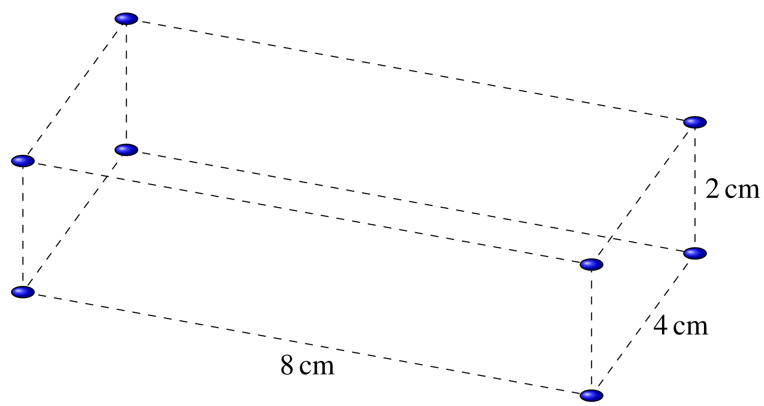
HW-7-v3—solutions for posting March-17/24
Problem 1—using cms for Box

Difficulty: ★☆☆

2 cm x 4 cm x 8 cm

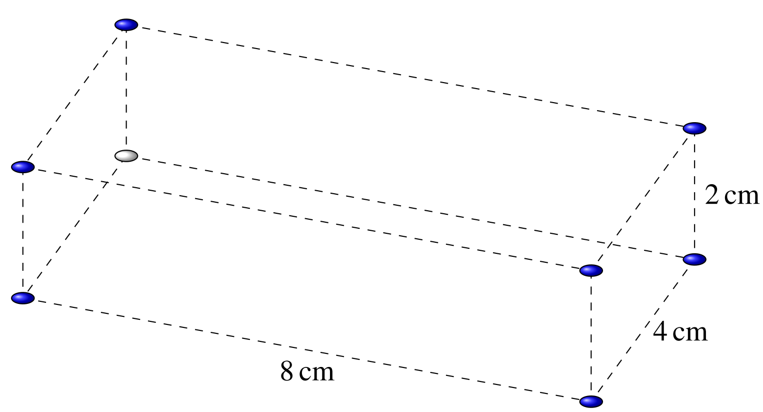
A rectangular box has dimensions ~~$2\,\text{m} \times 4\,\text{m} \times 8\,\text{m}$~~. At each corner of the box sits a point charge of value Q. *Leave your answers in terms of k and Q.*



a) What is the electric field at the center of the box?

b) What is the potential at the center of the box?

One of the point charges on the corners is removed.



c) What is the magnitude of the new electric field at the center of the box?

d) What is the new potential at the center of the box?

e) What is the work [redacted]?

you must do to remove that point charge ??

Solution:

a) Because of the symmetry at the center of the box we find that the electric field from each point charge cancels out with another point charge. We can find the electric field due to each point charge at the center and add the eight vectors together to get the total contribution. In doing this, we will see that the electric field at the center of the box is zero.

b) All of the point charges are the same distance away from the center of the box. So, to compute the potential at the center of the box we can just compute the potential due to one point charge and multiply by eight.

$$r = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}cm$$

$$V_1 = \frac{kQ}{r} = \frac{kQ}{\sqrt{21}} * 100$$ = 21.82 kQ in standard MKS units

$$V_{\text{total}} = 800V_1 = \boxed{\frac{800kQ}{\sqrt{21}}}$$ =174.6 kQ in standard MKS units

c) Originally, the charges diagonally opposite one another were cancelling another out. Once we remove one of the point charges then one of the point charges will no longer be cancelled out so the magnitude of the electric field will be the same as the electric field felt at the center due to one point charge.

$$r = \sqrt{21} * 10^{-2}m$$

$$|\vec{E}| = \frac{kQ}{r^2} = \boxed{\frac{kQ}{21}} * 10,000$$ = 476.2 kQ in MKS units

d) The potential at the center can be computed in the same manner as before, except now we only have seven point charges contributing to the potential.

$$\boxed{V = \frac{700kQ}{\sqrt{21}}}$$ =152.8 kQ Volts

e) To compute the work done in removing the charge we must compute the potential energy existing between the point that was removed and each of the other charges. This will give the potential difference which we can use to get the work done.

$$U_{12} = \frac{kQ^2}{2}$$

$$U_{13} = \frac{kQ^2}{4}$$

$$U_{14} = \frac{kQ^2}{\sqrt{2^2 + 4^2}} = \frac{kQ^2}{\sqrt{20}}$$

$$U_{15} = \frac{kQ^2}{8}$$

$$U_{16} = \frac{kQ^2}{\sqrt{2^2 + 8^2}} = \frac{kQ^2}{\sqrt{68}}$$

$$U_{17} = \frac{kQ^2}{\sqrt{4^2 + 8^2}} = \frac{kQ^2}{\sqrt{80}}$$

$$U_{18} = \frac{kQ^2}{\sqrt{2^2 + 4^2 + 8^2}} = \frac{kQ^2}{\sqrt{84}}$$

$$U_{\text{total}} = U_{12} + U_{13} + U_{14} + U_{15} + U_{16} + U_{17} + U_{18}$$

$$U_{\text{total}} = \left(\frac{kQ^2}{2} + \frac{kQ^2}{4} + \frac{kQ^2}{\sqrt{20}} + \frac{kQ^2}{8} + \frac{kQ^2}{\sqrt{68}} + \frac{kQ^2}{\sqrt{80}} + \frac{kQ^2}{\sqrt{84}}\right)$$ *100 in MKS units

$U_{\text{total}} \approx$ 144kQ^2

Note — The E field will push this positive charge away from the box so the work you must do is NEGATIVE

$$W_{16} = \frac{kQ^2}{\sqrt{2^2 + 8^2}} = \frac{kQ^2}{\sqrt{68}}$$

$$W_{17} = \frac{kQ^2}{\sqrt{4^2 + 8^2}} = \frac{kQ^2}{\sqrt{80}}$$

$$W_{18} = \frac{kQ^2}{\sqrt{2^2 + 4^2 + 8^2}} = \frac{kQ^2}{\sqrt{84}}$$

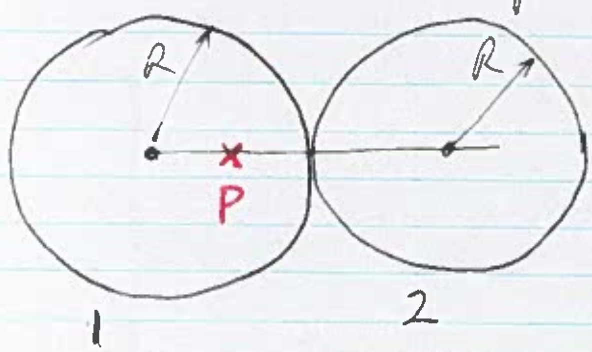
$$W_{\text{total}} = W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18}$$

$$W_{\text{total}} = \left(\frac{kQ^2}{2} + \frac{kQ^2}{4} + \frac{kQ^2}{\sqrt{20}} + \frac{kQ^2}{8} + \frac{kQ^2}{\sqrt{68}} + \frac{kQ^2}{\sqrt{80}} + \frac{kQ^2}{\sqrt{84}}\right) \text{ * 100 in MKS units}$$

$W_{\text{total}} \approx$ — 144kQ^2 in MKS units

Note — The E field will push this positive charge away from the box so the work you must do is NEGATIVE

2. Consider 2 solid spheres (unif dist chge).



$E_2(P) = E_{outside} = \frac{kQ_2}{(1.5R)^2}$

Sphere 2 acts like a point chge for $r > R$.

$E_1(P)$ is due to the charge enclosed inside 0.5R

Use Gauss's Law $\Rightarrow$ $\oint d\vec{A}_1 \cdot \vec{E}_1(P) = 4\pi r_1^2 E_1(r_1) = \frac{Q_{enc}}{\epsilon_0}$

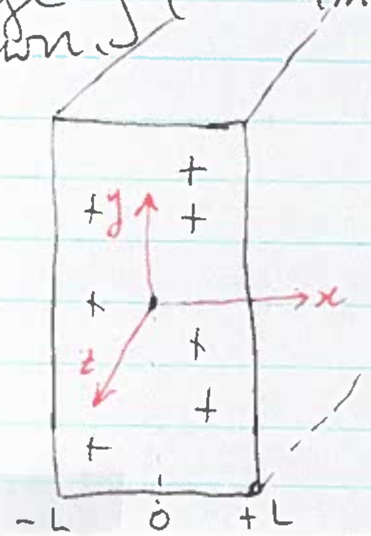
But $Q_{enc} = Q_1 \frac{\left(\frac{4\pi}{3} r_1^3\right)}{\frac{4\pi}{3} R^3} = Q_1 \left(\frac{r_1^3}{R^3}\right)$

$\Rightarrow E_1(P) = \frac{1}{4\pi\epsilon_0} Q_1 \left(\frac{r_1^3}{r_1^2} \frac{1}{R^3}\right) = \frac{kQ_1 r_1}{R^3}$

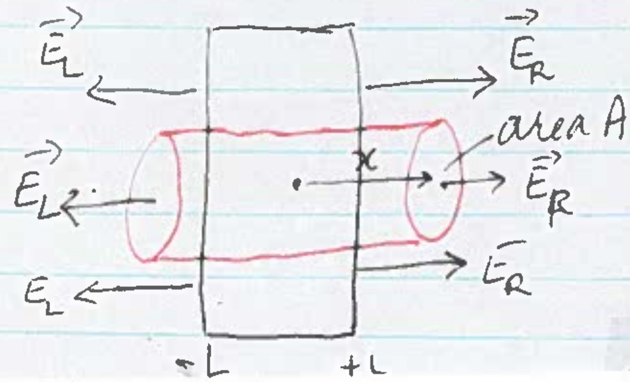
Since $\vec{E}_1 + \vec{E}_2 = \vec{0} \Rightarrow |E_1| = |E_2| \Rightarrow \frac{kQ_1}{R^3}(0.5R) = \frac{kQ_2}{2.25R^2}$

$\Rightarrow \frac{Q_1}{Q_2} = \frac{1}{0.5(2.25)} = \frac{1}{1.125} \Rightarrow \boxed{\frac{Q_2}{Q_1} = 1.125}$

(3) Consider a vertical slab of uniformly distributed charge ρ (coul/m^3) in (y, z) plane between $x = \pm L$ as shown.



(a) $E_0(x)$ for $|x| \geq L$



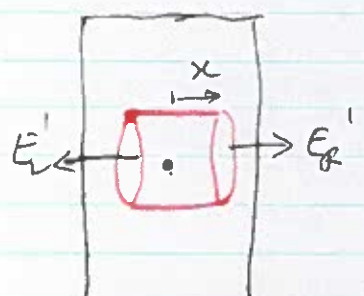
For large slab $\vec{E}_R$ is $\perp$ surface of plane –
$\vec{E}_R = E\,\hat{x}$
$\vec{E}_L = E(-\hat{x})$

Using Gauss's Law – for $|x| \geq L$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} = E_R A + E_L A = 2EA = \frac{\rho A(2L)}{\epsilon_0} = \frac{Q_{enc}}{\epsilon_0}$$

Hence $E(x) = \frac{\rho L}{\epsilon_0}$ in $\pm\hat{x}$ direction for $|x| \geq L$

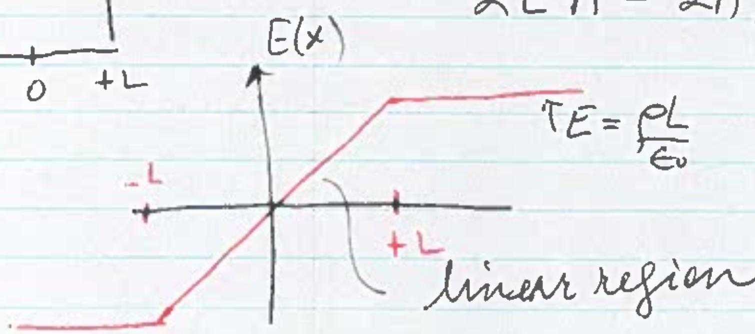
(b) Consider $|x| \leq L$ (inside the slab)



now $Q_{enc}' = \rho A(2x)$

$$\therefore \oint \vec{E}'\cdot d\vec{A} = E_R' A + E_L' A = \frac{\rho A\, 2x}{\epsilon_0}$$

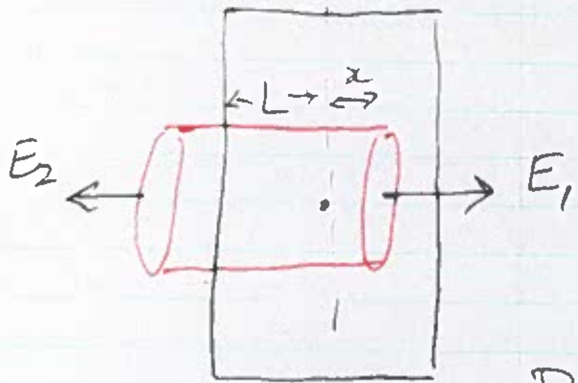
$$2E'A = 2A\frac{\rho x}{\epsilon_0} \Rightarrow \boxed{\vec{E}' = \frac{\rho x}{\epsilon_0}\hat{x}}$$



Suppose we take a non-symmetric Gaussian box with one side @ $x = 0$

Note E_R points to Right, E_L points to left $\Rightarrow \boxed{E(0) = 0}$

Suppose we use the following Gaussian surface for $x \le L$ case



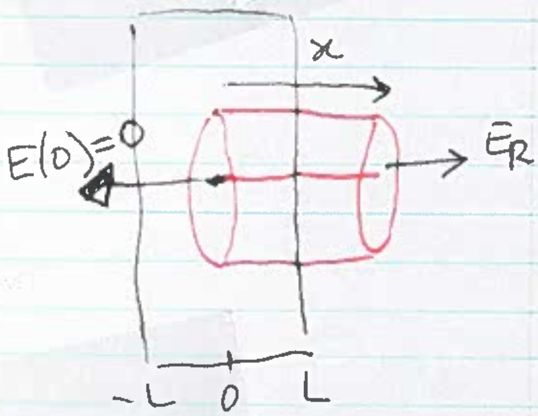
$$Q''_{enc} = \frac{\rho(L+x)}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{A} = E_1(x)A + E_2(L)A$$

$$\therefore \left[E_1(x) + E_2(L)\right]A = \frac{\rho L}{\epsilon_0} + \frac{\rho x}{\epsilon_0}$$

But $E_2(L) = \frac{\rho L}{\epsilon_0} = E_R(L)$

So we again find $\boxed{E_{inside}(x) = \frac{\rho x}{\epsilon_0} \quad |x| \le L}$



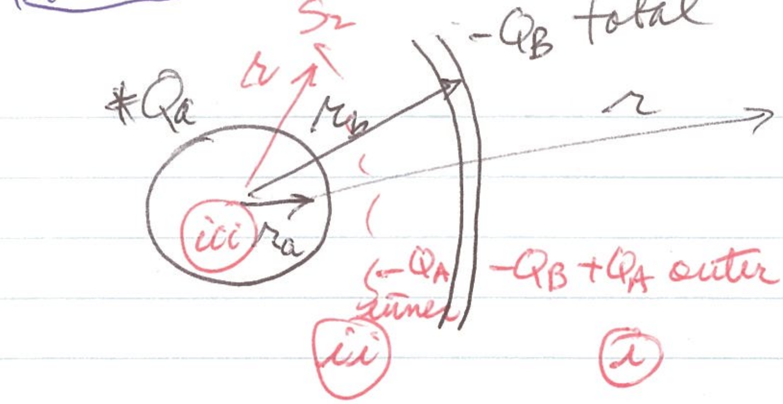
$$\oint \bar{E} \cdot d\bar{A} = E_R A + 0 = \frac{Q_{enc}}{\epsilon_0} = \frac{\rho A L}{\epsilon_0}$$

$$\Rightarrow E_R = \frac{\rho L}{\epsilon_0}$$

4. Tetrahedron Flux = E dot Area

(i) inward flux is Negative = -14.62 Nm^2/C
(ii) outward flux = 0.33 * | inward flux | = 4.87

(Q5 - 2024 March)



$$\oint_{S_1} \vec{E}_1(r) \cdot d\vec{A} = \frac{Q_A - Q_B}{\epsilon_0} \quad r > r_b$$

S_1 Gauss's Law

$$\Rightarrow \boxed{E_1(r > r_b) = \left(\frac{Q_A - Q_B}{4\pi\epsilon_0}\right)\frac{1}{r^2}}$$

(b) for $r < r_b$ but $r > r_a$

$$\oint_{S_2} \vec{E}_2(r) \cdot d\vec{A} = \frac{Q_A}{\epsilon_0}$$

$$r_a < r < r_b \Rightarrow \boxed{E_2(r) = k\frac{Q_A}{r^2}}$$

(c) calc Electric potential V

region (i) $r > r_b$

$$\int_r^\infty dV = -\int_r^\infty \vec{E}_1 \cdot d\vec{l} \quad \text{integrate outwards.} \quad d\vec{l} = +d\vec{r}$$

$$\underset{0}{V_i(\infty)} - V_i(r) = -\int_r^\infty k\left(\frac{Q_A - Q_B}{r^2}\right)dr = \left.\frac{Q_A - Q_B}{r}\right|_r^\infty$$

$$-V_i(r) = -k\frac{(Q_A - Q_B)}{r} \Rightarrow V_i(r) = +k\left(\frac{Q_A - Q_B}{r}\right)$$

$$\text{Hence } \boxed{V_i(r_b) = k\left(\frac{Q_A - Q_B}{r_b}\right)}$$

region (ii)

$r_a < r < r_b$

$$\int_r^{r_b} dV = -\int_r^{r_b} \vec{E}_2 \cdot d\vec{A} \quad \text{integrate outwards} \quad d\vec{l} = +d\vec{r}$$

$$= -\int_r^{r_b} k\frac{Q_A}{r^2}\,dr = \left.+k\frac{Q_A}{r}\right|_r^{r_b}$$

$$V(r_b) - V_{ii}(r) = kQ_A\left(\frac{1}{r_b} - \frac{1}{r}\right)$$

$$\therefore \left(\frac{\cancel{Q_A} - Q_B}{r_b}\right)k - V_{ii}(r) = \frac{k\cancel{Q_A}}{r_b} - \frac{kQ_A}{r}$$

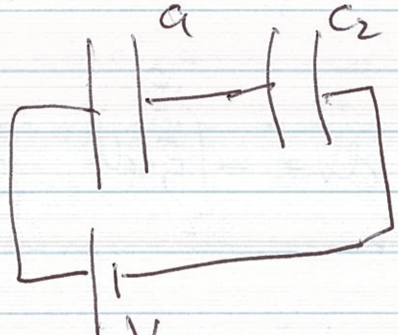
Hence $\boxed{V_{ii}(r) = \frac{kQ_A}{r} - \frac{Q_B k}{r_b}}$ for $r_a < r < r_b$

region (iii) $E_3 = 0$

Hence $V_{iii}(r)$ for $r < r_a$ is $V_{ii}(r_a)$

$$\boxed{V_{iii}(r) = \text{constant} = \frac{kQ_A}{r_a} - k\frac{Q_B}{r_b}}$$

6.



$U_0 =$

(a) Battery connected $\Rightarrow Q_1 = Q_2 \Rightarrow C_1V_1 = C_2V_2 \Rightarrow V_1 = V_2 = V/2$
no dielectric

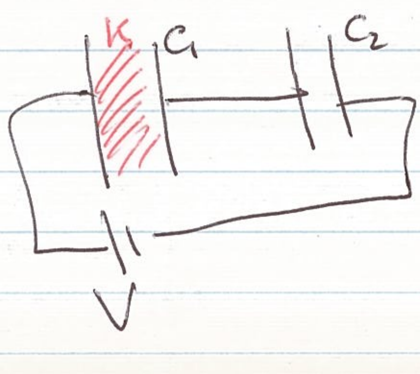
$$V = V_1 + V_2$$

$$U_0 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} = \frac{1}{2}Q^2\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q^2}{2}\left(\frac{2}{C}\right)$$

since $Q_1 = Q_2$

$$\Rightarrow \underline{U_0 = Q^2/C} = \frac{(CV/2)^2}{C} = \frac{CV^2}{4}.$$

Now we add a dielectric in $C_1 \Rightarrow C_1' = kC_1$



$$\frac{1}{C_T'} = \frac{1}{kC} + \frac{1}{C} = \frac{1}{C}\left(\frac{K+1}{K}\right)$$

$$C_T' = \frac{KC}{K+1}$$

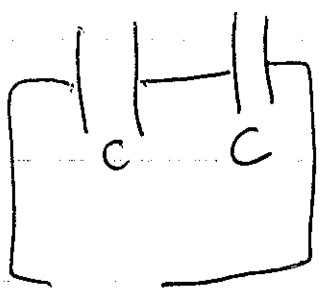
Hence $U = \frac{1}{2} C' V^2$ (Same Voltage as before - battery remains connected

$\Rightarrow U = \frac{1}{2} \frac{KC}{K+1} V^2$, $U_0 = \frac{CV^2}{4}$

$$\Rightarrow \boxed{\frac{U}{U_0} = \frac{KCV^2}{2(K+1)} \frac{4}{CV^2} = \frac{2K}{K+1}}$$ (battery stays connected)

(b) Since $K > 1$ $U > U_0$ (Energy increases)

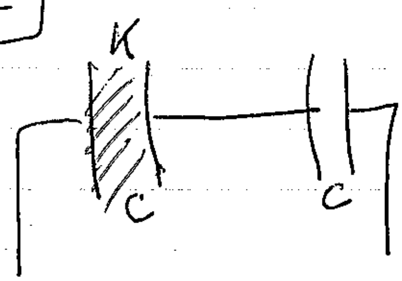
(c) Now we disconnect the battery so Q remains constant as we insert the dielectric



$U_0 = \frac{1}{4} C V^2$ $Q_1 = C \frac{V}{2} = Q_2 = Q$

$$\boxed{U_0 = Q^2/C}$$

Now we insert the dielectric



$Q_1'' = KCV_1''$

$Q_2'' = CV_2''$ $C_T' = \frac{KC}{K+1}$ $U = \frac{1}{2} \frac{Q^2}{C_T'}$

$U = \frac{1}{2} \frac{Q^2 (K+1)}{KC}$; $U_0 = \frac{Q^2}{C}$

$$\boxed{\frac{U}{U_0} = \frac{Q^2(K+1)}{2KC}\left(\frac{C}{Q^2}\right) = \frac{K+1}{2K}}$$

energy decreases when dielectric is inserted if Q remains constant.