Meaning

Math_226 (Spring 2023) Notes

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Contents

Chapter 1

10.1 (Part One): Sequences (Part One) (03/28/23)

1.1 Reminders

- The first MyLab homework is due on **Wednesday, March 28, 2023**.
  - Series (Part 0)
- The first **written homework** is going to be due on **Friday, March 31, 2023**.

1.2 Objectives

- We want to be able to derive the concept of a series and a sequence.
- We want to be able to understand where the idea of a series and sequence come from, especially from seemingly-ordinary objects.
- We want to explore the idea of a limit in relation to a sequence.
- We want to be able to discretely express a sequence.

1.3 Motivation

In former calculus classes, we have observed the idea of limits, derivatives, and integrals at face value. We know how to evaluate these different calculations, but what exactly do they mean in the context of math? How can we better observe what exactly happens in these calculations, and understand them outside the context of visualizing graphs or projectile motion.

1.4 A third...

Chapter 2

10.1 (Part Two): Sequences (Part Two) (03/29/23)

2.1 Reminders

- MyLab Math Assignment 1 - **Sequences (Part 0)** is due **tonight, March 29, 2023**.
- Written Homework 1 is due **Friday, March 31, 2023** at the **beginning of class**.
- Friday's lesson is going to be over **sections 4.6 and 10.1** and we will be talking about **Newton's Method.**

2.1.1 Course Philosophy

Remember that the entire point of this class is to develop an intuition and a larger understanding and appreciation of calculus.

"Calculus is just algebra with a tiny drop of limits"

2.2 Motivation

In the last class, we got our first taste of sequences by exploring the idea of a third and eventually relating it to the **geometric sequence**, which results in the formula of

$$a_n = \frac{1}{1-x}$$

In this class, we were essentially taking our "preview" of sequences, actually defining different aspects of our sequence, doing operations on sequences, and finally, exploring one of the most important ideas of sequences, which are limit convergence divergence, which led us to the famous $\varepsilon - N$ proof, which is also known as the **precise definition of convergence.**

2.3 Sequences

Definition 2.3.1: Sequences

Functions with a domain of natural numbers and a co-domain of real numbers.

$$f : \mathbb{N} \to \mathbb{R}$$

Although our intuition would tell us that a sequence is just a list or a collection of numbers, a sequence is more precisely just a function in which we input an **index** (a natural number) and we output a **term** (a real number). We, of course, then, collect these outputs, and this is what we generally see.

Example.

$$\{a_n\}$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{a_n\}_{n=0}^{\infty}$$
$$1, 2, 3, 4, \ldots$$
$$1.1, 2.2, 3.3, 4.4, \ldots$$

2.4 Convergence

Definition 2.4.1: Sequence Convergence

Informal Definition.
A sequence $\{a_n\}$ converges to a limit L if the terms get arbitrarily close to L as n gets sufficiently large, which is also known as

$$\lim_{n\to\infty} a_n = L$$

Formal Definition.
A sequence $\{a_n\}$ converges to a limit L if, for every $\varepsilon > 0$, where ε is the distance from the range to the limit L, there exists such a number that

$$|a_n - L| < \varepsilon \text{ for } n \geqslant N$$

The preceding expression is also known as the $\varepsilon - N$ proof.

2.5 Divergence

Definition 2.5.1: Sequence Divergence

Informal Definition.
A sequence diverges when it doesn't converge. If the sequence $\{a_n\}$ does not get arbitrarily close to limit L as n gets sufficiently large. This is also known as when the limit L **does not exist.**

$$\lim_{n\to\infty} a_n \text{ does not exist}$$

Formal Definition.
A sequence a_n diverges to (positive) infinity if, for every $M > 0$, there exists an N such that

$$a_n > M \text{ whenever } n > M$$

Additionally, a sequence a_n diverges to negative infinity, if, for every $M < 0$, there exists an N such that

$$a_n < M \text{ whenever } n > M$$

2.6 $\varepsilon - N$ Proof

In this proof, we are proving the existence of a limit when given a sequence a_n

2.7 Properties of Sequence Limits

Chapter 3

10.1: Sequences (Part 3) (03/31/23)

3.1 Summary of Lesson

In this lesson, we further develop the idea of convergence and divergence by **expanding it to non-elementary sequences**. We apply several new theorems as a result of this, such as the Sandwich Theorem. The homework, as a result, is all about convergence and divergence, testing on how well we are able to determine convergence and divergence given sequences including factorials, logarithms, and exponential functions.

3.2 Reminders

- The **third MyLab Math assignment** (Sequences (Part 2)) is due on **Tuesday, April 3rd**.
- The second written assignment is due on **Friday, April 7, 2023**.

3.3 Motivation

In the previous lesson, we learned about the **precise definition of convergence and divergence**. We learned about what exactly convergence and divergence means, and we were able to apply these concepts to various elementary sequences. However, what if we were given a sequence like

$$\{a_n\} = \frac{\cos(n)}{n}$$

Additionally, what if we were given a sequence like

$$\{a_n\} = \left(1 + \frac{x}{n}\right)^n$$

These sequences are undoubtedly more complex than our former examples and are not nearly as intuitive whenever it comes to actually solving them.

Therefore, we need to learn a few more **theorems** as well as **techniques** in order to determine convergence and divergence in more complex sequence functions.

Theorem 3.3.1 S

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ndwich Theorem Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of real numbers. If $a_n \leqslant b_n \leqslant c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to L} b_n = L$ also.

$$\text{let } \{a_n\}, \{b_n\}, \{c_n\} \text{ be sequences}$$
$$\text{if } a_n \leqslant b_n \leqslant c_n \text{ and}$$
$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} c_n = L, \text{ then}$$
$$\lim_{n\to\infty} b_n = L$$

Theorem 3.3.2 T

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e Continuous Function Theorem For Sequences (Theorem 3) Let $\{a_n\}$ be a sequence of real numbers $\mathbb{R}$. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n then $f(a_n) \to f(L)$.

Informal Definition.
Essentially, as long as we know that if a sequence a_n exists and approaches some limit L, we are able to say that if some function f is also continuous and the value $f(L)$ exists and that the function f is defined for all terms in the sequence a_n, then we know that the function with a domain of the terms of the sequence a_n, which we represent as $f(a_n)$ will approach the value of the function f at the limit L, which is also represented as the function $f(L)$.

3.3.1 Importance of the Continuous Function Theorem for Sequences

The implication of the continous function theorem for sequences, of course, is that we can derive smaller, more trivial functions from a larger function. For example, if we were trying to solve for

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}}$$

by using the **continuous function theorem for sequences**, we are able to separate this problem into a function, in which the domain is just a sequence a_n (or another function).

$$\text{let } f(x) = \sqrt{x}, \text{ let } x(n) = \frac{2n}{n+1}$$

Now, we are able to evaluate the limit of the inner function.

$$\lim_{n\to\infty} \frac{2n}{n+1} \Rightarrow \frac{2n/n}{n/n + 1/n} \Rightarrow \frac{2}{1} \Rightarrow 2$$

Knowing that the domain of the function $f(x)$ will always approach 2, we are able to evaluate the outer limit.

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} \Rightarrow \lim_{n\to\infty} \sqrt{2} = \sqrt{2}$$

Theorem 3.3.3 T

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eorem 4. (L'Hopital's Rule) Suppose that $f(x)$ is a function defined for all $x \geqslant n_0$ and that $\{a_n\}$ is a sequence of real numbers $\mathbb{R}$ such that $a_n = f(n)$ for $n \geqslant n_0$. Then,

$$\lim_{n\to\infty} a_n = L \text{ wherever } \lim_{n\to\infty} f(x) = L$$

Warning. We cannot say the same about the converse of this theorem. That is, we cannot say that as long as a sequence approaches infinity, that a function f with a domain of the terms of the sequence a_n will converge at a a limit L. For example, imagine that if we had some high-degree polynomial expression and we had a sequence that converged at 0. The function itself might have several roots, as it will intersect the x-axis numerous times at zero. Ultimately however, the polynomial might not approach 0, but value of the function at the terms a_n might suggest it does.

Theorem 3.3.4 T

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eorem 5. (Commonly Occuring Limits) The following six sequences converge to the following limits listed below:

1. $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1 \iff \lim_{n\to\infty} n^{1/n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1$ for all $(x > 0)$
4. $\lim_{n\to\infty} x^n = 0$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for all x
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

Chapter 4

4.6, 10.1: Finishing Sequences + Newton's Method (04/03/2023)

4.1 Reminders

- The **third MyLab Math assignment** (Sequences (Part 2)) is due on **Tuesday, April 3rd**.
- The second written assignment is due on **Monday, April 10th, 2023**.

4.2 Recall

In the previous section, remember how we essentially continued to enrich our understanding of sequences and limits. More specifically, we learned a few more theorems about sequence convergence and divergence.

First, let's start by actually talking about some patterns we see with the behavior of sequences. Recall that when we talk about sequences, we are essentially just talking about a function, in which the domain are the indices of the sequence (which indicate the position of the elements within the sequence) of which are natural numbers, that have a range of real numbers.

Convergence and divergence in a sequence refers to whether or not the sequence approaches some tangible, finite number as the domain of the sequence (the indices) approach infinity. This, of course, is the **informal definition of sequence convergence**.

The **formal definition of sequence convergence** states that, in order for a sequence to converge to some value L, there must exist some value ε after some point N in the sequence, in which all terms a_n of the sequence for $n > N$ satisfy

$$|a_n - L| < \varepsilon \text{ where } n > N$$

This basically states that, after some point in the sequence (denoted by the index N), all terms of the sequence a_n must fall within the range $(L - \varepsilon, L + \varepsilon)$, where ε, of course, represents any arbitrarily small number.

Likewise, we can state that a sequence **diverges** when this behavior doesn't occur, which makes sense, as if there exists no point in the sequence N where all consecutive terms all exist within some infinitesimially small range, then they simply cannot be approaching any number in the first place.

In addition to this basic idea of convergence, we also learn about some restrictions as well as some implications that arise from convergence.

For example, if we simply just want to determine whether or not a sequence converges, all we have to do is just solve for the limit of that sequence. We just treat the sequence a_n as some function $f(n)$ for all a_n. This, of course has a ton of different implications.

- The **Sandwich Theorem** states that, if we have three sequences $\{a_n\}\{b_n\}\{c_n\}$, and that the sequence a_n approaches L, and that the sequence c_n approaches L, if the sequence b_n is between a_n and c_n (that is, $a_n \leqslant b_n \leqslant c_n$), then $\lim_{n\to\infty} b_n = L$.

$$\text{let } \{a_n\}, \{b_n\}, \{c_n\}$$

$$\text{if } \lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n = L$$

$$\lim_{n\to\infty} a_n \leqslant \lim_{n\to\infty} b_n \leqslant \lim_{n\to\infty} c_n$$
$$\lim_{n\to\infty} b_n = L$$

- The **Continuous Function Theorem for Sequences** states that, if we have a function that is continuous for all values of $x \geqslant n_0$ (such that n_0 is the initial index of a sequence) and that there is a corresponding sequence a_n such that f is defined and continuous for all values a_n and is continuous at L, then we are able to say that the function of the domain a_n approaches the function value of the limit L.

  Essentially, from this informs us that if there is a function f that is defined for all values of a sequence a_n, as long as the a_n converges at a limit L, then function f of which its domain is an expression of na_n, then the function $f(a_n)$ must converge at $f(L)$. This allows us to state that if we are trying to find evaluate the limit of a function, we are able to dissect the function into an outer function $f(x)$ and an inner expression $x(n)$. The procedure from here is that we are able to find the limit of this inner function $x(n)$, substitute it as the domain of the outer function, then evaluate the final limit of the outer function.

- Additionally, like all other limits, we are able to evaluate the value at which a sequence converges to by using **L'Hopital's Rule**. The technical definition of L'Hopital's Rule states that if there exists a function $f(n) = a_n$, then the limit of the function $\lim_{n\to\infty} f(n) = L$ always implies that $\lim_{n\to\infty} a_n = L$. If we are able to evaluate the limit of the sequence as a limit of a function, then we are able to assert that the limit of the sequence is equal to the limit of that function. On the converse, however, we are unable to assert that if a function converges at a limit, then the sequence of that function will also converge at that limit. This all really just comes down to the idea that sequences are not functions.

- Finally, after this, there are just some important, well-known and already proven sequence convergences that I will be tested on. It is important to understand these for later, since they are pretty essential to solving many future sequence problems.

4.3 Motivation

For this lecture, we are literally just finishing up sequences and getting ready to move onto the next topic, series. We spent this class just learning about one final piece of the puzzle in series, which is **monotonicity** and the **monotonicity convergence theorem**.

4.4 Types of Sequences

Recall that through this section, we have thought about sequences in two different ways: **explicitly** and **recursively**.

4.4.1 Explicit Sequences

An explicit sequence is a sequence in which the value of the term of the sequence a_n is determined only by the index n. We don't need information about other terms in the sequence, as we are able to compute the value of any term of the sequence *solely* based on its position.

4.4.2 Examples of Explicit Sequences

$$a_n = n + 1,\ a_n = \frac{n}{2}$$

4.4.3 Recursive Sequences

A **recursive sequence** is a sequence in which the value of a term in the sequence is computed using the previous term of the sequence. It might be helpful to think of this as recursion in computer science, as we are literally just taking a previous output as input in this function. Generally, whenever we denote a recursive function, we will say that the term FOLLOWING the current term is equal to some expression of the current term.

4.4.4 Examples of Recursive Sequences

$$a_{n+1} = a_n + 1,\ a_{n+1} = \frac{a_n}{2}$$

4.5 Defining Divergence

Definition 4.5.1: Sequence Divergence (Towards Infinity)

A sequence $\{a_n\}$ diverges to positive infinity if for all M, there is an N such that $a_n > M$ for all $n > N$

Definition 4.1: S

quence Divergence (Towards Negative Infinity)

A sequence $\{a_n\}$ diverges to negative infinity if for all M, there is an N such that $a_n < N$ for all $n > N$.

When we read these definitions, we can imagine that the M is just some upper and lower bound or threshold. If, after some point in the sequence a_n, all subsequent terms a_n are all above or below a particular threshold, it makes sense that they would just approach infinity. This idea of divergence, however, all depends on the idea of **monotonicity**.

4.6 Monotonicity

Monotonicity defines a sequences general shape and behavior. A series is monotonous as long all of its values all pertain to some behavior.

4.6.1 Nondecreasing Monotonic Sequences

A **nondecreasing monotonic sequence** is a sequence in which, as the name suggests, the terms do not decrease. More precisely, we can say that

$$a_n \leqslant a_{n+1}$$

4.6.2 Nonincreasing Monotonic Sequences

A **nonincreasing monotonic sequence** is a sequence in which the terms do not increase. More precisely,

$$a_n \leqslant a_{n+1}$$

4.6.3 Monotonic Convergence

We can apply the idea of monotonicity, that is, the idea that a function bears only a single behavior into limits. In fact, there is actually an entire theorem dedicated to this idea.

Theorem 4.6.1 M
o

notonic Convergence Theorem If a sequence

$$\{a_n\}$$

is both bounded and monotonic, then it converges.
Informal Definition.
For a sequence is a nonincreasing, that is $a_n \geqslant a_{n+1}$, if all terms after some point N are less than or equal to the greatest possible lower bound L, then we can say that the bounded, monotonicaly nonincreasing sequence a_n converges at that greatest possible lower bound L.
Likewise, for a sequence that is nondecreasing, or $a_n \leqslant a_{n+1}$, if all terms after some point N are less than or equal to the least possible upper bound L, then we can say that the bounded monotonically nondecreasing sequence a_n converges at L.

4.7 Newton's Method

TODO

4.8 Introduction to Series

A series is just an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

Imagine that we are just taking a sequence a_n and then we are just trying to find the sum of the terms a_n for all values of n.
This, howver, is particularly hard to compute, since after all ,if we were trying to find the sum of an infintie amount of termsl ike in the following series

$$\sum_{n=1}^{\infty} a_n$$

would be impossible. In order to mediate this, we have a few tricks in which we can actaully break up series in order to the actual sum of the series.

4.8.1 Sequence of Partial Sums

In the previous section, we just introduced the idea of a sequence. A sequence, of course its composed of its terms a_n as well as its indices n. For sake of argument, however, let us consider having a sequence in which each term is the sum of the current term and all of the terms preceding it. For reference

$$\text{let } \{a_n\}$$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Notice that the sequence of partial sums for n terms looks **suspiciously** simialr to the idea of infinite series...because it they're the same thing! Whenever we are thinking of a series, we can just contextualize it as just being a sequence of partial sums for the nth term.
This recontextualization of the series as just a sequence of partial sums to the nth term (as opposed to the first term of the second term), is very **powerful**, since it will grant us a beter understanding of how exactly the the series works, how to compute operations with it, as well as working with series convergence and divergence.

Definition 4.2: S

quence of Partial Sums

Given a sequence of numbers $\{a_n\}$, an expresssion of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an **infinite series**. The number a_n is the n**th term** of the series. The sequence $\{S_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series, the number s_n begin the n**th partial sum.** If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L.

Definition 4.3: S

quence of Partial Sums (cont'd.)

Intuition.
Previously, we talked about this idea of contextualizing a series as the sum of a sequence up until a certain point. For example,

$$\sum_{n=1}^{2} a_n = a_1 + a_2$$

$$\sum_{n=1}^{3} a_n = a_1 + a_2 + a_3$$

This is all pretty cool, but lets apply this idea of finding the series of every single term in a sequence a_n and then putting these series (which, of course, really only refers to the sum of the series) **within a sequence**.

$$\{S_n\} = S_1 + S_2 + S_3 + \cdots + S_n$$

$$\{S_n\} = \sum_{n=1}^{1} a_n + \sum_{n=1}^{2} a_n + \sum_{n=1}^{3} a_n + \cdots + \sum_{n=1}^{n} a_n$$

Note, the sequence of partial sums is **not equivalent** to the series itself. (After all, the sequence of partial sums is literally just a sequence and additionally, the series we are analyzing is just a term within the sequence). The sequence of partial sums is just a collection of all possible series up until our given series n. This is useful, because we can think of the value or sum of the series as being equal to the limit of the sequence of its partial sums. This idea of thinking of the value of a series as the limit of sequence of partial sums at some point n is extremely powerful and allows us to work within the framework of series convergence and divergence.

4.8.2 Methodology for solving for the sum of a series

In order to solve for the sum of a series, we again have to consider numerous factors:

- The series $\sum_{n=1}^{\infty} a_n$ can also be represented as the point of convergence or limit for the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{n=1}^{1} a_n + \sum_{n=1}^{2} a_n + \cdot + \sum_{n=1}^{n} a_n$$

- We know how to compute the limit L or point of convergence of a sequence, as well as how to determine whether not a sequence actually converges or not.

From this, we can consider the following steps:

1. Let the terms of the series be a sequence.
2. Evaluate the partial sums for the first few terms.
3. Determine the "pattern" between these terms.
   - The "pattern" here is eventually going to transform into the "function" of a sequence. We are going to determine a rule between the partial sums.
4. Let this pattern be a function/algorithm for computing future terms.
5. Let this pattern be the function of a sequence of partial sums $\{S_n\}$
6. Determine whether or not the sequence of partial sums $\{S_n\}$ converges or diverges. If the sequence converges, then determine at what point it converges to. The point at which the sequence of partial sums converges to, or the result of the series at some point n, is going to be known as the **sum** of that series.

4.9 Example of Series Sums

Example (1). *Determine the sum of the series*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Step 1: Let the terms of the series be a sequence.

$$\text{let } \{a_n\} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$$

Step 2: Evaluate the partial sums for the first few terms of the sequence (which we made through the terms of the series).

$$\text{let } \{S_n\} \text{ be the sequence of partial sums for } a_n$$

$$\{S_n\} = S_1, S_2, S_3, \cdots, S_n$$

$$S_1 = 1 = 1$$

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$$

Example ((1) cont'd).

Step 3: Let the pattern in the sequence of partial sums be the function of the sequence of partial sums.

$$\text{let } S_n = \frac{2^n - 1}{2^{n-1}}$$

Step 4: Determine whether or not the sequence of partial sums converges or diverges. Find the limit if the sequence converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}}$$
$$\Rightarrow \lim_{n\to\infty} \frac{2^n}{2^{n-1}} - \lim_{n\to\infty} \frac{1}{2^{n-1}}$$
$$\Rightarrow 2 - 0$$
$$\lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = 2$$

Step 5: Let the limit or point of convergence of the sequence of partial sums be the sum of the series in question.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = 2$$

The sum of the series given by $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is 2.

4.9.1 Series Notation

Whenever we are trying to denote a sequence, we observe the following kinds of notation:

$$\text{let } a_n \text{ be some expression for } n$$
$$\sum_{n=1}^{\infty} a_n, \;\; \sum a_n$$

The left notation $\sum_{n=1}^{\infty} a_n$, of course, is more specific, letting us know what index we should start the series at as well as giving us a terminal point, which although strange, is just infinity in the case of the first example.

In the second case $\sum a_n$, we are just demonstarting that a_n is just a series, similar to how we just denote sequences with just curly braces $\{a_n\}$.

4.9.2 Introduction to Series Convergence and Series Divergence

Again, just to reiterate what we have learned in the past section, whenever we are rying to determine whether or not a series converges or diverges, we are essentially trying ot determine whether the sum or the evaluation of the series is going to be some finite number. We can always try to imagine this kind of like a Riemann sum beneath a curve

- Recall that whenever we were trying to solve a Riemann sum, we were always trying to estimate the area underneath a curve by using rectangles. In that same kind of sense, we are trying to see if the sum generated by the series, which is going to be one of these "rectangels" beneath the graph of a function of the series, is going to be finite.

So, in order to determine whether or not a series is convergent or divergent, we first have to transform the series into a sequence of partial sums, where the domain of course is going to be natural numbers that are determiend by the indicies of the sequence and the range is going to be determined by the sum of the sequence of hte function (which is distince from the sequence of partial sums) at that point.
Therefore, when the sequence of partial sums converges, this can be thought of as the series if the series just appproaches infinity.
Whenever we are trying to determien whether or not the series diverges, we just want to see if the sequence of partial sums that we obtain through the series actually converges or not. Whenever the sequence of partial sums does not converge, then it must diverge.

Chapter 5

10.2 (Part One): Infinite Series (Part One) (4/05/23)

5.1 Reminders

- The **fifth MyLab Math assignment: Infinite Series**, will be due on **Sunday, April 9th, 2023**.
- The second written assignment is due on **Monday, April 10th, 2023**.

5.2 Motivation

In the last section, we obtained a taste of what series were and how we are able to interpret them. Most prominently, we can imagine that a series can be considered as a term in a sequence, where all other terms in the sequence are just series up until a particular point. This concept is known as the **sequence of partial sums**. Whenever we are evaluating the sequence of partial sums, it is important that we understand this distinction.

Recall that we denote sequences with the following notation:

$$\sum_{n}^{k} a_n \text{ for the specific case}$$

$$\sum a_n \text{ for the general case}$$
$$a_1 + a_2 + a_3 + a_4 + \cdots$$

This is how we represent series as partial sums.

$$\text{let } \{S_n\} \text{ be a sequence of partial sums}$$

$$\text{let } \{s_n\} \text{ be a function denoting the partial sum of } n$$

$$\{S_n\} = S_1, S_2, S_3, S_4, \cdots$$

$$\{S_n\} = \sum_{n=1}^{1} s_n,\ \sum_{n=1}^{2} s_n,\ \sum_{n=1}^{3} s_n,\ \sum_{n=1}^{4} s_n,\ \cdots,\ \sum_{n=1}^{n} s_n$$

Whenever we are trying to determine **series convergence and divergence**, we have to be able to contextualize or infinite series as the convergence point of a **sequence of partial sums**. The convergence point of the infinite series, therefore must be the convergence point of the sequence of partial sums. This, of course, does present us with many different things to consider whenever we are evaluating for partial sums. We have seen how we can evaluate a series whenever it is just given to us in the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

We basically think of these terms as the sequence a_n, then we just create separate sequence S_n that represents the partial sums of the terms of the sequence a_n. Then, we have to represent this sequence S_n as some function or algorithm. After we have derived an algorithm for the sequence of partial sums S_n, we can determine the convergence of the sequence of partial sums S_n by evaluating

$$\lim_{n \to \infty} S_n$$

By finding the point of convergence for the sequence of partial sums, we have determined both whether or not the series converges, but also where the series converges to. By evaluating a series, we have found its **sum**.

5.2.1 What's on the Menu

Now that we have learned the basic idea of what series are, we need to understand that not all series are this nice. Unfortuantely, series, much like sequences, can become much more complicated and complex in their terms, and for this reason, we also need to learn some new techniques, methodologies, and patterns for which to evaluate these series (that is, whenever we are evaluating for a series, we are really just determining their point of convergence / limit).

5.3 Commonly Seen Series

5.3.1 Geometric Series

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Definition 5.1: G

ometric Series

Geometric series are series of the following form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as

$$\sum_{n=0}^{\infty} ar^n$$

Notation
The variable r is known as the **ratio** of the geometric series and a is known as the initial term.

- Of course, the reason why we call a the initial term of the geometric sequence is because whenever n is equal to the initial index of the series, 1

$$ar^{n-1} \Rightarrow ar^{1-1} \Rightarrow a$$

r can be either a positive or negative number, as observed in the two following series.

$$\text{let} \sum a_n, \ \text{let} \sum b_n$$

$$\sum a_n = \sum_{n=1}^{\infty} \frac{1}{2}(1^n)$$

$$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2}((-1)^n)$$

5.3.2 Geometric Series Convergence and Divergence

Recall that whenever we are trying to determine whether or not a series converges or diverges, we are trying to contextualize the series as a term within a **sequence of partial sums**.

$$\text{let } \{S_n\} \text{ be a sequence of partial sums}$$

$$\text{let } s_n \text{ be a function denoting the partial sum at some index } n$$

$$\{S_n\} = s_1, s_2, s_3, s_4, \cdots, s_n$$

$$\{S_n\} = \sum_{n=1}^{1} s_n, \sum_{n=1}^{2} s_n, \sum_{n=1}^{3} s_n, \sum_{n=1}^{4} s_n, \cdots, \sum_{n=1}^{n} s_n$$

From this, we know that the limit of the sequence of partial sums $\sum_{n=1}^{\infty} s_n$ is equal to the sum of a series. In the most general case of a geometric series, recall that we obtain the following form:

$$\text{let} \sum S_n \text{ be some geometric series (that is, some fucntion)}$$

$$\text{let the function of } \sum S_n \text{ be } ar^{n-1}$$

$$\sum S_n = S_1 + S_2 + S_3 + S_4 + \cdots + S_n \ \textbf{Write the } \boldsymbol{n}\textbf{th partial sum}$$

$$\Rightarrow \sum S_n = ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + \cdots a^{n-1}$$

$$\Rightarrow \sum S_n = ar^0 + ar^1 + ar^2 + ar^3 + \cdots a^{n-1}$$

$$\Rightarrow \sum S_n = a + ar + ar^2 + ar^3 + \cdots a^{n-1}$$

$$\Rightarrow r \sum S_n = r(ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + \cdots ar^{n-1}) \ \textbf{Multiply both sides by } \boldsymbol{r}$$

$$\Rightarrow S_n - r \sum S_n = (a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n)$$

Here, we now subtracting $r \sum S_n$ from S_n. We just introduce S_n.

$$S_n - r \sum S_n = a - ar^n$$

Notice all terms except for the initial term a and the terminating term a^n are left. Let's factor both sides now.

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \text{for all } r \text{ where } r \neq 1$$

Now that we have a general function to solve for the sum of the geometric series $\sum S_n$, let us try to determine what values of r that the geometric sequences converges and diverges.

$$\text{let } \sum S_n \text{ be geomtric series}$$

$$S_n = ar^{n+1}$$

$$\Rightarrow S_n = S_1 + S_2 + S_3 + \cdots + S_n$$

$$\Rightarrow S_n = ar^{1-1} + ar^{2-1} + ar^{3-1} + \cdots + ar^{n-1}$$

$$\Rightarrow S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$\Rightarrow r(S_n) = r(a + ar + ar^2 + \cdots + ar^{n-1})$$

$$\Rightarrow r(S_n) = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$\Rightarrow S_n - r(S_n) = S_n - (ar + ar^2 + ar^3 + \cdots + ar^n)$$

$$\Rightarrow (a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$

$$\Rightarrow S_n - r(S_n) = a - ar^n$$

$$\Rightarrow S_n(1 - r) = a(1 - r^n)$$

$$\Rightarrow S_n = \frac{a(1 - r^n)}{(1 - r)} \text{ for all } r \neq 1$$

This formula $S_n = \frac{a(1-r^n)}{(1-r)}$ represents the sum or value of any geometric series. However, how can we use this formula to determine the convergence and divergence of a geometric series?

5.3.3 Convergence and Divergence of the Geiometric Series

Let us observe the behavior of the sum of a geometric series.

Recall.

The sum of a geometric series is equal to the following equation:

$$\sum S_n = \frac{a(1-r^n)}{(1-r)}$$

Definition.

$$\text{let } \sum S_n \text{ be a geometric series}$$

$$S_n = \sum_{n=1}^{\infty} ar^{n-1}$$

In order to determine the convergence of a geometric series, we must determine the limit of the sum of the geometric series.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)}$$

Let us look at the dominate term of the function, r_n. Observe that when the absolute value of r is greater than one $|r| > 1$, the term r^n will approach infinity. However, whenever the absolute value of r is less than 1 $|r| < 1$, r^n will just approach 0, since all values less than one will only get smaller when exponentiated.

Therefore, we know that for the geometric series $\sum S_n$, its value or sum diverges when the ratio r is greater than one.

$$S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \infty \text{ when } |r| > 1$$

On the other hand, whenever the value of the ratio r is less than one or $|r| < 1$, we know that the ratio itself is only going to get smaller and smaller, eventually approaching 0, meaning that the value of the infinite series will converge.

$$S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} \text{ for } |r| < 1$$

$$\Rightarrow S_n = \lim_{n\to\infty} \frac{a(1-0)}{(1-r)} = \frac{a}{(1-r)}$$

5.3.4 Geometric Series Examples

Chapter 6

10.2 (Part Two): Infinite Series (Part Two) (04/07/23)

6.1 Reminders

- The **fifth MyLab Math assignment: Infinite Series**, will be due on **Sunday, April 9th, 2023**.
- The second written assignment is due on Monday, April 10th, 2023.

6.2 Summary

In this lecture, we essentially expanded on what we learned about series in the previous lecture. Recall how we learned about geometric series, their formation, as well as how we determine whether or not they converge or diverge. Most importantly, we also learned how to manipulate the values of a series in order to determine an equation for finding the sum of a geometric series.

All of this culminates into today's lesson, which is all about determinign the convergence of other types of series.

6.3 Recall

Over our introduction and talks about series, we have gone over different methods and algorithms for determining the value of a sequence as well as determining convergence and divergence.

6.3.1 Convergence and Divergence in Series

Remember, convergence and divergence in regards to series refers to the sum of the series as the terms of the series approach infinity. This can be represented through the following notation:

$$\sum_{n=1}^{\infty} S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

Given this general form of a how a series works, we can actually recontextualize the terms of the series as a sequence

$$\{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_n$$

This is important because we can actually create a new separate sequence $\{S_n\}$, where each term is a series up until that term in the sequence

$$\{S_n\} = S_1, S_2, S_3, S_4, \ldots, S_n$$

$$\{S_n\} = (a_1), (a_1 + a_2), (a_1 + a_2 + a_3), \ldots, (a_1 + a_2 + a_3 + \cdots + a_n)$$

This final term $S_n = (a_1 + a_2 + a_3 + \cdots + a_n)$ is equivalent to the series $\sum S_n$. The value of a series is equivalent to the nth term of a sequence of partial sums. In order to evaluate the value and the series convergence, then, is to just determine the convergence of the sequence of partial sums. The point at which the sequence of partial sums converges to is equivalent to the sum of the series.

Now, this pattern of course works for trivial and elementary series, since all it requires is the ability to find the pattern between the terms in the sequence of partial sums. We can just apply sequence convergence methods and tricks in order to compute this. But, what happens when we want to determine if a more complex sequence converges or diverges? What if, for example, we had a geometric sequence?

A geometric sequence is a sequence that occurs in the following form

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

A geometric series, thankfully, has a general equation that allows us to compute its sum and value.

$$\text{let } S_n \text{ be the value of a geometric sequence}$$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

Using this equation, we can deduce that all values of the ratio r that are greater than 1, or $|r| > 1$ are going to **diverge**, while all values of the ratio r that are less or equal to 1, or $|r| \leqslant 1$ are going to **converge** at the following value:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \frac{a}{(1-r)} \text{ when } |r| \leqslant 1$$

Otherwise, all other geometric sequences will **diverge.**

6.4 Motivation

Given now that we know how a general algorithm for computing the value and convergence/divergence of a series as well as how to find series convergence and divergence given a particular type of series, let us learn some new techniques in order to compute the value of a series as well as series convergence and series divergence.

6.5 Partial Fraction Decomposition for Series

TODO

6.6 Telescoping Series

Definition 6.1: T

lescoping Series

A series who general term t_n is of the general form (the recursive formula)

$$t_n = a_n - a_{n+1}$$

which form the sequence $\{a_n\}$ in which we find the difference of each consecutive term.
As a result, we will always just have **two terms left in the telescoping series**, the first term as well as the **final term** $n + 1$.
Note, that the cancellation technique we use to negate consecutive terms within the sequence is known as the **method of differences**.

6.6.1 Telescoping Series Examples

Example. *Find the sum of the following series:*

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}\right)$$

1. Like, as we solve with the elementary series cases, let us write out the terms of the series as a sequence and observe the pattern. If we don't see a pattern there, then we can observe a pattern between the partial sums of the series.

   let $\{a_n\}$ be a sequence for all terms of the sequence

$$\{a_n\} = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\Rightarrow \left(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{7}}\right),$$

$$\left(\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n+3}}\right), \cdots, \left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}\right)$$

2. Now, we are able to just find the limit of this sequence, since we can clearly see that the terms between the first and the last term are all going to cancel out through the method of difference.

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{\infty}-\frac{1}{\infty}\right)$$

$$\Rightarrow 0$$

3. Therefore, we can see that the nth term of the sequence is just going to evaluate to 0, so we take into account the value of the first term, since all the intermediate terms all cancel each other out.

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + 0 + 0 + 0 + \cdots + 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \frac{1}{2}$$

6.7 nth-term Divergence Test

How can we actually determine when a series converges

Theorem 6.7.1 T

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eorem 7

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } a_n \text{ approaches } 0.$$

Warning.
We cannot say the same for the **converse** of this. Just because i

Theorem 6.7.2 T

h

e nth-term divergence test The series

$$\sum_{n=1}^{\infty} a_n$$

diverges if the limit of the function of the series function a_n fails to exist or is not equal to 0.

$$\sum_{n=1}^{\infty} \text{ converges when } \lim_{n\to\infty} a_n \neq 0 \text{ or does not exist.}$$

6.8 Combining Series

Now that we have learned some general ideas about series, let us talk about some operations we can do with series.

Theorem 6.8.1 [

8

] Combining Series Theorem (Theorem 8)

$$\text{Let } \sum a_n = A \text{ and } \sum b_n = B \text{ be convergent series}$$

- Sum Rule:
$$\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$$
- Difference Rule:
$$\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$$
- Constant Multiple Rule:
$$\sum k a_n = k \sum a_n = kA \text{ for all values of } k$$

Proof. Series Sum Rule

$$\text{let } \sum a_n = A \text{ and } \sum b_n = B$$
$$A = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$
$$B = b_1 + b_2 + b_3 + b_4 + \cdots + b_n$$
$$\text{let } S_n \text{ be equal to } \sum (a_n + b_n)$$
$$S_n = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$
$$\Rightarrow S_n = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots (a_n + b_n)$$
$$\Rightarrow S_n = A_n + B_n$$

Proof. Series Difference Rule

$$\text{let } \sum a_n = A \text{ and } \sum b_n = B$$

$$A = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$B = b_1 + b_2 + b_3 + b_4 + \cdots + b_n$$

$$\text{let } S_n \text{ be equal to } \sum (a_n - b_n)$$

$$S_n = \sum a_n - \sum b_n$$

$$S_n = (a_1 + a_2 + a_3 + \cdots + a_n) - (b_1 + b_2 + b_3 + \cdots + b_n)$$

$$S_n = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots + (a_n - b_n)$$

$$S_n = A - B$$

Proof. Series Constant Multiple Rule

$$\text{let } \sum a_n = A \text{ and } k \text{ be some constant}$$

$$\sum a_n = A = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\text{let } S_n \text{ be equal to } \sum (k \cdot a_n)$$

$$S_n = k \cdot a_1 + k \cdot a_2 + k \cdot a_3 + \cdots + k \cdot a_n$$

$$S_n = k(a_1 + a_2 + a_3 + \cdots + a_n)$$

$$S_n = k \cdot A = k \sum a_n$$

Chapter 7

10.3: The Integral Test (04/05/23)

7.1 Reminders

- The fifth **MyLab Math: Infinite Series** is due on **Sunday, April 9th, 2023**.
- The second **Written Homework: Infinite Series** is due on **Monday, April 10th, 2023**.

7.2 Motivation

In the previous section, we expanded our knowledge and understanding of infinite series convergence and divergence by observing some other types of infinite series as well as how to approach them.

- Namely, we learned about the **nth term divergence test**, **combining series**, as well as other types of series manipulation, such as **telescoping series** and **partial fraction decomposition**.

7.2.1 Recap: nth term divergence test

The ***n*th term divergence test** is one of the most basic **theorems for series convergence**. Essentially, the theorem states that if the limit of the function of the series diverges to any number n where $n \neq 0$, then the series must diverge.

It is important to remember, however, that the contrapositive of the statement is not true, as just because a series a_n has a limit n where $n = 0$ does not mean that the series converges.

7.2.2 Recap: Combining Series

TODO

7.2.3 Recap: Telescoping Series

A **Telescoping Series** is a unique case of series, in which we are able to conceptualize the series as a sequence in which all of the terms between the first and last term cancel each other out through **method of difference**. The reason this works is because the actual structure of the series formula $a_n = f(n)$ for all n involves a difference of fractions, where the second term of this expression of fractions cancels itself out with the first term of the subsequent expression.

7.2.4 Recap: Partial Fraction Decomposition

TODO

7.3 Integral Test

7.3.1 Non-Decreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geqslant 0$ for all n. Then, each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$, so

$$s_1 \leqslant s_2 \leqslant s_3 \leqslant \cdots \leqslant s_n \leqslant s_{n+1} \leqslant \cdots$$

Since the partial sums form a non decreasing sequence, the Monotonic Sequence Theorem gives the following result.

Corollary 7.1 C

rollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Layman Definition.
Essentially, what we are saying is that, by the Monotonic Convergence Theorem, which states that positive, nondecreasing series are bounded by a least upper bound L that a series of nonnegative terms will only converge if and only if it is bounded by some upper bound.

7.3.2 Integral Test Definition

Theorem 7.3.1 T

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e Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geqslant N$, where N is a positive integer. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)$ both converge or both diverge.

7.3.3 Integral Test Proof

We establish the test for the case $N = 1$. The proof for general N is similar.

We start with the assumption that f is a decreasing function with $f_n = a_n$ for every n. This leads us to observe that the rectangles, which have the area $1 \times a_n$ (since the indices of series are only natural numbers $\mathbb{N}$), collectively enclose more area than under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x)dx \leqslant a_1 + a_2 + \cdots + a_n$$

If we orient the rectangles to be on the left of the value of the series at any index a_n while disregarding a_1, we actually obtain the following inequality

$$a_2 + a_3 + \cdots + a_n \leqslant \int_1^n f(x)dx$$

If we include the area of a_1 in this inequality, then it has to follow this format

$$a_1 + a_2 + a_3 + \cdots + a_n \leqslant a_1 + \int_1^n f(x)dx$$

Combining these inequalities yields this:

$$\int_1^{n+1} f(x)dx \leqslant a_1 + a_2 + a_3 + \cdots + a_N \leqslant a_1 + \int_1^n f(x)dx$$

This inequality yields the following implications: If $\int_1^\infty f(x)$ is finite, then we know that the series $\sum a_n$ must also be finite, since we know that the series must be smaller than than $a_1 + \int_1^n f(x)dx$. Additionally, if we know that $\int_1^\infty f(x)dx$ is infinite, we know that the series a_n must also be infinite, since we know that $\int_1^{n+1}$ must be smaller than the sum of the partial sums of the series. Hence, we know that the behavior of the series and the integral of the function of the series imply each other's behavior.

7.3.4 Integral Test Proof Intuition

The proof for the Integral Test requires a little bit of creativity, for sure. It mainly tests our intuitoin of how a series works geometrically and requires us to be able to discern a series as well as its function.

As most things for series and sequence convergence, the proof requires us to recall the intuition that we utilize for the direct comparison test as well as any other comparison test. That is, we need to be able to remember that if one function is lesser than another, then these **inequalities create implications**.

Recalling the direct comparison test, of course, we have to remember that whenever we have a situation with two series $\sum a_n$ and $\sum b_n$ like so

$$0 \leqslant \sum a_n \leqslant \sum b_n$$

We can see that if $\sum b_n$ converges, then a_n converges because we know that

$$a_1 + a_2 + a_3 + \cdots + a_n \leqslant b_1 + b_2 + b_3 + \cdots + b_n$$

and that if $\sum b_n$ was finite, there's no possible way that $\sum a_n$ could be infinite.

We apply similar principles to this whenever we derive the **Integral Test Theorem**. Instead of comparing two different series, however, we are comparing the **geometric constructions of both the series as well as the integral of the function of the series**.

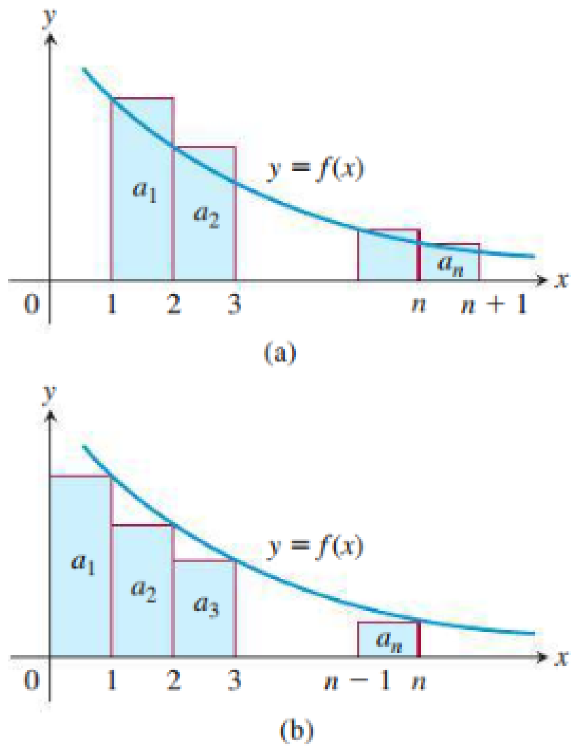

FIGURE 10.12 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\, dx$ both converge or both diverge.

First, we start the proof with some assertions:

- First, we need to assert that the sum of the values of the series is going to be greater than that of the integral of the function of the series.
- Then, we must assert that a section of the values of the series are always going to be less than the entire value of the integral.
- Finally, we can derive an inequality that essentially states that area produced by the entire integral is always going to be less than the sum of all values of the series, but additionally, some "shortened" version of the sum of the values of the series is always going to be less than some expression the integral.

By making the aforementioned assertions, we are able to more rigorously define the sum of the series as well as its relationship with the integral of the function of the series.

At the start of the proof, we must assert that the sum of the series is always going to exceed the value of the area produced by the integral of the function of the series.

$$\text{let } \sum a_n \text{ be a nonincreasing, positive series}$$

First, we define a function $a_n = f_n$ in which the function f is a function for all real numbers n. We know that, based on the definition of the series that

$$\sum a_n \leqslant \int_1^{n+1} f(x)dx$$

The reason why we set the bounds of the integral to be $n + 1$ is that, if we actually graph the integral, we can see that everything is "shifted forward", since of course, the value of a_1 is going to apply for all $1 \leqslant x = 1 \leqslant 2$.

Essentially, we are stating that the graph created by the series is always going to be greater in value than the integral, since the value of a_n is going to be maintained until the next real number.

Now, let us "shift the graph," we can see that if we align the values of each term with the corresponding index, that everything will actually fit, similar to a finite Riemann Sum. However, because we are only considering the bounds from 1, we must omit a_1, since the value of a_1 will actually be outside of our desired integral. Therefore,

$$a_2 + a_3 + a_4 + \cdots + a_n \leqslant \int_1^n f(x)dx$$

Because a_1 is not actually within the bounds in this case, we can rewrite the inequality as

$$a_1 + a_2 + a_3 + \cdots + a_n \leqslant a_1 + \int_1^n f(x)dx$$

If we combine these inequalities together, we can see that the sum of all values of the series a_2 through a_n must be less than the area of the integral of the function of the series, but the sum of all values of the series must also exceed the area of the integral.

$$\int_1^{n+1} f(x)dx \leqslant a_1 + a_2 + a_3 + \cdots + a_n \leqslant a_1 + \int_1^n f(x)dx$$

Therefore, the behavior of both the integral and the series must imply the other's behavior.

- If the sum of all values a_n of the series was finite, then we know that, for the integral to be smaller than the entire sum of values a_n must also be finite (since if it diverged and was infinite, then there would be no way that a finite number could possibly be bigger than an infinite number).

- If the sum of all values a_n of the series was infinite, then we know that in order for the entire area of the integral to be larger than the sum of the values of the series for a section of the domain for some n towards infinity, then the area of the integral would also need to be infinite.

- If the integral of the function of the series was finite, then we know that in order for the area of the integral to always be greater than some section of the sum of the values of the series from one point towards infinity, then the values of the series must also all be finite in order to be less than the integral.

- If the area of the integral was infinite, then we know that in order to satisfy the fact that the sum of all values of the series are always greater than the area of the integral, then the sum of the values of the series must also be infinite.

7.4 P-Series Test

7.4.1 Intuition

The intuition of the P-Series tests derives from the **Integral Test**.

Definition 7.1: P

Series Test

Given a series in the form of

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is some constant number, then we know that for $p > 1$, the series will diverge, while for all values $p \leqslant 1$, the series will converge.

Intuition.
Essentially, this all derives back to the **Integral Test** since, if we actually evaluate the integral for some value where $p > 1$, then we see that the antiderivative will yield a fraction with the exponentiated term in the denominator, meaning that if we were to solve the improper integral with infinity, then the antiderivative would converge, since it would just go to zero.
On the other hand, however, if we had a value of $p \leqslant 1$, then we observe that the antiderivative of this expression would either become some expression of the natural logarithm or, even worse, result in an antiderivative that was not a fraction, such as if $p = -1/2 \rightarrow P = 1/2$, where P is the antiderivative of p, which would mean that, when evaluating the improper integral, the antiderivative would diverge and approach infinity.

7.5 Error Estimation of the Integral Test

Whenever we were solving series such as the **Geometric Series** or the **Telescoping Series**, it was very easy for us to find the entire sum of these series. After all, with geometric series, there's a formula, and with the telescoping series, we can just cancel out terms. That being said though, the other series do not behave as nicely, as there's no easy way to actually derive their sum. Sure, we can just add up all the n terms, where n is some arbitrarily large number to get the sum of the first n terms S_n, but how can we actually gauge how accurate this is? By of course subtractin it from the sum S, of course.

First, let us declare some value R_n for remainder, which we can use as a way of gauging the accuracy of our estimate S_n compared to the actual sum of the series S.

$$R_n = S - S_n$$

If we take the entire value of the series, the sum S and subtract it from the value of the series up until the nth term S_n, we are just left with the sum of the remaining terms of the series.

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Now, let us think back to our visual intuitions of the **Integral Test**. We remember that we would "offset" the actual graph of the values of the series, making it so that the value of n through $n+1$ was actually $f(n+1)$. Under the same exact bounds, if we compare the value of the integral from n to $n+1$, we know that the value of the integral from all real numbers from n , or $f(n)$, to $n+1$, or $f(n+1)$, is going to be greater or equal to $f(n+1)$ for all real numbers between n and $n+1$. Let the difference between the value of the function and the value of the series at any point be the remainder R_n. Therefore, we can make the following assertions:

- First, we know that the area of all terms of the series from the domain $[n+1, \infty)$ are going to be less than the value of the integral from $[n, \infty]$.
- Second, we know that the area of the terms of the series is going to be greater than the value of the integral from the domain of $[n+1, \infty)$.

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leqslant \int_n^\infty f(x)dx$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geqslant \int_{n+1}^\infty f(x)dx$$

$$\Rightarrow \int_{n+1}^\infty f(x)dx \leqslant R_n \leqslant \int_n^\infty f(x)dx$$

$$\Rightarrow S_n + \int_{n+1}^{\infty} f(x)dx \leqslant R_n + S_n \leqslant S_n + \int_{n}^{\infty} f(x)dx$$
$$\Rightarrow S_n + \int_{n+1}^{\infty} f(x)dx \leqslant S \leqslant S_n + \int_{n}^{\infty} f(x)dx$$

Definition 7.2: B

unds for the Remainder in the Integral Test

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of all x for all $x \geqslant n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - S_n$ where S is the sum at which the series converges to and S_n is the nth sum of the series. Then, the remainder of $R_n = S - S_n$ satisfies the inequalities:

$$\Rightarrow \int_{n+1}^{\infty} f(x)dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)dx$$

The real question here is, so what? Knowing this inequality, we are able to solve for the sum of a series given any nth term of a series.

Example. *Estimate the sum of the series* $\sum \frac{1}{n^2}$ *given* $n = 10$.

Chapter 8

10.4: Comparision Tests (04/07/23)

8.1 Reminders

- The fifth **MyLab Math: Infinite Series** is due on **Sunday, April 9th, 2023**.
- The second **Written Homework: Infinite Series** is due on **Monday, April 10th, 2023**.

8.2 Motivation

In the previous lesson, we discussed a technique known as the **integral test for series convergence.** Essentially, we use this technique to determine whether or not a given series converges or diverges by comparing it to its integral. After all, whenever we do have a series, we can always think about it like a **Riemann Sum**, or at least, some botched version of a Riemann Sum. The function created by the series, however, can also be graphed as a function. What we end up doing is just graphing both the series as a series as well as the series as a function and then compare their behavior. If we can determine that the integral of the series as a function is finite, then we can reasonably say that the limit of the series must also be finite, and therefore must converge.

8.3 The Comparison Test

8.3.1 Intuition

Whether or not a sequence converges or diverges depends on its **tail**, or the movement of the sequence as the index n approaches infinity, which is also deonted as $n \to \infty$.

Definition 8.1: T

e Comparison Test (also known as the **Direct Comparison Test**)

The entire concept of the **Direct Comparison Test** stems from the idea of **tails**, that is, the behavior of the series as the domain $(D \in \mathbb{N}) \longrightarrow \infty$. The series $\sum_{n=1}^{\infty} a_n$ can only converge if and only if its tail $\sum_{n=k}^{\infty} a_n$ converges, where k is some term after the initial term n in the domain D.

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=k}^{\infty} a_k \text{ converges, where } k \text{ is some value after } n.$$

We can think of the difference between these two sums as

$$\sum_{n=1}^{\infty} a_n - \sum_{n=k}^{\infty} a_n = \sum_{n=1}^{k} a_n$$

From this, we know that if

$$0 \leqslant a_n \leqslant b_n \ \forall\ n \text{ after some point } N\text{, then}$$

1. If $\sum b_n$ converges, then this implies $\sum a_n$ also converges
2. If $\sum a_n$ diverges, then this implies that $\sum b_n$ also diverges (to ∞)

8.3.2 Comparison Test Proof

Example. *Assume that $\sum a_n$ diverges to $+\infty$. Show that $\sum b_n$ also diverges to $+\infty$.*

$$0 \leqslant a_n \leqslant b_n$$

$$\text{let } A_n = \sum_{k=1}^{n} a_k \text{ and } B_n = \sum_{k=1}^{n} b_k.$$

We must demonstrate that $\forall\ M, \exists\ N$, such that $B_n > M$, $\forall n > N$. (We must demonstrate that for all M, there exists such a value N that $B_n > M$ for all values of $n > N$, where M is some lower bound of B_n and n and N are indices of the series).

1. First, we create some "challenge" variable that defines all values of B_n. Let us state that our objective is to demonstrate that B_n is greater than some aribtrarily large value M at some point $n > N$.
2. Then, let us define the series $\sum a_n$ and its relationship to the series $\sum b_n$.
$$A_n = \sum_{n=1}^{\infty} a_n$$
$$A_n \leqslant B_n$$
3. Let us perform some algebra that defines the variables we are working with and how they are working in relation to each other.
$$A_n = a_1 + a_2 + a_3 + \cdots + a_n \leqslant b_1 + b_2 + b_3 + \cdots + b_n = B_n$$
$$A_n = a_1 + a_2 + a_3 + \cdots + a_n = \infty \leqslant B_n$$
4. Now, we use our algebra and make some conclusions.
   - Because we know that $\sum a_n \longrightarrow \infty$, then we know that at some point $n > N$, $\sum a_n > M\ \forall\ n > N$
   - In order to satisfy the inequality, this must mean that there exists some value N such that $\sum b_n > M\ \forall\ n > N$, which brings us back to our initial statement.

8.4 P-Series Test

Definition 8.2: P

Series Test

8.5 Limit Comparison Test

Theorem 8.5.1 L

i

mit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geqslant N$ (where N is an integer)

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

$$\text{let } a_n \text{ be a series, let } b_n \text{ be a series that behaves like } a_n$$

$$\text{let } c \text{ be the limit of } \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} c > 0, & \text{then } \sum a_n \Leftrightarrow \sum b_n \\ c = 0, & \text{then if } \sum b_n \text{ converges, } \sum a_n \text{ converges} \\ c = \infty, & \text{then if } \sum b_n \text{ diverges, } \sum a_n \text{ diverges} \end{cases}$$

Layman Definition.
The **Limit Comparison Test** is a **series convergence test** theorem that states that if we have a series $\sum a_n$ and we are able to compare to a series that we expect to have the same behavior $\sum b_n$, we can evaluate the limit of the ratio between $\sum a_n$ and $\sum b_n$

8.5.1 Limit Comparison Test Intuition

8.6 Limit Comparison Test Proof

8.6.1 Limit Comparison Part 1 Proof

Since we know that $\frac{c}{2} > 0$, since $c > 0$, there exists some integer N such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}, \text{ whenever } n > N$$

We draw this conclusion from the **definition of a limit for series**, which we recall as

$$\left|a_n - L\right| < \varepsilon \text{ for some } n \geqslant N$$

From this, we are able to expand the inequality to

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$$

$$\Rightarrow -\frac{c}{2} + c < \frac{a_n}{b_n} - c + c < \frac{c}{2} + c$$

$$\Rightarrow \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

$$\Rightarrow \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n$$

If $\sum b_n$ converges, then $\sum(\frac{3c}{2})b_n$ converges and $\sum a_n$ converges by the **Direct Comparsion Test**. If $\sum b_n$ diverges, then $\sum(\frac{c}{2})b_n$ diverges and $\sum a_n$ diverges by **Direct Comparison Test**.

8.6.2 Limit Comparison Part 2 Proof

8.6.3 Limit Comparison Part 3 Proof

Chapter 9

10.5: Absolute Convergence and the Ratio Test

9.1 Reminders (MATH_226)

- There are practice pracice problems on **MyLab Math** that have unalimited attempts and no due date.
- **MyLab Math 6: Integral Test** and **MyLab Math 7: Comparison Tests** are due on **Tuesday, April 11, 2023** and **Thursday, April 13, 2023**, respectively.
- There is a written homework that is due at the beginning of class on **Friday, April 14, 2023**.
- **MATH_226 Midterm 1** is on **Tuesday, April 18, 2023**

9.2 Reminders (MATH_230-1)

- **MyLab Math 6: Planes in Space** is due on **Thursday, April 13, 2023**.
- **Written Homework 1** is due on **Wednesday, April 12, 2023**.

9.3 Motivation

In the previou sclasses we have learned about when a series converges as well as how to test for that. However, in order to supplmeent our learning we need to learn about how to build functions from series

$$1 + x + x^2 + \cdots \Rightarrow \frac{1}{1-x},\ |x| \leqslant 1$$

9.4 Direct Comparison Test (Part 2)

Definition 9.1: D

rect Comparison Test (Theorem 10)

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leqslant a_n \leqslant b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges

Layman's Definition.
Essentially, it doesn't actually matter what a_n or b_n are, the idea is just being able to understand the implications of a series that exists inside of another series. Whenever the partial sums of one series are smaller than another, it makes sense that if the "smaller" series diverges, then the bigger series must also diverge. This applies, of course, only to series that diverge towards infinity.
On the other hand, if we know that the bigger series converges, than we know that the smaller series could not possibly diverge to infinity, since we know that the larger series is finite.

9.4.1 Direct Comparison Test Proof

The series $\sum a_n$ and $\sum b_n$ have non-negative terms. The Corrollary of Theorem 6 in Section 10.3 states that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number M. The partial sums $\sum_{n=1}^{N} a_n$ are all bounded from above by $M = \sum b_n$, since we know that

$$s_N = a_1 + a_2 + a_3 + \cdot + a_N \leqslant b_1 + b_2 + b_3 + \cdots b_N \leqslant \sum_{n=1}^{\infty} b_n = M$$

Since the partial sums of $\sum a_n$ are bounded from above, the Corrollary of Theorem 6 states implies that $\sum a_n$ converges. We conclude that when $\sum b_n$ converges, then so does $\sum a_n$.

In Part (2) of Theorem 10, where we assume that $\sum a_n$ diverges, the partial sums of $\sum b_n$ are not bounded from above. If they were, then the partial sums of $\sum a_n$ would also be bounded from above, since

$$a_1 + a_2 + a_3 + \cdots + a_N \leqslant b_1 + b_2 + b_3 + \cdots + b_N$$

and this would mean that $\sum a_n$ converges. We can conclude that if $\sum a_n$ diverges, then so does $\sum b_n$.

9.5 Limit Comparison Test

9.5.1 Intuition

Recall the **Constant Multiple Rule for Sequences**, which essentially stated that if there were two sequences $\{a_n\}$ and $\{ca_n\}$, where c is just a constant, then we know that, although $\{ca_n\} \neq \{a_n\}$, both sequences would just have the same behavior, since they'd basically just be the same sequences, just with different scales.

We can, of course, just apply this to series, since the series

$$\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$$

$$\Rightarrow \sum a_n\text{'s behavior} \Rightarrow c \sum a_n\text{'s behavior}$$

The idea here is that, what if we just thought of two sequences or series as scaled versions of the other. What if we could determine that scale? What would the scale imply?

Introduce the **Limit Comparison Test**.

Definition 9.2: L

mit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all $n > N$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} C > 0, \text{ then } \sum a_n\text{'s behavior} \Leftrightarrow \sum b_n\text{'s behavior} \\ 0, \text{ then } \sum b_n\text{'s conv.} \Rightarrow \text{implies } \sum a_n\text{'s conv.} \\ \infty, \text{ then } \sum a_n\text{'s conv.} \Rightarrow \text{implies } \sum b_n\text{'s conv.} \end{cases}$$

The contrapositive is also true,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} C > 0, \text{ then } \sum a_n\text{'s behavior} \Leftrightarrow \sum b_n\text{'s behavior} \\ 0, \text{ then } \sum a_n\text{'s div.} \Rightarrow \text{implies } \sum b_n\text{'s div.} \\ \infty, \text{ then } \sum b_n\text{'s div.} \Rightarrow \text{implies } \sum a_n\text{'s div.} \end{cases}$$

9.5.2 Intuition (Part 2.)

The main idea here is that, we want to analyze the ratio of a_n and b_n to determine which function is **greater than the other**. If some function is less than or equal to another function, and the greater function converges, then the lesser function must also converge logically. Vice versa, if the lesser function diverges, then the greater function must also diverge logically.

9.6 Absolute Convergence Test

Definition 9.3: A

solute Convergence

A series $\sum a_n$ **converges absolutely** (or is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$ also converges.

$$\sum a_n \text{ converges absolutely i.f.f. } \sum |a_n| \text{ also converges}$$

Theorem 9.6.1 T

h

e Absolute Convergence Test (Theorem 12) If $\sum |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Warning.
Remember that the **converse** is not true. A convergent series does not imply that the corresponding series of absolute values converges.

9.7 The Ratio Test and the Root Test

9.7.1 Underlying Intuition...

Recall that whenever we started to learn about the geometric series, we obsessed over the following notation

$$\sum_{n=1}^{\infty} a \cdot r^n$$

We talked about the idea of the varialbe r as a ratio that was **very related** to the series' convergence and divergence behavior.

- More specifically, as long as the ratio $|r| \leqslant 1$, then the value r^n would eventually converge to 0, which meant that the function $\frac{a(1-r^n)}{1-r}$ would converge.
- The contrapositive to this, that $|r| \geqslant 1$ meant that the value r^n would **diverge**.

This same idea applies here. If we were given a geometric series, for example, and we wanted to isolate this ratio r, there are a number of ways to do so.

We are able to isolate the variable r be analyzing **successive terms** with the following function:

$$\text{let } \sum a_n \text{ be a geometric series}$$

$$\Rightarrow \frac{ar^{n+1}}{ar^n} = r$$

as well as by isolating the variable r by analyzing the **nth root** of any term in the series:

$$\text{let } \sum a_n \text{ be a geometric series}$$

$$\Rightarrow \sqrt[n]{ar^n} = r$$

To take this a step further, let us take this discussion about the variable r outside the realm of purely geometric series. We are able to actually apply these ratios to non-specifically geometric functions, such as functions with fractions as well as factorials. The idea is that it does not matter whether or not there is some consistent ratio between all terms, but rather, that there is some ratio that we approach **at a limit**.

From this, we can observe the following equations

9.7.2 Ratio Test

Essentially, we want ot be able to analyze the ratio r in a series of the form

$$\sum_{n=1}^{\infty} ar^n$$

We want to analyze the r, since we can recall that as long as $|r| < 1$, then the series will converge.

Definition 9.4: R

tio Test

Given a series $\sum a_n$

$$\sum a_n = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2 \begin{cases} c < 1, \ \sum a_n \Rightarrow \textbf{absolutely convergent} \\ c > 1, \ \sum a_n \Rightarrow \textbf{divergent} \\ c = 1 \text{ or DNE }, \text{ the ratio test is inconclusive} \end{cases}$$

Chapter 10

10.6: Alternating Series and Conditional Convergence (04/12/23)

10.1 Reminders

- **MyLab 7: Comparison Tests** is due **tomorrow, Thursday, April 13, 2023**.
- **Written Assignment 3: Comparison Tests** is due on **Friday, April 14, 2023**.

10.2 Objectives

By the end of this section we want to be able to:

1. Be able to understand the conditions and underlying intuition of the **Alternating Series Test**
2. Be able to differentiate and define **absolutely converging** and **conditionally converging** series

10.3 Recall

TODO

10.4 Motivation

But what if we see a series that comes in this form:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

or this?

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots$$

As we will see, these are known as **Alternating Series**, and are particularly notable series because they have a particular methdology of assessing convergence and divergence.

10.5 Alternating Series Test

Theorem 10.5.1 A
l

ternating Series Test (Theorem 15)
The series denoted as

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges **if and only if** the following conditions are satisfied:

1. All values of u_n are positive
2. All values of u_n are **eventually nonincreasing**, meaning that $u_n \geqslant u_{n+1}$ for all $n \geqslant N$ for some integer N.
3. The values u_n eventually approach 0.

10.5.1 Alternating Series Intuition

Whenever we are working with alternating series, it is important to think of u_n as the terms, but rather, the partial sums of the terms at n. We want to emphasize the differences between u_n and u_{n+1}, thinking about the **difference between the partial sums** u_n.

If we were geometrically observing the values of u_n, we would imagine that the partial sums of the series are narrowing down, almost like a funnel. The idea here is that, if were were to calculate all partial sums of the series u_n, we would find that they all approach a single value, and therefore, converge. We need the values of u_n (the difference between the partial sums of a_n and a_{n+1}) to be positive because TODO.

The difference between the partial sums u_n and u_{n+1} also needs to be decreasing, forming this "cone" shape. If the differences between the partial sums is decreasing, then the difference will eventually approach 0, then we can state that the actual sum of the series will stop at some limit L.

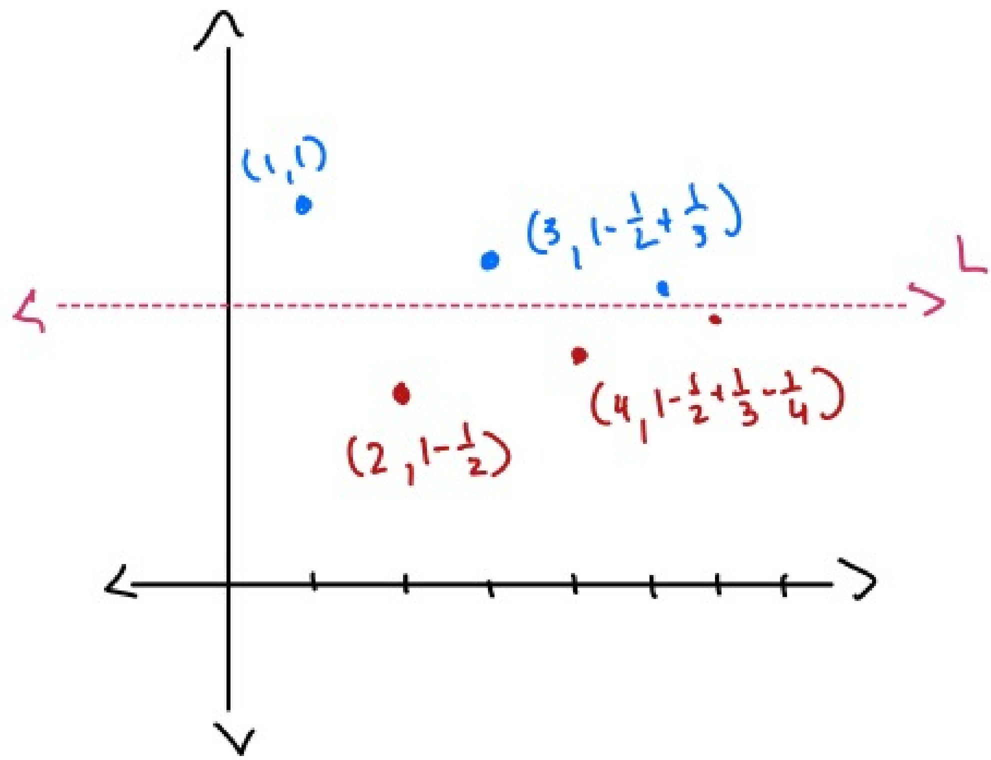

Figure 1. **(above)** Graphing out the partial sums of the series.

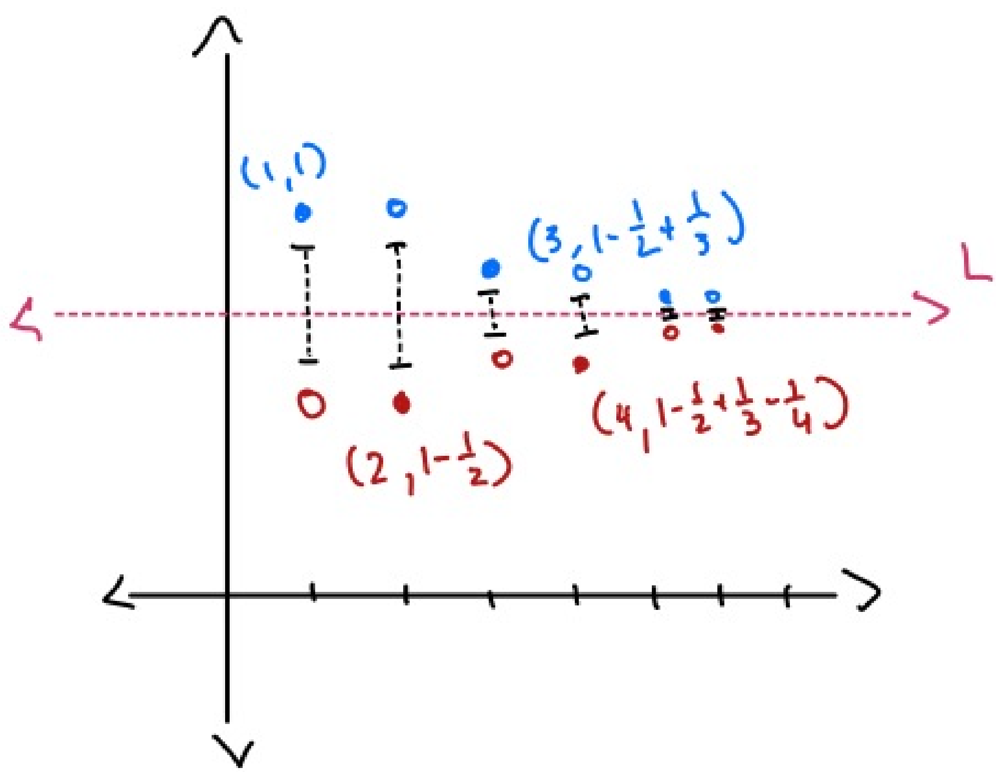

Figure 2. **(above)** Shifting the partial sums as to demonstrate the error estimation or remainder R_n for

between the partial sums. The most important thing to pay attention here for is the decreasing remainder R_n, since it demonstrates how the actual sum of the series actually stops decreasing at some point.

Corollary 10.1 A

ternating Harmonic Series Divergence

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is **conditionally convergent**, since we know that although the alternating harmonic series does diverge by using the **alternating series test**, we know that the absolute valu eof the alternating harmonic series is equal to the harmonic series, which we know diverges by the integral test.

$$\sum_{n=1}^{\infty} \frac{|(-1)^{n+1}|}{|n|} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \textbf{ which diverges}$$

Theorem 10.5.2 A

l

ternating Series Estimation Theorem (by Prof. Zaslow) Whenever a series is convegent by the **Alternating Series Test**, then we must know that the absolute value of the remainder $|R_n| \leqslant u_{n+1}$ for all $n \geqslant N$, where $|R_n|$ represents the difference between the partial sums u_n of the series.

$$|R_n| \leqslant u_{n+1}$$

10.5.2 Alternating Series Proof

TODO

10.5.3 Examples

Example (1). *Alternating Harmonic Series*

Example (2). *Non-Increasing Example*

10.6 Alternating Series Estimation Theorem

Theorem 10.6.1 A
l

ternating Series Estimation Theorem (Theorem 16) If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

satisfies all three conditions of the **Alternating Series Test (Theorem 15)**, then for $n \geqslant N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolutye value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1} and the remainder, $L - s_n$ has the same sign as the first unused term.

10.6.1 Alternating Series Estimation Theorem Intuition

TODO

10.6.2 Examples

Example (**3**).

Definition 10.2: C

nditional Convergence

A series that is convergent but not **absolutely convergent** is called **conditionally convergent**.

10.7.1 Conditional Convergence Intuition

Conditional convergence is mainly a concept that is utilized alongside **absolute convergence and the alternating series test**, as we figure out that through algebraic manipulation, also known as literally just utilizing the absolute value signs, or through rearrangement of the series, that the same series might not actually converge under different conditoins.

Essentially, if a series does not **absolutely converge**, then it is **conditionally converging**.

10.8 Rearranging Series

Theorem 10.8.1 R
e

arrangement Theorem for Absolutely Convergent Series (Theorem 17)
If

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely, and the sequene

$$b_1, b_2, b_3, \cdots, b_n, \cdots$$

is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

10.8.1 Rearranging Series Intuition

10.9 Summary of Tests to Determine Convergence or Divergence

1. The nth-Term Test for Divergence
2. Geometric Series
3. p-series
4. Series with nonnegative terms
5. Series with some negative terms
6. Alternating Series

10.10 Writing Proofs

10.10.1 Proofs Who?

Whenever we are writing proofs, it is important to consider these four elemnents:

- Definitions (Theorems, etc)
- Ideas
- Roadmap

10.11 Types of Proofs

In mathematics, there are four commonly used proof types:

- Direct Proofs
- Proofs by Contradiction

- Proofs by Induction
- Proofs by Contrapositive

Definition 10.3: D

rect Proofs

TODO

Definition 10.4: P

oofs by Contradiction

TODO

Definition 10.5: P

oofs by Induction

TODO

Definition 10.6: P

oofs by Contrapositive

TODO

Example. *Given that* $\{a_n\}$*is a nondecreasing, bounded sequence and that* L *is a least upper bound of the sequence* $\{a_n\}$*, prove that*

$$\lim_{n\to\infty} a_n = L$$

Chapter 11

Proof Writing (04/14/23)

11.1 Reminders for Math_226

- **Midterm 1** is on **Tuesday, April 18, 2023**
- **MyLab Math 8: Alternating Series and Conditional Convergence** is due on **Monday, April 17, 2023**.

11.2 Reminders for Math_230-1

- **MyLab 7: Cylinders and Quadric Surfaces**
- **MyLab 8: Polar Coordinates** is
- **Midterm 1** is on **Tuesday, April 25, 2023**

11.3 Motivation

11.4 Intro to Proofs

11.4.1 Types of Proofs

11.5 Scaffold of a Proof

-

11.6 Methodology

-

Chapter 12

Midterm 1 Study Guide

12.1 Reminders

Homework Spreadsheet Link

12.1.1 MATH_226-0 Reminders

- **MyLab Math 9: Alternating Series and Conditional Convergence** is due on **Monday, April 17, 2023**.
- **Midterm 1** is on **Tuesday, April 18, 2023** during **discussion**.
- **Written Homework 4** is due on **Friday, April 21, 2023**.

12.1.2 MATH_230-1 Reminders

- **MyLab Math 7: Cylinders and Quadric Surfaces** is due on **Tuesday, April 18, 2023**.
- **MyLab 8: Polar Coordinates** is due on **Thursday, April 20, 2023**.
- **Written Homework 2** is due on **Wednesday, April 19, 2023**.
- **Midterm 1** is on **Tuesday, April 25, 2023**.

12.2 Topics to Study

12.3 Sequence Convergence

12.4 Series Intuition

12.5 Geometric Series

12.6 Integral Test

12.7 Comparison Tests

12.8 Absolute Convergence and the Ratio + Root Test

12.9 Alternating Series Test and Conditional Convergence

12.10

Chapter 13

10.6: Strategies for Analyzing Convergence

13.1 Reminders

13.2 Motivation

Chapter 14

10.7 (Part One): Power Series:

14.1 Reminders

14.1.1 MATH_226 Reminders

1. **MyLab Math 10: Radius and Interval of Convergence** is due on **Friday, April 21, 2023**.
2. **Written Homework 4** is going to be due on **Monday, April 24, 2023**.

14.1.2 MATH_230 Reminders

1. **MyLab Math 8: Polar Coordinates** is due on **Thursday, April 20, 2023**.
2. **MyLab Math 9: Curves in Space and Their Tangents** is due on **Sunday, April 23, 2023**.
3. **MATH_230-1 Midterm 1** is **6 days away**. The test will be on **Tuesday, April 25, 2023**.

14.2 Objectives

1. Be able to identify and evaluate the coefficients of a power series as well as the center of the power series.
2. Represent certian elementary functions as power series.
3. Detemrine whether a power series converges at a point by applying convergence tests for infinite series

14.3 Motivation

In the past few chapters, we have learned about sequence and series convergence as well as their technical intricacies. We have been able to utilize our knowledge of sequences, series, and limits in order to develop a baseline understanding of how series and sequences worked, but we have hardly worked on the **applications of series**. Surprisingly, series show up everywhere, and most importantly, we can actually create generalizatoins about different series as well as their behaviors in order to apply them. The most notable way is applying the idea of series to polynomials, especially whenever we have series that after all, just look like **infinite polynomials**. I mean, examples that come to mind after all, are the geometric sequence, the alternating series, etc, which all have the common form of

$$\sum_{n=0}^{\infty} a_n \cdot r^n$$

$$\Rightarrow a_0 + a_1 \cdot r^1 + a_2 \cdot r^2 + a_3 \cdot r^3 + \cdots$$

The questions that we are trying to pose with power series are as follows:

- If a series looks like a polynomial, does it **behave** like a polynomial?

- If a series behaves like a polynomial, how can we apply our knowledge of algebraic manipulations of polynomials to series?
- What do these algebraic manipulations on power series suggest about the series itself? For example, how does adding one power series to another power series affect its convergence?
- How can we apply calculus to series?

14.4 Introduction to Power Series and Power Series Convergence

In the former section, we discussed the idea of the power series and how exactly we arrived at its conception.

- Recall that a power series is meant to be a **infinite series** that resembles an **infinitely-spanning polynomial**.
  - Recall that a polynomial is just any expression that utilizes terms of varying powers and degrees, such as the famous equation of the parabola:
  $$ax^2 + bx + c$$
- In the most primitive context, we can think of a polynomial as an expression of terms such that there is some constant c multiplied by some expression of x and y.
  - Think of the conic sections for example:
  $$a(x-h)^2 + b(y-k)^2 = c^2$$
  We can see that the polynomial here is blatantly just a **constant multiplied by some expression of a variable**.

Applying this to the idea of a series, let's think of a series that generates an **infinite polynomial** as a series in which we have some constant c multiplied by some expression of x that has a varying power of n.

- We will call the expression of x as the **center** of the power series. That is, much like how the expressions $(x-h)$ and $(y-k)$ denote the center of a circle, $(x-a)$ denotes the **center of convergence** of powre series.

Definition 14.1: P

wer Series

A **power series that is *centered* about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

such that c_n is just a constant term.

A **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

such that c_n is a constant and a is a constant.

Now, if we were to just **evaluate** this power series, letting $c_n = 1$ and letting $a = 0$, then we would obtain the following series.

Example. *Relating the Power Series Centered at 0 to the Geometric Series* let $c_n = 1$ and let $a = 0$

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n$$

Now if we expand this power series out...

$$\Rightarrow \sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + \cdots + x^n$$

$$\Rightarrow 1 + x + x^2 + \cdots + x^n$$

which looks **oddly** similar to the **geometric series** that we have introduced at the introduction of series... mainly becuase it is the same geometric series! The idea that the power series resembles the geometric series is a very importnat idea in the world of the power series, as it is important ot understand the idea of the ratio as well as **recognizing the patterns between the terms of an infinite polynomial**.
Recall that we know that the sum of an infinite geometric series is as follows

$$\sum_{n=0}^{\infty} a_n \cdot r^n = \frac{1}{1-r}$$

Although it seems weird, we must be able to asser that, since the power series $\sum_{n=0}^{\infty} x^n$ behaves exactly like the geometric series $\sum_{n=0}^{\infty} a_n \cdot r^n$ such that $r = x$, that they **must converge at the same place**.
Therefore, we know that the power series we have created must converge (or must have a total sum of)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \ \forall \ -1 < x < 1$$

According to what we know about the **geometric series**, then we know that the ratio $r = x$ must be less than 1, or at least, must have a value that is less than 1, such as to ensure that the parital sums of the series are constantly decreasing, as opposed to just increasing infinitely. Therefore, we know that the absolute value of x or $|x| < 1$.

14.4.1 Where Things Change... (Series aren't *actually* series...)

But don't get things crossed, although we are currently living in series-land, we must understand that **power series and eventually Taylor Series are not about the series themselves.** After all, we aren't actually able to write all of the partial sums of a series to infinity... we are just really good at understanding their behavior thanks to our immense knowledge of convergence therems and tests. That being said, though, in reality, we must really think of the power series, and any series here going forward, as **approximations** of the point of convergence, or the sum. We can think of any expression of partial sums as an approximation$P_n(x)$ of the true sum or point of convergence of the series.

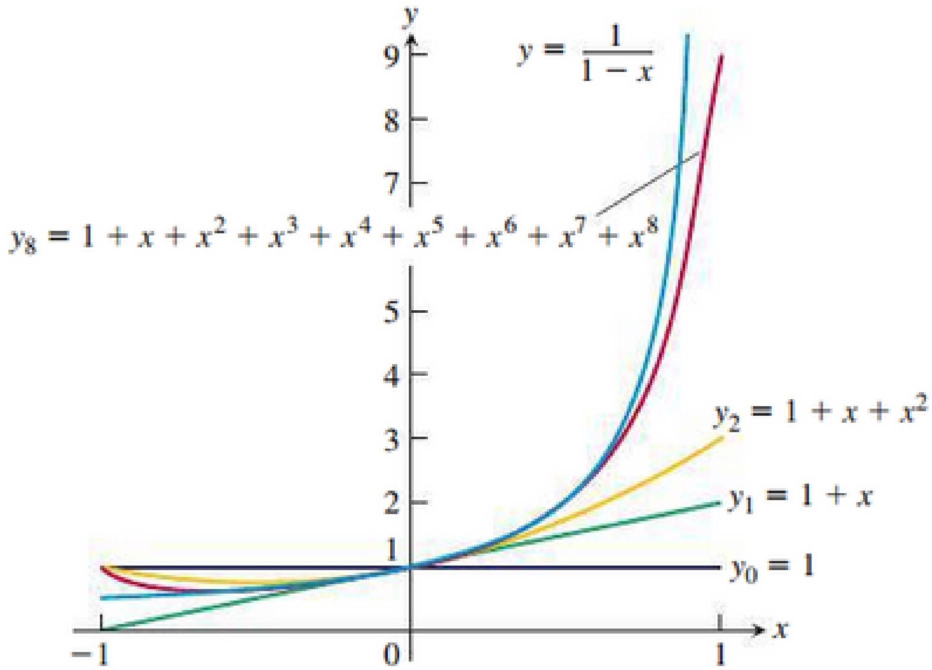

FIGURE 10.17 The graphs of $f(x) = 1/(1-x)$ in Example 1 and four of its polynomial approximations.

Within our image, we can see that the approxmation $P_n(x)$ improves with the addition of more terms. In the most simple sense, we are essentially just making the polynomial derived from the sequence more and more similar to the graph of the actual function. In this sense, if we were able to hypothetically add an **infinite** amount of terms to the power series, then we would be able to get an approxmation that is equal to the original function at some value x.

Example ((2))**.** *Find the components of the following series. Then, evaluate for what function the series converges to, what the radius of convergence is, as well as the infinite sum that represents the series.*

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n$$

Recall that whenever we are dealing with a power series, we are trying to think of the terms of the series as terms within a **polynomial**. The general form of a power series is as follows:

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

$$\Rightarrow c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots c_n(x-a)^n$$

Whenever we apply this general structure to the given power series, we can see that the center a of the power series must be 2. Additionally, we can see that $c_0 = 1$, $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{4}$, and $c_n = \left(-\frac{1}{2}\right)^n$. Therefore, we are able to translate this power series to the following infinite sum.

$$\Rightarrow \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$$

Because both of the terms of hte power series are functions of n, then we are able to rewrite the series in terms of one single ratio r:

$$\Rightarrow \sum_{n=0}^{\infty} \left(- \frac{(x-2)}{2} \right)^n$$

Given that the infinite sum has the form of a **geometric series**, we are able to determien its **radius of convergence** knowing that the $|r| < 1$.

$$\left| - \frac{x-2}{2} \right| < 1$$

$$\Rightarrow \left| \frac{x-2}{2} \right| < 1$$

$$\Rightarrow 0 < x < 4$$

We know that the series will converge only for values of $x \in (0, 4)$. In order to determine what the series actually converges to, we are able to apply the **sum of a geometric series formula**.

$$\frac{1}{1-r}$$

$$\Rightarrow \frac{1}{1 + \frac{x-2}{2}}$$

$$\Rightarrow \frac{2}{2 + x - 2}$$

$$\Rightarrow \frac{2}{x}$$

We know that our infinite sum is meant to represent the function $f(x) = \frac{2}{x}$ for all values $x \in (0, 4)$. The polynomial $P(n)$, which is equal to the series that we are given, gives aproximations to the function $f(x) = \frac{2}{x}$ for any value of x near 2. From this, we are able to determine that the following partial sums

$$P_0(x) = 1$$

$$\Rightarrow P_1(x) = 1 - \frac{1}{2}(x-2)$$

$$\Rightarrow P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2$$

are just approximations of the function $\frac{2}{x}$ that get more and more accurate with the addition of more terms. Note, though, power series are not just restricted to geometric series, as we can derive power series (and approximations of functions with polynomials) with other types of series.

14.5 The Convergence Theorem of Power Series

Our experiemntation with power series convergence leads us to the following theorem: **the Convergence Theorem of Power Series**, which offers us a general idea of how convergence works for power series– that is, how we can determine what values of x the power series converges to.

Theorem 14.5.1 C

o

nvergence Theorem of Power Series If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

converges at $x = c \neq 0$, then it converges absolutely $\forall$ such that $|x| < |c|$. If the series diveresg at $x = d$, then it diverges $\forall$ x such that $|x| > |d|$.

Essentially, this theorem tells us that if we have a power series that is centered at $x = 0$, then if the series converges at some nonzero value c, then x will converge for all values within the interval $-c < x < c$ and if we know that a power series diverges at $x = d$, then the series will diverge for all values of $x < d$ and $x > d$. Note that the theorem does *not* tell us whether or not that if $x = c$, then c is the *endpoint* of convergence. It merely states that any value below c will converge.

The other issue with this theorem, though, is that although it does give us a general idea of the behavior of a convergent power series, it only offers us informatoin on the behavior of a convergent power series that is *centered at* $a = 0$. If we want to understand how power series that are centered at nonzero values function, then we can look at **a Corollary to the Convergence Theorem of Power Series**. By looking to the **corollary of the the convergence theorem to power series**, we are able to describe all **three possible behaviors** of power series.

14.5.1 Corollary to the Convergence Theorem of Power Series

Definition 14.2: C

rollary to the Convergence Theorem of Power Series

The convergence of the series $\sum_{n=0}^{\infty}$ is described by three of the following behaviors

1. There is some positive number R such that the seires diverges for x for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. Series may or may not converge at endpoints $x = a - R$ and $x = a + R$
2. Converges absolutely $\forall$ x (this is the situation in which the radius $R = \infty$).
3. Series converges at $x = a$ and diverges $\forall$ $x \neq a$ (such that the radiu sof convergence is $R = 0$).

14.5.2 The Three Possible Behaviors of Power Series

From both the **convergence theorem of power series** as well as the **corollary of the convergence theorem of power series**, we can establish that there are three different types of behaviors of convergent power series.

1. $\sum a_n$ converges for $x = c$ (converges at a single point).
2. $\sum a_n$ converges $\forall$ $x \in (-\infty, \infty)$ (converges for all values of x).
3. $\sum a_n$ converges $\forall$ $x \in R$, such that R is some interval (converges for an interval).

In the corollary, we discuss the variable R. R represents the *radius of convergence*, which represents the distance between the center of the power series a from the endpoints of convergence. The interval of convergence for a power series centered at a is as follows:

$$|x - a| < R \text{ or } a - R < x < a + R$$

All values within the interval of convergence are **absolutely convergent**, whereas the endpoints are dpenedent on the series itself. This interval can be

- open
- closed
- or half-open

That being said, there is also possibility that the radius of convergence is *infinite*, which implies that the series converges for all values of x, or that the radius of convergence is equal to 0, which means that the power series only converges at some value $x = a$.

14.6 Testing Power Series for Convergence

Here is a computational algorithm for determining where a power series converges

Definition 14.3: H

w to Test a Power Series for Convergence

1. Use Ratio Test/Root Test to find hte largest open interval where the series converges absolutely such that

$$|x - a| < R \text{ or } a - R < x < a + R$$

2. If R is finite, test for convergence or divergence **at each endpoint** by using the Comparison Test, Integral Test, or Alternating Series Test.
3. If R is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally), since the nth term does not approach zero for those values of x.

14.7 Summary of the Behavior of Power Series

Recall that a power series is just a series that resembles an *infinite polynomial*. Power series usually possess some constant c_n as well as a *center of convergence* a.

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

Whenever we are working with power series, it is important to remember that we aren't necessarily thinking of the power series as an infinite series, but rather, an *approximation* of a polynomial. These polynomials are generally just the point at which the power series converges, as we can just think of each term (as well as the terms behind it) as approximations of this polynomial $f(x)$. Each approximation $P_n(x)$ improves with the addition of more terms. In our examples, we analyzed what forms power series can look like as well as what it means for a power series to **converge**. Because power series are an **expression** of x, then we have to determine what values of x satisfy the series in order to make it converge. We can determine this by analyzing the archetype of the power series itself (asking whether or not it resembles a Ratio Test/Root Test/Geometric Series Test), and then performing the test, then evaluating the result of that test for x. We will find that there are several situations of convergence for a power series, as the power series can converge at a single value of x, can converge for an interval of $-R < x - a < R$ (which can either be open, closed, or half-open), or can converge for all values of x. We also analyzed what it means for a power series to converge and diverge by looking at the **Covergence of Power Series Theorem** as well as the **Corollary to the Convergence of Power Series Theorem**, which essentially states that if we can determine that a power series converges for some nonzero value c, then the series must converge for all values such that $|x| < c$, although of course, it doesn't state that c is necessarily an endpoint of the interval of convergence. Similarly, we know that if a power series diverges for a value d, then the power series will diverge for all values of $|x| > |d|$.

We also generated a general method/ computational algorithm for determining whether or not a power series converges.

1. First, we want to apply the ratio/root test in order to evaluate where the given power series converges, which is generally in the form of
$$|x - a| < R \text{ or } a - R < x < a + R$$
2. Then, we want to determine the activity of the power series at the endpoints, since we can only determine the **open interval** of convergence. We can determine the behavior of the power series at the endpoints by using other convergence tests, such as the **Comparsion Test**, the **Integral Test**, as well as the **Alternating Series Test**.
3. We know, however, that all values outside of this interval will *always diverge*.

Chapter 15

10.7 (Part Two): Radius and Interval of Convergence

15.1 Reminders (as of (05/05/23))

15.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

15.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

15.2 Motivation

In the last sectoin, we introduced the idea of the **power series**, which is essentially just a series that mimics a **polynomial** in the form of

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 c_3 x^3 + \cdots + c_n x^n.$$

As we can see, term of of the power series are all diferentiated by their powers, which allows us to think of them and treat them **like polynomials**. In this new way of htinking, we are no longer thinking of series as just partial sums, but we're thinking of them as **approximations of more interesting functions**. For example, one of the easitest to remember power series is the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

This power series is just an estmation of the function e^x, which is also known as the **exponential function**. However, notice that the exponential function is a function of **two variables**, x and n, which means that there is a **domain** as well as a **range** that we are considering whenever we are expanding htis power seires as well as working with it. In this section, we are going to be investigating what it means for a power series to converge, as we can see that if there is a **domain** of a power series, is it bounded? If it is bounded, how do we find these bounds?

15.3 Objectives

1. Detemrine whether a power series converges at a point by applying convergence tests for infinite series
2. Determine the radius of convergence of a power series.
3. Completely determine the interval of convergence of a power series.

Chapter 16

10.7 (Part Three): Manipulation of Series (Part One)

16.1 Reminders (as of (05/05/23))

16.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

16.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

16.2 Objectives

1. Multiply, differentiate, and integrate convergent power series
2. Compose a convergent power series with a monomial function to generate a new convergent power series
3. Compute power series representation sof elementary functions by manipulating known power series.

16.3 Motivation

In the former section, we exposed ourselves to the world of **power series**, which are infinite series that resemble "infinite polynomials", meaning that we have an infinite number of terms of varying exponential degrees. Power series generally come in the general form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

such that c_n is just a constant and a is known as the *center of the interval of convergence*. We saw that the terms of the power series are interesting because they are able to represent *approximations* of functions $f(x)$. We have observed that these approximations, which represent the first n terms of a power series $P_n(x)$ beocme more accurate as we include more terms.

Finally, we observed the behavior of convergence and divergence within power series. We saw that because power series are multivariable expressions of x and n, that they are able to converge across numerous values of x. We observed that we can apply different *convergence tests*, such as the Geometric Series Test, the Ratio Test, as well as the Root Test in order to determine whether or not a given power series converges, and, more importantly, *where* the given power series converges. Once we determine where the power series converges, which is an open interval, we then have to determine the behavior of the power series at the endpoints of this interval, which we can assess using more convergence tests such as the Comparison Test, the Integral Test, as well as the Alternating Series Test.

Now that we have the basic principles of power series (suggesting what they are as well as their different behavior), let us now observe methods of which we are able to *manipulate power series* and work with them. In the previous section, we have alluded to adding, subtracting, multiplying, differentiating, and even integrating power series, and in this section, we are learning how to do this.

16.4 Operations on Power Series

We are able to perform numerous algebraic operatoins on two given power series. However, of course, this means that we are performing these operations on the **intersections of the two series' intervals of convergence**. Again, since we are merely just thinking of these series as polynomials, we can apply our intuition of operations on polynomials to operations of power series.

1. We multiply power series by distributing each term, much like FOILing
2. We add power series by adding like terms
3. We subtract power series by subtracting like terms
4. We divide power series similarly to as we divide two polynomials

Definition 16.1: S

ries Multiplication for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots a_{n-1} b_1 + a_n b_0 = \sum_{n=0}^{\infty} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x) \cdot B(x)$ for $|x| < R$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Evidently, it is a little tricky to find the coefficient of the resulting series c_n, since it requires us to *manually mutliply the terms and determine a general pattern.*

Example (1). *Multiply the power series*

$$\sum_{n=0}^{\infty} x_n \text{ and } \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) \qquad \text{Multiply second series . . .}$$

$$= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}_{\text{by } 1} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right)}_{\text{by } x^2} + \cdots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6} \cdots. \qquad \text{and gather the first four powers.}$$

16.5 Substituting Functions into Convergent Power Series

Much as we are able to multiply power series within their overlapping intervals of convergence, we are also able to substitute functions whose values are within the interval of convergence $|f(x)| < R$ into different power series.

Definition 16.2: T

eorem 20 (Subsituting Functions into Power Series)

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points $x$such that the range of the function $f(x)$ is within the radius of convergence, or $|f(x)| < R$.

Example ((2) Substituting Functions into Power Series)**.** *Determine the interval of convergence for the following power series only using subsitution.*

$$\sum_{n=0}^{\infty} (4x^2)^n$$

Whenever we are approaching this problem, we can observe what we do know. In the former section, we observed that the power series

$$\sum_{n=0}^{\infty} x^n$$

will converge for all values of $|x| < 1$, since we are able to think of the power series as a geometric series. We know that the power series, in this specific case, will converge to the value

$$\frac{1}{1-x}$$

Since we know that the function $4x^2$ is continuous, then we are able to substitute x for $4x^2$, and see that power series represents the function

$$\frac{1}{1-4x^2}$$

and, most importantly, will converge for the expression $|x| < 1 \to |4x^2| < 1$. From evaluating the inequality, we observe that the power series $\sum_{n=0}^{\infty} (4x^2)^n$ will converge at $|x| < \frac{1}{2}$.

In addition to being able to add power series, mulitply power series, and even substitute functions into power series, we are also able to **differentiate power series term by term for all values of x within their interval of convergence**.

16.5.1 Term-By-Term Differentation of Power Series

Definition 16.3: (

heorem 21) Term-By-Term Differentiation

If $\sum c_n(x-a)^n$has the radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ on the interval } a - R < x < a + R$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term. Here are the derivatives of a power series.

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$, which is consistent between *all derivatives.*

Essentially, we are able to find the **derivatives** of terms within a power series that are within the interval of convergence of that power series. We find the derivatives of the power series (that is, the expression that defines the power series) by literally finding the derivatives of each term and generating a new power series expression from it.

Note: Whenever we are evaluating the derivative of a power series, note how we change the **starting index**. The reason that this is, is that, whenever we actually derive the power series, the resulting function will always have an exponent of $n - a$, such that a is just some number. This means that at $n = 0$, the initial term will always be zero. Therefore, we just reindex the derivative of hte power series in order to avoid this initial zero.

Note: Term-by-term differentation does not work for *all* power series. For example, if we use the trigonometric series

$$\sum_{n=0}^{\infty} \frac{sin(n!x)}{n^2}$$

converges for all x. But if we differentaite term by term we obtain the following series

$$\sum_{n=1}^{\infty} \frac{n! \cos n!x}{n^2}$$

which diverges for all x. Note how these two series are **not** power series, since they are not a sum of *positive integer powers of* x .

In addition to being able to differentiate a power series, we are also able to *integrate* a power series and find antiderivatives that exist on the same interval of convergence.

16.5.2 Term-By-Term Integration

Definition 16.4: (

heorem 22) Term-By-Term Integration

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R (R > 0)$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

The consequences of being able to manipualte series like this is that we're able to find sequences that have the same exact radius of convergence, which allows us to discover functions that behave in similar ways (although, of course, not in the same exact way). In the given examples, we differentiate and integrate power series, starting with a power series, then integrating it, only to find it to resemble a more familar power series of which we know the point of convergence (or the fucntino that the power series represents), and then we can just integrate that function to find what the original power series represents.

It is important to understand that whenever we are integrating a power series, we are literally integrating all of the terms of the power series, similarly to how when we differentiate a power series, we are literally differentiating all of the terms of the power series. We are obtianing a series that, although is not going to yield the same sum as the original function, will behave similarly to the original function, albeit with different centers, for example. It is important for us to be able to freely differentiate and integrate series as well as multiply and divide them. The general form for differentiation and integration is as follows

$$\frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n = n c_n x^{n-1}$$

$$\frac{d^2}{dx^2} \sum_{n=0}^{\infty} c_n x^n = (n-1) n c_n x^{n-2}$$

As for integration:

$$\int c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

Chapter 17

10.7 (Part Four): Manipulation of Series (Part Two)

17.1 Reminders (as of (05/05/23))

17.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

17.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

17.2 Objectives

1. Multiply, differentiate, and integrate convergent power series
2. Compose a convergent power series with a monomial function to generate new power series
3. Compute power series rperesentations of several elementary functions by manipualting known convergent power series
4. Explain the connection between the coefficients of a power series and the sum of a power series

17.3 Motivation

Chapter 18

10.8 (Part One): Taylor Series

18.1 Reminders (as of (05/05/23))

18.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

18.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

18.2 Objectives

1. Compute the Taylor (as well as MacLaurin) seires generated by an infinitely differentiable fucntion centered at a given point.
2. Explore the relationships between the coefficient sof a power series and the sum of the same power series

18.3 Motivation

Chapter 19

10.8 (Part Two): Taylor Polynomials and Taylor Series Convergence

19.1 Reminders (as of (05/05/23))

19.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

19.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2.

19.2 Motivation

of Multiple Variables**MLM 12: Functions**

19.3 Reminders (as of (05/05/23))

19.3.1 MATH_226

1. **Midterm 2 is on Tuesday, May 15th, 2023, which is 10 days away.**
2. **Remember to revise MyLab 15: Applications of Taylor Series.**
3. **MyLab 16: Complex Numbers is due tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7 is due on Friday, May 12, 2023.**

19.3.2 MATH_230

1. **230 Midterm 2 is in 10 days on Tuesday, May 15, 2023.**
2. **MLM 12: Functions of Multiple Variables needs revisio n**
3. **MLM 13: Limits and Continuity in Higher Dimensions is due on Sunday, May 7, 2023.**

4. **MLM 14: Partial Derivatives is due on Tuesday, May 9, 2023. MLM 15: The Chain Rule is due on Thursday, May 11, 2023.**

5. **WHW 4 is due on Wednesday, May 10, 2023.**

19.4 Motivation

of Multiple Variables needs revisio n

MLM 13: Limits and Continuity in Higher Dimensions is due on Sunday, May 7, 2023.

MLM 14: Partial Derivatives is due on Tuesday, May 9, 2023. MLM 15: The Chain Rule is due on Thursday, May 11, 2023.

WHW 4 is due on Wednesday, May 10, 2023.

19.5 Motivation

19.6 Reminders (as of (05/05/23))

19.6.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

19.6.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

19.7 Motivation

Chapter 20

Convergence of Taylor Series ((05/05/23))

20.1 Reminders (as of (05/05/23))

20.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

20.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

20.2 Motivation

Chapter 21

Applications of Taylor Series

21.1 Reminders (as of (05/05/23))

21.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

21.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4.

21.2 Motivation

14: Partial Derivatives**MLM**

21.3 Reminders (as of (05/05/23))

21.3.1 MATH_226

1. **Midterm 2 is on Tuesday, May 15th, 2023, which is 10 days away.**
2. **Remember to revise MyLab 15: Applications of Taylor Series.**
3. **MyLab 16: Complex Numbers is due tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7 is due on Friday, May 12, 2023.**

21.3.2 MATH_230

1. **230 Midterm 2 is in 10 days on Tuesday, May 15, 2023.**
2. **MLM 12: Functions of Multiple Variables needs revisio n**
3. **MLM 13: Limits and Continuity in Higher Dimensions is due on Sunday, May 7, 2023.**
4. **MLM 14: Partial Derivatives is due on Tuesday, May 9, 2023. MLM 15: The Chain Rule is due on Thursday, May 11, 2023.**
5. **WHW 4 is due on Wednesday, May 10, 2023.**

21.4 Motivation

14: Partial Derivatives is due on Tuesday, May 9, 2023. MLM 15: The Chain Rule is due on Thursday, May 11, 2023.

WHW 4 is due on Wednesday, May 10, 2023.

21.5 Objectives

1. Explore advanced application sof Taylor Series representations to major topics in single-variable calculus

21.6 Motivation

Chapter 22

A7 (Part One): Complex Numbers (Part One)

22.1 Reminders (as of (05/05/23))

22.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

22.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

22.2 Objectives

1. Explain the significance of the complex number systems via the **Fundamental Theorem of Calcuslu**.
2. Compute the complex conjugate and absolute value of a ocmplex number
3. Algebraically manipulate algebraic expressions involving complex numbers

22.3 Motivation

Chapter 23

10.10, A7 (Part Two): Complex Numbers (Part Two)

23.1 Reminders (as of (05/05/23))

23.1.1 MATH_226

1. **Midterm 2** is on **Tuesday, May 15th, 2023**, which is **10 days away.**
2. Remember to revise **MyLab 15: Applications of Taylor Series**.
3. **MyLab 16: Complex Numbers** is due **tomorrow, Saturday, May 06, 2023.**
4. **Written Homework 7** is due on **Friday, May 12, 2023**.

23.1.2 MATH_230

1. **230 Midterm 2** is in **10 days** on **Tuesday, May 15, 2023**.
2. **MLM 12: Functions of Multiple Variables** needs revisio n
3. **MLM 13: Limits and Continuity in Higher Dimensions** is due on **Sunday, May 7, 2023**.aaaa
4. **MLM 14: Partial Derivatives** is due on **Tuesday, May 9, 2023.** MLM 15: The Chain Rule is due on **Thursday, May 11, 2023.**
5. **WHW 4** is due on **Wednesday, May 10, 2023**.

23.2 Objectives

1. Understand Euler's fomrula and the polar coordinate representations of complex numbers
2. Be able to visualize the ocmplex plane as well as points and expressions of the ocmplex plane
3. Be able to manipulate the exponential function and investigate its propertie sin reltaion to complex numbers
4. Establish roots of unity and **DeMoivre numbers and theoerems**.

23.3 Motivation

Chapter 24

19.1 (Part One): Vectors

24.1 Reminders

24.2 Motivation

Chapter 25

19.1 (Part Two): Inner Products

25.1 Reminders

25.2 Motivation

Chapter 26

19.2 (Part One): Functions as Vectors, Periodic Functions

26.1 Reminders

26.2 Motivation

Chapter 27

19.2 (Part Two): Fourier Series, Demos

27.1 Reminders

27.2 Motivation

Chapter 28

19.3: Fourier Series, Theory

28.1 Reminders

28.2 Motivation

Chapter 29

19.5 (Part One): Applications (Part One)

29.1 Reminders

29.1.1 MATH_226

1. **Written Homework 9: Second-Order Linear Differential Equations** is due on **Friday, 26, 2023**.

29.1.2 MATH_230

1. **Written Homework 5** is due on **Wednesday, May 24, 2023**.
2. **MyLab Math 17: Tangent Planes and Linearization** is due **tomorrow, May 23, 2023**.
3. **MyLab Math 18: Taylor's Formula** is due **tomorrow May 23, 2023**.

29.2 Motivation

Throughout the entire ocurse, we have touched on sequences, series, and how to use them in numerous different contexts, such as Fourier Series. Now, however, we want to take our series manipulations in another direction. For exapmle, what if we want to approximate the temperature, while considering all possible conditions?

29.3 Brief Introduction to Differntial Equations (17.1)

Definition 29.3.1: Second-Order Linear Differential Equations

A **second-order linear differential equation** is an equation that involves a function $y(x)$ as well as its derivatives, usually taking on the following form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x) \tag{29.1}$$

There are two different types of second-order linear differential equations:

- Homogeneous Linear Differential Equations
- Heterogeneous Linear Differential Equations

Definition 29.3.2: Homogeneous Linear Differential Equations

Given the equation (29.1), a **homogeneous linear differential equation** is an equation in which $G(x)$ is equal to zero for all values of x over an interval I. This results in the following differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \tag{29.2}$$

Definition 29.3.3: Heterogeneous Linear Differential Equations

By contrast, a **heterogeneous linear differential equation** is one in which $G(x) \neq 0$ for all values of x over an interval I.

29.4 Evaluating Differential Equations with Power Series (17.5)

Among the numerous ways we are able to evaluate differential equatoins, we can actually apply our familiarity with **power series** in order to evaluate differential equations.

Chapter 30

19.5 (Part Two): Applications (Part Two)

30.1 Reminders

30.2 Motivation