COMP_SCI 496: Graduate Algorithms Notes

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Contents

1. Chapter 0: A Prelude in Graph Theory and Combinatorics

This is a chapter that is devoted to Randy learning how graph theory works in finer granularity.

1.1. Subgraphs

Subgraph Given a graph $G = (V, E)$, a subgraph of G, which is denoted as $G' = (V', E')$ is a graph whose vertices V' are a subset of V and whose edges E' are a subset of E.

1.2. Induced Subgraphs

Induced Subgraph Given a graph $G = (V, E)$, an induced subgraph $G' = (V', E')$ is a subgraph of G that contains the following property:

- all edges $e' \in E'$ must have (both) endpoints be in V'

We define the subgraph induced by V', where V' is a set of vertices $V' \subseteq V$, as being the subgraph $G' = (V', E')$ in which all $e' \in E'$ must have both endpoints $r'_1, r'_2 \in V'$.

1.3. Graph Orientations

Orientation Given an undirected graph G, an *orientation* of the graph G would be the resulting graph in which we assign each edge a direction, resulting in a directed graph.

1.4. Tournaments

Tournaments Given a set of vertices V, a tournament of V, denoted as T on V, is an *orientation* of the vertices, such that the resulting graph is connected.

- Note, in a tournament, two endpoints i, j can only have a single edge between them, in which $i \to j$ or $j \to i$, but not both

1.5. Dominating Sets

Dominating Set A *dominating set* of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V - U$ has at least one neighbor in U.

2. Chapter 1: The Basic Method

2.1. The Probabilistic Method

- Powerful tool for tackling problems in discrete math
- Main idea of the method
  - **Objective.** We see kto prove that a structure with certain desired properties *exists*
  - We first define an appropriate probabilisty space of structures
  - then, we show that the desired properties hold in these structures with positive probability

2.1.1. Example 1

We note that the *Ramsey number* $R(k, \ell)$ is the smallest integer n such that in any wo-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k (ie, a ocmplete subgraph on k vertices all of whose edges are colored red) or there is a blue K_ℓ.

- Ramsey showed that $R(k, \ell)$ is finite for any two integers k and l.
- We can obtain a lower bound for hte diagonal Ramsey numbers $R(k, k)$
  - **Remark.** A *complete graph* K on n (which is denoted as K_n) is a graph in which each pair of distinct n vertices is connected together by an edge.

2.1.1.1. Prop. 1

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$. Thus $R(k, k) > \left\lfloor 2^{\frac{k}{2}} \right\rfloor$ for all $k \geq 3$

- Note, here $R(k, k)$ represents a graph R in which the induced monochromatic graphs must *both* be of size k, which is a stronger condition than $R(k, \ell)$

Proof. In Section 2.1.1.1, we first derive the following:

- We first seek out the probability that an induced graph K_n is monochromatic, which is equiv. to

$$\Pr[A_R] = \left(\frac{1}{2}\right)^{\binom{n}{2}} \to 2^{\binom{n}{2}} \leftrightarrow 2^{1-\binom{n}{2}} \tag{1}$$

- We next seek the probability that any induced graph (aka a n-combination of the original k-graph) is monochromatic

$$\binom{k}{n} \cdot 2^{1-\binom{n}{2}} \tag{2}$$

- **Observation 1.** We understand that the probability of this event occuring must be bounded by 1, thus leading us to the conclusion that

$$\binom{k}{n} \Pr[A_R] < 1 \tag{3}$$

  which must imply that the probability that event $\Pr[A_R]$ doesn't occur must be non-zero
  - What exactly is the *negation* of event A_R? If A_R denotes the event that the induced subgraph of K_k on R is monochromatic, then $\neg(A_R)$ must be the probability that a two-coloring of the graph R does *not* produce a monochromatic induced subgraph
  - We would denote this is as the Ramsey number of induced grpah sizes n and n, or $R(n, n)$. Given that we want the negation, then we know that the size must be greater than n_0, since the Ramsey number must be n_0
- **Observation 2.** We understand that if the size of the induced graph, denoted as $n \geq 3$ and if we take the size of the graph k to be $k = \left\lfloor 2^{\frac{k}{2}} \right\rfloor$, then we know that

$$\binom{k}{n} 2^{1-\binom{n}{2}} < \left(\frac{2^{1+\frac{n}{2}}}{n!}\right) \cdot \left(\frac{k^n}{2^{\frac{n^2}{2}}}\right) < 1 \tag{4}$$

Thus, $R(n, n) > \lfloor 2^{\frac{n}{2}} \rfloor \forall n \geq 3$

2.2. Optional Proof Notes

- Because there are $\binom{k}{n}$ choosings of the graph G, then it follows that at least one of these events occurring must be non-zero but strictly less than 1.
  - Thus, the inverse of this statement must be true- ($\exists$ a two-coloring of the graph G of k vertices G_k that doesn't have a monochromatic induced graph G_n)
  - This is represented as the Ramsey number $R(k, k)$, which semantically equates to smallest size of a graph that has a monochromatic edge-colored subgraph of size k
  - Given that we know that the Ramsey number refers to the smallest n for which there is a monochromatic induced subgraph of sizes k or ℓ, then it follows that in order for both monographic subgraph sto be of sizes k, then the graph must be at least the size of the graph whose Ramsey number $R(k, \ell)$ is n.

2.3. The Essence of the Probabilistic Method

Note the way that we evaluated this problem.

- **Objective.** Prove the existence of a good coloring K_n given a graph K.
  - We first defined what a "good" coloring was– which was a non-monochromatic graph formed from the induced graph of the two-colored graph.
  - Then, we showed that it *exists*, in a nonconstructive way
    - We defined a probability space of events, and we narrowed that probability space down to events that described structure sof particular properties
    - From there, we then just showed that the desired properties that we want will hold in this narrowed down probability space, with positive probability.

2.3.1. Why is this approach effective?

This approach is effective because the vast majority of probability spaces in combinatorial problems are *finite*

- Sure, we could use an algorithm to try and find such a structure with a particular property
- For example, if we wanted to actually find an edge two-coloring of K_n without a monochromatic induced graph, we could just iterate through all possible edge-colorings and find their induced grpahs.
  - Obviously, this is impractical (it's actually class $\mathbb{P}$ haha)
  - Although these problems could be solved using *exhaustive searches*, we want a faster way.
  - This is the difference between *constructivist* and *nonconstructivist* ideas in proofs
    - Although we don't have a deterministic way of forming the graph, we are able to define an algorithm that could potentially lead to the desired graph, which, which is more effective than just trying to deterministically create one
    - In the case of the Ramsey-number problem, it would be more effective to find a good coloring (a non-monochromatic induced graph) by just letting a fair coin toss decide on how to color the nodes

2.4. Second Look at the Probabilistic Method

2.4.1. Property S_k

We state that a tournament T has the property S_k if and only if, for every set of k Players, there is one that beats them all./

- Formally, this would mean that given a tournament $T = \langle V, E \rangle$ and subsets K of size k

$$\exists v \in T - K : (v, k) \forall k \in K \tag{5}$$

Claim. *Is it true that for every finite k that there exists a tournament T (on more than k vertices) with the property S_k?*

Proof. In order to prove this, let us consider a random tournament T.

Given this random tournament T, let's determine the probability that a node v in $T - K$ beats all of the nodes $j \in K$. This is a difficult probability to calculate, however, and it is this probability as the complement of its negation (that there isn't a node in $V - K$ that beats all the nodes $j \in K$).

Let us find probability that a fixed node v in $V - K$ beats all the nodes $j \in K$.

- **Remark.** Because T is a tournament, we know that if we're considering a vertex v, it must be connected to all of the nodes within the subset K. Thus, there is a $\frac{1}{2}$ probability with which the edge with endpoints $v, j : j \in K$ is directed $v \to j$.
  - Since there are k nodes in K and that the event of v beating a vertex j is independent, then we just find the product

$$\Pr(v \text{ beats them all}) \to \prod_{1}^{k} \left(\frac{1}{2}\right) \to \left(\frac{1}{2}\right)^{k} \leftrightarrow (2)^{-k} \tag{6}$$

From this, it follows that the probability that v doesn't beat them all is given by

$$\begin{aligned} \Pr(v \text{ does not beat them all}) = & \\ &= (1 - \Pr(v \text{ beats them all})) \\ &= \left(1 - 2^{-k}\right) \end{aligned} \tag{7}$$

Now, we simply just need to find the probability that *any* fixed v doesn't beat them all.

$$\begin{aligned} \Pr(\text{no vertex beats them all}) = & \\ &= (\text{number of possible } v \in V - K) \\ &\times \Pr(v \text{ doesnot beat them all}) \\ &= \prod_{v \in V - K} \Pr(v \text{ does not beat them all}) \\ &= \prod_{v \in V - K} \left(1 - 2^{-k}\right) \\ &= \left(1 - 2^{-k}\right)^{n-k} \end{aligned} \tag{8}$$

Finally, we just need to consider this scenario for all subsets K of size k in V

$$\begin{aligned} \sum_{\substack{K \subset V \\ |K| = k}} \Pr(\text{no vertex beats them all}) = & \\ &= \binom{n}{k} \Pr(\text{no vertex beats them all}) \\ &= \binom{n}{k} \left(1 - 2^{-k}\right)^{n-k} \end{aligned} \tag{9}$$