PHYS 170

Week 7: Kinematics I

Section 201 (Mon Wed Fri 12:00 – 13:00)

Intro remarks:

- Kinematics: Considers how (but not why) objects move
  - …in contrast to Kinetics (Weeks 9&10), which analyses forces causing accelerated motion & motion per se
- We will need calculus!
- …and vector analysis, and algebra, and trigonometry.

Rectilinear (1D) motion

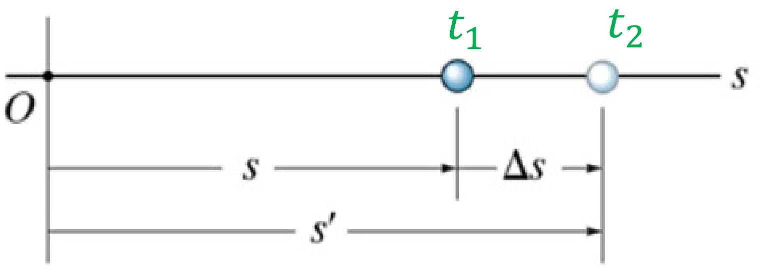
Text: 12.1-12.3

Content:

- Position, velocity, acceleration
- Velocity vs average velocity vs speed
- Connections $x(t) \Leftrightarrow v(t) \Leftrightarrow a(t)$
- Initial conditions: when and why do we need them?
- Special case: Motion with constant acceleration

Average velocity, Velocity, Speed & Average acceleration, Acceleration

- Position of the object can change as the time goes:



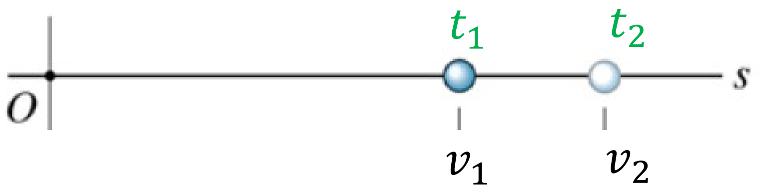
- Velocity: shows how fast the coordinate changes

- Units: m/s, ft/s, km/h…

$$\underbrace{v_{av} = \frac{\Delta s}{\Delta t} \qquad v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}}_{\text{Velocity (can be positive or negative)}}$$

$$\underbrace{v_{sp} = |v|}_{\text{Speed}}$$

- Velocity of the object can change as the time goes:



- Acceleration: shows how fast the velocity changes

- Units: m/s^2, ft/s^2…

$$a_{av} = \frac{\Delta v}{\Delta t} \qquad a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

SIGNS OF VELOCITY AND ACCELERATION

- Acceleration:

  - The object is speeding up if $\vec{a} \uparrow\uparrow \vec{v}$
  - The object is slowing down if $\vec{a} \uparrow\downarrow \vec{v}$

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$$

Assume $\Delta t = t_2 - t_1 = 1\ s$, and that the object is slowing down from 3 m/s at t_1 to 1 m/s at t_2.

Moves in:	v_1	v_2	a
$\longrightarrow\ +s$	3 m/s	1 m/s	-2 m/s²
$-s\ \longleftarrow$	-3 m/s	-1 m/s	2 m/s²

Indeed, a and v always have opposite signs.

When is the object slowing down?

When its speed decreases.

$$(\text{speed}) = \begin{cases} v, & \text{if } v > 0 \\ -v, & \text{if } v < 0 \end{cases}$$

$$\frac{d}{dt}(\text{speed}) = \begin{cases} \dfrac{d}{dt}(v) = a, & \text{if } v > 0 \\ \dfrac{d}{dt}(-v) = -a, & \text{if } v < 0 \end{cases}$$

$$d(\text{speed})/dt < 0 \ \text{ if: } \left[\begin{array}{l} a < 0 \ \text{ and } \ v > 0 \\ a > 0 \ \text{ and } \ v < 0 \end{array}\right.$$

… and a similar analysis for the case when an object is speeding up.

$s(t) \Rightarrow v(t) \Rightarrow a(t)$

$\rightarrow v\,!$

$$a\,ds = \frac{dv}{dt} \cdot ds = dv\left(\frac{ds}{dt}\right) = dv \cdot v$$

- $s(t) \Rightarrow v(t)$: $\quad v = \dfrac{ds}{dt} \equiv \dot{s}$

- $v(t) \Rightarrow a(t)$: $\quad a = \dfrac{dv}{dt} = \dot{v}$

- $s(t) \Rightarrow a(t)$: $\quad a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2 s}{dt^2} \equiv \ddot{s}$

W7-1. Prove that for <u>any 1D motion</u> (with arbitrary acceleration, not necessarily constant!)

$$\boxed{a\,ds = v\,dv}$$

This equation is convenient when you want to determine the velocity of a particle, and the acceleration is given as a function of position, $a = a(s)$.

$a(t) \Rightarrow v(t) \Rightarrow s(t)$

- $v(t) \Rightarrow s(t)$: $v = \dfrac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt \quad \Rightarrow \quad s(t) = \int v\,dt + C_s$

- $a(t) \Rightarrow v(t)$: $a = \dfrac{dv}{dt} \quad \Rightarrow \quad dv = a\,dt \quad \Rightarrow \quad v(t) = \int a\,dt + C_v$

- This shows that when we are computing $s(t)$ using known $v(t)$, or computing $v(t)$ using known $a(t)$, we need to specify an initial condition: for example, if we know that $s(t = t_0) = s_0$, we will be able to find C_s. Likewise, if we know that $v(t = t_0) = v_0$, we will be able to find C_v.

- Actually, it is more convenient to rewrite these equations using definite integrals, as it is done on the next slide

Example

- $v(t) \Rightarrow s(t)$:

$$v = \frac{ds}{dt} \Rightarrow ds = v\,dt \Rightarrow ?$$

$$\int_{s_0}^{s} ds = \int_{t_0}^{t} v\,dt \Rightarrow s\Big|_{s=s_0}^{s=s} = \int_{t_0}^{t} v(t)\,dt$$

$$s(t=t_0) = s_0$$

$$s - s_0 = \int_{t_0}^{t} v(t)\,dt$$

$$\boxed{s(t) = \overset{\text{const}}{s_0} + \int_{t_0}^{t} v(t)\,dt}$$

$$a(t) \Rightarrow v(t) \Rightarrow s(t)$$

...and, similarly, $v(t) = \int a(t)\,dt + C$

$$\begin{cases} ds = v\,dt = v(t)dt, \\ s(t_0) = s_0 \end{cases}$$

Then: $\int_{s_0}^{s} ds = \int_{t_0}^{t} v\,(t)\,dt$

$$s(t) - s_0 = \int_{t_0}^{t} v\,(t)\,dt$$

$$\boxed{s(t) = s_0 + \int_{t_0}^{t} v\,(t)\,dt}$$

$$\begin{cases} dv = a\,dt = a(t)dt, \\ v(t_0) = v_0 \end{cases}$$

Then: $\int_{v_0}^{v} dv = \int_{t_0}^{t} a\,(t)\,dt$

$$v(t) - v_0 = \int_{t_0}^{t} a\,(t)\,dt$$

$$\boxed{v(t) = v_0 + \int_{t_0}^{t} a\,(t)\,dt}$$

$$\begin{cases} ds = v\,dt = v(t)\,dt \\ s(t_0) = s_0 \end{cases}$$

Then: $\int ds = \int v(t)\,dt$

$$\boxed{s(t) = \int v(t)\,dt + C}$$

and we will choose C in such a way, that at $t = t_0$ we would get: $s(t_0) = s_0$.

Using definite integrals

Using indefinite integrals

Let's summarize:

- What we will do in this chapter: $s(t) \Leftrightarrow v(t) \Leftrightarrow a(t)$

- When integrating, we will need initial conditions (s_0, v_0)

- Toolkit:
  - Derivatives
  - Integration

- Notations:

$$\frac{ds}{dt} \equiv \dot{s}$$

$$\frac{d^2s}{dt^2} \equiv \ddot{s}$$

$$\frac{d^3s}{dt^3} \equiv \dddot{s}$$

...

Position

$$v(t) = \frac{ds}{dt} = \dot{s}$$

slope of the $s(t)$ graph

$$s(t) = s_0 + \int_{t_0}^{t} v(t)\, dt$$

area under the $v(t)$ graph

Velocity

$$a(t) = \frac{dv}{dt} = \ddot{s}$$

slope of the $v(t)$ graph

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\, dt$$

area under the $a(t)$ graph

Acceleration

What to expect:

- We will learn how, knowing one of the functions $s(t), v(t), a(t)$, to derive all other functions.
- We will work in 1D, 2D and 3D, and in a variety of coordinate systems.

$a = const$

- One specific kinematic case known to most of you from high school is motion with a constant acceleration, such as free fall. We will start with it.
- However, our accent will be on a general situation, when $\boxed{a = a(t).}$ It is only natural that objects are not bound to move keeping their acceleration constant; think of a car which first speeds up ($\vec{a} \uparrow\uparrow \vec{v}$), then goes with a constant speed ($\vec{a} = 0$), then slows down ($\vec{a} \uparrow\downarrow \vec{v}$). Here $\vec{a} \neq const$.
  - $j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}$: “jerk” (important for roller coasters)
  - $\sigma(t) = \frac{dj}{dt} = \frac{d^4 s}{dt^4}$: “snap” (and next ones are “crackle” and “pop”… seriously)
- Important to remember when integrating and differentiating:
  - Differentiation is about 1 point (local in time domain)
  - Integration requires knowing initial conditions (result depends on pre-history)
- Review calculus !!!

…and now let’s go.

Q: An object travels with velocity $v = 3t^2$ m/s, where t is measured in seconds. How far does it travel between $t = 1$ s and $t = 2$ s?

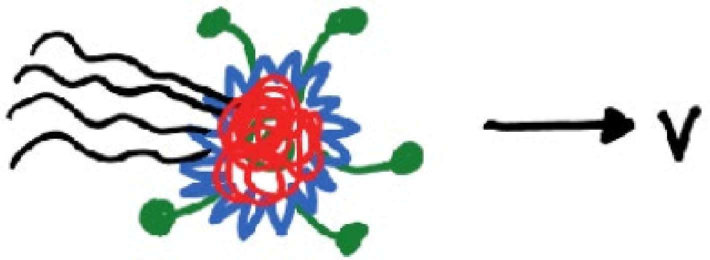
A. 12 m

B. 11 m

C. 9 m

D. 7 m

E. None of the above



$$s \leftrightarrows v \qquad v = \frac{ds}{dt} \rightarrow ds = v\,dt$$

$$\int_{s(t=1s)}^{s(t=2s)} ds = \int_{t=1s}^{t=2s} v(t)\,dt$$

$$s\Big|_{s(t=1s)}^{s=(2s)} = S(2s) - S(1s) = \Delta S = \int_{t=1}^{t=2} 3t^2 \cdot dt =$$

$$= \cancel{3} \cdot \frac{t^3}{\cancel{3}}\Big|_{t=1s}^{t=2s} = (2)^3 - (1)^3 =$$

$$= 8 - 1 = 7\,m$$

IMPORTANT SPECIAL CASE: Motion with constant acceleration

W7-2. Sometimes acceleration does not change. Important **example** is a free fall; near the surface of the Earth its magnitude is $g = 9.81$ m/s^2 and it acts downwards.

Prove that in this case (a is the acceleration, s_0 and v_0 are the object's position and velocity at $t = 0$):

$$s(t) = s_0 + v_0 t + \frac{a\,t^2}{2} \qquad (1) \checkmark$$

$$v(t) = v_0 + a\,t \qquad (2) \checkmark$$

$$v^2(t) = v_0^2 + 2a(s - s_0) \qquad (3) \checkmark$$

const

$$\int_{s_0}^{s} a\,ds = \int_{v_0}^{v} v\,dv$$

$$a \cdot s\,\Big|_{s_0}^{s} = \frac{v^2}{2}\Big|_{v_0}^{v}$$

$$2a(s - s_0) = v^2 - v_0^2$$

When it is more convenient to use Eq.(2), and when Eq.(3) to find the velocity?

a = const

$$(2) \quad a = \frac{dv}{dt} \rightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} a\,dt \rightarrow v - v_0 = a\int_{t_0=0}^{t} dt = a(t - t_0) \rightarrow v(t) = v_0 + a t$$

const

$$(1) \quad v = \frac{ds}{dt} \rightarrow \int_{s_0}^{s} ds = \int_{t_0=0}^{t} v\,dt \rightarrow s - s_0 \overset{(2)}{=} \int_{t_0=0}^{t} [v_0 + at]\,dt = \int_{t_0=0}^{t} v_0\,dt + \int_{t_0=0}^{t} a t\,dt =$$

$$= v_0 t\,\Big|_{t_0=0}^{t} + a\,\frac{t^2}{2}\Big|_{t_0=0}^{t} = v_0 t + \frac{a t^2}{2}$$

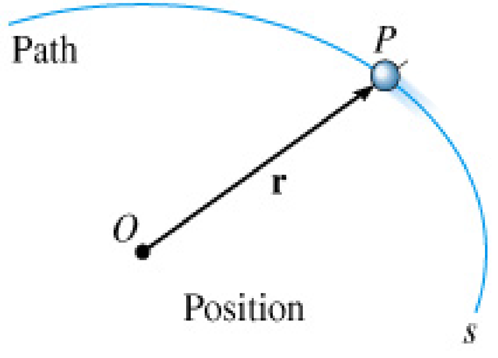
Curvilinear (2D, 3D) motion:
Cartesian components

Text: 12.4-12.5

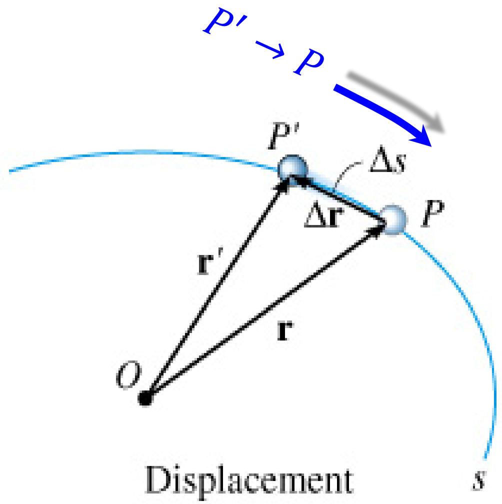
Content:

- Beyond 1D: Definitions of $s(t), v(t), a(t)$
- Graphical interpretation
- Equations of motion in Cartesian components

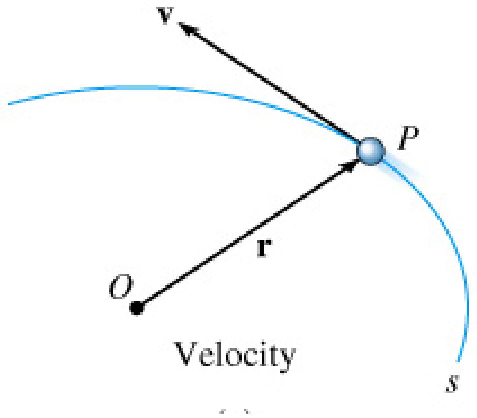
Velocity & Acceleration in 2D: Graphical Approach



- Trajectory: $P = r(t)$



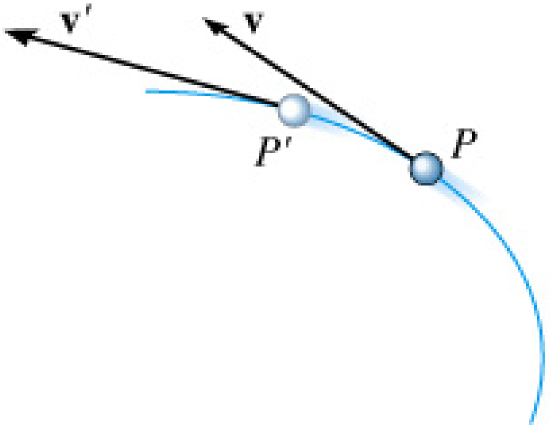
- Draw a small shift along the trajectory, $\Delta\vec{r}$
- $\Delta\vec{r}$ = displacement vector, Δs = its magnitude
- Let $P' \to P$: $\Delta\vec{r} \to d\vec{r}$, $\Delta s \to ds$



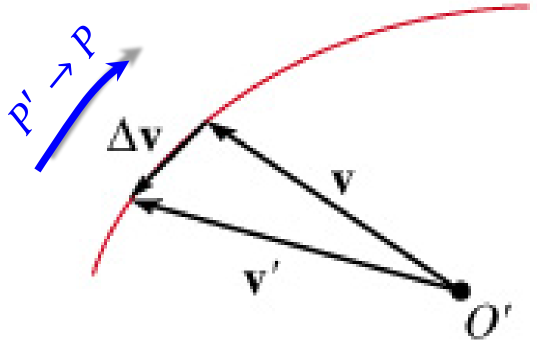
- Velocity:
  - Tangent to the trajectory
  - Magnitude: $v = ds/dt$
  - We can also say that:

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}}$$

Velocity & Acceleration in 2D: Graphical Approach

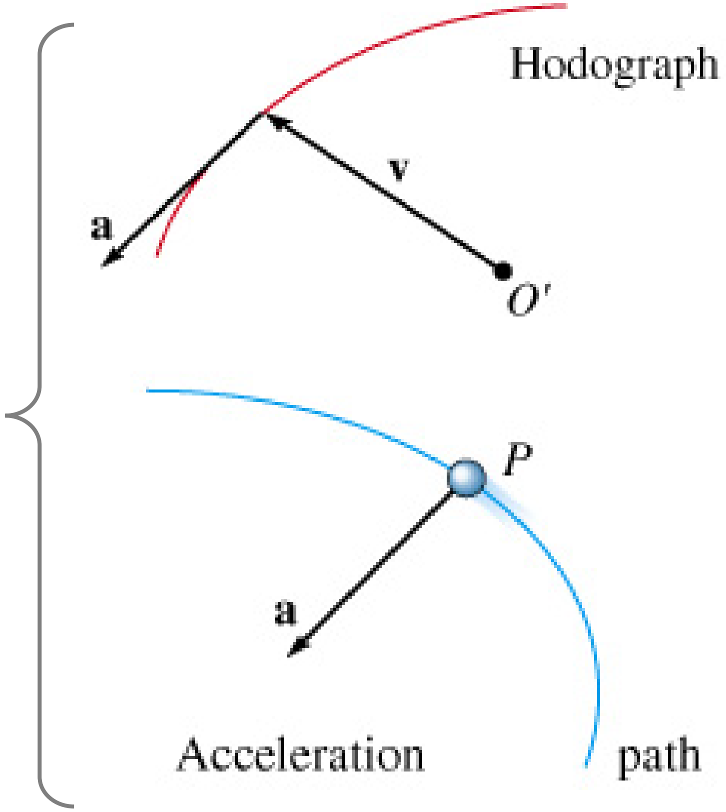


- Draw trajectory, $P = r(t)$, and add the velocity vectors at each point



- Draw the hodograph (the curve which the velocity arrowheads touch)
- $\Delta\vec{v} =$ difference between two close velocities (at moments t' and t)
- Let $t' \to t$: $\Delta\vec{v} \to d\vec{v}$
- Then we will get:

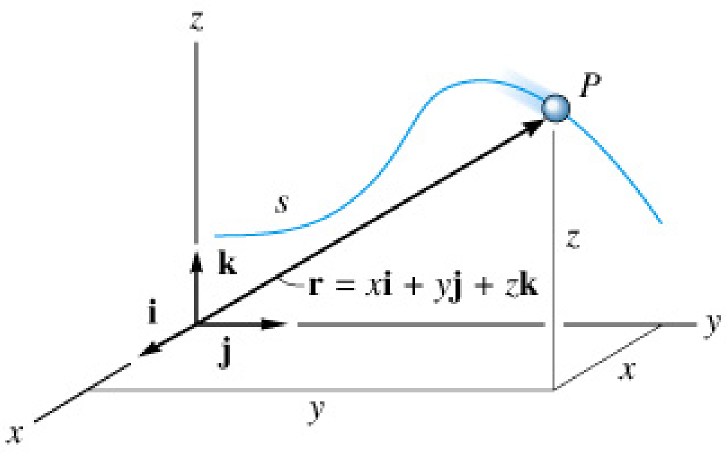
$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$



- Acceleration:
  - Tangent to the hodograph
  - Points inwards the trajectory
  - Must account for the change in the direction of v, and also for change in its magnitude!

Velocity & Acceleration in Rectangular Components

- These pictures are nice, but it is difficult to work with them. Let us come up with something else.



Position

- If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, what is $\dfrac{d\vec{r}}{dt} = \vec{v}$?

- Product rule: $\dfrac{d(ab)}{dt} = a\dfrac{db}{dt} + b\dfrac{da}{dt}$

- Then: $\dfrac{d(x\vec{i})}{dt} = \cancel{x\dfrac{d\vec{i}}{dt}} + \vec{i}\;\dfrac{dx}{dt}$

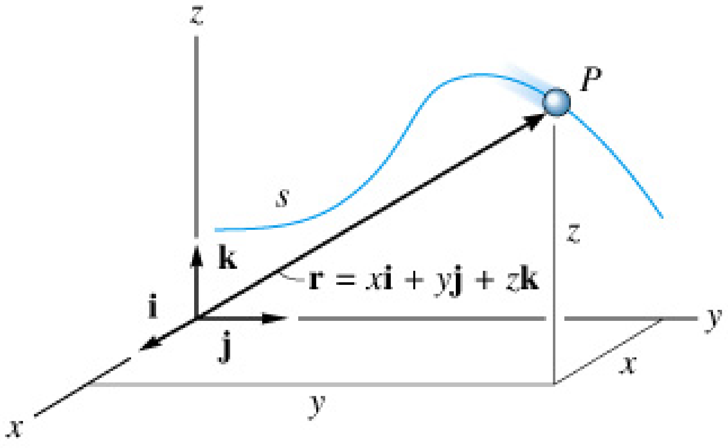
- Note that: $\dfrac{d\vec{i}}{dt} = 0$ ($\vec{i}$ does not change with t)

$$\boxed{\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}}$$

where

$$\boxed{v_x = \frac{dx}{dt} = \dot{x}, \qquad v_y = \frac{dy}{dt} = \dot{y}, \qquad v_z = \frac{dz}{dt} = \dot{z}}$$

Velocity & Acceleration in Rectangular Components



Position

- If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$:

$$\boxed{\begin{aligned}\vec{v} &= v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}\\ \vec{a} &= a_x\,\vec{i} + a_y\,\vec{j} + a_z\,\vec{k}\end{aligned}}$$

where:

$$v_x = \frac{dx}{dt} = \dot{x}, \quad v_y = \frac{dy}{dt} = \dot{y}, \quad v_z = \frac{dz}{dt} = \dot{z}$$

$$a_x = \frac{dv_x}{dt} = \ddot{x}, \quad a_y = \frac{dv_y}{dt} = \ddot{y}, \quad a_z = \frac{dv_z}{dt} = \ddot{z}$$

- Note: we now have three one-dimensional problems (which we already know how to work with!)
- We can use these algebraic equations to find the components of $\vec{r}(t), \vec{v}(t)$ and $\vec{a}(t)$
  - Q: These quasi-1D-problems are not completely independent. What connects them?