MATH 152 MATLAB Computer Lab 6

Random Walks, Eigenvalues and Eigenvectors

Instructions

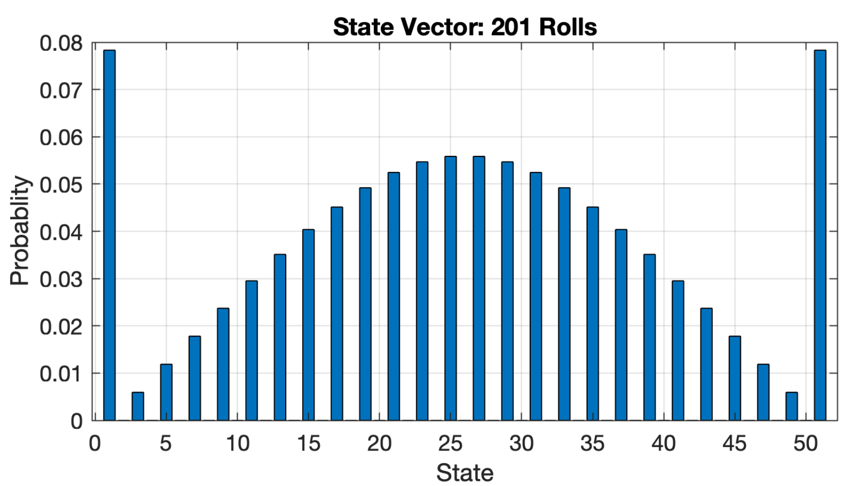
- Download `data6.mat` and upload to your MATLAB environment
- Save all variables to a file called `lab6.mat` and submit the file to Canvas
- Attend your scheduled lab section and visit MATLAB TA office hours for extra help

Exercise 1

You and a friend each start with 25 coins (50 coins total). On each turn a dice is rolled. If the number on the dice is even, then your friend gives you 1 coin. If the dice is odd, then you give your friend 1 coin. A player wins once they have collected all the coins. Load the file `data6.mat`. The matrix `P50` in the file is the transition matrix P for the coin game.

(a) Find the probability distribution after 201 rolls of the dice. In other words, find the state vector $\mathbf{x}_{201}$ where $\mathbf{x}_0 = [0, \ldots, 1, \ldots, 0]^T$ is the initial state vector with 1 at index 26 and all other entries 0. Store the vector $\mathbf{x}_{201}$ as `Ex1Avec`. We can visualize the state vector as a bar plot:

```
>> bar(Ex1Avec), xlabel('State'), ylabel('Probablity')
>> title('State Vector: 201 Rolls')
```



Notice that after an *odd* number of turns the probability that the game will be at an *even* state is 0.

(b) What is the probability of *someone* winning the 50 coin game after 500 turns? Save the probability as `Ex1Bnum`.

(c) Use trial and error to find the number of turns such that there is at least a 80% chance that *someone* has won the game. Save the value as `Ex1Cnum`.

(d) Suppose you start the game with 20 coins, and your friend starts with 30. This changes the initial state vector $\mathbf{x}_0$. What is the probability that you win the game *eventually* (after so many turns that the game is almost surely over). Save the probability as `Ex1Dnum`.

Exercise 2

Assume that Vancouver weather can be characterized as either sunny, rainy, snowy, or stormy. Suppose analysis of historical records indicates that:

1. If it is sunny today then:
    - 35% chance of sunny tomorrow.
    - 55% chance of rainy tomorrow.
    - 10% chance of snowy tomorrow.
2. If it is rainy today then:
    - 10% chance of sunny tomorrow.
    - 60% chance of rainy tomorrow.
    - 25% chance of snowy tomorrow.
3. If it is snowy today then:
    - 25% chance of sunny tomorrow.
    - 50% chance of rainy tomorrow.
    - 15% chance of snowy tomorrow.
4. If it is stormy today then:
    - 40% chance of sunny tomorrow.
    - 30% chance of rainy tomorrow.
    - 0% chance of snowy tomorrow.

Let us label the 4 weather states sun, rain, snow and storm as $1, 2, 3, 4$ respectively. Now define the vector

$$\mathbf{x}_n = \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ x_{3,n} \\ x_{4,n} \end{pmatrix}$$

as the state vector of the system on day n in the future. The entries of $\mathbf{x}_n$ represent the probability that the system is in any one of its states on day n. For example, the probability that it is raining on day n is $x_{2,n}$. If today is day 0 and we know that it is rainy today then $x_{2,0} = 1$.

This system can be modelled by a random walk. The transition matrix for this random walk with the given ordering of states is

$$P = \begin{pmatrix} 0.35 & 0.1 & ??? & 0.4 \\ ??? & 0.6 & ??? & 0.3 \\ ??? & 0.25 & 0.15 & 0.0 \\ ??? & ??? & ??? & ??? \end{pmatrix}$$

(a) Use the information above to figure out the missing entries in the transition matrix P. Save the matrix as `Ex2Amat`. Note the columns should add to 1:

```
>> sum(Ex2Amat)

ans =

     1     1     1     1
```

(b) Suppose today is a sunny day. What is the probability that it will be snowing in 6 days from now? Save the value as `Ex2Bnum`.

(c) If today is stormy, what is the weather probability distribution $\mathbf{x}_7$ one week later (ie. after 7 days)? Save this vector as `Ex2Cvec`.

(d) Suppose we want to predict the weather after a *very* long time. Given our model, would you expect (1) sun, (2) rain, (3) snow or (4) storm? Save your prediction as `Ex2Dnum`.

Exercise 3

Consider the matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0.0 & 0.2 \\ 0.1 & 0.6 & 0.3 & 0.0 \\ 0.0 & 0.2 & 0.4 & 0.2 \\ 0.1 & 0.0 & 0.3 & 0.6 \end{pmatrix}$$

The matrix P is the transition matrix of a random walk because the entries are non-negative and all the columns sum to 1:

```
>> P = [0.8 0.2 0.0 0.2; 0.1 0.6 0.3 0.0; 0.0 0.2 0.4 0.2; 0.1 0.0 0.3 0.6]
>> sum(P)

ans =

     1     1     1     1
```

(a) What are the eigenvalues of the matrix P? Save the eigenvalues as a 4×1 column vector with the variable name `Ex3Avec`.

(b) What are the eigenvectors of the matrix P? Save the eigenvectors as `Ex3Bmat` where each column is an eigenvector.

(c) The matrix P represents the transition matrix of a random walk with 4 states. After a *very large* number of steps through the random walk, what is the probability that you are in State 3? Save the probability as `Ex3Cnum`.

Exercise 4

The transition matrix P of a random walk with N states is a $N \times N$ matrix with non-negative entries such that each column sums to 1. The following command will generate a random 5×5 transition matrix `P`:

```
>> N = 5; P = rand(N); P = P./sum(P)
```

Do you see why the result P is a transition matrix? Also notice that the result will always have non-zero entries because the random number generator `rand` will never return 0. Run the command several times and, for each P, compute the eigenvalues and eigenvectors. What do you notice? Enter `"True"` or `"False"` for each statement below.

(a) An eigenvalue λ of any transition matrix P satisfies $|\lambda| \leq 1$. Save your answer as `Ex4Aword`. (If λ is real, then $|\lambda|$ denotes the absolute value. If λ is complex, then $|\lambda|$ denotes the modulus.)

(b) A real eigenvalue λ of a transition matrix P satisfies $\lambda \geq 0$. Save your answer as `Ex4Bword`.

(c) Every transition matrix P with non-zero entries has eigenvalue $\lambda = 1$ with multiplicity 1. Save your answer as `Ex4Cword`.

(d) There exists a transition matrix P (not necessarily with all non-zero entries) which has eigenvalue $\lambda = 1$ with multiplicity greater than 1. Save your answer as `Ex4Dword`.