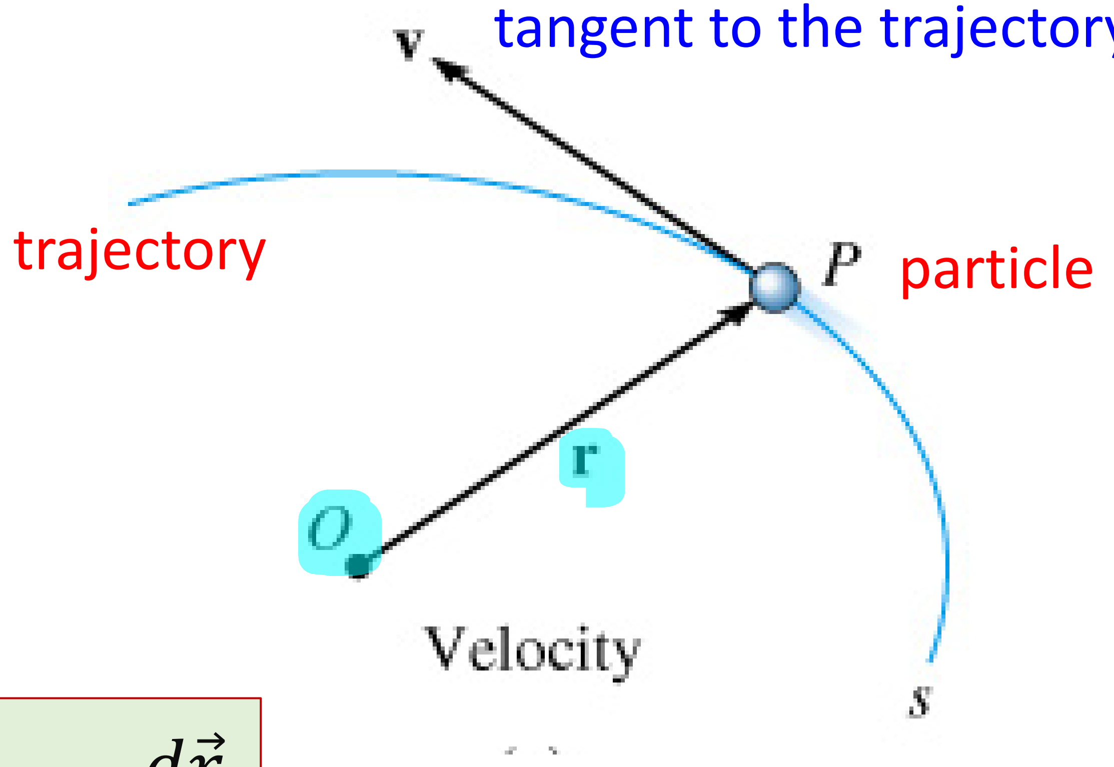
Exam: Thursday, March 7th , 6 pm

- (!) Read each problem carefully. It will pay off. (!)
- When studying for the midterm, don't read posted solutions. Consult them only to check your answers, or to have a hint on what to do if you are stuck.
- Study with someone, or explain your work to an imaginary partner. "When one teaches, two learn"
- Office hours:
  - Cancelled tomorrow (conflict with PHYS 158 midterm)
  - Replacement: Monday, March 4th, 5:00-6:00 pm

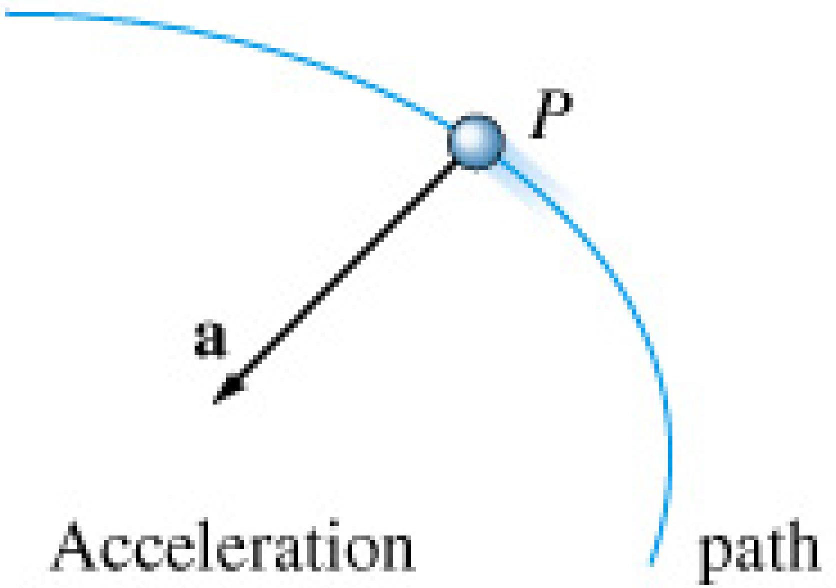
Last Time: Curvilinear motion in Cartesian coordinates

Velocity is always tangent to the trajectory



$$\vec{v} = \frac{d\vec{r}}{dt}$$

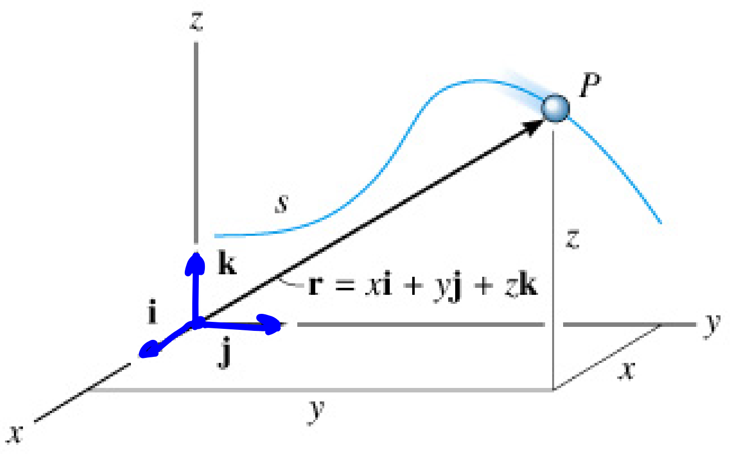
Acceleration always points inwards (into the trajectory)



$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Velocity & Acceleration in Rectangular Components

- These pictures are nice, but it is difficult to work with them. Let us come up with something else.



Position

- If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, what is $\dfrac{d\vec{r}}{dt} = \vec{v}$?

- Product rule: $\dfrac{d(ab)}{dt} = a\dfrac{db}{dt} + b\dfrac{da}{dt}$

- Then: $\dfrac{d(x\vec{i})}{dt} = x\overbrace{\cancel{\dfrac{d\vec{i}}{dt}}}^{=0} + \vec{i}\underbrace{\left(\dfrac{dx}{dt}\right)}_{v_x}$

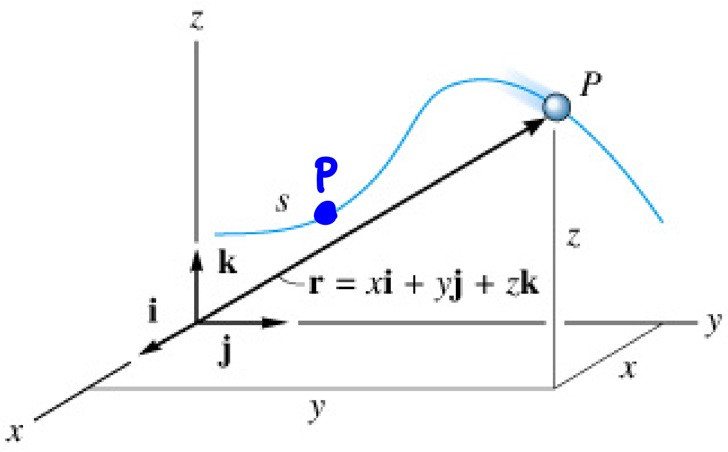
- Note that: $\dfrac{d\vec{i}}{dt} = 0$ ($\vec{i}$ does not change with t)

$$\boxed{\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}}$$

where

$$\boxed{v_x = \frac{dx}{dt} = \dot{x}, \qquad v_y = \frac{dy}{dt} = \dot{y}, \qquad v_z = \frac{dz}{dt} = \dot{z}}$$

Velocity & Acceleration in Rectangular Components



Position

- If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$:

$$x(t) \quad y(t) \quad z(t)$$

$$\boxed{\begin{aligned}\vec{v} &= v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}\\ \vec{a} &= a_x\,\vec{i} + a_y\,\vec{j} + a_z\,\vec{k}\end{aligned}}$$

where:

$$v_x = \frac{dx}{dt} = \dot{x}, \qquad v_y = \frac{dy}{dt} = \dot{y}, \qquad v_z = \frac{dz}{dt} = \dot{z}$$

$$a_x = \frac{dv_x}{dt} = \ddot{x}, \qquad a_y = \frac{dv_y}{dt} = \ddot{y}, \qquad a_z = \frac{dv_z}{dt} = \ddot{z}$$

- Note: we now have three one-dimensional problems (which we already know how to work with!)
- We can use these algebraic equations to find the components of $\vec{r}(t), \vec{v}(t)$ and $\vec{a}(t)$

  - Q: These quasi-1D-problems are not completely independent. What connects them?

- Q: Assume that the object's acceleration is expressed as:

$$\vec{a}(t) = \left(\underbrace{t}_{a_x}\,\vec{i} + \underbrace{t^2}_{a_y}\,\vec{j} + \underbrace{t^3}_{a_z}\,\vec{k}\right)\, m/s^2\,.$$

What is its acceleration at $t = 1$?

$$a(t) = \sqrt{a_x^2(t) + a_y(t)^2 + a_z^2(t)}$$

A. 1 m/s²

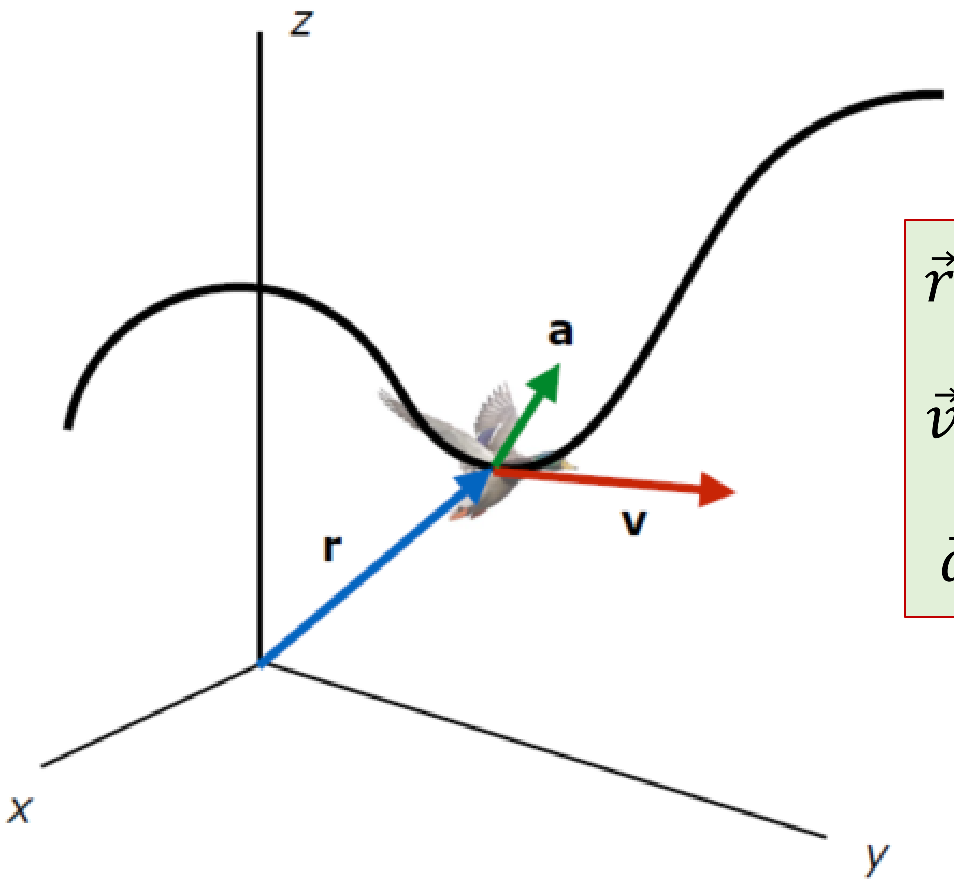
B. 2 m/s²

C. 3 m/s²

D. $\sqrt{2}$ m/s²

E. $\sqrt{3}$ m/s²

Velocity & Acceleration in 3D Cartesian coordinates (summary)



- Algebraic expressions for duck's position, velocity and acceleration:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$\vec{v}(t) = v_x(t)\vec{i} + v_y(t)\vec{j} + v_z(t)\,\vec{k}$$

$$\vec{a}(t) = a_x(t)\vec{i} + a_y(t)\vec{j} + a_z(t)\vec{k}$$

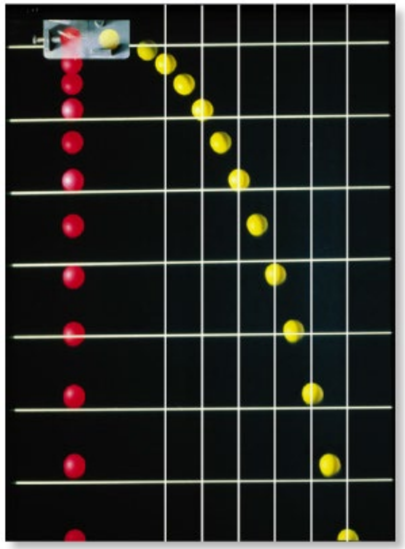
$$v_x = \frac{dx}{dt},$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},$$

etc.

- Graphical representation of duck's motion. Note:
  - $\vec{v}$ is tangent to the trajectory
  - $\vec{a}$ points "inwards" (concave side of the path)

Projectile Motion



Text: 12.6

Content:

- This is a 2D motion with a constant y-acceleration
- Independence of motion along different axes
- Trajectory equation

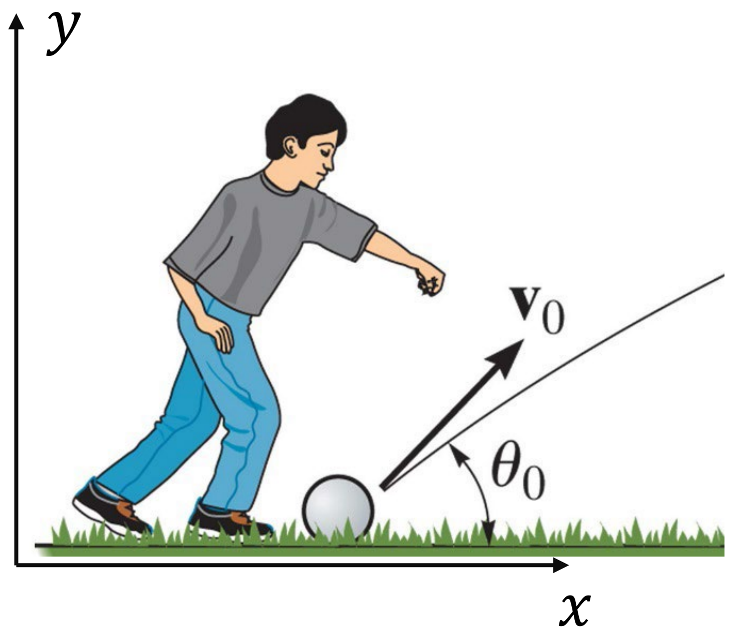
Our first example: Motion in the gravitational field of the Earth

- "Free fall": No other forces but gravity act on the object
- Motion unfolds in a plane => 2D problem (curvilinear, in general)
- Motion with constant acceleration => equations from Problem W7-2 apply!
  - ➢ $a_y = -g$ (if the positive y-direction is upwards)
  - ➢ $g = 9.81\ \text{m/s}^2 = 32.2\ \text{ft/s}^2$

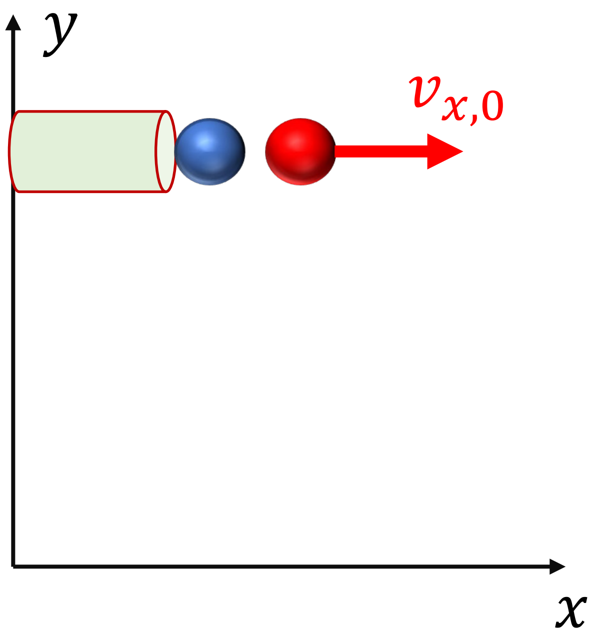
(SI) (FPS)

$a = \text{const}$

mg



Q: Two balls are released from the gun at the same moment of time. The blue ball just drops on the ground. The red ball is shoot with a horizontal velocity $v_{x,0}$. Which of them will reach the ground first?



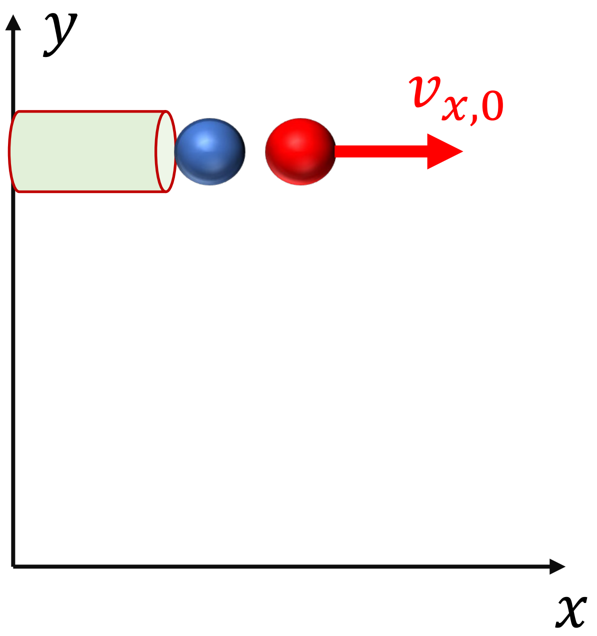
A. The blue ball

B. The red ball

C. Simultaneously

D. Not sure

Q: Two balls are released from the gun at the same moment of time. The blue ball just drops on the ground. The red ball is shoot with a horizontal velocity $v_{x,0}$. Which of them will reach the ground first?



https://www.youtube.com/watch?v=HGslBnCJVQg&ab_channel=NiteshBatra

PROJECTILE MOTION

For each component:

- $s(t) = s_0 + v(t)t + \frac{a\,t^2}{2}$;
- $v(t) = v_0 + a\,t$;
- $v^2(t) = v_0^2 + 2a(s - s_0)$

- Motion along each cartesian axis is described by its “own” equation => they are independent motions (though coupled through the same time, t)

- Along x:
  - $a_x = 0$: motion with constant velocity!
  - $v_x(t) = v_{0,x} + 0$
  - $x(t) = x_0 + v_{0,x}\,t + 0$ (*)

$\hookrightarrow t = t(x)$

- Along y:
  - $a_y = -g$: motion with constant (negative) acceleration!
  - $v_y(t) = v_{0,y} - g\,t$
  - $y(t) = y_0 + v_{0,y}\,t - \frac{g t^2}{2}$ (**)

$\vec{a}$

y

$y = y(x)$
parabola

From equation (*), we can find t as a function of x and plug it into equation (**). This will give us equation for y(x) = trajectory equation!

- Exercise: Do it!
- Check that $y(x)$ is a parabola *(a well-known result)*

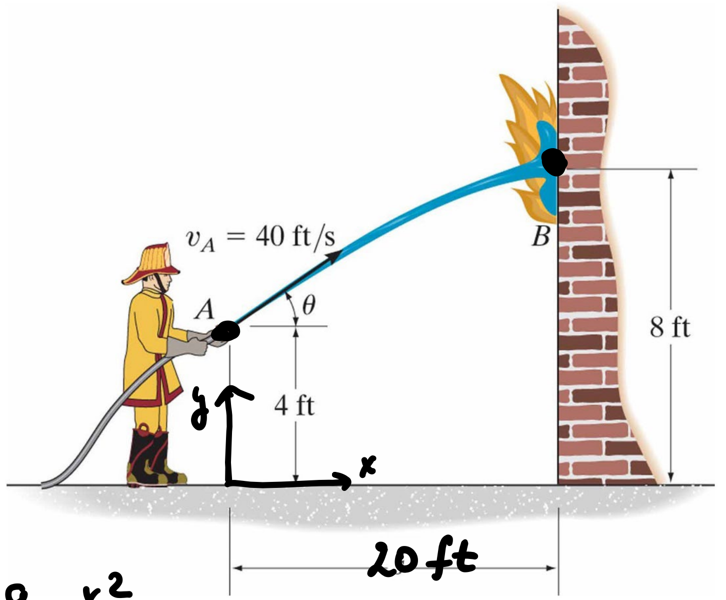
PROJECTILE MOTION: Trajectory equation (on your own)

- Along x:
  - $a_x = 0$: motion with constant velocity!
  - $v_x(t) = v_{0,x}$
  - $x(t) = x_0 + v_{0,x}\, t$ (*)

- Along y:
  - $a_y = -g$: motion with constant (negative) acceleration!
  - $v_y(t) = v_{0,y} - g\, t$
  - $y(t) = y_0 + v_{0,y} t - \frac{g t^2}{2}$ (**)

- $s(t) = s_0 + v_0 t + \dfrac{a\,t^2}{2}$;

W7-3. Water is discharged from the hose with a speed of 40 ft/s. Determine the two possible angles θ the firefighter can hold the hose so that the water strikes the building at B. Take $s = 20$ ft.



- Motion along x: $x_0 = 0$ $\quad v_{0x} = v_A \cos\theta$ $\quad a_x = 0$

$$x(t) = \underset{=}{x_0} + v_{0x} t = v_A \cos\theta \cdot t \longrightarrow t = \frac{x}{v_A \cos\theta}$$

- Motion along y: $y_0 = 4$ ft $\quad v_{0y} = v_A \sin\theta$ $\quad a_y = -g$

$$y(t) = y_0 + v_{0y} t + \frac{a_y t^2}{2} = \underset{=}{4} + v_A \sin\theta \cdot t - \frac{g t^2}{2} = 4 + \frac{\cancel{v_A} \sin\theta \cdot x}{\cancel{v_A} \cos\theta} - \frac{g}{2} \frac{x^2}{(v_A \cos\theta)^2} = y(x)$$

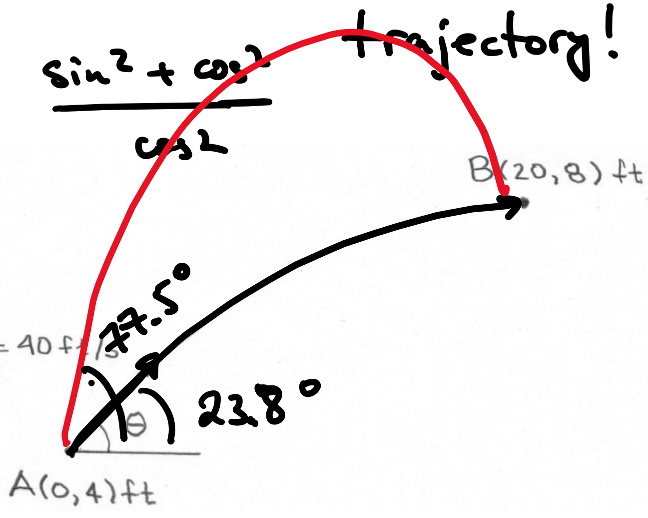
$x = 20,\ y = 8$:

$$8 = 4 + (20)\tan\theta - \frac{\overset{\text{" } g}{32.2}}{2} \frac{(20)^2}{(40)^2 (\cos\theta)^2}$$

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = \frac{\sin^2 + \cos^2}{\cos^2}$$

$$8 = 4 + (20)\tan\theta - \frac{32.2}{2} \frac{(20)^2}{(40)^2} [1 + \tan^2\theta]$$

$$\tan\theta \begin{cases} 4.529 \\ 0.4403 \end{cases} \qquad \theta \begin{cases} 77.5^\circ \\ 23.8^\circ \end{cases}$$



Curvilinear (2D, 3D) motion: Normal & Tangential components

Text: 12.7

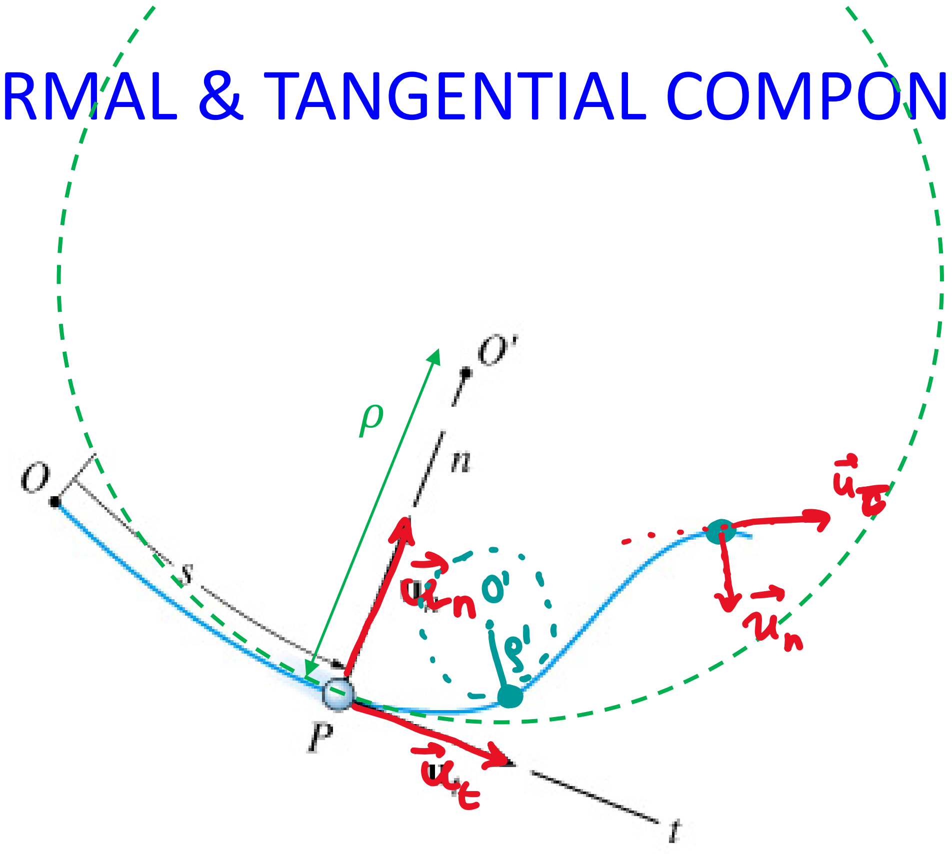
Content:

- Normal and tangential components
- Velocity: v_t
- Acceleration: a_t, a_n and a

Intro remarks

- Cartesian components: unit vectors $\vec{i}, \vec{j}, \vec{k}$ are static (do not move)
- Let us try something new: allow the coordinate system to move as the time goes.
- Will consider two such coordinate systems:
  - Normal & Tangential components (this week)
    - Unit vectors $\vec{u}_n, \vec{u}_t$ (will depend on time)
    - Convenient when you know the path along which the object moves (e.g. car moving along a curved road)
  - Polar & cylindrical coordinates (next week)
    - Unit vectors $\vec{u}_\theta, \vec{u}_r, \vec{u}_z$ (will depend on time)
    - Convenient when you want to describe motion in terms of radial distance from an origin and an angular position relative to some axis.

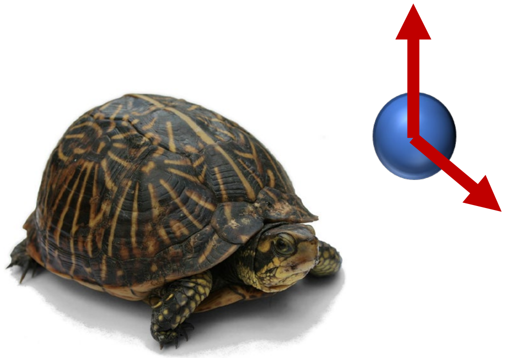
NORMAL & TANGENTIAL COMPONENTS



- Location of a particle on the trajectory defines two time-dependent unit vectors:
  - ➢ $\vec{u}_t$: tangent to the trajectory, pointing in the direction of motion
  - ➢ $\vec{u}_n$: normal to $\vec{u}_t$, pointing inwards, perpendicular to $\vec{u}_t$ (towards the *center of curvature*, O', along *radius of curvature*, ρ)
  - ➢ O' and ρ: Center and radius of an imaginary circle which would match your ds at that particular point

- We will also define the particle's position along the curve:

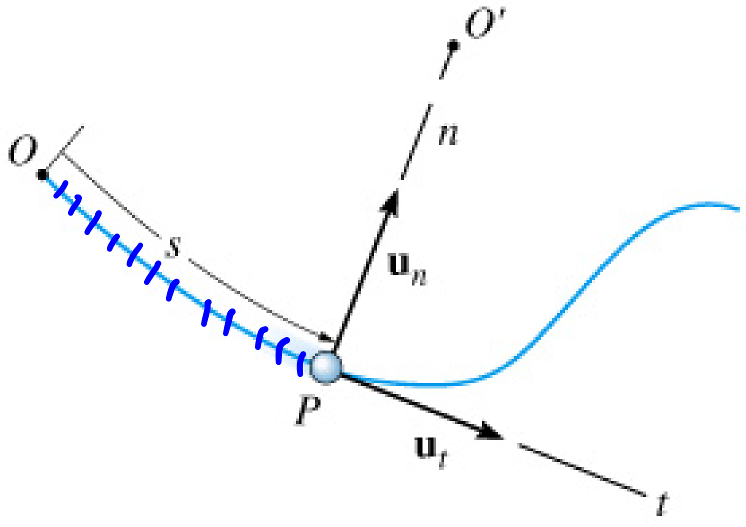
$$s = s(t)$$

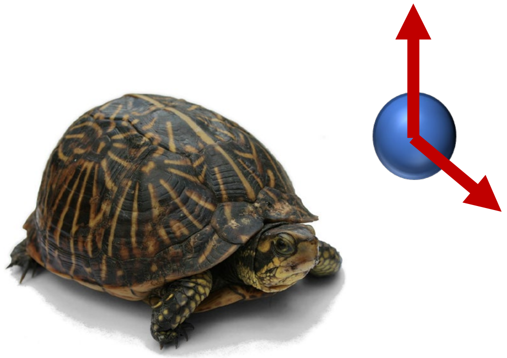


- Note that this coordinate system is carried by the particle, similarly to the shell carried by a turtle

- Q: What is the particle position vector in this coordinate system?

VELOCITY





- Particle's velocity vector is always tangent to its trajectory *(take two points on the trajectory, let the second tend to the first when $dt \to 0$, and you will get a tangent line)*

- That means that the normal component of the velocity is equal to zero:

$$\vec{v} = v_t \, \vec{u}_t + 0 \, \vec{u}_n$$

- The t-component of the velocity is

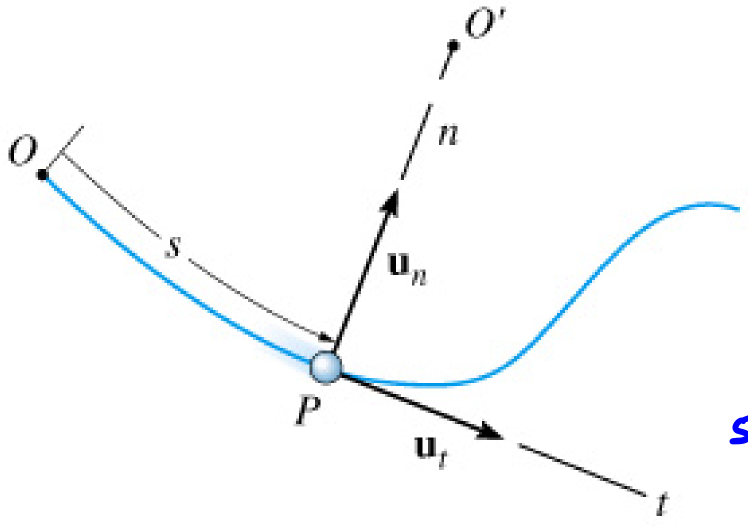
$$v_t = \frac{ds}{dt} = \dot{s}$$

(as if the particle travels along a 1D s-trajectory and "does not know" that the trajectory is curved)

- We get:

$$\boxed{\vec{v} = \frac{ds}{dt} \, \vec{u}_t}$$

ACCELERATION



Q: What can you say about acceleration in normal & tangential components? Consider a general situation.

50% A. I remember that acceleration always points inwards => it only has a normal component and no tangential component.

5% B. It only has a tangential component, since $\vec{a} = \frac{d\vec{v}}{dt}$, and $\vec{v}$ only has a tangential component.

41% C. Acceleration has both normal and tangential components

D. One cannot define acceleration in this coordinate system

E. Not sure

$$\boxed{\vec{v} = \frac{ds}{dt}\,\vec{u}_t}$$