ASSIGNMENT 1

In this assignment, you will see how piecewise defined functions appear naturally in applications. In this case, an interesting example arises from considering the gravitational force of a planet. You will also consider a small argument limit and a large argument limit.

Learning Goals:

- Use mathematical models to formalize ideas about natural phenomena.
- Proficiency with piecewise defined functions.
- Practice graphing (piecewise defined) functions (non-calculus methods).
- Use asymptotics to identify the approximate behaviour of functions in different regions of the domain.

Contributors

On the first page of your submission, list the student numbers and full names (with the last name in ***bold****) of all team members. Indicate members who have not contributed using the comment "(non-contributing)".*

Reflection question

Reflection questions encourage you to think about how mathematics is done. This is an important ingredient of success. Reflection questions contribute to your ***engagement grade****.*

1. To work productively as a team, it is helpful to have shared expectations. This question consists of prompts that form a "team contract", a document that guides how you will work together. Before answering the prompts, read the document "Group resources" that is posted along with the assignment on the Canvas course page.
   (a) What are your overarching "ground rules"? Come up with 4-6 specific expectations regarding communication (including how often and what medium), meetings (how often, how long, and where), preparation and attendance.
   (b) What actions will your team take if a member does not follow the ground rules? Be specific.
   (c) What happens if a team member does not fulfill an agreed-upon task for an assignment? Consider both how the team will handle any dropped tasks, as well as actions you will take as a team.

Assignment questions

The questions in this section contribute to your ***assignment grade****. Starred questions are at the level of Part 2 exam questions.*

2. (★★☆☆)
   We are going to consider the classical gravitational force associated with the planet Earth. Forces are vector quantities, which have magnitudes and directions, but we will focus on the scalar functions associated with these forces.

Let's assume Earth is a perfect sphere. Let's also assume the material making up Earth is uniformly distributed throughout this sphere, meaning the density of the Earth is uniform. (This is not true in reality as the core of the Earth is denser than the crust, for example.)

Suppose Earth has a radius r_E and that we measure distances r to any object from the centre of Earth, which we say is $r = 0$. Since we are interested in understanding the gravitational force of Earth alone, it is useful to imagine this object to be small and to have small mass since we will ignore the gravitational force produced by this object itself.

Physics tells us the gravitational force due to the mass of Earth on our small object will only depend on the distance r this object is from the centre of Earth. We would like to find a function F to describe the gravitational force as a function of the distance r.

For distances r that are greater than $r = r_E$, Newton's law tell us the gravitational force of Earth on our small object is proportional to the inverse of the squared distance r; that is, $F(r) = \alpha/r^2$.

If we imagine drilling a hole from the surface of Earth to its centre and placing our object at some depth in this hole a distance r from the centre, then an interesting application of Gauss's law tells us the gravitational force on our small object is proportional to its distance r from the centre of Earth; that is, $F(r) = \beta r$.

(a) Find the explicit expression of the function $F(r)$ that describes the gravitational force associated with Earth, where $r \geq 0$.

(b) Sketch a graph of $F(r)$ and write a few observations about this function and its graph. What do you know about the gravitational force at the surface of Earth and what does this tell you about the function $F(r)$ when $r = r_E$? What do you know about the relationship between the constants of proportionality α and β?

Hint: The function $F(r)$ will be a piecewise defined function; it has a different form for distances that place the object inside Earth than for objects outside Earth's surface.

3. (★★★☆)

Let us consider now a slightly more realistic model of the internal structure of Earth. In particular, imagine the material inside Earth is still distributed with spherical symmetry, meaning the density only depends on the distance from the centre of Earth, but the density of the material varies with this distance. Suppose there is a dense central core of radius equal to $0.4r_E$ of uniform (constant) high density, where r_E is the radius of Earth. This core is surrounded by a mantle of uniform (constant) density that is 60% of the density of the core.

Let's find the gravitational force on a small object placed at a point inside this Earth.

Since we are only looking inside the planet, we will measure distances from the centre of Earth as a fraction of Earth's radius. Call this fractional distance R and so R goes from 0 (the centre of Earth) to 1 (the surface of Earth). In terms of the usual distance r, we have $R = r/r_E$.

The gravitational force on a small object in the core is $F_c(R) = 1.6R$.

The expression for the gravitational force on a small object in the mantle is slightly more complicated: $F_m(R) = \dfrac{0.04}{R^2} + R$.

(a) Write down the function $F(R)$ describing the gravitational force an object will experience anywhere inside Earth given this model of Earth's internal structure.

(b) Sketch a graph of this function.

(c) If you drilled a narrow hole from the surface straight to the centre of Earth and dropped a small object into this hole, describe carefully how the object would accelerate as it fell towards the centre. (Remember Newton's Second Law says $F = ma$.)

(d) Some approximations were made to produce the numerical constants in the expressions for the gravitational forces in the core and in the mantle. How can you tell these constants are not given exactly based on properties of the function $F(R)$?

4. (★★★☆)

In 1970, science fiction author Larry Niven wrote *Ringworld*, in which there is a large artificial world constructed as a ring that has a diameter approximately the same size as Earth's orbit around the Sun, which means the circumference of Ringworld is about 1 billion kilometers. The total mass of this world is about 2×10^{27} kilograms (about 333 times the mass of Earth), which means it has a large gravitational field. For the purposes of this question, we will consider this mass to be uniformly distributed around the ring.

While it is tricky to calculate exactly this gravitational field at a general point in space around this world, the symmetry of Ringworld about an axis running through the centre of the ring and perpendicular to the plane of the ring makes it possible to calculate exactly the gravitational force a small object placed anywhere on this axis will experience.

The easiest place to calculate the gravitational force is at the centre of the ring: it must be exactly 0 there because of the symmetry since each bit of the ring has a corresponding bit of the ring of equal mass diametrically opposite to it on the ring. These antipodal bits of the ring exert equal and opposite gravitational forces on a small object at the centre, meaning the net force there is 0 (technically, the gravitational force is a vector, so this is the 0-vector, but we will focus on the magnitude of it, which is the scalar 0). This is true for all such bits of the ring and so their total contributions add up to 0 at the centre of the ring.

Now consider the axis that passes through the centre of the ring, and that is perpendicular to the plane of the ring. Think of this axis as a straight line acting as a coordinate line where the origin of the coordinate is at the centre of the ring. Let's call this coordinate z and say $z = 0$ is the centre. If $z > 0$, then we are looking at points in one direction along this axis, and when $z < 0$, we are considering points in the other direction. (The choice of positive direction is arbitrary.)

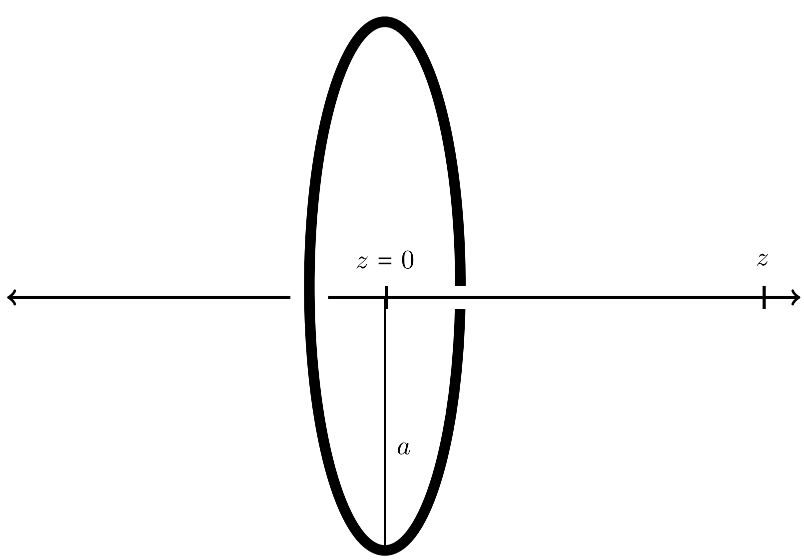

Figure 1: Ringworld and its Central Axis

To find the gravitational force at a point z on this axis requires adding up all the contributions of bits of the ring. This requires knowing how to integrate, which is a topic covered in a later course, so we will just use the final result of this calculation: the scalar function associated to the gravitational force at a point z on this central axis is given as

$$F(z) = \frac{Kz}{(a^2 + z^2)^{3/2}},$$

where $K > 0$ is a constant, and a is the radius of the ring (≈150 million kilometers). This force points along the axis towards the centre of the ring. (Why?) Note this result includes the fact $F(0) = 0$, which says the net gravitational force at the centre of the ring is 0.

(a) If you are very close to the centre of Ringworld along this central axis, what does the gravitational force look like? That is, consider the behaviour of the function $F(z)$ for small z – what simpler function does $F(z)$ look like in this case?

(b) If you are very far away from Ringworld along this central axis, what does the gravitational force look like? This is, consider the behaviour of the function $F(z)$ for large z – what simpler function does F(z) look like in this case? Comment on the reasonableness of using this simpler function to study the gravitational field at large distances from Ringworld.

(c) Even though you don't have an expression for the function describing the gravitational force at a general point in space around Ringworld, what do you expect this function to look like when you consider points at large distances from Ringworld in any direction? What is the reason for your expectation?

Practice questions

The questions below are for practice. Starred questions are at the level of Part 2 exam questions. They do not contribute to your grade, but you are strongly encouraged to work through them under exam conditions.

6. (★★★☆)

In the chemistry of enzymes and substrates, it is common to consider a simple set of reactions

$$\mathrm{E} + \mathrm{S} \underset{\mathrm{k}_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{ES} \xrightarrow{k_2} \mathrm{E} + \mathrm{P},$$

where k_1, k_{-1}, and k_2 are positive rate constants. Note the enzyme acts as a catalyst in this reaction and is ultimately returned unchanged when the product P is produced.

The *law of mass action* can be applied to this enzyme reaction to produce a set of differential equations that describe the kinetics of this set of reactions. We will discuss differential equations later on in MATH 100, and so only work with a function that arises as a consequence of this set of differential equations in this problem.

In many situations, the formation of the enzyme substrate ES occurs rapidly, but the rate of the forward reaction producing E + P, captured by the *turnover number* k_2, occurs on a slower time scale. If we assume the formation of ES is instantaneous, often referred to as a *quasi-steady state*, then it follows that the forward reaction $\mathrm{ES} \xrightarrow{k_2} \mathrm{E} + \mathrm{P}$ has a reaction rate that depends on the concentration [S] of the substrate, and on the presumed fixed initial constant concentration $[\mathrm{E}]_0$ of enzyme:

$$R([\mathrm{S}]) = \frac{k_2[\mathrm{E}]_0[\mathrm{S}]}{\beta + [\mathrm{S}]} = \frac{\alpha[\mathrm{S}]}{\beta + [\mathrm{S}]},$$

where α and β are positive constants.

(s) For small values of [S], find a simpler function that approximates the reaction rate $R([\mathrm{S}]$ and comment on the behaviour of the forward reaction for small values of [S].

(a) For large values of [S], find a simpler function that approximates the reaction rate $R([\mathrm{S}]$ and comment on the behaviour of the forward reaction for large values of [S].

(c) From a practical point of view, discuss what it might mean for concentrations of substrate to be considered “large”. Think in terms of the chemical reactions at play. How does a chemistry-based understanding of large [S] compare to a mathematician’s understanding of what it means for a variable to be large?