Lecture 17.

Gauss’s law.

Symmetry of the charge distribution.

Applications: E-field of:

- point charge
- infinite sheet of charge
- infinite line of charge
- charged sphere (if time permits)

Gauss's law

Last Time

$$\frac{1}{4\pi\varepsilon_0} = k$$

$$\boxed{\Phi_e = \frac{Q_{\text{in}}}{\varepsilon_0}}$$

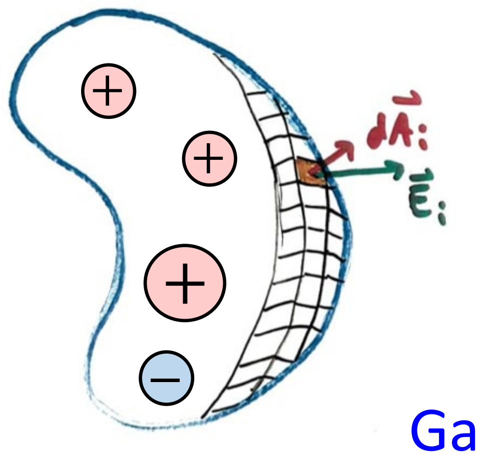
➢ Net electric flux through a closed surface = charge inside that surface/ε_0

- Now, let's recall the definition of the flux: $\Phi_e = \oint \vec{E} \cdot d\vec{A}$. We get:

$$\boxed{\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}}$$

➢ Q_{in}: total (net) charge inside a closed surface.

➢ The $\oint \ldots d\vec{A}$ notation: integrate the flux over a closed surface (enclosing the charge Q_{in})



Gauss's law relates electric field with charges that create this field

Q: Rank the magnitudes of the electric flux passing through the 4 surfaces.

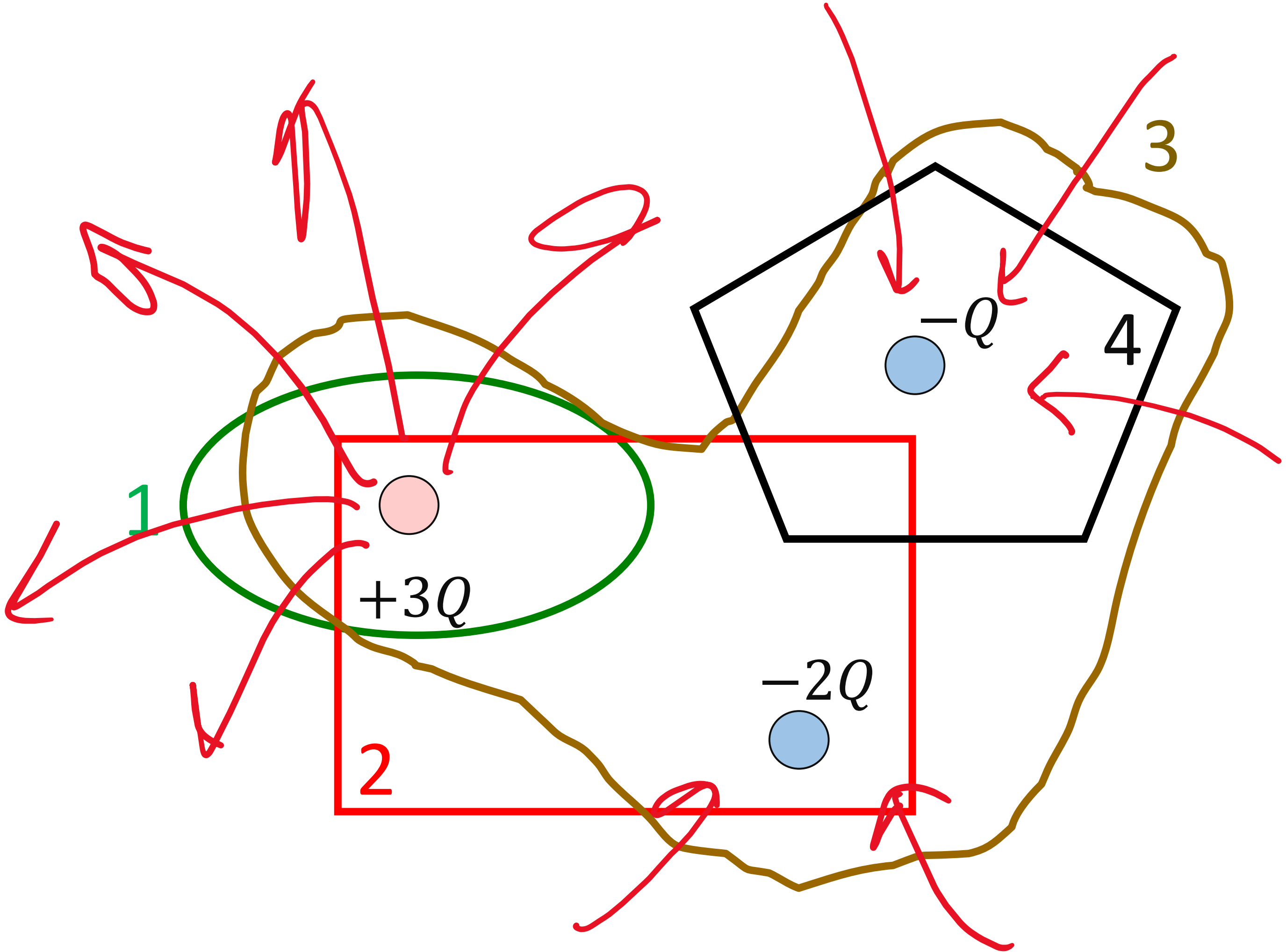
$$\phi_e = \frac{Q_{in}}{\varepsilon_0}$$

1: $+3Q$

2: $+Q$

3: 0

4: $-Q$



A. 1 = 2 = 3 = 4

B. 1 > 2 = 4 > 3

C. 3 > 2 > 1 > 4

D. 3 > 2 > 1 = 4

E. None of the above

Q: Rank the magnitudes of the electric flux passing through the 4 surfaces.

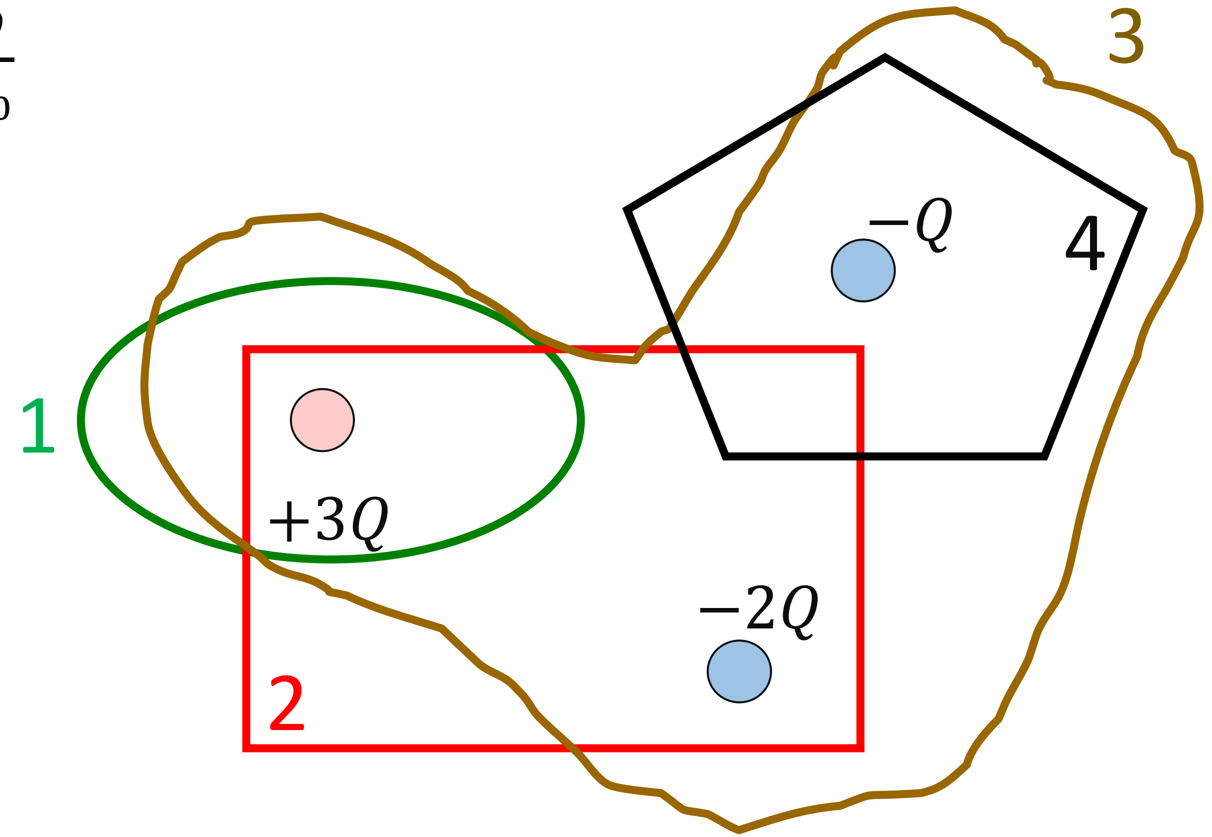
Surface 1: Net charge is $+3Q \Rightarrow |\Phi_e| = \frac{3Q}{\varepsilon_0}$

Surface 2: Net charge is $+Q \Rightarrow |\Phi_e| = \frac{Q}{\varepsilon_0}$

Surface 3: Net charge is $0 \Rightarrow |\Phi_e| = 0$

Surface 4: Net charge is $-Q \Rightarrow |\Phi_e| = \frac{Q}{\varepsilon_0}$

We are interested in the magnitude of the flux, so B. Note that the flux through region 2 is outward (+), while the flux through region 4 is inward (−).



A. 1 = 2 = 3 = 4

B. 1 > 2 = 4 > 3

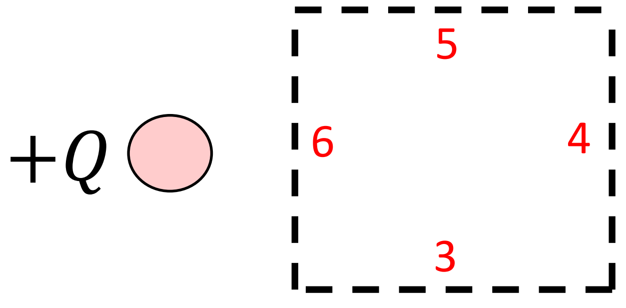
C. 3 > 2 > 1 > 4

D. 3 > 2 > 1 = 4

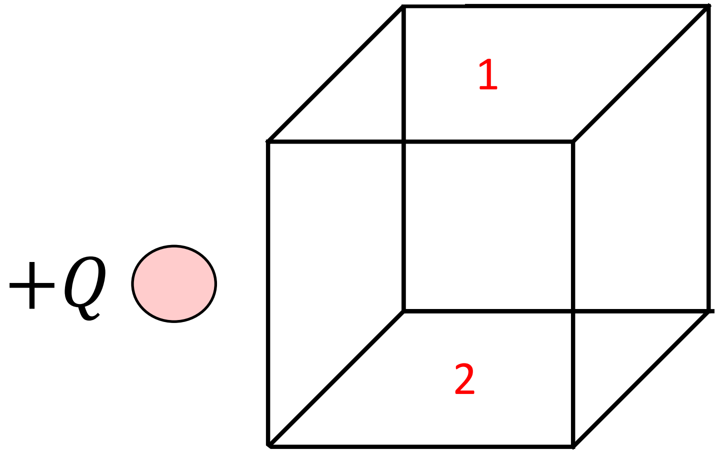
E. None of the above

Q: What is the sign (positive or negative) of the NET electric flux passing through the four side surfaces of this cube?

(Remember the cube has 6 sides. It has a TOP, BOTTOM and 4 SIDE surfaces)



Top view



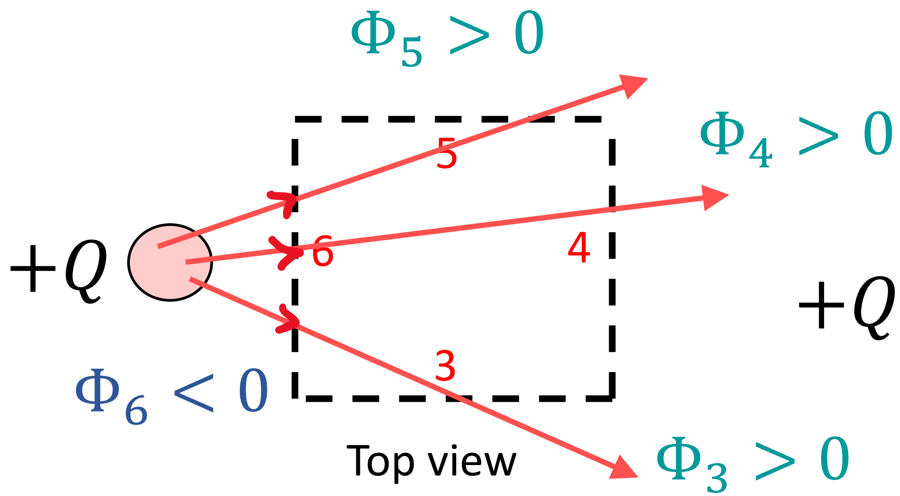
3-D view

A. Positive
B. Negative
C. Zero
D. Not enough information is given to answer

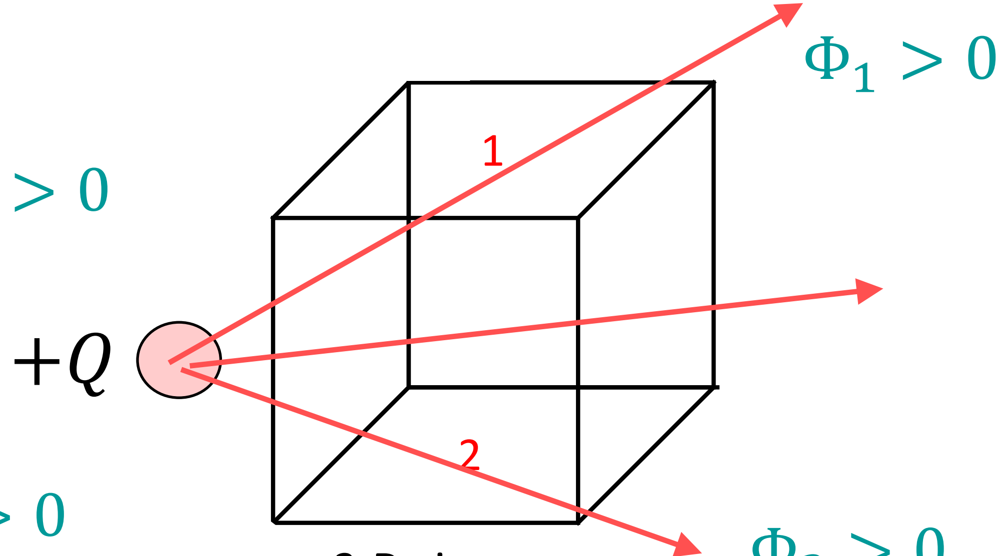
$$\Phi_{net} = \frac{Q_{in}}{\varepsilon_0} = 0$$

Q: What is the sign (positive or negative) of the NET electric flux passing through the four side surfaces of this cube?

(Remember the cube has 6 sides. It has a TOP, BOTTOM and 4 SIDE surfaces)



Top view



3-D view

A. Positive
B. Negative
C. Zero
D. Not enough information is given to answer

- $\Phi_1 > 0$ & $\Phi_2 > 0$ (since the flux is outwards)
- Since there is no charge inside: $(\Phi_1 + \Phi_2) + (\Phi_3 + \Phi_4 + \Phi_5 + \Phi_6) = 0$
- Hence, $\Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 < 0$

$\phi_{net} = \frac{Q_{in}}{\varepsilon_0} = 0$

How to make Gauss's law user-friendly?

$$\boxed{\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}}$$

where the integral is taken over an arbitrary surface enclosing the charge Q_{in}.

- Note that here $\vec{E}$ is the electric field at that surface.

- Gauss's law is useful only if we can calculate this integral easily. We could do it if:

- $\vec{E}$ is tangent to the surface: $\vec{E} \cdot d\vec{A} = E\, dA \cos 90^o = 0$. Then:

$$\Phi_e = 0$$

- $\vec{E}$ is normal to the surface, and constant at every point of that surface. Then:

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint dA = EA$$

How to make Gauss's law user-friendly?

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}$$

- $\vec{E}$ is tangent to the surface: $\vec{E} \cdot d\vec{A} = E\,dA \cos 90^o = 0$. Then:

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- $\vec{E}$ is normal to the surface, and constant at every point of that surface. Then:

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \oint E\,dA = E \oint dA = EA$$

- How can we choose the integration surface ("Gaussian surface") with such properties?

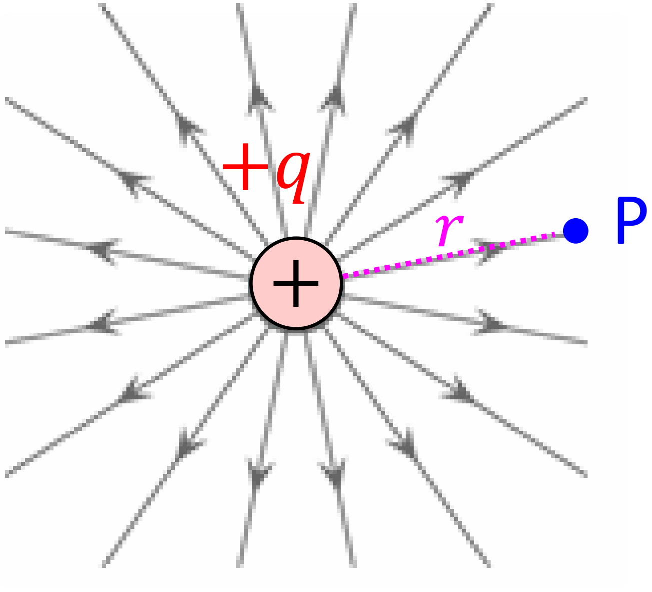
  (a) fixed angle between $\vec{E}$ and $d\vec{A}$, (b) same magnitude of the field, E, on the surface

- We can do it only in some convenient circumstances:
  1. The charge distribution has high symmetry;
  2. We can come up with a surface (*Gaussian surface*) that would match this symmetry.

- Remember that in the Gauss's law the integral is taken over any arbitrary surface enclosing the charge – the choice of the Gaussian surface is ours!

Using Gauss's law (example 1)

Q: Use Gauss's law to find electric field of a point charge $+q$ at a distance r from it.



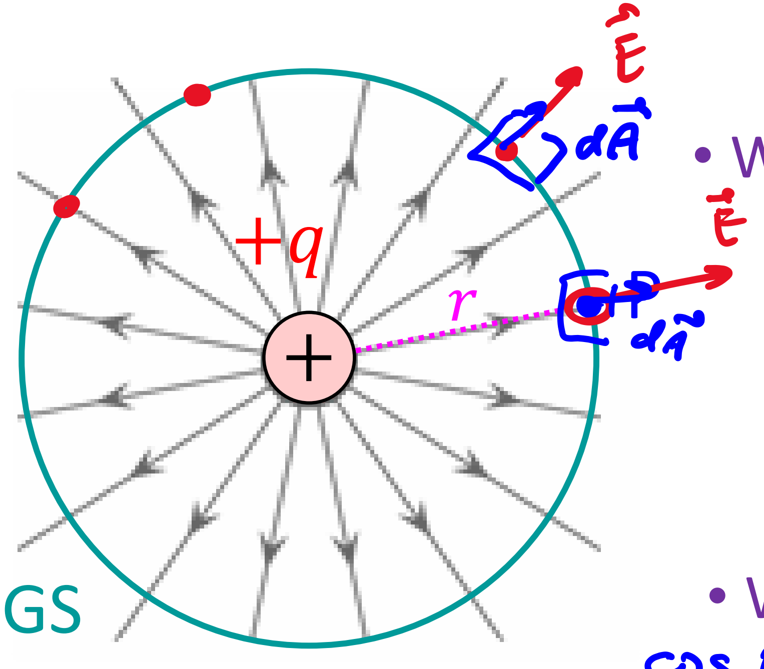
- Which Gaussian surface matches the symmetry of the charge?

- What is the flux through it?

- Use Gauss's law to set up the charge-field relationship.

Using Gauss's law (example 1)

Q: Use Gauss's law to find electric field of a point charge $+q$ at a distance r from it.



- Which Gaussian surface matches the symmetry of the charge?

$\vec{E} \cdot d\vec{A}$

  - Sphere centered on $+q$, passing through P
  - Same $\vec{E}$ at all points of this sphere, perpendicular to its surface

- What is the flux through it?

$\cos\theta = 90°$

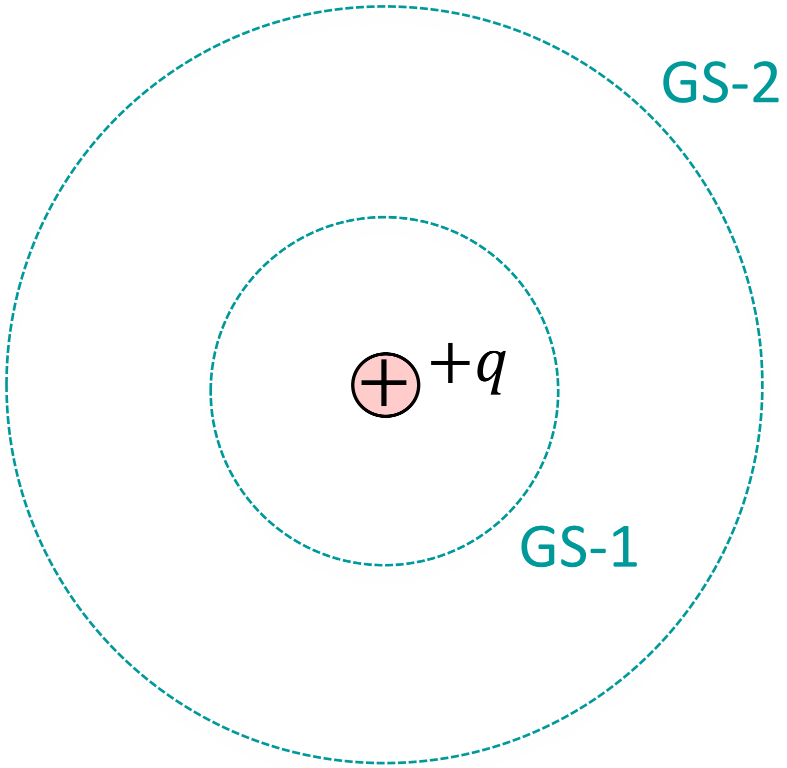
$$\Phi_e = \int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \int_{\text{sphere}} E\, dA = E \int_{\text{sphere}} dA = E A_{\text{sphere}} = 4\pi r^2\, E$$

- Use Gauss's law to set up the charge-field relationship.

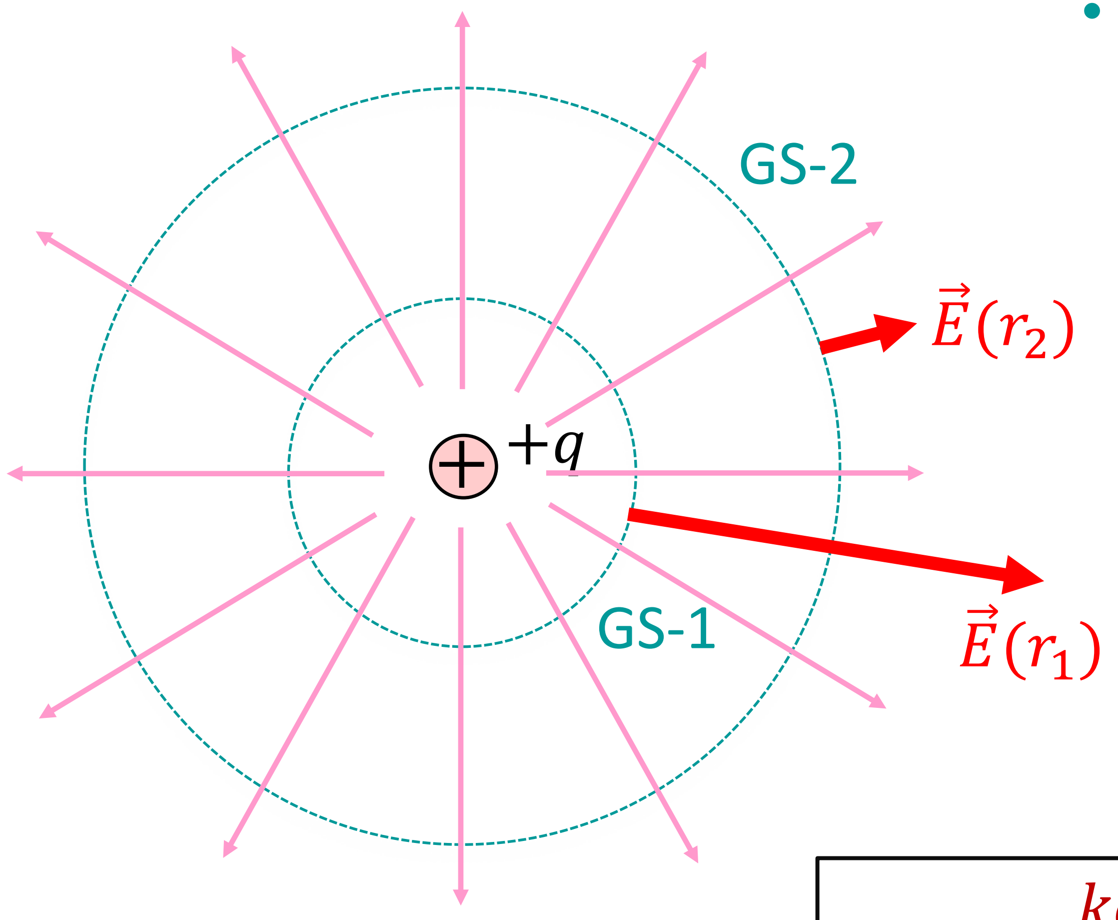
$$\Phi_e = 4\pi r^2\, E = \frac{q}{\varepsilon_0} \quad \Rightarrow \quad \boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{kq}{r^2}}$$

Coulomb's law

Q: How is that possible that the flux through these two Gaussian surfaces is equal to the same value, q/ε_0? They have different areas!



Q: How is that possible that the flux through these two Gaussian surfaces is equal to the same value, q/ε_0? They have different areas!



- Indeed, they have the same flux: the same number of electric field lines through them!

- The key to this is the magnitude of E-field:

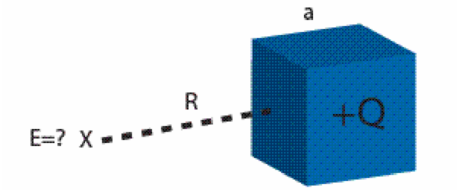
$$\Phi_e = \int_{\text{sphere}} \vec{E} \cdot d\vec{A} = E A_{\text{sphere}} = 4\pi r^2\, E$$

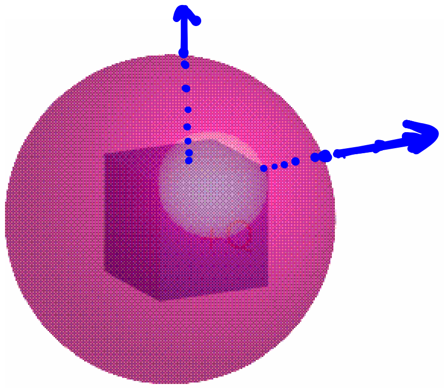
Larger distance away from the charge (and hence larger area over which the flux is distributed) is compensated by a smaller electric field magnitude

$$\boxed{E(r) = \frac{kq}{r^2}}$$

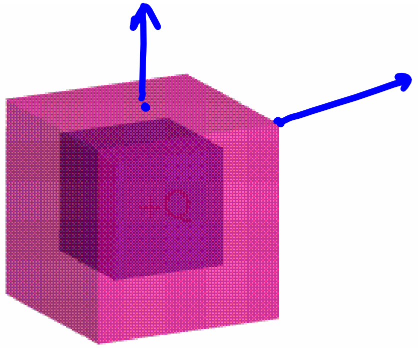
Symmetries

Q: You are asked to use Gauss's Law to calculate the electric field at a distance r away from a charged cube of dimension a. Which of the following Gaussian surfaces is best suited for this purpose?

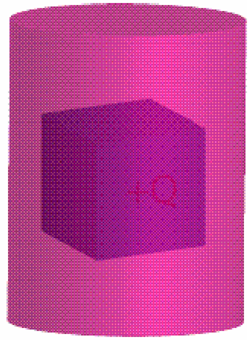




~~A~~



~~B~~



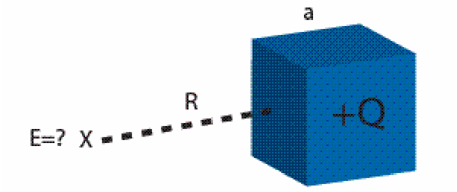
~~C~~

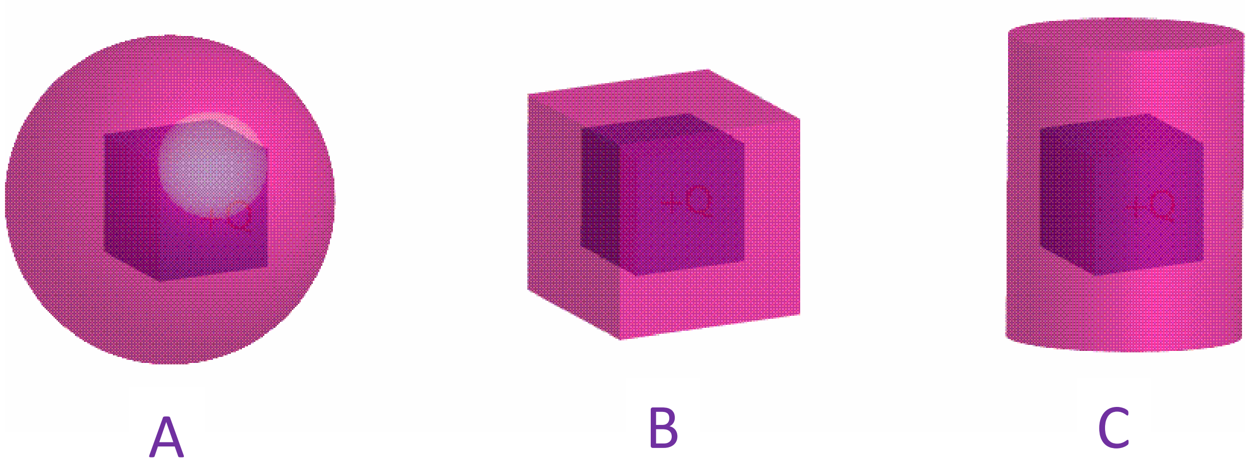
D. None of them

E. All of them

Symmetries

Q: You are asked to use Gauss's Law to calculate the electric field at a distance r away from a charged cube of dimension a. Which of the following Gaussian surfaces is best suited for this purpose?





D. None of them

E. All of them

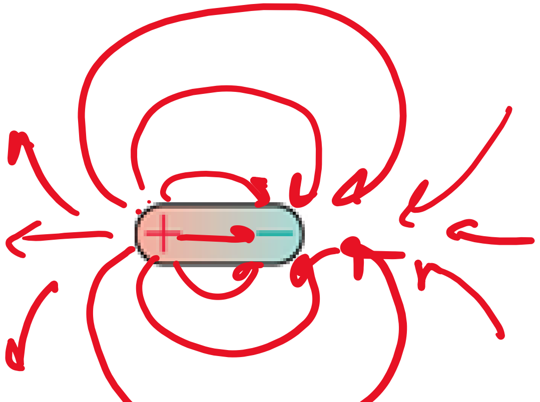
- What are the properties of a useful Gaussian surface?

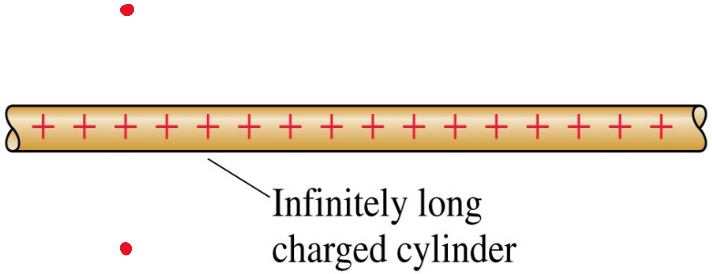
➢ It should, at least, clearly have the same value of E in all its points.

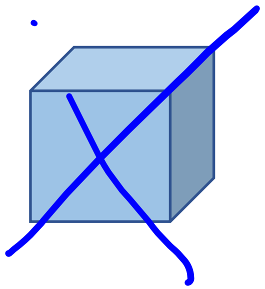
➢ To make it possible, the charge distribution should have "enough symmetry"

Gauss's law can be used to calculate electric fields only in cases of highly-symmetric charge distributions.

Q: Which of the objects below have an electric field that is symmetric enough to make Gauss's law useful?







A. Only electric dipole

B. Only point charge & insulating cube

C. Only long uniformly charged cylinder

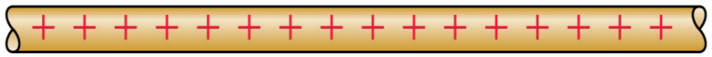
D. Only point charge

E. Only the point charge & long cylinder

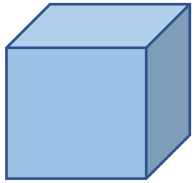
Gauss's law can be used to calculate electric fields only in cases of highly-symmetric charge distributions.

Q: Which of the objects below have an electric field that is symmetric enough to make Gauss's law useful?

- Directions: not equivalent



- Directions: not equivalent



- All points at a distance r from the point charge are equivalent => expect same field magnitude. Direction: radially away form the charge (there is no reason for it to be tilted in any direction)

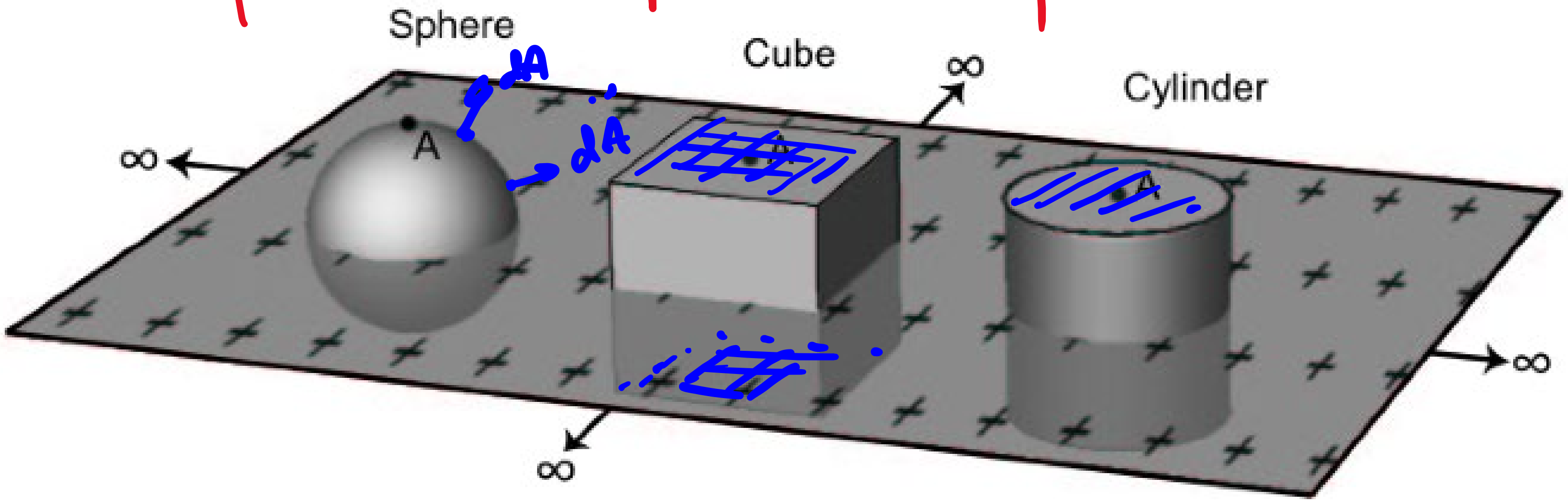
- All points at a distance r from the rod are equivalent => expect same field magnitude (no reason to be different below or above the rod). Direction: perpendicular to the rod (there is no reason for it to be tilted left or right)

D. Only point charge

E. Only the point charge & long cylinder

Gauss’s law can be used to calculate electric fields only if we manage to come up with a Gaussian surface that matches the symmetry of the charge distributions.

Q: To successfully apply Gauss’s law, which Gaussian surface will work for a large (infinite) sheet of charge?

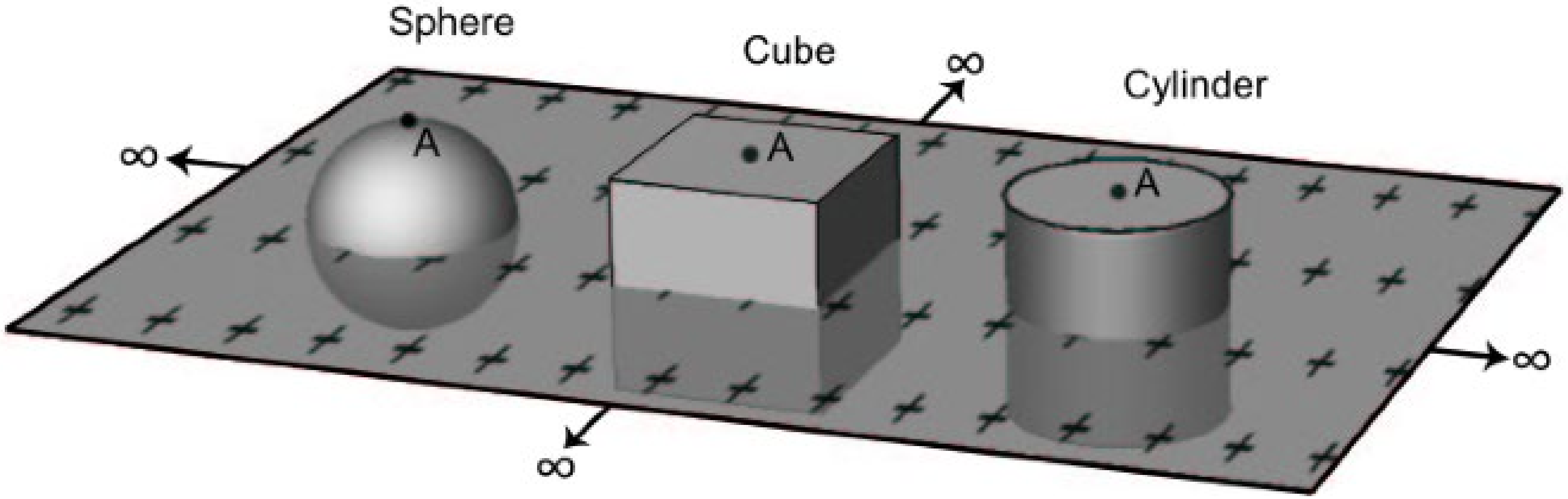


A. Only the sphere
B. Only the cube
C. Only the cylinder
D. Only the cylinder and the cube
E. All Gaussian surfaces will work

Gauss's law can be used to calculate electric fields only if we manage to come up with a Gaussian surface that matches the symmetry of the charge distributions.

Q: To successfully apply Gauss's law, which Gaussian surface will work for a large (infinite) sheet of charge?

- E-field will be the same at the flat surfaces of cube and cylinder parallel to the plane



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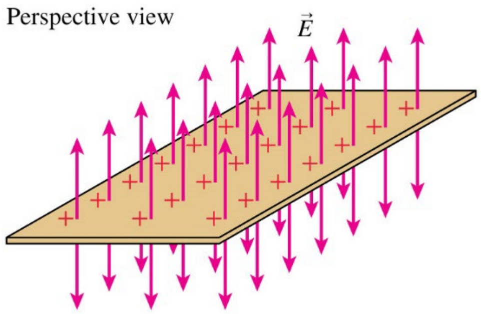


- Shape of electric field that we expect from symmetry arguments

Using Gauss's law (example 2)

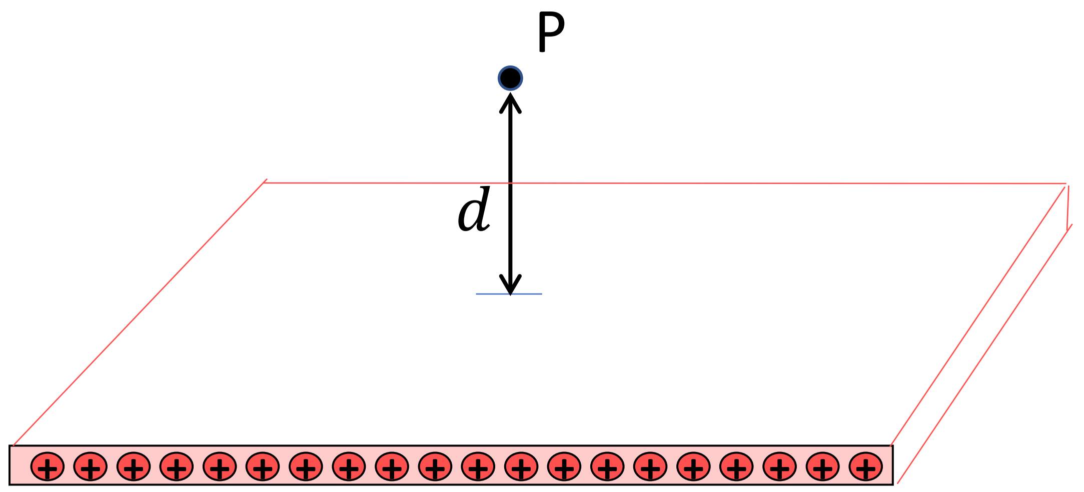
Q: Find electric field of an infinite charged plane at a vertical distance d from it. Assume uniform surface charge density $\sigma = \frac{Q}{A}$ (C/m^2)

1. Matching GS?

2. Flux: $\Phi_e = \oint \vec{E} \cdot d\vec{A} = ?$

3. Enclosed charge: $Q_{in} = ?$

4. Gauss's law: $\Phi_e = Q_{in}/\varepsilon_0$



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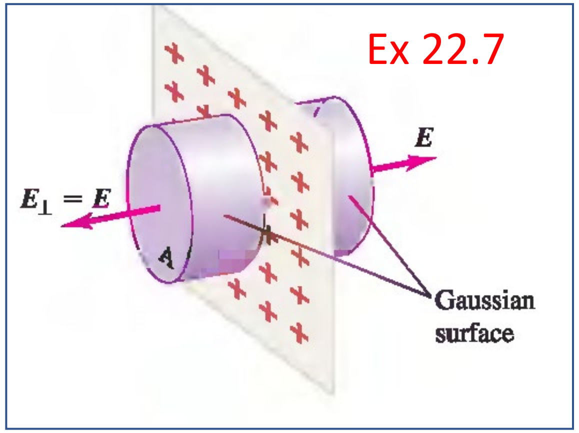
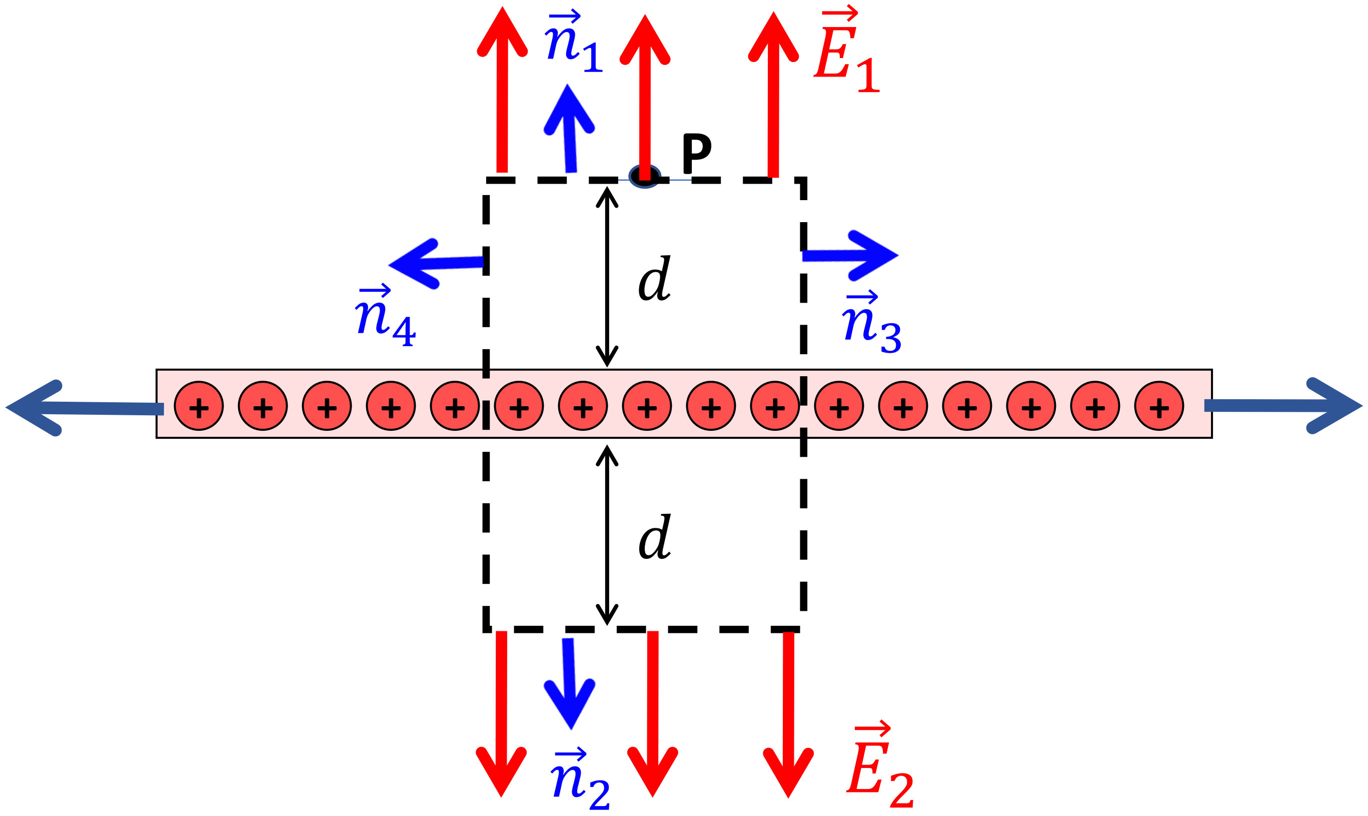


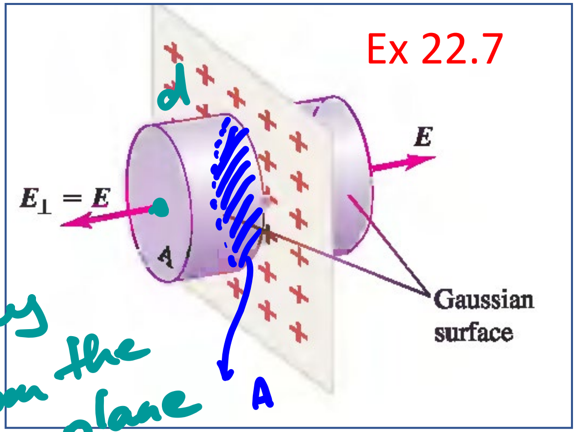
- Prism / cylinder, with horizontal sides above and below the sheet by d



- By symmetry, the electric field lines should be perpendicular to the sheet => no flux is lost through the vertical sides of the prism => all flux goes through the top and the bottom

Using Gauss's law (example 2)

Q: Find electric field of an infinite charged plane at a vertical distance d from it. Assume uniform surface charge density $\sigma = \frac{Q}{A_{plane}}$ (C/m^2)



1. Matching GS?

Prism / cylinder (only T = top, B = bottom)

$Q_{in} = \sigma A$

@d away from the plane

2. Flux: $\Phi_e = \oint \vec{E} \cdot d\vec{A} = ?$

$$\Phi_e = \int_{T,B} E\, dA \cos 0^0 = E \int_{T,B} dA = E\left(A_{top} + A_{bottom}\right) = 2E\cancel{A}$$

$$\frac{Q_{in}}{\varepsilon_0} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

3. Enclosed charge: $Q_{in} = ?$

Charge sitting on the area A: $Q_{in} = \sigma A$

4. Gauss's law: $\Phi_e = Q_{in}/\varepsilon_0$

$$2EA = \sigma A/\varepsilon_0$$

$$\boxed{E(d) = \frac{\sigma}{2\varepsilon_0}}$$

does not depend on d