Q}{\text{area}}$:

$$\sigma_{u,a} = \frac{Q_u}{2\pi r_a^2} = \frac{Q}{2\pi r_a^2(1+K)} \qquad \sigma_{u,b} = \frac{Q}{2\pi r_b^2(1+K)}$$

$$\sigma_{L,a} = \frac{Q_L}{2\pi r_a^2} = \frac{KQ}{2\pi r_a^2(1+K)} \qquad \sigma_{L,b} = \frac{KQ}{2\pi r_b^2(1+K)}$$