Math_226 Written Homework 1: Sequences

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March 30, 2023

Problem 1.1. Compute $|a_n - 2|$, simplifying your answer as much as possible.

$$\text{let } a_n = \frac{2n+5}{n+1}$$
$$\Rightarrow |a_n - 2|$$
$$\Rightarrow \left| \frac{2n+5}{n+1} - 2 \right|$$
$$\Rightarrow \left| \frac{2n+5}{n+1} - \frac{2(n+1)}{n+1} \right|$$
$$\Rightarrow \left| \frac{3}{n+1} \right| = |a_n - 2|$$

Problem 1.2. Take $r = 1$ in the definition of convergence. Find the first value of N which satisfies $|a_N - 2| < 1$.

Recall.

$$|a_n - L| < r \text{ such that } n > N$$

$$\text{let r } = 1, \text{ let } |a_N - 2| = \left| \frac{3}{n+1} \right|$$
$$\Rightarrow |a_N - 2| < 1$$
$$\Rightarrow \left| \frac{3}{n+1} \right| < 1$$
$$\Rightarrow -1 < \frac{3}{n+1} < 1$$
$$\Rightarrow -1(n+1) < 3 < 1(n+1)$$
$$\Rightarrow -n - 1 < 3 < n + 1$$
$$\Rightarrow -n - 1 < 3 \text{ and } 3 < n + 1$$

$$\Rightarrow -n < 4 \text{ and } 2 < n$$

$$\Rightarrow n > -4 \text{ and } n > 2$$

Because we know that $N \in \mathbb{N}$, then N cannot be -4. Therefore, we know that $N > 2$.

The first value N that satisfies $|a_n - 2| < 1$ must be 3.

Problem 1.3. Justify the fact that the sequence a_n is decreasing.

Let us consider the derivative of a_n. By considering the derivative, we are able to observe the relationship between the indices n of the sequence and the terms a_n, and, most importantly, whether or not a_n increases or decreases.

$$\text{let } a_n = f(x)$$

$$\text{for all } x \text{ where } x \text{ is a real number}$$

$$f(x) = \frac{2x+5}{x+1}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{2x+5}{x+1}\right)$$

$$\Rightarrow \left(\frac{(x+1)(2x+5)' - (2x+5)(x+1)'}{(x+1)^2}\right)$$

$$\Rightarrow \left(\frac{(x+1)(2) - (2x+5)(1)}{(x+1)^2}\right)$$

$$\Rightarrow \left(\frac{2x+2-2x-5}{(x+1)^2}\right)$$

$$\Rightarrow -\frac{3}{(x+1)^2} = f'(x)$$

Given that the derivative $f'(x) = -\frac{3}{(x+1)^2}$, we can see that the value of $f'(x) < 0$ for all x. Therefore, the range a_n of the function $a_n = f(n)$ decreases as the domain n increases.

Problem 1.4. Justify how all terms in the sequence beyond a_N also satisfy the definition

$$|a_n - 2| < 1$$

Solution.

By solving for the first term a_n that satisfies the $n > 2$, a_3, we observe

$$a_3 = \frac{2(3)+5}{3+1} = \frac{11}{4} < (2+1)$$

$$\Rightarrow (2-1) < \frac{11}{4} < (2+1)$$

Therefore,

$$\Rightarrow \left|\frac{11}{4} - 2\right| < 1$$

$$\Rightarrow |a_3 - 2| < 1$$

Given that $|a_3 - 2| < 1$ and that, from question (3), the derivative $f'(x)$ decreases, all terms a_n subsequent $n > 2$ and $n = 3$ must decrease and also be less than 1.

$$|a_n - 2| < |a_3 - 2| < 1$$

In addition, we must also observe that for all terms a_n in the sequence where $n > 2$, the difference $|a_n - 2|$ will always be greater than 0, since we know that the limit 2 will always be less than the function a_n for all n

$$2 = \frac{2(n+1)}{n+1} < \frac{2n+5}{n+1}$$

We can conclude that all of the terms a_n following $n = 3$ will lie within the bounds (2-1, 2+1), since their distance will always be less than 1 but will never be less than 0.

$$0 < |a_n - 2| < |a_3 - 2| < 1$$

$$\Rightarrow 0 < |a_n - 2| < |a_3 - 2| < 1$$

Problem 1.5.1. Repeat Questions (1-4) for $r = \frac{1}{100}$

$$\text{let } r = \frac{1}{100}$$

$$\text{let } (a_n - 2) = \frac{3}{n+1}$$

$$|a_n - 2| < r$$

$$\Rightarrow \left|\frac{3}{n+1}\right| < \frac{1}{100}$$

$$\Rightarrow -\frac{1}{100} < \frac{3}{n+1} < \frac{1}{100}$$

$$\Rightarrow -n-1 < 300 \text{ and } n+1 > 300$$

$$\Rightarrow n > -301 \text{ and } n > 299$$

Because $n \in \mathbb{N}$, we omit all negative numbers. Therefore

$$N > 299$$

By solving for the first term a_n that satisfies $N > 299$, a_{300}, we observe

$$a_{300} = \frac{2(300)+5}{300+1} = \frac{605}{301} = 2.009967... < (2 + \frac{1}{100})$$

$$\Rightarrow (2 - \frac{1}{100}) < \frac{605}{301} < (2 + \frac{1}{100})$$

Therefore,

$$\Rightarrow \left| \frac{605}{301} - 2 \right| < \frac{1}{100}$$

$$\Rightarrow |a_{300} - 2| < \frac{1}{100}$$

Note that we do not need to calculate the derivative of the function $a_n = f(x)$, since the function itself remains the same.

Given that $|a_{300} - 2| < \frac{1}{100}$ and that, from question (3), the derivative $f'(x)$ decreases, all terms a_n subsequent $n > 299$ and $n = 300$ must decrease and in turn be less than $\frac{1}{100}$.

$$|a_n - 2| < |a_{300} - 2| < \frac{1}{100}$$

In addition, we must also observe that for all terms a_n in the sequence where $n > 299$, the difference $|a_n - 2|$ will always be greater than 0, since we know that the limit 2 will always be less than the function a_n for all n

$$2 = \frac{2(n+1)}{n+1} < \frac{2n+5}{n+1}$$

We can conclude that all of the terms a_n following $n = 300$ will lie within the bounds $(2 - \frac{1}{100}, 2 + \frac{1}{100})$, since their distance will always be less than $\frac{1}{100}$ but will never be less than 0.

$$0 < |a_n - 2| < |a_{300} - 2| < \frac{1}{100}$$

$$\Rightarrow 0 < |a_n - 2| < |a_{300} - 2| < \frac{1}{100}$$

Problem 1.5.2. Repeat Questions (1-4) for $r = \frac{1}{1000}$

$$\text{let } r = \frac{1}{1000}$$

$$\text{let } (a_n - 2) = \frac{3}{n+1}$$

$$|a_n - 2| < r$$

$$\Rightarrow \left|\frac{3}{n+1}\right| < \frac{1}{1000}$$

$$\Rightarrow -\frac{1}{1000} < \frac{3}{n+1} < \frac{1}{1000}$$

$$\Rightarrow -n - 1 < 3000 \text{ and } n + 1 > 3000$$

$$\Rightarrow n > -3001 \text{ and } n > 2999$$

Because $n \in \mathbb{N}$, we omit all negative numbers. Therefore

$$N > 2999$$

By solving for the first term a_n that satisfies the $N > 2999$, a_{3000}, we observe

$$a_{3000} = \frac{2(3000) + 5}{3000 + 1} = \frac{6005}{3001} = 2.000999... < (2 + \frac{1}{1000})$$

$$\Rightarrow (2 - \frac{1}{1000}) < \frac{6005}{3001} < (2 + \frac{1}{1000})$$

Therefore,

$$\Rightarrow \left|\frac{6005}{3001} - 2\right| < \frac{1}{1000}$$

$$\Rightarrow |a_{3000} - 2| < \frac{1}{1000}$$

Again, we do not need to re-calculate the derivative of $a_n = f(x)$, since the function does not change.

Given that $|a_{3000} - 2| < \frac{1}{1000}$ and that, from question (3), the derivative $f'(x)$ decreases, all terms a_n subsequent $n > 2999$ and $n = 3000$ must decrease and in turn be less than $\frac{1}{1000}$.

$$|a_n - 2| < |a_{3000} - 2| < \frac{1}{1000}$$

In addition, we must also observe that for all terms a_n in the sequence where $n > 2999$, the difference $|a_n - 2|$ will always be greater than 0, since we know that the limit 2 will always be less than the function a_n for all n

$$2 = \frac{2(n+1)}{n+1} < \frac{2n+5}{n+1}$$

We can conclude that all of the terms a_n following $n = 3000$ will lie within the bounds $(2 - \frac{1}{1000}, 2 + \frac{1}{1000})$, since their distance will always be less than $\frac{1}{1000}$ but will never be less than 0.

$$0 < |a_n - 2| < |a_{3000} - 2| < \frac{1}{1000}$$

$$\Rightarrow 0 < |a_n - 2| < |a_{3000} - 2| < \frac{1}{1000}$$