Math_226 (Spring 2023) Notes

Randy Truong

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Contents

Chapter 1

10.1 (Part One): Sequences (Part One) (03/28/23)

1.1 Reminders

- The first MyLab homework is due on **Wednesday, March 28, 2023**.
  - Series (Part 0)
- The first **written homework** is going to be due on **Friday, March 31, 2023**.

1.2 Objectives

- We want to be able to derive the concept of a series and a sequence.
- We want to be able to understand where the idea of a series and sequence come from, especially from seemingly-ordinary objects.
- We want to explore the idea of a limit in relation to a sequence.
- We want to be able to discretely express a sequence.

1.3 Motivation

In former calculus classes, we have observed the idea of limits, derivatives, and integrals at face value. We know how to evaluate these different calculations, but what exactly do they mean in the context of math? How can we better observe what exactly happens in these calculations, and understand them outside the context of visualizing graphs or projectile motion.

1.4 A third...

Chapter 2

10.1 (Part Two): Sequences (Part Two) (03/29/23)

2.1 Reminders

- MyLab Math Assignment 1 - **Sequences (Part 0)** is due **tonight, March 29, 2023**.
- Written Homework 1 is due **Friday, March 31, 2023** at the **beginning of class**.
- Friday's lesson is going to be over **sections 4.6 and 10.1** and we will be talking about **Newton's Method.**

2.1.1 Course Philosophy

Remember that the entire point of this class is to develop an intuition and a larger understanding and appreciation of calculus.

"Calculus is just algebra with a tiny drop of limits"

2.2 Motivation

In the last class, we got our first taste of sequences by exploring the idea of a third and eventually relating it to the **geometric sequence**, which results in the formula of

$$a_n = \frac{1}{1-x}$$

In this class, we were essentially taking our "preview" of sequences, actually defining different aspects of our sequence, doing operations on sequences, and finally, exploring one of the most important ideas of sequences, which are limit convergence divergence, which led us to the famous $\varepsilon - N$ proof, which is also known as the **precise definition of convergence.**

2.3 Sequences

Definition 1. *Sequences*

A function with a domain of natural numbers and a co-domain of real numbers.

$$f : \mathbb{N} \to \mathbb{R}$$

Meaning.

Although our intuition would tell us that a sequence is just a list or a collection of numbers, a sequence is more precisely just a function in which we input an **index** (a natural number) and we output a **term** (a real number). We, of course, then, collect these outputs, and this is what we generally see.

Example.

$$\{a_n\}$$

$$\{a_n\}_{n=1}^{\infty}$$

$$\{a_n\}_{n=0}^{\infty}$$

$$1, 2, 3, 4, \ldots$$

$$1.1, 2.2, 3.3, 4.4, \ldots$$

2.4 Convergence

Definition 2. *Sequence Convergence*

Informal Definition.
A sequence $\{a_n\}$ converges to a limit L if the terms get arbitrarily close to L as n gets sufficiently large, which is also known as

$$\lim_{n\to\infty} a_n = L$$

Formal Definition.
A sequence $\{a_n\}$ converges to a limit L if, for every $\varepsilon > 0$, where ε is the distance from the range to the limit L, there exists such a number that

$$|a_n - L| < \varepsilon \text{ for } n \geq N$$

The preceding expression is also known as the $\varepsilon - N$ proof.

2.5 Divergence

Definition 3. *Sequence Divergence*

Informal Definition.
A sequence diverges when it doesn't converge. If the sequence $\{a_n\}$ does not get arbitrarily close to limit L as n gets sufficiently large. This is also known as when the limit L **does not exist.**

$$\lim_{n \to \infty} a_n \text{ does not exist}$$

Formal Definition.
A sequence a_n diverges to (positive) infinity if, for every $M > 0$, there exists an N such that

$$a_n > M \text{ whenever } n > M$$

Additionally, a sequence a_n diverges to negative infinity, if, for every $M < 0$, there exists an N such that

$$a_n < M \text{ whenever } n > M$$

2.6 $\varepsilon - N$ Proof

In this proof, we are proving the existence of a limit when given a sequence a_n

2.7 Properties of Sequence Limits

Chapter 3

10.1: Sequences (Part 3) (03/31/23)

3.1 Summary of Lesson

In this lesson, we further develop the idea of convergence and divergence by **expanding it to non-elementary sequences**. We apply several new theorems as a result of this, such as the Sandwich Theorem. The homework, as a result, is all about convergence and divergence, testing on how well we are able to determine convergence and divergence given sequences including factorials, logarithms, and exponential functions.

3.2 Reminders

- The **third MyLab Math assignment** (Sequences (Part 2)) is due on **Tuesday, April 3rd**.
- The second written assignment is due on **Friday, April 7, 2023**.

3.3 Motivation

In the previous lesson, we learned about the **precise definition of convergence and divergence**. We learned about what exactly convergence and divergence means, and we were able to apply these concepts to various elementary sequences. However, what if we were given a sequence like

$$\{a_n\} = \frac{\cos(n)}{n}$$

Additionally, what if we were given a sequence like

$$\{a_n\} = \left(1 + \frac{x}{n}\right)^n$$

These sequences are undoubtedly more complex than our former examples and are not nearly as intuitive whenever it comes to actually solving them.

Therefore, we need to learn a few more **theorems** as well as **techniques** in order to determine convergence and divergence in more complex sequence functions.

Theorem 1. *Sandwich Theorem*

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to L} b_n = L$ also.

$$\text{let } \{a_n\}, \{b_n\}, \{c_n\} \text{ be sequences}$$

$$\text{if } a_n \leq b_n \leq c_n \text{ and}$$

$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} c_n = L, \text{ then}$$

$$\lim_{n\to\infty} b_n = L$$

Theorem 2. *The Continuous Function Theorem For Sequences (Theorem 3)*

Let $\{a_n\}$ be a sequence of real numbers $\mathbb{R}$. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n then $f(a_n) \to f(L)$.

Informal Definition.
Essentially, as long as we know that if a sequence a_n exists and approaches some limit L, we are able to say that if some function f is also continuous and the value $f(L)$ exists and that the function f is defined for all terms in the sequence a_n, then we know that the function with a domain of the terms of the sequence a_n, which we represent as $f(a_n)$ will approach the value of the function f at the limit L, which is also represented as the function $f(L)$.

3.3.1 Importance of the Continuous Function Theorem for Sequences

The implication of the continous function theorem for sequences, of course, is that we can derive smaller, more trivial functions from a larger function. For example, if we were trying to solve for

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}}$$

by using the **continuous function theorem for sequences**, we are able to separate this problem into a function, in which the domain is just a sequence a_n (or another function).

$$\text{let } f(x) = \sqrt{x}, \text{ let } x(n) = \frac{2n}{n+1}$$

Now, we are able to evaluate the limit of the inner function.

$$\lim_{n\to\infty} \frac{2n}{n+1} \Rightarrow \frac{2n/n}{n/n+1/n} \Rightarrow \frac{2}{1} \Rightarrow 2$$

Knowing that the domain of the function $f(x)$ will always approach 2, we are able to evaluate the outer limit.

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} \Rightarrow \lim_{n\to\infty} \sqrt{2} = \sqrt{2}$$

Theorem 3. *Theorem 4. (L'Hopital's Rule)*

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers $\mathbb{R}$ such that $a_n = f(n)$ for $n \geq n_0$. Then,

$$\lim_{n\to\infty} a_n = L \text{ wherever } \lim_{n\to\infty} f(x) = L$$

Warning. We cannot say the same about the converse of this theorem. That is, we cannot say that as long as a sequence approaches infinity, that a function f with a domain of the terms of the sequence a_n will converge at a a limit L. For example, imagine that if we had some high-degree polynomial expression and we had a sequence that converged at 0. The function itself might have several roots, as it will intersect the x-axis numerous times at zero. Ultimately however, the polynomial might not approach 0, but value of the function at the terms a_n might suggest it does.

Theorem 4. *Theorem 5. (Commonly Occuring Limits)*

The following six sequences converge to the following limits listed below:

1. $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1 \iff \lim_{n\to\infty} n^{1/n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1$ for all $(x > 0)$
4. $\lim_{n\to\infty} x^n = 0$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for all x
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

Chapter 4

4.6, 10.1: Finishing Sequences + Newton's Method (04/03/2023)

4.1 Reminders

- The **third MyLab Math assignment** (Sequences (Part 2)) is due on **Tuesday, April 3rd**.
- The second written assignment is due on **Monday, April 10th, 2023**.

4.2 Recall

In the previous section, remember how we essentially continued to enrich our understanding of sequences and limits. More specifically, we learned a few more theorems about sequence convergence and divergence.

First, let's start by actually talking about some patterns we see with the behavior of sequences. Recall that when we talk about sequences, we are essentially just talking about a function, in which the domain are the indices of the sequence (which indicate the position of the elements within the sequence) of which are natural numbers, that have a range of real numbers.

Convergence and divergence in a sequence refers to whether or not the sequence approaches some tangible, finite number as the domain of the sequence (the indices) approach infinity. This, of course, is the **informal definition of sequence convergence**.

The **formal definition of sequence convergence** states that, in order for a sequence to converge to some value L, there must exist some value ε after some point N in the sequence, in which all terms a_n of the sequence for $n > N$

satisfy

$$|a_n - L| < \varepsilon \text{ where } n > N$$

This basically states that, after some point in the sequence (denoted by the index N), all terms of the sequence a_n must fall within the range $(L-\varepsilon, L+\varepsilon)$, where ε, of course, represents any arbitrarily small number.

Likewise, we can state that a sequence **diverges** when this behavior doesn't occur, which makes sense, as if there exists no point in the sequence N where all consecutive terms all exist within some infinitesmially small range, then they simply cannot be approaching any number in the first place.

In addition to this basic idea of convergence, we also learn about some restrictions as well as some implications that arise from convergence.

For example, if we simply just want to determine whether or not a sequence converges, all we have to do is just solve for the limit of that sequence. We just treat the sequence a_n as some function $f(n)$ for all a_n. This, of course has a ton of different implications.

- The **Sandwich Theorem** states that, if we have three sequences $\{a_n\}\{b_n\}\{c_n\}$, and that the sequence a_n approaches L, and that the sequence c_n approaches L, if the sequence b_n is between a_n and c_n (that is, $a_n \leq b_n \leq c_n$), then $\lim_{n\to\infty} b_n = L$.

$$\text{let } \{a_n\}, \{b_n\}, \{c_n\}$$

$$\text{if } \lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n = L$$

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$$

$$\lim_{n\to\infty} b_n = L$$

- The **Continuous Function Theorem for Sequences** states that, if we have a function that is continuous for all values of $x \geq n_0$ (such that n_0 is the initial index of a sequence) and that there is a corresponding sequence a_n such that f is defined and continuous for all values a_n and is continuous at L, then we are able to say that the function of the domain a_n approaches the function value of the limit L.

  Essentially, from this informs us that if there is a function f that is defined for all values of a sequence a_n, as long as the a_n converges at a limit L, then function f of which its domain is an expression of na_n, then the function $f(a_n)$ must converge at $f(L)$. This allows us to state that if we are trying to find evaluate the limit of a function, we are able to dissect the function into an outer function $f(x)$ and an inner expression $x(n)$. The procedure from here is that we are able to find the limit of this inner function $x(n)$, substitute it as the domain of the outer function, then evaluate the final limit of the outer function.

- Additionally, like all other limits, we are able to evaluate the value at which a sequence converges to by using **L'Hopital's Rule**. The technical definition of L'Hopital's Rule states that if there exists a function $f(n) = a_n$, then the limit of the function $\lim_{n\to\infty} f(n) = L$ always implies that $\lim_{n\to\infty} a_n = L$. If we are able to evaluate the limit of the sequence as a limit of a function, then we are able to assert that the limit of the sequence is equal to the limit of that function. On the converse, however, we are unable to assert that if a function converges at a limit, then the sequence of that function will also converge at that limit. This all really just comes down to the idea that sequences are not functions.
- Finally, after this, there are just some important, well-known and already proven sequence convergences that I will be tested on. It is important to understand these for later, since they are pretty essential to solving many future sequence problems.

4.3 Motivation

For this lecture, we are literally just finishing up sequences and getting ready to move onto the next topic, series. We spent this class just learning about one final piece of the puzzle in series, which is **monotonicity** and the **monotonicity convergence theorem**.

4.4 Types of Sequences

Recall that through this section, we have thought about sequences in two different ways: **explicitly** and **recursively**.

4.4.1 Explicit Sequences

An explicit sequence is a sequence in which the value of the term of the sequence a_n is determined only by the index n. We don't need information about other terms in the sequence, as we are able to compute the value of any term of the sequence *solely* based on its position.

4.4.2 Examples of Explicit Sequences

$$a_n = n + 1,\ a_n = \frac{n}{2}$$

4.4.3 Recursive Sequences

A **recursive sequence** is a sequence in which the value of a term in the sequence is computed using the previous term of the sequence. It might be helpful to think of this as recursion in computer science, as we are literally just taking a previous output as input in this function. Generally, whenever we denote a recursive function, we will say that the term FOLLOWING the current term is equal to some expression of the current term.

4.4.4 Examples of Recursive Sequences

$$a_{n+1} = a_n + 1,\ a_{n+1} = \frac{a_n}{2}$$

4.5 Defining Divergence

Definition 4. *Sequence Divergence (Towards Infinity)*

A sequence $\{a_n\}$ diverges to positive infinity if for all M, there is an N such that $a_n > M$ for all $n > N$

Definition 5. *Sequence Divergence (Towards Negative Infinity)*

A sequence $\{a_n\}$ diverges to negative infinity if for all M, there is an N such that $a_n < N$ for all $n > N$.

When we read these definitions, we can imagine that the M is just some upper and lower bound or threshold. If, after some point in the sequence a_n, all subsequent terms a_n are all above or below a particular threshold, it makes sense that they would just approach infinity. This idea of divergence, however, all depends on the idea of **monotonicity**.

4.6 Monotonicity

Monotonicity defines a sequences general shape and behavior. A series is monotonous as long all of its values all pertain to some behavior.

4.6.1 Nondecreasing Monotonic Sequences

A **nondecreasing monotonic sequence** is a sequence in which, as the name suggests, the terms do not decrease. More precisely, we can say that

$$a_n \leq a_{n+1}$$

4.6.2 Nonincreasing Monotonic Sequences

A **nonincreasing monotonic sequence** is a sequence in which the terms do not increase. More precisely,

$$a_n \leq a_{n+1}$$

4.6.3 Monotonic Convergence

We can apply the idea of monotonicity, that is, the idea that a function bears only a single behavior into limits. In fact, there is actually an entire theorem dedicated to this idea.

Theorem 5. *Monotonic Convergence Theorem*

If a sequence

$$\{a_n\}$$

is both bounded and monotonic, then it converges.
Informal Definition.
For a sequence is a nonincreasing, that is $a_n \geq a_{n+1}$, if all terms after some point N are less than or equal to the greatest possible lower bound L, then we can say that the bounded, monotonicaly nonincreasing sequence a_n converges at that greatest possible lower bound L.
Likewise, for a sequence that is nondecreasing, or $a_n \leq a_{n+1}$, if all terms after some point N are less than or equal to the least possible upper bound L, then we can say that the bounded monotonically nondecreasing sequence a_n converges at L.

4.7 Newton's Method

TODO

4.8 Introduction to Series

A series is just an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

Imagine that we are just taking a sequence a_n and then we are just trying to find the sum of the terms a_n for all values of n.
This, howver, is particularly hard to compute, since after all ,if we were trying to find the sum of an infintie amount of termsl ike in the following series

$$\sum_{n=1}^{\infty} a_n$$

would be impossible. In order to mediate this, we have a few tricks in which we can actaully break up series in order to the actual sum of the series.

4.8.1 Sequence of Partial Sums

In the previous section, we just introduced the idea of a sequence. A sequence, of course its composed of its terms a_n as well as its indices n. For sake of argument, however, let us consider having a sequence in which each term is the sum of the current term and all of the terms preceding it. For reference

$$\text{let } \{a_n\}$$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Notice that the sequence of partial sums for n terms looks **suspiciously** simialr to the idea of infinite series... because it they're the same thing! Whenever we are thinking of a series, we can just contextualize it as just being a sequence of partial sums for the nth term.

This recontextualization of the series as just a sequence of partial sums to the nth term (as opposed to the first term of the second term), is very **powerful**, since it will grant us a beter understanding of how exactly the the series works, how to compute operations with it, as well as working with series convergence and divergence.

Definition 6. *Sequence of Partial Sums*

Given a sequence of numbers $\{a_n\}$, an expresssion of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an **infinite series**. The number a_n is the n**th term** of the series. The sequence $\{S_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series, the number s_n begin the n**th partial sum.** If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L.

Definition 7. *Sequence of Partial Sums (cont'd.)*

Intuition.
Previously, we talked about this idea of contextualizing a series as the sum of a sequence up until a certain point. For example,

$$\sum_{n=1}^{2} a_n = a_1 + a_2$$

$$\sum_{n=1}^{3} a_n = a_1 + a_2 + a_3$$

This is all pretty cool, but lets apply this idea of finding the series of every single term in a sequence a_n and then putting these series (which, of course, really only refers to the sum of the series) **within a sequence**.

$$\{S_n\} = S_1 + S_2 + S_3 + \cdots + S_n$$

$$\{S_n\} = \sum_{n=1}^{1} a_n + \sum_{n=1}^{2} a_n + \sum_{n=1}^{3} a_n + \cdots + \sum_{n=1}^{n} a_n$$

Note, the sequence of partial sums is **not equivalent** to the series itself. (After all, the sequence of partial sums is literally just a sequence and additionally, the series we are analyzing is just a term within the sequence). The sequence of partial sums is just a collection of all possible series up until our given series n. This is useful, because we can think of the value or sum of the series as being equal to the limit of the sequence of its partial sums. This idea of thinking of the value of a series as the limit of sequence of partial sums at some point n is extremely powerful and allows us to work within the framework of series convergence and divergence.

4.8.2 Methodology for solving for the sum of a series

In order to solve for the sum of a series, we again have to consider numerous factors:

- The series $\sum_{n=1}^{\infty} a_n$ can also be represented as the point of convergence or limit for the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{n=1}^{1} a_n + \sum_{n=1}^{2} a_n + \cdot + \sum_{n=1}^{n} a_n$$

- We know how to compute the limit L or point of convergence of a sequence, as well as how to determine whether not a sequence actually converges or not.

From this, we can consider the following steps:

1. Let the terms of the series be a sequence.
2. Evaluate the partial sums for the first few terms.
3. Determine the "pattern" between these terms.
    - The "pattern" here is eventually going to transform into the "function" of a sequence. We are going to determine a rule between the partial sums.
4. Let this pattern be a function/algorithm for computing future terms.
5. Let this pattern be the function of a sequence of partial sums $\{S_n\}$
6. Determine whether or not the sequence of partial sums $\{S_n\}$ converges or diverges. If the sequence converges, then determine at what point it converges to. The point at which the sequence of partial sums converges to, or the result of the series at some point n, is going to be known as the **sum** of that series.

4.9 Example of Series Sums

Example (1). *Determine the sum of the series*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Step 1: Let the terms of the series be a sequence.

$$\text{let } \{a_n\} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$$

Step 2: Evaluate the partial sums for the first few terms of the sequence (which we made through the terms of the series).

$$\text{let } \{S_n\} \text{ be the sequence of partial sums for } a_n$$

$$\{S_n\} = S_1, S_2, S_3, \cdots, S_n$$
$$S_1 = 1 = 1$$
$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$
$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$$

Example ((1) **cont'd**).

Step 3: Let the pattern in the sequence of partial sums be the function of the sequence of partial sums.

$$\text{let } S_n = \frac{2^n - 1}{2^{n-1}}$$

Step 4: Determine whether or not the sequence of partial sums converges or diverges. Find the limit if the sequence converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2^n}{2^{n-1}} - \lim_{n\to\infty} \frac{1}{2^{n-1}}$$

$$\Rightarrow 2 - 0$$

$$\lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = 2$$

Step 5: Let the limit or point of convergence of the sequence of partial sums be the sum of the series in question.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = 2$$

The sum of the series given by $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is 2.

4.9.1 Series Notation

Whenever we are trying to denote a sequence, we observe the following kinds of notation:

$$\text{let } a_n \text{ be some expression for } n$$

$$\sum_{n=1}^{\infty} a_n, \ \sum a_n$$

The left notation $\sum_{n=1}^{\infty} a_n$, of course, is more specific, letting us know what index we should start the series at as well as giving us a terminal point, which although strange, is just infinity in the case of the first example.

In the second case $\sum a_n$, we are just demonstarting that a_n is just a series, similar to how we just denote sequences with just curly braces $\{a_n\}$.

4.9.2 Introduction to Series Convergence and Series Divergence

Again, just to reiterate what we have learned in the past section, whenever we are rying to determine whether or not a series converges or diverges, we are essentially trying ot determine whether the sum or the evaluation of the series is going to be some finite number. We can always try to imagine this kind of like a Riemann sum beneath a curve

- Recall that whenever we were trying to solve a Riemann sum, we were always trying to estimate the area underneath a curve by using rectangles. In that same kind of sense, we are trying to see if the sum generated by the series, which is going to be one of these "rectangels" beneath the graph of a function of the series, is going to be finite.

So, in order to determine whether or not a series is convergent or divergent, we first have to transform the series into a sequence of partial sums, where the domain of course is going to be natural numbers that are determiend by the indicies of the sequence and the range is going to be determined by the sum of the sequence of hte function (which is distince from the sequence of partial sums) at that point.
Therefore, when the sequence of partial sums converges, this can be thought of as the series if the series just appproaches infinity.
Whenever we are trying to determien whether or not the series diverges, we just want to see if the sequence of partial sums that we obtain through the series actually converges or not. Whenever the sequence of partial sums does not converge, then it must diverge.

Chapter 5

10.2 (Part One): Infinite Series (Part One) (4/05/23)

5.1 Reminders

- The **fifth MyLab Math assignment: Infinite Series**, will be due on **Sunday, April 9th, 2023**.
- The second written assignment is due on **Monday, April 10th, 2023**.

5.2 Motivation

In the last section, we obtained a taste of what series were and how we are able to interpret them. Most prominently, we can imagine that a series can be considered as a term in a sequence, where all other terms in the sequence are just series up until a particular point. This concept is known as the **sequence of partial sums**. Whenever we are evaluating the sequence of partial sums, it is important that we understand this distinction.

Recall that we denote sequences with the following notation:

$$\sum_{n}^{k} a_n \text{ for the specific case}$$

$$\sum a_n \text{ for the general case}$$

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

This is how we represent series as partial sums.

$$\text{let } \{S_n\} \text{ be a sequence of partial sums}$$

$$\text{let } \{s_n\} \text{ be a function denoting the partial sum of } n$$

$$\{S_n\} = S_1, S_2, S_3, S_4, \cdots$$

$$\{S_n\} = \sum_{n=1}^{1} s_n,\ \ \sum_{n=1}^{2} s_n,\ \ \sum_{n=1}^{3} s_n,\ \ \sum_{n=1}^{4} s_n,\ \ \cdots,\ \ \sum_{n=1}^{n} s_n$$

Whenever we are trying to determine **series convergence and divergence**, we have to be able to contextualize or infinite series as the convergence point of a **sequence of partial sums**. The convergence point of the infinite series, therefore must be the convergence point of the sequence of partial sums. This, of course, does present us with many different things to consider whenever we are evaluating for partial sums. We have seen how we can evaluate a series whenever it is just given to us in the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

We basically think of these terms as the sequence a_n, then we just create separate sequence S_n that represents the partial sums of the terms of the sequence a_n. Then, we have to represent this sequence S_n as some function or algorithm. After we have derived an algorithm for the sequence of partial sums S_n, we can determine the convergence of the sequence of partial sums S_n by evaluating

$$\lim_{n\to\infty} S_n$$

By finding the point of convergence for the sequence of partial sums, we have determined both whether or not the series converges, but also where the series converges to. By evaluating a series, we have found its **sum**.

5.2.1 What's on the Menu

Now that we have learned the basic idea of what series are, we need to understand that not all series are this nice. Unfortuantely, series, much like sequences, can become much more complicated and complex in their terms, and for this reason, we also need to learn some new techniques, methodologies, and patterns for which to evaluate these series (that is, whenever we are evaluating for a series, we are really just determining their point of convergence / limit).

5.3 Commonly Seen Series

5.3.1 Geometric Series

Definition 8 (10.2 (1)). *Geometric Series*

Geometric series are series of the following form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as

$$\sum_{n=0}^{\infty} ar^n$$

Notation
The variable r is known as the **ratio** of the geometric series and a is known as the initial term.

- Of course, the reason why we call a the initial term of the geometric sequence is because whenever n is equal to the initial index of the series, 1

$$ar^{n-1} \Rightarrow ar^{1-1} \Rightarrow a$$

r can be either a positive or negative number, as observed in the two following series.

$$\text{let} \sum a_n, \text{ let} \sum b_n$$

$$\sum a_n = \sum_{n=1}^{\infty} \frac{1}{2}(1^n)$$

$$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2}((-1)^n)$$

5.3.2 Geometric Series Convergence and Divergence

Recall that whenever we are trying to determine whether or not a series converges or diverges, we are trying to contextualize the series as a term within a **sequence of partial sums**.

$$\text{let } \{S_n\} \text{ be a sequence of partial sums}$$

$$\text{let } s_n \text{ be a function denoting the partial sum at some index } n$$

$$\{S_n\} = s_1, s_2, s_3, s_4, \cdots, s_n$$

$$\{S_n\} = \sum_{n=1}^{1} s_n, \sum_{n=1}^{2} s_n, \sum_{n=1}^{3} s_n, \sum_{n=1}^{4} s_n, \cdots, \sum_{n=1}^{n} s_n$$

From this, we know that the limit of the sequence of partial sums $\sum_{n=1}^{\infty} s_n$ is equal to the sum of a series.

In the most general case of a geometric series, recall that we obtain the following form:

$$\text{let } \sum S_n \text{ be some geometric series (that is, some fucntion)}$$

$$\text{let the function of } \sum S_n \text{ be } ar^{n-1}$$

$$\sum S_n = S_1 + S_2 + S_3 + S_4 + \cdots + S_n \textbf{ Write the } n\textbf{th partial sum}$$

$$\Rightarrow \sum S_n = ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + \cdots a^{n-1}$$

$$\Rightarrow \sum S_n = ar^0 + ar^1 + ar^2 + ar^3 + \cdots a^{n-1}$$

$$\Rightarrow \sum S_n = a + ar + ar^2 + ar^3 + \cdots a^{n-1}$$

$$\Rightarrow r\sum S_n = r(ar^{1-1}+ar^{2-1}+ar^{3-1}+ar^{4-1}+\cdots ar^{n-1}) \textbf{ Multiply both sides by } r$$

$$\Rightarrow S_n - r\sum S_n = (a+ar+ar^2+ar^3+\cdots+ar^{n-1}) - (ar+ar^2+ar^3+ar^4+\cdots+ar^n)$$

Here, we now subtracting $r\sum S_n$ from S_n. We just introduce S_n.

$$S_n - r\sum S_n = a - ar^n$$

Notice all terms except for the initial term a and the terminating term a^n are left. Let's factor both sides now.

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{(1-r)} \text{for all } r \text{ where } r \neq 1$$

Now that we have a general function to solve for the sum of the geometric series $\sum S_n$, let us try to determine what values of r that the geometric sequences converges and diverges.

$$\text{let } \sum S_n \text{ be geomtric series}$$

$$S_n = ar^{n+1}$$

$$\Rightarrow S_n = S_1 + S_2 + S_3 + \cdots + S_n$$

$$\Rightarrow S_n = ar^{1-1} + ar^{2-1} + ar^{3-1} + \cdots + ar^{n-1}$$

$$\Rightarrow S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$\Rightarrow r(S_n) = r(a + ar + ar^2 + \cdots + ar^{n-1})$$

$$\Rightarrow r(S_n) = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$\Rightarrow S_n - r(S_n) = S_n - (ar + ar^2 + ar^3 + \cdots + ar^n)$$

$$\Rightarrow (a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$

$$\Rightarrow S_n - r(S_n) = a - ar^n$$

$$\Rightarrow S_n(1 - r) = a(1 - r^n)$$

$$\Rightarrow S_n = \frac{a(1 - r^n)}{(1 - r)} \text{ for all } r \neq 1$$

This formula $S_n = \frac{a(1-r^n)}{(1-r)}$ represents the sum or value of any geometric series. However, how can we use this formula to determine the convergence and divergence of a geometric series?

5.3.3 Convergence and Divergence of the Geometric Series

Let us observe the behavior of the sum of a geometric series.

Recall.

The sum of a geometric series is equal to the following equation:

$$\sum S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Definition.

$$\text{let } \sum S_n \text{ be a geometric series}$$

$$S_n = \sum_{n=1}^{\infty} ar^{n-1}$$

In order to determine the convergence of a geometric series, we must determine the limit of the sum of the geometric series.

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{(1 - r)}$$

Let us look at the dominate term of the function, r_n. Observe that when the absolute value of r is greater than one $|r| > 1$, the term r^n will approach infinity. However, whenever the absolute value of r is less than 1 $|r| < 1$, r^n will just approach 0, since all values less than one will only get smaller when exponentiated.

Therefore, we know that for the geometric series $\sum S_n$, its value or sum diverges when the ratio r is greater than one.

$$S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \infty \text{ when } |r| > 1$$

On the other hand, whenever the value of the ratio r is less than one or $|r| < 1$, we know that the ratio itself is only going to get smaller and smaller, eventually approaching 0, meaning that the value of the infinite series will converge.

$$S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} \text{ for } |r| < 1$$

$$\Rightarrow S_n = \lim_{n\to\infty} \frac{a(1-0)}{(1-r)} = \frac{a}{(1-r)}$$

5.3.4 Geometric Series Examples

Chapter 6

10.2 (Part Two): Infinite Series (Part Two) (04/07/23)

6.1 Reminders

- The **fifth MyLab Math assignment: Infinite Series**, will be due on **Sunday, April 9th, 2023**.
- The second written assignment is due on Monday, April 10th, 2023.

6.2 Summary

In this lecture, we essentially expanded on what we learned about series in the previous lecture. Recall how we learned about geometric series, their formation, as well as how we determine whether or not they converge or diverge. Most importantly, we also learned how to manipulate the values of a series in order to determine an equation for finding the sum of a geometric series.

All of this culminates into today's lesson, which is all about determinign the convergence of other types of series.

6.3 Recall

Over our introduction and talks about series, we have gone over different methods and algorithms for determining the value of a sequence as well as determining convergence and divergence.

6.3.1 Convergence and Divergence in Series

Remember, convergence and divergence in regards to series refers to the sum of the series as the terms of the series approach infinity. This can be represented through the following notation:

$$\sum_{n=1}^{\infty} S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

Given this general form of a how a series works, we can actually recontextualize the terms of the series as a sequence

$$\{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_n$$

This is important because we can actually create a new separate sequence $\{S_n\}$, where each term is a series up until that term in the sequence

$$\{S_n\} = S_1, S_2, S_3, S_4, \ldots, S_n$$

$$\{S_n\} = (a_1), (a_1 + a_2), (a_1 + a_2 + a_3), \ldots, (a_1 + a_2 + a_3 + \cdots + a_n)$$

This final term $S_n = (a_1 + a_2 + a_3 + \cdots + a_n)$ is equivalent to the series $\sum S_n$. The value of a series is equivalent to the nth term of a sequence of partial sums. In order to evaluate the value and the series convergence, then, is to just determine the convergence of the sequence of partial sums. The point at which the sequence of partial sums converges to is equivalent to the sum of the series.

Now, this pattern of course works for trivial and elementary series, since all it requires is the ability to find the pattern between the terms in the sequence of partial sums. We can just apply sequence convergence methods and tricks in order to compute this. But, what happens when we want to determine if a more complex sequence converges or diverges? What if, for example, we had a geometric sequence?

A geometric sequence is a sequence that occurs in the following form

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

A geometric series, thankfully, has a general equation that allows us to compute its sum and value.

$$\text{let } S_n \text{ be the value of a geometric sequence}$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Using this equation, we can deduce that all values of the ratio r that are greater than 1, or $|r| > 1$ are going to **diverge**, while all values of the ratio r that are less or equal to 1, or $|r| \leq 1$ are going to **converge** at the following value:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \frac{a}{(1-r)} \text{ when } |r| \leq 1$$

Otherwise, all other geometric sequences will **diverge.**

6.4 Motivation

Given now that we know how a general algorithm for computing the value and convergence/divergence of a series as well as how to find series convergence and divergence given a particular type of series, let us learn some new techniques in order to compute the value of a series as well as series convergence and series divergence.

6.5 Partial Fraction Decomposition for Series

TODO

6.6 Telescoping Series

TODO

6.7 nth-term Divergence Test

How can we actually determine when a series converges

Theorem 6. *Theorem 7*

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } a_n \text{ approaches } 0.$$

Warning.
We cannot say the same for the **converse** of this. Just because i

Theorem 7. *The nth-term divergence test*

The series

$$\sum_{n=1}^{\infty} a_n$$

diverges if the limit of the function of the series function a_n fails to exist or is not equal to 0.

$$\sum_{n=1}^{\infty} \text{ converges when } \lim_{n\to\infty} a_n \neq 0 \text{ or does not exist.}$$

6.8 Combining Series

Now that we have learned some general ideas about series, let us talk about some operations we can do with series.

Theorem 8 (8)**.** *Combining Series Theorem (Theorem 8)*

$$\text{Let } \sum a_n = A \text{ and } \sum b_n = B \text{ be convergent series}$$

- Sum Rule:
$$\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$$
- Difference Rule:
$$\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$$
- Constant Multiple Rule:
$$\sum ka_n = k\sum a_n = kA \text{ for all values of } k$$

Proof. Series Sum Rule

$$\text{let} \sum a_n = A \text{ and} \sum b_n = B$$

$$A = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$B = b_1 + b_2 + b_3 + b_4 + \cdots + b_n$$

$$\text{let } S_n \text{ be equal to} \sum (a_n + b_n)$$

$$S_n = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$

$$\Rightarrow S_n = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots (a_n + b_n)$$

$$\Rightarrow S_n = A_n + B_n$$

Proof. Series Difference Rule

$$\text{let} \sum a_n = A \text{ and} \sum b_n = B$$

$$A = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$B = b_1 + b_2 + b_3 + b_4 + \cdots + b_n$$

$$\text{let } S_n \text{ be equal to} \sum (a_n - b_n)$$

$$S_n = \sum a_n - \sum b_n$$

$$S_n = (a_1 + a_2 + a_3 + \cdots + a_n) - (b_1 + b_2 + b_3 + \cdots + b_n)$$

$$S_n = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots + (a_n - b_n)$$

$$S_n = A - B$$

Proof. Series Constant Multiple Rule

$$\text{let} \sum a_n = A \text{ and } k \text{ be some constant}$$

$$\sum a_n = A = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\text{let } S_n \text{ be equal to} \sum (k \cdot a_n)$$

$$S_n = k \cdot a_1 + k \cdot a_2 + k \cdot a_3 + \cdots + k \cdot a_n$$

$$S_n = k(a_1 + a_2 + a_3 + \cdots + a_n)$$

$$S_n = k \cdot A = k \sum a_n$$

Chapter 7

10.3: The Integral Test (04/05/23)

7.1 Reminders

- The fifth **MyLab Math: Infinite Series** is due on **Sunday, April 9th, 2023**.
- The second **Written Homework: Infinite Series** is due on **Monday, April 10th, 2023**.

7.2 Motivation

In the previous section, we expanded our knowledge and understanding of infinite series convergence and divergence by observing some other types of infinite series as well as how to approach them.

- Namely, we learned about the **nth term divergence test**, **combining series**, as well as other types of series manipulation, such as **telescoping series** and **partial fraction decomposition**.

7.2.1 Recap: nth term divergence test

TODO

7.2.2 Recap: Combining Series

TODO

7.2.3 Recap: Telescoping Series

TODO

7.2.4 Recap: Partial Fraction Decomposition

TODO

7.3 Integral Test

Chapter 8

10.4: Comparision Tests (04/07/23)

8.1 Reminders

- The fifth **MyLab Math: Infinite Series** is due on **Sunday, April 9th, 2023**.
- The second **Written Homework: Infinite Series** is due on **Monday, April 10th, 2023**.

8.2 Motivation

In the previous lesson, we discussed a technique known as the **integral test for series convergence.** Essentially, we use this technique to determine whether or not a given series converges or diverges by comparing it to its integral. After all, whenever we do have a series, we can always think about it like a **Riemann Sum**, or at least, some botched version of a Riemann Sum. The function created by the series, however, can also be graphed as a function. What we end up doing is just graphing both the series as a series as well as the series as a function and then compare their behavior. If we can determine that the integral of the series as a function is finite, then we can reasonably say that the limit of the series must also be finite, and therefore must converge.

8.3 The Comparison Test

8.3.1 Intuition

Whether or not a sequence converges or diverges depends on its **tail**, or the movement of the sequence as the index n approaches infinity, which is also deonted as $n \to \infty$.

Definition 9. *The Comparison Test*

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=k}^{\infty} a_k$ converges, where k is some value after n

We can think of the difference between these two sums as

$$\sum_{n=1}^{\infty} a_n - \sum_{n=k}^{\infty} a_n = \sum_{n=1}^{k} a_n$$

since we can think of the resulting sum as the series terms a_n between n and k, or $1 < n < k$.
From this, we know that if

$$0 \leq a_n \leq b_n \ \forall n, \text{ then}$$

1. If $\sum b_n$ converges, then this implies $\sum a_n$ also converges
2. If $\sum a_n$ diverges, then this implies that $\sum b_n$ also diverges (to ∞)

8.3.2 Comparison Test Proof

Example. *Assume that $\sum a_n$ diverges to $+\infty$. Show that $\sum b_n$ also diverges to $+\infty$.*

$$0 \leq a_n \leq b_n$$

$$\text{let } A_n = \sum_{k=1}^{n} a_k \text{ and } B_n = \sum_{k=1}^{n} b_k.$$

We must demonstrate that $\forall M, \exists N$, such that $B_n > M, \forall n > N$. (We must demonstrate that for all M, there exists such a value N that $B_n > M$ for all values of $n > N$, where M is some lower bound of B_n and n and N are indices of the series).

8.4 P-Series Test

TODO

8.5 Limit Comparison Test

TODO

Chapter 9

10.5: Absolute Convergence and the Ratio Test

9.1 Reminders (MATH_226)

- There are practice pracice problems on **MyLab Math** that have unalimited attempts and no due date.
- **MyLab Math 6: Integral Test** and **MyLab Math 7: Comparison Tests** are due on **Tuesday, April 11, 2023** and **Thursday, April 13, 2023**, respectively.
- There is a written homework that is due at the beginning of class on **Friday, April 14, 2023**.
- **MATH_226 Midterm 1** is on **Tuesday, April 18, 2023**

9.2 Reminders (MATH_230-1)

- **MyLab Math 6: Planes in Space** is due on **Thursday, April 13, 2023**.
- **Written Homework 1** is due on **Wednesday, April 12, 2023**.

9.3 Motivation

In the previou sclasses we have learned about when a series converges as well as how to test for that. However, in order to supplmeent our learning we need to learn about how to build functions from series

$$1 + x + x^2 + \cdots \Rightarrow \frac{1}{1-x},\ |x| \leq 1$$

9.4 Direct Comparison Test (Part 2)

Definition 10. *Direct Comparison Test*

TODO

9.5 Limit Comparison Test

9.5.1 Intuition

Recall the **Constant Multiple Rule for Sequences**, which essentially stated that if there were two sequences $\{a_n\}$ and $\{ca_n\}$, where c is just a constant, then we know that, although $\{ca_n\} \neq \{a_n\}$, both sequences would just have the same behavior, since they'd basically just be the same sequences, just with different scales.

We can, of course, just apply this to series, since the series

$$\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$$

$$\Rightarrow \sum a_n\text{'s behavior} \Rightarrow c \sum a_n\text{'s behavior}$$

The idea here is that, what if we just thought of two sequences or series as scaled versions of the other. What if we could determine that scale? What would the scale imply?

Introduce the **Limit Comparison Test**.

Definition 11. *Limit Comparison Test*

Suppose $a_n > 0$ and $b_n > 0$ for all $n > N$.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \begin{cases} C > 0, \text{ then } \sum a_n\text{'s behavior} \Leftrightarrow \sum b_n\text{'s behavior} \\ 0, \text{ then } \sum b_n\text{'s conv.} \Rightarrow \text{implies } \sum a_n\text{'s conv.} \\ \infty, \text{ then } \sum a_n\text{'s conv.} \Rightarrow \text{implies } \sum b_n\text{'s conv.} \end{cases}$$

The contrapositive is also true,

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \begin{cases} C > 0, \text{ then } \sum a_n\text{'s behavior} \Leftrightarrow \sum b_n\text{'s behavior} \\ 0, \text{ then } \sum a_n\text{'s div.} \Rightarrow \text{implies } \sum b_n\text{'s div.} \\ \infty, \text{ then } \sum b_n\text{'s div.} \Rightarrow \text{implies } \sum a_n\text{'s div.} \end{cases}$$

9.5.2 Intuition (Part 2.)

The main idea here is that, we want to analyze the ratio of a_n and b_n to determine which function is **greater than the other**. If some function is less than or equal to another function, and the greater function converges, then the lesser function must also converge logically. Vice versa, if the lesser function diverges, then the greater function must also diverge logically.

9.6 Absolute Convergence Test

Definition 12. *Absolute Convergence*

9.7 The Ratio Test and the Root Test

9.7.1 Underlying Intuition...

Recall that whenever we started to learn about the geometric series, we obsessed over the following notation

$$\sum_{n=1}^{\infty} a \cdot r^n$$

We talked about the idea of the varialbe r as a ratio that was **very related** to the series' convergence and divergence behavior.

- More specifically, as long as the ratio $|r| \leq 1$, then the value r^n would eventually converge to 0, which meant that the function $\frac{a(1-r^n)}{1-r}$ would converge.
- The contrapositive to this, that $|r| \geq 1$ meant that the value r^n would **diverge**.

This same idea applies here. If we were given a geometric series, for example, and we wanted to isolate this ratio r, there are a number of ways to do so.

We are able to isolate the variable r be analyzing **successive terms** with the following function:

$$\text{let } \sum a_n \text{ be a geometric series}$$

$$\Rightarrow \frac{ar^{n+1}}{ar^n} = r$$

as well as by isolating the variable r by analyzing the **nth root** of any term in the series:

$$\text{let } \sum a_n \text{ be a geometric series}$$

$$\Rightarrow \sqrt[n]{ar^n} = r$$

To take this a step further, let us take this discussion about the variable r outside the realm of purely geometric series. We are able to actually apply these ratios to non-specifically geometric functions, such as functions with fractions as well as factorials. The idea is that it does not matter whether or not there is some consistent ratio between all terms, but rather, that there is some ratio that we approach **at a limit**.

From this, we can observe the following equations

9.7.2 Ratio Test

Essentially, we want ot be able to analyze the ratio r in a series of the form

$$\sum_{n=1}^{\infty} ar^n$$

We want to analyze the r, since we can recall that as long as $|r| < 1$, then the series will converge.

Chapter 10

10.6: Alternating Series and Conditional Convergence (04/12/23)

10.1 Reminders

- **MyLab 7: Comparison Tests** is due **tomorrow, Thursday, April 13, 2023**.
- **Written Assignment 3: Comparison Tests** is due on **Friday, April 14, 2023**.

10.2 Objectives

TODO

10.3 Recall

TODO

10.4 Motivation

But what if we see a series that comes in this form:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

or this?

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots$$

As we will see, these are known as **Alternating Series**, and are particularly notable series because they have a particular methdology of assessing convergence and divergence.

10.5 Alternating Series Test

Theorem 9. *Alternating Series Test (Theorem 15)*

The series denoted as

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges **if and only if** the following conditions are satisfied:

1. All values of u_n are positive
2. All values of u_n are **eventually nonincreasing**, meaning that $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N.
3. The values u_n eventually approach 0.

10.5.1 Alternating Series Intuition

Whenever we are working with alternating series, it is important to think of u_n as the terms, but rather, the partial sums of the terms at n. We want to emphasize the differences between u_n and u_{n+1}, thinking about the **difference between the partial sums** u_n.

If we were geometrically observing the values of u_n, we would imagine that the partial sums of the series are narrowing down, almost like a funnel. The idea here is that, if were were to calculate all partial sums of the series u_n, we would find that they all approach a single value, and therefore, converge. We need the values of u_n (the difference between the partial sums of a_n and a_{n+1}) to be positive because TODO.

The difference between the partial sums u_n and u_{n+1} also needs to be decreasing, forming this "cone" shape. If the differences between the partial sums is decreasing, then the difference will eventually approach 0, then we can state that the actual sum of the series will stop at some limit L.

TODO: Insert Graphic displaying the difference

Corollary .1. *Alternating Harmonic Series Divergence*

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is **conditionally convergent**, since we know that although the alternating harmonic series does diverge by using the **alternating series test**, we know that the absolute valu eof the alternating harmonic series is equal to the harmonic series, which we know diverges by the integral test.

$$\sum_{n=1}^{\infty} \frac{|(-1)^{n+1}|}{|n|} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \textbf{ which diverges}$$

Theorem 10. *Alternating Series Estimation Theorem (by Prof. Zaslow)*

Whenever a series is convegent by the **Alternating Series Test**, then we must know that the absolute value of the remainder $|R_n| \leq u_{n+1}$ for all $n \geq N$, where $|R_n|$ represents the difference between the partial sums u_n of the series.

$$|R_n| \leq u_{n+1}$$

10.5.2 Alternating Series Proof

TODO

10.5.3 Examples

Example (1). *Alternating Harmonic Series*

Example (2). *Non-Increasing Example*

10.6 Alternating Series Estimation Theorem

Theorem 11. *Alternating Series Estimation Theorem (Theorem 16)*

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

satisfies all three conditions of the **Alternating Series Test (Theorem 15)**, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolutye value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1} and the remainder, $L - s_n$ has the same sign as the first unused term.

10.6.1 Alternating Series Estimation Theorem Intuition

TODO

10.6.2 Examples

Example (**3**).

10.7 Conditional Convergence

Definition 13. *Conditional Convergence*

A series that is convergent but not **absolutely convergent** is called **conditionally convergent**.

10.7.1 Conditional Convergence Intuition

TODO

10.8 Rearranging Series

Theorem 12. *Rearrangement Theorem for Absolutely Convergent Series (Theorem 17)*

If

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely, and the sequene

$$b_1, b_2, b_3, \cdots, b_n, \cdots$$

is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

10.8.1 Rearranging Series Intuition

10.9 Summary of Tests to Determine Convergence or Divergence

1. The nth-Term Test for Divergence
2. Geometric Series
3. p-series
4. Series with nonnegative terms
5. Series with some negative terms
6. Alternating Series

Chapter 11

10.6: Strategies for Analyzing Convergence

11.1 Reminders

11.2 Motivation

Chapter 12

10.7 (Part One): Power Series

12.1 Reminders

12.2 Motivation

Chapter 13

10.7 (Part Two): Radius and Interval of Convergence

13.1 Reminders

13.2 Motivation

Chapter 14

10.7 (Part Three): Manipulation of Series (Part One)

14.1 Reminders

14.2 Motivation

Chapter 15

10.7 (Part Four): Manipulation of Series (Part Two)

15.1 Reminders

15.2 Motivation

Chapter 16

10.8 (Part One):

16.1 Reminders

16.2 Motivation

Chapter 17

10.8 (Part Two):

17.1 Reminders

17.2 Motivation

Chapter 18

10.9: Convergence of Taylor Series

18.1 Reminders

18.2 Motivation

Chapter 19

10.10: Applications of Taylor Series

19.1 Reminders

19.2 Motivation

Chapter 20

A7 (Part One): Complex Numbers (Part One)

20.1 Reminders

20.2 Motivation

Chapter 21

10.10, A7 (Part Two): Complex Numbers (Part Two)

21.1 Reminders

21.2 Motivation

Chapter 22

19.1 (Part One): Vectors

22.1 Reminders

22.2 Motivation

Chapter 23

19.1 (Part Two): Inner Products

23.1 Reminders

23.2 Motivation

Chapter 24

19.2 (Part One): Functions as Vectors, Periodic Functions

24.1 Reminders

24.2 Motivation

Chapter 25

19.2 (Part Two): Fourier Series, Demos

25.1 Reminders

25.2 Motivation

Chapter 26

19.3: Fourier Series, Theory

26.1 Reminders

26.2 Motivation

Chapter 27

19.5 (Part One): Applications (Part One)

27.1 Reminders

27.2 Motivation

Chapter 28

19.5 (Part Two): Applications (Part Two)

28.1 Reminders

28.2 Motivation