Physics 158 Written Homework 4–MH-Feb-9

Problem 1

Difficulty: ★☆☆
A 2 F capacitor, 0.25 H inductor, and a 100 Ω resistor are connected in series with a voltage source $v(t) = 25\cos(80t - \frac{\pi}{2})$

a) What is the impedence of this circuit?

b) What is the peak current?

c) What is the peak voltage across each element?

Solution:

a) We can recognize that the frequency, ω is 80Hz. The impedence can then be computed as

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} = \sqrt{100^2 + \left((0.25)(80) - \frac{1}{(2)(80)}\right)} = \boxed{101.98\,\Omega}$$

b) The peak current can be computed using the peak voltage and the impedence to be

$$I_{\text{peak}} = \frac{V_{\text{peak}}}{Z} = \frac{25\text{ V}}{101.98\,\Omega} = \boxed{0.245\text{ A}}$$

c) The peak voltage across each element can be computed as $V = I_{\text{peak}}X$.

- For the resistor:

$$V_R = I_{\text{peak}}R = (0.245\text{ A})(100\,\Omega) = \boxed{24.51\text{ V}}$$

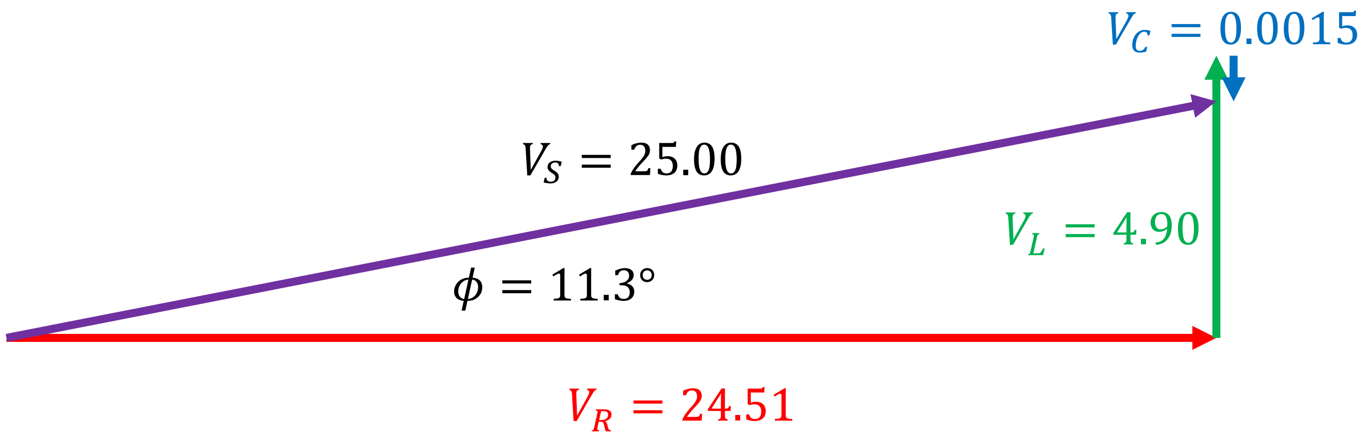
- For the capacitor:

$$X_C = \frac{1}{\omega C} = \frac{1}{(80)(2)} = 0.00625\,\Omega$$
$$V_C = I_{\text{peak}}X_C = (0.245)(0.00625)4 = \boxed{0.00153\text{ V}}$$

- For the inductor:

$$X_L = \omega L = (80)(0.25) = 20\,\Omega$$
$$V_L = I_{\text{peak}}X_L = (0.245)(20) = \boxed{4.90\text{ V}}$$

Q1(d) Phasor diagram



A circuit has an AC voltage in series with a resistor and capacitor connected in series. The AC voltage source has voltage amplitude 0.900 kV and angular frequency ω = 20.0 $\mathrm{rad/s}$. The voltage amplitude across the capacitor is 0.500 kV. The resistor has resistance R = 0.300 $\mathrm{k\Omega}$.

Part A

What is the voltage amplitude across the resistor?

Express your answer with the appropriate units.

ANSWER:

HW-4 Problem 2 Y&F 31-22

V_R = 748 V

Part B

What is the capacitance C of the capacitor?

Express your answer with the appropriate units.

ANSWER:

C = 249 $\mu\mathrm{F}$

Part C

Does the source voltage lag or lead the current?

ANSWER:

- ○ Leads the current
- ◉ Lags the current

Part D

What is the average rate at which the ac source supplies electrical energy to the circuit?

Express your answer with the appropriate units.

ANSWER:

P_{av} = 932 W

HW-4-Question 3 — T13

Difficulty: ★★☆

Mystery RLC circuit: You are given an RLC circuit with elements connected in series. Values of R, L and C are unknown. You have at your disposal a source of AC voltage with $V_{\text{RMS}} = 8\,\text{V}$ and a tunable frequency ω. You also have an Ammeter which measures the RMS current I_{RMS} and the power factor $\cos\phi$.

Suppose you measured I_{RMS} as a function of frequency and found that the maximum RMS current occurs at $\omega_0 =$ ~~12.5 kHz~~ 12,500 rad/s and is equal to $40\,\text{mA}$.

can also use f_0 = 12.5 kHz, w_0 = 78,540 rad/s

a) What is the resistance, R? What does this tell you about L and C?

b) What is the power factor at $\omega = \omega_0$?

c) In addition you find that at $\omega_1 =$ ~~17 kHz~~ 17,000 rad/s the power factor is 0.5. Based on this information, what are the values of L and C?

Solution:

The key formulas that will help solve this problem are

$$I_{\text{RMS}} = \frac{V_{\text{RMS}}}{\sqrt{R^2 + (L\omega - \frac{1}{C\omega})^2}}$$

and

$$\tan\phi = \frac{L\omega - \frac{1}{C\omega}}{R}$$

a) At resonance frequency, we will have $L\omega_0 = \frac{1}{C\omega_0}$. This simplifies our above equation for the current to be $I_{\text{RMS}} = \frac{V_{\text{RMS}}}{R}$. Rearranging we can solve for R to be

$$R = \frac{V_{\text{RMS}}}{I_{\text{RMS}}} = \frac{8\,\text{V}}{40\,\text{mA}} = \boxed{200\,\Omega}$$

b) At resonance, the circuit is purely resistive so $\phi = 0$. Therefore, $\boxed{\cos\phi = 1}$

c) At $\omega_1 = 17\,\text{kHz}$, $\cos\phi = 0.5$ so $\phi = 60°$.

$$\tan 60° = \sqrt{3} = \frac{L\omega_1 - \frac{1}{\omega_1 C}}{R}$$

$$L\omega_1 - \frac{1}{C\omega_1} = R\sqrt{3}$$

and from part a we have $\frac{1}{C\omega_0} = L\omega_0$

Solve for L:

$$L\left(\omega_1 - \frac{\omega_0^2}{\omega_1}\right) = R\sqrt{3} \Rightarrow L = \frac{R\sqrt{3}}{\omega_1 - \frac{\omega_0^2}{\omega_1}} = 44.3\,\text{mH}$$

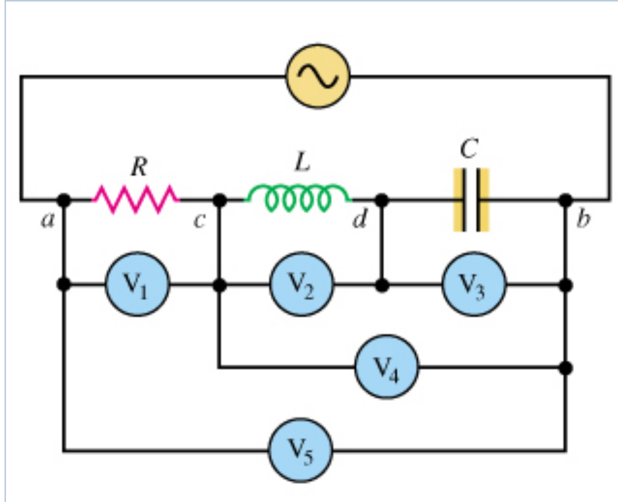
$$C = \frac{1}{L\omega_0^2} = 0.144\,\text{pF}$$

Using other units

f_0 = 12.5 kHz		
w_0 = 78,500 rad/s	12,500 rad/s	12.5 rad/s
f_1 = 17.0 kHz		
w_0 = 106,814 rad/s	17,000 rad/s	17.0 rad/s
C = 22.96 nF	144.3 nF	144.3 uF
L = 7.06 mH	44.4 mH	44.4 H

Five infinite-impedance voltmeters, calibrated to read rms values, are connected as shown in figure . Let $R = 200\ \Omega$, $L = 0.400\ \mathrm{H}$, $C = 6.00\ \mu\mathrm{F}$ and V_rms = 30.0 Volts

HW-4-Problem 4 Y&F 31-40
modified for V_source = 30.0 rms



Q4 $\quad V_s^{RMS} = 30.0\,V,\; R = 200\,\Omega,\; L = 0.40\,H,\; C = 6.0\,\mu F$

$\omega = 200\,\text{rad/s}$

(a) $\omega = 200\,\text{rad/s}$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$X_L = 200(0.4) = 80\,\Omega$

$X_C = \frac{1}{\omega C} = \frac{10^6}{200(6)} = 833.3\,\Omega$

$$Z = \sqrt{200^2 + (753.3)^2} = 779.4\,\Omega$$

Hence $I^{RMS} = \frac{V^{RMS}}{Z} = \frac{30.0}{779.4} = 0.0385\,A$

$\Rightarrow V_R^{RMS} = I^{RMS} R = 7.70\,V,\quad V_2 = V_L = 3.08\,V,\quad V_3 = V_C = I X_C = 32.1\,V$

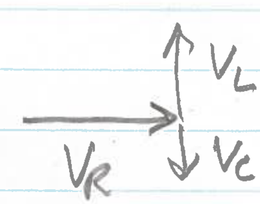
$V_4 = |V_L - V_C| = 29.0\,V,\quad V_5 = \sqrt{V_R^2 + (V_L - V_C)^2} = 30.0\,V = V_S^{RMS}$

(b) $\omega' = 1000\,\text{rad/s}$

$X_L = 1000(0.4) = 400\,\Omega,\quad X_C = \frac{1}{\omega C} = \frac{10^6}{10^3(6)} = 166.7\,\Omega$

$$Z = \sqrt{200^2 + (400 - 166.7)^2} = 307.3\,\Omega$$

$I'^{RMS} = 30.0/307.3 = 0.0976\,A$



$V_1 = V_R^{RMS} = I' R = 0.0976\,(200) = 19.52\,V$

$V_2 = I' X_L = 39.04\,V,\quad V_3 = V_C = 16.27\,V,\quad V_4 = |V_L - V_C|$

$V_4 = 22.77\,V$

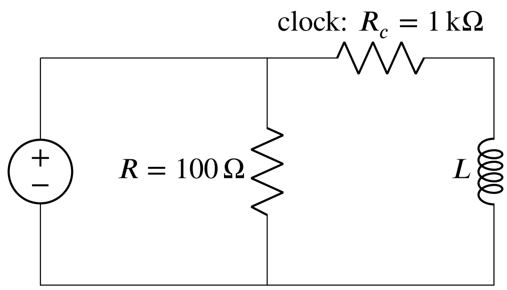
$V_5 = \sqrt{V_R^2 + (V_L - V_C)^2} = 29.99\,V$ as expected.

There is no need to convert $V_s^{RMS} = 30.0\,V$ into $V_s^{peak} = 30\sqrt{2} = 42.43$ Volts.

HW-4-Question 5 — T6

Difficulty: ★★☆

Using their newfound knowledge of LR circuits, a Phys 158 student came up with a clever idea for a prank. They want to design an alarm clock that will continue to ring for 10 seconds after the battery is removed. The alarm clock can be thought of as a 1 kΩ resistor which requires at least 1 Watt to operate. They designed the following circuit to achieve this.



a) What value should the battery be such that the power supplied to the alarm clock does not exceed 3 Watts?

b) What value of inductor should they use so that the alarm clock remains on for 10 seconds after the battery is disconnected?

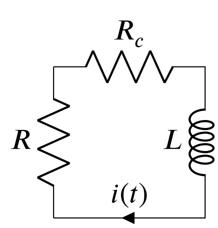
Solution:

a) For maximum power to be supplied to the clock, $P_c = 3\,\mathrm{W}$

$$P_C = i_C^2 R_C \Rightarrow i_C = \sqrt{\frac{P_C}{R_C}}$$

$$\varepsilon = V_C = i_C R_C = \sqrt{\frac{P_C}{R_C}} \cdot R_C = \sqrt{P_C R_C} = \sqrt{(3\,\mathrm{W})(1000\,\Omega)} = 54.8\,\mathrm{V}$$

b) We can start by writing out Kirchoff's loop voltage law for the circuit and then solving the resulting differential equation by separation of variables to get an expression for the current as a function of time:



$$iR + iR_C + L\frac{di}{dt} = 0$$

$$L\frac{di}{dt} = -i(R + R_C)$$
$$\frac{di}{dt} = -\frac{R + R_C}{L}i$$
$$\frac{di}{i} = -\frac{R + R_C}{L}dt$$
$$\int \frac{di}{i} = -\frac{R + R_C}{L}\int dt$$
$$\ln|i| = -\frac{R + R_C}{L}t + \text{Constant}$$
$$i(t) = e^{-\frac{R+R_C}{L}t + \text{Constant}} = i_0 e^{-\frac{R+R_C}{L}t}$$

Alternatively, we can get the same expression by thinking about it conceptually and computing the time constant.
We know that the current will initially want to stay the same because of the inductor and it will slowly decay to 0 so we know that the equation of the current should look like exponential decay and be of the form

$$i(t) = i_0 e^{-t/\tau}$$

We can then compute the time constant for an RL circuit as

$$\tau = \frac{L}{R_{eq}} = \frac{L}{R + R_C}$$

Plugging this in will yield the same expression as above.
The initial current, i_0, will be the current that was initially flowing through the inductor. We computed this in part (a) to be

$$i_0 = i_C = \sqrt{\frac{P_C}{R_C}} = \sqrt{\frac{3\,\mathrm{W}}{1000\,\Omega}} = 54.8\,\mathrm{mA}$$

Now we have a complete expression for the current as a function of time. We can get the power as a function of time as

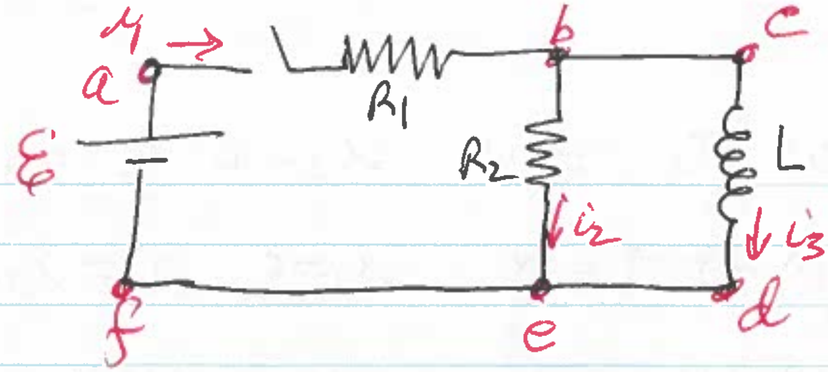
$$P(t) = i^2 R_C = i_0^2 R_C e^{-\frac{2(R+R_C)}{L}t}$$
$$i_0^2 R_C = \frac{P_C}{R_C} \cdot R_C = P_C$$
$$P(t) = P_C e^{-\frac{2(R+R_C)}{L}t}$$

We are told that the clock must have a minimum of 1 Watt and we want it to last for 10 seconds so we can set $P = 1$ and $t = 10$ and solve for L.

$$\frac{P}{P_C} = e^{-\frac{2(R+R_C)}{L}t}$$
$$\ln\left(\frac{P}{P_C}\right) = -\frac{2(R + R_C)}{L}t$$
$$L = -\frac{2(R + R_C)t}{\ln\left(\frac{P}{P_C}\right)} = -\frac{2(100\,\Omega + 1000\,\Omega)(10\,\mathrm{s})}{\ln\left(\frac{1\,\mathrm{W}}{3\,\mathrm{W}}\right)} = 20,025\,\mathrm{H}$$

HW4 Question 6

(c) (d)



① K1 junction $\Rightarrow$ $i_1 = i_2 + i_3$

K2 loop rule $\Rightarrow$

② $\mathcal{E} - i_1 R_1 - i_2 R_2 = 0$

③ $V_{be} = V_{cd} \Rightarrow i_2 R_2 = L \dfrac{di_3}{dt}$

$R_1 = 8.0\,\Omega$ $R_2 = 6.0\,\Omega$ $L = 0.2\,H$

$\mathcal{E} = 48.0$ Volts

From ① $i_2 = i_1 - i_3$

From ② $i_1 = (\mathcal{E} - i_2 R_2)/R_1$

substituting into ① we now have $i_2 = \dfrac{\mathcal{E} - i_2 R_2}{R_1} - i_3$

$$\Rightarrow i_2\left(1 + \frac{R_2}{R_1}\right) = \frac{\mathcal{E}}{R_1} - i_3$$

$$\Rightarrow \boxed{i_2 = \frac{\mathcal{E}}{R_1 + R_2} - \frac{i_3 R_1}{R_1 + R_2}} \Rightarrow \boxed{i_2 R_2 = \frac{\mathcal{E} R_2}{R_1 + R_2} - \frac{i_3 R_1 R_2}{R_1 + R_2}}$$

Now we substitute $i_2 R_2$ into ③

$$\Rightarrow L \frac{di_3}{dt} = \frac{R_1 R_2}{R_1 + R_2}\left(\frac{\mathcal{E}}{R_1} - i_3\right)$$

$$\Rightarrow \frac{di_3}{dt} = \frac{-R_1 R_2}{L(R_1 + R_2)}\left(i_3 - \frac{\mathcal{E}}{R_1}\right) \quad \text{or}$$

$$\boxed{\frac{di_3}{\left(i_3 - \frac{\mathcal{E}}{R_1}\right)} = -\left(\frac{R_1 R_2}{R_1 + R_2}\right)\frac{1}{L}\,dt} \Rightarrow$$

eqn 30.14 with $R' \rightarrow \dfrac{R_1 R_2}{R_1 + R_2}$ $\boxed{i_3(t) = \dfrac{\mathcal{E}}{R_1}\left(1 - e^{-\frac{R' t}{L}}\right)}$

(a) at $t = 0+$ $\quad i_3 = 0 \quad i_1 = i_2 = \dfrac{\mathcal{E}}{R_1 + R_2} = \dfrac{48.0}{14} = 3.43A$

(b) at $t = \infty$ $\quad i_2 = 0 \quad i_1 = i_3 = \dfrac{\mathcal{E}}{R_1} = \dfrac{48}{8} = 6.0\,A$

(c) $R' = R_1 R_2 / (R_1 + R_2) = \dfrac{6(8)}{14} = 3.43\,\Omega$

$$i_3(t') = i_3(\infty)\left(1 - e^{-\frac{R't}{L}}\right) = 6\left(1 - e^{-\frac{3.43}{0.2}t'}\right)$$

$$\frac{i_3(\infty)}{2} = i_3(\infty)\left(1 - e^{-17.15t'}\right) \Rightarrow t' \text{ when } i_3 = \frac{i_3(\infty)}{2}$$

$$\Rightarrow \quad 0.5 = \left(1 - e^{-17.15t'}\right)$$

$$\Rightarrow \quad e^{-17.15t'} = 0.50 \quad \Rightarrow \quad -17.15t' = \ln(0.50) = -0.693$$

$$\boxed{t' = 40.4\ \text{msec}}$$