PHYS 170

Week 8: Kinematics 2

Section 201 (Mon Wed Fri 12:00 – 13:00)

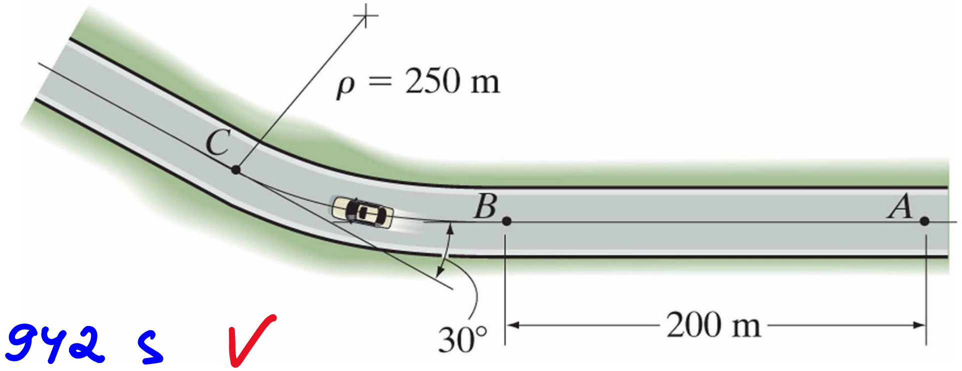
Exam: Thursday, March 9th , 6 pm

- Prepare a useful information sheet
  - 1 double-sided 8½ x 11 in hand-written sheet of your own notes
  - No sample problems or solutions. Can contain formulas, strategies, information…
  - Purpose: review and prioritization
  - You will hand it in, it won't be returned (make a copy?)

W7-5. The car is travelling at 25 m/s at A. The brakes are applied at and its speed is reduced by $t^{1/2}/4$ m/s² where t is in seconds.

a) Determine how long it takes the car to travel from A to C.

b) Determine the car's speed and acceleration when it reaches C.



$$s(t) = \left(25t - \frac{1}{15}\cdot t^{\frac{5}{2}}\right) = 200 + 130.9 \text{ m}$$

$t = 15.942$ s ✓

$t = 39.674$ s ✗

$$v(t) = 25 - \frac{1}{6}t^{3/2} \quad = 14.39\,\frac{m}{s}$$

$$a_t(t) = -\frac{1}{4}\sqrt{t} \; = 0.988\,\frac{m}{s^2} \qquad a_n(t) = \frac{v^2(t)}{\rho} = 0.8284\,\frac{m}{s^2} \qquad a = \sqrt{a_t^2 + a_n^2} = 1.30\,\frac{m}{s^2}$$

Last Time

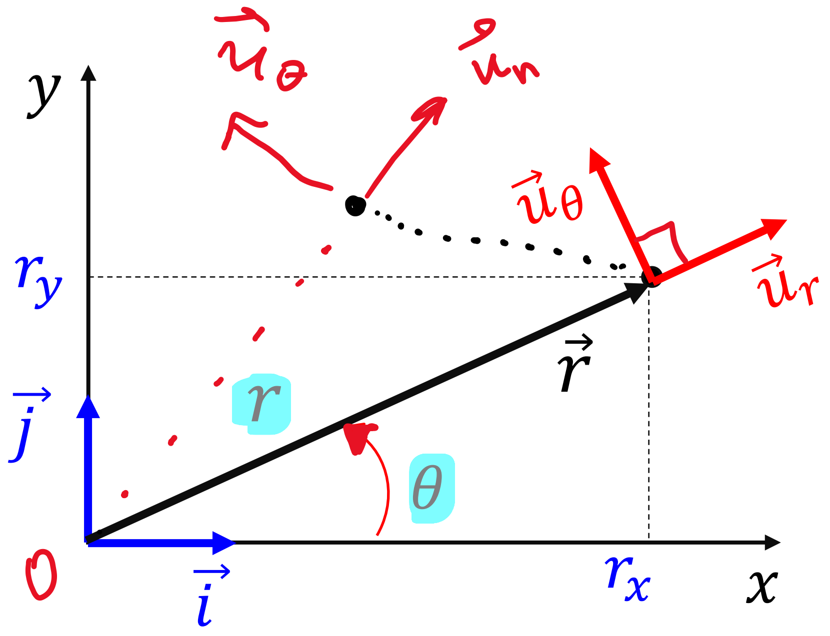
Polar Coordinates

Text: 12.8

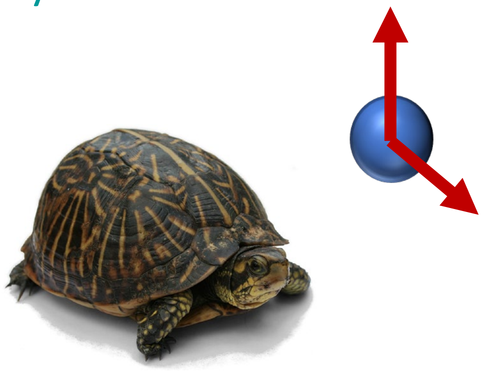
Content:

- Polar and cylindrical coordinate systems
- Velocity and acceleration in polar coordinates
- Applications

POLAR COORDINATES



Polar coordinates: One more coordinate system, in which the particle "carries" the coordinate system with it



- We can characterize a point in 2D by two Cartesian components:

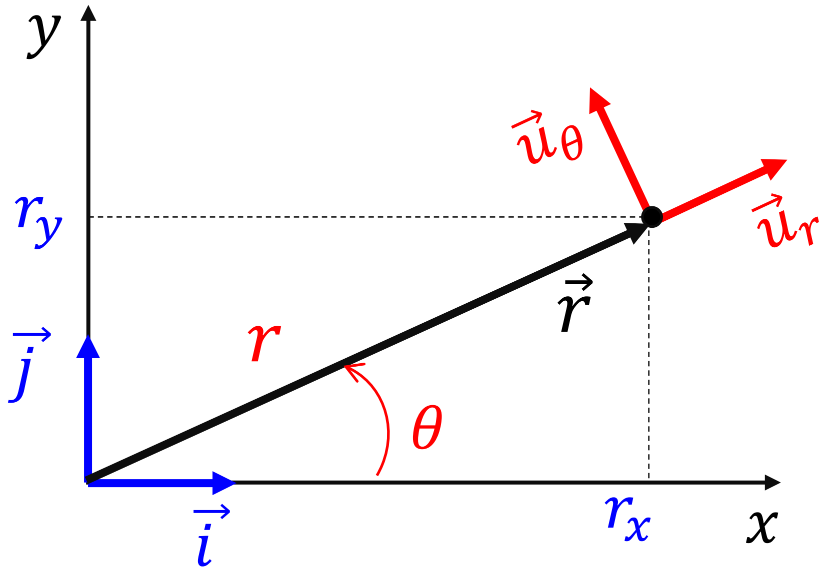
$$\vec{r} = r_x \vec{i} + r_y \vec{j}$$

- Another representation:

$$\vec{r} = (r\cos\theta)\,\vec{i} + (r\sin\theta)\,\vec{j}$$

- We can adopt r and θ as two new coordinates, and introduce the corresponding unit vectors, $\vec{u}_r$ and $\vec{u}_\theta$

POLAR COORDINATES



- Polar components are defined with the help of two numbers:
  - Radial coordinate (r)
  - Angular coordinate (θ), measured counterclockwise from some axis, usually the horizontal
- Polar unit vectors:
  - $\vec{u}_r$ (along r) and $\vec{u}_\theta$ (perpendicular to $\vec{u}_r$, in the direction of increasing θ).

❖ Between two representation:

- $r = \sqrt{r_x^2 + r_y^2}$
- $\theta = \arctan(r_y/r_x)$
- $r_x = r\cos\theta$
- $r_y = r\sin\theta$

- We can work out the following connections:

$(\vec{u}_r)_x = \underbrace{u_r}_{1} \cdot \cos\theta \qquad (\vec{u}_r)_y = \underbrace{u_r}_{1} \sin\theta$

❖ Between the unit vectors of these two systems:

$(\vec{u}_\theta)_x = -\underbrace{u_\theta}_{1} \sin\theta \qquad (\vec{u}_\theta) = \underbrace{u_\theta}_{1} \cos\theta$

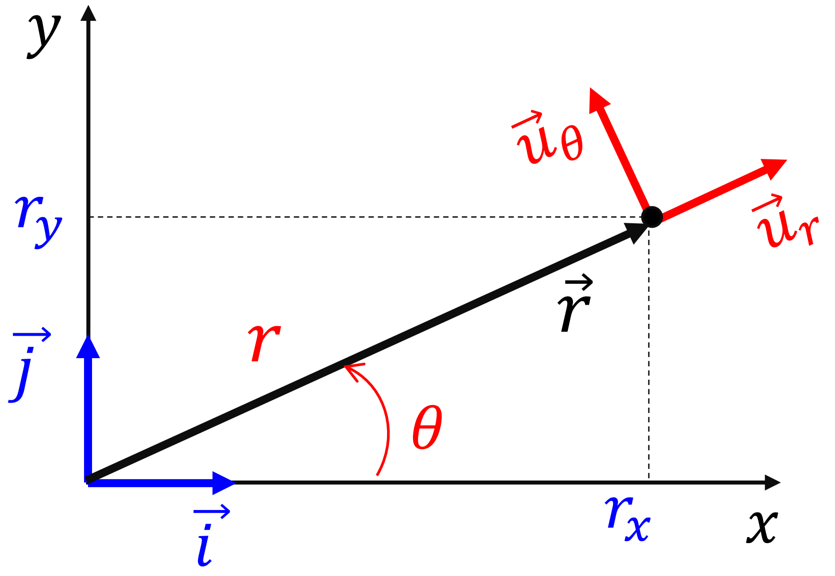
$$\vec{u}_r = \underline{\cos\theta}\,\vec{i} + \underline{\sin\theta}\,\vec{j}$$
$$\vec{u}_\theta = \underline{-\sin\theta}\,\vec{i} + \underline{\cos\theta}\,\vec{j}$$

$$\vec{i} = \cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta$$
$$\vec{j} = \sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta$$

POLAR COORDINATES

$$\vec{u}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\vec{u}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$



- Using these equations for $\vec{u}_r$ and $\vec{u}_\theta$, you can:
  - Show that, indeed, $\vec{u}_r \perp \vec{u}_\theta$ (Hint: calculate $\vec{u}_r \cdot \vec{u}_\theta$)
  - Prove that:

$$\boxed{\dot{\vec{u}}_r = \dot{\theta}\vec{u}_\theta} \qquad \dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_r \qquad (*)$$

on your own

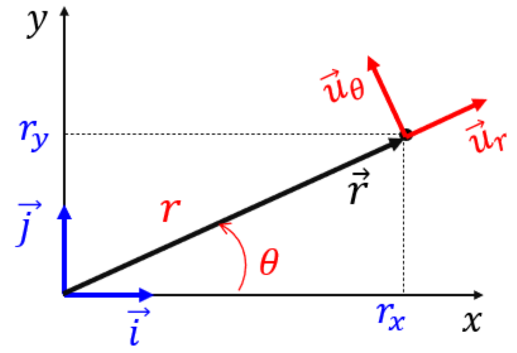
$$\frac{d}{dt}\vec{u}_r = \frac{d}{dt}\left[\cos\theta\cdot\vec{i} + \sin\theta\,\vec{j}\right] = \left(\frac{d\cos\theta}{dt}\cdot\vec{i} + \cos\theta\,\cancelto{0}{\frac{d\vec{i}}{dt}}\right) + \left(\frac{d\sin\theta}{dt}\cdot\vec{j} + \sin\theta\,\cancelto{0}{\frac{d\vec{j}}{dt}}\right)$$

$$= \vec{i}\,\frac{d\cos\theta}{dt} + \vec{j}\,\frac{d\sin\theta}{dt} = \vec{i}\left(-\sin\theta\cdot\dot{\theta}\right) + \vec{j}\left(\cos\theta\cdot\dot{\theta}\right) = \dot{\theta}\left[-\vec{i}\sin\theta + \vec{j}\cos\theta\right] = \dot{\theta}\vec{u}_\theta$$

$$\frac{d\cos\theta}{dt} = \frac{d}{dt}\cdot\frac{d\theta}{d\theta}\cos\theta = \frac{d\cos\theta}{d\theta}\,\frac{d\theta}{dt} = -\sin\theta\,\dot{\theta}$$

POLAR COORDINATES: $\vec{r}, \vec{v}, \vec{a}$

$$\dot{\vec{u}}_r = \dot{\theta}\vec{u}_\theta \qquad \dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_r \qquad (*)$$



- $\vec{r} = r \cdot \vec{u}_r$

- $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left(r \cdot \vec{u}_r\right) = \dfrac{dr}{dt} \cdot \vec{u}_r + r\,\dfrac{d\vec{u}_r}{dt} = \overset{v_r}{(\dot{r})}\,\vec{u}_r + \overset{v_\theta}{(r\dot{\theta})}\,\vec{u}_\theta$

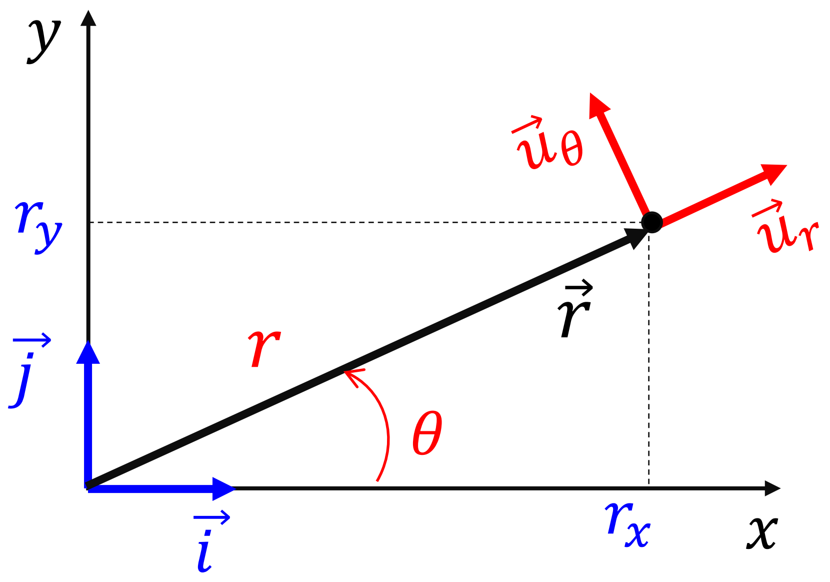
- $\vec{a} = \dfrac{d}{dt}\vec{v} = \dfrac{d}{dt}\left[\dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta\right] = \left(\ddot{r}\vec{u}_r + \dot{r}\dot{\vec{u}}_r\right) + \left(\dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}\dot{\vec{u}}_\theta\right)$

$$= \ddot{r}\vec{u}_r + \dot{r}\dot{\theta}\vec{u}_\theta + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta - r\dot{\theta}^2\vec{u}_r$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\vec{u}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\vec{u}_\theta$$

POLAR COORDINATES: $\vec{r}, \vec{v}, \vec{a}$ (Summary)

$$\dot{\vec{u}}_r = \dot{\theta}\vec{u}_\theta \qquad \dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_r \qquad (*)$$



$$\boxed{\vec{r} = r\vec{u}_r} \qquad (1)$$

- ...and using $(*)$ we can find $\vec{v}$ and $\vec{a}$:

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta} \qquad (2)$$

- Note that:

$$\boxed{v = \sqrt{v_r^2 + v_\theta^2}}$$

$\theta, \dot{\theta}, \ddot{\theta}, r, \dot{r}, \ddot{r}$

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2, \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

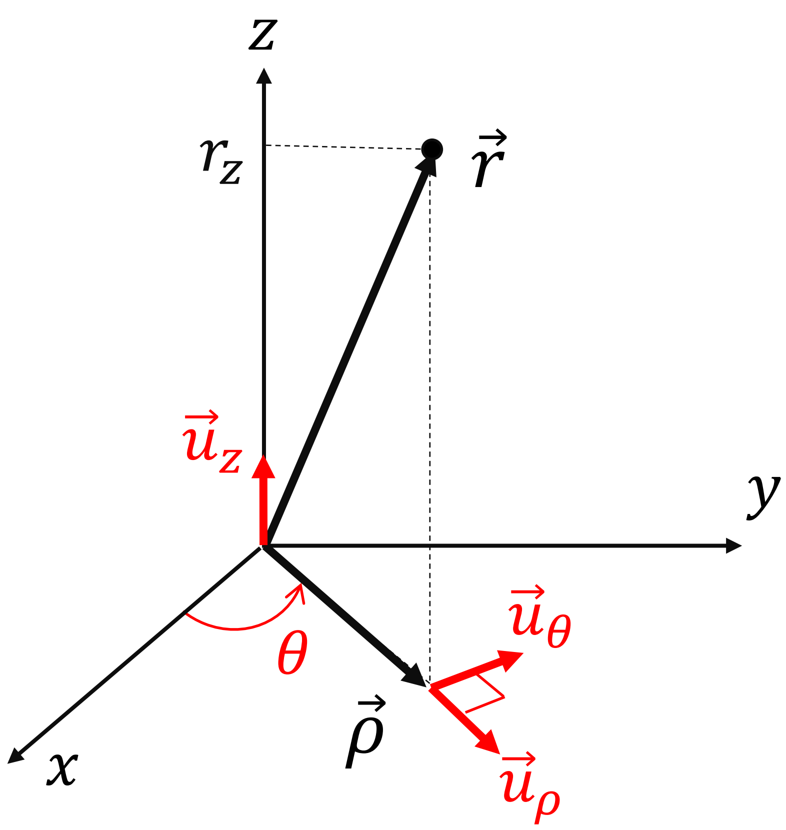
$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta} \qquad (3)$$

- Note that:

$$\boxed{a = \sqrt{a_r^2 + a_\theta^2}}$$

$\dot{\theta}$ = angular velocity
$\ddot{\theta}$ = angular acceleration

CYLINDRICAL COORDINATES



- Polar coordinates with a cartesian z-axis added:

$$\boxed{\vec{r} = \rho\vec{u}_\rho + z\vec{k}}$$

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}}$$

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\vec{u}_\theta + \ddot{z}\vec{k}}$$

DERIVATIVES $\dot{r}, \ddot{r}$

$r(\theta) = 15\cos^2\theta - e^{\theta}$

$\theta = \theta(t)$

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2, \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

- To find $\vec{v}$ and $\vec{a}$, we need to know $\dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta}$.
- Often r will be given as a function of θ, not as a function of time. How can we find $\dot{r}$ and $\ddot{r}$ then?

➢ Using chain rule

$$\dot{r} = \frac{dr(\theta)}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta}; \qquad \ddot{r} = \frac{d\dot{r}(\theta)}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \cdots$$

➢ Using $r = f(\theta)$ to connect derivatives

- Example: Path is given by $r^2 = 6\theta^3$, and $\theta, \dot{\theta}, \ddot{\theta}$ are known. Find $\dot{r}, \ddot{r}$.

$\dot{r}$: $\frac{d}{dt}\left[r^2 = 6\theta^3\right]$

$2r\cdot\dot{r} = 6\cdot 3\theta^2\cdot\dot{\theta}$

$\rightarrow r\dot{r} = 9\theta^2\dot{\theta}$

$\rightarrow \boxed{\dot{r} = 9\theta^2\dot{\theta}/r}$

$\ddot{r}$: $\frac{d}{dt}\left[r\dot{r} = 9\theta^2\dot{\theta}\right]$

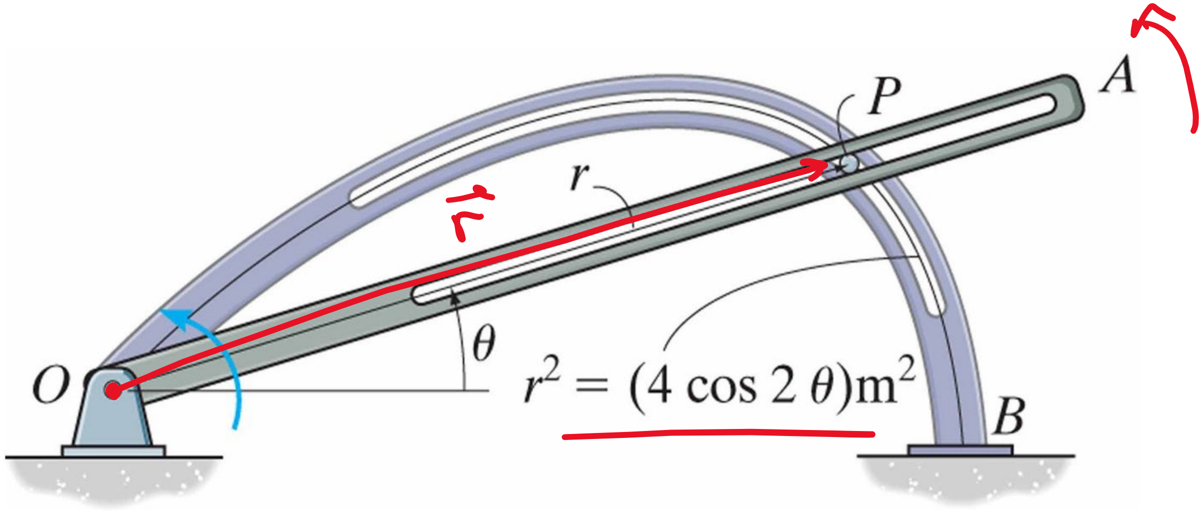
$\dot{r}\dot{r} + \ddot{r}r = 9\cdot 2\theta\dot{\theta}\cdot\dot{\theta} + 9\theta^2\ddot{\theta}$

$$\boxed{\ddot{r} = \frac{9\theta^2\ddot{\theta} + 18\dot{\theta}^2\theta - \dot{r}^2}{r}}$$

W8-1. The motion of ball is constrained by the curved slot in OB and by the slotted arm OA. OA rotates counterclockwise with angular speed $3t^{3/2}$ rad/s where t is in seconds and $\theta = 0$ when $t = 0$.

a) Determine the time when $\theta = 30°$.

b) Determine the radial and transverse components of the ball's velocity and acceleration when $\theta = 30°$.

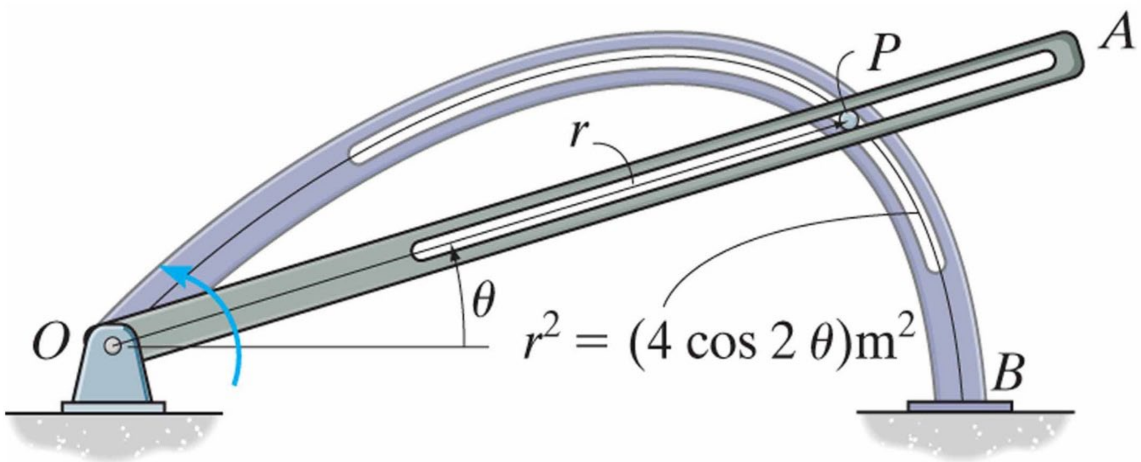


W8-1. The motion of ball is constrained by the curved slot in OB and by the slotted arm OA. OA rotates counterclockwise with angular speed $3t^{3/2}$ rad/s where t is in seconds and $\theta = 0$ when $t = 0$.

$\dot{\theta}$

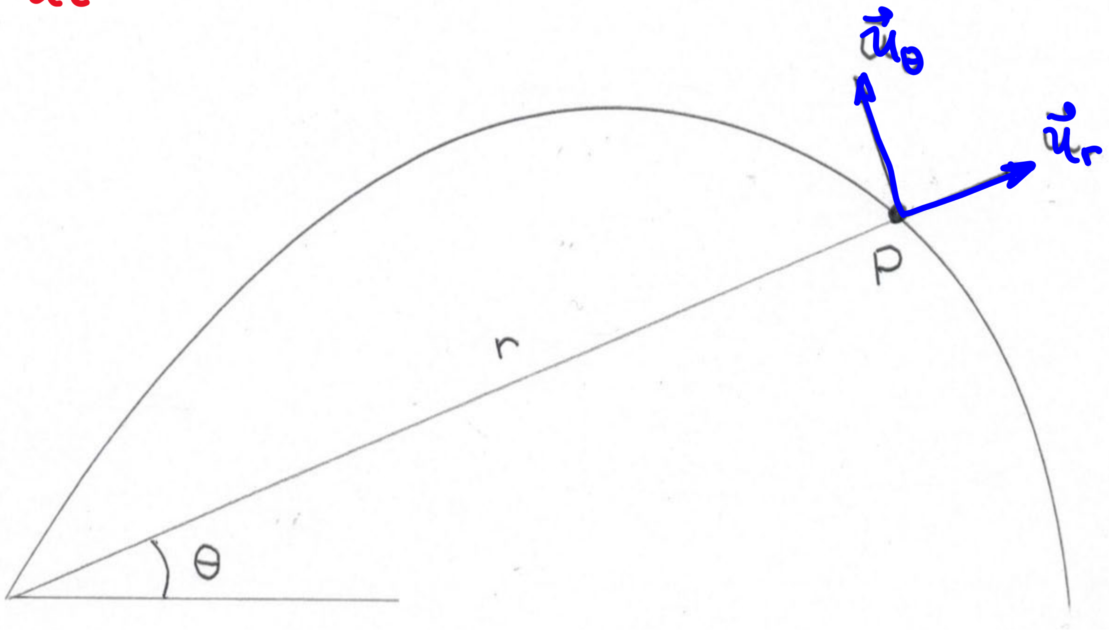
a) Determine the time when $\theta = 30°$.

b) Determine the radial and transverse components of the ball's velocity and acceleration when $\theta = 30°$.



$\underline{\underline{\theta = \theta(t)}}$: t: $\theta(t) = 30°$

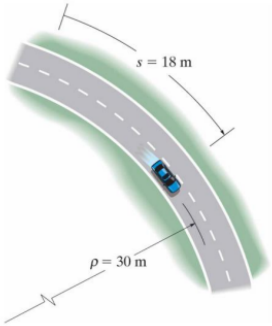
$\frac{d\theta}{dt} \xleftrightarrow{\int} \theta(t) \longrightarrow t: \theta = 30°$



W7-4. The car travels along the circular path. Starting from rest, its acceleration along the path is $0.5\, e^t$ m/s² where t is in seconds.

a) Determine how long it takes the car to travel 18 m.

b) Determine the car's speed and acceleration at this time.



W8-1. $\dot{\theta} = 3t^{3/2}$ rad/s where t is in seconds, and $\theta = 0$ when $t = 0$. Also, $r^2 = (4\cos 2\theta)m^2$.

a) Determine the time when $\theta = 30^o$.

b) Determine the radial and transverse components of the ball's velocity and acceleration when $\theta = 30^o$.

$$\frac{d\theta}{dt} = 3t^{3/2} \ \frac{\text{rad}}{s} \longrightarrow \int_{\theta=0}^{\theta} d\theta = \int_{t=0}^{t} 3t^{3/2}\,dt$$

$$\theta(t) = \underset{\overset{\shortparallel}{0}}{\theta_0} + \int_0^t 3t^{3/2}\,dt = 3\left.\frac{t^{5/2}}{5/2}\right|_{t=0}^{t=t} = \frac{6}{5}t^{5/2}$$

$$t: \quad \theta(t) = \frac{6}{5}t^{5/2} = \frac{\pi}{6}$$

$$t^{5/2} = \frac{5\pi}{36}$$

$$t = \left(\frac{5\pi}{36}\right)^{2/5} = 0.7177\ s$$