ASSIGNMENT 4

In this assignment, you will use quantitative and qualitative methods to explore the solutions to a non-linear differential equation. You will study solutions where the initial condition takes values near a steady state. You will use Euler's method to find numerical solutions that approximate the exact solutions of a differential equation.

Learning goals

- Use linear approximation to study solutions of a non-linear differential equation near stationary points.
- Build a stronger understanding of the relationship between the phase line for a differential equation and the the solutions to that differential equation.
- Practice graphing or sketching solutions to a differential equation.

The material in Chapters 12 and 13 of Keshet's Differential Calculus for the Life Sciences (DCLS) is relevant to this assignment. This e-book is linked in Canvas under Textbooks.

Contributors

*On the first page of your submission, list the student numbers and full names (with the last name in **bold**) of all team members. Indicate members who have not contributed using the comment "(non-contributing)".*

Reflection question

*Reflection questions encourage you to think about how mathematics is done. This is an important ingredient of success. Reflection questions contribute to your **engagement grade**.*

1. Review the team contract you constructed in Assignment 1. Then answer each of the following parts. Note that parts (b) and (c) require each contributing member of the team to contribute individual responses.
   (a) How faithfully did your team meet the conditions of your team contract?
   (b) As an individual member, what was your most helpful contribution to the team?
   (c) As an individual member, which other team member did you especially appreciate, and why?

Assignment questions

*The questions in this section contribute to your **assignment grade**. Stars indicate the difficulty of the questions, as described on Canvas.*

Consider the differential equation

$$\frac{dy}{dy} = y(y-1)(3-y). \tag{1}$$

The solutions of this differential equation will have various behaviours that depend on an initial condition $y(t_0) = y_0$.

In class, we have considered how to sketch solutions for a differential equation of this form by making use of a plot of $\frac{dy}{dt}$ versus y to plot the *direction field* (equivalently called the *slope field*) for this equation.

We have also made use of the *phase line* to understand the behaviour of the solutions near the *steady states* – places where $\frac{dy}{dt} = 0$.

In this assignment, you will make a more detailed study of the behaviour of solutions to this differential equation near its steady states.

2. (4 marks) ★★☆☆

   (a) Identify the steady states of equation (1). Create a plot of $\frac{dy}{dt}$ against y and label the steady states on this plot. Also, treat the horizontal axis as the phase line for this equation, and show the directions of flow for equation (1) on this phase line (the y-axis).

   (b) For each steady state $y = a$, sketch the solutions $y(t)$ of equation (1) for the set of initial conditions $y(0) = a - 0.25$, $y(0) = a$, and $y(0) = a + 0.25$. Use the same set of y versus t axes for sketching all the solutions together.

3. (8 marks) ★★★☆ To study equation (1) near a steady state, approximate the cubic

$$f(y) = y(y-1)(3-y) = -y^3 + 4y^2 - 3y$$

   by the linear approximation to $f(y)$ at that steady state. For example, if there is a steady state at $y = a$, then approximate $f(y)$ by $L(y; a) = f'(a)(y - a) + f(a)$.

   (a) Consider the differential equation

$$\frac{dy}{dt} = L(y; a) = f'(a)(y - a) + f(a) \qquad (2)$$

   for initial conditions $y(0) = y_0$ where y_0 is close to a. For each of the steady states $y = a$ for equation (1), find the linear approximation of $f(y)$ at that state and write down the corresponding version of the differential equation (2).

   Give the solutions $y(t)$ of this new, linearized differential equation (2) for the following set of initial conditions: $y(0) = a - 0.25$, $y(0) = a$, and $y(0) = a + 0.25$.

   (b) Plot all the solutions you found in 3(a) on the same set of axes. Compare the plots of the solutions you found in 3(a) to the solutions of equation (1) you sketched in 2(b) for the same initial conditions. Comment on the behaviour of the solutions you found in 3(a) as t gets large. Comment on how the solutions you found in 3(a) compare with the solutions for equation (1) you sketched in 2(b).

4. (5 marks) ★★★☆ While we do not have techniques to find an exact solution of the differential equation (1) with initial condition $y(0) = y_0$, we can find a numerical solution that approximates $y(t)$ using *Euler's Method*. (A detail discussion of Euler's Method is found in DCLS Chapter 12.3.)

   Euler's method approximates the initial value problem

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0,$$

   by taking the continuous time interval $0 \leq t \leq T$ and looking at a discrete set of times found by starting at time $t = 0$ and stepping along by a small constant time difference $\Delta t = (T - 0)/n = T/n$:

$$t_0 = 0,\ t_1 = 0+\Delta t,\ \ t_2 = t_1+\Delta t = 0+2\Delta t, ..., t_k = t_{k-1}+\Delta t = 0+k\Delta t, ..., t_n = t_{n-1}+\Delta t = 0+n\Delta t = n{\cdot}T/n = T,$$

   and calculating the numerical values

$$y_{k+1} = y_k + f(y_k)\Delta t,$$

   for $k = 0, 1, 2, 3, ..., n - 1$. Note, this iteration starts with the initial condition $y(0) = y_0$ being used to calculate y_1. How well this numerical solution approximates the true solution $y(t)$ depends on the size of Δt, amongst other things. We generally choose Δt to be small. The numerical solution can then be presented as a table of values with columns k, t_k, and y_k, where $k = 0, 1, 2, ..., n$; the choice of n depends on our choice of the time step Δt and the length of the time interval for which a numerical solution is needed. Euler's method can be easily implemented in a spreadsheet.

(a) For each steady state $y = a$, use Euler's method to find solutions of equation (1) for the set of initial conditions $y(0) = a - 0.25$, $y(0) = a$, and $y(0) = a + 0.25$ for a time interval $0 \le t \le 5$ using a time-step $t = 0.1$. Report **the last 10 values** of each of these solutions using tables of values. Using the same set of axes, plot all of these numerical solutions. Be sure to use a legend to label the individual solutions. Comment on the comparison of the numerical solutions you found to the qualitative sketches of solutions you did in 2(b).

(b) In 3(a), you found exact solutions to the differential equation (1) linearized at each steady state (equation (2)) for the same set of initial conditions you used in 4(a) to find numerical solutions using Euler's method. For each of the numerical solutions you found in 4(a), add a column to its table of values giving the value of the corresponding exact solution found in 3(a) for the 10 listed values in each table.

(c) On a separate set of axes for each steady state $y = a$, plot the numerical solutions and exact linearized solutions you found in 4(a) and 4(b) for the initial conditions $y(0) = a - 0.25$, $y(0) = a$, and $y(0) = a + 0.25$. Be sure to use a legend to label the individual solutions. Comment on what you see, giving particular consideration to the comments you made in response to 3(b).

Practice questions

The questions below are for practice. Starred questions are at the level of Part 2 exam questions. They do not contribute to your grade, but you are strongly encouraged to work through them under exam conditions.

5. ★★★☆

(a) Coffee in an urn is kept at 95 celsius. After pouring it in a cup, it cools to 90 celsius in 2 minutes. How long does it take for the coffee to cool to 65 celsius?

(b) Use Euler's method to find a numerical approximation to the function that describes the temperature of the coffee for twenty minutes after it is poured from the urn into a cup.

6. ★★★★ Consider the differential equation

$$\frac{dx}{dt} = \sqrt{|x|},$$

with initial condition $y(1/4) = 1/64$.

(a) Find a solution by guessing $x(t)$.

(b) Find a numerical solution of this initial value problem using Euler's method for $-1 \le t \le 1$ using time steps: (i) $\Delta t = 0.25$, (ii) $\Delta t = 0.1$, and (iii) $\Delta t = 0.01$.

(c) Find a numerical solution for the case where the initial conditions is (i) $y(0.1) = 0.0025$ and where it is (ii) $y(-0.1) = -0.0025$